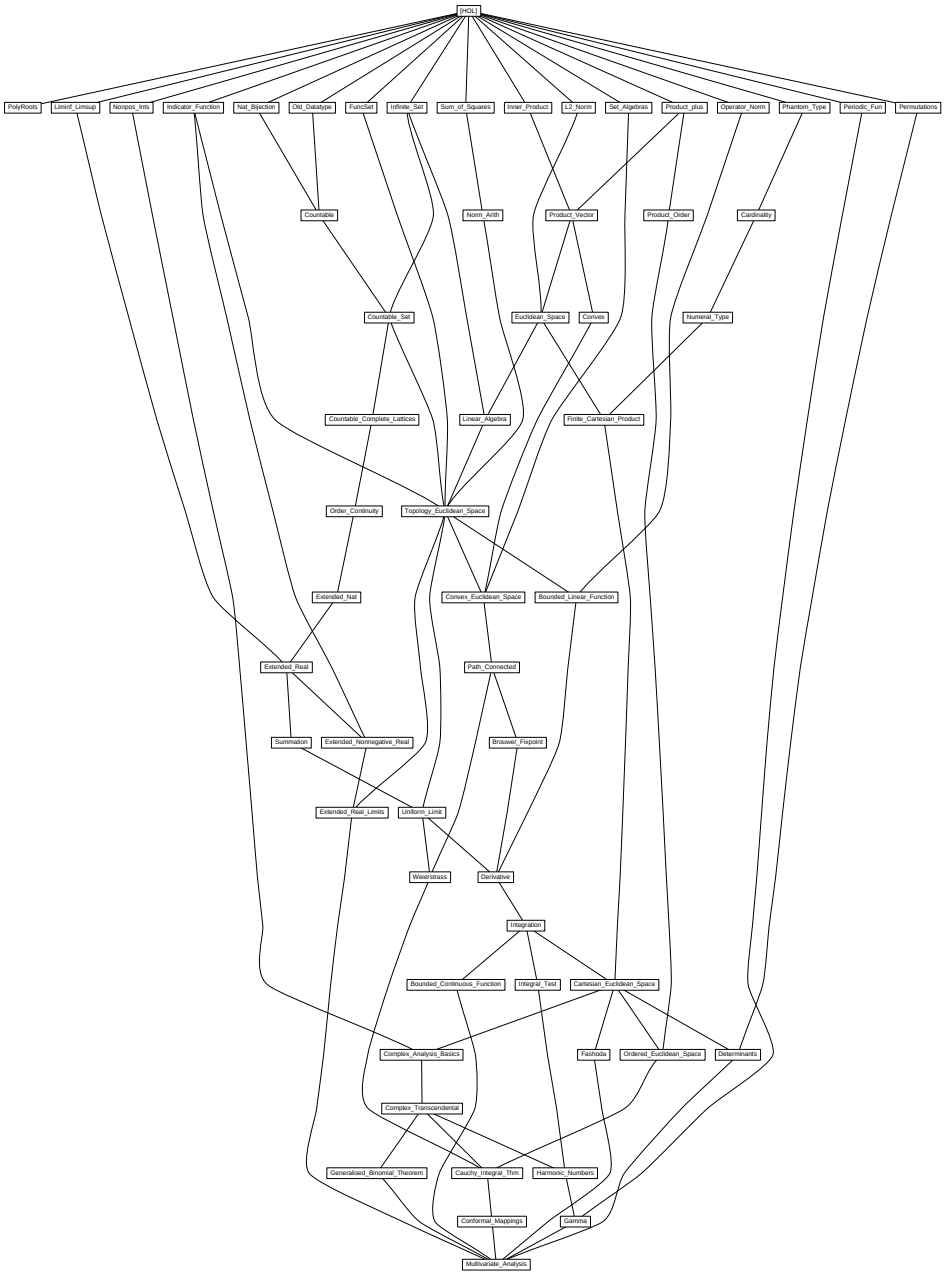
Multivariate Analysis

April 17, 2016

Contents


HOL1
PolyRoots
Limit_Limsup
Nonpos_Ints
Indicator_Function
Nat_Bijection
Old_Datatype
FuncSet
Infinite_Set
Sum_of_Squares
Inner_Product
L2_Norm
Set_Algebras
Product_plus
Operator_Norm
Phantom_Type
Periodic_Fun
Permutations
Countable
Norm_Arith
Product_Vector
Product_Order
Cardinality
Countable_Set
Euclidean_Space
Convex
Numeral_Type
Countable_Complete_Lattices
Linear_Algebra
Finite_Cartesian_Product
Order_Continuity
Topology_Euclidean_Space
Extended_Nat
Convex_Euclidean_Space
Bounded_Linear_Function
Extended_Real
Path_Connected
Summation
Extended_Nonnegative_Real
Brouwer_Fixpoint
Extended_Real_Limits
Uniform_Limit
Weierstrass
Derivative
Integration
Bounded_Continuous_Function
Integral_Test
Cartesian_Euclidean_Space
Complex_Analysis_Basics
Fashoda
Ordered_Euclidean_Space
Determinants
Complex_Transcendental
Generalised_Binomial_Theorem
Cauchy_Integral_Thm
Harmonic_Numbers
Conformal_Mappings
Gamma
Multivariate_Analysis

1 Indicator Function

theory *Indicator-Function*
imports *Complex-Main*
begin

definition *indicator S x* = (*if x* ∈ *S then 1 else 0*)

lemma *indicator-simps*[*simp*]:
x ∈ *S* ⟹ *indicator S x* = *1*
x ∉ *S* ⟹ *indicator S x* = *0*
unfolding *indicator-def* **by** *auto*

lemma *indicator-pos-le*[*intro*, *simp*]: (*0*::'*a*::*linordered-semidom*) ≤ *indicator S x*
and *indicator-le-1*[*intro*, *simp*]: *indicator S x* ≤ (*1*::'*a*::*linordered-semidom*)
unfolding *indicator-def* **by** *auto*

lemma *indicator-abs-le-1*: |*indicator S x*| ≤ (*1*::'*a*::*linordered-idom*)
unfolding *indicator-def* **by** *auto*

lemma *indicator-eq-0-iff*: *indicator A x* = (*0*::-::*zero-neq-one*) ⟷ *x* ∉ *A*
by (*auto simp*: *indicator-def*)

lemma *indicator-eq-1-iff*: *indicator A x* = (*1*::-::*zero-neq-one*) ⟷ *x* ∈ *A*
by (*auto simp*: *indicator-def*)

lemma *split-indicator*: *P* (*indicator S x*) ⟷ ((*x* ∈ *S* ⟶ *P 1*) ∧ (*x* ∉ *S* ⟶ *P 0*))
unfolding *indicator-def* **by** *auto*

lemma *split-indicator-asm*: *P* (*indicator S x*) ⟷ (¬ (*x* ∈ *S* ∧ ¬ *P 1* ∨ *x* ∉ *S* ∧ ¬ *P 0*))
unfolding *indicator-def* **by** *auto*

lemma *indicator-inter-arith*: *indicator* (*A* ∩ *B*) *x* = *indicator A x* ∗ (*indicator B x*::'*a*::*semiring-1*)
unfolding *indicator-def* **by** (*auto simp*: *min-def max-def*)

lemma *indicator-union-arith*: *indicator* (*A* ∪ *B*) *x* = *indicator A x* + *indicator B x* − *indicator A x* ∗ (*indicator B x*::'*a*::*ring-1*)
unfolding *indicator-def* **by** (*auto simp*: *min-def max-def*)

lemma *indicator-inter-min*: *indicator* (*A* ∩ *B*) *x* = *min* (*indicator A x*) (*indicator B x*::'*a*::*linordered-semidom*)
and *indicator-union-max*: *indicator* (*A* ∪ *B*) *x* = *max* (*indicator A x*) (*indicator B x*::'*a*::*linordered-semidom*)
unfolding *indicator-def* **by** (*auto simp*: *min-def max-def*)

lemma *indicator-disj-union*: *A* ∩ *B* = {} ⟹ *indicator* (*A* ∪ *B*) *x* = (*indicator*

A x + indicator B x::*'a*::*linordered-semidom*)
by (*auto split*: *split-indicator*)

lemma *indicator-compl*: *indicator* (− *A*) *x* = *1* − (*indicator A x*::*'a*::*ring-1*)
and *indicator-diff*: *indicator* (*A* − *B*) *x* = *indicator A x* ∗ (*1* − *indicator B x*::*'a*::*ring-1*)
unfolding *indicator-def* **by** (*auto simp*: *min-def max-def*)

lemma *indicator-times*: *indicator* (*A* × *B*) *x* = *indicator A* (*fst x*) ∗ (*indicator B* (*snd x*)::*'a*::*semiring-1*)
unfolding *indicator-def* **by** (*cases x*) *auto*

lemma *indicator-sum*: *indicator* (*A* <+> *B*) *x* = (*case x of Inl x* ⇒ *indicator A x* | *Inr x* ⇒ *indicator B x*)
unfolding *indicator-def* **by** (*cases x*) *auto*

lemma *indicator-image*: *inj f* ⟹ *indicator* (*f* ‘ *X*) (*f x*) = (*indicator X x*::-::*zero-neq-one*)
by (*auto simp*: *indicator-def inj-on-def*)

lemma *indicator-vimage*: *indicator* (*f* −‘ *A*) *x* = *indicator A* (*f x*)
by(*auto split*: *split-indicator*)

lemma
fixes *f* :: *'a* ⇒ *'b*::*semiring-1* **assumes** *finite A*
shows *setsum-mult-indicator*[*simp*]: (∑ *x* ∈ *A*. *f x* ∗ *indicator B x*) = (∑ *x* ∈ *A* ∩ *B*. *f x*)
and *setsum-indicator-mult*[*simp*]: (∑ *x* ∈ *A*. *indicator B x* ∗ *f x*) = (∑ *x* ∈ *A* ∩ *B*. *f x*)
unfolding *indicator-def*
using *assms* **by** (*auto intro*!: *setsum.mono-neutral-cong-right split*: *if-split-asm*)

lemma *setsum-indicator-eq-card*:
assumes *finite A*
shows (∑ *x* ∈ *A*. *indicator B x*) = *card* (*A Int B*)
using *setsum-mult-indicator*[*OF assms*, *of* %*x*. *1*::*nat*]
unfolding *card-eq-setsum* **by** *simp*

lemma *setsum-indicator-scaleR*[*simp*]:
finite A ⟹
(∑ *x* ∈ *A*. *indicator* (*B x*) (*g x*) ∗*R f x*) = (∑ *x* ∈ {*x*∈*A*. *g x* ∈ *B x*}. *f x*::*'a*::*real-vector*)
using *assms* **by** (*auto intro*!: *setsum.mono-neutral-cong-right split*: *if-split-asm simp*: *indicator-def*)

lemma *LIMSEQ-indicator-incseq*:
assumes *incseq A*
shows (λ*i*. *indicator* (*A i*) *x* :: *'a* :: {*topological-space*, *one*, *zero*}) ⟶ *indicator* (⋃ *i*. *A i*) *x*
proof *cases*

assume $\exists i.\ x \in A\ i$
then obtain i **where** $x \in A\ i$
by *auto*
then have
$\bigwedge n.\ (indicator\ (A\ (n + i))\ x :: 'a) = 1$
$(indicator\ (\bigcup i.\ A\ i)\ x :: 'a) = 1$
using *incseqD*[*OF* ‹*incseq A*›, *of i n* + *i* **for** *n*] ‹$x \in A\ i$› **by** (*auto simp*: *indicator-def*)
then show *?thesis*
by (*rule-tac LIMSEQ-offset*[*of* - *i*]) *simp*
qed (*auto simp*: *indicator-def*)

lemma *LIMSEQ-indicator-UN*:
$(\lambda k.\ indicator\ (\bigcup i{<}k.\ A\ i)\ x :: 'a :: \{topological\text{-}space, one, zero\}) \longrightarrow indicator\ (\bigcup i.\ A\ i)\ x$
proof −
have $(\lambda k.\ indicator\ (\bigcup i{<}k.\ A\ i)\ x{::}'a) \longrightarrow indicator\ (\bigcup k.\ \bigcup i{<}k.\ A\ i)\ x$
by (*intro LIMSEQ-indicator-incseq*) (*auto simp*: *incseq-def intro*: *less-le-trans*)
also have $(\bigcup k.\ \bigcup i{<}k.\ A\ i) = (\bigcup i.\ A\ i)$
by *auto*
finally show *?thesis* **.**
qed

lemma *LIMSEQ-indicator-decseq*:
assumes *decseq A*
shows $(\lambda i.\ indicator\ (A\ i)\ x :: 'a :: \{topological\text{-}space, one, zero\}) \longrightarrow indicator\ (\bigcap i.\ A\ i)\ x$
proof *cases*
assume $\exists i.\ x \notin A\ i$
then obtain i **where** $x \notin A\ i$
by *auto*
then have
$\bigwedge n.\ (indicator\ (A\ (n + i))\ x :: 'a) = 0$
$(indicator\ (\bigcap i.\ A\ i)\ x :: 'a) = 0$
using *decseqD*[*OF* ‹*decseq A*›, *of i n* + *i* **for** *n*] ‹$x \notin A\ i$› **by** (*auto simp*: *indicator-def*)
then show *?thesis*
by (*rule-tac LIMSEQ-offset*[*of* - *i*]) *simp*
qed (*auto simp*: *indicator-def*)

lemma *LIMSEQ-indicator-INT*:
$(\lambda k.\ indicator\ (\bigcap i{<}k.\ A\ i)\ x :: 'a :: \{topological\text{-}space, one, zero\}) \longrightarrow indicator\ (\bigcap i.\ A\ i)\ x$
proof −
have $(\lambda k.\ indicator\ (\bigcap i{<}k.\ A\ i)\ x{::}'a) \longrightarrow indicator\ (\bigcap k.\ \bigcap i{<}k.\ A\ i)\ x$
by (*intro LIMSEQ-indicator-decseq*) (*auto simp*: *decseq-def intro*: *less-le-trans*)
also have $(\bigcap k.\ \bigcap i{<}k.\ A\ i) = (\bigcap i.\ A\ i)$
by *auto*
finally show *?thesis* **.**

qed

lemma *indicator-add*:
$A \cap B = \{\} \Longrightarrow (indicator\ A\ x::-::monoid\text{-}add) + indicator\ B\ x = indicator\ (A \cup B)\ x$
unfolding *indicator-def* **by** *auto*

lemma *of-real-indicator*: *of-real* (*indicator A x*) = *indicator A x*
by (*simp split*: *split-indicator*)

lemma *real-of-nat-indicator*: *real* (*indicator A x* :: *nat*) = *indicator A x*
by (*simp split*: *split-indicator*)

lemma *abs-indicator*: $|indicator\ A\ x :: 'a::linordered\text{-}idom| = indicator\ A\ x$
by (*simp split*: *split-indicator*)

lemma *mult-indicator-subset*:
$A \subseteq B \Longrightarrow indicator\ A\ x * indicator\ B\ x = (indicator\ A\ x :: 'a::\{comm\text{-}semiring\text{-}1\})$
by (*auto split*: *split-indicator simp*: *fun-eq-iff*)

lemma *indicator-sums*:
assumes $\bigwedge i\ j.\ i \neq j \Longrightarrow A\ i \cap A\ j = \{\}$
shows $(\lambda i.\ indicator\ (A\ i)\ x::real)\ sums\ indicator\ (\bigcup i.\ A\ i)\ x$
proof *cases*
assume $\exists i.\ x \in A\ i$
then obtain *i* **where** *i*: $x \in A\ i$ **..**
with *assms* **have** $(\lambda i.\ indicator\ (A\ i)\ x::real)\ sums\ (\sum i \in \{i\}.\ indicator\ (A\ i)\ x)$
by (*intro sums-finite*) (*auto split*: *split-indicator*)
also have $(\sum i \in \{i\}.\ indicator\ (A\ i)\ x) = indicator\ (\bigcup i.\ A\ i)\ x$
using *i* **by** (*auto split*: *split-indicator*)
finally show *?thesis* **.**
qed *simp*

end

2 Old Datatype package: constructing datatypes from Cartesian Products and Disjoint Sums

theory *Old-Datatype*
imports *../Main*
keywords *old-datatype* :: *thy-decl*
begin

ML-file ~~/*src/HOL/Tools/datatype-realizer.ML*

2.1 The datatype universe

definition *Node* = {*p*. *EX f x k*. *p* = (*f* :: *nat* => *'b* + *nat*, *x* ::*'a* + *nat*) & *f k* = *Inr 0*}

typedef (*'a*, *'b*) *node* = *Node* :: ((*nat* => *'b* + *nat*) * (*'a* + *nat*)) *set*
morphisms *Rep-Node Abs-Node*
unfolding *Node-def* **by** *auto*

Datatypes will be represented by sets of type *node*

type-synonym *'a item* = (*'a*, *unit*) *node set*
type-synonym (*'a*, *'b*) *dtree* = (*'a*, *'b*) *node set*

definition *Push* :: [(*'b* + *nat*), *nat* => (*'b* + *nat*)] => (*nat* => (*'b* + *nat*))

where *Push* == (%*b h*. *case-nat b h*)

definition *Push-Node* :: [(*'b* + *nat*), (*'a*, *'b*) *node*] => (*'a*, *'b*) *node*
where *Push-Node* == (%*n x*. *Abs-Node* (*apfst* (*Push n*) (*Rep-Node x*)))

definition *Atom* :: (*'a* + *nat*) => (*'a*, *'b*) *dtree*
where *Atom* == (%*x*. {*Abs-Node*((%*k*. *Inr 0*, *x*))})
definition *Scons* :: [(*'a*, *'b*) *dtree*, (*'a*, *'b*) *dtree*] => (*'a*, *'b*) *dtree*
where *Scons M N* == (*Push-Node* (*Inr 1*) ‘ *M*) *Un* (*Push-Node* (*Inr* (*Suc 1*)) ‘ *N*)

definition *Leaf* :: *'a* => (*'a*, *'b*) *dtree*
where *Leaf* == *Atom o Inl*
definition *Numb* :: *nat* => (*'a*, *'b*) *dtree*
where *Numb* == *Atom o Inr*

definition *In0* :: (*'a*, *'b*) *dtree* => (*'a*, *'b*) *dtree*
where *In0*(*M*) == *Scons* (*Numb 0*) *M*
definition *In1* :: (*'a*, *'b*) *dtree* => (*'a*, *'b*) *dtree*
where *In1*(*M*) == *Scons* (*Numb 1*) *M*

definition *Lim* :: (*'b* => (*'a*, *'b*) *dtree*) => (*'a*, *'b*) *dtree*
where *Lim f* == $\bigcup${*z*. *?* *x*. *z* = *Push-Node* (*Inl x*) ‘ (*f x*)}

definition *ndepth* :: (*'a*, *'b*) *node* => *nat*
where *ndepth*(*n*) == (%(*f*,*x*). *LEAST k*. *f k* = *Inr 0*) (*Rep-Node n*)
definition *ntrunc* :: [*nat*, (*'a*, *'b*) *dtree*] => (*'a*, *'b*) *dtree*

where *ntrunc k N == {n. n:N & ndepth(n)<k}*

definition *uprod* :: *[('a, 'b) dtree set, ('a, 'b) dtree set]=> ('a, 'b) dtree set*
where *uprod A B == UN x:A. UN y:B. { Scons x y }*
definition *usum* :: *[('a, 'b) dtree set, ('a, 'b) dtree set]=> ('a, 'b) dtree set*
where *usum A B == In0`A Un In1`B*

definition *Split* :: *[[('a, 'b) dtree, ('a, 'b) dtree]=>'c, ('a, 'b) dtree] => 'c*
where *Split c M == THE u. EX x y. M = Scons x y & u = c x y*

definition *Case* :: *[[('a, 'b) dtree]=>'c, [('a, 'b) dtree]=>'c, ('a, 'b) dtree] => 'c*
where *Case c d M == THE u. (EX x . M = In0(x) & u = c(x)) | (EX y . M = In1(y) & u = d(y))*

definition *dprod* :: *[(('a, 'b) dtree * ('a, 'b) dtree)set, (('a, 'b) dtree * ('a, 'b) dtree)set]*
*=> (('a, 'b) dtree * ('a, 'b) dtree)set*
where *dprod r s == UN (x,x'):r. UN (y,y'):s. {(Scons x y, Scons x' y')}*

definition *dsum* :: *[(('a, 'b) dtree * ('a, 'b) dtree)set, (('a, 'b) dtree * ('a, 'b) dtree)set]*
*=> (('a, 'b) dtree * ('a, 'b) dtree)set*
where *dsum r s == (UN (x,x'):r. {(In0(x),In0(x'))}) Un (UN (y,y'):s. {(In1(y),In1(y'))})*

lemma *apfst-convE*:
[| q = apfst f p; !!x y. [| p = (x,y); q = (f(x),y) |] ==> R |] ==> R
by (*force simp add*: *apfst-def*)

lemma *Push-inject1*: *Push i f = Push j g ==> i=j*
apply (*simp add*: *Push-def fun-eq-iff*)
apply (*drule-tac x=0* **in** *spec*, *simp*)
done

lemma *Push-inject2*: *Push i f = Push j g ==> f=g*
apply (*auto simp add*: *Push-def fun-eq-iff*)
apply (*drule-tac x=Suc x* **in** *spec*, *simp*)
done

lemma *Push-inject*:
[| Push i f =Push j g; [| i=j; f=g |] ==> P |] ==> P

by (*blast dest*: *Push-inject1 Push-inject2*)

lemma *Push-neq-K0*: *Push* (*Inr* (*Suc k*)) *f* = (%*z*. *Inr 0*) ==> *P*
by (*auto simp add*: *Push-def fun-eq-iff split*: *nat.split-asm*)

lemmas *Abs-Node-inj* = *Abs-Node-inject* [*THEN* [*2*] *rev-iffD1*]

lemma *Node-K0-I*: (%*k*. *Inr 0*, *a*) : *Node*
by (*simp add*: *Node-def*)

lemma *Node-Push-I*: *p*: *Node* ==> *apfst* (*Push i*) *p* : *Node*
apply (*simp add*: *Node-def Push-def*)
apply (*fast intro*!: *apfst-conv nat.case*(*2*)[*THEN trans*])
done

2.2 Freeness: Distinctness of Constructors

lemma *Scons-not-Atom* [*iff*]: *Scons M N* ≠ *Atom*(*a*)
unfolding *Atom-def Scons-def Push-Node-def One-nat-def*
by (*blast intro*: *Node-K0-I Rep-Node* [*THEN Node-Push-I*]
dest!: *Abs-Node-inj*
elim!: *apfst-convE sym* [*THEN Push-neq-K0*])

lemmas *Atom-not-Scons* [*iff*] = *Scons-not-Atom* [*THEN not-sym*]

lemma *inj-Atom*: *inj*(*Atom*)
apply (*simp add*: *Atom-def*)
apply (*blast intro*!: *inj-onI Node-K0-I dest*!: *Abs-Node-inj*)
done
lemmas *Atom-inject* = *inj-Atom* [*THEN injD*]

lemma *Atom-Atom-eq* [*iff*]: (*Atom*(*a*)=*Atom*(*b*)) = (*a*=*b*)
by (*blast dest*!: *Atom-inject*)

lemma *inj-Leaf*: *inj*(*Leaf*)
apply (*simp add*: *Leaf-def o-def*)
apply (*rule inj-onI*)
apply (*erule Atom-inject* [*THEN Inl-inject*])
done

lemmas *Leaf-inject* [*dest*!] = *inj-Leaf* [*THEN injD*]

lemma *inj-Numb*: *inj*(*Numb*)
apply (*simp add*: *Numb-def o-def*)
apply (*rule inj-onI*)
apply (*erule Atom-inject* [*THEN Inr-inject*])
done

lemmas *Numb-inject* [*dest*!] = *inj-Numb* [*THEN injD*]

lemma *Push-Node-inject*:
[| *Push-Node i m* =*Push-Node j n*; [| *i*=*j*; *m*=*n* |] ==> *P*
|] ==> *P*
apply (*simp add*: *Push-Node-def*)
apply (*erule Abs-Node-inj* [*THEN apfst-convE*])
apply (*rule Rep-Node* [*THEN Node-Push-I*])+
apply (*erule sym* [*THEN apfst-convE*])
apply (*blast intro*: *Rep-Node-inject* [*THEN iffD1*] *trans sym elim*!: *Push-inject*)
done

lemma *Scons-inject-lemma1*: *Scons M N* <= *Scons M' N'* ==> *M*<=*M'*
unfolding *Scons-def One-nat-def*
by (*blast dest*!: *Push-Node-inject*)

lemma *Scons-inject-lemma2*: *Scons M N* <= *Scons M' N'* ==> *N*<=*N'*
unfolding *Scons-def One-nat-def*
by (*blast dest*!: *Push-Node-inject*)

lemma *Scons-inject1*: *Scons M N* = *Scons M' N'* ==> *M*=*M'*
apply (*erule equalityE*)
apply (*iprover intro*: *equalityI Scons-inject-lemma1*)
done

lemma *Scons-inject2*: *Scons M N* = *Scons M' N'* ==> *N*=*N'*
apply (*erule equalityE*)
apply (*iprover intro*: *equalityI Scons-inject-lemma2*)
done

lemma *Scons-inject*:
[| *Scons M N* = *Scons M' N'*; [| *M*=*M'*; *N*=*N'* |] ==> *P* |] ==> *P*
by (*iprover dest*: *Scons-inject1 Scons-inject2*)

lemma *Scons-Scons-eq* [*iff*]: (*Scons M N* = *Scons M' N'*) = (*M*=*M'* & *N*=*N'*)
by (*blast elim*!: *Scons-inject*)

lemma *Scons-not-Leaf* [*iff*]: *Scons M N* ≠ *Leaf*(*a*)
unfolding *Leaf-def o-def* **by** (*rule Scons-not-Atom*)

lemmas *Leaf-not-Scons* [*iff*] = *Scons-not-Leaf* [*THEN not-sym*]

lemma *Scons-not-Numb* [*iff*]: *Scons M N* ≠ *Numb*(*k*)
unfolding *Numb-def o-def* **by** (*rule Scons-not-Atom*)

lemmas *Numb-not-Scons* [*iff*] = *Scons-not-Numb* [*THEN not-sym*]

lemma *Leaf-not-Numb* [*iff*]: *Leaf*(*a*) ≠ *Numb*(*k*)
by (*simp add*: *Leaf-def Numb-def*)

lemmas *Numb-not-Leaf* [*iff*] = *Leaf-not-Numb* [*THEN not-sym*]

lemma *ndepth-K0*: *ndepth* (*Abs-Node*(%*k*. *Inr 0*, *x*)) = *0*
by (*simp add*: *ndepth-def Node-K0-I* [*THEN Abs-Node-inverse*] *Least-equality*)

lemma *ndepth-Push-Node-aux*:
case-nat (*Inr* (*Suc i*)) *f k* = *Inr 0* −−> *Suc*(*LEAST x*. *f x* = *Inr 0*) <= *k*
apply (*induct-tac k*, *auto*)
apply (*erule Least-le*)
done

lemma *ndepth-Push-Node*:
ndepth (*Push-Node* (*Inr* (*Suc i*)) *n*) = *Suc*(*ndepth*(*n*))
apply (*insert Rep-Node* [*of n*, *unfolded Node-def*])
apply (*auto simp add*: *ndepth-def Push-Node-def*
Rep-Node [*THEN Node-Push-I*, *THEN Abs-Node-inverse*])
apply (*rule Least-equality*)
apply (*auto simp add*: *Push-def ndepth-Push-Node-aux*)
apply (*erule LeastI*)
done

lemma *ntrunc-0* [*simp*]: *ntrunc 0 M* = {}
by (*simp add*: *ntrunc-def*)

lemma *ntrunc-Atom* [*simp*]: *ntrunc* (*Suc k*) (*Atom a*) = *Atom*(*a*)
by (*auto simp add*: *Atom-def ntrunc-def ndepth-K0*)

lemma *ntrunc-Leaf* [*simp*]: *ntrunc* (*Suc k*) (*Leaf a*) = *Leaf*(*a*)
unfolding *Leaf-def o-def* **by** (*rule ntrunc-Atom*)

lemma *ntrunc-Numb* [*simp*]: *ntrunc* (*Suc k*) (*Numb i*) = *Numb*(*i*)
unfolding *Numb-def o-def* **by** (*rule ntrunc-Atom*)

lemma *ntrunc-Scons* [*simp*]:
ntrunc (*Suc k*) (*Scons M N*) = *Scons* (*ntrunc k M*) (*ntrunc k N*)
unfolding *Scons-def ntrunc-def One-nat-def*
by (*auto simp add*: *ndepth-Push-Node*)

lemma *ntrunc-one-In0* [*simp*]: *ntrunc* (*Suc 0*) (*In0 M*) = {}
apply (*simp add*: *In0-def*)
apply (*simp add*: *Scons-def*)
done

lemma *ntrunc-In0* [*simp*]: *ntrunc* (*Suc*(*Suc k*)) (*In0 M*) = *In0* (*ntrunc* (*Suc k*) *M*)
by (*simp add*: *In0-def*)

lemma *ntrunc-one-In1* [*simp*]: *ntrunc* (*Suc 0*) (*In1 M*) = {}
apply (*simp add*: *In1-def*)
apply (*simp add*: *Scons-def*)
done

lemma *ntrunc-In1* [*simp*]: *ntrunc* (*Suc*(*Suc k*)) (*In1 M*) = *In1* (*ntrunc* (*Suc k*) *M*)
by (*simp add*: *In1-def*)

2.3 Set Constructions

lemma *uprodI* [*intro*!]: [| *M*:*A*; *N*:*B* |] ==> *Scons M N* : *uprod A B*
by (*simp add*: *uprod-def*)

lemma *uprodE* [*elim*!]:
[| *c* : *uprod A B*;
!!*x y*. [| *x*:*A*; *y*:*B*; *c* = *Scons x y* |] ==> *P*

$|]$ ==> P
by (*auto simp add*: *uprod-def*)

lemma *uprodE2*: [| *Scons M N* : *uprod A B*; [| M:A; N:B |] ==> P |] ==> P
by (*auto simp add*: *uprod-def*)

lemma *usum-In0I* [*intro*]: M:A ==> *In0*(M) : *usum A B*
by (*simp add*: *usum-def*)

lemma *usum-In1I* [*intro*]: N:B ==> *In1*(N) : *usum A B*
by (*simp add*: *usum-def*)

lemma *usumE* [*elim*!]:
[| u : *usum A B*;
!!x. [| x:A; u=*In0*(x) |] ==> P;
!!y. [| y:B; u=*In1*(y) |] ==> P
|] ==> P
by (*auto simp add*: *usum-def*)

lemma *In0-not-In1* [*iff*]: *In0*(M) ≠ *In1*(N)
unfolding *In0-def In1-def One-nat-def* **by** *auto*

lemmas *In1-not-In0* [*iff*] = *In0-not-In1* [*THEN not-sym*]

lemma *In0-inject*: *In0*(M) = *In0*(N) ==> M=N
by (*simp add*: *In0-def*)

lemma *In1-inject*: *In1*(M) = *In1*(N) ==> M=N
by (*simp add*: *In1-def*)

lemma *In0-eq* [*iff*]: (*In0 M* = *In0 N*) = (M=N)
by (*blast dest*!: *In0-inject*)

lemma *In1-eq* [*iff*]: (*In1 M* = *In1 N*) = (M=N)
by (*blast dest*!: *In1-inject*)

lemma *inj-In0*: *inj In0*
by (*blast intro*!: *inj-onI*)

lemma *inj-In1*: *inj In1*
by (*blast intro*!: *inj-onI*)

lemma *Lim-inject*: *Lim f = Lim g ==> f = g*
apply (*simp add*: *Lim-def*)
apply (*rule ext*)
apply (*blast elim*!: *Push-Node-inject*)
done

lemma *ntrunc-subsetI*: *ntrunc k M <= M*
by (*auto simp add*: *ntrunc-def*)

lemma *ntrunc-subsetD*: (!!*k*. *ntrunc k M <= N*) *==> M<=N*
by (*auto simp add*: *ntrunc-def*)

lemma *ntrunc-equality*: (!!*k*. *ntrunc k M = ntrunc k N*) *==> M=N*
apply (*rule equalityI*)
apply (*rule-tac* [!] *ntrunc-subsetD*)
apply (*rule-tac* [!] *ntrunc-subsetI* [*THEN* [*2*] *subset-trans*], *auto*)
done

lemma *ntrunc-o-equality*:
[| !!*k*. (*ntrunc*(*k*) *o h1*) = (*ntrunc*(*k*) *o h2*) |] *==> h1=h2*
apply (*rule ntrunc-equality* [*THEN ext*])
apply (*simp add*: *fun-eq-iff*)
done

lemma *uprod-mono*: [| *A<=A'*; *B<=B'* |] *==> uprod A B <= uprod A' B'*
by (*simp add*: *uprod-def*, *blast*)

lemma *usum-mono*: [| *A<=A'*; *B<=B'* |] *==> usum A B <= usum A' B'*
by (*simp add*: *usum-def*, *blast*)

lemma *Scons-mono*: [| *M<=M'*; *N<=N'* |] *==> Scons M N <= Scons M' N'*
by (*simp add*: *Scons-def*, *blast*)

lemma *In0-mono*: *M<=N ==> In0*(*M*) *<= In0*(*N*)
by (*simp add*: *In0-def Scons-mono*)

lemma *In1-mono*: *M<=N ==> In1*(*M*) *<= In1*(*N*)
by (*simp add*: *In1-def Scons-mono*)

lemma *Split* [*simp*]: *Split c* (*Scons M N*) = *c M N*
by (*simp add*: *Split-def*)

lemma *Case-In0* [*simp*]: *Case c d* (*In0 M*) = *c*(*M*)
by (*simp add*: *Case-def*)

lemma *Case-In1* [*simp*]: *Case c d* (*In1 N*) = *d*(*N*)
by (*simp add*: *Case-def*)

lemma *ntrunc-UN1*: *ntrunc k* (*UN x*. *f*(*x*)) = (*UN x*. *ntrunc k* (*f x*))
by (*simp add*: *ntrunc-def*, *blast*)

lemma *Scons-UN1-x*: *Scons* (*UN x*. *f x*) *M* = (*UN x*. *Scons* (*f x*) *M*)
by (*simp add*: *Scons-def*, *blast*)

lemma *Scons-UN1-y*: *Scons M* (*UN x*. *f x*) = (*UN x*. *Scons M* (*f x*))
by (*simp add*: *Scons-def*, *blast*)

lemma *In0-UN1*: *In0*(*UN x*. *f*(*x*)) = (*UN x*. *In0*(*f*(*x*)))
by (*simp add*: *In0-def Scons-UN1-y*)

lemma *In1-UN1*: *In1*(*UN x*. *f*(*x*)) = (*UN x*. *In1*(*f*(*x*)))
by (*simp add*: *In1-def Scons-UN1-y*)

lemma *dprodI* [*intro*!]:
[| (*M*,*M'*):*r*; (*N*,*N'*):*s* |] ==> (*Scons M N*, *Scons M' N'*) : *dprod r s*
by (*auto simp add*: *dprod-def*)

lemma *dprodE* [*elim*!]:
[| *c* : *dprod r s*;
!!*x y x' y'*. [| (*x*,*x'*) : *r*; (*y*,*y'*) : *s*;
c = (*Scons x y*, *Scons x' y'*) |] ==> *P*
|] ==> *P*
by (*auto simp add*: *dprod-def*)

lemma *dsum-In0I* [*intro*]: (*M*,*M'*):*r* ==> (*In0*(*M*), *In0*(*M'*)) : *dsum r s*
by (*auto simp add*: *dsum-def*)

lemma *dsum-In1I* [*intro*]: (*N*,*N'*):*s* ==> (*In1*(*N*), *In1*(*N'*)) : *dsum r s*
by (*auto simp add*: *dsum-def*)

lemma *dsumE* [*elim*!]:
[| *w* : *dsum r s*;
!!*x x'*. [| (*x*,*x'*) : *r*; *w* = (*In0*(*x*), *In0*(*x'*)) |] ==> *P*;
!!*y y'*. [| (*y*,*y'*) : *s*; *w* = (*In1*(*y*), *In1*(*y'*)) |] ==> *P*
|] ==> *P*
by (*auto simp add*: *dsum-def*)

lemma *dprod-mono*: [| *r*<=*r'*; *s*<=*s'* |] ==> *dprod r s* <= *dprod r' s'*
by *blast*

lemma *dsum-mono*: [| *r*<=*r'*; *s*<=*s'* |] ==> *dsum r s* <= *dsum r' s'*
by *blast*

lemma *dprod-Sigma*: (*dprod* (*A* × *B*) (*C* × *D*)) <= (*uprod A C*) × (*uprod B D*)
by *blast*

lemmas *dprod-subset-Sigma* = *subset-trans* [*OF dprod-mono dprod-Sigma*]

lemma *dprod-subset-Sigma2*:
(*dprod* (*Sigma A B*) (*Sigma C D*)) <= *Sigma* (*uprod A C*) (*Split* (%*x y*. *uprod* (*B x*) (*D y*)))
by *auto*

lemma *dsum-Sigma*: (*dsum* (*A* × *B*) (*C* × *D*)) <= (*usum A C*) × (*usum B D*)
by *blast*

lemmas *dsum-subset-Sigma* = *subset-trans* [*OF dsum-mono dsum-Sigma*]

lemma *Domain-dprod* [*simp*]: *Domain* (*dprod r s*) = *uprod* (*Domain r*) (*Domain s*)
by *auto*

lemma *Domain-dsum* [*simp*]: *Domain* (*dsum r s*) = *usum* (*Domain r*) (*Domain s*)
by *auto*

hides popular names

hide-type (**open**) *node item*
hide-const (**open**) *Push Node Atom Leaf Numb Lim Split Case*

ML-file ~~/*src*/*HOL*/*Tools*/*Old-Datatype*/*old-datatype.ML*
ML-file ~~/*src*/*HOL*/*Tools*/*inductive-realizer.ML*

end

3 Bijections between natural numbers and other types

theory *Nat-Bijection*
imports *Main*
begin

3.1 Type *nat* × *nat*

Triangle numbers: 0, 1, 3, 6, 10, 15, ...

definition *triangle* :: *nat* ⇒ *nat*
where *triangle n* = (*n* ∗ *Suc n*) *div 2*

lemma *triangle-0* [*simp*]: *triangle 0* = *0*
unfolding *triangle-def* **by** *simp*

lemma *triangle-Suc* [*simp*]: *triangle* (*Suc n*) = *triangle n* + *Suc n*
unfolding *triangle-def* **by** *simp*

definition *prod-encode* :: *nat* × *nat* ⇒ *nat*
where *prod-encode* = (λ(*m*, *n*). *triangle* (*m* + *n*) + *m*)

In this auxiliary function, *triangle k* + *m* is an invariant.

fun *prod-decode-aux* :: *nat* ⇒ *nat* ⇒ *nat* × *nat*
where
prod-decode-aux k m =
(*if m* ≤ *k then* (*m*, *k* − *m*) *else prod-decode-aux* (*Suc k*) (*m* − *Suc k*))

declare *prod-decode-aux.simps* [*simp del*]

definition *prod-decode* :: *nat* ⇒ *nat* × *nat*
where *prod-decode* = *prod-decode-aux 0*

lemma *prod-encode-prod-decode-aux*:

prod-encode (*prod-decode-aux k m*) = *triangle k* + *m*
apply (*induct k m rule*: *prod-decode-aux.induct*)
apply (*subst prod-decode-aux.simps*)
apply (*simp add*: *prod-encode-def*)
done

lemma *prod-decode-inverse* [*simp*]: *prod-encode* (*prod-decode n*) = *n*
unfolding *prod-decode-def* **by** (*simp add*: *prod-encode-prod-decode-aux*)

lemma *prod-decode-triangle-add*: *prod-decode* (*triangle k* + *m*) = *prod-decode-aux k m*
apply (*induct k arbitrary*: *m*)
apply (*simp add*: *prod-decode-def*)
apply (*simp only*: *triangle-Suc add.assoc*)
apply (*subst prod-decode-aux.simps*, *simp*)
done

lemma *prod-encode-inverse* [*simp*]: *prod-decode* (*prod-encode x*) = *x*
unfolding *prod-encode-def*
apply (*induct x*)
apply (*simp add*: *prod-decode-triangle-add*)
apply (*subst prod-decode-aux.simps*, *simp*)
done

lemma *inj-prod-encode*: *inj-on prod-encode A*
by (*rule inj-on-inverseI*, *rule prod-encode-inverse*)

lemma *inj-prod-decode*: *inj-on prod-decode A*
by (*rule inj-on-inverseI*, *rule prod-decode-inverse*)

lemma *surj-prod-encode*: *surj prod-encode*
by (*rule surjI*, *rule prod-decode-inverse*)

lemma *surj-prod-decode*: *surj prod-decode*
by (*rule surjI*, *rule prod-encode-inverse*)

lemma *bij-prod-encode*: *bij prod-encode*
by (*rule bijI* [*OF inj-prod-encode surj-prod-encode*])

lemma *bij-prod-decode*: *bij prod-decode*
by (*rule bijI* [*OF inj-prod-decode surj-prod-decode*])

lemma *prod-encode-eq*: *prod-encode x* = *prod-encode y* ⟷ *x* = *y*
by (*rule inj-prod-encode* [*THEN inj-eq*])

lemma *prod-decode-eq*: *prod-decode x* = *prod-decode y* ⟷ *x* = *y*
by (*rule inj-prod-decode* [*THEN inj-eq*])

Ordering properties

lemma *le-prod-encode-1*: $a \leq$ *prod-encode* (a, b)
unfolding *prod-encode-def* **by** *simp*

lemma *le-prod-encode-2*: $b \leq$ *prod-encode* (a, b)
unfolding *prod-encode-def* **by** (*induct b*, *simp-all*)

3.2 Type *nat* + *nat*

definition *sum-encode* :: *nat* + *nat* ⇒ *nat*
where
sum-encode x = (*case x of Inl a* ⇒ *2* ∗ *a* | *Inr b* ⇒ *Suc* (*2* ∗ *b*))

definition *sum-decode* :: *nat* ⇒ *nat* + *nat*
where
sum-decode n = (*if even n then Inl* (*n div 2*) *else Inr* (*n div 2*))

lemma *sum-encode-inverse* [*simp*]: *sum-decode* (*sum-encode x*) = *x*
unfolding *sum-decode-def sum-encode-def*
by (*induct x*) *simp-all*

lemma *sum-decode-inverse* [*simp*]: *sum-encode* (*sum-decode n*) = *n*
by (*simp add*: *even-two-times-div-two sum-decode-def sum-encode-def*)

lemma *inj-sum-encode*: *inj-on sum-encode A*
by (*rule inj-on-inverseI*, *rule sum-encode-inverse*)

lemma *inj-sum-decode*: *inj-on sum-decode A*
by (*rule inj-on-inverseI*, *rule sum-decode-inverse*)

lemma *surj-sum-encode*: *surj sum-encode*
by (*rule surjI*, *rule sum-decode-inverse*)

lemma *surj-sum-decode*: *surj sum-decode*
by (*rule surjI*, *rule sum-encode-inverse*)

lemma *bij-sum-encode*: *bij sum-encode*
by (*rule bijI* [*OF inj-sum-encode surj-sum-encode*])

lemma *bij-sum-decode*: *bij sum-decode*
by (*rule bijI* [*OF inj-sum-decode surj-sum-decode*])

lemma *sum-encode-eq*: *sum-encode x* = *sum-encode y* ⟷ *x* = *y*
by (*rule inj-sum-encode* [*THEN inj-eq*])

lemma *sum-decode-eq*: *sum-decode x* = *sum-decode y* ⟷ *x* = *y*
by (*rule inj-sum-decode* [*THEN inj-eq*])

3.3 Type *int*

definition *int-encode* :: *int* ⇒ *nat*

where
int-encode i = sum-encode (if 0 ≤ i then Inl (nat i) else Inr (nat (− i − 1)))

definition *int-decode* :: *nat ⇒ int*
where
int-decode n = (case sum-decode n of Inl a ⇒ int a | Inr b ⇒ − int b − 1)

lemma *int-encode-inverse* [*simp*]: *int-decode (int-encode x) = x*
unfolding *int-decode-def int-encode-def* **by** *simp*

lemma *int-decode-inverse* [*simp*]: *int-encode (int-decode n) = n*
unfolding *int-decode-def int-encode-def* **using** *sum-decode-inverse* [*of n*]
by (*cases sum-decode n*, *simp-all*)

lemma *inj-int-encode*: *inj-on int-encode A*
by (*rule inj-on-inverseI*, *rule int-encode-inverse*)

lemma *inj-int-decode*: *inj-on int-decode A*
by (*rule inj-on-inverseI*, *rule int-decode-inverse*)

lemma *surj-int-encode*: *surj int-encode*
by (*rule surjI*, *rule int-decode-inverse*)

lemma *surj-int-decode*: *surj int-decode*
by (*rule surjI*, *rule int-encode-inverse*)

lemma *bij-int-encode*: *bij int-encode*
by (*rule bijI* [*OF inj-int-encode surj-int-encode*])

lemma *bij-int-decode*: *bij int-decode*
by (*rule bijI* [*OF inj-int-decode surj-int-decode*])

lemma *int-encode-eq*: *int-encode x = int-encode y ⟷ x = y*
by (*rule inj-int-encode* [*THEN inj-eq*])

lemma *int-decode-eq*: *int-decode x = int-decode y ⟷ x = y*
by (*rule inj-int-decode* [*THEN inj-eq*])

3.4 Type *nat list*

fun *list-encode* :: *nat list ⇒ nat*
where
list-encode [] = 0
| *list-encode (x # xs) = Suc (prod-encode (x, list-encode xs))*

function *list-decode* :: *nat ⇒ nat list*
where
list-decode 0 = []
| *list-decode (Suc n) = (case prod-decode n of (x, y) ⇒ x # list-decode y)*

by *pat-completeness auto*

termination *list-decode*
apply (*relation measure id*, *simp-all*)
apply (*drule arg-cong* [**where** *f*=*prod-encode*])
apply (*drule sym*)
apply (*simp add*: *le-imp-less-Suc le-prod-encode-2*)
done

lemma *list-encode-inverse* [*simp*]: *list-decode* (*list-encode x*) = *x*
by (*induct x rule*: *list-encode.induct*) *simp-all*

lemma *list-decode-inverse* [*simp*]: *list-encode* (*list-decode n*) = *n*
apply (*induct n rule*: *list-decode.induct*, *simp*)
apply (*simp split*: *prod.split*)
apply (*simp add*: *prod-decode-eq* [*symmetric*])
done

lemma *inj-list-encode*: *inj-on list-encode A*
by (*rule inj-on-inverseI*, *rule list-encode-inverse*)

lemma *inj-list-decode*: *inj-on list-decode A*
by (*rule inj-on-inverseI*, *rule list-decode-inverse*)

lemma *surj-list-encode*: *surj list-encode*
by (*rule surjI*, *rule list-decode-inverse*)

lemma *surj-list-decode*: *surj list-decode*
by (*rule surjI*, *rule list-encode-inverse*)

lemma *bij-list-encode*: *bij list-encode*
by (*rule bijI* [*OF inj-list-encode surj-list-encode*])

lemma *bij-list-decode*: *bij list-decode*
by (*rule bijI* [*OF inj-list-decode surj-list-decode*])

lemma *list-encode-eq*: *list-encode x* = *list-encode y* ⟷ *x* = *y*
by (*rule inj-list-encode* [*THEN inj-eq*])

lemma *list-decode-eq*: *list-decode x* = *list-decode y* ⟷ *x* = *y*
by (*rule inj-list-decode* [*THEN inj-eq*])

3.5 Finite sets of naturals

3.5.1 Preliminaries

lemma *finite-vimage-Suc-iff*: *finite* (*Suc* −‘ *F*) ⟷ *finite F*
apply (*safe intro*!: *finite-vimageI inj-Suc*)
apply (*rule finite-subset* [**where** *B*=*insert 0* (*Suc* ‘ *Suc* −‘ *F*)])
apply (*rule subsetI*, *case-tac x*, *simp*, *simp*)

apply (*rule finite-insert* [*THEN iffD2*])
apply (*erule finite-imageI*)
done

lemma *vimage-Suc-insert-0*: *Suc −‘ insert 0 A = Suc −‘ A*
by *auto*

lemma *vimage-Suc-insert-Suc*:
Suc −‘ insert (Suc n) A = insert n (Suc −‘ A)
by *auto*

lemma *div2-even-ext-nat*:
fixes *x y* :: *nat*
assumes *x div 2 = y div 2*
and *even x ⟷ even y*
shows *x = y*
proof −
from ‹*even x ⟷ even y*› **have** *x mod 2 = y mod 2*
by (*simp only*: *even-iff-mod-2-eq-zero*) *auto*
with *assms* **have** *x div 2 * 2 + x mod 2 = y div 2 * 2 + y mod 2*
by *simp*
then show *?thesis*
by *simp*
qed

3.5.2 From sets to naturals

definition *set-encode* :: *nat set ⇒ nat*
where *set-encode = setsum* (*op ^ 2*)

lemma *set-encode-empty* [*simp*]: *set-encode {} = 0*
by (*simp add*: *set-encode-def*)

lemma *set-encode-inf*: ~ *finite A ⟹ set-encode A = 0*
by (*simp add*: *set-encode-def*)

lemma *set-encode-insert* [*simp*]:
⟦*finite A*; *n ∉ A*⟧ *⟹ set-encode* (*insert n A*) = *2^n + set-encode A*
by (*simp add*: *set-encode-def*)

lemma *even-set-encode-iff*: *finite A ⟹ even* (*set-encode A*) *⟷ 0 ∉ A*
unfolding *set-encode-def* **by** (*induct set*: *finite*, *auto*)

lemma *set-encode-vimage-Suc*: *set-encode* (*Suc −‘ A*) = *set-encode A div 2*
apply (*cases finite A*)
apply (*erule finite-induct*, *simp*)
apply (*case-tac x*)
apply (*simp add*: *even-set-encode-iff vimage-Suc-insert-0*)
apply (*simp add*: *finite-vimageI add.commute vimage-Suc-insert-Suc*)

apply (*simp add*: *set-encode-def finite-vimage-Suc-iff*)
done

lemmas *set-encode-div-2* = *set-encode-vimage-Suc* [*symmetric*]

3.5.3 From naturals to sets

definition *set-decode* :: *nat* ⇒ *nat set*
  where *set-decode x* = {*n*. *odd* (*x div 2* ^ *n*)}

lemma *set-decode-0* [*simp*]: *0* ∈ *set-decode x* ⟷ *odd x*
by (*simp add*: *set-decode-def*)

lemma *set-decode-Suc* [*simp*]:
  Suc n ∈ *set-decode x* ⟷ *n* ∈ *set-decode* (*x div 2*)
by (*simp add*: *set-decode-def div-mult2-eq*)

lemma *set-decode-zero* [*simp*]: *set-decode 0* = {}
by (*simp add*: *set-decode-def*)

lemma *set-decode-div-2*: *set-decode* (*x div 2*) = *Suc* −‘ *set-decode x*
by *auto*

lemma *set-decode-plus-power-2*:
  n ∉ *set-decode z* ⟹ *set-decode* (*2* ^ *n* + *z*) = *insert n* (*set-decode z*)
proof (*induct n arbitrary*: *z*)
  case *0* **show** *?case*
  proof (*rule set-eqI*)
    fix *q* **show** *q* ∈ *set-decode* (*2* ^ *0* + *z*) ⟷ *q* ∈ *insert 0* (*set-decode z*)
      by (*induct q*) (*insert 0*, *simp-all*)
  qed
next
  case (*Suc n*) **show** *?case*
  proof (*rule set-eqI*)
    fix *q* **show** *q* ∈ *set-decode* (*2* ^ *Suc n* + *z*) ⟷ *q* ∈ *insert* (*Suc n*) (*set-decode z*)
      by (*induct q*) (*insert Suc*, *simp-all*)
  qed
qed

lemma *finite-set-decode* [*simp*]: *finite* (*set-decode n*)
apply (*induct n rule*: *nat-less-induct*)
apply (*case-tac n* = *0*, *simp*)
apply (*drule-tac x*=*n div 2* **in** *spec*, *simp*)
apply (*simp add*: *set-decode-div-2*)
apply (*simp add*: *finite-vimage-Suc-iff*)
done

3.5.4 Proof of isomorphism

lemma *set-decode-inverse* [*simp*]: *set-encode* (*set-decode n*) = *n*
apply (*induct n rule*: *nat-less-induct*)
apply (*case-tac n* = *0*, *simp*)
apply (*drule-tac x*=*n div 2* **in** *spec*, *simp*)
apply (*simp add*: *set-decode-div-2 set-encode-vimage-Suc*)
apply (*erule div2-even-ext-nat*)
apply (*simp add*: *even-set-encode-iff*)
done

lemma *set-encode-inverse* [*simp*]: *finite A* ⟹ *set-decode* (*set-encode A*) = *A*
apply (*erule finite-induct*, *simp-all*)
apply (*simp add*: *set-decode-plus-power-2*)
done

lemma *inj-on-set-encode*: *inj-on set-encode* (*Collect finite*)
by (*rule inj-on-inverseI* [**where** *g*=*set-decode*], *simp*)

lemma *set-encode-eq*:
⟦*finite A*; *finite B*⟧ ⟹ *set-encode A* = *set-encode B* ⟷ *A* = *B*
by (*rule iffI*, *simp add*: *inj-onD* [*OF inj-on-set-encode*], *simp*)

lemma *subset-decode-imp-le*:
assumes *set-decode m* ⊆ *set-decode n*
shows *m* ≤ *n*
proof −
have *n* = *m* + *set-encode* (*set-decode n* − *set-decode m*)
proof −
obtain *A B* **where** *m* = *set-encode A finite A*
n = *set-encode B finite B*
by (*metis finite-set-decode set-decode-inverse*)
thus *?thesis* **using** *assms*
apply *auto*
apply (*simp add*: *set-encode-def add.commute setsum.subset-diff*)
done
qed
thus *?thesis*
by (*metis le-add1*)
qed

end

4 Encoding (almost) everything into natural numbers

theory *Countable*
imports *Old-Datatype* ~~/*src*/*HOL*/*Rat Nat-Bijection*
begin

4.1 The class of countable types

class *countable* =
  assumes *ex-inj*: $\exists$ *to-nat* :: *'a* $\Rightarrow$ *nat*. *inj to-nat*

lemma *countable-classI*:
  fixes *f* :: *'a* $\Rightarrow$ *nat*
  assumes $\bigwedge x\ y.\ f\ x = f\ y \Longrightarrow x = y$
  shows *OFCLASS*(*'a*, *countable-class*)
proof (*intro-classes*, *rule exI*)
  show *inj f*
    by (*rule injI* [*OF assms*]) *assumption*
qed

4.2 Conversion functions

definition *to-nat* :: *'a*::*countable* $\Rightarrow$ *nat* **where**
  to-nat = (*SOME f*. *inj f*)

definition *from-nat* :: *nat* $\Rightarrow$ *'a*::*countable* **where**
  from-nat = *inv* (*to-nat* :: *'a* $\Rightarrow$ *nat*)

lemma *inj-to-nat* [*simp*]: *inj to-nat*
  by (*rule exE-some* [*OF ex-inj*]) (*simp add*: *to-nat-def*)

lemma *inj-on-to-nat*[*simp*, *intro*]: *inj-on to-nat S*
  using *inj-to-nat* **by** (*auto simp*: *inj-on-def*)

lemma *surj-from-nat* [*simp*]: *surj from-nat*
  unfolding *from-nat-def* **by** (*simp add*: *inj-imp-surj-inv*)

lemma *to-nat-split* [*simp*]: *to-nat x* = *to-nat y* $\longleftrightarrow$ *x* = *y*
  using *injD* [*OF inj-to-nat*] **by** *auto*

lemma *from-nat-to-nat* [*simp*]:
  from-nat (*to-nat x*) = *x*
  by (*simp add*: *from-nat-def*)

4.3 Finite types are countable

subclass (**in** *finite*) *countable*
proof
  have *finite* (*UNIV*::*'a set*) **by** (*rule finite-UNIV*)
  with *finite-conv-nat-seg-image* [*of UNIV*::*'a set*]
  obtain *n* **and** *f* :: *nat* $\Rightarrow$ *'a*
    where *UNIV* = *f* ' {*i*. *i* < *n*} **by** *auto*
  then have *surj f* **unfolding** *surj-def* **by** *auto*
  then have *inj* (*inv f*) **by** (*rule surj-imp-inj-inv*)
  then show $\exists$ *to-nat* :: *'a* $\Rightarrow$ *nat*. *inj to-nat* **by** (*rule exI*[*of inj*])
qed

4.4 Automatically proving countability of old-style datatypes

context
begin

qualified inductive *finite-item* :: *'a Old-Datatype.item* ⇒ *bool* **where**
  undefined: *finite-item undefined*
| *In0*: *finite-item x* ⟹ *finite-item* (*Old-Datatype.In0 x*)
| *In1*: *finite-item x* ⟹ *finite-item* (*Old-Datatype.In1 x*)
| *Leaf*: *finite-item* (*Old-Datatype.Leaf a*)
| *Scons*: ⟦*finite-item x*; *finite-item y*⟧ ⟹ *finite-item* (*Old-Datatype.Scons x y*)

qualified function *nth-item* :: *nat* ⇒ (*'a*::*countable*) *Old-Datatype.item*
where
  nth-item 0 = *undefined*
| *nth-item* (*Suc n*) =
  (*case sum-decode n of*
    Inl i ⇒
    (*case sum-decode i of*
      Inl j ⇒ *Old-Datatype.In0* (*nth-item j*)
    | *Inr j* ⇒ *Old-Datatype.In1* (*nth-item j*))
  | *Inr i* ⇒
    (*case sum-decode i of*
      Inl j ⇒ *Old-Datatype.Leaf* (*from-nat j*)
    | *Inr j* ⇒
      (*case prod-decode j of*
        (*a*, *b*) ⇒ *Old-Datatype.Scons* (*nth-item a*) (*nth-item b*))))
by *pat-completeness auto*

lemma *le-sum-encode-Inl*: *x* ≤ *y* ⟹ *x* ≤ *sum-encode* (*Inl y*)
unfolding *sum-encode-def* **by** *simp*

lemma *le-sum-encode-Inr*: *x* ≤ *y* ⟹ *x* ≤ *sum-encode* (*Inr y*)
unfolding *sum-encode-def* **by** *simp*

qualified termination
by (*relation measure id*)
  (*auto simp add*: *sum-encode-eq* [*symmetric*] *prod-encode-eq* [*symmetric*]
    le-imp-less-Suc le-sum-encode-Inl le-sum-encode-Inr
    le-prod-encode-1 le-prod-encode-2)

lemma *nth-item-covers*: *finite-item x* ⟹ ∃ *n*. *nth-item n* = *x*
proof (*induct set*: *finite-item*)
  case *undefined*
  have *nth-item 0* = *undefined* **by** *simp*
  thus *?case* **..**
next
  case (*In0 x*)
  then obtain *n* **where** *nth-item n* = *x* **by** *fast*
 hence *nth-item* (*Suc* (*sum-encode* (*Inl* (*sum-encode* (*Inl n*))))) = *Old-Datatype.In0*

x **by** *simp*
thus *?case* **..**
next
case (*In1 x*)
then obtain *n* **where** *nth-item n = x* **by** *fast*
hence *nth-item* (*Suc* (*sum-encode* (*Inl* (*sum-encode* (*Inr n*))))) = *Old-Datatype.In1*
x **by** *simp*
thus *?case* **..**
next
case (*Leaf a*)
have *nth-item* (*Suc* (*sum-encode* (*Inr* (*sum-encode* (*Inl* (*to-nat a*)))))) = *Old-Datatype.Leaf*
a
by *simp*
thus *?case* **..**
next
case (*Scons x y*)
then obtain *i j* **where** *nth-item i = x* **and** *nth-item j = y* **by** *fast*
hence *nth-item*
(*Suc* (*sum-encode* (*Inr* (*sum-encode* (*Inr* (*prod-encode* (*i*, *j*))))))) = *Old-Datatype.Scons*
x y
by *simp*
thus *?case* **..**
qed

theorem *countable-datatype*:
fixes *Rep* :: *'b* ⇒ (*'a*::*countable*) *Old-Datatype.item*
fixes *Abs* :: (*'a*::*countable*) *Old-Datatype.item* ⇒ *'b*
fixes *rep-set* :: (*'a*::*countable*) *Old-Datatype.item* ⇒ *bool*
assumes *type*: *type-definition Rep Abs* (*Collect rep-set*)
assumes *finite-item*: ⋀*x*. *rep-set x* ⟹ *finite-item x*
shows *OFCLASS*(*'b*, *countable-class*)
proof
def *f* ≡ λ*y*. *LEAST n*. *nth-item n = Rep y*
{
fix *y* :: *'b*
have *rep-set* (*Rep y*)
using *type-definition.Rep* [*OF type*] **by** *simp*
hence *finite-item* (*Rep y*)
by (*rule finite-item*)
hence ∃ *n*. *nth-item n = Rep y*
by (*rule nth-item-covers*)
hence *nth-item* (*f y*) = *Rep y*
unfolding *f-def* **by** (*rule LeastI-ex*)
hence *Abs* (*nth-item* (*f y*)) = *y*
using *type-definition.Rep-inverse* [*OF type*] **by** *simp*
}
hence *inj f*
by (*rule inj-on-inverseI*)
thus ∃ *f*::*'b* ⇒ *nat*. *inj f*

by − (*rule exI*)
qed

ML ‹
fun old-countable-datatype-tac ctxt =
SUBGOAL (fn (goal, -) =>
let
val ty-name =
(case goal of
(- $ Const (@{const-name Pure.type}, Type (@{type-name itself}, [Type (n, -)])))) => n
| - => raise Match)
val typedef-info = hd (Typedef.get-info ctxt ty-name)
val typedef-thm = #type-definition (snd typedef-info)
val pred-name =
(case HOLogic.dest-Trueprop (Thm.concl-of typedef-thm) of
(- $ - $ - $ (- $ Const (n, -))) => n
| - => raise Match)
val induct-info = Inductive.the-inductive ctxt pred-name
val pred-names = #names (fst induct-info)
val induct-thms = #inducts (snd induct-info)
val alist = pred-names ~~ induct-thms
val induct-thm = the (AList.lookup (op =) alist pred-name)
val vars = rev (Term.add-vars (Thm.prop-of induct-thm) [])
val insts = vars |> map (fn (-, T) => try (Thm.cterm-of ctxt)
(Const (@{const-name Countable.finite-item}, T)))
val induct-thm' = Thm.instantiate' [] insts induct-thm
val rules = @{thms finite-item.intros}
in
SOLVED' (fn i => EVERY
[resolve-tac ctxt @{thms countable-datatype} i,
resolve-tac ctxt [typedef-thm] i,
eresolve-tac ctxt [induct-thm'] i,
REPEAT (resolve-tac ctxt rules i ORELSE assume-tac ctxt i)]) 1
end)
›

end

4.5 Automatically proving countability of datatypes

ML-file *../Tools/BNF/bnf-lfp-countable.ML*

ML ‹
fun countable-datatype-tac ctxt st =
(case try (fn () => HEADGOAL (old-countable-datatype-tac ctxt) st) () of
SOME res => res
| NONE => BNF-LFP-Countable.countable-datatype-tac ctxt st);

```
(* compatibility *)
fun countable-tac ctxt =
  SELECT-GOAL (countable-datatype-tac ctxt);
›

method-setup countable-datatype = ‹
  Scan.succeed (SIMPLE-METHOD o countable-datatype-tac)
› prove countable class instances for datatypes
```

4.6 More Countable types

Naturals

```
instance nat :: countable
  by (rule countable-classI [of id]) simp
```

Pairs

```
instance prod :: (countable, countable) countable
  by (rule countable-classI [of λ(x, y). prod-encode (to-nat x, to-nat y)])
    (auto simp add: prod-encode-eq)
```

Sums

```
instance sum :: (countable, countable) countable
  by (rule countable-classI [of (λx. case x of Inl a ⇒ to-nat (False, to-nat a)
                                  | Inr b ⇒ to-nat (True, to-nat b))])
    (simp split: sum.split-asm)
```

Integers

```
instance int :: countable
  by (rule countable-classI [of int-encode]) (simp add: int-encode-eq)
```

Options

```
instance option :: (countable) countable
  by countable-datatype
```

Lists

```
instance list :: (countable) countable
  by countable-datatype
```

String literals

```
instance String.literal :: countable
 by (rule countable-classI [of to-nat ∘ String.explode]) (auto simp add: explode-inject)
```

Functions

```
instance fun :: (finite, countable) countable
proof
  obtain xs :: 'a list where xs: set xs = UNIV
    using finite-list [OF finite-UNIV] ..
```

show $\exists$ *to-nat*::('*a* $\Rightarrow$ '*b*) $\Rightarrow$ *nat*. *inj to-nat*
proof
show *inj* (λ*f*. *to-nat* (*map f xs*))
by (*rule injI*, *simp add*: *xs fun-eq-iff*)
qed
qed

Typereps

instance *typerep* :: *countable*
by *countable-datatype*

4.7 The rationals are countably infinite

definition *nat-to-rat-surj* :: *nat* $\Rightarrow$ *rat* **where**
nat-to-rat-surj n = (*let* (*a*, *b*) = *prod-decode n in Fract* (*int-decode a*) (*int-decode b*))

lemma *surj-nat-to-rat-surj*: *surj nat-to-rat-surj*
unfolding *surj-def*
proof
fix *r*::*rat*
show $\exists$ *n*. *r* = *nat-to-rat-surj n*
proof (*cases r*)
fix *i j* **assume** [*simp*]: *r* = *Fract i j* **and** *j* > *0*
have *r* = (*let m* = *int-encode i*; *n* = *int-encode j in nat-to-rat-surj* (*prod-encode* (*m*, *n*)))
by (*simp add*: *Let-def nat-to-rat-surj-def*)
thus $\exists$ *n*. *r* = *nat-to-rat-surj n* **by**(*auto simp*: *Let-def*)
qed
qed

lemma *Rats-eq-range-nat-to-rat-surj*: $\mathbb{Q}$ = *range nat-to-rat-surj*
by (*simp add*: *Rats-def surj-nat-to-rat-surj*)

context *field-char-0*
begin

lemma *Rats-eq-range-of-rat-o-nat-to-rat-surj*:
$\mathbb{Q}$ = *range* (*of-rat* $\circ$ *nat-to-rat-surj*)
using *surj-nat-to-rat-surj*
by (*auto simp*: *Rats-def image-def surj-def*) (*blast intro*: *arg-cong*[**where** *f* = *of-rat*])

lemma *surj-of-rat-nat-to-rat-surj*:
r $\in$ $\mathbb{Q}$ $\Longrightarrow$ $\exists$ *n*. *r* = *of-rat* (*nat-to-rat-surj n*)
by (*simp add*: *Rats-eq-range-of-rat-o-nat-to-rat-surj image-def*)

end

instance *rat* :: *countable*
proof
  show ∃ *to-nat*::*rat* ⇒ *nat*. *inj to-nat*
  proof
    have *surj nat-to-rat-surj*
      by (*rule surj-nat-to-rat-surj*)
    then show *inj* (*inv nat-to-rat-surj*)
      by (*rule surj-imp-inj-inv*)
  qed
qed

end

5 Infinite Sets and Related Concepts

theory *Infinite-Set*
imports *Main*
begin

The set of natural numbers is infinite.

lemma *infinite-nat-iff-unbounded-le*: *infinite* (*S*::*nat set*) ⟷ (∀ *m*. ∃ *n*≥*m*. *n* ∈ *S*)
  using *frequently-cofinite*[*of* λ*x*. *x* ∈ *S*]
  by (*simp add*: *cofinite-eq-sequentially frequently-def eventually-sequentially*)

lemma *infinite-nat-iff-unbounded*: *infinite* (*S*::*nat set*) ⟷ (∀ *m*. ∃ *n*>*m*. *n* ∈ *S*)
  using *frequently-cofinite*[*of* λ*x*. *x* ∈ *S*]
  by (*simp add*: *cofinite-eq-sequentially frequently-def eventually-at-top-dense*)

lemma *finite-nat-iff-bounded*: *finite* (*S*::*nat set*) ⟷ (∃ *k*. *S* ⊆ {..<*k*})
  using *infinite-nat-iff-unbounded-le*[*of S*] **by** (*simp add*: *subset-eq*) (*metis not-le*)

lemma *finite-nat-iff-bounded-le*: *finite* (*S*::*nat set*) ⟷ (∃ *k*. *S* ⊆ {.. *k*})
  using *infinite-nat-iff-unbounded*[*of S*] **by** (*simp add*: *subset-eq*) (*metis not-le*)

lemma *finite-nat-bounded*: *finite* (*S*::*nat set*) ⟹ ∃ *k*. *S* ⊆ {..<*k*}
  by (*simp add*: *finite-nat-iff-bounded*)

For a set of natural numbers to be infinite, it is enough to know that for any number larger than some k, there is some larger number that is an element of the set.

lemma *unbounded-k-infinite*: ∀ *m*>*k*. ∃ *n*>*m*. *n* ∈ *S* ⟹ *infinite* (*S*::*nat set*)
apply (*clarsimp simp add*: *finite-nat-set-iff-bounded*)
apply (*drule-tac x*=*Suc* (*max m k*) **in** *spec*)
using *less-Suc-eq* **by** *fastforce*

lemma *nat-not-finite*: *finite* (*UNIV*::*nat set*) ⟹ *R*
  by *simp*

lemma *range-inj-infinite*:
inj (*f*::*nat* ⇒ *'a*) ⟹ *infinite* (*range f*)
proof
assume *finite* (*range f*) **and** *inj f*
then have *finite* (*UNIV*::*nat set*)
by (*rule finite-imageD*)
then show *False* **by** *simp*
qed

The set of integers is also infinite.

lemma *infinite-int-iff-infinite-nat-abs*: *infinite* (*S*::*int set*) ⟷ *infinite* ((*nat o abs*) ` *S*)
by (*auto simp*: *transfer-nat-int-set-relations o-def image-comp dest*: *finite-image-absD*)

proposition *infinite-int-iff-unbounded-le*: *infinite* (*S*::*int set*) ⟷ (∀ *m*. ∃ *n*. |*n*| ≥ *m* ∧ *n* ∈ *S*)
apply (*simp add*: *infinite-int-iff-infinite-nat-abs infinite-nat-iff-unbounded-le o-def image-def*)
apply (*metis abs-ge-zero nat-le-eq-zle le-nat-iff*)
done

proposition *infinite-int-iff-unbounded*: *infinite* (*S*::*int set*) ⟷ (∀ *m*. ∃ *n*. |*n*| > *m* ∧ *n* ∈ *S*)
apply (*simp add*: *infinite-int-iff-infinite-nat-abs infinite-nat-iff-unbounded o-def image-def*)
apply (*metis* (*full-types*) *nat-le-iff nat-mono not-le*)
done

proposition *finite-int-iff-bounded*: *finite* (*S*::*int set*) ⟷ (∃ *k*. *abs* ` *S* ⊆ {..<*k*})
using *infinite-int-iff-unbounded-le*[*of S*] **by** (*simp add*: *subset-eq*) (*metis not-le*)

proposition *finite-int-iff-bounded-le*: *finite* (*S*::*int set*) ⟷ (∃ *k*. *abs* ` *S* ⊆ {.. *k*})
using *infinite-int-iff-unbounded*[*of S*] **by** (*simp add*: *subset-eq*) (*metis not-le*)

5.1 Infinitely Many and Almost All

We often need to reason about the existence of infinitely many (resp., all but finitely many) objects satisfying some predicate, so we introduce corresponding binders and their proof rules.

lemma *not-INFM* [*simp*]: ¬ (*INFM x*. *P x*) ⟷ (*MOST x*. ¬ *P x*) **by** (*fact not-frequently*)
lemma *not-MOST* [*simp*]: ¬ (*MOST x*. *P x*) ⟷ (*INFM x*. ¬ *P x*) **by** (*fact not-eventually*)

lemma *INFM-const* [*simp*]: (*INFM x*::*'a*. *P*) ⟷ *P* ∧ *infinite* (*UNIV*::*'a set*)
by (*simp add*: *frequently-const-iff*)

lemma *MOST-const* [*simp*]: $(MOST\ x{::}'a.\ P) \longleftrightarrow P \vee finite\ (UNIV{::}'a\ set)$
by (*simp add*: *eventually-const-iff*)

lemma *INFM-imp-distrib*: $(INFM\ x.\ P\ x \longrightarrow Q\ x) \longleftrightarrow ((MOST\ x.\ P\ x) \longrightarrow (INFM\ x.\ Q\ x))$
by (*simp only*: *imp-conv-disj frequently-disj-iff not-eventually*)

lemma *MOST-imp-iff*: $MOST\ x.\ P\ x \Longrightarrow (MOST\ x.\ P\ x \longrightarrow Q\ x) \longleftrightarrow (MOST\ x.\ Q\ x)$
by (*auto intro*: *eventually-rev-mp eventually-mono*)

lemma *INFM-conjI*: $INFM\ x.\ P\ x \Longrightarrow MOST\ x.\ Q\ x \Longrightarrow INFM\ x.\ P\ x \wedge Q\ x$
by (*rule frequently-rev-mp*[*of P*]) (*auto elim*: *eventually-mono*)

Properties of quantifiers with injective functions.

lemma *INFM-inj*: $INFM\ x.\ P\ (f\ x) \Longrightarrow inj\ f \Longrightarrow INFM\ x.\ P\ x$
using *finite-vimageI*[*of* $\{x.\ P\ x\}$ *f*] **by** (*auto simp*: *frequently-cofinite*)

lemma *MOST-inj*: $MOST\ x.\ P\ x \Longrightarrow inj\ f \Longrightarrow MOST\ x.\ P\ (f\ x)$
using *finite-vimageI*[*of* $\{x.\ \neg\ P\ x\}$ *f*] **by** (*auto simp*: *eventually-cofinite*)

Properties of quantifiers with singletons.

lemma *not-INFM-eq* [*simp*]:
$\neg\ (INFM\ x.\ x = a)$
$\neg\ (INFM\ x.\ a = x)$
unfolding *frequently-cofinite* **by** *simp-all*

lemma *MOST-neq* [*simp*]:
$MOST\ x.\ x \neq a$
$MOST\ x.\ a \neq x$
unfolding *eventually-cofinite* **by** *simp-all*

lemma *INFM-neq* [*simp*]:
$(INFM\ x{::}'a.\ x \neq a) \longleftrightarrow infinite\ (UNIV{::}'a\ set)$
$(INFM\ x{::}'a.\ a \neq x) \longleftrightarrow infinite\ (UNIV{::}'a\ set)$
unfolding *frequently-cofinite* **by** *simp-all*

lemma *MOST-eq* [*simp*]:
$(MOST\ x{::}'a.\ x = a) \longleftrightarrow finite\ (UNIV{::}'a\ set)$
$(MOST\ x{::}'a.\ a = x) \longleftrightarrow finite\ (UNIV{::}'a\ set)$
unfolding *eventually-cofinite* **by** *simp-all*

lemma *MOST-eq-imp*:
$MOST\ x.\ x = a \longrightarrow P\ x$
$MOST\ x.\ a = x \longrightarrow P\ x$
unfolding *eventually-cofinite* **by** *simp-all*

Properties of quantifiers over the naturals.

lemma *MOST-nat*: $(\forall_\infty n.\ P\ (n{::}nat)) \longleftrightarrow (\exists\, m.\ \forall\, n{>}m.\ P\ n)$

by (*auto simp add*: *eventually-cofinite finite-nat-iff-bounded-le subset-eq not-le*[*symmetric*])

lemma *MOST-nat-le*: $(\forall_\infty n.\ P\ (n::nat)) \longleftrightarrow (\exists m.\ \forall n{\geq}m.\ P\ n)$
by (*auto simp add*: *eventually-cofinite finite-nat-iff-bounded subset-eq not-le*[*symmetric*])

lemma *INFM-nat*: $(\exists_\infty n.\ P\ (n::nat)) \longleftrightarrow (\forall m.\ \exists n{>}m.\ P\ n)$
by (*simp add*: *frequently-cofinite infinite-nat-iff-unbounded*)

lemma *INFM-nat-le*: $(\exists_\infty n.\ P\ (n::nat)) \longleftrightarrow (\forall m.\ \exists n{\geq}m.\ P\ n)$
by (*simp add*: *frequently-cofinite infinite-nat-iff-unbounded-le*)

lemma *MOST-INFM*: *infinite* $(UNIV::'a\ set) \Longrightarrow MOST\ x::'a.\ P\ x \Longrightarrow INFM\ x::'a.\ P\ x$
by (*simp add*: *eventually-frequently*)

lemma *MOST-Suc-iff*: $(MOST\ n.\ P\ (Suc\ n)) \longleftrightarrow (MOST\ n.\ P\ n)$
by (*simp add*: *cofinite-eq-sequentially eventually-sequentially-Suc*)

lemma
shows *MOST-SucI*: $MOST\ n.\ P\ n \Longrightarrow MOST\ n.\ P\ (Suc\ n)$
and *MOST-SucD*: $MOST\ n.\ P\ (Suc\ n) \Longrightarrow MOST\ n.\ P\ n$
by (*simp-all add*: *MOST-Suc-iff*)

lemma *MOST-ge-nat*: $MOST\ n::nat.\ m \leq n$
by (*simp add*: *cofinite-eq-sequentially eventually-ge-at-top*)

lemma *Inf-many-def*: *Inf-many* $P \longleftrightarrow$ *infinite* $\{x.\ P\ x\}$ **by** (*fact frequently-cofinite*)
lemma *Alm-all-def*: *Alm-all* $P \longleftrightarrow \neg\ (INFM\ x.\ \neg\ P\ x)$ **by** *simp*
lemma *INFM-iff-infinite*: $(INFM\ x.\ P\ x) \longleftrightarrow$ *infinite* $\{x.\ P\ x\}$ **by** (*fact frequently-cofinite*)
lemma *MOST-iff-cofinite*: $(MOST\ x.\ P\ x) \longleftrightarrow$ *finite* $\{x.\ \neg\ P\ x\}$ **by** (*fact eventually-cofinite*)
lemma *INFM-EX*: $(\exists_\infty x.\ P\ x) \Longrightarrow (\exists x.\ P\ x)$ **by** (*fact frequently-ex*)
lemma *ALL-MOST*: $\forall x.\ P\ x \Longrightarrow \forall_\infty x.\ P\ x$ **by** (*fact always-eventually*)
lemma *INFM-mono*: $\exists_\infty x.\ P\ x \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow Q\ x) \Longrightarrow \exists_\infty x.\ Q\ x$ **by** (*fact frequently-elim1*)
lemma *MOST-mono*: $\forall_\infty x.\ P\ x \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow Q\ x) \Longrightarrow \forall_\infty x.\ Q\ x$ **by** (*fact eventually-mono*)
lemma *INFM-disj-distrib*: $(\exists_\infty x.\ P\ x \vee Q\ x) \longleftrightarrow (\exists_\infty x.\ P\ x) \vee (\exists_\infty x.\ Q\ x)$ **by** (*fact frequently-disj-iff*)
lemma *MOST-rev-mp*: $\forall_\infty x.\ P\ x \Longrightarrow \forall_\infty x.\ P\ x \longrightarrow Q\ x \Longrightarrow \forall_\infty x.\ Q\ x$ **by** (*fact eventually-rev-mp*)
lemma *MOST-conj-distrib*: $(\forall_\infty x.\ P\ x \wedge Q\ x) \longleftrightarrow (\forall_\infty x.\ P\ x) \wedge (\forall_\infty x.\ Q\ x)$ **by** (*fact eventually-conj-iff*)
lemma *MOST-conjI*: $MOST\ x.\ P\ x \Longrightarrow MOST\ x.\ Q\ x \Longrightarrow MOST\ x.\ P\ x \wedge Q\ x$ **by** (*fact eventually-conj*)
lemma *INFM-finite-Bex-distrib*: *finite* $A \Longrightarrow (INFM\ y.\ \exists x{\in}A.\ P\ x\ y) \longleftrightarrow (\exists x{\in}A.\ INFM\ y.\ P\ x\ y)$ **by** (*fact frequently-bex-finite-distrib*)
lemma *MOST-finite-Ball-distrib*: *finite* $A \Longrightarrow (MOST\ y.\ \forall x{\in}A.\ P\ x\ y) \longleftrightarrow (\forall x{\in}A.\ MOST\ y.\ P\ x\ y)$ **by** (*fact eventually-ball-finite-distrib*)

lemma *INFM-E*: *INFM x. P x* $\Longrightarrow$ $(\bigwedge x.\ P\ x \Longrightarrow thesis) \Longrightarrow thesis$ **by** (*fact frequentlyE*)
lemma *MOST-I*: $(\bigwedge x.\ P\ x) \Longrightarrow$ *MOST x. P x* **by** (*rule eventuallyI*)
lemmas *MOST-iff-finiteNeg* = *MOST-iff-cofinite*

5.2 Enumeration of an Infinite Set

The set's element type must be wellordered (e.g. the natural numbers).

