Machine Words in Isabelle/HOL

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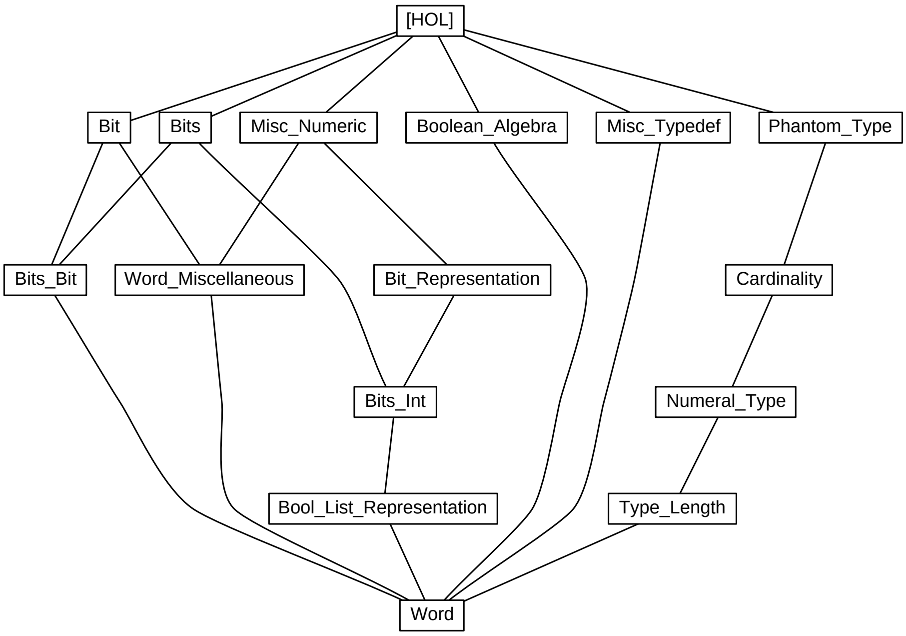
Abstract

A formalisation of generic, fixed size machine words in Isabelle/HOL. An earlier version of this formalisation is described in [1].

Contents


[HOL]
Bit
Bits
Misc_Numeric
Boolean_Algebra
Misc_Typedef
Phantom_Type
Bits_Bit
Word_Miscellaneous
Bit_Representation
Cardinality
Bits_Int
Numeral_Type
Bool_List_Representation
Type_Length
Word

1 A generic phantom type

```
theory Phantom-Type
imports Main
begin

datatype ('a, 'b) phantom = phantom (of-phantom: 'b)

lemma type-definition-phantom': type-definition of-phantom phantom UNIV
by(unfold-locales) simp-all

lemma phantom-comp-of-phantom [simp]: phantom ∘ of-phantom = id
  and of-phantom-comp-phantom [simp]: of-phantom ∘ phantom = id
by(simp-all add: o-def id-def)

syntax -Phantom :: type ⇒ logic ((1Phantom/(1'(-'))))
translations
  Phantom('t) => CONST phantom :: - ⇒ ('t, -) phantom

typed-print-translation ‹
  let
    fun phantom-tr' ctxt (Type (@{type-name fun}, [-, Type (@{type-name phantom}, [T, -])])) ts =
          list-comb
            (Syntax.const @{syntax-const -Phantom} $ Syntax-Phases.term-of-typ ctxt T, ts)
      | phantom-tr' - - - = raise Match;
  in [(@{const-syntax phantom}, phantom-tr')] end
›

lemma of-phantom-inject [simp]:
  of-phantom x = of-phantom y ⟷ x = y
by(cases x y rule: phantom.exhaust[case-product phantom.exhaust]) simp

end
```

2 Cardinality of types

```
theory Cardinality
imports Phantom-Type
begin
```

2.1 Preliminary lemmas

```
lemma (in type-definition) univ:
  UNIV = Abs ‘ A
proof
  show Abs ‘ A ⊆ UNIV by (rule subset-UNIV)
  show UNIV ⊆ Abs ‘ A
```

proof
fix *x* :: *'b*
have *x* = *Abs* (*Rep x*) **by** (*rule Rep-inverse* [*symmetric*])
moreover have *Rep x* ∈ *A* **by** (*rule Rep*)
ultimately show *x* ∈ *Abs* ' *A* **by** (*rule image-eqI*)
qed
qed

lemma (**in** *type-definition*) *card*: *card* (*UNIV* :: *'b set*) = *card A*
by (*simp add*: *univ card-image inj-on-def Abs-inject*)

lemma *finite-range-Some*: *finite* (*range* (*Some* :: *'a* ⇒ *'a option*)) = *finite* (*UNIV* :: *'a set*)
by(*auto dest*: *finite-imageD intro*: *inj-Some*)

lemma *infinite-literal*: ¬ *finite* (*UNIV* :: *String.literal set*)
proof −
have *inj STR* **by**(*auto intro*: *injI*)
thus *?thesis*
by(*auto simp add*: *type-definition.univ*[*OF type-definition-literal*] *infinite-UNIV-listI dest*: *finite-imageD*)
qed

2.2 Cardinalities of types

syntax *-type-card* :: *type* => *nat* ((*1CARD*/(*1'*(-'))))

translations *CARD*(*'t*) => *CONST card* (*CONST UNIV* :: *'t set*)

print-translation ‹
let
fun card-univ-tr' ctxt [*Const* (@{*const-syntax UNIV*}, *Type* (-, [*T*]))] =
Syntax.const @{*syntax-const -type-card*} $ *Syntax-Phases.term-of-typ ctxt T*
in [(@{*const-syntax card*}, *card-univ-tr'*)] *end*
›

lemma *card-prod* [*simp*]: *CARD*(*'a* × *'b*) = *CARD*(*'a*) ∗ *CARD*(*'b*)
unfolding *UNIV-Times-UNIV* [*symmetric*] **by** (*simp only*: *card-cartesian-product*)

lemma *card-UNIV-sum*: *CARD*(*'a* + *'b*) = (*if CARD*(*'a*) ≠ *0* ∧ *CARD*(*'b*) ≠ *0 then CARD*(*'a*) + *CARD*(*'b*) *else 0*)
unfolding *UNIV-Plus-UNIV*[*symmetric*]
by(*auto simp add*: *card-eq-0-iff card-Plus simp del*: *UNIV-Plus-UNIV*)

lemma *card-sum* [*simp*]: *CARD*(*'a* + *'b*) = *CARD*(*'a*::*finite*) + *CARD*(*'b*::*finite*)
by(*simp add*: *card-UNIV-sum*)

lemma *card-UNIV-option*: *CARD*(*'a option*) = (*if CARD*(*'a*) = *0 then 0 else CARD*(*'a*) + *1*)

```
proof −
  have (None :: 'a option) ∉ range Some by clarsimp
  thus ?thesis
    by (simp add: UNIV-option-conv card-eq-0-iff finite-range-Some card-image)
qed

lemma card-option [simp]: CARD('a option) = Suc CARD('a::finite)
by(simp add: card-UNIV-option)

lemma card-UNIV-set: CARD('a set) = (if CARD('a) = 0 then 0 else 2 ^ CARD('a))
by(simp add: Pow-UNIV[symmetric] card-eq-0-iff card-Pow del: Pow-UNIV)

lemma card-set [simp]: CARD('a set) = 2 ^ CARD('a::finite)
by(simp add: card-UNIV-set)

lemma card-nat [simp]: CARD(nat) = 0
  by (simp add: card-eq-0-iff)

lemma card-fun: CARD('a ⇒ 'b) = (if CARD('a) ≠ 0 ∧ CARD('b) ≠ 0 ∨
CARD('b) = 1 then CARD('b) ^ CARD('a) else 0)
proof −
  { assume 0 < CARD('a) and 0 < CARD('b)
    hence fina: finite (UNIV :: 'a set) and finb: finite (UNIV :: 'b set)
      by(simp-all only: card-ge-0-finite)
    from finite-distinct-list[OF finb] obtain bs
      where bs: set bs = (UNIV :: 'b set) and distb: distinct bs by blast
    from finite-distinct-list[OF fina] obtain as
      where as: set as = (UNIV :: 'a set) and dista: distinct as by blast
    have cb: CARD('b) = length bs
      unfolding bs[symmetric] distinct-card[OF distb] ..
    have ca: CARD('a) = length as
      unfolding as[symmetric] distinct-card[OF dista] ..
    let ?xs = map (λys. the o map-of (zip as ys)) (List.n-lists (length as) bs)
    have UNIV = set ?xs
    proof(rule UNIV-eq-I)
      fix f :: 'a ⇒ 'b
      from as have f = the ∘ map-of (zip as (map f as))
        by(auto simp add: map-of-zip-map)
      thus f ∈ set ?xs using bs by(auto simp add: set-n-lists)
    qed
    moreover have distinct ?xs unfolding distinct-map
    proof(intro conjI distinct-n-lists distb inj-onI)
      fix xs ys :: 'b list
      assume xs: xs ∈ set (List.n-lists (length as) bs)
        and ys: ys ∈ set (List.n-lists (length as) bs)
        and eq: the ∘ map-of (zip as xs) = the ∘ map-of (zip as ys)
      from xs ys have [simp]: length xs = length as length ys = length as
        by(simp-all add: length-n-lists-elem)
      have map-of (zip as xs) = map-of (zip as ys)
```

proof
fix *x*
from *as bs* **have** $\exists\, y.\ map\text{-}of\ (zip\ as\ xs)\ x = Some\ y$ $\exists\, y.\ map\text{-}of\ (zip\ as\ ys)\ x = Some\ y$
by(*simp-all add*: *map-of-zip-is-Some*[*symmetric*])
with *eq* **show** $map\text{-}of\ (zip\ as\ xs)\ x = map\text{-}of\ (zip\ as\ ys)\ x$
by(*auto dest*: *fun-cong*[**where** $x{=}x$])
qed
with *dista* **show** $xs = ys$ **by**(*simp add*: *map-of-zip-inject*)
qed
hence $card\ (set\ ?xs) = length\ ?xs$ **by**(*simp only*: *distinct-card*)
moreover have $length\ ?xs = length\ bs\ \hat{}\ length\ as$ **by**(*simp add*: *length-n-lists*)
ultimately have $CARD('a \Rightarrow 'b) = CARD('b)\ \hat{}\ CARD('a)$ **using** *cb ca* **by** *simp* **}**
moreover {
assume *cb*: $CARD('b) = 1$
then obtain *b* **where** *b*: $UNIV = \{b :: 'b\}$ **by**(*auto simp add*: *card-Suc-eq*)
have *eq*: $UNIV = \{\lambda x :: 'a.\ b :: 'b\}$
proof(*rule UNIV-eq-I*)
fix $x :: 'a \Rightarrow 'b$
{ fix *y*
have $x\ y \in UNIV$ **..**
hence $x\ y = b$ **unfolding** *b* **by** *simp* **}**
thus $x \in \{\lambda x.\ b\}$ **by**(*auto*)
qed
have $CARD('a \Rightarrow 'b) = 1$ **unfolding** *eq* **by** *simp* **}**
ultimately show *?thesis*
by(*auto simp del*: *One-nat-def*)(*auto simp add*: *card-eq-0-iff dest*: *finite-fun-UNIVD2 finite-fun-UNIVD1*)
qed

corollary *finite-UNIV-fun*:
$finite\ (UNIV :: ('a \Rightarrow 'b)\ set) \longleftrightarrow$
$finite\ (UNIV :: 'a\ set) \wedge finite\ (UNIV :: 'b\ set) \vee CARD('b) = 1$
(**is** $?lhs \longleftrightarrow ?rhs$)
proof −
have $?lhs \longleftrightarrow CARD('a \Rightarrow 'b) > 0$ **by**(*simp add*: *card-gt-0-iff*)
also have $\ldots \longleftrightarrow CARD('a) > 0 \wedge CARD('b) > 0 \vee CARD('b) = 1$
by(*simp add*: *card-fun*)
also have $\ldots = ?rhs$ **by**(*simp add*: *card-gt-0-iff*)
finally show *?thesis* **.**
qed

lemma *card-literal*: $CARD(String.literal) = 0$
by(*simp add*: *card-eq-0-iff infinite-literal*)

2.3 Classes with at least 1 and 2

Class finite already captures "at least 1"

lemma *zero-less-card-finite* [*simp*]: $0 < CARD('a::finite)$
  unfolding *neq0-conv* [*symmetric*] **by** *simp*

lemma *one-le-card-finite* [*simp*]: $Suc\ 0 \leq CARD('a::finite)$
  by (*simp add*: *less-Suc-eq-le* [*symmetric*])

Class for cardinality "at least 2"

class *card2* = *finite* +
  assumes *two-le-card*: $2 \leq CARD('a)$

lemma *one-less-card*: $Suc\ 0 < CARD('a::card2)$
  using *two-le-card* [**where** *'a='a*] **by** *simp*

lemma *one-less-int-card*: $1 < int\ CARD('a::card2)$
  using *one-less-card* [**where** *'a='a*] **by** *simp*

2.4 A type class for deciding finiteness of types

type-synonym *'a finite-UNIV* = (*'a*, *bool*) *phantom*

class *finite-UNIV* =
  fixes *finite-UNIV* :: (*'a*, *bool*) *phantom*
  assumes *finite-UNIV*: *finite-UNIV* = *Phantom*(*'a*) (*finite* (*UNIV* :: *'a set*))

lemma *finite-UNIV-code* [*code-unfold*]:
  finite (*UNIV* :: *'a* :: *finite-UNIV set*)
  $\longleftrightarrow$ *of-phantom* (*finite-UNIV* :: *'a finite-UNIV*)
by(*simp add*: *finite-UNIV*)

2.5 A type class for computing the cardinality of types

definition *is-list-UNIV* :: *'a list* ⇒ *bool*
where *is-list-UNIV xs* = (*let c* = *CARD*(*'a*) *in if c* = *0 then False else size* (*remdups xs*) = *c*)

lemma *is-list-UNIV-iff*: *is-list-UNIV xs* ⟷ *set xs* = *UNIV*
by(*auto simp add*: *is-list-UNIV-def Let-def card-eq-0-iff List.card-set*[*symmetric*]
  dest: *subst*[**where** *P=finite*, *OF - finite-set*] *card-eq-UNIV-imp-eq-UNIV*)

type-synonym *'a card-UNIV* = (*'a*, *nat*) *phantom*

class *card-UNIV* = *finite-UNIV* +
  fixes *card-UNIV* :: *'a card-UNIV*
  assumes *card-UNIV*: *card-UNIV* = *Phantom*(*'a*) *CARD*(*'a*)

2.6 Instantiations for *card-UNIV*

instantiation *nat* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*nat*) *False*
definition *card-UNIV* = *Phantom*(*nat*) *0*

instance by *intro-classes* (*simp-all add*: *finite-UNIV-nat-def card-UNIV-nat-def*)
end

instantiation *int* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*int*) *False*
definition *card-UNIV* = *Phantom*(*int*) *0*
instance by *intro-classes* (*simp-all add*: *card-UNIV-int-def finite-UNIV-int-def infinite-UNIV-int*)
end

instantiation *natural* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*natural*) *False*
definition *card-UNIV* = *Phantom*(*natural*) *0*
instance
by *standard*
(*auto simp add*: *finite-UNIV-natural-def card-UNIV-natural-def card-eq-0-iff*
type-definition.univ [*OF type-definition-natural*] *natural-eq-iff*
dest!: *finite-imageD intro*: *inj-onI*)
end

instantiation *integer* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*integer*) *False*
definition *card-UNIV* = *Phantom*(*integer*) *0*
instance
by *standard*
(*auto simp add*: *finite-UNIV-integer-def card-UNIV-integer-def card-eq-0-iff*
type-definition.univ [*OF type-definition-integer*] *infinite-UNIV-int*
dest!: *finite-imageD intro*: *inj-onI*)
end

instantiation *list* :: (*type*) *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*'a list*) *False*
definition *card-UNIV* = *Phantom*(*'a list*) *0*
instance by *intro-classes* (*simp-all add*: *card-UNIV-list-def finite-UNIV-list-def infinite-UNIV-listI*)
end

instantiation *unit* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*unit*) *True*
definition *card-UNIV* = *Phantom*(*unit*) *1*
instance by *intro-classes* (*simp-all add*: *card-UNIV-unit-def finite-UNIV-unit-def*)
end

instantiation *bool* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*bool*) *True*
definition *card-UNIV* = *Phantom*(*bool*) *2*
instance by(*intro-classes*)(*simp-all add*: *card-UNIV-bool-def finite-UNIV-bool-def*)
end

instantiation *char* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*char*) *True*
definition *card-UNIV* = *Phantom*(*char*) *256*
instance by *intro-classes* (*simp-all add*: *card-UNIV-char-def card-UNIV-char finite-UNIV-char-def*)
end

instantiation *prod* :: (*finite-UNIV*, *finite-UNIV*) *finite-UNIV* **begin**
definition *finite-UNIV* = *Phantom*('*a* × '*b*)
(*of-phantom* (*finite-UNIV* :: '*a finite-UNIV*) ∧ *of-phantom* (*finite-UNIV* :: '*b finite-UNIV*))
instance by *intro-classes* (*simp add*: *finite-UNIV-prod-def finite-UNIV finite-prod*)
end

instantiation *prod* :: (*card-UNIV*, *card-UNIV*) *card-UNIV* **begin**
definition *card-UNIV* = *Phantom*('*a* × '*b*)
(*of-phantom* (*card-UNIV* :: '*a card-UNIV*) ∗ *of-phantom* (*card-UNIV* :: '*b card-UNIV*))
instance by *intro-classes* (*simp add*: *card-UNIV-prod-def card-UNIV*)
end

instantiation *sum* :: (*finite-UNIV*, *finite-UNIV*) *finite-UNIV* **begin**
definition *finite-UNIV* = *Phantom*('*a* + '*b*)
(*of-phantom* (*finite-UNIV* :: '*a finite-UNIV*) ∧ *of-phantom* (*finite-UNIV* :: '*b finite-UNIV*))
instance
by *intro-classes* (*simp add*: *UNIV-Plus-UNIV*[*symmetric*] *finite-UNIV-sum-def finite-UNIV del*: *UNIV-Plus-UNIV*)
end

instantiation *sum* :: (*card-UNIV*, *card-UNIV*) *card-UNIV* **begin**
definition *card-UNIV* = *Phantom*('*a* + '*b*)
(*let ca* = *of-phantom* (*card-UNIV* :: '*a card-UNIV*);
cb = *of-phantom* (*card-UNIV* :: '*b card-UNIV*)
in if ca ≠ *0* ∧ *cb* ≠ *0 then ca* + *cb else 0*)
instance by *intro-classes* (*auto simp add*: *card-UNIV-sum-def card-UNIV card-UNIV-sum*)
end

instantiation *fun* :: (*finite-UNIV*, *card-UNIV*) *finite-UNIV* **begin**
definition *finite-UNIV* = *Phantom*('*a* ⇒ '*b*)
(*let cb* = *of-phantom* (*card-UNIV* :: '*b card-UNIV*)
in cb = *1* ∨ *of-phantom* (*finite-UNIV* :: '*a finite-UNIV*) ∧ *cb* ≠ *0*)
instance
by *intro-classes* (*auto simp add*: *finite-UNIV-fun-def Let-def card-UNIV finite-UNIV finite-UNIV-fun card-gt-0-iff*)
end

instantiation *fun* :: (*card-UNIV*, *card-UNIV*) *card-UNIV* **begin**
definition *card-UNIV* = *Phantom*('*a* ⇒ '*b*)
(*let ca* = *of-phantom* (*card-UNIV* :: '*a card-UNIV*);
cb = *of-phantom* (*card-UNIV* :: '*b card-UNIV*)

in if ca $\neq$ *0* $\wedge$ *cb* $\neq$ *0* $\vee$ *cb* = *1 then cb ^ ca else 0*)
instance by *intro-classes* (*simp add*: *card-UNIV-fun-def card-UNIV Let-def card-fun*)
end

instantiation *option* :: (*finite-UNIV*) *finite-UNIV* **begin**
definition *finite-UNIV* = *Phantom*('*a option*) (*of-phantom* (*finite-UNIV* :: '*a finite-UNIV*))
instance by *intro-classes* (*simp add*: *finite-UNIV-option-def finite-UNIV*)
end

instantiation *option* :: (*card-UNIV*) *card-UNIV* **begin**
definition *card-UNIV* = *Phantom*('*a option*)
(*let c* = *of-phantom* (*card-UNIV* :: '*a card-UNIV*) *in if c* $\neq$ *0 then Suc c else 0*)
instance by *intro-classes* (*simp add*: *card-UNIV-option-def card-UNIV card-UNIV-option*)
end

instantiation *String.literal* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*String.literal*) *False*
definition *card-UNIV* = *Phantom*(*String.literal*) *0*
instance
by *intro-classes* (*simp-all add*: *card-UNIV-literal-def finite-UNIV-literal-def infinite-literal card-literal*)
end

instantiation *set* :: (*finite-UNIV*) *finite-UNIV* **begin**
definition *finite-UNIV* = *Phantom*('*a set*) (*of-phantom* (*finite-UNIV* :: '*a finite-UNIV*))
instance by *intro-classes* (*simp add*: *finite-UNIV-set-def finite-UNIV Finite-Set.finite-set*)
end

instantiation *set* :: (*card-UNIV*) *card-UNIV* **begin**
definition *card-UNIV* = *Phantom*('*a set*)
(*let c* = *of-phantom* (*card-UNIV* :: '*a card-UNIV*) *in if c* = *0 then 0 else 2 ^ c*)
instance by *intro-classes* (*simp add*: *card-UNIV-set-def card-UNIV-set card-UNIV*)
end

lemma *UNIV-finite-1*: *UNIV* = *set* [*finite-1*.a_1]
by(*auto intro*: *finite-1.exhaust*)

lemma *UNIV-finite-2*: *UNIV* = *set* [*finite-2*.a_1, *finite-2*.a_2]
by(*auto intro*: *finite-2.exhaust*)

lemma *UNIV-finite-3*: *UNIV* = *set* [*finite-3*.a_1, *finite-3*.a_2, *finite-3*.a_3]
by(*auto intro*: *finite-3.exhaust*)

lemma *UNIV-finite-4*: *UNIV* = *set* [*finite-4*.a_1, *finite-4*.a_2, *finite-4*.a_3, *finite-4*.a_4]
by(*auto intro*: *finite-4.exhaust*)

lemma *UNIV-finite-5*:
UNIV = *set* [*finite-5*.a_1, *finite-5*.a_2, *finite-5*.a_3, *finite-5*.a_4, *finite-5*.a_5]

by(*auto intro*: *finite-5.exhaust*)

instantiation *Enum.finite-1* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*Enum.finite-1*) *True*
definition *card-UNIV* = *Phantom*(*Enum.finite-1*) *1*
instance
by *intro-classes* (*simp-all add*: *UNIV-finite-1 card-UNIV-finite-1-def finite-UNIV-finite-1-def*)
end

instantiation *Enum.finite-2* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*Enum.finite-2*) *True*
definition *card-UNIV* = *Phantom*(*Enum.finite-2*) *2*
instance
by *intro-classes* (*simp-all add*: *UNIV-finite-2 card-UNIV-finite-2-def finite-UNIV-finite-2-def*)
end

instantiation *Enum.finite-3* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*Enum.finite-3*) *True*
definition *card-UNIV* = *Phantom*(*Enum.finite-3*) *3*
instance
by *intro-classes* (*simp-all add*: *UNIV-finite-3 card-UNIV-finite-3-def finite-UNIV-finite-3-def*)
end

instantiation *Enum.finite-4* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*Enum.finite-4*) *True*
definition *card-UNIV* = *Phantom*(*Enum.finite-4*) *4*
instance
by *intro-classes* (*simp-all add*: *UNIV-finite-4 card-UNIV-finite-4-def finite-UNIV-finite-4-def*)
end

instantiation *Enum.finite-5* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*Enum.finite-5*) *True*
definition *card-UNIV* = *Phantom*(*Enum.finite-5*) *5*
instance
by *intro-classes* (*simp-all add*: *UNIV-finite-5 card-UNIV-finite-5-def finite-UNIV-finite-5-def*)
end

2.7 Code setup for sets

Implement $CARD('a)$ via *card-UNIV-class.card-UNIV* and provide implementations for *finite*, *card*, *op* ⊆, and *op* =if the calling context already provides *finite-UNIV* and *card-UNIV* instances. If we implemented the latter always via *card-UNIV-class.card-UNIV*, we would require instances of essentially all element types, i.e., a lot of instantiation proofs and – at run time – possibly slow dictionary constructions.

context
begin

qualified definition *card-UNIV′* :: *′a card-UNIV*
where [*code del*]: *card-UNIV′* = *Phantom(′a) CARD(′a)*

lemma *CARD-code* [*code-unfold*]:
CARD(′a) = *of-phantom* (*card-UNIV′* :: *′a card-UNIV*)
by(*simp add*: *card-UNIV′-def*)

lemma *card-UNIV′-code* [*code*]:
card-UNIV′ = *card-UNIV*
by(*simp add*: *card-UNIV card-UNIV′-def*)

end

lemma *card-Compl*:
finite A ⟹ *card* (− *A*) = *card* (*UNIV* :: *′a set*) − *card* (*A* :: *′a set*)
by (*metis Compl-eq-Diff-UNIV card-Diff-subset top-greatest*)

context fixes *xs* :: *′a* :: *finite-UNIV list*
begin

qualified definition *finite′* :: *′a set* ⇒ *bool*
where [*simp*, *code del*, *code-abbrev*]: *finite′* = *finite*

lemma *finite′-code* [*code*]:
finite′ (*set xs*) ⟷ *True*
finite′ (*List.coset xs*) ⟷ *of-phantom* (*finite-UNIV* :: *′a finite-UNIV*)
by(*simp-all add*: *card-gt-0-iff finite-UNIV*)

end

context fixes *xs* :: *′a* :: *card-UNIV list*
begin

qualified definition *card′* :: *′a set* ⇒ *nat*
where [*simp*, *code del*, *code-abbrev*]: *card′* = *card*

lemma *card′-code* [*code*]:
card′ (*set xs*) = *length* (*remdups xs*)
card′ (*List.coset xs*) = *of-phantom* (*card-UNIV* :: *′a card-UNIV*) − *length* (*remdups xs*)
by(*simp-all add*: *List.card-set card-Compl card-UNIV*)

qualified definition *subset′* :: *′a set* ⇒ *′a set* ⇒ *bool*
where [*simp*, *code del*, *code-abbrev*]: *subset′* = *op* ⊆

lemma *subset′-code* [*code*]:
subset′ A (*List.coset ys*) ⟷ (∀ *y* ∈ *set ys*. *y* ∉ *A*)
subset′ (*set ys*) *B* ⟷ (∀ *y* ∈ *set ys*. *y* ∈ *B*)

subset′ (*List.coset xs*) (*set ys*) ⟷ (*let n* = *CARD*(*′a*) *in n* > *0* ∧ *card*(*set* (*xs* @ *ys*)) = *n*)
by(*auto simp add*: *Let-def card-gt-0-iff dest*: *card-eq-UNIV-imp-eq-UNIV intro*: *arg-cong*[**where** *f*=*card*])
(*metis finite-compl finite-set rev-finite-subset*)

qualified definition *eq-set* :: *′a set* ⇒ *′a set* ⇒ *bool*
where [*simp*, *code del*, *code-abbrev*]: *eq-set* = *op* =

lemma *eq-set-code* [*code*]:
fixes *ys*
defines *rhs* ≡
let n = *CARD*(*′a*)
in if n = *0 then False else*
let xs′ = *remdups xs*; *ys′* = *remdups ys*
in length xs′ + *length ys′* = *n* ∧ (∀ *x* ∈ *set xs′*. *x* ∉ *set ys′*) ∧ (∀ *y* ∈ *set ys′*. *y* ∉ *set xs′*)
shows *eq-set* (*List.coset xs*) (*set ys*) ⟷ *rhs*
and *eq-set* (*set ys*) (*List.coset xs*) ⟷ *rhs*
and *eq-set* (*set xs*) (*set ys*) ⟷ (∀ *x* ∈ *set xs*. *x* ∈ *set ys*) ∧ (∀ *y* ∈ *set ys*. *y* ∈ *set xs*)
and *eq-set* (*List.coset xs*) (*List.coset ys*) ⟷ (∀ *x* ∈ *set xs*. *x* ∈ *set ys*) ∧ (∀ *y* ∈ *set ys*. *y* ∈ *set xs*)
proof *goal-cases*
{
case *1*
show *?case* (**is** *?lhs* ⟷ *?rhs*)
proof
show *?rhs* **if** *?lhs*
using *that*
by (*auto simp add*: *rhs-def Let-def List.card-set*[*symmetric*]
card-Un-Int[**where** *A*=*set xs* **and** *B*=− *set xs*] *card-UNIV*
Compl-partition card-gt-0-iff dest: *sym*)(*metis finite-compl finite-set*)
show *?lhs* **if** *?rhs*
proof −
have ⟦ ∀ *y*∈*set xs*. *y* ∉ *set ys*; ∀ *x*∈*set ys*. *x* ∉ *set xs* ⟧ ⟹ *set xs* ∩ *set ys* = {} **by** *blast*
with *that* **show** *?thesis*
by (*auto simp add*: *rhs-def Let-def List.card-set*[*symmetric*]
card-UNIV card-gt-0-iff card-Un-Int[**where** *A*=*set xs* **and** *B*=*set ys*]
dest: *card-eq-UNIV-imp-eq-UNIV split*: *if-split-asm*)
qed
qed
}
moreover
case *2*
ultimately show *?case* **unfolding** *eq-set-def* **by** *blast*
next
case *3*

```
  show ?case unfolding eq-set-def List.coset-def by blast
next
  case 4
  show ?case unfolding eq-set-def List.coset-def by blast
qed

end
```

Provide more informative exceptions than Match for non-rewritten cases. If generated code raises one these exceptions, then a code equation calls the mentioned operator for an element type that is not an instance of *card-UNIV* and is therefore not implemented via *card-UNIV-class.card-UNIV*. Constrain the element type with sort *card-UNIV* to change this.

```
lemma card-coset-error [code]:
  card (List.coset xs) =
   Code.abort (STR ''card (List.coset -) requires type class instance card-UNIV'')
     (λ-. card (List.coset xs))
by(simp)

lemma coset-subseteq-set-code [code]:
  List.coset xs ⊆ set ys ⟷
  (if xs = [] ∧ ys = [] then False
   else Code.abort
    (STR ''subset-eq (List.coset -) (List.set -) requires type class instance card-UNIV'')
     (λ-. List.coset xs ⊆ set ys))
by simp

notepad begin — test code setup
have List.coset [True] = set [False] ∧
      List.coset [] ⊆ List.set [True, False] ∧
      finite (List.coset [True])
  by eval
end

end
```

3 Numeral Syntax for Types

```
theory Numeral-Type
imports Cardinality
begin
```

3.1 Numeral Types

```
typedef num0 = UNIV :: nat set ..
typedef num1 = UNIV :: unit set ..

typedef 'a bit0 = {0 ..< 2 * int CARD('a::finite)}
```

proof
  show $0 \in \{0 ..< 2 * int\ CARD('a)\}$
    by *simp*
qed

typedef $'a\ bit1 = \{0 ..< 1 + 2 * int\ CARD('a::finite)\}$
proof
  show $0 \in \{0 ..< 1 + 2 * int\ CARD('a)\}$
    by *simp*
qed

lemma *card-num0* [*simp*]: $CARD\ (num0) = 0$
  unfolding *type-definition.card* [*OF type-definition-num0*]
  by *simp*

lemma *infinite-num0*: ¬ *finite* (*UNIV* :: *num0 set*)
  using *card-num0*[*unfolded card-eq-0-iff*]
  by *simp*

lemma *card-num1* [*simp*]: $CARD(num1) = 1$
  unfolding *type-definition.card* [*OF type-definition-num1*]
  by (*simp only*: *card-UNIV-unit*)

lemma *card-bit0* [*simp*]: $CARD('a\ bit0) = 2 * CARD('a::finite)$
  unfolding *type-definition.card* [*OF type-definition-bit0*]
  by *simp*

lemma *card-bit1* [*simp*]: $CARD('a\ bit1) = Suc\ (2 * CARD('a::finite))$
  unfolding *type-definition.card* [*OF type-definition-bit1*]
  by *simp*

instance *num1* :: *finite*
proof
  show *finite* (*UNIV*::*num1 set*)
    unfolding *type-definition.univ* [*OF type-definition-num1*]
    using *finite* **by** (*rule finite-imageI*)
qed

instance *bit0* :: (*finite*) *card2*
proof
  show *finite* (*UNIV*::*'a bit0 set*)
    unfolding *type-definition.univ* [*OF type-definition-bit0*]
    by *simp*
  show $2 \leq CARD('a\ bit0)$
    by *simp*
qed

instance *bit1* :: (*finite*) *card2*
proof

show *finite* (*UNIV*::'*a bit1 set*)
unfolding *type-definition.univ* [*OF type-definition-bit1*]
by *simp*
show $2 \leq$ *CARD*('*a bit1*)
by *simp*
qed

3.2 Locales for for modular arithmetic subtypes

locale *mod-type* =
fixes *n* :: *int*
and *Rep* :: '*a*::{*zero,one,plus,times,uminus,minus*} ⇒ *int*
and *Abs* :: *int* ⇒ '*a*::{*zero,one,plus,times,uminus,minus*}
assumes *type*: *type-definition Rep Abs* {*0..*<*n*}
and *size1*: *1* < *n*
and *zero-def*: *0* = *Abs 0*
and *one-def*: *1* = *Abs 1*
and *add-def*: *x* + *y* = *Abs* ((*Rep x* + *Rep y*) *mod n*)
and *mult-def*: *x* ∗ *y* = *Abs* ((*Rep x* ∗ *Rep y*) *mod n*)
and *diff-def*: *x* − *y* = *Abs* ((*Rep x* − *Rep y*) *mod n*)
and *minus-def*: − *x* = *Abs* ((− *Rep x*) *mod n*)
begin

lemma *size0*: *0* < *n*
using *size1* **by** *simp*

lemmas *definitions* =
zero-def one-def add-def mult-def minus-def diff-def

lemma *Rep-less-n*: *Rep x* < *n*
by (*rule type-definition.Rep* [*OF type*, *simplified*, *THEN conjunct2*])

lemma *Rep-le-n*: *Rep x* ≤ *n*
by (*rule Rep-less-n* [*THEN order-less-imp-le*])

lemma *Rep-inject-sym*: *x* = *y* ⟷ *Rep x* = *Rep y*
by (*rule type-definition.Rep-inject* [*OF type*, *symmetric*])

lemma *Rep-inverse*: *Abs* (*Rep x*) = *x*
by (*rule type-definition.Rep-inverse* [*OF type*])

lemma *Abs-inverse*: *m* ∈ {*0..*<*n*} ⟹ *Rep* (*Abs m*) = *m*
by (*rule type-definition.Abs-inverse* [*OF type*])

lemma *Rep-Abs-mod*: *Rep* (*Abs* (*m mod n*)) = *m mod n*
by (*simp add*: *Abs-inverse pos-mod-conj* [*OF size0*])

lemma *Rep-Abs-0*: *Rep* (*Abs 0*) = *0*
by (*simp add*: *Abs-inverse size0*)

lemma *Rep-0*: *Rep 0 = 0*
by (*simp add*: *zero-def Rep-Abs-0*)

lemma *Rep-Abs-1*: *Rep* (*Abs 1*) = *1*
by (*simp add*: *Abs-inverse size1*)

lemma *Rep-1*: *Rep 1 = 1*
by (*simp add*: *one-def Rep-Abs-1*)

lemma *Rep-mod*: *Rep x mod n = Rep x*
apply (*rule-tac x=x* **in** *type-definition.Abs-cases* [*OF type*])
apply (*simp add*: *type-definition.Abs-inverse* [*OF type*])
apply (*simp add*: *mod-pos-pos-trivial*)
done

lemmas *Rep-simps* =
Rep-inject-sym Rep-inverse Rep-Abs-mod Rep-mod Rep-Abs-0 Rep-Abs-1

lemma *comm-ring-1*: *OFCLASS*('*a*, *comm-ring-1-class*)
apply (*intro-classes*, *unfold definitions*)
apply (*simp-all add*: *Rep-simps zmod-simps field-simps*)
done

end

locale *mod-ring* = *mod-type n Rep Abs*
for *n* :: *int*
and *Rep* :: '*a*::{*comm-ring-1*} ⇒ *int*
and *Abs* :: *int* ⇒ '*a*::{*comm-ring-1*}
begin

lemma *of-nat-eq*: *of-nat k* = *Abs* (*int k mod n*)
apply (*induct k*)
apply (*simp add*: *zero-def*)
apply (*simp add*: *Rep-simps add-def one-def zmod-simps ac-simps*)
done

lemma *of-int-eq*: *of-int z* = *Abs* (*z mod n*)
apply (*cases z rule*: *int-diff-cases*)
apply (*simp add*: *Rep-simps of-nat-eq diff-def zmod-simps*)
done

lemma *Rep-numeral*:
Rep (*numeral w*) = *numeral w mod n*
using *of-int-eq* [*of numeral w*]
by (*simp add*: *Rep-inject-sym Rep-Abs-mod*)

lemma *iszero-numeral*:

iszero (*numeral w*::'*a*) $\longleftrightarrow$ *numeral w mod n* = *0*
by (*simp add*: *Rep-inject-sym Rep-numeral Rep-0 iszero-def*)

lemma *cases*:
assumes *1*: $\bigwedge z.\ [\![(x::'a) = \textit{of-int}\ z;\ 0 \le z;\ z < n]\!] \Longrightarrow P$
shows *P*
apply (*cases x rule*: *type-definition.Abs-cases* [*OF type*])
apply (*rule-tac z*=*y* **in** *1*)
apply (*simp-all add*: *of-int-eq mod-pos-pos-trivial*)
done

lemma *induct*:
$(\bigwedge z.\ [\![0 \le z;\ z < n]\!] \Longrightarrow P\ (\textit{of-int}\ z)) \Longrightarrow P\ (x::'a)$
by (*cases x rule*: *cases*) *simp*

end

3.3 Ring class instances

Unfortunately *ring-1* instance is not possible for *num1*, since 0 and 1 are not distinct.

instantiation *num1* :: {*comm-ring*,*comm-monoid-mult*,*numeral*}
begin

lemma *num1-eq-iff*: (*x*::*num1*) = (*y*::*num1*) $\longleftrightarrow$ *True*
by (*induct x*, *induct y*) *simp*

instance
by *standard* (*simp-all add*: *num1-eq-iff*)

end

instantiation
bit0 **and** *bit1* :: (*finite*) {*zero*,*one*,*plus*,*times*,*uminus*,*minus*}
begin

definition *Abs-bit0'* :: *int* ⇒ '*a bit0* **where**
Abs-bit0' x = *Abs-bit0* (*x mod int CARD*('*a bit0*))

definition *Abs-bit1'* :: *int* ⇒ '*a bit1* **where**
Abs-bit1' x = *Abs-bit1* (*x mod int CARD*('*a bit1*))

definition *0* = *Abs-bit0 0*
definition *1* = *Abs-bit0 1*
definition *x* + *y* = *Abs-bit0'* (*Rep-bit0 x* + *Rep-bit0 y*)
definition *x* ∗ *y* = *Abs-bit0'* (*Rep-bit0 x* ∗ *Rep-bit0 y*)
definition *x* − *y* = *Abs-bit0'* (*Rep-bit0 x* − *Rep-bit0 y*)
definition − *x* = *Abs-bit0'* (− *Rep-bit0 x*)

definition *0* = *Abs-bit1 0*
definition *1* = *Abs-bit1 1*
definition *x* + *y* = *Abs-bit1'* (*Rep-bit1 x* + *Rep-bit1 y*)
definition *x* ∗ *y* = *Abs-bit1'* (*Rep-bit1 x* ∗ *Rep-bit1 y*)
definition *x* − *y* = *Abs-bit1'* (*Rep-bit1 x* − *Rep-bit1 y*)
definition − *x* = *Abs-bit1'* (− *Rep-bit1 x*)

instance ..

end

interpretation *bit0*:
  mod-type int CARD(*'a*::*finite bit0*)
    Rep-bit0 :: *'a*::*finite bit0* ⇒ *int*
    Abs-bit0 :: *int* ⇒ *'a*::*finite bit0*
apply (*rule mod-type.intro*)
apply (*simp add*: *of-nat-mult type-definition-bit0*)
apply (*rule one-less-int-card*)
apply (*rule zero-bit0-def*)
apply (*rule one-bit0-def*)
apply (*rule plus-bit0-def* [*unfolded Abs-bit0'-def*])
apply (*rule times-bit0-def* [*unfolded Abs-bit0'-def*])
apply (*rule minus-bit0-def* [*unfolded Abs-bit0'-def*])
apply (*rule uminus-bit0-def* [*unfolded Abs-bit0'-def*])
done

interpretation *bit1*:
  mod-type int CARD(*'a*::*finite bit1*)
    Rep-bit1 :: *'a*::*finite bit1* ⇒ *int*
    Abs-bit1 :: *int* ⇒ *'a*::*finite bit1*
apply (*rule mod-type.intro*)
apply (*simp add*: *of-nat-mult type-definition-bit1*)
apply (*rule one-less-int-card*)
apply (*rule zero-bit1-def*)
apply (*rule one-bit1-def*)
apply (*rule plus-bit1-def* [*unfolded Abs-bit1'-def*])
apply (*rule times-bit1-def* [*unfolded Abs-bit1'-def*])
apply (*rule minus-bit1-def* [*unfolded Abs-bit1'-def*])
apply (*rule uminus-bit1-def* [*unfolded Abs-bit1'-def*])
done

instance *bit0* :: (*finite*) *comm-ring-1*
  by (*rule bit0.comm-ring-1*)

instance *bit1* :: (*finite*) *comm-ring-1*
  by (*rule bit1.comm-ring-1*)

interpretation *bit0*:
  mod-ring int CARD(*'a*::*finite bit0*)

Rep-bit0 :: *'a::finite bit0* ⇒ *int*
Abs-bit0 :: *int* ⇒ *'a::finite bit0*
..

interpretation *bit1*:
mod-ring int CARD('a::finite bit1)
Rep-bit1 :: *'a::finite bit1* ⇒ *int*
Abs-bit1 :: *int* ⇒ *'a::finite bit1*
..

Set up cases, induction, and arithmetic

lemmas *bit0-cases* [*case-names of-int*, *cases type*: *bit0*] = *bit0.cases*
lemmas *bit1-cases* [*case-names of-int*, *cases type*: *bit1*] = *bit1.cases*

lemmas *bit0-induct* [*case-names of-int*, *induct type*: *bit0*] = *bit0.induct*
lemmas *bit1-induct* [*case-names of-int*, *induct type*: *bit1*] = *bit1.induct*

lemmas *bit0-iszero-numeral* [*simp*] = *bit0.iszero-numeral*
lemmas *bit1-iszero-numeral* [*simp*] = *bit1.iszero-numeral*

lemmas [*simp*] = *eq-numeral-iff-iszero* [**where** *'a='a bit0*] **for** *dummy* :: *'a::finite*
lemmas [*simp*] = *eq-numeral-iff-iszero* [**where** *'a='a bit1*] **for** *dummy* :: *'a::finite*

3.4 Order instances

instantiation *bit0* **and** *bit1* :: (*finite*) *linorder* **begin**
definition $a < b \longleftrightarrow$ *Rep-bit0 a* < *Rep-bit0 b*
definition $a \leq b \longleftrightarrow$ *Rep-bit0 a* ≤ *Rep-bit0 b*
definition $a < b \longleftrightarrow$ *Rep-bit1 a* < *Rep-bit1 b*
definition $a \leq b \longleftrightarrow$ *Rep-bit1 a* ≤ *Rep-bit1 b*

instance
by(*intro-classes*)
(*auto simp add*: *less-eq-bit0-def less-bit0-def less-eq-bit1-def less-bit1-def Rep-bit0-inject Rep-bit1-inject*)
end

lemma (**in** *preorder*) *tranclp-less*: *op* $<^{++}$ = *op* <
by(*auto simp add*: *fun-eq-iff intro*: *less-trans elim*: *tranclp.induct*)

instance *bit0* **and** *bit1* :: (*finite*) *wellorder*
proof −
have *wf* {(*x* :: *'a bit0*, *y*). *x* < *y*}
by(*auto simp add*: *trancl-def tranclp-less intro*!: *finite-acyclic-wf acyclicI*)
thus *OFCLASS*(*'a bit0*, *wellorder-class*)
by(*rule wf-wellorderI*) *intro-classes*
next
have *wf* {(*x* :: *'a bit1*, *y*). *x* < *y*}
by(*auto simp add*: *trancl-def tranclp-less intro*!: *finite-acyclic-wf acyclicI*)

thus *OFCLASS('a bit1, wellorder-class)*
by(*rule wf-wellorderI*) *intro-classes*
qed

3.5 Code setup and type classes for code generation

Code setup for *num0* and *num1*

definition *Num0* :: *num0* **where** *Num0* = *Abs-num0 0*
code-datatype *Num0*

instantiation *num0* :: *equal* **begin**
definition *equal-num0* :: *num0* ⇒ *num0* ⇒ *bool*
where *equal-num0* = *op* =
instance by *intro-classes* (*simp add*: *equal-num0-def*)
end

lemma *equal-num0-code* [*code*]:
equal-class.equal Num0 Num0 = *True*
by(*rule equal-refl*)

code-datatype *1* :: *num1*

instantiation *num1* :: *equal* **begin**
definition *equal-num1* :: *num1* ⇒ *num1* ⇒ *bool*
where *equal-num1* = *op* =
instance by *intro-classes* (*simp add*: *equal-num1-def*)
end

lemma *equal-num1-code* [*code*]:
equal-class.equal (*1* :: *num1*) *1* = *True*
by(*rule equal-refl*)

instantiation *num1* :: *enum* **begin**
definition *enum-class.enum* = [*1* :: *num1*]
definition *enum-class.enum-all P* = *P* (*1* :: *num1*)
definition *enum-class.enum-ex P* = *P* (*1* :: *num1*)
instance
by *intro-classes*
(*auto simp add*: *enum-num1-def enum-all-num1-def enum-ex-num1-def num1-eq-iff Ball-def*,
(*metis* (*full-types*) *num1-eq-iff*)+)
end

instantiation *num0* **and** *num1* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*num0*) *False*
definition *card-UNIV* = *Phantom*(*num0*) *0*
definition *finite-UNIV* = *Phantom*(*num1*) *True*
definition *card-UNIV* = *Phantom*(*num1*) *1*
instance

by *intro-classes*
(*simp-all add*: *finite-UNIV-num0-def card-UNIV-num0-def infinite-num0 finite-UNIV-num1-def card-UNIV-num1-def*)
end

Code setup for *'a bit0* and *'a bit1*

declare
bit0.Rep-inverse[*code abstype*]
bit0.Rep-0[*code abstract*]
bit0.Rep-1[*code abstract*]

lemma *Abs-bit0'-code* [*code abstract*]:
Rep-bit0 (*Abs-bit0' x* :: *'a* :: *finite bit0*) = *x mod int* (*CARD*(*'a bit0*))
by(*auto simp add*: *Abs-bit0'-def intro*!: *Abs-bit0-inverse*)

lemma *inj-on-Abs-bit0*:
inj-on (*Abs-bit0* :: *int* ⇒ *'a bit0*) {*0*..<*2* ∗ *int CARD*(*'a* :: *finite*)}
by(*auto intro*: *inj-onI simp add*: *Abs-bit0-inject*)

declare
bit1.Rep-inverse[*code abstype*]
bit1.Rep-0[*code abstract*]
bit1.Rep-1[*code abstract*]

lemma *Abs-bit1'-code* [*code abstract*]:
Rep-bit1 (*Abs-bit1' x* :: *'a* :: *finite bit1*) = *x mod int* (*CARD*(*'a bit1*))
by(*auto simp add*: *Abs-bit1'-def intro*!: *Abs-bit1-inverse*)

lemma *inj-on-Abs-bit1*:
inj-on (*Abs-bit1* :: *int* ⇒ *'a bit1*) {*0*..<*1* + *2* ∗ *int CARD*(*'a* :: *finite*)}
by(*auto intro*: *inj-onI simp add*: *Abs-bit1-inject*)

instantiation *bit0* **and** *bit1* :: (*finite*) *equal* **begin**

definition *equal-class.equal x y* ⟷ *Rep-bit0 x* = *Rep-bit0 y*
definition *equal-class.equal x y* ⟷ *Rep-bit1 x* = *Rep-bit1 y*

instance
by *intro-classes* (*simp-all add*: *equal-bit0-def equal-bit1-def Rep-bit0-inject Rep-bit1-inject*)

end

instantiation *bit0* :: (*finite*) *enum* **begin**
definition (*enum-class.enum* :: *'a bit0 list*) = *map* (*Abs-bit0'* ∘ *int*) (*upt 0* (*CARD*(*'a bit0*)))
definition *enum-class.enum-all P* = (∀ *b* :: *'a bit0* ∈ *set enum-class.enum*. *P b*)
definition *enum-class.enum-ex P* = (∃ *b* :: *'a bit0* ∈ *set enum-class.enum*. *P b*)

instance

proof(*intro-classes*)
show *distinct* (*enum-class.enum* :: *'a bit0 list*)
by (*simp add*: *enum-bit0-def distinct-map inj-on-def Abs-bit0'-def Abs-bit0-inject mod-pos-pos-trivial*)

show *univ-eq*: (*UNIV* :: *'a bit0 set*) = *set enum-class.enum*
unfolding *enum-bit0-def type-definition.Abs-image*[*OF type-definition-bit0*, *symmetric*]
by(*simp add*: *image-comp* [*symmetric*] *inj-on-Abs-bit0 card-image image-int-atLeastLessThan*)
(*auto intro*!: *image-cong*[*OF refl*] *simp add*: *Abs-bit0'-def mod-pos-pos-trivial*)

fix *P* :: *'a bit0* ⇒ *bool*
show *enum-class.enum-all P* = *Ball UNIV P*
and *enum-class.enum-ex P* = *Bex UNIV P*
by(*simp-all add*: *enum-all-bit0-def enum-ex-bit0-def univ-eq*)
qed

end

instantiation *bit1* :: (*finite*) *enum* **begin**
definition (*enum-class.enum* :: *'a bit1 list*) = *map* (*Abs-bit1'* ∘ *int*) (*upt 0* (*CARD*(*'a bit1*)))
definition *enum-class.enum-all P* = (∀ *b* :: *'a bit1* ∈ *set enum-class.enum*. *P b*)
definition *enum-class.enum-ex P* = (∃ *b* :: *'a bit1* ∈ *set enum-class.enum*. *P b*)

instance
proof(*intro-classes*)
show *distinct* (*enum-class.enum* :: *'a bit1 list*)
by(*simp only*: *Abs-bit1'-def zmod-int*[*symmetric*] *enum-bit1-def distinct-map Suc-eq-plus1 card-bit1 o-apply inj-on-def*)
(*clarsimp simp add*: *Abs-bit1-inject*)

show *univ-eq*: (*UNIV* :: *'a bit1 set*) = *set enum-class.enum*
unfolding *enum-bit1-def type-definition.Abs-image*[*OF type-definition-bit1*, *symmetric*]
by(*simp add*: *image-comp* [*symmetric*] *inj-on-Abs-bit1 card-image image-int-atLeastLessThan*)
(*auto intro*!: *image-cong*[*OF refl*] *simp add*: *Abs-bit1'-def mod-pos-pos-trivial*)

fix *P* :: *'a bit1* ⇒ *bool*
show *enum-class.enum-all P* = *Ball UNIV P*
and *enum-class.enum-ex P* = *Bex UNIV P*
by(*simp-all add*: *enum-all-bit1-def enum-ex-bit1-def univ-eq*)
qed

end

instantiation *bit0* **and** *bit1* :: (*finite*) *finite-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*'a bit0*) *True*
definition *finite-UNIV* = *Phantom*(*'a bit1*) *True*

instance by *intro-classes* (*simp-all add*: *finite-UNIV-bit0-def finite-UNIV-bit1-def*)
end

instantiation *bit0* **and** *bit1* :: ({*finite*,*card-UNIV*}) *card-UNIV* **begin**
definition *card-UNIV* = *Phantom*('*a bit0*) (*2* ∗ *of-phantom* (*card-UNIV* :: '*a card-UNIV*))
definition *card-UNIV* = *Phantom*('*a bit1*) (*1* + *2* ∗ *of-phantom* (*card-UNIV* :: '*a card-UNIV*))
instance by *intro-classes* (*simp-all add*: *card-UNIV-bit0-def card-UNIV-bit1-def card-UNIV*)
end

3.6 Syntax

```
syntax
  -NumeralType :: num-token => type (-)
  -NumeralType0 :: type (0)
  -NumeralType1 :: type (1)

translations
  (type) 1 == (type) num1
  (type) 0 == (type) num0

parse-translation ‹
  let
    fun mk-bintype n =
      let
        fun mk-bit 0 = Syntax.const @{type-syntax bit0}
          | mk-bit 1 = Syntax.const @{type-syntax bit1};
        fun bin-of n =
          if n = 1 then Syntax.const @{type-syntax num1}
          else if n = 0 then Syntax.const @{type-syntax num0}
          else if n = ~1 then raise TERM (negative type numeral, [])
          else
            let val (q, r) = Integer.div-mod n 2;
            in mk-bit r $ bin-of q end;
      in bin-of n end;

    fun numeral-tr [Free (str, -)] = mk-bintype (the (Int.fromString str))
      | numeral-tr ts = raise TERM (numeral-tr, ts);

  in [(@{syntax-const -NumeralType}, K numeral-tr)] end;
›

print-translation ‹
  let
    fun int-of [] = 0
      | int-of (b :: bs) = b + 2 ∗ int-of bs;
```

```
  fun bin-of (Const (@{type-syntax num0}, -)) = []
    | bin-of (Const (@{type-syntax num1}, -)) = [1]
    | bin-of (Const (@{type-syntax bit0}, -) $ bs) = 0 :: bin-of bs
    | bin-of (Const (@{type-syntax bit1}, -) $ bs) = 1 :: bin-of bs
    | bin-of t = raise TERM (bin-of, [t]);

  fun bit-tr' b [t] =
        let
          val rev-digs = b :: bin-of t handle TERM - => raise Match
          val i = int-of rev-digs;
          val num = string-of-int (abs i);
        in
          Syntax.const @{syntax-const -NumeralType} $ Syntax.free num
        end
    | bit-tr' b - = raise Match;
 in
  [(@{type-syntax bit0}, K (bit-tr' 0)),
   (@{type-syntax bit1}, K (bit-tr' 1))]
 end;
⟩
```

3.7 Examples

lemma *CARD(0) = 0* **by** *simp*
lemma *CARD(17) = 17* **by** *simp*
lemma *8 ∗ 11 ˆ 3 − 6 = (2::5)* **by** *simp*

end

4 Assigning lengths to types by typeclasses

theory *Type-Length*
imports *~~/src/HOL/Library/Numeral-Type*
begin

The aim of this is to allow any type as index type, but to provide a default instantiation for numeral types. This independence requires some duplication with the definitions in *Numeral-Type*.

class *len0* =
  fixes *len-of* :: *'a itself ⇒ nat*

Some theorems are only true on words with length greater 0.

class *len* = *len0* +
  assumes *len-gt-0* [*iff*]: *0 < len-of TYPE ('a)*

instantiation *num0* **and** *num1* :: *len0*
begin

definition
len-num0: *len-of* (*x*::*num0 itself*) = *0*

definition
len-num1: *len-of* (*x*::*num1 itself*) = *1*

instance ..

end

instantiation *bit0* **and** *bit1* :: (*len0*) *len0*
begin

definition
len-bit0: *len-of* (*x*::'*a*::*len0 bit0 itself*) = *2* ∗ *len-of TYPE* ('*a*)

definition
len-bit1: *len-of* (*x*::'*a*::*len0 bit1 itself*) = *2* ∗ *len-of TYPE* ('*a*) + *1*

instance ..

end

lemmas *len-of-numeral-defs* [*simp*] = *len-num0 len-num1 len-bit0 len-bit1*

instance *num1* :: *len* **proof qed** *simp*
instance *bit0* :: (*len*) *len* **proof qed** *simp*
instance *bit1* :: (*len0*) *len* **proof qed** *simp*

end

5 Boolean Algebras

theory *Boolean-Algebra*
imports *Main*
begin

locale *boolean* =
fixes *conj* :: '*a* ⇒ '*a* ⇒ '*a* (**infixr** ⊓ *70*)
fixes *disj* :: '*a* ⇒ '*a* ⇒ '*a* (**infixr** ⊔ *65*)
fixes *compl* :: '*a* ⇒ '*a* (∼ - [*81*] *80*)
fixes *zero* :: '*a* (**0**)
fixes *one* :: '*a* (**1**)
assumes *conj-assoc*: (*x* ⊓ *y*) ⊓ *z* = *x* ⊓ (*y* ⊓ *z*)
assumes *disj-assoc*: (*x* ⊔ *y*) ⊔ *z* = *x* ⊔ (*y* ⊔ *z*)
assumes *conj-commute*: *x* ⊓ *y* = *y* ⊓ *x*
assumes *disj-commute*: *x* ⊔ *y* = *y* ⊔ *x*
assumes *conj-disj-distrib*: *x* ⊓ (*y* ⊔ *z*) = (*x* ⊓ *y*) ⊔ (*x* ⊓ *z*)
assumes *disj-conj-distrib*: *x* ⊔ (*y* ⊓ *z*) = (*x* ⊔ *y*) ⊓ (*x* ⊔ *z*)

assumes *conj-one-right* [*simp*]: $x \sqcap \mathbf{1} = x$
assumes *disj-zero-right* [*simp*]: $x \sqcup \mathbf{0} = x$
assumes *conj-cancel-right* [*simp*]: $x \sqcap \sim x = \mathbf{0}$
assumes *disj-cancel-right* [*simp*]: $x \sqcup \sim x = \mathbf{1}$
begin

sublocale *conj*: *abel-semigroup conj*
by *standard* (*fact conj-assoc conj-commute*)+

sublocale *disj*: *abel-semigroup disj*
by *standard* (*fact disj-assoc disj-commute*)+

lemmas *conj-left-commute* = *conj.left-commute*

lemmas *disj-left-commute* = *disj.left-commute*

lemmas *conj-ac* = *conj.assoc conj.commute conj.left-commute*
lemmas *disj-ac* = *disj.assoc disj.commute disj.left-commute*

lemma *dual*: *boolean disj conj compl one zero*
apply (*rule boolean.intro*)
apply (*rule disj-assoc*)
apply (*rule conj-assoc*)
apply (*rule disj-commute*)
apply (*rule conj-commute*)
apply (*rule disj-conj-distrib*)
apply (*rule conj-disj-distrib*)
apply (*rule disj-zero-right*)
apply (*rule conj-one-right*)
apply (*rule disj-cancel-right*)
apply (*rule conj-cancel-right*)
done

5.1 Complement

lemma *complement-unique*:
assumes *1*: $a \sqcap x = \mathbf{0}$
assumes *2*: $a \sqcup x = \mathbf{1}$
assumes *3*: $a \sqcap y = \mathbf{0}$
assumes *4*: $a \sqcup y = \mathbf{1}$
shows $x = y$
proof −
have $(a \sqcap x) \sqcup (x \sqcap y) = (a \sqcap y) \sqcup (x \sqcap y)$ **using** *1 3* **by** *simp*
hence $(x \sqcap a) \sqcup (x \sqcap y) = (y \sqcap a) \sqcup (y \sqcap x)$ **using** *conj-commute* **by** *simp*
hence $x \sqcap (a \sqcup y) = y \sqcap (a \sqcup x)$ **using** *conj-disj-distrib* **by** *simp*
hence $x \sqcap \mathbf{1} = y \sqcap \mathbf{1}$ **using** *2 4* **by** *simp*
thus $x = y$ **using** *conj-one-right* **by** *simp*
qed

lemma *compl-unique*: $[\![x \sqcap y = \mathbf{0}; x \sqcup y = \mathbf{1}]\!] \Longrightarrow \sim x = y$
by (*rule complement-unique* [*OF conj-cancel-right disj-cancel-right*])

lemma *double-compl* [*simp*]: $\sim (\sim x) = x$
proof (*rule compl-unique*)
  from *conj-cancel-right* **show** $\sim x \sqcap x = \mathbf{0}$ **by** (*simp only*: *conj-commute*)
  from *disj-cancel-right* **show** $\sim x \sqcup x = \mathbf{1}$ **by** (*simp only*: *disj-commute*)
qed

lemma *compl-eq-compl-iff* [*simp*]: $(\sim x = \sim y) = (x = y)$
by (*rule inj-eq* [*OF inj-on-inverseI*], *rule double-compl*)

5.2 Conjunction

lemma *conj-absorb* [*simp*]: $x \sqcap x = x$
proof −
  have $x \sqcap x = (x \sqcap x) \sqcup \mathbf{0}$ **using** *disj-zero-right* **by** *simp*
  also have ... $= (x \sqcap x) \sqcup (x \sqcap \sim x)$ **using** *conj-cancel-right* **by** *simp*
  also have ... $= x \sqcap (x \sqcup \sim x)$ **using** *conj-disj-distrib* **by** (*simp only*:)
  also have ... $= x \sqcap \mathbf{1}$ **using** *disj-cancel-right* **by** *simp*
  also have ... $= x$ **using** *conj-one-right* **by** *simp*
  finally show *?thesis* **.**
qed

lemma *conj-zero-right* [*simp*]: $x \sqcap \mathbf{0} = \mathbf{0}$
proof −
  have $x \sqcap \mathbf{0} = x \sqcap (x \sqcap \sim x)$ **using** *conj-cancel-right* **by** *simp*
  also have ... $= (x \sqcap x) \sqcap \sim x$ **using** *conj-assoc* **by** (*simp only*:)
  also have ... $= x \sqcap \sim x$ **using** *conj-absorb* **by** *simp*
  also have ... $= \mathbf{0}$ **using** *conj-cancel-right* **by** *simp*
  finally show *?thesis* **.**
qed

lemma *compl-one* [*simp*]: $\sim \mathbf{1} = \mathbf{0}$
by (*rule compl-unique* [*OF conj-zero-right disj-zero-right*])

lemma *conj-zero-left* [*simp*]: $\mathbf{0} \sqcap x = \mathbf{0}$
by (*subst conj-commute*) (*rule conj-zero-right*)

lemma *conj-one-left* [*simp*]: $\mathbf{1} \sqcap x = x$
by (*subst conj-commute*) (*rule conj-one-right*)

lemma *conj-cancel-left* [*simp*]: $\sim x \sqcap x = \mathbf{0}$
by (*subst conj-commute*) (*rule conj-cancel-right*)

lemma *conj-left-absorb* [*simp*]: $x \sqcap (x \sqcap y) = x \sqcap y$
by (*simp only*: *conj-assoc* [*symmetric*] *conj-absorb*)

lemma *conj-disj-distrib2*:

$(y \sqcup z) \sqcap x = (y \sqcap x) \sqcup (z \sqcap x)$
by (*simp only*: *conj-commute conj-disj-distrib*)

lemmas *conj-disj-distribs* =
conj-disj-distrib conj-disj-distrib2

5.3 Disjunction

lemma *disj-absorb* [*simp*]: $x \sqcup x = x$
by (*rule boolean.conj-absorb* [*OF dual*])

lemma *disj-one-right* [*simp*]: $x \sqcup \mathbf{1} = \mathbf{1}$
by (*rule boolean.conj-zero-right* [*OF dual*])

lemma *compl-zero* [*simp*]: $\sim \mathbf{0} = \mathbf{1}$
by (*rule boolean.compl-one* [*OF dual*])

lemma *disj-zero-left* [*simp*]: $\mathbf{0} \sqcup x = x$
by (*rule boolean.conj-one-left* [*OF dual*])

lemma *disj-one-left* [*simp*]: $\mathbf{1} \sqcup x = \mathbf{1}$
by (*rule boolean.conj-zero-left* [*OF dual*])

lemma *disj-cancel-left* [*simp*]: $\sim x \sqcup x = \mathbf{1}$
by (*rule boolean.conj-cancel-left* [*OF dual*])

lemma *disj-left-absorb* [*simp*]: $x \sqcup (x \sqcup y) = x \sqcup y$
by (*rule boolean.conj-left-absorb* [*OF dual*])

lemma *disj-conj-distrib2*:
$(y \sqcap z) \sqcup x = (y \sqcup x) \sqcap (z \sqcup x)$
by (*rule boolean.conj-disj-distrib2* [*OF dual*])

lemmas *disj-conj-distribs* =
disj-conj-distrib disj-conj-distrib2

5.4 De Morgan's Laws

lemma *de-Morgan-conj* [*simp*]: $\sim (x \sqcap y) = \sim x \sqcup \sim y$
proof (*rule compl-unique*)
have $(x \sqcap y) \sqcap (\sim x \sqcup \sim y) = ((x \sqcap y) \sqcap \sim x) \sqcup ((x \sqcap y) \sqcap \sim y)$
by (*rule conj-disj-distrib*)
also have ... $= (y \sqcap (x \sqcap \sim x)) \sqcup (x \sqcap (y \sqcap \sim y))$
by (*simp only*: *conj-ac*)
finally show $(x \sqcap y) \sqcap (\sim x \sqcup \sim y) = \mathbf{0}$
by (*simp only*: *conj-cancel-right conj-zero-right disj-zero-right*)
next
have $(x \sqcap y) \sqcup (\sim x \sqcup \sim y) = (x \sqcup (\sim x \sqcup \sim y)) \sqcap (y \sqcup (\sim x \sqcup \sim y))$
by (*rule disj-conj-distrib2*)
also have ... $= (\sim y \sqcup (x \sqcup \sim x)) \sqcap (\sim x \sqcup (y \sqcup \sim y))$

by (*simp only*: *disj-ac*)
finally show $(x \sqcap y) \sqcup (\sim x \sqcup \sim y) = \mathbf{1}$
by (*simp only*: *disj-cancel-right disj-one-right conj-one-right*)
qed

lemma *de-Morgan-disj* [*simp*]: $\sim (x \sqcup y) = \sim x \sqcap \sim y$
by (*rule boolean.de-Morgan-conj* [*OF dual*])

end

5.5 Symmetric Difference

locale *boolean-xor* = *boolean* +
fixes *xor* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixr** $\oplus$ *65*)
assumes *xor-def*: $x \oplus y = (x \sqcap \sim y) \sqcup (\sim x \sqcap y)$
begin

sublocale *xor*: *abel-semigroup xor*
proof
fix *x y z* :: $'a$
let *?t* $= (x \sqcap y \sqcap z) \sqcup (x \sqcap \sim y \sqcap \sim z) \sqcup$
$(\sim x \sqcap y \sqcap \sim z) \sqcup (\sim x \sqcap \sim y \sqcap z)$
have *?t* $\sqcup (z \sqcap x \sqcap \sim x) \sqcup (z \sqcap y \sqcap \sim y) =$
?t $\sqcup (x \sqcap y \sqcap \sim y) \sqcup (x \sqcap z \sqcap \sim z)$
by (*simp only*: *conj-cancel-right conj-zero-right*)
thus $(x \oplus y) \oplus z = x \oplus (y \oplus z)$
apply (*simp only*: *xor-def de-Morgan-disj de-Morgan-conj double-compl*)
apply (*simp only*: *conj-disj-distribs conj-ac disj-ac*)
done
show $x \oplus y = y \oplus x$
by (*simp only*: *xor-def conj-commute disj-commute*)
qed

lemmas *xor-assoc* = *xor.assoc*
lemmas *xor-commute* = *xor.commute*
lemmas *xor-left-commute* = *xor.left-commute*

lemmas *xor-ac* = *xor.assoc xor.commute xor.left-commute*

lemma *xor-def2*:
$x \oplus y = (x \sqcup y) \sqcap (\sim x \sqcup \sim y)$
by (*simp only*: *xor-def conj-disj-distribs*
disj-ac conj-ac conj-cancel-right disj-zero-left)

lemma *xor-zero-right* [*simp*]: $x \oplus \mathbf{0} = x$
by (*simp only*: *xor-def compl-zero conj-one-right conj-zero-right disj-zero-right*)

lemma *xor-zero-left* [*simp*]: $\mathbf{0} \oplus x = x$
by (*subst xor-commute*) (*rule xor-zero-right*)

lemma *xor-one-right* [*simp*]: $x \oplus \mathbf{1} = \sim x$
by (*simp only*: *xor-def compl-one conj-zero-right conj-one-right disj-zero-left*)

lemma *xor-one-left* [*simp*]: $\mathbf{1} \oplus x = \sim x$
by (*subst xor-commute*) (*rule xor-one-right*)

lemma *xor-self* [*simp*]: $x \oplus x = \mathbf{0}$
by (*simp only*: *xor-def conj-cancel-right conj-cancel-left disj-zero-right*)

lemma *xor-left-self* [*simp*]: $x \oplus (x \oplus y) = y$
by (*simp only*: *xor-assoc* [*symmetric*] *xor-self xor-zero-left*)

lemma *xor-compl-left* [*simp*]: $\sim x \oplus y = \sim (x \oplus y)$
apply (*simp only*: *xor-def de-Morgan-disj de-Morgan-conj double-compl*)
apply (*simp only*: *conj-disj-distribs*)
apply (*simp only*: *conj-cancel-right conj-cancel-left*)
apply (*simp only*: *disj-zero-left disj-zero-right*)
apply (*simp only*: *disj-ac conj-ac*)
done

lemma *xor-compl-right* [*simp*]: $x \oplus \sim y = \sim (x \oplus y)$
apply (*simp only*: *xor-def de-Morgan-disj de-Morgan-conj double-compl*)
apply (*simp only*: *conj-disj-distribs*)
apply (*simp only*: *conj-cancel-right conj-cancel-left*)
apply (*simp only*: *disj-zero-left disj-zero-right*)
apply (*simp only*: *disj-ac conj-ac*)
done

lemma *xor-cancel-right*: $x \oplus \sim x = \mathbf{1}$
by (*simp only*: *xor-compl-right xor-self compl-zero*)

lemma *xor-cancel-left*: $\sim x \oplus x = \mathbf{1}$
by (*simp only*: *xor-compl-left xor-self compl-zero*)

lemma *conj-xor-distrib*: $x \sqcap (y \oplus z) = (x \sqcap y) \oplus (x \sqcap z)$
proof −
  have $(x \sqcap y \sqcap \sim z) \sqcup (x \sqcap \sim y \sqcap z) =$
      $(y \sqcap x \sqcap \sim x) \sqcup (z \sqcap x \sqcap \sim x) \sqcup (x \sqcap y \sqcap \sim z) \sqcup (x \sqcap \sim y \sqcap z)$
    by (*simp only*: *conj-cancel-right conj-zero-right disj-zero-left*)
  thus $x \sqcap (y \oplus z) = (x \sqcap y) \oplus (x \sqcap z)$
    by (*simp* (*no-asm-use*) *only*:
      xor-def de-Morgan-disj de-Morgan-conj double-compl
      conj-disj-distribs conj-ac disj-ac)
qed

lemma *conj-xor-distrib2*: $(y \oplus z) \sqcap x = (y \sqcap x) \oplus (z \sqcap x)$
proof −
  have $x \sqcap (y \oplus z) = (x \sqcap y) \oplus (x \sqcap z)$

```
    by (rule conj-xor-distrib)
  thus (y ⊕ z) ⊓ x = (y ⊓ x) ⊕ (z ⊓ x)
    by (simp only: conj-commute)
qed

lemmas conj-xor-distribs = conj-xor-distrib conj-xor-distrib2

end

end
```

6 Syntactic classes for bitwise operations

```
theory Bits
imports Main
begin

class bit =
  fixes bitNOT :: 'a ⇒ 'a        (NOT - [70] 71)
    and bitAND :: 'a ⇒ 'a ⇒ 'a (infixr AND 64)
    and bitOR  :: 'a ⇒ 'a ⇒ 'a (infixr OR  59)
    and bitXOR :: 'a ⇒ 'a ⇒ 'a (infixr XOR 59)
```

We want the bitwise operations to bind slightly weaker than + and −, but ~~ to bind slightly stronger than ∗.

Testing and shifting operations.

```
class bits = bit +
  fixes test-bit :: 'a ⇒ nat ⇒ bool (infixl !! 100)
    and lsb      :: 'a ⇒ bool
    and set-bit  :: 'a ⇒ nat ⇒ bool ⇒ 'a
    and set-bits :: (nat ⇒ bool) ⇒ 'a (binder BITS  10)
    and shiftl   :: 'a ⇒ nat ⇒ 'a (infixl << 55)
    and shiftr   :: 'a ⇒ nat ⇒ 'a (infixl >> 55)

class bitss = bits +
  fixes msb      :: 'a ⇒ bool

end
```

7 The Field of Integers mod 2

```
theory Bit
imports Main
begin
```

7.1 Bits as a datatype

```
typedef bit = UNIV :: bool set
```

morphisms *set Bit*
..

instantiation *bit* :: {*zero*, *one*}
begin

definition *zero-bit-def*:
0 = *Bit False*

definition *one-bit-def*:
1 = *Bit True*

instance ..

end

old-rep-datatype *0*::*bit* *1*::*bit*
proof −
fix *P* **and** *x* :: *bit*
assume *P* (*0*::*bit*) **and** *P* (*1*::*bit*)
then have ∀ *b*. *P* (*Bit b*)
unfolding *zero-bit-def one-bit-def*
by (*simp add*: *all-bool-eq*)
then show *P x*
by (*induct x*) *simp*
next
show (*0*::*bit*) ≠ (*1*::*bit*)
unfolding *zero-bit-def one-bit-def*
by (*simp add*: *Bit-inject*)
qed

lemma *Bit-set-eq* [*simp*]:
Bit (*set b*) = *b*
by (*fact set-inverse*)

lemma *set-Bit-eq* [*simp*]:
set (*Bit P*) = *P*
by (*rule Bit-inverse*) *rule*

lemma *bit-eq-iff*:
x = *y* ⟷ (*set x* ⟷ *set y*)
by (*auto simp add*: *set-inject*)

lemma *Bit-inject* [*simp*]:
Bit P = *Bit Q* ⟷ (*P* ⟷ *Q*)
by (*auto simp add*: *Bit-inject*)

lemma *set* [*iff*]:
¬ *set 0*

set 1
by (*simp-all add*: *zero-bit-def one-bit-def Bit-inverse*)

lemma [*code*]:
set 0 $\longleftrightarrow$ *False*
set 1 $\longleftrightarrow$ *True*
by *simp-all*

lemma *set-iff*:
set b $\longleftrightarrow$ *b* = *1*
by (*cases b*) *simp-all*

lemma *bit-eq-iff-set*:
b = *0* $\longleftrightarrow$ $\neg$ *set b*
b = *1* $\longleftrightarrow$ *set b*
by (*simp-all add*: *bit-eq-iff*)

lemma *Bit* [*simp*, *code*]:
Bit False = *0*
Bit True = *1*
by (*simp-all add*: *zero-bit-def one-bit-def*)

lemma *bit-not-0-iff* [*iff*]:
(*x*::*bit*) $\neq$ *0* $\longleftrightarrow$ *x* = *1*
by (*simp add*: *bit-eq-iff*)

lemma *bit-not-1-iff* [*iff*]:
(*x*::*bit*) $\neq$ *1* $\longleftrightarrow$ *x* = *0*
by (*simp add*: *bit-eq-iff*)

lemma [*code*]:
HOL.equal 0 b $\longleftrightarrow$ $\neg$ *set b*
HOL.equal 1 b $\longleftrightarrow$ *set b*
by (*simp-all add*: *equal set-iff*)

7.2 Type *bit* forms a field

instantiation *bit* :: *field*
begin

definition *plus-bit-def*:
x + *y* = *case-bit y* (*case-bit 1 0 y*) *x*

definition *times-bit-def*:
x ∗ *y* = *case-bit 0 y x*

definition *uminus-bit-def* [*simp*]:
− *x* = (*x* :: *bit*)

definition *minus-bit-def* [*simp*]:
x − *y* = (*x* + *y* :: *bit*)

definition *inverse-bit-def* [*simp*]:
inverse x = (*x* :: *bit*)

definition *divide-bit-def* [*simp*]:
x div y = (*x* ∗ *y* :: *bit*)

lemmas *field-bit-defs* =
plus-bit-def times-bit-def minus-bit-def uminus-bit-def
divide-bit-def inverse-bit-def

instance
by *standard* (*auto simp*: *field-bit-defs split*: *bit.split*)

end

lemma *bit-add-self*: *x* + *x* = (*0* :: *bit*)
unfolding *plus-bit-def* **by** (*simp split*: *bit.split*)

lemma *bit-mult-eq-1-iff* [*simp*]: *x* ∗ *y* = (*1* :: *bit*) ⟷ *x* = *1* ∧ *y* = *1*
unfolding *times-bit-def* **by** (*simp split*: *bit.split*)

Not sure whether the next two should be simp rules.

lemma *bit-add-eq-0-iff*: *x* + *y* = (*0* :: *bit*) ⟷ *x* = *y*
unfolding *plus-bit-def* **by** (*simp split*: *bit.split*)

lemma *bit-add-eq-1-iff*: *x* + *y* = (*1* :: *bit*) ⟷ *x* ≠ *y*
unfolding *plus-bit-def* **by** (*simp split*: *bit.split*)

7.3 Numerals at type *bit*

All numerals reduce to either 0 or 1.

lemma *bit-minus1* [*simp*]: − *1* = (*1* :: *bit*)
by (*simp only*: *uminus-bit-def*)

lemma *bit-neg-numeral* [*simp*]: (− *numeral w* :: *bit*) = *numeral w*
by (*simp only*: *uminus-bit-def*)

lemma *bit-numeral-even* [*simp*]: *numeral* (*Num.Bit0 w*) = (*0* :: *bit*)
by (*simp only*: *numeral-Bit0 bit-add-self*)

lemma *bit-numeral-odd* [*simp*]: *numeral* (*Num.Bit1 w*) = (*1* :: *bit*)
by (*simp only*: *numeral-Bit1 bit-add-self add-0-left*)

7.4 Conversion from *bit*

context *zero-neq-one*

begin

definition *of-bit* :: *bit* ⇒ *'a*
where
of-bit b = *case-bit 0 1 b*

lemma *of-bit-eq* [*simp*, *code*]:
of-bit 0 = *0*
of-bit 1 = *1*
by (*simp-all add*: *of-bit-def*)

lemma *of-bit-eq-iff*:
of-bit x = *of-bit y* ⟷ *x* = *y*
by (*cases x*) (*cases y*, *simp-all*)+

end

context *semiring-1*
begin

lemma *of-nat-of-bit-eq*:
of-nat (*of-bit b*) = *of-bit b*
by (*cases b*) *simp-all*

end

context *ring-1*
begin

lemma *of-int-of-bit-eq*:
of-int (*of-bit b*) = *of-bit b*
by (*cases b*) *simp-all*

end

hide-const (**open**) *set*

end

8 Bit operations in $\mathcal{Z}_{\in}$

theory *Bits-Bit*
imports *Bits* ~~/*src*/*HOL*/*Library*/*Bit*
begin

instantiation *bit* :: *bit*
begin

primrec *bitNOT-bit* **where**

```
  NOT 0 = (1::bit)
  | NOT 1 = (0::bit)

primrec bitAND-bit where
  0 AND y = (0::bit)
  | 1 AND y = (y::bit)

primrec bitOR-bit where
  0 OR y = (y::bit)
  | 1 OR y = (1::bit)

primrec bitXOR-bit where
  0 XOR y = (y::bit)
  | 1 XOR y = (NOT y :: bit)

instance ..

end

lemmas bit-simps =
  bitNOT-bit.simps bitAND-bit.simps bitOR-bit.simps bitXOR-bit.simps

lemma bit-extra-simps [simp]:
  x AND 0 = (0::bit)
  x AND 1 = (x::bit)
  x OR 1 = (1::bit)
  x OR 0 = (x::bit)
  x XOR 1 = NOT (x::bit)
  x XOR 0 = (x::bit)
  by (cases x, auto)+

lemma bit-ops-comm:
  (x::bit) AND y = y AND x
  (x::bit) OR y = y OR x
  (x::bit) XOR y = y XOR x
  by (cases y, auto)+

lemma bit-ops-same [simp]:
  (x::bit) AND x = x
  (x::bit) OR x = x
  (x::bit) XOR x = 0
  by (cases x, auto)+

lemma bit-not-not [simp]: NOT (NOT (x::bit)) = x
  by (cases x) auto

lemma bit-or-def: (b::bit) OR c = NOT (NOT b AND NOT c)
  by (induct b, simp-all)
```

lemma *bit-xor-def*: (*b*::*bit*) *XOR c* = (*b AND NOT c*) *OR* (*NOT b AND c*)
by (*induct b*, *simp-all*)

lemma *bit-NOT-eq-1-iff* [*simp*]: *NOT* (*b*::*bit*) = *1* ⟷ *b* = *0*
by (*induct b*, *simp-all*)

lemma *bit-AND-eq-1-iff* [*simp*]: (*a*::*bit*) *AND b* = *1* ⟷ *a* = *1* ∧ *b* = *1*
by (*induct a*, *simp-all*)

end

9 Useful Numerical Lemmas

theory *Misc-Numeric*
imports *Main*
begin

lemma *mod-2-neq-1-eq-eq-0*:
fixes *k* :: *int*
shows *k mod 2* ≠ *1* ⟷ *k mod 2* = *0*
by (*fact not-mod-2-eq-1-eq-0*)

lemma *z1pmod2*:
fixes *b* :: *int*
shows (*2* ∗ *b* + *1*) *mod 2* = (*1*::*int*)
by *arith*

lemma *diff-le-eq′*:
a − *b* ≤ *c* ⟷ *a* ≤ *b* + (*c*::*int*)
by *arith*

lemma *emep1*:
fixes *n d* :: *int*
shows *even n* ⟹ *even d* ⟹ *0* ≤ *d* ⟹ (*n* + *1*) *mod d* = (*n mod d*) + *1*
by (*auto simp add*: *pos-zmod-mult-2 add.commute dvd-def*)

lemma *int-mod-ge*:
a < *n* ⟹ *0* < (*n* :: *int*) ⟹ *a* ≤ *a mod n*
by (*metis dual-order.trans le-cases mod-pos-pos-trivial pos-mod-conj*)

lemma *int-mod-ge′*:
b < *0* ⟹ *0* < (*n* :: *int*) ⟹ *b* + *n* ≤ *b mod n*
by (*metis add-less-same-cancel2 int-mod-ge mod-add-self2*)

lemma *int-mod-le′*:
(*0*::*int*) ≤ *b* − *n* ⟹ *b mod n* ≤ *b* − *n*
by (*metis minus-mod-self2 zmod-le-nonneg-dividend*)

lemma *zless2*:

0 < (2 :: int)
by (*fact zero-less-numeral*)

lemma *zless2p*:
0 < (2 ^ n :: int)
by *arith*

lemma *zle2p*:
0 ≤ (2 ^ n :: int)
by *arith*

lemma *m1mod2k*:
− 1 mod 2 ^ n = (2 ^ n − 1 :: int)
using *zless2p* **by** (*rule zmod-minus1*)

lemma *p1mod22k'*:
fixes *b :: int*
shows *(1 + 2 ∗ b) mod (2 ∗ 2 ^ n) = 1 + 2 ∗ (b mod 2 ^ n)*
using *zle2p* **by** (*rule pos-zmod-mult-2*)

lemma *p1mod22k*:
fixes *b :: int*
shows *(2 ∗ b + 1) mod (2 ∗ 2 ^ n) = 2 ∗ (b mod 2 ^ n) + 1*
by (*simp add: p1mod22k' add.commute*)

lemma *int-mod-lem*:
(0 :: int) < n ==> (0 <= b & b < n) = (b mod n = b)
apply *safe*
apply (*erule (1) mod-pos-pos-trivial*)
apply (*erule-tac [!] subst*)
apply *auto*
done

end

10 Integers as implict bit strings

theory *Bit-Representation*
imports *Misc-Numeric*
begin

10.1 Constructors and destructors for binary integers

definition *Bit :: int ⇒ bool ⇒ int* (**infixl** *BIT 90*)
where
k BIT b = (if b then 1 else 0) + k + k

lemma *Bit-B0*:
k BIT False = k + k

by (*unfold Bit-def*) *simp*

lemma *Bit-B1*:
k BIT True = k + k + 1
by (*unfold Bit-def*) *simp*

lemma *Bit-B0-2t*: *k BIT False = 2 ∗ k*
by (*rule trans*, *rule Bit-B0*) *simp*

lemma *Bit-B1-2t*: *k BIT True = 2 ∗ k + 1*
by (*rule trans*, *rule Bit-B1*) *simp*

definition *bin-last* :: *int* ⇒ *bool*
where
bin-last w ⟷ *w mod 2 = 1*

lemma *bin-last-odd*:
bin-last = odd
by (*rule ext*) (*simp add*: *bin-last-def even-iff-mod-2-eq-zero*)

definition *bin-rest* :: *int* ⇒ *int*
where
bin-rest w = w div 2

lemma *bin-rl-simp* [*simp*]:
bin-rest w BIT bin-last w = w
unfolding *bin-rest-def bin-last-def Bit-def*
using *mod-div-equality* [*of w 2*]
by (*cases w mod 2 = 0*, *simp-all*)

lemma *bin-rest-BIT* [*simp*]: *bin-rest* (*x BIT b*) = *x*
unfolding *bin-rest-def Bit-def*
by (*cases b*, *simp-all*)

lemma *bin-last-BIT* [*simp*]: *bin-last* (*x BIT b*) = *b*
unfolding *bin-last-def Bit-def*
by (*cases b*) *simp-all*

lemma *BIT-eq-iff* [*iff*]: *u BIT b = v BIT c* ⟷ *u = v* ∧ *b = c*
apply (*auto simp add*: *Bit-def*)
apply *arith*
apply *arith*
done

lemma *BIT-bin-simps* [*simp*]:
numeral k BIT False = numeral (*Num.Bit0 k*)
numeral k BIT True = numeral (*Num.Bit1 k*)
(− *numeral k*) *BIT False* = − *numeral* (*Num.Bit0 k*)
(− *numeral k*) *BIT True* = − *numeral* (*Num.BitM k*)

unfolding *numeral.simps numeral-BitM*
unfolding *Bit-def*
by (*simp-all del*: *arith-simps add-numeral-special diff-numeral-special*)

lemma *BIT-special-simps* [*simp*]:
shows *0 BIT False* = *0* **and** *0 BIT True* = *1*
and *1 BIT False* = *2* **and** *1 BIT True* = *3*
and (− *1*) *BIT False* = − *2* **and** (− *1*) *BIT True* = − *1*
unfolding *Bit-def* **by** *simp-all*

lemma *Bit-eq-0-iff*: *w BIT b* = *0* ⟷ *w* = *0* ∧ ¬ *b*
apply (*auto simp add*: *Bit-def*)
apply *arith*
done

lemma *Bit-eq-m1-iff*: *w BIT b* = −*1* ⟷ *w* = −*1* ∧ *b*
apply (*auto simp add*: *Bit-def*)
apply *arith*
done

lemma *BitM-inc*: *Num.BitM* (*Num.inc w*) = *Num.Bit1 w*
by (*induct w*, *simp-all*)

lemma *expand-BIT*:
numeral (*Num.Bit0 w*) = *numeral w BIT False*
numeral (*Num.Bit1 w*) = *numeral w BIT True*
− *numeral* (*Num.Bit0 w*) = (− *numeral w*) *BIT False*
− *numeral* (*Num.Bit1 w*) = (− *numeral* (*w* + *Num.One*)) *BIT True*
unfolding *add-One* **by** (*simp-all add*: *BitM-inc*)

lemma *bin-last-numeral-simps* [*simp*]:
¬ *bin-last 0*
bin-last 1
bin-last (− *1*)
bin-last Numeral1
¬ *bin-last* (*numeral* (*Num.Bit0 w*))
bin-last (*numeral* (*Num.Bit1 w*))
¬ *bin-last* (− *numeral* (*Num.Bit0 w*))
bin-last (− *numeral* (*Num.Bit1 w*))
by (*simp-all add*: *bin-last-def zmod-zminus1-eq-if*) (*auto simp add*: *divmod-def*)

lemma *bin-rest-numeral-simps* [*simp*]:
bin-rest 0 = *0*
bin-rest 1 = *0*
bin-rest (− *1*) = − *1*
bin-rest Numeral1 = *0*
bin-rest (*numeral* (*Num.Bit0 w*)) = *numeral w*
bin-rest (*numeral* (*Num.Bit1 w*)) = *numeral w*
bin-rest (− *numeral* (*Num.Bit0 w*)) = − *numeral w*

bin-rest (− *numeral* (*Num.Bit1 w*)) = − *numeral* (*w* + *Num.One*)
by (*simp-all add*: *bin-rest-def zdiv-zminus1-eq-if*) (*auto simp add*: *divmod-def*)

lemma *less-Bits*:
$v\ BIT\ b < w\ BIT\ c \longleftrightarrow v < w \lor v \le w \land \neg\ b \land c$
unfolding *Bit-def* **by** *auto*

lemma *le-Bits*:
$v\ BIT\ b \le w\ BIT\ c \longleftrightarrow v < w \lor v \le w \land (\neg\ b \lor c)$
unfolding *Bit-def* **by** *auto*

lemma *pred-BIT-simps* [*simp*]:
$x\ BIT\ False - 1 = (x - 1)\ BIT\ True$
$x\ BIT\ True - 1 = x\ BIT\ False$
by (*simp-all add*: *Bit-B0-2t Bit-B1-2t*)

lemma *succ-BIT-simps* [*simp*]:
$x\ BIT\ False + 1 = x\ BIT\ True$
$x\ BIT\ True + 1 = (x + 1)\ BIT\ False$
by (*simp-all add*: *Bit-B0-2t Bit-B1-2t*)

lemma *add-BIT-simps* [*simp*]:
$x\ BIT\ False + y\ BIT\ False = (x + y)\ BIT\ False$
$x\ BIT\ False + y\ BIT\ True = (x + y)\ BIT\ True$
$x\ BIT\ True + y\ BIT\ False = (x + y)\ BIT\ True$
$x\ BIT\ True + y\ BIT\ True = (x + y + 1)\ BIT\ False$
by (*simp-all add*: *Bit-B0-2t Bit-B1-2t*)

lemma *mult-BIT-simps* [*simp*]:
$x\ BIT\ False * y = (x * y)\ BIT\ False$
$x * y\ BIT\ False = (x * y)\ BIT\ False$
$x\ BIT\ True * y = (x * y)\ BIT\ False + y$
by (*simp-all add*: *Bit-B0-2t Bit-B1-2t algebra-simps*)

lemma *B-mod-2'*:
$X = 2 ==> (w\ BIT\ True)\ mod\ X = 1\ \&\ (w\ BIT\ False)\ mod\ X = 0$
apply (*simp* (*no-asm*) *only*: *Bit-B0 Bit-B1*)
apply *simp*
done

lemma *bin-ex-rl*: $EX\ w\ b.\ w\ BIT\ b = bin$
by (*metis bin-rl-simp*)

lemma *bin-exhaust*:
assumes Q: $\bigwedge x\ b.\ bin = x\ BIT\ b \Longrightarrow Q$
shows Q
apply (*insert bin-ex-rl* [*of bin*])
apply (*erule exE*)+
apply (*rule Q*)

```
  apply force
  done

primrec bin-nth where
  Z: bin-nth w 0 ⟷ bin-last w
  | Suc: bin-nth w (Suc n) ⟷ bin-nth (bin-rest w) n

lemma bin-abs-lem:
  bin = (w BIT b) ==> bin ~= -1 --> bin ~= 0 -->
    nat |w| < nat |bin|
  apply clarsimp
  apply (unfold Bit-def)
  apply (cases b)
   apply (clarsimp, arith)
  apply (clarsimp, arith)
  done

lemma bin-induct:
  assumes PPls: P 0
    and PMin: P (- 1)
    and PBit: !!bin bit. P bin ==> P (bin BIT bit)
  shows P bin
  apply (rule-tac P=P and a=bin and f1=nat o abs
                in wf-measure [THEN wf-induct])
  apply (simp add: measure-def inv-image-def)
  apply (case-tac x rule: bin-exhaust)
  apply (frule bin-abs-lem)
  apply (auto simp add : PPls PMin PBit)
  done

lemma Bit-div2 [simp]: (w BIT b) div 2 = w
  unfolding bin-rest-def [symmetric] by (rule bin-rest-BIT)

lemma bin-nth-eq-iff:
  bin-nth x = bin-nth y ⟷ x = y
proof -
  have bin-nth-lem [rule-format]: ALL y. bin-nth x = bin-nth y --> x = y
    apply (induct x rule: bin-induct)
      apply safe
      apply (erule rev-mp)
      apply (induct-tac y rule: bin-induct)
        apply safe
        apply (drule-tac x=0 in fun-cong, force)
       apply (erule notE, rule ext,
            drule-tac x=Suc x in fun-cong, force)
      apply (drule-tac x=0 in fun-cong, force)
     apply (erule rev-mp)
     apply (induct-tac y rule: bin-induct)
       apply safe
```

```
      apply (drule-tac x=0 in fun-cong, force)
     apply (erule notE, rule ext,
          drule-tac x=Suc x in fun-cong, force)
     apply (metis Bit-eq-m1-iff Z bin-last-BIT)
    apply (case-tac y rule: bin-exhaust)
    apply clarify
    apply (erule allE)
    apply (erule impE)
     prefer 2
     apply (erule conjI)
     apply (drule-tac x=0 in fun-cong, force)
    apply (rule ext)
    apply (drule-tac x=Suc x for x in fun-cong, force)
    done
  show ?thesis
  by (auto elim: bin-nth-lem)
qed

lemmas bin-eqI = ext [THEN bin-nth-eq-iff [THEN iffD1]]

lemma bin-eq-iff:
  x = y ⟷ (∀ n. bin-nth x n = bin-nth y n)
  using bin-nth-eq-iff by auto

lemma bin-nth-zero [simp]: ¬ bin-nth 0 n
  by (induct n) auto

lemma bin-nth-1 [simp]: bin-nth 1 n ⟷ n = 0
  by (cases n) simp-all

lemma bin-nth-minus1 [simp]: bin-nth (− 1) n
  by (induct n) auto

lemma bin-nth-0-BIT: bin-nth (w BIT b) 0 ⟷ b
  by auto

lemma bin-nth-Suc-BIT: bin-nth (w BIT b) (Suc n) = bin-nth w n
  by auto

lemma bin-nth-minus [simp]: 0 < n ==> bin-nth (w BIT b) n = bin-nth w (n
− 1)
  by (cases n) auto

lemma bin-nth-numeral:
  bin-rest x = y ⟹ bin-nth x (numeral n) = bin-nth y (pred-numeral n)
  by (simp add: numeral-eq-Suc)

lemmas bin-nth-numeral-simps [simp] =
  bin-nth-numeral [OF bin-rest-numeral-simps(2)]
```

bin-nth-numeral [*OF bin-rest-numeral-simps*(*5*)]
bin-nth-numeral [*OF bin-rest-numeral-simps*(*6*)]
bin-nth-numeral [*OF bin-rest-numeral-simps*(*7*)]
bin-nth-numeral [*OF bin-rest-numeral-simps*(*8*)]

lemmas *bin-nth-simps* =
bin-nth.Z bin-nth.Suc bin-nth-zero bin-nth-minus1
bin-nth-numeral-simps

10.2 Truncating binary integers

definition *bin-sign* :: *int* ⇒ *int*
where
bin-sign-def: *bin-sign k* = (*if k* ≥ *0 then 0 else* − *1*)

lemma *bin-sign-simps* [*simp*]:
bin-sign 0 = *0*
bin-sign 1 = *0*
bin-sign (− *1*) = − *1*
bin-sign (*numeral k*) = *0*
bin-sign (− *numeral k*) = −*1*
bin-sign (*w BIT b*) = *bin-sign w*
unfolding *bin-sign-def Bit-def*
by *simp-all*

lemma *bin-sign-rest* [*simp*]:
bin-sign (*bin-rest w*) = *bin-sign w*
by (*cases w rule*: *bin-exhaust*) *auto*

primrec *bintrunc* :: *nat* ⇒ *int* ⇒ *int* **where**
Z : *bintrunc 0 bin* = *0*
| *Suc* : *bintrunc* (*Suc n*) *bin* = *bintrunc n* (*bin-rest bin*) *BIT* (*bin-last bin*)

primrec *sbintrunc* :: *nat* => *int* => *int* **where**
Z : *sbintrunc 0 bin* = (*if bin-last bin then* −*1 else 0*)
| *Suc* : *sbintrunc* (*Suc n*) *bin* = *sbintrunc n* (*bin-rest bin*) *BIT* (*bin-last bin*)

lemma *sign-bintr*: *bin-sign* (*bintrunc n w*) = *0*
by (*induct n arbitrary*: *w*) *auto*

lemma *bintrunc-mod2p*: *bintrunc n w* = (*w mod 2* ˆ *n*)
apply (*induct n arbitrary*: *w*, *clarsimp*)
apply (*simp add*: *bin-last-def bin-rest-def Bit-def zmod-zmult2-eq*)
done

lemma *sbintrunc-mod2p*: *sbintrunc n w* = (*w* + *2* ˆ *n*) *mod 2* ˆ (*Suc n*) − *2* ˆ *n*
apply (*induct n arbitrary*: *w*)
apply *simp*
apply (*subst mod-add-left-eq*)

apply (*simp add*: *bin-last-def*)
apply *arith*
apply (*simp add*: *bin-last-def bin-rest-def Bit-def*)
apply (*clarsimp simp*: *mod-mult-mult1* [*symmetric*]
zmod-zdiv-equality [*THEN diff-eq-eq* [*THEN iffD2* [*THEN sym*]]])
apply (*rule trans* [*symmetric*, *OF* - *emep1*])
apply *auto*
done

10.3 Simplifications for (s)bintrunc

lemma *bintrunc-n-0* [*simp*]: *bintrunc n 0 = 0*
by (*induct n*) *auto*

lemma *sbintrunc-n-0* [*simp*]: *sbintrunc n 0 = 0*
by (*induct n*) *auto*

lemma *sbintrunc-n-minus1* [*simp*]: *sbintrunc n* (− *1*) = −*1*
by (*induct n*) *auto*

lemma *bintrunc-Suc-numeral*:
bintrunc (*Suc n*) *1* = *1*
bintrunc (*Suc n*) (− *1*) = *bintrunc n* (− *1*) *BIT True*
bintrunc (*Suc n*) (*numeral* (*Num.Bit0 w*)) = *bintrunc n* (*numeral w*) *BIT False*
bintrunc (*Suc n*) (*numeral* (*Num.Bit1 w*)) = *bintrunc n* (*numeral w*) *BIT True*
bintrunc (*Suc n*) (− *numeral* (*Num.Bit0 w*)) =
bintrunc n (− *numeral w*) *BIT False*
bintrunc (*Suc n*) (− *numeral* (*Num.Bit1 w*)) =
bintrunc n (− *numeral* (*w* + *Num.One*)) *BIT True*
by *simp-all*

lemma *sbintrunc-0-numeral* [*simp*]:
sbintrunc 0 1 = −*1*
sbintrunc 0 (*numeral* (*Num.Bit0 w*)) = *0*
sbintrunc 0 (*numeral* (*Num.Bit1 w*)) = −*1*
sbintrunc 0 (− *numeral* (*Num.Bit0 w*)) = *0*
sbintrunc 0 (− *numeral* (*Num.Bit1 w*)) = −*1*
by *simp-all*

lemma *sbintrunc-Suc-numeral*:
sbintrunc (*Suc n*) *1* = *1*
sbintrunc (*Suc n*) (*numeral* (*Num.Bit0 w*)) =
sbintrunc n (*numeral w*) *BIT False*
sbintrunc (*Suc n*) (*numeral* (*Num.Bit1 w*)) =
sbintrunc n (*numeral w*) *BIT True*
sbintrunc (*Suc n*) (− *numeral* (*Num.Bit0 w*)) =
sbintrunc n (− *numeral w*) *BIT False*
sbintrunc (*Suc n*) (− *numeral* (*Num.Bit1 w*)) =
sbintrunc n (− *numeral* (*w* + *Num.One*)) *BIT True*

by *simp-all*

lemma *bin-sign-lem*: (*bin-sign* (*sbintrunc n bin*) = −*1*) = *bin-nth bin n*
apply (*induct n arbitrary*: *bin*)
apply (*case-tac bin rule*: *bin-exhaust*, *case-tac b*, *auto*)
done

lemma *nth-bintr*: *bin-nth* (*bintrunc m w*) *n* = (*n* < *m* & *bin-nth w n*)
apply (*induct n arbitrary*: *w m*)
apply (*case-tac m*, *auto*)[*1*]
apply (*case-tac m*, *auto*)[*1*]
done

lemma *nth-sbintr*:
bin-nth (*sbintrunc m w*) *n* =
(*if n* < *m then bin-nth w n else bin-nth w m*)
apply (*induct n arbitrary*: *w m*)
apply (*case-tac m*)
apply *simp-all*
apply (*case-tac m*)
apply *simp-all*
done

lemma *bin-nth-Bit*:
bin-nth (*w BIT b*) *n* = (*n* = *0* & *b* | (*EX m*. *n* = *Suc m* & *bin-nth w m*))
by (*cases n*) *auto*

lemma *bin-nth-Bit0*:
bin-nth (*numeral* (*Num.Bit0 w*)) *n* ⟷
(∃ *m*. *n* = *Suc m* ∧ *bin-nth* (*numeral w*) *m*)
using *bin-nth-Bit* [**where** *w*=*numeral w* **and** *b*=*False*] **by** *simp*

lemma *bin-nth-Bit1*:
bin-nth (*numeral* (*Num.Bit1 w*)) *n* ⟷
n = *0* ∨ (∃ *m*. *n* = *Suc m* ∧ *bin-nth* (*numeral w*) *m*)
using *bin-nth-Bit* [**where** *w*=*numeral w* **and** *b*=*True*] **by** *simp*

lemma *bintrunc-bintrunc-l*:
n <= *m* ==> (*bintrunc m* (*bintrunc n w*) = *bintrunc n w*)
by (*rule bin-eqI*) (*auto simp add* : *nth-bintr*)

lemma *sbintrunc-sbintrunc-l*:
n <= *m* ==> (*sbintrunc m* (*sbintrunc n w*) = *sbintrunc n w*)
by (*rule bin-eqI*) (*auto simp*: *nth-sbintr*)

lemma *bintrunc-bintrunc-ge*:
n <= *m* ==> (*bintrunc n* (*bintrunc m w*) = *bintrunc n w*)
by (*rule bin-eqI*) (*auto simp*: *nth-bintr*)

lemma *bintrunc-bintrunc-min* [*simp*]:
bintrunc m (*bintrunc n w*) = *bintrunc* (*min m n*) *w*
apply (*rule bin-eqI*)
apply (*auto simp*: *nth-bintr*)
done

lemma *sbintrunc-sbintrunc-min* [*simp*]:
sbintrunc m (*sbintrunc n w*) = *sbintrunc* (*min m n*) *w*
apply (*rule bin-eqI*)
apply (*auto simp*: *nth-sbintr min.absorb1 min.absorb2*)
done

lemmas *bintrunc-Pls* =
bintrunc.Suc [**where** *bin=0*, *simplified bin-last-numeral-simps bin-rest-numeral-simps*]

lemmas *bintrunc-Min* [*simp*] =
bintrunc.Suc [**where** *bin=−1*, *simplified bin-last-numeral-simps bin-rest-numeral-simps*]

lemmas *bintrunc-BIT* [*simp*] =
bintrunc.Suc [**where** *bin=w BIT b*, *simplified bin-last-BIT bin-rest-BIT*] **for** *w b*

lemmas *bintrunc-Sucs* = *bintrunc-Pls bintrunc-Min bintrunc-BIT*
bintrunc-Suc-numeral

lemmas *sbintrunc-Suc-Pls* =
sbintrunc.Suc [**where** *bin=0*, *simplified bin-last-numeral-simps bin-rest-numeral-simps*]

lemmas *sbintrunc-Suc-Min* =
sbintrunc.Suc [**where** *bin=−1*, *simplified bin-last-numeral-simps bin-rest-numeral-simps*]

lemmas *sbintrunc-Suc-BIT* [*simp*] =
sbintrunc.Suc [**where** *bin=w BIT b*, *simplified bin-last-BIT bin-rest-BIT*] **for** *w*
b

lemmas *sbintrunc-Sucs* = *sbintrunc-Suc-Pls sbintrunc-Suc-Min sbintrunc-Suc-BIT*
sbintrunc-Suc-numeral

lemmas *sbintrunc-Pls* =
sbintrunc.Z [**where** *bin=0*,
simplified bin-last-numeral-simps bin-rest-numeral-simps]

lemmas *sbintrunc-Min* =
sbintrunc.Z [**where** *bin=−1*,
simplified bin-last-numeral-simps bin-rest-numeral-simps]

lemmas *sbintrunc-0-BIT-B0* [*simp*] =
sbintrunc.Z [**where** *bin=w BIT False*,
simplified bin-last-numeral-simps bin-rest-numeral-simps] **for** *w*

lemmas *sbintrunc-0-BIT-B1* [*simp*] =
sbintrunc.Z [**where** *bin=w BIT True*,
simplified bin-last-BIT bin-rest-numeral-simps] **for** *w*

lemmas *sbintrunc-0-simps* =
sbintrunc-Pls sbintrunc-Min sbintrunc-0-BIT-B0 sbintrunc-0-BIT-B1

lemmas *bintrunc-simps* = *bintrunc.Z bintrunc-Sucs*
lemmas *sbintrunc-simps* = *sbintrunc-0-simps sbintrunc-Sucs*

lemma *bintrunc-minus*:
0 < *n* ==> *bintrunc* (*Suc* (*n* − *1*)) *w* = *bintrunc n w*
by *auto*

lemma *sbintrunc-minus*:
0 < *n* ==> *sbintrunc* (*Suc* (*n* − *1*)) *w* = *sbintrunc n w*
by *auto*

lemmas *bintrunc-minus-simps* =
bintrunc-Sucs [*THEN* [*2*] *bintrunc-minus* [*symmetric*, *THEN trans*]]
lemmas *sbintrunc-minus-simps* =
sbintrunc-Sucs [*THEN* [*2*] *sbintrunc-minus* [*symmetric*, *THEN trans*]]

lemmas *thobini1* = *arg-cong* [**where** *f* = *%w. w BIT b*] **for** *b*

lemmas *bintrunc-BIT-I* = *trans* [*OF bintrunc-BIT thobini1*]
lemmas *bintrunc-Min-I* = *trans* [*OF bintrunc-Min thobini1*]

lemmas *bmsts* = *bintrunc-minus-simps*(*1*−*3*) [*THEN thobini1* [*THEN* [*2*] *trans*]]
lemmas *bintrunc-Pls-minus-I* = *bmsts*(*1*)
lemmas *bintrunc-Min-minus-I* = *bmsts*(*2*)
lemmas *bintrunc-BIT-minus-I* = *bmsts*(*3*)

lemma *bintrunc-Suc-lem*:
bintrunc (*Suc n*) *x* = *y* ==> *m* = *Suc n* ==> *bintrunc m x* = *y*
by *auto*

lemmas *bintrunc-Suc-Ialts* =
bintrunc-Min-I [*THEN bintrunc-Suc-lem*]
bintrunc-BIT-I [*THEN bintrunc-Suc-lem*]

lemmas *sbintrunc-BIT-I* = *trans* [*OF sbintrunc-Suc-BIT thobini1*]

lemmas *sbintrunc-Suc-Is* =
sbintrunc-Sucs(*1*−*3*) [*THEN thobini1* [*THEN* [*2*] *trans*]]

lemmas *sbintrunc-Suc-minus-Is* =
sbintrunc-minus-simps(*1*−*3*) [*THEN thobini1* [*THEN* [*2*] *trans*]]

lemma *sbintrunc-Suc-lem*:
sbintrunc (*Suc n*) *x* = *y* ==> *m* = *Suc n* ==> *sbintrunc m x* = *y*
by *auto*

lemmas *sbintrunc-Suc-Ialts* =
sbintrunc-Suc-Is [*THEN sbintrunc-Suc-lem*]

lemma *sbintrunc-bintrunc-lt*:
m > *n* ==> *sbintrunc n* (*bintrunc m w*) = *sbintrunc n w*
by (*rule bin-eqI*) (*auto simp*: *nth-sbintr nth-bintr*)

lemma *bintrunc-sbintrunc-le*:
m <= *Suc n* ==> *bintrunc m* (*sbintrunc n w*) = *bintrunc m w*
apply (*rule bin-eqI*)
apply (*auto simp*: *nth-sbintr nth-bintr*)
apply (*subgoal-tac x*=*n*, *safe*, *arith*+)[*1*]
apply (*subgoal-tac x*=*n*, *safe*, *arith*+)[*1*]
done

lemmas *bintrunc-sbintrunc* [*simp*] = *order-refl* [*THEN bintrunc-sbintrunc-le*]
lemmas *sbintrunc-bintrunc* [*simp*] = *lessI* [*THEN sbintrunc-bintrunc-lt*]
lemmas *bintrunc-bintrunc* [*simp*] = *order-refl* [*THEN bintrunc-bintrunc-l*]
lemmas *sbintrunc-sbintrunc* [*simp*] = *order-refl* [*THEN sbintrunc-sbintrunc-l*]

lemma *bintrunc-sbintrunc′* [*simp*]:
0 < *n* ⟹ *bintrunc n* (*sbintrunc* (*n* − *1*) *w*) = *bintrunc n w*
by (*cases n*) (*auto simp del*: *bintrunc.Suc*)

lemma *sbintrunc-bintrunc′* [*simp*]:
0 < *n* ⟹ *sbintrunc* (*n* − *1*) (*bintrunc n w*) = *sbintrunc* (*n* − *1*) *w*
by (*cases n*) (*auto simp del*: *bintrunc.Suc*)

lemma *bin-sbin-eq-iff*:
bintrunc (*Suc n*) *x* = *bintrunc* (*Suc n*) *y* ⟷
sbintrunc n x = *sbintrunc n y*
apply (*rule iffI*)
apply (*rule box-equals* [*OF - sbintrunc-bintrunc sbintrunc-bintrunc*])
apply *simp*
apply (*rule box-equals* [*OF - bintrunc-sbintrunc bintrunc-sbintrunc*])
apply *simp*
done

lemma *bin-sbin-eq-iff′*:
0 < *n* ⟹ *bintrunc n x* = *bintrunc n y* ⟷
sbintrunc (*n* − *1*) *x* = *sbintrunc* (*n* − *1*) *y*
by (*cases n*) (*simp-all add*: *bin-sbin-eq-iff del*: *bintrunc.Suc*)

lemmas *bintrunc-sbintruncS0* [*simp*] = *bintrunc-sbintrunc′* [*unfolded One-nat-def*]
lemmas *sbintrunc-bintruncS0* [*simp*] = *sbintrunc-bintrunc′* [*unfolded One-nat-def*]

lemmas *bintrunc-bintrunc-l′* = *le-add1* [*THEN bintrunc-bintrunc-l*]
lemmas *sbintrunc-sbintrunc-l′* = *le-add1* [*THEN sbintrunc-sbintrunc-l*]

lemmas *nat-non0-gr* =
trans [*OF iszero-def* [*THEN Not-eq-iff* [*THEN iffD2*]] *refl*]

lemma *bintrunc-numeral*:
bintrunc (*numeral k*) *x* =
bintrunc (*pred-numeral k*) (*bin-rest x*) *BIT bin-last x*
by (*simp add*: *numeral-eq-Suc*)

lemma *sbintrunc-numeral*:
sbintrunc (*numeral k*) *x* =
sbintrunc (*pred-numeral k*) (*bin-rest x*) *BIT bin-last x*
by (*simp add*: *numeral-eq-Suc*)

lemma *bintrunc-numeral-simps* [*simp*]:
bintrunc (*numeral k*) (*numeral* (*Num.Bit0 w*)) =
bintrunc (*pred-numeral k*) (*numeral w*) *BIT False*
bintrunc (*numeral k*) (*numeral* (*Num.Bit1 w*)) =
bintrunc (*pred-numeral k*) (*numeral w*) *BIT True*
bintrunc (*numeral k*) (− *numeral* (*Num.Bit0 w*)) =
bintrunc (*pred-numeral k*) (− *numeral w*) *BIT False*
bintrunc (*numeral k*) (− *numeral* (*Num.Bit1 w*)) =
bintrunc (*pred-numeral k*) (− *numeral* (*w* + *Num.One*)) *BIT True*
bintrunc (*numeral k*) *1* = *1*
by (*simp-all add*: *bintrunc-numeral*)

lemma *sbintrunc-numeral-simps* [*simp*]:
sbintrunc (*numeral k*) (*numeral* (*Num.Bit0 w*)) =
sbintrunc (*pred-numeral k*) (*numeral w*) *BIT False*
sbintrunc (*numeral k*) (*numeral* (*Num.Bit1 w*)) =
sbintrunc (*pred-numeral k*) (*numeral w*) *BIT True*
sbintrunc (*numeral k*) (− *numeral* (*Num.Bit0 w*)) =
sbintrunc (*pred-numeral k*) (− *numeral w*) *BIT False*
sbintrunc (*numeral k*) (− *numeral* (*Num.Bit1 w*)) =
sbintrunc (*pred-numeral k*) (− *numeral* (*w* + *Num.One*)) *BIT True*
sbintrunc (*numeral k*) *1* = *1*
by (*simp-all add*: *sbintrunc-numeral*)

lemma *no-bintr-alt1*: *bintrunc n* = (λ*w*. *w mod 2* ˆ *n* :: *int*)
by (*rule ext*) (*rule bintrunc-mod2p*)

lemma *range-bintrunc*: *range* (*bintrunc n*) = {*i*. *0* <= *i* & *i* < *2* ˆ *n*}
apply (*unfold no-bintr-alt1*)
apply (*auto simp add*: *image-iff*)

apply (*rule exI*)
apply (*auto intro*: *int-mod-lem* [*THEN iffD1*, *symmetric*])
done

lemma *no-sbintr-alt2*:
sbintrunc n = (%*w*. (*w* + *2* ˆ *n*) *mod 2* ˆ *Suc n* − *2* ˆ *n* :: *int*)
by (*rule ext*) (*simp add* : *sbintrunc-mod2p*)

lemma *range-sbintrunc*:
range (*sbintrunc n*) = {*i*. − (*2* ˆ *n*) <= *i* & *i* < *2* ˆ *n*}
apply (*unfold no-sbintr-alt2*)
apply (*auto simp add*: *image-iff eq-diff-eq*)
apply (*rule exI*)
apply (*auto intro*: *int-mod-lem* [*THEN iffD1*, *symmetric*])
done

lemma *sb-inc-lem*:
(*a*::*int*) + *2*ˆ*k* < *0* ⟹ *a* + *2*ˆ*k* + *2*ˆ(*Suc k*) <= (*a* + *2*ˆ*k*) *mod 2*ˆ(*Suc k*)
apply (*erule int-mod-ge′* [**where** *n* = *2* ˆ (*Suc k*) **and** *b* = *a* + *2* ˆ *k*, *simplified zless2p*])
apply (*rule TrueI*)
done

lemma *sb-inc-lem′*:
(*a*::*int*) < − (*2*ˆ*k*) ⟹ *a* + *2*ˆ*k* + *2*ˆ(*Suc k*) <= (*a* + *2*ˆ*k*) *mod 2*ˆ(*Suc k*)
by (*rule sb-inc-lem*) *simp*

lemma *sbintrunc-inc*:
x < − (*2*ˆ*n*) ==> *x* + *2*ˆ(*Suc n*) <= *sbintrunc n x*
unfolding *no-sbintr-alt2* **by** (*drule sb-inc-lem′*) *simp*

lemma *sb-dec-lem*:
(*0*::*int*) ≤ − (*2* ˆ *k*) + *a* ⟹ (*a* + *2* ˆ *k*) *mod* (*2* ∗ *2* ˆ *k*) ≤ − (*2* ˆ *k*) + *a*
using *int-mod-le′*[**where** *n* = *2* ˆ (*Suc k*) **and** *b* = *a* + *2* ˆ *k*] **by** *simp*

lemma *sb-dec-lem′*:
(*2*::*int*) ˆ *k* ≤ *a* ⟹ (*a* + *2* ˆ *k*) *mod* (*2* ∗ *2* ˆ *k*) ≤ − (*2* ˆ *k*) + *a*
by (*rule sb-dec-lem*) *simp*

lemma *sbintrunc-dec*:
x >= (*2* ˆ *n*) ==> *x* − *2* ˆ (*Suc n*) >= *sbintrunc n x*
unfolding *no-sbintr-alt2* **by** (*drule sb-dec-lem′*) *simp*

lemmas *zmod-uminus′* = *zminus-zmod* [**where** *m*=*c*] **for** *c*
lemmas *zpower-zmod′* = *power-mod* [**where** *b*=*c* **and** *n*=*k*] **for** *c k*

lemmas *brdmod1s′* [*symmetric*] =
mod-add-left-eq mod-add-right-eq
mod-diff-left-eq mod-diff-right-eq

mod-mult-left-eq mod-mult-right-eq

lemmas *brdmods′* [*symmetric*] =
zpower-zmod′ [*symmetric*]
trans [*OF mod-add-left-eq mod-add-right-eq*]
trans [*OF mod-diff-left-eq mod-diff-right-eq*]
trans [*OF mod-mult-right-eq mod-mult-left-eq*]
zmod-uminus′ [*symmetric*]
mod-add-left-eq [**where** *b* = *1*::*int*]
mod-diff-left-eq [**where** *b* = *1*::*int*]

lemmas *bintr-arith1s* =
brdmod1s′ [**where** *c*=*2*ˆ*n*::*int*, *folded bintrunc-mod2p*] **for** *n*
lemmas *bintr-ariths* =
brdmods′ [**where** *c*=*2*ˆ*n*::*int*, *folded bintrunc-mod2p*] **for** *n*

lemmas *m2pths* = *pos-mod-sign pos-mod-bound* [*OF zless2p*]

lemma *bintr-ge0*: *0* ≤ *bintrunc n w*
by (*simp add*: *bintrunc-mod2p*)

lemma *bintr-lt2p*: *bintrunc n w* < *2* ˆ *n*
by (*simp add*: *bintrunc-mod2p*)

lemma *bintr-Min*: *bintrunc n* (− *1*) = *2* ˆ *n* − *1*
by (*simp add*: *bintrunc-mod2p m1mod2k*)

lemma *sbintr-ge*: − (*2* ˆ *n*) ≤ *sbintrunc n w*
by (*simp add*: *sbintrunc-mod2p*)

lemma *sbintr-lt*: *sbintrunc n w* < *2* ˆ *n*
by (*simp add*: *sbintrunc-mod2p*)

lemma *sign-Pls-ge-0*:
(*bin-sign bin* = *0*) = (*bin* >= (*0* :: *int*))
unfolding *bin-sign-def* **by** *simp*

lemma *sign-Min-lt-0*:
(*bin-sign bin* = −*1*) = (*bin* < (*0* :: *int*))
unfolding *bin-sign-def* **by** *simp*

lemma *bin-rest-trunc*:
(*bin-rest* (*bintrunc n bin*)) = *bintrunc* (*n* − *1*) (*bin-rest bin*)
by (*induct n arbitrary*: *bin*) *auto*

lemma *bin-rest-power-trunc*:
(*bin-rest* ˆˆ *k*) (*bintrunc n bin*) =
bintrunc (*n* − *k*) ((*bin-rest* ˆˆ *k*) *bin*)
by (*induct k*) (*auto simp*: *bin-rest-trunc*)

lemma *bin-rest-trunc-i*:
bintrunc n (*bin-rest bin*) = *bin-rest* (*bintrunc* (*Suc n*) *bin*)
by *auto*

lemma *bin-rest-strunc*:
bin-rest (*sbintrunc* (*Suc n*) *bin*) = *sbintrunc n* (*bin-rest bin*)
by (*induct n arbitrary*: *bin*) *auto*

lemma *bintrunc-rest* [*simp*]:
bintrunc n (*bin-rest* (*bintrunc n bin*)) = *bin-rest* (*bintrunc n bin*)
apply (*induct n arbitrary*: *bin*, *simp*)
apply (*case-tac bin rule*: *bin-exhaust*)
apply (*auto simp*: *bintrunc-bintrunc-l*)
done

lemma *sbintrunc-rest* [*simp*]:
sbintrunc n (*bin-rest* (*sbintrunc n bin*)) = *bin-rest* (*sbintrunc n bin*)
apply (*induct n arbitrary*: *bin*, *simp*)
apply (*case-tac bin rule*: *bin-exhaust*)
apply (*auto simp*: *bintrunc-bintrunc-l split*: *bool.splits*)
done

lemma *bintrunc-rest'*:
bintrunc n o bin-rest o bintrunc n = *bin-rest o bintrunc n*
by (*rule ext*) *auto*

lemma *sbintrunc-rest'* :
sbintrunc n o bin-rest o sbintrunc n = *bin-rest o sbintrunc n*
by (*rule ext*) *auto*

lemma *rco-lem*:
f o g o f = *g o f* ==> *f o* (*g o f*) ^^ *n* = *g* ^^ *n o f*
apply (*rule ext*)
apply (*induct-tac n*)
apply (*simp-all* (*no-asm*))
apply (*drule fun-cong*)
apply (*unfold o-def*)
apply (*erule trans*)
apply *simp*
done

lemmas *rco-bintr* = *bintrunc-rest'*
[*THEN rco-lem* [*THEN fun-cong*], *unfolded o-def*]
lemmas *rco-sbintr* = *sbintrunc-rest'*
[*THEN rco-lem* [*THEN fun-cong*], *unfolded o-def*]