Could be generalized to $enumerate'\ S\ n = (SOME\ t.\ t \in s \wedge finite\ \{s \in S.\ s < t\} \wedge card\ \{s \in S.\ s < t\} = n)$.

primrec (**in** *wellorder*) *enumerate* :: *'a set* $\Rightarrow$ *nat* $\Rightarrow$ *'a*
where
  enumerate-0: *enumerate S 0* = $(LEAST\ n.\ n \in S)$
| *enumerate-Suc*: *enumerate S* $(Suc\ n)$ = *enumerate* $(S - \{LEAST\ n.\ n \in S\})\ n$

lemma *enumerate-Suc'*: *enumerate S* $(Suc\ n)$ = *enumerate* $(S - \{$*enumerate S 0*$\})\ n$
  by *simp*

lemma *enumerate-in-set*: *infinite S* $\Longrightarrow$ *enumerate S n* $\in$ *S*
  apply (*induct n arbitrary*: *S*)
   apply (*fastforce intro*: *LeastI dest*!: *infinite-imp-nonempty*)
  apply *simp*
  apply (*metis DiffE infinite-remove*)
  done

declare *enumerate-0* [*simp del*] *enumerate-Suc* [*simp del*]

lemma *enumerate-step*: *infinite S* $\Longrightarrow$ *enumerate S n* $<$ *enumerate S* $(Suc\ n)$
  apply (*induct n arbitrary*: *S*)
   apply (*rule order-le-neq-trans*)
    apply (*simp add*: *enumerate-0 Least-le enumerate-in-set*)
   apply (*simp only*: *enumerate-Suc'*)
   apply (*subgoal-tac enumerate* $(S - \{$*enumerate S 0*$\})\ 0 \in S - \{$*enumerate S 0*$\}$)
    apply (*blast intro*: *sym*)
   apply (*simp add*: *enumerate-in-set del*: *Diff-iff*)
  apply (*simp add*: *enumerate-Suc'*)
  done

lemma *enumerate-mono*: *m* $<$ *n* $\Longrightarrow$ *infinite S* $\Longrightarrow$ *enumerate S m* $<$ *enumerate S n*
  apply (*erule less-Suc-induct*)
  apply (*auto intro*: *enumerate-step*)
  done

lemma *le-enumerate*:

```
  assumes S: infinite S
  shows n ≤ enumerate S n
  using S
proof (induct n)
  case 0
  then show ?case by simp
next
  case (Suc n)
  then have n ≤ enumerate S n by simp
  also note enumerate-mono[of n Suc n, OF - ‹infinite S›]
  finally show ?case by simp
qed

lemma enumerate-Suc'':
  fixes S :: 'a::wellorder set
  assumes infinite S
  shows enumerate S (Suc n) = (LEAST s. s ∈ S ∧ enumerate S n < s)
  using assms
proof (induct n arbitrary: S)
  case 0
  then have ∀ s ∈ S. enumerate S 0 ≤ s
    by (auto simp: enumerate.simps intro: Least-le)
  then show ?case
    unfolding enumerate-Suc' enumerate-0[of S − {enumerate S 0}]
    by (intro arg-cong[where f = Least] ext) auto
next
  case (Suc n S)
  show ?case
    using enumerate-mono[OF zero-less-Suc ‹infinite S›, of n] ‹infinite S›
    apply (subst (1 2) enumerate-Suc')
    apply (subst Suc)
    using ‹infinite S›
    apply simp
    apply (intro arg-cong[where f = Least] ext)
    apply (auto simp: enumerate-Suc'[symmetric])
    done
qed

lemma enumerate-Ex:
  assumes S: infinite (S::nat set)
  shows s ∈ S ⟹ ∃ n. enumerate S n = s
proof (induct s rule: less-induct)
  case (less s)
  show ?case
  proof cases
    let ?y = Max {s'∈S. s' < s}
    assume ∃ y∈S. y < s
    then have y: ⋀x. ?y < x ⟷ (∀ s'∈S. s' < s ⟶ s' < x)
      by (subst Max-less-iff) auto
```

```
    then have y-in: ?y ∈ {s'∈S. s' < s}
      by (intro Max-in) auto
    with less.hyps[of ?y] obtain n where enumerate S n = ?y
      by auto
    with S have enumerate S (Suc n) = s
      by (auto simp: y less enumerate-Suc'' intro!: Least-equality)
    then show ?case by auto
  next
    assume *: ¬ (∃ y∈S. y < s)
    then have ∀ t∈S. s ≤ t by auto
    with ‹s ∈ S› show ?thesis
      by (auto intro!: exI[of - 0] Least-equality simp: enumerate-0)
  qed
qed

lemma bij-enumerate:
  fixes S :: nat set
  assumes S: infinite S
  shows bij-betw (enumerate S) UNIV S
proof −
  have ⋀n m. n ≠ m ⟹ enumerate S n ≠ enumerate S m
    using enumerate-mono[OF - ‹infinite S›] by (auto simp: neq-iff)
  then have inj (enumerate S)
    by (auto simp: inj-on-def)
  moreover have ∀ s ∈ S. ∃ i. enumerate S i = s
    using enumerate-Ex[OF S] by auto
  moreover note ‹infinite S›
  ultimately show ?thesis
    unfolding bij-betw-def by (auto intro: enumerate-in-set)
qed

end
```

6 Countable sets

```
theory Countable-Set
imports Countable Infinite-Set
begin
```

6.1 Predicate for countable sets

```
definition countable :: 'a set ⇒ bool where
  countable S ⟷ (∃ f::'a ⇒ nat. inj-on f S)

lemma countableE:
  assumes S: countable S obtains f :: 'a ⇒ nat where inj-on f S
  using S by (auto simp: countable-def)

lemma countableI: inj-on (f::'a ⇒ nat) S ⟹ countable S
```

by (*auto simp*: *countable-def*)

lemma *countableI'*: *inj-on* (*f*::'*a* ⇒ '*b*::*countable*) *S* ⟹ *countable S*
using *comp-inj-on*[*of f S to-nat*] **by** (*auto intro*: *countableI*)

lemma *countableE-bij*:
assumes *S*: *countable S* **obtains** *f* :: *nat* ⇒ '*a* **and** *C* :: *nat set* **where** *bij-betw f C S*
using *S* **by** (*blast elim*: *countableE dest*: *inj-on-imp-bij-betw bij-betw-inv*)

lemma *countableI-bij*: *bij-betw f* (*C*::*nat set*) *S* ⟹ *countable S*
by (*blast intro*: *countableI bij-betw-inv-into bij-betw-imp-inj-on*)

lemma *countable-finite*: *finite S* ⟹ *countable S*
by (*blast dest*: *finite-imp-inj-to-nat-seg countableI*)

lemma *countableI-bij1*: *bij-betw f A B* ⟹ *countable A* ⟹ *countable B*
by (*blast elim*: *countableE-bij intro*: *bij-betw-trans countableI-bij*)

lemma *countableI-bij2*: *bij-betw f B A* ⟹ *countable A* ⟹ *countable B*
by (*blast elim*: *countableE-bij intro*: *bij-betw-trans bij-betw-inv-into countableI-bij*)

lemma *countable-iff-bij*[*simp*]: *bij-betw f A B* ⟹ *countable A* ⟷ *countable B*
by (*blast intro*: *countableI-bij1 countableI-bij2*)

lemma *countable-subset*: *A* ⊆ *B* ⟹ *countable B* ⟹ *countable A*
by (*auto simp*: *countable-def intro*: *subset-inj-on*)

lemma *countableI-type*[*intro*, *simp*]: *countable* (*A*:: '*a* :: *countable set*)
using *countableI*[*of to-nat A*] **by** *auto*

6.2 Enumerate a countable set

lemma *countableE-infinite*:
assumes *countable S infinite S*
obtains *e* :: '*a* ⇒ *nat* **where** *bij-betw e S UNIV*
proof −
obtain *f* :: '*a* ⇒ *nat* **where** *inj-on f S*
using ‹*countable S*› **by** (*rule countableE*)
then have *bij-betw f S* (*f*‘*S*)
unfolding *bij-betw-def* **by** *simp*
moreover
from ‹*inj-on f S*› ‹*infinite S*› **have** *inf-fS*: *infinite* (*f*‘*S*)
by (*auto dest*: *finite-imageD*)
then have *bij-betw* (*the-inv-into UNIV* (*enumerate* (*f*‘*S*))) (*f*‘*S*) *UNIV*
by (*intro bij-betw-the-inv-into bij-enumerate*)
ultimately have *bij-betw* (*the-inv-into UNIV* (*enumerate* (*f*‘*S*)) ∘ *f*) *S UNIV*
by (*rule bij-betw-trans*)
then show *thesis* **..**

qed

lemma *countable-enum-cases*:
assumes *countable S*
obtains (*finite*) *f* :: *'a* ⇒ *nat* **where** *finite S bij-betw f S* {..<*card S*}
| (*infinite*) *f* :: *'a* ⇒ *nat* **where** *infinite S bij-betw f S UNIV*
using *ex-bij-betw-finite-nat*[*of S*] *countableE-infinite* ‹*countable S*›
by (*cases finite S*) (*auto simp add*: *atLeast0LessThan*)

definition *to-nat-on* :: *'a set* ⇒ *'a* ⇒ *nat* **where**
to-nat-on S = (*SOME f*. *if finite S then bij-betw f S* {..< *card S*} *else bij-betw f S UNIV*)

definition *from-nat-into* :: *'a set* ⇒ *nat* ⇒ *'a* **where**
from-nat-into S n = (*if n* ∈ *to-nat-on S* ‘ *S then inv-into S* (*to-nat-on S*) *n else SOME s*. *s*∈*S*)

lemma *to-nat-on-finite*: *finite S* ⟹ *bij-betw* (*to-nat-on S*) *S* {..< *card S*}
using *ex-bij-betw-finite-nat* **unfolding** *to-nat-on-def*
by (*intro someI2-ex*[**where** *Q*=λ*f*. *bij-betw f S* {..<*card S*}]) (*auto simp add*: *atLeast0LessThan*)

lemma *to-nat-on-infinite*: *countable S* ⟹ *infinite S* ⟹ *bij-betw* (*to-nat-on S*) *S UNIV*
using *countableE-infinite* **unfolding** *to-nat-on-def*
by (*intro someI2-ex*[**where** *Q*=λ*f*. *bij-betw f S UNIV*]) *auto*

lemma *bij-betw-from-nat-into-finite*: *finite S* ⟹ *bij-betw* (*from-nat-into S*) {..< *card S*} *S*
unfolding *from-nat-into-def*[*abs-def*]
using *to-nat-on-finite*[*of S*]
apply (*subst bij-betw-cong*)
apply (*split if-split*)
apply (*simp add*: *bij-betw-def*)
apply (*auto cong*: *bij-betw-cong*
intro: *bij-betw-inv-into to-nat-on-finite*)
done

lemma *bij-betw-from-nat-into*: *countable S* ⟹ *infinite S* ⟹ *bij-betw* (*from-nat-into S*) *UNIV S*
unfolding *from-nat-into-def*[*abs-def*]
using *to-nat-on-infinite*[*of S*, *unfolded bij-betw-def*]
by (*auto cong*: *bij-betw-cong intro*: *bij-betw-inv-into to-nat-on-infinite*)

lemma *inj-on-to-nat-on*[*intro*]: *countable A* ⟹ *inj-on* (*to-nat-on A*) *A*
using *to-nat-on-infinite*[*of A*] *to-nat-on-finite*[*of A*]
by (*cases finite A*) (*auto simp*: *bij-betw-def*)

lemma *to-nat-on-inj*[*simp*]:

countable A $\Longrightarrow$ *a* $\in$ *A* $\Longrightarrow$ *b* $\in$ *A* $\Longrightarrow$ *to-nat-on A a* = *to-nat-on A b* $\longleftrightarrow$ *a* = *b*
using *inj-on-to-nat-on*[*of A*] **by** (*auto dest*: *inj-onD*)

lemma *from-nat-into-to-nat-on*[*simp*]: *countable A* $\Longrightarrow$ *a* $\in$ *A* $\Longrightarrow$ *from-nat-into A* (*to-nat-on A a*) = *a*
by (*auto simp*: *from-nat-into-def intro*!: *inv-into-f-f*)

lemma *subset-range-from-nat-into*: *countable A* $\Longrightarrow$ *A* $\subseteq$ *range* (*from-nat-into A*)
by (*auto intro*: *from-nat-into-to-nat-on*[*symmetric*])

lemma *from-nat-into*: *A* $\neq$ {} $\Longrightarrow$ *from-nat-into A n* $\in$ *A*
unfolding *from-nat-into-def* **by** (*metis equals0I inv-into-into someI-ex*)

lemma *range-from-nat-into-subset*: *A* $\neq$ {} $\Longrightarrow$ *range* (*from-nat-into A*) $\subseteq$ *A*
using *from-nat-into*[*of A*] **by** *auto*

lemma *range-from-nat-into*[*simp*]: *A* $\neq$ {} $\Longrightarrow$ *countable A* $\Longrightarrow$ *range* (*from-nat-into A*) = *A*
by (*metis equalityI range-from-nat-into-subset subset-range-from-nat-into*)

lemma *image-to-nat-on*: *countable A* $\Longrightarrow$ *infinite A* $\Longrightarrow$ *to-nat-on A* ' *A* = *UNIV*
using *to-nat-on-infinite*[*of A*] **by** (*simp add*: *bij-betw-def*)

lemma *to-nat-on-surj*: *countable A* $\Longrightarrow$ *infinite A* $\Longrightarrow$ $\exists$ *a*$\in$*A*. *to-nat-on A a* = *n*
by (*metis* (*no-types*) *image-iff iso-tuple-UNIV-I image-to-nat-on*)

lemma *to-nat-on-from-nat-into*[*simp*]: *n* $\in$ *to-nat-on A* ' *A* $\Longrightarrow$ *to-nat-on A* (*from-nat-into A n*) = *n*
by (*simp add*: *f-inv-into-f from-nat-into-def*)

lemma *to-nat-on-from-nat-into-infinite*[*simp*]:
countable A $\Longrightarrow$ *infinite A* $\Longrightarrow$ *to-nat-on A* (*from-nat-into A n*) = *n*
by (*metis image-iff to-nat-on-surj to-nat-on-from-nat-into*)

lemma *from-nat-into-inj*:
countable A $\Longrightarrow$ *m* $\in$ *to-nat-on A* ' *A* $\Longrightarrow$ *n* $\in$ *to-nat-on A* ' *A* $\Longrightarrow$
from-nat-into A m = *from-nat-into A n* $\longleftrightarrow$ *m* = *n*
by (*subst to-nat-on-inj*[*symmetric*, *of A*]) *auto*

lemma *from-nat-into-inj-infinite*[*simp*]:
countable A $\Longrightarrow$ *infinite A* $\Longrightarrow$ *from-nat-into A m* = *from-nat-into A n* $\longleftrightarrow$ *m* = *n*
using *image-to-nat-on*[*of A*] *from-nat-into-inj*[*of A m n*] **by** *simp*

lemma *eq-from-nat-into-iff*:
countable A $\Longrightarrow$ *x* $\in$ *A* $\Longrightarrow$ *i* $\in$ *to-nat-on A* ' *A* $\Longrightarrow$ *x* = *from-nat-into A i* $\longleftrightarrow$ *i* = *to-nat-on A x*
by *auto*

lemma *from-nat-into-surj*: *countable A* $\Longrightarrow$ *a* $\in$ *A* $\Longrightarrow$ $\exists$ *n. from-nat-into A n = a*
by (*rule exI*[*of - to-nat-on A a*]) *simp*

lemma *from-nat-into-inject*[*simp*]:
A $\neq$ {} $\Longrightarrow$ *countable A* $\Longrightarrow$ *B* $\neq$ {} $\Longrightarrow$ *countable B* $\Longrightarrow$ *from-nat-into A = from-nat-into B* $\longleftrightarrow$ *A = B*
by (*metis range-from-nat-into*)

lemma *inj-on-from-nat-into*: *inj-on from-nat-into* ({*A. A* $\neq$ {} $\wedge$ *countable A*})
unfolding *inj-on-def* **by** *auto*

6.3 Closure properties of countability

lemma *countable-SIGMA*[*intro*, *simp*]:
countable I $\Longrightarrow$ ($\bigwedge$*i. i* $\in$ *I* $\Longrightarrow$ *countable* (*A i*)) $\Longrightarrow$ *countable* (*SIGMA i* : *I. A i*)
by (*intro countableI'*[*of* λ(*i*, *a*). (*to-nat-on I i*, *to-nat-on* (*A i*) *a*)]) (*auto simp*: *inj-on-def*)

lemma *countable-image*[*intro*, *simp*]:
assumes *countable A*
shows *countable* (*f*`*A*)
proof −
obtain *g* :: *'a* $\Rightarrow$ *nat* **where** *inj-on g A*
using *assms* **by** (*rule countableE*)
moreover have *inj-on* (*inv-into A f*) (*f*`*A*) *inv-into A f* ` *f* ` *A* $\subseteq$ *A*
by (*auto intro*: *inj-on-inv-into inv-into-into*)
ultimately show *?thesis*
by (*blast dest*: *comp-inj-on subset-inj-on intro*: *countableI*)
qed

lemma *countable-image-inj-on*: *countable* (*f* ` *A*) $\Longrightarrow$ *inj-on f A* $\Longrightarrow$ *countable A*
by (*metis countable-image the-inv-into-onto*)

lemma *countable-UN*[*intro*, *simp*]:
fixes *I* :: *'i set* **and** *A* :: *'i* => *'a set*
assumes *I*: *countable I*
assumes *A*: $\bigwedge$*i. i* $\in$ *I* $\Longrightarrow$ *countable* (*A i*)
shows *countable* ($\bigcup$ *i*$\in$*I. A i*)
proof −
have ($\bigcup$ *i*$\in$*I. A i*) = *snd* ` (*SIGMA i* : *I. A i*) **by** (*auto simp*: *image-iff*)
then show *?thesis* **by** (*simp add*: *assms*)
qed

lemma *countable-Un*[*intro*]: *countable A* $\Longrightarrow$ *countable B* $\Longrightarrow$ *countable* (*A* $\cup$ *B*)
by (*rule countable-UN*[*of* {*True*, *False*} λ*True* $\Rightarrow$ *A* | *False* $\Rightarrow$ *B*, *simplified*])
(*simp split*: *bool.split*)

lemma *countable-Un-iff*[*simp*]: *countable* $(A \cup B) \longleftrightarrow$ *countable* $A \wedge$ *countable* B
by (*metis countable-Un countable-subset inf-sup-ord*(*3*,*4*))

lemma *countable-Plus*[*intro*, *simp*]:
countable $A \Longrightarrow$ *countable* $B \Longrightarrow$ *countable* $(A <+> B)$
by (*simp add*: *Plus-def*)

lemma *countable-empty*[*intro*, *simp*]: *countable* $\{\}$
by (*blast intro*: *countable-finite*)

lemma *countable-insert*[*intro*, *simp*]: *countable* $A \Longrightarrow$ *countable* (*insert a A*)
using *countable-Un*[*of* $\{a\}$ *A*] **by** (*auto simp*: *countable-finite*)

lemma *countable-Int1*[*intro*, *simp*]: *countable* $A \Longrightarrow$ *countable* $(A \cap B)$
by (*force intro*: *countable-subset*)

lemma *countable-Int2*[*intro*, *simp*]: *countable* $B \Longrightarrow$ *countable* $(A \cap B)$
by (*blast intro*: *countable-subset*)

lemma *countable-INT*[*intro*, *simp*]: $i \in I \Longrightarrow$ *countable* $(A\ i) \Longrightarrow$ *countable* $(\bigcap i \in I.\ A\ i)$
by (*blast intro*: *countable-subset*)

lemma *countable-Diff*[*intro*, *simp*]: *countable* $A \Longrightarrow$ *countable* $(A - B)$
by (*blast intro*: *countable-subset*)

lemma *countable-insert-eq* [*simp*]: *countable* (*insert x A*) = *countable A*
by *auto* (*metis Diff-insert-absorb countable-Diff insert-absorb*)

lemma *countable-vimage*: $B \subseteq$ *range f* $\Longrightarrow$ *countable* $(f -` B) \Longrightarrow$ *countable B*
by (*metis Int-absorb2 assms countable-image image-vimage-eq*)

lemma *surj-countable-vimage*: *surj f* $\Longrightarrow$ *countable* $(f -` B) \Longrightarrow$ *countable B*
by (*metis countable-vimage top-greatest*)

lemma *countable-Collect*[*simp*]: *countable* $A \Longrightarrow$ *countable* $\{a \in A.\ \varphi\ a\}$
by (*metis Collect-conj-eq Int-absorb Int-commute Int-def countable-Int1*)

lemma *countable-Image*:
assumes $\bigwedge y.\ y \in Y \Longrightarrow$ *countable* $(X `` \{y\})$
assumes *countable Y*
shows *countable* $(X `` Y)$
proof −
have *countable* $(X `` (\bigcup y \in Y.\ \{y\}))$
unfolding *Image-UN* **by** (*intro countable-UN assms*)
then show *?thesis* **by** *simp*
qed

lemma *countable-relpow*:
fixes $X :: 'a\ rel$
assumes *Image-X*: $\bigwedge Y.\ countable\ Y \Longrightarrow countable\ (X\ ``\ Y)$
assumes Y: *countable Y*
shows *countable* $((X\ \hat{}\hat{}\ i)\ ``\ Y)$
using Y **by** (*induct i arbitrary*: Y) (*auto simp*: *relcomp-Image Image-X*)

lemma *countable-funpow*:
fixes $f :: 'a\ set \Rightarrow 'a\ set$
assumes $\bigwedge A.\ countable\ A \Longrightarrow countable\ (f\ A)$
and *countable A*
shows *countable* $((f\ \hat{}\hat{}\ n)\ A)$
by(*induction n*)(*simp-all add*: *assms*)

lemma *countable-rtrancl*:
$(\bigwedge Y.\ countable\ Y \Longrightarrow countable\ (X\ ``\ Y)) \Longrightarrow countable\ Y \Longrightarrow countable\ (X\hat{}*\ ``\ Y)$
unfolding *rtrancl-is-UN-relpow UN-Image* **by** (*intro countable-UN countableI-type countable-relpow*)

lemma *countable-lists*[*intro*, *simp*]:
assumes A: *countable A* **shows** *countable* (*lists A*)
proof −
have *countable* (*lists* (*range* (*from-nat-into A*)))
by (*auto simp*: *lists-image*)
with A **show** *?thesis*
by (*auto dest*: *subset-range-from-nat-into countable-subset lists-mono*)
qed

lemma *Collect-finite-eq-lists*: *Collect finite* = *set* ' *lists UNIV*
using *finite-list* **by** *auto*

lemma *countable-Collect-finite*: *countable* (*Collect* (*finite*::$'a$::*countable set*⇒*bool*))
by (*simp add*: *Collect-finite-eq-lists*)

lemma *countable-rat*: *countable* $\mathbb{Q}$
unfolding *Rats-def* **by** *auto*

lemma *Collect-finite-subset-eq-lists*: $\{A.\ finite\ A \wedge A \subseteq T\} = set\ `\ lists\ T$
using *finite-list* **by** (*auto simp*: *lists-eq-set*)

lemma *countable-Collect-finite-subset*:
countable T ⟹ *countable* $\{A.\ finite\ A \wedge A \subseteq T\}$
unfolding *Collect-finite-subset-eq-lists* **by** *auto*

lemma *countable-set-option* [*simp*]: *countable* (*set-option x*)
by(*cases x*) *auto*

6.4 Misc lemmas

lemma *infinite-countable-subset'*:
assumes X: *infinite* X **shows** $\exists\, C{\subseteq}X.$ *countable* $C \wedge$ *infinite* C
proof −
from *infinite-countable-subset*[*OF* X] **guess** f **..**
then show *?thesis*
by (*intro exI*[*of* - *range* f]) (*auto simp*: *range-inj-infinite*)
qed

lemma *countable-all*:
assumes S: *countable* S
shows $(\forall\, s{\in}S.\ P\ s) \longleftrightarrow (\forall\, n{::}nat.$ *from-nat-into* $S\ n \in S \longrightarrow P$ (*from-nat-into* $S\ n$))
using S[*THEN subset-range-from-nat-into*] **by** *auto*

lemma *finite-sequence-to-countable-set*:
assumes *countable* X **obtains** F **where** $\bigwedge i.$ $F\ i \subseteq X$ $\bigwedge i.$ $F\ i \subseteq F$ (*Suc* i) $\bigwedge i.$ *finite* ($F\ i$) $(\bigcup i.\ F\ i) = X$
proof − **show** *thesis*
apply (*rule that*[*of* $\lambda i.$ *if* $X = \{\}$ *then* $\{\}$ *else from-nat-into* X ‘ $\{..i\}$])
apply (*auto simp*: *image-iff Ball-def intro*: *from-nat-into split*: *if-split-asm*)
proof −
fix $x\ n$ **assume** $x \in X$ $\forall\, i\ m.\ m \leq i \longrightarrow x \neq$ *from-nat-into* $X\ m$
with *from-nat-into-surj*[*OF* ‹*countable* X› ‹$x \in X$›]
show *False*
by *auto*
qed
qed

lemma *transfer-countable*[*transfer-rule*]:
bi-unique $R \Longrightarrow$ *rel-fun* (*rel-set* R) *op* = *countable countable*
by (*rule rel-funI*, *erule* (*1*) *bi-unique-rel-set-lemma*)
(*auto dest*: *countable-image-inj-on*)

6.5 Uncountable

abbreviation *uncountable* **where**
uncountable $A \equiv \neg$ *countable* A

lemma *uncountable-def*: *uncountable* $A \longleftrightarrow A \neq \{\} \wedge \neg\, (\exists f{::}(nat \Rightarrow 'a).$ *range* $f = A$)
by (*auto intro*: *inj-on-inv-into simp*: *countable-def*)
(*metis all-not-in-conv inj-on-iff-surj subset-UNIV*)

lemma *uncountable-bij-betw*: *bij-betw* $f\ A\ B \Longrightarrow$ *uncountable* $B \Longrightarrow$ *uncountable* A
unfolding *bij-betw-def* **by** (*metis countable-image*)

lemma *uncountable-infinite*: *uncountable* $A \Longrightarrow$ *infinite* A

by (*metis countable-finite*)

lemma *uncountable-minus-countable*:
uncountable A $\Longrightarrow$ *countable B* $\Longrightarrow$ *uncountable* $(A - B)$
using *countable-Un*[*of B A − B*] *assms* **by** *auto*

lemma *countable-Diff-eq* [*simp*]: *countable* $(A - \{x\})$ = *countable A*
by (*meson countable-Diff countable-empty countable-insert uncountable-minus-countable*)

end

7 Pi and Function Sets

theory *FuncSet*
imports *Hilbert-Choice Main*
begin

definition *Pi* :: *'a set* ⇒ (*'a* ⇒ *'b set*) ⇒ (*'a* ⇒ *'b*) *set*
where *Pi A B* = {*f*. ∀ *x*. *x* ∈ *A* ⟶ *f x* ∈ *B x*}

definition *extensional* :: *'a set* ⇒ (*'a* ⇒ *'b*) *set*
where *extensional A* = {*f*. ∀ *x*. *x* ∉ *A* ⟶ *f x* = *undefined*}

definition *restrict* :: (*'a* ⇒ *'b*) ⇒ *'a set* ⇒ *'a* ⇒ *'b*
where *restrict f A* = (λ*x*. *if x* ∈ *A then f x else undefined*)

abbreviation *funcset* :: *'a set* ⇒ *'b set* ⇒ (*'a* ⇒ *'b*) *set* (**infixr** → *60*)
where *A* → *B* ≡ *Pi A* (λ-. *B*)

syntax (*ASCII*)
-Pi :: *pttrn* ⇒ *'a set* ⇒ *'b set* ⇒ (*'a* ⇒ *'b*) *set* ((*3PI -:-./ -*) *10*)
-lam :: *pttrn* ⇒ *'a set* ⇒ *'a* ⇒ *'b* ⇒ (*'a* ⇒ *'b*) ((*3%-:-./ -*) [*0,0,3*] *3*)
syntax
-Pi :: *pttrn* ⇒ *'a set* ⇒ *'b set* ⇒ (*'a* ⇒ *'b*) *set* ((*3*Π *-∈-./ -*) *10*)
-lam :: *pttrn* ⇒ *'a set* ⇒ (*'a* ⇒ *'b*) ⇒ (*'a* ⇒ *'b*) ((*3*λ*-∈-./ -*) [*0,0,3*] *3*)
translations
Π *x*∈*A*. *B* ⇌ *CONST Pi A* (λ*x*. *B*)
λ*x*∈*A*. *f* ⇌ *CONST restrict* (λ*x*. *f*) *A*

definition *compose* :: *'a set* ⇒ (*'b* ⇒ *'c*) ⇒ (*'a* ⇒ *'b*) ⇒ (*'a* ⇒ *'c*)
where *compose A g f* = (λ*x*∈*A*. *g* (*f x*))

7.1 Basic Properties of *Pi*

lemma *Pi-I*[*intro*!]: (⋀*x*. *x* ∈ *A* ⟹ *f x* ∈ *B x*) ⟹ *f* ∈ *Pi A B*
by (*simp add*: *Pi-def*)

lemma *Pi-I'*[*simp*]: (⋀*x*. *x* ∈ *A* ⟶ *f x* ∈ *B x*) ⟹ *f* ∈ *Pi A B*
by (*simp add*:*Pi-def*)

lemma *funcsetI*: $(\bigwedge x.\ x \in A \Longrightarrow f\ x \in B) \Longrightarrow f \in A \to B$
by (*simp add*: *Pi-def*)

lemma *Pi-mem*: $f \in Pi\ A\ B \Longrightarrow x \in A \Longrightarrow f\ x \in B\ x$
by (*simp add*: *Pi-def*)

lemma *Pi-iff*: $f \in Pi\ I\ X \longleftrightarrow (\forall\, i{\in}I.\ f\ i \in X\ i)$
unfolding *Pi-def* **by** *auto*

lemma *PiE* [*elim*]: $f \in Pi\ A\ B \Longrightarrow (f\ x \in B\ x \Longrightarrow Q) \Longrightarrow (x \notin A \Longrightarrow Q) \Longrightarrow Q$
by (*auto simp*: *Pi-def*)

lemma *Pi-cong*: $(\bigwedge w.\ w \in A \Longrightarrow f\ w = g\ w) \Longrightarrow f \in Pi\ A\ B \longleftrightarrow g \in Pi\ A\ B$
by (*auto simp*: *Pi-def*)

lemma *funcset-id* [*simp*]: $(\lambda x.\ x) \in A \to A$
by *auto*

lemma *funcset-mem*: $f \in A \to B \Longrightarrow x \in A \Longrightarrow f\ x \in B$
by (*simp add*: *Pi-def*)

lemma *funcset-image*: $f \in A \to B \Longrightarrow f\ `\ A \subseteq B$
by *auto*

lemma *image-subset-iff-funcset*: $F\ `\ A \subseteq B \longleftrightarrow F \in A \to B$
by *auto*

lemma *Pi-eq-empty*[*simp*]: $(\Pi\ x \in A.\ B\ x) = \{\} \longleftrightarrow (\exists\, x{\in}A.\ B\ x = \{\})$
apply (*simp add*: *Pi-def*)
apply *auto*

Converse direction requires Axiom of Choice to exhibit a function picking an element from each non-empty $B\ x$

apply (*drule-tac* $x = \lambda u.\ SOME\ y.\ y \in B\ u$ **in** *spec*)
apply *auto*
apply (*cut-tac* $P = \lambda y.\ y \in B\ x$ **in** *some-eq-ex*)
apply *auto*
done

lemma *Pi-empty* [*simp*]: $Pi\ \{\}\ B = UNIV$
by (*simp add*: *Pi-def*)

lemma *Pi-Int*: $Pi\ I\ E \cap Pi\ I\ F = (\Pi\ i{\in}I.\ E\ i \cap F\ i)$
by *auto*

lemma *Pi-UN*:
fixes $A :: nat \Rightarrow 'i \Rightarrow 'a\ set$
assumes *finite I*

and *mono*: $\bigwedge i\ n\ m.\ i \in I \Longrightarrow n \leq m \Longrightarrow A\ n\ i \subseteq A\ m\ i$
shows $(\bigcup n.\ Pi\ I\ (A\ n)) = (\Pi\ i{\in}I.\ \bigcup n.\ A\ n\ i)$
proof (*intro set-eqI iffI*)
fix f
assume $f \in (\Pi\ i{\in}I.\ \bigcup n.\ A\ n\ i)$
then have $\forall\, i{\in}I.\ \exists\, n.\ f\ i \in A\ n\ i$
by *auto*
from *bchoice*[*OF this*] **obtain** n **where** n: $\bigwedge i.\ i \in I \Longrightarrow f\ i \in (A\ (n\ i)\ i)$
by *auto*
obtain k **where** k: $\bigwedge i.\ i \in I \Longrightarrow n\ i \leq k$
using ‹*finite I*› *finite-nat-set-iff-bounded-le*[*of n'I*] **by** *auto*
have $f \in Pi\ I\ (A\ k)$
proof (*intro Pi-I*)
fix i
assume $i \in I$
from *mono*[*OF this*, *of n i k*] k[*OF this*] n[*OF this*]
show $f\ i \in A\ k\ i$ **by** *auto*
qed
then show $f \in (\bigcup n.\ Pi\ I\ (A\ n))$
by *auto*
qed *auto*

lemma *Pi-UNIV* [*simp*]: $A \to UNIV = UNIV$
by (*simp add*: *Pi-def*)

Covariance of Pi-sets in their second argument

lemma *Pi-mono*: $(\bigwedge x.\ x \in A \Longrightarrow B\ x \subseteq C\ x) \Longrightarrow Pi\ A\ B \subseteq Pi\ A\ C$
by *auto*

Contravariance of Pi-sets in their first argument

lemma *Pi-anti-mono*: $A' \subseteq A \Longrightarrow Pi\ A\ B \subseteq Pi\ A'\ B$
by *auto*

lemma *prod-final*:
assumes *1*: $fst \circ f \in Pi\ A\ B$
and *2*: $snd \circ f \in Pi\ A\ C$
shows $f \in (\Pi\ z \in A.\ B\ z \times C\ z)$
proof (*rule Pi-I*)
fix z
assume z: $z \in A$
have $f\ z = (fst\ (f\ z),\ snd\ (f\ z))$
by *simp*
also have $\ldots \in B\ z \times C\ z$
by (*metis SigmaI PiE o-apply 1 2 z*)
finally show $f\ z \in B\ z \times C\ z$ **.**
qed

lemma *Pi-split-domain*[*simp*]: $x \in Pi\ (I \cup J)\ X \longleftrightarrow x \in Pi\ I\ X \wedge x \in Pi\ J\ X$
by (*auto simp*: *Pi-def*)

lemma *Pi-split-insert-domain*[*simp*]: $x \in Pi\ (insert\ i\ I)\ X \longleftrightarrow x \in Pi\ I\ X \wedge x\ i \in X\ i$
by (*auto simp*: *Pi-def*)

lemma *Pi-cancel-fupd-range*[*simp*]: $i \notin I \Longrightarrow x \in Pi\ I\ (B(i := b)) \longleftrightarrow x \in Pi\ I\ B$
by (*auto simp*: *Pi-def*)

lemma *Pi-cancel-fupd*[*simp*]: $i \notin I \Longrightarrow x(i := a) \in Pi\ I\ B \longleftrightarrow x \in Pi\ I\ B$
by (*auto simp*: *Pi-def*)

lemma *Pi-fupd-iff*: $i \in I \Longrightarrow f \in Pi\ I\ (B(i := A)) \longleftrightarrow f \in Pi\ (I - \{i\})\ B \wedge f\ i \in A$
apply *auto*
apply (*drule-tac x*=*x* **in** *Pi-mem*)
apply (*simp-all split*: *if-split-asm*)
apply (*drule-tac x*=*i* **in** *Pi-mem*)
apply (*auto dest*!: *Pi-mem*)
done

7.2 Composition With a Restricted Domain: *compose*

lemma *funcset-compose*: $f \in A \to B \Longrightarrow g \in B \to C \Longrightarrow compose\ A\ g\ f \in A \to C$
by (*simp add*: *Pi-def compose-def restrict-def*)

lemma *compose-assoc*:
assumes $f \in A \to B$
and $g \in B \to C$
and $h \in C \to D$
shows $compose\ A\ h\ (compose\ A\ g\ f) = compose\ A\ (compose\ B\ h\ g)\ f$
using *assms* **by** (*simp add*: *fun-eq-iff Pi-def compose-def restrict-def*)

lemma *compose-eq*: $x \in A \Longrightarrow compose\ A\ g\ f\ x = g\ (f\ x)$
by (*simp add*: *compose-def restrict-def*)

lemma *surj-compose*: $f\ `\ A = B \Longrightarrow g\ `\ B = C \Longrightarrow compose\ A\ g\ f\ `\ A = C$
by (*auto simp add*: *image-def compose-eq*)

7.3 Bounded Abstraction: *restrict*

lemma *restrict-cong*: $I = J \Longrightarrow (\bigwedge i.\ i \in J =simp=> f\ i = g\ i) \Longrightarrow restrict\ f\ I = restrict\ g\ J$
by (*auto simp*: *restrict-def fun-eq-iff simp-implies-def*)

lemma *restrict-in-funcset*: $(\bigwedge x.\ x \in A \Longrightarrow f\ x \in B) \Longrightarrow (\lambda x{\in}A.\ f\ x) \in A \to B$
by (*simp add*: *Pi-def restrict-def*)

lemma *restrictI*[*intro*!]: $(\bigwedge x.\ x \in A \Longrightarrow f\ x \in B\ x) \Longrightarrow (\lambda x{\in}A.\ f\ x) \in Pi\ A\ B$

by (*simp add*: *Pi-def restrict-def*)

lemma *restrict-apply*[*simp*]: $(\lambda y{\in}A.\ f\ y)\ x = (\mathit{if}\ x \in A\ \mathit{then}\ f\ x\ \mathit{else}\ \mathit{undefined})$
by (*simp add*: *restrict-def*)

lemma *restrict-apply'*: $x \in A \Longrightarrow (\lambda y{\in}A.\ f\ y)\ x = f\ x$
by *simp*

lemma *restrict-ext*: $(\bigwedge x.\ x \in A \Longrightarrow f\ x = g\ x) \Longrightarrow (\lambda x{\in}A.\ f\ x) = (\lambda x{\in}A.\ g\ x)$
by (*simp add*: *fun-eq-iff Pi-def restrict-def*)

lemma *restrict-UNIV*: *restrict f UNIV* $= f$
by (*simp add*: *restrict-def*)

lemma *inj-on-restrict-eq* [*simp*]: *inj-on* (*restrict f A*) A = *inj-on f A*
by (*simp add*: *inj-on-def restrict-def*)

lemma *Id-compose*: $f \in A \to B \Longrightarrow f \in \mathit{extensional}\ A \Longrightarrow \mathit{compose}\ A\ (\lambda y{\in}B.\ y)\ f = f$
by (*auto simp add*: *fun-eq-iff compose-def extensional-def Pi-def*)

lemma *compose-Id*: $g \in A \to B \Longrightarrow g \in \mathit{extensional}\ A \Longrightarrow \mathit{compose}\ A\ g\ (\lambda x{\in}A.\ x) = g$
by (*auto simp add*: *fun-eq-iff compose-def extensional-def Pi-def*)

lemma *image-restrict-eq* [*simp*]: (*restrict f A*) ‘ A = f ‘ A
by (*auto simp add*: *restrict-def*)

lemma *restrict-restrict*[*simp*]: *restrict* (*restrict f A*) B = *restrict f* $(A \cap B)$
unfolding *restrict-def* **by** (*simp add*: *fun-eq-iff*)

lemma *restrict-fupd*[*simp*]: $i \notin I \Longrightarrow \mathit{restrict}\ (f\ (i := x))\ I = \mathit{restrict}\ f\ I$
by (*auto simp*: *restrict-def*)

lemma *restrict-upd*[*simp*]: $i \notin I \Longrightarrow (\mathit{restrict}\ f\ I)(i := y) = \mathit{restrict}\ (f(i := y))\ (\mathit{insert}\ i\ I)$
by (*auto simp*: *fun-eq-iff*)

lemma *restrict-Pi-cancel*: *restrict x I* $\in$ *Pi I A* $\longleftrightarrow$ $x \in$ *Pi I A*
by (*auto simp*: *restrict-def Pi-def*)

7.4 Bijections Between Sets

The definition of *bij-betw* is in *Fun.thy*, but most of the theorems belong here, or need at least *Hilbert-Choice*.

lemma *bij-betwI*:
assumes $f \in A \to B$
and $g \in B \to A$
and *g-f*: $\bigwedge x.\ x{\in}A \Longrightarrow g\ (f\ x) = x$

and *f-g*: $\bigwedge y.\ y \in B \Longrightarrow f\ (g\ y) = y$
shows *bij-betw f A B*
unfolding *bij-betw-def*
proof
show *inj-on f A*
by (*metis g-f inj-on-def*)
have $f\ `\ A \subseteq B$
using ‹$f \in A \to B$› **by** *auto*
moreover
have $B \subseteq f\ `\ A$
by *auto* (*metis Pi-mem* ‹$g \in B \to A$› *f-g image-iff*)
ultimately show $f\ `\ A = B$
by *blast*
qed

lemma *bij-betw-imp-funcset*: *bij-betw f A B* $\Longrightarrow f \in A \to B$
by (*auto simp add*: *bij-betw-def*)

lemma *inj-on-compose*: *bij-betw f A B* $\Longrightarrow$ *inj-on g B* $\Longrightarrow$ *inj-on* (*compose A g f*) *A*
by (*auto simp add*: *bij-betw-def inj-on-def compose-eq*)

lemma *bij-betw-compose*: *bij-betw f A B* $\Longrightarrow$ *bij-betw g B C* $\Longrightarrow$ *bij-betw* (*compose A g f*) *A C*
apply (*simp add*: *bij-betw-def compose-eq inj-on-compose*)
apply (*auto simp add*: *compose-def image-def*)
done

lemma *bij-betw-restrict-eq* [*simp*]: *bij-betw* (*restrict f A*) *A B* = *bij-betw f A B*
by (*simp add*: *bij-betw-def*)

7.5 Extensionality

lemma *extensional-empty*[*simp*]: *extensional* {} = {$\lambda x.$ *undefined*}
unfolding *extensional-def* **by** *auto*

lemma *extensional-arb*: $f \in$ *extensional A* $\Longrightarrow x \notin A \Longrightarrow f\ x =$ *undefined*
by (*simp add*: *extensional-def*)

lemma *restrict-extensional* [*simp*]: *restrict f A* $\in$ *extensional A*
by (*simp add*: *restrict-def extensional-def*)

lemma *compose-extensional* [*simp*]: *compose A f g* $\in$ *extensional A*
by (*simp add*: *compose-def*)

lemma *extensionalityI*:
assumes $f \in$ *extensional A*
and $g \in$ *extensional A*
and $\bigwedge x.\ x \in A \Longrightarrow f\ x = g\ x$

shows $f = g$
using *assms* **by** (*force simp add*: *fun-eq-iff extensional-def*)

lemma *extensional-restrict*: $f \in \mathit{extensional}\ A \Longrightarrow \mathit{restrict}\ f\ A = f$
by (*rule extensionalityI*[*OF restrict-extensional*]) *auto*

lemma *extensional-subset*: $f \in \mathit{extensional}\ A \Longrightarrow A \subseteq B \Longrightarrow f \in \mathit{extensional}\ B$
unfolding *extensional-def* **by** *auto*

lemma *inv-into-funcset*: $f\ `\ A = B \Longrightarrow (\lambda x{\in}B.\ \mathit{inv\text{-}into}\ A\ f\ x) \in B \to A$
by (*unfold inv-into-def*) (*fast intro*: *someI2*)

lemma *compose-inv-into-id*: *bij-betw f A B* $\Longrightarrow$ *compose A* ($\lambda y{\in}B.$ *inv-into A f y*) *f* $= (\lambda x{\in}A.\ x)$
apply (*simp add*: *bij-betw-def compose-def*)
apply (*rule restrict-ext*, *auto*)
done

lemma *compose-id-inv-into*: $f\ `\ A = B \Longrightarrow \mathit{compose}\ B\ f\ (\lambda y{\in}B.\ \mathit{inv\text{-}into}\ A\ f\ y) = (\lambda x{\in}B.\ x)$
apply (*simp add*: *compose-def*)
apply (*rule restrict-ext*)
apply (*simp add*: *f-inv-into-f*)
done

lemma *extensional-insert*[*intro*, *simp*]:
assumes $a \in \mathit{extensional}\ (\mathit{insert}\ i\ I)$
shows $a(i := b) \in \mathit{extensional}\ (\mathit{insert}\ i\ I)$
using *assms* **unfolding** *extensional-def* **by** *auto*

lemma *extensional-Int*[*simp*]: *extensional I* $\cap$ *extensional I′* $=$ *extensional* $(I \cap I')$
unfolding *extensional-def* **by** *auto*

lemma *extensional-UNIV*[*simp*]: *extensional UNIV* $=$ *UNIV*
by (*auto simp*: *extensional-def*)

lemma *restrict-extensional-sub*[*intro*]: $A \subseteq B \Longrightarrow \mathit{restrict}\ f\ A \in \mathit{extensional}\ B$
unfolding *restrict-def extensional-def* **by** *auto*

lemma *extensional-insert-undefined*[*intro*, *simp*]:
$a \in \mathit{extensional}\ (\mathit{insert}\ i\ I) \Longrightarrow a(i := \mathit{undefined}) \in \mathit{extensional}\ I$
unfolding *extensional-def* **by** *auto*

lemma *extensional-insert-cancel*[*intro*, *simp*]:
$a \in \mathit{extensional}\ I \Longrightarrow a \in \mathit{extensional}\ (\mathit{insert}\ i\ I)$
unfolding *extensional-def* **by** *auto*

7.6 Cardinality

lemma *card-inj*: $f \in A \to B \Longrightarrow$ *inj-on* $f\ A \Longrightarrow$ *finite* $B \Longrightarrow$ *card* $A \leq$ *card* B
by (*rule card-inj-on-le*) *auto*

lemma *card-bij*:
assumes $f \in A \to B$ *inj-on* $f\ A$
and $g \in B \to A$ *inj-on* $g\ B$
and *finite* A *finite* B
shows *card* $A =$ *card* B
using *assms* **by** (*blast intro*: *card-inj order-antisym*)

7.7 Extensional Function Spaces

definition $PiE :: 'a\ set \Rightarrow ('a \Rightarrow 'b\ set) \Rightarrow ('a \Rightarrow 'b)\ set$
where $PiE\ S\ T = Pi\ S\ T \cap extensional\ S$

abbreviation $Pi_E\ A\ B \equiv PiE\ A\ B$

syntax (*ASCII*)
-PiE :: *pttrn* $\Rightarrow$ $'a\ set \Rightarrow 'b\ set \Rightarrow ('a \Rightarrow 'b)\ set$ ((*3PIE -:-./ -*) *10*)
syntax
-PiE :: *pttrn* $\Rightarrow$ $'a\ set \Rightarrow 'b\ set \Rightarrow ('a \Rightarrow 'b)\ set$ ((*3*Π_E *-∈-./ -*) *10*)
translations
$\Pi_E\ x{\in}A.\ B \rightleftharpoons CONST\ Pi_E\ A\ (\lambda x.\ B)$

abbreviation *extensional-funcset* :: $'a\ set \Rightarrow 'b\ set \Rightarrow ('a \Rightarrow 'b)\ set$ (**infixr** $\to_E$ *60*)
where $A \to_E B \equiv (\Pi_E\ i{\in}A.\ B)$

lemma *extensional-funcset-def*: *extensional-funcset* $S\ T = (S \to T) \cap$ *extensional* S
by (*simp add*: *PiE-def*)

lemma *PiE-empty-domain*[*simp*]: $PiE\ \{\}\ T = \{\lambda x.\ undefined\}$
unfolding *PiE-def* **by** *simp*

lemma *PiE-UNIV-domain*: $PiE\ UNIV\ T = Pi\ UNIV\ T$
unfolding *PiE-def* **by** *simp*

lemma *PiE-empty-range*[*simp*]: $i \in I \Longrightarrow F\ i = \{\} \Longrightarrow (\Pi_E\ i{\in}I.\ F\ i) = \{\}$
unfolding *PiE-def* **by** *auto*

lemma *PiE-eq-empty-iff*: $Pi_E\ I\ F = \{\} \longleftrightarrow (\exists\, i{\in}I.\ F\ i = \{\})$
proof
assume $Pi_E\ I\ F = \{\}$
show $\exists\, i{\in}I.\ F\ i = \{\}$
proof (*rule ccontr*)
assume $\neg$ *?thesis*
then have $\forall\, i.\ \exists\, y.\ (i \in I \longrightarrow y \in F\ i) \wedge (i \notin I \longrightarrow y = undefined)$

by *auto*
from *choice*[*OF this*]
obtain *f* **where** $\forall x.\ (x \in I \longrightarrow f\ x \in F\ x) \wedge (x \notin I \longrightarrow f\ x = undefined)$ **..**
then have $f \in Pi_E\ I\ F$
by (*auto simp*: *extensional-def PiE-def*)
with ‹$Pi_E\ I\ F = \{\}$› **show** *False*
by *auto*
qed
qed (*auto simp*: *PiE-def*)

lemma *PiE-arb*: $f \in PiE\ S\ T \Longrightarrow x \notin S \Longrightarrow f\ x = undefined$
unfolding *PiE-def* **by** *auto* (*auto dest*!: *extensional-arb*)

lemma *PiE-mem*: $f \in PiE\ S\ T \Longrightarrow x \in S \Longrightarrow f\ x \in T\ x$
unfolding *PiE-def* **by** *auto*

lemma *PiE-fun-upd*: $y \in T\ x \Longrightarrow f \in PiE\ S\ T \Longrightarrow f(x := y) \in PiE\ (insert\ x\ S)\ T$
unfolding *PiE-def extensional-def* **by** *auto*

lemma *fun-upd-in-PiE*: $x \notin S \Longrightarrow f \in PiE\ (insert\ x\ S)\ T \Longrightarrow f(x := undefined) \in PiE\ S\ T$
unfolding *PiE-def extensional-def* **by** *auto*

lemma *PiE-insert-eq*: $PiE\ (insert\ x\ S)\ T = (\lambda(y,\ g).\ g(x := y))\ `\ (T\ x \times PiE\ S\ T)$
proof −
{
fix *f* **assume** $f \in PiE\ (insert\ x\ S)\ T\ x \notin S$
with *assms* **have** $f \in (\lambda(y,\ g).\ g(x := y))\ `\ (T\ x \times PiE\ S\ T)$
by (*auto intro*!: *image-eqI*[**where** $x=(f\ x, f(x := undefined))$] *intro*: *fun-upd-in-PiE PiE-mem*)
}
moreover
{
fix *f* **assume** $f \in PiE\ (insert\ x\ S)\ T\ x \in S$
with *assms* **have** $f \in (\lambda(y,\ g).\ g(x := y))\ `\ (T\ x \times PiE\ S\ T)$
by (*auto intro*!: *image-eqI*[**where** $x=(f\ x,\ f)$] *intro*: *fun-upd-in-PiE PiE-mem simp*: *insert-absorb*)
}
ultimately show *?thesis*
using *assms* **by** (*auto intro*: *PiE-fun-upd*)
qed

lemma *PiE-Int*: $Pi_E\ I\ A \cap Pi_E\ I\ B = Pi_E\ I\ (\lambda x.\ A\ x \cap B\ x)$
by (*auto simp*: *PiE-def*)

lemma *PiE-cong*: $(\bigwedge i.\ i \in I \Longrightarrow A\ i = B\ i) \Longrightarrow Pi_E\ I\ A = Pi_E\ I\ B$
unfolding *PiE-def* **by** (*auto simp*: *Pi-cong*)

lemma *PiE-E* [*elim*]:
assumes $f \in PiE\ A\ B$
obtains $x \in A$ **and** $f\ x \in B\ x$
| $x \notin A$ **and** $f\ x = undefined$
using *assms* **by** (*auto simp*: *Pi-def PiE-def extensional-def*)

lemma *PiE-I*[*intro*!]:
$(\bigwedge x.\ x \in A \Longrightarrow f\ x \in B\ x) \Longrightarrow (\bigwedge x.\ x \notin A \Longrightarrow f\ x = undefined) \Longrightarrow f \in PiE\ A\ B$
by (*simp add*: *PiE-def extensional-def*)

lemma *PiE-mono*: $(\bigwedge x.\ x \in A \Longrightarrow B\ x \subseteq C\ x) \Longrightarrow PiE\ A\ B \subseteq PiE\ A\ C$
by *auto*

lemma *PiE-iff*: $f \in PiE\ I\ X \longleftrightarrow (\forall i{\in}I.\ f\ i \in X\ i) \wedge f \in extensional\ I$
by (*simp add*: *PiE-def Pi-iff*)

lemma *PiE-restrict*[*simp*]: $f \in PiE\ A\ B \Longrightarrow restrict\ f\ A = f$
by (*simp add*: *extensional-restrict PiE-def*)

lemma *restrict-PiE*[*simp*]: $restrict\ f\ I \in PiE\ I\ S \longleftrightarrow f \in Pi\ I\ S$
by (*auto simp*: *PiE-iff*)

lemma *PiE-eq-subset*:
assumes *ne*: $\bigwedge i.\ i \in I \Longrightarrow F\ i \neq \{\}$ $\bigwedge i.\ i \in I \Longrightarrow F'\ i \neq \{\}$
and *eq*: $Pi_E\ I\ F = Pi_E\ I\ F'$
and $i \in I$
shows $F\ i \subseteq F'\ i$
proof
fix x
assume $x \in F\ i$
with *ne* **have** $\forall j.\ \exists y.\ (j \in I \longrightarrow y \in F\ j \wedge (i = j \longrightarrow x = y)) \wedge (j \notin I \longrightarrow y = undefined)$
by *auto*
from *choice*[*OF this*] **obtain** f
where f: $\forall j.\ (j \in I \longrightarrow f\ j \in F\ j \wedge (i = j \longrightarrow x = f\ j)) \wedge (j \notin I \longrightarrow f\ j = undefined)$ **..**
then have $f \in Pi_E\ I\ F$
by (*auto simp*: *extensional-def PiE-def*)
then have $f \in Pi_E\ I\ F'$
using *assms* **by** *simp*
then show $x \in F'\ i$
using f ‹$i \in I$› **by** (*auto simp*: *PiE-def*)
qed

lemma *PiE-eq-iff-not-empty*:
assumes *ne*: $\bigwedge i.\ i \in I \Longrightarrow F\ i \neq \{\}$ $\bigwedge i.\ i \in I \Longrightarrow F'\ i \neq \{\}$
shows $Pi_E\ I\ F = Pi_E\ I\ F' \longleftrightarrow (\forall i{\in}I.\ F\ i = F'\ i)$

proof (*intro iffI ballI*)
fix i
assume *eq*: $Pi_E\ I\ F = Pi_E\ I\ F'$
assume i: $i \in I$
show $F\ i = F'\ i$
using *PiE-eq-subset*[*of I F F'*, *OF ne eq i*]
using *PiE-eq-subset*[*of I F' F*, *OF ne*(*2*,*1*) *eq*[*symmetric*] *i*]
by *auto*
qed (*auto simp*: *PiE-def*)

lemma *PiE-eq-iff*:
$Pi_E\ I\ F = Pi_E\ I\ F' \longleftrightarrow (\forall\, i{\in}I.\ F\ i = F'\ i) \vee ((\exists\, i{\in}I.\ F\ i = \{\}) \wedge (\exists\, i{\in}I.\ F'\ i = \{\}))$
proof (*intro iffI disjCI*)
assume *eq*[*simp*]: $Pi_E\ I\ F = Pi_E\ I\ F'$
assume $\neg\ ((\exists\, i{\in}I.\ F\ i = \{\}) \wedge (\exists\, i{\in}I.\ F'\ i = \{\}))$
then have $(\forall\, i{\in}I.\ F\ i \neq \{\}) \wedge (\forall\, i{\in}I.\ F'\ i \neq \{\})$
using *PiE-eq-empty-iff*[*of I F*] *PiE-eq-empty-iff*[*of I F'*] **by** *auto*
with *PiE-eq-iff-not-empty*[*of I F F'*] **show** $\forall\, i{\in}I.\ F\ i = F'\ i$
by *auto*
next
assume $(\forall\, i{\in}I.\ F\ i = F'\ i) \vee (\exists\, i{\in}I.\ F\ i = \{\}) \wedge (\exists\, i{\in}I.\ F'\ i = \{\})$
then show $Pi_E\ I\ F = Pi_E\ I\ F'$
using *PiE-eq-empty-iff*[*of I F*] *PiE-eq-empty-iff*[*of I F'*] **by** (*auto simp*: *PiE-def*)
qed

lemma *extensional-funcset-fun-upd-restricts-rangeI*:
$\forall\, y \in S.\ f\ x \neq f\ y \Longrightarrow f \in (insert\ x\ S) \rightarrow_E T \Longrightarrow f(x := undefined) \in S \rightarrow_E (T - \{f\ x\})$
unfolding *extensional-funcset-def extensional-def*
apply *auto*
apply (*case-tac* $x = xa$)
apply *auto*
done

lemma *extensional-funcset-fun-upd-extends-rangeI*:
assumes $a \in T$ $f \in S \rightarrow_E (T - \{a\})$
shows $f(x := a) \in insert\ x\ S \rightarrow_E T$
using *assms* **unfolding** *extensional-funcset-def extensional-def* **by** *auto*

7.7.1 Injective Extensional Function Spaces

lemma *extensional-funcset-fun-upd-inj-onI*:
assumes $f \in S \rightarrow_E (T - \{a\})$
and *inj-on f S*
shows *inj-on* $(f(x := a))\ S$
using *assms*
unfolding *extensional-funcset-def* **by** (*auto intro*!: *inj-on-fun-updI*)

lemma *extensional-funcset-extend-domain-inj-on-eq*:
assumes $x \notin S$
shows $\{f.\ f \in (insert\ x\ S) \to_E T \wedge inj\text{-}on\ f\ (insert\ x\ S)\} =$
$(\lambda(y,\ g).\ g(x{:=}y))\ `\ \{(y,\ g).\ y \in T \wedge g \in S \to_E (T - \{y\}) \wedge inj\text{-}on\ g\ S\}$
using *assms*
apply (*auto del*: *PiE-I PiE-E*)
apply (*auto intro*: *extensional-funcset-fun-upd-inj-onI*
extensional-funcset-fun-upd-extends-rangeI del: *PiE-I PiE-E*)
apply (*auto simp add*: *image-iff inj-on-def*)
apply (*rule-tac x*=*xa x* **in** *exI*)
apply (*auto intro*: *PiE-mem del*: *PiE-I PiE-E*)
apply (*rule-tac x*=*xa*($x := undefined$) **in** *exI*)
apply (*auto intro*!: *extensional-funcset-fun-upd-restricts-rangeI*)
apply (*auto dest*!: *PiE-mem split*: *if-split-asm*)
done

lemma *extensional-funcset-extend-domain-inj-onI*:
assumes $x \notin S$
shows *inj-on* $(\lambda(y,\ g).\ g(x := y))\ \{(y,\ g).\ y \in T \wedge g \in S \to_E (T - \{y\}) \wedge$
inj-on g S$\}$
using *assms*
apply (*auto intro*!: *inj-onI*)
apply (*metis fun-upd-same*)
apply (*metis assms PiE-arb fun-upd-triv fun-upd-upd*)
done

7.7.2 Cardinality

lemma *finite-PiE*: *finite S* $\Longrightarrow$ $(\bigwedge i.\ i \in S \Longrightarrow$ *finite* $(T\ i)) \Longrightarrow$ *finite* $(\Pi_E\ i \in S.$
$T\ i)$
by (*induct S arbitrary*: *T rule*: *finite-induct*) (*simp-all add*: *PiE-insert-eq*)

lemma *inj-combinator*: $x \notin S \Longrightarrow$ *inj-on* $(\lambda(y,\ g).\ g(x := y))\ (T\ x \times Pi_E\ S\ T)$
proof (*safe intro*!: *inj-onI ext*)
fix *f y g z*
assume $x \notin S$
assume *fg*: $f \in Pi_E\ S\ T\ g \in Pi_E\ S\ T$
assume $f(x := y) = g(x := z)$
then have $*$: $\bigwedge i.\ (f(x := y))\ i = (g(x := z))\ i$
unfolding *fun-eq-iff* **by** *auto*
from *this*[*of x*] **show** $y = z$ **by** *simp*
fix *i* **from** $*$[*of i*] ‹$x \notin S$› *fg* **show** $f\ i = g\ i$
by (*auto split*: *if-split-asm simp*: *PiE-def extensional-def*)
qed

lemma *card-PiE*: *finite S* $\Longrightarrow$ *card* $(\Pi_E\ i \in S.\ T\ i) = (\prod\ i{\in}S.\ card\ (T\ i))$
proof (*induct rule*: *finite-induct*)
case *empty*
then show *?case* **by** *auto*

next
case (*insert x S*)
then show *?case*
by (*simp add*: *PiE-insert-eq inj-combinator card-image card-cartesian-product*)
qed

end

8 Square root of sum of squares

theory *L2-Norm*
imports *NthRoot*
begin

definition
setL2 f A = *sqrt* ($\sum i{\in}A.$ $(f\ i)^2$)

lemma *setL2-cong*:
$[\![A = B;\ \bigwedge x.\ x \in B \Longrightarrow f\ x = g\ x]\!] \Longrightarrow$ *setL2 f A* = *setL2 g B*
unfolding *setL2-def* **by** *simp*

lemma *strong-setL2-cong*:
$[\![A = B;\ \bigwedge x.\ x \in B$ =*simp*=> $f\ x = g\ x]\!] \Longrightarrow$ *setL2 f A* = *setL2 g B*
unfolding *setL2-def simp-implies-def* **by** *simp*

lemma *setL2-infinite* [*simp*]: ¬ *finite A* $\Longrightarrow$ *setL2 f A* = *0*
unfolding *setL2-def* **by** *simp*

lemma *setL2-empty* [*simp*]: *setL2 f* {} = *0*
unfolding *setL2-def* **by** *simp*

lemma *setL2-insert* [*simp*]:
$[\![$*finite F*; *a* ∉ *F*$]\!] \Longrightarrow$
setL2 f (*insert a F*) = *sqrt* ($(f\ a)^2$ + $(setL2\ f\ F)^2$)
unfolding *setL2-def* **by** (*simp add*: *setsum-nonneg*)

lemma *setL2-nonneg* [*simp*]: *0* ≤ *setL2 f A*
unfolding *setL2-def* **by** (*simp add*: *setsum-nonneg*)

lemma *setL2-0'*: ∀ *a*∈*A*. *f a* = *0* $\Longrightarrow$ *setL2 f A* = *0*
unfolding *setL2-def* **by** *simp*

lemma *setL2-constant*: *setL2* (λ*x*. *y*) *A* = *sqrt* (*of-nat* (*card A*)) * |*y*|
unfolding *setL2-def* **by** (*simp add*: *real-sqrt-mult*)

lemma *setL2-mono*:
assumes $\bigwedge i.\ i \in K \Longrightarrow f\ i \le g\ i$
assumes $\bigwedge i.\ i \in K \Longrightarrow 0 \le f\ i$
shows *setL2 f K* ≤ *setL2 g K*

unfolding *setL2-def*
by (*simp add*: *setsum-nonneg setsum-mono power-mono assms*)

lemma *setL2-strict-mono*:
assumes *finite K* **and** $K \neq \{\}$
assumes $\bigwedge i.\ i \in K \Longrightarrow f\ i < g\ i$
assumes $\bigwedge i.\ i \in K \Longrightarrow 0 \leq f\ i$
shows *setL2 f K* < *setL2 g K*
unfolding *setL2-def*
by (*simp add*: *setsum-strict-mono power-strict-mono assms*)

lemma *setL2-right-distrib*:
$0 \leq r \Longrightarrow r * setL2\ f\ A = setL2\ (\lambda x.\ r * f\ x)\ A$
unfolding *setL2-def*
apply (*simp add*: *power-mult-distrib*)
apply (*simp add*: *setsum-right-distrib* [*symmetric*])
apply (*simp add*: *real-sqrt-mult setsum-nonneg*)
done

lemma *setL2-left-distrib*:
$0 \leq r \Longrightarrow setL2\ f\ A * r = setL2\ (\lambda x.\ f\ x * r)\ A$
unfolding *setL2-def*
apply (*simp add*: *power-mult-distrib*)
apply (*simp add*: *setsum-left-distrib* [*symmetric*])
apply (*simp add*: *real-sqrt-mult setsum-nonneg*)
done

lemma *setL2-eq-0-iff*: *finite A* $\Longrightarrow setL2\ f\ A = 0 \longleftrightarrow (\forall x{\in}A.\ f\ x = 0)$
unfolding *setL2-def*
by (*simp add*: *setsum-nonneg setsum-nonneg-eq-0-iff*)

lemma *setL2-triangle-ineq*:
shows $setL2\ (\lambda i.\ f\ i + g\ i)\ A \leq setL2\ f\ A + setL2\ g\ A$
proof (*cases finite A*)
case *False*
thus *?thesis* **by** *simp*
next
case *True*
thus *?thesis*
proof (*induct set*: *finite*)
case *empty*
show *?case* **by** *simp*
next
case (*insert x F*)
hence $sqrt\ ((f\ x + g\ x)^2 + (setL2\ (\lambda i.\ f\ i + g\ i)\ F)^2) \leq$
$sqrt\ ((f\ x + g\ x)^2 + (setL2\ f\ F + setL2\ g\ F)^2)$
by (*intro real-sqrt-le-mono add-left-mono power-mono insert*
setL2-nonneg add-increasing zero-le-power2)
also have

```
    ... ≤ sqrt ((f x)^2 + (setL2 f F)^2) + sqrt ((g x)^2 + (setL2 g F)^2)
      by (rule real-sqrt-sum-squares-triangle-ineq)
   finally show ?case
     using insert by simp
 qed
qed

lemma sqrt-sum-squares-le-sum:
 ⟦0 ≤ x; 0 ≤ y⟧ ⟹ sqrt (x^2 + y^2) ≤ x + y
 apply (rule power2-le-imp-le)
 apply (simp add: power2-sum)
 apply simp
 done

lemma setL2-le-setsum [rule-format]:
 (∀ i∈A. 0 ≤ f i) ⟶ setL2 f A ≤ setsum f A
 apply (cases finite A)
 apply (induct set: finite)
 apply simp
 apply clarsimp
 apply (erule order-trans [OF sqrt-sum-squares-le-sum])
 apply simp
 apply simp
 apply simp
 done

lemma sqrt-sum-squares-le-sum-abs: sqrt (x^2 + y^2) ≤ |x| + |y|
 apply (rule power2-le-imp-le)
 apply (simp add: power2-sum)
 apply simp
 done

lemma setL2-le-setsum-abs: setL2 f A ≤ (∑ i∈A. |f i|)
 apply (cases finite A)
 apply (induct set: finite)
 apply simp
 apply simp
 apply (rule order-trans [OF sqrt-sum-squares-le-sum-abs])
 apply simp
 apply simp
 done

lemma setL2-mult-ineq-lemma:
 fixes a b c d :: real
 shows 2 * (a * c) * (b * d) ≤ a^2 * d^2 + b^2 * c^2
proof -
 have 0 ≤ (a * d - b * c)^2 by simp
 also have ... = a^2 * d^2 + b^2 * c^2 - 2 * (a * d) * (b * c)
   by (simp only: power2-diff power-mult-distrib)
```

also have ... $= a^2 * d^2 + b^2 * c^2 - 2 * (a * c) * (b * d)$
by *simp*
finally show $2 * (a * c) * (b * d) \leq a^2 * d^2 + b^2 * c^2$
by *simp*
qed

lemma *setL2-mult-ineq*: $(\sum i{\in}A.\ |f\ i| * |g\ i|) \leq setL2\ f\ A * setL2\ g\ A$
apply (*cases finite A*)
apply (*induct set*: *finite*)
apply *simp*
apply (*rule power2-le-imp-le*, *simp*)
apply (*rule order-trans*)
apply (*rule power-mono*)
apply (*erule add-left-mono*)
apply (*simp add*: *setsum-nonneg*)
apply (*simp add*: *power2-sum*)
apply (*simp add*: *power-mult-distrib*)
apply (*simp add*: *distrib-left distrib-right*)
apply (*rule ord-le-eq-trans*)
apply (*rule setL2-mult-ineq-lemma*)
apply *simp-all*
done

lemma *member-le-setL2*: $[\![finite\ A;\ i \in A]\!] \Longrightarrow f\ i \leq setL2\ f\ A$
apply (*rule-tac s=insert i* $(A - \{i\})$ **and** *t=A* **in** *subst*)
apply *fast*
apply (*subst setL2-insert*)
apply *simp*
apply *simp*
apply *simp*
done

end

9 Inner Product Spaces and the Gradient Derivative

theory *Inner-Product*
imports ~~/*src/HOL/Complex-Main*
begin

9.1 Real inner product spaces

Temporarily relax type constraints for *open*, *uniformity*, *dist*, and *norm*.

setup ‹*Sign.add-const-constraint*
(@{*const-name open*}, *SOME* @{*typ 'a::open set* ⇒ *bool*})›

setup ‹*Sign.add-const-constraint*

(@{*const-name dist*}, *SOME* @{*typ 'a::dist ⇒ 'a ⇒ real*})›

setup ‹*Sign.add-const-constraint*
(@{*const-name uniformity*}, *SOME* @{*typ ('a::uniformity × 'a) filter*})›

setup ‹*Sign.add-const-constraint*
(@{*const-name norm*}, *SOME* @{*typ 'a::norm ⇒ real*})›

class *real-inner* = *real-vector* + *sgn-div-norm* + *dist-norm* + *uniformity-dist* + *open-uniformity* +
fixes *inner* :: *'a ⇒ 'a ⇒ real*
assumes *inner-commute*: *inner x y = inner y x*
and *inner-add-left*: *inner (x + y) z = inner x z + inner y z*
and *inner-scaleR-left* [*simp*]: *inner (scaleR r x) y = r * (inner x y)*
and *inner-ge-zero* [*simp*]: *0 ≤ inner x x*
and *inner-eq-zero-iff* [*simp*]: *inner x x = 0 ⟷ x = 0*
and *norm-eq-sqrt-inner*: *norm x = sqrt (inner x x)*
begin

lemma *inner-zero-left* [*simp*]: *inner 0 x = 0*
using *inner-add-left* [*of 0 0 x*] **by** *simp*

lemma *inner-minus-left* [*simp*]: *inner (- x) y = - inner x y*
using *inner-add-left* [*of x - x y*] **by** *simp*

lemma *inner-diff-left*: *inner (x - y) z = inner x z - inner y z*
using *inner-add-left* [*of x - y z*] **by** *simp*

lemma *inner-setsum-left*: *inner (∑ x∈A. f x) y = (∑ x∈A. inner (f x) y)*
by (*cases finite A*, *induct set*: *finite*, *simp-all add*: *inner-add-left*)

Transfer distributivity rules to right argument.

lemma *inner-add-right*: *inner x (y + z) = inner x y + inner x z*
using *inner-add-left* [*of y z x*] **by** (*simp only*: *inner-commute*)

lemma *inner-scaleR-right* [*simp*]: *inner x (scaleR r y) = r * (inner x y)*
using *inner-scaleR-left* [*of r y x*] **by** (*simp only*: *inner-commute*)

lemma *inner-zero-right* [*simp*]: *inner x 0 = 0*
using *inner-zero-left* [*of x*] **by** (*simp only*: *inner-commute*)

lemma *inner-minus-right* [*simp*]: *inner x (- y) = - inner x y*
using *inner-minus-left* [*of y x*] **by** (*simp only*: *inner-commute*)

lemma *inner-diff-right*: *inner x (y - z) = inner x y - inner x z*
using *inner-diff-left* [*of y z x*] **by** (*simp only*: *inner-commute*)

lemma *inner-setsum-right*: *inner x (∑ y∈A. f y) = (∑ y∈A. inner x (f y))*
using *inner-setsum-left* [*of f A x*] **by** (*simp only*: *inner-commute*)

lemmas *inner-add* [*algebra-simps*] = *inner-add-left inner-add-right*
lemmas *inner-diff* [*algebra-simps*] = *inner-diff-left inner-diff-right*
lemmas *inner-scaleR* = *inner-scaleR-left inner-scaleR-right*

Legacy theorem names

lemmas *inner-left-distrib* = *inner-add-left*
lemmas *inner-right-distrib* = *inner-add-right*
lemmas *inner-distrib* = *inner-left-distrib inner-right-distrib*

lemma *inner-gt-zero-iff* [*simp*]: $0 < \mathit{inner}\ x\ x \longleftrightarrow x \neq 0$
by (*simp add*: *order-less-le*)

lemma *power2-norm-eq-inner*: $(\mathit{norm}\ x)^2 = \mathit{inner}\ x\ x$
by (*simp add*: *norm-eq-sqrt-inner*)

Identities involving real multiplication and division.

lemma *inner-mult-left*: $\mathit{inner}\ (\mathit{of\text{-}real}\ m * a)\ b = m * (\mathit{inner}\ a\ b)$
by (*metis real-inner-class.inner-scaleR-left scaleR-conv-of-real*)

lemma *inner-mult-right*: $\mathit{inner}\ a\ (\mathit{of\text{-}real}\ m * b) = m * (\mathit{inner}\ a\ b)$
by (*metis real-inner-class.inner-scaleR-right scaleR-conv-of-real*)

lemma *inner-mult-left'*: $\mathit{inner}\ (a * \mathit{of\text{-}real}\ m)\ b = m * (\mathit{inner}\ a\ b)$
by (*simp add*: *of-real-def*)

lemma *inner-mult-right'*: $\mathit{inner}\ a\ (b * \mathit{of\text{-}real}\ m) = (\mathit{inner}\ a\ b) * m$
by (*simp add*: *of-real-def real-inner-class.inner-scaleR-right*)

lemma *Cauchy-Schwarz-ineq*:
$(\mathit{inner}\ x\ y)^2 \leq \mathit{inner}\ x\ x * \mathit{inner}\ y\ y$
proof (*cases*)
assume $y = 0$
thus *?thesis* **by** *simp*
next
assume *y*: $y \neq 0$
let *?r* = $\mathit{inner}\ x\ y\ /\ \mathit{inner}\ y\ y$
have $0 \leq \mathit{inner}\ (x - \mathit{scaleR}\ ?r\ y)\ (x - \mathit{scaleR}\ ?r\ y)$
by (*rule inner-ge-zero*)
also have ... = $\mathit{inner}\ x\ x - \mathit{inner}\ y\ x * ?r$
by (*simp add*: *inner-diff*)
also have ... = $\mathit{inner}\ x\ x - (\mathit{inner}\ x\ y)^2\ /\ \mathit{inner}\ y\ y$
by (*simp add*: *power2-eq-square inner-commute*)
finally have $0 \leq \mathit{inner}\ x\ x - (\mathit{inner}\ x\ y)^2\ /\ \mathit{inner}\ y\ y$.
hence $(\mathit{inner}\ x\ y)^2\ /\ \mathit{inner}\ y\ y \leq \mathit{inner}\ x\ x$
by (*simp add*: *le-diff-eq*)
thus $(\mathit{inner}\ x\ y)^2 \leq \mathit{inner}\ x\ x * \mathit{inner}\ y\ y$
by (*simp add*: *pos-divide-le-eq y*)
qed

lemma *Cauchy-Schwarz-ineq2*:
$|inner\ x\ y| \leq norm\ x * norm\ y$
proof (*rule power2-le-imp-le*)
have $(inner\ x\ y)^2 \leq inner\ x\ x * inner\ y\ y$
using *Cauchy-Schwarz-ineq* **.**
thus $|inner\ x\ y|^2 \leq (norm\ x * norm\ y)^2$
by (*simp add*: *power-mult-distrib power2-norm-eq-inner*)
show $0 \leq norm\ x * norm\ y$
unfolding *norm-eq-sqrt-inner*
by (*intro mult-nonneg-nonneg real-sqrt-ge-zero inner-ge-zero*)
qed

lemma *norm-cauchy-schwarz*: $inner\ x\ y \leq norm\ x * norm\ y$
using *Cauchy-Schwarz-ineq2* [*of x y*] **by** *auto*

subclass *real-normed-vector*
proof
fix $a :: real$ **and** $x\ y :: 'a$
show $norm\ x = 0 \longleftrightarrow x = 0$
unfolding *norm-eq-sqrt-inner* **by** *simp*
show $norm\ (x + y) \leq norm\ x + norm\ y$
proof (*rule power2-le-imp-le*)
have $inner\ x\ y \leq norm\ x * norm\ y$
by (*rule norm-cauchy-schwarz*)
thus $(norm\ (x + y))^2 \leq (norm\ x + norm\ y)^2$
unfolding *power2-sum power2-norm-eq-inner*
by (*simp add*: *inner-add inner-commute*)
show $0 \leq norm\ x + norm\ y$
unfolding *norm-eq-sqrt-inner* **by** *simp*
qed
have $sqrt\ (a^2 * inner\ x\ x) = |a| * sqrt\ (inner\ x\ x)$
by (*simp add*: *real-sqrt-mult-distrib*)
then show $norm\ (a *_R x) = |a| * norm\ x$
unfolding *norm-eq-sqrt-inner*
by (*simp add*: *power2-eq-square mult.assoc*)
qed

end

lemma *inner-divide-left*:
fixes $a :: 'a :: \{real\text{-}inner, real\text{-}div\text{-}algebra\}$
shows $inner\ (a\ /\ of\text{-}real\ m)\ b = (inner\ a\ b)\ /\ m$
by (*metis* (*no-types*) *divide-inverse inner-commute inner-scaleR-right mult.left-neutral mult.right-neutral mult-scaleR-right of-real-inverse scaleR-conv-of-real times-divide-eq-left*)

lemma *inner-divide-right*:
fixes $a :: 'a :: \{real\text{-}inner, real\text{-}div\text{-}algebra\}$
shows $inner\ a\ (b\ /\ of\text{-}real\ m) = (inner\ a\ b)\ /\ m$

by (*metis inner-commute inner-divide-left*)

Re-enable constraints for *open*, *uniformity*, *dist*, and *norm*.

setup ‹*Sign.add-const-constraint*
(@{*const-name open*}, *SOME* @{*typ 'a*::*topological-space set* ⇒ *bool*})›

setup ‹*Sign.add-const-constraint*
(@{*const-name uniformity*}, *SOME* @{*typ* ('*a*::*uniform-space* × '*a*) *filter*})›

setup ‹*Sign.add-const-constraint*
(@{*const-name dist*}, *SOME* @{*typ 'a*::*metric-space* ⇒ '*a* ⇒ *real*})›

setup ‹*Sign.add-const-constraint*
(@{*const-name norm*}, *SOME* @{*typ 'a*::*real-normed-vector* ⇒ *real*})›

lemma *bounded-bilinear-inner*:
bounded-bilinear (*inner*::'*a*::*real-inner* ⇒ '*a* ⇒ *real*)
proof
fix *x y z* :: '*a* **and** *r* :: *real*
show *inner* (*x* + *y*) *z* = *inner x z* + *inner y z*
by (*rule inner-add-left*)
show *inner x* (*y* + *z*) = *inner x y* + *inner x z*
by (*rule inner-add-right*)
show *inner* (*scaleR r x*) *y* = *scaleR r* (*inner x y*)
unfolding *real-scaleR-def* **by** (*rule inner-scaleR-left*)
show *inner x* (*scaleR r y*) = *scaleR r* (*inner x y*)
unfolding *real-scaleR-def* **by** (*rule inner-scaleR-right*)
show ∃ *K*. ∀ *x y*::'*a*. *norm* (*inner x y*) ≤ *norm x* ∗ *norm y* ∗ *K*
proof
show ∀ *x y*::'*a*. *norm* (*inner x y*) ≤ *norm x* ∗ *norm y* ∗ *1*
by (*simp add*: *Cauchy-Schwarz-ineq2*)
qed
qed

lemmas *tendsto-inner* [*tendsto-intros*] =
bounded-bilinear.tendsto [*OF bounded-bilinear-inner*]

lemmas *isCont-inner* [*simp*] =
bounded-bilinear.isCont [*OF bounded-bilinear-inner*]

lemmas *has-derivative-inner* [*derivative-intros*] =
bounded-bilinear.FDERIV [*OF bounded-bilinear-inner*]

lemmas *bounded-linear-inner-left* =
bounded-bilinear.bounded-linear-left [*OF bounded-bilinear-inner*]

lemmas *bounded-linear-inner-right* =
bounded-bilinear.bounded-linear-right [*OF bounded-bilinear-inner*]

lemmas *bounded-linear-inner-left-comp* = *bounded-linear-inner-left*[*THEN bounded-linear-compose*]

lemmas *bounded-linear-inner-right-comp* = *bounded-linear-inner-right*[*THEN bounded-linear-compose*]

lemmas *has-derivative-inner-right* [*derivative-intros*] =
bounded-linear.has-derivative [*OF bounded-linear-inner-right*]

lemmas *has-derivative-inner-left* [*derivative-intros*] =
bounded-linear.has-derivative [*OF bounded-linear-inner-left*]

lemma *differentiable-inner* [*simp*]:
f differentiable (*at x within s*) $\Longrightarrow$ *g differentiable at x within s* $\Longrightarrow$ (λx. *inner* (*f x*) (*g x*)) *differentiable at x within s*
unfolding *differentiable-def* **by** (*blast intro*: *has-derivative-inner*)

9.2 Class instances

instantiation *real* :: *real-inner*
begin

definition *inner-real-def* [*simp*]: *inner* = *op* $*$

instance
proof
fix *x y z r* :: *real*
show *inner x y* = *inner y x*
unfolding *inner-real-def* **by** (*rule mult.commute*)
show *inner* (*x* + *y*) *z* = *inner x z* + *inner y z*
unfolding *inner-real-def* **by** (*rule distrib-right*)
show *inner* (*scaleR r x*) *y* = *r* $*$ *inner x y*
unfolding *inner-real-def real-scaleR-def* **by** (*rule mult.assoc*)
show *0* $\leq$ *inner x x*
unfolding *inner-real-def* **by** *simp*
show *inner x x* = *0* $\longleftrightarrow$ *x* = *0*
unfolding *inner-real-def* **by** *simp*
show *norm x* = *sqrt* (*inner x x*)
unfolding *inner-real-def* **by** *simp*
qed

end

instantiation *complex* :: *real-inner*
begin

definition *inner-complex-def*:
inner x y = *Re x* $*$ *Re y* + *Im x* $*$ *Im y*

instance
proof

fix $x\ y\ z$ :: *complex* **and** r :: *real*
show *inner x y = inner y x*
unfolding *inner-complex-def* **by** (*simp add*: *mult.commute*)
show *inner (x + y) z = inner x z + inner y z*
unfolding *inner-complex-def* **by** (*simp add*: *distrib-right*)
show *inner (scaleR r x) y = r * inner x y*
unfolding *inner-complex-def* **by** (*simp add*: *distrib-left*)
show *0 ≤ inner x x*
unfolding *inner-complex-def* **by** *simp*
show *inner x x = 0 ⟷ x = 0*
unfolding *inner-complex-def*
by (*simp add*: *add-nonneg-eq-0-iff complex-Re-Im-cancel-iff*)
show *norm x = sqrt (inner x x)*
unfolding *inner-complex-def complex-norm-def*
by (*simp add*: *power2-eq-square*)
qed

end

lemma *complex-inner-1* [*simp*]: *inner 1 x = Re x*
unfolding *inner-complex-def* **by** *simp*

lemma *complex-inner-1-right* [*simp*]: *inner x 1 = Re x*
unfolding *inner-complex-def* **by** *simp*

lemma *complex-inner-ii-left* [*simp*]: *inner ii x = Im x*
unfolding *inner-complex-def* **by** *simp*

lemma *complex-inner-ii-right* [*simp*]: *inner x ii = Im x*
unfolding *inner-complex-def* **by** *simp*

9.3 Gradient derivative

definition
gderiv ::
['*a*::*real-inner* ⇒ *real*, '*a*, '*a*] ⇒ *bool*
((*GDERIV* (-)/ (-)/ :> (-)) [*1000*, *1000*, *60*] *60*)
where
GDERIV f x :> D ⟷ FDERIV f x :> (λh. inner h D)

lemma *gderiv-deriv* [*simp*]: *GDERIV f x :> D ⟷ DERIV f x :> D*
by (*simp only*: *gderiv-def has-field-derivative-def inner-real-def mult-commute-abs*)

lemma *GDERIV-DERIV-compose*:
⟦*GDERIV f x :> df*; *DERIV g (f x) :> dg*⟧
⟹ *GDERIV (λx. g (f x)) x :> scaleR dg df*
unfolding *gderiv-def has-field-derivative-def*
apply (*drule* (*1*) *has-derivative-compose*)
apply (*simp add*: *ac-simps*)

done

lemma *has-derivative-subst*: $[\![FDERIV\ f\ x :> df;\ df = d]\!] \Longrightarrow FDERIV\ f\ x :> d$
by *simp*

lemma *GDERIV-subst*: $[\![GDERIV\ f\ x :> df;\ df = d]\!] \Longrightarrow GDERIV\ f\ x :> d$
by *simp*

lemma *GDERIV-const*: $GDERIV\ (\lambda x.\ k)\ x :> 0$
unfolding *gderiv-def inner-zero-right* **by** (*rule has-derivative-const*)

lemma *GDERIV-add*:
$[\![GDERIV\ f\ x :> df;\ GDERIV\ g\ x :> dg]\!]$
$\Longrightarrow GDERIV\ (\lambda x.\ f\ x + g\ x)\ x :> df + dg$
unfolding *gderiv-def inner-add-right* **by** (*rule has-derivative-add*)

lemma *GDERIV-minus*:
$GDERIV\ f\ x :> df \Longrightarrow GDERIV\ (\lambda x.\ -\ f\ x)\ x :> -\ df$
unfolding *gderiv-def inner-minus-right* **by** (*rule has-derivative-minus*)

lemma *GDERIV-diff*:
$[\![GDERIV\ f\ x :> df;\ GDERIV\ g\ x :> dg]\!]$
$\Longrightarrow GDERIV\ (\lambda x.\ f\ x - g\ x)\ x :> df - dg$
unfolding *gderiv-def inner-diff-right* **by** (*rule has-derivative-diff*)

lemma *GDERIV-scaleR*:
$[\![DERIV\ f\ x :> df;\ GDERIV\ g\ x :> dg]\!]$
$\Longrightarrow GDERIV\ (\lambda x.\ scaleR\ (f\ x)\ (g\ x))\ x$
$:> (scaleR\ (f\ x)\ dg + scaleR\ df\ (g\ x))$
unfolding *gderiv-def has-field-derivative-def inner-add-right inner-scaleR-right*
apply (*rule has-derivative-subst*)
apply (*erule* (*1*) *has-derivative-scaleR*)
apply (*simp add*: *ac-simps*)
done

lemma *GDERIV-mult*:
$[\![GDERIV\ f\ x :> df;\ GDERIV\ g\ x :> dg]\!]$
$\Longrightarrow GDERIV\ (\lambda x.\ f\ x * g\ x)\ x :> scaleR\ (f\ x)\ dg + scaleR\ (g\ x)\ df$
unfolding *gderiv-def*
apply (*rule has-derivative-subst*)
apply (*erule* (*1*) *has-derivative-mult*)
apply (*simp add*: *inner-add ac-simps*)
done

lemma *GDERIV-inverse*:
$[\![GDERIV\ f\ x :> df;\ f\ x \neq 0]\!]$
$\Longrightarrow GDERIV\ (\lambda x.\ inverse\ (f\ x))\ x :> -\ (inverse\ (f\ x))^2 *_R df$
apply (*erule GDERIV-DERIV-compose*)
apply (*erule DERIV-inverse* [*folded numeral-2-eq-2*])

done

lemma *GDERIV-norm*:
assumes $x \neq 0$ **shows** *GDERIV* $(\lambda x.\ norm\ x)\ x :> sgn\ x$
proof −
have *1*: *FDERIV* $(\lambda x.\ inner\ x\ x)\ x :> (\lambda h.\ inner\ x\ h + inner\ h\ x)$
by (*intro has-derivative-inner has-derivative-ident*)
have *2*: $(\lambda h.\ inner\ x\ h + inner\ h\ x) = (\lambda h.\ inner\ h\ (scaleR\ 2\ x))$
by (*simp add*: *fun-eq-iff inner-commute*)
have $0 < inner\ x\ x$ **using** ‹$x \neq 0$› **by** *simp*
then have *3*: *DERIV sqrt* $(inner\ x\ x) :> (inverse\ (sqrt\ (inner\ x\ x))\ /\ 2)$
by (*rule DERIV-real-sqrt*)
have *4*: $(inverse\ (sqrt\ (inner\ x\ x))\ /\ 2) *_R 2 *_R x = sgn\ x$
by (*simp add*: *sgn-div-norm norm-eq-sqrt-inner*)
show *?thesis*
unfolding *norm-eq-sqrt-inner*
apply (*rule GDERIV-subst* [*OF* - *4*])
apply (*rule GDERIV-DERIV-compose* [**where** *g*=*sqrt* **and** *df*=*scaleR 2 x*])
apply (*subst gderiv-def*)
apply (*rule has-derivative-subst* [*OF* - *2*])
apply (*rule 1*)
apply (*rule 3*)
done
qed

lemmas *has-derivative-norm* = *GDERIV-norm* [*unfolded gderiv-def*]

end

10 Additive group operations on product types

theory *Product-plus*
imports *Main*
begin

10.1 Operations

instantiation *prod* :: (*zero*, *zero*) *zero*
begin

definition *zero-prod-def*: $0 = (0, 0)$

instance ..
end

instantiation *prod* :: (*plus*, *plus*) *plus*
begin

definition *plus-prod-def*:

$x + y = (fst\ x + fst\ y,\ snd\ x + snd\ y)$

instance ..
end

instantiation *prod* :: (*minus*, *minus*) *minus*
begin

definition *minus-prod-def*:
$x - y = (fst\ x - fst\ y,\ snd\ x - snd\ y)$

instance ..
end

instantiation *prod* :: (*uminus*, *uminus*) *uminus*
begin

definition *uminus-prod-def*:
$- x = (- fst\ x,\ - snd\ x)$

instance ..
end

lemma *fst-zero* [*simp*]: $fst\ 0 = 0$
unfolding *zero-prod-def* **by** *simp*

lemma *snd-zero* [*simp*]: $snd\ 0 = 0$
unfolding *zero-prod-def* **by** *simp*

lemma *fst-add* [*simp*]: $fst\ (x + y) = fst\ x + fst\ y$
unfolding *plus-prod-def* **by** *simp*

lemma *snd-add* [*simp*]: $snd\ (x + y) = snd\ x + snd\ y$
unfolding *plus-prod-def* **by** *simp*

lemma *fst-diff* [*simp*]: $fst\ (x - y) = fst\ x - fst\ y$
unfolding *minus-prod-def* **by** *simp*

lemma *snd-diff* [*simp*]: $snd\ (x - y) = snd\ x - snd\ y$
unfolding *minus-prod-def* **by** *simp*

lemma *fst-uminus* [*simp*]: $fst\ (- x) = - fst\ x$
unfolding *uminus-prod-def* **by** *simp*

lemma *snd-uminus* [*simp*]: $snd\ (- x) = - snd\ x$
unfolding *uminus-prod-def* **by** *simp*

lemma *add-Pair* [*simp*]: $(a, b) + (c, d) = (a + c, b + d)$
unfolding *plus-prod-def* **by** *simp*

lemma *diff-Pair* [*simp*]: $(a, b) - (c, d) = (a - c, b - d)$
unfolding *minus-prod-def* **by** *simp*

lemma *uminus-Pair* [*simp*, *code*]: $- (a, b) = (- a, - b)$
unfolding *uminus-prod-def* **by** *simp*

10.2 Class instances

instance *prod* :: (*semigroup-add*, *semigroup-add*) *semigroup-add*
by *standard* (*simp add*: *prod-eq-iff add.assoc*)

instance *prod* :: (*ab-semigroup-add*, *ab-semigroup-add*) *ab-semigroup-add*
by *standard* (*simp add*: *prod-eq-iff add.commute*)

instance *prod* :: (*monoid-add*, *monoid-add*) *monoid-add*
by *standard* (*simp-all add*: *prod-eq-iff*)

instance *prod* :: (*comm-monoid-add*, *comm-monoid-add*) *comm-monoid-add*
by *standard* (*simp add*: *prod-eq-iff*)

instance *prod* :: (*cancel-semigroup-add*, *cancel-semigroup-add*) *cancel-semigroup-add*
by *standard* (*simp-all add*: *prod-eq-iff*)

instance *prod* :: (*cancel-ab-semigroup-add*, *cancel-ab-semigroup-add*) *cancel-ab-semigroup-add*
by *standard* (*simp-all add*: *prod-eq-iff diff-diff-eq*)

instance *prod* :: (*cancel-comm-monoid-add*, *cancel-comm-monoid-add*) *cancel-comm-monoid-add*
..