10.4 Splitting and concatenation

primrec *bin-split* :: *nat* ⇒ *int* ⇒ *int* × *int* **where**
Z: *bin-split 0 w* = (*w*, *0*)
| *Suc*: *bin-split* (*Suc n*) *w* = (*let* (*w1*, *w2*) = *bin-split n* (*bin-rest w*)
in (*w1*, *w2 BIT bin-last w*))

lemma [*code*]:
bin-split (*Suc n*) *w* = (*let* (*w1*, *w2*) = *bin-split n* (*bin-rest w*) *in* (*w1*, *w2 BIT bin-last w*))
bin-split 0 w = (*w*, *0*)
by *simp-all*

primrec *bin-cat* :: *int* ⇒ *nat* ⇒ *int* ⇒ *int* **where**
Z: *bin-cat w 0 v* = *w*
| *Suc*: *bin-cat w* (*Suc n*) *v* = *bin-cat w n* (*bin-rest v*) *BIT bin-last v*

end

11 Bitwise Operations on Binary Integers

theory *Bits-Int*
imports *Bits Bit-Representation*
begin

11.1 Logical operations

bit-wise logical operations on the int type

instantiation *int* :: *bit*
begin

definition *int-not-def*:
bitNOT = (λ*x*::*int*. − *x* − *1*)

function *bitAND-int* **where**
bitAND-int x y =
(*if x* = *0 then 0 else if x* = −*1 then y else*
(*bin-rest x AND bin-rest y*) *BIT* (*bin-last x* ∧ *bin-last y*))
by *pat-completeness simp*

termination
by (*relation measure* (*nat o abs o fst*), *simp-all add*: *bin-rest-def*)

declare *bitAND-int.simps* [*simp del*]

definition *int-or-def*:
bitOR = (λ*x y*::*int*. *NOT* (*NOT x AND NOT y*))

definition *int-xor-def*:

bitXOR = (λ*x y*::*int*. (*x AND NOT y*) *OR* (*NOT x AND y*))

instance ..

end

11.1.1 Basic simplification rules

lemma *int-not-BIT* [*simp*]:
NOT (*w BIT b*) = (*NOT w*) *BIT* (¬ *b*)
unfolding *int-not-def Bit-def* **by** (*cases b*, *simp-all*)

lemma *int-not-simps* [*simp*]:
NOT (*0*::*int*) = −*1*
NOT (*1*::*int*) = −*2*
NOT (− *1*::*int*) = *0*
NOT (*numeral w*::*int*) = − *numeral* (*w* + *Num.One*)
NOT (− *numeral* (*Num.Bit0 w*)::*int*) = *numeral* (*Num.BitM w*)
NOT (− *numeral* (*Num.Bit1 w*)::*int*) = *numeral* (*Num.Bit0 w*)
unfolding *int-not-def* **by** *simp-all*

lemma *int-not-not* [*simp*]: *NOT* (*NOT* (*x*::*int*)) = *x*
unfolding *int-not-def* **by** *simp*

lemma *int-and-0* [*simp*]: (*0*::*int*) *AND x* = *0*
by (*simp add*: *bitAND-int.simps*)

lemma *int-and-m1* [*simp*]: (−*1*::*int*) *AND x* = *x*
by (*simp add*: *bitAND-int.simps*)

lemma *int-and-Bits* [*simp*]:
(*x BIT b*) *AND* (*y BIT c*) = (*x AND y*) *BIT* (*b* ∧ *c*)
by (*subst bitAND-int.simps*, *simp add*: *Bit-eq-0-iff Bit-eq-m1-iff*)

lemma *int-or-zero* [*simp*]: (*0*::*int*) *OR x* = *x*
unfolding *int-or-def* **by** *simp*

lemma *int-or-minus1* [*simp*]: (−*1*::*int*) *OR x* = −*1*
unfolding *int-or-def* **by** *simp*

lemma *int-or-Bits* [*simp*]:
(*x BIT b*) *OR* (*y BIT c*) = (*x OR y*) *BIT* (*b* ∨ *c*)
unfolding *int-or-def* **by** *simp*

lemma *int-xor-zero* [*simp*]: (*0*::*int*) *XOR x* = *x*
unfolding *int-xor-def* **by** *simp*

lemma *int-xor-Bits* [*simp*]:
(*x BIT b*) *XOR* (*y BIT c*) = (*x XOR y*) *BIT* ((*b* ∨ *c*) ∧ ¬ (*b* ∧ *c*))

unfolding *int-xor-def* **by** *auto*

11.1.2 Binary destructors

lemma *bin-rest-NOT* [*simp*]: *bin-rest* (*NOT x*) = *NOT* (*bin-rest x*)
by (*cases x rule*: *bin-exhaust*, *simp*)

lemma *bin-last-NOT* [*simp*]: *bin-last* (*NOT x*) $\longleftrightarrow$ $\neg$ *bin-last x*
by (*cases x rule*: *bin-exhaust*, *simp*)

lemma *bin-rest-AND* [*simp*]: *bin-rest* (*x AND y*) = *bin-rest x AND bin-rest y*
by (*cases x rule*: *bin-exhaust*, *cases y rule*: *bin-exhaust*, *simp*)

lemma *bin-last-AND* [*simp*]: *bin-last* (*x AND y*) $\longleftrightarrow$ *bin-last x* $\wedge$ *bin-last y*
by (*cases x rule*: *bin-exhaust*, *cases y rule*: *bin-exhaust*, *simp*)

lemma *bin-rest-OR* [*simp*]: *bin-rest* (*x OR y*) = *bin-rest x OR bin-rest y*
by (*cases x rule*: *bin-exhaust*, *cases y rule*: *bin-exhaust*, *simp*)

lemma *bin-last-OR* [*simp*]: *bin-last* (*x OR y*) $\longleftrightarrow$ *bin-last x* $\vee$ *bin-last y*
by (*cases x rule*: *bin-exhaust*, *cases y rule*: *bin-exhaust*, *simp*)

lemma *bin-rest-XOR* [*simp*]: *bin-rest* (*x XOR y*) = *bin-rest x XOR bin-rest y*
by (*cases x rule*: *bin-exhaust*, *cases y rule*: *bin-exhaust*, *simp*)

lemma *bin-last-XOR* [*simp*]: *bin-last* (*x XOR y*) $\longleftrightarrow$ (*bin-last x* $\vee$ *bin-last y*) $\wedge$
$\neg$ (*bin-last x* $\wedge$ *bin-last y*)
by (*cases x rule*: *bin-exhaust*, *cases y rule*: *bin-exhaust*, *simp*)

lemma *bin-nth-ops*:
!!*x y*. *bin-nth* (*x AND y*) *n* = (*bin-nth x n* & *bin-nth y n*)
!!*x y*. *bin-nth* (*x OR y*) *n* = (*bin-nth x n* | *bin-nth y n*)
!!*x y*. *bin-nth* (*x XOR y*) *n* = (*bin-nth x n* ~= *bin-nth y n*)
!!*x*. *bin-nth* (*NOT x*) *n* = (~ *bin-nth x n*)
by (*induct n*) *auto*

11.1.3 Derived properties

lemma *int-xor-minus1* [*simp*]: (−*1*::*int*) *XOR x* = *NOT x*
by (*auto simp add*: *bin-eq-iff bin-nth-ops*)

lemma *int-xor-extra-simps* [*simp*]:
w XOR (*0*::*int*) = *w*
w XOR (−*1*::*int*) = *NOT w*
by (*auto simp add*: *bin-eq-iff bin-nth-ops*)

lemma *int-or-extra-simps* [*simp*]:
w OR (*0*::*int*) = *w*
w OR (−*1*::*int*) = −*1*
by (*auto simp add*: *bin-eq-iff bin-nth-ops*)

lemma *int-and-extra-simps* [*simp*]:
w AND (*0*::*int*) = *0*
w AND (−*1*::*int*) = *w*
by (*auto simp add*: *bin-eq-iff bin-nth-ops*)

lemma *bin-ops-comm*:
shows
int-and-comm: !!*y*::*int*. *x AND y* = *y AND x* **and**
int-or-comm: !!*y*::*int*. *x OR y* = *y OR x* **and**
int-xor-comm: !!*y*::*int*. *x XOR y* = *y XOR x*
by (*auto simp add*: *bin-eq-iff bin-nth-ops*)

lemma *bin-ops-same* [*simp*]:
(*x*::*int*) *AND x* = *x*
(*x*::*int*) *OR x* = *x*
(*x*::*int*) *XOR x* = *0*
by (*auto simp add*: *bin-eq-iff bin-nth-ops*)

lemmas *bin-log-esimps* =
int-and-extra-simps int-or-extra-simps int-xor-extra-simps
int-and-0 int-and-m1 int-or-zero int-or-minus1 int-xor-zero int-xor-minus1

lemma *bbw-ao-absorb*:
!!*y*::*int*. *x AND* (*y OR x*) = *x* & *x OR* (*y AND x*) = *x*
by (*auto simp add*: *bin-eq-iff bin-nth-ops*)

lemma *bbw-ao-absorbs-other*:
x AND (*x OR y*) = *x* ∧ (*y AND x*) *OR x* = (*x*::*int*)
(*y OR x*) *AND x* = *x* ∧ *x OR* (*x AND y*) = (*x*::*int*)
(*x OR y*) *AND x* = *x* ∧ (*x AND y*) *OR x* = (*x*::*int*)
by (*auto simp add*: *bin-eq-iff bin-nth-ops*)

lemmas *bbw-ao-absorbs* [*simp*] = *bbw-ao-absorb bbw-ao-absorbs-other*

lemma *int-xor-not*:
!!*y*::*int*. (*NOT x*) *XOR y* = *NOT* (*x XOR y*) &
x XOR (*NOT y*) = *NOT* (*x XOR y*)
by (*auto simp add*: *bin-eq-iff bin-nth-ops*)

lemma *int-and-assoc*:
(*x AND y*) *AND* (*z*::*int*) = *x AND* (*y AND z*)
by (*auto simp add*: *bin-eq-iff bin-nth-ops*)

lemma *int-or-assoc*:
(*x OR y*) *OR* (*z*::*int*) = *x OR* (*y OR z*)

by (*auto simp add*: *bin-eq-iff bin-nth-ops*)

lemma *int-xor-assoc*:
(*x XOR y*) *XOR* (*z*::*int*) = *x XOR* (*y XOR z*)
by (*auto simp add*: *bin-eq-iff bin-nth-ops*)

lemmas *bbw-assocs* = *int-and-assoc int-or-assoc int-xor-assoc*

lemma *bbw-lcs* [*simp*]:
(*y*::*int*) *AND* (*x AND z*) = *x AND* (*y AND z*)
(*y*::*int*) *OR* (*x OR z*) = *x OR* (*y OR z*)
(*y*::*int*) *XOR* (*x XOR z*) = *x XOR* (*y XOR z*)
by (*auto simp add*: *bin-eq-iff bin-nth-ops*)

lemma *bbw-not-dist*:
!!*y*::*int*. *NOT* (*x OR y*) = (*NOT x*) *AND* (*NOT y*)
!!*y*::*int*. *NOT* (*x AND y*) = (*NOT x*) *OR* (*NOT y*)
by (*auto simp add*: *bin-eq-iff bin-nth-ops*)

lemma *bbw-oa-dist*:
!!*y z*::*int*. (*x AND y*) *OR z* =
(*x OR z*) *AND* (*y OR z*)
by (*auto simp add*: *bin-eq-iff bin-nth-ops*)

lemma *bbw-ao-dist*:
!!*y z*::*int*. (*x OR y*) *AND z* =
(*x AND z*) *OR* (*y AND z*)
by (*auto simp add*: *bin-eq-iff bin-nth-ops*)

11.1.4 Simplification with numerals

Cases for *0* and *−1* are already covered by other simp rules.

lemma *bin-rl-eqI*: ⟦*bin-rest x* = *bin-rest y*; *bin-last x* = *bin-last y*⟧ ⟹ *x* = *y*
by (*metis* (*mono-tags*) *BIT-eq-iff bin-ex-rl bin-last-BIT bin-rest-BIT*)

lemma *bin-rest-neg-numeral-BitM* [*simp*]:
bin-rest (− *numeral* (*Num.BitM w*)) = − *numeral w*
by (*simp only*: *BIT-bin-simps* [*symmetric*] *bin-rest-BIT*)

lemma *bin-last-neg-numeral-BitM* [*simp*]:
bin-last (− *numeral* (*Num.BitM w*))
by (*simp only*: *BIT-bin-simps* [*symmetric*] *bin-last-BIT*)

FIXME: The rule sets below are very large (24 rules for each operator). Is there a simpler way to do this?

lemma *int-and-numerals* [*simp*]:
numeral (*Num.Bit0 x*) *AND numeral* (*Num.Bit0 y*) = (*numeral x AND numeral y*) *BIT False*

numeral (Num.Bit0 x) AND numeral (Num.Bit1 y) = (numeral x AND numeral y) BIT False
numeral (Num.Bit1 x) AND numeral (Num.Bit0 y) = (numeral x AND numeral y) BIT False
numeral (Num.Bit1 x) AND numeral (Num.Bit1 y) = (numeral x AND numeral y) BIT True
numeral (Num.Bit0 x) AND − numeral (Num.Bit0 y) = (numeral x AND − numeral y) BIT False
numeral (Num.Bit0 x) AND − numeral (Num.Bit1 y) = (numeral x AND − numeral (y + Num.One)) BIT False
numeral (Num.Bit1 x) AND − numeral (Num.Bit0 y) = (numeral x AND − numeral y) BIT False
numeral (Num.Bit1 x) AND − numeral (Num.Bit1 y) = (numeral x AND − numeral (y + Num.One)) BIT True
− numeral (Num.Bit0 x) AND numeral (Num.Bit0 y) = (− numeral x AND numeral y) BIT False
− numeral (Num.Bit0 x) AND numeral (Num.Bit1 y) = (− numeral x AND numeral y) BIT False
− numeral (Num.Bit1 x) AND numeral (Num.Bit0 y) = (− numeral (x + Num.One) AND numeral y) BIT False
− numeral (Num.Bit1 x) AND numeral (Num.Bit1 y) = (− numeral (x + Num.One) AND numeral y) BIT True
− numeral (Num.Bit0 x) AND − numeral (Num.Bit0 y) = (− numeral x AND − numeral y) BIT False
− numeral (Num.Bit0 x) AND − numeral (Num.Bit1 y) = (− numeral x AND − numeral (y + Num.One)) BIT False
− numeral (Num.Bit1 x) AND − numeral (Num.Bit0 y) = (− numeral (x + Num.One) AND − numeral y) BIT False
− numeral (Num.Bit1 x) AND − numeral (Num.Bit1 y) = (− numeral (x + Num.One) AND − numeral (y + Num.One)) BIT True
(1::int) AND numeral (Num.Bit0 y) = 0
(1::int) AND numeral (Num.Bit1 y) = 1
(1::int) AND − numeral (Num.Bit0 y) = 0
(1::int) AND − numeral (Num.Bit1 y) = 1
numeral (Num.Bit0 x) AND (1::int) = 0
numeral (Num.Bit1 x) AND (1::int) = 1
− numeral (Num.Bit0 x) AND (1::int) = 0
− numeral (Num.Bit1 x) AND (1::int) = 1
by (*rule bin-rl-eqI*, *simp*, *simp*)+

lemma *int-or-numerals* [*simp*]:
numeral (Num.Bit0 x) OR numeral (Num.Bit0 y) = (numeral x OR numeral y) BIT False
numeral (Num.Bit0 x) OR numeral (Num.Bit1 y) = (numeral x OR numeral y) BIT True
numeral (Num.Bit1 x) OR numeral (Num.Bit0 y) = (numeral x OR numeral y) BIT True
numeral (Num.Bit1 x) OR numeral (Num.Bit1 y) = (numeral x OR numeral y) BIT True

numeral (Num.Bit0 x) OR − numeral (Num.Bit0 y) = (numeral x OR − numeral y) BIT False
numeral (Num.Bit0 x) OR − numeral (Num.Bit1 y) = (numeral x OR − numeral (y + Num.One)) BIT True
numeral (Num.Bit1 x) OR − numeral (Num.Bit0 y) = (numeral x OR − numeral y) BIT True
numeral (Num.Bit1 x) OR − numeral (Num.Bit1 y) = (numeral x OR − numeral (y + Num.One)) BIT True
− numeral (Num.Bit0 x) OR numeral (Num.Bit0 y) = (− numeral x OR numeral y) BIT False
− numeral (Num.Bit0 x) OR numeral (Num.Bit1 y) = (− numeral x OR numeral y) BIT True
− numeral (Num.Bit1 x) OR numeral (Num.Bit0 y) = (− numeral (x + Num.One) OR numeral y) BIT True
− numeral (Num.Bit1 x) OR numeral (Num.Bit1 y) = (− numeral (x + Num.One) OR numeral y) BIT True
− numeral (Num.Bit0 x) OR − numeral (Num.Bit0 y) = (− numeral x OR − numeral y) BIT False
− numeral (Num.Bit0 x) OR − numeral (Num.Bit1 y) = (− numeral x OR − numeral (y + Num.One)) BIT True
− numeral (Num.Bit1 x) OR − numeral (Num.Bit0 y) = (− numeral (x + Num.One) OR − numeral y) BIT True
− numeral (Num.Bit1 x) OR − numeral (Num.Bit1 y) = (− numeral (x + Num.One) OR − numeral (y + Num.One)) BIT True
(1::int) OR numeral (Num.Bit0 y) = numeral (Num.Bit1 y)
(1::int) OR numeral (Num.Bit1 y) = numeral (Num.Bit1 y)
(1::int) OR − numeral (Num.Bit0 y) = − numeral (Num.BitM y)
(1::int) OR − numeral (Num.Bit1 y) = − numeral (Num.Bit1 y)
numeral (Num.Bit0 x) OR (1::int) = numeral (Num.Bit1 x)
numeral (Num.Bit1 x) OR (1::int) = numeral (Num.Bit1 x)
− numeral (Num.Bit0 x) OR (1::int) = − numeral (Num.BitM x)
− numeral (Num.Bit1 x) OR (1::int) = − numeral (Num.Bit1 x)
by (*rule bin-rl-eqI*, *simp*, *simp*)+

lemma *int-xor-numerals* [*simp*]:
numeral (Num.Bit0 x) XOR numeral (Num.Bit0 y) = (numeral x XOR numeral y) BIT False
numeral (Num.Bit0 x) XOR numeral (Num.Bit1 y) = (numeral x XOR numeral y) BIT True
numeral (Num.Bit1 x) XOR numeral (Num.Bit0 y) = (numeral x XOR numeral y) BIT True
numeral (Num.Bit1 x) XOR numeral (Num.Bit1 y) = (numeral x XOR numeral y) BIT False
numeral (Num.Bit0 x) XOR − numeral (Num.Bit0 y) = (numeral x XOR − numeral y) BIT False
numeral (Num.Bit0 x) XOR − numeral (Num.Bit1 y) = (numeral x XOR − numeral (y + Num.One)) BIT True
numeral (Num.Bit1 x) XOR − numeral (Num.Bit0 y) = (numeral x XOR − numeral y) BIT True

numeral (Num.Bit1 x) XOR − numeral (Num.Bit1 y) = (numeral x XOR − numeral (y + Num.One)) BIT False
− numeral (Num.Bit0 x) XOR numeral (Num.Bit0 y) = (− numeral x XOR numeral y) BIT False
− numeral (Num.Bit0 x) XOR numeral (Num.Bit1 y) = (− numeral x XOR numeral y) BIT True
− numeral (Num.Bit1 x) XOR numeral (Num.Bit0 y) = (− numeral (x + Num.One) XOR numeral y) BIT True
− numeral (Num.Bit1 x) XOR numeral (Num.Bit1 y) = (− numeral (x + Num.One) XOR numeral y) BIT False
− numeral (Num.Bit0 x) XOR − numeral (Num.Bit0 y) = (− numeral x XOR − numeral y) BIT False
− numeral (Num.Bit0 x) XOR − numeral (Num.Bit1 y) = (− numeral x XOR − numeral (y + Num.One)) BIT True
− numeral (Num.Bit1 x) XOR − numeral (Num.Bit0 y) = (− numeral (x + Num.One) XOR − numeral y) BIT True
− numeral (Num.Bit1 x) XOR − numeral (Num.Bit1 y) = (− numeral (x + Num.One) XOR − numeral (y + Num.One)) BIT False
(1::int) XOR numeral (Num.Bit0 y) = numeral (Num.Bit1 y)
(1::int) XOR numeral (Num.Bit1 y) = numeral (Num.Bit0 y)
(1::int) XOR − numeral (Num.Bit0 y) = − numeral (Num.BitM y)
(1::int) XOR − numeral (Num.Bit1 y) = − numeral (Num.Bit0 (y + Num.One))
numeral (Num.Bit0 x) XOR (1::int) = numeral (Num.Bit1 x)
numeral (Num.Bit1 x) XOR (1::int) = numeral (Num.Bit0 x)
− numeral (Num.Bit0 x) XOR (1::int) = − numeral (Num.BitM x)
− numeral (Num.Bit1 x) XOR (1::int) = − numeral (Num.Bit0 (x + Num.One))
by (*rule bin-rl-eqI*, *simp*, *simp*)+

11.1.5 Interactions with arithmetic

lemma *plus-and-or* [*rule-format*]:
ALL y::int. (x AND y) + (x OR y) = x + y
apply (*induct x rule*: *bin-induct*)
apply *clarsimp*
apply *clarsimp*
apply *clarsimp*
apply (*case-tac y rule*: *bin-exhaust*)
apply *clarsimp*
apply (*unfold Bit-def*)
apply *clarsimp*
apply (*erule-tac x = x* **in** *allE*)
apply *simp*
done

lemma *le-int-or*:
bin-sign (y::int) = 0 ==> x <= x OR y
apply (*induct y arbitrary*: *x rule*: *bin-induct*)
apply *clarsimp*
apply *clarsimp*

apply (*case-tac x rule*: *bin-exhaust*)
apply (*case-tac b*)
apply (*case-tac* [!] *bit*)
apply (*auto simp*: *le-Bits*)
done

lemmas *int-and-le* =
xtrans(*3*) [*OF bbw-ao-absorbs* (*2*) [*THEN conjunct2*, *symmetric*] *le-int-or*]

lemma *bin-add-not*: *x* + *NOT x* = (−*1*::*int*)
apply (*induct x rule*: *bin-induct*)
apply *clarsimp*
apply *clarsimp*
apply (*case-tac bit*, *auto*)
done

11.1.6 Truncating results of bit-wise operations

lemma *bin-trunc-ao*:
!!*x y*. (*bintrunc n x*) *AND* (*bintrunc n y*) = *bintrunc n* (*x AND y*)
!!*x y*. (*bintrunc n x*) *OR* (*bintrunc n y*) = *bintrunc n* (*x OR y*)
by (*auto simp add*: *bin-eq-iff bin-nth-ops nth-bintr*)

lemma *bin-trunc-xor*:
!!*x y*. *bintrunc n* (*bintrunc n x XOR bintrunc n y*) =
bintrunc n (*x XOR y*)
by (*auto simp add*: *bin-eq-iff bin-nth-ops nth-bintr*)

lemma *bin-trunc-not*:
!!*x*. *bintrunc n* (*NOT* (*bintrunc n x*)) = *bintrunc n* (*NOT x*)
by (*auto simp add*: *bin-eq-iff bin-nth-ops nth-bintr*)

lemma *bintr-bintr-i*:
x = *bintrunc n y* ==> *bintrunc n x* = *bintrunc n y*
by *auto*

lemmas *bin-trunc-and* = *bin-trunc-ao*(*1*) [*THEN bintr-bintr-i*]
lemmas *bin-trunc-or* = *bin-trunc-ao*(*2*) [*THEN bintr-bintr-i*]

11.2 Setting and clearing bits

primrec
bin-sc :: *nat* => *bool* => *int* => *int*
where
Z: *bin-sc 0 b w* = *bin-rest w BIT b*
| *Suc*: *bin-sc* (*Suc n*) *b w* = *bin-sc n b* (*bin-rest w*) *BIT bin-last w*

lemma *bin-nth-sc* [*simp*]:
bin-nth (*bin-sc n b w*) *n* ⟷ *b*
by (*induct n arbitrary*: *w*) *auto*

lemma *bin-sc-sc-same* [*simp*]:
bin-sc n c (*bin-sc n b w*) = *bin-sc n c w*
by (*induct n arbitrary*: *w*) *auto*

lemma *bin-sc-sc-diff*:
m ~= *n* ==>
bin-sc m c (*bin-sc n b w*) = *bin-sc n b* (*bin-sc m c w*)
apply (*induct n arbitrary*: *w m*)
apply (*case-tac* [!] *m*)
apply *auto*
done

lemma *bin-nth-sc-gen*:
bin-nth (*bin-sc n b w*) *m* = (*if m* = *n then b else bin-nth w m*)
by (*induct n arbitrary*: *w m*) (*case-tac* [!] *m*, *auto*)

lemma *bin-sc-nth* [*simp*]:
(*bin-sc n* (*bin-nth w n*) *w*) = *w*
by (*induct n arbitrary*: *w*) *auto*

lemma *bin-sign-sc* [*simp*]:
bin-sign (*bin-sc n b w*) = *bin-sign w*
by (*induct n arbitrary*: *w*) *auto*

lemma *bin-sc-bintr* [*simp*]:
bintrunc m (*bin-sc n x* (*bintrunc m* (*w*))) = *bintrunc m* (*bin-sc n x w*)
apply (*induct n arbitrary*: *w m*)
apply (*case-tac* [!] *w rule*: *bin-exhaust*)
apply (*case-tac* [!] *m*, *auto*)
done

lemma *bin-clr-le*:
bin-sc n False w <= *w*
apply (*induct n arbitrary*: *w*)
apply (*case-tac* [!] *w rule*: *bin-exhaust*)
apply (*auto simp*: *le-Bits*)
done

lemma *bin-set-ge*:
bin-sc n True w >= *w*
apply (*induct n arbitrary*: *w*)
apply (*case-tac* [!] *w rule*: *bin-exhaust*)
apply (*auto simp*: *le-Bits*)
done

```
lemma bintr-bin-clr-le:
  bintrunc n (bin-sc m False w) <= bintrunc n w
  apply (induct n arbitrary: w m)
   apply simp
  apply (case-tac w rule: bin-exhaust)
  apply (case-tac m)
   apply (auto simp: le-Bits)
  done

lemma bintr-bin-set-ge:
  bintrunc n (bin-sc m True w) >= bintrunc n w
  apply (induct n arbitrary: w m)
   apply simp
  apply (case-tac w rule: bin-exhaust)
  apply (case-tac m)
   apply (auto simp: le-Bits)
  done

lemma bin-sc-FP [simp]: bin-sc n False 0 = 0
  by (induct n) auto

lemma bin-sc-TM [simp]: bin-sc n True (- 1) = - 1
  by (induct n) auto

lemmas bin-sc-simps = bin-sc.Z bin-sc.Suc bin-sc-TM bin-sc-FP

lemma bin-sc-minus:
  0 < n ==> bin-sc (Suc (n - 1)) b w = bin-sc n b w
  by auto

lemmas bin-sc-Suc-minus =
  trans [OF bin-sc-minus [symmetric] bin-sc.Suc]

lemma bin-sc-numeral [simp]:
  bin-sc (numeral k) b w =
    bin-sc (pred-numeral k) b (bin-rest w) BIT bin-last w
  by (simp add: numeral-eq-Suc)
```

11.3 Splitting and concatenation

```
definition bin-rcat :: nat ⇒ int list ⇒ int
where
  bin-rcat n = foldl (λu v. bin-cat u n v) 0

fun bin-rsplit-aux :: nat ⇒ nat ⇒ int ⇒ int list ⇒ int list
where
  bin-rsplit-aux n m c bs =
    (if m = 0 | n = 0 then bs else
      let (a, b) = bin-split n c
```

```
    in bin-rsplit-aux n (m − n) a (b # bs))

definition bin-rsplit :: nat ⇒ nat × int ⇒ int list
where
  bin-rsplit n w = bin-rsplit-aux n (fst w) (snd w) []

fun bin-rsplitl-aux :: nat ⇒ nat ⇒ int ⇒ int list ⇒ int list
where
  bin-rsplitl-aux n m c bs =
    (if m = 0 | n = 0 then bs else
      let (a, b) = bin-split (min m n) c
      in bin-rsplitl-aux n (m − n) a (b # bs))

definition bin-rsplitl :: nat ⇒ nat × int ⇒ int list
where
  bin-rsplitl n w = bin-rsplitl-aux n (fst w) (snd w) []

declare bin-rsplit-aux.simps [simp del]
declare bin-rsplitl-aux.simps [simp del]

lemma bin-sign-cat:
  bin-sign (bin-cat x n y) = bin-sign x
  by (induct n arbitrary: y) auto

lemma bin-cat-Suc-Bit:
  bin-cat w (Suc n) (v BIT b) = bin-cat w n v BIT b
  by auto

lemma bin-nth-cat:
  bin-nth (bin-cat x k y) n =
    (if n < k then bin-nth y n else bin-nth x (n − k))
  apply (induct k arbitrary: n y)
   apply clarsimp
  apply (case-tac n, auto)
  done

lemma bin-nth-split:
  bin-split n c = (a, b) ==>
    (ALL k. bin-nth a k = bin-nth c (n + k)) &
    (ALL k. bin-nth b k = (k < n & bin-nth c k))
  apply (induct n arbitrary: b c)
   apply clarsimp
  apply (clarsimp simp: Let-def split: prod.split-asm)
  apply (case-tac k)
  apply auto
  done

lemma bin-cat-assoc:
  bin-cat (bin-cat x m y) n z = bin-cat x (m + n) (bin-cat y n z)
```

by (*induct n arbitrary*: *z*) *auto*

lemma *bin-cat-assoc-sym*:
bin-cat x m (*bin-cat y n z*) = *bin-cat* (*bin-cat x* (*m* − *n*) *y*) (*min m n*) *z*
apply (*induct n arbitrary*: *z m*, *clarsimp*)
apply (*case-tac m*, *auto*)
done

lemma *bin-cat-zero* [*simp*]: *bin-cat 0 n w* = *bintrunc n w*
by (*induct n arbitrary*: *w*) *auto*

lemma *bintr-cat1*:
bintrunc (*k* + *n*) (*bin-cat a n b*) = *bin-cat* (*bintrunc k a*) *n b*
by (*induct n arbitrary*: *b*) *auto*

lemma *bintr-cat*: *bintrunc m* (*bin-cat a n b*) =
bin-cat (*bintrunc* (*m* − *n*) *a*) *n* (*bintrunc* (*min m n*) *b*)
by (*rule bin-eqI*) (*auto simp*: *bin-nth-cat nth-bintr*)

lemma *bintr-cat-same* [*simp*]:
bintrunc n (*bin-cat a n b*) = *bintrunc n b*
by (*auto simp add* : *bintr-cat*)

lemma *cat-bintr* [*simp*]:
bin-cat a n (*bintrunc n b*) = *bin-cat a n b*
by (*induct n arbitrary*: *b*) *auto*

lemma *split-bintrunc*:
bin-split n c = (*a*, *b*) ==> *b* = *bintrunc n c*
by (*induct n arbitrary*: *b c*) (*auto simp*: *Let-def split*: *prod.split-asm*)

lemma *bin-cat-split*:
bin-split n w = (*u*, *v*) ==> *w* = *bin-cat u n v*
by (*induct n arbitrary*: *v w*) (*auto simp*: *Let-def split*: *prod.split-asm*)

lemma *bin-split-cat*:
bin-split n (*bin-cat v n w*) = (*v*, *bintrunc n w*)
by (*induct n arbitrary*: *w*) *auto*

lemma *bin-split-zero* [*simp*]: *bin-split n 0* = (*0*, *0*)
by (*induct n*) *auto*

lemma *bin-split-minus1* [*simp*]:
bin-split n (− *1*) = (− *1*, *bintrunc n* (− *1*))
by (*induct n*) *auto*

lemma *bin-split-trunc*:
bin-split (*min m n*) *c* = (*a*, *b*) ==>
bin-split n (*bintrunc m c*) = (*bintrunc* (*m* − *n*) *a*, *b*)

apply (*induct n arbitrary*: *m b c*, *clarsimp*)
apply (*simp add*: *bin-rest-trunc Let-def split*: *prod.split-asm*)
apply (*case-tac m*)
apply (*auto simp*: *Let-def split*: *prod.split-asm*)
done

lemma *bin-split-trunc1*:
bin-split n c = (*a*, *b*) ==>
bin-split n (*bintrunc m c*) = (*bintrunc* (*m* − *n*) *a*, *bintrunc m b*)
apply (*induct n arbitrary*: *m b c*, *clarsimp*)
apply (*simp add*: *bin-rest-trunc Let-def split*: *prod.split-asm*)
apply (*case-tac m*)
apply (*auto simp*: *Let-def split*: *prod.split-asm*)
done

lemma *bin-cat-num*:
bin-cat a n b = *a* ∗ *2* ˆ *n* + *bintrunc n b*
apply (*induct n arbitrary*: *b*, *clarsimp*)
apply (*simp add*: *Bit-def*)
done

lemma *bin-split-num*:
bin-split n b = (*b div 2* ˆ *n*, *b mod 2* ˆ *n*)
apply (*induct n arbitrary*: *b*, *simp*)
apply (*simp add*: *bin-rest-def zdiv-zmult2-eq*)
apply (*case-tac b rule*: *bin-exhaust*)
apply *simp*
apply (*simp add*: *Bit-def mod-mult-mult1 p1mod22k*)
done

11.4 Miscellaneous lemmas

lemma *nth-2p-bin*:
bin-nth (*2* ˆ *n*) *m* = (*m* = *n*)
apply (*induct n arbitrary*: *m*)
apply *clarsimp*
apply *safe*
apply (*case-tac m*)
apply (*auto simp*: *Bit-B0-2t* [*symmetric*])
done

lemma *ex-eq-or*:
(*EX m*. *n* = *Suc m* & (*m* = *k* | *P m*)) = (*n* = *Suc k* | (*EX m*. *n* = *Suc m* & *P m*))
by *auto*

lemma *power-BIT*: *2* ˆ (*Suc n*) − *1* = (*2* ˆ *n* − *1*) *BIT True*

```
  unfolding Bit-B1
  by (induct n) simp-all

lemma mod-BIT:
  bin BIT bit mod 2 ^ Suc n = (bin mod 2 ^ n) BIT bit
proof -
  have bin mod 2 ^ n < 2 ^ n by simp
  then have bin mod 2 ^ n ≤ 2 ^ n - 1 by simp
  then have 2 * (bin mod 2 ^ n) ≤ 2 * (2 ^ n - 1)
    by (rule mult-left-mono) simp
  then have 2 * (bin mod 2 ^ n) + 1 < 2 * 2 ^ n by simp
  then show ?thesis
    by (auto simp add: Bit-def mod-mult-mult1 mod-add-left-eq [of 2 * bin]
      mod-pos-pos-trivial)
qed

lemma AND-mod:
  fixes x :: int
  shows x AND 2 ^ n - 1 = x mod 2 ^ n
proof (induct x arbitrary: n rule: bin-induct)
  case 1
  then show ?case
    by simp
next
  case 2
  then show ?case
    by (simp, simp add: m1mod2k)
next
  case (3 bin bit)
  show ?case
  proof (cases n)
    case 0
    then show ?thesis by simp
  next
    case (Suc m)
    with 3 show ?thesis
      by (simp only: power-BIT mod-BIT int-and-Bits) simp
  qed
qed

end
```

12 Bool lists and integers

```
theory Bool-List-Representation
imports Main Bits-Int
begin

definition map2 :: ('a ⇒ 'b ⇒ 'c) ⇒ 'a list ⇒ 'b list ⇒ 'c list
```

where
map2 f as bs = map (case-prod f) (zip as bs)

lemma *map2-Nil* [*simp*, *code*]:
map2 f [] ys = []
unfolding *map2-def* **by** *auto*

lemma *map2-Nil2* [*simp*, *code*]:
map2 f xs [] = []
unfolding *map2-def* **by** *auto*

lemma *map2-Cons* [*simp*, *code*]:
map2 f (x # xs) (y # ys) = f x y # map2 f xs ys
unfolding *map2-def* **by** *auto*

12.1 Operations on lists of booleans

primrec *bl-to-bin-aux* :: *bool list* ⇒ *int* ⇒ *int*
where
Nil: *bl-to-bin-aux [] w = w*
| *Cons*: *bl-to-bin-aux (b # bs) w =*
bl-to-bin-aux bs (w BIT b)

definition *bl-to-bin* :: *bool list* ⇒ *int*
where
bl-to-bin-def: *bl-to-bin bs = bl-to-bin-aux bs 0*

primrec *bin-to-bl-aux* :: *nat* ⇒ *int* ⇒ *bool list* ⇒ *bool list*
where
Z: *bin-to-bl-aux 0 w bl = bl*
| *Suc*: *bin-to-bl-aux (Suc n) w bl =*
bin-to-bl-aux n (bin-rest w) ((bin-last w) # bl)

definition *bin-to-bl* :: *nat* ⇒ *int* ⇒ *bool list*
where
bin-to-bl-def : *bin-to-bl n w = bin-to-bl-aux n w []*

primrec *bl-of-nth* :: *nat* ⇒ (*nat* ⇒ *bool*) ⇒ *bool list*
where
Suc: *bl-of-nth (Suc n) f = f n # bl-of-nth n f*
| *Z*: *bl-of-nth 0 f = []*

primrec *takefill* :: *'a* ⇒ *nat* ⇒ *'a list* ⇒ *'a list*
where
Z: *takefill fill 0 xs = []*
| *Suc*: *takefill fill (Suc n) xs = (*
case xs of [] => fill # takefill fill n xs
| *y # ys => y # takefill fill n ys)*

12.2 Arithmetic in terms of bool lists

Arithmetic operations in terms of the reversed bool list, assuming input list(s) the same length, and don't extend them.

primrec *rbl-succ* :: *bool list* => *bool list*
where
Nil: *rbl-succ Nil* = *Nil*
| *Cons*: *rbl-succ* (*x* # *xs*) = (*if x then False* # *rbl-succ xs else True* # *xs*)

primrec *rbl-pred* :: *bool list* => *bool list*
where
Nil: *rbl-pred Nil* = *Nil*
| *Cons*: *rbl-pred* (*x* # *xs*) = (*if x then False* # *xs else True* # *rbl-pred xs*)

primrec *rbl-add* :: *bool list* => *bool list* => *bool list*
where
— result is length of first arg, second arg may be longer
Nil: *rbl-add Nil x* = *Nil*
| *Cons*: *rbl-add* (*y* # *ys*) *x* = (*let ws* = *rbl-add ys* (*tl x*) *in*
(*y* ~= *hd x*) # (*if hd x* & *y then rbl-succ ws else ws*))

primrec *rbl-mult* :: *bool list* => *bool list* => *bool list*
where
— result is length of first arg, second arg may be longer
Nil: *rbl-mult Nil x* = *Nil*
| *Cons*: *rbl-mult* (*y* # *ys*) *x* = (*let ws* = *False* # *rbl-mult ys x in*
if y then rbl-add ws x else ws)

lemma *butlast-power*:
(*butlast* ^^ *n*) *bl* = *take* (*length bl* − *n*) *bl*
by (*induct n*) (*auto simp*: *butlast-take*)

lemma *bin-to-bl-aux-zero-minus-simp* [*simp*]:
0 < *n* ⟹ *bin-to-bl-aux n 0 bl* =
bin-to-bl-aux (*n* − *1*) *0* (*False* # *bl*)
by (*cases n*) *auto*

lemma *bin-to-bl-aux-minus1-minus-simp* [*simp*]:
0 < *n* ==> *bin-to-bl-aux n* (− *1*) *bl* =
bin-to-bl-aux (*n* − *1*) (− *1*) (*True* # *bl*)
by (*cases n*) *auto*

lemma *bin-to-bl-aux-one-minus-simp* [*simp*]:
0 < *n* ⟹ *bin-to-bl-aux n 1 bl* =
bin-to-bl-aux (*n* − *1*) *0* (*True* # *bl*)
by (*cases n*) *auto*

lemma *bin-to-bl-aux-Bit-minus-simp* [*simp*]:
0 < *n* ==> *bin-to-bl-aux n* (*w BIT b*) *bl* =

bin-to-bl-aux (*n* − *1*) *w* (*b* # *bl*)
by (*cases n*) *auto*

lemma *bin-to-bl-aux-Bit0-minus-simp* [*simp*]:
0 < *n* ==> *bin-to-bl-aux n* (*numeral* (*Num.Bit0 w*)) *bl* =
bin-to-bl-aux (*n* − *1*) (*numeral w*) (*False* # *bl*)
by (*cases n*) *auto*

lemma *bin-to-bl-aux-Bit1-minus-simp* [*simp*]:
0 < *n* ==> *bin-to-bl-aux n* (*numeral* (*Num.Bit1 w*)) *bl* =
bin-to-bl-aux (*n* − *1*) (*numeral w*) (*True* # *bl*)
by (*cases n*) *auto*

Link between bin and bool list.

lemma *bl-to-bin-aux-append*:
bl-to-bin-aux (*bs* @ *cs*) *w* = *bl-to-bin-aux cs* (*bl-to-bin-aux bs w*)
by (*induct bs arbitrary*: *w*) *auto*

lemma *bin-to-bl-aux-append*:
bin-to-bl-aux n w bs @ *cs* = *bin-to-bl-aux n w* (*bs* @ *cs*)
by (*induct n arbitrary*: *w bs*) *auto*

lemma *bl-to-bin-append*:
bl-to-bin (*bs* @ *cs*) = *bl-to-bin-aux cs* (*bl-to-bin bs*)
unfolding *bl-to-bin-def* **by** (*rule bl-to-bin-aux-append*)

lemma *bin-to-bl-aux-alt*:
bin-to-bl-aux n w bs = *bin-to-bl n w* @ *bs*
unfolding *bin-to-bl-def* **by** (*simp add* : *bin-to-bl-aux-append*)

lemma *bin-to-bl-0* [*simp*]: *bin-to-bl 0 bs* = []
unfolding *bin-to-bl-def* **by** *auto*

lemma *size-bin-to-bl-aux*:
size (*bin-to-bl-aux n w bs*) = *n* + *length bs*
by (*induct n arbitrary*: *w bs*) *auto*

lemma *size-bin-to-bl* [*simp*]: *size* (*bin-to-bl n w*) = *n*
unfolding *bin-to-bl-def* **by** (*simp add* : *size-bin-to-bl-aux*)

lemma *bin-bl-bin′*:
bl-to-bin (*bin-to-bl-aux n w bs*) =
bl-to-bin-aux bs (*bintrunc n w*)
by (*induct n arbitrary*: *w bs*) (*auto simp add* : *bl-to-bin-def*)

lemma *bin-bl-bin* [*simp*]: *bl-to-bin* (*bin-to-bl n w*) = *bintrunc n w*
unfolding *bin-to-bl-def bin-bl-bin′* **by** *auto*

lemma *bl-bin-bl′*:

```
  bin-to-bl (n + length bs) (bl-to-bin-aux bs w) =
    bin-to-bl-aux n w bs
  apply (induct bs arbitrary: w n)
   apply auto
    apply (simp-all only : add-Suc [symmetric])
    apply (auto simp add : bin-to-bl-def)
  done

lemma bl-bin-bl [simp]: bin-to-bl (length bs) (bl-to-bin bs) = bs
  unfolding bl-to-bin-def
  apply (rule box-equals)
    apply (rule bl-bin-bl')
   prefer 2
   apply (rule bin-to-bl-aux.Z)
  apply simp
  done

lemma bl-to-bin-inj:
  bl-to-bin bs = bl-to-bin cs ==> length bs = length cs ==> bs = cs
  apply (rule-tac box-equals)
    defer
    apply (rule bl-bin-bl)
   apply (rule bl-bin-bl)
  apply simp
  done

lemma bl-to-bin-False [simp]: bl-to-bin (False # bl) = bl-to-bin bl
  unfolding bl-to-bin-def by auto

lemma bl-to-bin-Nil [simp]: bl-to-bin [] = 0
  unfolding bl-to-bin-def by auto

lemma bin-to-bl-zero-aux:
  bin-to-bl-aux n 0 bl = replicate n False @ bl
  by (induct n arbitrary: bl) (auto simp: replicate-app-Cons-same)

lemma bin-to-bl-zero: bin-to-bl n 0 = replicate n False
  unfolding bin-to-bl-def by (simp add: bin-to-bl-zero-aux)

lemma bin-to-bl-minus1-aux:
  bin-to-bl-aux n (- 1) bl = replicate n True @ bl
  by (induct n arbitrary: bl) (auto simp: replicate-app-Cons-same)

lemma bin-to-bl-minus1: bin-to-bl n (- 1) = replicate n True
  unfolding bin-to-bl-def by (simp add: bin-to-bl-minus1-aux)

lemma bl-to-bin-rep-F:
  bl-to-bin (replicate n False @ bl) = bl-to-bin bl
  apply (simp add: bin-to-bl-zero-aux [symmetric] bin-bl-bin')
```

```
  apply (simp add: bl-to-bin-def)
  done

lemma bin-to-bl-trunc [simp]:
  n <= m ==> bin-to-bl n (bintrunc m w) = bin-to-bl n w
  by (auto intro: bl-to-bin-inj)

lemma bin-to-bl-aux-bintr:
  bin-to-bl-aux n (bintrunc m bin) bl =
    replicate (n − m) False @ bin-to-bl-aux (min n m) bin bl
  apply (induct n arbitrary: m bin bl)
   apply clarsimp
  apply clarsimp
  apply (case-tac m)
   apply (clarsimp simp: bin-to-bl-zero-aux)
   apply (erule thin-rl)
   apply (induct-tac n)
    apply auto
  done

lemma bin-to-bl-bintr:
  bin-to-bl n (bintrunc m bin) =
    replicate (n − m) False @ bin-to-bl (min n m) bin
  unfolding bin-to-bl-def by (rule bin-to-bl-aux-bintr)

lemma bl-to-bin-rep-False: bl-to-bin (replicate n False) = 0
  by (induct n) auto

lemma len-bin-to-bl-aux:
  length (bin-to-bl-aux n w bs) = n + length bs
  by (fact size-bin-to-bl-aux)

lemma len-bin-to-bl [simp]: length (bin-to-bl n w) = n
  by (fact size-bin-to-bl)

lemma sign-bl-bin':
  bin-sign (bl-to-bin-aux bs w) = bin-sign w
  by (induct bs arbitrary: w) auto

lemma sign-bl-bin: bin-sign (bl-to-bin bs) = 0
  unfolding bl-to-bin-def by (simp add : sign-bl-bin')

lemma bl-sbin-sign-aux:
  hd (bin-to-bl-aux (Suc n) w bs) =
    (bin-sign (sbintrunc n w) = −1)
  apply (induct n arbitrary: w bs)
   apply clarsimp
   apply (cases w rule: bin-exhaust)
   apply simp
```

done

lemma *bl-sbin-sign*:
hd (*bin-to-bl* (*Suc n*) *w*) = (*bin-sign* (*sbintrunc n w*) = −*1*)
unfolding *bin-to-bl-def* **by** (*rule bl-sbin-sign-aux*)

lemma *bin-nth-of-bl-aux*:
bin-nth (*bl-to-bin-aux bl w*) *n* =
(*n* < *size bl* & *rev bl* ! *n* | *n* >= *length bl* & *bin-nth w* (*n* − *size bl*))
apply (*induct bl arbitrary*: *w*)
apply *clarsimp*
apply *clarsimp*
apply (*cut-tac x*=*n* **and** *y*=*size bl* **in** *linorder-less-linear*)
apply (*erule disjE*, *simp add*: *nth-append*)+
apply *auto*
done

lemma *bin-nth-of-bl*: *bin-nth* (*bl-to-bin bl*) *n* = (*n* < *length bl* & *rev bl* ! *n*)
unfolding *bl-to-bin-def* **by** (*simp add* : *bin-nth-of-bl-aux*)

lemma *bin-nth-bl*: *n* < *m* ⟹ *bin-nth w n* = *nth* (*rev* (*bin-to-bl m w*)) *n*
apply (*induct n arbitrary*: *m w*)
apply *clarsimp*
apply (*case-tac m*, *clarsimp*)
apply (*clarsimp simp*: *bin-to-bl-def*)
apply (*simp add*: *bin-to-bl-aux-alt*)
apply *clarsimp*
apply (*case-tac m*, *clarsimp*)
apply (*clarsimp simp*: *bin-to-bl-def*)
apply (*simp add*: *bin-to-bl-aux-alt*)
done

lemma *nth-rev*:
n < *length xs* ⟹ *rev xs* ! *n* = *xs* ! (*length xs* − *1* − *n*)
apply (*induct xs*)
apply *simp*
apply (*clarsimp simp add* : *nth-append nth.simps split add* : *nat.split*)
apply (*rule-tac f* = λ*n*. *xs* ! *n* **in** *arg-cong*)
apply *arith*
done

lemma *nth-rev-alt*: *n* < *length ys* ⟹ *ys* ! *n* = *rev ys* ! (*length ys* − *Suc n*)
by (*simp add*: *nth-rev*)

lemma *nth-bin-to-bl-aux*:
n < *m* + *length bl* ⟹ (*bin-to-bl-aux m w bl*) ! *n* =
(*if n* < *m then bin-nth w* (*m* − *1* − *n*) *else bl* ! (*n* − *m*))
apply (*induct m arbitrary*: *w n bl*)
apply *clarsimp*

```
  apply clarsimp
  apply (case-tac w rule: bin-exhaust)
  apply simp
  done

lemma nth-bin-to-bl: n < m ==> (bin-to-bl m w) ! n = bin-nth w (m − Suc n)
  unfolding bin-to-bl-def by (simp add : nth-bin-to-bl-aux)

lemma bl-to-bin-lt2p-aux:
  bl-to-bin-aux bs w < (w + 1) ∗ (2 ^ length bs)
  apply (induct bs arbitrary: w)
   apply clarsimp
  apply clarsimp
  apply (drule meta-spec, erule xtrans(8) [rotated], simp add: Bit-def)+
  done

lemma bl-to-bin-lt2p-drop:
  bl-to-bin bs < 2 ^ length (dropWhile Not bs)
proof (induct bs)
  case (Cons b bs) with bl-to-bin-lt2p-aux[where w=1]
  show ?case unfolding bl-to-bin-def by simp
qed simp

lemma bl-to-bin-lt2p: bl-to-bin bs < 2 ^ length bs
  by (metis bin-bl-bin bintr-lt2p bl-bin-bl)

lemma bl-to-bin-ge2p-aux:
  bl-to-bin-aux bs w >= w ∗ (2 ^ length bs)
  apply (induct bs arbitrary: w)
   apply clarsimp
  apply clarsimp
   apply (drule meta-spec, erule order-trans [rotated],
          simp add: Bit-B0-2t Bit-B1-2t algebra-simps)+
   apply (simp add: Bit-def)
  done

lemma bl-to-bin-ge0: bl-to-bin bs >= 0
  apply (unfold bl-to-bin-def)
  apply (rule xtrans(4))
   apply (rule bl-to-bin-ge2p-aux)
  apply simp
  done

lemma butlast-rest-bin:
  butlast (bin-to-bl n w) = bin-to-bl (n − 1) (bin-rest w)
  apply (unfold bin-to-bl-def)
  apply (cases w rule: bin-exhaust)
  apply (cases n, clarsimp)
  apply clarsimp
```

```
  apply (auto simp add: bin-to-bl-aux-alt)
  done

lemma butlast-bin-rest:
  butlast bl = bin-to-bl (length bl − Suc 0) (bin-rest (bl-to-bin bl))
  using butlast-rest-bin [where w=bl-to-bin bl and n=length bl] by simp

lemma butlast-rest-bl2bin-aux:
  bl ~= [] ⟹
    bl-to-bin-aux (butlast bl) w = bin-rest (bl-to-bin-aux bl w)
  by (induct bl arbitrary: w) auto

lemma butlast-rest-bl2bin:
  bl-to-bin (butlast bl) = bin-rest (bl-to-bin bl)
  apply (unfold bl-to-bin-def)
  apply (cases bl)
   apply (auto simp add: butlast-rest-bl2bin-aux)
  done

lemma trunc-bl2bin-aux:
  bintrunc m (bl-to-bin-aux bl w) =
    bl-to-bin-aux (drop (length bl − m) bl) (bintrunc (m − length bl) w)
proof (induct bl arbitrary: w)
  case Nil show ?case by simp
next
  case (Cons b bl) show ?case
  proof (cases m − length bl)
    case 0 then have Suc (length bl) − m = Suc (length bl − m) by simp
    with Cons show ?thesis by simp
  next
    case (Suc n) then have *: m − Suc (length bl) = n by simp
    with Suc Cons show ?thesis by simp
  qed
qed

lemma trunc-bl2bin:
  bintrunc m (bl-to-bin bl) = bl-to-bin (drop (length bl − m) bl)
  unfolding bl-to-bin-def by (simp add : trunc-bl2bin-aux)

lemma trunc-bl2bin-len [simp]:
  bintrunc (length bl) (bl-to-bin bl) = bl-to-bin bl
  by (simp add: trunc-bl2bin)

lemma bl2bin-drop:
  bl-to-bin (drop k bl) = bintrunc (length bl − k) (bl-to-bin bl)
  apply (rule trans)
   prefer 2
   apply (rule trunc-bl2bin [symmetric])
  apply (cases k <= length bl)
```

 apply *auto*
done

lemma *nth-rest-power-bin*:
bin-nth ((*bin-rest* ^^ *k*) *w*) *n* = *bin-nth* *w* (*n* + *k*)
apply (*induct k arbitrary*: *n*, *clarsimp*)
apply *clarsimp*
apply (*simp only*: *bin-nth.Suc* [*symmetric*] *add-Suc*)
done

lemma *take-rest-power-bin*:
m <= *n* ==> *take m* (*bin-to-bl n w*) = *bin-to-bl m* ((*bin-rest* ^^ (*n* − *m*)) *w*)
apply (*rule nth-equalityI*)
 apply *simp*
apply (*clarsimp simp add*: *nth-bin-to-bl nth-rest-power-bin*)
done

lemma *hd-butlast*: *size xs* > *1* ==> *hd* (*butlast xs*) = *hd xs*
by (*cases xs*) *auto*

lemma *last-bin-last'*:
size xs > *0* $\Longrightarrow$ *last xs* $\longleftrightarrow$ *bin-last* (*bl-to-bin-aux xs w*)
by (*induct xs arbitrary*: *w*) *auto*

lemma *last-bin-last*:
size xs > *0* ==> *last xs* $\longleftrightarrow$ *bin-last* (*bl-to-bin xs*)
unfolding *bl-to-bin-def* **by** (*erule last-bin-last'*)

lemma *bin-last-last*:
bin-last w $\longleftrightarrow$ *last* (*bin-to-bl* (*Suc n*) *w*)
apply (*unfold bin-to-bl-def*)
apply *simp*
apply (*auto simp add*: *bin-to-bl-aux-alt*)
done

lemma *bl-xor-aux-bin*:
map2 (%*x y*. *x* ~= *y*) (*bin-to-bl-aux n v bs*) (*bin-to-bl-aux n w cs*) =
  bin-to-bl-aux n (*v XOR w*) (*map2* (%*x y*. *x* ~= *y*) *bs cs*)
apply (*induct n arbitrary*: *v w bs cs*)
 apply *simp*
apply (*case-tac v rule*: *bin-exhaust*)
apply (*case-tac w rule*: *bin-exhaust*)
apply *clarsimp*
apply (*case-tac b*)
apply *auto*
done

lemma *bl-or-aux-bin*:
  map2 (*op* |) (*bin-to-bl-aux n v bs*) (*bin-to-bl-aux n w cs*) =
    bin-to-bl-aux n (*v OR w*) (*map2* (*op* |) *bs cs*)
  apply (*induct n arbitrary*: *v w bs cs*)
   apply *simp*
  apply (*case-tac v rule*: *bin-exhaust*)
  apply (*case-tac w rule*: *bin-exhaust*)
  apply *clarsimp*
  done

lemma *bl-and-aux-bin*:
  map2 (*op* &) (*bin-to-bl-aux n v bs*) (*bin-to-bl-aux n w cs*) =
    bin-to-bl-aux n (*v AND w*) (*map2* (*op* &) *bs cs*)
  apply (*induct n arbitrary*: *v w bs cs*)
   apply *simp*
  apply (*case-tac v rule*: *bin-exhaust*)
  apply (*case-tac w rule*: *bin-exhaust*)
  apply *clarsimp*
  done

lemma *bl-not-aux-bin*:
  map Not (*bin-to-bl-aux n w cs*) =
    bin-to-bl-aux n (*NOT w*) (*map Not cs*)
  apply (*induct n arbitrary*: *w cs*)
   apply *clarsimp*
  apply *clarsimp*
  done

lemma *bl-not-bin*: *map Not* (*bin-to-bl n w*) = *bin-to-bl n* (*NOT w*)
  unfolding *bin-to-bl-def* **by** (*simp add*: *bl-not-aux-bin*)

lemma *bl-and-bin*:
  map2 (*op* ∧) (*bin-to-bl n v*) (*bin-to-bl n w*) = *bin-to-bl n* (*v AND w*)
  unfolding *bin-to-bl-def* **by** (*simp add*: *bl-and-aux-bin*)

lemma *bl-or-bin*:
  map2 (*op* ∨) (*bin-to-bl n v*) (*bin-to-bl n w*) = *bin-to-bl n* (*v OR w*)
  unfolding *bin-to-bl-def* **by** (*simp add*: *bl-or-aux-bin*)

lemma *bl-xor-bin*:
  map2 (λ*x y*. *x* ≠ *y*) (*bin-to-bl n v*) (*bin-to-bl n w*) = *bin-to-bl n* (*v XOR w*)
  unfolding *bin-to-bl-def* **by** (*simp only*: *bl-xor-aux-bin map2-Nil*)

lemma *drop-bin2bl-aux*:
  drop m (*bin-to-bl-aux n bin bs*) =
    bin-to-bl-aux (*n* − *m*) *bin* (*drop* (*m* − *n*) *bs*)
  apply (*induct n arbitrary*: *m bin bs*, *clarsimp*)
  apply *clarsimp*
  apply (*case-tac bin rule*: *bin-exhaust*)

```
  apply (case-tac m <= n, simp)
  apply (case-tac m − n, simp)
  apply simp
  apply (rule-tac f = %nat. drop nat bs in arg-cong)
  apply simp
  done

lemma drop-bin2bl: drop m (bin-to-bl n bin) = bin-to-bl (n − m) bin
  unfolding bin-to-bl-def by (simp add : drop-bin2bl-aux)

lemma take-bin2bl-lem1:
  take m (bin-to-bl-aux m w bs) = bin-to-bl m w
  apply (induct m arbitrary: w bs, clarsimp)
  apply clarsimp
  apply (simp add: bin-to-bl-aux-alt)
  apply (simp add: bin-to-bl-def)
  apply (simp add: bin-to-bl-aux-alt)
  done

lemma take-bin2bl-lem:
  take m (bin-to-bl-aux (m + n) w bs) =
    take m (bin-to-bl (m + n) w)
  apply (induct n arbitrary: w bs)
   apply (simp-all (no-asm) add: bin-to-bl-def take-bin2bl-lem1)
  apply simp
  done

lemma bin-split-take:
  bin-split n c = (a, b) ⟹
    bin-to-bl m a = take m (bin-to-bl (m + n) c)
  apply (induct n arbitrary: b c)
   apply clarsimp
  apply (clarsimp simp: Let-def split: prod.split-asm)
  apply (simp add: bin-to-bl-def)
  apply (simp add: take-bin2bl-lem)
  done

lemma bin-split-take1:
  k = m + n ==> bin-split n c = (a, b) ==>
    bin-to-bl m a = take m (bin-to-bl k c)
  by (auto elim: bin-split-take)

lemma nth-takefill: m < n ⟹
    takefill fill n l ! m = (if m < length l then l ! m else fill)
  apply (induct n arbitrary: m l, clarsimp)
  apply clarsimp
  apply (case-tac m)
   apply (simp split: list.split)
  apply (simp split: list.split)
```

done

lemma *takefill-alt*:
takefill fill n l = take n l @ replicate (n − length l) fill
by (*induct n arbitrary*: *l*) (*auto split*: *list.split*)

lemma *takefill-replicate* [*simp*]:
takefill fill n (*replicate m fill*) = *replicate n fill*
by (*simp add* : *takefill-alt replicate-add* [*symmetric*])

lemma *takefill-le'*:
$n = m + k \Longrightarrow$ *takefill x m* (*takefill x n l*) = *takefill x m l*
by (*induct m arbitrary*: *l n*) (*auto split*: *list.split*)

lemma *length-takefill* [*simp*]: *length* (*takefill fill n l*) = *n*
by (*simp add* : *takefill-alt*)

lemma *take-takefill'*:
!!*w n*. *n* = *k* + *m* ==> *take k* (*takefill fill n w*) = *takefill fill k w*
by (*induct k*) (*auto split add* : *list.split*)

lemma *drop-takefill*:
!!*w*. *drop k* (*takefill fill* (*m* + *k*) *w*) = *takefill fill m* (*drop k w*)
by (*induct k*) (*auto split add* : *list.split*)

lemma *takefill-le* [*simp*]:
$m \le n \Longrightarrow$ *takefill x m* (*takefill x n l*) = *takefill x m l*
by (*auto simp*: *le-iff-add takefill-le'*)

lemma *take-takefill* [*simp*]:
$m \le n \Longrightarrow$ *take m* (*takefill fill n w*) = *takefill fill m w*
by (*auto simp*: *le-iff-add take-takefill'*)

lemma *takefill-append*:
takefill fill (*m* + *length xs*) (*xs* @ *w*) = *xs* @ (*takefill fill m w*)
by (*induct xs*) *auto*

lemma *takefill-same'*:
l = *length xs* ==> *takefill fill l xs* = *xs*
by (*induct xs arbitrary*: *l*, *auto*)

lemmas *takefill-same* [*simp*] = *takefill-same'* [*OF refl*]

lemma *takefill-bintrunc*:
takefill False n bl = *rev* (*bin-to-bl n* (*bl-to-bin* (*rev bl*)))
apply (*rule nth-equalityI*)
 apply *simp*
apply (*clarsimp simp*: *nth-takefill nth-rev nth-bin-to-bl bin-nth-of-bl*)
done

lemma *bl-bin-bl-rtf*:
  bin-to-bl n (bl-to-bin bl) = rev (takefill False n (rev bl))
  by *(simp add : takefill-bintrunc)*

lemma *bl-bin-bl-rep-drop*:
  bin-to-bl n (bl-to-bin bl) =
    replicate (n − length bl) False @ drop (length bl − n) bl
  by *(simp add: bl-bin-bl-rtf takefill-alt rev-take)*

lemma *tf-rev*:
  n + k = m + length bl ==> takefill x m (rev (takefill y n bl)) =
    rev (takefill y m (rev (takefill x k (rev bl))))
  apply *(rule nth-equalityI)*
   apply *(auto simp add: nth-takefill nth-rev)*
  apply *(rule-tac f = %n. bl ! n* **in** *arg-cong)*
  apply *arith*
  done

lemma *takefill-minus*:
  0 < n ==> takefill fill (Suc (n − 1)) w = takefill fill n w
  by *auto*

lemmas *takefill-Suc-cases =*
  list.cases [THEN takefill.Suc [THEN trans]]

lemmas *takefill-Suc-Nil = takefill-Suc-cases (1)*
lemmas *takefill-Suc-Cons = takefill-Suc-cases (2)*

lemmas *takefill-minus-simps = takefill-Suc-cases [THEN [2]*
  takefill-minus [symmetric, THEN trans]]

lemma *takefill-numeral-Nil [simp]*:
  takefill fill (numeral k) [] = fill # takefill fill (pred-numeral k) []
  by *(simp add: numeral-eq-Suc)*

lemma *takefill-numeral-Cons [simp]*:
  takefill fill (numeral k) (x # xs) = x # takefill fill (pred-numeral k) xs
  by *(simp add: numeral-eq-Suc)*

lemma *bl-to-bin-aux-cat*:
  !!nv v. bl-to-bin-aux bs (bin-cat w nv v) =
    bin-cat w (nv + length bs) (bl-to-bin-aux bs v)
  apply *(induct bs)*
   apply *simp*
  apply *(simp add: bin-cat-Suc-Bit [symmetric] del: bin-cat.simps)*
  done

lemma *bin-to-bl-aux-cat*:
  !!*w bs*. *bin-to-bl-aux* (*nv* + *nw*) (*bin-cat v nw w*) *bs* =
    bin-to-bl-aux nv v (*bin-to-bl-aux nw w bs*)
  by (*induct nw*) *auto*

lemma *bl-to-bin-aux-alt*:
  bl-to-bin-aux bs w = *bin-cat w* (*length bs*) (*bl-to-bin bs*)
  using *bl-to-bin-aux-cat* [**where** *nv* = *0* **and** *v* = *0*]
  unfolding *bl-to-bin-def* [*symmetric*] **by** *simp*

lemma *bin-to-bl-cat*:
  bin-to-bl (*nv* + *nw*) (*bin-cat v nw w*) =
    bin-to-bl-aux nv v (*bin-to-bl nw w*)
  unfolding *bin-to-bl-def* **by** (*simp add*: *bin-to-bl-aux-cat*)

lemmas *bl-to-bin-aux-app-cat* =
  trans [*OF bl-to-bin-aux-append bl-to-bin-aux-alt*]

lemmas *bin-to-bl-aux-cat-app* =
  trans [*OF bin-to-bl-aux-cat bin-to-bl-aux-alt*]

lemma *bl-to-bin-app-cat*:
  bl-to-bin (*bsa* @ *bs*) = *bin-cat* (*bl-to-bin bsa*) (*length bs*) (*bl-to-bin bs*)
  by (*simp only*: *bl-to-bin-aux-app-cat bl-to-bin-def*)

lemma *bin-to-bl-cat-app*:
  bin-to-bl (*n* + *nw*) (*bin-cat w nw wa*) = *bin-to-bl n w* @ *bin-to-bl nw wa*
  by (*simp only*: *bin-to-bl-def bin-to-bl-aux-cat-app*)

lemma *bl-to-bin-app-cat-alt*:
  bin-cat (*bl-to-bin cs*) *n w* = *bl-to-bin* (*cs* @ *bin-to-bl n w*)
  by (*simp add* : *bl-to-bin-app-cat*)

lemma *mask-lem*: (*bl-to-bin* (*True* # *replicate n False*)) =
    (*bl-to-bin* (*replicate n True*)) + *1*
  apply (*unfold bl-to-bin-def*)
  apply (*induct n*)
   apply *simp*
  apply (*simp only*: *Suc-eq-plus1 replicate-add*
                append-Cons [*symmetric*] *bl-to-bin-aux-append*)
  apply (*simp add*: *Bit-B0-2t Bit-B1-2t*)
  done

lemma *length-bl-of-nth* [*simp*]: *length* (*bl-of-nth n f*) = *n*
  by (*induct n*) *auto*

lemma *nth-bl-of-nth* [*simp*]:
$m < n \Longrightarrow$ *rev* (*bl-of-nth n f*) ! $m = f\ m$
apply (*induct n*)
apply *simp*
apply (*clarsimp simp add* : *nth-append*)
apply (*rule-tac f* = *f* **in** *arg-cong*)
apply *simp*
done

lemma *bl-of-nth-inj*:
(!!*k*. $k < n$ ==> *f k* = *g k*) ==> *bl-of-nth n f* = *bl-of-nth n g*
by (*induct n*) *auto*

lemma *bl-of-nth-nth-le*:
$n \le$ *length xs* $\Longrightarrow$ *bl-of-nth n* (*nth* (*rev xs*)) = *drop* (*length xs* − *n*) *xs*
apply (*induct n arbitrary*: *xs*, *clarsimp*)
apply *clarsimp*
apply (*rule trans* [*OF* - *hd-Cons-tl*])
apply (*frule Suc-le-lessD*)
apply (*simp add*: *nth-rev trans* [*OF drop-Suc drop-tl*, *symmetric*])
apply (*subst hd-drop-conv-nth*)
apply *force*
apply *simp-all*
apply (*rule-tac f* = %*n*. *drop n xs* **in** *arg-cong*)
apply *simp*
done

lemma *bl-of-nth-nth* [*simp*]: *bl-of-nth* (*length xs*) (*op* ! (*rev xs*)) = *xs*
by (*simp add*: *bl-of-nth-nth-le*)

lemma *size-rbl-pred*: *length* (*rbl-pred bl*) = *length bl*
by (*induct bl*) *auto*

lemma *size-rbl-succ*: *length* (*rbl-succ bl*) = *length bl*
by (*induct bl*) *auto*

lemma *size-rbl-add*:
!!*cl*. *length* (*rbl-add bl cl*) = *length bl*
by (*induct bl*) (*auto simp*: *Let-def size-rbl-succ*)

lemma *size-rbl-mult*:
!!*cl*. *length* (*rbl-mult bl cl*) = *length bl*
by (*induct bl*) (*auto simp add* : *Let-def size-rbl-add*)

lemmas *rbl-sizes* [*simp*] =
size-rbl-pred size-rbl-succ size-rbl-add size-rbl-mult

lemmas *rbl-Nils* =
rbl-pred.Nil rbl-succ.Nil rbl-add.Nil rbl-mult.Nil

```
lemma rbl-pred:
  rbl-pred (rev (bin-to-bl n bin)) = rev (bin-to-bl n (bin − 1))
  apply (induct n arbitrary: bin, simp)
  apply (unfold bin-to-bl-def)
  apply clarsimp
  apply (case-tac bin rule: bin-exhaust)
  apply (case-tac b)
   apply (clarsimp simp: bin-to-bl-aux-alt)+
  done

lemma rbl-succ:
  rbl-succ (rev (bin-to-bl n bin)) = rev (bin-to-bl n (bin + 1))
  apply (induct n arbitrary: bin, simp)
  apply (unfold bin-to-bl-def)
  apply clarsimp
  apply (case-tac bin rule: bin-exhaust)
  apply (case-tac b)
   apply (clarsimp simp: bin-to-bl-aux-alt)+
  done

lemma rbl-add:
  !!bina binb. rbl-add (rev (bin-to-bl n bina)) (rev (bin-to-bl n binb)) =
    rev (bin-to-bl n (bina + binb))
  apply (induct n, simp)
  apply (unfold bin-to-bl-def)
  apply clarsimp
  apply (case-tac bina rule: bin-exhaust)
  apply (case-tac binb rule: bin-exhaust)
  apply (case-tac b)
   apply (case-tac [!] ba)
     apply (auto simp: rbl-succ bin-to-bl-aux-alt Let-def ac-simps)
  done

lemma rbl-add-app2:
  !!blb. length blb >= length bla ==>
    rbl-add bla (blb @ blc) = rbl-add bla blb
  apply (induct bla, simp)
  apply clarsimp
  apply (case-tac blb, clarsimp)
  apply (clarsimp simp: Let-def)
  done

lemma rbl-add-take2:
  !!blb. length blb >= length bla ==>
    rbl-add bla (take (length bla) blb) = rbl-add bla blb
  apply (induct bla, simp)
  apply clarsimp
  apply (case-tac blb, clarsimp)
```

```
  apply (clarsimp simp: Let-def)
  done

lemma rbl-add-long:
  m >= n ==> rbl-add (rev (bin-to-bl n bina)) (rev (bin-to-bl m binb)) =
    rev (bin-to-bl n (bina + binb))
  apply (rule box-equals [OF - rbl-add-take2 rbl-add])
   apply (rule-tac f = rbl-add (rev (bin-to-bl n bina)) in arg-cong)
   apply (rule rev-swap [THEN iffD1])
   apply (simp add: rev-take drop-bin2bl)
  apply simp
  done

lemma rbl-mult-app2:
  !!blb. length blb >= length bla ==>
    rbl-mult bla (blb @ blc) = rbl-mult bla blb
  apply (induct bla, simp)
  apply clarsimp
  apply (case-tac blb, clarsimp)
  apply (clarsimp simp: Let-def rbl-add-app2)
  done

lemma rbl-mult-take2:
  length blb >= length bla ==>
    rbl-mult bla (take (length bla) blb) = rbl-mult bla blb
  apply (rule trans)
   apply (rule rbl-mult-app2 [symmetric])
   apply simp
  apply (rule-tac f = rbl-mult bla in arg-cong)
  apply (rule append-take-drop-id)
  done

lemma rbl-mult-gt1:
  m >= length bl ==> rbl-mult bl (rev (bin-to-bl m binb)) =
    rbl-mult bl (rev (bin-to-bl (length bl) binb))
  apply (rule trans)
   apply (rule rbl-mult-take2 [symmetric])
   apply simp-all
  apply (rule-tac f = rbl-mult bl in arg-cong)
  apply (rule rev-swap [THEN iffD1])
  apply (simp add: rev-take drop-bin2bl)
  done

lemma rbl-mult-gt:
  m > n ==> rbl-mult (rev (bin-to-bl n bina)) (rev (bin-to-bl m binb)) =
    rbl-mult (rev (bin-to-bl n bina)) (rev (bin-to-bl n binb))
  by (auto intro: trans [OF rbl-mult-gt1])

lemmas rbl-mult-Suc = lessI [THEN rbl-mult-gt]
```

```
lemma rbbl-Cons:
  b # rev (bin-to-bl n x) = rev (bin-to-bl (Suc n) (x BIT b))
  apply (unfold bin-to-bl-def)
  apply simp
  apply (simp add: bin-to-bl-aux-alt)
  done

lemma rbl-mult: !!bina binb.
    rbl-mult (rev (bin-to-bl n bina)) (rev (bin-to-bl n binb)) =
    rev (bin-to-bl n (bina * binb))
  apply (induct n)
   apply simp
  apply (unfold bin-to-bl-def)
  apply clarsimp
  apply (case-tac bina rule: bin-exhaust)
  apply (case-tac binb rule: bin-exhaust)
  apply (case-tac b)
   apply (case-tac [!] ba)
     apply (auto simp: bin-to-bl-aux-alt Let-def)
     apply (auto simp: rbbl-Cons rbl-mult-Suc rbl-add)
  done

lemma rbl-add-split:
  P (rbl-add (y # ys) (x # xs)) =
    (ALL ws. length ws = length ys --> ws = rbl-add ys xs -->
   (y --> ((x --> P (False # rbl-succ ws)) & (~ x --> P (True # ws)))) &
    (~ y --> P (x # ws)))
  apply (auto simp add: Let-def)
   apply (case-tac [!] y)
     apply auto
  done

lemma rbl-mult-split:
  P (rbl-mult (y # ys) xs) =
    (ALL ws. length ws = Suc (length ys) --> ws = False # rbl-mult ys xs -->

    (y --> P (rbl-add ws xs)) & (~ y --> P ws))
  by (clarsimp simp add : Let-def)
```

12.3 Repeated splitting or concatenation

```
lemma sclem:
  size (concat (map (bin-to-bl n) xs)) = length xs * n
  by (induct xs) auto

lemma bin-cat-foldl-lem:
  foldl (%u. bin-cat u n) x xs =
    bin-cat x (size xs * n) (foldl (%u. bin-cat u n) y xs)
```

apply (*induct xs arbitrary*: *x*)
 apply *simp*
apply (*simp* (*no-asm*))
apply (*frule asm-rl*)
apply (*drule meta-spec*)
apply (*erule trans*)
apply (*drule-tac x* = *bin-cat y n a* **in** *meta-spec*)
apply (*simp add* : *bin-cat-assoc-sym min.absorb2*)
done

lemma *bin-rcat-bl*:
(*bin-rcat n wl*) = *bl-to-bin* (*concat* (*map* (*bin-to-bl n*) *wl*))
apply (*unfold bin-rcat-def*)
apply (*rule sym*)
apply (*induct wl*)
 apply (*auto simp add* : *bl-to-bin-append*)
apply (*simp add* : *bl-to-bin-aux-alt sclem*)
apply (*simp add* : *bin-cat-foldl-lem* [*symmetric*])
done

lemmas *bin-rsplit-aux-simps* = *bin-rsplit-aux.simps bin-rsplitl-aux.simps*
lemmas *rsplit-aux-simps* = *bin-rsplit-aux-simps*

lemmas *th-if-simp1* = *if-split* [**where** *P* = *op* = *l*, *THEN iffD1*, *THEN conjunct1*, *THEN mp*] **for** *l*
lemmas *th-if-simp2* = *if-split* [**where** *P* = *op* = *l*, *THEN iffD1*, *THEN conjunct2*, *THEN mp*] **for** *l*

lemmas *rsplit-aux-simp1s* = *rsplit-aux-simps* [*THEN th-if-simp1*]

lemmas *rsplit-aux-simp2ls* = *rsplit-aux-simps* [*THEN th-if-simp2*]

lemmas *bin-rsplit-aux-simp2s* [*simp*] = *rsplit-aux-simp2ls* [*unfolded Let-def*]
lemmas *rbscl* = *bin-rsplit-aux-simp2s* (*2*)

lemmas *rsplit-aux-0-simps* [*simp*] =
rsplit-aux-simp1s [*OF disjI1*] *rsplit-aux-simp1s* [*OF disjI2*]

lemma *bin-rsplit-aux-append*:
bin-rsplit-aux n m c (*bs* @ *cs*) = *bin-rsplit-aux n m c bs* @ *cs*
apply (*induct n m c bs rule*: *bin-rsplit-aux.induct*)
apply (*subst bin-rsplit-aux.simps*)
apply (*subst bin-rsplit-aux.simps*)
apply (*clarsimp split*: *prod.split*)
done

lemma *bin-rsplitl-aux-append*:
bin-rsplitl-aux n m c (*bs* @ *cs*) = *bin-rsplitl-aux n m c bs* @ *cs*
apply (*induct n m c bs rule*: *bin-rsplitl-aux.induct*)

apply (*subst bin-rsplitl-aux.simps*)
apply (*subst bin-rsplitl-aux.simps*)
apply (*clarsimp split*: *prod.split*)
done

lemmas *rsplit-aux-apps* [**where** *bs* = []] =
bin-rsplit-aux-append bin-rsplitl-aux-append

lemmas *rsplit-def-auxs* = *bin-rsplit-def bin-rsplitl-def*

lemmas *rsplit-aux-alts* = *rsplit-aux-apps*
[*unfolded append-Nil rsplit-def-auxs* [*symmetric*]]

lemma *bin-split-minus*: *0* < *n* ==> *bin-split* (*Suc* (*n* − *1*)) *w* = *bin-split n w*
by *auto*

lemmas *bin-split-minus-simp* =
bin-split.Suc [*THEN* [*2*] *bin-split-minus* [*symmetric*, *THEN trans*]]

lemma *bin-split-pred-simp* [*simp*]:
(*0*::*nat*) < *numeral bin* ⟹
bin-split (*numeral bin*) *w* =
(*let* (*w1*, *w2*) = *bin-split* (*numeral bin* − *1*) (*bin-rest w*)
in (*w1*, *w2 BIT bin-last w*))
by (*simp only*: *bin-split-minus-simp*)

lemma *bin-rsplit-aux-simp-alt*:
bin-rsplit-aux n m c bs =
(*if m* = *0* ∨ *n* = *0*
then bs
else let (*a*, *b*) = *bin-split n c in bin-rsplit n* (*m* − *n*, *a*) @ *b* # *bs*)
unfolding *bin-rsplit-aux.simps* [*of n m c bs*]
apply *simp*
apply (*subst rsplit-aux-alts*)
apply (*simp add*: *bin-rsplit-def*)
done

lemmas *bin-rsplit-simp-alt* =
trans [*OF bin-rsplit-def bin-rsplit-aux-simp-alt*]

lemmas *bthrs* = *bin-rsplit-simp-alt* [*THEN* [*2*] *trans*]

lemma *bin-rsplit-size-sign′* [*rule-format*] :
⟦*n* > *0*; *rev sw* = *bin-rsplit n* (*nw*, *w*)⟧ ⟹
(*ALL v*: *set sw*. *bintrunc n v* = *v*)
apply (*induct sw arbitrary*: *nw w*)
apply *clarsimp*
apply *clarsimp*
apply (*drule bthrs*)

```
  apply (simp (no-asm-use) add: Let-def split: prod.split-asm if-split-asm)
  apply clarify
  apply (drule split-bintrunc)
  apply simp
  done

lemmas bin-rsplit-size-sign = bin-rsplit-size-sign' [OF asm-rl
  rev-rev-ident [THEN trans] set-rev [THEN equalityD2 [THEN subsetD]]]

lemma bin-nth-rsplit [rule-format] :
  n > 0 ==> m < n ==> (ALL w k nw. rev sw = bin-rsplit n (nw, w) -->
       k < size sw --> bin-nth (sw ! k) m = bin-nth w (k * n + m))
  apply (induct sw)
   apply clarsimp
  apply clarsimp
  apply (drule bthrs)
  apply (simp (no-asm-use) add: Let-def split: prod.split-asm if-split-asm)
  apply clarify
  apply (erule allE, erule impE, erule exI)
  apply (case-tac k)
   apply clarsimp
   prefer 2
   apply clarsimp
   apply (erule allE)
   apply (erule (1) impE)
   apply (drule bin-nth-split, erule conjE, erule allE,
          erule trans, simp add : ac-simps)+
  done

lemma bin-rsplit-all:
  0 < nw ==> nw <= n ==> bin-rsplit n (nw, w) = [bintrunc n w]
  unfolding bin-rsplit-def
  by (clarsimp dest!: split-bintrunc simp: rsplit-aux-simp2ls split: prod.split)

lemma bin-rsplit-l [rule-format] :
  ALL bin. bin-rsplitl n (m, bin) = bin-rsplit n (m, bintrunc m bin)
  apply (rule-tac a = m in wf-less-than [THEN wf-induct])
  apply (simp (no-asm) add : bin-rsplitl-def bin-rsplit-def)
  apply (rule allI)
  apply (subst bin-rsplitl-aux.simps)
  apply (subst bin-rsplit-aux.simps)
  apply (clarsimp simp: Let-def split: prod.split)
  apply (drule bin-split-trunc)
  apply (drule sym [THEN trans], assumption)
  apply (subst rsplit-aux-alts(1))
  apply (subst rsplit-aux-alts(2))
  apply clarsimp
  unfolding bin-rsplit-def bin-rsplitl-def
  apply simp
```

```
  done

lemma bin-rsplit-rcat [rule-format] :
  n > 0 --> bin-rsplit n (n * size ws, bin-rcat n ws) = map (bintrunc n) ws
  apply (unfold bin-rsplit-def bin-rcat-def)
  apply (rule-tac xs = ws in rev-induct)
   apply clarsimp
  apply clarsimp
  apply (subst rsplit-aux-alts)
  unfolding bin-split-cat
  apply simp
  done

lemma bin-rsplit-aux-len-le [rule-format] :
  ∀ ws m. n ≠ 0 ⟶ ws = bin-rsplit-aux n nw w bs ⟶
    length ws ≤ m ⟷ nw + length bs * n ≤ m * n
proof -
  { fix i j j' k k' m :: nat and R
    assume d: (i::nat) ≤ j ∨ m < j'
    assume R1: i * k ≤ j * k ⟹ R
    assume R2: Suc m * k' ≤ j' * k' ⟹ R
    have R using d
      apply safe
       apply (rule R1, erule mult-le-mono1)
      apply (rule R2, erule Suc-le-eq [THEN iffD2 [THEN mult-le-mono1]])
      done
  } note A = this
  { fix sc m n lb :: nat
    have (0::nat) < sc ⟹ sc − n + (n + lb * n) ≤ m * n ⟷ sc + lb * n ≤
m * n
      apply safe
       apply arith
      apply (case-tac sc >= n)
       apply arith
      apply (insert linorder-le-less-linear [of m lb])
      apply (erule-tac k2=n and k'2=n in A)
       apply arith
      apply simp
      done
  } note B = this
  show ?thesis
    apply (induct n nw w bs rule: bin-rsplit-aux.induct)
    apply (subst bin-rsplit-aux.simps)
    apply (simp add: B Let-def split: prod.split)
    done
qed

lemma bin-rsplit-len-le:
  n ≠ 0 --> ws = bin-rsplit n (nw, w) --> (length ws <= m) = (nw <= m *
```

```
n)
  unfolding bin-rsplit-def by (clarsimp simp add : bin-rsplit-aux-len-le)

lemma bin-rsplit-aux-len:
  n ≠ 0 ⟹ length (bin-rsplit-aux n nw w cs) =
    (nw + n − 1) div n + length cs
  apply (induct n nw w cs rule: bin-rsplit-aux.induct)
  apply (subst bin-rsplit-aux.simps)
  apply (clarsimp simp: Let-def split: prod.split)
  apply (erule thin-rl)
  apply (case-tac m)
  apply simp
  apply (case-tac m <= n)
  apply auto
  done

lemma bin-rsplit-len:
  n≠0 ==> length (bin-rsplit n (nw, w)) = (nw + n − 1) div n
  unfolding bin-rsplit-def by (clarsimp simp add : bin-rsplit-aux-len)

lemma bin-rsplit-aux-len-indep:
  n ≠ 0 ⟹ length bs = length cs ⟹
    length (bin-rsplit-aux n nw v bs) =
    length (bin-rsplit-aux n nw w cs)
proof (induct n nw w cs arbitrary: v bs rule: bin-rsplit-aux.induct)
  case (1 n m w cs v bs) show ?case
  proof (cases m = 0)
    case True then show ?thesis using ‹length bs = length cs› by simp
  next
    case False
    from 1.hyps ‹m ≠ 0› ‹n ≠ 0› have hyp: ⋀v bs. length bs = Suc (length cs)
⟹
      length (bin-rsplit-aux n (m − n) v bs) =
      length (bin-rsplit-aux n (m − n) (fst (bin-split n w)) (snd (bin-split n w) #
cs))
    by auto
    show ?thesis using ‹length bs = length cs› ‹n ≠ 0›
      by (auto simp add: bin-rsplit-aux-simp-alt Let-def bin-rsplit-len
        split: prod.split)
  qed
qed

lemma bin-rsplit-len-indep:
  n≠0 ==> length (bin-rsplit n (nw, v)) = length (bin-rsplit n (nw, w))
  apply (unfold bin-rsplit-def)
  apply (simp (no-asm))
  apply (erule bin-rsplit-aux-len-indep)
  apply (rule refl)
  done
```

Even more bit operations

instantiation *int* :: *bitss*
begin

definition [*iff*]:
  i !! $n \longleftrightarrow$ *bin-nth* i n

definition
  lsb $i = (i :: int)$!! *0*

definition
  set-bit i n b = *bin-sc* n b i

definition
  set-bits f =
  (*if* $\exists n. \forall n' \geq n. \neg f\ n'$ *then*
    let n = *LEAST* $n. \forall n' \geq n. \neg f\ n'$
    in *bl-to-bin* (*rev* (*map* f [*0*..<n]))
   else if $\exists n. \forall n' \geq n. f\ n'$ *then*
    let n = *LEAST* $n. \forall n' \geq n. f\ n'$
    in *sbintrunc* n (*bl-to-bin* (*True* # *rev* (*map* f [*0*..<n])))
   else *0* :: *int*)

definition
  shiftl x $n = (x :: int) * 2$ ^ n

definition
  shiftr x $n = (x :: int)$ *div* *2* ^ n

definition
  msb $x \longleftrightarrow (x :: int) < 0$

instance ..

end

end

13 Type Definition Theorems

theory *Misc-Typedef*
imports *Main*
begin

14 More lemmas about normal type definitions

lemma
  tdD1: *type-definition Rep Abs A* $\Longrightarrow \forall x.$ *Rep* $x \in A$ **and**

tdD2: *type-definition Rep Abs A* $\Longrightarrow \forall x.$ *Abs* (*Rep x*) = *x* **and**
tdD3: *type-definition Rep Abs A* $\Longrightarrow \forall y.\ y \in A \longrightarrow$ *Rep* (*Abs y*) = *y*
by (*auto simp*: *type-definition-def*)

lemma *td-nat-int*:
type-definition int nat (*Collect* (*op* <= *0*))
unfolding *type-definition-def* **by** *auto*

context *type-definition*
begin

declare *Rep* [*iff*] *Rep-inverse* [*simp*] *Rep-inject* [*simp*]

lemma *Abs-eqD*: *Abs x* = *Abs y* ==> *x* ∈ *A* ==> *y* ∈ *A* ==> *x* = *y*
by (*simp add*: *Abs-inject*)

lemma *Abs-inverse'*:
r : *A* ==> *Abs r* = *a* ==> *Rep a* = *r*
by (*safe elim*!: *Abs-inverse*)

lemma *Rep-comp-inverse*:
Rep o f = *g* ==> *Abs o g* = *f*
using *Rep-inverse* **by** *auto*

lemma *Rep-eqD* [*elim*!]: *Rep x* = *Rep y* ==> *x* = *y*
by *simp*

lemma *Rep-inverse'*: *Rep a* = *r* ==> *Abs r* = *a*
by (*safe intro*!: *Rep-inverse*)

lemma *comp-Abs-inverse*:
f o Abs = *g* ==> *g o Rep* = *f*
using *Rep-inverse* **by** *auto*

lemma *set-Rep*:
A = *range Rep*
proof (*rule set-eqI*)
fix *x*
show (*x* ∈ *A*) = (*x* ∈ *range Rep*)
by (*auto dest*: *Abs-inverse* [*of x*, *symmetric*])
qed

lemma *set-Rep-Abs*: *A* = *range* (*Rep o Abs*)
proof (*rule set-eqI*)
fix *x*
show (*x* ∈ *A*) = (*x* ∈ *range* (*Rep o Abs*))
by (*auto dest*: *Abs-inverse* [*of x*, *symmetric*])
qed

lemma *Abs-inj-on*: *inj-on Abs A*
unfolding *inj-on-def*
by (*auto dest*: *Abs-inject* [*THEN iffD1*])

lemma *image*: *Abs ' A = UNIV*
by (*auto intro*!: *image-eqI*)

lemmas *td-thm* = *type-definition-axioms*

lemma *fns1*:
Rep o fa = fr o Rep | fa o Abs = Abs o fr ==> Abs o fr o Rep = fa
by (*auto dest*: *Rep-comp-inverse elim*: *comp-Abs-inverse simp*: *o-assoc*)

lemmas *fns1a* = *disjI1* [*THEN fns1*]
lemmas *fns1b* = *disjI2* [*THEN fns1*]

lemma *fns4*:
Rep o fa o Abs = fr ==>
Rep o fa = fr o Rep & fa o Abs = Abs o fr
by *auto*

end

interpretation *nat-int*: *type-definition int nat Collect* (*op <= 0*)
by (*rule td-nat-int*)

declare
nat-int.Rep-cases [*cases del*]
nat-int.Abs-cases [*cases del*]
nat-int.Rep-induct [*induct del*]
nat-int.Abs-induct [*induct del*]

14.1 Extended form of type definition predicate

lemma *td-conds*:
norm o norm = norm ==> (*fr o norm = norm o fr*) =
(*norm o fr o norm = fr o norm & norm o fr o norm = norm o fr*)
apply *safe*
apply (*simp-all add*: *comp-assoc*)
apply (*simp-all add*: *o-assoc*)
done

lemma *fn-comm-power*:
fa o tr = tr o fr ==> fa ^^ n o tr = tr o fr ^^ n
apply (*rule ext*)
apply (*induct n*)
apply (*auto dest*: *fun-cong*)
done

lemmas *fn-comm-power′* =
ext [*THEN fn-comm-power*, *THEN fun-cong*, *unfolded o-def*]

locale *td-ext* = *type-definition* +
fixes *norm*
assumes *eq-norm*: $\bigwedge x$. *Rep* (*Abs x*) = *norm x*
begin

lemma *Abs-norm* [*simp*]:
Abs (*norm x*) = *Abs x*
using *eq-norm* [*of x*] **by** (*auto elim*: *Rep-inverse′*)

lemma *td-th*:
g o Abs = *f* ==> *f* (*Rep x*) = *g x*
by (*drule comp-Abs-inverse* [*symmetric*]) *simp*

lemma *eq-norm′*: *Rep o Abs* = *norm*
by (*auto simp*: *eq-norm*)

lemma *norm-Rep* [*simp*]: *norm* (*Rep x*) = *Rep x*
by (*auto simp*: *eq-norm′ intro*: *td-th*)

lemmas *td* = *td-thm*

lemma *set-iff-norm*: *w* : *A* ⟷ *w* = *norm w*
by (*auto simp*: *set-Rep-Abs eq-norm′ eq-norm* [*symmetric*])

lemma *inverse-norm*:
(*Abs n* = *w*) = (*Rep w* = *norm n*)
apply (*rule iffI*)
apply (*clarsimp simp add*: *eq-norm*)
apply (*simp add*: *eq-norm′* [*symmetric*])
done

lemma *norm-eq-iff*:
(*norm x* = *norm y*) = (*Abs x* = *Abs y*)
by (*simp add*: *eq-norm′* [*symmetric*])

lemma *norm-comps*:
Abs o norm = *Abs*
norm o Rep = *Rep*
norm o norm = *norm*
by (*auto simp*: *eq-norm′* [*symmetric*] *o-def*)

lemmas *norm-norm* [*simp*] = *norm-comps*

lemma *fns5*:
Rep o fa o Abs = *fr* ==>

```
  fr o norm = fr & norm o fr = fr
  by (fold eq-norm') auto


lemma fns2:
  Abs o fr o Rep = fa ==>
   (norm o fr o norm = fr o norm) = (Rep o fa = fr o Rep)
  apply (fold eq-norm')
  apply safe
   prefer 2
   apply (simp add: o-assoc)
  apply (rule ext)
  apply (drule-tac x=Rep x in fun-cong)
  apply auto
  done

lemma fns3:
  Abs o fr o Rep = fa ==>
   (norm o fr o norm = norm o fr) = (fa o Abs = Abs o fr)
  apply (fold eq-norm')
  apply safe
   prefer 2
   apply (simp add: comp-assoc)
  apply (rule ext)
  apply (drule-tac f=a o b for a b in fun-cong)
  apply simp
  done

lemma fns:
  fr o norm = norm o fr ==>
    (fa o Abs = Abs o fr) = (Rep o fa = fr o Rep)
  apply safe
   apply (frule fns1b)
   prefer 2
   apply (frule fns1a)
   apply (rule fns3 [THEN iffD1])
     prefer 3
     apply (rule fns2 [THEN iffD1])
       apply (simp-all add: comp-assoc)
   apply (simp-all add: o-assoc)
  done

lemma range-norm:
  range (Rep o Abs) = A
  by (simp add: set-Rep-Abs)

end

lemmas td-ext-def' =
```

td-ext-def [*unfolded type-definition-def td-ext-axioms-def*]

end

15 Miscellaneous lemmas, of at least doubtful value

theory *Word-Miscellaneous*
imports *Main* ~~/*src*/*HOL*/*Library*/*Bit Misc-Numeric*
begin

lemma *power-minus-simp*:
$0 < n \Longrightarrow a \hat{} n = a * a \hat{} (n - 1)$
by (*auto dest*: *gr0-implies-Suc*)

lemma *funpow-minus-simp*:
$0 < n \Longrightarrow f \hat{}\hat{} n = f \circ f \hat{}\hat{} (n - 1)$
by (*auto dest*: *gr0-implies-Suc*)

lemma *power-numeral*:
a ^ *numeral k* = *a* ∗ *a* ^ (*pred-numeral k*)
by (*simp add*: *numeral-eq-Suc*)

lemma *funpow-numeral* [*simp*]:
f ^^ *numeral k* = *f* ∘ *f* ^^ (*pred-numeral k*)
by (*simp add*: *numeral-eq-Suc*)

lemma *replicate-numeral* [*simp*]:
replicate (*numeral k*) *x* = *x* # *replicate* (*pred-numeral k*) *x*
by (*simp add*: *numeral-eq-Suc*)

lemma *rco-alt*: (*f o g*) ^^ *n o f* = *f o* (*g o f*) ^^ *n*
apply (*rule ext*)
apply (*induct n*)
apply (*simp-all add*: *o-def*)
done

lemma *list-exhaust-size-gt0*:
assumes *y*: $\bigwedge$*a list*. *y* = *a* # *list* ⟹ *P*
shows *0* < *length y* ⟹ *P*
apply (*cases y*, *simp*)
apply (*rule y*)
apply *fastforce*
done

lemma *list-exhaust-size-eq0*:
assumes *y*: *y* = [] ⟹ *P*
shows *length y* = *0* ⟹ *P*
apply (*cases y*)
apply (*rule y*, *simp*)

apply *simp*
done

lemma *size-Cons-lem-eq*:
y = xa # list ==> size y = Suc k ==> size list = k
by *auto*

lemmas *ls-splits = prod.split prod.split-asm if-split-asm*

lemma *not-B1-is-B0*: $y \neq (1::bit) \Longrightarrow y = (0::bit)$
by (*cases y*) *auto*

lemma *B1-ass-B0*:
assumes *y*: $y = (0::bit) \Longrightarrow y = (1::bit)$
shows $y = (1::bit)$
apply (*rule classical*)
apply (*drule not-B1-is-B0*)
apply (*erule y*)
done

— simplifications for specific word lengths
lemmas *n2s-ths* [*THEN eq-reflection*] = *add-2-eq-Suc add-2-eq-Suc′*

lemmas *s2n-ths = n2s-ths* [*symmetric*]

lemma *and-len*: *xs = ys ==> xs = ys & length xs = length ys*
by *auto*

lemma *size-if*: *size* (*if p then xs else ys*) = (*if p then size xs else size ys*)
by *auto*

lemma *tl-if*: *tl* (*if p then xs else ys*) = (*if p then tl xs else tl ys*)
by *auto*

lemma *hd-if*: *hd* (*if p then xs else ys*) = (*if p then hd xs else hd ys*)
by *auto*

lemma *if-Not-x*: (*if p then ~ x else x*) = (*p* = (*~ x*))
by *auto*

lemma *if-x-Not*: (*if p then x else ~ x*) = (*p = x*)
by *auto*

lemma *if-same-and*: (*If p x y & If p u v*) = (*if p then x & u else y & v*)
by *auto*

lemma *if-same-eq*: (*If p x y* = (*If p u v*)) = (*if p then x* = (*u*) *else y* = (*v*))
by *auto*

lemma *if-same-eq-not*:
(*If p x y* = (~ *If p u v*)) = (*if p then x* = (~*u*) *else y* = (~*v*))
by *auto*

lemma *if-Cons*: (*if p then x # xs else y # ys*) = *If p x y # If p xs ys*
by *auto*

lemma *if-single*:
(*if xc then* [*xab*] *else* [*an*]) = [*if xc then xab else an*]
by *auto*

lemma *if-bool-simps*:
If p True y = (*p* | *y*) & *If p False y* = (~*p* & *y*) &
If p y True = (*p* --> *y*) & *If p y False* = (*p* & *y*)
by *auto*

lemmas *if-simps* = *if-x-Not if-Not-x if-cancel if-True if-False if-bool-simps*

lemmas *seqr* = *eq-reflection* [**where** *x* = *size w*] **for** *w*

lemma *the-elemI*: *y* = {*x*} ==> *the-elem y* = *x*
by *simp*

lemma *nonemptyE*: *S* ~= {} ==> (!!*x*. *x* : *S* ==> *R*) ==> *R* **by** *auto*

lemma *gt-or-eq-0*: *0* < *y* ∨ *0* = (*y*::*nat*) **by** *arith*

lemmas *xtr1* = *xtrans*(*1*)
lemmas *xtr2* = *xtrans*(*2*)
lemmas *xtr3* = *xtrans*(*3*)
lemmas *xtr4* = *xtrans*(*4*)
lemmas *xtr5* = *xtrans*(*5*)
lemmas *xtr6* = *xtrans*(*6*)
lemmas *xtr7* = *xtrans*(*7*)
lemmas *xtr8* = *xtrans*(*8*)

lemmas *nat-simps* = *diff-add-inverse2 diff-add-inverse*
lemmas *nat-iffs* = *le-add1 le-add2*

lemma *sum-imp-diff*: *j* = *k* + *i* ==> *j* − *i* = (*k* :: *nat*) **by** *arith*

lemmas *pos-mod-sign2* = *zless2* [*THEN pos-mod-sign* [**where** *b* = *2*::*int*]]
lemmas *pos-mod-bound2* = *zless2* [*THEN pos-mod-bound* [**where** *b* = *2*::*int*]]

lemma *nmod2*: *n mod* (*2*::*int*) = *0* | *n mod 2* = *1*
by *arith*

lemmas *eme1p* = *emep1* [*simplified add.commute*]

lemma *le-diff-eq′*: $(a \le c - b) = (b + a \le (c::int))$ **by** *arith*

lemma *less-diff-eq′*: $(a < c - b) = (b + a < (c::int))$ **by** *arith*

lemma *diff-less-eq′*: $(a - b < c) = (a < b + (c::int))$ **by** *arith*

lemmas *m1mod22k* = *mult-pos-pos* [*OF zless2 zless2p*, *THEN zmod-minus1*]

lemma *z1pdiv2*:
$(2 * b + 1)$ *div* $2 = (b::int)$ **by** *arith*

lemmas *zdiv-le-dividend* = *xtr3* [*OF div-by-1* [*symmetric*] *zdiv-mono2*,
simplified int-one-le-iff-zero-less, *simplified*]

lemma *axxbyy*:
a + m + m = b + n + n ==> (a = 0 | a = 1) ==> (b = 0 | b = 1) ==>
a = b & m = (n :: int) **by** *arith*

lemma *axxmod2*:
(1 + x + x) mod 2 = (1 :: int) & (0 + x + x) mod 2 = (0 :: int) **by** *arith*

lemma *axxdiv2*:
(1 + x + x) div 2 = (x :: int) & (0 + x + x) div 2 = (x :: int) **by** *arith*

lemmas *iszero-minus* = *trans* [*THEN trans*,
OF iszero-def neg-equal-0-iff-equal iszero-def [*symmetric*]]

lemmas *zadd-diff-inverse* = *trans* [*OF diff-add-cancel* [*symmetric*] *add.commute*]

lemmas *add-diff-cancel2* = *add.commute* [*THEN diff-eq-eq* [*THEN iffD2*]]

lemmas *rdmods* [*symmetric*] = *mod-minus-eq*
mod-diff-left-eq mod-diff-right-eq mod-add-left-eq
mod-add-right-eq mod-mult-right-eq mod-mult-left-eq

lemma *mod-plus-right*:
((a + x) mod m = (b + x) mod m) = (a mod m = b mod (m :: nat))
apply (*induct x*)
 apply (*simp-all add*: *mod-Suc*)
apply *arith*
done

lemma *nat-minus-mod*: *(n − n mod m) mod m = (0 :: nat)*
by (*induct n*) (*simp-all add* : *mod-Suc*)

lemmas *nat-minus-mod-plus-right* = *trans* [*OF nat-minus-mod mod-0* [*symmetric*],
THEN mod-plus-right [*THEN iffD2*], *simplified*]

```
lemmas push-mods' = mod-add-eq
  mod-mult-eq mod-diff-eq
  mod-minus-eq

lemmas push-mods = push-mods' [THEN eq-reflection]
lemmas pull-mods = push-mods [symmetric] rdmods [THEN eq-reflection]
lemmas mod-simps =
  mod-mult-self2-is-0 [THEN eq-reflection]
  mod-mult-self1-is-0 [THEN eq-reflection]
  mod-mod-trivial [THEN eq-reflection]

lemma nat-mod-eq:
  !!b. b < n ==> a mod n = b mod n ==> a mod n = (b :: nat)
  by (induct a) auto

lemmas nat-mod-eq' = refl [THEN [2] nat-mod-eq]

lemma nat-mod-lem:
  (0 :: nat) < n ==> b < n = (b mod n = b)
  apply safe
   apply (erule nat-mod-eq')
  apply (erule subst)
  apply (erule mod-less-divisor)
  done

lemma mod-nat-add:
  (x :: nat) < z ==> y < z ==>
   (x + y) mod z = (if x + y < z then x + y else x + y − z)
  apply (rule nat-mod-eq)
   apply auto
  apply (rule trans)
   apply (rule le-mod-geq)
   apply simp
  apply (rule nat-mod-eq')
  apply arith
  done

lemma mod-nat-sub:
  (x :: nat) < z ==> (x − y) mod z = x − y
  by (rule nat-mod-eq') arith

lemma int-mod-eq:
  (0 :: int) <= b ==> b < n ==> a mod n = b mod n ==> a mod n = b
  by (metis mod-pos-pos-trivial)

lemmas int-mod-eq' = mod-pos-pos-trivial

lemma int-mod-le: (0::int) <= a ==> a mod n <= a
  by (fact Divides.semiring-numeral-div-class.mod-less-eq-dividend)
```

lemma *mod-add-if-z*:
(*x* :: *int*) < *z* ==> *y* < *z* ==> *0* <= *y* ==> *0* <= *x* ==> *0* <= *z* ==>
(*x* + *y*) *mod z* = (*if x* + *y* < *z then x* + *y else x* + *y* − *z*)
by (*auto intro*: *int-mod-eq*)

lemma *mod-sub-if-z*:
(*x* :: *int*) < *z* ==> *y* < *z* ==> *0* <= *y* ==> *0* <= *x* ==> *0* <= *z* ==>
(*x* − *y*) *mod z* = (*if y* <= *x then x* − *y else x* − *y* + *z*)
by (*auto intro*: *int-mod-eq*)

lemmas *zmde* = *zmod-zdiv-equality* [*THEN diff-eq-eq* [*THEN iffD2*], *symmetric*]
lemmas *mcl* = *mult-cancel-left* [*THEN iffD1*, *THEN make-pos-rule*]

lemma *zdiv-mult-self*: *m* ~= (*0* :: *int*) ==> (*a* + *m* ∗ *n*) *div m* = *a div m* + *n*
apply (*rule mcl*)
prefer *2*
apply (*erule asm-rl*)
apply (*simp add*: *zmde ring-distribs*)
done

lemma *mod-power-lem*:
a > *1* ==> *a* ^ *n mod a* ^ *m* = (*if m* <= *n then 0 else* (*a* :: *int*) ^ *n*)
apply *clarsimp*
apply *safe*
apply (*simp add*: *dvd-eq-mod-eq-0* [*symmetric*])
apply (*drule le-iff-add* [*THEN iffD1*])
apply (*force simp*: *power-add*)
apply (*rule mod-pos-pos-trivial*)
apply (*simp*)
apply (*rule power-strict-increasing*)
apply *auto*
done

lemma *pl-pl-rels*:
a + *b* = *c* + *d* ==>
a >= *c* & *b* <= *d* | *a* <= *c* & *b* >= (*d* :: *nat*) **by** *arith*

lemmas *pl-pl-rels'* = *add.commute* [*THEN* [*2*] *trans*, *THEN pl-pl-rels*]

lemma *minus-eq*: (*m* − *k* = *m*) = (*k* = *0* | *m* = (*0* :: *nat*)) **by** *arith*

lemma *pl-pl-mm*: (*a* :: *nat*) + *b* = *c* + *d* ==> *a* − *c* = *d* − *b* **by** *arith*

lemmas *pl-pl-mm'* = *add.commute* [*THEN* [*2*] *trans*, *THEN pl-pl-mm*]

lemmas *dme* = *box-equals* [*OF div-mod-equality add-0-right add-0-right*]
lemmas *dtle* = *xtr3* [*OF dme* [*symmetric*] *le-add1*]

```
lemmas th2 = order-trans [OF order-refl [THEN [2] mult-le-mono] dtle]

lemma td-gal:
  0 < c ==> (a >= b * c) = (a div c >= (b :: nat))
  apply safe
   apply (erule (1) xtr4 [OF div-le-mono div-mult-self-is-m])
  apply (erule th2)
  done

lemmas td-gal-lt = td-gal [simplified not-less [symmetric], simplified]

lemma div-mult-le: (a :: nat) div b * b <= a
  by (fact dtle)

lemmas sdl = split-div-lemma [THEN iffD1, symmetric]

lemma given-quot: f > (0 :: nat) ==> (f * l + (f - 1)) div f = l
  by (rule sdl, assumption) (simp (no-asm))

lemma given-quot-alt: f > (0 :: nat) ==> (l * f + f - Suc 0) div f = l
  apply (frule given-quot)
  apply (rule trans)
   prefer 2
   apply (erule asm-rl)
  apply (rule-tac f=%n. n div f in arg-cong)
  apply (simp add : ac-simps)
  done

lemma diff-mod-le: (a::nat) < d ==> b dvd d ==> a - a mod b <= d - b
  apply (unfold dvd-def)
  apply clarify
  apply (case-tac k)
   apply clarsimp
  apply clarify
  apply (cases b > 0)
   apply (drule mult.commute [THEN xtr1])
   apply (frule (1) td-gal-lt [THEN iffD1])
   apply (clarsimp simp: le-simps)
   apply (rule mult-div-cancel [THEN [2] xtr4])
   apply (rule mult-mono)
      apply auto
  done

lemma less-le-mult':
  w * c < b * c ==> 0 ≤ c ==> (w + 1) * c ≤ b * (c::int)
  apply (rule mult-right-mono)
   apply (rule zless-imp-add1-zle)
   apply (erule (1) mult-right-less-imp-less)
  apply assumption
```

done

lemma *less-le-mult*:
w ∗ *c* < *b* ∗ *c* ⟹ *0* ≤ *c* ⟹ *w* ∗ *c* + *c* ≤ *b* ∗ (*c*::*int*)
using *less-le-mult'* [*of w c b*] **by** (*simp add*: *algebra-simps*)

lemmas *less-le-mult-minus* = *iffD2* [*OF le-diff-eq less-le-mult*,
simplified left-diff-distrib]

lemma *gen-minus*: *0* < *n* ==> *f n* = *f* (*Suc* (*n* − *1*))
by *auto*

lemma *mpl-lem*: *j* <= (*i* :: *nat*) ==> *k* < *j* ==> *i* − *j* + *k* < *i* **by** *arith*

lemma *nonneg-mod-div*:
0 <= *a* ==> *0* <= *b* ==> *0* <= (*a mod b* :: *int*) & *0* <= *a div b*
apply (*cases b* = *0*, *clarsimp*)
apply (*auto intro*: *pos-imp-zdiv-nonneg-iff* [*THEN iffD2*])
done

declare *iszero-0* [*intro*]

lemma *min-pm* [*simp*]:
min a b + (*a* − *b*) = (*a* :: *nat*)
by *arith*

lemma *min-pm1* [*simp*]:
a − *b* + *min a b* = (*a* :: *nat*)
by *arith*

lemma *rev-min-pm* [*simp*]:
min b a + (*a* − *b*) = (*a* :: *nat*)
by *arith*

lemma *rev-min-pm1* [*simp*]:
a − *b* + *min b a* = (*a* :: *nat*)
by *arith*

lemma *min-minus* [*simp*]:
min m (*m* − *k*) = (*m* − *k* :: *nat*)
by *arith*

lemma *min-minus'* [*simp*]:
min (*m* − *k*) *m* = (*m* − *k* :: *nat*)
by *arith*

end

16 A type of finite bit strings

theory *Word*
imports
Type-Length
~~/src/HOL/Library/Boolean-Algebra
Bits-Bit
Bool-List-Representation
Misc-Typedef
Word-Miscellaneous
begin

See `Examples/WordExamples.thy` for examples.