instance *prod* :: (*group-add*, *group-add*) *group-add*
by *standard* (*simp-all add*: *prod-eq-iff*)

instance *prod* :: (*ab-group-add*, *ab-group-add*) *ab-group-add*
by *standard* (*simp-all add*: *prod-eq-iff*)

lemma *fst-setsum*: *fst* $(\sum x{\in}A.\ f\ x) = (\sum x{\in}A.\ fst\ (f\ x))$
proof (*cases finite A*)
case *True*
then show *?thesis* **by** *induct simp-all*
next
case *False*
then show *?thesis* **by** *simp*
qed

lemma *snd-setsum*: *snd* $(\sum x{\in}A.\ f\ x) = (\sum x{\in}A.\ snd\ (f\ x))$
proof (*cases finite A*)
case *True*
then show *?thesis* **by** *induct simp-all*

next
case *False*
then show *?thesis* **by** *simp*
qed

lemma *setsum-prod*: $(\sum x{\in}A.\ (f\ x,\ g\ x)) = (\sum x{\in}A.\ f\ x,\ \sum x{\in}A.\ g\ x)$
proof (*cases finite A*)
case *True*
then show *?thesis* **by** *induct* (*simp-all add*: *zero-prod-def*)
next
case *False*
then show *?thesis* **by** (*simp add*: *zero-prod-def*)
qed

end

11 Cartesian Products as Vector Spaces

theory *Product-Vector*
imports *Inner-Product Product-plus*
begin

11.1 Product is a real vector space

instantiation *prod* :: (*real-vector*, *real-vector*) *real-vector*
begin

definition *scaleR-prod-def*:
scaleR r A = (*scaleR r* (*fst A*), *scaleR r* (*snd A*))

lemma *fst-scaleR* [*simp*]: *fst* (*scaleR r A*) = *scaleR r* (*fst A*)
unfolding *scaleR-prod-def* **by** *simp*

lemma *snd-scaleR* [*simp*]: *snd* (*scaleR r A*) = *scaleR r* (*snd A*)
unfolding *scaleR-prod-def* **by** *simp*

lemma *scaleR-Pair* [*simp*]: *scaleR r* (*a*, *b*) = (*scaleR r a*, *scaleR r b*)
unfolding *scaleR-prod-def* **by** *simp*

instance
proof
fix *a b* :: *real* **and** *x y* :: *'a* × *'b*
show *scaleR a* (*x* + *y*) = *scaleR a x* + *scaleR a y*
by (*simp add*: *prod-eq-iff scaleR-right-distrib*)
show *scaleR* (*a* + *b*) *x* = *scaleR a x* + *scaleR b x*
by (*simp add*: *prod-eq-iff scaleR-left-distrib*)
show *scaleR a* (*scaleR b x*) = *scaleR* (*a* ∗ *b*) *x*
by (*simp add*: *prod-eq-iff*)
show *scaleR 1 x* = *x*

by (*simp add*: *prod-eq-iff*)
qed

end

11.2 Product is a metric space

instantiation *prod* :: (*metric-space*, *metric-space*) *dist*
begin

definition *dist-prod-def*[*code del*]:
dist x y = *sqrt* ((*dist* (*fst x*) (*fst y*))^2 + (*dist* (*snd x*) (*snd y*))^2)

instance ..
end

instantiation *prod* :: (*metric-space*, *metric-space*) *uniformity-dist*
begin

definition [*code del*]:
(*uniformity* :: (('*a* × '*b*) × ('*a* × '*b*)) *filter*) =
(*INF e*:{*0* <..}. *principal* {(*x*, *y*). *dist x y* < *e*})

instance
by *standard* (*rule uniformity-prod-def*)
end

declare *uniformity-Abort*[**where** '*a*='*a* :: *metric-space* × '*b* :: *metric-space*, *code*]

instantiation *prod* :: (*metric-space*, *metric-space*) *metric-space*
begin

lemma *dist-Pair-Pair*: *dist* (*a*, *b*) (*c*, *d*) = *sqrt* ((*dist a c*)^2 + (*dist b d*)^2)
unfolding *dist-prod-def* **by** *simp*

lemma *dist-fst-le*: *dist* (*fst x*) (*fst y*) ≤ *dist x y*
unfolding *dist-prod-def* **by** (*rule real-sqrt-sum-squares-ge1*)

lemma *dist-snd-le*: *dist* (*snd x*) (*snd y*) ≤ *dist x y*
unfolding *dist-prod-def* **by** (*rule real-sqrt-sum-squares-ge2*)

instance
proof
fix *x y* :: '*a* × '*b*
show *dist x y* = *0* ⟷ *x* = *y*
unfolding *dist-prod-def prod-eq-iff* **by** *simp*
next
fix *x y z* :: '*a* × '*b*
show *dist x y* ≤ *dist x z* + *dist y z*

```
    unfolding dist-prod-def
    by (intro order-trans [OF - real-sqrt-sum-squares-triangle-ineq]
        real-sqrt-le-mono add-mono power-mono dist-triangle2 zero-le-dist)
next
  fix S :: ('a × 'b) set
  have ∗: open S ⟷ (∀ x∈S. ∃ e>0. ∀ y. dist y x < e ⟶ y ∈ S)
  proof
    assume open S show ∀ x∈S. ∃ e>0. ∀ y. dist y x < e ⟶ y ∈ S
    proof
      fix x assume x ∈ S
      obtain A B where open A open B x ∈ A × B A × B ⊆ S
        using ‹open S› and ‹x ∈ S› by (rule open-prod-elim)
      obtain r where r: 0 < r ∀ y. dist y (fst x) < r ⟶ y ∈ A
        using ‹open A› and ‹x ∈ A × B› unfolding open-dist by auto
      obtain s where s: 0 < s ∀ y. dist y (snd x) < s ⟶ y ∈ B
        using ‹open B› and ‹x ∈ A × B› unfolding open-dist by auto
      let ?e = min r s
      have 0 < ?e ∧ (∀ y. dist y x < ?e ⟶ y ∈ S)
      proof (intro allI impI conjI)
        show 0 < min r s by (simp add: r(1) s(1))
      next
        fix y assume dist y x < min r s
        hence dist y x < r and dist y x < s
          by simp-all
        hence dist (fst y) (fst x) < r and dist (snd y) (snd x) < s
          by (auto intro: le-less-trans dist-fst-le dist-snd-le)
        hence fst y ∈ A and snd y ∈ B
          by (simp-all add: r(2) s(2))
        hence y ∈ A × B by (induct y, simp)
        with ‹A × B ⊆ S› show y ∈ S ..
      qed
      thus ∃ e>0. ∀ y. dist y x < e ⟶ y ∈ S ..
    qed
  next
    assume ∗: ∀ x∈S. ∃ e>0. ∀ y. dist y x < e ⟶ y ∈ S show open S
    proof (rule open-prod-intro)
      fix x assume x ∈ S
      then obtain e where 0 < e and S: ∀ y. dist y x < e ⟶ y ∈ S
        using ∗ by fast
      def r ≡ e / sqrt 2 and s ≡ e / sqrt 2
      from ‹0 < e› have 0 < r and 0 < s
        unfolding r-def s-def by simp-all
      from ‹0 < e› have e = sqrt (r^2 + s^2)
        unfolding r-def s-def by (simp add: power-divide)
      def A ≡ {y. dist (fst x) y < r} and B ≡ {y. dist (snd x) y < s}
      have open A and open B
        unfolding A-def B-def by (simp-all add: open-ball)
      moreover have x ∈ A × B
        unfolding A-def B-def mem-Times-iff
```

```
      using ‹0 < r› and ‹0 < s› by simp
    moreover have A × B ⊆ S
    proof (clarify)
      fix a b assume a ∈ A and b ∈ B
      hence dist a (fst x) < r and dist b (snd x) < s
        unfolding A-def B-def by (simp-all add: dist-commute)
      hence dist (a, b) x < e
        unfolding dist-prod-def ‹e = sqrt (r² + s²)›
        by (simp add: add-strict-mono power-strict-mono)
      thus (a, b) ∈ S
        by (simp add: S)
    qed
    ultimately show ∃A B. open A ∧ open B ∧ x ∈ A × B ∧ A × B ⊆ S by
fast
  qed
 qed
 show open S = (∀x∈S. ∀F (x', y) in uniformity. x' = x ⟶ y ∈ S)
  unfolding ∗ eventually-uniformity-metric
  by (simp del: split-paired-All add: dist-prod-def dist-commute)
qed

end

declare [[code abort: dist::('a::metric-space∗'b::metric-space)⇒('a∗'b) ⇒ real]]

lemma Cauchy-fst: Cauchy X ⟹ Cauchy (λn. fst (X n))
 unfolding Cauchy-def by (fast elim: le-less-trans [OF dist-fst-le])

lemma Cauchy-snd: Cauchy X ⟹ Cauchy (λn. snd (X n))
 unfolding Cauchy-def by (fast elim: le-less-trans [OF dist-snd-le])

lemma Cauchy-Pair:
 assumes Cauchy X and Cauchy Y
 shows Cauchy (λn. (X n, Y n))
proof (rule metric-CauchyI)
 fix r :: real assume 0 < r
 hence 0 < r / sqrt 2 (is 0 < ?s) by simp
 obtain M where M: ∀m≥M. ∀n≥M. dist (X m) (X n) < ?s
  using metric-CauchyD [OF ‹Cauchy X› ‹0 < ?s›] ..
 obtain N where N: ∀m≥N. ∀n≥N. dist (Y m) (Y n) < ?s
  using metric-CauchyD [OF ‹Cauchy Y› ‹0 < ?s›] ..
 have ∀m≥max M N. ∀n≥max M N. dist (X m, Y m) (X n, Y n) < r
  using M N by (simp add: real-sqrt-sum-squares-less dist-Pair-Pair)
 then show ∃n0. ∀m≥n0. ∀n≥n0. dist (X m, Y m) (X n, Y n) < r ..
qed
```

11.3 Product is a complete metric space

```
instance prod :: (complete-space, complete-space) complete-space
```

proof
fix $X :: nat \Rightarrow 'a \times 'b$ **assume** *Cauchy X*
have *1*: $(\lambda n.\ fst\ (X\ n)) \longrightarrow lim\ (\lambda n.\ fst\ (X\ n))$
using *Cauchy-fst* [*OF* ‹*Cauchy X*›]
by (*simp add*: *Cauchy-convergent-iff convergent-LIMSEQ-iff*)
have *2*: $(\lambda n.\ snd\ (X\ n)) \longrightarrow lim\ (\lambda n.\ snd\ (X\ n))$
using *Cauchy-snd* [*OF* ‹*Cauchy X*›]
by (*simp add*: *Cauchy-convergent-iff convergent-LIMSEQ-iff*)
have $X \longrightarrow (lim\ (\lambda n.\ fst\ (X\ n)),\ lim\ (\lambda n.\ snd\ (X\ n)))$
using *tendsto-Pair* [*OF 1 2*] **by** *simp*
then show *convergent X*
by (*rule convergentI*)
qed

11.4 Product is a normed vector space

instantiation *prod* :: (*real-normed-vector*, *real-normed-vector*) *real-normed-vector*
begin

definition *norm-prod-def*[*code del*]:
$norm\ x = sqrt\ ((norm\ (fst\ x))^2 + (norm\ (snd\ x))^2)$

definition *sgn-prod-def*:
$sgn\ (x::'a \times 'b) = scaleR\ (inverse\ (norm\ x))\ x$

lemma *norm-Pair*: $norm\ (a,\ b) = sqrt\ ((norm\ a)^2 + (norm\ b)^2)$
unfolding *norm-prod-def* **by** *simp*

instance
proof
fix $r :: real$ **and** $x\ y :: 'a \times 'b$
show $norm\ x = 0 \longleftrightarrow x = 0$
unfolding *norm-prod-def*
by (*simp add*: *prod-eq-iff*)
show $norm\ (x + y) \leq norm\ x + norm\ y$
unfolding *norm-prod-def*
apply (*rule order-trans* [*OF - real-sqrt-sum-squares-triangle-ineq*])
apply (*simp add*: *add-mono power-mono norm-triangle-ineq*)
done
show $norm\ (scaleR\ r\ x) = |r| * norm\ x$
unfolding *norm-prod-def*
apply (*simp add*: *power-mult-distrib*)
apply (*simp add*: *distrib-left* [*symmetric*])
apply (*simp add*: *real-sqrt-mult-distrib*)
done
show $sgn\ x = scaleR\ (inverse\ (norm\ x))\ x$
by (*rule sgn-prod-def*)
show $dist\ x\ y = norm\ (x - y)$
unfolding *dist-prod-def norm-prod-def*

by (*simp add*: *dist-norm*)
qed

end

declare [[*code abort*: *norm*::('*a*::*real-normed-vector*∗'*b*::*real-normed-vector*) ⇒ *real*]]

instance *prod* :: (*banach*, *banach*) *banach* **..**

11.4.1 Pair operations are linear

lemma *bounded-linear-fst*: *bounded-linear fst*
using *fst-add fst-scaleR*
by (*rule bounded-linear-intro* [**where** *K*=*1*], *simp add*: *norm-prod-def*)

lemma *bounded-linear-snd*: *bounded-linear snd*
using *snd-add snd-scaleR*
by (*rule bounded-linear-intro* [**where** *K*=*1*], *simp add*: *norm-prod-def*)

lemmas *bounded-linear-fst-comp* = *bounded-linear-fst*[*THEN bounded-linear-compose*]

lemmas *bounded-linear-snd-comp* = *bounded-linear-snd*[*THEN bounded-linear-compose*]

lemma *bounded-linear-Pair*:
assumes *f*: *bounded-linear f*
assumes *g*: *bounded-linear g*
shows *bounded-linear* (λ*x*. (*f x*, *g x*))
proof
interpret *f*: *bounded-linear f* **by** *fact*
interpret *g*: *bounded-linear g* **by** *fact*
fix *x y* **and** *r* :: *real*
show (*f* (*x* + *y*), *g* (*x* + *y*)) = (*f x*, *g x*) + (*f y*, *g y*)
by (*simp add*: *f.add g.add*)
show (*f* (*r* ∗*R* *x*), *g* (*r* ∗*R* *x*)) = *r* ∗*R* (*f x*, *g x*)
by (*simp add*: *f.scaleR g.scaleR*)
obtain *Kf* **where** *0* < *Kf* **and** *norm-f*: ⋀*x*. *norm* (*f x*) ≤ *norm x* ∗ *Kf*
using *f.pos-bounded* **by** *fast*
obtain *Kg* **where** *0* < *Kg* **and** *norm-g*: ⋀*x*. *norm* (*g x*) ≤ *norm x* ∗ *Kg*
using *g.pos-bounded* **by** *fast*
have ∀*x*. *norm* (*f x*, *g x*) ≤ *norm x* ∗ (*Kf* + *Kg*)
apply (*rule allI*)
apply (*simp add*: *norm-Pair*)
apply (*rule order-trans* [*OF sqrt-add-le-add-sqrt*], *simp*, *simp*)
apply (*simp add*: *distrib-left*)
apply (*rule add-mono* [*OF norm-f norm-g*])
done
then show ∃*K*. ∀*x*. *norm* (*f x*, *g x*) ≤ *norm x* ∗ *K* **..**
qed

11.4.2 Frechet derivatives involving pairs

lemma *has-derivative-Pair* [*derivative-intros*]:
assumes *f*: (*f has-derivative f'*) (*at x within s*) **and** *g*: (*g has-derivative g'*) (*at x within s*)
shows ((λ*x*. (*f x*, *g x*)) *has-derivative* (λ*h*. (*f' h*, *g' h*))) (*at x within s*)
proof (*rule has-derivativeI-sandwich*[*of 1*])
show *bounded-linear* (λ*h*. (*f' h*, *g' h*))
using *f g* **by** (*intro bounded-linear-Pair has-derivative-bounded-linear*)
let *?Rf* = λ*y*. *f y* − *f x* − *f'* (*y* − *x*)
let *?Rg* = λ*y*. *g y* − *g x* − *g'* (*y* − *x*)
let *?R* = λ*y*. ((*f y*, *g y*) − (*f x*, *g x*) − (*f'* (*y* − *x*), *g'* (*y* − *x*)))

show ((λ*y*. *norm* (*?Rf y*) / *norm* (*y* − *x*) + *norm* (*?Rg y*) / *norm* (*y* − *x*)) ⟶ *0*) (*at x within s*)
using *f g* **by** (*intro tendsto-add-zero*) (*auto simp*: *has-derivative-iff-norm*)

fix *y* :: *'a* **assume** *y* ≠ *x*
show *norm* (*?R y*) / *norm* (*y* − *x*) ≤ *norm* (*?Rf y*) / *norm* (*y* − *x*) + *norm* (*?Rg y*) / *norm* (*y* − *x*)
unfolding *add-divide-distrib* [*symmetric*]
by (*simp add*: *norm-Pair divide-right-mono order-trans* [*OF sqrt-add-le-add-sqrt*])
qed *simp*

lemmas *has-derivative-fst* [*derivative-intros*] = *bounded-linear.has-derivative* [*OF bounded-linear-fst*]
lemmas *has-derivative-snd* [*derivative-intros*] = *bounded-linear.has-derivative* [*OF bounded-linear-snd*]

lemma *has-derivative-split* [*derivative-intros*]:
((λ*p*. *f* (*fst p*) (*snd p*)) *has-derivative f'*) *F* ⟹ ((λ(*a*, *b*). *f a b*) *has-derivative f'*) *F*
unfolding *split-beta'* **.**

11.5 Product is an inner product space

instantiation *prod* :: (*real-inner*, *real-inner*) *real-inner*
begin

definition *inner-prod-def*:
inner x y = *inner* (*fst x*) (*fst y*) + *inner* (*snd x*) (*snd y*)

lemma *inner-Pair* [*simp*]: *inner* (*a*, *b*) (*c*, *d*) = *inner a c* + *inner b d*
unfolding *inner-prod-def* **by** *simp*

instance
proof
fix *r* :: *real*
fix *x y z* :: *'a*::*real-inner* × *'b*::*real-inner*
show *inner x y* = *inner y x*

unfolding *inner-prod-def*
by (*simp add*: *inner-commute*)
show *inner* (*x* + *y*) *z* = *inner x z* + *inner y z*
unfolding *inner-prod-def*
by (*simp add*: *inner-add-left*)
show *inner* (*scaleR r x*) *y* = *r* ∗ *inner x y*
unfolding *inner-prod-def*
by (*simp add*: *distrib-left*)
show *0* ≤ *inner x x*
unfolding *inner-prod-def*
by (*intro add-nonneg-nonneg inner-ge-zero*)
show *inner x x* = *0* ⟷ *x* = *0*
unfolding *inner-prod-def prod-eq-iff*
by (*simp add*: *add-nonneg-eq-0-iff*)
show *norm x* = *sqrt* (*inner x x*)
unfolding *norm-prod-def inner-prod-def*
by (*simp add*: *power2-norm-eq-inner*)
qed

end

lemma *inner-Pair-0*: *inner x* (*0*, *b*) = *inner* (*snd x*) *b inner x* (*a*, *0*) = *inner* (*fst x*) *a*
by (*cases x*, *simp*)+

lemma
fixes *x* :: *'a*::*real-normed-vector*
shows *norm-Pair1* [*simp*]: *norm* (*0*,*x*) = *norm x*
and *norm-Pair2* [*simp*]: *norm* (*x*,*0*) = *norm x*
by (*auto simp*: *norm-Pair*)

lemma *norm-commute*: *norm* (*x*,*y*) = *norm* (*y*,*x*)
by (*simp add*: *norm-Pair*)

lemma *norm-fst-le*: *norm x* ≤ *norm* (*x*,*y*)
by (*metis dist-fst-le fst-conv fst-zero norm-conv-dist*)

lemma *norm-snd-le*: *norm y* ≤ *norm* (*x*,*y*)
by (*metis dist-snd-le snd-conv snd-zero norm-conv-dist*)

end

12 Finite-Dimensional Inner Product Spaces

theory *Euclidean-Space*
imports
L2-Norm
~~/*src*/*HOL*/*Library*/*Inner-Product*
~~/*src*/*HOL*/*Library*/*Product-Vector*

begin

12.1 Type class of Euclidean spaces

class *euclidean-space* = *real-inner* +
  fixes *Basis* :: *'a set*
  assumes *nonempty-Basis* [*simp*]: *Basis* ≠ {}
  assumes *finite-Basis* [*simp*]: *finite Basis*
  assumes *inner-Basis*:
    ⟦*u* ∈ *Basis*; *v* ∈ *Basis*⟧ ⟹ *inner u v* = (*if u* = *v then 1 else 0*)
  assumes *euclidean-all-zero-iff*:
    (∀ *u*∈*Basis*. *inner x u* = *0*) ⟷ (*x* = *0*)

abbreviation *dimension* :: (*'a*::*euclidean-space*) *itself* ⇒ *nat* **where**
  dimension TYPE(*'a*) ≡ *card* (*Basis* :: *'a set*)

syntax *-type-dimension* :: *type* => *nat* ((*1DIM*/(*1'*(*-'*))))

translations *DIM*(*'t*) == *CONST dimension* (*TYPE*(*'t*))

lemma (**in** *euclidean-space*) *norm-Basis*[*simp*]: *u* ∈ *Basis* ⟹ *norm u* = *1*
  unfolding *norm-eq-sqrt-inner* **by** (*simp add*: *inner-Basis*)

lemma (**in** *euclidean-space*) *inner-same-Basis*[*simp*]: *u* ∈ *Basis* ⟹ *inner u u* = *1*
  by (*simp add*: *inner-Basis*)

lemma (**in** *euclidean-space*) *inner-not-same-Basis*: *u* ∈ *Basis* ⟹ *v* ∈ *Basis* ⟹ *u* ≠ *v* ⟹ *inner u v* = *0*
  by (*simp add*: *inner-Basis*)

lemma (**in** *euclidean-space*) *sgn-Basis*: *u* ∈ *Basis* ⟹ *sgn u* = *u*
  unfolding *sgn-div-norm* **by** (*simp add*: *scaleR-one*)

lemma (**in** *euclidean-space*) *Basis-zero* [*simp*]: *0* ∉ *Basis*
proof
  assume *0* ∈ *Basis* **thus** *False*
    using *inner-Basis* [*of 0 0*] **by** *simp*
qed

lemma (**in** *euclidean-space*) *nonzero-Basis*: *u* ∈ *Basis* ⟹ *u* ≠ *0*
  by *clarsimp*

lemma (**in** *euclidean-space*) *SOME-Basis*: (*SOME i*. *i* ∈ *Basis*) ∈ *Basis*
  by (*metis ex-in-conv nonempty-Basis someI-ex*)

lemma (**in** *euclidean-space*) *inner-setsum-left-Basis*[*simp*]:
    b ∈ *Basis* ⟹ *inner* (∑ *i*∈*Basis*. *f i* *$_R$ *i*) *b* = *f b*
 by (*simp add*: *inner-setsum-left inner-Basis if-distrib comm-monoid-add-class.setsum.If-cases*)

lemma (**in** *euclidean-space*) *euclidean-eqI*:
assumes *b*: $\bigwedge b.\ b \in Basis \Longrightarrow inner\ x\ b = inner\ y\ b$ **shows** $x = y$
proof −
from *b* **have** $\forall\, b{\in}Basis.\ inner\ (x - y)\ b = 0$
by (*simp add*: *inner-diff-left*)
then show $x = y$
by (*simp add*: *euclidean-all-zero-iff*)
qed

lemma (**in** *euclidean-space*) *euclidean-eq-iff*:
$x = y \longleftrightarrow (\forall\, b{\in}Basis.\ inner\ x\ b = inner\ y\ b)$
by (*auto intro*: *euclidean-eqI*)

lemma (**in** *euclidean-space*) *euclidean-representation-setsum*:
$(\sum i{\in}Basis.\ f\ i\ *_R\ i) = b \longleftrightarrow (\forall\, i{\in}Basis.\ f\ i = inner\ b\ i)$
by (*subst euclidean-eq-iff*) *simp*

lemma (**in** *euclidean-space*) *euclidean-representation-setsum'*:
$b = (\sum i{\in}Basis.\ f\ i\ *_R\ i) \longleftrightarrow (\forall\, i{\in}Basis.\ f\ i = inner\ b\ i)$
by (*auto simp add*: *euclidean-representation-setsum*[*symmetric*])

lemma (**in** *euclidean-space*) *euclidean-representation*: $(\sum b{\in}Basis.\ inner\ x\ b\ *_R\ b) = x$
unfolding *euclidean-representation-setsum* **by** *simp*

lemma (**in** *euclidean-space*) *choice-Basis-iff*:
fixes $P :: 'a \Rightarrow real \Rightarrow bool$
shows $(\forall\, i{\in}Basis.\ \exists\, x.\ P\ i\ x) \longleftrightarrow (\exists\, x.\ \forall\, i{\in}Basis.\ P\ i\ (inner\ x\ i))$
unfolding *bchoice-iff*
proof *safe*
fix *f* **assume** $\forall\, i{\in}Basis.\ P\ i\ (f\ i)$
then show $\exists\, x.\ \forall\, i{\in}Basis.\ P\ i\ (inner\ x\ i)$
by (*auto intro*!: *exI*[*of* - $\sum i{\in}Basis.\ f\ i\ *_R\ i$])
qed *auto*

lemma (**in** *euclidean-space*) *euclidean-representation-setsum-fun*:
$(\lambda x.\ \sum b{\in}Basis.\ inner\ (f\ x)\ b\ *_R\ b) = f$
by (*rule ext*) (*simp add*: *euclidean-representation-setsum*)

lemma *euclidean-isCont*:
assumes $\bigwedge b.\ b \in Basis \Longrightarrow isCont\ (\lambda x.\ (inner\ (f\ x)\ b)\ *_R\ b)\ x$
shows *isCont f x*
apply (*subst euclidean-representation-setsum-fun* [*symmetric*])
apply (*rule isCont-setsum*)
apply (*blast intro*: *assms*)
done

lemma *DIM-positive*: $0 < DIM('a{::}euclidean\text{-}space)$

by (*simp add*: *card-gt-0-iff*)

12.2 Subclass relationships

instance *euclidean-space* ⊆ *perfect-space*
proof
fix *x* :: *'a* **show** ¬ *open* {*x*}
proof
assume *open* {*x*}
then obtain *e* **where** *0* < *e* **and** *e*: ∀ *y*. *dist y x* < *e* ⟶ *y* = *x*
unfolding *open-dist* **by** *fast*
def *y* ≡ *x* + *scaleR* (*e*/*2*) (*SOME b*. *b* ∈ *Basis*)
have [*simp*]: (*SOME b*. *b* ∈ *Basis*) ∈ *Basis*
by (*rule someI-ex*) (*auto simp*: *ex-in-conv*)
from ‹*0* < *e*› **have** *y* ≠ *x*
unfolding *y-def* **by** (*auto intro*!: *nonzero-Basis*)
from ‹*0* < *e*› **have** *dist y x* < *e*
unfolding *y-def* **by** (*simp add*: *dist-norm*)
from ‹*y* ≠ *x*› **and** ‹*dist y x* < *e*› **show** *False*
using *e* **by** *simp*
qed
qed

12.3 Class instances

12.3.1 Type *real*

instantiation *real* :: *euclidean-space*
begin

definition
[*simp*]: *Basis* = {*1*::*real*}

instance
by *standard auto*

end

lemma *DIM-real*[*simp*]: *DIM*(*real*) = *1*
by *simp*

12.3.2 Type *complex*

instantiation *complex* :: *euclidean-space*
begin

definition *Basis-complex-def*:
Basis = {*1*, *ii*}

instance

by *standard* (*auto simp add*: *Basis-complex-def intro*: *complex-eqI split*: *if-split-asm*)

end

lemma *DIM-complex*[*simp*]: *DIM*(*complex*) = *2*
unfolding *Basis-complex-def* **by** *simp*

12.3.3 Type $'a \times 'b$

instantiation *prod* :: (*euclidean-space*, *euclidean-space*) *euclidean-space*
begin

definition
Basis = (λ*u*. (*u*, *0*)) ' *Basis* ∪ (λ*v*. (*0*, *v*)) ' *Basis*

lemma *setsum-Basis-prod-eq*:
fixes *f*::('*a*∗'*b*)⇒('*a*∗'*b*)
shows *setsum f Basis* = *setsum* (λ*i*. *f* (*i*, *0*)) *Basis* + *setsum* (λ*i*. *f* (*0*, *i*)) *Basis*
proof −
have *inj-on* (λ*u*. (*u*::'*a*, *0*::'*b*)) *Basis inj-on* (λ*u*. (*0*::'*a*, *u*::'*b*)) *Basis*
by (*auto intro*!: *inj-onI Pair-inject*)
thus *?thesis*
unfolding *Basis-prod-def*
by (*subst setsum.union-disjoint*) (*auto simp*: *Basis-prod-def setsum.reindex*)
qed

instance proof
show (*Basis* :: ('*a* × '*b*) *set*) ≠ {}
unfolding *Basis-prod-def* **by** *simp*
next
show *finite* (*Basis* :: ('*a* × '*b*) *set*)
unfolding *Basis-prod-def* **by** *simp*
next
fix *u v* :: '*a* × '*b*
assume *u* ∈ *Basis* **and** *v* ∈ *Basis*
thus *inner u v* = (*if u* = *v then 1 else 0*)
unfolding *Basis-prod-def inner-prod-def*
by (*auto simp add*: *inner-Basis split*: *if-split-asm*)
next
fix *x* :: '*a* × '*b*
show (∀ *u*∈*Basis*. *inner x u* = *0*) ⟷ *x* = *0*
unfolding *Basis-prod-def ball-Un ball-simps*
by (*simp add*: *inner-prod-def prod-eq-iff euclidean-all-zero-iff*)
qed

lemma *DIM-prod*[*simp*]: *DIM*('*a* × '*b*) = *DIM*('*a*) + *DIM*('*b*)
unfolding *Basis-prod-def*
by (*subst card-Un-disjoint*) (*auto intro*!: *card-image arg-cong2*[**where** *f*=*op* +]

inj-onI)

end

end

13 Elementary linear algebra on Euclidean spaces

theory *Linear-Algebra*
imports
Euclidean-Space
~~/src/HOL/Library/Infinite-Set
begin

lemma *cond-application-beta*: (*if b then f else g*) *x* = (*if b then f x else g x*)
by *auto*

notation *inner* (**infix** · *70*)

lemma *square-bound-lemma*:
fixes *x* :: *real*
shows *x* < (*1* + *x*) ∗ (*1* + *x*)
proof −
have (*x* + *1/2*)^2 + *3/4* > *0*
using *zero-le-power2*[*of x+1/2*] **by** *arith*
then show *?thesis*
by (*simp add*: *field-simps power2-eq-square*)
qed

lemma *square-continuous*:
fixes *e* :: *real*
shows *e* > *0* ⟹ ∃ *d*. *0* < *d* ∧ (∀ *y*. |*y* − *x*| < *d* ⟶ |*y* ∗ *y* − *x* ∗ *x*| < *e*)
using *isCont-power*[*OF continuous-ident*, *of x*, *unfolded isCont-def LIM-eq*, *rule-format*, *of e 2*]
by (*force simp add*: *power2-eq-square*)

Hence derive more interesting properties of the norm.

lemma *norm-eq-0-dot*: *norm x* = *0* ⟷ *x* · *x* = (*0*::*real*)
by *simp*

lemma *norm-triangle-sub*:
fixes *x y* :: *'a*::*real-normed-vector*
shows *norm x* ≤ *norm y* + *norm* (*x* − *y*)
using *norm-triangle-ineq*[*of y x* − *y*] **by** (*simp add*: *field-simps*)

lemma *norm-le*: *norm x* ≤ *norm y* ⟷ *x* · *x* ≤ *y* · *y*
by (*simp add*: *norm-eq-sqrt-inner*)

lemma *norm-lt*: *norm x* < *norm y* ⟷ *x* · *x* < *y* · *y*

by (*simp add*: *norm-eq-sqrt-inner*)

lemma *norm-eq*: *norm x* = *norm y* ⟷ *x* · *x* = *y* · *y*
apply (*subst order-eq-iff*)
apply (*auto simp*: *norm-le*)
done

lemma *norm-eq-1*: *norm x* = *1* ⟷ *x* · *x* = *1*
by (*simp add*: *norm-eq-sqrt-inner*)

Squaring equations and inequalities involving norms.

lemma *dot-square-norm*: *x* · *x* = (*norm x*)^2
by (*simp only*: *power2-norm-eq-inner*)

lemma *norm-eq-square*: *norm x* = *a* ⟷ *0* ≤ *a* ∧ *x* · *x* = *a*^{2}
by (*auto simp add*: *norm-eq-sqrt-inner*)

lemma *norm-le-square*: *norm x* ≤ *a* ⟷ *0* ≤ *a* ∧ *x* · *x* ≤ *a*^{2}
apply (*simp add*: *dot-square-norm abs-le-square-iff*[*symmetric*])
using *norm-ge-zero*[*of x*]
apply *arith*
done

lemma *norm-ge-square*: *norm x* ≥ *a* ⟷ *a* ≤ *0* ∨ *x* · *x* ≥ *a*^{2}
apply (*simp add*: *dot-square-norm abs-le-square-iff*[*symmetric*])
using *norm-ge-zero*[*of x*]
apply *arith*
done

lemma *norm-lt-square*: *norm x* < *a* ⟷ *0* < *a* ∧ *x* · *x* < *a*^{2}
by (*metis not-le norm-ge-square*)

lemma *norm-gt-square*: *norm x* > *a* ⟷ *a* < *0* ∨ *x* · *x* > *a*^{2}
by (*metis norm-le-square not-less*)

Dot product in terms of the norm rather than conversely.

lemmas *inner-simps* = *inner-add-left inner-add-right inner-diff-right inner-diff-left inner-scaleR-left inner-scaleR-right*

lemma *dot-norm*: *x* · *y* = ((*norm* (*x* + *y*))^2 − (*norm x*)^2 − (*norm y*)^2) / *2*
unfolding *power2-norm-eq-inner inner-simps inner-commute* **by** *auto*

lemma *dot-norm-neg*: *x* · *y* = (((*norm x*)^2 + (*norm y*)^2) − (*norm* (*x* − *y*))^2) / *2*
unfolding *power2-norm-eq-inner inner-simps inner-commute*
by (*auto simp add*: *algebra-simps*)

Equality of vectors in terms of *op* · products.

lemma *vector-eq*: *x* = *y* ⟷ *x* · *x* = *x* · *y* ∧ *y* · *y* = *x* · *x*
(**is** *?lhs* ⟷ *?rhs*)

proof
assume *?lhs*
then show *?rhs* **by** *simp*
next
assume *?rhs*
then have $x \cdot x - x \cdot y = 0 \wedge x \cdot y - y \cdot y = 0$
by *simp*
then have $x \cdot (x - y) = 0 \wedge y \cdot (x - y) = 0$
by (*simp add*: *inner-diff inner-commute*)
then have $(x - y) \cdot (x - y) = 0$
by (*simp add*: *field-simps inner-diff inner-commute*)
then show $x = y$ **by** *simp*
qed

lemma *norm-triangle-half-r*:
$norm\ (y - x1) < e\ /\ 2 \Longrightarrow norm\ (y - x2) < e\ /\ 2 \Longrightarrow norm\ (x1 - x2) < e$
using *dist-triangle-half-r* **unfolding** *dist-norm*[*symmetric*] **by** *auto*

lemma *norm-triangle-half-l*:
assumes $norm\ (x - y) < e\ /\ 2$
and $norm\ (x' - y) < e\ /\ 2$
shows $norm\ (x - x') < e$
using *dist-triangle-half-l*[*OF assms*[*unfolded dist-norm*[*symmetric*]]]
unfolding *dist-norm*[*symmetric*] **.**

lemma *norm-triangle-le*: $norm\ x + norm\ y \leq e \Longrightarrow norm\ (x + y) \leq e$
by (*rule norm-triangle-ineq* [*THEN order-trans*])

lemma *norm-triangle-lt*: $norm\ x + norm\ y < e \Longrightarrow norm\ (x + y) < e$
by (*rule norm-triangle-ineq* [*THEN le-less-trans*])

lemma *setsum-clauses*:
shows $setsum\ f\ \{\} = 0$
and *finite* $S \Longrightarrow setsum\ f\ (insert\ x\ S) = (if\ x \in S\ then\ setsum\ f\ S\ else\ f\ x + setsum\ f\ S)$
by (*auto simp add*: *insert-absorb*)

lemma *setsum-norm-le*:
fixes $f :: 'a \Rightarrow 'b$::*real-normed-vector*
assumes *fg*: $\forall x \in S.\ norm\ (f\ x) \leq g\ x$
shows $norm\ (setsum\ f\ S) \leq setsum\ g\ S$
by (*rule order-trans* [*OF norm-setsum setsum-mono*]) (*simp add*: *fg*)

lemma *setsum-norm-bound*:
fixes $f :: 'a \Rightarrow 'b$::*real-normed-vector*
assumes *K*: $\forall x \in S.\ norm\ (f\ x) \leq K$
shows $norm\ (setsum\ f\ S) \leq of\text{-}nat\ (card\ S) * K$
using *setsum-norm-le*[*OF K*] *setsum-constant*[*symmetric*]
by *simp*

lemma *setsum-group*:
assumes *fS*: *finite S* **and** *fT*: *finite T* **and** *fST*: $f \text{ ` } S \subseteq T$
shows $setsum\ (\lambda y.\ setsum\ g\ \{x.\ x \in S \wedge f\ x = y\})\ T = setsum\ g\ S$
apply (*subst setsum-image-gen*[*OF fS*, *of g f*])
apply (*rule setsum.mono-neutral-right*[*OF fT fST*])
apply (*auto intro*: *setsum.neutral*)
done

lemma *vector-eq-ldot*: $(\forall x.\ x \cdot y = x \cdot z) \longleftrightarrow y = z$
proof
assume $\forall x.\ x \cdot y = x \cdot z$
then have $\forall x.\ x \cdot (y - z) = 0$
by (*simp add*: *inner-diff*)
then have $(y - z) \cdot (y - z) = 0$ **..**
then show $y = z$ **by** *simp*
qed *simp*

lemma *vector-eq-rdot*: $(\forall z.\ x \cdot z = y \cdot z) \longleftrightarrow x = y$
proof
assume $\forall z.\ x \cdot z = y \cdot z$
then have $\forall z.\ (x - y) \cdot z = 0$
by (*simp add*: *inner-diff*)
then have $(x - y) \cdot (x - y) = 0$ **..**
then show $x = y$ **by** *simp*
qed *simp*

13.1 Orthogonality.

context *real-inner*
begin

definition $orthogonal\ x\ y \longleftrightarrow x \cdot y = 0$

lemma *orthogonal-clauses*:
$orthogonal\ a\ 0$
$orthogonal\ a\ x \Longrightarrow orthogonal\ a\ (c *_R x)$
$orthogonal\ a\ x \Longrightarrow orthogonal\ a\ (- x)$
$orthogonal\ a\ x \Longrightarrow orthogonal\ a\ y \Longrightarrow orthogonal\ a\ (x + y)$
$orthogonal\ a\ x \Longrightarrow orthogonal\ a\ y \Longrightarrow orthogonal\ a\ (x - y)$
$orthogonal\ 0\ a$
$orthogonal\ x\ a \Longrightarrow orthogonal\ (c *_R x)\ a$
$orthogonal\ x\ a \Longrightarrow orthogonal\ (- x)\ a$
$orthogonal\ x\ a \Longrightarrow orthogonal\ y\ a \Longrightarrow orthogonal\ (x + y)\ a$
$orthogonal\ x\ a \Longrightarrow orthogonal\ y\ a \Longrightarrow orthogonal\ (x - y)\ a$
unfolding *orthogonal-def inner-add inner-diff* **by** *auto*

end

lemma *orthogonal-commute*: *orthogonal x y* $\longleftrightarrow$ *orthogonal y x*
by (*simp add*: *orthogonal-def inner-commute*)

13.2 Linear functions.

lemma *linear-iff*:
linear f $\longleftrightarrow$ $(\forall\, x\ y.\ f\ (x + y) = f\ x + f\ y) \wedge (\forall\, c\ x.\ f\ (c *_R x) = c *_R f\ x)$
(**is** *linear f* $\longleftrightarrow$ *?rhs*)
proof
assume *linear f*
then interpret *f*: *linear f* **.**
show *?rhs* **by** (*simp add*: *f.add f.scaleR*)
next
assume *?rhs*
then show *linear f* **by** *unfold-locales simp-all*
qed

lemma *linear-compose-cmul*: *linear f* $\Longrightarrow$ *linear* $(\lambda x.\ c *_R f\ x)$
by (*simp add*: *linear-iff algebra-simps*)

lemma *linear-compose-neg*: *linear f* $\Longrightarrow$ *linear* $(\lambda x.\ -\ f\ x)$
by (*simp add*: *linear-iff*)

lemma *linear-compose-add*: *linear f* $\Longrightarrow$ *linear g* $\Longrightarrow$ *linear* $(\lambda x.\ f\ x + g\ x)$
by (*simp add*: *linear-iff algebra-simps*)

lemma *linear-compose-sub*: *linear f* $\Longrightarrow$ *linear g* $\Longrightarrow$ *linear* $(\lambda x.\ f\ x - g\ x)$
by (*simp add*: *linear-iff algebra-simps*)

lemma *linear-compose*: *linear f* $\Longrightarrow$ *linear g* $\Longrightarrow$ *linear* $(g \circ f)$
by (*simp add*: *linear-iff*)

lemma *linear-id*: *linear id*
by (*simp add*: *linear-iff id-def*)

lemma *linear-zero*: *linear* $(\lambda x.\ 0)$
by (*simp add*: *linear-iff*)

lemma *linear-compose-setsum*:
assumes *lS*: $\forall\, a \in S.$ *linear* $(f\ a)$
shows *linear* $(\lambda x.$ *setsum* $(\lambda a.\ f\ a\ x)\ S)$
proof (*cases finite S*)
case *True*
then show *?thesis*
using *lS* **by** *induct* (*simp-all add*: *linear-zero linear-compose-add*)
next
case *False*
then show *?thesis*
by (*simp add*: *linear-zero*)

qed

lemma *linear-0*: *linear f* $\Longrightarrow$ *f 0 = 0*
unfolding *linear-iff*
apply *clarsimp*
apply (*erule allE*[**where** *x=0*::*'a*])
apply *simp*
done

lemma *linear-cmul*: *linear f* $\Longrightarrow$ *f* (*c* $*_R$ *x*) = *c* $*_R$ *f x*
by (*rule linear.scaleR*)

lemma *linear-neg*: *linear f* $\Longrightarrow$ *f* (− *x*) = − *f x*
using *linear-cmul* [**where** *c=−1*] **by** *simp*

lemma *linear-add*: *linear f* $\Longrightarrow$ *f* (*x* + *y*) = *f x* + *f y*
by (*metis linear-iff*)

lemma *linear-sub*: *linear f* $\Longrightarrow$ *f* (*x* − *y*) = *f x* − *f y*
using *linear-add* [*of f x* − *y*] **by** (*simp add*: *linear-neg*)

lemma *linear-setsum*:
assumes *f*: *linear f*
shows *f* (*setsum g S*) = *setsum* (*f* ∘ *g*) *S*
proof (*cases finite S*)
case *True*
then show *?thesis*
by *induct* (*simp-all add*: *linear-0* [*OF f*] *linear-add* [*OF f*])
next
case *False*
then show *?thesis*
by (*simp add*: *linear-0* [*OF f*])
qed

lemma *linear-setsum-mul*:
assumes *lin*: *linear f*
shows *f* (*setsum* (*λi. c i* $*_R$ *v i*) *S*) = *setsum* (*λi. c i* $*_R$ *f* (*v i*)) *S*
using *linear-setsum*[*OF lin*, *of λi. c i* $*_R$ *v i* , *unfolded o-def*] *linear-cmul*[*OF lin*]
by *simp*

lemma *linear-injective-0*:
assumes *lin*: *linear f*
shows *inj f* ⟷ (∀ *x*. *f x* = *0* ⟶ *x* = *0*)
proof −
have *inj f* ⟷ (∀ *x y*. *f x* = *f y* ⟶ *x* = *y*)
by (*simp add*: *inj-on-def*)
also have ... ⟷ (∀ *x y*. *f x* − *f y* = *0* ⟶ *x* − *y* = *0*)
by *simp*

also have ... $\longleftrightarrow$ $(\forall\ x\ y.\ f\ (x - y) = 0 \longrightarrow x - y = 0)$
by (*simp add*: *linear-sub*[*OF lin*])
also have ... $\longleftrightarrow$ $(\forall\ x.\ f\ x = 0 \longrightarrow x = 0)$
by *auto*
finally show *?thesis* **.**
qed

lemma *linear-scaleR* [*simp*]: *linear* $(\lambda x.\ scaleR\ c\ x)$
by (*simp add*: *linear-iff scaleR-add-right*)

lemma *linear-scaleR-left* [*simp*]: *linear* $(\lambda r.\ scaleR\ r\ x)$
by (*simp add*: *linear-iff scaleR-add-left*)

lemma *injective-scaleR*: $c \neq 0 \Longrightarrow inj\ (\lambda x::'a::real\text{-}vector.\ scaleR\ c\ x)$
by (*simp add*: *inj-on-def*)

lemma *linear-add-cmul*:
assumes *linear f*
shows $f\ (a *_R x + b *_R y) = a *_R f\ x + b *_R f\ y$
using *linear-add*[*of f*] *linear-cmul*[*of f*] *assms* **by** *simp*

lemma *linear-componentwise*:
fixes $f::\ 'a::euclidean\text{-}space \Rightarrow 'b::real\text{-}inner$
assumes *lf*: *linear f*
shows $(f\ x) \cdot j = (\sum i{\in}Basis.\ (x{\cdot}i) * (f\ i{\cdot}j))$ (**is** *?lhs* = *?rhs*)
proof −
have *?rhs* = $(\sum i{\in}Basis.\ (x{\cdot}i) *_R (f\ i)){\cdot}j$
by (*simp add*: *inner-setsum-left*)
then show *?thesis*
unfolding *linear-setsum-mul*[*OF lf*, *symmetric*]
unfolding *euclidean-representation* **..**
qed

13.3 Bilinear functions.

definition *bilinear f* $\longleftrightarrow$ $(\forall x.\ linear\ (\lambda y.\ f\ x\ y)) \wedge (\forall y.\ linear\ (\lambda x.\ f\ x\ y))$

lemma *bilinear-ladd*: *bilinear h* $\Longrightarrow h\ (x + y)\ z = h\ x\ z + h\ y\ z$
by (*simp add*: *bilinear-def linear-iff*)

lemma *bilinear-radd*: *bilinear h* $\Longrightarrow h\ x\ (y + z) = h\ x\ y + h\ x\ z$
by (*simp add*: *bilinear-def linear-iff*)

lemma *bilinear-lmul*: *bilinear h* $\Longrightarrow h\ (c *_R x)\ y = c *_R h\ x\ y$
by (*simp add*: *bilinear-def linear-iff*)

lemma *bilinear-rmul*: *bilinear h* $\Longrightarrow h\ x\ (c *_R y) = c *_R h\ x\ y$
by (*simp add*: *bilinear-def linear-iff*)

lemma *bilinear-lneg*: *bilinear h* $\Longrightarrow$ *h* (− *x*) *y* = − *h x y*
by (*drule bilinear-lmul* [*of* - − *1*]) *simp*

lemma *bilinear-rneg*: *bilinear h* $\Longrightarrow$ *h x* (− *y*) = − *h x y*
by (*drule bilinear-rmul* [*of* - - − *1*]) *simp*

lemma (**in** *ab-group-add*) *eq-add-iff*: *x* = *x* + *y* $\longleftrightarrow$ *y* = *0*
using *add-left-imp-eq*[*of x y 0*] **by** *auto*

lemma *bilinear-lzero*:
assumes *bilinear h*
shows *h 0 x* = *0*
using *bilinear-ladd* [*OF assms*, *of 0 0 x*] **by** (*simp add*: *eq-add-iff field-simps*)

lemma *bilinear-rzero*:
assumes *bilinear h*
shows *h x 0* = *0*
using *bilinear-radd* [*OF assms*, *of x 0 0*] **by** (*simp add*: *eq-add-iff field-simps*)

lemma *bilinear-lsub*: *bilinear h* $\Longrightarrow$ *h* (*x* − *y*) *z* = *h x z* − *h y z*
using *bilinear-ladd* [*of h x* − *y*] **by** (*simp add*: *bilinear-lneg*)

lemma *bilinear-rsub*: *bilinear h* $\Longrightarrow$ *h z* (*x* − *y*) = *h z x* − *h z y*
using *bilinear-radd* [*of h* - *x* − *y*] **by** (*simp add*: *bilinear-rneg*)

lemma *bilinear-setsum*:
assumes *bh*: *bilinear h*
and *fS*: *finite S*
and *fT*: *finite T*
shows *h* (*setsum f S*) (*setsum g T*) = *setsum* (λ(*i*,*j*). *h* (*f i*) (*g j*)) (*S* × *T*)
proof −
have *h* (*setsum f S*) (*setsum g T*) = *setsum* (λ*x*. *h* (*f x*) (*setsum g T*)) *S*
apply (*rule linear-setsum*[*unfolded o-def*])
using *bh fS*
apply (*auto simp add*: *bilinear-def*)
done
also have ... = *setsum* (λ*x*. *setsum* (λ*y*. *h* (*f x*) (*g y*)) *T*) *S*
apply (*rule setsum.cong*, *simp*)
apply (*rule linear-setsum*[*unfolded o-def*])
using *bh fT*
apply (*auto simp add*: *bilinear-def*)
done
finally show *?thesis*
unfolding *setsum.cartesian-product* **.**
qed

13.4 Adjoints.

definition *adjoint f* = (*SOME f'*. ∀ *x y*. *f x* · *y* = *x* · *f' y*)

lemma *adjoint-unique*:
assumes $\forall x\ y.\ inner\ (f\ x)\ y = inner\ x\ (g\ y)$
shows $adjoint\ f = g$
unfolding *adjoint-def*
proof (*rule some-equality*)
show $\forall x\ y.\ inner\ (f\ x)\ y = inner\ x\ (g\ y)$
by (*rule assms*)
next
fix h
assume $\forall x\ y.\ inner\ (f\ x)\ y = inner\ x\ (h\ y)$
then have $\forall x\ y.\ inner\ x\ (g\ y) = inner\ x\ (h\ y)$
using *assms* **by** *simp*
then have $\forall x\ y.\ inner\ x\ (g\ y - h\ y) = 0$
by (*simp add*: *inner-diff-right*)
then have $\forall y.\ inner\ (g\ y - h\ y)\ (g\ y - h\ y) = 0$
by *simp*
then have $\forall y.\ h\ y = g\ y$
by *simp*
then show $h = g$ **by** (*simp add*: *ext*)
qed

TODO: The following lemmas about adjoints should hold for any Hilbert space (i.e. complete inner product space). (see http://en.wikipedia.org/wiki/Hermitian_adjoint)

lemma *adjoint-works*:
fixes $f :: 'n$::*euclidean-space* $\Rightarrow$ $'m$::*euclidean-space*
assumes *lf*: *linear f*
shows $x \cdot adjoint\ f\ y = f\ x \cdot y$
proof −
have $\forall y.\ \exists w.\ \forall x.\ f\ x \cdot y = x \cdot w$
proof (*intro allI exI*)
fix $y :: 'm$ **and** x
let $?w = (\sum i{\in}Basis.\ (f\ i \cdot y) *_R i) :: 'n$
have $f\ x \cdot y = f\ (\sum i{\in}Basis.\ (x \cdot i) *_R i) \cdot y$
by (*simp add*: *euclidean-representation*)
also have $\ldots = (\sum i{\in}Basis.\ (x \cdot i) *_R f\ i) \cdot y$
unfolding *linear-setsum*[*OF lf*]
by (*simp add*: *linear-cmul*[*OF lf*])
finally show $f\ x \cdot y = x \cdot ?w$
by (*simp add*: *inner-setsum-left inner-setsum-right mult.commute*)
qed
then show *?thesis*
unfolding *adjoint-def choice-iff*
by (*intro someI2-ex*[**where** $Q{=}\lambda f'.\ x \cdot f'\ y = f\ x \cdot y$]) *auto*
qed

lemma *adjoint-clauses*:
fixes $f :: 'n$::*euclidean-space* $\Rightarrow$ $'m$::*euclidean-space*

assumes *lf*: *linear f*
shows $x \cdot$ *adjoint f y* $=$ *f x* $\cdot$ *y*
and *adjoint f y* $\cdot$ *x* $=$ *y* $\cdot$ *f x*
by (*simp-all add*: *adjoint-works*[*OF lf*] *inner-commute*)

lemma *adjoint-linear*:
fixes *f* :: *'n*::*euclidean-space* $\Rightarrow$ *'m*::*euclidean-space*
assumes *lf*: *linear f*
shows *linear* (*adjoint f*)
by (*simp add*: *lf linear-iff euclidean-eq-iff*[**where** *'a='n*] *euclidean-eq-iff*[**where** *'a='m*]
adjoint-clauses[*OF lf*] *inner-distrib*)

lemma *adjoint-adjoint*:
fixes *f* :: *'n*::*euclidean-space* $\Rightarrow$ *'m*::*euclidean-space*
assumes *lf*: *linear f*
shows *adjoint* (*adjoint f*) $=$ *f*
by (*rule adjoint-unique*, *simp add*: *adjoint-clauses* [*OF lf*])

13.5 Interlude: Some properties of real sets

lemma *seq-mono-lemma*:
assumes $\forall$ (*n*::*nat*) $\geq$ *m*. (*d n* :: *real*) $<$ *e n*
and $\forall$ *n* $\geq$ *m*. *e n* $\leq$ *e m*
shows $\forall$ *n* $\geq$ *m*. *d n* $<$ *e m*
using *assms*
apply *auto*
apply (*erule-tac x=n* **in** *allE*)
apply (*erule-tac x=n* **in** *allE*)
apply *auto*
done

lemma *infinite-enumerate*:
assumes *fS*: *infinite S*
shows $\exists$ *r*. *subseq r* $\wedge$ ($\forall$ *n*. *r n* $\in$ *S*)
unfolding *subseq-def*
using *enumerate-in-set*[*OF fS*] *enumerate-mono*[*of - - S*] *fS* **by** *auto*

lemma *approachable-lt-le*: ($\exists$ (*d*::*real*) $>$ *0*. $\forall$ *x*. *f x* $<$ *d* $\longrightarrow$ *P x*) $\longleftrightarrow$ ($\exists$ *d*$>$*0*. $\forall$ *x*. *f x* $\leq$ *d* $\longrightarrow$ *P x*)
apply *auto*
apply (*rule-tac x=d/2* **in** *exI*)
apply *auto*
done

lemma *approachable-lt-le2*: — like the above, but pushes aside an extra formula
($\exists$ (*d*::*real*) $>$ *0*. $\forall$ *x*. *Q x* $\longrightarrow$ *f x* $<$ *d* $\longrightarrow$ *P x*) $\longleftrightarrow$ ($\exists$ *d*$>$*0*. $\forall$ *x*. *f x* $\leq$ *d* $\longrightarrow$ *Q x* $\longrightarrow$ *P x*)
apply *auto*

apply (*rule-tac x=d/2* **in** *exI*, *auto*)
done

lemma *triangle-lemma*:
fixes $x\ y\ z$ :: *real*
assumes x: $0 \leq x$
and y: $0 \leq y$
and z: $0 \leq z$
and xy: $x^2 \leq y^2 + z^2$
shows $x \leq y + z$
proof −
have $y^2 + z^2 \leq y^2 + 2 * y * z + z^2$
using $z\ y$ **by** *simp*
with xy **have** th: $x^2 \leq (y + z)^2$
by (*simp add*: *power2-eq-square field-simps*)
from $y\ z$ **have** yz: $y + z \geq 0$
by *arith*
from *power2-le-imp-le*[*OF th yz*] **show** *?thesis* **.**
qed

13.6 A generic notion of "hull" (convex, affine, conic hull and closure).

definition *hull* :: ('*a set* ⇒ *bool*) ⇒ '*a set* ⇒ '*a set* (**infixl** *hull* 75)
where $S\ hull\ s = \bigcap \{t.\ S\ t \wedge s \subseteq t\}$

lemma *hull-same*: $S\ s \Longrightarrow S\ hull\ s = s$
unfolding *hull-def* **by** *auto*

lemma *hull-in*: $(\bigwedge T.\ Ball\ T\ S \Longrightarrow S\ (\bigcap T)) \Longrightarrow S\ (S\ hull\ s)$
unfolding *hull-def Ball-def* **by** *auto*

lemma *hull-eq*: $(\bigwedge T.\ Ball\ T\ S \Longrightarrow S\ (\bigcap T)) \Longrightarrow (S\ hull\ s) = s \longleftrightarrow S\ s$
using *hull-same*[*of S s*] *hull-in*[*of S s*] **by** *metis*

lemma *hull-hull*: $S\ hull\ (S\ hull\ s) = S\ hull\ s$
unfolding *hull-def* **by** *blast*

lemma *hull-subset*[*intro*]: $s \subseteq (S\ hull\ s)$
unfolding *hull-def* **by** *blast*

lemma *hull-mono*: $s \subseteq t \Longrightarrow (S\ hull\ s) \subseteq (S\ hull\ t)$
unfolding *hull-def* **by** *blast*

lemma *hull-antimono*: $\forall x.\ S\ x \longrightarrow T\ x \Longrightarrow (T\ hull\ s) \subseteq (S\ hull\ s)$
unfolding *hull-def* **by** *blast*

lemma *hull-minimal*: $s \subseteq t \Longrightarrow S\ t \Longrightarrow (S\ hull\ s) \subseteq t$
unfolding *hull-def* **by** *blast*

lemma *subset-hull*: $S\ t \Longrightarrow S\ hull\ s \subseteq t \longleftrightarrow s \subseteq t$
unfolding *hull-def* **by** *blast*

lemma *hull-UNIV*: $S\ hull\ UNIV = UNIV$
unfolding *hull-def* **by** *auto*

lemma *hull-unique*: $s \subseteq t \Longrightarrow S\ t \Longrightarrow (\bigwedge t'.\ s \subseteq t' \Longrightarrow S\ t' \Longrightarrow t \subseteq t') \Longrightarrow (S\ hull\ s = t)$
unfolding *hull-def* **by** *auto*

lemma *hull-induct*: $(\bigwedge x.\ x \in S \Longrightarrow P\ x) \Longrightarrow Q\ \{x.\ P\ x\} \Longrightarrow \forall x \in Q\ hull\ S.\ P\ x$
using *hull-minimal*[*of S* $\{x.\ P\ x\}$ *Q*]
by (*auto simp add*: *subset-eq*)

lemma *hull-inc*: $x \in S \Longrightarrow x \in P\ hull\ S$
by (*metis hull-subset subset-eq*)

lemma *hull-union-subset*: $(S\ hull\ s) \cup (S\ hull\ t) \subseteq (S\ hull\ (s \cup t))$
unfolding *Un-subset-iff* **by** (*metis hull-mono Un-upper1 Un-upper2*)

lemma *hull-union*:
assumes *T*: $\bigwedge T.\ Ball\ T\ S \Longrightarrow S\ (\bigcap T)$
shows $S\ hull\ (s \cup t) = S\ hull\ (S\ hull\ s \cup S\ hull\ t)$
apply *rule*
apply (*rule hull-mono*)
unfolding *Un-subset-iff*
apply (*metis hull-subset Un-upper1 Un-upper2 subset-trans*)
apply (*rule hull-minimal*)
apply (*metis hull-union-subset*)
apply (*metis hull-in T*)
done

lemma *hull-redundant-eq*: $a \in (S\ hull\ s) \longleftrightarrow S\ hull\ (insert\ a\ s) = S\ hull\ s$
unfolding *hull-def* **by** *blast*

lemma *hull-redundant*: $a \in (S\ hull\ s) \Longrightarrow S\ hull\ (insert\ a\ s) = S\ hull\ s$
by (*metis hull-redundant-eq*)

13.7 Archimedean properties and useful consequences

Bernoulli's inequality

proposition *Bernoulli-inequality*:
fixes $x :: real$
assumes $-1 \leq x$
shows $1 + n * x \leq (1 + x)\ \hat{}\ n$
proof (*induct n*)
case *0*
then show *?case* **by** *simp*

next
case (*Suc n*)
have *1 + Suc n * x ≤ 1 + (Suc n)*x + n * x^2*
by (*simp add*: *algebra-simps*)
also have *... = (1 + x) * (1 + n*x)*
by (*auto simp*: *power2-eq-square algebra-simps of-nat-Suc*)
also have *... ≤ (1 + x) ^ Suc n*
using *Suc.hyps assms mult-left-mono* **by** *fastforce*
finally show *?case* **.**
qed

corollary *Bernoulli-inequality-even*:
fixes *x* :: *real*
assumes *even n*
shows *1 + n * x ≤ (1 + x) ^ n*
proof (*cases −1 ≤ x ∨ n=0*)
case *True*
then show *?thesis*
by (*auto simp*: *Bernoulli-inequality*)
next
case *False*
then have *real n ≥ 1*
by *simp*
with *False* **have** *n * x ≤ −1*
by (*metis linear minus-zero mult.commute mult.left-neutral mult-left-mono-neg neg-le-iff-le order-trans zero-le-one*)
then have *1 + n * x ≤ 0*
by *auto*
also have *... ≤ (1 + x) ^ n*
using *assms*
using *zero-le-even-power* **by** *blast*
finally show *?thesis* **.**
qed

corollary *real-arch-pow*:
fixes *x* :: *real*
assumes *x*: *1 < x*
shows ∃ *n*. *y < x^n*
proof −
from *x* **have** *x0*: *x − 1 > 0*
by *arith*
from *reals-Archimedean3*[*OF x0*, *rule-format*, *of y*]
obtain *n* :: *nat* **where** *n*: *y < real n * (x − 1)* **by** *metis*
from *x0* **have** *x00*: *x− 1 ≥ −1* **by** *arith*
from *Bernoulli-inequality*[*OF x00*, *of n*] *n*
have *y < x^n* **by** *auto*
then show *?thesis* **by** *metis*
qed

corollary *real-arch-pow-inv*:
fixes $x\ y$ :: *real*
assumes *y*: $y > 0$
and *x1*: $x < 1$
shows $\exists n.\ x \hat{} n < y$
proof (*cases* $x > 0$)
case *True*
with *x1* **have** *ix*: $1 < 1/x$ **by** (*simp add*: *field-simps*)
from *real-arch-pow*[*OF ix*, *of* $1/y$]
obtain *n* **where** *n*: $1/y < (1/x) \hat{} n$ **by** *blast*
then show *?thesis* **using** *y* ‹$x > 0$›
by (*auto simp add*: *field-simps*)
next
case *False*
with *y x1* **show** *?thesis*
apply *auto*
apply (*rule exI*[**where** *x*=*1*])
apply *auto*
done
qed

lemma *forall-pos-mono*:
$(\bigwedge d\ e{::}real.\ d < e \Longrightarrow P\ d \Longrightarrow P\ e) \Longrightarrow$
$(\bigwedge n{::}nat.\ n \neq 0 \Longrightarrow P\ (inverse\ (real\ n))) \Longrightarrow (\bigwedge e.\ 0 < e \Longrightarrow P\ e)$
by (*metis real-arch-inverse*)

lemma *forall-pos-mono-1*:
$(\bigwedge d\ e{::}real.\ d < e \Longrightarrow P\ d \Longrightarrow P\ e) \Longrightarrow$
$(\bigwedge n.\ P\ (inverse\ (real\ (Suc\ n)))) \Longrightarrow 0 < e \Longrightarrow P\ e$
apply (*rule forall-pos-mono*)
apply *auto*
apply (*metis Suc-pred of-nat-Suc*)
done

13.8 A bit of linear algebra.

definition (**in** *real-vector*) *subspace* :: $'a\ set \Rightarrow bool$
where $subspace\ S \longleftrightarrow 0 \in S \wedge (\forall x \in S.\ \forall y \in S.\ x + y \in S) \wedge (\forall c.\ \forall x \in S.\ c *_R x \in S)$

definition (**in** *real-vector*) *span* $S = (subspace\ hull\ S)$
definition (**in** *real-vector*) *dependent* $S \longleftrightarrow (\exists a \in S.\ a \in span\ (S - \{a\}))$
abbreviation (**in** *real-vector*) *independent* $s \equiv \neg\ dependent\ s$

Closure properties of subspaces.

lemma *subspace-UNIV*[*simp*]: *subspace UNIV*
by (*simp add*: *subspace-def*)

lemma (**in** *real-vector*) *subspace-0*: $subspace\ S \Longrightarrow 0 \in S$

by (*metis subspace-def*)

lemma (**in** *real-vector*) *subspace-add*: *subspace S* $\Longrightarrow$ $x \in S \Longrightarrow y \in S \Longrightarrow x + y \in S$
by (*metis subspace-def*)

lemma (**in** *real-vector*) *subspace-mul*: *subspace S* $\Longrightarrow$ $x \in S \Longrightarrow c *_R x \in S$
by (*metis subspace-def*)

lemma *subspace-neg*: *subspace S* $\Longrightarrow$ $x \in S \Longrightarrow - x \in S$
by (*metis scaleR-minus1-left subspace-mul*)

lemma *subspace-sub*: *subspace S* $\Longrightarrow$ $x \in S \Longrightarrow y \in S \Longrightarrow x - y \in S$
using *subspace-add* [*of S x* $-$ *y*] **by** (*simp add*: *subspace-neg*)

lemma (**in** *real-vector*) *subspace-setsum*:
assumes *sA*: *subspace A*
and *f*: $\forall x{\in}B.\ f\ x \in A$
shows *setsum f B* $\in A$
proof (*cases finite B*)
case *True*
then show *?thesis*
using *f* **by** *induct* (*simp-all add*: *subspace-0* [*OF sA*] *subspace-add* [*OF sA*])
qed (*simp add*: *subspace-0* [*OF sA*])

lemma *subspace-linear-image*:
assumes *lf*: *linear f*
and *sS*: *subspace S*
shows *subspace* (*f* ‘ *S*)
using *lf sS linear-0*[*OF lf*]
unfolding *linear-iff subspace-def*
apply (*auto simp add*: *image-iff*)
apply (*rule-tac x*=*x* + *y* **in** *bexI*)
apply *auto*
apply (*rule-tac x*=*c* $*_R$ *x* **in** *bexI*)
apply *auto*
done

lemma *subspace-linear-vimage*: *linear f* $\Longrightarrow$ *subspace S* $\Longrightarrow$ *subspace* (*f* $-$‘ *S*)
by (*auto simp add*: *subspace-def linear-iff linear-0*[*of f*])

lemma *subspace-linear-preimage*: *linear f* $\Longrightarrow$ *subspace S* $\Longrightarrow$ *subspace* $\{x.\ f\ x \in S\}$
by (*auto simp add*: *subspace-def linear-iff linear-0*[*of f*])

lemma *subspace-trivial*: *subspace* $\{0\}$
by (*simp add*: *subspace-def*)

lemma (**in** *real-vector*) *subspace-inter*: *subspace A* $\Longrightarrow$ *subspace B* $\Longrightarrow$ *subspace*

$(A \cap B)$
by (*simp add*: *subspace-def*)

lemma *subspace-Times*: *subspace A* $\Longrightarrow$ *subspace B* $\Longrightarrow$ *subspace* $(A \times B)$
unfolding *subspace-def zero-prod-def* **by** *simp*

Properties of span.

lemma (**in** *real-vector*) *span-mono*: $A \subseteq B \Longrightarrow$ *span A* $\subseteq$ *span B*
by (*metis span-def hull-mono*)

lemma (**in** *real-vector*) *subspace-span*: *subspace* (*span S*)
unfolding *span-def*
apply (*rule hull-in*)
apply (*simp only*: *subspace-def Inter-iff Int-iff subset-eq*)
apply *auto*
done

lemma (**in** *real-vector*) *span-clauses*:
$a \in S \Longrightarrow a \in$ *span S*
$0 \in$ *span S*
$x \in$ *span S* $\Longrightarrow y \in$ *span S* $\Longrightarrow x + y \in$ *span S*
$x \in$ *span S* $\Longrightarrow c *_R x \in$ *span S*
by (*metis span-def hull-subset subset-eq*) (*metis subspace-span subspace-def*)+

lemma *span-unique*:
$S \subseteq T \Longrightarrow$ *subspace T* $\Longrightarrow (\bigwedge T'.\ S \subseteq T' \Longrightarrow$ *subspace* $T' \Longrightarrow T \subseteq T') \Longrightarrow$ *span S* $= T$
unfolding *span-def* **by** (*rule hull-unique*)

lemma *span-minimal*: $S \subseteq T \Longrightarrow$ *subspace T* $\Longrightarrow$ *span S* $\subseteq T$
unfolding *span-def* **by** (*rule hull-minimal*)

lemma (**in** *real-vector*) *span-induct*:
assumes *x*: $x \in$ *span S*
and *P*: *subspace P*
and *SP*: $\bigwedge x.\ x \in S \Longrightarrow x \in P$
shows $x \in P$
proof −
from *SP* **have** *SP'*: $S \subseteq P$
by (*simp add*: *subset-eq*)
from *x hull-minimal*[**where** *S*=*subspace*, *OF SP' P*, *unfolded span-def*[*symmetric*]]
show $x \in P$
by (*metis subset-eq*)
qed

lemma *span-empty*[*simp*]: *span* $\{\} = \{0\}$
apply (*simp add*: *span-def*)
apply (*rule hull-unique*)
apply (*auto simp add*: *subspace-def*)

done

lemma (**in** *real-vector*) *independent-empty* [*iff*]: *independent* {}
by (*simp add*: *dependent-def*)

lemma *dependent-single*[*simp*]: *dependent* $\{x\} \longleftrightarrow x = 0$
unfolding *dependent-def* **by** *auto*

lemma (**in** *real-vector*) *independent-mono*: *independent* $A \Longrightarrow B \subseteq A \Longrightarrow$ *independent* B
apply (*clarsimp simp add*: *dependent-def span-mono*)
apply (*subgoal-tac span* $(B - \{a\}) \leq$ *span* $(A - \{a\})$)
apply *force*
apply (*rule span-mono*)
apply *auto*
done

lemma (**in** *real-vector*) *span-subspace*: $A \subseteq B \Longrightarrow B \leq$ *span* $A \Longrightarrow$ *subspace* $B \Longrightarrow$ *span* $A = B$
by (*metis order-antisym span-def hull-minimal*)

lemma (**in** *real-vector*) *span-induct'*:
assumes *SP*: $\forall x \in S.\ P\ x$
and *P*: *subspace* $\{x.\ P\ x\}$
shows $\forall x \in$ *span* $S.\ P\ x$
using *span-induct SP P* **by** *blast*

inductive-set (**in** *real-vector*) *span-induct-alt-help* **for** $S :: 'a$ *set*
where
span-induct-alt-help-0: $0 \in$ *span-induct-alt-help* S
| *span-induct-alt-help-S*:
$x \in S \Longrightarrow z \in$ *span-induct-alt-help* $S \Longrightarrow$
$(c *_R x + z) \in$ *span-induct-alt-help* S

lemma *span-induct-alt'*:
assumes *h0*: $h\ 0$
and *hS*: $\bigwedge c\ x\ y.\ x \in S \Longrightarrow h\ y \Longrightarrow h\ (c *_R x + y)$
shows $\forall x \in$ *span* $S.\ h\ x$
proof −
{
fix $x :: 'a$
assume *x*: $x \in$ *span-induct-alt-help* S
have $h\ x$
apply (*rule span-induct-alt-help.induct*[*OF x*])
apply (*rule h0*)
apply (*rule hS*)
apply *assumption*
apply *assumption*
done

```
 }
 note th0 = this
 {
  fix x
  assume x: x ∈ span S
  have x ∈ span-induct-alt-help S
  proof (rule span-induct[where x=x and S=S])
   show x ∈ span S by (rule x)
  next
   fix x
   assume xS: x ∈ S
   from span-induct-alt-help-S[OF xS span-induct-alt-help-0, of 1]
   show x ∈ span-induct-alt-help S
    by simp
  next
   have 0 ∈ span-induct-alt-help S by (rule span-induct-alt-help-0)
   moreover
   {
    fix x y
    assume h: x ∈ span-induct-alt-help S y ∈ span-induct-alt-help S
    from h have (x + y) ∈ span-induct-alt-help S
     apply (induct rule: span-induct-alt-help.induct)
     apply simp
     unfolding add.assoc
     apply (rule span-induct-alt-help-S)
     apply assumption
     apply simp
     done
   }
   moreover
   {
    fix c x
    assume xt: x ∈ span-induct-alt-help S
    then have (c *R x) ∈ span-induct-alt-help S
     apply (induct rule: span-induct-alt-help.induct)
     apply (simp add: span-induct-alt-help-0)
     apply (simp add: scaleR-right-distrib)
     apply (rule span-induct-alt-help-S)
     apply assumption
     apply simp
     done }
   ultimately show subspace (span-induct-alt-help S)
    unfolding subspace-def Ball-def by blast
  qed
 }
 with th0 show ?thesis by blast
qed

lemma span-induct-alt:
```

assumes *h0*: *h 0*
and *hS*: $\bigwedge c\ x\ y.\ x \in S \Longrightarrow h\ y \Longrightarrow h\ (c *_R x + y)$
and *x*: $x \in span\ S$
shows *h x*
using *span-induct-alt'*[*of h S*] *h0 hS x* **by** *blast*

Individual closure properties.

lemma *span-span*: *span* (*span A*) = *span A*
unfolding *span-def hull-hull* **..**

lemma (**in** *real-vector*) *span-superset*: $x \in S \Longrightarrow x \in span\ S$
by (*metis span-clauses*(*1*))

lemma (**in** *real-vector*) *span-0*: *0* ∈ *span S*
by (*metis subspace-span subspace-0*)

lemma *span-inc*: $S \subseteq span\ S$
by (*metis subset-eq span-superset*)

lemma (**in** *real-vector*) *dependent-0*:
assumes *0* ∈ *A*
shows *dependent A*
unfolding *dependent-def*
using *assms span-0*
by *auto*

lemma (**in** *real-vector*) *span-add*: $x \in span\ S \Longrightarrow y \in span\ S \Longrightarrow x + y \in span\ S$
by (*metis subspace-add subspace-span*)

lemma (**in** *real-vector*) *span-mul*: $x \in span\ S \Longrightarrow c *_R x \in span\ S$
by (*metis subspace-span subspace-mul*)

lemma *span-neg*: $x \in span\ S \Longrightarrow - x \in span\ S$
by (*metis subspace-neg subspace-span*)

lemma *span-sub*: $x \in span\ S \Longrightarrow y \in span\ S \Longrightarrow x - y \in span\ S$
by (*metis subspace-span subspace-sub*)

lemma (**in** *real-vector*) *span-setsum*: $\forall x{\in}A.\ f\ x \in span\ S \Longrightarrow setsum\ f\ A \in span\ S$
by (*rule subspace-setsum* [*OF subspace-span*])

lemma *span-add-eq*: $x \in span\ S \Longrightarrow x + y \in span\ S \longleftrightarrow y \in span\ S$
by (*metis add-minus-cancel scaleR-minus1-left subspace-def subspace-span*)

Mapping under linear image.

lemma *span-linear-image*:
assumes *lf*: *linear f*

shows *span* (*f* ‘ *S*) = *f* ‘ *span S*
proof (*rule span-unique*)
show *f* ‘ *S* ⊆ *f* ‘ *span S*
by (*intro image-mono span-inc*)
show *subspace* (*f* ‘ *span S*)
using *lf subspace-span* **by** (*rule subspace-linear-image*)
next
fix *T*
assume *f* ‘ *S* ⊆ *T* **and** *subspace T*
then show *f* ‘ *span S* ⊆ *T*
unfolding *image-subset-iff-subset-vimage*
by (*intro span-minimal subspace-linear-vimage lf*)
qed

lemma *span-union*: *span* (*A* ∪ *B*) = (λ(*a*, *b*). *a* + *b*) ‘ (*span A* × *span B*)
proof (*rule span-unique*)
show *A* ∪ *B* ⊆ (λ(*a*, *b*). *a* + *b*) ‘ (*span A* × *span B*)
by *safe* (*force intro*: *span-clauses*)+
next
have *linear* (λ(*a*, *b*). *a* + *b*)
by (*simp add*: *linear-iff scaleR-add-right*)
moreover have *subspace* (*span A* × *span B*)
by (*intro subspace-Times subspace-span*)
ultimately show *subspace* ((λ(*a*, *b*). *a* + *b*) ‘ (*span A* × *span B*))
by (*rule subspace-linear-image*)
next
fix *T*
assume *A* ∪ *B* ⊆ *T* **and** *subspace T*
then show (λ(*a*, *b*). *a* + *b*) ‘ (*span A* × *span B*) ⊆ *T*
by (*auto intro*!: *subspace-add elim*: *span-induct*)
qed

The key breakdown property.

lemma *span-singleton*: *span* {*x*} = *range* (λ*k*. *k* ∗*R* *x*)
proof (*rule span-unique*)
show {*x*} ⊆ *range* (λ*k*. *k* ∗*R* *x*)
by (*fast intro*: *scaleR-one* [*symmetric*])
show *subspace* (*range* (λ*k*. *k* ∗*R* *x*))
unfolding *subspace-def*
by (*auto intro*: *scaleR-add-left* [*symmetric*])
next
fix *T*
assume {*x*} ⊆ *T* **and** *subspace T*
then show *range* (λ*k*. *k* ∗*R* *x*) ⊆ *T*
unfolding *subspace-def* **by** *auto*
qed

lemma *span-insert*: *span* (*insert a S*) = {*x*. ∃*k*. (*x* − *k* ∗*R* *a*) ∈ *span S*}
proof −

```
  have span ({a} ∪ S) = {x. ∃k. (x − k *R a) ∈ span S}
    unfolding span-union span-singleton
    apply safe
    apply (rule-tac x=k in exI, simp)
    apply (erule rev-image-eqI [OF SigmaI [OF rangeI]])
    apply auto
    done
  then show ?thesis by simp
qed

lemma span-breakdown:
  assumes bS: b ∈ S
    and aS: a ∈ span S
  shows ∃k. a − k *R b ∈ span (S − {b})
  using assms span-insert [of b S − {b}]
  by (simp add: insert-absorb)

lemma span-breakdown-eq: x ∈ span (insert a S) ⟷ (∃k. x − k *R a ∈ span S)
  by (simp add: span-insert)
```

Hence some "reversal" results.

```
lemma in-span-insert:
  assumes a: a ∈ span (insert b S)
    and na: a ∉ span S
  shows b ∈ span (insert a S)
proof −
  from a obtain k where k: a − k *R b ∈ span S
    unfolding span-insert by fast
  show ?thesis
  proof (cases k = 0)
    case True
    with k have a ∈ span S by simp
    with na show ?thesis by simp
  next
    case False
    from k have (− inverse k) *R (a − k *R b) ∈ span S
      by (rule span-mul)
    then have b − inverse k *R a ∈ span S
      using ‹k ≠ 0› by (simp add: scaleR-diff-right)
    then show ?thesis
      unfolding span-insert by fast
  qed
qed

lemma in-span-delete:
  assumes a: a ∈ span S
    and na: a ∉ span (S − {b})
  shows b ∈ span (insert a (S − {b}))
  apply (rule in-span-insert)
```

apply (*rule set-rev-mp*)
apply (*rule a*)
apply (*rule span-mono*)
apply *blast*
apply (*rule na*)
done

Transitivity property.

lemma *span-redundant*: $x \in span\ S \Longrightarrow span\ (insert\ x\ S) = span\ S$
unfolding *span-def* **by** (*rule hull-redundant*)

lemma *span-trans*:
assumes *x*: $x \in span\ S$
and *y*: $y \in span\ (insert\ x\ S)$
shows $y \in span\ S$
using *assms* **by** (*simp only*: *span-redundant*)

lemma *span-insert-0*[*simp*]: $span\ (insert\ 0\ S) = span\ S$
by (*simp only*: *span-redundant span-0*)

An explicit expansion is sometimes needed.

lemma *span-explicit*:
$span\ P = \{y.\ \exists S\ u.\ finite\ S \wedge S \subseteq P \wedge setsum\ (\lambda v.\ u\ v *_R v)\ S = y\}$
(**is** - = *?E* **is** - = $\{y.\ ?h\ y\}$ **is** - = $\{y.\ \exists S\ u.\ ?Q\ S\ u\ y\}$)
proof −
{
fix *x*
assume *?h x*
then obtain *S u* **where** *finite S* **and** $S \subseteq P$ **and** $setsum\ (\lambda v.\ u\ v *_R v)\ S = x$
by *blast*
then have $x \in span\ P$
by (*auto intro*: *span-setsum span-mul span-superset*)
}
moreover
have $\forall x \in span\ P.\ ?h\ x$
proof (*rule span-induct-alt'*)
show *?h 0*
by (*rule exI*[**where** *x*={}], *simp*)
next
fix *c x y*
assume *x*: $x \in P$
assume *hy*: *?h y*
from *hy* **obtain** *S u* **where** *fS*: *finite S* **and** *SP*: $S \subseteq P$
and *u*: $setsum\ (\lambda v.\ u\ v *_R v)\ S = y$ **by** *blast*
let *?S* = *insert x S*
let *?u* = $\lambda y.\ if\ y = x\ then\ (if\ x \in S\ then\ u\ y + c\ else\ c)\ else\ u\ y$
from *fS SP x* **have** *th0*: *finite* (*insert x S*) *insert x S* $\subseteq P$
by *blast*+

```
    have ?Q ?S ?u (c*R x + y)
    proof cases
      assume xS: x ∈ S
      have setsum (λv. ?u v *R v) ?S = (∑v∈S - {x}. u v *R v) + (u x + c)
*R x
        using xS by (simp add: setsum.remove [OF fS xS] insert-absorb)
      also have ... = (∑v∈S. u v *R v) + c *R x
        by (simp add: setsum.remove [OF fS xS] algebra-simps)
      also have ... = c*R x + y
        by (simp add: add.commute u)
      finally have setsum (λv. ?u v *R v) ?S = c*R x + y .
      then show ?thesis using th0 by blast
    next
      assume xS: x ∉ S
      have th00: (∑v∈S. (if v = x then c else u v) *R v) = y
        unfolding u[symmetric]
        apply (rule setsum.cong)
        using xS
        apply auto
        done
      show ?thesis using fS xS th0
        by (simp add: th00 add.commute cong del: if-weak-cong)
    qed
    then show ?h (c*R x + y)
      by fast
  qed
  ultimately show ?thesis by blast
qed

lemma dependent-explicit:
  dependent P ⟷ (∃S u. finite S ∧ S ⊆ P ∧ (∃v∈S. u v ≠ 0 ∧ setsum (λv. u
v *R v) S = 0))
  (is ?lhs = ?rhs)
proof -
  {
    assume dP: dependent P
    then obtain a S u where aP: a ∈ P and fS: finite S
      and SP: S ⊆ P - {a} and ua: setsum (λv. u v *R v) S = a
      unfolding dependent-def span-explicit by blast
    let ?S = insert a S
    let ?u = λy. if y = a then - 1 else u y
    let ?v = a
    from aP SP have aS: a ∉ S
      by blast
    from fS SP aP have th0: finite ?S ?S ⊆ P ?v ∈ ?S ?u ?v ≠ 0
      by auto
    have s0: setsum (λv. ?u v *R v) ?S = 0
      using fS aS
      apply simp
```

```
    apply (subst (2) ua[symmetric])
    apply (rule setsum.cong)
    apply auto
    done
  with th0 have ?rhs by fast
 }
 moreover
 {
  fix S u v
  assume fS: finite S
    and SP: S ⊆ P
    and vS: v ∈ S
    and uv: u v ≠ 0
    and u: setsum (λv. u v *R v) S = 0
  let ?a = v
  let ?S = S - {v}
  let ?u = λi. (- u i) / u v
  have th0: ?a ∈ P finite ?S ?S ⊆ P
    using fS SP vS by auto
  have setsum (λv. ?u v *R v) ?S =
    setsum (λv. (- (inverse (u ?a))) *R (u v *R v)) S - ?u v *R v
    using fS vS uv by (simp add: setsum-diff1 field-simps)
  also have ... = ?a
    unfolding scaleR-right.setsum [symmetric] u using uv by simp
  finally have setsum (λv. ?u v *R v) ?S = ?a .
  with th0 have ?lhs
    unfolding dependent-def span-explicit
    apply -
    apply (rule bexI[where x= ?a])
    apply (simp-all del: scaleR-minus-left)
    apply (rule exI[where x= ?S])
    apply (auto simp del: scaleR-minus-left)
    done
 }
 ultimately show ?thesis by blast
qed


lemma span-finite:
 assumes fS: finite S
 shows span S = {y. ∃ u. setsum (λv. u v *R v) S = y}
 (is - = ?rhs)
proof -
 {
  fix y
  assume y: y ∈ span S
  from y obtain S' u where fS': finite S'
    and SS': S' ⊆ S
    and u: setsum (λv. u v *R v) S' = y
```

```
    unfolding span-explicit by blast
   let ?u = λx. if x ∈ S' then u x else 0
   have setsum (λv. ?u v *R v) S = setsum (λv. u v *R v) S'
    using SS' fS by (auto intro!: setsum.mono-neutral-cong-right)
   then have setsum (λv. ?u v *R v) S = y by (metis u)
   then have y ∈ ?rhs by auto
  }
  moreover
  {
   fix y u
   assume u: setsum (λv. u v *R v) S = y
   then have y ∈ span S using fS unfolding span-explicit by auto
  }
  ultimately show ?thesis by blast
qed
```

This is useful for building a basis step-by-step.

```
lemma independent-insert:
  independent (insert a S) ⟷
    (if a ∈ S then independent S else independent S ∧ a ∉ span S)
  (is ?lhs ⟷ ?rhs)
proof (cases a ∈ S)
  case True
  then show ?thesis
    using insert-absorb[OF True] by simp
next
  case False
  show ?thesis
  proof
    assume i: ?lhs
    then show ?rhs
      using False
      apply simp
      apply (rule conjI)
      apply (rule independent-mono)
      apply assumption
      apply blast
      apply (simp add: dependent-def)
      done
  next
    assume i: ?rhs
    show ?lhs
      using i False
      apply (auto simp add: dependent-def)
      by (metis in-span-insert insert-Diff-if insert-Diff-single insert-absorb)
  qed
qed
```

The degenerate case of the Exchange Lemma.

lemma *spanning-subset-independent*:
assumes BA: $B \subseteq A$
and iA: *independent* A
and AsB: $A \subseteq$ *span* B
shows $A = B$
proof
show $B \subseteq A$ **by** (*rule* BA)

from *span-mono*[*OF* BA] *span-mono*[*OF* AsB]
have sAB: *span* A = *span* B **unfolding** *span-span* **by** *blast*

{
fix x
assume x: $x \in A$
from iA **have** *th0*: $x \notin$ *span* $(A - \{x\})$
unfolding *dependent-def* **using** x **by** *blast*
from x **have** xsA: $x \in$ *span* A
by (*blast intro*: *span-superset*)
have $A - \{x\} \subseteq A$ **by** *blast*
then have *th1*: *span* $(A - \{x\}) \subseteq$ *span* A
by (*metis span-mono*)
{
assume xB: $x \notin B$
from xB BA **have** $B \subseteq A - \{x\}$
by *blast*
then have *span* $B \subseteq$ *span* $(A - \{x\})$
by (*metis span-mono*)
with *th1 th0* sAB **have** $x \notin$ *span* A
by *blast*
with x **have** *False*
by (*metis span-superset*)
}
then have $x \in B$ **by** *blast*
}
then show $A \subseteq B$ **by** *blast*
qed

The general case of the Exchange Lemma, the key to what follows.

lemma *exchange-lemma*:
assumes f:*finite* t
and i: *independent* s
and sp: $s \subseteq$ *span* t
shows $\exists t'.$ *card* $t' =$ *card* $t \wedge$ *finite* $t' \wedge s \subseteq t' \wedge t' \subseteq s \cup t \wedge s \subseteq$ *span* t'
using f i sp
proof (*induct card* $(t - s)$ *arbitrary*: s t *rule*: *less-induct*)
case *less*
note ft = ‹*finite* t› **and** s = ‹*independent* s› **and** sp = ‹$s \subseteq$ *span* t›
let *?P* = $\lambda t'.$ *card* $t' =$ *card* $t \wedge$ *finite* $t' \wedge s \subseteq t' \wedge t' \subseteq s \cup t \wedge s \subseteq$ *span* t'
let *?ths* = $\exists t'.$ *?P* t'

```
 {
  assume s ⊆ t
  then have ?ths
   by (metis ft Un-commute sp sup-ge1)
 }
 moreover
 {
  assume st: t ⊆ s
  from spanning-subset-independent[OF st s sp] st ft span-mono[OF st]
  have ?ths
   by (metis Un-absorb sp)
 }
 moreover
 {
  assume st: ¬ s ⊆ t ¬ t ⊆ s
  from st(2) obtain b where b: b ∈ t b ∉ s
   by blast
  from b have t - {b} - s ⊂ t - s
   by blast
  then have cardlt: card (t - {b} - s) < card (t - s)
   using ft by (auto intro: psubset-card-mono)
  from b ft have ct0: card t ≠ 0
   by auto
  have ?ths
  proof cases
   assume stb: s ⊆ span (t - {b})
   from ft have ftb: finite (t - {b})
    by auto
   from less(1)[OF cardlt ftb s stb]
   obtain u where u: card u = card (t - {b}) s ⊆ u u ⊆ s ∪ (t - {b}) s ⊆
span u
    and fu: finite u by blast
   let ?w = insert b u
   have th0: s ⊆ insert b u
    using u by blast
   from u(3) b have u ⊆ s ∪ t
    by blast
   then have th1: insert b u ⊆ s ∪ t
    using u b by blast
   have bu: b ∉ u
    using b u by blast
   from u(1) ft b have card u = (card t - 1)
    by auto
   then have th2: card (insert b u) = card t
    using card-insert-disjoint[OF fu bu] ct0 by auto
   from u(4) have s ⊆ span u .
   also have ... ⊆ span (insert b u)
    by (rule span-mono) blast
   finally have th3: s ⊆ span (insert b u) .
```

```
    from th0 th1 th2 th3 fu have th: ?P ?w
      by blast
    from th show ?thesis by blast
  next
    assume stb: ¬ s ⊆ span (t − {b})
    from stb obtain a where a: a ∈ s a ∉ span (t − {b})
      by blast
    have ab: a ≠ b
      using a b by blast
    have at: a ∉ t
      using a ab span-superset[of a t− {b}] by auto
    have mlt: card ((insert a (t − {b})) − s) < card (t − s)
      using cardlt ft a b by auto
    have ft': finite (insert a (t − {b}))
      using ft by auto
    {
      fix x
      assume xs: x ∈ s
      have t: t ⊆ insert b (insert a (t − {b}))
        using b by auto
      from b(1) have b ∈ span t
        by (simp add: span-superset)
      have bs: b ∈ span (insert a (t − {b}))
        apply (rule in-span-delete)
        using a sp unfolding subset-eq
        apply auto
        done
      from xs sp have x ∈ span t
        by blast
      with span-mono[OF t] have x: x ∈ span (insert b (insert a (t − {b}))) ..
      from span-trans[OF bs x] have x ∈ span (insert a (t − {b})) .
    }
    then have sp': s ⊆ span (insert a (t − {b}))
      by blast
    from less(1)[OF mlt ft' s sp'] obtain u where u:
      card u = card (insert a (t − {b}))
      finite u s ⊆ u u ⊆ s ∪ insert a (t − {b})
      s ⊆ span u by blast
    from u a b ft at ct0 have ?P u
      by auto
    then show ?thesis by blast
  qed
 }
 ultimately show ?ths by blast
qed
```

This implies corresponding size bounds.

```
lemma independent-span-bound:
 assumes f: finite t
```

and *i*: *independent s*
and *sp*: $s \subseteq$ *span t*
shows *finite s* $\wedge$ *card s* $\leq$ *card t*
by (*metis exchange-lemma*[*OF f i sp*] *finite-subset card-mono*)

lemma *finite-Atleast-Atmost-nat*[*simp*]: *finite* {*f x* |*x*. *x*$\in$ (*UNIV*::'*a*::*finite set*)}
proof −
have *eq*: {*f x* |*x*. *x*$\in$ *UNIV*} = *f* ' *UNIV*
by *auto*
show *?thesis* **unfolding** *eq*
apply (*rule finite-imageI*)
apply (*rule finite*)
done
qed

13.9 Euclidean Spaces as Typeclass

lemma *independent-Basis*: *independent Basis*
unfolding *dependent-def*
apply (*subst span-finite*)
apply *simp*
apply *clarify*
apply (*drule-tac f*=*inner a* **in** *arg-cong*)
apply (*simp add*: *inner-Basis inner-setsum-right eq-commute*)
done

lemma *span-Basis* [*simp*]: *span Basis* = *UNIV*
unfolding *span-finite* [*OF finite-Basis*]
by (*fast intro*: *euclidean-representation*)

lemma *in-span-Basis*: *x* $\in$ *span Basis*
unfolding *span-Basis* **..**

lemma *Basis-le-norm*: *b* $\in$ *Basis* $\Longrightarrow$ |*x* $\cdot$ *b*| $\leq$ *norm x*
by (*rule order-trans* [*OF Cauchy-Schwarz-ineq2*]) *simp*

lemma *norm-bound-Basis-le*: *b* $\in$ *Basis* $\Longrightarrow$ *norm x* $\leq$ *e* $\Longrightarrow$ |*x* $\cdot$ *b*| $\leq$ *e*
by (*metis Basis-le-norm order-trans*)

lemma *norm-bound-Basis-lt*: *b* $\in$ *Basis* $\Longrightarrow$ *norm x* $<$ *e* $\Longrightarrow$ |*x* $\cdot$ *b*| $<$ *e*
by (*metis Basis-le-norm le-less-trans*)

lemma *norm-le-l1*: *norm x* $\leq$ ($\sum$ *b*$\in$*Basis*. |*x* $\cdot$ *b*|)
apply (*subst euclidean-representation*[*of x*, *symmetric*])
apply (*rule order-trans*[*OF norm-setsum*])
apply (*auto intro*!: *setsum-mono*)
done

lemma *setsum-norm-allsubsets-bound*:

fixes $f :: 'a \Rightarrow 'n$::*euclidean-space*
assumes *fP*: *finite P*
and *fPs*: $\bigwedge Q.\ Q \subseteq P \Longrightarrow norm\ (setsum\ f\ Q) \leq e$
shows $(\sum x{\in}P.\ norm\ (f\ x)) \leq 2 * real\ DIM('n) * e$
proof −
have $(\sum x{\in}P.\ norm\ (f\ x)) \leq (\sum x{\in}P.\ \sum b{\in}Basis.\ |f\ x \cdot b|)$
by (*rule setsum-mono*) (*rule norm-le-l1*)
also have $(\sum x{\in}P.\ \sum b{\in}Basis.\ |f\ x \cdot b|) = (\sum b{\in}Basis.\ \sum x{\in}P.\ |f\ x \cdot b|)$
by (*rule setsum.commute*)
also have $\ldots \leq of\text{-}nat\ (card\ (Basis :: 'n\ set)) * (2 * e)$
proof (*rule setsum-bounded-above*)
fix $i :: 'n$
assume *i*: $i \in Basis$
have $norm\ (\sum x{\in}P.\ |f\ x \cdot i|) \leq$
$norm\ ((\sum x{\in}P \cap -\ \{x.\ f\ x \cdot i < 0\}.\ f\ x) \cdot i) + norm\ ((\sum x{\in}P \cap \{x.\ f\ x \cdot i < 0\}.\ f\ x) \cdot i)$
by (*simp add*: *abs-real-def setsum.If-cases*[*OF fP*] *setsum-negf norm-triangle-ineq4 inner-setsum-left*
del: *real-norm-def*)
also have $\ldots \leq e + e$
unfolding *real-norm-def*
by (*intro add-mono norm-bound-Basis-le i fPs*) *auto*
finally show $(\sum x{\in}P.\ |f\ x \cdot i|) \leq 2{*}e$ **by** *simp*
qed
also have $\ldots = 2 * real\ DIM('n) * e$ **by** *simp*
finally show *?thesis* **.**
qed

13.10 Linearity and Bilinearity continued

lemma *linear-bounded*:
fixes $f :: 'a$::*euclidean-space* $\Rightarrow 'b$::*real-normed-vector*
assumes *lf*: *linear f*
shows $\exists B.\ \forall x.\ norm\ (f\ x) \leq B * norm\ x$
proof
let *?B* $= \sum b{\in}Basis.\ norm\ (f\ b)$
show $\forall x.\ norm\ (f\ x) \leq$ *?B* $* norm\ x$
proof
fix $x :: 'a$
let *?g* $= \lambda b.\ (x \cdot b) *_R f\ b$
have $norm\ (f\ x) = norm\ (f\ (\sum b{\in}Basis.\ (x \cdot b) *_R b))$
unfolding *euclidean-representation* **..**
also have $\ldots = norm\ (setsum$ *?g Basis*$)$
by (*simp add*: *linear-setsum* [*OF lf*] *linear-cmul* [*OF lf*])
finally have *th0*: $norm\ (f\ x) = norm\ (setsum$ *?g Basis*$)$ **.**
have *th*: $\forall b{\in}Basis.\ norm\ (?g\ b) \leq norm\ (f\ b) * norm\ x$
proof
fix $i :: 'a$
assume *i*: $i \in Basis$

```
      from Basis-le-norm[OF i, of x]
      show norm (?g i) ≤ norm (f i) * norm x
        unfolding norm-scaleR
        apply (subst mult.commute)
        apply (rule mult-mono)
        apply (auto simp add: field-simps)
        done
    qed
    from setsum-norm-le[of - ?g, OF th]
    show norm (f x) ≤ ?B * norm x
      unfolding th0 setsum-left-distrib by metis
  qed
qed

lemma linear-conv-bounded-linear:
  fixes f :: 'a::euclidean-space ⇒ 'b::real-normed-vector
  shows linear f ⟷ bounded-linear f
proof
  assume linear f
  then interpret f: linear f .
  show bounded-linear f
  proof
    have ∃ B. ∀ x. norm (f x) ≤ B * norm x
      using ‹linear f› by (rule linear-bounded)
    then show ∃ K. ∀ x. norm (f x) ≤ norm x * K
      by (simp add: mult.commute)
  qed
next
  assume bounded-linear f
  then interpret f: bounded-linear f .
  show linear f ..
qed

lemmas linear-linear = linear-conv-bounded-linear[symmetric]

lemma linear-bounded-pos:
  fixes f :: 'a::euclidean-space ⇒ 'b::real-normed-vector
  assumes lf: linear f
  shows ∃ B > 0. ∀ x. norm (f x) ≤ B * norm x
proof −
  have ∃ B > 0. ∀ x. norm (f x) ≤ norm x * B
    using lf unfolding linear-conv-bounded-linear
    by (rule bounded-linear.pos-bounded)
  then show ?thesis
    by (simp only: mult.commute)
qed

lemma bounded-linearI':
  fixes f ::'a::euclidean-space ⇒ 'b::real-normed-vector
```

assumes $\bigwedge x\ y.\ f\ (x + y) = f\ x + f\ y$
and $\bigwedge c\ x.\ f\ (c *_R x) = c *_R f\ x$
shows *bounded-linear f*
unfolding *linear-conv-bounded-linear*[*symmetric*]
by (*rule linearI*[*OF assms*])

lemma *bilinear-bounded*:
fixes *h* :: *'m*::*euclidean-space* ⇒ *'n*::*euclidean-space* ⇒ *'k*::*real-normed-vector*
assumes *bh*: *bilinear h*
shows $\exists B.\ \forall x\ y.\ norm\ (h\ x\ y) \leq B * norm\ x * norm\ y$
proof (*clarify intro*!: *exI*[*of* - $\sum i{\in}Basis.\ \sum j{\in}Basis.\ norm\ (h\ i\ j)$])
fix *x* :: *'m*
fix *y* :: *'n*
have *norm* (*h x y*) = *norm* (*h* (*setsum* (λ*i*. (*x* · *i*) *$*_R$ *i*) *Basis*) (*setsum* (λ*i*. (*y* · *i*) *$*_R$ *i*) *Basis*))
apply (*subst euclidean-representation*[**where** *'a*=*'m*])
apply (*subst euclidean-representation*[**where** *'a*=*'n*])
apply *rule*
done
also have ... = *norm* (*setsum* (λ (*i*,*j*). *h* ((*x* · *i*) *$*_R$ *i*) ((*y* · *j*) *$*_R$ *j*)) (*Basis* × *Basis*))
unfolding *bilinear-setsum*[*OF bh finite-Basis finite-Basis*] **..**
finally have *th*: *norm* (*h x y*) = ... **.**
show *norm* (*h x y*) ≤ ($\sum i{\in}Basis.\ \sum j{\in}Basis.\ norm\ (h\ i\ j)$) * *norm x* * *norm y*
apply (*auto simp add*: *setsum-left-distrib th setsum.cartesian-product*)
apply (*rule setsum-norm-le*)
apply *simp*
apply (*auto simp add*: *bilinear-rmul*[*OF bh*] *bilinear-lmul*[*OF bh*] *field-simps simp del*: *scaleR-scaleR*)
apply (*rule mult-mono*)
apply (*auto simp add*: *zero-le-mult-iff Basis-le-norm*)
apply (*rule mult-mono*)
apply (*auto simp add*: *zero-le-mult-iff Basis-le-norm*)
done
qed

lemma *bilinear-conv-bounded-bilinear*:
fixes *h* :: *'a*::*euclidean-space* ⇒ *'b*::*euclidean-space* ⇒ *'c*::*real-normed-vector*
shows *bilinear h* ⟷ *bounded-bilinear h*
proof
assume *bilinear h*
show *bounded-bilinear h*
proof
fix *x y z*
show *h* (*x* + *y*) *z* = *h x z* + *h y z*
using ‹*bilinear h*› **unfolding** *bilinear-def linear-iff* **by** *simp*
next
fix *x y z*

```
    show h x (y + z) = h x y + h x z
      using ‹bilinear h› unfolding bilinear-def linear-iff by simp
  next
    fix r x y
    show h (scaleR r x) y = scaleR r (h x y)
      using ‹bilinear h› unfolding bilinear-def linear-iff
      by simp
  next
    fix r x y
    show h x (scaleR r y) = scaleR r (h x y)
      using ‹bilinear h› unfolding bilinear-def linear-iff
      by simp
  next
    have ∃ B. ∀ x y. norm (h x y) ≤ B * norm x * norm y
      using ‹bilinear h› by (rule bilinear-bounded)
    then show ∃ K. ∀ x y. norm (h x y) ≤ norm x * norm y * K
      by (simp add: ac-simps)
  qed
next
  assume bounded-bilinear h
  then interpret h: bounded-bilinear h .
  show bilinear h
    unfolding bilinear-def linear-conv-bounded-linear
    using h.bounded-linear-left h.bounded-linear-right by simp
qed

lemma bilinear-bounded-pos:
  fixes h :: 'a::euclidean-space ⇒ 'b::euclidean-space ⇒ 'c::real-normed-vector
  assumes bh: bilinear h
  shows ∃ B > 0. ∀ x y. norm (h x y) ≤ B * norm x * norm y
proof −
  have ∃ B > 0. ∀ x y. norm (h x y) ≤ norm x * norm y * B
    using bh [unfolded bilinear-conv-bounded-bilinear]
    by (rule bounded-bilinear.pos-bounded)
  then show ?thesis
    by (simp only: ac-simps)
qed
```

13.11 We continue.

```
lemma independent-bound:
  fixes S :: 'a::euclidean-space set
  shows independent S ⟹ finite S ∧ card S ≤ DIM('a)
  using independent-span-bound[OF finite-Basis, of S] by auto

corollary
  fixes S :: 'a::euclidean-space set
  assumes independent S
 shows independent-imp-finite: finite S and independent-card-le:card S ≤ DIM('a)
```

using *assms independent-bound* **by** *auto*

lemma *dependent-biggerset*:
fixes $S :: 'a$::*euclidean-space set*
shows (*finite* $S \Longrightarrow$ *card* $S >$ *DIM*$('a)) \Longrightarrow$ *dependent* S
by (*metis independent-bound not-less*)

Hence we can create a maximal independent subset.

lemma *maximal-independent-subset-extend*:
fixes $S :: 'a$::*euclidean-space set*
assumes *sv*: $S \subseteq V$
and *iS*: *independent* S
shows $\exists B.\ S \subseteq B \wedge B \subseteq V \wedge$ *independent* $B \wedge V \subseteq$ *span* B
using *sv iS*
proof (*induct DIM*$('a) -$ *card* S *arbitrary*: S *rule*: *less-induct*)
case *less*
note *sv* = ‹$S \subseteq V$› **and** *i* = ‹*independent* S›
let *?P* = $\lambda B.\ S \subseteq B \wedge B \subseteq V \wedge$ *independent* $B \wedge V \subseteq$ *span* B
let *?ths* = $\exists x.$ *?P* x
let *?d* = *DIM*$('a)$
show *?ths*
proof (*cases* $V \subseteq$ *span* S)
case *True*
then show *?thesis*
using *sv i* **by** *blast*
next
case *False*
then obtain *a* **where** *a*: $a \in V$ $a \notin$ *span* S
by *blast*
from *a* **have** *aS*: $a \notin S$
by (*auto simp add*: *span-superset*)
have *th0*: *insert a S* $\subseteq V$
using *a sv* **by** *blast*
from *independent-insert*[*of a S*] *i a*
have *th1*: *independent* (*insert a S*)
by *auto*
have *mlt*: *?d* $-$ *card* (*insert a S*) $<$ *?d* $-$ *card S*
using *aS a independent-bound*[*OF th1*] **by** *auto*

from *less*(*1*)[*OF mlt th0 th1*]
obtain *B* **where** *B*: *insert a S* $\subseteq B$ $B \subseteq V$ *independent B* $V \subseteq$ *span B*
by *blast*
from *B* **have** *?P B* **by** *auto*
then show *?thesis* **by** *blast*
qed
qed

lemma *maximal-independent-subset*:
$\exists (B ::$ ($'a$::*euclidean-space*) *set*). $B \subseteq V \wedge$ *independent* $B \wedge V \subseteq$ *span* B

by (*metis maximal-independent-subset-extend*[*of* {}:: ('*a*::*euclidean-space*) *set*] *empty-subsetI independent-empty*)

Notion of dimension.

definition *dim V* = (*SOME n*. ∃ *B*. *B* ⊆ *V* ∧ *independent B* ∧ *V* ⊆ *span B* ∧ *card B* = *n*)

lemma *basis-exists*:
∃ *B*. (*B* :: ('*a*::*euclidean-space*) *set*) ⊆ *V* ∧ *independent B* ∧ *V* ⊆ *span B* ∧ (*card B* = *dim V*)
unfolding *dim-def some-eq-ex*[*of* λ*n*. ∃ *B*. *B* ⊆ *V* ∧ *independent B* ∧ *V* ⊆ *span B* ∧ (*card B* = *n*)]
using *maximal-independent-subset*[*of V*] *independent-bound*
by *auto*

corollary *dim-le-card*:
fixes *s* :: '*a*::*euclidean-space set*
shows *finite s* ⟹ *dim s* ≤ *card s*
by (*metis basis-exists card-mono*)

Consequences of independence or spanning for cardinality.

lemma *independent-card-le-dim*:
fixes *B* :: '*a*::*euclidean-space set*
assumes *B* ⊆ *V*
and *independent B*
shows *card B* ≤ *dim V*
proof −
from *basis-exists*[*of V*] ‹*B* ⊆ *V*›
obtain *B'* **where** *independent B'*
and *B* ⊆ *span B'*
and *card B'* = *dim V*
by *blast*
with *independent-span-bound*[*OF* - ‹*independent B*› ‹*B* ⊆ *span B'*›] *independent-bound*[*of B'*]
show *?thesis* **by** *auto*
qed

lemma *span-card-ge-dim*:
fixes *B* :: '*a*::*euclidean-space set*
shows *B* ⊆ *V* ⟹ *V* ⊆ *span B* ⟹ *finite B* ⟹ *dim V* ≤ *card B*
by (*metis basis-exists*[*of V*] *independent-span-bound subset-trans*)

lemma *basis-card-eq-dim*:
fixes *V* :: '*a*::*euclidean-space set*
shows *B* ⊆ *V* ⟹ *V* ⊆ *span B* ⟹ *independent B* ⟹ *finite B* ∧ *card B* = *dim V*
by (*metis order-eq-iff independent-card-le-dim span-card-ge-dim independent-bound*)

lemma *dim-unique*:

fixes $B :: 'a::euclidean\text{-}space\ set$
shows $B \subseteq V \Longrightarrow V \subseteq span\ B \Longrightarrow independent\ B \Longrightarrow card\ B = n \Longrightarrow dim\ V = n$
by (*metis basis-card-eq-dim*)

More lemmas about dimension.

lemma *dim-UNIV*: $dim\ (UNIV :: 'a::euclidean\text{-}space\ set) = DIM('a)$
using *independent-Basis*
by (*intro dim-unique*[*of Basis*]) *auto*

lemma *dim-subset*:
fixes $S :: 'a::euclidean\text{-}space\ set$
shows $S \subseteq T \Longrightarrow dim\ S \leq dim\ T$
using *basis-exists*[*of T*] *basis-exists*[*of S*]
by (*metis independent-card-le-dim subset-trans*)

lemma *dim-subset-UNIV*:
fixes $S :: 'a::euclidean\text{-}space\ set$
shows $dim\ S \leq DIM('a)$
by (*metis dim-subset subset-UNIV dim-UNIV*)

Converses to those.

lemma *card-ge-dim-independent*:
fixes $B :: 'a::euclidean\text{-}space\ set$
assumes BV: $B \subseteq V$
and iB: *independent B*
and dVB: $dim\ V \leq card\ B$
shows $V \subseteq span\ B$
proof
fix *a*
assume aV: $a \in V$
{
assume aB: $a \notin span\ B$
then have iaB: *independent* (*insert a B*)
using *iB aV BV* **by** (*simp add*: *independent-insert*)
from *aV BV* **have** *th0*: *insert a B* $\subseteq V$
by *blast*
from *aB* **have** $a \notin B$
by (*auto simp add*: *span-superset*)
with *independent-card-le-dim*[*OF th0 iaB*] *dVB independent-bound*[*OF iB*]
have *False* **by** *auto*
}
then show $a \in span\ B$ **by** *blast*
qed

lemma *card-le-dim-spanning*:
assumes BV: $(B:: ('a::euclidean\text{-}space)\ set) \subseteq V$
and VB: $V \subseteq span\ B$
and fB: *finite B*

```
    and dVB: dim V ≥ card B
  shows independent B
proof −
  {
    fix a
    assume a: a ∈ B a ∈ span (B − {a})
    from a fB have c0: card B ≠ 0
      by auto
    from a fB have cb: card (B − {a}) = card B − 1
      by auto
    from BV a have th0: B − {a} ⊆ V
      by blast
    {
      fix x
      assume x: x ∈ V
      from a have eq: insert a (B − {a}) = B
        by blast
      from x VB have x': x ∈ span B
        by blast
      from span-trans[OF a(2), unfolded eq, OF x']
      have x ∈ span (B − {a}) .
    }
    then have th1: V ⊆ span (B − {a})
      by blast
    have th2: finite (B − {a})
      using fB by auto
    from span-card-ge-dim[OF th0 th1 th2]
    have c: dim V ≤ card (B − {a}) .
    from c c0 dVB cb have False by simp
  }
  then show ?thesis
    unfolding dependent-def by blast
qed

lemma card-eq-dim:
  fixes B :: 'a::euclidean-space set
  shows B ⊆ V ⟹ card B = dim V ⟹ finite B ⟹ independent B ⟷ V ⊆
span B
  by (metis order-eq-iff card-le-dim-spanning card-ge-dim-independent)
```

More general size bound lemmas.

```
lemma independent-bound-general:
  fixes S :: 'a::euclidean-space set
  shows independent S ⟹ finite S ∧ card S ≤ dim S
  by (metis independent-card-le-dim independent-bound subset-refl)

lemma dependent-biggerset-general:
  fixes S :: 'a::euclidean-space set
  shows (finite S ⟹ card S > dim S) ⟹ dependent S
```

using *independent-bound-general*[*of S*] **by** (*metis linorder-not-le*)

lemma *dim-span* [*simp*]:
fixes *S* :: *'a*::*euclidean-space set*
shows *dim* (*span S*) = *dim S*
proof −
have *th0*: *dim S* ≤ *dim* (*span S*)
by (*auto simp add*: *subset-eq intro*: *dim-subset span-superset*)
from *basis-exists*[*of S*]
obtain *B* **where** *B*: *B* ⊆ *S independent B S* ⊆ *span B card B* = *dim S*
by *blast*
from *B* **have** *fB*: *finite B card B* = *dim S*
using *independent-bound* **by** *blast*+
have *bSS*: *B* ⊆ *span S*
using *B*(*1*) **by** (*metis subset-eq span-inc*)
have *sssB*: *span S* ⊆ *span B*
using *span-mono*[*OF B*(*3*)] **by** (*simp add*: *span-span*)
from *span-card-ge-dim*[*OF bSS sssB fB*(*1*)] *th0* **show** *?thesis*
using *fB*(*2*) **by** *arith*
qed

lemma *subset-le-dim*:
fixes *S* :: *'a*::*euclidean-space set*
shows *S* ⊆ *span T* ⟹ *dim S* ≤ *dim T*
by (*metis dim-span dim-subset*)

lemma *span-eq-dim*:
fixes *S* :: *'a*::*euclidean-space set*
shows *span S* = *span T* ⟹ *dim S* = *dim T*
by (*metis dim-span*)

lemma *spans-image*:
assumes *lf*: *linear f*
and *VB*: *V* ⊆ *span B*
shows *f* ‘ *V* ⊆ *span* (*f* ‘ *B*)
unfolding *span-linear-image*[*OF lf*] **by** (*metis VB image-mono*)

lemma *dim-image-le*:
fixes *f* :: *'a*::*euclidean-space* ⇒ *'b*::*euclidean-space*
assumes *lf*: *linear f*
shows *dim* (*f* ‘ *S*) ≤ *dim* (*S*)
proof −
from *basis-exists*[*of S*] **obtain** *B* **where**
B: *B* ⊆ *S independent B S* ⊆ *span B card B* = *dim S* **by** *blast*
from *B* **have** *fB*: *finite B card B* = *dim S*
using *independent-bound* **by** *blast*+
have *dim* (*f* ‘ *S*) ≤ *card* (*f* ‘ *B*)
apply (*rule span-card-ge-dim*)
using *lf B fB*

```
    apply (auto simp add: span-linear-image spans-image subset-image-iff)
    done
  also have ... ≤ dim S
    using card-image-le[OF fB(1)] fB by simp
  finally show ?thesis .
qed
```

Relation between bases and injectivity/surjectivity of map.

```
lemma spanning-surjective-image:
  assumes us: UNIV ⊆ span S
    and lf: linear f
    and sf: surj f
  shows UNIV ⊆ span (f ‘ S)
proof −
  have UNIV ⊆ f ‘ UNIV
    using sf by (auto simp add: surj-def)
  also have ... ⊆ span (f ‘ S)
    using spans-image[OF lf us] .
  finally show ?thesis .
qed

lemma independent-injective-image:
  assumes iS: independent S
    and lf: linear f
    and fi: inj f
  shows independent (f ‘ S)
proof −
  {
    fix a
    assume a: a ∈ S f a ∈ span (f ‘ S − {f a})
    have eq: f ‘ S − {f a} = f ‘ (S − {a})
      using fi by (auto simp add: inj-on-def)
    from a have f a ∈ f ‘ span (S − {a})
      unfolding eq span-linear-image[OF lf, of S − {a}] by blast
    then have a ∈ span (S − {a})
      using fi by (auto simp add: inj-on-def)
    with a(1) iS have False
      by (simp add: dependent-def)
  }
  then show ?thesis
    unfolding dependent-def by blast
qed
```

Picking an orthogonal replacement for a spanning set.

```
lemma vector-sub-project-orthogonal:
  fixes b x :: 'a::euclidean-space
  shows b • (x − ((b • x) / (b • b)) *R b) = 0
  unfolding inner-simps by auto
```

lemma *pairwise-orthogonal-insert*:
assumes *pairwise orthogonal S*
and $\bigwedge y.\ y \in S \Longrightarrow$ *orthogonal x y*
shows *pairwise orthogonal* (*insert x S*)
using *assms* **unfolding** *pairwise-def*
by (*auto simp add*: *orthogonal-commute*)

lemma *basis-orthogonal*:
fixes *B* :: *'a*::*real-inner set*
assumes *fB*: *finite B*
shows $\exists\, C.$ *finite C* $\wedge$ *card C* $\leq$ *card B* $\wedge$ *span C* = *span B* $\wedge$ *pairwise orthogonal C*
(**is** $\exists\, C.$ *?P B C*)
using *fB*
proof (*induct rule*: *finite-induct*)
case *empty*
then show *?case*
apply (*rule exI*[**where** *x*={}])
apply (*auto simp add*: *pairwise-def*)
done
next
case (*insert a B*)
note *fB* = ‹*finite B*› **and** *aB* = ‹*a* $\notin$ *B*›
from ‹$\exists\, C.$ *finite C* $\wedge$ *card C* $\leq$ *card B* $\wedge$ *span C* = *span B* $\wedge$ *pairwise orthogonal C*›
obtain *C* **where** *C*: *finite C card C* $\leq$ *card B*
span C = *span B pairwise orthogonal C* **by** *blast*
let *?a* = *a* − *setsum* ($\lambda x.$ (*x* · *a* / (*x* · *x*)) $*_R$ *x*) *C*
let *?C* = *insert ?a C*
from *C*(*1*) **have** *fC*: *finite ?C*
by *simp*
from *fB aB C*(*1*,*2*) **have** *cC*: *card ?C* $\leq$ *card* (*insert a B*)
by (*simp add*: *card-insert-if*)
{
fix *x k*
have *th0*: $\bigwedge$(*a*::*'a*) *b c*. *a* − (*b* − *c*) = *c* + (*a* − *b*)
by (*simp add*: *field-simps*)
have *x* − *k* $*_R$ (*a* − ($\sum$ *x*∈*C*. (*x* · *a* / (*x* · *x*)) $*_R$ *x*)) ∈ *span C* ⟷ *x* − *k* $*_R$ *a* ∈ *span C*
apply (*simp only*: *scaleR-right-diff-distrib th0*)
apply (*rule span-add-eq*)
apply (*rule span-mul*)
apply (*rule span-setsum*)
apply *clarify*
apply (*rule span-mul*)
apply (*rule span-superset*)
apply *assumption*
done
}

```
  then have SC: span ?C = span (insert a B)
    unfolding set-eq-iff span-breakdown-eq C(3)[symmetric] by auto
  {
    fix y
    assume yC: y ∈ C
    then have Cy: C = insert y (C − {y})
      by blast
    have fth: finite (C − {y})
      using C by simp
    have orthogonal ?a y
      unfolding orthogonal-def
      unfolding inner-diff inner-setsum-left right-minus-eq
      unfolding setsum.remove [OF ‹finite C› ‹y ∈ C›]
      apply (clarsimp simp add: inner-commute[of y a])
      apply (rule setsum.neutral)
      apply clarsimp
      apply (rule C(4)[unfolded pairwise-def orthogonal-def, rule-format])
      using ‹y ∈ C› by auto
  }
  with ‹pairwise orthogonal C› have CPO: pairwise orthogonal ?C
    by (rule pairwise-orthogonal-insert)
  from fC cC SC CPO have ?P (insert a B) ?C
    by blast
  then show ?case by blast
qed

lemma orthogonal-basis-exists:
  fixes V :: ('a::euclidean-space) set
  shows ∃ B. independent B ∧ B ⊆ span V ∧ V ⊆ span B ∧ (card B = dim V)
∧ pairwise orthogonal B
proof −
  from basis-exists[of V] obtain B where
    B: B ⊆ V independent B V ⊆ span B card B = dim V
    by blast
  from B have fB: finite B card B = dim V
    using independent-bound by auto
  from basis-orthogonal[OF fB(1)] obtain C where
    C: finite C card C ≤ card B span C = span B pairwise orthogonal C
    by blast
  from C B have CSV: C ⊆ span V
    by (metis span-inc span-mono subset-trans)
  from span-mono[OF B(3)] C have SVC: span V ⊆ span C
    by (simp add: span-span)
  from card-le-dim-spanning[OF CSV SVC C(1)] C(2,3) fB
  have iC: independent C
    by (simp add: dim-span)
  from C fB have card C ≤ dim V
    by simp
  moreover have dim V ≤ card C
```

```
    using span-card-ge-dim[OF CSV SVC C(1)]
    by (simp add: dim-span)
  ultimately have CdV: card C = dim V
    using C(1) by simp
  from C B CSV CdV iC show ?thesis
    by auto
qed

lemma span-eq: span S = span T ⟷ S ⊆ span T ∧ T ⊆ span S
  using span-inc[unfolded subset-eq] using span-mono[of T span S] span-mono[of
S span T]
  by (auto simp add: span-span)
```

Low-dimensional subset is in a hyperplane (weak orthogonal complement).

```
lemma span-not-univ-orthogonal:
  fixes S :: 'a::euclidean-space set
  assumes sU: span S ≠ UNIV
  shows ∃ a::'a. a ≠ 0 ∧ (∀ x ∈ span S. a · x = 0)
proof −
  from sU obtain a where a: a ∉ span S
    by blast
  from orthogonal-basis-exists obtain B where
    B: independent B B ⊆ span S S ⊆ span B card B = dim S pairwise orthogonal
B
    by blast
  from B have fB: finite B card B = dim S
    using independent-bound by auto
  from span-mono[OF B(2)] span-mono[OF B(3)]
  have sSB: span S = span B
    by (simp add: span-span)
  let ?a = a − setsum (λb. (a · b / (b · b)) *R b) B
  have setsum (λb. (a · b / (b · b)) *R b) B ∈ span S
    unfolding sSB
    apply (rule span-setsum)
    apply clarsimp
    apply (rule span-mul)
    apply (rule span-superset)
    apply assumption
    done
  with a have a0:?a ≠ 0
    by auto
  have ∀ x∈span B. ?a · x = 0
  proof (rule span-induct')
    show subspace {x. ?a · x = 0}
      by (auto simp add: subspace-def inner-add)
  next
    {
      fix x
      assume x: x ∈ B
```

```
    from x have B': B = insert x (B - {x})
      by blast
    have fth: finite (B - {x})
      using fB by simp
    have ?a · x = 0
      apply (subst B')
      using fB fth
      unfolding setsum-clauses(2)[OF fth]
      apply simp unfolding inner-simps
      apply (clarsimp simp add: inner-add inner-setsum-left)
      apply (rule setsum.neutral, rule ballI)
      unfolding inner-commute
      apply (auto simp add: x field-simps
        intro: B(5)[unfolded pairwise-def orthogonal-def, rule-format])
      done
  }
  then show ∀ x ∈ B. ?a · x = 0
    by blast
 qed
 with a0 show ?thesis
  unfolding sSB by (auto intro: exI[where x=?a])
qed

lemma span-not-univ-subset-hyperplane:
 fixes S :: 'a::euclidean-space set
 assumes SU: span S ≠ UNIV
 shows ∃ a. a ≠0 ∧ span S ⊆ {x. a · x = 0}
 using span-not-univ-orthogonal[OF SU] by auto

lemma lowdim-subset-hyperplane:
 fixes S :: 'a::euclidean-space set
 assumes d: dim S < DIM('a)
 shows ∃ a::'a. a ≠ 0 ∧ span S ⊆ {x. a · x = 0}
proof -
 {
  assume span S = UNIV
  then have dim (span S) = dim (UNIV :: ('a) set)
    by simp
  then have dim S = DIM('a)
    by (simp add: dim-span dim-UNIV)
  with d have False by arith
 }
 then have th: span S ≠ UNIV
  by blast
 from span-not-univ-subset-hyperplane[OF th] show ?thesis .
qed
```

We can extend a linear basis-basis injection to the whole set.

```
lemma linear-indep-image-lemma:
```

```
  assumes lf: linear f
    and fB: finite B
    and ifB: independent (f ‘ B)
    and fi: inj-on f B
    and xsB: x ∈ span B
    and fx: f x = 0
  shows x = 0
  using fB ifB fi xsB fx
proof (induct arbitrary: x rule: finite-induct[OF fB])
  case 1
  then show ?case by auto
next
  case (2 a b x)
  have fb: finite b using 2.prems by simp
  have th0: f ‘ b ⊆ f ‘ (insert a b)
    apply (rule image-mono)
    apply blast
    done
  from independent-mono[ OF 2.prems(2) th0]
  have ifb: independent (f ‘ b) .
  have fib: inj-on f b
    apply (rule subset-inj-on [OF 2.prems(3)])
    apply blast
    done
  from span-breakdown[of a insert a b, simplified, OF 2.prems(4)]
  obtain k where k: x − k∗R a ∈ span (b − {a})
    by blast
  have f (x − k∗R a) ∈ span (f ‘ b)
    unfolding span-linear-image[OF lf]
    apply (rule imageI)
    using k span-mono[of b − {a} b]
    apply blast
    done
  then have f x − k∗R f a ∈ span (f ‘ b)
    by (simp add: linear-sub[OF lf] linear-cmul[OF lf])
  then have th: −k ∗R f a ∈ span (f ‘ b)
    using 2.prems(5) by simp
  have xsb: x ∈ span b
  proof (cases k = 0)
    case True
    with k have x ∈ span (b − {a}) by simp
    then show ?thesis using span-mono[of b − {a} b]
      by blast
  next
    case False
    with span-mul[OF th, of − 1/ k]
    have th1: f a ∈ span (f ‘ b)
      by auto
    from inj-on-image-set-diff[OF 2.prems(3), of insert a b {a}, symmetric]
```

```
    have tha: f ‘ insert a b − f ‘ {a} = f ‘ (insert a b − {a}) by blast
    from 2.prems(2) [unfolded dependent-def bex-simps(8), rule-format, of f a]
    have f a ∉ span (f ‘ b) using tha
      using 2.hyps(2)
      2.prems(3) by auto
    with th1 have False by blast
    then show ?thesis by blast
  qed
  from 2.hyps(3)[OF fb ifb fib xsb 2.prems(5)] show x = 0 .
qed
```

We can extend a linear mapping from basis.

```
lemma linear-independent-extend-lemma:
  fixes f :: 'a::real-vector ⇒ 'b::real-vector
  assumes fi: finite B
    and ib: independent B
  shows ∃ g.
    (∀ x∈ span B. ∀ y∈ span B. g (x + y) = g x + g y) ∧
    (∀ x∈ span B. ∀ c. g (c∗R x) = c ∗R g x) ∧
    (∀ x∈ B. g x = f x)
  using ib fi
proof (induct rule: finite-induct[OF fi])
  case 1
  then show ?case by auto
next
  case (2 a b)
  from 2.prems 2.hyps have ibf: independent b finite b
    by (simp-all add: independent-insert)
  from 2.hyps(3)[OF ibf] obtain g where
    g: ∀ x∈span b. ∀ y∈span b. g (x + y) = g x + g y
    ∀ x∈span b. ∀ c. g (c ∗R x) = c ∗R g x ∀ x∈b. g x = f x by blast
  let ?h = λz. SOME k. (z − k ∗R a) ∈ span b
  {
    fix z
    assume z: z ∈ span (insert a b)
    have th0: z − ?h z ∗R a ∈ span b
      apply (rule someI-ex)
      unfolding span-breakdown-eq[symmetric]
      apply (rule z)
      done
    {
      fix k
      assume k: z − k ∗R a ∈ span b
      have eq: z − ?h z ∗R a − (z − k∗R a) = (k − ?h z) ∗R a
        by (simp add: field-simps scaleR-left-distrib [symmetric])
      from span-sub[OF th0 k] have khz: (k − ?h z) ∗R a ∈ span b
        by (simp add: eq)
      {
        assume k ≠ ?h z
```

```
      then have k0: k - ?h z ≠ 0 by simp
      from k0 span-mul[OF khz, of 1 /(k - ?h z)]
      have a ∈ span b by simp
      with 2.prems(1) 2.hyps(2) have False
        by (auto simp add: dependent-def)
    }
    then have k = ?h z by blast
  }
  with th0 have z - ?h z *R a ∈ span b ∧ (∀k. z - k *R a ∈ span b ⟶ k =
?h z)
    by blast
 }
 note h = this
 let ?g = λz. ?h z *R f a + g (z - ?h z *R a)
 {
  fix x y
  assume x: x ∈ span (insert a b)
    and y: y ∈ span (insert a b)
  have tha: ⋀(x::'a) y a k l. (x + y) - (k + l) *R a = (x - k *R a) + (y - l
*R a)
    by (simp add: algebra-simps)
  have addh: ?h (x + y) = ?h x + ?h y
    apply (rule conjunct2[OF h, rule-format, symmetric])
    apply (rule span-add[OF x y])
    unfolding tha
    apply (metis span-add x y conjunct1[OF h, rule-format])
    done
  have ?g (x + y) = ?g x + ?g y
    unfolding addh tha
    g(1)[rule-format,OF conjunct1[OF h, OF x] conjunct1[OF h, OF y]]
    by (simp add: scaleR-left-distrib)}
 moreover
 {
  fix x :: 'a
  fix c :: real
  assume x: x ∈ span (insert a b)
  have tha: ⋀(x::'a) c k a. c *R x - (c * k) *R a = c *R (x - k *R a)
    by (simp add: algebra-simps)
  have hc: ?h (c *R x) = c * ?h x
    apply (rule conjunct2[OF h, rule-format, symmetric])
    apply (metis span-mul x)
    apply (metis tha span-mul x conjunct1[OF h])
    done
  have ?g (c *R x) = c*R ?g x
    unfolding hc tha g(2)[rule-format, OF conjunct1[OF h, OF x]]
    by (simp add: algebra-simps)
 }
 moreover
 {
```

```
    fix x
    assume x: x ∈ insert a b
    {
      assume xa: x = a
      have ha1: 1 = ?h a
        apply (rule conjunct2[OF h, rule-format])
        apply (metis span-superset insertI1)
        using conjunct1[OF h, OF span-superset, OF insertI1]
        apply (auto simp add: span-0)
        done
      from xa ha1[symmetric] have ?g x = f x
        apply simp
        using g(2)[rule-format, OF span-0, of 0]
        apply simp
        done
    }
    moreover
    {
      assume xb: x ∈ b
      have h0: 0 = ?h x
        apply (rule conjunct2[OF h, rule-format])
        apply (metis  span-superset x)
        apply simp
        apply (metis span-superset xb)
        done
      have ?g x = f x
        by (simp add: h0[symmetric] g(3)[rule-format, OF xb])
    }
    ultimately have ?g x = f x
      using x by blast
  }
  ultimately show ?case
    apply −
    apply (rule exI[where x=?g])
    apply blast
    done
qed

lemma linear-independent-extend:
  fixes B :: 'a::euclidean-space set
  assumes iB: independent B
  shows ∃ g. linear g ∧ (∀ x∈B. g x = f x)
proof −
  from maximal-independent-subset-extend[of B UNIV] iB
  obtain C where C: B ⊆ C independent C ⋀x. x ∈ span C
    by auto

  from C(2) independent-bound[of C] linear-independent-extend-lemma[of C f]
  obtain g where g:
```

```
    (∀ x∈ span C. ∀ y∈ span C. g (x + y) = g x + g y) ∧
     (∀ x∈ span C. ∀ c. g (c∗R x) = c ∗R g x) ∧
     (∀ x∈ C. g x = f x) by blast
  from g show ?thesis
    unfolding linear-iff
    using C
    apply clarsimp
    apply blast
    done
qed
```

Can construct an isomorphism between spaces of same dimension.

```
lemma subspace-isomorphism:
  fixes S :: 'a::euclidean-space set
    and T :: 'b::euclidean-space set
  assumes s: subspace S
    and t: subspace T
    and d: dim S = dim T
  shows ∃f. linear f ∧ f ‘ S = T ∧ inj-on f S
proof −
  from basis-exists[of S] independent-bound
  obtain B where B: B ⊆ S independent B S ⊆ span B card B = dim S and fB:
finite B
    by blast
  from basis-exists[of T] independent-bound
  obtain C where C: C ⊆ T independent C T ⊆ span C card C = dim T and
fC: finite C
    by blast
  from B(4) C(4) card-le-inj[of B C] d
  obtain f where f: f ‘ B ⊆ C inj-on f B using ‹finite B› ‹finite C›
    by auto
  from linear-independent-extend[OF B(2)]
  obtain g where g: linear g ∀ x∈ B. g x = f x
    by blast
  from inj-on-iff-eq-card[OF fB, of f] f(2) have card (f ‘ B) = card B
    by simp
  with B(4) C(4) have ceq: card (f ‘ B) = card C
    using d by simp
  have g ‘ B = f ‘ B
    using g(2) by (auto simp add: image-iff)
  also have ... = C using card-subset-eq[OF fC f(1) ceq] .
  finally have gBC: g ‘ B = C .
  have gi: inj-on g B
    using f(2) g(2) by (auto simp add: inj-on-def)
  note g0 = linear-indep-image-lemma[OF g(1) fB, unfolded gBC, OF C(2) gi]
  {
    fix x y
    assume x: x ∈ S and y: y ∈ S and gxy: g x = g y
    from B(3) x y have x′: x ∈ span B and y′: y ∈ span B
```

```
    by blast+
  from gxy have th0: g (x − y) = 0
    by (simp add: linear-sub[OF g(1)])
  have th1: x − y ∈ span B
    using x′ y′ by (metis span-sub)
  have x = y
    using g0[OF th1 th0] by simp
 }
 then have giS: inj-on g S
  unfolding inj-on-def by blast
 from span-subspace[OF B(1,3) s] have g ‘ S = span (g ‘ B)
  by (simp add: span-linear-image[OF g(1)])
 also have ... = span C unfolding gBC ..
 also have ... = T using span-subspace[OF C(1,3) t] .
 finally have gS: g ‘ S = T .
 from g(1) gS giS show ?thesis
  by blast
qed
```

Linear functions are equal on a subspace if they are on a spanning set.

```
lemma subspace-kernel:
 assumes lf: linear f
 shows subspace {x. f x = 0}
 apply (simp add: subspace-def)
 apply (simp add: linear-add[OF lf] linear-cmul[OF lf] linear-0[OF lf])
 done

lemma linear-eq-0-span:
 assumes lf: linear f and f0: ∀ x∈B. f x = 0
 shows ∀ x ∈ span B. f x = 0
 using f0 subspace-kernel[OF lf]
 by (rule span-induct′)

lemma linear-eq-0:
 assumes lf: linear f
   and SB: S ⊆ span B
   and f0: ∀ x∈B. f x = 0
 shows ∀ x ∈ S. f x = 0
 by (metis linear-eq-0-span[OF lf] subset-eq SB f0)

lemma linear-eq:
 assumes lf: linear f
   and lg: linear g
   and S: S ⊆ span B
   and fg: ∀ x∈ B. f x = g x
 shows ∀ x∈ S. f x = g x
proof −
 let ?h = λx. f x − g x
 from fg have fg′: ∀ x∈ B. ?h x = 0 by simp
```

from *linear-eq-0*[*OF linear-compose-sub*[*OF lf lg*] *S fg'*]
show *?thesis* **by** *simp*
qed

lemma *linear-eq-stdbasis*:
fixes *f* :: *'a*::*euclidean-space* ⇒ -
assumes *lf*: *linear f*
and *lg*: *linear g*
and *fg*: ∀ *b*∈*Basis*. *f b* = *g b*
shows *f* = *g*
using *linear-eq*[*OF lf lg*, *of* - *Basis*] *fg* **by** *auto*

Similar results for bilinear functions.

lemma *bilinear-eq*:
assumes *bf*: *bilinear f*
and *bg*: *bilinear g*
and *SB*: *S* ⊆ *span B*
and *TC*: *T* ⊆ *span C*
and *fg*: ∀ *x*∈ *B*. ∀ *y*∈ *C*. *f x y* = *g x y*
shows ∀ *x*∈*S*. ∀ *y*∈*T*. *f x y* = *g x y*
proof −
let *?P* = {*x*. ∀ *y*∈ *span C*. *f x y* = *g x y*}
from *bf bg* **have** *sp*: *subspace ?P*
unfolding *bilinear-def linear-iff subspace-def bf bg*
by (*auto simp add*: *span-0 bilinear-lzero*[*OF bf*] *bilinear-lzero*[*OF bg*] *span-add Ball-def*
intro: *bilinear-ladd*[*OF bf*])

have ∀ *x* ∈ *span B*. ∀ *y*∈ *span C*. *f x y* = *g x y*
apply (*rule span-induct'* [*OF* - *sp*])
apply (*rule ballI*)
apply (*rule span-induct'*)
apply (*simp add*: *fg*)
apply (*auto simp add*: *subspace-def*)
using *bf bg* **unfolding** *bilinear-def linear-iff*
apply (*auto simp add*: *span-0 bilinear-rzero*[*OF bf*] *bilinear-rzero*[*OF bg*] *span-add Ball-def*
intro: *bilinear-ladd*[*OF bf*])
done
then show *?thesis*
using *SB TC* **by** *auto*
qed

lemma *bilinear-eq-stdbasis*:
fixes *f* :: *'a*::*euclidean-space* ⇒ *'b*::*euclidean-space* ⇒ -
assumes *bf*: *bilinear f*
and *bg*: *bilinear g*
and *fg*: ∀ *i*∈*Basis*. ∀ *j*∈*Basis*. *f i j* = *g i j*
shows *f* = *g*

using *bilinear-eq*[*OF bf bg equalityD2*[*OF span-Basis*] *equalityD2*[*OF span-Basis*] *fg*] **by** *blast*

Detailed theorems about left and right invertibility in general case.

lemma *linear-injective-left-inverse*:
fixes $f :: 'a$::*euclidean-space* $\Rightarrow$ $'b$::*euclidean-space*
assumes *lf*: *linear f*
and *fi*: *inj f*
shows $\exists g.$ *linear* $g \wedge g \circ f = id$
proof −
from *linear-independent-extend*[*OF independent-injective-image*, *OF independent-Basis*, *OF lf fi*]
obtain $h :: 'b \Rightarrow 'a$ **where** *h*: *linear h* $\forall x \in f$ ' *Basis*. *h x* = *inv f x*
by *blast*
from *h*(*2*) **have** *th*: $\forall i \in$*Basis*. $(h \circ f)\ i = id\ i$
using *inv-o-cancel*[*OF fi*, *unfolded fun-eq-iff id-def o-def*]
by *auto*
from *linear-eq-stdbasis*[*OF linear-compose*[*OF lf h*(*1*)] *linear-id th*]
have $h \circ f = id$ **.**
then show *?thesis*
using *h*(*1*) **by** *blast*
qed

lemma *linear-surjective-right-inverse*:
fixes $f :: 'a$::*euclidean-space* $\Rightarrow$ $'b$::*euclidean-space*
assumes *lf*: *linear f*
and *sf*: *surj f*
shows $\exists g.$ *linear* $g \wedge f \circ g = id$
proof −
from *linear-independent-extend*[*OF independent-Basis*[**where** $'a='b$],*of inv f*]
obtain $h :: 'b \Rightarrow 'a$ **where** *h*: *linear h* $\forall x \in$*Basis*. *h x* = *inv f x*
by *blast*
from *h*(*2*) **have** *th*: $\forall i \in$*Basis*. $(f \circ h)\ i = id\ i$
using *sf* **by** (*auto simp add*: *surj-iff-all*)
from *linear-eq-stdbasis*[*OF linear-compose*[*OF h*(*1*) *lf*] *linear-id th*]
have $f \circ h = id$ **.**
then show *?thesis*
using *h*(*1*) **by** *blast*
qed

An injective map $'a \Rightarrow 'b$ is also surjective.

lemma *linear-injective-imp-surjective*:
fixes $f :: 'a$::*euclidean-space* $\Rightarrow$ $'a$::*euclidean-space*
assumes *lf*: *linear f*
and *fi*: *inj f*
shows *surj f*
proof −
let *?U* = *UNIV* :: $'a$ *set*
from *basis-exists*[*of ?U*] **obtain** *B*

```
    where B: B ⊆ ?U independent B ?U ⊆ span B card B = dim ?U
    by blast
  from B(4) have d: dim ?U = card B
    by simp
  have th: ?U ⊆ span (f ‘ B)
    apply (rule card-ge-dim-independent)
    apply blast
    apply (rule independent-injective-image[OF B(2) lf fi])
    apply (rule order-eq-refl)
    apply (rule sym)
    unfolding d
    apply (rule card-image)
    apply (rule subset-inj-on[OF fi])
    apply blast
    done
  from th show ?thesis
    unfolding span-linear-image[OF lf] surj-def
    using B(3) by blast
qed
```

And vice versa.

```
lemma surjective-iff-injective-gen:
  assumes fS: finite S
    and fT: finite T
    and c: card S = card T
    and ST: f ‘ S ⊆ T
  shows (∀ y ∈ T. ∃ x ∈ S. f x = y) ⟷ inj-on f S
  (is ?lhs ⟷ ?rhs)
proof
  assume h: ?lhs
  {
    fix x y
    assume x: x ∈ S
    assume y: y ∈ S
    assume f: f x = f y
    from x fS have S0: card S ≠ 0
      by auto
    have x = y
    proof (rule ccontr)
      assume xy: ¬ ?thesis
      have th: card S ≤ card (f ‘ (S − {y}))
        unfolding c
        apply (rule card-mono)
        apply (rule finite-imageI)
        using fS apply simp
        using h xy x y f unfolding subset-eq image-iff
        apply auto
        apply (case-tac xa = f x)
        apply (rule bexI[where x=x])
```

```
      apply auto
      done
    also have  ... ≤ card (S − {y})
      apply (rule card-image-le)
      using fS by simp
    also have ... ≤ card S − 1 using y fS by simp
    finally show False using S0 by arith
  qed
 }
 then show ?rhs
  unfolding inj-on-def by blast
next
 assume h: ?rhs
 have f ‘ S = T
  apply (rule card-subset-eq[OF fT ST])
  unfolding card-image[OF h]
  apply (rule c)
  done
 then show ?lhs by blast
qed

lemma linear-surjective-imp-injective:
 fixes f :: 'a::euclidean-space ⇒ 'a::euclidean-space
 assumes lf: linear f
  and sf: surj f
 shows inj f
proof −
 let ?U = UNIV :: 'a set
 from basis-exists[of ?U] obtain B
  where B: B ⊆ ?U independent B ?U ⊆ span B and d: card B = dim ?U
  by blast
 {
  fix x
  assume x: x ∈ span B
  assume fx: f x = 0
  from B(2) have fB: finite B
   using independent-bound by auto
  have fBi: independent (f ‘ B)
   apply (rule card-le-dim-spanning[of f ‘ B ?U])
   apply blast
   using sf B(3)
   unfolding span-linear-image[OF lf] surj-def subset-eq image-iff
   apply blast
   using fB apply blast
   unfolding d[symmetric]
   apply (rule card-image-le)
   apply (rule fB)
   done
  have th0: dim ?U ≤ card (f ‘ B)
```

apply (*rule span-card-ge-dim*)
apply *blast*
unfolding *span-linear-image*[*OF lf*]
apply (*rule subset-trans*[**where** $B = f$ ‘ *UNIV*])
using *sf* **unfolding** *surj-def*
apply *blast*
apply (*rule image-mono*)
apply (*rule B*(*3*))
apply (*metis finite-imageI fB*)
done
moreover have *card* (f ‘ B) $\leq$ *card* B
by (*rule card-image-le*, *rule fB*)
ultimately have *th1*: *card* B = *card* (f ‘ B)
unfolding d **by** *arith*
have *fiB*: *inj-on f B*
unfolding *surjective-iff-injective-gen*[*OF fB finite-imageI*[*OF fB*] *th1 subset-refl*, *symmetric*]
by *blast*
from *linear-indep-image-lemma*[*OF lf fB fBi fiB x*] *fx*
have $x = 0$ **by** *blast*
}
then show *?thesis*
unfolding *linear-injective-0*[*OF lf*]
using *B*(*3*)
by *blast*
qed

Hence either is enough for isomorphism.

lemma *left-right-inverse-eq*:
assumes *fg*: $f \circ g = id$
and *gh*: $g \circ h = id$
shows $f = h$
proof −
have $f = f \circ (g \circ h)$
unfolding *gh* **by** *simp*
also have ... = $(f \circ g) \circ h$
by (*simp add*: *o-assoc*)
finally show $f = h$
unfolding *fg* **by** *simp*
qed

lemma *isomorphism-expand*:
$f \circ g = id \wedge g \circ f = id \longleftrightarrow (\forall x.\ f\ (g\ x) = x) \wedge (\forall x.\ g\ (f\ x) = x)$
by (*simp add*: *fun-eq-iff o-def id-def*)

lemma *linear-injective-isomorphism*:
fixes f :: *'a*::*euclidean-space* ⇒ *'a*::*euclidean-space*
assumes *lf*: *linear f*
and *fi*: *inj f*

shows $\exists f'.\ \mathit{linear}\ f' \wedge (\forall x.\ f'\ (f\ x) = x) \wedge (\forall x.\ f\ (f'\ x) = x)$
unfolding *isomorphism-expand*[*symmetric*]
using *linear-surjective-right-inverse*[*OF lf linear-injective-imp-surjective*[*OF lf fi*]]
linear-injective-left-inverse[*OF lf fi*]
by (*metis left-right-inverse-eq*)

lemma *linear-surjective-isomorphism*:
fixes $f :: 'a$::*euclidean-space* $\Rightarrow$ $'a$::*euclidean-space*
assumes *lf*: *linear f*
and *sf*: *surj f*
shows $\exists f'.\ \mathit{linear}\ f' \wedge (\forall x.\ f'\ (f\ x) = x) \wedge (\forall x.\ f\ (f'\ x) = x)$
unfolding *isomorphism-expand*[*symmetric*]
using *linear-surjective-right-inverse*[*OF lf sf*]
linear-injective-left-inverse[*OF lf linear-surjective-imp-injective*[*OF lf sf*]]
by (*metis left-right-inverse-eq*)

Left and right inverses are the same for $'a \Rightarrow 'a$.

lemma *linear-inverse-left*:
fixes $f :: 'a$::*euclidean-space* $\Rightarrow$ $'a$::*euclidean-space*
assumes *lf*: *linear f*
and *lf'*: *linear f'*
shows $f \circ f' = id \longleftrightarrow f' \circ f = id$
proof −
{
fix $f\ f'$:: $'a \Rightarrow 'a$
assume *lf*: *linear f linear f'*
assume *f*: $f \circ f' = id$
from *f* **have** *sf*: *surj f*
apply (*auto simp add*: *o-def id-def surj-def*)
apply *metis*
done
from *linear-surjective-isomorphism*[*OF lf*(*1*) *sf*] *lf f*
have $f' \circ f = id$
unfolding *fun-eq-iff o-def id-def* **by** *metis*
}
then show *?thesis*
using *lf lf'* **by** *metis*
qed

Moreover, a one-sided inverse is automatically linear.

lemma *left-inverse-linear*:
fixes $f :: 'a$::*euclidean-space* $\Rightarrow$ $'a$::*euclidean-space*
assumes *lf*: *linear f*
and *gf*: $g \circ f = id$
shows *linear g*
proof −
from *gf* **have** *fi*: *inj f*
apply (*auto simp add*: *inj-on-def o-def id-def fun-eq-iff*)
apply *metis*

```
    done
  from linear-injective-isomorphism[OF lf fi]
  obtain h :: 'a ⇒ 'a where h: linear h ∀x. h (f x) = x ∀x. f (h x) = x
    by blast
  have h = g
    apply (rule ext) using gf h(2,3)
    apply (simp add: o-def id-def fun-eq-iff)
    apply metis
    done
  with h(1) show ?thesis by blast
qed

lemma inj-linear-imp-inv-linear:
  fixes f :: 'a::euclidean-space ⇒ 'a::euclidean-space
  assumes linear f inj f shows linear (inv f)
using assms inj-iff left-inverse-linear by blast
```

13.12 Infinity norm

```
definition infnorm (x::'a::euclidean-space) = Sup {|x · b| |b. b ∈ Basis}

lemma infnorm-set-image:
  fixes x :: 'a::euclidean-space
  shows {|x · i| |i. i ∈ Basis} = (λi. |x · i|) ' Basis
  by blast

lemma infnorm-Max:
  fixes x :: 'a::euclidean-space
  shows infnorm x = Max ((λi. |x · i|) ' Basis)
  by (simp add: infnorm-def infnorm-set-image cSup-eq-Max)

lemma infnorm-set-lemma:
  fixes x :: 'a::euclidean-space
  shows finite {|x · i| |i. i ∈ Basis}
    and {|x · i| |i. i ∈ Basis} ≠ {}
  unfolding infnorm-set-image
  by auto

lemma infnorm-pos-le:
  fixes x :: 'a::euclidean-space
  shows 0 ≤ infnorm x
  by (simp add: infnorm-Max Max-ge-iff ex-in-conv)

lemma infnorm-triangle:
  fixes x :: 'a::euclidean-space
  shows infnorm (x + y) ≤ infnorm x + infnorm y
proof −
  have ∗: ⋀a b c d :: real. |a| ≤ c ⟹ |b| ≤ d ⟹ |a + b| ≤ c + d
    by simp
```

show *?thesis*
by (*auto simp*: *infnorm-Max inner-add-left intro*!: ∗)
qed

lemma *infnorm-eq-0*:
fixes *x* :: *'a*::*euclidean-space*
shows *infnorm x* = *0* ⟷ *x* = *0*
proof −
have *infnorm x* ≤ *0* ⟷ *x* = *0*
unfolding *infnorm-Max* **by** (*simp add*: *euclidean-all-zero-iff*)
then show *?thesis*
using *infnorm-pos-le*[*of x*] **by** *simp*
qed

lemma *infnorm-0*: *infnorm 0* = *0*
by (*simp add*: *infnorm-eq-0*)

lemma *infnorm-neg*: *infnorm* (− *x*) = *infnorm x*
unfolding *infnorm-def*
apply (*rule cong*[*of Sup Sup*])
apply *blast*
apply *auto*
done

lemma *infnorm-sub*: *infnorm* (*x* − *y*) = *infnorm* (*y* − *x*)
proof −
have *y* − *x* = − (*x* − *y*) **by** *simp*
then show *?thesis*
by (*metis infnorm-neg*)
qed

lemma *real-abs-sub-infnorm*: |*infnorm x* − *infnorm y*| ≤ *infnorm* (*x* − *y*)
proof −
have *th*: ⋀(*nx*::*real*) *n ny*. *nx* ≤ *n* + *ny* ⟹ *ny* ≤ *n* + *nx* ⟹ |*nx* − *ny*| ≤ *n*
by *arith*
from *infnorm-triangle*[*of x* − *y y*] *infnorm-triangle*[*of x* − *y* −*x*]
have *ths*: *infnorm x* ≤ *infnorm* (*x* − *y*) + *infnorm y*
infnorm y ≤ *infnorm* (*x* − *y*) + *infnorm x*
by (*simp-all add*: *field-simps infnorm-neg*)
from *th*[*OF ths*] **show** *?thesis* **.**
qed

lemma *real-abs-infnorm*: |*infnorm x*| = *infnorm x*
using *infnorm-pos-le*[*of x*] **by** *arith*

lemma *Basis-le-infnorm*:
fixes *x* :: *'a*::*euclidean-space*
shows *b* ∈ *Basis* ⟹ |*x* ∙ *b*| ≤ *infnorm x*
by (*simp add*: *infnorm-Max*)

lemma *infnorm-mul*: *infnorm* $(a *_R x) = |a| * infnorm\ x$
unfolding *infnorm-Max*
proof (*safe intro*!: *Max-eqI*)
let *?B* = $(\lambda i.\ |x \cdot i|)$ ' *Basis*
{
fix $b :: 'a$
assume $b \in Basis$
then show $|a *_R x \cdot b| \leq |a| * Max\ ?B$
by (*simp add*: *abs-mult mult-left-mono*)
next
from *Max-in*[*of ?B*] **obtain** b **where** $b \in Basis\ Max\ ?B = |x \cdot b|$
by (*auto simp del*: *Max-in*)
then show $|a| * Max\ ((\lambda i.\ |x \cdot i|)\ `\ Basis) \in (\lambda i.\ |a *_R x \cdot i|)\ `\ Basis$
by (*intro image-eqI*[**where** $x{=}b$]) (*auto simp*: *abs-mult*)
}
qed *simp*

lemma *infnorm-mul-lemma*: *infnorm* $(a *_R x) \leq |a| * infnorm\ x$
unfolding *infnorm-mul* **..**

lemma *infnorm-pos-lt*: *infnorm* $x > 0 \longleftrightarrow x \neq 0$
using *infnorm-pos-le*[*of x*] *infnorm-eq-0*[*of x*] **by** *arith*

Prove that it differs only up to a bound from Euclidean norm.

lemma *infnorm-le-norm*: *infnorm* $x \leq norm\ x$
by (*simp add*: *Basis-le-norm infnorm-Max*)

lemma (**in** *euclidean-space*) *euclidean-inner*: *inner* $x\ y = (\sum b{\in}Basis.\ (x \cdot b) * (y \cdot b))$
by (*subst* (*1 2*) *euclidean-representation* [*symmetric*])
(*simp add*: *inner-setsum-right inner-Basis ac-simps*)

lemma *norm-le-infnorm*:
fixes $x :: 'a{::}euclidean\text{-}space$
shows $norm\ x \leq sqrt\ DIM('a) * infnorm\ x$
proof −
let *?d* = $DIM('a)$
have *real ?d* ≥ 0
by *simp*
then have *d2*: $(sqrt\ (real\ ?d))^2 = real\ ?d$
by (*auto intro*: *real-sqrt-pow2*)
have *th*: $sqrt\ (real\ ?d) * infnorm\ x \geq 0$
by (*simp add*: *zero-le-mult-iff infnorm-pos-le*)
have *th1*: $x \cdot x \leq (sqrt\ (real\ ?d) * infnorm\ x)^2$
unfolding *power-mult-distrib d2*
apply (*subst euclidean-inner*)
apply (*subst power2-abs*[*symmetric*])
apply (*rule order-trans*[*OF setsum-bounded-above*[**where** $K{=}|infnorm\ x|^2$]])

```
    apply (auto simp add: power2-eq-square[symmetric])
    apply (subst power2-abs[symmetric])
    apply (rule power-mono)
    apply (auto simp: infnorm-Max)
    done
  from real-le-lsqrt[OF inner-ge-zero th th1]
  show ?thesis
    unfolding norm-eq-sqrt-inner id-def .
qed

lemma tendsto-infnorm [tendsto-intros]:
  assumes (f ⟶ a) F
  shows ((λx. infnorm (f x)) ⟶ infnorm a) F
proof (rule tendsto-compose [OF LIM-I assms])
  fix r :: real
  assume r > 0
  then show ∃s>0. ∀x. x ≠ a ∧ norm (x − a) < s ⟶ norm (infnorm x −
infnorm a) < r
    by (metis real-norm-def le-less-trans real-abs-sub-infnorm infnorm-le-norm)
qed
```

Equality in Cauchy-Schwarz and triangle inequalities.

```
lemma norm-cauchy-schwarz-eq: x • y = norm x * norm y ⟷ norm x *R y =
norm y *R x
  (is ?lhs ⟷ ?rhs)
proof −
  {
    assume h: x = 0
    then have ?thesis by simp
  }
  moreover
  {
    assume h: y = 0
    then have ?thesis by simp
  }
  moreover
  {
    assume x: x ≠ 0 and y: y ≠ 0
    from inner-eq-zero-iff[of norm y *R x − norm x *R y]
    have ?rhs ⟷
      (norm y * (norm y * norm x * norm x − norm x * (x • y)) −
        norm x * (norm y * (y • x) − norm x * norm y * norm y) = 0)
      using x y
      unfolding inner-simps
     unfolding power2-norm-eq-inner[symmetric] power2-eq-square right-minus-eq
      apply (simp add: inner-commute)
      apply (simp add: field-simps)
      apply metis
      done
```

```
    also have ... ⟷ (2 * norm x * norm y * (norm x * norm y - x · y) = 0)
using x y
      by (simp add: field-simps inner-commute)
    also have ... ⟷ ?lhs using x y
      apply simp
      apply metis
      done
    finally have ?thesis by blast
  }
  ultimately show ?thesis by blast
qed

lemma norm-cauchy-schwarz-abs-eq:
  |x · y| = norm x * norm y ⟷
    norm x *R y = norm y *R x ∨ norm x *R y = - norm y *R x
  (is ?lhs ⟷ ?rhs)
proof -
  have th: ⋀(x::real) a. a ≥ 0 ⟹ |x| = a ⟷ x = a ∨ x = - a
    by arith
  have ?rhs ⟷ norm x *R y = norm y *R x ∨ norm (- x) *R y = norm y *R
(- x)
    by simp
  also have ... ⟷(x · y = norm x * norm y ∨ (- x) · y = norm x * norm y)
    unfolding norm-cauchy-schwarz-eq[symmetric]
    unfolding norm-minus-cancel norm-scaleR ..
  also have ... ⟷ ?lhs
    unfolding th[OF mult-nonneg-nonneg, OF norm-ge-zero[of x] norm-ge-zero[of
y]] inner-simps
    by auto
  finally show ?thesis ..
qed

lemma norm-triangle-eq:
  fixes x y :: 'a::real-inner
  shows norm (x + y) = norm x + norm y ⟷ norm x *R y = norm y *R x
proof -
  {
    assume x: x = 0 ∨ y = 0
    then have ?thesis
      by (cases x = 0) simp-all
  }
  moreover
  {
    assume x: x ≠ 0 and y: y ≠ 0
    then have norm x ≠ 0 norm y ≠ 0
      by simp-all
    then have n: norm x > 0 norm y > 0
      using norm-ge-zero[of x] norm-ge-zero[of y] by arith+
    have th: ⋀(a::real) b c. a + b + c ≠ 0 ⟹ a = b + c ⟷ a^2 = (b + c)^2
```

by *algebra*
have *norm* $(x + y) = norm\ x + norm\ y \longleftrightarrow (norm\ (x + y))^2 = (norm\ x + norm\ y)^2$
apply (*rule th*)
using *n norm-ge-zero*[*of* $x + y$]
apply *arith*
done
also have $\ldots \longleftrightarrow norm\ x *_R y = norm\ y *_R x$
unfolding *norm-cauchy-schwarz-eq*[*symmetric*]
unfolding *power2-norm-eq-inner inner-simps*
by (*simp add*: *power2-norm-eq-inner*[*symmetric*] *power2-eq-square inner-commute field-simps*)
finally have *?thesis* **.**
}
ultimately show *?thesis* **by** *blast*
qed

13.13 Collinearity

definition *collinear* :: *'a*::*real-vector set* ⇒ *bool*
where *collinear* $S \longleftrightarrow (\exists u.\ \forall x \in S.\ \forall\ y \in S.\ \exists c.\ x - y = c *_R u)$

lemma *collinear-empty* [*iff*]: *collinear* {}
by (*simp add*: *collinear-def*)

lemma *collinear-sing* [*iff*]: *collinear* $\{x\}$
by (*simp add*: *collinear-def*)

lemma *collinear-2* [*iff*]: *collinear* $\{x, y\}$
apply (*simp add*: *collinear-def*)
apply (*rule exI*[**where** *x*=$x - y$])
apply *auto*
apply (*rule exI*[**where** *x*=*1*], *simp*)
apply (*rule exI*[**where** *x*=$-$ *1*], *simp*)
done

lemma *collinear-lemma*: *collinear* $\{0, x, y\} \longleftrightarrow x = 0 \vee y = 0 \vee (\exists c.\ y = c *_R x)$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof −
{
assume $x = 0 \vee y = 0$
then have *?thesis*
by (*cases* $x = 0$) (*simp-all add*: *collinear-2 insert-commute*)
}
moreover
{
assume *x*: $x \neq 0$ **and** *y*: $y \neq 0$
have *?thesis*

```
    proof
      assume h: ?lhs
      then obtain u where u: ∀ x∈ {0,x,y}. ∀ y∈ {0,x,y}. ∃ c. x − y = c *R u
        unfolding collinear-def by blast
      from u[rule-format, of x 0] u[rule-format, of y 0]
      obtain cx and cy where
        cx: x = cx *R u and cy: y = cy *R u
        by auto
      from cx x have cx0: cx ≠ 0 by auto
      from cy y have cy0: cy ≠ 0 by auto
      let ?d = cy / cx
      from cx cy cx0 have y = ?d *R x
        by simp
      then show ?rhs using x y by blast
    next
      assume h: ?rhs
      then obtain c where c: y = c *R x
        using x y by blast
      show ?lhs
        unfolding collinear-def c
        apply (rule exI[where x=x])
        apply auto
        apply (rule exI[where x=− 1], simp)
        apply (rule exI[where x= −c], simp)
        apply (rule exI[where x=1], simp)
        apply (rule exI[where x=1 − c], simp add: scaleR-left-diff-distrib)
        apply (rule exI[where x=c − 1], simp add: scaleR-left-diff-distrib)
        done
    qed
  }
  ultimately show ?thesis by blast
qed

lemma norm-cauchy-schwarz-equal: |x · y| = norm x * norm y ⟷ collinear {0,
x, y}
  unfolding norm-cauchy-schwarz-abs-eq
  apply (cases x=0, simp-all add: collinear-2)
  apply (cases y=0, simp-all add: collinear-2 insert-commute)
  unfolding collinear-lemma
  apply simp
  apply (subgoal-tac norm x ≠ 0)
  apply (subgoal-tac norm y ≠ 0)
  apply (rule iffI)
  apply (cases norm x *R y = norm y *R x)
  apply (rule exI[where x=(1/norm x) * norm y])
  apply (drule sym)
  unfolding scaleR-scaleR[symmetric]
  apply (simp add: field-simps)
  apply (rule exI[where x=(1/norm x) * − norm y])
```

apply *clarify*
apply (*drule sym*)
unfolding *scaleR-scaleR*[*symmetric*]
apply (*simp add*: *field-simps*)
apply (*erule exE*)
apply (*erule ssubst*)
unfolding *scaleR-scaleR*
unfolding *norm-scaleR*
apply (*subgoal-tac norm x* $*$ *c* $=$ $|c|$ $*$ *norm x* $\vee$ *norm x* $*$ *c* $=$ $-$ $|c|$ $*$ *norm x*)
apply (*auto simp add*: *field-simps*)
done

end

14 A decision procedure for universal multivariate real arithmetic with addition, multiplication and ordering using semidefinite programming

theory *Sum-of-Squares*
imports *Complex-Main*
begin

ML-file *positivstellensatz.ML*
ML-file *Sum-of-Squares/sum-of-squares.ML*
ML-file *Sum-of-Squares/positivstellensatz-tools.ML*
ML-file *Sum-of-Squares/sos-wrapper.ML*

end

15 General linear decision procedure for normed spaces

theory *Norm-Arith*
imports ~~/*src/HOL/Library/Sum-of-Squares*
begin

lemma *norm-cmul-rule-thm*:
fixes *x* :: *'a*::*real-normed-vector*
shows *b* $\geq$ *norm x* $\Longrightarrow$ $|c|$ $*$ *b* $\geq$ *norm* (*scaleR c x*)
unfolding *norm-scaleR*
apply (*erule mult-left-mono*)
apply *simp*
done

lemma *norm-add-rule-thm*:
fixes *x1 x2* :: *'a*::*real-normed-vector*
shows *norm x1* ≤ *b1* ⟹ *norm x2* ≤ *b2* ⟹ *norm* (*x1* + *x2*) ≤ *b1* + *b2*
by (*rule order-trans* [*OF norm-triangle-ineq add-mono*])

lemma *ge-iff-diff-ge-0*:
fixes *a* :: *'a*::*linordered-ring*
shows *a* ≥ *b* ≡ *a* − *b* ≥ *0*
by (*simp add*: *field-simps*)

lemma *pth-1*:
fixes *x* :: *'a*::*real-normed-vector*
shows *x* ≡ *scaleR 1 x* **by** *simp*

lemma *pth-2*:
fixes *x* :: *'a*::*real-normed-vector*
shows *x* − *y* ≡ *x* + −*y*
by (*atomize* (*full*)) *simp*

lemma *pth-3*:
fixes *x* :: *'a*::*real-normed-vector*
shows − *x* ≡ *scaleR* (−*1*) *x*
by *simp*

lemma *pth-4*:
fixes *x* :: *'a*::*real-normed-vector*
shows *scaleR 0 x* ≡ *0*
and *scaleR c 0* = (*0*::*'a*)
by *simp-all*

lemma *pth-5*:
fixes *x* :: *'a*::*real-normed-vector*
shows *scaleR c* (*scaleR d x*) ≡ *scaleR* (*c* ∗ *d*) *x*
by *simp*

lemma *pth-6*:
fixes *x* :: *'a*::*real-normed-vector*
shows *scaleR c* (*x* + *y*) ≡ *scaleR c x* + *scaleR c y*
by (*simp add*: *scaleR-right-distrib*)

lemma *pth-7*:
fixes *x* :: *'a*::*real-normed-vector*
shows *0* + *x* ≡ *x*
and *x* + *0* ≡ *x*
by *simp-all*

lemma *pth-8*:
fixes *x* :: *'a*::*real-normed-vector*
shows *scaleR c x* + *scaleR d x* ≡ *scaleR* (*c* + *d*) *x*

by (*simp add*: *scaleR-left-distrib*)

lemma *pth-9*:
fixes *x* :: *'a*::*real-normed-vector*
shows (*scaleR c x* + *z*) + *scaleR d x* ≡ *scaleR* (*c* + *d*) *x* + *z*
and *scaleR c x* + (*scaleR d x* + *z*) ≡ *scaleR* (*c* + *d*) *x* + *z*
and (*scaleR c x* + *w*) + (*scaleR d x* + *z*) ≡ *scaleR* (*c* + *d*) *x* + (*w* + *z*)
by (*simp-all add*: *algebra-simps*)

lemma *pth-a*:
fixes *x* :: *'a*::*real-normed-vector*
shows *scaleR 0 x* + *y* ≡ *y*
by *simp*

lemma *pth-b*:
fixes *x* :: *'a*::*real-normed-vector*
shows *scaleR c x* + *scaleR d y* ≡ *scaleR c x* + *scaleR d y*
and (*scaleR c x* + *z*) + *scaleR d y* ≡ *scaleR c x* + (*z* + *scaleR d y*)
and *scaleR c x* + (*scaleR d y* + *z*) ≡ *scaleR c x* + (*scaleR d y* + *z*)
and (*scaleR c x* + *w*) + (*scaleR d y* + *z*) ≡ *scaleR c x* + (*w* + (*scaleR d y* + *z*))
by (*simp-all add*: *algebra-simps*)

lemma *pth-c*:
fixes *x* :: *'a*::*real-normed-vector*
shows *scaleR c x* + *scaleR d y* ≡ *scaleR d y* + *scaleR c x*
and (*scaleR c x* + *z*) + *scaleR d y* ≡ *scaleR d y* + (*scaleR c x* + *z*)
and *scaleR c x* + (*scaleR d y* + *z*) ≡ *scaleR d y* + (*scaleR c x* + *z*)
and (*scaleR c x* + *w*) + (*scaleR d y* + *z*) ≡ *scaleR d y* + ((*scaleR c x* + *w*) + *z*)
by (*simp-all add*: *algebra-simps*)

lemma *pth-d*:
fixes *x* :: *'a*::*real-normed-vector*
shows *x* + *0* ≡ *x*
by *simp*

lemma *norm-imp-pos-and-ge*:
fixes *x* :: *'a*::*real-normed-vector*
shows *norm x* ≡ *n* ⟹ *norm x* ≥ *0* ∧ *n* ≥ *norm x*
by *atomize auto*

lemma *real-eq-0-iff-le-ge-0*:
fixes *x* :: *real*
shows *x* = *0* ≡ *x* ≥ *0* ∧ − *x* ≥ *0*
by *arith*

lemma *norm-pths*:
fixes *x* :: *'a*::*real-normed-vector*

shows $x = y \longleftrightarrow norm\ (x - y) \leq 0$
and $x \neq y \longleftrightarrow \neg\ (norm\ (x - y) \leq 0)$
using *norm-ge-zero*[*of* $x - y$] **by** *auto*

lemmas *arithmetic-simps* =
arith-simps
add-numeral-special
add-neg-numeral-special
mult-1-left
mult-1-right

ML-file *normarith.ML*

method-setup *norm* = ‹
Scan.succeed (*SIMPLE-METHOD'* o *NormArith.norm-arith-tac*)
› *prove simple linear statements about vector norms*

Hence more metric properties.

lemma *dist-triangle-add*:
fixes $x\ y\ x'\ y'$ :: *'a*::*real-normed-vector*
shows $dist\ (x + y)\ (x' + y') \leq dist\ x\ x' + dist\ y\ y'$
by *norm*

lemma *dist-triangle-add-half*:
fixes $x\ x'\ y\ y'$ :: *'a*::*real-normed-vector*
shows $dist\ x\ x' < e\ /\ 2 \Longrightarrow dist\ y\ y' < e\ /\ 2 \Longrightarrow dist(x + y)\ (x' + y') < e$
by *norm*

end

16 Elementary topology in Euclidean space.

theory *Topology-Euclidean-Space*
imports
~~/*src*/*HOL*/*Library*/*Indicator-Function*
~~/*src*/*HOL*/*Library*/*Countable-Set*
~~/*src*/*HOL*/*Library*/*FuncSet*
Linear-Algebra
Norm-Arith
begin

lemma *image-affinity-interval*:
fixes c :: *'a*::*ordered-real-vector*
shows $((\lambda x.\ m *_R x + c)\ `\ \{a..b\}) = (if\ \{a..b\}=\{\}\ then\ \{\}$
else if $0 <= m$ *then* $\{m *_R a + c\ ..\ m *_R b + c\}$
else $\{m *_R b + c\ ..\ m *_R a + c\}$)
apply (*case-tac* $m=0$, *force*)
apply (*auto simp*: *scaleR-left-mono*)

apply (*rule-tac x=inverse m* $*_R$ (*x*−*c*) **in** *rev-image-eqI*, *auto simp*: *pos-le-divideR-eq le-diff-eq scaleR-left-mono-neg*)
apply (*metis diff-le-eq inverse-inverse-eq order.not-eq-order-implies-strict pos-le-divideR-eq positive-imp-inverse-positive*)
apply (*rule-tac x=inverse m* $*_R$ (*x*−*c*) **in** *rev-image-eqI*, *auto simp*: *not-le neg-le-divideR-eq diff-le-eq*)
using *le-diff-eq scaleR-le-cancel-left-neg*
apply *fastforce*
done

lemma *countable-PiE*:
finite I $\Longrightarrow$ ($\bigwedge i.\ i \in I \Longrightarrow$ *countable* (*F i*)) $\Longrightarrow$ *countable* (*PiE I F*)
by (*induct I arbitrary*: *F rule*: *finite-induct*) (*auto simp*: *PiE-insert-eq*)

lemma *continuous-on-cases*:
closed s $\Longrightarrow$ *closed t* $\Longrightarrow$ *continuous-on s f* $\Longrightarrow$ *continuous-on t g* $\Longrightarrow$
$\forall x.\ (x{\in}s \wedge \neg\ P\ x) \vee (x \in t \wedge P\ x) \longrightarrow f\ x = g\ x \Longrightarrow$
continuous-on (*s* $\cup$ *t*) ($\lambda x.$ *if P x then f x else g x*)
by (*rule continuous-on-If*) *auto*

16.1 Topological Basis

context *topological-space*
begin

definition *topological-basis B* $\longleftrightarrow$
($\forall\, b{\in}B.$ *open b*) $\wedge$ ($\forall\, x.$ *open x* $\longrightarrow$ ($\exists\, B'.\ B' \subseteq B \wedge \bigcup B' = x$))

lemma *topological-basis*:
topological-basis B $\longleftrightarrow$ ($\forall\, x.$ *open x* $\longleftrightarrow$ ($\exists\, B'.\ B' \subseteq B \wedge \bigcup B' = x$))
unfolding *topological-basis-def*
apply *safe*
apply *fastforce*
apply *fastforce*
apply (*erule-tac x=x* **in** *allE*)
apply *simp*
apply (*rule-tac x*={*x*} **in** *exI*)
apply *auto*
done

lemma *topological-basis-iff*:
assumes $\bigwedge B'.\ B' \in B \Longrightarrow$ *open B'*
shows *topological-basis B* $\longleftrightarrow$ ($\forall\, O'.$ *open O'* $\longrightarrow$ ($\forall\, x{\in}O'.\ \exists\, B'{\in}B.\ x \in B' \wedge B' \subseteq O'$))
(**is** - $\longleftrightarrow$ *?rhs*)
proof *safe*
fix *O'* **and** *x*::'*a*
assume *H*: *topological-basis B open O' x* $\in$ *O'*
then have ($\exists\, B'{\subseteq}B.\ \bigcup B' = O'$) **by** (*simp add*: *topological-basis-def*)

then obtain B' **where** $B' \subseteq B\ O' = \bigcup B'$ **by** *auto*
then show $\exists B' \in B.\ x \in B' \wedge B' \subseteq O'$ **using** H **by** *auto*
next
assume H: *?rhs*
show *topological-basis B*
using *assms* **unfolding** *topological-basis-def*
proof *safe*
fix O' :: *'a set*
assume *open* O'
with H **obtain** f **where** $\forall x \in O'.\ f\ x \in B \wedge x \in f\ x \wedge f\ x \subseteq O'$
by (*force intro*: *bchoice simp*: *Bex-def*)
then show $\exists B' \subseteq B.\ \bigcup B' = O'$
by (*auto intro*: *exI*[**where** $x = \{f\ x\ |x.\ x \in O'\}$])
qed
qed

lemma *topological-basisI*:
assumes $\bigwedge B'.\ B' \in B \Longrightarrow$ *open* B'
and $\bigwedge O'\ x.$ *open* $O' \Longrightarrow x \in O' \Longrightarrow \exists B' \in B.\ x \in B' \wedge B' \subseteq O'$
shows *topological-basis B*
using *assms* **by** (*subst topological-basis-iff*) *auto*

lemma *topological-basisE*:
fixes O'
assumes *topological-basis B*
and *open* O'
and $x \in O'$
obtains B' **where** $B' \in B\ x \in B'\ B' \subseteq O'$
proof *atomize-elim*
from *assms* **have** $\bigwedge B'.\ B' \in B \Longrightarrow$ *open* B'
by (*simp add*: *topological-basis-def*)
with *topological-basis-iff assms*
show $\exists B'.\ B' \in B \wedge x \in B' \wedge B' \subseteq O'$
using *assms* **by** (*simp add*: *Bex-def*)
qed

lemma *topological-basis-open*:
assumes *topological-basis B*
and $X \in B$
shows *open X*
using *assms* **by** (*simp add*: *topological-basis-def*)

lemma *topological-basis-imp-subbasis*:
assumes B: *topological-basis B*
shows *open* = *generate-topology B*
proof (*intro ext iffI*)
fix S :: *'a set*
assume *open S*
with B **obtain** B' **where** $B' \subseteq B\ S = \bigcup B'$

```
    unfolding topological-basis-def by blast
  then show generate-topology B S
    by (auto intro: generate-topology.intros dest: topological-basis-open)
next
  fix S :: 'a set
  assume generate-topology B S
  then show open S
    by induct (auto dest: topological-basis-open[OF B])
qed

lemma basis-dense:
  fixes B :: 'a set set
    and f :: 'a set ⇒ 'a
  assumes topological-basis B
    and choosefrom-basis: ⋀B'. B' ≠ {} ⟹ f B' ∈ B'
  shows ∀X. open X ⟶ X ≠ {} ⟶ (∃B' ∈ B. f B' ∈ X)
proof (intro allI impI)
  fix X :: 'a set
  assume open X and X ≠ {}
  from topological-basisE[OF ‹topological-basis B› ‹open X› choosefrom-basis[OF
‹X ≠ {}›]]
  obtain B' where B' ∈ B f X ∈ B' B' ⊆ X .
  then show ∃B'∈B. f B' ∈ X
    by (auto intro!: choosefrom-basis)
qed

end

lemma topological-basis-prod:
  assumes A: topological-basis A
    and B: topological-basis B
  shows topological-basis ((λ(a, b). a × b) ` (A × B))
  unfolding topological-basis-def
proof (safe, simp-all del: ex-simps add: subset-image-iff ex-simps(1)[symmetric])
  fix S :: ('a × 'b) set
  assume open S
  then show ∃X⊆A × B. (⋃(a,b)∈X. a × b) = S
  proof (safe intro!: exI[of - {x∈A × B. fst x × snd x ⊆ S}])
    fix x y
    assume (x, y) ∈ S
    from open-prod-elim[OF ‹open S› this]
    obtain a b where a: open a x ∈ a and b: open b y ∈ b and a × b ⊆ S
      by (metis mem-Sigma-iff)
    moreover
    from A a obtain A0 where A0 ∈ A x ∈ A0 A0 ⊆ a
      by (rule topological-basisE)
    moreover
    from B b obtain B0 where B0 ∈ B y ∈ B0 B0 ⊆ b
      by (rule topological-basisE)
```

ultimately show $(x, y) \in (\bigcup (a, b) \in \{X \in A \times B.\ fst\ X \times snd\ X \subseteq S\}.\ a \times b)$
by (*intro UN-I*[*of* (*A0*, *B0*)]) *auto*
qed *auto*
qed (*metis A B topological-basis-open open-Times*)