16.1 Type definition

typedef (**overloaded**) *'a word* = {(*0*::*int*) ..< *2* ^ *len-of TYPE*(*'a*::*len0*)}
morphisms *uint Abs-word* **by** *auto*

lemma *uint-nonnegative*:
0 ≤ *uint w*
using *word.uint* [*of w*] **by** *simp*

lemma *uint-bounded*:
fixes *w* :: *'a*::*len0 word*
shows *uint w* < *2* ^ *len-of TYPE*(*'a*)
using *word.uint* [*of w*] **by** *simp*

lemma *uint-idem*:
fixes *w* :: *'a*::*len0 word*
shows *uint w mod 2* ^ *len-of TYPE*(*'a*) = *uint w*
using *uint-nonnegative uint-bounded* **by** (*rule mod-pos-pos-trivial*)

lemma *word-uint-eq-iff*:
a = *b* ⟷ *uint a* = *uint b*
by (*simp add*: *uint-inject*)

lemma *word-uint-eqI*:
uint a = *uint b* ⟹ *a* = *b*
by (*simp add*: *word-uint-eq-iff*)

definition *word-of-int* :: *int* ⇒ *'a*::*len0 word*
where
— representation of words using unsigned or signed bins, only difference in these is the type class
word-of-int k = *Abs-word* (*k mod 2* ^ *len-of TYPE*(*'a*))

lemma *uint-word-of-int*:
uint (*word-of-int k* :: *'a*::*len0 word*) = *k mod 2* ^ *len-of TYPE*(*'a*)
by (*auto simp add*: *word-of-int-def intro*: *Abs-word-inverse*)

lemma *word-of-int-uint*:
  word-of-int (*uint w*) = *w*
  by (*simp add*: *word-of-int-def uint-idem uint-inverse*)

lemma *split-word-all*:
  $(\bigwedge x::'a::len0\ word.\ PROP\ P\ x) \equiv (\bigwedge x.\ PROP\ P\ (word\text{-}of\text{-}int\ x))$
proof
  fix *x* :: *'a word*
  assume $\bigwedge x.\ PROP\ P\ (word\text{-}of\text{-}int\ x)$
  then have *PROP P* (*word-of-int* (*uint x*)) **.**
  then show *PROP P x* **by** (*simp add*: *word-of-int-uint*)
qed

16.2 Type conversions and casting

definition *sint* :: *'a::len word* ⇒ *int*
where
  — treats the most-significant-bit as a sign bit
  sint-uint: *sint w* = *sbintrunc* (*len-of TYPE* (*'a*) − *1*) (*uint w*)

definition *unat* :: *'a::len0 word* ⇒ *nat*
where
  unat w = *nat* (*uint w*)

definition *uints* :: *nat* ⇒ *int set*
where
  — the sets of integers representing the words
  uints n = *range* (*bintrunc n*)

definition *sints* :: *nat* ⇒ *int set*
where
  sints n = *range* (*sbintrunc* (*n* − *1*))

lemma *uints-num*:
  $uints\ n = \{i.\ 0 \le i \wedge i < 2\ \hat{}\ n\}$
  by (*simp add*: *uints-def range-bintrunc*)

lemma *sints-num*:
  $sints\ n = \{i.\ -(2\ \hat{}\ (n - 1)) \le i \wedge i < 2\ \hat{}\ (n - 1)\}$
  by (*simp add*: *sints-def range-sbintrunc*)

definition *unats* :: *nat* ⇒ *nat set*
where
  $unats\ n = \{i.\ i < 2\ \hat{}\ n\}$

definition *norm-sint* :: *nat* ⇒ *int* ⇒ *int*
where
  $norm\text{-}sint\ n\ w = (w + 2\ \hat{}\ (n - 1))\ mod\ 2\ \hat{}\ n - 2\ \hat{}\ (n - 1)$

definition *scast* :: *'a*::*len word* ⇒ *'b*::*len word*
where
— cast a word to a different length
scast w = *word-of-int* (*sint w*)

definition *ucast* :: *'a*::*len0 word* ⇒ *'b*::*len0 word*
where
ucast w = *word-of-int* (*uint w*)

instantiation *word* :: (*len0*) *size*
begin

definition
word-size: *size* (*w* :: *'a word*) = *len-of TYPE*(*'a*)

instance ..

end

lemma *word-size-gt-0* [*iff*]:
0 < *size* (*w*::*'a*::*len word*)
by (*simp add*: *word-size*)

lemmas *lens-gt-0* = *word-size-gt-0 len-gt-0*

lemma *lens-not-0* [*iff*]:
shows *size* (*w*::*'a*::*len word*) ≠ *0*
and *len-of TYPE*(*'a*::*len*) ≠ *0*
by *auto*

definition *source-size* :: (*'a*::*len0 word* ⇒ *'b*) ⇒ *nat*
where
— whether a cast (or other) function is to a longer or shorter length
[*code del*]: *source-size c* = (*let arb* = *undefined*; *x* = *c arb in size arb*)

definition *target-size* :: (*'a* ⇒ *'b*::*len0 word*) ⇒ *nat*
where
[*code del*]: *target-size c* = *size* (*c undefined*)

definition *is-up* :: (*'a*::*len0 word* ⇒ *'b*::*len0 word*) ⇒ *bool*
where
is-up c ⟷ *source-size c* ≤ *target-size c*

definition *is-down* :: (*'a* :: *len0 word* ⇒ *'b* :: *len0 word*) ⇒ *bool*
where
is-down c ⟷ *target-size c* ≤ *source-size c*

definition *of-bl* :: *bool list* ⇒ *'a*::*len0 word*

where
of-bl bl = word-of-int (bl-to-bin bl)

definition *to-bl :: 'a::len0 word ⇒ bool list*
where
to-bl w = bin-to-bl (len-of TYPE ('a)) (uint w)

definition *word-reverse :: 'a::len0 word ⇒ 'a word*
where
word-reverse w = of-bl (rev (to-bl w))

definition *word-int-case :: (int ⇒ 'b) ⇒ 'a::len0 word ⇒ 'b*
where
word-int-case f w = f (uint w)

translations
case x of XCONST of-int y => b == CONST word-int-case (%y. b) x
case x of (XCONST of-int :: 'a) y => b => CONST word-int-case (%y. b) x

16.3 Correspondence relation for theorem transfer

definition *cr-word :: int ⇒ 'a::len0 word ⇒ bool*
where
cr-word = (λx y. word-of-int x = y)

lemma *Quotient-word*:
Quotient (λx y. bintrunc (len-of TYPE('a)) x = bintrunc (len-of TYPE('a)) y)
word-of-int uint (cr-word :: - ⇒ 'a::len0 word ⇒ bool)
unfolding *Quotient-alt-def cr-word-def*
by (*simp add*: *no-bintr-alt1 word-of-int-uint*) (*simp add*: *word-of-int-def Abs-word-inject*)

lemma *reflp-word*:
reflp (λx y. bintrunc (len-of TYPE('a::len0)) x = bintrunc (len-of TYPE('a)) y)
by (*simp add*: *reflp-def*)

setup-lifting *Quotient-word reflp-word*

TODO: The next lemma could be generated automatically.

lemma *uint-transfer* [*transfer-rule*]:
(*rel-fun pcr-word op =*) (*bintrunc* (*len-of TYPE*('*a*)))
(*uint :: 'a::len0 word ⇒ int*)
unfolding *rel-fun-def word.pcr-cr-eq cr-word-def*
by (*simp add*: *no-bintr-alt1 uint-word-of-int*)

16.4 Basic code generation setup

definition *Word :: int ⇒ 'a::len0 word*
where

[*code-post*]: *Word* = *word-of-int*

lemma [*code abstype*]:
Word (*uint w*) = *w*
by (*simp add*: *Word-def word-of-int-uint*)

declare *uint-word-of-int* [*code abstract*]

instantiation *word* :: (*len0*) *equal*
begin

definition *equal-word* :: *'a word* ⇒ *'a word* ⇒ *bool*
where
equal-word k l ⟷ *HOL.equal* (*uint k*) (*uint l*)

instance proof
qed (*simp add*: *equal equal-word-def word-uint-eq-iff*)

end

notation *fcomp* (**infixl** ∘> *60*)
notation *scomp* (**infixl** ∘→ *60*)

instantiation *word* :: ({*len0*, *typerep*}) *random*
begin

definition
random-word i = *Random.range i* ∘→ (λ*k*. *Pair* (
let j = *word-of-int* (*int-of-integer* (*integer-of-natural k*)) :: *'a word*
in (*j*, λ-::*unit*. *Code-Evaluation.term-of j*)))

instance ..

end

no-notation *fcomp* (**infixl** ∘> *60*)
no-notation *scomp* (**infixl** ∘→ *60*)

16.5 Type-definition locale instantiations

lemmas *uint-0* = *uint-nonnegative*
lemmas *uint-lt* = *uint-bounded*
lemmas *uint-mod-same* = *uint-idem*

lemma *td-ext-uint*:
td-ext (*uint* :: *'a word* ⇒ *int*) *word-of-int* (*uints* (*len-of TYPE*(*'a*::*len0*)))
(λ*w*::*int*. *w mod 2* ˆ *len-of TYPE*(*'a*))
apply (*unfold td-ext-def'*)
apply (*simp add*: *uints-num word-of-int-def bintrunc-mod2p*)

apply (*simp add*: *uint-mod-same uint-0 uint-lt*
word.uint-inverse word.Abs-word-inverse int-mod-lem)
done

interpretation *word-uint*:
td-ext uint::'*a*::*len0 word* ⇒ *int*
word-of-int
uints (*len-of TYPE*('*a*::*len0*))
λ*w*. *w mod 2* ^ *len-of TYPE*('*a*::*len0*)
by (*fact td-ext-uint*)

lemmas *td-uint* = *word-uint.td-thm*
lemmas *int-word-uint* = *word-uint.eq-norm*

lemma *td-ext-ubin*:
td-ext (*uint* :: '*a word* ⇒ *int*) *word-of-int* (*uints* (*len-of TYPE*('*a*::*len0*)))
(*bintrunc* (*len-of TYPE*('*a*)))
by (*unfold no-bintr-alt1*) (*fact td-ext-uint*)

interpretation *word-ubin*:
td-ext uint::'*a*::*len0 word* ⇒ *int*
word-of-int
uints (*len-of TYPE*('*a*::*len0*))
bintrunc (*len-of TYPE*('*a*::*len0*))
by (*fact td-ext-ubin*)

16.6 Arithmetic operations

lift-definition *word-succ* :: '*a*::*len0 word* ⇒ '*a word* **is** λ*x*. *x* + *1*
by (*metis bintr-ariths*(*6*))

lift-definition *word-pred* :: '*a*::*len0 word* ⇒ '*a word* **is** λ*x*. *x* − *1*
by (*metis bintr-ariths*(*7*))

instantiation *word* :: (*len0*) {*neg-numeral*, *Divides.div*, *comm-monoid-mult*, *comm-ring*}
begin

lift-definition *zero-word* :: '*a word* **is** *0* **.**

lift-definition *one-word* :: '*a word* **is** *1* **.**

lift-definition *plus-word* :: '*a word* ⇒ '*a word* ⇒ '*a word* **is** *op* +
by (*metis bintr-ariths*(*2*))

lift-definition *minus-word* :: '*a word* ⇒ '*a word* ⇒ '*a word* **is** *op* −
by (*metis bintr-ariths*(*3*))

lift-definition *uminus-word* :: '*a word* ⇒ '*a word* **is** *uminus*
by (*metis bintr-ariths*(*5*))

lift-definition *times-word* :: *'a word* ⇒ *'a word* ⇒ *'a word* **is** *op* ∗
  by (*metis bintr-ariths*(*4*))

definition
  word-div-def: *a div b* = *word-of-int* (*uint a div uint b*)

definition
  word-mod-def: *a mod b* = *word-of-int* (*uint a mod uint b*)

instance
  by *standard* (*transfer*, *simp add*: *algebra-simps*)+

end

Legacy theorems:

lemma *word-arith-wis* [*code*]: **shows**
  word-add-def: *a* + *b* = *word-of-int* (*uint a* + *uint b*) **and**
  word-sub-wi: *a* − *b* = *word-of-int* (*uint a* − *uint b*) **and**
  word-mult-def: *a* ∗ *b* = *word-of-int* (*uint a* ∗ *uint b*) **and**
  word-minus-def: − *a* = *word-of-int* (− *uint a*) **and**
  word-succ-alt: *word-succ a* = *word-of-int* (*uint a* + *1*) **and**
  word-pred-alt: *word-pred a* = *word-of-int* (*uint a* − *1*) **and**
  word-0-wi: *0* = *word-of-int 0* **and**
  word-1-wi: *1* = *word-of-int 1*
  unfolding *plus-word-def minus-word-def times-word-def uminus-word-def*
  unfolding *word-succ-def word-pred-def zero-word-def one-word-def*
  by *simp-all*

lemmas *arths* =
  bintr-ariths [*THEN word-ubin.norm-eq-iff* [*THEN iffD1*], *folded word-ubin.eq-norm*]

lemma *wi-homs*:
  shows
  wi-hom-add: *word-of-int a* + *word-of-int b* = *word-of-int* (*a* + *b*) **and**
  wi-hom-sub: *word-of-int a* − *word-of-int b* = *word-of-int* (*a* − *b*) **and**
  wi-hom-mult: *word-of-int a* ∗ *word-of-int b* = *word-of-int* (*a* ∗ *b*) **and**
  wi-hom-neg: − *word-of-int a* = *word-of-int* (− *a*) **and**
  wi-hom-succ: *word-succ* (*word-of-int a*) = *word-of-int* (*a* + *1*) **and**
  wi-hom-pred: *word-pred* (*word-of-int a*) = *word-of-int* (*a* − *1*)
  by (*transfer*, *simp*)+

lemmas *wi-hom-syms* = *wi-homs* [*symmetric*]

lemmas *word-of-int-homs* = *wi-homs word-0-wi word-1-wi*

lemmas *word-of-int-hom-syms* = *word-of-int-homs* [*symmetric*]

instance *word* :: (*len*) *comm-ring-1*

proof
  have *0 < len-of TYPE('a)* **by** (*rule len-gt-0*)
  then show (*0::'a word*) $\neq$ *1*
    by − (*transfer*, *auto simp add*: *gr0-conv-Suc*)
qed

lemma *word-of-nat*: *of-nat n* = *word-of-int* (*int n*)
  by (*induct n*) (*auto simp add* : *word-of-int-hom-syms*)

lemma *word-of-int*: *of-int* = *word-of-int*
  apply (*rule ext*)
  apply (*case-tac x rule*: *int-diff-cases*)
  apply (*simp add*: *word-of-nat wi-hom-sub*)
  done

definition *udvd* :: *'a::len word* => *'a::len word* => *bool* (**infixl** *udvd 50*)
where
  a udvd b = (*EX n>=0. uint b* = *n* ∗ *uint a*)

16.7 Ordering

instantiation *word* :: (*len0*) *linorder*
begin

definition
  word-le-def: *a* ≤ *b* ⟷ *uint a* ≤ *uint b*

definition
  word-less-def: *a* < *b* ⟷ *uint a* < *uint b*

instance
  by *standard* (*auto simp*: *word-less-def word-le-def*)

end

definition *word-sle* :: *'a* :: *len word* => *'a word* => *bool* ((-/ <=s -) [*50*, *51*] *50*)
where
  a <=*s b* = (*sint a* <= *sint b*)

definition *word-sless* :: *'a* :: *len word* => *'a word* => *bool* ((-/ <s -) [*50*, *51*] *50*)
where
  (*x* <*s y*) = (*x* <=*s y* & *x* ~= *y*)

16.8 Bit-wise operations

instantiation *word* :: (*len0*) *bits*
begin

lift-definition *bitNOT-word* :: *'a word* ⇒ *'a word* **is** *bitNOT*
  by (*metis bin-trunc-not*)

lift-definition *bitAND-word* :: $'a$ *word* ⇒ $'a$ *word* ⇒ $'a$ *word* **is** *bitAND*
by (*metis bin-trunc-and*)

lift-definition *bitOR-word* :: $'a$ *word* ⇒ $'a$ *word* ⇒ $'a$ *word* **is** *bitOR*
by (*metis bin-trunc-or*)

lift-definition *bitXOR-word* :: $'a$ *word* ⇒ $'a$ *word* ⇒ $'a$ *word* **is** *bitXOR*
by (*metis bin-trunc-xor*)

definition
word-test-bit-def: *test-bit a* = *bin-nth* (*uint a*)

definition
word-set-bit-def: *set-bit a n x* =
word-of-int (*bin-sc n x* (*uint a*))

definition
word-set-bits-def: (*BITS n. f n*) = *of-bl* (*bl-of-nth* (*len-of TYPE* ($'a$)) *f*)

definition
word-lsb-def: *lsb a* ⟷ *bin-last* (*uint a*)

definition *shiftl1* :: $'a$ *word* ⇒ $'a$ *word*
where
shiftl1 w = *word-of-int* (*uint w BIT False*)

definition *shiftr1* :: $'a$ *word* ⇒ $'a$ *word*
where
— shift right as unsigned or as signed, ie logical or arithmetic
shiftr1 w = *word-of-int* (*bin-rest* (*uint w*))

definition
shiftl-def: *w* << *n* = (*shiftl1* ^^ *n*) *w*

definition
shiftr-def: *w* >> *n* = (*shiftr1* ^^ *n*) *w*

instance ..

end

lemma [*code*]: **shows**
word-not-def: *NOT* (*a*::$'a$::*len0 word*) = *word-of-int* (*NOT* (*uint a*)) **and**
word-and-def: (*a*::$'a$ *word*) *AND b* = *word-of-int* (*uint a AND uint b*) **and**
word-or-def: (*a*::$'a$ *word*) *OR b* = *word-of-int* (*uint a OR uint b*) **and**
word-xor-def: (*a*::$'a$ *word*) *XOR b* = *word-of-int* (*uint a XOR uint b*)
unfolding *bitNOT-word-def bitAND-word-def bitOR-word-def bitXOR-word-def*
by *simp-all*

instantiation *word* :: (*len*) *bitss*
begin

definition
word-msb-def:
msb a ⟷ *bin-sign* (*sint a*) = −*1*

instance ..

end

definition *setBit* :: *'a* :: *len0 word* => *nat* => *'a word*
where
setBit w n = *set-bit w n True*

definition *clearBit* :: *'a* :: *len0 word* => *nat* => *'a word*
where
clearBit w n = *set-bit w n False*

16.9 Shift operations

definition *sshiftr1* :: *'a* :: *len word* => *'a word*
where
sshiftr1 w = *word-of-int* (*bin-rest* (*sint w*))

definition *bshiftr1* :: *bool* => *'a* :: *len word* => *'a word*
where
bshiftr1 b w = *of-bl* (*b* # *butlast* (*to-bl w*))

definition *sshiftr* :: *'a* :: *len word* => *nat* => *'a word* (**infixl** >>> *55*)
where
w >>> *n* = (*sshiftr1* ^^ *n*) *w*

definition *mask* :: *nat* => *'a::len word*
where
mask n = (*1* << *n*) − *1*

definition *revcast* :: *'a* :: *len0 word* => *'b* :: *len0 word*
where
revcast w = *of-bl* (*takefill False* (*len-of TYPE*(*'b*)) (*to-bl w*))

definition *slice1* :: *nat* => *'a* :: *len0 word* => *'b* :: *len0 word*
where
slice1 n w = *of-bl* (*takefill False n* (*to-bl w*))

definition *slice* :: *nat* => *'a* :: *len0 word* => *'b* :: *len0 word*
where
slice n w = *slice1* (*size w* − *n*) *w*

16.10 Rotation

definition *rotater1* :: *'a list => 'a list*
where
rotater1 ys =
(*case ys of* [] => [] | *x* # *xs* => *last ys* # *butlast ys*)

definition *rotater* :: *nat => 'a list => 'a list*
where
rotater n = *rotater1* ^^ *n*

definition *word-rotr* :: *nat => 'a* :: *len0 word => 'a* :: *len0 word*
where
word-rotr n w = *of-bl* (*rotater n* (*to-bl w*))

definition *word-rotl* :: *nat => 'a* :: *len0 word => 'a* :: *len0 word*
where
word-rotl n w = *of-bl* (*rotate n* (*to-bl w*))

definition *word-roti* :: *int => 'a* :: *len0 word => 'a* :: *len0 word*
where
word-roti i w = (*if i* >= *0 then word-rotr* (*nat i*) *w*
else word-rotl (*nat* (− *i*)) *w*)

16.11 Split and cat operations

definition *word-cat* :: *'a* :: *len0 word => 'b* :: *len0 word => 'c* :: *len0 word*
where
word-cat a b = *word-of-int* (*bin-cat* (*uint a*) (*len-of TYPE* (*'b*)) (*uint b*))

definition *word-split* :: *'a* :: *len0 word =>* (*'b* :: *len0 word*) ∗ (*'c* :: *len0 word*)
where
word-split a =
(*case bin-split* (*len-of TYPE* (*'c*)) (*uint a*) *of*
(*u*, *v*) => (*word-of-int u*, *word-of-int v*))

definition *word-rcat* :: *'a* :: *len0 word list => 'b* :: *len0 word*
where
word-rcat ws =
word-of-int (*bin-rcat* (*len-of TYPE* (*'a*)) (*map uint ws*))

definition *word-rsplit* :: *'a* :: *len0 word => 'b* :: *len word list*
where
word-rsplit w =
map word-of-int (*bin-rsplit* (*len-of TYPE* (*'b*)) (*len-of TYPE* (*'a*), *uint w*))

definition *max-word* :: *'a::len word* — Largest representable machine integer.
where
max-word = *word-of-int* (*2* ^ *len-of TYPE*(*'a*) − *1*)

lemmas *of-nth-def* = *word-set-bits-def*

16.12 Theorems about typedefs

lemma *sint-sbintrunc′*:
sint (*word-of-int bin* :: *′a word*) =
(*sbintrunc* (*len-of TYPE* (*′a* :: *len*) − *1*) *bin*)
unfolding *sint-uint*
by (*auto simp*: *word-ubin.eq-norm sbintrunc-bintrunc-lt*)

lemma *uint-sint*:
uint w = *bintrunc* (*len-of TYPE*(*′a*)) (*sint* (*w* :: *′a* :: *len word*))
unfolding *sint-uint* **by** (*auto simp*: *bintrunc-sbintrunc-le*)

lemma *bintr-uint*:
fixes *w* :: *′a*::*len0 word*
shows *len-of TYPE*(*′a*) ≤ *n* ⟹ *bintrunc n* (*uint w*) = *uint w*
apply (*subst word-ubin.norm-Rep* [*symmetric*])
apply (*simp only*: *bintrunc-bintrunc-min word-size*)
apply (*simp add*: *min.absorb2*)
done

lemma *wi-bintr*:
len-of TYPE(*′a*::*len0*) ≤ *n* ⟹
word-of-int (*bintrunc n w*) = (*word-of-int w* :: *′a word*)
by (*clarsimp simp add*: *word-ubin.norm-eq-iff* [*symmetric*] *min.absorb1*)

lemma *td-ext-sbin*:
td-ext (*sint* :: *′a word* ⇒ *int*) *word-of-int* (*sints* (*len-of TYPE*(*′a*::*len*)))
(*sbintrunc* (*len-of TYPE*(*′a*) − *1*))
apply (*unfold td-ext-def′ sint-uint*)
apply (*simp add* : *word-ubin.eq-norm*)
apply (*cases len-of TYPE*(*′a*))
apply (*auto simp add* : *sints-def*)
apply (*rule sym* [*THEN trans*])
apply (*rule word-ubin.Abs-norm*)
apply (*simp only*: *bintrunc-sbintrunc*)
apply (*drule sym*)
apply *simp*
done

lemma *td-ext-sint*:
td-ext (*sint* :: *′a word* ⇒ *int*) *word-of-int* (*sints* (*len-of TYPE*(*′a*::*len*)))
(λ*w*. (*w* + *2* ^ (*len-of TYPE*(*′a*) − *1*)) *mod 2* ^ *len-of TYPE*(*′a*) −
2 ^ (*len-of TYPE*(*′a*) − *1*))
using *td-ext-sbin* [**where** *?′a* = *′a*] **by** (*simp add*: *no-sbintr-alt2*)

interpretation *word-sint*:

td-ext sint ::'*a*::*len word* => *int*
word-of-int
sints (*len-of TYPE*('*a*::*len*))
%*w*. (*w* + 2^(*len-of TYPE*('*a*::*len*) − 1)) *mod* 2^*len-of TYPE*('*a*::*len*) − 2 ^ (*len-of TYPE*('*a*::*len*) − 1)
by (*rule td-ext-sint*)

interpretation *word-sbin*:
td-ext sint ::'*a*::*len word* => *int*
word-of-int
sints (*len-of TYPE*('*a*::*len*))
sbintrunc (*len-of TYPE*('*a*::*len*) − 1)
by (*rule td-ext-sbin*)

lemmas *int-word-sint* = *td-ext-sint* [*THEN td-ext.eq-norm*]

lemmas *td-sint* = *word-sint.td*

lemma *to-bl-def'*:
(*to-bl* :: '*a* :: *len0 word* => *bool list*) =
bin-to-bl (*len-of TYPE*('*a*)) *o uint*
by (*auto simp*: *to-bl-def*)

lemmas *word-reverse-no-def* [*simp*] = *word-reverse-def* [*of numeral w*] **for** *w*

lemma *uints-mod*: *uints n* = *range* (λ*w*. *w mod* 2 ^ *n*)
by (*fact uints-def* [*unfolded no-bintr-alt1*])

lemma *word-numeral-alt*:
numeral b = *word-of-int* (*numeral b*)
by (*induct b*, *simp-all only*: *numeral.simps word-of-int-homs*)

declare *word-numeral-alt* [*symmetric*, *code-abbrev*]

lemma *word-neg-numeral-alt*:
− *numeral b* = *word-of-int* (− *numeral b*)
by (*simp only*: *word-numeral-alt wi-hom-neg*)

declare *word-neg-numeral-alt* [*symmetric*, *code-abbrev*]

lemma *word-numeral-transfer* [*transfer-rule*]:
(*rel-fun op* = *pcr-word*) *numeral numeral*
(*rel-fun op* = *pcr-word*) (− *numeral*) (− *numeral*)
apply (*simp-all add*: *rel-fun-def word.pcr-cr-eq cr-word-def*)
using *word-numeral-alt* [*symmetric*] *word-neg-numeral-alt* [*symmetric*] **by** *blast*+

lemma *uint-bintrunc* [*simp*]:
uint (*numeral bin* :: '*a word*) =
bintrunc (*len-of TYPE* ('*a* :: *len0*)) (*numeral bin*)

unfolding *word-numeral-alt* **by** (*rule word-ubin.eq-norm*)

lemma *uint-bintrunc-neg* [*simp*]: *uint* (− *numeral bin* :: ′*a word*) =
bintrunc (*len-of TYPE* (′*a* :: *len0*)) (− *numeral bin*)
by (*simp only*: *word-neg-numeral-alt word-ubin.eq-norm*)

lemma *sint-sbintrunc* [*simp*]:
sint (*numeral bin* :: ′*a word*) =
sbintrunc (*len-of TYPE* (′*a* :: *len*) − *1*) (*numeral bin*)
by (*simp only*: *word-numeral-alt word-sbin.eq-norm*)

lemma *sint-sbintrunc-neg* [*simp*]: *sint* (− *numeral bin* :: ′*a word*) =
sbintrunc (*len-of TYPE* (′*a* :: *len*) − *1*) (− *numeral bin*)
by (*simp only*: *word-neg-numeral-alt word-sbin.eq-norm*)

lemma *unat-bintrunc* [*simp*]:
unat (*numeral bin* :: ′*a* :: *len0 word*) =
nat (*bintrunc* (*len-of TYPE*(′*a*)) (*numeral bin*))
by (*simp only*: *unat-def uint-bintrunc*)

lemma *unat-bintrunc-neg* [*simp*]:
unat (− *numeral bin* :: ′*a* :: *len0 word*) =
nat (*bintrunc* (*len-of TYPE*(′*a*)) (− *numeral bin*))
by (*simp only*: *unat-def uint-bintrunc-neg*)

lemma *size-0-eq*: *size* (*w* :: ′*a* :: *len0 word*) = *0* ⟹ *v* = *w*
apply (*unfold word-size*)
apply (*rule word-uint.Rep-eqD*)
apply (*rule box-equals*)
defer
apply (*rule word-ubin.norm-Rep*)+
apply *simp*
done

lemma *uint-ge-0* [*iff*]: *0* ≤ *uint* (*x*::′*a*::*len0 word*)
using *word-uint.Rep* [*of x*] **by** (*simp add*: *uints-num*)

lemma *uint-lt2p* [*iff*]: *uint* (*x*::′*a*::*len0 word*) < *2* ^ *len-of TYPE*(′*a*)
using *word-uint.Rep* [*of x*] **by** (*simp add*: *uints-num*)

lemma *sint-ge*: − (*2* ^ (*len-of TYPE*(′*a*) − *1*)) ≤ *sint* (*x*::′*a*::*len word*)
using *word-sint.Rep* [*of x*] **by** (*simp add*: *sints-num*)

lemma *sint-lt*: *sint* (*x*::′*a*::*len word*) < *2* ^ (*len-of TYPE*(′*a*) − *1*)
using *word-sint.Rep* [*of x*] **by** (*simp add*: *sints-num*)

lemma *sign-uint-Pls* [*simp*]:
bin-sign (*uint x*) = *0*
by (*simp add*: *sign-Pls-ge-0*)

lemma *uint-m2p-neg*: *uint* (*x*::′*a*::*len0 word*) − *2* ˆ *len-of TYPE*(′*a*) < *0*
  by (*simp only*: *diff-less-0-iff-less uint-lt2p*)

lemma *uint-m2p-not-non-neg*:
  ¬ *0* ≤ *uint* (*x*::′*a*::*len0 word*) − *2* ˆ *len-of TYPE*(′*a*)
  by (*simp only*: *not-le uint-m2p-neg*)

lemma *lt2p-lem*:
  len-of TYPE(′*a*) ≤ *n* ⟹ *uint* (*w* :: ′*a*::*len0 word*) < *2* ˆ *n*
  by (*metis bintr-uint bintrunc-mod2p int-mod-lem zless2p*)

lemma *uint-le-0-iff* [*simp*]: *uint x* ≤ *0* ⟷ *uint x* = *0*
  by (*fact uint-ge-0* [*THEN leD*, *THEN linorder-antisym-conv1*])

lemma *uint-nat*: *uint w* = *int* (*unat w*)
  unfolding *unat-def* **by** *auto*

lemma *uint-numeral*:
  uint (*numeral b* :: ′*a* :: *len0 word*) = *numeral b mod 2* ˆ *len-of TYPE*(′*a*)
  unfolding *word-numeral-alt*
  by (*simp only*: *int-word-uint*)

lemma *uint-neg-numeral*:
  uint (− *numeral b* :: ′*a* :: *len0 word*) = − *numeral b mod 2* ˆ *len-of TYPE*(′*a*)
  unfolding *word-neg-numeral-alt*
  by (*simp only*: *int-word-uint*)

lemma *unat-numeral*:
  unat (*numeral b*::′*a*::*len0 word*) = *numeral b mod 2* ˆ *len-of TYPE* (′*a*)
  apply (*unfold unat-def*)
  apply (*clarsimp simp only*: *uint-numeral*)
  apply (*rule nat-mod-distrib* [*THEN trans*])
    apply (*rule zero-le-numeral*)
   apply (*simp-all add*: *nat-power-eq*)
  done

lemma *sint-numeral*: *sint* (*numeral b* :: ′*a* :: *len word*) = (*numeral b* +
    2 ˆ (*len-of TYPE*(′*a*) − *1*)) *mod 2* ˆ *len-of TYPE*(′*a*) −
    2 ˆ (*len-of TYPE*(′*a*) − *1*)
  unfolding *word-numeral-alt* **by** (*rule int-word-sint*)

lemma *word-of-int-0* [*simp*, *code-post*]:
  word-of-int 0 = *0*
  unfolding *word-0-wi* **..**

lemma *word-of-int-1* [*simp*, *code-post*]:
  word-of-int 1 = *1*
  unfolding *word-1-wi* **..**

lemma *word-of-int-neg-1* [*simp*]: *word-of-int* (− *1*) = − *1*
by (*simp add*: *wi-hom-syms*)

lemma *word-of-int-numeral* [*simp*] :
(*word-of-int* (*numeral bin*) :: *'a* :: *len0 word*) = (*numeral bin*)
unfolding *word-numeral-alt* **..**

lemma *word-of-int-neg-numeral* [*simp*]:
(*word-of-int* (− *numeral bin*) :: *'a* :: *len0 word*) = (− *numeral bin*)
unfolding *word-numeral-alt wi-hom-syms* **..**

lemma *word-int-case-wi*:
word-int-case f (*word-of-int i* :: *'b word*) =
f (*i mod 2* ˆ *len-of TYPE*(*'b*::*len0*))
unfolding *word-int-case-def* **by** (*simp add*: *word-uint.eq-norm*)

lemma *word-int-split*:
P (*word-int-case f x*) =
(*ALL i. x* = (*word-of-int i* :: *'b* :: *len0 word*) &
0 <= *i* & *i* < *2* ˆ *len-of TYPE*(*'b*) −−> *P* (*f i*))
unfolding *word-int-case-def*
by (*auto simp*: *word-uint.eq-norm mod-pos-pos-trivial*)

lemma *word-int-split-asm*:
P (*word-int-case f x*) =
(~ (*EX n. x* = (*word-of-int n* :: *'b*::*len0 word*) &
0 <= *n* & *n* < *2* ˆ *len-of TYPE*(*'b*::*len0*) & ~ *P* (*f n*)))
unfolding *word-int-case-def*
by (*auto simp*: *word-uint.eq-norm mod-pos-pos-trivial*)

lemmas *uint-range'* = *word-uint.Rep* [*unfolded uints-num mem-Collect-eq*]
lemmas *sint-range'* = *word-sint.Rep* [*unfolded One-nat-def sints-num mem-Collect-eq*]

lemma *uint-range-size*: *0* <= *uint w* & *uint w* < *2* ˆ *size w*
unfolding *word-size* **by** (*rule uint-range'*)

lemma *sint-range-size*:
− (*2* ˆ (*size w* − *Suc 0*)) <= *sint w* & *sint w* < *2* ˆ (*size w* − *Suc 0*)
unfolding *word-size* **by** (*rule sint-range'*)

lemma *sint-above-size*: *2* ˆ (*size* (*w*::*'a*::*len word*) − *1*) ≤ *x* ⟹ *sint w* < *x*
unfolding *word-size* **by** (*rule less-le-trans* [*OF sint-lt*])

lemma *sint-below-size*:
x ≤ − (*2* ˆ (*size* (*w*::*'a*::*len word*) − *1*)) ⟹ *x* ≤ *sint w*
unfolding *word-size* **by** (*rule order-trans* [*OF - sint-ge*])

16.13 Testing bits

lemma *test-bit-eq-iff*: (*test-bit* (*u*::′*a*::*len0 word*) = *test-bit v*) = (*u* = *v*)
unfolding *word-test-bit-def* **by** (*simp add*: *bin-nth-eq-iff*)

lemma *test-bit-size* [*rule-format*] : (*w*::′*a*::*len0 word*) !! *n* −−> *n* < *size w*
apply (*unfold word-test-bit-def*)
apply (*subst word-ubin.norm-Rep* [*symmetric*])
apply (*simp only*: *nth-bintr word-size*)
apply *fast*
done

lemma *word-eq-iff*:
fixes *x y* :: ′*a*::*len0 word*
shows *x* = *y* ⟷ (∀ *n*<*len-of TYPE*(′*a*). *x* !! *n* = *y* !! *n*)
unfolding *uint-inject* [*symmetric*] *bin-eq-iff word-test-bit-def* [*symmetric*]
by (*metis test-bit-size* [*unfolded word-size*])

lemma *word-eqI* [*rule-format*]:
fixes *u* :: ′*a*::*len0 word*
shows (*ALL n*. *n* < *size u* −−> *u* !! *n* = *v* !! *n*) ⟹ *u* = *v*
by (*simp add*: *word-size word-eq-iff*)

lemma *word-eqD*: (*u*::′*a*::*len0 word*) = *v* ⟹ *u* !! *x* = *v* !! *x*
by *simp*

lemma *test-bit-bin*′: *w* !! *n* = (*n* < *size w* & *bin-nth* (*uint w*) *n*)
unfolding *word-test-bit-def word-size*
by (*simp add*: *nth-bintr* [*symmetric*])

lemmas *test-bit-bin* = *test-bit-bin*′ [*unfolded word-size*]

lemma *bin-nth-uint-imp*:
bin-nth (*uint* (*w*::′*a*::*len0 word*)) *n* ⟹ *n* < *len-of TYPE*(′*a*)
apply (*rule nth-bintr* [*THEN iffD1*, *THEN conjunct1*])
apply (*subst word-ubin.norm-Rep*)
apply *assumption*
done

lemma *bin-nth-sint*:
fixes *w* :: ′*a*::*len word*
shows *len-of TYPE*(′*a*) ≤ *n* ⟹
bin-nth (*sint w*) *n* = *bin-nth* (*sint w*) (*len-of TYPE*(′*a*) − *1*)
apply (*subst word-sbin.norm-Rep* [*symmetric*])
apply (*auto simp add*: *nth-sbintr*)
done

lemma *td-bl*:
type-definition (*to-bl* :: ′*a*::*len0 word* => *bool list*)

```
          of-bl
          {bl. length bl = len-of TYPE('a)}
  apply (unfold type-definition-def of-bl-def to-bl-def)
  apply (simp add: word-ubin.eq-norm)
  apply safe
  apply (drule sym)
  apply simp
  done

interpretation word-bl:
  type-definition to-bl :: 'a::len0 word => bool list
          of-bl
          {bl. length bl = len-of TYPE('a::len0)}
  by (fact td-bl)

lemmas word-bl-Rep' = word-bl.Rep [unfolded mem-Collect-eq, iff]

lemma word-size-bl: size w = size (to-bl w)
  unfolding word-size by auto

lemma to-bl-use-of-bl:
  (to-bl w = bl) = (w = of-bl bl ∧ length bl = length (to-bl w))
  by (fastforce elim!: word-bl.Abs-inverse [unfolded mem-Collect-eq])

lemma to-bl-word-rev: to-bl (word-reverse w) = rev (to-bl w)
  unfolding word-reverse-def by (simp add: word-bl.Abs-inverse)

lemma word-rev-rev [simp] : word-reverse (word-reverse w) = w
  unfolding word-reverse-def by (simp add : word-bl.Abs-inverse)

lemma word-rev-gal: word-reverse w = u ⟹ word-reverse u = w
  by (metis word-rev-rev)

lemma word-rev-gal': u = word-reverse w ⟹ w = word-reverse u
  by simp

lemma length-bl-gt-0 [iff]: 0 < length (to-bl (x::'a::len word))
  unfolding word-bl-Rep' by (rule len-gt-0)

lemma bl-not-Nil [iff]: to-bl (x::'a::len word) ≠ []
  by (fact length-bl-gt-0 [unfolded length-greater-0-conv])

lemma length-bl-neq-0 [iff]: length (to-bl (x::'a::len word)) ≠ 0
  by (fact length-bl-gt-0 [THEN gr-implies-not0])

lemma hd-bl-sign-sint: hd (to-bl w) = (bin-sign (sint w) = −1)
  apply (unfold to-bl-def sint-uint)
  apply (rule trans [OF - bl-sbin-sign])
  apply simp
```

done

lemma *of-bl-drop'*:
lend = length bl − len-of TYPE ('a :: len0) ⟹
of-bl (drop lend bl) = (of-bl bl :: 'a word)
apply (*unfold of-bl-def*)
apply (*clarsimp simp add : trunc-bl2bin [symmetric]*)
done

lemma *test-bit-of-bl*:
(*of-bl bl::'a::len0 word*) !! *n* = (*rev bl* ! *n* ∧ *n* < *len-of TYPE('a)* ∧ *n* < *length bl*)
apply (*unfold of-bl-def word-test-bit-def*)
apply (*auto simp add: word-size word-ubin.eq-norm nth-bintr bin-nth-of-bl*)
done

lemma *no-of-bl*:
(*numeral bin ::'a::len0 word*) = *of-bl* (*bin-to-bl* (*len-of TYPE ('a)*) (*numeral bin*))
unfolding *of-bl-def* **by** *simp*

lemma *uint-bl*: *to-bl w = bin-to-bl (size w) (uint w)*
unfolding *word-size to-bl-def* **by** *auto*

lemma *to-bl-bin*: *bl-to-bin (to-bl w) = uint w*
unfolding *uint-bl* **by** (*simp add : word-size*)

lemma *to-bl-of-bin*:
to-bl (word-of-int bin::'a::len0 word) = bin-to-bl (len-of TYPE('a)) bin
unfolding *uint-bl* **by** (*clarsimp simp add: word-ubin.eq-norm word-size*)

lemma *to-bl-numeral* [*simp*]:
to-bl (numeral bin::'a::len0 word) =
bin-to-bl (len-of TYPE('a)) (numeral bin)
unfolding *word-numeral-alt* **by** (*rule to-bl-of-bin*)

lemma *to-bl-neg-numeral* [*simp*]:
to-bl (− numeral bin::'a::len0 word) =
bin-to-bl (len-of TYPE('a)) (− numeral bin)
unfolding *word-neg-numeral-alt* **by** (*rule to-bl-of-bin*)

lemma *to-bl-to-bin* [*simp*] : *bl-to-bin (to-bl w) = uint w*
unfolding *uint-bl* **by** (*simp add : word-size*)

lemma *uint-bl-bin*:
fixes *x :: 'a::len0 word*
shows *bl-to-bin (bin-to-bl (len-of TYPE('a)) (uint x)) = uint x*
by (*rule trans [OF bin-bl-bin word-ubin.norm-Rep]*)

lemma *uints-unats*: *uints n* = *int* ' *unats n*
apply (*unfold unats-def uints-num*)
apply *safe*
apply (*rule-tac image-eqI*)
apply (*erule-tac nat-0-le* [*symmetric*])
apply *auto*
apply (*erule-tac nat-less-iff* [*THEN iffD2*])
apply (*rule-tac* [*2*] *zless-nat-eq-int-zless* [*THEN iffD1*])
apply (*auto simp add* : *nat-power-eq of-nat-power*)
done

lemma *unats-uints*: *unats n* = *nat* ' *uints n*
by (*auto simp add* : *uints-unats image-iff*)

lemmas *bintr-num* = *word-ubin.norm-eq-iff*
[*of numeral a numeral b*, *symmetric*, *folded word-numeral-alt*] **for** *a b*
lemmas *sbintr-num* = *word-sbin.norm-eq-iff*
[*of numeral a numeral b*, *symmetric*, *folded word-numeral-alt*] **for** *a b*

lemma *num-of-bintr'*:
bintrunc (*len-of TYPE*('*a* :: *len0*)) (*numeral a*) = (*numeral b*) ⟹
numeral a = (*numeral b* :: '*a word*)
unfolding *bintr-num* **by** (*erule subst*, *simp*)

lemma *num-of-sbintr'*:
sbintrunc (*len-of TYPE*('*a* :: *len*) − *1*) (*numeral a*) = (*numeral b*) ⟹
numeral a = (*numeral b* :: '*a word*)
unfolding *sbintr-num* **by** (*erule subst*, *simp*)

lemma *num-abs-bintr*:
(*numeral x* :: '*a word*) =
word-of-int (*bintrunc* (*len-of TYPE*('*a*::*len0*)) (*numeral x*))
by (*simp only*: *word-ubin.Abs-norm word-numeral-alt*)

lemma *num-abs-sbintr*:
(*numeral x* :: '*a word*) =
word-of-int (*sbintrunc* (*len-of TYPE*('*a*::*len*) − *1*) (*numeral x*))
by (*simp only*: *word-sbin.Abs-norm word-numeral-alt*)

lemma *ucast-id*: *ucast w* = *w*
unfolding *ucast-def* **by** *auto*

lemma *scast-id*: *scast w* = *w*
unfolding *scast-def* **by** *auto*

lemma *ucast-bl*: *ucast w* = *of-bl* (*to-bl w*)
unfolding *ucast-def of-bl-def uint-bl*

by (*auto simp add* : *word-size*)

lemma *nth-ucast*:
(*ucast w*::′*a*::*len0 word*) !! *n* = (*w* !! *n* & *n* < *len-of TYPE*(′*a*))
apply (*unfold ucast-def test-bit-bin*)
apply (*simp add*: *word-ubin.eq-norm nth-bintr word-size*)
apply (*fast elim*!: *bin-nth-uint-imp*)
done

lemma *ucast-bintr* [*simp*]:
ucast (*numeral w* ::′*a*::*len0 word*) =
word-of-int (*bintrunc* (*len-of TYPE*(′*a*)) (*numeral w*))
unfolding *ucast-def* **by** *simp*

lemma *scast-sbintr* [*simp*]:
scast (*numeral w* ::′*a*::*len word*) =
word-of-int (*sbintrunc* (*len-of TYPE*(′*a*) − *Suc 0*) (*numeral w*))
unfolding *scast-def* **by** *simp*

lemma *source-size*: *source-size* (*c*::′*a*::*len0 word* ⇒ -) = *len-of TYPE*(′*a*)
unfolding *source-size-def word-size Let-def* **..**

lemma *target-size*: *target-size* (*c*::- ⇒ ′*b*::*len0 word*) = *len-of TYPE*(′*b*)
unfolding *target-size-def word-size Let-def* **..**

lemma *is-down*:
fixes *c* :: ′*a*::*len0 word* ⇒ ′*b*::*len0 word*
shows *is-down c* ⟷ *len-of TYPE*(′*b*) ≤ *len-of TYPE*(′*a*)
unfolding *is-down-def source-size target-size* **..**

lemma *is-up*:
fixes *c* :: ′*a*::*len0 word* ⇒ ′*b*::*len0 word*
shows *is-up c* ⟷ *len-of TYPE*(′*a*) ≤ *len-of TYPE*(′*b*)
unfolding *is-up-def source-size target-size* **..**

lemmas *is-up-down* = *trans* [*OF is-up is-down* [*symmetric*]]

lemma *down-cast-same* [*OF refl*]: *uc* = *ucast* ⟹ *is-down uc* ⟹ *uc* = *scast*
apply (*unfold is-down*)
apply *safe*
apply (*rule ext*)
apply (*unfold ucast-def scast-def uint-sint*)
apply (*rule word-ubin.norm-eq-iff* [*THEN iffD1*])
apply *simp*
done

lemma *word-rev-tf*:
to-bl (*of-bl bl*::'*a*::*len0 word*) =
rev (*takefill False* (*len-of TYPE*('*a*)) (*rev bl*))
unfolding *of-bl-def uint-bl*
by (*clarsimp simp add*: *bl-bin-bl-rtf word-ubin.eq-norm word-size*)

lemma *word-rep-drop*:
to-bl (*of-bl bl*::'*a*::*len0 word*) =
replicate (*len-of TYPE*('*a*) − *length bl*) *False* @
drop (*length bl* − *len-of TYPE*('*a*)) *bl*
by (*simp add*: *word-rev-tf takefill-alt rev-take*)

lemma *to-bl-ucast*:
to-bl (*ucast* (*w*::'*b*::*len0 word*) ::'*a*::*len0 word*) =
replicate (*len-of TYPE*('*a*) − *len-of TYPE*('*b*)) *False* @
drop (*len-of TYPE*('*b*) − *len-of TYPE*('*a*)) (*to-bl w*)
apply (*unfold ucast-bl*)
apply (*rule trans*)
apply (*rule word-rep-drop*)
apply *simp*
done

lemma *ucast-up-app* [*OF refl*]:
uc = *ucast* ⟹ *source-size uc* + *n* = *target-size uc* ⟹
to-bl (*uc w*) = *replicate n False* @ (*to-bl w*)
by (*auto simp add* : *source-size target-size to-bl-ucast*)

lemma *ucast-down-drop* [*OF refl*]:
uc = *ucast* ⟹ *source-size uc* = *target-size uc* + *n* ⟹
to-bl (*uc w*) = *drop n* (*to-bl w*)
by (*auto simp add* : *source-size target-size to-bl-ucast*)

lemma *scast-down-drop* [*OF refl*]:
sc = *scast* ⟹ *source-size sc* = *target-size sc* + *n* ⟹
to-bl (*sc w*) = *drop n* (*to-bl w*)
apply (*subgoal-tac sc* = *ucast*)
apply *safe*
apply *simp*
apply (*erule ucast-down-drop*)
apply (*rule down-cast-same* [*symmetric*])
apply (*simp add* : *source-size target-size is-down*)
done

lemma *sint-up-scast* [*OF refl*]:
sc = *scast* ⟹ *is-up sc* ⟹ *sint* (*sc w*) = *sint w*
apply (*unfold is-up*)
apply *safe*
apply (*simp add*: *scast-def word-sbin.eq-norm*)
apply (*rule box-equals*)

```
    prefer 3
    apply (rule word-sbin.norm-Rep)
   apply (rule sbintrunc-sbintrunc-l)
   defer
   apply (subst word-sbin.norm-Rep)
   apply (rule refl)
  apply simp
  done

lemma uint-up-ucast [OF refl]:
  uc = ucast ⟹ is-up uc ⟹ uint (uc w) = uint w
  apply (unfold is-up)
  apply safe
  apply (rule bin-eqI)
  apply (fold word-test-bit-def)
  apply (auto simp add: nth-ucast)
  apply (auto simp add: test-bit-bin)
  done

lemma ucast-up-ucast [OF refl]:
  uc = ucast ⟹ is-up uc ⟹ ucast (uc w) = ucast w
  apply (simp (no-asm) add: ucast-def)
  apply (clarsimp simp add: uint-up-ucast)
  done

lemma scast-up-scast [OF refl]:
  sc = scast ⟹ is-up sc ⟹ scast (sc w) = scast w
  apply (simp (no-asm) add: scast-def)
  apply (clarsimp simp add: sint-up-scast)
  done

lemma ucast-of-bl-up [OF refl]:
  w = of-bl bl ⟹ size bl <= size w ⟹ ucast w = of-bl bl
  by (auto simp add : nth-ucast word-size test-bit-of-bl intro!: word-eqI)

lemmas ucast-up-ucast-id = trans [OF ucast-up-ucast ucast-id]
lemmas scast-up-scast-id = trans [OF scast-up-scast scast-id]

lemmas isduu = is-up-down [where c = ucast, THEN iffD2]
lemmas isdus = is-up-down [where c = scast, THEN iffD2]
lemmas ucast-down-ucast-id = isduu [THEN ucast-up-ucast-id]
lemmas scast-down-scast-id = isdus [THEN ucast-up-ucast-id]

lemma up-ucast-surj:
  is-up (ucast :: 'b::len0 word => 'a::len0 word) ⟹
   surj (ucast :: 'a word => 'b word)
  by (rule surjI, erule ucast-up-ucast-id)

lemma up-scast-surj:
```

is-up (*scast* :: *'b::len word => 'a::len word*) ⟹
surj (*scast* :: *'a word => 'b word*)
by (*rule surjI*, *erule scast-up-scast-id*)

lemma *down-scast-inj*:
is-down (*scast* :: *'b::len word => 'a::len word*) ⟹
inj-on (*ucast* :: *'a word => 'b word*) *A*
by (*rule inj-on-inverseI*, *erule scast-down-scast-id*)

lemma *down-ucast-inj*:
is-down (*ucast* :: *'b::len0 word => 'a::len0 word*) ⟹
inj-on (*ucast* :: *'a word => 'b word*) *A*
by (*rule inj-on-inverseI*, *erule ucast-down-ucast-id*)

lemma *of-bl-append-same*: *of-bl* (*X* @ *to-bl w*) = *w*
by (*rule word-bl.Rep-eqD*) (*simp add*: *word-rep-drop*)

lemma *ucast-down-wi* [*OF refl*]:
uc = *ucast* ⟹ *is-down uc* ⟹ *uc* (*word-of-int x*) = *word-of-int x*
apply (*unfold is-down*)
apply (*clarsimp simp add*: *ucast-def word-ubin.eq-norm*)
apply (*rule word-ubin.norm-eq-iff* [*THEN iffD1*])
apply (*erule bintrunc-bintrunc-ge*)
done

lemma *ucast-down-no* [*OF refl*]:
uc = *ucast* ⟹ *is-down uc* ⟹ *uc* (*numeral bin*) = *numeral bin*
unfolding *word-numeral-alt* **by** *clarify* (*rule ucast-down-wi*)

lemma *ucast-down-bl* [*OF refl*]:
uc = *ucast* ⟹ *is-down uc* ⟹ *uc* (*of-bl bl*) = *of-bl bl*
unfolding *of-bl-def* **by** *clarify* (*erule ucast-down-wi*)

lemmas *slice-def'* = *slice-def* [*unfolded word-size*]
lemmas *test-bit-def'* = *word-test-bit-def* [*THEN fun-cong*]

lemmas *word-log-defs* = *word-and-def word-or-def word-xor-def word-not-def*

16.14 Word Arithmetic

lemma *word-less-alt*: (*a* < *b*) = (*uint a* < *uint b*)
by (*fact word-less-def*)

lemma *signed-linorder*: *class.linorder word-sle word-sless*
by *standard* (*unfold word-sle-def word-sless-def*, *auto*)

interpretation *signed*: *linorder word-sle word-sless*
by (*rule signed-linorder*)

lemma *udvdI*:
  $0 \leq n \Longrightarrow uint\ b = n * uint\ a \Longrightarrow a\ udvd\ b$
  by (*auto simp*: *udvd-def*)

lemmas *word-div-no* [*simp*] = *word-div-def* [*of numeral a numeral b*] **for** *a b*

lemmas *word-mod-no* [*simp*] = *word-mod-def* [*of numeral a numeral b*] **for** *a b*

lemmas *word-less-no* [*simp*] = *word-less-def* [*of numeral a numeral b*] **for** *a b*

lemmas *word-le-no* [*simp*] = *word-le-def* [*of numeral a numeral b*] **for** *a b*

lemmas *word-sless-no* [*simp*] = *word-sless-def* [*of numeral a numeral b*] **for** *a b*

lemmas *word-sle-no* [*simp*] = *word-sle-def* [*of numeral a numeral b*] **for** *a b*

lemma *word-m1-wi*: − *1* = *word-of-int* (− *1*)
  using *word-neg-numeral-alt* [*of Num.One*] **by** *simp*

lemma *word-0-bl* [*simp*]: *of-bl* [] = *0*
  unfolding *of-bl-def* **by** *simp*

lemma *word-1-bl*: *of-bl* [*True*] = *1*
  unfolding *of-bl-def* **by** (*simp add*: *bl-to-bin-def*)

lemma *uint-eq-0* [*simp*]: *uint 0* = *0*
  unfolding *word-0-wi word-ubin.eq-norm* **by** *simp*

lemma *of-bl-0* [*simp*]: *of-bl* (*replicate n False*) = *0*
  by (*simp add*: *of-bl-def bl-to-bin-rep-False*)

lemma *to-bl-0* [*simp*]:
  to-bl (*0*::*'a*::*len0 word*) = *replicate* (*len-of TYPE*(*'a*)) *False*
  unfolding *uint-bl*
  by (*simp add*: *word-size bin-to-bl-zero*)

lemma *uint-0-iff*:
  $uint\ x = 0 \longleftrightarrow x = 0$
  by (*simp add*: *word-uint-eq-iff*)

lemma *unat-0-iff*:
  $unat\ x = 0 \longleftrightarrow x = 0$
  unfolding *unat-def* **by** (*auto simp add* : *nat-eq-iff uint-0-iff*)

lemma *unat-0* [*simp*]:
  unat 0 = *0*
  unfolding *unat-def* **by** *auto*

lemma *size-0-same'*:

size w = 0 ⟹ w = (v :: 'a :: len0 word)
apply (*unfold word-size*)
apply (*rule box-equals*)
defer
apply (*rule word-uint.Rep-inverse*)+
apply (*rule word-ubin.norm-eq-iff* [*THEN iffD1*])
apply *simp*
done

lemmas *size-0-same = size-0-same'* [*unfolded word-size*]

lemmas *unat-eq-0 = unat-0-iff*
lemmas *unat-eq-zero = unat-0-iff*

lemma *unat-gt-0*: (*0 < unat x*) = (*x ~= 0*)
by (*auto simp*: *unat-0-iff* [*symmetric*])

lemma *ucast-0* [*simp*]: *ucast 0 = 0*
unfolding *ucast-def* **by** *simp*

lemma *sint-0* [*simp*]: *sint 0 = 0*
unfolding *sint-uint* **by** *simp*

lemma *scast-0* [*simp*]: *scast 0 = 0*
unfolding *scast-def* **by** *simp*

lemma *sint-n1* [*simp*] : *sint* (− *1*) = − *1*
unfolding *word-m1-wi word-sbin.eq-norm* **by** *simp*

lemma *scast-n1* [*simp*]: *scast* (− *1*) = − *1*
unfolding *scast-def* **by** *simp*

lemma *uint-1* [*simp*]: *uint* (*1::'a::len word*) = *1*
by (*simp only*: *word-1-wi word-ubin.eq-norm*) (*simp add*: *bintrunc-minus-simps*(*4*))

lemma *unat-1* [*simp*]: *unat* (*1::'a::len word*) = *1*
unfolding *unat-def* **by** *simp*

lemma *ucast-1* [*simp*]: *ucast* (*1::'a::len word*) = *1*
unfolding *ucast-def* **by** *simp*