16.2 Countable Basis

locale *countable-basis* =
fixes $B :: 'a::topological\text{-}space\ set\ set$
assumes *is-basis*: *topological-basis B*
and *countable-basis*: *countable B*
begin

lemma *open-countable-basis-ex*:
assumes *open X*
shows $\exists B' \subseteq B.\ X = \bigcup B'$
using *assms countable-basis is-basis*
unfolding *topological-basis-def* **by** *blast*

lemma *open-countable-basisE*:
assumes *open X*
obtains B' **where** $B' \subseteq B\ X = \bigcup B'$
using *assms open-countable-basis-ex*
by (*atomize-elim*) *simp*

lemma *countable-dense-exists*:
$\exists D::'a\ set.\ countable\ D \wedge (\forall X.\ open\ X \longrightarrow X \neq \{\} \longrightarrow (\exists d \in D.\ d \in X))$
proof −
let *?f* = $(\lambda B'.\ SOME\ x.\ x \in B')$
have *countable* (*?f* ‘ *B*) **using** *countable-basis* **by** *simp*
with *basis-dense*[*OF is-basis*, *of ?f*] **show** *?thesis*
by (*intro exI*[**where** *x*=*?f* ‘ *B*]) (*metis* (*mono-tags*) *all-not-in-conv imageI someI*)
qed

lemma *countable-dense-setE*:
obtains $D :: 'a\ set$
where *countable D* $\bigwedge X.\ open\ X \Longrightarrow X \neq \{\} \Longrightarrow \exists d \in D.\ d \in X$
using *countable-dense-exists* **by** *blast*

end

lemma (**in** *first-countable-topology*) *first-countable-basisE*:
obtains *A* **where** *countable A* $\bigwedge a.\ a \in A \Longrightarrow x \in a$ $\bigwedge a.\ a \in A \Longrightarrow open\ a$
$\bigwedge S.\ open\ S \Longrightarrow x \in S \Longrightarrow (\exists a \in A.\ a \subseteq S)$
using *first-countable-basis*[*of x*]
apply *atomize-elim*
apply (*elim exE*)

apply (*rule-tac x=range A* **in** *exI*)
apply *auto*
done

lemma (**in** *first-countable-topology*) *first-countable-basis-Int-stableE*:
obtains A **where** *countable* A $\bigwedge a.\ a \in A \Longrightarrow x \in a$ $\bigwedge a.\ a \in A \Longrightarrow$ *open* a
$\bigwedge S.$ *open* $S \Longrightarrow x \in S \Longrightarrow (\exists\, a{\in}A.\ a \subseteq S)$
$\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow a \cap b \in A$
proof *atomize-elim*
obtain A' **where** A':
countable A'
$\bigwedge a.\ a \in A' \Longrightarrow x \in a$
$\bigwedge a.\ a \in A' \Longrightarrow$ *open* a
$\bigwedge S.$ *open* $S \Longrightarrow x \in S \Longrightarrow \exists\, a{\in}A'.\ a \subseteq S$
by (*rule first-countable-basisE*) *blast*
def $A \equiv (\lambda N.\ \bigcap((\lambda n.$ *from-nat-into* $A'\ n)$ ‘ $N))$ ‘ (*Collect finite*::*nat set set*)
then show $\exists\, A.$ *countable* $A \wedge (\forall\, a.\ a \in A \longrightarrow x \in a) \wedge (\forall\, a.\ a \in A \longrightarrow$ *open* $a) \wedge$
$(\forall\, S.$ *open* $S \longrightarrow x \in S \longrightarrow (\exists\, a{\in}A.\ a \subseteq S)) \wedge (\forall\, a\ b.\ a \in A \longrightarrow b \in A \longrightarrow a \cap b \in A)$
proof (*safe intro*!: *exI*[**where** $x{=}A$])
show *countable* A
unfolding *A-def* **by** (*intro countable-image countable-Collect-finite*)
fix a
assume $a \in A$
then show $x \in a$ *open* a
using $A'(4)$[*OF open-UNIV*] **by** (*auto simp*: *A-def intro*: A' *from-nat-into*)
next
let *?int* $= \lambda N.\ \bigcap$(*from-nat-into* A' ‘ N)
fix $a\ b$
assume $a \in A\ b \in A$
then obtain $N\ M$ **where** $a =$ *?int* $N\ b =$ *?int* M *finite* $(N \cup M)$
by (*auto simp*: *A-def*)
then show $a \cap b \in A$
by (*auto simp*: *A-def intro*!: *image-eqI*[**where** $x{=}N \cup M$])
next
fix S
assume *open* $S\ x \in S$
then obtain a **where** a: $a{\in}A'\ a \subseteq S$ **using** A' **by** *blast*
then show $\exists\, a{\in}A.\ a \subseteq S$ **using** $a\ A'$
by (*intro bexI*[**where** $x{=}a$]) (*auto simp*: *A-def intro*: *image-eqI*[**where** $x{=}\{$*to-nat-on* $A'\ a\}$])
qed
qed

lemma (**in** *topological-space*) *first-countableI*:
assumes *countable* A
and *1*: $\bigwedge a.\ a \in A \Longrightarrow x \in a$ $\bigwedge a.\ a \in A \Longrightarrow$ *open* a
and *2*: $\bigwedge S.$ *open* $S \Longrightarrow x \in S \Longrightarrow \exists\, a{\in}A.\ a \subseteq S$

shows $\exists A{::}nat \Rightarrow 'a\ set.\ (\forall i.\ x \in A\ i \wedge open\ (A\ i)) \wedge (\forall S.\ open\ S \wedge x \in S \longrightarrow (\exists i.\ A\ i \subseteq S))$
proof (*safe intro*!: *exI*[*of* - *from-nat-into A*])
fix *i*
have $A \neq \{\}$ **using** *2*[*of UNIV*] **by** *auto*
show $x \in$ *from-nat-into A i open* (*from-nat-into A i*)
using *range-from-nat-into-subset*[*OF* ‹$A \neq \{\}$›] *1* **by** *auto*
next
fix *S*
assume *open S x*∈*S* **from** *2*[*OF this*]
show $\exists i.$ *from-nat-into A i* $\subseteq S$
using *subset-range-from-nat-into*[*OF* ‹*countable A*›] **by** *auto*
qed

instance *prod* :: (*first-countable-topology*, *first-countable-topology*) *first-countable-topology*
proof
fix $x :: 'a \times 'b$
obtain *A* **where** *A*:
countable A
$\bigwedge a.\ a \in A \Longrightarrow fst\ x \in a$
$\bigwedge a.\ a \in A \Longrightarrow open\ a$
$\bigwedge S.\ open\ S \Longrightarrow fst\ x \in S \Longrightarrow \exists a{\in}A.\ a \subseteq S$
by (*rule first-countable-basisE*[*of fst x*]) *blast*
obtain *B* **where** *B*:
countable B
$\bigwedge a.\ a \in B \Longrightarrow snd\ x \in a$
$\bigwedge a.\ a \in B \Longrightarrow open\ a$
$\bigwedge S.\ open\ S \Longrightarrow snd\ x \in S \Longrightarrow \exists a{\in}B.\ a \subseteq S$
by (*rule first-countable-basisE*[*of snd x*]) *blast*
show $\exists A{::}nat \Rightarrow ('a \times 'b)\ set.$
$(\forall i.\ x \in A\ i \wedge open\ (A\ i)) \wedge (\forall S.\ open\ S \wedge x \in S \longrightarrow (\exists i.\ A\ i \subseteq S))$
proof (*rule first-countableI*[*of* $(\lambda(a, b).\ a \times b)$ ‘ $(A \times B)$], *safe*)
fix *a b*
assume *x*: $a \in A\ b \in B$
with *A*(*2*, *3*)[*of a*] *B*(*2*, *3*)[*of b*] **show** $x \in a \times b$ **and** *open* $(a \times b)$
unfolding *mem-Times-iff*
by (*auto intro*: *open-Times*)
next
fix *S*
assume *open S* $x \in S$
then obtain $a'\ b'$ **where** $a'b'$: *open* a' *open* b' $x \in a' \times b'$ $a' \times b' \subseteq S$
by (*rule open-prod-elim*)
moreover
from $a'b'$ *A*(*4*)[*of* a'] *B*(*4*)[*of* b']
obtain *a b* **where** $a \in A\ a \subseteq a'\ b \in B\ b \subseteq b'$
by *auto*
ultimately
show $\exists a{\in}(\lambda(a, b).\ a \times b)$ ‘ $(A \times B).\ a \subseteq S$
by (*auto intro*!: *bexI*[*of* - $a \times b$] *bexI*[*of* - *a*] *bexI*[*of* - *b*])

qed (*simp add*: *A B*)
qed

class *second-countable-topology* = *topological-space* +
assumes *ex-countable-subbasis*:
∃ *B*::'*a*::*topological-space set set*. *countable B* ∧ *open* = *generate-topology B*
begin

lemma *ex-countable-basis*: ∃ *B*::'*a set set*. *countable B* ∧ *topological-basis B*
proof −
from *ex-countable-subbasis* **obtain** *B* **where** *B*: *countable B open* = *generate-topology B*
by *blast*
let *?B* = *Inter* ' {*b*. *finite b* ∧ *b* ⊆ *B* }

show *?thesis*
proof (*intro exI conjI*)
show *countable ?B*
by (*intro countable-image countable-Collect-finite-subset B*)
{
fix *S*
assume *open S*
then have ∃ *B'*⊆{*b*. *finite b* ∧ *b* ⊆ *B*}. (⋃ *b*∈*B'*. ⋂ *b*) = *S*
unfolding *B*
proof *induct*
case *UNIV*
show *?case* **by** (*intro exI*[*of* - {{}}]) *simp*
next
case (*Int a b*)
then obtain *x y* **where** *x*: *a* = *UNION x Inter* ⋀*i*. *i* ∈ *x* ⟹ *finite i* ∧ *i* ⊆ *B*
and *y*: *b* = *UNION y Inter* ⋀*i*. *i* ∈ *y* ⟹ *finite i* ∧ *i* ⊆ *B*
by *blast*
show *?case*
unfolding *x y Int-UN-distrib2*
by (*intro exI*[*of* - {*i* ∪ *j*| *i j*. *i* ∈ *x* ∧ *j* ∈ *y*}]) (*auto dest*: *x*(*2*) *y*(*2*))
next
case (*UN K*)
then have ∀ *k*∈*K*. ∃ *B'*⊆{*b*. *finite b* ∧ *b* ⊆ *B*}. *UNION B' Inter* = *k* **by** *auto*
then obtain *k* **where**
∀ *ka*∈*K*. *k ka* ⊆ {*b*. *finite b* ∧ *b* ⊆ *B*} ∧ *UNION* (*k ka*) *Inter* = *ka*
unfolding *bchoice-iff* **..**
then show ∃ *B'*⊆{*b*. *finite b* ∧ *b* ⊆ *B*}. *UNION B' Inter* = ⋃ *K*
by (*intro exI*[*of* - *UNION K k*]) *auto*
next
case (*Basis S*)
then show *?case*
by (*intro exI*[*of* - {{*S*}}]) *auto*

```
    qed
    then have (∃ B'⊆Inter ` {b. finite b ∧ b ⊆ B}. ⋃ B' = S)
      unfolding subset-image-iff by blast }
  then show topological-basis ?B
    unfolding topological-space-class.topological-basis-def
    by (safe intro!: topological-space-class.open-Inter)
      (simp-all add: B generate-topology.Basis subset-eq)
 qed
qed

end

sublocale second-countable-topology <
  countable-basis SOME B. countable B ∧ topological-basis B
  using someI-ex[OF ex-countable-basis]
  by unfold-locales safe

instance prod :: (second-countable-topology, second-countable-topology) second-countable-topology
proof
  obtain A :: 'a set set where countable A topological-basis A
    using ex-countable-basis by auto
  moreover
  obtain B :: 'b set set where countable B topological-basis B
    using ex-countable-basis by auto
  ultimately show ∃ B::('a × 'b) set set. countable B ∧ open = generate-topology
B
    by (auto intro!: exI[of - (λ(a, b). a × b) ` (A × B)] topological-basis-prod
      topological-basis-imp-subbasis)
qed

instance second-countable-topology ⊆ first-countable-topology
proof
  fix x :: 'a
  def B ≡ SOME B::'a set set. countable B ∧ topological-basis B
  then have B: countable B topological-basis B
    using countable-basis is-basis
    by (auto simp: countable-basis is-basis)
  then show ∃ A::nat ⇒ 'a set.
    (∀ i. x ∈ A i ∧ open (A i)) ∧ (∀ S. open S ∧ x ∈ S ⟶ (∃ i. A i ⊆ S))
    by (intro first-countableI[of {b∈B. x ∈ b}])
      (fastforce simp: topological-space-class.topological-basis-def)+
qed
```

16.3 Polish spaces

Textbooks define Polish spaces as completely metrizable. We assume the topology to be complete for a given metric.

```
class polish-space = complete-space + second-countable-topology
```

16.4 General notion of a topology as a value

definition *istopology L* $\longleftrightarrow$
$L\ \{\} \wedge (\forall S\ T.\ L\ S \longrightarrow L\ T \longrightarrow L\ (S \cap T)) \wedge (\forall K.\ Ball\ K\ L \longrightarrow L\ (\bigcup K))$

typedef *'a topology* = $\{L::('a\ set) \Rightarrow bool.\ istopology\ L\}$
morphisms *openin topology*
unfolding *istopology-def* **by** *blast*

lemma *istopology-openin*[*intro*]: *istopology*(*openin U*)
using *openin*[*of U*] **by** *blast*

lemma *topology-inverse'*: *istopology U* $\Longrightarrow$ *openin* (*topology U*) = *U*
using *topology-inverse*[*unfolded mem-Collect-eq*] **.**

lemma *topology-inverse-iff*: *istopology U* $\longleftrightarrow$ *openin* (*topology U*) = *U*
using *topology-inverse*[*of U*] *istopology-openin*[*of topology U*] **by** *auto*

lemma *topology-eq*: *T1* = *T2* $\longleftrightarrow$ ($\forall S.$ *openin T1 S* $\longleftrightarrow$ *openin T2 S*)
proof
assume *T1* = *T2*
then show $\forall S.$ *openin T1 S* $\longleftrightarrow$ *openin T2 S* **by** *simp*
next
assume *H*: $\forall S.$ *openin T1 S* $\longleftrightarrow$ *openin T2 S*
then have *openin T1* = *openin T2* **by** (*simp add*: *fun-eq-iff*)
then have *topology* (*openin T1*) = *topology* (*openin T2*) **by** *simp*
then show *T1* = *T2* **unfolding** *openin-inverse* **.**
qed

Infer the "universe" from union of all sets in the topology.

definition *topspace T* = $\bigcup\{S.\ openin\ T\ S\}$

16.4.1 Main properties of open sets

lemma *openin-clauses*:
fixes *U* :: *'a topology*
shows
openin U {}
$\bigwedge S\ T.$ *openin U S* $\Longrightarrow$ *openin U T* $\Longrightarrow$ *openin U* (*S*$\cap$*T*)
$\bigwedge K.$ ($\forall S \in K.$ *openin U S*) $\Longrightarrow$ *openin U* ($\bigcup K$)
using *openin*[*of U*] **unfolding** *istopology-def mem-Collect-eq* **by** *fast+*

lemma *openin-subset*[*intro*]: *openin U S* $\Longrightarrow$ *S* $\subseteq$ *topspace U*
unfolding *topspace-def* **by** *blast*

lemma *openin-empty*[*simp*]: *openin U* {}
by (*rule openin-clauses*)

lemma *openin-Int*[*intro*]: *openin U S* $\Longrightarrow$ *openin U T* $\Longrightarrow$ *openin U* (*S* $\cap$ *T*)
by (*rule openin-clauses*)

lemma *openin-Union*[*intro*]: $(\bigwedge S.\ S \in K \Longrightarrow \mathit{openin}\ U\ S) \Longrightarrow \mathit{openin}\ U\ (\bigcup K)$
using *openin-clauses* **by** *blast*

lemma *openin-Un*[*intro*]: $\mathit{openin}\ U\ S \Longrightarrow \mathit{openin}\ U\ T \Longrightarrow \mathit{openin}\ U\ (S \cup T)$
using *openin-Union*[*of* $\{S,T\}$ U] **by** *auto*

lemma *openin-topspace*[*intro*, *simp*]: $\mathit{openin}\ U\ (\mathit{topspace}\ U)$
by (*force simp add*: *openin-Union topspace-def*)

lemma *openin-subopen*: $\mathit{openin}\ U\ S \longleftrightarrow (\forall x \in S.\ \exists T.\ \mathit{openin}\ U\ T \wedge x \in T \wedge T \subseteq S)$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
assume *?lhs*
then show *?rhs* **by** *auto*
next
assume *H*: *?rhs*
let *?t* $= \bigcup \{T.\ \mathit{openin}\ U\ T \wedge T \subseteq S\}$
have *openin U ?t* **by** (*force simp add*: *openin-Union*)
also have *?t* $= S$ **using** *H* **by** *auto*
finally show *openin U S* **.**
qed

16.4.2 Closed sets

definition $\mathit{closedin}\ U\ S \longleftrightarrow S \subseteq \mathit{topspace}\ U \wedge \mathit{openin}\ U\ (\mathit{topspace}\ U - S)$

lemma *closedin-subset*: $\mathit{closedin}\ U\ S \Longrightarrow S \subseteq \mathit{topspace}\ U$
by (*metis closedin-def*)

lemma *closedin-empty*[*simp*]: $\mathit{closedin}\ U\ \{\}$
by (*simp add*: *closedin-def*)

lemma *closedin-topspace*[*intro*, *simp*]: $\mathit{closedin}\ U\ (\mathit{topspace}\ U)$
by (*simp add*: *closedin-def*)

lemma *closedin-Un*[*intro*]: $\mathit{closedin}\ U\ S \Longrightarrow \mathit{closedin}\ U\ T \Longrightarrow \mathit{closedin}\ U\ (S \cup T)$
by (*auto simp add*: *Diff-Un closedin-def*)

lemma *Diff-Inter*[*intro*]: $A - \bigcap S = \bigcup \{A - s | s.\ s \in S\}$
by *auto*

lemma *closedin-Inter*[*intro*]:
assumes *Ke*: $K \neq \{\}$
and *Kc*: $\bigwedge S.\ S \in K \Longrightarrow \mathit{closedin}\ U\ S$
shows $\mathit{closedin}\ U\ (\bigcap K)$
using *Ke Kc* **unfolding** *closedin-def Diff-Inter* **by** *auto*

lemma *closedin-INT*[*intro*]:
assumes $A \neq \{\}$ $\bigwedge x.\ x \in A \Longrightarrow$ *closedin* U $(B\ x)$
shows *closedin* U $(\bigcap x{\in}A.\ B\ x)$
apply (*rule closedin-Inter*)
using *assms*
apply *auto*
done

lemma *closedin-Int*[*intro*]: *closedin* $U\ S \Longrightarrow$ *closedin* $U\ T \Longrightarrow$ *closedin* U $(S \cap T)$
using *closedin-Inter*[*of* $\{S,T\}$ U] **by** *auto*

lemma *openin-closedin-eq*: *openin* $U\ S \longleftrightarrow S \subseteq$ *topspace* $U \wedge$ *closedin* U (*topspace* $U - S$)
apply (*auto simp add*: *closedin-def Diff-Diff-Int inf-absorb2*)
apply (*metis openin-subset subset-eq*)
done

lemma *openin-closedin*: $S \subseteq$ *topspace* $U \Longrightarrow$ (*openin* $U\ S \longleftrightarrow$ *closedin* U (*topspace* $U - S$))
by (*simp add*: *openin-closedin-eq*)

lemma *openin-diff*[*intro*]:
assumes *oS*: *openin* $U\ S$
and *cT*: *closedin* $U\ T$
shows *openin* U $(S - T)$
proof −
have $S - T = S \cap$ (*topspace* $U - T$) **using** *openin-subset*[*of* $U\ S$] *oS cT*
by (*auto simp add*: *topspace-def openin-subset*)
then show *?thesis* **using** *oS cT*
by (*auto simp add*: *closedin-def*)
qed

lemma *closedin-diff*[*intro*]:
assumes *oS*: *closedin* $U\ S$
and *cT*: *openin* $U\ T$
shows *closedin* U $(S - T)$
proof −
have $S - T = S \cap$ (*topspace* $U - T$)
using *closedin-subset*[*of* $U\ S$] *oS cT* **by** (*auto simp add*: *topspace-def*)
then show *?thesis*
using *oS cT* **by** (*auto simp add*: *openin-closedin-eq*)
qed

16.4.3 Subspace topology

definition *subtopology* $U\ V =$ *topology* $(\lambda T.\ \exists S.\ T = S \cap V \wedge$ *openin* $U\ S)$

lemma *istopology-subtopology*: *istopology* ($\lambda T.\ \exists S.\ T = S \cap V \wedge$ *openin* $U\ S$)
(**is** *istopology ?L*)
proof −
have *?L* {} **by** *blast*
{
fix *A B*
assume *A*: *?L A* **and** *B*: *?L B*
from *A B* **obtain** *Sa* **and** *Sb* **where** *Sa*: *openin U Sa A* = *Sa* ∩ *V* **and** *Sb*: *openin U Sb B* = *Sb* ∩ *V*
by *blast*
have *A* ∩ *B* = (*Sa* ∩ *Sb*) ∩ *V openin U* (*Sa* ∩ *Sb*)
using *Sa Sb* **by** *blast+*
then have *?L* (*A* ∩ *B*) **by** *blast*
}
moreover
{
fix *K*
assume *K*: *K* ⊆ *Collect ?L*
have *th0*: *Collect ?L* = (λ*S*. *S* ∩ *V*) ‘ *Collect* (*openin U*)
by *blast*
from *K*[*unfolded th0 subset-image-iff*]
obtain *Sk* **where** *Sk*: *Sk* ⊆ *Collect* (*openin U*) *K* = (λ*S*. *S* ∩ *V*) ‘ *Sk*
by *blast*
have ⋃ *K* = (⋃ *Sk*) ∩ *V*
using *Sk* **by** *auto*
moreover have *openin U* (⋃ *Sk*)
using *Sk* **by** (*auto simp add*: *subset-eq*)
ultimately have *?L* (⋃ *K*) **by** *blast*
}
ultimately show *?thesis*
unfolding *subset-eq mem-Collect-eq istopology-def* **by** *auto*
qed

lemma *openin-subtopology*: *openin* (*subtopology U V*) *S* ⟷ (∃ *T*. *openin U T* ∧ *S* = *T* ∩ *V*)
unfolding *subtopology-def topology-inverse′*[*OF istopology-subtopology*]
by *auto*

lemma *topspace-subtopology*: *topspace* (*subtopology U V*) = *topspace U* ∩ *V*
by (*auto simp add*: *topspace-def openin-subtopology*)

lemma *closedin-subtopology*: *closedin* (*subtopology U V*) *S* ⟷ (∃ *T*. *closedin U T* ∧ *S* = *T* ∩ *V*)
unfolding *closedin-def topspace-subtopology*
by (*auto simp add*: *openin-subtopology*)

lemma *openin-subtopology-refl*: *openin* (*subtopology U V*) *V* ⟷ *V* ⊆ *topspace U*
unfolding *openin-subtopology*

by *auto* (*metis IntD1 in-mono openin-subset*)

lemma *subtopology-superset*:
assumes *UV*: *topspace U* ⊆ *V*
shows *subtopology U V* = *U*
proof −
{
fix *S*
{
fix *T*
assume *T*: *openin U T S* = *T* ∩ *V*
from *T openin-subset*[*OF T*(*1*)] *UV* **have** *eq*: *S* = *T*
by *blast*
have *openin U S*
unfolding *eq* **using** *T* **by** *blast*
}
moreover
{
assume *S*: *openin U S*
then have ∃ *T*. *openin U T* ∧ *S* = *T* ∩ *V*
using *openin-subset*[*OF S*] *UV* **by** *auto*
}
ultimately have (∃ *T*. *openin U T* ∧ *S* = *T* ∩ *V*) ⟷ *openin U S*
by *blast*
}
then show *?thesis*
unfolding *topology-eq openin-subtopology* **by** *blast*
qed

lemma *subtopology-topspace*[*simp*]: *subtopology U* (*topspace U*) = *U*
by (*simp add*: *subtopology-superset*)

lemma *subtopology-UNIV*[*simp*]: *subtopology U UNIV* = *U*
by (*simp add*: *subtopology-superset*)

lemma *openin-subtopology-empty*:
openin (*subtopology U* {}) *s* ⟷ *s* = {}
by (*metis Int-empty-right openin-empty openin-subtopology*)

lemma *closedin-subtopology-empty*:
closedin (*subtopology U* {}) *s* ⟷ *s* = {}
by (*metis Int-empty-right closedin-empty closedin-subtopology*)

lemma *closedin-subtopology-refl*:
closedin (*subtopology U u*) *u* ⟷ *u* ⊆ *topspace U*
by (*metis closedin-def closedin-topspace inf.absorb-iff2 le-inf-iff topspace-subtopology*)

lemma *openin-imp-subset*:
openin (*subtopology U s*) *t* ⟹ *t* ⊆ *s*

by (*metis Int-iff openin-subtopology subsetI*)

lemma *closedin-imp-subset*:
closedin (*subtopology U s*) *t* ⟹ *t* ⊆ *s*
by (*simp add*: *closedin-def topspace-subtopology*)

lemma *openin-subtopology-Un*:
openin (*subtopology U t*) *s* ∧ *openin* (*subtopology U u*) *s*
⟹ *openin* (*subtopology U* (*t* ∪ *u*)) *s*
by (*simp add*: *openin-subtopology*) *blast*

16.4.4 The standard Euclidean topology

definition *euclidean* :: *'a*::*topological-space topology*
where *euclidean* = *topology open*

lemma *open-openin*: *open S* ⟷ *openin euclidean S*
unfolding *euclidean-def*
apply (*rule cong*[**where** *x*=*S* **and** *y*=*S*])
apply (*rule topology-inverse*[*symmetric*])
apply (*auto simp add*: *istopology-def*)
done

lemma *topspace-euclidean*: *topspace euclidean* = *UNIV*
apply (*simp add*: *topspace-def*)
apply (*rule set-eqI*)
apply (*auto simp add*: *open-openin*[*symmetric*])
done

lemma *topspace-euclidean-subtopology*[*simp*]: *topspace* (*subtopology euclidean S*)
= *S*
by (*simp add*: *topspace-euclidean topspace-subtopology*)

lemma *closed-closedin*: *closed S* ⟷ *closedin euclidean S*
by (*simp add*: *closed-def closedin-def topspace-euclidean open-openin Compl-eq-Diff-UNIV*)

lemma *open-subopen*: *open S* ⟷ (∀ *x*∈*S*. ∃ *T*. *open T* ∧ *x* ∈ *T* ∧ *T* ⊆ *S*)
by (*simp add*: *open-openin openin-subopen*[*symmetric*])

lemma *openin-subtopology-self* [*simp*]: *openin* (*subtopology euclidean S*) *S*
by (*metis openin-topspace topspace-euclidean-subtopology*)

Basic "localization" results are handy for connectedness.

lemma *openin-open*: *openin* (*subtopology euclidean U*) *S* ⟷ (∃ *T*. *open T* ∧ (*S*
= *U* ∩ *T*))
by (*auto simp add*: *openin-subtopology open-openin*[*symmetric*])

lemma *openin-open-Int*[*intro*]: *open S* ⟹ *openin* (*subtopology euclidean U*) (*U*
∩ *S*)

by (*auto simp add*: *openin-open*)

lemma *open-openin-trans*[*trans*]:
open S $\Longrightarrow$ *open T* $\Longrightarrow$ $T \subseteq S \Longrightarrow$ *openin* (*subtopology euclidean S*) *T*
by (*metis Int-absorb1 openin-open-Int*)

lemma *open-subset*: $S \subseteq T \Longrightarrow$ *open S* $\Longrightarrow$ *openin* (*subtopology euclidean T*) *S*
by (*auto simp add*: *openin-open*)

lemma *closedin-closed*: *closedin* (*subtopology euclidean U*) *S* $\longleftrightarrow$ ($\exists\, T.$ *closed T* $\wedge$ $S = U \cap T$)
by (*simp add*: *closedin-subtopology closed-closedin Int-ac*)

lemma *closedin-closed-Int*: *closed S* $\Longrightarrow$ *closedin* (*subtopology euclidean U*) ($U \cap S$)
by (*metis closedin-closed*)

lemma *closed-closedin-trans*:
closed S $\Longrightarrow$ *closed T* $\Longrightarrow$ $T \subseteq S \Longrightarrow$ *closedin* (*subtopology euclidean S*) *T*
by (*metis closedin-closed inf.absorb2*)

lemma *closed-subset*: $S \subseteq T \Longrightarrow$ *closed S* $\Longrightarrow$ *closedin* (*subtopology euclidean T*) *S*
by (*auto simp add*: *closedin-closed*)

lemma *openin-euclidean-subtopology-iff*:
fixes *S U* :: *'a*::*metric-space set*
shows *openin* (*subtopology euclidean U*) *S* $\longleftrightarrow$
$S \subseteq U \wedge (\forall\, x{\in}S.\ \exists\, e{>}0.\ \forall\, x'{\in}U.\ dist\ x'\ x < e \longrightarrow x'{\in}\ S)$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
assume *?lhs*
then show *?rhs*
unfolding *openin-open open-dist* **by** *blast*
next
def $T \equiv \{x.\ \exists\, a{\in}S.\ \exists\, d{>}0.\ (\forall\, y{\in}U.\ dist\ y\ a < d \longrightarrow y \in S) \wedge dist\ x\ a < d\}$
have *1*: $\forall\, x{\in}T.\ \exists\, e{>}0.\ \forall\, y.\ dist\ y\ x < e \longrightarrow y \in T$
unfolding *T-def*
apply *clarsimp*
apply (*rule-tac* $x{=}d - dist\ x\ a$ **in** *exI*)
apply (*clarsimp simp add*: *less-diff-eq*)
by (*metis dist-commute dist-triangle-lt*)
assume *?rhs* **then have** *2*: $S = U \cap T$
unfolding *T-def*
by *auto* (*metis dist-self*)
from *1 2* **show** *?lhs*
unfolding *openin-open open-dist* **by** *fast*
qed

lemma *connected-openin*:
  connected s $\longleftrightarrow$
  $\sim$($\exists$ *e1 e2*. *openin* (*subtopology euclidean s*) *e1* $\wedge$
    openin (*subtopology euclidean s*) *e2* $\wedge$
    s $\subseteq$ *e1* $\cup$ *e2* $\wedge$ *e1* $\cap$ *e2* = {} $\wedge$ *e1* $\neq$ {} $\wedge$ *e2* $\neq$ {})
 apply (*simp add*: *connected-def openin-open*, *safe*)
 apply (*simp-all*, *blast*+) — slow
 done

lemma *connected-openin-eq*:
  connected s $\longleftrightarrow$
  $\sim$($\exists$ *e1 e2*. *openin* (*subtopology euclidean s*) *e1* $\wedge$
    openin (*subtopology euclidean s*) *e2* $\wedge$
    e1 $\cup$ *e2* = *s* $\wedge$ *e1* $\cap$ *e2* = {} $\wedge$
    e1 $\neq$ {} $\wedge$ *e2* $\neq$ {})
 apply (*simp add*: *connected-openin*, *safe*)
 apply *blast*
 by (*metis Int-lower1 Un-subset-iff openin-open subset-antisym*)

lemma *connected-closedin*:
  connected s $\longleftrightarrow$
  $\sim$($\exists$ *e1 e2*.
    closedin (*subtopology euclidean s*) *e1* $\wedge$
    closedin (*subtopology euclidean s*) *e2* $\wedge$
    s $\subseteq$ *e1* $\cup$ *e2* $\wedge$ *e1* $\cap$ *e2* = {} $\wedge$
    e1 $\neq$ {} $\wedge$ *e2* $\neq$ {})
proof −
 { fix *A B x x'*
  assume *s-sub*: *s* $\subseteq$ *A* $\cup$ *B*
    and *disj*: *A* $\cap$ *B* $\cap$ *s* = {}
    and *x*: *x* $\in$ *s x* $\in$ *B* **and** *x'*: *x'* $\in$ *s x'* $\in$ *A*
    and *cl*: *closed A closed B*
  assume $\forall$ *e1*. ($\forall$ *T*. *closed T* $\longrightarrow$ *e1* $\neq$ *s* $\cap$ *T*) $\vee$ ($\forall$ *e2*. *e1* $\cap$ *e2* = {} $\longrightarrow$ *s* $\subseteq$ *e1* $\cup$ *e2* $\longrightarrow$ ($\forall$ *T*. *closed T* $\longrightarrow$ *e2* $\neq$ *s* $\cap$ *T*) $\vee$ *e1* = {} $\vee$ *e2* = {})
  then have $\bigwedge$*C D*. *s* $\cap$ *C* = {} $\vee$ *s* $\cap$ *D* = {} $\vee$ *s* $\cap$ (*C* $\cap$ (*s* $\cap$ *D*)) $\neq$ {} $\vee$ $\neg$ *s* $\subseteq$ *s* $\cap$ (*C* $\cup$ *D*) $\vee$ $\neg$ *closed C* $\vee$ $\neg$ *closed D*
    by (*metis* (*no-types*) *Int-Un-distrib Int-assoc*)
  moreover have *s* $\cap$ (*A* $\cap$ *B*) = {} *s* $\cap$ (*A* $\cup$ *B*) = *s s* $\cap$ *B* $\neq$ {}
    using *disj s-sub x* **by** *blast*+
  ultimately have *s* $\cap$ *A* = {}
    using *cl* **by** (*metis inf.left-commute inf-bot-right order-refl*)
  then have *False*
    using *x'* **by** *blast*
 } note $*$ = *this*
 show *?thesis*
  apply (*simp add*: *connected-closed closedin-closed*)
  apply (*safe*; *simp*)
  apply *blast*
  apply (*blast intro*: $*$)

done
qed

lemma *connected-closedin-eq*:
connected s ⟷
~(∃ *e1 e2*.
closedin (*subtopology euclidean s*) *e1* ∧
closedin (*subtopology euclidean s*) *e2* ∧
e1 ∪ *e2* = *s* ∧ *e1* ∩ *e2* = {} ∧
e1 ≠ {} ∧ *e2* ≠ {})
apply (*simp add*: *connected-closedin*, *safe*)
apply *blast*
by (*metis Int-lower1 Un-subset-iff closedin-closed subset-antisym*)

These "transitivity" results are handy too

lemma *openin-trans*[*trans*]:
openin (*subtopology euclidean T*) *S* ⟹ *openin* (*subtopology euclidean U*) *T* ⟹
openin (*subtopology euclidean U*) *S*
unfolding *open-openin openin-open* **by** *blast*

lemma *openin-open-trans*: *openin* (*subtopology euclidean T*) *S* ⟹ *open T* ⟹ *open S*
by (*auto simp add*: *openin-open intro*: *openin-trans*)

lemma *closedin-trans*[*trans*]:
closedin (*subtopology euclidean T*) *S* ⟹ *closedin* (*subtopology euclidean U*) *T* ⟹
closedin (*subtopology euclidean U*) *S*
by (*auto simp add*: *closedin-closed closed-closedin closed-Inter Int-assoc*)

lemma *closedin-closed-trans*: *closedin* (*subtopology euclidean T*) *S* ⟹ *closed T* ⟹ *closed S*
by (*auto simp add*: *closedin-closed intro*: *closedin-trans*)

lemma *openin-subtopology-Int-subset*:
⟦*openin* (*subtopology euclidean u*) (*u* ∩ *S*); *v* ⊆ *u*⟧ ⟹ *openin* (*subtopology euclidean v*) (*v* ∩ *S*)
by (*auto simp*: *openin-subtopology*)

lemma *openin-open-eq*: *open s* ⟹ (*openin* (*subtopology euclidean s*) *t* ⟷ *open t* ∧ *t* ⊆ *s*)
using *open-subset openin-open-trans openin-subset* **by** *fastforce*

16.5 Open and closed balls

definition *ball* :: *'a*::*metric-space* ⇒ *real* ⇒ *'a set*
where *ball x e* = {*y*. *dist x y* < *e*}

definition *cball* :: *'a*::*metric-space* ⇒ *real* ⇒ *'a set*

where *cball x e* = {*y*. *dist x y* ≤ *e*}

definition *sphere* :: *'a*::*metric-space* ⇒ *real* ⇒ *'a set*
where *sphere x e* = {*y*. *dist x y* = *e*}

lemma *mem-ball* [*simp*]: *y* ∈ *ball x e* ⟷ *dist x y* < *e*
by (*simp add*: *ball-def*)

lemma *mem-cball* [*simp*]: *y* ∈ *cball x e* ⟷ *dist x y* ≤ *e*
by (*simp add*: *cball-def*)

lemma *mem-sphere* [*simp*]: *y* ∈ *sphere x e* ⟷ *dist x y* = *e*
by (*simp add*: *sphere-def*)

lemma *ball-trivial* [*simp*]: *ball x 0* = {}
by (*simp add*: *ball-def*)

lemma *cball-trivial* [*simp*]: *cball x 0* = {*x*}
by (*simp add*: *cball-def*)

lemma *mem-ball-0* [*simp*]:
fixes *x* :: *'a*::*real-normed-vector*
shows *x* ∈ *ball 0 e* ⟷ *norm x* < *e*
by (*simp add*: *dist-norm*)

lemma *mem-cball-0* [*simp*]:
fixes *x* :: *'a*::*real-normed-vector*
shows *x* ∈ *cball 0 e* ⟷ *norm x* ≤ *e*
by (*simp add*: *dist-norm*)

lemma *centre-in-ball* [*simp*]: *x* ∈ *ball x e* ⟷ *0* < *e*
by *simp*

lemma *centre-in-cball* [*simp*]: *x* ∈ *cball x e* ⟷ *0* ≤ *e*
by *simp*

lemma *ball-subset-cball* [*simp*,*intro*]: *ball x e* ⊆ *cball x e*
by (*simp add*: *subset-eq*)

lemma *sphere-cball* [*simp*,*intro*]: *sphere z r* ⊆ *cball z r*
by *force*

lemma *subset-ball*[*intro*]: *d* ≤ *e* ⟹ *ball x d* ⊆ *ball x e*
by (*simp add*: *subset-eq*)

lemma *subset-cball*[*intro*]: *d* ≤ *e* ⟹ *cball x d* ⊆ *cball x e*
by (*simp add*: *subset-eq*)

lemma *ball-max-Un*: *ball a* (*max r s*) = *ball a r* ∪ *ball a s*

by (*simp add*: *set-eq-iff*) *arith*

lemma *ball-min-Int*: *ball a* (*min r s*) = *ball a r* ∩ *ball a s*
by (*simp add*: *set-eq-iff*)

lemma *cball-diff-eq-sphere*: *cball a r* − *ball a r* = {*x*. *dist x a* = *r*}
by (*auto simp*: *cball-def ball-def dist-commute*)

lemma *image-add-ball* [*simp*]:
fixes *a* :: 'a::*real-normed-vector*
shows *op* + *b* ‘ *ball a r* = *ball* (*a*+*b*) *r*
apply (*intro equalityI subsetI*)
apply (*force simp*: *dist-norm*)
apply (*rule-tac x*=*x*−*b* **in** *image-eqI*)
apply (*auto simp*: *dist-norm algebra-simps*)
done

lemma *image-add-cball* [*simp*]:
fixes *a* :: 'a::*real-normed-vector*
shows *op* + *b* ‘ *cball a r* = *cball* (*a*+*b*) *r*
apply (*intro equalityI subsetI*)
apply (*force simp*: *dist-norm*)
apply (*rule-tac x*=*x*−*b* **in** *image-eqI*)
apply (*auto simp*: *dist-norm algebra-simps*)
done

lemma *open-ball* [*intro*, *simp*]: *open* (*ball x e*)
proof −
have *open* (*dist x* −‘ {..<*e*})
by (*intro open-vimage open-lessThan continuous-intros*)
also have *dist x* −‘ {..<*e*} = *ball x e*
by *auto*
finally show *?thesis* **.**
qed

lemma *open-contains-ball*: *open S* ⟷ (∀ *x*∈*S*. ∃ *e*>0. *ball x e* ⊆ *S*)
unfolding *open-dist subset-eq mem-ball Ball-def dist-commute* **..**

lemma *openI* [*intro?*]: (⋀*x*. *x*∈*S* ⟹ ∃ *e*>0. *ball x e* ⊆ *S*) ⟹ *open S*
by (*auto simp*: *open-contains-ball*)

lemma *openE*[*elim?*]:
assumes *open S x*∈*S*
obtains *e* **where** *e*>0 *ball x e* ⊆ *S*
using *assms* **unfolding** *open-contains-ball* **by** *auto*

lemma *open-contains-ball-eq*: *open S* ⟹ *x*∈*S* ⟷ (∃ *e*>0. *ball x e* ⊆ *S*)
by (*metis open-contains-ball subset-eq centre-in-ball*)

lemma *openin-contains-ball*:
  openin (*subtopology euclidean t*) *s* ⟷
  s ⊆ *t* ∧ (∀ *x* ∈ *s*. ∃ *e*. *0* < *e* ∧ *ball x e* ∩ *t* ⊆ *s*)
  (**is** *?lhs* = *?rhs*)
proof
 assume *?lhs*
 then show *?rhs*
  apply (*simp add*: *openin-open*)
 apply (*metis Int-commute Int-mono inf.cobounded2 open-contains-ball order-refl subsetCE*)
  done
next
 assume *?rhs*
 then show *?lhs*
  apply (*simp add*: *openin-euclidean-subtopology-iff*)
  by (*metis* (*no-types*) *Int-iff dist-commute inf.absorb-iff2 mem-ball*)
qed

lemma *openin-contains-cball*:
  openin (*subtopology euclidean t*) *s* ⟷
    s ⊆ *t* ∧
    (∀ *x* ∈ *s*. ∃ *e*. *0* < *e* ∧ *cball x e* ∩ *t* ⊆ *s*)
apply (*simp add*: *openin-contains-ball*)
apply (*rule iffI*)
apply (*auto dest*!: *bspec*)
apply (*rule-tac x*=*e*/*2* **in** *exI*)
apply *force*+
done

lemma *ball-eq-empty*[*simp*]: *ball x e* = {} ⟷ *e* ≤ *0*
 unfolding *mem-ball set-eq-iff*
 apply (*simp add*: *not-less*)
 apply (*metis zero-le-dist order-trans dist-self*)
 done

lemma *ball-empty*: *e* ≤ *0* ⟹ *ball x e* = {} **by** *simp*

lemma *euclidean-dist-l2*:
 fixes *x y* :: *'a* :: *euclidean-space*
 shows *dist x y* = *setL2* (λ*i*. *dist* (*x* ∙ *i*) (*y* ∙ *i*)) *Basis*
 unfolding *dist-norm norm-eq-sqrt-inner setL2-def*
 by (*subst euclidean-inner*) (*simp add*: *power2-eq-square inner-diff-left*)

lemma *eventually-nhds-ball*: *d* > *0* ⟹ *eventually* (λ*x*. *x* ∈ *ball z d*) (*nhds z*)
 by (*rule eventually-nhds-in-open*) *simp-all*

lemma *eventually-at-ball*: *d* > *0* ⟹ *eventually* (λ*t*. *t* ∈ *ball z d* ∧ *t* ∈ *A*) (*at z within A*)
 unfolding *eventually-at* **by** (*intro exI*[*of* - *d*]) (*simp-all add*: *dist-commute*)

lemma *eventually-at-ball'*: $d > 0 \Longrightarrow$ *eventually* $(\lambda t.\ t \in ball\ z\ d \wedge t \neq z \wedge t \in A)$ $(at\ z\ within\ A)$
unfolding *eventually-at* **by** (*intro exI*[*of* - *d*]) (*simp-all add*: *dist-commute*)

16.6 Boxes

abbreviation *One* :: *'a*::*euclidean-space*
where $One \equiv \sum Basis$

definition (**in** *euclidean-space*) *eucl-less* (**infix** $<e$ *50*)
where $eucl\text{-}less\ a\ b \longleftrightarrow (\forall i{\in}Basis.\ a \cdot i < b \cdot i)$

definition *box-eucl-less*: $box\ a\ b = \{x.\ a <e\ x \wedge x <e\ b\}$
definition $cbox\ a\ b = \{x.\ \forall i{\in}Basis.\ a \cdot i \leq x \cdot i \wedge x \cdot i \leq b \cdot i\}$

lemma *box-def*: $box\ a\ b = \{x.\ \forall i{\in}Basis.\ a \cdot i < x \cdot i \wedge x \cdot i < b \cdot i\}$
and *in-box-eucl-less*: $x \in box\ a\ b \longleftrightarrow a <e\ x \wedge x <e\ b$
and *mem-box*: $x \in box\ a\ b \longleftrightarrow (\forall i{\in}Basis.\ a \cdot i < x \cdot i \wedge x \cdot i < b \cdot i)$
$x \in cbox\ a\ b \longleftrightarrow (\forall i{\in}Basis.\ a \cdot i \leq x \cdot i \wedge x \cdot i \leq b \cdot i)$
by (*auto simp*: *box-eucl-less eucl-less-def cbox-def*)

lemma *cbox-Pair-eq*: $cbox\ (a,\ c)\ (b,\ d) = cbox\ a\ b \times cbox\ c\ d$
by (*force simp*: *cbox-def Basis-prod-def*)

lemma *cbox-Pair-iff* [*iff*]: $(x,\ y) \in cbox\ (a,\ c)\ (b,\ d) \longleftrightarrow x \in cbox\ a\ b \wedge y \in cbox\ c\ d$
by (*force simp*: *cbox-Pair-eq*)

lemma *cbox-Pair-eq-0*: $cbox\ (a,\ c)\ (b,\ d) = \{\} \longleftrightarrow cbox\ a\ b = \{\} \vee cbox\ c\ d = \{\}$
by (*force simp*: *cbox-Pair-eq*)

lemma *swap-cbox-Pair* [*simp*]: $prod.swap\ `\ cbox\ (c,\ a)\ (d,\ b) = cbox\ (a,c)\ (b,d)$
by *auto*

lemma *mem-box-real*[*simp*]:
$(x{::}real) \in box\ a\ b \longleftrightarrow a < x \wedge x < b$
$(x{::}real) \in cbox\ a\ b \longleftrightarrow a \leq x \wedge x \leq b$
by (*auto simp*: *mem-box*)

lemma *box-real*[*simp*]:
fixes *a b*:: *real*
shows $box\ a\ b = \{a <..< b\}$ $cbox\ a\ b = \{a\ ..\ b\}$
by *auto*

lemma *box-Int-box*:
fixes *a* :: *'a*::*euclidean-space*
shows $box\ a\ b \cap box\ c\ d =$

box ($\sum i{\in}Basis.\ max\ (a{\cdot}i)\ (c{\cdot}i) *_R i$) ($\sum i{\in}Basis.\ min\ (b{\cdot}i)\ (d{\cdot}i) *_R i$)
unfolding *set-eq-iff* **and** *Int-iff* **and** *mem-box* **by** *auto*

lemma *rational-boxes*:
fixes $x :: 'a{::}euclidean\text{-}space$
assumes $e > 0$
shows $\exists\, a\ b.\ (\forall\, i{\in}Basis.\ a \cdot i \in \mathbb{Q} \wedge b \cdot i \in \mathbb{Q}\ ) \wedge x \in box\ a\ b \wedge box\ a\ b \subseteq ball\ x\ e$
proof −
def $e' \equiv e\ /\ (2 * sqrt\ (real\ (DIM\ ('a))))$
then have *e*: $e' > 0$
using *assms* **by** (*auto simp*: *DIM-positive*)
have $\forall\, i.\ \exists\, y.\ y \in \mathbb{Q} \wedge y < x \cdot i \wedge x \cdot i - y < e'$ (**is** $\forall\, i.\ ?th\ i$)
proof
fix *i*
from *Rats-dense-in-real*[*of* $x \cdot i - e'\ x \cdot i$] *e*
show *?th i* **by** *auto*
qed
from *choice*[*OF this*] **obtain** *a* **where**
a: $\forall\, xa.\ a\ xa \in \mathbb{Q} \wedge a\ xa < x \cdot xa \wedge x \cdot xa - a\ xa < e'$ **..**
have $\forall\, i.\ \exists\, y.\ y \in \mathbb{Q} \wedge x \cdot i < y \wedge y - x \cdot i < e'$ (**is** $\forall\, i.\ ?th\ i$)
proof
fix *i*
from *Rats-dense-in-real*[*of* $x \cdot i\ x \cdot i + e'$] *e*
show *?th i* **by** *auto*
qed
from *choice*[*OF this*] **obtain** *b* **where**
b: $\forall\, xa.\ b\ xa \in \mathbb{Q} \wedge x \cdot xa < b\ xa \wedge b\ xa - x \cdot xa < e'$ **..**
let $?a = \sum i{\in}Basis.\ a\ i *_R i$ **and** $?b = \sum i{\in}Basis.\ b\ i *_R i$
show *?thesis*
proof (*rule exI*[*of* - *?a*], *rule exI*[*of* - *?b*], *safe*)
fix $y :: 'a$
assume $*$: $y \in box\ ?a\ ?b$
have $dist\ x\ y = sqrt\ (\sum i{\in}Basis.\ (dist\ (x \cdot i)\ (y \cdot i))^2)$
unfolding *setL2-def*[*symmetric*] **by** (*rule euclidean-dist-l2*)
also have $\ldots < sqrt\ (\sum (i::'a){\in}Basis.\ e\hat{}2\ /\ real\ (DIM('a)))$
proof (*rule real-sqrt-less-mono*, *rule setsum-strict-mono*)
fix $i :: 'a$
assume *i*: $i \in Basis$
have $a\ i < y{\cdot}i \wedge y{\cdot}i < b\ i$
using $*\ i$ **by** (*auto simp*: *box-def*)
moreover have $a\ i < x{\cdot}i\ x{\cdot}i - a\ i < e'$
using *a* **by** *auto*
moreover have $x{\cdot}i < b\ i\ b\ i - x{\cdot}i < e'$
using *b* **by** *auto*
ultimately have $|x{\cdot}i - y{\cdot}i| < 2 * e'$
by *auto*
then have $dist\ (x \cdot i)\ (y \cdot i) < e/sqrt\ (real\ (DIM('a)))$
unfolding *e'-def* **by** (*auto simp*: *dist-real-def*)

```
      then have (dist (x · i) (y · i))² < (e/sqrt (real (DIM('a))))²
        by (rule power-strict-mono) auto
      then show (dist (x · i) (y · i))² < e² / real DIM('a)
        by (simp add: power-divide)
    qed auto
    also have ... = e
      using ‹0 < e› by simp
    finally show y ∈ ball x e
      by (auto simp: ball-def)
  qed (insert a b, auto simp: box-def)
qed

lemma open-UNION-box:
  fixes M :: 'a::euclidean-space set
  assumes open M
  defines a' ≡ λf :: 'a ⇒ real × real. (∑(i::'a)∈Basis. fst (f i) *R i)
  defines b' ≡ λf :: 'a ⇒ real × real. (∑(i::'a)∈Basis. snd (f i) *R i)
  defines I ≡ {f∈Basis →E ℚ × ℚ. box (a' f) (b' f) ⊆ M}
  shows M = (⋃f∈I. box (a' f) (b' f))
proof −
  have x ∈ (⋃f∈I. box (a' f) (b' f)) if x ∈ M for x
  proof −
    obtain e where e: e > 0 ball x e ⊆ M
      using openE[OF ‹open M› ‹x ∈ M›] by auto
    moreover obtain a b where ab:
      x ∈ box a b
      ∀i ∈ Basis. a · i ∈ ℚ
      ∀i∈Basis. b · i ∈ ℚ
      box a b ⊆ ball x e
      using rational-boxes[OF e(1)] by metis
    ultimately show ?thesis
      by (intro UN-I[of λi∈Basis. (a · i, b · i)])
        (auto simp: euclidean-representation I-def a'-def b'-def)
  qed
  then show ?thesis by (auto simp: I-def)
qed

lemma box-eq-empty:
  fixes a :: 'a::euclidean-space
  shows (box a b = {} ⟷ (∃i∈Basis. b•i ≤ a•i)) (is ?th1)
    and (cbox a b = {} ⟷ (∃i∈Basis. b•i < a•i)) (is ?th2)
proof −
  {
    fix i x
    assume i: i∈Basis and as:b•i ≤ a•i and x:x∈box a b
    then have a · i < x · i ∧ x · i < b · i
      unfolding mem-box by (auto simp: box-def)
    then have a•i < b•i by auto
    then have False using as by auto
```

```
  }
  moreover
  {
    assume as: ∀ i∈Basis. ¬ (b•i ≤ a•i)
    let ?x = (1/2) *R (a + b)
    {
      fix i :: 'a
      assume i: i ∈ Basis
      have a•i < b•i
        using as[THEN bspec[where x=i]] i by auto
      then have a•i < ((1/2) *R (a+b)) • i ((1/2) *R (a+b)) • i < b•i
        by (auto simp: inner-add-left)
    }
    then have box a b ≠ {}
      using mem-box(1)[of ?x a b] by auto
  }
  ultimately show ?th1 by blast

  {
    fix i x
    assume i: i ∈ Basis and as:b•i < a•i and x:x∈cbox a b
    then have a • i ≤ x • i ∧ x • i ≤ b • i
      unfolding mem-box by auto
    then have a•i ≤ b•i by auto
    then have False using as by auto
  }
  moreover
  {
    assume as:∀ i∈Basis. ¬ (b•i < a•i)
    let ?x = (1/2) *R (a + b)
    {
      fix i :: 'a
      assume i:i ∈ Basis
      have a•i ≤ b•i
        using as[THEN bspec[where x=i]] i by auto
      then have a•i ≤ ((1/2) *R (a+b)) • i ((1/2) *R (a+b)) • i ≤ b•i
        by (auto simp: inner-add-left)
    }
    then have cbox a b ≠ {}
      using mem-box(2)[of ?x a b] by auto
  }
  ultimately show ?th2 by blast
qed

lemma box-ne-empty:
  fixes a :: 'a::euclidean-space
  shows cbox a b ≠ {} ⟷ (∀ i∈Basis. a•i ≤ b•i)
  and box a b ≠ {} ⟷ (∀ i∈Basis. a•i < b•i)
  unfolding box-eq-empty[of a b] by fastforce+
```

lemma
fixes *a* :: *'a*::*euclidean-space*
shows *cbox-sing*: *cbox a a* = {*a*}
and *box-sing*: *box a a* = {}
unfolding *set-eq-iff mem-box eq-iff* [*symmetric*]
by (*auto intro*!: *euclidean-eqI*[**where** *'a*=*'a*])
(*metis all-not-in-conv nonempty-Basis*)

lemma *subset-box-imp*:
fixes *a* :: *'a*::*euclidean-space*
shows (∀ *i*∈*Basis*. *a*•*i* ≤ *c*•*i* ∧ *d*•*i* ≤ *b*•*i*) ⟹ *cbox c d* ⊆ *cbox a b*
and (∀ *i*∈*Basis*. *a*•*i* < *c*•*i* ∧ *d*•*i* < *b*•*i*) ⟹ *cbox c d* ⊆ *box a b*
and (∀ *i*∈*Basis*. *a*•*i* ≤ *c*•*i* ∧ *d*•*i* ≤ *b*•*i*) ⟹ *box c d* ⊆ *cbox a b*
and (∀ *i*∈*Basis*. *a*•*i* ≤ *c*•*i* ∧ *d*•*i* ≤ *b*•*i*) ⟹ *box c d* ⊆ *box a b*
unfolding *subset-eq*[*unfolded Ball-def*] **unfolding** *mem-box*
by (*best intro*: *order-trans less-le-trans le-less-trans less-imp-le*)+

lemma *box-subset-cbox*:
fixes *a* :: *'a*::*euclidean-space*
shows *box a b* ⊆ *cbox a b*
unfolding *subset-eq* [*unfolded Ball-def*] *mem-box*
by (*fast intro*: *less-imp-le*)

lemma *subset-box*:
fixes *a* :: *'a*::*euclidean-space*
shows *cbox c d* ⊆ *cbox a b* ⟷ (∀ *i*∈*Basis*. *c*•*i* ≤ *d*•*i*) −−> (∀ *i*∈*Basis*. *a*•*i* ≤ *c*•*i* ∧ *d*•*i* ≤ *b*•*i*) (**is** *?th1*)
and *cbox c d* ⊆ *box a b* ⟷ (∀ *i*∈*Basis*. *c*•*i* ≤ *d*•*i*) −−> (∀ *i*∈*Basis*. *a*•*i* < *c*•*i* ∧ *d*•*i* < *b*•*i*) (**is** *?th2*)
and *box c d* ⊆ *cbox a b* ⟷ (∀ *i*∈*Basis*. *c*•*i* < *d*•*i*) −−> (∀ *i*∈*Basis*. *a*•*i* ≤ *c*•*i* ∧ *d*•*i* ≤ *b*•*i*) (**is** *?th3*)
and *box c d* ⊆ *box a b* ⟷ (∀ *i*∈*Basis*. *c*•*i* < *d*•*i*) −−> (∀ *i*∈*Basis*. *a*•*i* ≤ *c*•*i* ∧ *d*•*i* ≤ *b*•*i*) (**is** *?th4*)
proof −
show *?th1*
unfolding *subset-eq* **and** *Ball-def* **and** *mem-box*
by (*auto intro*: *order-trans*)
show *?th2*
unfolding *subset-eq* **and** *Ball-def* **and** *mem-box*
by (*auto intro*: *le-less-trans less-le-trans order-trans less-imp-le*)
{
assume *as*: *box c d* ⊆ *cbox a b* ∀ *i*∈*Basis*. *c*•*i* < *d*•*i*
then have *box c d* ≠ {}
unfolding *box-eq-empty* **by** *auto*
fix *i* :: *'a*
assume *i*: *i* ∈ *Basis*

{

```
    let ?x = (∑j∈Basis. (if j=i then ((min (a•j) (d•j))+c•j)/2 else (c•j+d•j)/2)
*R j)::'a
     assume as2: a•i > c•i
     {
      fix j :: 'a
      assume j: j ∈ Basis
      then have c • j < ?x • j ∧ ?x • j < d • j
       apply (cases j = i)
       using as(2)[THEN bspec[where x=j]] i
       apply (auto simp add: as2)
       done
     }
     then have ?x∈box c d
      using i unfolding mem-box by auto
     moreover
     have ?x ∉ cbox a b
      unfolding mem-box
      apply auto
      apply (rule-tac x=i in bexI)
      using as(2)[THEN bspec[where x=i]] and as2 i
      apply auto
      done
     ultimately have False using as by auto
    }
    then have a•i ≤ c•i by (rule ccontr) auto
    moreover
    {
     let ?x = (∑j∈Basis. (if j=i then ((max (b•j) (c•j))+d•j)/2 else (c•j+d•j)/2)
*R j)::'a
     assume as2: b•i < d•i
     {
      fix j :: 'a
      assume j∈Basis
      then have d • j > ?x • j ∧ ?x • j > c • j
       apply (cases j = i)
       using as(2)[THEN bspec[where x=j]]
       apply (auto simp add: as2)
       done
     }
     then have ?x∈box c d
      unfolding mem-box by auto
     moreover
     have ?x∉cbox a b
      unfolding mem-box
      apply auto
      apply (rule-tac x=i in bexI)
      using as(2)[THEN bspec[where x=i]] and as2 using i
      apply auto
      done
```

```
    ultimately have False using as by auto
   }
   then have b•i ≥ d•i by (rule ccontr) auto
   ultimately
   have a•i ≤ c•i ∧ d•i ≤ b•i by auto
  } note part1 = this
  show ?th3
   unfolding subset-eq and Ball-def and mem-box
   apply (rule, rule, rule, rule)
   apply (rule part1)
   unfolding subset-eq and Ball-def and mem-box
   prefer 4
   apply auto
   apply (erule-tac x=xa in allE, erule-tac x=xa in allE, fastforce)+
   done
  {
   assume as: box c d ⊆ box a b ∀ i∈Basis. c•i < d•i
   fix i :: 'a
   assume i:i∈Basis
   from as(1) have box c d ⊆ cbox a b
    using box-subset-cbox[of a b] by auto
   then have a•i ≤ c•i ∧ d•i ≤ b•i
    using part1 and as(2) using i by auto
  } note * = this
  show ?th4
   unfolding subset-eq and Ball-def and mem-box
   apply (rule, rule, rule, rule)
   apply (rule *)
   unfolding subset-eq and Ball-def and mem-box
   prefer 4
   apply auto
   apply (erule-tac x=xa in allE, simp)+
   done
qed

lemma inter-interval:
  fixes a :: 'a::euclidean-space
  shows cbox a b ∩ cbox c d =
    cbox (∑ i∈Basis. max (a•i) (c•i) *R i) (∑ i∈Basis. min (b•i) (d•i) *R i)
  unfolding set-eq-iff and Int-iff and mem-box
  by auto

lemma disjoint-interval:
  fixes a::'a::euclidean-space
  shows cbox a b ∩ cbox c d = {} ⟷ (∃ i∈Basis. (b•i < a•i ∨ d•i < c•i ∨ b•i <
c•i ∨ d•i < a•i)) (is ?th1)
    and cbox a b ∩ box c d = {} ⟷ (∃ i∈Basis. (b•i < a•i ∨ d•i ≤ c•i ∨ b•i ≤
c•i ∨ d•i ≤ a•i)) (is ?th2)
    and box a b ∩ cbox c d = {} ⟷ (∃ i∈Basis. (b•i ≤ a•i ∨ d•i < c•i ∨ b•i ≤
```

$c{\cdot}i \vee d{\cdot}i \leq a{\cdot}i)$) (**is** *?th3*)
  and *box a b ∩ box c d = {}* ⟷ (∃ *i*∈*Basis*. (*b*·*i* ≤ *a*·*i* ∨ *d*·*i* ≤ *c*·*i* ∨ *b*·*i* ≤ *c*·*i* ∨ *d*·*i* ≤ *a*·*i*)) (**is** *?th4*)
proof −
 let *?z* = (∑ *i*∈*Basis*. (((*max* (*a*·*i*) (*c*·*i*)) + (*min* (*b*·*i*) (*d*·*i*))) / *2*) ∗$_R$ *i*)::′*a*
 have ∗∗: ⋀*P Q*. (⋀*i* :: ′*a*. *i* ∈ *Basis* ⟹ *Q ?z i* ⟹ *P i*) ⟹
    (⋀*i x* :: ′*a*. *i* ∈ *Basis* ⟹ *P i* ⟹ *Q x i*) ⟹ (∀ *x*. ∃ *i*∈*Basis*. *Q x i*) ⟷ (∃ *i*∈*Basis*. *P i*)
  by *blast*
 note ∗ = *set-eq-iff Int-iff empty-iff mem-box ball-conj-distrib*[*symmetric*] *eq-False ball-simps*(*10*)
 show *?th1* **unfolding** ∗ **by** (*intro* ∗∗) *auto*
 show *?th2* **unfolding** ∗ **by** (*intro* ∗∗) *auto*
 show *?th3* **unfolding** ∗ **by** (*intro* ∗∗) *auto*
 show *?th4* **unfolding** ∗ **by** (*intro* ∗∗) *auto*
qed

lemma *UN-box-eq-UNIV*: (⋃ *i*::*nat*. *box* (− (*real i* ∗$_R$ *One*)) (*real i* ∗$_R$ *One*)) = *UNIV*
proof −
 have |*x* · *b*| < *real-of-int* (⌈*Max* ((λ*b*. |*x* · *b*|)‘*Basis*)⌉ + *1*)
  if [*simp*]: *b* ∈ *Basis* **for** *x b* :: ′*a*
 proof −
  have |*x* · *b*| ≤ *real-of-int* ⌈|*x* · *b*|⌉
   by (*rule le-of-int-ceiling*)
  also have ... ≤ *real-of-int* ⌈*Max* ((λ*b*. |*x* · *b*|)‘*Basis*)⌉
   by (*auto intro*!: *ceiling-mono*)
  also have ... < *real-of-int* (⌈*Max* ((λ*b*. |*x* · *b*|)‘*Basis*)⌉ + *1*)
   by *simp*
  finally show *?thesis* **.**
 qed
 then have ∃ *n*::*nat*. ∀ *b*∈*Basis*. |*x* · *b*| < *real n* **for** *x* :: ′*a*
  by (*metis order.strict-trans reals-Archimedean2*)
 moreover have ⋀*x b*::′*a*. ⋀*n*::*nat*. |*x* · *b*| < *real n* ⟷ − *real n* < *x* · *b* ∧ *x* · *b* < *real n*
  by *auto*
 ultimately show *?thesis*
  by (*auto simp*: *box-def inner-setsum-left inner-Basis setsum.If-cases*)
qed

Intervals in general, including infinite and mixtures of open and closed.

definition *is-interval* (*s*::(′*a*::*euclidean-space*) *set*) ⟷
 (∀ *a*∈*s*. ∀ *b*∈*s*. ∀ *x*. (∀ *i*∈*Basis*. ((*a*·*i* ≤ *x*·*i* ∧ *x*·*i* ≤ *b*·*i*) ∨ (*b*·*i* ≤ *x*·*i* ∧ *x*·*i* ≤ *a*·*i*))) ⟶ *x* ∈ *s*)

lemma *is-interval-cbox*: *is-interval* (*cbox a* (*b*::′*a*::*euclidean-space*)) (**is** *?th1*)
 and *is-interval-box*: *is-interval* (*box a b*) (**is** *?th2*)
 unfolding *is-interval-def mem-box Ball-def atLeastAtMost-iff*
 by (*meson order-trans le-less-trans less-le-trans less-trans*)+

lemma *is-interval-empty* [*iff*]: *is-interval* {}
unfolding *is-interval-def* **by** *simp*

lemma *is-interval-univ* [*iff*]: *is-interval UNIV*
unfolding *is-interval-def* **by** *simp*

lemma *mem-is-intervalI*:
assumes *is-interval s*
assumes $a \in s$ $b \in s$
assumes $\bigwedge i.\ i \in Basis \Longrightarrow a \cdot i \leq x \cdot i \wedge x \cdot i \leq b \cdot i \vee b \cdot i \leq x \cdot i \wedge x \cdot i \leq a \cdot i$
shows $x \in s$
by (*rule assms*(*1*)[*simplified is-interval-def*, *rule-format*, *OF assms*(*2*,*3*,*4*)])

lemma *interval-subst*:
fixes *S*::*'a*::*euclidean-space set*
assumes *is-interval S*
assumes $x \in S$ y $j \in S$
assumes $j \in Basis$
shows $(\sum i{\in}Basis.\ (if\ i = j\ then\ y\ i \cdot i\ else\ x \cdot i) *_R i) \in S$
by (*rule mem-is-intervalI*[*OF assms*(*1*,*2*)]) (*auto simp*: *assms*)

lemma *mem-box-componentwiseI*:
fixes *S*::*'a*::*euclidean-space set*
assumes *is-interval S*
assumes $\bigwedge i.\ i \in Basis \Longrightarrow x \cdot i \in ((\lambda x.\ x \cdot i)\ `\ S)$
shows $x \in S$
proof −
from *assms* **have** $\forall i \in Basis.\ \exists s \in S.\ x \cdot i = s \cdot i$
by *auto*
with *finite-Basis* **obtain** *s* **and** *bs*::*'a list* **where**
s: $\bigwedge i.\ i \in Basis \Longrightarrow x \cdot i = s\ i \cdot i$ $\bigwedge i.\ i \in Basis \Longrightarrow s\ i \in S$ **and**
bs: *set bs* = *Basis distinct bs*
by (*metis finite-distinct-list*)
from *nonempty-Basis s* **obtain** *j* **where** *j*: $j \in Basis$ $s\ j \in S$ **by** *blast*
def $y \equiv$ *rec-list*
(*s j*)
(λj - Y. $(\sum i{\in}Basis.\ (if\ i = j\ then\ s\ i \cdot i\ else\ Y \cdot i) *_R i)$)
have $x = (\sum i{\in}Basis.\ (if\ i \in set\ bs\ then\ s\ i \cdot i\ else\ s\ j \cdot i) *_R i)$
using *bs* **by** (*auto simp add*: *s*(*1*)[*symmetric*] *euclidean-representation*)
also have [*symmetric*]: *y bs* = ...
using *bs*(*2*) *bs*(*1*)[*THEN equalityD1*]
by (*induct bs*) (*auto simp*: *y-def euclidean-representation intro*!: *euclidean-eqI*[**where** *'a*=*'a*])
also have *y bs* ∈ *S*
using *bs*(*1*)[*THEN equalityD1*]
apply (*induct bs*)
apply (*auto simp*: *y-def j*)

apply (*rule interval-subst*[*OF assms*(*1*)])
apply (*auto simp*: *s*)
done
finally show *?thesis* **.**
qed

16.7 Connectedness

lemma *connected-local*:
connected S ⟷
¬ (∃ *e1 e2*.
openin (*subtopology euclidean S*) *e1* ∧
openin (*subtopology euclidean S*) *e2* ∧
S ⊆ *e1* ∪ *e2* ∧
e1 ∩ *e2* = {} ∧
e1 ≠ {} ∧
e2 ≠ {})
unfolding *connected-def openin-open*
by *safe blast+*

lemma *exists-diff*:
fixes *P* :: *'a set* ⇒ *bool*
shows (∃ *S*. *P* (− *S*)) ⟷ (∃ *S*. *P S*) (**is** *?lhs* ⟷ *?rhs*)
proof −
{
assume *?lhs*
then have *?rhs* **by** *blast*
}
moreover
{
fix *S*
assume *H*: *P S*
have *S* = − (− *S*) **by** *auto*
with *H* **have** *P* (− (− *S*)) **by** *metis*
}
ultimately show *?thesis* **by** *metis*
qed

lemma *connected-clopen*: *connected S* ⟷
(∀ *T*. *openin* (*subtopology euclidean S*) *T* ∧
closedin (*subtopology euclidean S*) *T* ⟶ *T* = {} ∨ *T* = *S*) (**is** *?lhs* ⟷ *?rhs*)
proof −
have ¬ *connected S* ⟷
(∃ *e1 e2*. *open e1* ∧ *open* (− *e2*) ∧ *S* ⊆ *e1* ∪ (− *e2*) ∧ *e1* ∩ (− *e2*) ∩ *S* = {}
∧ *e1* ∩ *S* ≠ {} ∧ (− *e2*) ∩ *S* ≠ {})
unfolding *connected-def openin-open closedin-closed*
by (*metis double-complement*)
then have *th0*: *connected S* ⟷
¬ (∃ *e2 e1*. *closed e2* ∧ *open e1* ∧ *S* ⊆ *e1* ∪ (− *e2*) ∧ *e1* ∩ (− *e2*) ∩ *S* = {}

$\wedge$ *e1* $\cap$ *S* $\neq$ {} $\wedge$ (− *e2*) $\cap$ *S* $\neq$ {})
(**is** - $\longleftrightarrow$ $\neg$ ($\exists$ *e2 e1*. *?P e2 e1*))
apply (*simp add*: *closed-def*)
apply *metis*
done
have *th1*: *?rhs* $\longleftrightarrow$ $\neg$ ($\exists$ *t'* *t*. *closed t'*$\wedge$*t* = *S*$\cap$*t'* $\wedge$ *t*$\neq${} $\wedge$ *t*$\neq$*S* $\wedge$ ($\exists$ *t'*. *open t'* $\wedge$ *t* = *S* $\cap$ *t'*))
(**is** - $\longleftrightarrow$ $\neg$ ($\exists$ *t'* *t*. *?Q t' t*))
unfolding *connected-def openin-open closedin-closed* **by** *auto*
{
fix *e2*
{
fix *e1*
have *?P e2 e1* $\longleftrightarrow$ ($\exists$ *t*. *closed e2* $\wedge$ *t* = *S*$\cap$*e2* $\wedge$ *open e1* $\wedge$ *t* = *S*$\cap$*e1* $\wedge$ *t*$\neq${} $\wedge$ *t* $\neq$ *S*)
by *auto*
}
then have ($\exists$ *e1*. *?P e2 e1*) $\longleftrightarrow$ ($\exists$ *t*. *?Q e2 t*)
by *metis*
}
then have $\forall$ *e2*. ($\exists$ *e1*. *?P e2 e1*) $\longleftrightarrow$ ($\exists$ *t*. *?Q e2 t*)
by *blast*
then show *?thesis*
unfolding *th0 th1* **by** *simp*
qed

16.8 Limit points

definition (**in** *topological-space*) *islimpt*:: *'a* $\Rightarrow$ *'a set* $\Rightarrow$ *bool* (**infixr** *islimpt 60*)
where *x islimpt S* $\longleftrightarrow$ ($\forall$ *T*. *x*$\in$*T* $\longrightarrow$ *open T* $\longrightarrow$ ($\exists$ *y*$\in$*S*. *y*$\in$*T* $\wedge$ *y*$\neq$*x*))

lemma *islimptI*:
assumes $\bigwedge$*T*. *x* $\in$ *T* $\Longrightarrow$ *open T* $\Longrightarrow$ $\exists$ *y*$\in$*S*. *y* $\in$ *T* $\wedge$ *y* $\neq$ *x*
shows *x islimpt S*
using *assms* **unfolding** *islimpt-def* **by** *auto*

lemma *islimptE*:
assumes *x islimpt S* **and** *x* $\in$ *T* **and** *open T*
obtains *y* **where** *y* $\in$ *S* **and** *y* $\in$ *T* **and** *y* $\neq$ *x*
using *assms* **unfolding** *islimpt-def* **by** *auto*

lemma *islimpt-iff-eventually*: *x islimpt S* $\longleftrightarrow$ $\neg$ *eventually* (λ*y*. *y* $\notin$ *S*) (*at x*)
unfolding *islimpt-def eventually-at-topological* **by** *auto*

lemma *islimpt-subset*: *x islimpt S* $\Longrightarrow$ *S* $\subseteq$ *T* $\Longrightarrow$ *x islimpt T*
unfolding *islimpt-def* **by** *fast*

lemma *islimpt-approachable*:
fixes *x* :: *'a*::*metric-space*

shows *x islimpt S* $\longleftrightarrow$ $(\forall\, e{>}0.\ \exists\, x'{\in}S.\ x' \neq x \wedge dist\ x'\ x < e)$
unfolding *islimpt-iff-eventually eventually-at* **by** *fast*

lemma *islimpt-approachable-le*:
fixes *x* :: *'a*::*metric-space*
shows *x islimpt S* $\longleftrightarrow$ $(\forall\, e{>}0.\ \exists\, x'{\in}\ S.\ x' \neq x \wedge dist\ x'\ x \leq e)$
unfolding *islimpt-approachable*
using *approachable-lt-le* [**where** $f{=}\lambda y.\ dist\ y\ x$ **and** $P{=}\lambda y.\ y \notin S \vee y = x$,
THEN arg-cong [**where** *f*=*Not*]]
by (*simp add*: *Bex-def conj-commute conj-left-commute*)

lemma *islimpt-UNIV-iff*: *x islimpt UNIV* $\longleftrightarrow$ $\neg$ *open* $\{x\}$
unfolding *islimpt-def* **by** (*safe*, *fast*, *case-tac* $T = \{x\}$, *fast*, *fast*)

lemma *islimpt-punctured*: *x islimpt S* = *x islimpt* $(S{-}\{x\})$
unfolding *islimpt-def* **by** *blast*

A perfect space has no isolated points.

lemma *islimpt-UNIV* [*simp*, *intro*]: (*x*::*'a*::*perfect-space*) *islimpt UNIV*
unfolding *islimpt-UNIV-iff* **by** (*rule not-open-singleton*)

lemma *perfect-choose-dist*:
fixes *x* :: *'a*::{*perfect-space*, *metric-space*}
shows $0 < r \Longrightarrow \exists\, a.\ a \neq x \wedge dist\ a\ x < r$
using *islimpt-UNIV* [*of x*]
by (*simp add*: *islimpt-approachable*)

lemma *closed-limpt*: *closed S* $\longleftrightarrow$ $(\forall\, x.\ x\ islimpt\ S \longrightarrow x \in S)$
unfolding *closed-def*
apply (*subst open-subopen*)
apply (*simp add*: *islimpt-def subset-eq*)
apply (*metis ComplE ComplI*)
done

lemma *islimpt-EMPTY*[*simp*]: $\neg$ *x islimpt* $\{\}$
unfolding *islimpt-def* **by** *auto*

lemma *finite-set-avoid*:
fixes *a* :: *'a*::*metric-space*
assumes *fS*: *finite S*
shows $\exists\, d{>}0.\ \forall\, x{\in}S.\ x \neq a \longrightarrow d \leq dist\ a\ x$
proof (*induct rule*: *finite-induct*[*OF fS*])
case *1*
then show *?case* **by** (*auto intro*: *zero-less-one*)
next
case (*2 x F*)
from *2* **obtain** *d* **where** *d*: $d > 0\ \forall\, x{\in}F.\ x \neq a \longrightarrow d \leq dist\ a\ x$
by *blast*
show *?case*

```
  proof (cases x = a)
    case True
    then show ?thesis using d by auto
  next
    case False
    let ?d = min d (dist a x)
    have dp: ?d > 0
      using False d(1) by auto
    from d have d': ∀x∈F. x ≠ a ⟶ ?d ≤ dist a x
      by auto
    with dp False show ?thesis
      by (auto intro!: exI[where x=?d])
  qed
qed

lemma islimpt-Un: x islimpt (S ∪ T) ⟷ x islimpt S ∨ x islimpt T
  by (simp add: islimpt-iff-eventually eventually-conj-iff)

lemma discrete-imp-closed:
  fixes S :: 'a::metric-space set
  assumes e: 0 < e
    and d: ∀x ∈ S. ∀y ∈ S. dist y x < e ⟶ y = x
  shows closed S
proof -
  {
    fix x
    assume C: ∀e>0. ∃x'∈S. x' ≠ x ∧ dist x' x < e
    from e have e2: e/2 > 0 by arith
    from C[rule-format, OF e2] obtain y where y: y ∈ S y ≠ x dist y x < e/2
      by blast
    let ?m = min (e/2) (dist x y)
    from e2 y(2) have mp: ?m > 0
      by simp
    from C[rule-format, OF mp] obtain z where z: z ∈ S z ≠ x dist z x < ?m
      by blast
    have th: dist z y < e using z y
      by (intro dist-triangle-lt [where z=x], simp)
    from d[rule-format, OF y(1) z(1) th] y z
    have False by (auto simp add: dist-commute)}
  then show ?thesis
    by (metis islimpt-approachable closed-limpt [where 'a='a])
qed

lemma closed-of-nat-image: closed (of-nat ` A :: 'a :: real-normed-algebra-1 set)
  by (rule discrete-imp-closed[of 1]) (auto simp: dist-of-nat)

lemma closed-of-int-image: closed (of-int ` A :: 'a :: real-normed-algebra-1 set)
  by (rule discrete-imp-closed[of 1]) (auto simp: dist-of-int)
```

lemma *closed-Nats* [*simp*]: *closed* ($\mathbb{N}$ :: *'a* :: *real-normed-algebra-1 set*)
unfolding *Nats-def* **by** (*rule closed-of-nat-image*)

lemma *closed-Ints* [*simp*]: *closed* ($\mathbb{Z}$ :: *'a* :: *real-normed-algebra-1 set*)
unfolding *Ints-def* **by** (*rule closed-of-int-image*)

16.9 Interior of a Set

definition *interior S* = $\bigcup \{T.\ open\ T \wedge T \subseteq S\}$

lemma *interiorI* [*intro?*]:
assumes *open T* **and** $x \in T$ **and** $T \subseteq S$
shows $x \in$ *interior S*
using *assms* **unfolding** *interior-def* **by** *fast*

lemma *interiorE* [*elim?*]:
assumes $x \in$ *interior S*
obtains *T* **where** *open T* **and** $x \in T$ **and** $T \subseteq S$
using *assms* **unfolding** *interior-def* **by** *fast*

lemma *open-interior* [*simp*, *intro*]: *open* (*interior S*)
by (*simp add*: *interior-def open-Union*)

lemma *interior-subset*: *interior S* $\subseteq$ *S*
by (*auto simp add*: *interior-def*)

lemma *interior-maximal*: $T \subseteq S \Longrightarrow$ *open T* $\Longrightarrow T \subseteq$ *interior S*
by (*auto simp add*: *interior-def*)

lemma *interior-open*: *open S* $\Longrightarrow$ *interior S* = *S*
by (*intro equalityI interior-subset interior-maximal subset-refl*)

lemma *interior-eq*: *interior S* = *S* $\longleftrightarrow$ *open S*
by (*metis open-interior interior-open*)

lemma *open-subset-interior*: *open S* $\Longrightarrow S \subseteq$ *interior T* $\longleftrightarrow S \subseteq T$
by (*metis interior-maximal interior-subset subset-trans*)

lemma *interior-empty* [*simp*]: *interior* {} = {}
using *open-empty* **by** (*rule interior-open*)

lemma *interior-UNIV* [*simp*]: *interior UNIV* = *UNIV*
using *open-UNIV* **by** (*rule interior-open*)

lemma *interior-interior* [*simp*]: *interior* (*interior S*) = *interior S*
using *open-interior* **by** (*rule interior-open*)

lemma *interior-mono*: $S \subseteq T \Longrightarrow$ *interior S* $\subseteq$ *interior T*
by (*auto simp add*: *interior-def*)

lemma *interior-unique*:
assumes $T \subseteq S$ **and** *open* T
assumes $\bigwedge T'.\ T' \subseteq S \Longrightarrow$ *open* $T' \Longrightarrow T' \subseteq T$
shows *interior* $S = T$
by (*intro equalityI assms interior-subset open-interior interior-maximal*)

lemma *interior-singleton* [*simp*]:
fixes a :: $'a$::*perfect-space* **shows** *interior* $\{a\} = \{\}$
apply (*rule interior-unique*, *simp-all*)
using *not-open-singleton subset-singletonD* **by** *fastforce*

lemma *interior-Int* [*simp*]: *interior* $(S \cap T) =$ *interior* $S \cap$ *interior* T
by (*intro equalityI Int-mono Int-greatest interior-mono Int-lower1*
Int-lower2 interior-maximal interior-subset open-Int open-interior)

lemma *mem-interior*: $x \in$ *interior* $S \longleftrightarrow (\exists\, e{>}0.$ *ball* $x\ e \subseteq S)$
using *open-contains-ball-eq* [**where** S=*interior* S]
by (*simp add*: *open-subset-interior*)

lemma *eventually-nhds-in-nhd*: $x \in$ *interior* $s \Longrightarrow$ *eventually* $(\lambda y.\ y \in s)$ (*nhds* x)
using *interior-subset*[*of s*] **by** (*subst eventually-nhds*) *blast*

lemma *interior-limit-point* [*intro*]:
fixes x :: $'a$::*perfect-space*
assumes *x*: $x \in$ *interior* S
shows x *islimpt* S
using x *islimpt-UNIV* [*of x*]
unfolding *interior-def islimpt-def*
apply (*clarsimp*, *rename-tac* $T\ T'$)
apply (*drule-tac* $x{=}T \cap T'$ **in** *spec*)
apply (*auto simp add*: *open-Int*)
done

lemma *interior-closed-Un-empty-interior*:
assumes *cS*: *closed* S
and *iT*: *interior* $T = \{\}$
shows *interior* $(S \cup T) =$ *interior* S
proof
show *interior* $S \subseteq$ *interior* $(S \cup T)$
by (*rule interior-mono*) (*rule Un-upper1*)
show *interior* $(S \cup T) \subseteq$ *interior* S
proof
fix x
assume $x \in$ *interior* $(S \cup T)$
then obtain R **where** *open* R $x \in R$ $R \subseteq S \cup T$ **..**
show $x \in$ *interior* S
proof (*rule ccontr*)

```
      assume x ∉ interior S
      with ‹x ∈ R› ‹open R› obtain y where y ∈ R − S
        unfolding interior-def by fast
      from ‹open R› ‹closed S› have open (R − S)
        by (rule open-Diff)
      from ‹R ⊆ S ∪ T› have R − S ⊆ T
        by fast
     from ‹y ∈ R − S› ‹open (R − S)› ‹R − S ⊆ T› ‹interior T = {}› show False
        unfolding interior-def by fast
    qed
  qed
qed

lemma interior-Times: interior (A × B) = interior A × interior B
proof (rule interior-unique)
  show interior A × interior B ⊆ A × B
    by (intro Sigma-mono interior-subset)
  show open (interior A × interior B)
    by (intro open-Times open-interior)
  fix T
  assume T ⊆ A × B and open T
  then show T ⊆ interior A × interior B
  proof safe
    fix x y
    assume (x, y) ∈ T
    then obtain C D where open C open D C × D ⊆ T x ∈ C y ∈ D
      using ‹open T› unfolding open-prod-def by fast
    then have open C open D C ⊆ A D ⊆ B x ∈ C y ∈ D
      using ‹T ⊆ A × B› by auto
    then show x ∈ interior A and y ∈ interior B
      by (auto intro: interiorI)
  qed
qed

lemma interior-Ici:
  fixes x :: 'a :: {dense-linorder, linorder-topology}
  assumes b < x
  shows interior { x ..} = { x <..}
proof (rule interior-unique)
  fix T assume T ⊆ {x ..} open T
  moreover have x ∉ T
  proof
    assume x ∈ T
    obtain y where y < x {y <.. x} ⊆ T
      using open-left[OF ‹open T› ‹x ∈ T› ‹b < x›] by auto
    with dense[OF ‹y < x›] obtain z where z ∈ T z < x
      by (auto simp: subset-eq Ball-def)
    with ‹T ⊆ {x ..}› show False by auto
  qed
```

ultimately show $T \subseteq \{x <..\}$
by (*auto simp*: *subset-eq less-le*)
qed *auto*

lemma *interior-Iic*:
fixes $x :: 'a :: \{$*dense-linorder*, *linorder-topology*$\}$
assumes $x < b$
shows *interior* $\{.. x\} = \{..< x\}$
proof (*rule interior-unique*)
fix T **assume** $T \subseteq \{.. x\}$ *open* T
moreover have $x \notin T$
proof
assume $x \in T$
obtain y **where** $x < y$ $\{x ..< y\} \subseteq T$
using *open-right*[*OF* ‹*open* T› ‹$x \in T$› ‹$x < b$›] **by** *auto*
with *dense*[*OF* ‹$x < y$›] **obtain** z **where** $z \in T$ $x < z$
by (*auto simp*: *subset-eq Ball-def less-le*)
with ‹$T \subseteq \{.. x\}$› **show** *False* **by** *auto*
qed
ultimately show $T \subseteq \{..< x\}$
by (*auto simp*: *subset-eq less-le*)
qed *auto*