16.15 Transferring goals from words to ints

lemma *word-ths*:
shows
word-succ-p1: *word-succ a = a + 1* **and**
word-pred-m1: *word-pred a = a − 1* **and**
word-pred-succ: *word-pred* (*word-succ a*) = *a* **and**
word-succ-pred: *word-succ* (*word-pred a*) = *a* **and**

word-mult-succ: *word-succ a* ∗ *b* = *b* + *a* ∗ *b*
by (*transfer*, *simp add*: *algebra-simps*)+

lemma *uint-cong*: *x* = *y* ⟹ *uint x* = *uint y*
by *simp*

lemma *uint-word-ariths*:
fixes *a b* :: ′*a*::*len0 word*
shows *uint* (*a* + *b*) = (*uint a* + *uint b*) *mod 2* ˆ *len-of TYPE*(′*a*::*len0*)
and *uint* (*a* − *b*) = (*uint a* − *uint b*) *mod 2* ˆ *len-of TYPE*(′*a*)
and *uint* (*a* ∗ *b*) = *uint a* ∗ *uint b mod 2* ˆ *len-of TYPE*(′*a*)
and *uint* (− *a*) = − *uint a mod 2* ˆ *len-of TYPE*(′*a*)
and *uint* (*word-succ a*) = (*uint a* + *1*) *mod 2* ˆ *len-of TYPE*(′*a*)
and *uint* (*word-pred a*) = (*uint a* − *1*) *mod 2* ˆ *len-of TYPE*(′*a*)
and *uint* (*0* :: ′*a word*) = *0 mod 2* ˆ *len-of TYPE*(′*a*)
and *uint* (*1* :: ′*a word*) = *1 mod 2* ˆ *len-of TYPE*(′*a*)
by (*simp-all add*: *word-arith-wis* [*THEN trans* [*OF uint-cong int-word-uint*]])

lemma *uint-word-arith-bintrs*:
fixes *a b* :: ′*a*::*len0 word*
shows *uint* (*a* + *b*) = *bintrunc* (*len-of TYPE*(′*a*)) (*uint a* + *uint b*)
and *uint* (*a* − *b*) = *bintrunc* (*len-of TYPE*(′*a*)) (*uint a* − *uint b*)
and *uint* (*a* ∗ *b*) = *bintrunc* (*len-of TYPE*(′*a*)) (*uint a* ∗ *uint b*)
and *uint* (− *a*) = *bintrunc* (*len-of TYPE*(′*a*)) (− *uint a*)
and *uint* (*word-succ a*) = *bintrunc* (*len-of TYPE*(′*a*)) (*uint a* + *1*)
and *uint* (*word-pred a*) = *bintrunc* (*len-of TYPE*(′*a*)) (*uint a* − *1*)
and *uint* (*0* :: ′*a word*) = *bintrunc* (*len-of TYPE*(′*a*)) *0*
and *uint* (*1* :: ′*a word*) = *bintrunc* (*len-of TYPE*(′*a*)) *1*
by (*simp-all add*: *uint-word-ariths bintrunc-mod2p*)

lemma *sint-word-ariths*:
fixes *a b* :: ′*a*::*len word*
shows *sint* (*a* + *b*) = *sbintrunc* (*len-of TYPE*(′*a*) − *1*) (*sint a* + *sint b*)
and *sint* (*a* − *b*) = *sbintrunc* (*len-of TYPE*(′*a*) − *1*) (*sint a* − *sint b*)
and *sint* (*a* ∗ *b*) = *sbintrunc* (*len-of TYPE*(′*a*) − *1*) (*sint a* ∗ *sint b*)
and *sint* (− *a*) = *sbintrunc* (*len-of TYPE*(′*a*) − *1*) (− *sint a*)
and *sint* (*word-succ a*) = *sbintrunc* (*len-of TYPE*(′*a*) − *1*) (*sint a* + *1*)
and *sint* (*word-pred a*) = *sbintrunc* (*len-of TYPE*(′*a*) − *1*) (*sint a* − *1*)
and *sint* (*0* :: ′*a word*) = *sbintrunc* (*len-of TYPE*(′*a*) − *1*) *0*
and *sint* (*1* :: ′*a word*) = *sbintrunc* (*len-of TYPE*(′*a*) − *1*) *1*
by (*simp-all add*: *uint-word-arith-bintrs*
[*THEN uint-sint* [*symmetric*, *THEN trans*],
unfolded uint-sint bintr-arith1s bintr-ariths
len-gt-0 [*THEN bin-sbin-eq-iff* ′] *word-sbin.norm-Rep*])

lemmas *uint-div-alt* = *word-div-def* [*THEN trans* [*OF uint-cong int-word-uint*]]
lemmas *uint-mod-alt* = *word-mod-def* [*THEN trans* [*OF uint-cong int-word-uint*]]

lemma *word-pred-0-n1*: *word-pred 0* = *word-of-int* (− *1*)

unfolding *word-pred-m1* **by** *simp*

lemma *succ-pred-no* [*simp*]:
word-succ (*numeral w*) = *numeral w* + *1*
word-pred (*numeral w*) = *numeral w* − *1*
word-succ (− *numeral w*) = − *numeral w* + *1*
word-pred (− *numeral w*) = − *numeral w* − *1*
unfolding *word-succ-p1 word-pred-m1* **by** *simp-all*

lemma *word-sp-01* [*simp*] :
word-succ (− *1*) = *0* & *word-succ 0* = *1* & *word-pred 0* = − *1* & *word-pred 1* = *0*
unfolding *word-succ-p1 word-pred-m1* **by** *simp-all*

lemma *word-of-int-Ex*:
∃ *y*. *x* = *word-of-int y*
by (*rule-tac x*=*uint x* **in** *exI*) *simp*

16.16 Order on fixed-length words

lemma *word-zero-le* [*simp*] :
0 <= (*y* :: *'a* :: *len0 word*)
unfolding *word-le-def* **by** *auto*

lemma *word-m1-ge* [*simp*] : *word-pred 0* >= *y*
unfolding *word-le-def*
by (*simp only* : *word-pred-0-n1 word-uint.eq-norm m1mod2k*) *auto*

lemma *word-n1-ge* [*simp*]: *y* ≤ (−*1*::*'a*::*len0 word*)
unfolding *word-le-def*
by (*simp only*: *word-m1-wi word-uint.eq-norm m1mod2k*) *auto*

lemmas *word-not-simps* [*simp*] =
word-zero-le [*THEN leD*] *word-m1-ge* [*THEN leD*] *word-n1-ge* [*THEN leD*]

lemma *word-gt-0*: *0* < *y* ⟷ *0* ≠ (*y* :: *'a* :: *len0 word*)
by (*simp add*: *less-le*)

lemmas *word-gt-0-no* [*simp*] = *word-gt-0* [*of numeral y*] **for** *y*

lemma *word-sless-alt*: (*a* <*s b*) = (*sint a* < *sint b*)
unfolding *word-sle-def word-sless-def*
by (*auto simp add*: *less-le*)

lemma *word-le-nat-alt*: (*a* <= *b*) = (*unat a* <= *unat b*)
unfolding *unat-def word-le-def*
by (*rule nat-le-eq-zle* [*symmetric*]) *simp*

lemma *word-less-nat-alt*: (*a* < *b*) = (*unat a* < *unat b*)
unfolding *unat-def word-less-alt*
by (*rule nat-less-eq-zless* [*symmetric*]) *simp*

lemma *wi-less*:
(*word-of-int n* < (*word-of-int m* :: *'a* :: *len0 word*)) =
(*n mod 2 ^ len-of TYPE('a)* < *m mod 2 ^ len-of TYPE('a)*)
unfolding *word-less-alt* **by** (*simp add*: *word-uint.eq-norm*)

lemma *wi-le*:
(*word-of-int n* <= (*word-of-int m* :: *'a* :: *len0 word*)) =
(*n mod 2 ^ len-of TYPE('a)* <= *m mod 2 ^ len-of TYPE('a)*)
unfolding *word-le-def* **by** (*simp add*: *word-uint.eq-norm*)

lemma *udvd-nat-alt*: *a udvd b* = (*EX n*>=*0*. *unat b* = *n* ∗ *unat a*)
apply (*unfold udvd-def*)
apply *safe*
apply (*simp add*: *unat-def nat-mult-distrib*)
apply (*simp add*: *uint-nat of-nat-mult*)
apply (*rule exI*)
apply *safe*
prefer *2*
apply (*erule notE*)
apply (*rule refl*)
apply *force*
done

lemma *udvd-iff-dvd*: *x udvd y* ⟷ *unat x dvd unat y*
unfolding *dvd-def udvd-nat-alt* **by** *force*

lemmas *unat-mono* = *word-less-nat-alt* [*THEN iffD1*]

lemma *unat-minus-one*:
assumes *w* ≠ *0*
shows *unat* (*w* − *1*) = *unat w* − *1*
proof −
have *0* ≤ *uint w* **by** (*fact uint-nonnegative*)
moreover from *assms* **have** *0* ≠ *uint w* **by** (*simp add*: *uint-0-iff*)
ultimately have *1* ≤ *uint w* **by** *arith*
from *uint-lt2p* [*of w*] **have** *uint w* − *1* < *2 ^ len-of TYPE('a)* **by** *arith*
with ‹*1* ≤ *uint w*› **have** (*uint w* − *1*) *mod 2 ^ len-of TYPE('a)* = *uint w* − *1*
by (*auto intro*: *mod-pos-pos-trivial*)
with ‹*1* ≤ *uint w*› **have** *nat* ((*uint w* − *1*) *mod 2 ^ len-of TYPE('a)*) = *nat* (*uint w*) − *1*
by *auto*
then show *?thesis*
by (*simp only*: *unat-def int-word-uint word-arith-wis mod-diff-right-eq* [*symmetric*])
qed

lemma *measure-unat*: *p ~= 0 ⟹ unat (p − 1) < unat p*
by (*simp add*: *unat-minus-one*) (*simp add*: *unat-0-iff* [*symmetric*])

lemmas *uint-add-ge0* [*simp*] = *add-nonneg-nonneg* [*OF uint-ge-0 uint-ge-0*]
lemmas *uint-mult-ge0* [*simp*] = *mult-nonneg-nonneg* [*OF uint-ge-0 uint-ge-0*]

lemma *uint-sub-lt2p* [*simp*]:
uint (x :: 'a :: len0 word) − uint (y :: 'b :: len0 word) <
2 ^ len-of TYPE('a)
using *uint-ge-0* [*of y*] *uint-lt2p* [*of x*] **by** *arith*

16.17 Conditions for the addition (etc) of two words to overflow

lemma *uint-add-lem*:
(uint x + uint y < 2 ^ len-of TYPE('a)) =
(uint (x + y :: 'a :: len0 word) = uint x + uint y)
by (*unfold uint-word-ariths*) (*auto intro*!: *trans* [*OF - int-mod-lem*])

lemma *uint-mult-lem*:
(uint x ∗ uint y < 2 ^ len-of TYPE('a)) =
(uint (x ∗ y :: 'a :: len0 word) = uint x ∗ uint y)
by (*unfold uint-word-ariths*) (*auto intro*!: *trans* [*OF - int-mod-lem*])

lemma *uint-sub-lem*:
(uint x >= uint y) = (uint (x − y) = uint x − uint y)
by (*unfold uint-word-ariths*) (*auto intro*!: *trans* [*OF - int-mod-lem*])

lemma *uint-add-le*: *uint (x + y) <= uint x + uint y*
unfolding *uint-word-ariths* **by** (*metis uint-add-ge0 zmod-le-nonneg-dividend*)

lemma *uint-sub-ge*: *uint (x − y) >= uint x − uint y*
unfolding *uint-word-ariths* **by** (*metis int-mod-ge uint-sub-lt2p zless2p*)

lemma *mod-add-if-z*:
(x :: int) < z ==> y < z ==> 0 <= y ==> 0 <= x ==> 0 <= z ==>
(x + y) mod z = (if x + y < z then x + y else x + y − z)
by (*auto intro*: *int-mod-eq*)

lemma *uint-plus-if'*:
uint ((a::'a word) + b) =
(if uint a + uint b < 2 ^ len-of TYPE('a::len0) then uint a + uint b
else uint a + uint b − 2 ^ len-of TYPE('a))
using *mod-add-if-z* [*of uint a - uint b*] **by** (*simp add*: *uint-word-ariths*)

lemma *mod-sub-if-z*:
(x :: int) < z ==> y < z ==> 0 <= y ==> 0 <= x ==> 0 <= z ==>
(x − y) mod z = (if y <= x then x − y else x − y + z)
by (*auto intro*: *int-mod-eq*)

lemma *uint-sub-if'*:
  uint ((*a*::'*a word*) − *b*) =
  (*if uint b* ≤ *uint a then uint a* − *uint b*
   else uint a − *uint b* + *2* ^ *len-of TYPE*('*a*::*len0*))
  using *mod-sub-if-z* [*of uint a - uint b*] **by** (*simp add*: *uint-word-ariths*)

16.18 Definition of *uint-arith*

lemma *word-of-int-inverse*:
  word-of-int r = *a* ⟹ *0* <= *r* ⟹ *r* < *2* ^ *len-of TYPE*('*a*) ⟹
   uint (*a*::'*a*::*len0 word*) = *r*
  apply (*erule word-uint.Abs-inverse'* [*rotated*])
  apply (*simp add*: *uints-num*)
  done

lemma *uint-split*:
  fixes *x*::'*a*::*len0 word*
  shows *P* (*uint x*) =
      (*ALL i. word-of-int i* = *x* & *0* <= *i* & *i* < *2*^*len-of TYPE*('*a*) −−> *P i*)
  apply (*fold word-int-case-def*)
  apply (*auto dest*!: *word-of-int-inverse simp*: *int-word-uint mod-pos-pos-trivial*
        split: *word-int-split*)
  done

lemma *uint-split-asm*:
  fixes *x*::'*a*::*len0 word*
  shows *P* (*uint x*) =
      (~(*EX i. word-of-int i* = *x* & *0* <= *i* & *i* < *2*^*len-of TYPE*('*a*) & ~ *P i*))
  by (*auto dest*!: *word-of-int-inverse*
       simp: *int-word-uint mod-pos-pos-trivial*
       split: *uint-split*)

lemmas *uint-splits* = *uint-split uint-split-asm*

lemmas *uint-arith-simps* =
  word-le-def word-less-alt
  word-uint.Rep-inject [*symmetric*]
  uint-sub-if' uint-plus-if'

lemma *power-False-cong*: *False* ⟹ *a* ^ *b* = *c* ^ *d*
  by *auto*

ML ‹
fun uint-arith-simpset ctxt =
  ctxt addsimps @{*thms uint-arith-simps*}
    delsimps @{*thms word-uint.Rep-inject*}

```
  |> fold Splitter.add-split @{thms if-split-asm}
  |> fold Simplifier.add-cong @{thms power-False-cong}

fun uint-arith-tacs ctxt =
 let
  fun arith-tac' n t =
   Arith-Data.arith-tac ctxt n t
    handle Cooper.COOPER - => Seq.empty;
 in
  [ clarify-tac ctxt 1,
   full-simp-tac (uint-arith-simpset ctxt) 1,
   ALLGOALS (full-simp-tac
    (put-simpset HOL-ss ctxt
     |> fold Splitter.add-split @{thms uint-splits}
     |> fold Simplifier.add-cong @{thms power-False-cong})),
   rewrite-goals-tac ctxt @{thms word-size},
   ALLGOALS (fn n => REPEAT (resolve-tac ctxt [allI, impI] n) THEN
           REPEAT (eresolve-tac ctxt [conjE] n) THEN
           REPEAT (dresolve-tac ctxt @{thms word-of-int-inverse} n
               THEN assume-tac ctxt n
               THEN assume-tac ctxt n)),
   TRYALL arith-tac' ]
 end

fun uint-arith-tac ctxt = SELECT-GOAL (EVERY (uint-arith-tacs ctxt))
⟩
```

method-setup *uint-arith* =
⟨*Scan.succeed* (*SIMPLE-METHOD'* *o* *uint-arith-tac*)⟩
solving word arithmetic via integers and arith

16.19 More on overflows and monotonicity

lemma *no-plus-overflow-uint-size*:
((*x* :: *'a* :: *len0 word*) <= *x* + *y*) = (*uint x* + *uint y* < *2* ^ *size x*)
unfolding *word-size* **by** *uint-arith*

lemmas *no-olen-add* = *no-plus-overflow-uint-size* [*unfolded word-size*]

lemma *no-ulen-sub*: ((*x* :: *'a* :: *len0 word*) >= *x* − *y*) = (*uint y* <= *uint x*)
by *uint-arith*

lemma *no-olen-add'*:
fixes *x* :: *'a::len0 word*
shows ($x \leq y + x$) = (*uint y* + *uint x* < *2* ^ *len-of TYPE('a)*)
by (*simp add*: *ac-simps no-olen-add*)

lemmas *olen-add-eqv* = *trans* [*OF no-olen-add no-olen-add'* [*symmetric*]]

lemmas *uint-plus-simple-iff* = *trans* [*OF no-olen-add uint-add-lem*]
lemmas *uint-plus-simple* = *uint-plus-simple-iff* [*THEN iffD1*]
lemmas *uint-minus-simple-iff* = *trans* [*OF no-ulen-sub uint-sub-lem*]
lemmas *uint-minus-simple-alt* = *uint-sub-lem* [*folded word-le-def*]
lemmas *word-sub-le-iff* = *no-ulen-sub* [*folded word-le-def*]
lemmas *word-sub-le* = *word-sub-le-iff* [*THEN iffD2*]

lemma *word-less-sub1*:
(*x* :: *'a* :: *len word*) ~= *0* ⟹ (*1* < *x*) = (*0* < *x* − *1*)
by *uint-arith*

lemma *word-le-sub1*:
(*x* :: *'a* :: *len word*) ~= *0* ⟹ (*1* <= *x*) = (*0* <= *x* − *1*)
by *uint-arith*

lemma *sub-wrap-lt*:
((*x* :: *'a* :: *len0 word*) < *x* − *z*) = (*x* < *z*)
by *uint-arith*

lemma *sub-wrap*:
((*x* :: *'a* :: *len0 word*) <= *x* − *z*) = (*z* = *0* | *x* < *z*)
by *uint-arith*

lemma *plus-minus-not-NULL-ab*:
(*x* :: *'a* :: *len0 word*) <= *ab* − *c* ⟹ *c* <= *ab* ⟹ *c* ~= *0* ⟹ *x* + *c* ~= *0*
by *uint-arith*

lemma *plus-minus-no-overflow-ab*:
(*x* :: *'a* :: *len0 word*) <= *ab* − *c* ⟹ *c* <= *ab* ⟹ *x* <= *x* + *c*
by *uint-arith*

lemma *le-minus'*:
(*a* :: *'a* :: *len0 word*) + *c* <= *b* ⟹ *a* <= *a* + *c* ⟹ *c* <= *b* − *a*
by *uint-arith*

lemma *le-plus'*:
(*a* :: *'a* :: *len0 word*) <= *b* ⟹ *c* <= *b* − *a* ⟹ *a* + *c* <= *b*
by *uint-arith*

lemmas *le-plus* = *le-plus'* [*rotated*]

lemmas *le-minus* = *leD* [*THEN thin-rl*, *THEN le-minus'*]

lemma *word-plus-mono-right*:
(*y* :: *'a* :: *len0 word*) <= *z* ⟹ *x* <= *x* + *z* ⟹ *x* + *y* <= *x* + *z*
by *uint-arith*

lemma *word-less-minus-cancel*:
y − *x* < *z* − *x* ⟹ *x* <= *z* ⟹ (*y* :: *'a* :: *len0 word*) < *z*

by *uint-arith*

lemma *word-less-minus-mono-left*:
$(y :: 'a :: len0\ word) < z \Longrightarrow x <= y \Longrightarrow y - x < z - x$
by *uint-arith*

lemma *word-less-minus-mono*:
$a < c \Longrightarrow d < b \Longrightarrow a - b < a \Longrightarrow c - d < c$
$\Longrightarrow a - b < c - (d::'a::len\ word)$
by *uint-arith*

lemma *word-le-minus-cancel*:
$y - x <= z - x \Longrightarrow x <= z \Longrightarrow (y :: 'a :: len0\ word) <= z$
by *uint-arith*

lemma *word-le-minus-mono-left*:
$(y :: 'a :: len0\ word) <= z \Longrightarrow x <= y \Longrightarrow y - x <= z - x$
by *uint-arith*

lemma *word-le-minus-mono*:
$a <= c \Longrightarrow d <= b \Longrightarrow a - b <= a \Longrightarrow c - d <= c$
$\Longrightarrow a - b <= c - (d::'a::len\ word)$
by *uint-arith*

lemma *plus-le-left-cancel-wrap*:
$(x :: 'a :: len0\ word) + y' < x \Longrightarrow x + y < x \Longrightarrow (x + y' < x + y) = (y' < y)$
by *uint-arith*

lemma *plus-le-left-cancel-nowrap*:
$(x :: 'a :: len0\ word) <= x + y' \Longrightarrow x <= x + y \Longrightarrow$
$(x + y' < x + y) = (y' < y)$
by *uint-arith*

lemma *word-plus-mono-right2*:
$(a :: 'a :: len0\ word) <= a + b \Longrightarrow c <= b \Longrightarrow a <= a + c$
by *uint-arith*

lemma *word-less-add-right*:
$(x :: 'a :: len0\ word) < y - z \Longrightarrow z <= y \Longrightarrow x + z < y$
by *uint-arith*

lemma *word-less-sub-right*:
$(x :: 'a :: len0\ word) < y + z \Longrightarrow y <= x \Longrightarrow x - y < z$
by *uint-arith*

lemma *word-le-plus-either*:
$(x :: 'a :: len0\ word) <= y \mid x <= z \Longrightarrow y <= y + z \Longrightarrow x <= y + z$
by *uint-arith*

lemma *word-less-nowrapI*:
  (*x* :: *'a* :: *len0 word*) < *z* − *k* ⟹ *k* <= *z* ⟹ *0* < *k* ⟹ *x* < *x* + *k*
  by *uint-arith*

lemma *inc-le*: (*i* :: *'a* :: *len word*) < *m* ⟹ *i* + *1* <= *m*
  by *uint-arith*

lemma *inc-i*:
  (*1* :: *'a* :: *len word*) <= *i* ⟹ *i* < *m* ⟹ *1* <= (*i* + *1*) & *i* + *1* <= *m*
  by *uint-arith*

lemma *udvd-incr-lem*:
  up < *uq* ⟹ *up* = *ua* + *n* ∗ *uint K* ⟹
    uq = *ua* + *n'* ∗ *uint K* ⟹ *up* + *uint K* <= *uq*
  apply *clarsimp*

  apply (*drule less-le-mult*)
  apply *safe*
  done

lemma *udvd-incr'*:
  p < *q* ⟹ *uint p* = *ua* + *n* ∗ *uint K* ⟹
    uint q = *ua* + *n'* ∗ *uint K* ⟹ *p* + *K* <= *q*
  apply (*unfold word-less-alt word-le-def*)
  apply (*drule* (*2*) *udvd-incr-lem*)
  apply (*erule uint-add-le* [*THEN order-trans*])
  done

lemma *udvd-decr'*:
  p < *q* ⟹ *uint p* = *ua* + *n* ∗ *uint K* ⟹
    uint q = *ua* + *n'* ∗ *uint K* ⟹ *p* <= *q* − *K*
  apply (*unfold word-less-alt word-le-def*)
  apply (*drule* (*2*) *udvd-incr-lem*)
  apply (*drule le-diff-eq* [*THEN iffD2*])
  apply (*erule order-trans*)
  apply (*rule uint-sub-ge*)
  done

lemmas *udvd-incr-lem0* = *udvd-incr-lem* [**where** *ua*=*0*, *unfolded add-0-left*]
lemmas *udvd-incr0* = *udvd-incr'* [**where** *ua*=*0*, *unfolded add-0-left*]
lemmas *udvd-decr0* = *udvd-decr'* [**where** *ua*=*0*, *unfolded add-0-left*]

lemma *udvd-minus-le'*:
  xy < *k* ⟹ *z udvd xy* ⟹ *z udvd k* ⟹ *xy* <= *k* − *z*
  apply (*unfold udvd-def*)
  apply *clarify*
  apply (*erule* (*2*) *udvd-decr0*)
  done

lemma *udvd-incr2-K*:
p < *a* + *s* ⟹ *a* <= *a* + *s* ⟹ *K udvd s* ⟹ *K udvd p* − *a* ⟹ *a* <= *p* ⟹
0 < *K* ⟹ *p* <= *p* + *K* & *p* + *K* <= *a* + *s*
using [[*simproc del*: *linordered-ring-less-cancel-factor*]]
apply (*unfold udvd-def*)
apply *clarify*
apply (*simp add*: *uint-arith-simps split*: *if-split-asm*)
prefer *2*
apply (*insert uint-range′* [*of s*])[*1*]
apply *arith*
apply (*drule add.commute* [*THEN xtr1*])
apply (*simp add*: *diff-less-eq* [*symmetric*])
apply (*drule less-le-mult*)
apply *arith*
apply *simp*
done

lemma *word-succ-rbl*:
to-bl w = *bl* ⟹ *to-bl* (*word-succ w*) = (*rev* (*rbl-succ* (*rev bl*)))
apply (*unfold word-succ-def*)
apply *clarify*
apply (*simp add*: *to-bl-of-bin*)
apply (*simp add*: *to-bl-def rbl-succ*)
done

lemma *word-pred-rbl*:
to-bl w = *bl* ⟹ *to-bl* (*word-pred w*) = (*rev* (*rbl-pred* (*rev bl*)))
apply (*unfold word-pred-def*)
apply *clarify*
apply (*simp add*: *to-bl-of-bin*)
apply (*simp add*: *to-bl-def rbl-pred*)
done

lemma *word-add-rbl*:
to-bl v = *vbl* ⟹ *to-bl w* = *wbl* ⟹
to-bl (*v* + *w*) = (*rev* (*rbl-add* (*rev vbl*) (*rev wbl*)))
apply (*unfold word-add-def*)
apply *clarify*
apply (*simp add*: *to-bl-of-bin*)
apply (*simp add*: *to-bl-def rbl-add*)
done

lemma *word-mult-rbl*:
to-bl v = *vbl* ⟹ *to-bl w* = *wbl* ⟹
to-bl (*v* ∗ *w*) = (*rev* (*rbl-mult* (*rev vbl*) (*rev wbl*)))
apply (*unfold word-mult-def*)
apply *clarify*
apply (*simp add*: *to-bl-of-bin*)

apply (*simp add*: *to-bl-def rbl-mult*)
done

lemma *rtb-rbl-ariths*:
rev (*to-bl w*) = *ys* ⟹ *rev* (*to-bl* (*word-succ w*)) = *rbl-succ ys*
rev (*to-bl w*) = *ys* ⟹ *rev* (*to-bl* (*word-pred w*)) = *rbl-pred ys*
rev (*to-bl v*) = *ys* ⟹ *rev* (*to-bl w*) = *xs* ⟹ *rev* (*to-bl* (*v* ∗ *w*)) = *rbl-mult ys xs*
rev (*to-bl v*) = *ys* ⟹ *rev* (*to-bl w*) = *xs* ⟹ *rev* (*to-bl* (*v* + *w*)) = *rbl-add ys xs*
by (*auto simp*: *rev-swap* [*symmetric*] *word-succ-rbl*
word-pred-rbl word-mult-rbl word-add-rbl)

16.20 Arithmetic type class instantiations

lemmas *word-le-0-iff* [*simp*] =
word-zero-le [*THEN leD*, *THEN linorder-antisym-conv1*]

lemma *word-of-int-nat*:
0 <= *x* ⟹ *word-of-int x* = *of-nat* (*nat x*)
by (*simp add*: *of-nat-nat word-of-int*)

lemma *iszero-word-no* [*simp*]:
iszero (*numeral bin* :: ′*a* :: *len word*) =
iszero (*bintrunc* (*len-of TYPE*(′*a*)) (*numeral bin*))
using *word-ubin.norm-eq-iff* [**where** ′*a*=′*a*, *of numeral bin 0*]
by (*simp add*: *iszero-def* [*symmetric*])

Use *iszero* to simplify equalities between word numerals.

lemmas *word-eq-numeral-iff-iszero* [*simp*] =
eq-numeral-iff-iszero [**where** ′*a*=′*a*::*len word*]

16.21 Word and nat

lemma *td-ext-unat* [*OF refl*]:
n = *len-of TYPE* (′*a* :: *len*) ⟹
td-ext (*unat* :: ′*a word* => *nat*) *of-nat*
(*unats n*) (%*i*. *i mod 2* ˆ *n*)
apply (*unfold td-ext-def*′ *unat-def word-of-nat unats-uints*)
apply (*auto intro*!: *imageI simp add* : *word-of-int-hom-syms*)
apply (*erule word-uint.Abs-inverse* [*THEN arg-cong*])
apply (*simp add*: *int-word-uint nat-mod-distrib nat-power-eq*)
done

lemmas *unat-of-nat* = *td-ext-unat* [*THEN td-ext.eq-norm*]

interpretation *word-unat*:
td-ext unat::′*a*::*len word* => *nat*
of-nat

unats (*len-of TYPE*('*a*::*len*))
%*i*. *i mod 2 ^ len-of TYPE*('*a*::*len*)
by (*rule td-ext-unat*)

lemmas *td-unat* = *word-unat.td-thm*

lemmas *unat-lt2p* [*iff*] = *word-unat.Rep* [*unfolded unats-def mem-Collect-eq*]

lemma *unat-le*: *y* <= *unat* (*z* :: '*a* :: *len word*) ⟹ *y* : *unats* (*len-of TYPE* ('*a*))
apply (*unfold unats-def*)
apply *clarsimp*
apply (*rule xtrans*, *rule unat-lt2p*, *assumption*)
done

lemma *word-nchotomy*:
ALL w. *EX n*. (*w* :: '*a* :: *len word*) = *of-nat n* & *n* < *2 ^ len-of TYPE* ('*a*)
apply (*rule allI*)
apply (*rule word-unat.Abs-cases*)
apply (*unfold unats-def*)
apply *auto*
done

lemma *of-nat-eq*:
fixes *w* :: '*a*::*len word*
shows (*of-nat n* = *w*) = (∃ *q*. *n* = *unat w* + *q* ∗ *2 ^ len-of TYPE*('*a*))
apply (*rule trans*)
apply (*rule word-unat.inverse-norm*)
apply (*rule iffI*)
apply (*rule mod-eqD*)
apply *simp*
apply *clarsimp*
done

lemma *of-nat-eq-size*:
(*of-nat n* = *w*) = (*EX q*. *n* = *unat w* + *q* ∗ *2 ^ size w*)
unfolding *word-size* **by** (*rule of-nat-eq*)

lemma *of-nat-0*:
(*of-nat m* = (*0*::'*a*::*len word*)) = (∃ *q*. *m* = *q* ∗ *2 ^ len-of TYPE*('*a*))
by (*simp add*: *of-nat-eq*)

lemma *of-nat-2p* [*simp*]:
of-nat (*2 ^ len-of TYPE*('*a*)) = (*0*::'*a*::*len word*)
by (*fact mult-1* [*symmetric*, *THEN iffD2* [*OF of-nat-0 exI*]])

lemma *of-nat-gt-0*: *of-nat k* ~= *0* ⟹ *0* < *k*
by (*cases k*) *auto*

lemma *of-nat-neq-0*:

0 < k ⟹ k < 2 ^ len-of TYPE ('a :: len) ⟹ of-nat k ~= (0 :: 'a word)
by (*clarsimp simp add : of-nat-0*)

lemma *Abs-fnat-hom-add*:
of-nat a + of-nat b = of-nat (a + b)
by *simp*

lemma *Abs-fnat-hom-mult*:
of-nat a ∗ of-nat b = (of-nat (a ∗ b) :: 'a :: len word)
by (*simp add*: *word-of-nat wi-hom-mult*)

lemma *Abs-fnat-hom-Suc*:
word-succ (of-nat a) = of-nat (Suc a)
by (*simp add*: *word-of-nat wi-hom-succ ac-simps*)

lemma *Abs-fnat-hom-0*: *(0::'a::len word) = of-nat 0*
by *simp*

lemma *Abs-fnat-hom-1*: *(1::'a::len word) = of-nat (Suc 0)*
by *simp*

lemmas *Abs-fnat-homs =*
Abs-fnat-hom-add Abs-fnat-hom-mult Abs-fnat-hom-Suc
Abs-fnat-hom-0 Abs-fnat-hom-1

lemma *word-arith-nat-add*:
a + b = of-nat (unat a + unat b)
by *simp*

lemma *word-arith-nat-mult*:
a ∗ b = of-nat (unat a ∗ unat b)
by (*simp add*: *of-nat-mult*)

lemma *word-arith-nat-Suc*:
word-succ a = of-nat (Suc (unat a))
by (*subst Abs-fnat-hom-Suc [symmetric]*) *simp*

lemma *word-arith-nat-div*:
a div b = of-nat (unat a div unat b)
by (*simp add*: *word-div-def word-of-nat zdiv-int uint-nat*)

lemma *word-arith-nat-mod*:
a mod b = of-nat (unat a mod unat b)
by (*simp add*: *word-mod-def word-of-nat zmod-int uint-nat*)

lemmas *word-arith-nat-defs =*
word-arith-nat-add word-arith-nat-mult
word-arith-nat-Suc Abs-fnat-hom-0
Abs-fnat-hom-1 word-arith-nat-div

```
  word-arith-nat-mod

lemma unat-cong: x = y ⟹ unat x = unat y
  by simp

lemmas unat-word-ariths = word-arith-nat-defs
  [THEN trans [OF unat-cong unat-of-nat]]

lemmas word-sub-less-iff = word-sub-le-iff
  [unfolded linorder-not-less [symmetric] Not-eq-iff]

lemma unat-add-lem:
  (unat x + unat y < 2 ^ len-of TYPE('a)) =
    (unat (x + y :: 'a :: len word) = unat x + unat y)
  unfolding unat-word-ariths
  by (auto intro!: trans [OF - nat-mod-lem])

lemma unat-mult-lem:
  (unat x * unat y < 2 ^ len-of TYPE('a)) =
    (unat (x * y :: 'a :: len word) = unat x * unat y)
  unfolding unat-word-ariths
  by (auto intro!: trans [OF - nat-mod-lem])

lemmas unat-plus-if' = trans [OF unat-word-ariths(1) mod-nat-add, simplified]

lemma le-no-overflow:
  x <= b ⟹ a <= a + b ⟹ x <= a + (b :: 'a :: len0 word)
  apply (erule order-trans)
  apply (erule olen-add-eqv [THEN iffD1])
  done

lemmas un-ui-le = trans [OF word-le-nat-alt [symmetric] word-le-def]

lemma unat-sub-if-size:
  unat (x − y) = (if unat y <= unat x
   then unat x − unat y
   else unat x + 2 ^ size x − unat y)
  apply (unfold word-size)
  apply (simp add: un-ui-le)
  apply (auto simp add: unat-def uint-sub-if')
   apply (rule nat-diff-distrib)
    prefer 3
    apply (simp add: algebra-simps)
    apply (rule nat-diff-distrib [THEN trans])
      prefer 3
      apply (subst nat-add-distrib)
        prefer 3
        apply (simp add: nat-power-eq)
       apply auto
```

apply *uint-arith*
done

lemmas *unat-sub-if'* = *unat-sub-if-size* [*unfolded word-size*]

lemma *unat-div*: *unat* ((*x* :: '*a* :: *len word*) *div y*) = *unat x div unat y*
apply (*simp add* : *unat-word-ariths*)
apply (*rule unat-lt2p* [*THEN xtr7*, *THEN nat-mod-eq'*])
apply (*rule div-le-dividend*)
done

lemma *unat-mod*: *unat* ((*x* :: '*a* :: *len word*) *mod y*) = *unat x mod unat y*
apply (*clarsimp simp add* : *unat-word-ariths*)
apply (*cases unat y*)
prefer *2*
apply (*rule unat-lt2p* [*THEN xtr7*, *THEN nat-mod-eq'*])
apply (*rule mod-le-divisor*)
apply *auto*
done

lemma *uint-div*: *uint* ((*x* :: '*a* :: *len word*) *div y*) = *uint x div uint y*
unfolding *uint-nat* **by** (*simp add* : *unat-div zdiv-int*)

lemma *uint-mod*: *uint* ((*x* :: '*a* :: *len word*) *mod y*) = *uint x mod uint y*
unfolding *uint-nat* **by** (*simp add* : *unat-mod zmod-int*)

16.22 Definition of *unat-arith* tactic

lemma *unat-split*:
fixes *x*::'*a*::*len word*
shows *P* (*unat x*) =
(*ALL n*. *of-nat n* = *x* & *n* < *2*^*len-of TYPE*('*a*) --> *P n*)
by (*auto simp*: *unat-of-nat*)

lemma *unat-split-asm*:
fixes *x*::'*a*::*len word*
shows *P* (*unat x*) =
(~(*EX n*. *of-nat n* = *x* & *n* < *2*^*len-of TYPE*('*a*) & ~ *P n*))
by (*auto simp*: *unat-of-nat*)

lemmas *of-nat-inverse* =
word-unat.Abs-inverse' [*rotated*, *unfolded unats-def*, *simplified*]

lemmas *unat-splits* = *unat-split unat-split-asm*

lemmas *unat-arith-simps* =
word-le-nat-alt word-less-nat-alt
word-unat.Rep-inject [*symmetric*]
unat-sub-if' unat-plus-if' unat-div unat-mod

ML ‹
```
fun unat-arith-simpset ctxt =
  ctxt addsimps @{thms unat-arith-simps}
     delsimps @{thms word-unat.Rep-inject}
     |> fold Splitter.add-split @{thms if-split-asm}
     |> fold Simplifier.add-cong @{thms power-False-cong}

fun unat-arith-tacs ctxt =
  let
    fun arith-tac' n t =
      Arith-Data.arith-tac ctxt n t
        handle Cooper.COOPER - => Seq.empty;
  in
    [ clarify-tac ctxt 1,
      full-simp-tac (unat-arith-simpset ctxt) 1,
      ALLGOALS (full-simp-tac
        (put-simpset HOL-ss ctxt
          |> fold Splitter.add-split @{thms unat-splits}
          |> fold Simplifier.add-cong @{thms power-False-cong})),
      rewrite-goals-tac ctxt @{thms word-size},
      ALLGOALS (fn n => REPEAT (resolve-tac ctxt [allI, impI] n) THEN
                       REPEAT (eresolve-tac ctxt [conjE] n) THEN
                       REPEAT (dresolve-tac ctxt @{thms of-nat-inverse} n THEN
assume-tac ctxt n)),
      TRYALL arith-tac' ]
  end

fun unat-arith-tac ctxt = SELECT-GOAL (EVERY (unat-arith-tacs ctxt))
```
›

method-setup *unat-arith* =
‹*Scan.succeed (SIMPLE-METHOD' o unat-arith-tac)*›
solving word arithmetic via natural numbers and arith

lemma *no-plus-overflow-unat-size*:
$((x :: 'a :: len\ word) <= x + y) = (unat\ x + unat\ y < 2\ \hat{}\ size\ x)$
unfolding *word-size* **by** *unat-arith*

lemmas *no-olen-add-nat* = *no-plus-overflow-unat-size* [*unfolded word-size*]

lemmas *unat-plus-simple* = *trans* [*OF no-olen-add-nat unat-add-lem*]

lemma *word-div-mult*:
$(0 :: 'a :: len\ word) < y \Longrightarrow unat\ x * unat\ y < 2\ \hat{}\ len\text{-}of\ TYPE('a) \Longrightarrow$
$x * y\ div\ y = x$
apply *unat-arith*
apply *clarsimp*

apply (*subst unat-mult-lem* [*THEN iffD1*])
apply *auto*
done

lemma *div-lt′*: (*i* :: ′*a* :: *len word*) <= *k div x* ⟹
unat i ∗ *unat x* < *2* ˆ *len-of TYPE*(′*a*)
apply *unat-arith*
apply *clarsimp*
apply (*drule mult-le-mono1*)
apply (*erule order-le-less-trans*)
apply (*rule xtr7* [*OF unat-lt2p div-mult-le*])
done

lemmas *div-lt′′* = *order-less-imp-le* [*THEN div-lt′*]

lemma *div-lt-mult*: (*i* :: ′*a* :: *len word*) < *k div x* ⟹ *0* < *x* ⟹ *i* ∗ *x* < *k*
apply (*frule div-lt′′* [*THEN unat-mult-lem* [*THEN iffD1*]])
apply (*simp add*: *unat-arith-simps*)
apply (*drule* (*1*) *mult-less-mono1*)
apply (*erule order-less-le-trans*)
apply (*rule div-mult-le*)
done

lemma *div-le-mult*:
(*i* :: ′*a* :: *len word*) <= *k div x* ⟹ *0* < *x* ⟹ *i* ∗ *x* <= *k*
apply (*frule div-lt′* [*THEN unat-mult-lem* [*THEN iffD1*]])
apply (*simp add*: *unat-arith-simps*)
apply (*drule mult-le-mono1*)
apply (*erule order-trans*)
apply (*rule div-mult-le*)
done

lemma *div-lt-uint′*:
(*i* :: ′*a* :: *len word*) <= *k div x* ⟹ *uint i* ∗ *uint x* < *2* ˆ *len-of TYPE*(′*a*)
apply (*unfold uint-nat*)
apply (*drule div-lt′*)
by (*metis of-nat-less-iff of-nat-mult of-nat-numeral of-nat-power*)

lemmas *div-lt-uint′′* = *order-less-imp-le* [*THEN div-lt-uint′*]

lemma *word-le-exists′*:
(*x* :: ′*a* :: *len0 word*) <= *y* ⟹
(*EX z*. *y* = *x* + *z* & *uint x* + *uint z* < *2* ˆ *len-of TYPE*(′*a*))
apply (*rule exI*)
apply (*rule conjI*)
apply (*rule zadd-diff-inverse*)
apply *uint-arith*
done

lemmas *plus-minus-not-NULL* = *order-less-imp-le* [*THEN plus-minus-not-NULL-ab*]

lemmas *plus-minus-no-overflow* =
  order-less-imp-le [*THEN plus-minus-no-overflow-ab*]

lemmas *mcs* = *word-less-minus-cancel word-less-minus-mono-left*
  word-le-minus-cancel word-le-minus-mono-left

lemmas *word-l-diffs* = *mcs* [**where** $y = w + x$, *unfolded add-diff-cancel*] **for** *w x*
lemmas *word-diff-ls* = *mcs* [**where** $z = w + x$, *unfolded add-diff-cancel*] **for** *w x*
lemmas *word-plus-mcs* = *word-diff-ls* [**where** $y = v + x$, *unfolded add-diff-cancel*]
for *v x*

lemmas *le-unat-uoi* = *unat-le* [*THEN word-unat.Abs-inverse*]

lemmas *thd* = *refl* [*THEN* [*2*] *split-div-lemma* [*THEN iffD2*], *THEN conjunct1*]

lemmas *uno-simps* [*THEN le-unat-uoi*] = *mod-le-divisor div-le-dividend dtle*

lemma *word-mod-div-equality*:
  (*n div b*) ∗ *b* + (*n mod b*) = (*n* :: *'a* :: *len word*)
  apply (*unfold word-less-nat-alt word-arith-nat-defs*)
  apply (*cut-tac y=unat b* **in** *gt-or-eq-0*)
  apply (*erule disjE*)
   apply (*simp only*: *mod-div-equality uno-simps Word.word-unat.Rep-inverse*)
  apply *simp*
  done

lemma *word-div-mult-le*: *a div b* ∗ *b* <= (*a*::*'a*::*len word*)
  apply (*unfold word-le-nat-alt word-arith-nat-defs*)
  apply (*cut-tac y=unat b* **in** *gt-or-eq-0*)
  apply (*erule disjE*)
   apply (*simp only*: *div-mult-le uno-simps Word.word-unat.Rep-inverse*)
  apply *simp*
  done

lemma *word-mod-less-divisor*: *0* < *n* ⟹ *m mod n* < (*n* :: *'a* :: *len word*)
  apply (*simp only*: *word-less-nat-alt word-arith-nat-defs*)
  apply (*clarsimp simp add* : *uno-simps*)
  done

lemma *word-of-int-power-hom*:
  word-of-int a ˆ *n* = (*word-of-int* (*a* ˆ *n*) :: *'a* :: *len word*)
  by (*induct n*) (*simp-all add*: *wi-hom-mult* [*symmetric*])

lemma *word-arith-power-alt*:
  a ˆ *n* = (*word-of-int* (*uint a* ˆ *n*) :: *'a* :: *len word*)
  by (*simp add* : *word-of-int-power-hom* [*symmetric*])

lemma *of-bl-length-less*:
length x = k ⟹ k < len-of TYPE('a) ⟹ (of-bl x :: 'a :: len word) < 2 ^ k
apply (*unfold of-bl-def word-less-alt word-numeral-alt*)
apply *safe*
apply (*simp* (*no-asm*) *add*: *word-of-int-power-hom word-uint.eq-norm*
del: *word-of-int-numeral*)
apply (*simp add*: *mod-pos-pos-trivial*)
apply (*subst mod-pos-pos-trivial*)
apply (*rule bl-to-bin-ge0*)
apply (*rule order-less-trans*)
apply (*rule bl-to-bin-lt2p*)
apply *simp*
apply (*rule bl-to-bin-lt2p*)
done

16.23 Cardinality, finiteness of set of words

instance *word* :: (*len0*) *finite*
by *standard* (*simp add*: *type-definition.univ* [*OF type-definition-word*])

lemma *card-word*: *CARD('a::len0 word) = 2 ^ len-of TYPE('a)*
by (*simp add*: *type-definition.card* [*OF type-definition-word*] *nat-power-eq*)

lemma *card-word-size*:
card (UNIV :: 'a :: len0 word set) = (2 ^ size (x :: 'a word))
unfolding *word-size* **by** (*rule card-word*)

16.24 Bitwise Operations on Words

lemmas *bin-log-bintrs = bin-trunc-not bin-trunc-xor bin-trunc-and bin-trunc-or*

lemmas *wils1 = bin-log-bintrs* [*THEN word-ubin.norm-eq-iff* [*THEN iffD1*],
folded word-ubin.eq-norm, *THEN eq-reflection*]

lemmas *word-log-binary-defs* =
word-and-def word-or-def word-xor-def

lemma *word-wi-log-defs*:
NOT word-of-int a = word-of-int (NOT a)
word-of-int a AND word-of-int b = word-of-int (a AND b)
word-of-int a OR word-of-int b = word-of-int (a OR b)
word-of-int a XOR word-of-int b = word-of-int (a XOR b)
by (*transfer*, *rule refl*)+

lemma *word-no-log-defs* [*simp*]:

NOT (*numeral a*) = *word-of-int* (*NOT* (*numeral a*))
NOT (− *numeral a*) = *word-of-int* (*NOT* (− *numeral a*))
numeral a AND numeral b = *word-of-int* (*numeral a AND numeral b*)
numeral a AND − *numeral b* = *word-of-int* (*numeral a AND* − *numeral b*)
− *numeral a AND numeral b* = *word-of-int* (− *numeral a AND numeral b*)
− *numeral a AND* − *numeral b* = *word-of-int* (− *numeral a AND* − *numeral b*)
numeral a OR numeral b = *word-of-int* (*numeral a OR numeral b*)
numeral a OR − *numeral b* = *word-of-int* (*numeral a OR* − *numeral b*)
− *numeral a OR numeral b* = *word-of-int* (− *numeral a OR numeral b*)
− *numeral a OR* − *numeral b* = *word-of-int* (− *numeral a OR* − *numeral b*)
numeral a XOR numeral b = *word-of-int* (*numeral a XOR numeral b*)
numeral a XOR − *numeral b* = *word-of-int* (*numeral a XOR* − *numeral b*)
− *numeral a XOR numeral b* = *word-of-int* (− *numeral a XOR numeral b*)
− *numeral a XOR* − *numeral b* = *word-of-int* (− *numeral a XOR* − *numeral b*)
by (*transfer*, *rule refl*)+

Special cases for when one of the arguments equals 1.

lemma *word-bitwise-1-simps* [*simp*]:
NOT (*1*::′*a*::*len0 word*) = −*2*
1 AND numeral b = *word-of-int* (*1 AND numeral b*)
1 AND − *numeral b* = *word-of-int* (*1 AND* − *numeral b*)
numeral a AND 1 = *word-of-int* (*numeral a AND 1*)
− *numeral a AND 1* = *word-of-int* (− *numeral a AND 1*)
1 OR numeral b = *word-of-int* (*1 OR numeral b*)
1 OR − *numeral b* = *word-of-int* (*1 OR* − *numeral b*)
numeral a OR 1 = *word-of-int* (*numeral a OR 1*)
− *numeral a OR 1* = *word-of-int* (− *numeral a OR 1*)
1 XOR numeral b = *word-of-int* (*1 XOR numeral b*)
1 XOR − *numeral b* = *word-of-int* (*1 XOR* − *numeral b*)
numeral a XOR 1 = *word-of-int* (*numeral a XOR 1*)
− *numeral a XOR 1* = *word-of-int* (− *numeral a XOR 1*)
by (*transfer*, *simp*)+

Special cases for when one of the arguments equals -1.