16.10 Closure of a Set

definition *closure* $S = S \cup \{x \mid x.\ x$ *islimpt* $S\}$

lemma *interior-closure*: *interior* $S = -$ (*closure* $(-S)$)
unfolding *interior-def closure-def islimpt-def* **by** *auto*

lemma *closure-interior*: *closure* $S = -$ *interior* $(-S)$
unfolding *interior-closure* **by** *simp*

lemma *closed-closure*[*simp*, *intro*]: *closed* (*closure* S)
unfolding *closure-interior* **by** (*simp add*: *closed-Compl*)

lemma *closure-subset*: $S \subseteq$ *closure* S
unfolding *closure-def* **by** *simp*

lemma *closure-hull*: *closure* $S =$ *closed hull* S
unfolding *hull-def closure-interior interior-def* **by** *auto*

lemma *closure-eq*: *closure* $S = S \longleftrightarrow$ *closed* S
unfolding *closure-hull* **using** *closed-Inter* **by** (*rule hull-eq*)

lemma *closure-closed* [*simp*]: *closed* $S \Longrightarrow$ *closure* $S = S$
unfolding *closure-eq* **.**

lemma *closure-closure* [*simp*]: *closure* (*closure* S) = *closure* S

unfolding *closure-hull* **by** (*rule hull-hull*)

lemma *closure-mono*: $S \subseteq T \Longrightarrow$ *closure* $S \subseteq$ *closure* T
unfolding *closure-hull* **by** (*rule hull-mono*)

lemma *closure-minimal*: $S \subseteq T \Longrightarrow$ *closed* $T \Longrightarrow$ *closure* $S \subseteq T$
unfolding *closure-hull* **by** (*rule hull-minimal*)

lemma *closure-unique*:
assumes $S \subseteq T$
and *closed* T
and $\bigwedge T'.\ S \subseteq T' \Longrightarrow$ *closed* $T' \Longrightarrow T \subseteq T'$
shows *closure* $S = T$
using *assms* **unfolding** *closure-hull* **by** (*rule hull-unique*)

lemma *closure-empty* [*simp*]: *closure* $\{\} = \{\}$
using *closed-empty* **by** (*rule closure-closed*)

lemma *closure-UNIV* [*simp*]: *closure UNIV* $=$ *UNIV*
using *closed-UNIV* **by** (*rule closure-closed*)

lemma *closure-union* [*simp*]: *closure* $(S \cup T) =$ *closure* $S \cup$ *closure* T
unfolding *closure-interior* **by** *simp*

lemma *closure-eq-empty* [*iff*]: *closure* $S = \{\} \longleftrightarrow S = \{\}$
using *closure-empty closure-subset*[*of* S]
by *blast*

lemma *closure-subset-eq*: *closure* $S \subseteq S \longleftrightarrow$ *closed* S
using *closure-eq*[*of* S] *closure-subset*[*of* S]
by *simp*

lemma *open-Int-closure-eq-empty*:
open $S \Longrightarrow (S \cap$ *closure* $T) = \{\} \longleftrightarrow S \cap T = \{\}$
using *open-subset-interior*[*of* $S - T$]
using *interior-subset*[*of* $- T$]
unfolding *closure-interior*
by *auto*

lemma *open-inter-closure-subset*:
open $S \Longrightarrow (S \cap (closure\ T)) \subseteq closure(S \cap T)$
proof
fix x
assume *as*: *open* $S\ x \in S \cap$ *closure* T
{
assume $*$: x *islimpt* T
have x *islimpt* $(S \cap T)$
proof (*rule islimptI*)
fix A

```
    assume x ∈ A open A
    with as have x ∈ A ∩ S open (A ∩ S)
      by (simp-all add: open-Int)
    with ∗ obtain y where y ∈ T y ∈ A ∩ S y ≠ x
      by (rule islimptE)
    then have y ∈ S ∩ T y ∈ A ∧ y ≠ x
      by simp-all
    then show ∃ y∈(S ∩ T). y ∈ A ∧ y ≠ x ..
  qed
 }
 then show x ∈ closure (S ∩ T) using as
  unfolding closure-def
  by blast
qed

lemma closure-complement: closure (− S) = − interior S
 unfolding closure-interior by simp

lemma interior-complement: interior (− S) = − closure S
 unfolding closure-interior by simp

lemma closure-Times: closure (A × B) = closure A × closure B
proof (rule closure-unique)
 show A × B ⊆ closure A × closure B
  by (intro Sigma-mono closure-subset)
 show closed (closure A × closure B)
  by (intro closed-Times closed-closure)
 fix T
 assume A × B ⊆ T and closed T
 then show closure A × closure B ⊆ T
  apply (simp add: closed-def open-prod-def, clarify)
  apply (rule ccontr)
  apply (drule-tac x=(a, b) in bspec, simp, clarify, rename-tac C D)
  apply (simp add: closure-interior interior-def)
  apply (drule-tac x=C in spec)
  apply (drule-tac x=D in spec)
  apply auto
  done
qed

lemma islimpt-in-closure: (x islimpt S) = (x:closure(S−{x}))
 unfolding closure-def using islimpt-punctured by blast

lemma connected-imp-connected-closure: connected s ⟹ connected (closure s)
   by (rule connectedI) (meson closure-subset open-Int open-Int-closure-eq-empty
subset-trans connectedD)

lemma limpt-of-limpts:
    fixes x :: 'a::metric-space
```

shows *x islimpt {y. y islimpt s} ⟹ x islimpt s*
apply (*clarsimp simp add*: *islimpt-approachable*)
apply (*drule-tac x=e/2* **in** *spec*)
apply (*auto simp*: *simp del*: *less-divide-eq-numeral1*)
apply (*drule-tac x=dist x' x* **in** *spec*)
apply (*auto simp*: *zero-less-dist-iff simp del*: *less-divide-eq-numeral1*)
apply (*erule rev-bexI*)
by (*metis dist-commute dist-triangle-half-r less-trans less-irrefl*)

lemma *closed-limpts*: *closed {x::'a::metric-space. x islimpt s}*
using *closed-limpt limpt-of-limpts* **by** *blast*

lemma *limpt-of-closure*:
fixes *x* :: *'a::metric-space*
shows *x islimpt closure s ⟷ x islimpt s*
by (*auto simp*: *closure-def islimpt-Un dest*: *limpt-of-limpts*)

lemma *closedin-limpt*:
closedin (*subtopology euclidean t*) *s ⟷ s ⊆ t ∧* (*∀ x. x islimpt s ∧ x ∈ t ⟶ x ∈ s*)
apply (*simp add*: *closedin-closed*, *safe*)
apply (*simp add*: *closed-limpt islimpt-subset*)
apply (*rule-tac x=closure s* **in** *exI*)
apply *simp*
apply (*force simp*: *closure-def*)
done

lemma *closedin-closed-eq*:
closed s ⟹ (*closedin* (*subtopology euclidean s*) *t ⟷ closed t ∧ t ⊆ s*)
by (*meson closedin-limpt closed-subset closedin-closed-trans*)

lemma *bdd-below-closure*:
fixes *A* :: *real set*
assumes *bdd-below A*
shows *bdd-below* (*closure A*)
proof −
from *assms* **obtain** *m* **where** *⋀x. x ∈ A ⟹ m ≤ x* **unfolding** *bdd-below-def*
by *auto*
hence *A ⊆ {m..}* **by** *auto*
hence *closure A ⊆ {m..}* **using** *closed-real-atLeast* **by** (*rule closure-minimal*)
thus *?thesis* **unfolding** *bdd-below-def* **by** *auto*
qed

16.11 Connected components, considered as a connectedness relation or a set

definition
connected-component s x y ≡ ∃ t. connected t ∧ t ⊆ s ∧ x ∈ t ∧ y ∈ t

abbreviation
connected-component-set s x ≡ *Collect* (*connected-component s x*)

lemma *connected-componentI*:
⟦*connected t*; *t* ⊆ *s*; *x* ∈ *t*; *y* ∈ *t*⟧ ⟹ *connected-component s x y*
by (*auto simp*: *connected-component-def*)

lemma *connected-component-in*: *connected-component s x y* ⟹ *x* ∈ *s* ∧ *y* ∈ *s*
by (*auto simp*: *connected-component-def*)

lemma *connected-component-refl*: *x* ∈ *s* ⟹ *connected-component s x x*
apply (*auto simp*: *connected-component-def*)
using *connected-sing* **by** *blast*

lemma *connected-component-refl-eq* [*simp*]: *connected-component s x x* ⟷ *x* ∈ *s*
by (*auto simp*: *connected-component-refl*) (*auto simp*: *connected-component-def*)

lemma *connected-component-sym*: *connected-component s x y* ⟹ *connected-component s y x*
by (*auto simp*: *connected-component-def*)

lemma *connected-component-trans*:
⟦*connected-component s x y*; *connected-component s y z*⟧ ⟹ *connected-component s x z*
unfolding *connected-component-def*
by (*metis Int-iff Un-iff Un-subset-iff equals0D connected-Un*)

lemma *connected-component-of-subset*: ⟦*connected-component s x y*; *s* ⊆ *t*⟧ ⟹ *connected-component t x y*
by (*auto simp*: *connected-component-def*)

lemma *connected-component-Union*: *connected-component-set s x* = ⋃{*t*. *connected t* ∧ *x* ∈ *t* ∧ *t* ⊆ *s*}
by (*auto simp*: *connected-component-def*)

lemma *connected-connected-component* [*iff*]: *connected* (*connected-component-set s x*)
by (*auto simp*: *connected-component-Union intro*: *connected-Union*)

lemma *connected-iff-eq-connected-component-set*: *connected s* ⟷ (∀ *x* ∈ *s*. *connected-component-set s x* = *s*)
proof (*cases s*={})
case *True* **then show** *?thesis* **by** *simp*
next
case *False*
then obtain *x* **where** *x* ∈ *s* **by** *auto*
show *?thesis*
proof
assume *connected s*

then show $\forall x \in s.$ *connected-component-set* $s\ x = s$
by (*force simp*: *connected-component-def*)
next
assume $\forall x \in s.$ *connected-component-set* $s\ x = s$
then show *connected* s
by (*metis* ‹$x \in s$› *connected-connected-component*)
qed
qed

lemma *connected-component-subset*: *connected-component-set* $s\ x \subseteq s$
using *connected-component-in* **by** *blast*

lemma *connected-component-eq-self*: $[\![$*connected* s; $x \in s]\!] \Longrightarrow$ *connected-component-set* $s\ x = s$
by (*simp add*: *connected-iff-eq-connected-component-set*)

lemma *connected-iff-connected-component*:
connected $s \longleftrightarrow (\forall x \in s.\ \forall y \in s.$ *connected-component* $s\ x\ y)$
using *connected-component-in* **by** (*auto simp*: *connected-iff-eq-connected-component-set*)

lemma *connected-component-maximal*:
$[\![x \in t$; *connected* t; $t \subseteq s]\!] \Longrightarrow t \subseteq$ (*connected-component-set* $s\ x$)
using *connected-component-eq-self connected-component-of-subset* **by** *blast*

lemma *connected-component-mono*:
$s \subseteq t \Longrightarrow$ (*connected-component-set* $s\ x$) $\subseteq$ (*connected-component-set* $t\ x$)
by (*simp add*: *Collect-mono connected-component-of-subset*)

lemma *connected-component-eq-empty* [*simp*]: *connected-component-set* $s\ x = \{\}$ $\longleftrightarrow (x \notin s)$
using *connected-component-refl* **by** (*fastforce simp*: *connected-component-in*)

lemma *connected-component-set-empty* [*simp*]: *connected-component-set* $\{\}\ x = \{\}$
using *connected-component-eq-empty* **by** *blast*

lemma *connected-component-eq*:
$y \in$ *connected-component-set* $s\ x$
$\Longrightarrow$ (*connected-component-set* $s\ y =$ *connected-component-set* $s\ x$)
by (*metis* (*no-types*, *lifting*) *Collect-cong connected-component-sym connected-component-trans mem-Collect-eq*)

lemma *closed-connected-component*:
assumes s: *closed* s **shows** *closed* (*connected-component-set* $s\ x$)
proof (*cases* $x \in s$)
case *False* **then show** *?thesis*
by (*metis connected-component-eq-empty closed-empty*)
next
case *True*

show *?thesis*
unfolding *closure-eq* [*symmetric*]
proof
show *closure* (*connected-component-set s x*) ⊆ *connected-component-set s x*
apply (*rule connected-component-maximal*)
apply (*simp add*: *closure-def True*)
apply (*simp add*: *connected-imp-connected-closure*)
apply (*simp add*: *s closure-minimal connected-component-subset*)
done
next
show *connected-component-set s x* ⊆ *closure* (*connected-component-set s x*)
by (*simp add*: *closure-subset*)
qed
qed

lemma *connected-component-disjoint*:
(*connected-component-set s a*) ∩ (*connected-component-set s b*) = {} ⟷
a ∉ *connected-component-set s b*
apply (*auto simp*: *connected-component-eq*)
using *connected-component-eq connected-component-sym* **by** *blast*

lemma *connected-component-nonoverlap*:
(*connected-component-set s a*) ∩ (*connected-component-set s b*) = {} ⟷
(*a* ∉ *s* ∨ *b* ∉ *s* ∨ *connected-component-set s a* ≠ *connected-component-set s b*)
apply (*auto simp*: *connected-component-in*)
using *connected-component-refl-eq* **apply** *blast*
apply (*metis connected-component-eq mem-Collect-eq*)
apply (*metis connected-component-eq mem-Collect-eq*)
done

lemma *connected-component-overlap*:
(*connected-component-set s a* ∩ *connected-component-set s b* ≠ {}) =
(*a* ∈ *s* ∧ *b* ∈ *s* ∧ *connected-component-set s a* = *connected-component-set s b*)
by (*auto simp*: *connected-component-nonoverlap*)

lemma *connected-component-sym-eq*: *connected-component s x y* ⟷ *connected-component s y x*
using *connected-component-sym* **by** *blast*

lemma *connected-component-eq-eq*:
connected-component-set s x = *connected-component-set s y* ⟷
x ∉ *s* ∧ *y* ∉ *s* ∨ *x* ∈ *s* ∧ *y* ∈ *s* ∧ *connected-component s x y*
apply (*case-tac y* ∈ *s*)
apply (*simp add*:)
apply (*metis connected-component-eq connected-component-eq-empty connected-component-refl-eq mem-Collect-eq*)
apply (*case-tac x* ∈ *s*)
apply (*simp add*:)
apply (*metis connected-component-eq-empty*)

using *connected-component-eq-empty* **by** *blast*

lemma *connected-iff-connected-component-eq*:
connected s ⟷
(∀ *x* ∈ *s*. ∀ *y* ∈ *s*. *connected-component-set s x* = *connected-component-set s y*)
by (*simp add*: *connected-component-eq-eq connected-iff-connected-component*)

lemma *connected-component-idemp*:
connected-component-set (*connected-component-set s x*) *x* = *connected-component-set s x*
apply (*rule subset-antisym*)
apply (*simp add*: *connected-component-subset*)
by (*metis connected-component-eq-empty connected-component-maximal connected-component-refl-eq connected-connected-component mem-Collect-eq set-eq-subset*)

lemma *connected-component-unique*:
⟦*x* ∈ *c*; *c* ⊆ *s*; *connected c*;
⋀*c'*. *x* ∈ *c'* ∧ *c'* ⊆ *s* ∧ *connected c'*
⟹ *c'* ⊆ *c*⟧
⟹ *connected-component-set s x* = *c*
apply (*rule subset-antisym*)
apply (*meson connected-component-maximal connected-component-subset connected-connected-component contra-subsetD*)
by (*simp add*: *connected-component-maximal*)

lemma *joinable-connected-component-eq*:
⟦*connected t*; *t* ⊆ *s*;
connected-component-set s x ∩ *t* ≠ {};
connected-component-set s y ∩ *t* ≠ {}⟧
⟹ *connected-component-set s x* = *connected-component-set s y*
apply (*simp add*: *ex-in-conv* [*symmetric*])
apply (*rule connected-component-eq*)
by (*metis* (*no-types*, *hide-lams*) *connected-component-eq-eq connected-component-in connected-component-maximal subsetD mem-Collect-eq*)

lemma *Union-connected-component*: ⋃(*connected-component-set s* ‘ *s*) = *s*
apply (*rule subset-antisym*)
apply (*simp add*: *SUP-least connected-component-subset*)
using *connected-component-refl-eq*
by *force*

lemma *complement-connected-component-unions*:
s − *connected-component-set s x* =
⋃(*connected-component-set s* ‘ *s* − {*connected-component-set s x*})
apply (*subst Union-connected-component* [*symmetric*], *auto*)
apply (*metis connected-component-eq-eq connected-component-in*)

by (*metis connected-component-eq mem-Collect-eq*)

lemma *connected-component-intermediate-subset*:
⟦*connected-component-set u a* ⊆ *t*; *t* ⊆ *u*⟧
⟹ *connected-component-set t a* = *connected-component-set u a*
apply (*case-tac a* ∈ *u*)
apply (*simp add*: *connected-component-maximal connected-component-mono subset-antisym*)
using *connected-component-eq-empty* **by** *blast*

16.12 The set of connected components of a set

definition *components*:: *'a*::*topological-space set* ⇒ *'a set set* **where**
components s ≡ *connected-component-set s* ' *s*

lemma *components-iff*: *s* ∈ *components u* ⟷ (∃ *x*. *x* ∈ *u* ∧ *s* = *connected-component-set u x*)
by (*auto simp*: *components-def*)

lemma *Union-components* [*simp*]: ⋃(*components u*) = *u*
apply (*rule subset-antisym*)
using *Union-connected-component components-def* **apply** *fastforce*
apply (*metis Union-connected-component components-def set-eq-subset*)
done

lemma *pairwise-disjoint-components*: *pairwise* (λ*X Y*. *X* ∩ *Y* = {}) (*components u*)
apply (*simp add*: *pairwise-def*)
apply (*auto simp*: *components-iff*)
apply (*metis connected-component-eq-eq connected-component-in*)+
done

lemma *in-components-nonempty*: *c* ∈ *components s* ⟹ *c* ≠ {}
by (*metis components-iff connected-component-eq-empty*)

lemma *in-components-subset*: *c* ∈ *components s* ⟹ *c* ⊆ *s*
using *Union-components* **by** *blast*

lemma *in-components-connected*: *c* ∈ *components s* ⟹ *connected c*
by (*metis components-iff connected-connected-component*)

lemma *in-components-maximal*:
c ∈ *components s* ⟷
(*c* ≠ {} ∧ *c* ⊆ *s* ∧ *connected c* ∧ (∀ *d*. *d* ≠ {} ∧ *c* ⊆ *d* ∧ *d* ⊆ *s* ∧ *connected d* ⟶ *d* = *c*))
apply (*rule iffI*)
apply (*simp add*: *in-components-nonempty in-components-connected*)
apply (*metis* (*full-types*) *components-iff connected-component-eq-self connected-component-intermediate-subset connected-component-refl in-components-subset mem-Collect-eq rev-subsetD*)
by (*metis bot.extremum-uniqueI components-iff connected-component-eq-empty*

connected-component-maximal connected-component-subset connected-connected-component subset-emptyI)

lemma *joinable-components-eq*:
connected t ∧ *t* ⊆ *s* ∧ *c1* ∈ *components s* ∧ *c2* ∈ *components s* ∧ *c1* ∩ *t* ≠ {} ∧ *c2* ∩ *t* ≠ {} ⟹ *c1* = *c2*
by (*metis* (*full-types*) *components-iff joinable-connected-component-eq*)

lemma *closed-components*: ⟦*closed s*; *c* ∈ *components s*⟧ ⟹ *closed c*
by (*metis closed-connected-component components-iff*)

lemma *components-nonoverlap*:
⟦*c* ∈ *components s*; *c'* ∈ *components s*⟧ ⟹ (*c* ∩ *c'* = {}) ⟷ (*c* ≠ *c'*)
apply (*auto simp*: *in-components-nonempty components-iff*)
using *connected-component-refl* **apply** *blast*
apply (*metis connected-component-eq-eq connected-component-in*)
by (*metis connected-component-eq mem-Collect-eq*)

lemma *components-eq*: ⟦*c* ∈ *components s*; *c'* ∈ *components s*⟧ ⟹ (*c* = *c'* ⟷ *c* ∩ *c'* ≠ {})
by (*metis components-nonoverlap*)

lemma *components-eq-empty* [*simp*]: *components s* = {} ⟷ *s* = {}
by (*simp add*: *components-def*)

lemma *components-empty* [*simp*]: *components* {} = {}
by *simp*

lemma *connected-eq-connected-components-eq*: *connected s* ⟷ (∀ *c* ∈ *components s*. ∀ *c'* ∈ *components s*. *c* = *c'*)
by (*metis* (*no-types*, *hide-lams*) *components-iff connected-component-eq-eq connected-iff-connected-component*

lemma *components-eq-sing-iff*: *components s* = {*s*} ⟷ *connected s* ∧ *s* ≠ {}
apply (*rule iffI*)
using *in-components-connected* **apply** *fastforce*
apply *safe*
using *Union-components* **apply** *fastforce*
apply (*metis components-iff connected-component-eq-self*)
using *in-components-maximal* **by** *auto*

lemma *components-eq-sing-exists*: (∃ *a*. *components s* = {*a*}) ⟷ *connected s* ∧ *s* ≠ {}
apply (*rule iffI*)
using *connected-eq-connected-components-eq* **apply** *fastforce*
by (*metis components-eq-sing-iff*)

lemma *connected-eq-components-subset-sing*: *connected s* ⟷ *components s* ⊆ {*s*}
by (*metis Union-components components-empty components-eq-sing-iff connected-empty insert-subset order-refl subset-singletonD*)

lemma *connected-eq-components-subset-sing-exists*: *connected s* $\longleftrightarrow$ ($\exists$ *a. components s* $\subseteq$ {*a*})
by (*metis components-eq-sing-exists connected-eq-components-subset-sing empty-iff subset-iff subset-singletonD*)

lemma *in-components-self*: *s* $\in$ *components s* $\longleftrightarrow$ *connected s* $\wedge$ *s* $\neq$ {}
by (*metis components-empty components-eq-sing-iff empty-iff in-components-connected insertI1*)

lemma *components-maximal*: ⟦*c* $\in$ *components s*; *connected t*; *t* $\subseteq$ *s*; *c* $\cap$ *t* $\neq$ {}⟧ $\Longrightarrow$ *t* $\subseteq$ *c*
apply (*simp add*: *components-def ex-in-conv* [*symmetric*], *clarify*)
by (*meson connected-component-def connected-component-trans*)

lemma *exists-component-superset*: ⟦*t* $\subseteq$ *s*; *s* $\neq$ {}; *connected t*⟧ $\Longrightarrow$ $\exists$ *c. c* $\in$ *components s* $\wedge$ *t* $\subseteq$ *c*
apply (*case-tac t* = {})
apply *force*
by (*metis components-def ex-in-conv connected-component-maximal contra-subsetD image-eqI*)

lemma *components-intermediate-subset*: ⟦*s* $\in$ *components u*; *s* $\subseteq$ *t*; *t* $\subseteq$ *u*⟧ $\Longrightarrow$ *s* $\in$ *components t*
apply (*auto simp*: *components-iff*)
by (*metis connected-component-eq-empty connected-component-intermediate-subset*)

lemma *in-components-unions-complement*: *c* $\in$ *components s* $\Longrightarrow$ *s* − *c* = $\bigcup$(*components s* − {*c*})
by (*metis complement-connected-component-unions components-def components-iff*)

lemma *connected-intermediate-closure*:
assumes *cs*: *connected s* **and** *st*: *s* $\subseteq$ *t* **and** *ts*: *t* $\subseteq$ *closure s*
shows *connected t*
proof (*rule connectedI*)
fix *A B*
assume *A*: *open A* **and** *B*: *open B* **and** *Alap*: *A* $\cap$ *t* $\neq$ {} **and** *Blap*: *B* $\cap$ *t* $\neq$ {}
and *disj*: *A* $\cap$ *B* $\cap$ *t* = {} **and** *cover*: *t* $\subseteq$ *A* $\cup$ *B*
have *disjs*: *A* $\cap$ *B* $\cap$ *s* = {}
using *disj st* **by** *auto*
have *A* $\cap$ *closure s* $\neq$ {}
using *Alap Int-absorb1 ts* **by** *blast*
then have *Alaps*: *A* $\cap$ *s* $\neq$ {}
by (*simp add*: *A open-Int-closure-eq-empty*)
have *B* $\cap$ *closure s* $\neq$ {}
using *Blap Int-absorb1 ts* **by** *blast*
then have *Blaps*: *B* $\cap$ *s* $\neq$ {}
by (*simp add*: *B open-Int-closure-eq-empty*)

then show *False*
using *cs* [*unfolded connected-def*] *A B disjs Alaps Blaps cover st*
by *blast*
qed

lemma *closedin-connected-component*: *closedin* (*subtopology euclidean s*) (*connected-component-set s x*)
proof (*cases connected-component-set s x* = {})
case *True* **then show** *?thesis*
by (*metis closedin-empty*)
next
case *False*
then obtain *y* **where** *y*: *connected-component s x y*
by *blast*
have *1*: *connected-component-set s x* ⊆ *s* ∩ *closure* (*connected-component-set s x*)
by (*auto simp*: *closure-def connected-component-in*)
have *2*: *connected-component s x y* ⟹ *s* ∩ *closure* (*connected-component-set s x*) ⊆ *connected-component-set s x*
apply (*rule connected-component-maximal*)
apply (*simp add*:)
using *closure-subset connected-component-in* **apply** *fastforce*
using *1 connected-intermediate-closure* **apply** *blast*+
done
show *?thesis* **using** *y*
apply (*simp add*: *Topology-Euclidean-Space.closedin-closed*)
using *1 2* **by** *auto*
qed

16.13 Frontier (aka boundary)

definition *frontier S* = *closure S* − *interior S*

lemma *frontier-closed* [*iff*]: *closed* (*frontier S*)
by (*simp add*: *frontier-def closed-Diff*)

lemma *frontier-closures*: *frontier S* = (*closure S*) ∩ (*closure*(− *S*))
by (*auto simp add*: *frontier-def interior-closure*)

lemma *frontier-straddle*:
fixes *a* :: *'a*::*metric-space*
shows *a* ∈ *frontier S* ⟷ (∀ *e*>*0*. (∃ *x*∈*S*. *dist a x* < *e*) ∧ (∃ *x*. *x* ∉ *S* ∧ *dist a x* < *e*))
unfolding *frontier-def closure-interior*
by (*auto simp add*: *mem-interior subset-eq ball-def*)

lemma *frontier-subset-closed*: *closed S* ⟹ *frontier S* ⊆ *S*
by (*metis frontier-def closure-closed Diff-subset*)

lemma *frontier-empty* [*simp*]: *frontier* {} = {}
by (*simp add*: *frontier-def*)

lemma *frontier-subset-eq*: *frontier S* ⊆ *S* ⟷ *closed S*
proof −
{
assume *frontier S* ⊆ *S*
then have *closure S* ⊆ *S*
using *interior-subset* **unfolding** *frontier-def* **by** *auto*
then have *closed S*
using *closure-subset-eq* **by** *auto*
}
then show *?thesis* **using** *frontier-subset-closed*[*of S*] **..**
qed

lemma *frontier-complement* [*simp*]: *frontier* (− *S*) = *frontier S*
by (*auto simp add*: *frontier-def closure-complement interior-complement*)

lemma *frontier-disjoint-eq*: *frontier S* ∩ *S* = {} ⟷ *open S*
using *frontier-complement frontier-subset-eq*[*of* − *S*]
unfolding *open-closed* **by** *auto*

lemma *frontier-UNIV* [*simp*]: *frontier UNIV* = {}
using *frontier-complement frontier-empty* **by** *fastforce*

16.14 Filters and the "eventually true" quantifier

definition *indirection* :: *'a*::*real-normed-vector* ⇒ *'a* ⇒ *'a filter*
(**infixr** *indirection 70*)
where *a indirection v* = *at a within* {*b*. ∃ *c*≥*0*. *b* − *a* = *scaleR c v*}

Identify Trivial limits, where we can't approach arbitrarily closely.

lemma *trivial-limit-within*: *trivial-limit* (*at a within S*) ⟷ ¬ *a islimpt S*
proof
assume *trivial-limit* (*at a within S*)
then show ¬ *a islimpt S*
unfolding *trivial-limit-def*
unfolding *eventually-at-topological*
unfolding *islimpt-def*
apply (*clarsimp simp add*: *set-eq-iff*)
apply (*rename-tac T*, *rule-tac x*=*T* **in** *exI*)
apply (*clarsimp*, *drule-tac x*=*y* **in** *bspec*, *simp-all*)
done
next
assume ¬ *a islimpt S*
then show *trivial-limit* (*at a within S*)
unfolding *trivial-limit-def eventually-at-topological islimpt-def*
by *metis*
qed

lemma *trivial-limit-at-iff*: *trivial-limit* (*at a*) ⟷ ¬ *a islimpt UNIV*
using *trivial-limit-within* [*of a UNIV*] **by** *simp*

lemma *trivial-limit-at*:
fixes *a* :: *'a*::*perfect-space*
shows ¬ *trivial-limit* (*at a*)
by (*rule at-neq-bot*)

lemma *trivial-limit-at-infinity*:
¬ *trivial-limit* (*at-infinity* :: (*'a*::{*real-normed-vector*,*perfect-space*}) *filter*)
unfolding *trivial-limit-def eventually-at-infinity*
apply *clarsimp*
apply (*subgoal-tac* ∃ *x*::*'a*. *x* ≠ *0*, *clarify*)
apply (*rule-tac x*=*scaleR* (*b* / *norm x*) *x* **in** *exI*, *simp*)
apply (*cut-tac islimpt-UNIV* [*of 0*::*'a*, *unfolded islimpt-def*])
apply (*drule-tac x*=*UNIV* **in** *spec*, *simp*)
done

lemma *not-trivial-limit-within*: ¬ *trivial-limit* (*at x within S*) = (*x* ∈ *closure* (*S* − {*x*}))
using *islimpt-in-closure*
by (*metis trivial-limit-within*)

lemma *at-within-eq-bot-iff*: (*at c within A* = *bot*) ⟷ (*c* ∉ *closure* (*A* − {*c*}))
using *not-trivial-limit-within*[*of c A*] **by** *blast*

Some property holds "sufficiently close" to the limit point.

lemma *trivial-limit-eventually*: *trivial-limit net* ⟹ *eventually P net*
by *simp*

lemma *trivial-limit-eq*: *trivial-limit net* ⟷ (∀ *P*. *eventually P net*)
by (*simp add*: *filter-eq-iff*)

16.15 Limits

lemma *Lim*:
(*f* ⟶ *l*) *net* ⟷
trivial-limit net ∨
(∀ *e*>*0*. *eventually* (λ*x*. *dist* (*f x*) *l* < *e*) *net*)
unfolding *tendsto-iff trivial-limit-eq* **by** *auto*

Show that they yield usual definitions in the various cases.

lemma *Lim-within-le*: (*f* ⟶ *l*)(*at a within S*) ⟷
(∀ *e*>*0*. ∃ *d*>*0*. ∀ *x*∈*S*. *0* < *dist x a* ∧ *dist x a* ≤ *d* ⟶ *dist* (*f x*) *l* < *e*)
by (*auto simp add*: *tendsto-iff eventually-at-le*)

lemma *Lim-within*: (*f* ⟶ *l*) (*at a within S*) ⟷
(∀ *e* >*0*. ∃ *d*>*0*. ∀ *x* ∈ *S*. *0* < *dist x a* ∧ *dist x a* < *d* ⟶ *dist* (*f x*) *l* < *e*)

by (*auto simp add*: *tendsto-iff eventually-at*)

corollary *Lim-withinI* [*intro?*]:
assumes $\bigwedge e.\ e > 0 \Longrightarrow \exists\, d{>}0.\ \forall\, x \in S.\ 0 < dist\ x\ a \wedge dist\ x\ a < d \longrightarrow dist\ (f\ x)\ l \leq e$
shows $(f \longrightarrow l)\ (at\ a\ within\ S)$
apply (*simp add*: *Lim-within*, *clarify*)
apply (*rule ex-forward* [*OF assms* [*OF half-gt-zero*]], *auto*)
done

lemma *Lim-at*: $(f \longrightarrow l)\ (at\ a) \longleftrightarrow$
$(\forall\, e{>}0.\ \exists\, d{>}0.\ \forall\, x.\ 0 < dist\ x\ a \wedge dist\ x\ a < d \longrightarrow dist\ (f\ x)\ l < e)$
by (*auto simp add*: *tendsto-iff eventually-at*)

lemma *Lim-at-infinity*:
$(f \longrightarrow l)\ at\text{-}infinity \longleftrightarrow (\forall\, e{>}0.\ \exists\, b.\ \forall\, x.\ norm\ x \geq b \longrightarrow dist\ (f\ x)\ l < e)$
by (*auto simp add*: *tendsto-iff eventually-at-infinity*)

corollary *Lim-at-infinityI* [*intro?*]:
assumes $\bigwedge e.\ e > 0 \Longrightarrow \exists\, B.\ \forall\, x.\ norm\ x \geq B \longrightarrow dist\ (f\ x)\ l \leq e$
shows $(f \longrightarrow l)\ at\text{-}infinity$
apply (*simp add*: *Lim-at-infinity*, *clarify*)
apply (*rule ex-forward* [*OF assms* [*OF half-gt-zero*]], *auto*)
done

lemma *Lim-eventually*: $eventually\ (\lambda x.\ f\ x = l)\ net \Longrightarrow (f \longrightarrow l)\ net$
by (*rule topological-tendstoI*, *auto elim*: *eventually-mono*)

lemma *Lim-transform-within-set*:
fixes $a\ l :: 'a{::}real\text{-}normed\text{-}vector$
shows $[\![(f \longrightarrow l)\ (at\ a\ within\ s);\ eventually\ (\lambda x.\ x \in s \longleftrightarrow x \in t)\ (at\ a)]\!]$
$\Longrightarrow (f \longrightarrow l)\ (at\ a\ within\ t)$
apply (*clarsimp simp*: *eventually-at Lim-within*)
apply (*drule-tac x=e* **in** *spec*, *clarify*)
apply (*rename-tac k*)
apply (*rule-tac x=min d k* **in** *exI*)
apply (*simp add*:)
done

lemma *Lim-transform-within-set-eq*:
fixes $a\ l :: 'a{::}real\text{-}normed\text{-}vector$
shows $eventually\ (\lambda x.\ x \in s \longleftrightarrow x \in t)\ (at\ a)$
$\Longrightarrow ((f \longrightarrow l)\ (at\ a\ within\ s) \longleftrightarrow (f \longrightarrow l)\ (at\ a\ within\ t))$
by (*force intro*: *Lim-transform-within-set elim*: *eventually-mono*)

The expected monotonicity property.

lemma *Lim-Un*:
assumes $(f \longrightarrow l)\ (at\ x\ within\ S)\ (f \longrightarrow l)\ (at\ x\ within\ T)$
shows $(f \longrightarrow l)\ (at\ x\ within\ (S \cup T))$

using *assms* **unfolding** *at-within-union* **by** (*rule filterlim-sup*)

lemma *Lim-Un-univ*:
(*f* ⟶ *l*) (*at x within S*) ⟹ (*f* ⟶ *l*) (*at x within T*) ⟹
S ∪ *T* = *UNIV* ⟹ (*f* ⟶ *l*) (*at x*)
by (*metis Lim-Un*)

Interrelations between restricted and unrestricted limits.

lemma *Lim-at-imp-Lim-at-within*:
(*f* ⟶ *l*) (*at x*) ⟹ (*f* ⟶ *l*) (*at x within S*)
by (*metis order-refl filterlim-mono subset-UNIV at-le*)

lemma *eventually-within-interior*:
assumes *x* ∈ *interior S*
shows *eventually P* (*at x within S*) ⟷ *eventually P* (*at x*)
(**is** *?lhs* = *?rhs*)
proof
from *assms* **obtain** *T* **where** *T*: *open T x* ∈ *T T* ⊆ *S* **..**
{
assume *?lhs*
then obtain *A* **where** *open A* **and** *x* ∈ *A* **and** ∀ *y*∈*A*. *y* ≠ *x* ⟶ *y* ∈ *S* ⟶ *P y*
unfolding *eventually-at-topological*
by *auto*
with *T* **have** *open* (*A* ∩ *T*) **and** *x* ∈ *A* ∩ *T* **and** ∀ *y* ∈ *A* ∩ *T*. *y* ≠ *x* ⟶ *P y*
by *auto*
then show *?rhs*
unfolding *eventually-at-topological* **by** *auto*
next
assume *?rhs*
then show *?lhs*
by (*auto elim*: *eventually-mono simp*: *eventually-at-filter*)
}
qed

lemma *at-within-interior*:
x ∈ *interior S* ⟹ *at x within S* = *at x*
unfolding *filter-eq-iff* **by** (*intro allI eventually-within-interior*)

lemma *Lim-within-LIMSEQ*:
fixes *a* :: *'a*::*first-countable-topology*
assumes ∀ *S*. (∀ *n*. *S n* ≠ *a* ∧ *S n* ∈ *T*) ∧ *S* ⟶ *a* ⟶ (λ*n*. *X* (*S n*)) ⟶ *L*
shows (*X* ⟶ *L*) (*at a within T*)
using *assms* **unfolding** *tendsto-def* [**where** *l*=*L*]
by (*simp add*: *sequentially-imp-eventually-within*)

lemma *Lim-right-bound*:
fixes *f* :: *'a* :: {*linorder-topology*, *conditionally-complete-linorder*, *no-top*} ⇒

```
    'b::{linorder-topology, conditionally-complete-linorder}
  assumes mono: ⋀a b. a ∈ I ⟹ b ∈ I ⟹ x < a ⟹ a ≤ b ⟹ f a ≤ f b
    and bnd: ⋀a. a ∈ I ⟹ x < a ⟹ K ≤ f a
  shows (f ⟶ Inf (f ‘ ({x<..} ∩ I))) (at x within ({x<..} ∩ I))
proof (cases {x<..} ∩ I = {})
  case True
  then show ?thesis by simp
next
  case False
  show ?thesis
  proof (rule order-tendstoI)
    fix a
    assume a: a < Inf (f ‘ ({x<..} ∩ I))
    {
      fix y
      assume y ∈ {x<..} ∩ I
      with False bnd have Inf (f ‘ ({x<..} ∩ I)) ≤ f y
        by (auto intro!: cInf-lower bdd-belowI2)
      with a have a < f y
        by (blast intro: less-le-trans)
    }
    then show eventually (λx. a < f x) (at x within ({x<..} ∩ I))
      by (auto simp: eventually-at-filter intro: exI[of - 1] zero-less-one)
  next
    fix a
    assume Inf (f ‘ ({x<..} ∩ I)) < a
    from cInf-lessD[OF - this] False obtain y where y: x < y y ∈ I f y < a
      by auto
    then have eventually (λx. x ∈ I ⟶ f x < a) (at-right x)
      unfolding eventually-at-right[OF ‹x < y›] by (metis less-imp-le le-less-trans
mono)
    then show eventually (λx. f x < a) (at x within ({x<..} ∩ I))
      unfolding eventually-at-filter by eventually-elim simp
  qed
qed
```

Another limit point characterization.

```
lemma islimpt-sequential:
  fixes x :: 'a::first-countable-topology
  shows x islimpt S ⟷ (∃f. (∀n::nat. f n ∈ S - {x}) ∧ (f ⟶ x) sequentially)
    (is ?lhs = ?rhs)
proof
  assume ?lhs
  from countable-basis-at-decseq[of x] obtain A where A:
      ⋀i. open (A i)
      ⋀i. x ∈ A i
      ⋀S. open S ⟹ x ∈ S ⟹ eventually (λi. A i ⊆ S) sequentially
    by blast
  def f ≡ λn. SOME y. y ∈ S ∧ y ∈ A n ∧ x ≠ y
```

```
  {
    fix n
    from ‹?lhs› have ∃ y. y ∈ S ∧ y ∈ A n ∧ x ≠ y
      unfolding islimpt-def using A(1,2)[of n] by auto
    then have f n ∈ S ∧ f n ∈ A n ∧ x ≠ f n
      unfolding f-def by (rule someI-ex)
    then have f n ∈ S f n ∈ A n x ≠ f n by auto
  }
  then have ∀ n. f n ∈ S − {x} by auto
  moreover have (λn. f n) ⟶ x
  proof (rule topological-tendstoI)
    fix S
    assume open S x ∈ S
    from A(3)[OF this] ‹⋀n. f n ∈ A n›
    show eventually (λx. f x ∈ S) sequentially
      by (auto elim!: eventually-mono)
  qed
  ultimately show ?rhs by fast
next
  assume ?rhs
  then obtain f :: nat ⇒ 'a where f: ⋀n. f n ∈ S − {x} and lim: f ⟶ x
    by auto
  show ?lhs
    unfolding islimpt-def
  proof safe
    fix T
    assume open T x ∈ T
    from lim[THEN topological-tendstoD, OF this] f
    show ∃ y∈S. y ∈ T ∧ y ≠ x
      unfolding eventually-sequentially by auto
  qed
qed

lemma Lim-null:
  fixes f :: 'a ⇒ 'b::real-normed-vector
  shows (f ⟶ l) net ⟷ ((λx. f(x) − l) ⟶ 0) net
  by (simp add: Lim dist-norm)

lemma Lim-null-comparison:
  fixes f :: 'a ⇒ 'b::real-normed-vector
  assumes eventually (λx. norm (f x) ≤ g x) net (g ⟶ 0) net
  shows (f ⟶ 0) net
  using assms(2)
proof (rule metric-tendsto-imp-tendsto)
  show eventually (λx. dist (f x) 0 ≤ dist (g x) 0) net
    using assms(1) by (rule eventually-mono) (simp add: dist-norm)
qed

lemma Lim-transform-bound:
```

fixes $f :: 'a \Rightarrow 'b$::*real-normed-vector*
and $g :: 'a \Rightarrow 'c$::*real-normed-vector*
assumes *eventually* $(\lambda n.\ norm\ (f\ n) \leq norm\ (g\ n))$ *net*
and $(g \longrightarrow 0)$ *net*
shows $(f \longrightarrow 0)$ *net*
using *assms*(*1*) *tendsto-norm-zero* [*OF assms*(*2*)]
by (*rule Lim-null-comparison*)

lemma *lim-null-mult-right-bounded*:
fixes $f :: 'a \Rightarrow 'b$::*real-normed-div-algebra*
assumes *f*: $(f \longrightarrow 0)\ F$ **and** *g*: *eventually* $(\lambda x.\ norm(g\ x) \leq B)\ F$
shows $((\lambda z.\ f\ z * g\ z) \longrightarrow 0)\ F$
proof −
have ∗: $((\lambda x.\ norm\ (f\ x) * B) \longrightarrow 0)\ F$
by (*simp add*: *f tendsto-mult-left-zero tendsto-norm-zero*)
have $((\lambda x.\ norm\ (f\ x) * norm\ (g\ x)) \longrightarrow 0)\ F$
apply (*rule Lim-null-comparison* [*OF* - ∗])
apply (*simp add*: *eventually-mono* [*OF g*] *mult-left-mono*)
done
then show *?thesis*
by (*subst tendsto-norm-zero-iff* [*symmetric*]) (*simp add*: *norm-mult*)
qed

lemma *lim-null-mult-left-bounded*:
fixes $f :: 'a \Rightarrow 'b$::*real-normed-div-algebra*
assumes *g*: *eventually* $(\lambda x.\ norm(g\ x) \leq B)\ F$ **and** *f*: $(f \longrightarrow 0)\ F$
shows $((\lambda z.\ g\ z * f\ z) \longrightarrow 0)\ F$
proof −
have ∗: $((\lambda x.\ B * norm\ (f\ x)) \longrightarrow 0)\ F$
by (*simp add*: *f tendsto-mult-right-zero tendsto-norm-zero*)
have $((\lambda x.\ norm\ (g\ x) * norm\ (f\ x)) \longrightarrow 0)\ F$
apply (*rule Lim-null-comparison* [*OF* - ∗])
apply (*simp add*: *eventually-mono* [*OF g*] *mult-right-mono*)
done
then show *?thesis*
by (*subst tendsto-norm-zero-iff* [*symmetric*]) (*simp add*: *norm-mult*)
qed

Deducing things about the limit from the elements.

lemma *Lim-in-closed-set*:
assumes *closed S*
and *eventually* $(\lambda x.\ f(x) \in S)$ *net*
and ¬ *trivial-limit net* $(f \longrightarrow l)$ *net*
shows $l \in S$
proof (*rule ccontr*)
assume $l \notin S$
with ‹*closed S*› **have** *open* $(-\ S)\ l \in -\ S$
by (*simp-all add*: *open-Compl*)
with *assms*(*4*) **have** *eventually* $(\lambda x.\ f\ x \in -\ S)$ *net*

by (*rule topological-tendstoD*)
with *assms*(*2*) **have** *eventually* (λ*x*. *False*) *net*
by (*rule eventually-elim2*) *simp*
with *assms*(*3*) **show** *False*
by (*simp add*: *eventually-False*)
qed

Need to prove closed(cball(x,e)) before deducing this as a corollary.

lemma *Lim-dist-ubound*:
assumes ¬(*trivial-limit net*)
and (*f* ⟶ *l*) *net*
and *eventually* (λ*x*. *dist a* (*f x*) ≤ *e*) *net*
shows *dist a l* ≤ *e*
using *assms* **by** (*fast intro*: *tendsto-le tendsto-intros*)

lemma *Lim-norm-ubound*:
fixes *f* :: *'a* ⇒ *'b*::*real-normed-vector*
assumes ¬(*trivial-limit net*) (*f* ⟶ *l*) *net eventually* (λ*x*. *norm*(*f x*) ≤ *e*) *net*
shows *norm*(*l*) ≤ *e*
using *assms* **by** (*fast intro*: *tendsto-le tendsto-intros*)

lemma *Lim-norm-lbound*:
fixes *f* :: *'a* ⇒ *'b*::*real-normed-vector*
assumes ¬ *trivial-limit net*
and (*f* ⟶ *l*) *net*
and *eventually* (λ*x*. *e* ≤ *norm* (*f x*)) *net*
shows *e* ≤ *norm l*
using *assms* **by** (*fast intro*: *tendsto-le tendsto-intros*)

Limit under bilinear function

lemma *Lim-bilinear*:
assumes (*f* ⟶ *l*) *net*
and (*g* ⟶ *m*) *net*
and *bounded-bilinear h*
shows ((λ*x*. *h* (*f x*) (*g x*)) ⟶ (*h l m*)) *net*
using ‹*bounded-bilinear h*› ‹(*f* ⟶ *l*) *net*› ‹(*g* ⟶ *m*) *net*›
by (*rule bounded-bilinear.tendsto*)

These are special for limits out of the same vector space.

lemma *Lim-within-id*: (*id* ⟶ *a*) (*at a within s*)
unfolding *id-def* **by** (*rule tendsto-ident-at*)

lemma *Lim-at-id*: (*id* ⟶ *a*) (*at a*)
unfolding *id-def* **by** (*rule tendsto-ident-at*)

lemma *Lim-at-zero*:
fixes *a* :: *'a*::*real-normed-vector*
and *l* :: *'b*::*topological-space*
shows (*f* ⟶ *l*) (*at a*) ⟷ ((λ*x*. *f*(*a* + *x*)) ⟶ *l*) (*at 0*)

using *LIM-offset-zero LIM-offset-zero-cancel* **..**

It's also sometimes useful to extract the limit point from the filter.

abbreviation *netlimit* :: *'a::t2-space filter* ⇒ *'a*
where *netlimit F* ≡ *Lim F* (λ*x*. *x*)

lemma *netlimit-within*: ¬ *trivial-limit* (*at a within S*) ⟹ *netlimit* (*at a within S*) = *a*
by (*rule tendsto-Lim*) (*auto intro*: *tendsto-intros*)

lemma *netlimit-at*:
fixes *a* :: *'a::*{*perfect-space*,*t2-space*}
shows *netlimit* (*at a*) = *a*
using *netlimit-within* [*of a UNIV*] **by** *simp*

lemma *lim-within-interior*:
x ∈ *interior S* ⟹ (*f* ⟶ *l*) (*at x within S*) ⟷ (*f* ⟶ *l*) (*at x*)
by (*metis at-within-interior*)

lemma *netlimit-within-interior*:
fixes *x* :: *'a::*{*t2-space*,*perfect-space*}
assumes *x* ∈ *interior S*
shows *netlimit* (*at x within S*) = *x*
using *assms* **by** (*metis at-within-interior netlimit-at*)

lemma *netlimit-at-vector*:
fixes *a* :: *'a::real-normed-vector*
shows *netlimit* (*at a*) = *a*
proof (*cases* ∃ *x*. *x* ≠ *a*)
case *True* **then obtain** *x* **where** *x*: *x* ≠ *a* **..**
have ¬ *trivial-limit* (*at a*)
unfolding *trivial-limit-def eventually-at dist-norm*
apply *clarsimp*
apply (*rule-tac x*=*a* + *scaleR* (*d* / *2*) (*sgn* (*x* − *a*)) **in** *exI*)
apply (*simp add*: *norm-sgn sgn-zero-iff x*)
done
then show *?thesis*
by (*rule netlimit-within* [*of a UNIV*])
qed *simp*

Useful lemmas on closure and set of possible sequential limits.

lemma *closure-sequential*:
fixes *l* :: *'a::first-countable-topology*
shows *l* ∈ *closure S* ⟷ (∃ *x*. (∀ *n*. *x n* ∈ *S*) ∧ (*x* ⟶ *l*) *sequentially*)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
moreover
{

```
    assume l ∈ S
    then have ?rhs using tendsto-const[of l sequentially] by auto
  }
  moreover
  {
    assume l islimpt S
    then have ?rhs unfolding islimpt-sequential by auto
  }
  ultimately show ?rhs
    unfolding closure-def by auto
next
  assume ?rhs
  then show ?lhs unfolding closure-def islimpt-sequential by auto
qed

lemma closed-sequential-limits:
  fixes S :: 'a::first-countable-topology set
  shows closed S ⟷ (∀ x l. (∀ n. x n ∈ S) ∧ (x ⟶ l) sequentially ⟶ l ∈ S)
by (metis closure-sequential closure-subset-eq subset-iff)

lemma closure-approachable:
  fixes S :: 'a::metric-space set
  shows x ∈ closure S ⟷ (∀ e>0. ∃ y∈S. dist y x < e)
  apply (auto simp add: closure-def islimpt-approachable)
  apply (metis dist-self)
  done

lemma closed-approachable:
  fixes S :: 'a::metric-space set
  shows closed S ⟹ (∀ e>0. ∃ y∈S. dist y x < e) ⟷ x ∈ S
  by (metis closure-closed closure-approachable)

lemma closure-contains-Inf:
  fixes S :: real set
  assumes S ≠ {} bdd-below S
  shows Inf S ∈ closure S
proof −
  have ∗: ∀ x∈S. Inf S ≤ x
    using cInf-lower[of - S] assms by metis
  {
    fix e :: real
    assume e > 0
    then have Inf S < Inf S + e by simp
    with assms obtain x where x ∈ S x < Inf S + e
      by (subst (asm) cInf-less-iff) auto
    with ∗ have ∃ x∈S. dist x (Inf S) < e
      by (intro bexI[of - x]) (auto simp add: dist-real-def)
  }
  then show ?thesis unfolding closure-approachable by auto
```

qed

lemma *closed-contains-Inf*:
fixes S :: *real set*
shows $S \neq \{\} \Longrightarrow$ *bdd-below* $S \Longrightarrow$ *closed* $S \Longrightarrow$ *Inf* $S \in S$
by (*metis closure-contains-Inf closure-closed assms*)

lemma *closed-subset-contains-Inf*:
fixes A C :: *real set*
shows *closed* $C \Longrightarrow A \subseteq C \Longrightarrow A \neq \{\} \Longrightarrow$ *bdd-below* $A \Longrightarrow$ *Inf* $A \in C$
by (*metis closure-contains-Inf closure-minimal subset-eq*)

lemma *atLeastAtMost-subset-contains-Inf*:
fixes A :: *real set* **and** a b :: *real*
shows $A \neq \{\} \Longrightarrow a \leq b \Longrightarrow A \subseteq \{a..b\} \Longrightarrow$ *Inf* $A \in \{a..b\}$
by (*rule closed-subset-contains-Inf*)
(*auto intro*: *closed-real-atLeastAtMost intro*!: *bdd-belowI*[*of A a*])

lemma *not-trivial-limit-within-ball*:
$\neg$ *trivial-limit* (*at x within S*) $\longleftrightarrow$ ($\forall e{>}0.\ S \cap$ *ball x e* $-\ \{x\} \neq \{\}$)
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
show *?rhs* **if** *?lhs*
proof −
{
fix e :: *real*
assume $e > 0$
then obtain y **where** $y \in S - \{x\}$ **and** *dist* $y\ x < e$
using ‹*?lhs*› *not-trivial-limit-within*[*of x S*] *closure-approachable*[*of x S* − $\{x\}$]
by *auto*
then have $y \in S \cap$ *ball* $x\ e - \{x\}$
unfolding *ball-def* **by** (*simp add*: *dist-commute*)
then have $S \cap$ *ball* $x\ e - \{x\} \neq \{\}$ **by** *blast*
}
then show *?thesis* **by** *auto*
qed
show *?lhs* **if** *?rhs*
proof −
{
fix e :: *real*
assume $e > 0$
then obtain y **where** $y \in S \cap$ *ball* $x\ e - \{x\}$
using ‹*?rhs*› **by** *blast*
then have $y \in S - \{x\}$ **and** *dist* $y\ x < e$
unfolding *ball-def* **by** (*simp-all add*: *dist-commute*)
then have $\exists y \in S - \{x\}.$ *dist* $y\ x < e$
by *auto*
}

then show *?thesis*
using *not-trivial-limit-within*[*of x S*] *closure-approachable*[*of x S − {x}*]
by *auto*
qed
qed

16.16 Infimum Distance

definition *infdist x A* = (*if A* = {} *then 0 else INF a*:*A*. *dist x a*)

lemma *bdd-below-infdist*[*intro*, *simp*]: *bdd-below* (*dist x*‘*A*)
by (*auto intro*!: *zero-le-dist*)

lemma *infdist-notempty*: *A* ≠ {} ⟹ *infdist x A* = (*INF a*:*A*. *dist x a*)
by (*simp add*: *infdist-def*)

lemma *infdist-nonneg*: *0* ≤ *infdist x A*
by (*auto simp add*: *infdist-def intro*: *cINF-greatest*)

lemma *infdist-le*: *a* ∈ *A* ⟹ *infdist x A* ≤ *dist x a*
by (*auto intro*: *cINF-lower simp add*: *infdist-def*)

lemma *infdist-le2*: *a* ∈ *A* ⟹ *dist x a* ≤ *d* ⟹ *infdist x A* ≤ *d*
by (*auto intro*!: *cINF-lower2 simp add*: *infdist-def*)

lemma *infdist-zero*[*simp*]: *a* ∈ *A* ⟹ *infdist a A* = *0*
by (*auto intro*!: *antisym infdist-nonneg infdist-le2*)

lemma *infdist-triangle*: *infdist x A* ≤ *infdist y A* + *dist x y*
proof (*cases A* = {})
case *True*
then show *?thesis* **by** (*simp add*: *infdist-def*)
next
case *False*
then obtain *a* **where** *a* ∈ *A* **by** *auto*
have *infdist x A* ≤ *Inf* {*dist x y* + *dist y a* |*a*. *a* ∈ *A*}
proof (*rule cInf-greatest*)
from ‹*A* ≠ {}› **show** {*dist x y* + *dist y a* |*a*. *a* ∈ *A*} ≠ {}
by *simp*
fix *d*
assume *d* ∈ {*dist x y* + *dist y a* |*a*. *a* ∈ *A*}
then obtain *a* **where** *d*: *d* = *dist x y* + *dist y a a* ∈ *A*
by *auto*
show *infdist x A* ≤ *d*
unfolding *infdist-notempty*[*OF* ‹*A* ≠ {}›]
proof (*rule cINF-lower2*)
show *a* ∈ *A* **by** *fact*
show *dist x a* ≤ *d*
unfolding *d* **by** (*rule dist-triangle*)

```
    qed simp
  qed
  also have ... = dist x y + infdist y A
  proof (rule cInf-eq, safe)
    fix a
    assume a ∈ A
    then show dist x y + infdist y A ≤ dist x y + dist y a
      by (auto intro: infdist-le)
  next
    fix i
    assume inf: ⋀d. d ∈ {dist x y + dist y a |a. a ∈ A} ⟹ i ≤ d
    then have i − dist x y ≤ infdist y A
      unfolding infdist-notempty[OF ‹A ≠ {}›] using ‹a ∈ A›
      by (intro cINF-greatest) (auto simp: field-simps)
    then show i ≤ dist x y + infdist y A
      by simp
  qed
  finally show ?thesis by simp
qed

lemma in-closure-iff-infdist-zero:
  assumes A ≠ {}
  shows x ∈ closure A ⟷ infdist x A = 0
proof
  assume x ∈ closure A
  show infdist x A = 0
  proof (rule ccontr)
    assume infdist x A ≠ 0
    with infdist-nonneg[of x A] have infdist x A > 0
      by auto
    then have ball x (infdist x A) ∩ closure A = {}
      apply auto
       apply (metis ‹x ∈ closure A› closure-approachable dist-commute infdist-le
not-less)
      done
    then have x ∉ closure A
      by (metis ‹0 < infdist x A› centre-in-ball disjoint-iff-not-equal)
    then show False using ‹x ∈ closure A› by simp
  qed
next
  assume x: infdist x A = 0
  then obtain a where a ∈ A
    by atomize-elim (metis all-not-in-conv assms)
  show x ∈ closure A
    unfolding closure-approachable
    apply safe
  proof (rule ccontr)
    fix e :: real
    assume e > 0
```

```
    assume ¬ (∃ y∈A. dist y x < e)
    then have infdist x A ≥ e using ‹a ∈ A›
      unfolding infdist-def
      by (force simp: dist-commute intro: cINF-greatest)
    with x ‹e > 0› show False by auto
  qed
qed

lemma in-closed-iff-infdist-zero:
  assumes closed A A ≠ {}
  shows x ∈ A ⟷ infdist x A = 0
proof −
  have x ∈ closure A ⟷ infdist x A = 0
    by (rule in-closure-iff-infdist-zero) fact
  with assms show ?thesis by simp
qed

lemma tendsto-infdist [tendsto-intros]:
  assumes f: (f ⟶ l) F
  shows ((λx. infdist (f x) A) ⟶ infdist l A) F
proof (rule tendstoI)
  fix e ::real
  assume e > 0
  from tendstoD[OF f this]
  show eventually (λx. dist (infdist (f x) A) (infdist l A) < e) F
  proof (eventually-elim)
    fix x
    from infdist-triangle[of l A f x] infdist-triangle[of f x A l]
    have dist (infdist (f x) A) (infdist l A) ≤ dist (f x) l
      by (simp add: dist-commute dist-real-def)
    also assume dist (f x) l < e
    finally show dist (infdist (f x) A) (infdist l A) < e .
  qed
qed
```

Some other lemmas about sequences.

```
lemma sequentially-offset:
  assumes eventually (λi. P i) sequentially
  shows eventually (λi. P (i + k)) sequentially
  using assms by (rule eventually-sequentially-seg [THEN iffD2])

lemma seq-offset-neg:
  (f ⟶ l) sequentially ⟹ ((λi. f(i − k)) ⟶ l) sequentially
  apply (erule filterlim-compose)
 apply (simp add: filterlim-def le-sequentially eventually-filtermap eventually-sequentially)
  apply arith
  done

lemma seq-harmonic: ((λn. inverse (real n)) ⟶ 0) sequentially
```

using *LIMSEQ-inverse-real-of-nat* **by** (*rule LIMSEQ-imp-Suc*)

16.17 More properties of closed balls

lemma *closed-cball* [*iff*]: *closed* (*cball x e*)
proof −
have *closed* (*dist x* −‘ {..*e*})
by (*intro closed-vimage closed-atMost continuous-intros*)
also have *dist x* −‘ {..*e*} = *cball x e*
by *auto*
finally show *?thesis* **.**
qed

lemma *open-contains-cball*: *open S* ⟷ (∀ *x*∈*S*. ∃ *e*>*0*. *cball x e* ⊆ *S*)
proof −
{
fix *x* **and** *e*::*real*
assume *x*∈*S e*>*0 ball x e* ⊆ *S*
then have ∃ *d*>*0*. *cball x d* ⊆ *S* **unfolding** *subset-eq* **by** (*rule-tac x*=*e*/*2* **in** *exI*, *auto*)
}
moreover
{
fix *x* **and** *e*::*real*
assume *x*∈*S e*>*0 cball x e* ⊆ *S*
then have ∃ *d*>*0*. *ball x d* ⊆ *S*
unfolding *subset-eq*
apply(*rule-tac x*=*e*/*2* **in** *exI*)
apply *auto*
done
}
ultimately show *?thesis*
unfolding *open-contains-ball* **by** *auto*
qed

lemma *open-contains-cball-eq*: *open S* ⟹ (∀ *x*. *x* ∈ *S* ⟷ (∃ *e*>*0*. *cball x e* ⊆ *S*))
by (*metis open-contains-cball subset-eq order-less-imp-le centre-in-cball*)

lemma *mem-interior-cball*: *x* ∈ *interior S* ⟷ (∃ *e*>*0*. *cball x e* ⊆ *S*)
apply (*simp add*: *interior-def*, *safe*)
apply (*force simp add*: *open-contains-cball*)
apply (*rule-tac x*=*ball x e* **in** *exI*)
apply (*simp add*: *subset-trans* [*OF ball-subset-cball*])
done

lemma *islimpt-ball*:
fixes *x y* :: *'a*::{*real-normed-vector*,*perfect-space*}
shows *y islimpt ball x e* ⟷ *0* < *e* ∧ *y* ∈ *cball x e*

```
  (is ?lhs ⟷ ?rhs)
proof
  show ?rhs if ?lhs
  proof
    {
      assume e ≤ 0
      then have ∗: ball x e = {}
        using ball-eq-empty[of x e] by auto
      have False using ‹?lhs›
        unfolding ∗ using islimpt-EMPTY[of y] by auto
    }
    then show e > 0 by (metis not-less)
    show y ∈ cball x e
      using closed-cball[of x e] islimpt-subset[of y ball x e cball x e]
        ball-subset-cball[of x e] ‹?lhs›
      unfolding closed-limpt by auto
  qed
  show ?lhs if ?rhs
  proof −
    from that have e > 0 by auto
    {
      fix d :: real
      assume d > 0
      have ∃x′∈ball x e. x′ ≠ y ∧ dist x′ y < d
      proof (cases d ≤ dist x y)
        case True
        then show ∃x′∈ball x e. x′ ≠ y ∧ dist x′ y < d
        proof (cases x = y)
          case True
          then have False
            using ‹d ≤ dist x y› ‹d>0› by auto
          then show ∃x′∈ball x e. x′ ≠ y ∧ dist x′ y < d
            by auto
        next
          case False
          have dist x (y − (d / (2 ∗ dist y x)) ∗R (y − x)) =
            norm (x − y + (d / (2 ∗ norm (y − x))) ∗R (y − x))
         unfolding mem-cball mem-ball dist-norm diff-diff-eq2 diff-add-eq[symmetric]
            by auto
          also have ... = |− 1 + d / (2 ∗ norm (x − y))| ∗ norm (x − y)
            using scaleR-left-distrib[of − 1 d / (2 ∗ norm (y − x)), symmetric, of
y − x]
            unfolding scaleR-minus-left scaleR-one
            by (auto simp add: norm-minus-commute)
          also have ... = |− norm (x − y) + d / 2|
            unfolding abs-mult-pos[of norm (x − y), OF norm-ge-zero[of x − y]]
            unfolding distrib-right using ‹x≠y› by auto
          also have ... ≤ e − d/2 using ‹d ≤ dist x y› and ‹d>0› and ‹?rhs›
            by (auto simp add: dist-norm)
```

```
      finally have y − (d / (2 ∗ dist y x)) ∗R (y − x) ∈ ball x e using ‹d>0›
        by auto
      moreover
      have (d / (2∗dist y x)) ∗R (y − x) ≠ 0
        using ‹x≠y›[unfolded dist-nz] ‹d>0› unfolding scaleR-eq-0-iff
        by (auto simp add: dist-commute)
      moreover
      have dist (y − (d / (2 ∗ dist y x)) ∗R (y − x)) y < d
        unfolding dist-norm
        apply simp
        unfolding norm-minus-cancel
        using ‹d > 0› ‹x≠y›[unfolded dist-nz] dist-commute[of x y]
        unfolding dist-norm
        apply auto
        done
      ultimately show ∃x′∈ball x e. x′ ≠ y ∧ dist x′ y < d
        apply (rule-tac x = y − (d / (2∗dist y x)) ∗R (y − x) in bexI)
        apply auto
        done
    qed
   next
    case False
    then have d > dist x y by auto
    show ∃x′ ∈ ball x e. x′ ≠ y ∧ dist x′ y < d
    proof (cases x = y)
     case True
     obtain z where ∗∗: z ≠ y dist z y < min e d
       using perfect-choose-dist[of min e d y]
       using ‹d > 0› ‹e>0› by auto
     show ∃x′∈ball x e. x′ ≠ y ∧ dist x′ y < d
       unfolding ‹x = y›
       using ‹z ≠ y› ∗∗
       apply (rule-tac x=z in bexI)
       apply (auto simp add: dist-commute)
       done
    next
     case False
     then show ∃x′∈ball x e. x′ ≠ y ∧ dist x′ y < d
       using ‹d>0› ‹d > dist x y› ‹?rhs›
       apply (rule-tac x=x in bexI)
       apply auto
       done
    qed
   qed
  }
  then show ?thesis
   unfolding mem-cball islimpt-approachable mem-ball by auto
 qed
qed
```

lemma *closure-ball-lemma*:
fixes $x\ y :: 'a$::*real-normed-vector*
assumes $x \neq y$
shows y *islimpt ball* x (*dist* $x\ y$)
proof (*rule islimptI*)
fix T
assume $y \in T$ *open* T
then obtain r **where** $0 < r\ \forall z.$ *dist* $z\ y < r \longrightarrow z \in T$
unfolding *open-dist* **by** *fast*

def $k \equiv$ *min* 1 (r / (2 $*$ *dist* $x\ y$))
def $z \equiv y +$ *scaleR* k ($x - y$)
have *z-def2*: $z = x +$ *scaleR* (1 $-$ k) ($y - x$)
unfolding *z-def* **by** (*simp add*: *algebra-simps*)
have *dist* $z\ y < r$
unfolding *z-def k-def* **using** ‹$0 < r$›
by (*simp add*: *dist-norm min-def*)
then have $z \in T$
using ‹$\forall z.$ *dist* $z\ y < r \longrightarrow z \in T$› **by** *simp*
have *dist* $x\ z <$ *dist* $x\ y$
unfolding *z-def2 dist-norm*
apply (*simp add*: *norm-minus-commute*)
apply (*simp only*: *dist-norm* [*symmetric*])
apply (*subgoal-tac* $|1 - k| *$ *dist* $x\ y < 1 *$ *dist* $x\ y$, *simp*)
apply (*rule mult-strict-right-mono*)
apply (*simp add*: *k-def* ‹$0 < r$› ‹$x \neq y$›)
apply (*simp add*: ‹$x \neq y$›)
done
then have $z \in$ *ball* x (*dist* $x\ y$)
by *simp*
have $z \neq y$
unfolding *z-def k-def* **using** ‹$x \neq y$› ‹$0 < r$›
by (*simp add*: *min-def*)
show $\exists z{\in}$*ball* x (*dist* $x\ y$). $z \in T \wedge z \neq y$
using ‹$z \in$ *ball* x (*dist* $x\ y$)› ‹$z \in T$› ‹$z \neq y$›
by *fast*
qed

lemma *closure-ball* [*simp*]:
fixes $x :: 'a$::*real-normed-vector*
shows $0 < e \Longrightarrow$ *closure* (*ball* $x\ e$) = *cball* $x\ e$
apply (*rule equalityI*)
apply (*rule closure-minimal*)
apply (*rule ball-subset-cball*)
apply (*rule closed-cball*)
apply (*rule subsetI*, *rename-tac* y)
apply (*simp add*: *le-less* [**where** *'a*=*real*])
apply (*erule disjE*)

apply (*rule subsetD* [*OF closure-subset*], *simp*)
apply (*simp add*: *closure-def*)
apply *clarify*
apply (*rule closure-ball-lemma*)
apply (*simp add*: *zero-less-dist-iff*)
done

lemma *interior-cball* [*simp*]:
fixes x :: $'a$::{*real-normed-vector*, *perfect-space*}
shows *interior* (*cball x e*) = *ball x e*
proof (*cases e* ≥ *0*)
case *False* **note** *cs* = *this*
from *cs* **have** *null*: *ball x e* = {}
using *ball-empty*[*of e x*] **by** *auto*
moreover
{
fix *y*
assume *y* ∈ *cball x e*
then have *False*
by (*metis ball-eq-empty null cs dist-eq-0-iff dist-le-zero-iff empty-subsetI mem-cball subset-antisym subset-ball*)
}
then have *cball x e* = {} **by** *auto*
then have *interior* (*cball x e*) = {}
using *interior-empty* **by** *auto*
ultimately show *?thesis* **by** *blast*
next
case *True* **note** *cs* = *this*
have *ball x e* ⊆ *cball x e*
using *ball-subset-cball* **by** *auto*
moreover
{
fix *S y*
assume *as*: *S* ⊆ *cball x e open S y*∈*S*
then obtain *d* **where** *d*>*0* **and** *d*: ∀ *x'*. *dist x' y* < *d* ⟶ *x'* ∈ *S*
unfolding *open-dist* **by** *blast*
then obtain *xa* **where** *xa-y*: *xa* ≠ *y* **and** *xa*: *dist xa y* < *d*
using *perfect-choose-dist* [*of d*] **by** *auto*
have *xa* ∈ *S*
using *d*[*THEN spec*[**where** *x* = *xa*]]
using *xa* **by** (*auto simp add*: *dist-commute*)
then have *xa-cball*: *xa* ∈ *cball x e*
using *as*(*1*) **by** *auto*
then have *y* ∈ *ball x e*
proof (*cases x* = *y*)
case *True*
then have *e* > *0* **using** *cs order.order-iff-strict xa-cball xa-y* **by** *fastforce*
then show *y* ∈ *ball x e*

```
      using ‹x = y › by simp
    next
      case False
      have dist (y + (d / 2 / dist y x) *R (y − x)) y < d
        unfolding dist-norm
        using ‹d>0› norm-ge-zero[of y − x] ‹x ≠ y› by auto
      then have *: y + (d / 2 / dist y x) *R (y − x) ∈ cball x e
        using d as(1)[unfolded subset-eq] by blast
      have y − x ≠ 0 using ‹x ≠ y› by auto
      hence **:d / (2 * norm (y − x)) > 0
        unfolding zero-less-norm-iff[symmetric] using ‹d>0› by auto
      have dist (y + (d / 2 / dist y x) *R (y − x)) x =
        norm (y + (d / (2 * norm (y − x))) *R y − (d / (2 * norm (y − x))) *R
x − x)
        by (auto simp add: dist-norm algebra-simps)
      also have ... = norm ((1 + d / (2 * norm (y − x))) *R (y − x))
        by (auto simp add: algebra-simps)
      also have ... = |1 + d / (2 * norm (y − x))| * norm (y − x)
        using ** by auto
      also have ... = (dist y x) + d/2
        using ** by (auto simp add: distrib-right dist-norm)
      finally have e ≥ dist x y +d/2
        using *[unfolded mem-cball] by (auto simp add: dist-commute)
      then show y ∈ ball x e
        unfolding mem-ball using ‹d>0› by auto
    qed
  }
  then have ∀ S ⊆ cball x e. open S ⟶ S ⊆ ball x e
    by auto
  ultimately show ?thesis
    using interior-unique[of ball x e cball x e]
    using open-ball[of x e]
    by auto
qed

lemma interior-ball [simp]: interior (ball x e) = ball x e
  by (simp add: interior-open)

lemma frontier-ball [simp]:
  fixes a :: 'a::real-normed-vector
  shows 0 < e ⟹ frontier (ball a e) = sphere a e
  by (force simp: frontier-def)

lemma frontier-cball [simp]:
  fixes a :: 'a::{real-normed-vector, perfect-space}
  shows frontier (cball a e) = sphere a e
  by (force simp: frontier-def)

lemma cball-eq-empty [simp]: cball x e = {} ⟷ e < 0
```

apply (*simp add*: *set-eq-iff not-le*)
apply (*metis zero-le-dist dist-self order-less-le-trans*)
done

lemma *cball-empty* [*simp*]: $e < 0 \Longrightarrow cball\ x\ e = \{\}$
by (*simp add*: *cball-eq-empty*)

lemma *cball-eq-sing*:
fixes $x :: 'a$::{*metric-space*,*perfect-space*}
shows $cball\ x\ e = \{x\} \longleftrightarrow e = 0$
proof (*rule linorder-cases*)
assume *e*: $0 < e$
obtain *a* **where** $a \neq x\ dist\ a\ x < e$
using *perfect-choose-dist* [*OF e*] **by** *auto*
then have $a \neq x\ dist\ x\ a \leq e$
by (*auto simp add*: *dist-commute*)
with *e* **show** *?thesis* **by** (*auto simp add*: *set-eq-iff*)
qed *auto*

lemma *cball-sing*:
fixes $x :: 'a$::*metric-space*
shows $e = 0 \Longrightarrow cball\ x\ e = \{x\}$
by (*auto simp add*: *set-eq-iff*)

lemma *ball-divide-subset*: $d \geq 1 \Longrightarrow ball\ x\ (e/d) \subseteq ball\ x\ e$
apply (*cases* $e \leq 0$)
apply (*simp add*: *ball-empty divide-simps*)
apply (*rule subset-ball*)
apply (*simp add*: *divide-simps*)
done

lemma *ball-divide-subset-numeral*: $ball\ x\ (e\ /\ numeral\ w) \subseteq ball\ x\ e$
using *ball-divide-subset one-le-numeral* **by** *blast*

lemma *cball-divide-subset*: $d \geq 1 \Longrightarrow cball\ x\ (e/d) \subseteq cball\ x\ e$
apply (*cases* $e < 0$)
apply (*simp add*: *divide-simps*)
apply (*rule subset-cball*)
apply (*metis divide-1 frac-le not-le order-refl zero-less-one*)
done

lemma *cball-divide-subset-numeral*: $cball\ x\ (e\ /\ numeral\ w) \subseteq cball\ x\ e$
using *cball-divide-subset one-le-numeral* **by** *blast*

16.18 Boundedness

definition (**in** *metric-space*) *bounded* :: $'a\ set \Rightarrow bool$
where $bounded\ S \longleftrightarrow (\exists x\ e.\ \forall y{\in}S.\ dist\ x\ y \leq e)$

lemma *bounded-subset-cball*: *bounded S* $\longleftrightarrow$ $(\exists\, e\ x.\ S \subseteq cball\ x\ e \wedge 0 \leq e)$
unfolding *bounded-def subset-eq* **by** *auto* (*meson order-trans zero-le-dist*)

lemma *bounded-subset-ballD*:
assumes *bounded S* **shows** $\exists\, r.\ 0 < r \wedge S \subseteq ball\ x\ r$
proof −
obtain *e*::*real* **and** *y* **where** $S \subseteq cball\ y\ e\ \ 0 \leq e$
using *assms* **by** (*auto simp*: *bounded-subset-cball*)
then show *?thesis*
apply (*rule-tac x*=*dist x y* + *e* + *1* **in** *exI*)
apply (*simp add*: *add.commute add-pos-nonneg*)
apply (*erule subset-trans*)
apply (*clarsimp simp add*: *cball-def*)
by (*metis add-le-cancel-right add-strict-increasing dist-commute dist-triangle-le zero-less-one*)
qed

lemma *bounded-any-center*: *bounded S* $\longleftrightarrow$ $(\exists\, e.\ \forall\, y{\in}S.\ dist\ a\ y \leq e)$
unfolding *bounded-def*
by *auto* (*metis add.commute add-le-cancel-right dist-commute dist-triangle-le*)

lemma *bounded-iff*: *bounded S* $\longleftrightarrow$ $(\exists\, a.\ \forall\, x{\in}S.\ norm\ x \leq a)$
unfolding *bounded-any-center* [**where** *a*=*0*]
by (*simp add*: *dist-norm*)

lemma *bdd-above-norm*: *bdd-above* (*norm* ' *X*) $\longleftrightarrow$ *bounded X*
by (*simp add*: *bounded-iff bdd-above-def*)

lemma *bounded-realI*:
assumes $\forall\, x{\in}s.\ |x{::}real| \leq B$
shows *bounded s*
unfolding *bounded-def dist-real-def*
by (*metis abs-minus-commute assms diff-0-right*)

lemma *bounded-empty* [*simp*]: *bounded* {}
by (*simp add*: *bounded-def*)

lemma *bounded-subset*: *bounded T* $\Longrightarrow$ $S \subseteq T \Longrightarrow$ *bounded S*
by (*metis bounded-def subset-eq*)

lemma *bounded-interior*[*intro*]: *bounded S* $\Longrightarrow$ *bounded*(*interior S*)
by (*metis bounded-subset interior-subset*)

lemma *bounded-closure*[*intro*]:
assumes *bounded S*
shows *bounded* (*closure S*)
proof −
from *assms* **obtain** *x* **and** *a* **where** *a*: $\forall\, y{\in}S.\ dist\ x\ y \leq a$
unfolding *bounded-def* **by** *auto*

```
  {
    fix y
    assume y ∈ closure S
    then obtain f where f: ∀ n. f n ∈ S  (f ⟶ y) sequentially
      unfolding closure-sequential by auto
    have ∀ n. f n ∈ S ⟶ dist x (f n) ≤ a using a by simp
    then have eventually (λn. dist x (f n) ≤ a) sequentially
      by (simp add: f(1))
    have dist x y ≤ a
      apply (rule Lim-dist-ubound [of sequentially f])
      apply (rule trivial-limit-sequentially)
      apply (rule f(2))
      apply fact
      done
  }
  then show ?thesis
    unfolding bounded-def by auto
qed

lemma bounded-cball[simp,intro]: bounded (cball x e)
  apply (simp add: bounded-def)
  apply (rule-tac x=x in exI)
  apply (rule-tac x=e in exI)
  apply auto
  done

lemma bounded-ball[simp,intro]: bounded (ball x e)
  by (metis ball-subset-cball bounded-cball bounded-subset)

lemma bounded-Un[simp]: bounded (S ∪ T) ⟷ bounded S ∧ bounded T
  apply (auto simp add: bounded-def)
  by (metis Un-iff add-le-cancel-left dist-triangle le-max-iff-disj max.order-iff)

lemma bounded-Union[intro]: finite F ⟹ ∀ S∈F. bounded S ⟹ bounded (⋃ F)
  by (induct rule: finite-induct[of F]) auto

lemma bounded-UN [intro]: finite A ⟹ ∀ x∈A. bounded (B x) ⟹ bounded
(⋃ x∈A. B x)
  by (induct set: finite) auto

lemma bounded-insert [simp]: bounded (insert x S) ⟷ bounded S
proof −
  have ∀ y∈{x}. dist x y ≤ 0
    by simp
  then have bounded {x}
    unfolding bounded-def by fast
  then show ?thesis
    by (metis insert-is-Un bounded-Un)
qed
```

lemma *finite-imp-bounded* [*intro*]: *finite S* $\Longrightarrow$ *bounded S*
by (*induct set*: *finite*) *simp-all*

lemma *bounded-pos*: *bounded S* $\longleftrightarrow$ ($\exists$ *b*>*0*. $\forall$ *x*$\in$ *S*. *norm x* $\leq$ *b*)
apply (*simp add*: *bounded-iff*)
apply (*subgoal-tac* $\bigwedge$*x* (*y*::*real*). *0* < *1* + |*y*| $\wedge$ (*x* $\leq$ *y* $\longrightarrow$ *x* $\leq$ *1* + |*y*|))
apply *metis*
apply *arith*
done

lemma *bounded-pos-less*: *bounded S* $\longleftrightarrow$ ($\exists$ *b*>*0*. $\forall$ *x*$\in$ *S*. *norm x* < *b*)
apply (*simp add*: *bounded-pos*)
apply (*safe*; *rule-tac x*=*b*+*1* **in** *exI*; *force*)
done

lemma *Bseq-eq-bounded*:
fixes *f* :: *nat* $\Rightarrow$ *'a*::*real-normed-vector*
shows *Bseq f* $\longleftrightarrow$ *bounded* (*range f*)
unfolding *Bseq-def bounded-pos* **by** *auto*

lemma *bounded-Int*[*intro*]: *bounded S* $\vee$ *bounded T* $\Longrightarrow$ *bounded* (*S* $\cap$ *T*)
by (*metis Int-lower1 Int-lower2 bounded-subset*)

lemma *bounded-diff*[*intro*]: *bounded S* $\Longrightarrow$ *bounded* (*S* − *T*)
by (*metis Diff-subset bounded-subset*)

lemma *not-bounded-UNIV*[*simp*]:
¬ *bounded* (*UNIV* :: *'a*::{*real-normed-vector*, *perfect-space*} *set*)
proof (*auto simp add*: *bounded-pos not-le*)
obtain *x* :: *'a* **where** *x* ≠ *0*
using *perfect-choose-dist* [*OF zero-less-one*] **by** *fast*
fix *b* :: *real*
assume *b*: *b* >*0*
have *b1*: *b* +*1* ≥ *0*
using *b* **by** *simp*
with ‹*x* ≠ *0*› **have** *b* < *norm* (*scaleR* (*b* + *1*) (*sgn x*))
by (*simp add*: *norm-sgn*)
then show $\exists$ *x*::*'a*. *b* < *norm x* **..**
qed

corollary *cobounded-imp-unbounded*:
fixes *S* :: *'a*::{*real-normed-vector*, *perfect-space*} *set*
shows *bounded* (− *S*) $\Longrightarrow$ ~ (*bounded S*)
using *bounded-Un* [*of S* −*S*] **by** (*simp add*: *sup-compl-top*)

lemma *bounded-linear-image*:
assumes *bounded S*
and *bounded-linear f*

```
  shows bounded (f ‘ S)
proof −
  from assms(1) obtain b where b: b > 0 ∀ x∈S. norm x ≤ b
    unfolding bounded-pos by auto
  from assms(2) obtain B where B: B > 0 ∀ x. norm (f x) ≤ B ∗ norm x
    using bounded-linear.pos-bounded by (auto simp add: ac-simps)
  {
    fix x
    assume x ∈ S
    then have norm x ≤ b
      using b by auto
    then have norm (f x) ≤ B ∗ b
      using B(2)
      apply (erule-tac x=x in allE)
      apply (metis B(1) B(2) order-trans mult-le-cancel-left-pos)
      done
  }
  then show ?thesis
    unfolding bounded-pos
    apply (rule-tac x=b∗B in exI)
    using b B by (auto simp add: mult.commute)
qed

lemma bounded-scaling:
  fixes S :: 'a::real-normed-vector set
  shows bounded S ⟹ bounded ((λx. c ∗R x) ‘ S)
  apply (rule bounded-linear-image)
  apply assumption
  apply (rule bounded-linear-scaleR-right)
  done

lemma bounded-translation:
  fixes S :: 'a::real-normed-vector set
  assumes bounded S
  shows bounded ((λx. a + x) ‘ S)
proof −
  from assms obtain b where b: b > 0 ∀ x∈S. norm x ≤ b
    unfolding bounded-pos by auto
  {
    fix x
    assume x ∈ S
    then have norm (a + x) ≤ b + norm a
      using norm-triangle-ineq[of a x] b by auto
  }
  then show ?thesis
    unfolding bounded-pos
    using norm-ge-zero[of a] b(1) and add-strict-increasing[of b 0 norm a]
    by (auto intro!: exI[of - b + norm a])
qed
```

lemma *bounded-translation-minus*:
fixes S :: *'a::real-normed-vector set*
shows *bounded* $S \Longrightarrow$ *bounded* $((\lambda x.\ x - a)\ `\ S)$
using *bounded-translation* [*of* $S\ -a$] **by** *simp*

lemma *bounded-uminus* [*simp*]:
fixes X :: *'a::real-normed-vector set*
shows *bounded* $(uminus\ `\ X) \longleftrightarrow$ *bounded* X
by (*auto simp*: *bounded-def dist-norm*; *rule-tac* $x{=}-x$ **in** *exI*; *force simp add*: *add.commute norm-minus-commute*)

Some theorems on sups and infs using the notion "bounded".

lemma *bounded-real*: *bounded* $(S$::*real set*$) \longleftrightarrow (\exists\, a.\ \forall\, x{\in}S.\ |x| \leq a)$
by (*simp add*: *bounded-iff*)

lemma *bounded-imp-bdd-above*: *bounded* $S \Longrightarrow$ *bdd-above* $(S$ :: *real set*$)$
by (*auto simp*: *bounded-def bdd-above-def dist-real-def*)
(*metis abs-le-D1 abs-minus-commute diff-le-eq*)

lemma *bounded-imp-bdd-below*: *bounded* $S \Longrightarrow$ *bdd-below* $(S$ :: *real set*$)$
by (*auto simp*: *bounded-def bdd-below-def dist-real-def*)
(*metis abs-le-D1 add.commute diff-le-eq*)

lemma *bounded-inner-imp-bdd-above*:
assumes *bounded s*
shows *bdd-above* $((\lambda x.\ x \cdot a)\ `\ s)$
by (*simp add*: *assms bounded-imp-bdd-above bounded-linear-image bounded-linear-inner-left*)

lemma *bounded-inner-imp-bdd-below*:
assumes *bounded s*
shows *bdd-below* $((\lambda x.\ x \cdot a)\ `\ s)$
by (*simp add*: *assms bounded-imp-bdd-below bounded-linear-image bounded-linear-inner-left*)

lemma *bounded-has-Sup*:
fixes S :: *real set*
assumes *bounded S*
and $S \neq \{\}$
shows $\forall\, x{\in}S.\ x \leq Sup\ S$
and $\forall\, b.\ (\forall\, x{\in}S.\ x \leq b) \longrightarrow Sup\ S \leq b$
proof
show $\forall\, b.\ (\forall\, x{\in}S.\ x \leq b) \longrightarrow Sup\ S \leq b$
using *assms* **by** (*metis cSup-least*)
qed (*metis cSup-upper assms*(*1*) *bounded-imp-bdd-above*)

lemma *Sup-insert*:
fixes S :: *real set*
shows *bounded* $S \Longrightarrow Sup\ (insert\ x\ S) = (if\ S = \{\}\ then\ x\ else\ max\ x\ (Sup\ S))$
by (*auto simp*: *bounded-imp-bdd-above sup-max cSup-insert-If*)

lemma *Sup-insert-finite*:
fixes $S :: 'a$::*conditionally-complete-linorder set*
shows *finite* $S \Longrightarrow$ *Sup* (*insert* x S) = (*if* $S = \{\}$ *then* x *else max* x (*Sup* S))
by (*simp add*: *cSup-insert sup-max*)

lemma *bounded-has-Inf*:
fixes S :: *real set*
assumes *bounded* S
and $S \neq \{\}$
shows $\forall x{\in}S.\ x \geq$ *Inf* S
and $\forall b.\ (\forall x{\in}S.\ x \geq b) \longrightarrow$ *Inf* $S \geq b$
proof
show $\forall b.\ (\forall x{\in}S.\ x \geq b) \longrightarrow$ *Inf* $S \geq b$
using *assms* **by** (*metis cInf-greatest*)
qed (*metis cInf-lower assms*(*1*) *bounded-imp-bdd-below*)

lemma *Inf-insert*:
fixes S :: *real set*
shows *bounded* $S \Longrightarrow$ *Inf* (*insert* x S) = (*if* $S = \{\}$ *then* x *else min* x (*Inf* S))
by (*auto simp*: *bounded-imp-bdd-below inf-min cInf-insert-If*)

lemma *Inf-insert-finite*:
fixes $S :: 'a$::*conditionally-complete-linorder set*
shows *finite* $S \Longrightarrow$ *Inf* (*insert* x S) = (*if* $S = \{\}$ *then* x *else min* x (*Inf* S))
by (*simp add*: *cInf-eq-Min*)

lemma *finite-imp-less-Inf*:
fixes $a :: 'a$::*conditionally-complete-linorder*
shows $[\![$*finite* X; $x \in X$; $\bigwedge x.\ x{\in}X \Longrightarrow a < x]\!] \Longrightarrow a <$ *Inf* X
by (*induction X rule*: *finite-induct*) (*simp-all add*: *cInf-eq-Min Inf-insert-finite*)

lemma *finite-less-Inf-iff*:
fixes $a :: 'a ::$ *conditionally-complete-linorder*
shows $[\![$*finite* X; $X \neq \{\}]\!] \Longrightarrow a <$ *Inf* $X \longleftrightarrow (\forall x \in X.\ a < x)$
by (*auto simp*: *cInf-eq-Min*)

lemma *finite-imp-Sup-less*:
fixes $a :: 'a$::*conditionally-complete-linorder*
shows $[\![$*finite* X; $x \in X$; $\bigwedge x.\ x{\in}X \Longrightarrow a > x]\!] \Longrightarrow a >$ *Sup* X
by (*induction X rule*: *finite-induct*) (*simp-all add*: *cSup-eq-Max Sup-insert-finite*)

lemma *finite-Sup-less-iff*:
fixes $a :: 'a ::$ *conditionally-complete-linorder*
shows $[\![$*finite* X; $X \neq \{\}]\!] \Longrightarrow a >$ *Sup* $X \longleftrightarrow (\forall x \in X.\ a > x)$
by (*auto simp*: *cSup-eq-Max*)