lemma *word-bitwise-m1-simps* [*simp*]:
NOT (−*1*::′*a*::*len0 word*) = *0*
(−*1*::′*a*::*len0 word*) *AND x* = *x*
x AND (−*1*::′*a*::*len0 word*) = *x*
(−*1*::′*a*::*len0 word*) *OR x* = −*1*
x OR (−*1*::′*a*::*len0 word*) = −*1*
(−*1*::′*a*::*len0 word*) *XOR x* = *NOT x*
x XOR (−*1*::′*a*::*len0 word*) = *NOT x*
by (*transfer*, *simp*)+

lemma *uint-or*: *uint* (*x OR y*) = (*uint x*) *OR* (*uint y*)
by (*transfer*, *simp add*: *bin-trunc-ao*)

lemma *uint-and*: *uint* (*x AND y*) = (*uint x*) *AND* (*uint y*)
by (*transfer*, *simp add*: *bin-trunc-ao*)

lemma *test-bit-wi* [*simp*]:
(*word-of-int x*::′*a*::*len0 word*) !! *n* ⟷ *n* < *len-of TYPE*(′*a*) ∧ *bin-nth x n*
unfolding *word-test-bit-def*
by (*simp add*: *word-ubin.eq-norm nth-bintr*)

lemma *word-test-bit-transfer* [*transfer-rule*]:
(*rel-fun pcr-word* (*rel-fun op* = *op* =))
(λ*x n*. *n* < *len-of TYPE*(′*a*) ∧ *bin-nth x n*) (*test-bit* :: ′*a*::*len0 word* ⇒ -)
unfolding *rel-fun-def word.pcr-cr-eq cr-word-def* **by** *simp*

lemma *word-ops-nth-size*:
n < *size* (*x*::′*a*::*len0 word*) ⟹
(*x OR y*) !! *n* = (*x* !! *n* | *y* !! *n*) &
(*x AND y*) !! *n* = (*x* !! *n* & *y* !! *n*) &
(*x XOR y*) !! *n* = (*x* !! *n* ~= *y* !! *n*) &
(*NOT x*) !! *n* = (~ *x* !! *n*)
unfolding *word-size* **by** *transfer* (*simp add*: *bin-nth-ops*)

lemma *word-ao-nth*:
fixes *x* :: ′*a*::*len0 word*
shows (*x OR y*) !! *n* = (*x* !! *n* | *y* !! *n*) &
(*x AND y*) !! *n* = (*x* !! *n* & *y* !! *n*)
by *transfer* (*auto simp add*: *bin-nth-ops*)

lemma *test-bit-numeral* [*simp*]:
(*numeral w* :: ′*a*::*len0 word*) !! *n* ⟷
n < *len-of TYPE*(′*a*) ∧ *bin-nth* (*numeral w*) *n*
by *transfer* (*rule refl*)

lemma *test-bit-neg-numeral* [*simp*]:
(− *numeral w* :: ′*a*::*len0 word*) !! *n* ⟷
n < *len-of TYPE*(′*a*) ∧ *bin-nth* (− *numeral w*) *n*
by *transfer* (*rule refl*)

lemma *test-bit-1* [*simp*]: (*1*::′*a*::*len word*) !! *n* ⟷ *n* = *0*
by *transfer auto*

lemma *nth-0* [*simp*]: ~ (*0*::′*a*::*len0 word*) !! *n*
by *transfer simp*

lemma *nth-minus1* [*simp*]: (−*1*::′*a*::*len0 word*) !! *n* ⟷ *n* < *len-of TYPE*(′*a*)
by *transfer simp*

lemmas *bwsimps* =
wi-hom-add
word-wi-log-defs

lemma *word-bw-assocs*:
  fixes *x* :: *'a::len0 word*
  shows
  (*x AND y*) *AND z* = *x AND y AND z*
  (*x OR y*) *OR z* = *x OR y OR z*
  (*x XOR y*) *XOR z* = *x XOR y XOR z*
  by (*auto simp*: *word-eq-iff word-ops-nth-size* [*unfolded word-size*])

lemma *word-bw-comms*:
  fixes *x* :: *'a::len0 word*
  shows
  x AND y = *y AND x*
  x OR y = *y OR x*
  x XOR y = *y XOR x*
  by (*auto simp*: *word-eq-iff word-ops-nth-size* [*unfolded word-size*])

lemma *word-bw-lcs*:
  fixes *x* :: *'a::len0 word*
  shows
  y AND x AND z = *x AND y AND z*
  y OR x OR z = *x OR y OR z*
  y XOR x XOR z = *x XOR y XOR z*
  by (*auto simp*: *word-eq-iff word-ops-nth-size* [*unfolded word-size*])

lemma *word-log-esimps* [*simp*]:
  fixes *x* :: *'a::len0 word*
  shows
  x AND 0 = *0*
  x AND −*1* = *x*
  x OR 0 = *x*
  x OR −*1* = −*1*
  x XOR 0 = *x*
  x XOR −*1* = *NOT x*
  0 AND x = *0*
  −*1 AND x* = *x*
  0 OR x = *x*
  −*1 OR x* = −*1*
  0 XOR x = *x*
  −*1 XOR x* = *NOT x*
  by (*auto simp*: *word-eq-iff word-ops-nth-size* [*unfolded word-size*])

lemma *word-not-dist*:
  fixes *x* :: *'a::len0 word*
  shows
  NOT (*x OR y*) = *NOT x AND NOT y*
  NOT (*x AND y*) = *NOT x OR NOT y*
  by (*auto simp*: *word-eq-iff word-ops-nth-size* [*unfolded word-size*])

lemma *word-bw-same*:
fixes x :: $'a$::*len0 word*
shows
x *AND* $x = x$
x *OR* $x = x$
x *XOR* $x = 0$
by (*auto simp*: *word-eq-iff word-ops-nth-size* [*unfolded word-size*])

lemma *word-ao-absorbs* [*simp*]:
fixes x :: $'a$::*len0 word*
shows
x *AND* (y *OR* x) $= x$
x *OR* y *AND* $x = x$
x *AND* (x *OR* y) $= x$
y *AND* x *OR* $x = x$
(y *OR* x) *AND* $x = x$
x *OR* x *AND* $y = x$
(x *OR* y) *AND* $x = x$
x *AND* y *OR* $x = x$
by (*auto simp*: *word-eq-iff word-ops-nth-size* [*unfolded word-size*])

lemma *word-not-not* [*simp*]:
NOT NOT (x::$'a$::*len0 word*) $= x$
by (*auto simp*: *word-eq-iff word-ops-nth-size* [*unfolded word-size*])

lemma *word-ao-dist*:
fixes x :: $'a$::*len0 word*
shows (x *OR* y) *AND* $z = x$ *AND* z *OR* y *AND* z
by (*auto simp*: *word-eq-iff word-ops-nth-size* [*unfolded word-size*])

lemma *word-oa-dist*:
fixes x :: $'a$::*len0 word*
shows x *AND* y *OR* z = (x *OR* z) *AND* (y *OR* z)
by (*auto simp*: *word-eq-iff word-ops-nth-size* [*unfolded word-size*])

lemma *word-add-not* [*simp*]:
fixes x :: $'a$::*len0 word*
shows x + *NOT* $x = -1$
by *transfer* (*simp add*: *bin-add-not*)

lemma *word-plus-and-or* [*simp*]:
fixes x :: $'a$::*len0 word*
shows (x *AND* y) + (x *OR* y) $= x + y$
by *transfer* (*simp add*: *plus-and-or*)

lemma *leoa*:
fixes x :: $'a$::*len0 word*
shows (w = (x *OR* y)) $\Longrightarrow$ (y = (w *AND* y)) **by** *auto*
lemma *leao*:

fixes *x' :: 'a::len0 word*
shows $(w' = (x' \; AND \; y')) \Longrightarrow (x' = (x' \; OR \; w'))$ **by** *auto*

lemma *word-ao-equiv*:
fixes *w w' :: 'a::len0 word*
shows $(w = w \; OR \; w') = (w' = w \; AND \; w')$
by (*auto intro*: *leoa leao*)

lemma *le-word-or2*: *x <= x OR (y::'a::len0 word)*
unfolding *word-le-def uint-or*
by (*auto intro*: *le-int-or*)

lemmas *le-word-or1 = xtr3 [OF word-bw-comms (2) le-word-or2]*
lemmas *word-and-le1 = xtr3 [OF word-ao-absorbs (4) [symmetric] le-word-or2]*
lemmas *word-and-le2 = xtr3 [OF word-ao-absorbs (8) [symmetric] le-word-or2]*

lemma *bl-word-not*: *to-bl (NOT w) = map Not (to-bl w)*
unfolding *to-bl-def word-log-defs bl-not-bin*
by (*simp add*: *word-ubin.eq-norm*)

lemma *bl-word-xor*: *to-bl (v XOR w) = map2 op ~= (to-bl v) (to-bl w)*
unfolding *to-bl-def word-log-defs bl-xor-bin*
by (*simp add*: *word-ubin.eq-norm*)

lemma *bl-word-or*: *to-bl (v OR w) = map2 op | (to-bl v) (to-bl w)*
unfolding *to-bl-def word-log-defs bl-or-bin*
by (*simp add*: *word-ubin.eq-norm*)

lemma *bl-word-and*: *to-bl (v AND w) = map2 op & (to-bl v) (to-bl w)*
unfolding *to-bl-def word-log-defs bl-and-bin*
by (*simp add*: *word-ubin.eq-norm*)

lemma *word-lsb-alt*: *lsb (w::'a::len0 word) = test-bit w 0*
by (*auto simp*: *word-test-bit-def word-lsb-def*)

lemma *word-lsb-1-0 [simp]*: *lsb (1::'a::len word) & ~ lsb (0::'b::len0 word)*
unfolding *word-lsb-def uint-eq-0 uint-1* **by** *simp*

lemma *word-lsb-last*: *lsb (w::'a::len word) = last (to-bl w)*
apply (*unfold word-lsb-def uint-bl bin-to-bl-def*)
apply (*rule-tac bin=uint w* **in** *bin-exhaust*)
apply (*cases size w*)
 apply *auto*
 apply (*auto simp add*: *bin-to-bl-aux-alt*)
done

lemma *word-lsb-int*: *lsb w = (uint w mod 2 = 1)*
unfolding *word-lsb-def bin-last-def* **by** *auto*

lemma *word-msb-sint*: *msb w* = (*sint w* < *0*)
unfolding *word-msb-def sign-Min-lt-0* **..**

lemma *msb-word-of-int*:
msb (*word-of-int x*::'*a*::*len word*) = *bin-nth x* (*len-of TYPE*('*a*) − *1*)
unfolding *word-msb-def* **by** (*simp add*: *word-sbin.eq-norm bin-sign-lem*)

lemma *word-msb-numeral* [*simp*]:
msb (*numeral w*::'*a*::*len word*) = *bin-nth* (*numeral w*) (*len-of TYPE*('*a*) − *1*)
unfolding *word-numeral-alt* **by** (*rule msb-word-of-int*)

lemma *word-msb-neg-numeral* [*simp*]:
msb (− *numeral w*::'*a*::*len word*) = *bin-nth* (− *numeral w*) (*len-of TYPE*('*a*) − *1*)
unfolding *word-neg-numeral-alt* **by** (*rule msb-word-of-int*)

lemma *word-msb-0* [*simp*]: ¬ *msb* (*0*::'*a*::*len word*)
unfolding *word-msb-def* **by** *simp*

lemma *word-msb-1* [*simp*]: *msb* (*1*::'*a*::*len word*) ⟷ *len-of TYPE*('*a*) = *1*
unfolding *word-1-wi msb-word-of-int eq-iff* [**where** '*a*=*nat*]
by (*simp add*: *Suc-le-eq*)

lemma *word-msb-nth*:
msb (*w*::'*a*::*len word*) = *bin-nth* (*uint w*) (*len-of TYPE*('*a*) − *1*)
unfolding *word-msb-def sint-uint* **by** (*simp add*: *bin-sign-lem*)

lemma *word-msb-alt*: *msb* (*w*::'*a*::*len word*) = *hd* (*to-bl w*)
apply (*unfold word-msb-nth uint-bl*)
apply (*subst hd-conv-nth*)
apply (*rule length-greater-0-conv* [*THEN iffD1*])
apply *simp*
apply (*simp add* : *nth-bin-to-bl word-size*)
done

lemma *word-set-nth* [*simp*]:
set-bit w n (*test-bit w n*) = (*w*::'*a*::*len0 word*)
unfolding *word-test-bit-def word-set-bit-def* **by** *auto*

lemma *bin-nth-uint'*:
bin-nth (*uint w*) *n* = (*rev* (*bin-to-bl* (*size w*) (*uint w*)) ! *n* & *n* < *size w*)
apply (*unfold word-size*)
apply (*safe elim*!: *bin-nth-uint-imp*)
apply (*frule bin-nth-uint-imp*)
apply (*fast dest*!: *bin-nth-bl*)+
done

lemmas *bin-nth-uint* = *bin-nth-uint'* [*unfolded word-size*]

lemma *test-bit-bl*: *w* !! *n* = (*rev* (*to-bl w*) ! *n* & *n* < *size w*)
  unfolding *to-bl-def word-test-bit-def word-size*
  by (*rule bin-nth-uint*)

lemma *to-bl-nth*: *n* < *size w* ⟹ *to-bl w* ! *n* = *w* !! (*size w* − *Suc n*)
  apply (*unfold test-bit-bl*)
  apply *clarsimp*
  apply (*rule trans*)
   apply (*rule nth-rev-alt*)
   apply (*auto simp add*: *word-size*)
  done

lemma *test-bit-set*:
  fixes *w* :: *'a*::*len0 word*
  shows (*set-bit w n x*) !! *n* = (*n* < *size w* & *x*)
  unfolding *word-size word-test-bit-def word-set-bit-def*
  by (*clarsimp simp add* : *word-ubin.eq-norm nth-bintr*)

lemma *test-bit-set-gen*:
  fixes *w* :: *'a*::*len0 word*
  shows *test-bit* (*set-bit w n x*) *m* =
        (*if m* = *n then n* < *size w* & *x else test-bit w m*)
  apply (*unfold word-size word-test-bit-def word-set-bit-def*)
  apply (*clarsimp simp add*: *word-ubin.eq-norm nth-bintr bin-nth-sc-gen*)
  apply (*auto elim*!: *test-bit-size* [*unfolded word-size*]
              simp add: *word-test-bit-def* [*symmetric*])
  done

lemma *of-bl-rep-False*: *of-bl* (*replicate n False* @ *bs*) = *of-bl bs*
  unfolding *of-bl-def bl-to-bin-rep-F* **by** *auto*

lemma *msb-nth*:
  fixes *w* :: *'a*::*len word*
  shows *msb w* = *w* !! (*len-of TYPE*(*'a*) − *1*)
  unfolding *word-msb-nth word-test-bit-def* **by** *simp*

lemmas *msb0* = *len-gt-0* [*THEN diff-Suc-less*, *THEN word-ops-nth-size* [*unfolded word-size*]]
lemmas *msb1* = *msb0* [**where** *i* = *0*]
lemmas *word-ops-msb* = *msb1* [*unfolded msb-nth* [*symmetric*, *unfolded One-nat-def*]]

lemmas *lsb0* = *len-gt-0* [*THEN word-ops-nth-size* [*unfolded word-size*]]
lemmas *word-ops-lsb* = *lsb0* [*unfolded word-lsb-alt*]

lemma *td-ext-nth* [*OF refl refl refl*, *unfolded word-size*]:
  n = *size* (*w*::*'a*::*len0 word*) ⟹ *ofn* = *set-bits* ⟹ [*w*, *ofn g*] = *l* ⟹
   td-ext test-bit ofn {*f*. *ALL i*. *f i* −−> *i* < *n*} (%*h i*. *h i* & *i* < *n*)
  apply (*unfold word-size td-ext-def'*)
  apply *safe*

apply (*rule-tac* [*3*] *ext*)
apply (*rule-tac* [*4*] *ext*)
apply (*unfold word-size of-nth-def test-bit-bl*)
apply *safe*
defer
apply (*clarsimp simp*: *word-bl.Abs-inverse*)+
apply (*rule word-bl.Rep-inverse'*)
apply (*rule sym* [*THEN trans*])
apply (*rule bl-of-nth-nth*)
apply *simp*
apply (*rule bl-of-nth-inj*)
apply (*clarsimp simp add* : *test-bit-bl word-size*)
done

interpretation *test-bit*:
td-ext op !! :: *'a*::*len0 word* => *nat* => *bool*
set-bits
{*f*. ∀ *i*. *f i* ⟶ *i* < *len-of TYPE*(*'a*::*len0*)}
(λ*h i*. *h i* ∧ *i* < *len-of TYPE*(*'a*::*len0*))
by (*rule td-ext-nth*)

lemmas *td-nth* = *test-bit.td-thm*

lemma *word-set-set-same* [*simp*]:
fixes *w* :: *'a*::*len0 word*
shows *set-bit* (*set-bit w n x*) *n y* = *set-bit w n y*
by (*rule word-eqI*) (*simp add* : *test-bit-set-gen word-size*)

lemma *word-set-set-diff*:
fixes *w* :: *'a*::*len0 word*
assumes *m* ~= *n*
shows *set-bit* (*set-bit w m x*) *n y* = *set-bit* (*set-bit w n y*) *m x*
by (*rule word-eqI*) (*clarsimp simp add*: *test-bit-set-gen word-size assms*)

lemma *nth-sint*:
fixes *w* :: *'a*::*len word*
defines *l* ≡ *len-of TYPE* (*'a*)
shows *bin-nth* (*sint w*) *n* = (*if n* < *l* − *1 then w* !! *n else w* !! (*l* − *1*))
unfolding *sint-uint l-def*
by (*clarsimp simp add*: *nth-sbintr word-test-bit-def* [*symmetric*])

lemma *word-lsb-numeral* [*simp*]:
lsb (*numeral bin* :: *'a* :: *len word*) ⟷ *bin-last* (*numeral bin*)
unfolding *word-lsb-alt test-bit-numeral* **by** *simp*

lemma *word-lsb-neg-numeral* [*simp*]:
lsb (− *numeral bin* :: *'a* :: *len word*) ⟷ *bin-last* (− *numeral bin*)
unfolding *word-lsb-alt test-bit-neg-numeral* **by** *simp*

lemma *set-bit-word-of-int*:
set-bit (*word-of-int x*) *n b* = *word-of-int* (*bin-sc n b x*)
unfolding *word-set-bit-def*
apply (*rule word-eqI*)
apply (*simp add*: *word-size bin-nth-sc-gen word-ubin.eq-norm nth-bintr*)
done

lemma *word-set-numeral* [*simp*]:
set-bit (*numeral bin*::′*a*::*len0 word*) *n b* =
word-of-int (*bin-sc n b* (*numeral bin*))
unfolding *word-numeral-alt* **by** (*rule set-bit-word-of-int*)

lemma *word-set-neg-numeral* [*simp*]:
set-bit (− *numeral bin*::′*a*::*len0 word*) *n b* =
word-of-int (*bin-sc n b* (− *numeral bin*))
unfolding *word-neg-numeral-alt* **by** (*rule set-bit-word-of-int*)

lemma *word-set-bit-0* [*simp*]:
set-bit 0 n b = *word-of-int* (*bin-sc n b 0*)
unfolding *word-0-wi* **by** (*rule set-bit-word-of-int*)

lemma *word-set-bit-1* [*simp*]:
set-bit 1 n b = *word-of-int* (*bin-sc n b 1*)
unfolding *word-1-wi* **by** (*rule set-bit-word-of-int*)

lemma *setBit-no* [*simp*]:
setBit (*numeral bin*) *n* = *word-of-int* (*bin-sc n True* (*numeral bin*))
by (*simp add*: *setBit-def*)

lemma *clearBit-no* [*simp*]:
clearBit (*numeral bin*) *n* = *word-of-int* (*bin-sc n False* (*numeral bin*))
by (*simp add*: *clearBit-def*)

lemma *to-bl-n1*:
to-bl (−*1*::′*a*::*len0 word*) = *replicate* (*len-of TYPE* (′*a*)) *True*
apply (*rule word-bl.Abs-inverse*′)
apply *simp*
apply (*rule word-eqI*)
apply (*clarsimp simp add*: *word-size*)
apply (*auto simp add*: *word-bl.Abs-inverse test-bit-bl word-size*)
done

lemma *word-msb-n1* [*simp*]: *msb* (−*1*::′*a*::*len word*)
unfolding *word-msb-alt to-bl-n1* **by** *simp*

lemma *word-set-nth-iff*:
(*set-bit w n b* = *w*) = (*w* !! *n* = *b* | *n* >= *size* (*w*::′*a*::*len0 word*))
apply (*rule iffI*)
apply (*rule disjCI*)

```
   apply (drule word-eqD)
   apply (erule sym [THEN trans])
   apply (simp add: test-bit-set)
  apply (erule disjE)
   apply clarsimp
  apply (rule word-eqI)
  apply (clarsimp simp add : test-bit-set-gen)
  apply (drule test-bit-size)
  apply force
  done

lemma test-bit-2p:
  (word-of-int (2 ^ n)::'a::len word) !! m ⟷ m = n ∧ m < len-of TYPE('a)
  unfolding word-test-bit-def
  by (auto simp add: word-ubin.eq-norm nth-bintr nth-2p-bin)

lemma nth-w2p:
  ((2::'a::len word) ^ n) !! m ⟷ m = n ∧ m < len-of TYPE('a::len)
  unfolding test-bit-2p [symmetric] word-of-int [symmetric]
  by (simp add: of-int-power)

lemma uint-2p:
  (0::'a::len word) < 2 ^ n ⟹ uint (2 ^ n::'a::len word) = 2 ^ n
  apply (unfold word-arith-power-alt)
  apply (case-tac len-of TYPE ('a))
   apply clarsimp
  apply (case-tac nat)
   apply clarsimp
   apply (case-tac n)
    apply clarsimp
   apply clarsimp
  apply (drule word-gt-0 [THEN iffD1])
  apply (safe intro!: word-eqI)
  apply (auto simp add: nth-2p-bin)
  apply (erule notE)
  apply (simp (no-asm-use) add: uint-word-of-int word-size)
  apply (subst mod-pos-pos-trivial)
  apply simp
  apply (rule power-strict-increasing)
  apply simp-all
  done

lemma word-of-int-2p: (word-of-int (2 ^ n) :: 'a :: len word) = 2 ^ n
  apply (unfold word-arith-power-alt)
  apply (case-tac len-of TYPE ('a))
   apply clarsimp
  apply (case-tac nat)
   apply (rule word-ubin.norm-eq-iff [THEN iffD1])
   apply (rule box-equals)
```

apply (*rule-tac* [*2*] *bintr-ariths* (*1*))+
apply *simp*
apply *simp*
done

lemma *bang-is-le*: *x* !! *m* ⟹ *2* ^ *m* <= (*x* :: ′*a* :: *len word*)
apply (*rule xtr3*)
apply (*rule-tac* [*2*] *y* = *x* **in** *le-word-or2*)
apply (*rule word-eqI*)
apply (*auto simp add*: *word-ao-nth nth-w2p word-size*)
done

lemma *word-clr-le*:
fixes *w* :: ′*a*::*len0 word*
shows *w* >= *set-bit w n False*
apply (*unfold word-set-bit-def word-le-def word-ubin.eq-norm*)
apply (*rule order-trans*)
apply (*rule bintr-bin-clr-le*)
apply *simp*
done

lemma *word-set-ge*:
fixes *w* :: ′*a*::*len word*
shows *w* <= *set-bit w n True*
apply (*unfold word-set-bit-def word-le-def word-ubin.eq-norm*)
apply (*rule order-trans* [*OF - bintr-bin-set-ge*])
apply *simp*
done

16.25 Shifting, Rotating, and Splitting Words

lemma *shiftl1-wi* [*simp*]: *shiftl1* (*word-of-int w*) = *word-of-int* (*w BIT False*)
unfolding *shiftl1-def*
apply (*simp add*: *word-ubin.norm-eq-iff* [*symmetric*] *word-ubin.eq-norm*)
apply (*subst refl* [*THEN bintrunc-BIT-I*, *symmetric*])
apply (*subst bintrunc-bintrunc-min*)
apply *simp*
done

lemma *shiftl1-numeral* [*simp*]:
shiftl1 (*numeral w*) = *numeral* (*Num.Bit0 w*)
unfolding *word-numeral-alt shiftl1-wi* **by** *simp*

lemma *shiftl1-neg-numeral* [*simp*]:
shiftl1 (− *numeral w*) = − *numeral* (*Num.Bit0 w*)
unfolding *word-neg-numeral-alt shiftl1-wi* **by** *simp*

lemma *shiftl1-0* [*simp*] : *shiftl1 0* = *0*
unfolding *shiftl1-def* **by** *simp*

lemma *shiftl1-def-u*: *shiftl1 w* = *word-of-int* (*uint w BIT False*)
by (*simp only*: *shiftl1-def*)

lemma *shiftl1-def-s*: *shiftl1 w* = *word-of-int* (*sint w BIT False*)
unfolding *shiftl1-def Bit-B0 wi-hom-syms* **by** *simp*

lemma *shiftr1-0* [*simp*]: *shiftr1 0* = *0*
unfolding *shiftr1-def* **by** *simp*

lemma *sshiftr1-0* [*simp*]: *sshiftr1 0* = *0*
unfolding *sshiftr1-def* **by** *simp*

lemma *sshiftr1-n1* [*simp*] : *sshiftr1* (− *1*) = − *1*
unfolding *sshiftr1-def* **by** *simp*

lemma *shiftl-0* [*simp*] : (*0*::′*a*::*len0 word*) << *n* = *0*
unfolding *shiftl-def* **by** (*induct n*) *auto*

lemma *shiftr-0* [*simp*] : (*0*::′*a*::*len0 word*) >> *n* = *0*
unfolding *shiftr-def* **by** (*induct n*) *auto*

lemma *sshiftr-0* [*simp*] : *0* >>> *n* = *0*
unfolding *sshiftr-def* **by** (*induct n*) *auto*

lemma *sshiftr-n1* [*simp*] : −*1* >>> *n* = −*1*
unfolding *sshiftr-def* **by** (*induct n*) *auto*

lemma *nth-shiftl1*: *shiftl1 w* !! *n* = (*n* < *size w* & *n* > *0* & *w* !! (*n* − *1*))
apply (*unfold shiftl1-def word-test-bit-def*)
apply (*simp add*: *nth-bintr word-ubin.eq-norm word-size*)
apply (*cases n*)
 apply *auto*
done

lemma *nth-shiftl′* [*rule-format*]:
ALL n. ((*w*::′*a*::*len0 word*) << *m*) !! *n* = (*n* < *size w* & *n* >= *m* & *w* !! (*n* − *m*))
apply (*unfold shiftl-def*)
apply (*induct m*)
 apply (*force elim*!: *test-bit-size*)
apply (*clarsimp simp add* : *nth-shiftl1 word-size*)
apply *arith*
done

lemmas *nth-shiftl* = *nth-shiftl′* [*unfolded word-size*]

lemma *nth-shiftr1*: *shiftr1 w* !! *n* = *w* !! *Suc n*
apply (*unfold shiftr1-def word-test-bit-def*)

```
  apply (simp add: nth-bintr word-ubin.eq-norm)
  apply safe
  apply (drule bin-nth.Suc [THEN iffD2, THEN bin-nth-uint-imp])
  apply simp
  done

lemma nth-shiftr:
  ⋀n. ((w::'a::len0 word) >> m) !! n = w !! (n + m)
  apply (unfold shiftr-def)
  apply (induct m)
   apply (auto simp add : nth-shiftr1)
  done

lemma uint-shiftr1: uint (shiftr1 w) = bin-rest (uint w)
  apply (unfold shiftr1-def word-ubin.eq-norm bin-rest-trunc-i)
  apply (subst bintr-uint [symmetric, OF order-refl])
  apply (simp only : bintrunc-bintrunc-l)
  apply simp
  done

lemma nth-sshiftr1:
  sshiftr1 w !! n = (if n = size w - 1 then w !! n else w !! Suc n)
  apply (unfold sshiftr1-def word-test-bit-def)
  apply (simp add: nth-bintr word-ubin.eq-norm
                   bin-nth.Suc [symmetric] word-size
             del: bin-nth.simps)
  apply (simp add: nth-bintr uint-sint del : bin-nth.simps)
  apply (auto simp add: bin-nth-sint)
  done

lemma nth-sshiftr [rule-format] :
  ALL n. sshiftr w m !! n = (n < size w &
    (if n + m >= size w then w !! (size w - 1) else w !! (n + m)))
  apply (unfold sshiftr-def)
  apply (induct-tac m)
   apply (simp add: test-bit-bl)
  apply (clarsimp simp add: nth-sshiftr1 word-size)
  apply safe
       apply arith
      apply arith
     apply (erule thin-rl)
     apply (case-tac n)
      apply safe
      apply simp
     apply simp
    apply (erule thin-rl)
    apply (case-tac n)
```

```
    apply safe
    apply simp
   apply simp
  apply arith+
  done

lemma shiftr1-div-2: uint (shiftr1 w) = uint w div 2
  apply (unfold shiftr1-def bin-rest-def)
  apply (rule word-uint.Abs-inverse)
  apply (simp add: uints-num pos-imp-zdiv-nonneg-iff)
  apply (rule xtr7)
   prefer 2
   apply (rule zdiv-le-dividend)
    apply auto
  done

lemma sshiftr1-div-2: sint (sshiftr1 w) = sint w div 2
  apply (unfold sshiftr1-def bin-rest-def [symmetric])
  apply (simp add: word-sbin.eq-norm)
  apply (rule trans)
   defer
   apply (subst word-sbin.norm-Rep [symmetric])
   apply (rule refl)
  apply (subst word-sbin.norm-Rep [symmetric])
  apply (unfold One-nat-def)
  apply (rule sbintrunc-rest)
  done

lemma shiftr-div-2n: uint (shiftr w n) = uint w div 2 ^ n
  apply (unfold shiftr-def)
  apply (induct n)
   apply simp
  apply (simp add: shiftr1-div-2 mult.commute
                 zdiv-zmult2-eq [symmetric])
  done

lemma sshiftr-div-2n: sint (sshiftr w n) = sint w div 2 ^ n
  apply (unfold sshiftr-def)
  apply (induct n)
   apply simp
  apply (simp add: sshiftr1-div-2 mult.commute
                 zdiv-zmult2-eq [symmetric])
  done
```

16.25.1 shift functions in terms of lists of bools

```
lemmas bshiftr1-numeral [simp] =
  bshiftr1-def [where w=numeral w, unfolded to-bl-numeral] for w
```

lemma *bshiftr1-bl*: *to-bl* (*bshiftr1 b w*) = *b* # *butlast* (*to-bl w*)
unfolding *bshiftr1-def* **by** (*rule word-bl.Abs-inverse*) *simp*

lemma *shiftl1-of-bl*: *shiftl1* (*of-bl bl*) = *of-bl* (*bl* @ [*False*])
by (*simp add*: *of-bl-def bl-to-bin-append*)

lemma *shiftl1-bl*: *shiftl1* (*w*::'*a*::*len0 word*) = *of-bl* (*to-bl w* @ [*False*])
proof −
have *shiftl1 w* = *shiftl1* (*of-bl* (*to-bl w*)) **by** *simp*
also have ... = *of-bl* (*to-bl w* @ [*False*]) **by** (*rule shiftl1-of-bl*)
finally show *?thesis* **.**
qed

lemma *bl-shiftl1*:
to-bl (*shiftl1* (*w* :: '*a* :: *len word*)) = *tl* (*to-bl w*) @ [*False*]
apply (*simp add*: *shiftl1-bl word-rep-drop drop-Suc drop-Cons'*)
apply (*fast intro*!: *Suc-leI*)
done

lemma *bl-shiftl1'*:
to-bl (*shiftl1 w*) = *tl* (*to-bl w* @ [*False*])
unfolding *shiftl1-bl*
by (*simp add*: *word-rep-drop drop-Suc del*: *drop-append*)

lemma *shiftr1-bl*: *shiftr1 w* = *of-bl* (*butlast* (*to-bl w*))
apply (*unfold shiftr1-def uint-bl of-bl-def*)
apply (*simp add*: *butlast-rest-bin word-size*)
apply (*simp add*: *bin-rest-trunc* [*symmetric*, *unfolded One-nat-def*])
done

lemma *bl-shiftr1*:
to-bl (*shiftr1* (*w* :: '*a* :: *len word*)) = *False* # *butlast* (*to-bl w*)
unfolding *shiftr1-bl*
by (*simp add* : *word-rep-drop len-gt-0* [*THEN Suc-leI*])

lemma *bl-shiftr1'*:
to-bl (*shiftr1 w*) = *butlast* (*False* # *to-bl w*)
apply (*rule word-bl.Abs-inverse'*)
apply (*simp del*: *butlast.simps*)
apply (*simp add*: *shiftr1-bl of-bl-def*)
done

lemma *shiftl1-rev*:
shiftl1 w = *word-reverse* (*shiftr1* (*word-reverse w*))
apply (*unfold word-reverse-def*)
apply (*rule word-bl.Rep-inverse'* [*symmetric*])
apply (*simp add*: *bl-shiftl1' bl-shiftr1' word-bl.Abs-inverse*)

```
  apply (cases to-bl w)
   apply auto
  done

lemma shiftl-rev:
  shiftl w n = word-reverse (shiftr (word-reverse w) n)
  apply (unfold shiftl-def shiftr-def)
  apply (induct n)
   apply (auto simp add : shiftl1-rev)
  done

lemma rev-shiftl: word-reverse w << n = word-reverse (w >> n)
  by (simp add: shiftl-rev)

lemma shiftr-rev: w >> n = word-reverse (word-reverse w << n)
  by (simp add: rev-shiftl)

lemma rev-shiftr: word-reverse w >> n = word-reverse (w << n)
  by (simp add: shiftr-rev)

lemma bl-sshiftr1:
  to-bl (sshiftr1 (w :: 'a :: len word)) = hd (to-bl w) # butlast (to-bl w)
  apply (unfold sshiftr1-def uint-bl word-size)
  apply (simp add: butlast-rest-bin word-ubin.eq-norm)
  apply (simp add: sint-uint)
  apply (rule nth-equalityI)
   apply clarsimp
  apply clarsimp
  apply (case-tac i)
   apply (simp-all add: hd-conv-nth length-0-conv [symmetric]
                        nth-bin-to-bl bin-nth.Suc [symmetric]
                        nth-sbintr
                   del: bin-nth.Suc)
   apply force
  apply (rule impI)
  apply (rule-tac f = bin-nth (uint w) in arg-cong)
  apply simp
  done

lemma drop-shiftr:
  drop n (to-bl ((w :: 'a :: len word) >> n)) = take (size w − n) (to-bl w)
  apply (unfold shiftr-def)
  apply (induct n)
   prefer 2
   apply (simp add: drop-Suc bl-shiftr1 butlast-drop [symmetric])
   apply (rule butlast-take [THEN trans])
  apply (auto simp: word-size)
  done
```

lemma *drop-sshiftr*:
drop n (*to-bl* ((*w* :: '*a* :: *len word*) >>> *n*)) = *take* (*size w* − *n*) (*to-bl w*)
apply (*unfold sshiftr-def*)
apply (*induct n*)
prefer *2*
apply (*simp add*: *drop-Suc bl-sshiftr1 butlast-drop* [*symmetric*])
apply (*rule butlast-take* [*THEN trans*])
apply (*auto simp*: *word-size*)
done

lemma *take-shiftr*:
n ≤ *size w* ⟹ *take n* (*to-bl* (*w* >> *n*)) = *replicate n False*
apply (*unfold shiftr-def*)
apply (*induct n*)
prefer *2*
apply (*simp add*: *bl-shiftr1' length-0-conv* [*symmetric*] *word-size*)
apply (*rule take-butlast* [*THEN trans*])
apply (*auto simp*: *word-size*)
done

lemma *take-sshiftr'* [*rule-format*] :
n <= *size* (*w* :: '*a* :: *len word*) −−> *hd* (*to-bl* (*w* >>> *n*)) = *hd* (*to-bl w*) &
take n (*to-bl* (*w* >>> *n*)) = *replicate n* (*hd* (*to-bl w*))
apply (*unfold sshiftr-def*)
apply (*induct n*)
prefer *2*
apply (*simp add*: *bl-sshiftr1*)
apply (*rule impI*)
apply (*rule take-butlast* [*THEN trans*])
apply (*auto simp*: *word-size*)
done

lemmas *hd-sshiftr* = *take-sshiftr'* [*THEN conjunct1*]
lemmas *take-sshiftr* = *take-sshiftr'* [*THEN conjunct2*]

lemma *atd-lem*: *take n xs* = *t* ⟹ *drop n xs* = *d* ⟹ *xs* = *t* @ *d*
by (*auto intro*: *append-take-drop-id* [*symmetric*])

lemmas *bl-shiftr* = *atd-lem* [*OF take-shiftr drop-shiftr*]
lemmas *bl-sshiftr* = *atd-lem* [*OF take-sshiftr drop-sshiftr*]

lemma *shiftl-of-bl*: *of-bl bl* << *n* = *of-bl* (*bl* @ *replicate n False*)
unfolding *shiftl-def*
by (*induct n*) (*auto simp*: *shiftl1-of-bl replicate-app-Cons-same*)

lemma *shiftl-bl*:
(*w*::'*a*::*len0 word*) << (*n*::*nat*) = *of-bl* (*to-bl w* @ *replicate n False*)
proof −
have *w* << *n* = *of-bl* (*to-bl w*) << *n* **by** *simp*

also have . . . = *of-bl* (*to-bl w* @ *replicate n False*) **by** (*rule shiftl-of-bl*)
finally show *?thesis* **.**
qed

lemmas *shiftl-numeral* [*simp*] = *shiftl-def* [**where** *w*=*numeral w*] **for** *w*

lemma *bl-shiftl*:
to-bl (*w* << *n*) = *drop n* (*to-bl w*) @ *replicate* (*min* (*size w*) *n*) *False*
by (*simp add*: *shiftl-bl word-rep-drop word-size*)

lemma *shiftl-zero-size*:
fixes *x* :: *'a*::*len0 word*
shows *size x* <= *n* ⟹ *x* << *n* = *0*
apply (*unfold word-size*)
apply (*rule word-eqI*)
apply (*clarsimp simp add*: *shiftl-bl word-size test-bit-of-bl nth-append*)
done

lemma *shiftl1-2t*: *shiftl1* (*w* :: *'a* :: *len word*) = *2* ∗ *w*
by (*simp add*: *shiftl1-def Bit-def wi-hom-mult* [*symmetric*])

lemma *shiftl1-p*: *shiftl1* (*w* :: *'a* :: *len word*) = *w* + *w*
by (*simp add*: *shiftl1-2t*)

lemma *shiftl-t2n*: *shiftl* (*w* :: *'a* :: *len word*) *n* = *2* ˆ *n* ∗ *w*
unfolding *shiftl-def*
by (*induct n*) (*auto simp*: *shiftl1-2t*)

lemma *shiftr1-bintr* [*simp*]:
(*shiftr1* (*numeral w*) :: *'a* :: *len0 word*) =
word-of-int (*bin-rest* (*bintrunc* (*len-of TYPE* (*'a*)) (*numeral w*)))
unfolding *shiftr1-def word-numeral-alt*
by (*simp add*: *word-ubin.eq-norm*)

lemma *sshiftr1-sbintr* [*simp*]:
(*sshiftr1* (*numeral w*) :: *'a* :: *len word*) =
word-of-int (*bin-rest* (*sbintrunc* (*len-of TYPE* (*'a*) − *1*) (*numeral w*)))
unfolding *sshiftr1-def word-numeral-alt*
by (*simp add*: *word-sbin.eq-norm*)

lemma *shiftr-no* [*simp*]:

(*numeral w*::*'a*::*len0 word*) >> *n* = *word-of-int*
((*bin-rest* ˆˆ *n*) (*bintrunc* (*len-of TYPE*(*'a*)) (*numeral w*)))
apply (*rule word-eqI*)
apply (*auto simp*: *nth-shiftr nth-rest-power-bin nth-bintr word-size*)
done

lemma *sshiftr-no* [*simp*]:

(*numeral w*::'*a*::*len word*) >>> *n* = *word-of-int*
((*bin-rest* ^^ *n*) (*sbintrunc* (*len-of TYPE*('*a*) − *1*) (*numeral w*)))
apply (*rule word-eqI*)
apply (*auto simp*: *nth-sshiftr nth-rest-power-bin nth-sbintr word-size*)
apply (*subgoal-tac na* + *n* = *len-of TYPE*('*a*) − *Suc 0*, *simp*, *simp*)+
done

lemma *shiftr1-bl-of*:
length bl ≤ *len-of TYPE*('*a*) ⟹
shiftr1 (*of-bl bl*::'*a*::*len0 word*) = *of-bl* (*butlast bl*)
by (*clarsimp simp*: *shiftr1-def of-bl-def butlast-rest-bl2bin*
word-ubin.eq-norm trunc-bl2bin)

lemma *shiftr-bl-of*:
length bl ≤ *len-of TYPE*('*a*) ⟹
(*of-bl bl*::'*a*::*len0 word*) >> *n* = *of-bl* (*take* (*length bl* − *n*) *bl*)
apply (*unfold shiftr-def*)
apply (*induct n*)
apply *clarsimp*
apply *clarsimp*
apply (*subst shiftr1-bl-of*)
apply *simp*
apply (*simp add*: *butlast-take*)
done

lemma *shiftr-bl*:
(*x*::'*a*::*len0 word*) >> *n* ≡ *of-bl* (*take* (*len-of TYPE*('*a*) − *n*) (*to-bl x*))
using *shiftr-bl-of* [**where** '*a*='*a*, *of to-bl x*] **by** *simp*

lemma *msb-shift*:
msb (*w*::'*a*::*len word*) ⟷ (*w* >> (*len-of TYPE*('*a*) − *1*)) ≠ *0*
apply (*unfold shiftr-bl word-msb-alt*)
apply (*simp add*: *word-size Suc-le-eq take-Suc*)
apply (*cases hd* (*to-bl w*))
apply (*auto simp*: *word-1-bl*
of-bl-rep-False [**where** *n*=*1* **and** *bs*=[], *simplified*])
done

lemma *zip-replicate*:
n ≥ *length ys* ⟹ *zip* (*replicate n x*) *ys* = *map* (λ*y*. (*x*, *y*)) *ys*
apply (*induct ys arbitrary*: *n*, *simp-all*)
apply (*case-tac n*, *simp-all*)
done

lemma *align-lem-or* [*rule-format*] :
ALL x m. *length x* = *n* + *m* −−> *length y* = *n* + *m* −−>

drop m x = replicate n False --> take m y = replicate m False -->
map2 op | x y = take m x @ drop m y
apply (*induct-tac y*)
apply *force*
apply *clarsimp*
apply (*case-tac x*, *force*)
apply (*case-tac m*, *auto*)
apply (*drule-tac t=length xs* **for** *xs* **in** *sym*)
apply (*clarsimp simp*: *map2-def zip-replicate o-def*)
done

lemma *align-lem-and* [*rule-format*] :
ALL x m. length x = n + m --> length y = n + m -->
drop m x = replicate n False --> take m y = replicate m False -->
map2 op & x y = replicate (n + m) False
apply (*induct-tac y*)
apply *force*
apply *clarsimp*
apply (*case-tac x*, *force*)
apply (*case-tac m*, *auto*)
apply (*drule-tac t=length xs* **for** *xs* **in** *sym*)
apply (*clarsimp simp*: *map2-def zip-replicate o-def map-replicate-const*)
done

lemma *aligned-bl-add-size* [*OF refl*]:
size x − n = m ⟹ n <= size x ⟹ drop m (to-bl x) = replicate n False ⟹
take m (to-bl y) = replicate m False ⟹
to-bl (x + y) = take m (to-bl x) @ drop m (to-bl y)
apply (*subgoal-tac x AND y = 0*)
prefer *2*
apply (*rule word-bl.Rep-eqD*)
apply (*simp add*: *bl-word-and*)
apply (*rule align-lem-and* [*THEN trans*])
apply (*simp-all add*: *word-size*)[*5*]
apply *simp*
apply (*subst word-plus-and-or* [*symmetric*])
apply (*simp add* : *bl-word-or*)
apply (*rule align-lem-or*)
apply (*simp-all add*: *word-size*)
done

16.25.2 Mask

lemma *nth-mask* [*OF refl*, *simp*]:
m = mask n ⟹ test-bit m i = (i < n & i < size m)
apply (*unfold mask-def test-bit-bl*)
apply (*simp only*: *word-1-bl* [*symmetric*] *shiftl-of-bl*)
apply (*clarsimp simp add*: *word-size*)
apply (*simp only*: *of-bl-def mask-lem word-of-int-hom-syms add-diff-cancel2*)

apply (*fold of-bl-def*)
apply (*simp add*: *word-1-bl*)
apply (*rule test-bit-of-bl* [*THEN trans*, *unfolded test-bit-bl word-size*])
apply *auto*
done

lemma *mask-bl*: *mask n* = *of-bl* (*replicate n True*)
by (*auto simp add* : *test-bit-of-bl word-size intro*: *word-eqI*)

lemma *mask-bin*: *mask n* = *word-of-int* (*bintrunc n* (− *1*))
by (*auto simp add*: *nth-bintr word-size intro*: *word-eqI*)

lemma *and-mask-bintr*: *w AND mask n* = *word-of-int* (*bintrunc n* (*uint w*))
apply (*rule word-eqI*)
apply (*simp add*: *nth-bintr word-size word-ops-nth-size*)
apply (*auto simp add*: *test-bit-bin*)
done

lemma *and-mask-wi*: *word-of-int i AND mask n* = *word-of-int* (*bintrunc n i*)
by (*auto simp add*: *nth-bintr word-size word-ops-nth-size word-eq-iff*)

lemma *and-mask-no*: *numeral i AND mask n* = *word-of-int* (*bintrunc n* (*numeral i*))
unfolding *word-numeral-alt* **by** (*rule and-mask-wi*)

lemma *bl-and-mask′*:
to-bl (*w AND mask n* :: *′a* :: *len word*) =
replicate (*len-of TYPE*(*′a*) − *n*) *False* @
drop (*len-of TYPE*(*′a*) − *n*) (*to-bl w*)
apply (*rule nth-equalityI*)
apply *simp*
apply (*clarsimp simp add*: *to-bl-nth word-size*)
apply (*simp add*: *word-size word-ops-nth-size*)
apply (*auto simp add*: *word-size test-bit-bl nth-append nth-rev*)
done

lemma *and-mask-mod-2p*: *w AND mask n* = *word-of-int* (*uint w mod 2* ˆ *n*)
by (*simp only*: *and-mask-bintr bintrunc-mod2p*)

lemma *and-mask-lt-2p*: *uint* (*w AND mask n*) < *2* ˆ *n*
apply (*simp add*: *and-mask-bintr word-ubin.eq-norm*)
apply (*simp add*: *bintrunc-mod2p*)
apply (*rule xtr8*)
prefer *2*
apply (*rule pos-mod-bound*)
apply *auto*
done

lemma *eq-mod-iff*: $0 < (n::int) \Longrightarrow b = b \bmod n \longleftrightarrow 0 \leq b \wedge b < n$

by (*simp add*: *int-mod-lem eq-sym-conv*)

lemma *mask-eq-iff*: (*w AND mask n*) = *w* ⟷ *uint w* < *2* ˆ *n*
apply (*simp add*: *and-mask-bintr*)
apply (*simp add*: *word-ubin.inverse-norm*)
apply (*simp add*: *eq-mod-iff bintrunc-mod2p min-def*)
apply (*fast intro*!: *lt2p-lem*)
done

lemma *and-mask-dvd*: *2* ˆ *n dvd uint w* = (*w AND mask n* = *0*)
apply (*simp add*: *dvd-eq-mod-eq-0 and-mask-mod-2p*)
apply (*simp add*: *word-uint.norm-eq-iff* [*symmetric*] *word-of-int-homs*
del: *word-of-int-0*)
apply (*subst word-uint.norm-Rep* [*symmetric*])
apply (*simp only*: *bintrunc-bintrunc-min bintrunc-mod2p* [*symmetric*] *min-def*)
apply *auto*
done

lemma *and-mask-dvd-nat*: *2* ˆ *n dvd unat w* = (*w AND mask n* = *0*)
apply (*unfold unat-def*)
apply (*rule trans* [*OF* - *and-mask-dvd*])
apply (*unfold dvd-def*)
apply *auto*
apply (*drule uint-ge-0* [*THEN nat-int.Abs-inverse′* [*simplified*], *symmetric*])
apply (*simp add* : *of-nat-mult of-nat-power*)
apply (*simp add* : *nat-mult-distrib nat-power-eq*)
done

lemma *word-2p-lem*:
n < *size w* ⟹ *w* < *2* ˆ *n* = (*uint* (*w* :: ′*a* :: *len word*) < *2* ˆ *n*)
apply (*unfold word-size word-less-alt word-numeral-alt*)
apply (*clarsimp simp add*: *word-of-int-power-hom word-uint.eq-norm*
mod-pos-pos-trivial
simp del: *word-of-int-numeral*)
done

lemma *less-mask-eq*: *x* < *2* ˆ *n* ⟹ *x AND mask n* = (*x* :: ′*a* :: *len word*)
apply (*unfold word-less-alt word-numeral-alt*)
apply (*clarsimp simp add*: *and-mask-mod-2p word-of-int-power-hom*
word-uint.eq-norm
simp del: *word-of-int-numeral*)
apply (*drule xtr8* [*rotated*])
apply (*rule int-mod-le*)
apply (*auto simp add* : *mod-pos-pos-trivial*)
done

lemmas *mask-eq-iff-w2p* = *trans* [*OF mask-eq-iff word-2p-lem* [*symmetric*]]

lemmas *and-mask-less′* = *iffD2* [*OF word-2p-lem and-mask-lt-2p*, *simplified word-size*]

lemma *and-mask-less-size*: $n < size\ x \Longrightarrow x\ AND\ mask\ n < 2\hat{}n$
unfolding *word-size* **by** (*erule and-mask-less'*)

lemma *word-mod-2p-is-mask* [*OF refl*]:
$c = 2\ \hat{}\ n \Longrightarrow c > 0 \Longrightarrow x\ mod\ c = (x :: 'a :: len\ word)\ AND\ mask\ n$
by (*clarsimp simp add*: *word-mod-def uint-2p and-mask-mod-2p*)

lemma *mask-eqs*:
$(a\ AND\ mask\ n) + b\ AND\ mask\ n = a + b\ AND\ mask\ n$
$a + (b\ AND\ mask\ n)\ AND\ mask\ n = a + b\ AND\ mask\ n$
$(a\ AND\ mask\ n) - b\ AND\ mask\ n = a - b\ AND\ mask\ n$
$a - (b\ AND\ mask\ n)\ AND\ mask\ n = a - b\ AND\ mask\ n$
$a * (b\ AND\ mask\ n)\ AND\ mask\ n = a * b\ AND\ mask\ n$
$(b\ AND\ mask\ n) * a\ AND\ mask\ n = b * a\ AND\ mask\ n$
$(a\ AND\ mask\ n) + (b\ AND\ mask\ n)\ AND\ mask\ n = a + b\ AND\ mask\ n$
$(a\ AND\ mask\ n) - (b\ AND\ mask\ n)\ AND\ mask\ n = a - b\ AND\ mask\ n$
$(a\ AND\ mask\ n) * (b\ AND\ mask\ n)\ AND\ mask\ n = a * b\ AND\ mask\ n$
$-\ (a\ AND\ mask\ n)\ AND\ mask\ n = -\ a\ AND\ mask\ n$
word-succ $(a\ AND\ mask\ n)\ AND\ mask\ n =$ *word-succ* $a\ AND\ mask\ n$
word-pred $(a\ AND\ mask\ n)\ AND\ mask\ n =$ *word-pred* $a\ AND\ mask\ n$
using *word-of-int-Ex* [**where** $x=a$] *word-of-int-Ex* [**where** $x=b$]
by (*auto simp*: *and-mask-wi bintr-ariths bintr-arith1s word-of-int-homs*)

lemma *mask-power-eq*:
$(x\ AND\ mask\ n)\ \hat{}\ k\ AND\ mask\ n = x\ \hat{}\ k\ AND\ mask\ n$
using *word-of-int-Ex* [**where** $x=x$]
by (*clarsimp simp*: *and-mask-wi word-of-int-power-hom bintr-ariths*)

16.25.3 Revcast

lemmas *revcast-def'* = *revcast-def* [*simplified*]
lemmas *revcast-def''* = *revcast-def'* [*simplified word-size*]
lemmas *revcast-no-def* [*simp*] = *revcast-def'* [**where** *w=numeral w*, *unfolded word-size*]
for *w*

lemma *to-bl-revcast*:
to-bl (*revcast w* :: *'a* :: *len0 word*) =
takefill False (*len-of TYPE* (*'a*)) (*to-bl w*)
apply (*unfold revcast-def' word-size*)
apply (*rule word-bl.Abs-inverse*)
apply *simp*
done

lemma *revcast-rev-ucast* [*OF refl refl refl*]:
cs = [*rc*, *uc*] $\Longrightarrow$ *rc* = *revcast* (*word-reverse w*) $\Longrightarrow$ *uc* = *ucast w* $\Longrightarrow$
rc = *word-reverse uc*
apply (*unfold ucast-def revcast-def' Let-def word-reverse-def*)
apply (*clarsimp simp add* : *to-bl-of-bin takefill-bintrunc*)

apply (*simp add* : *word-bl.Abs-inverse word-size*)
done

lemma *revcast-ucast*: *revcast w* = *word-reverse* (*ucast* (*word-reverse w*))
using *revcast-rev-ucast* [*of word-reverse w*] **by** *simp*

lemma *ucast-revcast*: *ucast w* = *word-reverse* (*revcast* (*word-reverse w*))
by (*fact revcast-rev-ucast* [*THEN word-rev-gal'*])

lemma *ucast-rev-revcast*: *ucast* (*word-reverse w*) = *word-reverse* (*revcast w*)
by (*fact revcast-ucast* [*THEN word-rev-gal'*])

— linking revcast and cast via shift

lemmas *wsst-TYs* = *source-size target-size word-size*

lemma *revcast-down-uu* [*OF refl*]:
rc = *revcast* ⟹ *source-size rc* = *target-size rc* + *n* ⟹
rc (*w* :: *'a* :: *len word*) = *ucast* (*w* >> *n*)
apply (*simp add*: *revcast-def'*)
apply (*rule word-bl.Rep-inverse'*)
apply (*rule trans*, *rule ucast-down-drop*)
prefer *2*
apply (*rule trans*, *rule drop-shiftr*)
apply (*auto simp*: *takefill-alt wsst-TYs*)
done

lemma *revcast-down-us* [*OF refl*]:
rc = *revcast* ⟹ *source-size rc* = *target-size rc* + *n* ⟹
rc (*w* :: *'a* :: *len word*) = *ucast* (*w* >>> *n*)
apply (*simp add*: *revcast-def'*)
apply (*rule word-bl.Rep-inverse'*)
apply (*rule trans*, *rule ucast-down-drop*)
prefer *2*
apply (*rule trans*, *rule drop-sshiftr*)
apply (*auto simp*: *takefill-alt wsst-TYs*)
done

lemma *revcast-down-su* [*OF refl*]:
rc = *revcast* ⟹ *source-size rc* = *target-size rc* + *n* ⟹
rc (*w* :: *'a* :: *len word*) = *scast* (*w* >> *n*)
apply (*simp add*: *revcast-def'*)
apply (*rule word-bl.Rep-inverse'*)
apply (*rule trans*, *rule scast-down-drop*)
prefer *2*
apply (*rule trans*, *rule drop-shiftr*)
apply (*auto simp*: *takefill-alt wsst-TYs*)
done

lemma *revcast-down-ss* [*OF refl*]:
rc = *revcast* ⟹ *source-size rc* = *target-size rc* + *n* ⟹
rc (*w* :: ′*a* :: *len word*) = *scast* (*w* >>> *n*)
apply (*simp add*: *revcast-def* ′)
apply (*rule word-bl.Rep-inverse*′)
apply (*rule trans*, *rule scast-down-drop*)
prefer *2*
apply (*rule trans*, *rule drop-sshiftr*)
apply (*auto simp*: *takefill-alt wsst-TYs*)
done

lemma *cast-down-rev*:
uc = *ucast* ⟹ *source-size uc* = *target-size uc* + *n* ⟹
uc w = *revcast* ((*w* :: ′*a* :: *len word*) << *n*)
apply (*unfold shiftl-rev*)
apply *clarify*
apply (*simp add*: *revcast-rev-ucast*)
apply (*rule word-rev-gal*′)
apply (*rule trans* [*OF - revcast-rev-ucast*])
apply (*rule revcast-down-uu* [*symmetric*])
apply (*auto simp add*: *wsst-TYs*)
done

lemma *revcast-up* [*OF refl*]:
rc = *revcast* ⟹ *source-size rc* + *n* = *target-size rc* ⟹
rc w = (*ucast w* :: ′*a* :: *len word*) << *n*
apply (*simp add*: *revcast-def* ′)
apply (*rule word-bl.Rep-inverse*′)
apply (*simp add*: *takefill-alt*)
apply (*rule bl-shiftl* [*THEN trans*])
apply (*subst ucast-up-app*)
apply (*auto simp add*: *wsst-TYs*)
done

lemmas *rc1* = *revcast-up* [*THEN*
revcast-rev-ucast [*symmetric*, *THEN trans*, *THEN word-rev-gal*, *symmetric*]]
lemmas *rc2* = *revcast-down-uu* [*THEN*
revcast-rev-ucast [*symmetric*, *THEN trans*, *THEN word-rev-gal*, *symmetric*]]

lemmas *ucast-up* =
rc1 [*simplified rev-shiftr* [*symmetric*] *revcast-ucast* [*symmetric*]]
lemmas *ucast-down* =
rc2 [*simplified rev-shiftr revcast-ucast* [*symmetric*]]

16.25.4 Slices

lemma *slice1-no-bin* [*simp*]:

slice1 n (*numeral w* :: ′*b word*) = *of-bl* (*takefill False n* (*bin-to-bl* (*len-of TYPE*(′*b* :: *len0*)) (*numeral w*)))
by (*simp add*: *slice1-def*)

lemma *slice-no-bin* [*simp*]:
slice n (*numeral w* :: ′*b word*) = *of-bl* (*takefill False* (*len-of TYPE*(′*b* :: *len0*) − *n*)
(*bin-to-bl* (*len-of TYPE*(′*b* :: *len0*)) (*numeral w*)))
by (*simp add*: *slice-def word-size*)

lemma *slice1-0* [*simp*] : *slice1 n 0* = *0*
unfolding *slice1-def* **by** *simp*

lemma *slice-0* [*simp*] : *slice n 0* = *0*
unfolding *slice-def* **by** *auto*

lemma *slice-take*′: *slice n w* = *of-bl* (*take* (*size w* − *n*) (*to-bl w*))
unfolding *slice-def*′ *slice1-def*
by (*simp add* : *takefill-alt word-size*)

lemmas *slice-take* = *slice-take*′ [*unfolded word-size*]

— shiftr to a word of the same size is just slice, slice is just shiftr then ucast
lemmas *shiftr-slice* = *trans* [*OF shiftr-bl* [*THEN meta-eq-to-obj-eq*] *slice-take* [*symmetric*]]

lemma *slice-shiftr*: *slice n w* = *ucast* (*w* >> *n*)
apply (*unfold slice-take shiftr-bl*)
apply (*rule ucast-of-bl-up* [*symmetric*])
apply (*simp add*: *word-size*)
done

lemma *nth-slice*:
(*slice n w* :: ′*a* :: *len0 word*) !! *m* =
(*w* !! (*m* + *n*) & *m* < *len-of TYPE* (′*a*))
unfolding *slice-shiftr*
by (*simp add* : *nth-ucast nth-shiftr*)

lemma *slice1-down-alt*′:
sl = *slice1 n w* ⟹ *fs* = *size sl* ⟹ *fs* + *k* = *n* ⟹
to-bl sl = *takefill False fs* (*drop k* (*to-bl w*))
unfolding *slice1-def word-size of-bl-def uint-bl*
by (*clarsimp simp*: *word-ubin.eq-norm bl-bin-bl-rep-drop drop-takefill*)

lemma *slice1-up-alt*′:
sl = *slice1 n w* ⟹ *fs* = *size sl* ⟹ *fs* = *n* + *k* ⟹
to-bl sl = *takefill False fs* (*replicate k False* @ (*to-bl w*))
apply (*unfold slice1-def word-size of-bl-def uint-bl*)
apply (*clarsimp simp*: *word-ubin.eq-norm bl-bin-bl-rep-drop*
takefill-append [*symmetric*])

apply (*rule-tac f* = %*k*. *takefill False* (*len-of TYPE*('*a*))
(*replicate k False* @ *bin-to-bl* (*len-of TYPE*('*b*)) (*uint w*)) **in** *arg-cong*)
apply *arith*
done

lemmas *sd1* = *slice1-down-alt'* [*OF refl refl*, *unfolded word-size*]
lemmas *su1* = *slice1-up-alt'* [*OF refl refl*, *unfolded word-size*]
lemmas *slice1-down-alt* = *le-add-diff-inverse* [*THEN sd1*]
lemmas *slice1-up-alts* =
le-add-diff-inverse [*symmetric*, *THEN su1*]
le-add-diff-inverse2 [*symmetric*, *THEN su1*]

lemma *ucast-slice1*: *ucast w* = *slice1* (*size w*) *w*
unfolding *slice1-def ucast-bl*
by (*simp add* : *takefill-same'* *word-size*)

lemma *ucast-slice*: *ucast w* = *slice 0 w*
unfolding *slice-def* **by** (*simp add* : *ucast-slice1*)

lemma *slice-id*: *slice 0 t* = *t*
by (*simp only*: *ucast-slice* [*symmetric*] *ucast-id*)

lemma *revcast-slice1* [*OF refl*]:
rc = *revcast w* ⟹ *slice1* (*size rc*) *w* = *rc*
unfolding *slice1-def revcast-def'* **by** (*simp add* : *word-size*)

lemma *slice1-tf-tf'*:
to-bl (*slice1 n w* :: '*a* :: *len0 word*) =
rev (*takefill False* (*len-of TYPE*('*a*)) (*rev* (*takefill False n* (*to-bl w*))))
unfolding *slice1-def* **by** (*rule word-rev-tf*)

lemmas *slice1-tf-tf* = *slice1-tf-tf'* [*THEN word-bl.Rep-inverse'*, *symmetric*]

lemma *rev-slice1*:
n + *k* = *len-of TYPE*('*a*) + *len-of TYPE*('*b*) ⟹
slice1 n (*word-reverse w* :: '*b* :: *len0 word*) =
word-reverse (*slice1 k w* :: '*a* :: *len0 word*)
apply (*unfold word-reverse-def slice1-tf-tf*)
apply (*rule word-bl.Rep-inverse'*)
apply (*rule rev-swap* [*THEN iffD1*])
apply (*rule trans* [*symmetric*])
apply (*rule tf-rev*)
apply (*simp add*: *word-bl.Abs-inverse*)
apply (*simp add*: *word-bl.Abs-inverse*)
done

lemma *rev-slice*:
n + *k* + *len-of TYPE*('*a*::*len0*) = *len-of TYPE*('*b*::*len0*) ⟹
slice n (*word-reverse* (*w*::'*b word*)) = *word-reverse* (*slice k w*::'*a word*)

apply (*unfold slice-def word-size*)
apply (*rule rev-slice1*)
apply *arith*
done

lemmas *sym-notr* =
not-iff [*THEN iffD2*, *THEN not-sym*, *THEN not-iff* [*THEN iffD1*]]