16.19 Compactness

16.19.1 Bolzano-Weierstrass property

lemma *heine-borel-imp-bolzano-weierstrass*:
assumes *compact s*
and *infinite t*
and $t \subseteq s$
shows $\exists x \in s.\ x$ *islimpt* t
proof (*rule ccontr*)
assume $\neg\ (\exists x \in s.\ x$ *islimpt* $t)$
then obtain f **where** f: $\forall x{\in}s.\ x \in f\,x \wedge$ *open* $(f\,x) \wedge (\forall y{\in}t.\ y \in f\,x \longrightarrow y = x)$
unfolding *islimpt-def*
using *bchoice*[*of s* $\lambda\ x\ T.\ x \in T \wedge$ *open* $T \wedge (\forall y{\in}t.\ y \in T \longrightarrow y = x)$]
by *auto*
obtain g **where** g: $g \subseteq \{t.\ \exists x.\ x \in s \wedge t = f\,x\}$ *finite* g $s \subseteq \bigcup g$
using *assms*(*1*)[*unfolded compact-eq-heine-borel*, *THEN spec*[**where** x=$\{t.\ \exists x.\ x{\in}s \wedge t = f\,x\}$]]
using f **by** *auto*
from g(*1*,*3*) **have** g':$\forall x{\in}g.\ \exists xa \in s.\ x = f\,xa$
by *auto*
{
fix $x\ y$
assume $x \in t\ y \in t\ f\,x = f\,y$
then have $x \in f\,x\ \ y \in f\,x \longrightarrow y = x$
using f[*THEN bspec*[**where** x=x]] **and** ‹$t \subseteq s$› **by** *auto*
then have $x = y$
using ‹$f\,x = f\,y$› **and** f[*THEN bspec*[**where** x=y]] **and** ‹$y \in t$› **and** ‹$t \subseteq s$›
by *auto*
}
then have *inj-on f t*
unfolding *inj-on-def* **by** *simp*
then have *infinite* $(f\ `\ t)$
using *assms*(*2*) **using** *finite-imageD* **by** *auto*
moreover
{
fix x
assume $x \in t\ f\,x \notin g$
from g(*3*) *assms*(*3*) ‹$x \in t$› **obtain** h **where** $h \in g$ **and** $x \in h$
by *auto*
then obtain y **where** $y \in s\ h = f\,y$
using g'[*THEN bspec*[**where** x=h]] **by** *auto*
then have $y = x$
using f[*THEN bspec*[**where** x=y]] **and** ‹$x{\in}t$› **and** ‹$x{\in}h$›[*unfolded* ‹$h = f\,y$›]
by *auto*
then have *False*
using ‹$f\,x \notin g$› ‹$h \in g$› **unfolding** ‹$h = f\,y$›
by *auto*
}

then have f ‘ $t \subseteq g$ **by** *auto*
ultimately show *False*
using $g(2)$ **using** *finite-subset* **by** *auto*
qed

lemma *acc-point-range-imp-convergent-subsequence*:
fixes $l :: 'a :: first\text{-}countable\text{-}topology$
assumes l: $\forall\, U.\ l{\in}U \longrightarrow open\ U \longrightarrow infinite\ (U \cap range\ f)$
shows $\exists\, r.\ subseq\ r \wedge (f \circ r) \longrightarrow l$
proof −
from *countable-basis-at-decseq*[*of l*]
obtain A **where** A:
$\bigwedge i.\ open\ (A\ i)$
$\bigwedge i.\ l \in A\ i$
$\bigwedge S.\ open\ S \Longrightarrow l \in S \Longrightarrow eventually\ (\lambda i.\ A\ i \subseteq S)\ sequentially$
by *blast*
def $s \equiv \lambda n\ i.\ SOME\ j.\ i < j \wedge f\ j \in A\ (Suc\ n)$
{
fix $n\ i$
have $infinite\ (A\ (Suc\ n) \cap range\ f - f`\{..\ i\})$
using $l\ A$ **by** *auto*
then have $\exists\, x.\ x \in A\ (Suc\ n) \cap range\ f - f`\{..\ i\}$
unfolding *ex-in-conv* **by** (*intro notI*) *simp*
then have $\exists\, j.\ f\ j \in A\ (Suc\ n) \wedge j \notin \{..\ i\}$
by *auto*
then have $\exists\, a.\ i < a \wedge f\ a \in A\ (Suc\ n)$
by (*auto simp*: *not-le*)
then have $i < s\ n\ i\ f\ (s\ n\ i) \in A\ (Suc\ n)$
unfolding *s-def* **by** (*auto intro*: *someI2-ex*)
}
note $s = this$
def $r \equiv rec\text{-}nat\ (s\ 0\ 0)\ s$
have $subseq\ r$
by (*auto simp*: *r-def s subseq-Suc-iff*)
moreover
have $(\lambda n.\ f\ (r\ n)) \longrightarrow l$
proof (*rule topological-tendstoI*)
fix S
assume $open\ S\ l \in S$
with $A(3)$ **have** $eventually\ (\lambda i.\ A\ i \subseteq S)\ sequentially$
by *auto*
moreover
{
fix i
assume $Suc\ 0 \leq i$
then have $f\ (r\ i) \in A\ i$
by (*cases i*) (*simp-all add*: *r-def s*)
}
then have $eventually\ (\lambda i.\ f\ (r\ i) \in A\ i)\ sequentially$

```
    by (auto simp: eventually-sequentially)
  ultimately show eventually (λi. f (r i) ∈ S) sequentially
    by eventually-elim auto
 qed
 ultimately show ∃r. subseq r ∧ (f ∘ r) ⟶ l
  by (auto simp: convergent-def comp-def)
qed

lemma sequence-infinite-lemma:
 fixes f :: nat ⇒ 'a::t1-space
 assumes ∀n. f n ≠ l
  and (f ⟶ l) sequentially
 shows infinite (range f)
proof
 assume finite (range f)
 then have closed (range f)
  by (rule finite-imp-closed)
 then have open (− range f)
  by (rule open-Compl)
 from assms(1) have l ∈ − range f
  by auto
 from assms(2) have eventually (λn. f n ∈ − range f) sequentially
  using ‹open (− range f)› ‹l ∈ − range f›
  by (rule topological-tendstoD)
 then show False
  unfolding eventually-sequentially
  by auto
qed

lemma closure-insert:
 fixes x :: 'a::t1-space
 shows closure (insert x s) = insert x (closure s)
 apply (rule closure-unique)
 apply (rule insert-mono [OF closure-subset])
 apply (rule closed-insert [OF closed-closure])
 apply (simp add: closure-minimal)
 done

lemma islimpt-insert:
 fixes x :: 'a::t1-space
 shows x islimpt (insert a s) ⟷ x islimpt s
proof
 assume ∗: x islimpt (insert a s)
 show x islimpt s
 proof (rule islimptI)
  fix t
  assume t: x ∈ t open t
  show ∃y∈s. y ∈ t ∧ y ≠ x
  proof (cases x = a)
```

```
      case True
      obtain y where y ∈ insert a s y ∈ t y ≠ x
        using ∗ t by (rule islimptE)
      with ‹x = a› show ?thesis by auto
    next
      case False
      with t have t′: x ∈ t − {a} open (t − {a})
        by (simp-all add: open-Diff)
      obtain y where y ∈ insert a s y ∈ t − {a} y ≠ x
        using ∗ t′ by (rule islimptE)
      then show ?thesis by auto
    qed
  qed
next
  assume x islimpt s
  then show x islimpt (insert a s)
    by (rule islimpt-subset) auto
qed

lemma islimpt-finite:
  fixes x :: 'a::t1-space
  shows finite s ⟹ ¬ x islimpt s
  by (induct set: finite) (simp-all add: islimpt-insert)

lemma islimpt-Un-finite:
  fixes x :: 'a::t1-space
  shows finite s ⟹ x islimpt (s ∪ t) ⟷ x islimpt t
  by (simp add: islimpt-Un islimpt-finite)

lemma islimpt-eq-acc-point:
  fixes l :: 'a :: t1-space
  shows l islimpt S ⟷ (∀ U. l∈U ⟶ open U ⟶ infinite (U ∩ S))
proof (safe intro!: islimptI)
  fix U
  assume l islimpt S l ∈ U open U finite (U ∩ S)
  then have l islimpt S l ∈ (U − (U ∩ S − {l})) open (U − (U ∩ S − {l}))
    by (auto intro: finite-imp-closed)
  then show False
    by (rule islimptE) auto
next
  fix T
  assume ∗: ∀ U. l∈U ⟶ open U ⟶ infinite (U ∩ S) l ∈ T open T
  then have infinite (T ∩ S − {l})
    by auto
  then have ∃ x. x ∈ (T ∩ S − {l})
    unfolding ex-in-conv by (intro notI) simp
  then show ∃ y∈S. y ∈ T ∧ y ≠ l
    by auto
qed
```

lemma *islimpt-range-imp-convergent-subsequence*:
fixes $l :: 'a :: \{t1\text{-}space, first\text{-}countable\text{-}topology\}$
assumes *l*: l *islimpt* (*range* f)
shows $\exists r.$ *subseq* $r \wedge (f \circ r) \longrightarrow l$
using l **unfolding** *islimpt-eq-acc-point*
by (*rule acc-point-range-imp-convergent-subsequence*)

lemma *sequence-unique-limpt*:
fixes $f :: nat \Rightarrow 'a::t2\text{-}space$
assumes $(f \longrightarrow l)$ *sequentially*
and l' *islimpt* (*range* f)
shows $l' = l$
proof (*rule ccontr*)
assume $l' \neq l$
obtain $s\ t$ **where** *open* s *open* t $l' \in s$ $l \in t$ $s \cap t = \{\}$
using *hausdorff* [*OF* ‹$l' \neq l$›] **by** *auto*
have *eventually* $(\lambda n.\ f\ n \in t)$ *sequentially*
using *assms*(*1*) ‹*open t*› ‹$l \in t$› **by** (*rule topological-tendstoD*)
then obtain N **where** $\forall n{\geq}N.\ f\ n \in t$
unfolding *eventually-sequentially* **by** *auto*

have $UNIV = \{..{<}N\} \cup \{N..\}$
by *auto*
then have l' *islimpt* $(f\ `\ (\{..{<}N\} \cup \{N..\}))$
using *assms*(*2*) **by** *simp*
then have l' *islimpt* $(f\ `\ \{..{<}N\} \cup f\ `\ \{N..\})$
by (*simp add*: *image-Un*)
then have l' *islimpt* $(f\ `\ \{N..\})$
by (*simp add*: *islimpt-Un-finite*)
then obtain y **where** $y \in f\ `\ \{N..\}$ $y \in s$ $y \neq l'$
using ‹$l' \in s$› ‹*open s*› **by** (*rule islimptE*)
then obtain n **where** $N \leq n$ $f\ n \in s$ $f\ n \neq l'$
by *auto*
with ‹$\forall n{\geq}N.\ f\ n \in t$› **have** $f\ n \in s \cap t$
by *simp*
with ‹$s \cap t = \{\}$› **show** *False*
by *simp*
qed

lemma *bolzano-weierstrass-imp-closed*:
fixes $s :: 'a::\{first\text{-}countable\text{-}topology,t2\text{-}space\}$ *set*
assumes $\forall t.$ *infinite* $t \wedge t \subseteq s \longrightarrow (\exists x \in s.\ x$ *islimpt* $t)$
shows *closed* s
proof −
{
fix $x\ l$
assume *as*: $\forall n::nat.\ x\ n \in s$ $(x \longrightarrow l)$ *sequentially*
then have $l \in s$

```
    proof (cases ∀ n. x n ≠ l)
      case False
      then show l∈s using as(1) by auto
    next
      case True note cas = this
      with as(2) have infinite (range x)
        using sequence-infinite-lemma[of x l] by auto
      then obtain l' where l'∈s l' islimpt (range x)
        using assms[THEN spec[where x=range x]] as(1) by auto
      then show l∈s using sequence-unique-limpt[of x l l']
        using as cas by auto
    qed
  }
  then show ?thesis
    unfolding closed-sequential-limits by fast
qed

lemma compact-imp-bounded:
  assumes compact U
  shows bounded U
proof −
  have compact U ∀ x∈U. open (ball x 1) U ⊆ (⋃ x∈U. ball x 1)
    using assms by auto
  then obtain D where D: D ⊆ U finite D U ⊆ (⋃ x∈D. ball x 1)
    by (rule compactE-image)
  from ‹finite D› have bounded (⋃ x∈D. ball x 1)
    by (simp add: bounded-UN)
  then show bounded U using ‹U ⊆ (⋃ x∈D. ball x 1)›
    by (rule bounded-subset)
qed
```

In particular, some common special cases.

```
lemma compact-Un [intro]:
  assumes compact s
    and compact t
  shows  compact (s ∪ t)
proof (rule compactI)
  fix f
  assume ∗: Ball f open s ∪ t ⊆ ⋃ f
  from ∗ ‹compact s› obtain s' where s' ⊆ f ∧ finite s' ∧ s ⊆ ⋃ s'
    unfolding compact-eq-heine-borel by (auto elim!: allE[of - f])
  moreover
  from ∗ ‹compact t› obtain t' where t' ⊆ f ∧ finite t' ∧ t ⊆ ⋃ t'
    unfolding compact-eq-heine-borel by (auto elim!: allE[of - f])
  ultimately show ∃ f'⊆f. finite f' ∧ s ∪ t ⊆ ⋃ f'
    by (auto intro!: exI[of - s' ∪ t'])
qed

lemma compact-Union [intro]: finite S ⟹ (⋀ T. T ∈ S ⟹ compact T) ⟹
```

compact ($\bigcup S$)
by (*induct set*: *finite*) *auto*

lemma *compact-UN* [*intro*]:
finite A $\Longrightarrow$ ($\bigwedge x.\ x \in A \Longrightarrow$ *compact* (*B x*)) $\Longrightarrow$ *compact* ($\bigcup x{\in}A.\ B\ x$)
by (*rule compact-Union*) *auto*

lemma *closed-Int-compact* [*intro*]:
assumes *closed s*
and *compact t*
shows *compact* ($s \cap t$)
using *compact-Int-closed* [*of t s*] *assms*
by (*simp add*: *Int-commute*)

lemma *compact-Int* [*intro*]:
fixes *s t* :: *'a* :: *t2-space set*
assumes *compact s*
and *compact t*
shows *compact* ($s \cap t$)
using *assms* **by** (*intro compact-Int-closed compact-imp-closed*)

lemma *compact-sing* [*simp*]: *compact* {*a*}
unfolding *compact-eq-heine-borel* **by** *auto*

lemma *compact-insert* [*simp*]:
assumes *compact s*
shows *compact* (*insert x s*)
proof −
have *compact* ({*x*} ∪ *s*)
using *compact-sing assms* **by** (*rule compact-Un*)
then show *?thesis* **by** *simp*
qed

lemma *finite-imp-compact*: *finite s* $\Longrightarrow$ *compact s*
by (*induct set*: *finite*) *simp-all*

lemma *open-delete*:
fixes *s* :: *'a*::*t1-space set*
shows *open s* $\Longrightarrow$ *open* ($s - \{x\}$)
by (*simp add*: *open-Diff*)

lemma *openin-delete*:
fixes *a* :: *'a* :: *t1-space*
shows *openin* (*subtopology euclidean u*) *s*
$\Longrightarrow$ *openin* (*subtopology euclidean u*) ($s - \{a\}$)
by (*metis Int-Diff open-delete openin-open*)

Compactness expressed with filters

lemma *closure-iff-nhds-not-empty*:

```
  x ∈ closure X ⟷ (∀ A. ∀ S⊆A. open S ⟶ x ∈ S ⟶ X ∩ A ≠ {})
proof safe
  assume x: x ∈ closure X
  fix S A
  assume open S x ∈ S X ∩ A = {} S ⊆ A
  then have x ∉ closure (−S)
    by (auto simp: closure-complement subset-eq[symmetric] intro: interiorI)
  with x have x ∈ closure X − closure (−S)
    by auto
  also have ... ⊆ closure (X ∩ S)
    using ‹open S› open-inter-closure-subset[of S X] by (simp add: closed-Compl
ac-simps)
  finally have X ∩ S ≠ {} by auto
  then show False using ‹X ∩ A = {}› ‹S ⊆ A› by auto
next
  assume ∀ A S. S ⊆ A ⟶ open S ⟶ x ∈ S ⟶ X ∩ A ≠ {}
  from this[THEN spec, of − X, THEN spec, of − closure X]
  show x ∈ closure X
    by (simp add: closure-subset open-Compl)
qed

lemma compact-filter:
  compact U ⟷ (∀ F. F ≠ bot ⟶ eventually (λx. x ∈ U) F ⟶ (∃ x∈U. inf
(nhds x) F ≠ bot))
proof (intro allI iffI impI compact-fip[THEN iffD2] notI)
  fix F
  assume compact U
  assume F: F ≠ bot eventually (λx. x ∈ U) F
  then have U ≠ {}
    by (auto simp: eventually-False)

  def Z ≡ closure ‘ {A. eventually (λx. x ∈ A) F}
  then have ∀ z∈Z. closed z
    by auto
  moreover
  have ev-Z: ⋀z. z ∈ Z ⟹ eventually (λx. x ∈ z) F
   unfolding Z-def by (auto elim: eventually-mono intro: set-mp[OF closure-subset])
  have (∀ B ⊆ Z. finite B ⟶ U ∩ ⋂B ≠ {})
  proof (intro allI impI)
    fix B assume finite B B ⊆ Z
    with ‹finite B› ev-Z F(2) have eventually (λx. x ∈ U ∩ (⋂B)) F
      by (auto simp: eventually-ball-finite-distrib eventually-conj-iff)
    with F show U ∩ ⋂B ≠ {}
      by (intro notI) (simp add: eventually-False)
  qed
  ultimately have U ∩ ⋂Z ≠ {}
    using ‹compact U› unfolding compact-fip by blast
  then obtain x where x ∈ U and x: ⋀z. z ∈ Z ⟹ x ∈ z
    by auto
```

have $\bigwedge P$. *eventually* P (*inf* (*nhds* x) F) $\Longrightarrow$ $P \neq$ *bot*
unfolding *eventually-inf eventually-nhds*
proof *safe*
fix P Q R S
assume *eventually* R F *open* S $x \in S$
with *open-Int-closure-eq-empty*[*of* S $\{x.\ R\ x\}$] x[*of closure* $\{x.\ R\ x\}$]
have $S \cap \{x.\ R\ x\} \neq \{\}$ **by** (*auto simp*: *Z-def*)
moreover assume *Ball* S Q $\forall x.\ Q\ x \wedge R\ x \longrightarrow$ *bot* x
ultimately show *False* **by** (*auto simp*: *set-eq-iff*)
qed
with ‹$x \in U$› **show** $\exists x{\in}U.$ *inf* (*nhds* x) $F \neq$ *bot*
by (*metis eventually-bot*)
next
fix A
assume A: $\forall a{\in}A.$ *closed* a $\forall B{\subseteq}A.$ *finite* $B \longrightarrow U \cap \bigcap B \neq \{\}$ $U \cap \bigcap A = \{\}$
def $F \equiv$ *INF* a:*insert* U A. *principal* a
have $F \neq$ *bot*
unfolding *F-def*
proof (*rule INF-filter-not-bot*)
fix X **assume** $X \subseteq$ *insert* U A *finite* X
moreover with *A*(*2*)[*THEN spec*, *of* $X - \{U\}$] **have** $U \cap \bigcap (X - \{U\}) \neq \{\}$
by *auto*
ultimately show (*INF* a:X. *principal* a) $\neq$ *bot*
by (*auto simp add*: *INF-principal-finite principal-eq-bot-iff*)
qed
moreover
have $F \leq$ *principal* U
unfolding *F-def* **by** *auto*
then have *eventually* ($\lambda x.\ x \in U$) F
by (*auto simp*: *le-filter-def eventually-principal*)
moreover
assume $\forall F.\ F \neq$ *bot* $\longrightarrow$ *eventually* ($\lambda x.\ x \in U$) $F \longrightarrow (\exists x{\in}U.$ *inf* (*nhds* x) $F \neq$ *bot*)
ultimately obtain x **where** $x \in U$ **and** x: *inf* (*nhds* x) $F \neq$ *bot*
by *auto*

{ fix V **assume** $V \in A$
then have $F \leq$ *principal* V
unfolding *F-def* **by** (*intro INF-lower2*[*of* V]) *auto*
then have V: *eventually* ($\lambda x.\ x \in V$) F
by (*auto simp*: *le-filter-def eventually-principal*)
have $x \in$ *closure* V
unfolding *closure-iff-nhds-not-empty*
proof (*intro impI allI*)
fix S A
assume *open* S $x \in S$ $S \subseteq A$
then have *eventually* ($\lambda x.\ x \in A$) (*nhds* x)

by (*auto simp*: *eventually-nhds*)
with *V* **have** *eventually* (λ*x*. *x* ∈ *V* ∩ *A*) (*inf* (*nhds x*) *F*)
by (*auto simp*: *eventually-inf*)
with *x* **show** *V* ∩ *A* ≠ {}
by (*auto simp del*: *Int-iff simp add*: *trivial-limit-def*)
qed
then have *x* ∈ *V*
using ‹*V* ∈ *A*› *A*(*1*) **by** *simp*
}
with ‹*x*∈*U*› **have** *x* ∈ *U* ∩ ⋂ *A* **by** *auto*
with ‹*U* ∩ ⋂ *A* = {}› **show** *False* **by** *auto*
qed

definition *countably-compact U* ⟷
(∀ *A*. *countable A* ⟶ (∀ *a*∈*A*. *open a*) ⟶ *U* ⊆ ⋃ *A* ⟶ (∃ *T*⊆*A*. *finite T* ∧ *U* ⊆ ⋃ *T*))

lemma *countably-compactE*:
assumes *countably-compact s* **and** ∀ *t*∈*C*. *open t* **and** *s* ⊆ ⋃ *C countable C*
obtains *C'* **where** *C'* ⊆ *C* **and** *finite C'* **and** *s* ⊆ ⋃ *C'*
using *assms* **unfolding** *countably-compact-def* **by** *metis*

lemma *countably-compactI*:
assumes ⋀*C*. ∀ *t*∈*C*. *open t* ⟹ *s* ⊆ ⋃ *C* ⟹ *countable C* ⟹ (∃ *C'*⊆*C*. *finite C'* ∧ *s* ⊆ ⋃ *C'*)
shows *countably-compact s*
using *assms* **unfolding** *countably-compact-def* **by** *metis*

lemma *compact-imp-countably-compact*: *compact U* ⟹ *countably-compact U*
by (*auto simp*: *compact-eq-heine-borel countably-compact-def*)

lemma *countably-compact-imp-compact*:
assumes *countably-compact U*
and *ccover*: *countable B* ∀ *b*∈*B*. *open b*
and *basis*: ⋀*T x*. *open T* ⟹ *x* ∈ *T* ⟹ *x* ∈ *U* ⟹ ∃ *b*∈*B*. *x* ∈ *b* ∧ *b* ∩ *U* ⊆ *T*
shows *compact U*
using ‹*countably-compact U*›
unfolding *compact-eq-heine-borel countably-compact-def*
proof *safe*
fix *A*
assume *A*: ∀ *a*∈*A*. *open a U* ⊆ ⋃ *A*
assume ∗: ∀ *A*. *countable A* ⟶ (∀ *a*∈*A*. *open a*) ⟶ *U* ⊆ ⋃ *A* ⟶ (∃ *T*⊆*A*. *finite T* ∧ *U* ⊆ ⋃ *T*)

moreover def *C* ≡ {*b*∈*B*. ∃ *a*∈*A*. *b* ∩ *U* ⊆ *a*}
ultimately have *countable C* ∀ *a*∈*C*. *open a*
unfolding *C-def* **using** *ccover* **by** *auto*
moreover

have $\bigcup A \cap U \subseteq \bigcup C$
proof *safe*
fix *x a*
assume $x \in U$ $x \in a$ $a \in A$
with *basis*[*of a x*] A **obtain** b **where** $b \in B$ $x \in b$ $b \cap U \subseteq a$
by *blast*
with ‹$a \in A$› **show** $x \in \bigcup C$
unfolding *C-def* **by** *auto*
qed
then have $U \subseteq \bigcup C$ **using** ‹$U \subseteq \bigcup A$› **by** *auto*
ultimately obtain T **where** T: $T \subseteq C$ *finite* T $U \subseteq \bigcup T$
using $*$ **by** *metis*
then have $\forall t \in T.\ \exists a \in A.\ t \cap U \subseteq a$
by (*auto simp*: *C-def*)
then obtain f **where** $\forall t \in T.\ f\ t \in A \wedge t \cap U \subseteq f\ t$
unfolding *bchoice-iff Bex-def* **..**
with T **show** $\exists T \subseteq A.$ *finite* $T \wedge U \subseteq \bigcup T$
unfolding *C-def* **by** (*intro exI*[*of - f*‘T]) *fastforce*
qed

lemma *countably-compact-imp-compact-second-countable*:
countably-compact $U \Longrightarrow$ *compact* (U :: *'a* :: *second-countable-topology set*)
proof (*rule countably-compact-imp-compact*)
fix T **and** x :: *'a*
assume *open* T $x \in T$
from *topological-basisE*[*OF is-basis this*] **obtain** b **where**
$b \in$ (*SOME B. countable B* $\wedge$ *topological-basis B*) $x \in b$ $b \subseteq T$ **.**
then show $\exists b \in$*SOME B. countable B* $\wedge$ *topological-basis B*. $x \in b \wedge b \cap U \subseteq T$
by *blast*
qed (*insert countable-basis topological-basis-open*[*OF is-basis*], *auto*)

lemma *countably-compact-eq-compact*:
countably-compact $U \longleftrightarrow$ *compact* (U :: *'a* :: *second-countable-topology set*)
using *countably-compact-imp-compact-second-countable compact-imp-countably-compact*
by *blast*

16.19.2 Sequential compactness

definition *seq-compact* :: *'a*::*topological-space set* ⇒ *bool*
where *seq-compact* $S \longleftrightarrow$
$(\forall f.\ (\forall n.\ f\ n \in S) \longrightarrow (\exists l \in S.\ \exists r.$ *subseq* $r \wedge ((f \circ r) \longrightarrow l)$ *sequentially*))

lemma *seq-compactI*:
assumes $\bigwedge f.\ \forall n.\ f\ n \in S \Longrightarrow \exists l \in S.\ \exists r.$ *subseq* $r \wedge ((f \circ r) \longrightarrow l)$ *sequentially*
shows *seq-compact* S
unfolding *seq-compact-def* **using** *assms* **by** *fast*

lemma *seq-compactE*:

assumes *seq-compact* S $\forall n.\ f\ n \in S$
obtains $l\ r$ **where** $l \in S$ *subseq* r $((f \circ r) \longrightarrow l)$ *sequentially*
using *assms* **unfolding** *seq-compact-def* **by** *fast*

lemma *closed-sequentially*:
assumes *closed* s **and** $\forall n.\ f\ n \in s$ **and** $f \longrightarrow l$
shows $l \in s$
proof (*rule ccontr*)
assume $l \notin s$
with ‹*closed* s› **and** ‹$f \longrightarrow l$› **have** *eventually* $(\lambda n.\ f\ n \in - s)$ *sequentially*
by (*fast intro*: *topological-tendstoD*)
with ‹$\forall n.\ f\ n \in s$› **show** *False*
by *simp*
qed

lemma *seq-compact-Int-closed*:
assumes *seq-compact* s **and** *closed* t
shows *seq-compact* $(s \cap t)$
proof (*rule seq-compactI*)
fix f **assume** $\forall n{::}nat.\ f\ n \in s \cap t$
hence $\forall n.\ f\ n \in s$ **and** $\forall n.\ f\ n \in t$
by *simp-all*
from ‹*seq-compact* s› **and** ‹$\forall n.\ f\ n \in s$›
obtain $l\ r$ **where** $l \in s$ **and** r: *subseq* r **and** l: $(f \circ r) \longrightarrow l$
by (*rule seq-compactE*)
from ‹$\forall n.\ f\ n \in t$› **have** $\forall n.\ (f \circ r)\ n \in t$
by *simp*
from ‹*closed* t› **and** *this* **and** l **have** $l \in t$
by (*rule closed-sequentially*)
with ‹$l \in s$› **and** r **and** l **show** $\exists l{\in}s \cap t.\ \exists r.$ *subseq* $r \wedge (f \circ r) \longrightarrow l$
by *fast*
qed

lemma *seq-compact-closed-subset*:
assumes *closed* s **and** $s \subseteq t$ **and** *seq-compact* t
shows *seq-compact* s
using *assms seq-compact-Int-closed* [*of* t s] **by** (*simp add*: *Int-absorb1*)

lemma *seq-compact-imp-countably-compact*:
fixes U :: *'a* :: *first-countable-topology set*
assumes *seq-compact* U
shows *countably-compact* U
proof (*safe intro*!: *countably-compactI*)
fix A
assume A: $\forall a{\in}A.$ *open* a $U \subseteq \bigcup A$ *countable* A
have *subseq*: $\bigwedge X.$ *range* $X \subseteq U \Longrightarrow \exists r\ x.\ x \in U \wedge$ *subseq* $r \wedge (X \circ r) \longrightarrow x$
using ‹*seq-compact* U› **by** (*fastforce simp*: *seq-compact-def subset-eq*)
show $\exists T{\subseteq}A.$ *finite* $T \wedge U \subseteq \bigcup T$

proof *cases*
assume *finite A*
with *A* **show** *?thesis* **by** *auto*
next
assume *infinite A*
then have $A \neq \{\}$ **by** *auto*
show *?thesis*
proof (*rule ccontr*)
assume $\neg$ ($\exists$ $T \subseteq A$. *finite* $T \wedge U \subseteq \bigcup T$)
then have $\forall T.\ \exists x.\ T \subseteq A \wedge$ *finite* $T \longrightarrow (x \in U - \bigcup T)$
by *auto*
then obtain X' **where** T: $\bigwedge T.\ T \subseteq A \Longrightarrow$ *finite* $T \Longrightarrow X'\ T \in U - \bigcup T$
by *metis*
def $X \equiv \lambda n.\ X'$ (*from-nat-into A* ‘ {.. *n*})
have X: $\bigwedge n.\ X\ n \in U - (\bigcup i \leq n.$ *from-nat-into A i*)
using ‹$A \neq \{\}$› **unfolding** *X-def* **by** (*intro T*) (*auto intro*: *from-nat-into*)
then have *range* $X \subseteq U$
by *auto*
with *subseq*[*of X*] **obtain** *r x* **where** $x \in U$ **and** *r*: *subseq r* ($X \circ r$) $\longrightarrow$ *x*
by *auto*
from ‹$x \in U$› ‹$U \subseteq \bigcup A$› *from-nat-into-surj*[*OF* ‹*countable A*›]
obtain *n* **where** $x \in$ *from-nat-into A n* **by** *auto*
with *r*(*2*) *A*(*1*) *from-nat-into*[*OF* ‹$A \neq \{\}$›, *of n*]
have *eventually* ($\lambda i.\ X\ (r\ i) \in$ *from-nat-into A n*) *sequentially*
unfolding *tendsto-def* **by** (*auto simp*: *comp-def*)
then obtain *N* **where** $\bigwedge i.\ N \leq i \Longrightarrow X\ (r\ i) \in$ *from-nat-into A n*
by (*auto simp*: *eventually-sequentially*)
moreover from *X* **have** $\bigwedge i.\ n \leq r\ i \Longrightarrow X\ (r\ i) \notin$ *from-nat-into A n*
by *auto*
moreover from ‹*subseq r*›[*THEN seq-suble*, *of max n N*] **have** $\exists i.\ n \leq r\ i \wedge N \leq i$
by (*auto intro*!: *exI*[*of - max n N*])
ultimately show *False*
by *auto*
qed
qed
qed

lemma *compact-imp-seq-compact*:
fixes *U* :: *'a* :: *first-countable-topology set*
assumes *compact U*
shows *seq-compact U*
unfolding *seq-compact-def*
proof *safe*
fix *X* :: *nat* $\Rightarrow$ *'a*
assume $\forall n.\ X\ n \in U$
then have *eventually* ($\lambda x.\ x \in U$) (*filtermap X sequentially*)
by (*auto simp*: *eventually-filtermap*)

moreover
have *filtermap X sequentially* $\neq$ *bot*
by (*simp add*: *trivial-limit-def eventually-filtermap*)
ultimately
obtain *x* **where** $x \in U$ **and** *x*: *inf* (*nhds x*) (*filtermap X sequentially*) $\neq$ *bot* (**is** *?F* $\neq$ -)
using ‹*compact U*› **by** (*auto simp*: *compact-filter*)

from *countable-basis-at-decseq*[*of x*]
obtain *A* **where** *A*:
$\bigwedge i.$ *open* (*A i*)
$\bigwedge i.$ $x \in A\ i$
$\bigwedge S.$ *open S* $\Longrightarrow$ $x \in S$ $\Longrightarrow$ *eventually* ($\lambda i.$ $A\ i \subseteq S$) *sequentially*
by *blast*
def $s \equiv \lambda n\ i.$ *SOME j*. $i < j \wedge X\ j \in A$ (*Suc n*)
{
fix *n i*
have $\exists\, a.$ $i < a \wedge X\ a \in A$ (*Suc n*)
proof (*rule ccontr*)
assume $\neg$ ($\exists\, a{>}i.$ $X\ a \in A$ (*Suc n*))
then have $\bigwedge a.$ *Suc i* $\leq a \Longrightarrow X\ a \notin A$ (*Suc n*)
by *auto*
then have *eventually* ($\lambda x.$ $x \notin A$ (*Suc n*)) (*filtermap X sequentially*)
by (*auto simp*: *eventually-filtermap eventually-sequentially*)
moreover have *eventually* ($\lambda x.$ $x \in A$ (*Suc n*)) (*nhds x*)
using *A*(*1*,*2*)[*of Suc n*] **by** (*auto simp*: *eventually-nhds*)
ultimately have *eventually* ($\lambda x.$ *False*) *?F*
by (*auto simp add*: *eventually-inf*)
with *x* **show** *False*
by (*simp add*: *eventually-False*)
qed
then have $i < s\ n\ i\ X\ (s\ n\ i) \in A$ (*Suc n*)
unfolding *s-def* **by** (*auto intro*: *someI2-ex*)
}
note $s =$ *this*
def $r \equiv$ *rec-nat* (*s 0 0*) *s*
have *subseq r*
by (*auto simp*: *r-def s subseq-Suc-iff*)
moreover
have ($\lambda n.$ $X\ (r\ n)$) $\longrightarrow$ *x*
proof (*rule topological-tendstoI*)
fix *S*
assume *open S x* $\in$ *S*
with *A*(*3*) **have** *eventually* ($\lambda i.$ $A\ i \subseteq S$) *sequentially*
by *auto*
moreover
{
fix *i*
assume *Suc 0* $\leq$ *i*

```
      then have X (r i) ∈ A i
        by (cases i) (simp-all add: r-def s)
    }
    then have eventually (λi. X (r i) ∈ A i) sequentially
      by (auto simp: eventually-sequentially)
    ultimately show eventually (λi. X (r i) ∈ S) sequentially
      by eventually-elim auto
  qed
  ultimately show ∃x ∈ U. ∃r. subseq r ∧ (X ∘ r) ⟶ x
    using ‹x ∈ U› by (auto simp: convergent-def comp-def)
qed

lemma countably-compact-imp-acc-point:
  assumes countably-compact s
    and countable t
    and infinite t
    and t ⊆ s
  shows ∃x∈s. ∀ U. x∈U ∧ open U ⟶ infinite (U ∩ t)
proof (rule ccontr)
  def C ≡ (λF. interior (F ∪ (− t))) ‘ {F. finite F ∧ F ⊆ t }
  note ‹countably-compact s›
  moreover have ∀ t∈C. open t
    by (auto simp: C-def)
  moreover
  assume ¬ (∃ x∈s. ∀ U. x∈U ∧ open U ⟶ infinite (U ∩ t))
  then have s: ⋀x. x ∈ s ⟹ ∃ U. x∈U ∧ open U ∧ finite (U ∩ t) by metis
  have s ⊆ ⋃ C
    using ‹t ⊆ s›
    unfolding C-def
    apply (safe dest!: s)
    apply (rule-tac a=U ∩ t in UN-I)
    apply (auto intro!: interiorI simp add: finite-subset)
    done
  moreover
  from ‹countable t› have countable C
    unfolding C-def by (auto intro: countable-Collect-finite-subset)
  ultimately
  obtain D where D ⊆ C finite D s ⊆ ⋃ D
    by (rule countably-compactE)
  then obtain E where E: E ⊆ {F. finite F ∧ F ⊆ t } finite E
    and s: s ⊆ (⋃F∈E. interior (F ∪ (− t)))
    by (metis (lifting) finite-subset-image C-def)
  from s ‹t ⊆ s› have t ⊆ ⋃ E
    using interior-subset by blast
  moreover have finite (⋃ E)
    using E by auto
  ultimately show False using ‹infinite t›
    by (auto simp: finite-subset)
qed
```

lemma *countable-acc-point-imp-seq-compact*:
fixes $s :: 'a$::*first-countable-topology set*
assumes $\forall\, t.$ *infinite* $t \wedge$ *countable* $t \wedge t \subseteq s \longrightarrow$
$(\exists\, x{\in}s.\ \forall\, U.\ x{\in}U \wedge$ *open* $U \longrightarrow$ *infinite* $(U \cap t))$
shows *seq-compact* s
proof −
{
fix $f ::$ *nat* $\Rightarrow 'a$
assume f: $\forall\, n.\ f\ n \in s$
have $\exists\, l{\in}s.\ \exists\, r.$ *subseq* $r \wedge ((f \circ r) \longrightarrow l)$ *sequentially*
proof (*cases finite* (*range* f))
case *True*
obtain l **where** *infinite* $\{n.\ f\ n = f\ l\}$
using *pigeonhole-infinite*[*OF - True*] **by** *auto*
then obtain r **where** *subseq* r **and** *fr*: $\forall\, n.\ f\ (r\ n) = f\ l$
using *infinite-enumerate* **by** *blast*
then have *subseq* $r \wedge (f \circ r) \longrightarrow f\ l$
by (*simp add*: *fr o-def*)
with f **show** $\exists\, l{\in}s.\ \exists\, r.$ *subseq* $r \wedge (f \circ r) \longrightarrow l$
by *auto*
next
case *False*
with f *assms* **have** $\exists\, x{\in}s.\ \forall\, U.\ x{\in}U \wedge$ *open* $U \longrightarrow$ *infinite* $(U \cap$ *range* $f)$
by *auto*
then obtain l **where** $l \in s\ \forall\, U.\ l{\in}U \wedge$ *open* $U \longrightarrow$ *infinite* $(U \cap$ *range* $f)$
..
from *this*(*2*) **have** $\exists\, r.$ *subseq* $r \wedge ((f \circ r) \longrightarrow l)$ *sequentially*
using *acc-point-range-imp-convergent-subsequence*[*of l f*] **by** *auto*
with ‹$l \in s$› **show** $\exists\, l{\in}s.\ \exists\, r.$ *subseq* $r \wedge ((f \circ r) \longrightarrow l)$ *sequentially* **..**
qed
}
then show *?thesis*
unfolding *seq-compact-def* **by** *auto*
qed

lemma *seq-compact-eq-countably-compact*:
fixes $U :: 'a ::$ *first-countable-topology set*
shows *seq-compact* $U \longleftrightarrow$ *countably-compact* U
using
countable-acc-point-imp-seq-compact
countably-compact-imp-acc-point
seq-compact-imp-countably-compact
by *metis*

lemma *seq-compact-eq-acc-point*:
fixes $s :: 'a ::$ *first-countable-topology set*
shows *seq-compact* $s \longleftrightarrow$
$(\forall\, t.$ *infinite* $t \wedge$ *countable* $t \wedge t \subseteq s$ −−> $(\exists\, x{\in}s.\ \forall\, U.\ x{\in}U \wedge$ *open* $U \longrightarrow$

infinite $(U \cap t)$))
using
countable-acc-point-imp-seq-compact[*of s*]
countably-compact-imp-acc-point[*of s*]
seq-compact-imp-countably-compact[*of s*]
by *metis*

lemma *seq-compact-eq-compact*:
fixes $U :: 'a :: second\text{-}countable\text{-}topology\ set$
shows *seq-compact U* $\longleftrightarrow$ *compact U*
using *seq-compact-eq-countably-compact countably-compact-eq-compact* **by** *blast*

lemma *bolzano-weierstrass-imp-seq-compact*:
fixes $s :: 'a::\{t1\text{-}space, first\text{-}countable\text{-}topology\}\ set$
shows $\forall t.\ infinite\ t \wedge t \subseteq s \longrightarrow (\exists x \in s.\ x\ islimpt\ t) \Longrightarrow seq\text{-}compact\ s$
by (*rule countable-acc-point-imp-seq-compact*) (*metis islimpt-eq-acc-point*)

16.19.3 Totally bounded

lemma *cauchy-def*: *Cauchy s* $\longleftrightarrow$ $(\forall e{>}0.\ \exists N.\ \forall m\ n.\ m \geq N \wedge n \geq N \longrightarrow dist(s\ m)(s\ n) < e)$
unfolding *Cauchy-def* **by** *metis*

lemma *seq-compact-imp-totally-bounded*:
assumes *seq-compact s*
shows $\forall e{>}0.\ \exists k.\ finite\ k \wedge k \subseteq s \wedge s \subseteq (\bigcup x{\in}k.\ ball\ x\ e)$
proof −
{ **fix** $e::real$ **assume** $e > 0$ **assume** $*$: $\bigwedge k.\ finite\ k \Longrightarrow k \subseteq s \Longrightarrow \neg\ s \subseteq (\bigcup x{\in}k.\ ball\ x\ e)$
let $?Q = \lambda x\ n\ r.\ r \in s \wedge (\forall m < (n::nat).\ \neg\ (dist\ (x\ m)\ r < e))$
have $\exists x.\ \forall n::nat.\ ?Q\ x\ n\ (x\ n)$
proof (*rule dependent-wellorder-choice*)
fix $n\ x$ **assume** $\bigwedge y.\ y < n \Longrightarrow ?Q\ x\ y\ (x\ y)$
then have $\neg\ s \subseteq (\bigcup x{\in}x\ `\ \{0..{<}n\}.\ ball\ x\ e)$
using $*$[*of x* ` $\{0\ ..{<}\ n\}$] **by** (*auto simp*: *subset-eq*)
then obtain z **where** $z{:}z{\in}s\ z \notin (\bigcup x{\in}x\ `\ \{0..{<}n\}.\ ball\ x\ e)$
unfolding *subset-eq* **by** *auto*
show $\exists r.\ ?Q\ x\ n\ r$
using z **by** *auto*
qed *simp*
then obtain x **where** $\forall n::nat.\ x\ n \in s$ **and** $x{:}\bigwedge n\ m.\ m < n \Longrightarrow \neg\ (dist\ (x\ m)\ (x\ n) < e)$
by *blast*
then obtain $l\ r$ **where** $l \in s$ **and** $r{:}subseq\ r$ **and** $((x \circ r) \longrightarrow l)\ sequentially$
using *assms* **by** (*metis seq-compact-def*)
from *this*(*3*) **have** *Cauchy* $(x \circ r)$
using *LIMSEQ-imp-Cauchy* **by** *auto*
then obtain $N::nat$ **where** $\bigwedge m\ n.\ N \leq m \Longrightarrow N \leq n \Longrightarrow dist\ ((x \circ r)\ m)\ ((x \circ r)\ n) < e$

```
    unfolding cauchy-def using ‹e > 0› by blast
  then have False
    using x[of r N r (N+1)] r by (auto simp: subseq-def) }
 then show ?thesis
  by metis
qed
```

16.19.4 Heine-Borel theorem

```
lemma seq-compact-imp-heine-borel:
 fixes s :: 'a :: metric-space set
 assumes seq-compact s
 shows compact s
proof −
 from seq-compact-imp-totally-bounded[OF ‹seq-compact s›]
 obtain f where f: ∀ e>0. finite (f e) ∧ f e ⊆ s ∧ s ⊆ (⋃ x∈f e. ball x e)
  unfolding choice-iff' ..
 def K ≡ (λ(x, r). ball x r) ' ((⋃ e ∈ ℚ ∩ {0 <..}. f e) × ℚ)
 have countably-compact s
  using ‹seq-compact s› by (rule seq-compact-imp-countably-compact)
 then show compact s
 proof (rule countably-compact-imp-compact)
  show countable K
   unfolding K-def using f
   by (auto intro: countable-finite countable-subset countable-rat
        intro!: countable-image countable-SIGMA countable-UN)
  show ∀ b∈K. open b by (auto simp: K-def)
 next
  fix T x
  assume T: open T x ∈ T and x: x ∈ s
  from openE[OF T] obtain e where 0 < e ball x e ⊆ T
   by auto
  then have 0 < e / 2 ball x (e / 2) ⊆ T
   by auto
  from Rats-dense-in-real[OF ‹0 < e / 2›] obtain r where r ∈ ℚ 0 < r r < e
/ 2
   by auto
  from f[rule-format, of r] ‹0 < r› ‹x ∈ s› obtain k where k ∈ f r x ∈ ball k r
   by auto
  from ‹r ∈ ℚ› ‹0 < r› ‹k ∈ f r› have ball k r ∈ K
   by (auto simp: K-def)
  then show ∃ b∈K. x ∈ b ∧ b ∩ s ⊆ T
  proof (rule bexI[rotated], safe)
   fix y
   assume y ∈ ball k r
   with ‹r < e / 2› ‹x ∈ ball k r› have dist x y < e
    by (intro dist-triangle-half-r [of k - e]) (auto simp: dist-commute)
   with ‹ball x e ⊆ T› show y ∈ T
    by auto
```

next
show $x \in ball\ k\ r$ **by** *fact*
qed
qed
qed

lemma *compact-eq-seq-compact-metric*:
$compact\ (s :: 'a{::}metric\text{-}space\ set) \longleftrightarrow seq\text{-}compact\ s$
using *compact-imp-seq-compact seq-compact-imp-heine-borel* **by** *blast*

lemma *compact-def*:
$compact\ (S :: 'a{::}metric\text{-}space\ set) \longleftrightarrow$
$(\forall f.\ (\forall n.\ f\ n \in S) \longrightarrow (\exists l{\in}S.\ \exists r.\ subseq\ r \wedge (f \circ r) \longrightarrow l))$
unfolding *compact-eq-seq-compact-metric seq-compact-def* **by** *auto*

16.19.5 Complete the chain of compactness variants

lemma *compact-eq-bolzano-weierstrass*:
fixes $s :: 'a{::}metric\text{-}space\ set$
shows $compact\ s \longleftrightarrow (\forall t.\ infinite\ t \wedge t \subseteq s \dashrightarrow (\exists x \in s.\ x\ islimpt\ t))$
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
using *heine-borel-imp-bolzano-weierstrass*[*of s*] **by** *auto*
next
assume *?rhs*
then show *?lhs*
unfolding *compact-eq-seq-compact-metric* **by** (*rule bolzano-weierstrass-imp-seq-compact*)
qed

lemma *bolzano-weierstrass-imp-bounded*:
$\forall t.\ infinite\ t \wedge t \subseteq s \longrightarrow (\exists x \in s.\ x\ islimpt\ t) \Longrightarrow bounded\ s$
using *compact-imp-bounded* **unfolding** *compact-eq-bolzano-weierstrass* **.**

16.20 Metric spaces with the Heine-Borel property

A metric space (or topological vector space) is said to have the Heine-Borel property if every closed and bounded subset is compact.

class *heine-borel* = *metric-space* +
assumes *bounded-imp-convergent-subsequence*:
$bounded\ (range\ f) \Longrightarrow \exists l\ r.\ subseq\ r \wedge ((f \circ r) \longrightarrow l)\ sequentially$

lemma *bounded-closed-imp-seq-compact*:
fixes $s{::}'a{::}heine\text{-}borel\ set$
assumes *bounded s*
and *closed s*
shows *seq-compact s*
proof (*unfold seq-compact-def*, *clarify*)

fix $f :: nat \Rightarrow 'a$
assume f: $\forall n.\ f\ n \in s$
with ‹*bounded s*› **have** *bounded* (*range f*)
by (*auto intro*: *bounded-subset*)
obtain $l\ r$ **where** r: *subseq r* **and** l: $((f \circ r) \longrightarrow l)$ *sequentially*
using *bounded-imp-convergent-subsequence* [*OF* ‹*bounded* (*range f*)›] **by** *auto*
from f **have** fr: $\forall n.\ (f \circ r)\ n \in s$
by *simp*
have $l \in s$ **using** ‹*closed s*› *fr l*
by (*rule closed-sequentially*)
show $\exists l{\in}s.\ \exists r.$ *subseq r* $\wedge$ $((f \circ r) \longrightarrow l)$ *sequentially*
using ‹$l \in s$› *r l* **by** *blast*
qed

lemma *compact-eq-bounded-closed*:
fixes s :: $'a$::*heine-borel set*
shows *compact s* $\longleftrightarrow$ *bounded s* $\wedge$ *closed s*
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
using *compact-imp-closed compact-imp-bounded*
by *blast*
next
assume *?rhs*
then show *?lhs*
using *bounded-closed-imp-seq-compact*[*of s*]
unfolding *compact-eq-seq-compact-metric*
by *auto*
qed

lemma *compact-closure* [*simp*]:
fixes S :: $'a$::*heine-borel set*
shows *compact*(*closure S*) $\longleftrightarrow$ *bounded S*
by (*meson bounded-closure bounded-subset closed-closure closure-subset compact-eq-bounded-closed*)

lemma *compact-components*:
fixes s :: $'a$::*heine-borel set*
shows $[\![$*compact s*; $c \in$ *components s*$]\!] \Longrightarrow$ *compact c*
by (*meson bounded-subset closed-components in-components-subset compact-eq-bounded-closed*)

lemma *not-compact-UNIV*[*simp*]:
fixes s :: $'a$::{*real-normed-vector*,*perfect-space*,*heine-borel*} *set*
shows $\sim$ *compact* (*UNIV*::$'a$ *set*)
by (*simp add*: *compact-eq-bounded-closed*)

lemma *bounded-increasing-convergent*:
fixes s :: *nat* $\Rightarrow$ *real*

shows *bounded* $\{s\ n|\ n.\ True\} \Longrightarrow \forall n.\ s\ n \leq s\ (Suc\ n) \Longrightarrow \exists l.\ s \longrightarrow l$
using *Bseq-mono-convergent*[*of s*] *incseq-Suc-iff*[*of s*]
by (*auto simp*: *image-def Bseq-eq-bounded convergent-def incseq-def*)

instance *real* :: *heine-borel*
proof
fix f :: *nat* ⇒ *real*
assume *f*: *bounded* (*range f*)
obtain *r* **where** *r*: *subseq r monoseq* (*f* ∘ *r*)
unfolding *comp-def* **by** (*metis seq-monosub*)
then have *Bseq* (*f* ∘ *r*)
unfolding *Bseq-eq-bounded* **using** *f* **by** (*force intro*: *bounded-subset*)
with *r* **show** ∃ *l r*. *subseq r* ∧ (*f* ∘ *r*) ⟶ *l*
using *Bseq-monoseq-convergent*[*of f* ∘ *r*] **by** (*auto simp*: *convergent-def*)
qed

lemma *compact-lemma-general*:
fixes *f* :: *nat* ⇒ *'a*
fixes *proj*::*'a* ⇒ *'b* ⇒ *'c*::*heine-borel* (**infixl** *proj 60*)
fixes *unproj*:: (*'b* ⇒ *'c*) ⇒ *'a*
assumes *finite-basis*: *finite basis*
assumes *bounded-proj*: ⋀*k*. *k* ∈ *basis* ⟹ *bounded* ((λ*x*. *x proj k*) ‘ *range f*)
assumes *proj-unproj*: ⋀*e k*. *k* ∈ *basis* ⟹ (*unproj e*) *proj k* = *e k*
assumes *unproj-proj*: ⋀*x*. *unproj* (λ*k*. *x proj k*) = *x*
shows ∀ *d*⊆*basis*. ∃ *l*::*'a*. ∃ *r*.
subseq r ∧ (∀ *e*>*0*. *eventually* (λ*n*. ∀ *i*∈*d*. *dist* (*f* (*r n*) *proj i*) (*l proj i*) < *e*) *sequentially*)
proof *safe*
fix *d* :: *'b set*
assume *d*: *d* ⊆ *basis*
with *finite-basis* **have** *finite d*
by (*blast intro*: *finite-subset*)
from *this d* **show** ∃ *l*::*'a*. ∃ *r*. *subseq r* ∧
(∀ *e*>*0*. *eventually* (λ*n*. ∀ *i*∈*d*. *dist* (*f* (*r n*) *proj i*) (*l proj i*) < *e*) *sequentially*)
proof (*induct d*)
case *empty*
then show *?case*
unfolding *subseq-def* **by** *auto*
next
case (*insert k d*)
have *k*[*intro*]: *k* ∈ *basis*
using *insert* **by** *auto*
have *s'*: *bounded* ((λ*x*. *x proj k*) ‘ *range f*)
using *k*
by (*rule bounded-proj*)
obtain *l1*::*'a* **and** *r1* **where** *r1*: *subseq r1*
and *lr1*: ∀ *e* > *0*. *eventually* (λ*n*. ∀ *i*∈*d*. *dist* (*f* (*r1 n*) *proj i*) (*l1 proj i*) < *e*) *sequentially*
using *insert*(*3*) **using** *insert*(*4*) **by** *auto*

```
    have f': ∀ n. f (r1 n) proj k ∈ (λx. x proj k) ‘ range f
      by simp
    have bounded (range (λi. f (r1 i) proj k))
      by (metis (lifting) bounded-subset f' image-subsetI s')
     then obtain l2 r2 where r2:subseq r2 and lr2:((λi. f (r1 (r2 i)) proj k)
⟶ l2) sequentially
      using bounded-imp-convergent-subsequence[of λi. f (r1 i) proj k]
      by (auto simp: o-def)
    def r ≡ r1 ∘ r2
    have r:subseq r
      using r1 and r2 unfolding r-def o-def subseq-def by auto
    moreover
    def l ≡ unproj (λi. if i = k then l2 else l1 proj i)::'a
    {
      fix e::real
      assume e > 0
      from lr1 ‹e > 0› have N1: eventually (λn. ∀ i∈d. dist (f (r1 n) proj i) (l1
proj i) < e) sequentially
        by blast
      from lr2 ‹e > 0› have N2:eventually (λn. dist (f (r1 (r2 n)) proj k) l2 <
e) sequentially
        by (rule tendstoD)
      from r2 N1 have N1': eventually (λn. ∀ i∈d. dist (f (r1 (r2 n)) proj i) (l1
proj i) < e) sequentially
        by (rule eventually-subseq)
       have eventually (λn. ∀ i∈(insert k d). dist (f (r n) proj i) (l proj i) < e)
sequentially
        using N1' N2
      by eventually-elim (insert insert.prems, auto simp: l-def r-def o-def proj-unproj)
    }
    ultimately show ?case by auto
  qed
qed

lemma compact-lemma:
  fixes f :: nat ⇒ 'a::euclidean-space
  assumes bounded (range f)
  shows ∀ d⊆Basis. ∃ l::'a. ∃ r.
     subseq r ∧ (∀ e>0. eventually (λn. ∀ i∈d. dist (f (r n) · i) (l · i) < e)
sequentially)
  by (rule compact-lemma-general[where unproj=λe. ∑ i∈Basis. e i *R i])
     (auto intro!: assms bounded-linear-inner-left bounded-linear-image
       simp: euclidean-representation)

instance euclidean-space ⊆ heine-borel
proof
  fix f :: nat ⇒ 'a
  assume f: bounded (range f)
  then obtain l::'a and r where r: subseq r
```

```
  and l: ∀ e>0. eventually (λn. ∀ i∈Basis. dist (f (r n) • i) (l • i) < e) sequentially
   using compact-lemma [OF f] by blast
 {
  fix e::real
  assume e > 0
  hence e / real-of-nat DIM('a) > 0 by (simp add: DIM-positive)
 with l have eventually (λn. ∀ i∈Basis. dist (f (r n) • i) (l • i) < e / (real-of-nat
DIM('a))) sequentially
    by simp
  moreover
  {
   fix n
   assume n: ∀ i∈Basis. dist (f (r n) • i) (l • i) < e / (real-of-nat DIM('a))
   have dist (f (r n)) l ≤ (∑ i∈Basis. dist (f (r n) • i) (l • i))
    apply (subst euclidean-dist-l2)
    using zero-le-dist
    apply (rule setL2-le-setsum)
    done
   also have ... < (∑ i∈(Basis::'a set). e / (real-of-nat DIM('a)))
    apply (rule setsum-strict-mono)
    using n
    apply auto
    done
   finally have dist (f (r n)) l < e
    by auto
  }
  ultimately have eventually (λn. dist (f (r n)) l < e) sequentially
   by (rule eventually-mono)
 }
 then have *: ((f ∘ r) ⟶ l) sequentially
  unfolding o-def tendsto-iff by simp
 with r show ∃ l r. subseq r ∧ ((f ∘ r) ⟶ l) sequentially
  by auto
qed

lemma bounded-fst: bounded s ⟹ bounded (fst ' s)
 unfolding bounded-def
 by (metis (erased, hide-lams) dist-fst-le image-iff order-trans)

lemma bounded-snd: bounded s ⟹ bounded (snd ' s)
 unfolding bounded-def
 by (metis (no-types, hide-lams) dist-snd-le image-iff order.trans)

instance prod :: (heine-borel, heine-borel) heine-borel
proof
 fix f :: nat ⇒ 'a × 'b
 assume f: bounded (range f)
 then have bounded (fst ' range f)
  by (rule bounded-fst)
```

then have *s1*: *bounded* (*range* (*fst* ∘ *f*))
by (*simp add*: *image-comp*)
obtain *l1 r1* **where** *r1*: *subseq r1* **and** *l1*: (λ*n*. *fst* (*f* (*r1 n*))) ⟶ *l1*
using *bounded-imp-convergent-subsequence* [*OF s1*] **unfolding** *o-def* **by** *fast*
from *f* **have** *s2*: *bounded* (*range* (*snd* ∘ *f* ∘ *r1*))
by (*auto simp add*: *image-comp intro*: *bounded-snd bounded-subset*)
obtain *l2 r2* **where** *r2*: *subseq r2* **and** *l2*: ((λ*n*. *snd* (*f* (*r1* (*r2 n*)))) ⟶ *l2*) *sequentially*
using *bounded-imp-convergent-subsequence* [*OF s2*]
unfolding *o-def* **by** *fast*
have *l1'*: ((λ*n*. *fst* (*f* (*r1* (*r2 n*)))) ⟶ *l1*) *sequentially*
using *LIMSEQ-subseq-LIMSEQ* [*OF l1 r2*] **unfolding** *o-def* **.**
have *l*: ((*f* ∘ (*r1* ∘ *r2*)) ⟶ (*l1*, *l2*)) *sequentially*
using *tendsto-Pair* [*OF l1' l2*] **unfolding** *o-def* **by** *simp*
have *r*: *subseq* (*r1* ∘ *r2*)
using *r1 r2* **unfolding** *subseq-def* **by** *simp*
show ∃ *l r*. *subseq r* ∧ ((*f* ∘ *r*) ⟶ *l*) *sequentially*
using *l r* **by** *fast*
qed

16.20.1 Completeness

lemma (**in** *metric-space*) *completeI*:
assumes ⋀*f*. ∀ *n*. *f n* ∈ *s* ⟹ *Cauchy f* ⟹ ∃ *l*∈*s*. *f* ⟶ *l*
shows *complete s*
using *assms* **unfolding** *complete-def* **by** *fast*

lemma (**in** *metric-space*) *completeE*:
assumes *complete s* **and** ∀ *n*. *f n* ∈ *s* **and** *Cauchy f*
obtains *l* **where** *l* ∈ *s* **and** *f* ⟶ *l*
using *assms* **unfolding** *complete-def* **by** *fast*

lemma *compact-imp-complete*:
fixes *s* :: 'a::*metric-space set*
assumes *compact s*
shows *complete s*
proof −
{
fix *f*
assume *as*: (∀ *n*::*nat*. *f n* ∈ *s*) *Cauchy f*
from *as*(*1*) **obtain** *l r* **where** *lr*: *l*∈*s subseq r* (*f* ∘ *r*) ⟶ *l*
using *assms* **unfolding** *compact-def* **by** *blast*

note *lr'* = *seq-suble* [*OF lr*(*2*)]
{
fix *e* :: *real*
assume *e* > *0*
from *as*(*2*) **obtain** *N* **where** *N*:∀ *m n*. *N* ≤ *m* ∧ *N* ≤ *n* ⟶ *dist* (*f m*) (*f*

```
n) < e/2
      unfolding cauchy-def
      using ‹e > 0›
      apply (erule-tac x=e/2 in allE)
      apply auto
      done
    from lr(3)[unfolded lim-sequentially, THEN spec[where x=e/2]]
    obtain M where M:∀ n≥M. dist ((f ∘ r) n) l < e/2
      using ‹e > 0› by auto
    {
      fix n :: nat
      assume n: n ≥ max N M
      have dist ((f ∘ r) n) l < e/2
        using n M by auto
      moreover have r n ≥ N
        using lr'[of n] n by auto
      then have dist (f n) ((f ∘ r) n) < e / 2
        using N and n by auto
      ultimately have dist (f n) l < e
        using dist-triangle-half-r[of f (r n) f n e l]
        by (auto simp add: dist-commute)
    }
    then have ∃ N. ∀ n≥N. dist (f n) l < e by blast
  }
  then have ∃ l∈s. (f ⟶ l) sequentially using ‹l∈s›
    unfolding lim-sequentially by auto
 }
 then show ?thesis unfolding complete-def by auto
qed

lemma nat-approx-posE:
 fixes e::real
 assumes 0 < e
 obtains n :: nat where 1 / (Suc n) < e
proof atomize-elim
 have 1 / real (Suc (nat ⌈1/e⌉)) < 1 / ⌈1/e⌉
   by (rule divide-strict-left-mono) (auto simp: ‹0 < e›)
 also have 1 / ⌈1/e⌉ ≤ 1 / (1/e)
   by (rule divide-left-mono) (auto simp: ‹0 < e› ceiling-correct)
 also have ... = e by simp
 finally show ∃ n. 1 / real (Suc n) < e ..
qed

lemma compact-eq-totally-bounded:
  compact s ⟷ complete s ∧ (∀ e>0. ∃ k. finite k ∧ s ⊆ (⋃ x∈k. ball x e))
   (is - ⟷ ?rhs)
proof
 assume assms: ?rhs
 then obtain k where k: ⋀e. 0 < e ⟹ finite (k e) ⋀e. 0 < e ⟹ s ⊆ (⋃ x∈k
```

```
e. ball x e)
  by (auto simp: choice-iff')

 show compact s
 proof cases
  assume s = {}
  then show compact s by (simp add: compact-def)
 next
  assume s ≠ {}
  show ?thesis
   unfolding compact-def
  proof safe
   fix f :: nat ⇒ 'a
   assume f: ∀ n. f n ∈ s

   def e ≡ λn. 1 / (2 * Suc n)
   then have [simp]: ⋀n. 0 < e n by auto
   def B ≡ λn U. SOME b. infinite {n. f n ∈ b} ∧ (∃ x. b ⊆ ball x (e n) ∩ U)
   {
    fix n U
    assume infinite {n. f n ∈ U}
    then have ∃ b∈k (e n). infinite {i∈{n. f n ∈ U}. f i ∈ ball b (e n)}
     using k f by (intro pigeonhole-infinite-rel) (auto simp: subset-eq)
    then obtain a where
     a ∈ k (e n)
     infinite {i ∈ {n. f n ∈ U}. f i ∈ ball a (e n)} ..
    then have ∃ b. infinite {i. f i ∈ b} ∧ (∃ x. b ⊆ ball x (e n) ∩ U)
     by (intro exI[of - ball a (e n) ∩ U] exI[of - a]) (auto simp: ac-simps)
    from someI-ex[OF this]
    have infinite {i. f i ∈ B n U} ∃ x. B n U ⊆ ball x (e n) ∩ U
     unfolding B-def by auto
   }
   note B = this

   def F ≡ rec-nat (B 0 UNIV) B
   {
    fix n
    have infinite {i. f i ∈ F n}
     by (induct n) (auto simp: F-def B)
   }
   then have F: ⋀n. ∃ x. F (Suc n) ⊆ ball x (e n) ∩ F n
    using B by (simp add: F-def)
   then have F-dec: ⋀m n. m ≤ n ⟹ F n ⊆ F m
    using decseq-SucI[of F] by (auto simp: decseq-def)

   obtain sel where sel: ⋀k i. i < sel k i ⋀k i. f (sel k i) ∈ F k
   proof (atomize-elim, unfold all-conj-distrib[symmetric], intro choice allI)
    fix k i
    have infinite ({n. f n ∈ F k} - {.. i})
```

```
        using ‹infinite {n. f n ∈ F k}› by auto
      from infinite-imp-nonempty[OF this]
      show ∃x>i. f x ∈ F k
        by (simp add: set-eq-iff not-le conj-commute)
    qed

    def t ≡ rec-nat (sel 0 0) (λn i. sel (Suc n) i)
    have subseq t
      unfolding subseq-Suc-iff by (simp add: t-def sel)
    moreover have ∀i. (f ∘ t) i ∈ s
      using f by auto
    moreover
    {
      fix n
      have (f ∘ t) n ∈ F n
        by (cases n) (simp-all add: t-def sel)
    }
    note t = this

    have Cauchy (f ∘ t)
    proof (safe intro!: metric-CauchyI exI elim!: nat-approx-posE)
      fix r :: real and N n m
      assume 1 / Suc N < r Suc N ≤ n Suc N ≤ m
      then have (f ∘ t) n ∈ F (Suc N) (f ∘ t) m ∈ F (Suc N) 2 * e N < r
        using F-dec t by (auto simp: e-def field-simps of-nat-Suc)
      with F[of N] obtain x where dist x ((f ∘ t) n) < e N dist x ((f ∘ t) m)
< e N
        by (auto simp: subset-eq)
      with dist-triangle[of (f ∘ t) m (f ∘ t) n x] ‹2 * e N < r›
      show dist ((f ∘ t) m) ((f ∘ t) n) < r
        by (simp add: dist-commute)
    qed

    ultimately show ∃l∈s. ∃r. subseq r ∧ (f ∘ r) ⟶ l
      using assms unfolding complete-def by blast
  qed
 qed
qed (metis compact-imp-complete compact-imp-seq-compact seq-compact-imp-totally-bounded)

lemma cauchy: Cauchy s ⟷ (∀e>0.∃ N::nat. ∀n≥N. dist(s n)(s N) < e) (is
?lhs = ?rhs)
proof −
 {
  assume ?rhs
  {
    fix e::real
    assume e>0
    with ‹?rhs› obtain N where N:∀n≥N. dist (s n) (s N) < e/2
      by (erule-tac x=e/2 in allE) auto
```

```
    {
      fix n m
      assume nm:N ≤ m ∧ N ≤ n
      then have dist (s m) (s n) < e using N
        using dist-triangle-half-l[of s m s N e s n]
        by blast
    }
    then have ∃N. ∀m n. N ≤ m ∧ N ≤ n ⟶ dist (s m) (s n) < e
      by blast
  }
  then have ?lhs
    unfolding cauchy-def
    by blast
 }
 then show ?thesis
   unfolding cauchy-def
   using dist-triangle-half-l
   by blast
qed

lemma cauchy-imp-bounded:
 assumes Cauchy s
 shows bounded (range s)
proof −
 from assms obtain N :: nat where ∀m n. N ≤ m ∧ N ≤ n ⟶ dist (s m) (s
n) < 1
   unfolding cauchy-def
   apply (erule-tac x= 1 in allE)
   apply auto
   done
 then have N:∀n. N ≤ n ⟶ dist (s N) (s n) < 1 by auto
 moreover
 have bounded (s ‘ {0..N})
   using finite-imp-bounded[of s ‘ {1..N}] by auto
 then obtain a where a:∀x∈s ‘ {0..N}. dist (s N) x ≤ a
   unfolding bounded-any-center [where a=s N] by auto
 ultimately show ?thesis
   unfolding bounded-any-center [where a=s N]
   apply (rule-tac x=max a 1 in exI)
   apply auto
   apply (erule-tac x=y in allE)
   apply (erule-tac x=y in ballE)
   apply auto
   done
qed

instance heine-borel < complete-space
proof
 fix f :: nat ⇒ 'a assume Cauchy f
```

then have *bounded* (*range f*)
by (*rule cauchy-imp-bounded*)
then have *compact* (*closure* (*range f*))
unfolding *compact-eq-bounded-closed* **by** *auto*
then have *complete* (*closure* (*range f*))
by (*rule compact-imp-complete*)
moreover have $\forall n.\ f\ n \in$ *closure* (*range f*)
using *closure-subset* [*of range f*] **by** *auto*
ultimately have $\exists l{\in}$*closure* (*range f*). (*f* $\longrightarrow$ *l*) *sequentially*
using ‹*Cauchy f*› **unfolding** *complete-def* **by** *auto*
then show *convergent f*
unfolding *convergent-def* **by** *auto*
qed

instance *euclidean-space* $\subseteq$ *banach* **..**

lemma *complete-UNIV*: *complete* (*UNIV* :: ('*a*::*complete-space*) *set*)
proof (*rule completeI*)
fix *f* :: *nat* $\Rightarrow$ '*a* **assume** *Cauchy f*
then have *convergent f* **by** (*rule Cauchy-convergent*)
then show $\exists l{\in}$*UNIV*. *f* $\longrightarrow$ *l* **unfolding** *convergent-def* **by** *simp*
qed

lemma *complete-imp-closed*:
fixes *s* :: '*a*::*metric-space set*
assumes *complete s*
shows *closed s*
proof (*unfold closed-sequential-limits*, *clarify*)
fix *f x* **assume** $\forall n.\ f\ n \in s$ **and** *f* $\longrightarrow$ *x*
from ‹*f* $\longrightarrow$ *x*› **have** *Cauchy f*
by (*rule LIMSEQ-imp-Cauchy*)
with ‹*complete s*› **and** ‹$\forall n.\ f\ n \in s$› **obtain** *l* **where** $l \in s$ **and** *f* $\longrightarrow$ *l*
by (*rule completeE*)
from ‹*f* $\longrightarrow$ *x*› **and** ‹*f* $\longrightarrow$ *l*› **have** $x = l$
by (*rule LIMSEQ-unique*)
with ‹$l \in s$› **show** $x \in s$
by *simp*
qed

lemma *complete-Int-closed*:
fixes *s* :: '*a*::*metric-space set*
assumes *complete s* **and** *closed t*
shows *complete* ($s \cap t$)
proof (*rule completeI*)
fix *f* **assume** $\forall n.\ f\ n \in s \cap t$ **and** *Cauchy f*
then have $\forall n.\ f\ n \in s$ **and** $\forall n.\ f\ n \in t$
by *simp-all*
from ‹*complete s*› **obtain** *l* **where** $l \in s$ **and** *f* $\longrightarrow$ *l*
using ‹$\forall n.\ f\ n \in s$› **and** ‹*Cauchy f*› **by** (*rule completeE*)

from ‹*closed t*› **and** ‹$\forall n.\ f\ n \in t$› **and** ‹$f \longrightarrow l$› **have** $l \in t$
by (*rule closed-sequentially*)
with ‹$l \in s$› **and** ‹$f \longrightarrow l$› **show** $\exists l{\in}s \cap t.\ f \longrightarrow l$
by *fast*
qed

lemma *complete-closed-subset*:
fixes $s :: 'a{::}$*metric-space set*
assumes *closed s* **and** $s \subseteq t$ **and** *complete t*
shows *complete s*
using *assms complete-Int-closed* [*of t s*] **by** (*simp add*: *Int-absorb1*)

lemma *complete-eq-closed*:
fixes $s :: ('a{::}$*complete-space*$)$ *set*
shows *complete s* $\longleftrightarrow$ *closed s*
proof
assume *closed s* **then show** *complete s*
using *subset-UNIV complete-UNIV* **by** (*rule complete-closed-subset*)
next
assume *complete s* **then show** *closed s*
by (*rule complete-imp-closed*)
qed

lemma *convergent-eq-cauchy*:
fixes $s ::$ *nat* $\Rightarrow 'a{::}$*complete-space*
shows $(\exists l.\ (s \longrightarrow l)$ *sequentially*$) \longleftrightarrow$ *Cauchy s*
unfolding *Cauchy-convergent-iff convergent-def* **..**

lemma *convergent-imp-bounded*:
fixes $s ::$ *nat* $\Rightarrow 'a{::}$*metric-space*
shows $(s \longrightarrow l)$ *sequentially* $\Longrightarrow$ *bounded* (*range s*)
by (*intro cauchy-imp-bounded LIMSEQ-imp-Cauchy*)

lemma *compact-cball*[*simp*]:
fixes $x :: 'a{::}$*heine-borel*
shows *compact* (*cball x e*)
using *compact-eq-bounded-closed bounded-cball closed-cball*
by *blast*

lemma *compact-frontier-bounded*[*intro*]:
fixes $s :: 'a{::}$*heine-borel set*
shows *bounded s* $\Longrightarrow$ *compact* (*frontier s*)
unfolding *frontier-def*
using *compact-eq-bounded-closed*
by *blast*

lemma *compact-frontier*[*intro*]:
fixes $s :: 'a{::}$*heine-borel set*
shows *compact s* $\Longrightarrow$ *compact* (*frontier s*)

using *compact-eq-bounded-closed compact-frontier-bounded*
by *blast*

corollary *compact-sphere*:
fixes *a* :: *'a*::{*real-normed-vector*,*perfect-space*,*heine-borel*}
shows *compact* (*sphere a r*)
using *compact-frontier* [*of cball a r*] **by** *simp*

lemma *frontier-subset-compact*:
fixes *s* :: *'a*::*heine-borel set*
shows *compact s* ⟹ *frontier s* ⊆ *s*
using *frontier-subset-closed compact-eq-bounded-closed*
by *blast*

16.21 Relations among convergence and absolute convergence for power series.

lemma *summable-imp-bounded*:
fixes *f* :: *nat* ⇒ *'a*::*real-normed-vector*
shows *summable f* ⟹ *bounded* (*range f*)
by (*frule summable-LIMSEQ-zero*) (*simp add*: *convergent-imp-bounded*)

lemma *summable-imp-sums-bounded*:
summable f ⟹ *bounded* (*range* (λ*n*. *setsum f* {..<*n*}))
by (*auto simp*: *summable-def sums-def dest*: *convergent-imp-bounded*)

lemma *power-series-conv-imp-absconv-weak*:
fixes *a*:: *nat* ⇒ *'a*::{*real-normed-div-algebra*,*banach*} **and** *w* :: *'a*
assumes *sum*: *summable* (λ*n*. *a n* ∗ *z* ^ *n*) **and** *no*: *norm w* < *norm z*
shows *summable* (λ*n*. *of-real*(*norm*(*a n*)) ∗ *w* ^ *n*)
proof −
obtain *M* **where** *M*: ⋀*x*. *norm* (*a x* ∗ *z* ^ *x*) ≤ *M*
using *summable-imp-bounded* [*OF sum*] **by** (*force simp add*: *bounded-iff*)
then have ∗: *summable* (λ*n*. *norm* (*a n*) ∗ *norm w* ^ *n*)
by (*rule-tac M*=*M* **in** *Abel-lemma*) (*auto simp*: *norm-mult norm-power intro*: *no*)
show *?thesis*
apply (*rule series-comparison-complex* [*of* (λ*n*. *of-real*(*norm*(*a n*) ∗ *norm w* ^ *n*))])
apply (*simp only*: *summable-complex-of-real* ∗)
apply (*auto simp*: *norm-mult norm-power*)
done
qed

16.22 Bounded closed nest property (proof does not use Heine-Borel)

lemma *bounded-closed-nest*:
fixes *s* :: *nat* ⇒ (*'a*::*heine-borel*) *set*

```
  assumes ∀n. closed (s n)
    and ∀n. s n ≠ {}
    and ∀m n. m ≤ n ⟶ s n ⊆ s m
    and bounded (s 0)
  shows ∃a. ∀n. a ∈ s n
proof −
  from assms(2) obtain x where x: ∀n. x n ∈ s n
    using choice[of λn x. x ∈ s n] by auto
  from assms(4,1) have seq-compact (s 0)
    by (simp add: bounded-closed-imp-seq-compact)
  then obtain l r where lr: l ∈ s 0 subseq r (x ∘ r) ⟶ l
    using x and assms(3) unfolding seq-compact-def by blast
  have ∀n. l ∈ s n
  proof
    fix n :: nat
    have closed (s n)
      using assms(1) by simp
    moreover have ∀i. (x ∘ r) i ∈ s i
      using x and assms(3) and lr(2) [THEN seq-suble] by auto
    then have ∀i. (x ∘ r) (i + n) ∈ s n
      using assms(3) by (fast intro!: le-add2)
    moreover have (λi. (x ∘ r) (i + n)) ⟶ l
      using lr(3) by (rule LIMSEQ-ignore-initial-segment)
    ultimately show l ∈ s n
      by (rule closed-sequentially)
  qed
  then show ?thesis ..
qed
```

Decreasing case does not even need compactness, just completeness.

```
lemma decreasing-closed-nest:
  fixes s :: nat ⇒ ('a::complete-space) set
  assumes
    ∀n. closed (s n)
    ∀n. s n ≠ {}
    ∀m n. m ≤ n ⟶ s n ⊆ s m
    ∀e>0. ∃n. ∀x∈s n. ∀y∈s n. dist x y < e
  shows ∃a. ∀n. a ∈ s n
proof −
  have ∀n. ∃x. x ∈ s n
    using assms(2) by auto
  then have ∃t. ∀n. t n ∈ s n
    using choice[of λn x. x ∈ s n] by auto
  then obtain t where t: ∀n. t n ∈ s n by auto
  {
    fix e :: real
    assume e > 0
    then obtain N where N:∀x∈s N. ∀y∈s N. dist x y < e
      using assms(4) by auto
```

```
    {
      fix m n :: nat
      assume N ≤ m ∧ N ≤ n
      then have t m ∈ s N t n ∈ s N
        using assms(3) t unfolding  subset-eq t by blast+
      then have dist (t m) (t n) < e
        using N by auto
    }
    then have ∃N. ∀m n. N ≤ m ∧ N ≤ n ⟶ dist (t m) (t n) < e
      by auto
  }
  then have Cauchy t
    unfolding cauchy-def by auto
  then obtain l where l:(t ⟶ l) sequentially
    using complete-UNIV unfolding complete-def by auto
  {
    fix n :: nat
    {
      fix e :: real
      assume e > 0
      then obtain N :: nat where N: ∀n≥N. dist (t n) l < e
        using l[unfolded lim-sequentially] by auto
      have t (max n N) ∈ s n
        using assms(3)
        unfolding subset-eq
        apply (erule-tac x=n in allE)
        apply (erule-tac x=max n N in allE)
        using t
        apply auto
        done
      then have ∃y∈s n. dist y l < e
        apply (rule-tac x=t (max n N) in bexI)
        using N
        apply auto
        done
    }
    then have l ∈ s n
      using closed-approachable[of s n l] assms(1) by auto
  }
  then show ?thesis by auto
qed
```

Strengthen it to the intersection actually being a singleton.

```
lemma decreasing-closed-nest-sing:
  fixes s :: nat ⇒ 'a::complete-space set
  assumes
    ∀n. closed(s n)
    ∀n. s n ≠ {}
    ∀m n. m ≤ n ⟶ s n ⊆ s m
```

```
    ∀ e>0. ∃ n. ∀ x ∈ (s n). ∀ y∈(s n). dist x y < e
  shows ∃ a. ⋂(range s) = {a}
proof −
  obtain a where a: ∀ n. a ∈ s n
    using decreasing-closed-nest[of s] using assms by auto
  {
    fix b
    assume b: b ∈ ⋂(range s)
    {
      fix e :: real
      assume e > 0
      then have dist a b < e
        using assms(4) and b and a by blast
    }
    then have dist a b = 0
      by (metis dist-eq-0-iff dist-nz less-le)
  }
  with a have ⋂(range s) = {a}
    unfolding image-def by auto
  then show ?thesis ..
qed
```

Cauchy-type criteria for uniform convergence.

```
lemma uniformly-convergent-eq-cauchy:
  fixes s::nat ⇒ 'b ⇒ 'a::complete-space
  shows
    (∃ l. ∀ e>0. ∃ N. ∀ n x. N ≤ n ∧ P x ⟶ dist(s n x)(l x) < e) ⟷
      (∀ e>0. ∃ N. ∀ m n x. N ≤ m ∧ N ≤ n ∧ P x ⟶ dist (s m x) (s n x) < e)
  (is ?lhs = ?rhs)
proof
  assume ?lhs
  then obtain l where l:∀ e>0. ∃ N. ∀ n x. N ≤ n ∧ P x ⟶ dist (s n x) (l x)
< e
    by auto
  {
    fix e :: real
    assume e > 0
    then obtain N :: nat where N: ∀ n x. N ≤ n ∧ P x ⟶ dist (s n x) (l x) <
e / 2
      using l[THEN spec[where x=e/2]] by auto
    {
      fix n m :: nat and x :: 'b
      assume N ≤ m ∧ N ≤ n ∧ P x
      then have dist (s m x) (s n x) < e
        using N[THEN spec[where x=m], THEN spec[where x=x]]
        using N[THEN spec[where x=n], THEN spec[where x=x]]
        using dist-triangle-half-l[of s m x l x e s n x] by auto
    }
    then have ∃ N. ∀ m n x. N ≤ m ∧ N ≤ n ∧ P x  −−> dist (s m x) (s n x)
```

```
< e  by auto
  }
  then show ?rhs by auto
next
  assume ?rhs
  then have ∀ x. P x ⟶ Cauchy (λn. s n x)
    unfolding cauchy-def
    apply auto
    apply (erule-tac x=e in allE)
    apply auto
    done
  then obtain l where l: ∀ x. P x ⟶ ((λn. s n x) ⟶⟶ l x) sequentially
    unfolding convergent-eq-cauchy[symmetric]
    using choice[of λx l. P x ⟶ ((λn. s n x) ⟶⟶ l) sequentially]
    by auto
  {
    fix e :: real
    assume e > 0
    then obtain N where N:∀ m n x. N ≤ m ∧ N ≤ n ∧ P x ⟶ dist (s m x)
(s n x) < e/2
      using ‹?rhs›[THEN spec[where x=e/2]] by auto
    {
      fix x
      assume P x
      then obtain M where M:∀ n≥M. dist (s n x) (l x) < e/2
        using l[THEN spec[where x=x], unfolded lim-sequentially] and ‹e > 0›
        by (auto elim!: allE[where x=e/2])
      fix n :: nat
      assume n ≥ N
      then have dist(s n x)(l x) < e
        using ‹P x›and N[THEN spec[where x=n], THEN spec[where x=N+M],
THEN spec[where x=x]]
         using M[THEN spec[where x=N+M]] and dist-triangle-half-l[of s n x s
(N+M) x e l x]
        by (auto simp add: dist-commute)
    }
    then have ∃ N. ∀ n x. N ≤ n ∧ P x ⟶ dist(s n x)(l x) < e
      by auto
  }
  then show ?lhs by auto
qed

lemma uniformly-cauchy-imp-uniformly-convergent:
  fixes s :: nat ⇒ 'a ⇒ 'b::complete-space
  assumes ∀ e>0.∃ N. ∀ m (n::nat) x. N ≤ m ∧ N ≤ n ∧ P x --> dist(s m x)(s
n x) < e
    and ∀ x. P x --> (∀ e>0. ∃ N. ∀ n. N ≤ n ⟶ dist(s n x)(l x) < e)
  shows ∀ e>0. ∃ N. ∀ n x. N ≤ n ∧ P x ⟶ dist(s n x)(l x) < e
proof −
```

obtain l' **where** l:$\forall\, e{>}0.\ \exists\, N.\ \forall\, n\ x.\ N \leq n \wedge P\ x \longrightarrow dist\ (s\ n\ x)\ (l'\ x) < e$
using *assms*(*1*) **unfolding** *uniformly-convergent-eq-cauchy*[*symmetric*] **by** *auto*
moreover
{
fix x
assume $P\ x$
then have $l\ x = l'\ x$
using *tendsto-unique*[*OF trivial-limit-sequentially*, *of* $\lambda n.\ s\ n\ x$ l x l' x]
using l **and** *assms*(*2*) **unfolding** *lim-sequentially* **by** *blast*
}
ultimately show *?thesis* **by** *auto*
qed

16.23 Continuity

Derive the epsilon-delta forms, which we often use as "definitions"

lemma *continuous-within-eps-delta*:
continuous (*at x within s*) $f \longleftrightarrow (\forall\, e{>}0.\ \exists\, d{>}0.\ \forall\, x'{\in}\ s.$ *dist* $x'\ x < d$ $--{>}$ *dist* $(f\ x')\ (f\ x) < e$)
unfolding *continuous-within* **and** *Lim-within*
apply *auto*
apply (*metis dist-nz dist-self*)
apply *blast*
done

corollary *continuous-at-eps-delta*:
continuous (*at x*) $f \longleftrightarrow (\forall\, e > 0.\ \exists\, d > 0.\ \forall\, x'.$ *dist* $x'\ x < d \longrightarrow$ *dist* $(f\ x')\ (f\ x) < e$)
using *continuous-within-eps-delta* [*of x UNIV f*] **by** *simp*

lemma *continuous-at-right-real-increasing*:
fixes f :: *real* $\Rightarrow$ *real*
assumes *nondecF*: $\bigwedge x\ y.\ x \leq y \Longrightarrow f\ x \leq f\ y$
shows *continuous* (*at-right a*) $f \longleftrightarrow (\forall\, e{>}0.\ \exists\, d{>}0.\ f\ (a + d) - f\ a < e)$
apply (*simp add*: *greaterThan-def dist-real-def continuous-within Lim-within-le*)
apply (*intro all-cong ex-cong*)
apply *safe*
apply (*erule-tac x*=$a + d$ **in** *allE*)
apply *simp*
apply (*simp add*: *nondecF field-simps*)
apply (*drule nondecF*)
apply *simp*
done

lemma *continuous-at-left-real-increasing*:
assumes *nondecF*: $\bigwedge\ x\ y.\ x \leq y \Longrightarrow f\ x \leq ((f\ y) ::$ *real*$)$
shows (*continuous* (*at-left* $(a ::$ *real*$)$) f) $= (\forall\, e > 0.\ \exists\, delta > 0.\ f\ a - f\ (a - delta) < e)$
apply (*simp add*: *lessThan-def dist-real-def continuous-within Lim-within-le*)

apply (*intro all-cong ex-cong*)
apply *safe*
apply (*erule-tac x=a − d* **in** *allE*)
apply *simp*
apply (*simp add*: *nondecF field-simps*)
apply (*cut-tac x=a − d* **and** *y=x* **in** *nondecF*)
apply *simp-all*
done

Versions in terms of open balls.