— problem posed by TPHOLs referee: criterion for overflow of addition of signed integers

lemma *sofl-test*:
(*sint* (*x* :: ′*a* :: *len word*) + *sint y* = *sint* (*x* + *y*)) =
((((*x*+*y*) *XOR x*) *AND* ((*x*+*y*) *XOR y*)) >> (*size x* − *1*) = *0*)
apply (*unfold word-size*)
apply (*cases len-of TYPE*(′*a*), *simp*)
apply (*subst msb-shift* [*THEN sym-notr*])
apply (*simp add*: *word-ops-msb*)
apply (*simp add*: *word-msb-sint*)
apply *safe*
apply *simp-all*
apply (*unfold sint-word-ariths*)
apply (*unfold word-sbin.set-iff-norm* [*symmetric*] *sints-num*)
apply *safe*
apply (*insert sint-range′* [**where** *x*=*x*])
apply (*insert sint-range′* [**where** *x*=*y*])
defer
apply (*simp* (*no-asm*), *arith*)
apply (*simp* (*no-asm*), *arith*)
defer
defer
apply (*simp* (*no-asm*), *arith*)
apply (*simp* (*no-asm*), *arith*)
apply (*rule notI* [*THEN notnotD*],
drule leI not-le-imp-less,
drule sbintrunc-inc sbintrunc-dec,
simp)+
done

16.26 Split and cat

lemmas *word-split-bin′* = *word-split-def*
lemmas *word-cat-bin′* = *word-cat-def*

lemma *word-rsplit-no*:
(*word-rsplit* (*numeral bin* :: ′*b* :: *len0 word*) :: ′*a word list*) =
map word-of-int (*bin-rsplit* (*len-of TYPE*(′*a* :: *len*))
(*len-of TYPE*(′*b*), *bintrunc* (*len-of TYPE*(′*b*)) (*numeral bin*)))
unfolding *word-rsplit-def* **by** (*simp add*: *word-ubin.eq-norm*)

lemmas *word-rsplit-no-cl* [*simp*] = *word-rsplit-no*
[*unfolded bin-rsplitl-def bin-rsplit-l* [*symmetric*]]

lemma *test-bit-cat*:
wc = *word-cat a b* ⟹ *wc* !! *n* = (*n* < *size wc* &
(*if n* < *size b then b* !! *n else a* !! (*n* − *size b*)))
apply (*unfold word-cat-bin′ test-bit-bin*)
apply (*auto simp add* : *word-ubin.eq-norm nth-bintr bin-nth-cat word-size*)
apply (*erule bin-nth-uint-imp*)
done

lemma *word-cat-bl*: *word-cat a b* = *of-bl* (*to-bl a* @ *to-bl b*)
apply (*unfold of-bl-def to-bl-def word-cat-bin′*)
apply (*simp add*: *bl-to-bin-app-cat*)
done

lemma *of-bl-append*:
(*of-bl* (*xs* @ *ys*) :: ′*a* :: *len word*) = *of-bl xs* ∗ *2*ˆ(*length ys*) + *of-bl ys*
apply (*unfold of-bl-def*)
apply (*simp add*: *bl-to-bin-app-cat bin-cat-num*)
apply (*simp add*: *word-of-int-power-hom* [*symmetric*] *word-of-int-hom-syms*)
done

lemma *of-bl-False* [*simp*]:
of-bl (*False*#*xs*) = *of-bl xs*
by (*rule word-eqI*)
(*auto simp add*: *test-bit-of-bl nth-append*)

lemma *of-bl-True* [*simp*]:
(*of-bl* (*True*#*xs*)::′*a*::*len word*) = *2*ˆ*length xs* + *of-bl xs*
by (*subst of-bl-append* [**where** *xs*=[*True*], *simplified*])
(*simp add*: *word-1-bl*)

lemma *of-bl-Cons*:
of-bl (*x*#*xs*) = *of-bool x* ∗ *2*ˆ*length xs* + *of-bl xs*
by (*cases x*) *simp-all*

lemma *split-uint-lem*: *bin-split n* (*uint* (*w* :: ′*a* :: *len0 word*)) = (*a*, *b*) ⟹
a = *bintrunc* (*len-of TYPE*(′*a*) − *n*) *a* & *b* = *bintrunc* (*len-of TYPE*(′*a*)) *b*
apply (*frule word-ubin.norm-Rep* [*THEN ssubst*])
apply (*drule bin-split-trunc1*)
apply (*drule sym* [*THEN trans*])
apply *assumption*
apply *safe*
done

lemma *word-split-bl′*:
std = *size c* − *size b* ⟹ (*word-split c* = (*a*, *b*)) ⟹

(*a* = *of-bl* (*take std* (*to-bl c*)) & *b* = *of-bl* (*drop std* (*to-bl c*)))
apply (*unfold word-split-bin′*)
apply *safe*
defer
apply (*clarsimp split*: *prod.splits*)
apply *hypsubst-thin*
apply (*drule word-ubin.norm-Rep* [*THEN ssubst*])
apply (*drule split-bintrunc*)
apply (*simp add* : *of-bl-def bl2bin-drop word-size*
word-ubin.norm-eq-iff [*symmetric*] *min-def del* : *word-ubin.norm-Rep*)
apply (*clarsimp split*: *prod.splits*)
apply (*frule split-uint-lem* [*THEN conjunct1*])
apply (*unfold word-size*)
apply (*cases len-of TYPE*(′*a*) >= *len-of TYPE*(′*b*))
defer
apply *simp*
apply (*simp add* : *of-bl-def to-bl-def*)
apply (*subst bin-split-take1* [*symmetric*])
prefer *2*
apply *assumption*
apply *simp*
apply (*erule thin-rl*)
apply (*erule arg-cong* [*THEN trans*])
apply (*simp add* : *word-ubin.norm-eq-iff* [*symmetric*])
done

lemma *word-split-bl*: *std* = *size c* − *size b* ⟹
(*a* = *of-bl* (*take std* (*to-bl c*)) & *b* = *of-bl* (*drop std* (*to-bl c*))) ⟷
word-split c = (*a*, *b*)
apply (*rule iffI*)
defer
apply (*erule* (*1*) *word-split-bl′*)
apply (*case-tac word-split c*)
apply (*auto simp add* : *word-size*)
apply (*frule word-split-bl′* [*rotated*])
apply (*auto simp add* : *word-size*)
done

lemma *word-split-bl-eq*:
(*word-split* (*c*::′*a*::*len word*) :: (′*c* :: *len0 word* ∗ ′*d* :: *len0 word*)) =
(*of-bl* (*take* (*len-of TYPE*(′*a*::*len*) − *len-of TYPE*(′*d*::*len0*)) (*to-bl c*)),
of-bl (*drop* (*len-of TYPE*(′*a*) − *len-of TYPE*(′*d*)) (*to-bl c*)))
apply (*rule word-split-bl* [*THEN iffD1*])
apply (*unfold word-size*)
apply (*rule refl conjI*)+
done

— keep quantifiers for use in simplification
lemma *test-bit-split′*:

word-split c = (*a*, *b*) −−> (*ALL n m*. *b* !! *n* = (*n* < *size b* & *c* !! *n*) &
  a !! *m* = (*m* < *size a* & *c* !! (*m* + *size b*)))
apply (*unfold word-split-bin′ test-bit-bin*)
apply (*clarify*)
apply (*clarsimp simp*: *word-ubin.eq-norm nth-bintr word-size split*: *prod.splits*)
apply (*drule bin-nth-split*)
apply *safe*
    apply (*simp-all add*: *add.commute*)
 apply (*erule bin-nth-uint-imp*)+
done

lemma *test-bit-split*:
word-split c = (*a*, *b*) ⟹
  (∀ *n*::*nat*. *b* !! *n* ⟷ *n* < *size b* ∧ *c* !! *n*) ∧ (∀ *m*::*nat*. *a* !! *m* ⟷ *m* < *size a* ∧ *c* !! (*m* + *size b*))
by (*simp add*: *test-bit-split′*)

lemma *test-bit-split-eq*: *word-split c* = (*a*, *b*) ⟷
((*ALL n*::*nat*. *b* !! *n* = (*n* < *size b* & *c* !! *n*)) &
  (*ALL m*::*nat*. *a* !! *m* = (*m* < *size a* & *c* !! (*m* + *size b*))))
apply (*rule-tac iffI*)
 apply (*rule-tac conjI*)
  apply (*erule test-bit-split* [*THEN conjunct1*])
 apply (*erule test-bit-split* [*THEN conjunct2*])
apply (*case-tac word-split c*)
apply (*frule test-bit-split*)
apply (*erule trans*)
apply (*fastforce intro* ! : *word-eqI simp add* : *word-size*)
done

— this odd result is analogous to *ucast-id*, result to the length given by the result type

lemma *word-cat-id*: *word-cat a b* = *b*
unfolding *word-cat-bin′* **by** (*simp add*: *word-ubin.inverse-norm*)

— limited hom result
lemma *word-cat-hom*:
len-of TYPE(′*a*::*len0*) <= *len-of TYPE*(′*b*::*len0*) + *len-of TYPE* (′*c*::*len0*)
⟹
(*word-cat* (*word-of-int w* :: ′*b word*) (*b* :: ′*c word*) :: ′*a word*) =
word-of-int (*bin-cat w* (*size b*) (*uint b*))
apply (*unfold word-cat-def word-size*)
apply (*clarsimp simp add*: *word-ubin.norm-eq-iff* [*symmetric*]
   word-ubin.eq-norm bintr-cat min.absorb1)
done

lemma *word-cat-split-alt*:
size w <= *size u* + *size v* ⟹ *word-split w* = (*u*, *v*) ⟹ *word-cat u v* = *w*

apply (*rule word-eqI*)
apply (*drule test-bit-split*)
apply (*clarsimp simp add* : *test-bit-cat word-size*)
apply *safe*
apply *arith*
done

lemmas *word-cat-split-size* = *sym* [*THEN* [*2*] *word-cat-split-alt* [*symmetric*]]

16.26.1 Split and slice

lemma *split-slices*:
word-split w = (*u*, *v*) ⟹ *u* = *slice* (*size v*) *w* & *v* = *slice 0 w*
apply (*drule test-bit-split*)
apply (*rule conjI*)
apply (*rule word-eqI*, *clarsimp simp*: *nth-slice word-size*)+
done

lemma *slice-cat1* [*OF refl*]:
wc = *word-cat a b* ⟹ *size wc* >= *size a* + *size b* ⟹ *slice* (*size b*) *wc* = *a*
apply *safe*
apply (*rule word-eqI*)
apply (*simp add*: *nth-slice test-bit-cat word-size*)
done

lemmas *slice-cat2* = *trans* [*OF slice-id word-cat-id*]

lemma *cat-slices*:
a = *slice n c* ⟹ *b* = *slice 0 c* ⟹ *n* = *size b* ⟹
size a + *size b* >= *size c* ⟹ *word-cat a b* = *c*
apply *safe*
apply (*rule word-eqI*)
apply (*simp add*: *nth-slice test-bit-cat word-size*)
apply *safe*
apply *arith*
done

lemma *word-split-cat-alt*:
w = *word-cat u v* ⟹ *size u* + *size v* <= *size w* ⟹ *word-split w* = (*u*, *v*)
apply (*case-tac word-split w*)
apply (*rule trans*, *assumption*)
apply (*drule test-bit-split*)
apply *safe*
apply (*rule word-eqI*, *clarsimp simp*: *test-bit-cat word-size*)+
done

lemmas *word-cat-bl-no-bin* [*simp*] =
word-cat-bl [**where** *a*=*numeral a* **and** *b*=*numeral b*,
unfolded to-bl-numeral]

for *a b*

lemmas *word-split-bl-no-bin* [*simp*] =
word-split-bl-eq [**where** *c*=*numeral c*, *unfolded to-bl-numeral*] **for** *c*

this odd result arises from the fact that the statement of the result implies that the decoded words are of the same type, and therefore of the same length, as the original word

lemma *word-rsplit-same*: *word-rsplit w* = [*w*]
unfolding *word-rsplit-def* **by** (*simp add* : *bin-rsplit-all*)

lemma *word-rsplit-empty-iff-size*:
(*word-rsplit w* = []) = (*size w* = *0*)
unfolding *word-rsplit-def bin-rsplit-def word-size*
by (*simp add*: *bin-rsplit-aux-simp-alt Let-def split*: *prod.split*)

lemma *test-bit-rsplit*:
sw = *word-rsplit w* ⟹ *m* < *size* (*hd sw* :: *'a* :: *len word*) ⟹
k < *length sw* ⟹ (*rev sw* ! *k*) !! *m* = (*w* !! (*k* ∗ *size* (*hd sw*) + *m*))
apply (*unfold word-rsplit-def word-test-bit-def*)
apply (*rule trans*)
apply (*rule-tac f* = *%x. bin-nth x m* **in** *arg-cong*)
apply (*rule nth-map* [*symmetric*])
apply *simp*
apply (*rule bin-nth-rsplit*)
apply *simp-all*
apply (*simp add* : *word-size rev-map*)
apply (*rule trans*)
defer
apply (*rule map-ident* [*THEN fun-cong*])
apply (*rule refl* [*THEN map-cong*])
apply (*simp add* : *word-ubin.eq-norm*)
apply (*erule bin-rsplit-size-sign* [*OF len-gt-0 refl*])
done

lemma *word-rcat-bl*: *word-rcat wl* = *of-bl* (*concat* (*map to-bl wl*))
unfolding *word-rcat-def to-bl-def' of-bl-def*
by (*clarsimp simp add* : *bin-rcat-bl*)

lemma *size-rcat-lem'*:
size (*concat* (*map to-bl wl*)) = *length wl* ∗ *size* (*hd wl*)
unfolding *word-size* **by** (*induct wl*) *auto*

lemmas *size-rcat-lem* = *size-rcat-lem'* [*unfolded word-size*]

lemmas *td-gal-lt-len* = *len-gt-0* [*THEN td-gal-lt*]

lemma *nth-rcat-lem*:
n < *length* (*wl*::*'a word list*) ∗ *len-of TYPE*(*'a*::*len*) ⟹

rev (*concat* (*map to-bl wl*)) ! *n* =
rev (*to-bl* (*rev wl* ! (*n div len-of TYPE*('*a*)))) ! (*n mod len-of TYPE*('*a*))
apply (*induct wl*)
apply *clarsimp*
apply (*clarsimp simp add* : *nth-append size-rcat-lem*)
apply (*simp* (*no-asm-use*) *only*: *mult-Suc* [*symmetric*]
td-gal-lt-len less-Suc-eq-le mod-div-equality')
apply *clarsimp*
done

lemma *test-bit-rcat*:
sw = *size* (*hd wl* :: '*a* :: *len word*) ⟹ *rc* = *word-rcat wl* ⟹ *rc* !! *n* =
(*n* < *size rc* & *n div sw* < *size wl* & (*rev wl*) ! (*n div sw*) !! (*n mod sw*))
apply (*unfold word-rcat-bl word-size*)
apply (*clarsimp simp add* :
test-bit-of-bl size-rcat-lem word-size td-gal-lt-len)
apply *safe*
apply (*auto simp add* :
test-bit-bl word-size td-gal-lt-len [*THEN iffD2*, *THEN nth-rcat-lem*])
done

lemma *foldl-eq-foldr*:
foldl op + *x xs* = *foldr op* + (*x* # *xs*) (*0* :: '*a* :: *comm-monoid-add*)
by (*induct xs arbitrary*: *x*) (*auto simp add* : *add.assoc*)

lemmas *test-bit-cong* = *arg-cong* [**where** *f* = *test-bit*, *THEN fun-cong*]

lemmas *test-bit-rsplit-alt* =
trans [*OF nth-rev-alt* [*THEN test-bit-cong*]
test-bit-rsplit [*OF refl asm-rl diff-Suc-less*]]

— lazy way of expressing that u and v, and su and sv, have same types
lemma *word-rsplit-len-indep* [*OF refl refl refl refl*]:
[*u*,*v*] = *p* ⟹ [*su*,*sv*] = *q* ⟹ *word-rsplit u* = *su* ⟹
word-rsplit v = *sv* ⟹ *length su* = *length sv*
apply (*unfold word-rsplit-def*)
apply (*auto simp add* : *bin-rsplit-len-indep*)
done

lemma *length-word-rsplit-size*:
n = *len-of TYPE* ('*a* :: *len*) ⟹
(*length* (*word-rsplit w* :: '*a word list*) <= *m*) = (*size w* <= *m* ∗ *n*)
apply (*unfold word-rsplit-def word-size*)
apply (*clarsimp simp add* : *bin-rsplit-len-le*)
done

lemmas *length-word-rsplit-lt-size* =
length-word-rsplit-size [*unfolded Not-eq-iff linorder-not-less* [*symmetric*]]

lemma *length-word-rsplit-exp-size*:
n = *len-of TYPE* (*'a* :: *len*) $\Longrightarrow$
length (*word-rsplit w* :: *'a word list*) = (*size w* + *n* − *1*) *div n*
unfolding *word-rsplit-def* **by** (*clarsimp simp add* : *word-size bin-rsplit-len*)

lemma *length-word-rsplit-even-size*:
n = *len-of TYPE* (*'a* :: *len*) $\Longrightarrow$ *size w* = *m* ∗ *n* $\Longrightarrow$
length (*word-rsplit w* :: *'a word list*) = *m*
by (*clarsimp simp add* : *length-word-rsplit-exp-size given-quot-alt*)

lemmas *length-word-rsplit-exp-size'* = *refl* [*THEN length-word-rsplit-exp-size*]

lemmas *tdle* = *iffD2* [*OF split-div-lemma refl*, *THEN conjunct1*]
lemmas *dtle* = *xtr4* [*OF tdle mult.commute*]

lemma *word-rcat-rsplit*: *word-rcat* (*word-rsplit w*) = *w*
apply (*rule word-eqI*)
apply (*clarsimp simp add* : *test-bit-rcat word-size*)
apply (*subst refl* [*THEN test-bit-rsplit*])
apply (*simp-all add*: *word-size*
refl [*THEN length-word-rsplit-size* [*simplified not-less* [*symmetric*], *simplified*]])
apply *safe*
apply (*erule xtr7*, *rule len-gt-0* [*THEN dtle*])+
done

lemma *size-word-rsplit-rcat-size*:
⟦*word-rcat* (*ws*::*'a*::*len word list*) = (*frcw*::*'b*::*len0 word*);
size frcw = *length ws* ∗ *len-of TYPE*(*'a*)⟧
$\Longrightarrow$ *length* (*word-rsplit frcw*::*'a word list*) = *length ws*
apply (*clarsimp simp add* : *word-size length-word-rsplit-exp-size'*)
apply (*fast intro*: *given-quot-alt*)
done

lemma *msrevs*:
fixes *n*::*nat*
shows *0* < *n* $\Longrightarrow$ (*k* ∗ *n* + *m*) *div n* = *m div n* + *k*
and (*k* ∗ *n* + *m*) *mod n* = *m mod n*
by (*auto simp*: *add.commute*)

lemma *word-rsplit-rcat-size* [*OF refl*]:
word-rcat (*ws* :: *'a* :: *len word list*) = *frcw* $\Longrightarrow$
size frcw = *length ws* ∗ *len-of TYPE* (*'a*) $\Longrightarrow$ *word-rsplit frcw* = *ws*
apply (*frule size-word-rsplit-rcat-size*, *assumption*)
apply (*clarsimp simp add* : *word-size*)
apply (*rule nth-equalityI*, *assumption*)
apply *clarsimp*
apply (*rule word-eqI* [*rule-format*])
apply (*rule trans*)

apply (*rule test-bit-rsplit-alt*)
apply (*clarsimp simp*: *word-size*)+
apply (*rule trans*)
apply (*rule test-bit-rcat* [*OF refl refl*])
apply (*simp add*: *word-size*)
apply (*subst nth-rev*)
apply *arith*
apply (*simp add*: *le0* [*THEN* [*2*] *xtr7*, *THEN diff-Suc-less*])
apply *safe*
apply (*simp add*: *diff-mult-distrib*)
apply (*rule mpl-lem*)
apply (*cases size ws*)
apply *simp-all*
done

16.27 Rotation

lemmas *rotater-0'* [*simp*] = *rotater-def* [**where** *n* = *0*, *simplified*]

lemmas *word-rot-defs* = *word-roti-def word-rotr-def word-rotl-def*

lemma *rotate-eq-mod*:
m mod length xs = *n mod length xs* ⟹ *rotate m xs* = *rotate n xs*
apply (*rule box-equals*)
defer
apply (*rule rotate-conv-mod* [*symmetric*])+
apply *simp*
done

lemmas *rotate-eqs* =
trans [*OF rotate0* [*THEN fun-cong*] *id-apply*]
rotate-rotate [*symmetric*]
rotate-id
rotate-conv-mod
rotate-eq-mod

16.27.1 Rotation of list to right

lemma *rotate1-rl'*: *rotater1* (*l* @ [*a*]) = *a* # *l*
unfolding *rotater1-def* **by** (*cases l*) *auto*

lemma *rotate1-rl* [*simp*] : *rotater1* (*rotate1 l*) = *l*
apply (*unfold rotater1-def*)
apply (*cases l*)
apply (*case-tac* [*2*] *list*)
apply *auto*
done

lemma *rotate1-lr* [*simp*] : *rotate1* (*rotater1 l*) = *l*
unfolding *rotater1-def* **by** (*cases l*) *auto*

lemma *rotater1-rev'*: *rotater1* (*rev xs*) = *rev* (*rotate1 xs*)
 apply (*cases xs*)
 apply (*simp add* : *rotater1-def*)
 apply (*simp add* : *rotate1-rl'*)
 done

lemma *rotater-rev'*: *rotater n* (*rev xs*) = *rev* (*rotate n xs*)
 unfolding *rotater-def* **by** (*induct n*) (*auto intro*: *rotater1-rev'*)

lemma *rotater-rev*: *rotater n ys* = *rev* (*rotate n* (*rev ys*))
 using *rotater-rev'* [**where** *xs* = *rev ys*] **by** *simp*

lemma *rotater-drop-take*:
 rotater n xs =
 drop (*length xs* − *n mod length xs*) *xs* @
 take (*length xs* − *n mod length xs*) *xs*
 by (*clarsimp simp add* : *rotater-rev rotate-drop-take rev-take rev-drop*)

lemma *rotater-Suc* [*simp*] :
 rotater (*Suc n*) *xs* = *rotater1* (*rotater n xs*)
 unfolding *rotater-def* **by** *auto*

lemma *rotate-inv-plus* [*rule-format*] :
 ALL k. *k* = *m* + *n* −−> *rotater k* (*rotate n xs*) = *rotater m xs* &
 rotate k (*rotater n xs*) = *rotate m xs* &
 rotater n (*rotate k xs*) = *rotate m xs* &
 rotate n (*rotater k xs*) = *rotater m xs*
 unfolding *rotater-def rotate-def*
 by (*induct n*) (*auto intro*: *funpow-swap1* [*THEN trans*])

lemmas *rotate-inv-rel* = *le-add-diff-inverse2* [*symmetric*, *THEN rotate-inv-plus*]

lemmas *rotate-inv-eq* = *order-refl* [*THEN rotate-inv-rel*, *simplified*]

lemmas *rotate-lr* [*simp*] = *rotate-inv-eq* [*THEN conjunct1*]
lemmas *rotate-rl* [*simp*] = *rotate-inv-eq* [*THEN conjunct2*, *THEN conjunct1*]

lemma *rotate-gal*: (*rotater n xs* = *ys*) = (*rotate n ys* = *xs*)
 by *auto*

lemma *rotate-gal'*: (*ys* = *rotater n xs*) = (*xs* = *rotate n ys*)
 by *auto*

lemma *length-rotater* [*simp*]:
 length (*rotater n xs*) = *length xs*
 by (*simp add* : *rotater-rev*)

lemma *restrict-to-left*:

assumes *x* = *y*
shows (*x* = *z*) = (*y* = *z*)
using *assms* **by** *simp*

lemmas *rrs0* = *rotate-eqs* [*THEN restrict-to-left*,
simplified rotate-gal [*symmetric*] *rotate-gal′* [*symmetric*]]
lemmas *rrs1* = *rrs0* [*THEN refl* [*THEN rev-iffD1*]]
lemmas *rotater-eqs* = *rrs1* [*simplified length-rotater*]
lemmas *rotater-0* = *rotater-eqs* (*1*)
lemmas *rotater-add* = *rotater-eqs* (*2*)

16.27.2 map, map2, commuting with rotate(r)

lemma *butlast-map*:
xs ~= [] ⟹ *butlast* (*map f xs*) = *map f* (*butlast xs*)
by (*induct xs*) *auto*

lemma *rotater1-map*: *rotater1* (*map f xs*) = *map f* (*rotater1 xs*)
unfolding *rotater1-def*
by (*cases xs*) (*auto simp add*: *last-map butlast-map*)

lemma *rotater-map*:
rotater n (*map f xs*) = *map f* (*rotater n xs*)
unfolding *rotater-def*
by (*induct n*) (*auto simp add* : *rotater1-map*)

lemma *but-last-zip* [*rule-format*] :
ALL ys. *length xs* = *length ys* --> *xs* ~= [] -->
last (*zip xs ys*) = (*last xs*, *last ys*) &
butlast (*zip xs ys*) = *zip* (*butlast xs*) (*butlast ys*)
apply (*induct xs*)
apply *auto*
apply ((*case-tac ys*, *auto simp*: *neq-Nil-conv*)[*1*])+
done

lemma *but-last-map2* [*rule-format*] :
ALL ys. *length xs* = *length ys* --> *xs* ~= [] -->
last (*map2 f xs ys*) = *f* (*last xs*) (*last ys*) &
butlast (*map2 f xs ys*) = *map2 f* (*butlast xs*) (*butlast ys*)
apply (*induct xs*)
apply *auto*
apply (*unfold map2-def*)
apply ((*case-tac ys*, *auto simp*: *neq-Nil-conv*)[*1*])+
done

lemma *rotater1-zip*:
length xs = *length ys* ⟹
rotater1 (*zip xs ys*) = *zip* (*rotater1 xs*) (*rotater1 ys*)
apply (*unfold rotater1-def*)

apply (*cases xs*)
 apply *auto*
 apply ((*case-tac ys*, *auto simp*: *neq-Nil-conv but-last-zip*)[*1*])+
done

lemma *rotater1-map2*:
length xs = *length ys* ⟹
 rotater1 (*map2 f xs ys*) = *map2 f* (*rotater1 xs*) (*rotater1 ys*)
unfolding *map2-def* **by** (*simp add*: *rotater1-map rotater1-zip*)

lemmas *lrth* =
box-equals [*OF asm-rl length-rotater* [*symmetric*]
 length-rotater [*symmetric*],
 THEN rotater1-map2]

lemma *rotater-map2*:
length xs = *length ys* ⟹
 rotater n (*map2 f xs ys*) = *map2 f* (*rotater n xs*) (*rotater n ys*)
by (*induct n*) (*auto intro*!: *lrth*)

lemma *rotate1-map2*:
length xs = *length ys* ⟹
 rotate1 (*map2 f xs ys*) = *map2 f* (*rotate1 xs*) (*rotate1 ys*)
apply (*unfold map2-def*)
apply (*cases xs*)
 apply (*cases ys*, *auto*)+
done

lemmas *lth* = *box-equals* [*OF asm-rl length-rotate* [*symmetric*]
length-rotate [*symmetric*], *THEN rotate1-map2*]

lemma *rotate-map2*:
length xs = *length ys* ⟹
 rotate n (*map2 f xs ys*) = *map2 f* (*rotate n xs*) (*rotate n ys*)
by (*induct n*) (*auto intro*!: *lth*)

— corresponding equalities for word rotation

lemma *to-bl-rotl*:
to-bl (*word-rotl n w*) = *rotate n* (*to-bl w*)
by (*simp add*: *word-bl.Abs-inverse′ word-rotl-def*)

lemmas *blrs0* = *rotate-eqs* [*THEN to-bl-rotl* [*THEN trans*]]

lemmas *word-rotl-eqs* =
blrs0 [*simplified word-bl-Rep′ word-bl.Rep-inject to-bl-rotl* [*symmetric*]]

lemma *to-bl-rotr*:

to-bl (*word-rotr n w*) = *rotater n* (*to-bl w*)
by (*simp add*: *word-bl.Abs-inverse′ word-rotr-def*)

lemmas *brrs0* = *rotater-eqs* [*THEN to-bl-rotr* [*THEN trans*]]

lemmas *word-rotr-eqs* =
brrs0 [*simplified word-bl-Rep′ word-bl.Rep-inject to-bl-rotr* [*symmetric*]]

declare *word-rotr-eqs* (*1*) [*simp*]
declare *word-rotl-eqs* (*1*) [*simp*]

lemma
word-rot-rl [*simp*]:
word-rotl k (*word-rotr k v*) = *v* **and**
word-rot-lr [*simp*]:
word-rotr k (*word-rotl k v*) = *v*
by (*auto simp add*: *to-bl-rotr to-bl-rotl word-bl.Rep-inject* [*symmetric*])

lemma
word-rot-gal:
(*word-rotr n v* = *w*) = (*word-rotl n w* = *v*) **and**
word-rot-gal′:
(*w* = *word-rotr n v*) = (*v* = *word-rotl n w*)
by (*auto simp*: *to-bl-rotr to-bl-rotl word-bl.Rep-inject* [*symmetric*]
dest: *sym*)

lemma *word-rotr-rev*:
word-rotr n w = *word-reverse* (*word-rotl n* (*word-reverse w*))
by (*simp only*: *word-bl.Rep-inject* [*symmetric*] *to-bl-word-rev*
to-bl-rotr to-bl-rotl rotater-rev)

lemma *word-roti-0* [*simp*]: *word-roti 0 w* = *w*
by (*unfold word-rot-defs*) *auto*

lemmas *abl-cong* = *arg-cong* [**where** *f* = *of-bl*]

lemma *word-roti-add*:
word-roti (*m* + *n*) *w* = *word-roti m* (*word-roti n w*)
proof –
have *rotater-eq-lem*:
⋀*m n xs*. *m* = *n* ⟹ *rotater m xs* = *rotater n xs*
by *auto*

have *rotate-eq-lem*:
⋀*m n xs*. *m* = *n* ⟹ *rotate m xs* = *rotate n xs*
by *auto*

note *rpts* [*symmetric*] =
rotate-inv-plus [*THEN conjunct1*]

```
    rotate-inv-plus [THEN conjunct2, THEN conjunct1]
    rotate-inv-plus [THEN conjunct2, THEN conjunct2, THEN conjunct1]
    rotate-inv-plus [THEN conjunct2, THEN conjunct2, THEN conjunct2]

  note rrp = trans [symmetric, OF rotate-rotate rotate-eq-lem]
  note rrrp = trans [symmetric, OF rotater-add [symmetric] rotater-eq-lem]

  show ?thesis
  apply (unfold word-rot-defs)
  apply (simp only: split: if-split)
  apply (safe intro!: abl-cong)
  apply (simp-all only: to-bl-rotl [THEN word-bl.Rep-inverse']
                  to-bl-rotl
                  to-bl-rotr [THEN word-bl.Rep-inverse']
                  to-bl-rotr)
  apply (rule rrp rrrp rpts,
         simp add: nat-add-distrib [symmetric]
                  nat-diff-distrib [symmetric])+
  done
qed

lemma word-roti-conv-mod': word-roti n w = word-roti (n mod int (size w)) w
  apply (unfold word-rot-defs)
  apply (cut-tac y=size w in gt-or-eq-0)
  apply (erule disjE)
   apply simp-all
  apply (safe intro!: abl-cong)
   apply (rule rotater-eqs)
   apply (simp add: word-size nat-mod-distrib)
  apply (simp add: rotater-add [symmetric] rotate-gal [symmetric])
  apply (rule rotater-eqs)
  apply (simp add: word-size nat-mod-distrib)
  apply (rule of-nat-eq-0-iff [THEN iffD1])
 apply (auto simp add: not-le mod-eq-0-iff-dvd zdvd-int nat-add-distrib [symmetric])
  using mod-mod-trivial zmod-eq-dvd-iff
  apply blast
  done

lemmas word-roti-conv-mod = word-roti-conv-mod' [unfolded word-size]
```

16.27.3 "Word rotation commutes with bit-wise operations

```
locale word-rotate
begin

lemmas word-rot-defs' = to-bl-rotl to-bl-rotr

lemmas blwl-syms [symmetric] = bl-word-not bl-word-and bl-word-or bl-word-xor
```

```
lemmas lbl-lbl = trans [OF word-bl-Rep' word-bl-Rep' [symmetric]]

lemmas ths-map2 [OF lbl-lbl] = rotate-map2 rotater-map2

lemmas ths-map [where xs = to-bl v] = rotate-map rotater-map for v

lemmas th1s [simplified word-rot-defs' [symmetric]] = ths-map2 ths-map

lemma word-rot-logs:
  word-rotl n (NOT v) = NOT word-rotl n v
  word-rotr n (NOT v) = NOT word-rotr n v
  word-rotl n (x AND y) = word-rotl n x AND word-rotl n y
  word-rotr n (x AND y) = word-rotr n x AND word-rotr n y
  word-rotl n (x OR y) = word-rotl n x OR word-rotl n y
  word-rotr n (x OR y) = word-rotr n x OR word-rotr n y
  word-rotl n (x XOR y) = word-rotl n x XOR word-rotl n y
  word-rotr n (x XOR y) = word-rotr n x XOR word-rotr n y
  by (rule word-bl.Rep-eqD,
      rule word-rot-defs' [THEN trans],
      simp only: blwl-syms [symmetric],
      rule th1s [THEN trans],
      rule refl)+
end

lemmas word-rot-logs = word-rotate.word-rot-logs

lemmas bl-word-rotl-dt = trans [OF to-bl-rotl rotate-drop-take,
  simplified word-bl-Rep']

lemmas bl-word-rotr-dt = trans [OF to-bl-rotr rotater-drop-take,
  simplified word-bl-Rep']

lemma bl-word-roti-dt':
  n = nat ((− i) mod int (size (w :: 'a :: len word))) ⟹
    to-bl (word-roti i w) = drop n (to-bl w) @ take n (to-bl w)
  apply (unfold word-roti-def)
  apply (simp add: bl-word-rotl-dt bl-word-rotr-dt word-size)
  apply safe
   apply (simp add: zmod-zminus1-eq-if)
   apply safe
    apply (simp add: nat-mult-distrib)
   apply (simp add: nat-diff-distrib [OF pos-mod-sign pos-mod-conj
                                  [THEN conjunct2, THEN order-less-imp-le]]
                 nat-mod-distrib)
  apply (simp add: nat-mod-distrib)
  done

lemmas bl-word-roti-dt = bl-word-roti-dt' [unfolded word-size]
```

lemmas *word-rotl-dt* = *bl-word-rotl-dt* [*THEN word-bl.Rep-inverse′* [*symmetric*]]
lemmas *word-rotr-dt* = *bl-word-rotr-dt* [*THEN word-bl.Rep-inverse′* [*symmetric*]]
lemmas *word-roti-dt* = *bl-word-roti-dt* [*THEN word-bl.Rep-inverse′* [*symmetric*]]

lemma *word-rotx-0* [*simp*] : *word-rotr i 0* = *0* & *word-rotl i 0* = *0*
by (*simp add* : *word-rotr-dt word-rotl-dt replicate-add* [*symmetric*])

lemma *word-roti-0′* [*simp*] : *word-roti n 0* = *0*
unfolding *word-roti-def* **by** *auto*

lemmas *word-rotr-dt-no-bin′* [*simp*] =
word-rotr-dt [**where** *w*=*numeral w*, *unfolded to-bl-numeral*] **for** *w*

lemmas *word-rotl-dt-no-bin′* [*simp*] =
word-rotl-dt [**where** *w*=*numeral w*, *unfolded to-bl-numeral*] **for** *w*

declare *word-roti-def* [*simp*]

16.28 Maximum machine word

lemma *word-int-cases*:
obtains *n* **where** (*x* ::′*a*::*len0 word*) = *word-of-int n* **and** *0* ≤ *n* **and** *n* < *2*ˆ*len-of TYPE*(′*a*)
by (*cases x rule*: *word-uint.Abs-cases*) (*simp add*: *uints-num*)

lemma *word-nat-cases* [*cases type*: *word*]:
obtains *n* **where** (*x* ::′*a*::*len word*) = *of-nat n* **and** *n* < *2*ˆ*len-of TYPE*(′*a*)
by (*cases x rule*: *word-unat.Abs-cases*) (*simp add*: *unats-def*)

lemma *max-word-eq*: (*max-word*::′*a*::*len word*) = *2*ˆ*len-of TYPE*(′*a*) − *1*
by (*simp add*: *max-word-def word-of-int-hom-syms word-of-int-2p*)

lemma *max-word-max* [*simp*,*intro*!]: *n* ≤ *max-word*
by (*cases n rule*: *word-int-cases*)
(*simp add*: *max-word-def word-le-def int-word-uint mod-pos-pos-trivial del*: *minus-mod-self1*)

lemma *word-of-int-2p-len*: *word-of-int* (*2* ˆ *len-of TYPE*(′*a*)) = (*0*::′*a*::*len0 word*)
by (*subst word-uint.Abs-norm* [*symmetric*]) *simp*

lemma *word-pow-0*:
(*2*::′*a*::*len word*) ˆ *len-of TYPE*(′*a*) = *0*
proof −
have *word-of-int* (*2* ˆ *len-of TYPE*(′*a*)) = (*0*::′*a word*)
by (*rule word-of-int-2p-len*)
thus *?thesis* **by** (*simp add*: *word-of-int-2p*)
qed

lemma *max-word-wrap*: *x* + *1* = *0* ⟹ *x* = *max-word*
  apply (*simp add*: *max-word-eq*)
  apply *uint-arith*
  apply *auto*
  apply (*simp add*: *word-pow-0*)
  done

lemma *max-word-minus*:
  max-word = (−*1*::′*a*::*len word*)
proof −
  have −*1* + *1* = (*0*::′*a word*) **by** *simp*
  thus *?thesis* **by** (*rule max-word-wrap* [*symmetric*])
qed

lemma *max-word-bl* [*simp*]:
  to-bl (*max-word*::′*a*::*len word*) = *replicate* (*len-of TYPE*(′*a*)) *True*
  by (*subst max-word-minus to-bl-n1*)+ *simp*

lemma *max-test-bit* [*simp*]:
  (*max-word*::′*a*::*len word*) !! *n* = (*n* < *len-of TYPE*(′*a*))
  by (*auto simp add*: *test-bit-bl word-size*)

lemma *word-and-max* [*simp*]:
  x AND max-word = *x*
  by (*rule word-eqI*) (*simp add*: *word-ops-nth-size word-size*)

lemma *word-or-max* [*simp*]:
  x OR max-word = *max-word*
  by (*rule word-eqI*) (*simp add*: *word-ops-nth-size word-size*)

lemma *word-ao-dist2*:
  x AND (*y OR z*) = *x AND y OR x AND* (*z*::′*a*::*len0 word*)
  by (*rule word-eqI*) (*auto simp add*: *word-ops-nth-size word-size*)

lemma *word-oa-dist2*:
  x OR y AND z = (*x OR y*) *AND* (*x OR* (*z*::′*a*::*len0 word*))
  by (*rule word-eqI*) (*auto simp add*: *word-ops-nth-size word-size*)

lemma *word-and-not* [*simp*]:
  x AND NOT x = (*0*::′*a*::*len0 word*)
  by (*rule word-eqI*) (*auto simp add*: *word-ops-nth-size word-size*)

lemma *word-or-not* [*simp*]:
  x OR NOT x = *max-word*
  by (*rule word-eqI*) (*auto simp add*: *word-ops-nth-size word-size*)

lemma *word-boolean*:
  boolean (*op AND*) (*op OR*) *bitNOT 0 max-word*

apply (*rule boolean.intro*)
apply (*rule word-bw-assocs*)
apply (*rule word-bw-assocs*)
apply (*rule word-bw-comms*)
apply (*rule word-bw-comms*)
apply (*rule word-ao-dist2*)
apply (*rule word-oa-dist2*)
apply (*rule word-and-max*)
apply (*rule word-log-esimps*)
apply (*rule word-and-not*)
apply (*rule word-or-not*)
done

interpretation *word-bool-alg*:
boolean op AND op OR bitNOT 0 max-word
by (*rule word-boolean*)

lemma *word-xor-and-or*:
x XOR y = x AND NOT y OR NOT x AND (*y*::*'a*::*len0 word*)
by (*rule word-eqI*) (*auto simp add*: *word-ops-nth-size word-size*)

interpretation *word-bool-alg*:
boolean-xor op AND op OR bitNOT 0 max-word op XOR
apply (*rule boolean-xor.intro*)
apply (*rule word-boolean*)
apply (*rule boolean-xor-axioms.intro*)
apply (*rule word-xor-and-or*)
done

lemma *shiftr-x-0* [*iff*]:
(*x*::*'a*::*len0 word*) >> *0* = *x*
by (*simp add*: *shiftr-bl*)

lemma *shiftl-x-0* [*simp*]:
(*x* :: *'a* :: *len word*) << *0* = *x*
by (*simp add*: *shiftl-t2n*)

lemma *shiftl-1* [*simp*]:
(*1*::*'a*::*len word*) << *n* = *2*^*n*
by (*simp add*: *shiftl-t2n*)

lemma *uint-lt-0* [*simp*]:
uint x < *0* = *False*
by (*simp add*: *linorder-not-less*)

lemma *shiftr1-1* [*simp*]:
shiftr1 (*1*::*'a*::*len word*) = *0*
unfolding *shiftr1-def* **by** *simp*

lemma *shiftr-1* [*simp*]:
  (*1*::′*a*::*len word*) >> *n* = (*if n* = *0 then 1 else 0*)
  by (*induct n*) (*auto simp*: *shiftr-def*)

lemma *word-less-1* [*simp*]:
  ((*x*::′*a*::*len word*) < *1*) = (*x* = *0*)
  by (*simp add*: *word-less-nat-alt unat-0-iff*)

lemma *to-bl-mask*:
  to-bl (*mask n* :: ′*a*::*len word*) =
  replicate (*len-of TYPE*(′*a*) − *n*) *False* @
    replicate (*min* (*len-of TYPE*(′*a*)) *n*) *True*
  by (*simp add*: *mask-bl word-rep-drop min-def*)

lemma *map-replicate-True*:
  n = *length xs* ⟹
    map (λ(*x*,*y*). *x* & *y*) (*zip xs* (*replicate n True*)) = *xs*
  by (*induct xs arbitrary*: *n*) *auto*

lemma *map-replicate-False*:
  n = *length xs* ⟹ *map* (λ(*x*,*y*). *x* & *y*)
    (*zip xs* (*replicate n False*)) = *replicate n False*
  by (*induct xs arbitrary*: *n*) *auto*

lemma *bl-and-mask*:
  fixes *w* :: ′*a*::*len word*
  fixes *n*
  defines *n*′ ≡ *len-of TYPE*(′*a*) − *n*
  shows *to-bl* (*w AND mask n*) = *replicate n*′ *False* @ *drop n*′ (*to-bl w*)
proof −
  note [*simp*] = *map-replicate-True map-replicate-False*
  have *to-bl* (*w AND mask n*) =
        map2 op & (*to-bl w*) (*to-bl* (*mask n*::′*a*::*len word*))
    by (*simp add*: *bl-word-and*)
  also
  have *to-bl w* = *take n*′ (*to-bl w*) @ *drop n*′ (*to-bl w*) **by** *simp*
  also
  have *map2 op* & . . . (*to-bl* (*mask n*::′*a*::*len word*)) =
        replicate n′ *False* @ *drop n*′ (*to-bl w*)
    unfolding *to-bl-mask n*′*-def map2-def*
    by (*subst zip-append*) *auto*
  finally
  show *?thesis* **.**
qed

lemma *drop-rev-takefill*:
  length xs ≤ *n* ⟹
    drop (*n* − *length xs*) (*rev* (*takefill False n* (*rev xs*))) = *xs*
  by (*simp add*: *takefill-alt rev-take*)

lemma *map-nth-0* [*simp*]:
  map (*op* !! (*0*::'*a*::*len0 word*)) *xs* = *replicate* (*length xs*) *False*
  by (*induct xs*) *auto*

lemma *uint-plus-if-size*:
  uint (*x* + *y*) =
  (*if uint x* + *uint y* < *2*ˆ*size x then*
      uint x + *uint y*
   else
      uint x + *uint y* − *2*ˆ*size x*)
  by (*simp add*: *word-arith-wis int-word-uint mod-add-if-z*
      word-size)

lemma *unat-plus-if-size*:
  unat (*x* + (*y*::'*a*::*len word*)) =
  (*if unat x* + *unat y* < *2*ˆ*size x then*
      unat x + *unat y*
   else
      unat x + *unat y* − *2*ˆ*size x*)
  apply (*subst word-arith-nat-defs*)
  apply (*subst unat-of-nat*)
  apply (*simp add*: *mod-nat-add word-size*)
  done

lemma *word-neq-0-conv*:
  fixes *w* :: '*a* :: *len word*
  shows ($w \neq 0$) = ($0 < w$)
  unfolding *word-gt-0* **by** *simp*

lemma *max-lt*:
  unat (*max a b div c*) = *unat* (*max a b*) *div unat* (*c*:: '*a* :: *len word*)
  by (*fact unat-div*)

lemma *uint-sub-if-size*:
  uint (*x* − *y*) =
  (*if uint y* ≤ *uint x then*
      uint x − *uint y*
   else
      uint x − *uint y* + *2*ˆ*size x*)
  by (*simp add*: *word-arith-wis int-word-uint mod-sub-if-z*
      word-size)

lemma *unat-sub*:
  b <= *a* ⟹ *unat* (*a* − *b*) = *unat a* − *unat b*
  by (*simp add*: *unat-def uint-sub-if-size word-le-def nat-diff-distrib*)

lemmas *word-less-sub1-numberof* [*simp*] = *word-less-sub1* [*of numeral w*] **for** *w*
lemmas *word-le-sub1-numberof* [*simp*] = *word-le-sub1* [*of numeral w*] **for** *w*

lemma *word-of-int-minus*:
  word-of-int $(2\hat{\ }len\text{-}of\ TYPE('a) - i) = (word\text{-}of\text{-}int\ (-i)::'a::len\ word)$
proof −
  have *x*: $2\hat{\ }len\text{-}of\ TYPE('a) - i = -i + 2\hat{\ }len\text{-}of\ TYPE('a)$ **by** *simp*
  show *?thesis*
    apply (*subst x*)
    apply (*subst word-uint.Abs-norm* [*symmetric*], *subst mod-add-self2*)
    apply *simp*
    done
qed

lemmas *word-of-int-inj* =
  word-uint.Abs-inject [*unfolded uints-num*, *simplified*]

lemma *word-le-less-eq*:
  $(x ::'z::len\ word) \le y = (x = y \vee x < y)$
  by (*auto simp add*: *order-class.le-less*)

lemma *mod-plus-cong*:
  assumes *1*: $(b::int) = b'$
    and *2*: $x\ mod\ b' = x'\ mod\ b'$
    and *3*: $y\ mod\ b' = y'\ mod\ b'$
    and *4*: $x' + y' = z'$
  shows $(x + y)\ mod\ b = z'\ mod\ b'$
proof −
  from *1 2*[*symmetric*] *3*[*symmetric*] **have** $(x + y)\ mod\ b = (x'\ mod\ b' + y'\ mod\ b')\ mod\ b'$
    by (*simp add*: *mod-add-eq*[*symmetric*])
  also have $\ldots = (x' + y')\ mod\ b'$
    by (*simp add*: *mod-add-eq*[*symmetric*])
  finally show *?thesis* **by** (*simp add*: *4*)
qed

lemma *mod-minus-cong*:
  assumes *1*: $(b::int) = b'$
    and *2*: $x\ mod\ b' = x'\ mod\ b'$
    and *3*: $y\ mod\ b' = y'\ mod\ b'$
    and *4*: $x' - y' = z'$
  shows $(x - y)\ mod\ b = z'\ mod\ b'$
  using *assms*
  apply (*subst mod-diff-left-eq*)
  apply (*subst mod-diff-right-eq*)
  apply (*simp add*: *mod-diff-left-eq* [*symmetric*] *mod-diff-right-eq* [*symmetric*])
  done

lemma *word-induct-less*:
  $[\![P\ (0::'a::len\ word); \bigwedge n.\ [\![n < m; P\ n]\!] \Longrightarrow P\ (1 + n)]\!] \Longrightarrow P\ m$
  apply (*cases m*)

apply *atomize*
apply (*erule rev-mp*)+
apply (*rule-tac x=m* **in** *spec*)
apply (*induct-tac n*)
apply *simp*
apply *clarsimp*
apply (*erule impE*)
apply *clarsimp*
apply (*erule-tac x=n* **in** *allE*)
apply (*erule impE*)
apply (*simp add*: *unat-arith-simps*)
apply (*clarsimp simp*: *unat-of-nat*)
apply *simp*
apply (*erule-tac x=of-nat na* **in** *allE*)
apply (*erule impE*)
apply (*simp add*: *unat-arith-simps*)
apply (*clarsimp simp*: *unat-of-nat*)
apply *simp*
done

lemma *word-induct*:
⟦*P* (*0*::*'a*::*len word*); ⋀*n*. *P n* ⟹ *P* (*1* + *n*)⟧ ⟹ *P m*
by (*erule word-induct-less*, *simp*)

lemma *word-induct2* [*induct type*]:
⟦*P 0*; ⋀*n*. ⟦*1* + *n* ≠ *0*; *P n*⟧ ⟹ *P* (*1* + *n*)⟧ ⟹ *P* (*n*::*'b*::*len word*)
apply (*rule word-induct*, *simp*)
apply (*case-tac 1*+*n* = *0*, *auto*)
done

16.29 Recursion combinator for words

definition *word-rec* :: *'a* ⇒ (*'b*::*len word* ⇒ *'a* ⇒ *'a*) ⇒ *'b word* ⇒ *'a*
where
word-rec forZero forSuc n = *rec-nat forZero* (*forSuc* ∘ *of-nat*) (*unat n*)

lemma *word-rec-0*: *word-rec z s 0* = *z*
by (*simp add*: *word-rec-def*)

lemma *word-rec-Suc*:
1 + *n* ≠ (*0*::*'a*::*len word*) ⟹ *word-rec z s* (*1* + *n*) = *s n* (*word-rec z s n*)
apply (*simp add*: *word-rec-def unat-word-ariths*)
apply (*subst nat-mod-eq'*)
apply (*metis Suc-eq-plus1-left Suc-lessI of-nat-2p unat-1 unat-lt2p word-arith-nat-add*)
apply *simp*
done

lemma *word-rec-Pred*:
n ≠ *0* ⟹ *word-rec z s n* = *s* (*n* − *1*) (*word-rec z s* (*n* − *1*))

apply (*rule subst*[**where** *t=n* **and** *s=1* + (*n* − *1*)])
apply *simp*
apply (*subst word-rec-Suc*)
apply *simp*
apply *simp*
done

lemma *word-rec-in*:
f (*word-rec z* (λ-. *f*) *n*) = *word-rec* (*f z*) (λ-. *f*) *n*
by (*induct n*) (*simp-all add*: *word-rec-0 word-rec-Suc*)

lemma *word-rec-in2*:
f n (*word-rec z f n*) = *word-rec* (*f 0 z*) (*f* ∘ *op* + *1*) *n*
by (*induct n*) (*simp-all add*: *word-rec-0 word-rec-Suc*)

lemma *word-rec-twice*:
m ≤ *n* ⟹ *word-rec z f n* = *word-rec* (*word-rec z f* (*n* − *m*)) (*f* ∘ *op* + (*n* − *m*)) *m*
apply (*erule rev-mp*)
apply (*rule-tac x=z* **in** *spec*)
apply (*rule-tac x=f* **in** *spec*)
apply (*induct n*)
apply (*simp add*: *word-rec-0*)
apply *clarsimp*
apply (*rule-tac t=1* + *n* − *m* **and** *s=1* + (*n* − *m*) **in** *subst*)
apply *simp*
apply (*case-tac 1* + (*n* − *m*) = *0*)
apply (*simp add*: *word-rec-0*)
apply (*rule-tac f* = *word-rec a b* **for** *a b* **in** *arg-cong*)
apply (*rule-tac t=m* **and** *s=m* + (*1* + (*n* − *m*)) **in** *subst*)
apply *simp*
apply (*simp* (*no-asm-use*))
apply (*simp add*: *word-rec-Suc word-rec-in2*)
apply (*erule impE*)
apply *uint-arith*
apply (*drule-tac x=x* ∘ *op* + *1* **in** *spec*)
apply (*drule-tac x=x 0 xa* **in** *spec*)
apply *simp*
apply (*rule-tac t=*λ*a. x* (*1* + (*n* − *m* + *a*)) **and** *s=*λ*a. x* (*1* + (*n* − *m*) + *a*)
in *subst*)
apply (*clarsimp simp add*: *fun-eq-iff*)
apply (*rule-tac t=*(*1* + (*n* − *m* + *xb*)) **and** *s=1* + (*n* − *m*) + *xb* **in** *subst*)
apply *simp*
apply (*rule refl*)
apply (*rule refl*)
done

lemma *word-rec-id*: *word-rec z* (λ-. *id*) *n* = *z*
by (*induct n*) (*auto simp add*: *word-rec-0 word-rec-Suc*)

lemma *word-rec-id-eq*: $\forall m < n.\ f\ m = id \Longrightarrow word\text{-}rec\ z\ f\ n = z$
apply (*erule rev-mp*)
apply (*induct n*)
 apply (*auto simp add*: *word-rec-0 word-rec-Suc*)
 apply (*drule spec*, *erule mp*)
 apply *uint-arith*
apply (*drule-tac x=n* **in** *spec*, *erule impE*)
 apply *uint-arith*
apply *simp*
done

lemma *word-rec-max*:
 $\forall m \geq n.\ m \neq -1 \longrightarrow f\ m = id \Longrightarrow word\text{-}rec\ z\ f\ (-1) = word\text{-}rec\ z\ f\ n$
apply (*subst word-rec-twice*[**where** *n=−1* **and** *m=−1 − n*])
 apply *simp*
apply *simp*
apply (*rule word-rec-id-eq*)
apply *clarsimp*
apply (*drule spec*, *rule mp*, *erule mp*)
 apply (*rule word-plus-mono-right2*[*OF - order-less-imp-le*])
  prefer *2*
  apply *assumption*
 apply *simp*
apply (*erule contrapos-pn*)
apply *simp*
apply (*drule arg-cong*[**where** $f = \lambda x.\ x - n$])
apply *simp*
done

lemma *unatSuc*:
 $1 + n \neq (0::'a::len\ word) \Longrightarrow unat\ (1 + n) = Suc\ (unat\ n)$
 by *unat-arith*

declare *bin-to-bl-def* [*simp*]

ML-file *Tools/word-lib.ML*
ML-file *Tools/smt-word.ML*

hide-const (**open**) *Word*

end

References

[1] Jeremy Dawson. Isabelle theories for machine words. In Michael Goldsmith and Bill Roscoe, editors, *Seventh International Workshop on Automated Verification of Critical Systems (AVOCS'07)*, Electronic Notes

in Theoretical Computer Science, page 15, Oxford, September 2007. Elsevier. to appear.