lemma *continuous-within-ball*:
continuous (*at x within s*) *f* ⟷
(∀ *e* > *0*. ∃ *d* > *0*. *f* ‘ (*ball x d* ∩ *s*) ⊆ *ball* (*f x*) *e*)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
{
fix *e* :: *real*
assume *e* > *0*
then obtain *d* **where** *d*: *d*>*0* ∀ *xa*∈*s*. *0* < *dist xa x* ∧ *dist xa x* < *d* ⟶ *dist* (*f xa*) (*f x*) < *e*
using ‹*?lhs*›[*unfolded continuous-within Lim-within*] **by** *auto*
{
fix *y*
assume *y* ∈ *f* ‘ (*ball x d* ∩ *s*)
then have *y* ∈ *ball* (*f x*) *e*
using *d*(*2*)
apply (*auto simp add*: *dist-commute*)
apply (*erule-tac x=xa* **in** *ballE*)
apply *auto*
using ‹*e* > *0*›
apply *auto*
done
}
then have ∃ *d*>*0*. *f* ‘ (*ball x d* ∩ *s*) ⊆ *ball* (*f x*) *e*
using ‹*d* > *0*›
unfolding *subset-eq ball-def* **by** (*auto simp add*: *dist-commute*)
}
then show *?rhs* **by** *auto*
next
assume *?rhs*
then show *?lhs*
unfolding *continuous-within Lim-within ball-def subset-eq*
apply (*auto simp add*: *dist-commute*)
apply (*erule-tac x=e* **in** *allE*)
apply *auto*
done
qed

lemma *continuous-at-ball*:
continuous (*at x*) *f* ⟷ (∀ *e*>*0*. ∃ *d*>*0*. *f* ‘ (*ball x d*) ⊆ *ball* (*f x*) *e*) (**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
unfolding *continuous-at Lim-at subset-eq Ball-def Bex-def image-iff mem-ball*
apply *auto*
apply (*erule-tac x*=*e* **in** *allE*)
apply *auto*
apply (*rule-tac x*=*d* **in** *exI*)
apply *auto*
apply (*erule-tac x*=*xa* **in** *allE*)
apply (*auto simp add*: *dist-commute*)
done
next
assume *?rhs*
then show *?lhs*
unfolding *continuous-at Lim-at subset-eq Ball-def Bex-def image-iff mem-ball*
apply *auto*
apply (*erule-tac x*=*e* **in** *allE*)
apply *auto*
apply (*rule-tac x*=*d* **in** *exI*)
apply *auto*
apply (*erule-tac x*=*f xa* **in** *allE*)
apply (*auto simp add*: *dist-commute*)
done
qed

Define setwise continuity in terms of limits within the set.

lemma *continuous-on-iff*:
continuous-on s f ⟷
(∀ *x*∈*s*. ∀ *e*>*0*. ∃ *d*>*0*. ∀ *x*′∈*s*. *dist x*′ *x* < *d* ⟶ *dist* (*f x*′) (*f x*) < *e*)
unfolding *continuous-on-def Lim-within*
by (*metis dist-pos-lt dist-self*)

lemma *continuous-within-E*:
assumes *continuous* (*at x within s*) *f e*>*0*
obtains *d* **where** *d*>*0* ⋀*x*′. ⟦*x*′∈ *s*; *dist x*′ *x* ≤ *d*⟧ ⟹ *dist* (*f x*′) (*f x*) < *e*
using *assms* **apply** (*simp add*: *continuous-within-eps-delta*)
apply (*drule spec* [*of* - *e*], *clarify*)
apply (*rule-tac d*=*d*/*2* **in** *that*, *auto*)
done

lemma *continuous-onI* [*intro?*]:
assumes ⋀*x e*. ⟦*e* > *0*; *x* ∈ *s*⟧ ⟹ ∃ *d*>*0*. ∀ *x*′∈*s*. *dist x*′ *x* < *d* ⟶ *dist* (*f x*′) (*f x*) ≤ *e*
shows *continuous-on s f*
apply (*simp add*: *continuous-on-iff*, *clarify*)

apply (*rule ex-forward* [*OF assms* [*OF half-gt-zero*]], *auto*)
done

lemma *uniformly-continuous-on-def*:
fixes $f :: 'a::metric\text{-}space \Rightarrow 'b::metric\text{-}space$
shows *uniformly-continuous-on s f* $\longleftrightarrow$
$(\forall e{>}0.\ \exists d{>}0.\ \forall x{\in}s.\ \forall x'{\in}s.\ dist\ x'\ x < d \longrightarrow dist\ (f\ x')\ (f\ x) < e)$
unfolding *uniformly-continuous-on-uniformity*
uniformity-dist filterlim-INF filterlim-principal eventually-inf-principal
by (*force simp*: *Ball-def uniformity-dist*[*symmetric*] *eventually-uniformity-metric*)

Some simple consequential lemmas.

lemma *continuous-onE*:
assumes *continuous-on s f x*$\in$*s e*$>$*0*
obtains *d* **where** *d*$>$*0* $\bigwedge x'.\ [\![x' \in s;\ dist\ x'\ x \le d]\!] \Longrightarrow dist\ (f\ x')\ (f\ x) < e$
using *assms*
apply (*simp add*: *continuous-on-iff*)
apply (*elim ballE allE*)
apply (*auto intro*: *that* [**where** *d*=*d*/*2* **for** *d*])
done

lemma *uniformly-continuous-onE*:
assumes *uniformly-continuous-on s f 0 < e*
obtains *d* **where** *d*$>$*0* $\bigwedge x\ x'.\ [\![x{\in}s;\ x'{\in}s;\ dist\ x'\ x < d]\!] \Longrightarrow dist\ (f\ x')\ (f\ x) <$ *e*
using *assms*
by (*auto simp*: *uniformly-continuous-on-def*)

lemma *continuous-at-imp-continuous-within*:
continuous (*at x*) *f* $\Longrightarrow$ *continuous* (*at x within s*) *f*
unfolding *continuous-within continuous-at* **using** *Lim-at-imp-Lim-at-within* **by** *auto*

lemma *Lim-trivial-limit*: *trivial-limit net* $\Longrightarrow$ (*f* $\longrightarrow$ *l*) *net*
by *simp*

lemmas *continuous-on* = *continuous-on-def* — legacy theorem name

lemma *continuous-within-subset*:
continuous (*at x within s*) *f* $\Longrightarrow$ *t* $\subseteq$ *s* $\Longrightarrow$ *continuous* (*at x within t*) *f*
unfolding *continuous-within* **by**(*metis tendsto-within-subset*)

lemma *continuous-on-interior*:
continuous-on s f $\Longrightarrow$ *x* $\in$ *interior s* $\Longrightarrow$ *continuous* (*at x*) *f*
by (*metis continuous-on-eq-continuous-at continuous-on-subset interiorE*)

lemma *continuous-on-eq*:
$[\![$*continuous-on s f*; $\bigwedge x.\ x \in s \Longrightarrow f\ x = g\ x]\!] \Longrightarrow$ *continuous-on s g*
unfolding *continuous-on-def tendsto-def eventually-at-topological*

by *simp*

Characterization of various kinds of continuity in terms of sequences.

lemma *continuous-within-sequentially*:
fixes f :: *'a::metric-space* ⇒ *'b::topological-space*
shows *continuous* (*at a within s*) f ⟷
$(\forall x.\ (\forall n{::}nat.\ x\ n \in s) \wedge (x \longrightarrow a)\ sequentially$
$\longrightarrow ((f \circ x) \longrightarrow f\ a)\ sequentially)$
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
{
fix x :: *nat* ⇒ *'a*
assume x: $\forall n.\ x\ n \in s\ \forall e{>}0.\ eventually\ (\lambda n.\ dist\ (x\ n)\ a < e)\ sequentially$
fix T :: *'b set*
assume *open* T **and** $f\ a \in T$
with ‹*?lhs*› **obtain** d **where** $d{>}0$ **and** $d{:}\forall x{\in}s.\ 0 < dist\ x\ a \wedge dist\ x\ a < d$
⟶ $f\ x \in T$
unfolding *continuous-within tendsto-def eventually-at* **by** *auto*
have *eventually* $(\lambda n.\ dist\ (x\ n)\ a < d)$ *sequentially*
using $x(2)$ ‹$d{>}0$› **by** *simp*
then have *eventually* $(\lambda n.\ (f \circ x)\ n \in T)$ *sequentially*
proof *eventually-elim*
case (*elim n*)
then show *?case*
using $d\ x(1)$ ‹$f\ a \in T$› **by** *auto*
qed
}
then show *?rhs*
unfolding *tendsto-iff tendsto-def* **by** *simp*
next
assume *?rhs*
then show *?lhs*
unfolding *continuous-within tendsto-def* [**where** $l{=}f\ a$]
by (*simp add*: *sequentially-imp-eventually-within*)
qed

lemma *continuous-at-sequentially*:
fixes f :: *'a::metric-space* ⇒ *'b::topological-space*
shows *continuous* (*at a*) f ⟷
$(\forall x.\ (x \longrightarrow a)\ sequentially \longrightarrow ((f \circ x) \longrightarrow f\ a)\ sequentially)$
using *continuous-within-sequentially*[*of a UNIV f*] **by** *simp*

lemma *continuous-on-sequentially*:
fixes f :: *'a::metric-space* ⇒ *'b::topological-space*
shows *continuous-on s f* ⟷
$(\forall x.\ \forall a \in s.\ (\forall n.\ x(n) \in s) \wedge (x \longrightarrow a)\ sequentially$
$\longrightarrow ((f \circ x) \longrightarrow f\ a)\ sequentially)$
(**is** *?lhs* = *?rhs*)

proof
assume *?rhs*
then show *?lhs*
using *continuous-within-sequentially*[*of - s f*]
unfolding *continuous-on-eq-continuous-within*
by *auto*
next
assume *?lhs*
then show *?rhs*
unfolding *continuous-on-eq-continuous-within*
using *continuous-within-sequentially*[*of - s f*]
by *auto*
qed

lemma *uniformly-continuous-on-sequentially*:
uniformly-continuous-on s f $\longleftrightarrow$ $(\forall\, x\ y.\ (\forall\, n.\ x\ n \in s) \wedge (\forall\, n.\ y\ n \in s) \wedge$
$(\lambda n.\ dist\ (x\ n)\ (y\ n)) \longrightarrow 0 \longrightarrow (\lambda n.\ dist\ (f(x\ n))\ (f(y\ n))) \longrightarrow 0)$ (**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
{
fix *x y*
assume *x*: $\forall\, n.\ x\ n \in s$
and *y*: $\forall\, n.\ y\ n \in s$
and *xy*: $((\lambda n.\ dist\ (x\ n)\ (y\ n)) \longrightarrow 0)$ *sequentially*
{
fix *e* :: *real*
assume $e > 0$
then obtain *d* **where** $d > 0$ **and** *d*: $\forall\, x{\in}s.\ \forall\, x'{\in}s.\ dist\ x'\ x < d \longrightarrow dist\ (f\ x')\ (f\ x) < e$
using ‹*?lhs*›[*unfolded uniformly-continuous-on-def*, *THEN spec*[**where** *x*=*e*]]
by *auto*
obtain *N* **where** *N*: $\forall\, n{\geq}N.\ dist\ (x\ n)\ (y\ n) < d$
using *xy*[*unfolded lim-sequentially dist-norm*] **and** ‹$d{>}0$› **by** *auto*
{
fix *n*
assume $n{\geq}N$
then have $dist\ (f\ (x\ n))\ (f\ (y\ n)) < e$
using *N*[*THEN spec*[**where** *x*=*n*]]
using *d*[*THEN bspec*[**where** *x*=*x n*], *THEN bspec*[**where** *x*=*y n*]]
using *x* **and** *y*
unfolding *dist-commute*
by *simp*
}
then have $\exists\, N.\ \forall\, n{\geq}N.\ dist\ (f\ (x\ n))\ (f\ (y\ n)) < e$
by *auto*
}
then have $((\lambda n.\ dist\ (f(x\ n))\ (f(y\ n))) \longrightarrow 0)$ *sequentially*
unfolding *lim-sequentially* **and** *dist-real-def* **by** *auto*

```
  }
  then show ?rhs by auto
next
  assume ?rhs
  {
    assume ¬ ?lhs
    then obtain e where e > 0 ∀ d>0. ∃ x∈s. ∃ x'∈s. dist x' x < d ∧ ¬ dist (f
x') (f x) < e
      unfolding uniformly-continuous-on-def by auto
    then obtain fa where fa:
      ∀ x. 0 < x ⟶ fst (fa x) ∈ s ∧ snd (fa x) ∈ s ∧ dist (fst (fa x)) (snd (fa x))
< x ∧ ¬ dist (f (fst (fa x))) (f (snd (fa x))) < e
      using choice[of λ d x. d>0 ⟶ fst x ∈ s ∧ snd x ∈ s ∧ dist (snd x) (fst x)
< d ∧ ¬ dist (f (snd x)) (f (fst x)) < e]
      unfolding Bex-def
      by (auto simp add: dist-commute)
    def x ≡ λn::nat. fst (fa (inverse (real n + 1)))
    def y ≡ λn::nat. snd (fa (inverse (real n + 1)))
    have xyn: ∀ n. x n ∈ s ∧ y n ∈ s
      and xy0: ∀ n. dist (x n) (y n) < inverse (real n + 1)
      and fxy:∀ n. ¬ dist (f (x n)) (f (y n)) < e
      unfolding x-def and y-def using fa
      by auto
    {
      fix e :: real
      assume e > 0
     then obtain N :: nat where N ≠ 0 and N: 0 < inverse (real N) ∧ inverse
(real N) < e
       unfolding real-arch-inverse[of e] by auto
      {
        fix n :: nat
        assume n ≥ N
        then have inverse (real n + 1) < inverse (real N)
          using of-nat-0-le-iff and ‹N≠0› by auto
        also have ... < e using N by auto
        finally have inverse (real n + 1) < e by auto
        then have dist (x n) (y n) < e
          using xy0[THEN spec[where x=n]] by auto
      }
      then have ∃ N. ∀ n≥N. dist (x n) (y n) < e by auto
    }
    then have ∀ e>0. ∃ N. ∀ n≥N. dist (f (x n)) (f (y n)) < e
      using ‹?rhs›[THEN spec[where x=x], THEN spec[where x=y]] and xyn
      unfolding lim-sequentially dist-real-def by auto
    then have False using fxy and ‹e>0› by auto
  }
  then show ?lhs
    unfolding uniformly-continuous-on-def by blast
qed
```

The usual transformation theorems.

lemma *continuous-transform-within*:
fixes $f\ g :: 'a::metric\text{-}space \Rightarrow 'b::topological\text{-}space$
assumes *continuous* $(at\ x\ within\ s)\ f$
and $0 < d$
and $x \in s$
and $\bigwedge x'.\ [\![x' \in s;\ dist\ x'\ x < d]\!] \Longrightarrow f\ x' = g\ x'$
shows *continuous* $(at\ x\ within\ s)\ g$
using *assms*
unfolding *continuous-within*
by (*force simp add*: *intro*: *Lim-transform-within*)

16.23.1 Structural rules for pointwise continuity

lemma *continuous-infdist*[*continuous-intros*]:
assumes *continuous F f*
shows *continuous F* $(\lambda x.\ infdist\ (f\ x)\ A)$
using *assms* **unfolding** *continuous-def* **by** (*rule tendsto-infdist*)

lemma *continuous-infnorm*[*continuous-intros*]:
continuous F f $\Longrightarrow$ *continuous F* $(\lambda x.\ infnorm\ (f\ x))$
unfolding *continuous-def* **by** (*rule tendsto-infnorm*)

lemma *continuous-inner*[*continuous-intros*]:
assumes *continuous F f*
and *continuous F g*
shows *continuous F* $(\lambda x.\ inner\ (f\ x)\ (g\ x))$
using *assms* **unfolding** *continuous-def* **by** (*rule tendsto-inner*)

lemmas *continuous-at-inverse* = *isCont-inverse*

16.23.2 Structural rules for setwise continuity

lemma *continuous-on-infnorm*[*continuous-intros*]:
continuous-on s f $\Longrightarrow$ *continuous-on s* $(\lambda x.\ infnorm\ (f\ x))$
unfolding *continuous-on* **by** (*fast intro*: *tendsto-infnorm*)

lemma *continuous-on-inner*[*continuous-intros*]:
fixes $g :: 'a::topological\text{-}space \Rightarrow 'b::real\text{-}inner$
assumes *continuous-on s f*
and *continuous-on s g*
shows *continuous-on s* $(\lambda x.\ inner\ (f\ x)\ (g\ x))$
using *bounded-bilinear-inner assms*
by (*rule bounded-bilinear.continuous-on*)

16.23.3 Structural rules for uniform continuity

lemma *uniformly-continuous-on-dist*[*continuous-intros*]:
fixes $f\ g :: 'a::metric\text{-}space \Rightarrow 'b::metric\text{-}space$

assumes *uniformly-continuous-on s f*
and *uniformly-continuous-on s g*
shows *uniformly-continuous-on s* (λ*x*. *dist* (*f x*) (*g x*))
proof −
{
fix *a b c d* :: *'b*
have |*dist a b* − *dist c d*| ≤ *dist a c* + *dist b d*
using *dist-triangle2* [*of a b c*] *dist-triangle2* [*of b c d*]
using *dist-triangle3* [*of c d a*] *dist-triangle* [*of a d b*]
by *arith*
} note *le* = *this*
{
fix *x y*
assume *f*: (λ*n*. *dist* (*f* (*x n*)) (*f* (*y n*))) ⟶ *0*
assume *g*: (λ*n*. *dist* (*g* (*x n*)) (*g* (*y n*))) ⟶ *0*
have (λ*n*. |*dist* (*f* (*x n*)) (*g* (*x n*)) − *dist* (*f* (*y n*)) (*g* (*y n*))|) ⟶ *0*
by (*rule Lim-transform-bound* [*OF - tendsto-add-zero* [*OF f g*]],
simp add: *le*)
}
then show *?thesis*
using *assms* **unfolding** *uniformly-continuous-on-sequentially*
unfolding *dist-real-def* **by** *simp*
qed

lemma *uniformly-continuous-on-norm*[*continuous-intros*]:
fixes *f* :: *'a* :: *metric-space* ⇒ *'b* :: *real-normed-vector*
assumes *uniformly-continuous-on s f*
shows *uniformly-continuous-on s* (λ*x*. *norm* (*f x*))
unfolding *norm-conv-dist* **using** *assms*
by (*intro uniformly-continuous-on-dist uniformly-continuous-on-const*)

lemma (**in** *bounded-linear*) *uniformly-continuous-on*[*continuous-intros*]:
fixes *g* :: -::*metric-space* ⇒ -
assumes *uniformly-continuous-on s g*
shows *uniformly-continuous-on s* (λ*x*. *f* (*g x*))
using *assms* **unfolding** *uniformly-continuous-on-sequentially*
unfolding *dist-norm tendsto-norm-zero-iff diff*[*symmetric*]
by (*auto intro*: *tendsto-zero*)

lemma *uniformly-continuous-on-cmul*[*continuous-intros*]:
fixes *f* :: *'a*::*metric-space* ⇒ *'b*::*real-normed-vector*
assumes *uniformly-continuous-on s f*
shows *uniformly-continuous-on s* (λ*x*. *c* ∗*R* *f*(*x*))
using *bounded-linear-scaleR-right assms*
by (*rule bounded-linear.uniformly-continuous-on*)

lemma *dist-minus*:
fixes *x y* :: *'a*::*real-normed-vector*
shows *dist* (− *x*) (− *y*) = *dist x y*

unfolding *dist-norm minus-diff-minus norm-minus-cancel* **..**

lemma *uniformly-continuous-on-minus*[*continuous-intros*]:
fixes *f* :: *'a*::*metric-space* ⇒ *'b*::*real-normed-vector*
shows *uniformly-continuous-on s f* ⟹ *uniformly-continuous-on s* (λ*x*. − *f x*)
unfolding *uniformly-continuous-on-def dist-minus* **.**

lemma *uniformly-continuous-on-add*[*continuous-intros*]:
fixes *f g* :: *'a*::*metric-space* ⇒ *'b*::*real-normed-vector*
assumes *uniformly-continuous-on s f*
and *uniformly-continuous-on s g*
shows *uniformly-continuous-on s* (λ*x*. *f x* + *g x*)
using *assms*
unfolding *uniformly-continuous-on-sequentially*
unfolding *dist-norm tendsto-norm-zero-iff add-diff-add*
by (*auto intro*: *tendsto-add-zero*)

lemma *uniformly-continuous-on-diff*[*continuous-intros*]:
fixes *f* :: *'a*::*metric-space* ⇒ *'b*::*real-normed-vector*
assumes *uniformly-continuous-on s f*
and *uniformly-continuous-on s g*
shows *uniformly-continuous-on s* (λ*x*. *f x* − *g x*)
using *assms uniformly-continuous-on-add* [*of s f* − *g*]
by (*simp add*: *fun-Compl-def uniformly-continuous-on-minus*)

lemmas *continuous-at-compose* = *isCont-o*

Continuity in terms of open preimages.

lemma *continuous-at-open*:
continuous (*at x*) *f* ⟷ (∀ *t*. *open t* ∧ *f x* ∈ *t* −−> (∃ *s*. *open s* ∧ *x* ∈ *s* ∧ (∀ *x'* ∈ *s*. (*f x'*) ∈ *t*)))
unfolding *continuous-within-topological* [*of x UNIV f*]
unfolding *imp-conjL*
by (*intro all-cong imp-cong ex-cong conj-cong refl*) *auto*

lemma *continuous-imp-tendsto*:
assumes *continuous* (*at x0*) *f*
and *x* ⟶ *x0*
shows (*f* ∘ *x*) ⟶ (*f x0*)
proof (*rule topological-tendstoI*)
fix *S*
assume *open S f x0* ∈ *S*
then obtain *T* **where** *T-def*: *open T x0* ∈ *T* ∀ *x*∈*T*. *f x* ∈ *S*
using *assms continuous-at-open* **by** *metis*
then have *eventually* (λ*n*. *x n* ∈ *T*) *sequentially*
using *assms T-def* **by** (*auto simp*: *tendsto-def*)
then show *eventually* (λ*n*. (*f* ∘ *x*) *n* ∈ *S*) *sequentially*
using *T-def* **by** (*auto elim*!: *eventually-mono*)
qed

lemma *continuous-on-open*:
continuous-on s f $\longleftrightarrow$
($\forall t$. *openin* (*subtopology euclidean* (*f* ' *s*)) *t* $\longrightarrow$
openin (*subtopology euclidean s*) $\{x \in s.\ f\ x \in t\}$)
unfolding *continuous-on-open-invariant openin-open Int-def vimage-def Int-commute*
by (*simp add*: *imp-ex imageI conj-commute eq-commute cong*: *conj-cong*)

Similarly in terms of closed sets.

lemma *continuous-on-closed*:
continuous-on s f $\longleftrightarrow$
($\forall t$. *closedin* (*subtopology euclidean* (*f* ' *s*)) *t* $\longrightarrow$
closedin (*subtopology euclidean s*) $\{x \in s.\ f\ x \in t\}$)
unfolding *continuous-on-closed-invariant closedin-closed Int-def vimage-def Int-commute*
by (*simp add*: *imp-ex imageI conj-commute eq-commute cong*: *conj-cong*)

Half-global and completely global cases.

lemma *continuous-openin-preimage*:
assumes *continuous-on s f open t*
shows *openin* (*subtopology euclidean s*) $\{x \in s.\ f\ x \in t\}$
proof −
have ∗: $\forall x.\ x \in s \wedge f\ x \in t \longleftrightarrow x \in s \wedge f\ x \in (t \cap f\ `\ s)$
by *auto*
have *openin* (*subtopology euclidean* (*f* ' *s*)) $(t \cap f\ `\ s)$
using *openin-open-Int*[*of t f* ' *s*, *OF assms*(*2*)] **unfolding** *openin-open* **by** *auto*
then show *?thesis*
using *assms*(*1*)[*unfolded continuous-on-open*, *THEN spec*[**where** *x*=*t* ∩ *f* ' *s*]]
using ∗ **by** *auto*
qed

lemma *continuous-closedin-preimage*:
assumes *continuous-on s f* **and** *closed t*
shows *closedin* (*subtopology euclidean s*) $\{x \in s.\ f\ x \in t\}$
proof −
have ∗: $\forall x.\ x \in s \wedge f\ x \in t \longleftrightarrow x \in s \wedge f\ x \in (t \cap f\ `\ s)$
by *auto*
have *closedin* (*subtopology euclidean* (*f* ' *s*)) $(t \cap f\ `\ s)$
using *closedin-closed-Int*[*of t f* ' *s*, *OF assms*(*2*)] **unfolding** *Int-commute*
by *auto*
then show *?thesis*
using *assms*(*1*)[*unfolded continuous-on-closed*, *THEN spec*[**where** *x*=*t* ∩ *f* ' *s*]]
using ∗ **by** *auto*
qed

lemma *continuous-open-preimage*:
assumes *continuous-on s f*
and *open s*
and *open t*

shows *open* $\{x \in s.\ f\ x \in t\}$
proof−
obtain T **where** T: *open* T $\{x \in s.\ f\ x \in t\} = s \cap T$
using *continuous-openin-preimage*[*OF assms*(*1*,*3*)] **unfolding** *openin-open* **by** *auto*
then show *?thesis*
using *open-Int*[*of s T*, *OF assms*(*2*)] **by** *auto*
qed

lemma *continuous-closed-preimage*:
assumes *continuous-on s f*
and *closed s*
and *closed t*
shows *closed* $\{x \in s.\ f\ x \in t\}$
proof−
obtain T **where** *closed* T $\{x \in s.\ f\ x \in t\} = s \cap T$
using *continuous-closedin-preimage*[*OF assms*(*1*,*3*)]
unfolding *closedin-closed* **by** *auto*
then show *?thesis* **using** *closed-Int*[*of s T*, *OF assms*(*2*)] **by** *auto*
qed

lemma *continuous-open-preimage-univ*:
$\forall x.$ *continuous* (*at x*) $f \Longrightarrow$ *open* $s \Longrightarrow$ *open* $\{x.\ f\ x \in s\}$
using *continuous-open-preimage*[*of UNIV f s*] *open-UNIV continuous-at-imp-continuous-on*
by *auto*

lemma *continuous-closed-preimage-univ*:
$(\forall x.$ *continuous* (*at x*) $f) \Longrightarrow$ *closed* $s \Longrightarrow$ *closed* $\{x.\ f\ x \in s\}$
using *continuous-closed-preimage*[*of UNIV f s*] *closed-UNIV continuous-at-imp-continuous-on*
by *auto*

lemma *continuous-open-vimage*: $\forall x.$ *continuous* (*at x*) $f \Longrightarrow$ *open* $s \Longrightarrow$ *open* (f $-`$ s)
unfolding *vimage-def* **by** (*rule continuous-open-preimage-univ*)

lemma *continuous-closed-vimage*: $\forall x.$ *continuous* (*at x*) $f \Longrightarrow$ *closed* $s \Longrightarrow$ *closed* (f $-`$ s)
unfolding *vimage-def* **by** (*rule continuous-closed-preimage-univ*)

lemma *interior-image-subset*:
assumes $\forall x.$ *continuous* (*at x*) f
and *inj f*
shows *interior* (f ` $s$) $\subseteq$ $f$ ` (*interior s*)
proof
fix x **assume** $x \in$ *interior* (f ` s)
then obtain T **where** *as*: *open* T $x \in T$ $T \subseteq f$ ` s **..**
then have $x \in f$ ` s **by** *auto*
then obtain y **where** y: $y \in s$ $x = f\ y$ **by** *auto*
have *open* (*vimage f T*)

using *assms(1)* ‹*open T*› **by** (*rule continuous-open-vimage*)
moreover have $y \in$ *vimage f T*
using ‹$x = f\ y$› ‹$x \in T$› **by** *simp*
moreover have *vimage f T* $\subseteq s$
using ‹$T \subseteq$ *image f s*› ‹*inj f*› **unfolding** *inj-on-def subset-eq* **by** *auto*
ultimately have $y \in$ *interior s* **..**
with ‹$x = f\ y$› **show** $x \in f$ ‘ *interior s* **..**
qed

Equality of continuous functions on closure and related results.

lemma *continuous-closedin-preimage-constant*:
fixes f :: - ⇒ *'b*::*t1-space*
shows *continuous-on s f* ⟹ *closedin* (*subtopology euclidean s*) $\{x \in s.\ f\ x = a\}$
using *continuous-closedin-preimage*[*of s f* $\{a\}$] **by** *auto*

lemma *continuous-closed-preimage-constant*:
fixes f :: - ⇒ *'b*::*t1-space*
shows *continuous-on s f* ⟹ *closed s* ⟹ *closed* $\{x \in s.\ f\ x = a\}$
using *continuous-closed-preimage*[*of s f* $\{a\}$] **by** *auto*

lemma *continuous-constant-on-closure*:
fixes f :: - ⇒ *'b*::*t1-space*
assumes *continuous-on* (*closure S*) *f*
and $\bigwedge x.\ x \in S \Longrightarrow f\ x = a$
and $x \in$ *closure S*
shows $f\ x = a$
using *continuous-closed-preimage-constant*[*of closure S f a*]
assms closure-minimal[*of S* $\{x \in$ *closure S*. $f\ x = a\}$] *closure-subset*
unfolding *subset-eq*
by *auto*

lemma *image-closure-subset*:
assumes *continuous-on* (*closure s*) *f*
and *closed t*
and (f ‘ s) $\subseteq t$
shows f ‘ (*closure s*) $\subseteq t$
proof −
have $s \subseteq \{x \in$ *closure s*. $f\ x \in t\}$
using *assms(3) closure-subset* **by** *auto*
moreover have *closed* $\{x \in$ *closure s*. $f\ x \in t\}$
using *continuous-closed-preimage*[*OF assms(1)*] **and** *assms(2)* **by** *auto*
ultimately have *closure s* = $\{x \in$ *closure s* . $f\ x \in t\}$
using *closure-minimal*[*of s* $\{x \in$ *closure s*. $f\ x \in t\}$] **by** *auto*
then show *?thesis* **by** *auto*
qed

lemma *continuous-on-closure-norm-le*:
fixes f :: *'a*::*metric-space* ⇒ *'b*::*real-normed-vector*
assumes *continuous-on* (*closure s*) *f*

```
    and ∀ y ∈ s. norm(f y) ≤ b
    and x ∈ (closure s)
  shows norm (f x) ≤ b
proof −
  have ∗: f ‘ s ⊆ cball 0 b
    using assms(2)[unfolded mem-cball-0[symmetric]] by auto
  show ?thesis
    using image-closure-subset[OF assms(1) closed-cball[of 0 b] ∗] assms(3)
    unfolding subset-eq
    apply (erule-tac x=f x in ballE)
    apply (auto simp add: dist-norm)
    done
qed

lemma isCont-indicator:
  fixes x :: 'a::t2-space
  shows isCont (indicator A :: 'a ⇒ real) x = (x ∉ frontier A)
proof auto
  fix x
  assume cts-at: isCont (indicator A :: 'a ⇒ real) x and fr: x ∈ frontier A
  with continuous-at-open have 1: ∀ V::real set. open V ∧ indicator A x ∈ V ⟶
    (∃ U::'a set. open U ∧ x ∈ U ∧ (∀ y∈U. indicator A y ∈ V)) by auto
  show False
  proof (cases x ∈ A)
    assume x: x ∈ A
    hence indicator A x ∈ ({0<..<2} :: real set) by simp
    hence ∃ U. open U ∧ x ∈ U ∧ (∀ y∈U. indicator A y ∈ ({0<..<2} :: real set))
      using 1 open-greaterThanLessThan by blast
    then guess U .. note U = this
    hence ∀ y∈U. indicator A y > (0::real)
      unfolding greaterThanLessThan-def by auto
    hence U ⊆ A using indicator-eq-0-iff by force
    hence x ∈ interior A using U interiorI by auto
    thus ?thesis using fr unfolding frontier-def by simp
  next
    assume x: x ∉ A
    hence indicator A x ∈ ({−1<..<1} :: real set) by simp
    hence ∃ U. open U ∧ x ∈ U ∧ (∀ y∈U. indicator A y ∈ ({−1<..<1} :: real
set))
      using 1 open-greaterThanLessThan by blast
    then guess U .. note U = this
    hence ∀ y∈U. indicator A y < (1::real)
      unfolding greaterThanLessThan-def by auto
    hence U ⊆ −A by auto
    hence x ∈ interior (−A) using U interiorI by auto
    thus ?thesis using fr interior-complement unfolding frontier-def by auto
  qed
next
  assume nfr: x ∉ frontier A
```

hence $x \in$ *interior* $A \vee x \in$ *interior* $(-A)$
by (*auto simp*: *frontier-def closure-interior*)
thus *isCont* ((*indicator* A)::$'a \Rightarrow$ *real*) x
proof
assume *int*: $x \in$ *interior* A
hence $\exists\, U.$ *open* $U \wedge x \in U \wedge U \subseteq A$ **unfolding** *interior-def* **by** *auto*
then guess U **.. note** $U =$ *this*
hence $\forall\, y{\in}U.$ *indicator* $A\ y = (1$::*real*) **unfolding** *indicator-def* **by** *auto*
hence *continuous-on* U (*indicator* A) **by** (*simp add*: *continuous-on-const indicator-eq-1-iff*)
thus *?thesis* **using** U *continuous-on-eq-continuous-at* **by** *auto*
next
assume *ext*: $x \in$ *interior* $(-A)$
hence $\exists\, U.$ *open* $U \wedge x \in U \wedge U \subseteq -A$ **unfolding** *interior-def* **by** *auto*
then guess U **.. note** $U =$ *this*
hence $\forall\, y{\in}U.$ *indicator* $A\ y = (0$::*real*) **unfolding** *indicator-def* **by** *auto*
hence *continuous-on* U (*indicator* A) **by** (*smt* U *continuous-on-topological indicator-def*)
thus *?thesis* **using** U *continuous-on-eq-continuous-at* **by** *auto*
qed
qed

16.24 Theorems relating continuity and uniform continuity to closures

lemma *continuous-on-closure*:
continuous-on (*closure* S) $f \longleftrightarrow$
$(\forall\, x\ e.\ x \in$ *closure* $S \wedge 0 < e$
$\longrightarrow (\exists\, d.\ 0 < d \wedge (\forall\, y.\ y \in S \wedge$ *dist* $y\ x < d \longrightarrow$ *dist* $(f\ y)\ (f\ x) < e)))$
(**is** *?lhs* $=$ *?rhs*)
proof
assume *?lhs* **then show** *?rhs*
unfolding *continuous-on-iff* **by** (*metis Un-iff closure-def*)
next
assume R [*rule-format*]: *?rhs*
show *?lhs*
proof
fix x **and** e::*real*
assume $0 < e$ **and** x: $x \in$ *closure* S
obtain δ::*real* **where** $\delta > 0$
and δ: $\bigwedge y.\ [\![y \in S;$ *dist* $y\ x < \delta]\!] \Longrightarrow$ *dist* $(f\ y)\ (f\ x) < e/2$
using R [*of* $x\ e/2$] ‹$0 < e$› x **by** *auto*
have *dist* $(f\ y)\ (f\ x) \leq e$ **if** y: $y \in$ *closure* S **and** *dyx*: *dist* $y\ x < \delta/2$ **for** y
proof −
obtain δ'::*real* **where** $\delta' > 0$
and δ': $\bigwedge z.\ [\![z \in S;$ *dist* $z\ y < \delta']\!] \Longrightarrow$ *dist* $(f\ z)\ (f\ y) < e/2$
using R [*of* $y\ e/2$] ‹$0 < e$› y **by** *auto*
obtain z **where** $z \in S$ **and** z: *dist* $z\ y <$ *min* $\delta'\ \delta\ /\ 2$
using *closure-approachable* y

by (*metis* ‹$0 < \delta'$› ‹$0 < \delta$› *divide-pos-pos min-less-iff-conj zero-less-numeral*)
have *dist* (*f z*) (*f y*) < *e*/2
apply (*rule* δ' [*OF* ‹$z \in S$›])
using *z* ‹$0 < \delta'$› **by** *linarith*
moreover have *dist* (*f z*) (*f x*) < *e*/2
apply (*rule* δ [*OF* ‹$z \in S$›])
using *z* ‹$0 < \delta$› *dist-commute*[*of y z*] *dist-triangle-half-r* [*of y*] *dyx* **by** *auto*

ultimately show *?thesis*
by (*metis dist-commute dist-triangle-half-l less-imp-le*)
qed
then show $\exists d{>}0.\ \forall x'{\in}$*closure S*. *dist x' x* < *d* ⟶ *dist* (*f x'*) (*f x*) ≤ *e*
by (*rule-tac x*=$\delta/2$ **in** *exI*) (*simp add*: ‹$\delta > 0$›)
qed
qed

lemma *continuous-on-closure-sequentially*:
fixes *f* :: *'a*::*metric-space* ⇒ *'b* :: *metric-space*
shows
continuous-on (*closure S*) *f* ⟷
($\forall x\ a.\ a \in$ *closure S* ∧ ($\forall n.\ x\ n \in S$) ∧ *x* ⟶ *a* ⟶ (*f* ∘ *x*) ⟶ *f a*)
(**is** *?lhs* = *?rhs*)
proof −
have *continuous-on* (*closure S*) *f* ⟷
($\forall x \in$ *closure S*. *continuous* (*at x within S*) *f*)
by (*force simp*: *continuous-on-closure Topology-Euclidean-Space.continuous-within-eps-delta*)
also have ... = *?rhs*
by (*force simp*: *continuous-within-sequentially*)
finally show *?thesis* **.**
qed

lemma *uniformly-continuous-on-closure*:
fixes *f* :: *'a*::*metric-space* ⇒ *'b*::*metric-space*
assumes *ucont*: *uniformly-continuous-on S f*
and *cont*: *continuous-on* (*closure S*) *f*
shows *uniformly-continuous-on* (*closure S*) *f*
unfolding *uniformly-continuous-on-def*
proof (*intro allI impI*)
fix *e*::*real*
assume *0* < *e*
then obtain *d*::*real*
where *d*>*0*
and *d*: $\bigwedge x\ x'.\ [\![x{\in}S;\ x'{\in}S;\ dist\ x'\ x < d]\!] \Longrightarrow dist\ (f\ x')\ (f\ x) < e/3$
using *ucont* [*unfolded uniformly-continuous-on-def*, *rule-format*, *of e*/*3*] **by** *auto*
show $\exists d{>}0.\ \forall x{\in}$*closure S*. $\forall x'{\in}$*closure S*. *dist x' x* < *d* ⟶ *dist* (*f x'*) (*f x*) < *e*
proof (*rule exI* [**where** *x*=*d*/*3*], *clarsimp simp*: ‹$d > 0$›)
fix *x y*

```
    assume x: x ∈ closure S and y: y ∈ closure S and dyx: dist y x * 3 < d
    obtain d1::real where d1 > 0
          and d1: ⋀w. ⟦w ∈ closure S; dist w x < d1⟧ ⟹ dist (f w) (f x) < e/3
      using cont [unfolded continuous-on-iff, rule-format, of x e/3] ‹0 < e› x by
auto
     obtain x' where x' ∈ S and x': dist x' x < min d1 (d / 3)
       using closure-approachable [of x S]
      by (metis ‹0 < d1› ‹0 < d› divide-pos-pos min-less-iff-conj x zero-less-numeral)

    obtain d2::real where d2 > 0
          and d2: ∀ w ∈ closure S. dist w y < d2 ⟶ dist (f w) (f y) < e/3
      using cont [unfolded continuous-on-iff, rule-format, of y e/3] ‹0 < e› y by
auto
     obtain y' where y' ∈ S and y': dist y' y < min d2 (d / 3)
       using closure-approachable [of y S]
      by (metis ‹0 < d2› ‹0 < d› divide-pos-pos min-less-iff-conj y zero-less-numeral)
     have dist x' x < d/3 using x' by auto
     moreover have dist x y < d/3
       by (metis dist-commute dyx less-divide-eq-numeral1(1))
     moreover have dist y y' < d/3
       by (metis (no-types) dist-commute min-less-iff-conj y')
     ultimately have dist x' y' < d/3 + d/3 + d/3
       by (meson dist-commute-lessI dist-triangle-lt add-strict-mono)
     then have dist x' y' < d by simp
     then have dist (f x') (f y') < e/3
       by (rule d [OF ‹y' ∈ S› ‹x' ∈ S›])
     moreover have dist (f x') (f x) < e/3 using ‹x' ∈ S› closure-subset x' d1
       by (simp add: closure-def)
     moreover have dist (f y') (f y) < e/3 using ‹y' ∈ S› closure-subset y' d2
       by (simp add: closure-def)
     ultimately have dist (f y) (f x) < e/3 + e/3 + e/3
       by (meson dist-commute-lessI dist-triangle-lt add-strict-mono)
    then show dist (f y) (f x) < e by simp
  qed
qed
```

16.25 Quotient maps

```
lemma quotient-map-imp-continuous-open:
  assumes t: f ‘ s ⊆ t
      and ope: ⋀u. u ⊆ t
            ⟹ (openin (subtopology euclidean s) {x. x ∈ s ∧ f x ∈ u} ⟷
                openin (subtopology euclidean t) u)
    shows continuous-on s f
proof −
  have [simp]: {x ∈ s. f x ∈ f ‘ s} = s by auto
  show ?thesis
    using ope [OF t]
    apply (simp add: continuous-on-open)
```

by (*metis* (*no-types*, *lifting*) *ope openin-imp-subset openin-trans*)
qed

lemma *quotient-map-imp-continuous-closed*:
assumes t: f ‘ $s \subseteq t$
and *ope*: $\bigwedge u.\ u \subseteq t$
$\Longrightarrow$ (*closedin* (*subtopology euclidean s*) $\{x.\ x \in s \wedge f\,x \in u\} \longleftrightarrow$
closedin (*subtopology euclidean t*) u)
shows *continuous-on s f*
proof −
have [*simp*]: $\{x \in s.\ f\,x \in f$ ‘ $s\} = s$ **by** *auto*
show *?thesis*
using *ope* [*OF t*]
apply (*simp add*: *continuous-on-closed*)
by (*metis* (*no-types*, *lifting*) *ope closedin-imp-subset closedin-subtopology-refl closedin-trans openin-subtopology-refl openin-subtopology-self*)
qed

lemma *open-map-imp-quotient-map*:
assumes *contf*: *continuous-on s f*
and t: $t \subseteq f$ ‘ s
and *ope*: $\bigwedge t.$ *openin* (*subtopology euclidean s*) t
$\Longrightarrow$ *openin* (*subtopology euclidean* (f ‘ s)) (f ‘ t)
shows *openin* (*subtopology euclidean s*) $\{x \in s.\ f\,x \in t\} =$
openin (*subtopology euclidean* (f ‘ s)) t
proof −
have $t =$ *image f* $\{x.\ x \in s \wedge f\,x \in t\}$
using t **by** *blast*
then show *?thesis*
using *ope contf continuous-on-open* **by** *fastforce*
qed

lemma *closed-map-imp-quotient-map*:
assumes *contf*: *continuous-on s f*
and t: $t \subseteq f$ ‘ s
and *ope*: $\bigwedge t.$ *closedin* (*subtopology euclidean s*) t
$\Longrightarrow$ *closedin* (*subtopology euclidean* (f ‘ s)) (f ‘ t)
shows *openin* (*subtopology euclidean s*) $\{x \in s.\ f\,x \in t\} \longleftrightarrow$
openin (*subtopology euclidean* (f ‘ s)) t
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then have $*$: *closedin* (*subtopology euclidean s*) ($s - \{x \in s.\ f\,x \in t\}$)
using *closedin-diff* **by** *fastforce*
have [*simp*]: (f ‘ $s - f$ ‘ ($s - \{x \in s.\ f\,x \in t\}$)) = t
using t **by** *blast*
show *?rhs*
using *ope* [*OF* $*$, *unfolded closedin-def*] **by** *auto*
next

assume *?rhs*
with *contf* **show** *?lhs*
by (*auto simp*: *continuous-on-open*)
qed

lemma *continuous-right-inverse-imp-quotient-map*:
assumes *contf*: *continuous-on s f* **and** *imf*: f ' $s \subseteq t$
and *contg*: *continuous-on t g* **and** *img*: g ' $t \subseteq s$
and *fg* [*simp*]: $\bigwedge y.\ y \in t \Longrightarrow f(g\ y) = y$
and *u*: $u \subseteq t$
shows *openin* (*subtopology euclidean s*) $\{x.\ x \in s \wedge f\ x \in u\} \longleftrightarrow$
openin (*subtopology euclidean t*) *u*
(**is** *?lhs* = *?rhs*)
proof −
have *f*: $\bigwedge z.$ *openin* (*subtopology euclidean* (f ' s)) $z \Longrightarrow$
openin (*subtopology euclidean s*) $\{x \in s.\ f\ x \in z\}$
and *g*: $\bigwedge z.$ *openin* (*subtopology euclidean* (g ' t)) $z \Longrightarrow$
openin (*subtopology euclidean t*) $\{x \in t.\ g\ x \in z\}$
using *contf contg* **by** (*auto simp*: *continuous-on-open*)
show *?thesis*
proof
have $\{x \in t.\ g\ x \in g$ ' $t \wedge g\ x \in s \wedge f\ (g\ x) \in u\} = \{x \in t.\ f\ (g\ x) \in u\}$
using *imf img* **by** *blast*
also have ... = *u*
using *u* **by** *auto*
finally have [*simp*]: $\{x \in t.\ g\ x \in g$ ' $t \wedge g\ x \in s \wedge f\ (g\ x) \in u\} = u$.
assume *?lhs*
then have $*$: *openin* (*subtopology euclidean* (g ' t)) (g ' $t \cap \{x \in s.\ f\ x \in u\}$)
by (*meson img openin-Int openin-subtopology-Int-subset openin-subtopology-self*)
show *?rhs*
using *g* [*OF* $*$] **by** *simp*
next
assume *rhs*: *?rhs*
show *?lhs*
apply (*rule f*)
by (*metis fg image-eqI image-subset-iff imf img openin-subopen openin-subtopology-self openin-trans rhs*)
qed
qed

lemma *continuous-left-inverse-imp-quotient-map*:
assumes *continuous-on s f*
and *continuous-on* (f ' s) *g*
and $\bigwedge x.\ x \in s \Longrightarrow g(f\ x) = x$
and $u \subseteq f$ ' s
shows *openin* (*subtopology euclidean s*) $\{x.\ x \in s \wedge f\ x \in u\} \longleftrightarrow$
openin (*subtopology euclidean* (f ' s)) *u*
apply (*rule continuous-right-inverse-imp-quotient-map*)
using *assms*

apply *force+*
done

16.26 A function constant on a set

definition *constant-on* (**infixl** (*constant'-on*) *50*)
  where f *constant-on* $A \equiv \exists\, y.\ \forall\, x{\in}A.\ f\ x = y$

lemma *constant-on-subset*: $[\![f$ *constant-on* $A;\ B \subseteq A]\!] \Longrightarrow f$ *constant-on* B
  unfolding *constant-on-def* **by** *blast*

lemma *injective-not-constant*:
  fixes S :: *'a*::{*perfect-space*} *set*
  shows $[\![$*open* S; *inj-on* $f\ S$; f *constant-on* $S]\!] \Longrightarrow S = \{\}$
unfolding *constant-on-def*
by (*metis equals0I inj-on-contraD islimpt-UNIV islimpt-def*)

lemma *constant-on-closureI*:
  fixes f :: - ⇒ *'b*::*t1-space*
  assumes *cof*: f *constant-on* S **and** *contf*: *continuous-on* (*closure* S) f
    shows f *constant-on* (*closure* S)
using *continuous-constant-on-closure* [*OF contf*] *cof* **unfolding** *constant-on-def*
by *metis*

Making a continuous function avoid some value in a neighbourhood.

lemma *continuous-within-avoid*:
  fixes f :: *'a*::*metric-space* ⇒ *'b*::*t1-space*
  assumes *continuous* (*at* x *within* s) f
    and $f\ x \neq a$
  shows $\exists\, e{>}0.\ \forall\, y \in s.$ *dist* $x\ y < e$ −−> $f\ y \neq a$
proof −
  obtain U **where** *open* U **and** $f\ x \in U$ **and** $a \notin U$
    using *t1-space* [*OF* ‹$f\ x \neq a$›] **by** *fast*
  have (f ⟶ $f\ x$) (*at* x *within* s)
    using *assms*(*1*) **by** (*simp add*: *continuous-within*)
  then have *eventually* (λy. $f\ y \in U$) (*at* x *within* s)
    using ‹*open* U› **and** ‹$f\ x \in U$›
    unfolding *tendsto-def* **by** *fast*
  then have *eventually* (λy. $f\ y \neq a$) (*at* x *within* s)
    using ‹$a \notin U$› **by** (*fast elim*: *eventually-mono*)
  then show *?thesis*
    using ‹$f\ x \neq a$› **by** (*auto simp*: *dist-commute zero-less-dist-iff eventually-at*)
qed

lemma *continuous-at-avoid*:
  fixes f :: *'a*::*metric-space* ⇒ *'b*::*t1-space*
  assumes *continuous* (*at* x) f
    and $f\ x \neq a$
  shows $\exists\, e{>}0.\ \forall\, y.$ *dist* $x\ y < e \longrightarrow f\ y \neq a$

using *assms continuous-within-avoid*[*of x UNIV f a*] **by** *simp*

lemma *continuous-on-avoid*:
fixes f :: *'a*::*metric-space* ⇒ *'b*::*t1-space*
assumes *continuous-on s f*
and $x \in s$
and $f\,x \neq a$
shows $\exists\, e{>}0.\ \forall\, y \in s.\ dist\ x\ y < e \longrightarrow f\ y \neq a$
using *assms*(*1*)[*unfolded continuous-on-eq-continuous-within*, *THEN bspec*[**where** *x*=*x*],
OF assms(*2*)] *continuous-within-avoid*[*of x s f a*]
using *assms*(*3*)
by *auto*

lemma *continuous-on-open-avoid*:
fixes f :: *'a*::*metric-space* ⇒ *'b*::*t1-space*
assumes *continuous-on s f*
and *open s*
and $x \in s$
and $f\,x \neq a$
shows $\exists\, e{>}0.\ \forall\, y.\ dist\ x\ y < e \longrightarrow f\ y \neq a$
using *assms*(*1*)[*unfolded continuous-on-eq-continuous-at*[*OF assms*(*2*)], *THEN bspec*[**where** *x*=*x*], *OF assms*(*3*)]
using *continuous-at-avoid*[*of x f a*] *assms*(*4*)
by *auto*

Proving a function is constant by proving open-ness of level set.

lemma *continuous-levelset-openin-cases*:
fixes f :: - ⇒ *'b*::*t1-space*
shows *connected s* ⟹ *continuous-on s f* ⟹
openin (*subtopology euclidean s*) $\{x \in s.\ f\,x = a\}$
⟹ $(\forall\, x \in s.\ f\,x \neq a) \vee (\forall\, x \in s.\ f\,x = a)$
unfolding *connected-clopen*
using *continuous-closedin-preimage-constant* **by** *auto*

lemma *continuous-levelset-openin*:
fixes f :: - ⇒ *'b*::*t1-space*
shows *connected s* ⟹ *continuous-on s f* ⟹
openin (*subtopology euclidean s*) $\{x \in s.\ f\,x = a\}$ ⟹
$(\exists\, x \in s.\ f\,x = a) \implies (\forall\, x \in s.\ f\,x = a)$
using *continuous-levelset-openin-cases*[*of s f*]
by *meson*

lemma *continuous-levelset-open*:
fixes f :: - ⇒ *'b*::*t1-space*
assumes *connected s*
and *continuous-on s f*
and *open* $\{x \in s.\ f\,x = a\}$
and $\exists\, x \in s.\ \ f\,x = a$

shows $\forall x \in s.\ f\ x = a$
using *continuous-levelset-openin*[*OF assms*(*1*,*2*), *of a*, *unfolded openin-open*]
using *assms* (*3*,*4*)
by *fast*

Some arithmetical combinations (more to prove).

lemma *open-scaling*[*intro*]:
fixes s :: *'a*::*real-normed-vector set*
assumes $c \neq 0$
and *open s*
shows *open*$((\lambda x.\ c *_R x)\ `\ s)$
proof −
{
fix x
assume $x \in s$
then obtain e **where** $e>0$
and e:$\forall x'.$ *dist* $x'\ x < e \longrightarrow x' \in s$ **using** *assms*(*2*)[*unfolded open-dist*, *THEN bspec*[**where** x=x]]
by *auto*
have $e * |c| > 0$
using *assms*(*1*)[*unfolded zero-less-abs-iff*[*symmetric*]] ‹$e>0$› **by** *auto*
moreover
{
fix y
assume *dist* $y\ (c *_R x) < e * |c|$
then have *norm* $((1\ /\ c) *_R y - x) < e$
unfolding *dist-norm*
using *norm-scaleR*[*of c* $(1\ /\ c) *_R y - x$, *unfolded scaleR-right-diff-distrib*, *unfolded scaleR-scaleR*] *assms*(*1*)
assms(*1*)[*unfolded zero-less-abs-iff*[*symmetric*]] **by** (*simp del*:*zero-less-abs-iff*)
then have $y \in op *_R c\ `\ s$
using *rev-image-eqI*[*of* $(1\ /\ c) *_R y$ *s y op* $*_R$ *c*]
using *e*[*THEN spec*[**where** x=$(1\ /\ c) *_R y$]]
using *assms*(*1*)
unfolding *dist-norm scaleR-scaleR*
by *auto*
}
ultimately have $\exists e>0.\ \forall x'.$ *dist* $x'\ (c *_R x) < e \longrightarrow x' \in op *_R c\ `\ s$
apply (*rule-tac* x=$e * |c|$ **in** *exI*)
apply *auto*
done
}
then show *?thesis* **unfolding** *open-dist* **by** *auto*
qed

lemma *minus-image-eq-vimage*:
fixes A :: *'a*::*ab-group-add set*
shows $(\lambda x.\ -\ x)\ `\ A = (\lambda x.\ -\ x) -`\ A$
by (*auto intro*!: *image-eqI* [**where** f=$\lambda x.\ -\ x$])

lemma *open-negations*:
fixes $s :: 'a::real\text{-}normed\text{-}vector\ set$
shows $open\ s \Longrightarrow open\ ((\lambda x.\ -\ x)\ `\ s)$
using *open-scaling* [*of* $-\ 1\ s$] **by** *simp*

lemma *open-translation*:
fixes $s :: 'a::real\text{-}normed\text{-}vector\ set$
assumes *open s*
shows $open((\lambda x.\ a\ +\ x)\ `\ s)$
proof −
{
fix *x*
have $continuous\ (at\ x)\ (\lambda x.\ x\ -\ a)$
by (*intro continuous-diff continuous-ident continuous-const*)
}
moreover have $\{x.\ x\ -\ a \in s\} = op\ +\ a\ `\ s$
by *force*
ultimately show *?thesis* **using** *continuous-open-preimage-univ*[*of* $\lambda x.\ x\ -\ a\ s$]
using *assms* **by** *auto*
qed

lemma *open-affinity*:
fixes $s :: 'a::real\text{-}normed\text{-}vector\ set$
assumes *open s* $c \neq 0$
shows $open\ ((\lambda x.\ a\ +\ c\ *_R\ x)\ `\ s)$
proof −
have $*$: $(\lambda x.\ a\ +\ c\ *_R\ x) = (\lambda x.\ a\ +\ x) \circ (\lambda x.\ c\ *_R\ x)$
unfolding *o-def* **..**
have $op\ +\ a\ `\ op\ *_R\ c\ `\ s = (op\ +\ a \circ op\ *_R\ c)\ `\ s$
by *auto*
then show *?thesis*
using *assms open-translation*[*of* $op\ *_R\ c\ `\ s\ a$]
unfolding $*$
by *auto*
qed

lemma *interior-translation*:
fixes $s :: 'a::real\text{-}normed\text{-}vector\ set$
shows $interior\ ((\lambda x.\ a\ +\ x)\ `\ s) = (\lambda x.\ a\ +\ x)\ `\ (interior\ s)$
proof (*rule set-eqI*, *rule*)
fix *x*
assume $x \in interior\ (op\ +\ a\ `\ s)$
then obtain *e* **where** $e > 0$ **and** *e*: $ball\ x\ e \subseteq op\ +\ a\ `\ s$
unfolding *mem-interior* **by** *auto*
then have $ball\ (x\ -\ a)\ e \subseteq s$
unfolding *subset-eq Ball-def mem-ball dist-norm*
by (*auto simp add*: *diff-diff-eq*)
then show $x \in op\ +\ a\ `\ interior\ s$

```
    unfolding image-iff
    apply (rule-tac x=x − a in bexI)
    unfolding mem-interior
    using ‹e > 0›
    apply auto
    done
next
  fix x
  assume x ∈ op + a ‘ interior s
  then obtain y e where e > 0 and e: ball y e ⊆ s and y: x = a + y
    unfolding image-iff Bex-def mem-interior by auto
  {
    fix z
    have ∗: a + y − z = y + a − z by auto
    assume z ∈ ball x e
    then have z − a ∈ s
      using e[unfolded subset-eq, THEN bspec[where x=z − a]]
      unfolding mem-ball dist-norm y group-add-class.diff-diff-eq2 ∗
      by auto
    then have z ∈ op + a ‘ s
      unfolding image-iff by (auto intro!: bexI[where x=z − a])
  }
  then have ball x e ⊆ op + a ‘ s
    unfolding subset-eq by auto
  then show x ∈ interior (op + a ‘ s)
    unfolding mem-interior using ‹e > 0› by auto
qed
```

Topological properties of linear functions.

```
lemma linear-lim-0:
  assumes bounded-linear f
  shows (f ⟶ 0) (at (0))
proof −
  interpret f: bounded-linear f by fact
  have (f ⟶ f 0) (at 0)
    using tendsto-ident-at by (rule f.tendsto)
  then show ?thesis unfolding f.zero .
qed

lemma linear-continuous-at:
  assumes bounded-linear f
  shows continuous (at a) f
  unfolding continuous-at using assms
  apply (rule bounded-linear.tendsto)
  apply (rule tendsto-ident-at)
  done

lemma linear-continuous-within:
  bounded-linear f ⟹ continuous (at x within s) f
```

using *continuous-at-imp-continuous-within*[*of x f s*] **using** *linear-continuous-at*[*of f*] **by** *auto*

lemma *linear-continuous-on*:
bounded-linear f ⟹ *continuous-on s f*
using *continuous-at-imp-continuous-on*[*of s f*] **using** *linear-continuous-at*[*of f*]
by *auto*

Also bilinear functions, in composition form.

lemma *bilinear-continuous-at-compose*:
continuous (*at x*) *f* ⟹ *continuous* (*at x*) *g* ⟹ *bounded-bilinear h* ⟹
continuous (*at x*) (λ*x*. *h* (*f x*) (*g x*))
unfolding *continuous-at*
using *Lim-bilinear*[*of f f x* (*at x*) *g g x h*]
by *auto*

lemma *bilinear-continuous-within-compose*:
continuous (*at x within s*) *f* ⟹ *continuous* (*at x within s*) *g* ⟹ *bounded-bilinear h* ⟹
continuous (*at x within s*) (λ*x*. *h* (*f x*) (*g x*))
by (*rule Limits.bounded-bilinear.continuous*)

lemma *bilinear-continuous-on-compose*:
continuous-on s f ⟹ *continuous-on s g* ⟹ *bounded-bilinear h* ⟹
continuous-on s (λ*x*. *h* (*f x*) (*g x*))
by (*rule Limits.bounded-bilinear.continuous-on*)

Preservation of compactness and connectedness under continuous function.

lemma *compact-eq-openin-cover*:
compact S ⟷
(∀ *C*. (∀ *c*∈*C*. *openin* (*subtopology euclidean S*) *c*) ∧ *S* ⊆ ⋃ *C* ⟶
(∃ *D*⊆*C*. *finite D* ∧ *S* ⊆ ⋃ *D*))
proof *safe*
fix *C*
assume *compact S* **and** ∀ *c*∈*C*. *openin* (*subtopology euclidean S*) *c* **and** *S* ⊆ ⋃ *C*
then have ∀ *c*∈{*T*. *open T* ∧ *S* ∩ *T* ∈ *C*}. *open c* **and** *S* ⊆ ⋃{*T*. *open T* ∧ *S* ∩ *T* ∈ *C*}
unfolding *openin-open* **by** *force*+
with ‹*compact S*› **obtain** *D* **where** *D* ⊆ {*T*. *open T* ∧ *S* ∩ *T* ∈ *C*} **and** *finite D* **and** *S* ⊆ ⋃ *D*
by (*rule compactE*)
then have *image* (λ*T*. *S* ∩ *T*) *D* ⊆ *C* ∧ *finite* (*image* (λ*T*. *S* ∩ *T*) *D*) ∧ *S* ⊆ ⋃(*image* (λ*T*. *S* ∩ *T*) *D*)
by *auto*
then show ∃ *D*⊆*C*. *finite D* ∧ *S* ⊆ ⋃ *D* **..**
next
assume *1*: ∀ *C*. (∀ *c*∈*C*. *openin* (*subtopology euclidean S*) *c*) ∧ *S* ⊆ ⋃ *C* ⟶
(∃ *D*⊆*C*. *finite D* ∧ *S* ⊆ ⋃ *D*)
show *compact S*

```
  proof (rule compactI)
    fix C
    let ?C = image (λT. S ∩ T) C
    assume ∀ t∈C. open t and S ⊆ ⋃ C
    then have (∀ c∈?C. openin (subtopology euclidean S) c) ∧ S ⊆ ⋃ ?C
      unfolding openin-open by auto
    with 1 obtain D where D ⊆ ?C and finite D and S ⊆ ⋃ D
      by metis
    let ?D = inv-into C (λT. S ∩ T) ` D
    have ?D ⊆ C ∧ finite ?D ∧ S ⊆ ⋃ ?D
    proof (intro conjI)
      from ‹D ⊆ ?C› show ?D ⊆ C
        by (fast intro: inv-into-into)
      from ‹finite D› show finite ?D
        by (rule finite-imageI)
      from ‹S ⊆ ⋃ D› show S ⊆ ⋃ ?D
        apply (rule subset-trans)
        apply clarsimp
        apply (frule subsetD [OF ‹D ⊆ ?C›, THEN f-inv-into-f])
        apply (erule rev-bexI, fast)
        done
    qed
    then show ∃ D⊆C. finite D ∧ S ⊆ ⋃ D ..
  qed
qed

lemma connected-continuous-image:
  assumes continuous-on s f
    and connected s
  shows connected(f ` s)
proof −
  {
    fix T
    assume as:
      T ≠ {}
      T ≠ f ` s
      openin (subtopology euclidean (f ` s)) T
      closedin (subtopology euclidean (f ` s)) T
    have {x ∈ s. f x ∈ T} = {} ∨ {x ∈ s. f x ∈ T} = s
      using assms(1)[unfolded continuous-on-open, THEN spec[where x=T]]
      using assms(1)[unfolded continuous-on-closed, THEN spec[where x=T]]
      using assms(2)[unfolded connected-clopen, THEN spec[where x={x ∈ s. f x
∈ T}]] as(3,4) by auto
    then have False using as(1,2)
      using as(4)[unfolded closedin-def topspace-euclidean-subtopology] by auto
  }
  then show ?thesis
    unfolding connected-clopen by auto
qed
```

lemma *connected-linear-image*:
fixes f :: $'a$::*euclidean-space* ⇒ $'b$::*real-normed-vector*
assumes *linear* f **and** *connected* s
shows *connected* (f ‘ s)
using *connected-continuous-image assms linear-continuous-on linear-conv-bounded-linear*
by *blast*

Continuity implies uniform continuity on a compact domain.

lemma *compact-uniformly-continuous*:
fixes f :: $'a$ :: *metric-space* ⇒ $'b$ :: *metric-space*
assumes f: *continuous-on* s f
and s: *compact* s
shows *uniformly-continuous-on* s f
unfolding *uniformly-continuous-on-def*
proof (*cases*, *safe*)
fix e :: *real*
assume $0 < e$ $s \neq \{\}$
def [*simp*]: $R \equiv \{(y,\ d).\ y \in s \wedge 0 < d \wedge ball\ y\ d \cap s \subseteq \{x \in s.\ f\ x \in ball\ (f\ y)\ (e/2)\ \}\ \}$
let *?b* = $(\lambda(y,\ d).\ ball\ y\ (d/2))$
have $(\forall r{\in}R.\ open\ (?b\ r))\ s \subseteq (\bigcup r{\in}R.\ ?b\ r)$
proof *safe*
fix y
assume $y \in s$
from *continuous-openin-preimage*[*OF f open-ball*]
obtain T **where** *open* T **and** T: $\{x \in s.\ f\ x \in ball\ (f\ y)\ (e/2)\} = T \cap s$
unfolding *openin-subtopology open-openin* **by** *metis*
then obtain d **where** *ball* y $d \subseteq T$ $0 < d$
using ‹$0 < e$› ‹$y \in s$› **by** (*auto elim*!: *openE*)
with T ‹$y \in s$› **show** $y \in (\bigcup r{\in}R.\ ?b\ r)$
by (*intro UN-I*[*of* $(y,\ d)$]) *auto*
qed *auto*
with s **obtain** D **where** D: *finite* D $D \subseteq R$ $s \subseteq (\bigcup (y,\ d){\in}D.\ ball\ y\ (d/2))$
by (*rule compactE-image*)
with ‹$s \neq \{\}$› **have** [*simp*]: $\bigwedge x.\ x < Min\ (snd\ `\ D) \longleftrightarrow (\forall (y,\ d){\in}D.\ x < d)$
by (*subst Min-gr-iff*) *auto*
show $\exists\, d{>}0.\ \forall x{\in}s.\ \forall x'{\in}s.\ dist\ x'\ x < d \longrightarrow dist\ (f\ x')\ (f\ x) < e$
proof (*rule*, *safe*)
fix x x'
assume *in-s*: $x' \in s$ $x \in s$
with D **obtain** y d **where** x: $x \in ball\ y\ (d/2)$ $(y,\ d) \in D$
by *blast*
moreover assume *dist* x $x' < Min\ (snd`D)\ /\ 2$
ultimately have *dist* y $x' < d$
by (*intro dist-triangle-half-r*[*of* x - d]) (*auto simp*: *dist-commute*)
with D x *in-s* **show** *dist* $(f\ x)$ $(f\ x') < e$
by (*intro dist-triangle-half-r*[*of* f y - e]) (*auto simp*: *dist-commute subset-eq*)
qed (*insert* D, *auto*)

qed *auto*

A uniformly convergent limit of continuous functions is continuous.

lemma *continuous-uniform-limit*:
fixes $f :: 'a \Rightarrow 'b::metric\text{-}space \Rightarrow 'c::metric\text{-}space$
assumes $\neg$ *trivial-limit F*
and *eventually* $(\lambda n.\ continuous\text{-}on\ s\ (f\ n))\ F$
and $\forall e>0.$ *eventually* $(\lambda n.\ \forall x\in s.\ dist\ (f\ n\ x)\ (g\ x) < e)\ F$
shows *continuous-on s g*
proof −
{
fix *x* **and** *e* :: *real*
assume $x\in s\ e>0$
have *eventually* $(\lambda n.\ \forall x\in s.\ dist\ (f\ n\ x)\ (g\ x) < e\ /\ 3)\ F$
using ‹$e>0$› *assms*(3)[*THEN spec*[**where** $x=e/3$]] **by** *auto*
from *eventually-happens* [*OF eventually-conj* [*OF this assms*(2)]]
obtain *n* **where** $n{:}\forall x\in s.\ dist\ (f\ n\ x)\ (g\ x) < e\ /\ 3$ *continuous-on s* $(f\ n)$
using *assms*(1) **by** *blast*
have $e\ /\ 3 > 0$ **using** ‹$e>0$› **by** *auto*
then obtain *d* **where** $d>0$ **and** $d{:}\forall x'\in s.\ dist\ x'\ x < d \longrightarrow dist\ (f\ n\ x')\ (f\ n\ x) < e\ /\ 3$
using *n*(2)[*unfolded continuous-on-iff*, *THEN bspec*[**where** $x=x$], *OF* ‹$x\in s$›, *THEN spec*[**where** $x=e/3$]] **by** *blast*
{
fix *y*
assume $y \in s$ **and** $dist\ y\ x < d$
then have $dist\ (f\ n\ y)\ (f\ n\ x) < e\ /\ 3$
by (*rule d* [*rule-format*])
then have $dist\ (f\ n\ y)\ (g\ x) < 2 * e\ /\ 3$
using *dist-triangle* [*of f n y g x f n x*]
using *n*(1)[*THEN bspec*[**where** $x=x$], *OF* ‹$x\in s$›]
by *auto*
then have $dist\ (g\ y)\ (g\ x) < e$
using *n*(1)[*THEN bspec*[**where** $x=y$], *OF* ‹$y\in s$›]
using *dist-triangle3* [*of g y g x f n y*]
by *auto*
}
then have $\exists d>0.\ \forall x'\in s.\ dist\ x'\ x < d \longrightarrow dist\ (g\ x')\ (g\ x) < e$
using ‹$d>0$› **by** *auto*
}
then show *?thesis*
unfolding *continuous-on-iff* **by** *auto*
qed

16.27 Topological stuff lifted from and dropped to R

lemma *open-real*:
fixes *s* :: *real set*
shows *open s* $\longleftrightarrow$ $(\forall x \in s.\ \exists e>0.\ \forall x'.\ |x' - x| < e \longrightarrow x' \in s)$

unfolding *open-dist dist-norm* **by** *simp*

lemma *islimpt-approachable-real*:
fixes s :: *real set*
shows x *islimpt* $s \longleftrightarrow (\forall e{>}0.\ \exists x'{\in}\ s.\ x' \neq x \wedge |x' - x| < e)$
unfolding *islimpt-approachable dist-norm* **by** *simp*

lemma *closed-real*:
fixes s :: *real set*
shows *closed* $s \longleftrightarrow (\forall x.\ (\forall e{>}0.\ \exists x' \in s.\ x' \neq x \wedge |x' - x| < e) \longrightarrow x \in s)$
unfolding *closed-limpt islimpt-approachable dist-norm* **by** *simp*

lemma *continuous-at-real-range*:
fixes f :: *'a::real-normed-vector* $\Rightarrow$ *real*
shows *continuous* (*at* x) $f \longleftrightarrow (\forall e{>}0.\ \exists d{>}0.\ \forall x'.\ norm(x' - x) < d \longrightarrow |f\ x' - f\ x| < e)$
unfolding *continuous-at*
unfolding *Lim-at*
unfolding *dist-norm*
apply *auto*
apply (*erule-tac* $x{=}e$ **in** *allE*)
apply *auto*
apply (*rule-tac* $x{=}d$ **in** *exI*)
apply *auto*
apply (*erule-tac* $x{=}x'$ **in** *allE*)
apply *auto*
apply (*erule-tac* $x{=}e$ **in** *allE*)
apply *auto*
done

lemma *continuous-on-real-range*:
fixes f :: *'a::real-normed-vector* $\Rightarrow$ *real*
shows *continuous-on* s $f \longleftrightarrow$
$(\forall x \in s.\ \forall e{>}0.\ \exists d{>}0.\ (\forall x' \in s.\ norm(x' - x) < d \longrightarrow |f\ x' - f\ x| < e))$
unfolding *continuous-on-iff dist-norm* **by** *simp*

Hence some handy theorems on distance, diameter etc. of/from a set.

lemma *distance-attains-sup*:
assumes *compact* s $s \neq \{\}$
shows $\exists x{\in}s.\ \forall y{\in}s.\ dist\ a\ y \leq dist\ a\ x$
proof (*rule continuous-attains-sup* [*OF assms*])
{
fix x
assume $x{\in}s$
have (*dist a* $\longrightarrow$ *dist a x*) (*at x within s*)
by (*intro tendsto-dist tendsto-const tendsto-ident-at*)
}
then show *continuous-on s* (*dist a*)
unfolding *continuous-on* **..**

qed

For *minimal* distance, we only need closure, not compactness.

lemma *distance-attains-inf*:
fixes *a* :: *'a*::*heine-borel*
assumes *closed s* **and** $s \neq \{\}$
obtains *x* **where** $x{\in}s$ $\bigwedge y.\ y \in s \Longrightarrow dist\ a\ x \leq dist\ a\ y$
proof −
from *assms* **obtain** *b* **where** $b \in s$ **by** *auto*
let *?B* = $s \cap cball\ a\ (dist\ b\ a)$
have *?B* $\neq \{\}$ **using** ‹$b \in s$›
by (*auto simp*: *dist-commute*)
moreover have *continuous-on ?B* (*dist a*)
by (*auto intro*!: *continuous-at-imp-continuous-on continuous-dist continuous-ident continuous-const*)
moreover have *compact ?B*
by (*intro closed-Int-compact* ‹*closed s*› *compact-cball*)
ultimately obtain *x* **where** $x \in$ *?B* $\forall\, y{\in}$*?B*. *dist a x* $\leq$ *dist a y*
by (*metis continuous-attains-inf*)
with *that* **show** *?thesis* **by** *fastforce*
qed

16.28 Cartesian products

lemma *bounded-Times*:
assumes *bounded s bounded t*
shows *bounded* ($s \times t$)
proof −
obtain *x y a b* **where** $\forall\, z{\in}s$. *dist x z* $\leq a$ $\forall\, z{\in}t$. *dist y z* $\leq b$
using *assms* [*unfolded bounded-def*] **by** *auto*
then have $\forall\, z{\in}s \times t$. *dist* (*x*, *y*) *z* $\leq$ *sqrt* ($a^2 + b^2$)
by (*auto simp add*: *dist-Pair-Pair real-sqrt-le-mono add-mono power-mono*)
then show *?thesis* **unfolding** *bounded-any-center* [**where** *a*=(*x*, *y*)] **by** *auto*
qed

lemma *mem-Times-iff*: $x \in A \times B \longleftrightarrow fst\ x \in A \wedge snd\ x \in B$
by (*induct x*) *simp*

lemma *seq-compact-Times*: *seq-compact s* $\Longrightarrow$ *seq-compact t* $\Longrightarrow$ *seq-compact* ($s \times t$)
unfolding *seq-compact-def*
apply *clarify*
apply (*drule-tac x*=*fst* $\circ$ *f* **in** *spec*)
apply (*drule mp*, *simp add*: *mem-Times-iff*)
apply (*clarify*, *rename-tac l1 r1*)
apply (*drule-tac x*=*snd* $\circ$ *f* $\circ$ *r1* **in** *spec*)
apply (*drule mp*, *simp add*: *mem-Times-iff*)
apply (*clarify*, *rename-tac l2 r2*)
apply (*rule-tac x*=(*l1*, *l2*) **in** *rev-bexI*, *simp*)

apply (*rule-tac x=r1 ∘ r2* **in** *exI*)
apply (*rule conjI*, *simp add*: *subseq-def*)
apply (*drule-tac f=r2* **in** *LIMSEQ-subseq-LIMSEQ*, *assumption*)
apply (*drule* (*1*) *tendsto-Pair*) **back**
apply (*simp add*: *o-def*)
done

lemma *compact-Times*:
assumes *compact s compact t*
shows *compact* (*s* × *t*)
proof (*rule compactI*)
fix *C*
assume *C*: ∀ *t*∈*C*. *open t s* × *t* ⊆ ⋃ *C*
have ∀ *x*∈*s*. ∃ *a*. *open a* ∧ *x* ∈ *a* ∧ (∃ *d*⊆*C*. *finite d* ∧ *a* × *t* ⊆ ⋃ *d*)
proof
fix *x*
assume *x* ∈ *s*
have ∀ *y*∈*t*. ∃ *a b c*. *c* ∈ *C* ∧ *open a* ∧ *open b* ∧ *x* ∈ *a* ∧ *y* ∈ *b* ∧ *a* × *b* ⊆ *c*
(**is** ∀ *y*∈*t*. *?P y*)
proof
fix *y*
assume *y* ∈ *t*
with ‹*x* ∈ *s*› *C* **obtain** *c* **where** *c* ∈ *C* (*x*, *y*) ∈ *c open c* **by** *auto*
then show *?P y* **by** (*auto elim*!: *open-prod-elim*)
qed
then obtain *a b c* **where** *b*: ⋀*y*. *y* ∈ *t* ⟹ *open* (*b y*)
and *c*: ⋀*y*. *y* ∈ *t* ⟹ *c y* ∈ *C* ∧ *open* (*a y*) ∧ *open* (*b y*) ∧ *x* ∈ *a y* ∧ *y* ∈ *b*
y ∧ *a y* × *b y* ⊆ *c y*
by *metis*
then have ∀ *y*∈*t*. *open* (*b y*) *t* ⊆ (⋃ *y*∈*t*. *b y*) **by** *auto*
from *compactE-image*[*OF* ‹*compact t*› *this*] **obtain** *D* **where** *D*: *D* ⊆ *t finite*
D t ⊆ (⋃ *y*∈*D*. *b y*)
by *auto*
moreover from *D c* **have** (⋂ *y*∈*D*. *a y*) × *t* ⊆ (⋃ *y*∈*D*. *c y*)
by (*fastforce simp*: *subset-eq*)
ultimately show ∃ *a*. *open a* ∧ *x* ∈ *a* ∧ (∃ *d*⊆*C*. *finite d* ∧ *a* × *t* ⊆ ⋃ *d*)
using *c* **by** (*intro exI*[*of - c*‘*D*] *exI*[*of - ⋂* (*a*‘*D*)] *conjI*) (*auto intro*!: *open-INT*)
qed
then obtain *a d* **where** *a*: ∀ *x*∈*s*. *open* (*a x*) *s* ⊆ (⋃ *x*∈*s*. *a x*)
and *d*: ⋀*x*. *x* ∈ *s* ⟹ *d x* ⊆ *C* ∧ *finite* (*d x*) ∧ *a x* × *t* ⊆ ⋃ *d x*
unfolding *subset-eq UN-iff* **by** *metis*
moreover
from *compactE-image*[*OF* ‹*compact s*› *a*]
obtain *e* **where** *e*: *e* ⊆ *s finite e* **and** *s*: *s* ⊆ (⋃ *x*∈*e*. *a x*)
by *auto*
moreover
{
from *s* **have** *s* × *t* ⊆ (⋃ *x*∈*e*. *a x* × *t*)
by *auto*

```
    also have ... ⊆ (⋃x∈e. ⋃ d x)
      using d ‹e ⊆ s› by (intro UN-mono) auto
    finally have s × t ⊆ (⋃x∈e. ⋃ d x) .
  }
  ultimately show ∃ C'⊆C. finite C' ∧ s × t ⊆ ⋃ C'
    by (intro exI[of - (⋃x∈e. d x)]) (auto simp add: subset-eq)
qed
```

Hence some useful properties follow quite easily.

```
lemma compact-scaling:
  fixes s :: 'a::real-normed-vector set
  assumes compact s
  shows compact ((λx. c *R x) ` s)
proof −
  let ?f = λx. scaleR c x
  have *: bounded-linear ?f by (rule bounded-linear-scaleR-right)
  show ?thesis
    using compact-continuous-image[of s ?f] continuous-at-imp-continuous-on[of s
?f]
    using linear-continuous-at[OF *] assms
    by auto
qed

lemma compact-negations:
  fixes s :: 'a::real-normed-vector set
  assumes compact s
  shows compact ((λx. − x) ` s)
  using compact-scaling [OF assms, of − 1] by auto

lemma compact-sums:
  fixes s t :: 'a::real-normed-vector set
  assumes compact s
    and compact t
  shows compact {x + y | x y. x ∈ s ∧ y ∈ t}
proof −
  have *: {x + y | x y. x ∈ s ∧ y ∈ t} = (λz. fst z + snd z) ` (s × t)
    apply auto
    unfolding image-iff
    apply (rule-tac x=(xa, y) in bexI)
    apply auto
    done
  have continuous-on (s × t) (λz. fst z + snd z)
    unfolding continuous-on by (rule ballI) (intro tendsto-intros)
  then show ?thesis
    unfolding * using compact-continuous-image compact-Times [OF assms] by
auto
qed

lemma compact-differences:
```

fixes $s\ t :: 'a$::*real-normed-vector set*
assumes *compact s*
and *compact t*
shows *compact* $\{x - y \mid x\ y.\ x \in s \wedge y \in t\}$
proof−
have $\{x - y \mid x\ y.\ x{\in}s \wedge y \in t\} = \{x + y \mid x\ y.\ x \in s \wedge y \in (uminus\ `\ t)\}$
apply *auto*
apply (*rule-tac x*= *xa* **in** *exI*)
apply *auto*
done
then show *?thesis*
using *compact-sums*[*OF assms*(*1*) *compact-negations*[*OF assms*(*2*)]] **by** *auto*
qed

lemma *compact-translation*:
fixes $s :: 'a$::*real-normed-vector set*
assumes *compact s*
shows *compact* $((\lambda x.\ a + x)\ `\ s)$
proof −
have $\{x + y \mid x\ y.\ x \in s \wedge y \in \{a\}\} = (\lambda x.\ a + x)\ `\ s$
by *auto*
then show *?thesis*
using *compact-sums*[*OF assms compact-sing*[*of a*]] **by** *auto*
qed

lemma *compact-affinity*:
fixes $s :: 'a$::*real-normed-vector set*
assumes *compact s*
shows *compact* $((\lambda x.\ a + c *_R x)\ `\ s)$
proof −
have $op + a\ `\ op *_R\ c\ `\ s = (\lambda x.\ a + c *_R x)\ `\ s$
by *auto*
then show *?thesis*
using *compact-translation*[*OF compact-scaling*[*OF assms*], *of a c*] **by** *auto*
qed

Hence we get the following.

lemma *compact-sup-maxdistance*:
fixes $s :: 'a$::*metric-space set*
assumes *compact s*
and $s \neq \{\}$
shows $\exists x{\in}s.\ \exists y{\in}s.\ \forall u{\in}s.\ \forall v{\in}s.\ dist\ u\ v \leq dist\ x\ y$
proof −
have *compact* $(s \times s)$
using ‹*compact s*› **by** (*intro compact-Times*)
moreover have $s \times s \neq \{\}$
using ‹$s \neq \{\}$› **by** *auto*
moreover have *continuous-on* $(s \times s)$ $(\lambda x.\ dist\ (fst\ x)\ (snd\ x))$
by (*intro continuous-at-imp-continuous-on ballI continuous-intros*)

ultimately show *?thesis*
using *continuous-attains-sup*[*of* $s \times s$ $\lambda x.$ *dist* (*fst* x) (*snd* x)] **by** *auto*
qed

We can state this in terms of diameter of a set.

definition *diameter* :: $'a$::*metric-space set* $\Rightarrow$ *real* **where**
diameter S = (*if* $S = \{\}$ *then* 0 *else SUP* (x,y):$S \times S$. *dist* x y)

lemma *diameter-bounded-bound*:
fixes s :: $'a$ :: *metric-space set*
assumes s: *bounded* s $x \in s$ $y \in s$
shows *dist* x $y \leq$ *diameter* s
proof −
from s **obtain** z d **where** z: $\bigwedge x.\ x \in s \Longrightarrow$ *dist* z $x \leq d$
unfolding *bounded-def* **by** *auto*
have *bdd-above* (*case-prod dist* ‘ ($s \times s$))
proof (*intro bdd-aboveI*, *safe*)
fix a b
assume $a \in s$ $b \in s$
with z[*of* a] z[*of* b] *dist-triangle*[*of* a b z]
show *dist* a $b \leq 2 * d$
by (*simp add*: *dist-commute*)
qed
moreover have (x,y) $\in s \times s$ **using** s **by** *auto*
ultimately have *dist* x $y \leq$ (*SUP* (x,y):$s \times s$. *dist* x y)
by (*rule cSUP-upper2*) *simp*
with ‹$x \in s$› **show** *?thesis*
by (*auto simp add*: *diameter-def*)
qed

lemma *diameter-lower-bounded*:
fixes s :: $'a$ :: *metric-space set*
assumes s: *bounded* s
and d: $0 < d$ $d <$ *diameter* s
shows $\exists\, x \in s.\ \exists\, y \in s.\ d <$ *dist* x y
proof (*rule ccontr*)
assume *contr*: ¬ *?thesis*
moreover have $s \neq \{\}$
using d **by** (*auto simp add*: *diameter-def*)
ultimately have *diameter* $s \leq d$
by (*auto simp*: *not-less diameter-def intro*!: *cSUP-least*)
with ‹$d <$ *diameter* s› **show** *False* **by** *auto*
qed

lemma *diameter-bounded*:
assumes *bounded* s
shows $\forall\, x \in s.\ \forall\, y \in s.$ *dist* x $y \leq$ *diameter* s
and $\forall\, d > 0.\ d <$ *diameter* $s \longrightarrow (\exists\, x \in s.\ \exists\, y \in s.$ *dist* x $y > d)$
using *diameter-bounded-bound*[*of* s] *diameter-lower-bounded*[*of* s] *assms*

by *auto*

lemma *diameter-compact-attained*:
assumes *compact s*
and $s \neq \{\}$
shows $\exists x{\in}s.\ \exists y{\in}s.\ dist\ x\ y = diameter\ s$
proof −
have *b*: *bounded s* **using** *assms*(*1*)
by (*rule compact-imp-bounded*)
then obtain *x y* **where** *xys*: $x{\in}s\ y{\in}s$
and *xy*: $\forall u{\in}s.\ \forall v{\in}s.\ dist\ u\ v \leq dist\ x\ y$
using *compact-sup-maxdistance*[*OF assms*] **by** *auto*
then have $diameter\ s \leq dist\ x\ y$
unfolding *diameter-def*
apply *clarsimp*
apply (*rule cSUP-least*)
apply *fast*+
done
then show *?thesis*
by (*metis b diameter-bounded-bound order-antisym xys*)
qed

Related results with closure as the conclusion.

lemma *closed-scaling*:
fixes *s* :: *'a*::*real-normed-vector set*
assumes *closed s*
shows *closed* $((\lambda x.\ c *_R x)\ `\ s)$
proof (*cases* $c = 0$)
case *True* **then show** *?thesis*
by (*auto simp add*: *image-constant-conv*)
next
case *False*
from *assms* **have** *closed* $((\lambda x.\ inverse\ c *_R x) -`\ s)$
by (*simp add*: *continuous-closed-vimage*)
also have $(\lambda x.\ inverse\ c *_R x) -`\ s = (\lambda x.\ c *_R x)\ `\ s$
using ‹$c \neq 0$› **by** (*auto elim*: *image-eqI* [*rotated*])
finally show *?thesis* **.**
qed

lemma *closed-negations*:
fixes *s* :: *'a*::*real-normed-vector set*
assumes *closed s*
shows *closed* $((\lambda x.\ -x)\ `\ s)$
using *closed-scaling*[*OF assms*, *of* $-\ 1$] **by** *simp*

lemma *compact-closed-sums*:
fixes *s* :: *'a*::*real-normed-vector set*
assumes *compact s* **and** *closed t*
shows *closed* $\{x + y \mid x\ y.\ x \in s \wedge y \in t\}$

proof −
let *?S* = {*x* + *y* |*x y*. *x* ∈ *s* ∧ *y* ∈ *t*}
{
fix *x l*
assume *as*: ∀ *n*. *x n* ∈ *?S* (*x* ⟶ *l*) *sequentially*
from *as*(*1*) **obtain** *f* **where** *f*: ∀ *n*. *x n* = *fst* (*f n*) + *snd* (*f n*) ∀ *n*. *fst* (*f n*) ∈ *s* ∀ *n*. *snd* (*f n*) ∈ *t*
using *choice*[*of* λ*n y*. *x n* = (*fst y*) + (*snd y*) ∧ *fst y* ∈ *s* ∧ *snd y* ∈ *t*] **by** *auto*
obtain *l*′ *r* **where** *l*′∈*s* **and** *r*: *subseq r* **and** *lr*: (((λ*n*. *fst* (*f n*)) ∘ *r*) ⟶ *l*′) *sequentially*
using *assms*(*1*)[*unfolded compact-def*, *THEN spec*[**where** *x*=λ *n*. *fst* (*f n*)]]
using *f*(*2*) **by** *auto*
have ((λ*n*. *snd* (*f* (*r n*))) ⟶ *l* − *l*′) *sequentially*
using *tendsto-diff*[*OF LIMSEQ-subseq-LIMSEQ*[*OF as*(*2*) *r*] *lr*] **and** *f*(*1*)
unfolding *o-def*
by *auto*
then have *l* − *l*′ ∈ *t*
using *assms*(*2*)[*unfolded closed-sequential-limits*,
THEN spec[**where** *x*=λ *n*. *snd* (*f* (*r n*))],
THEN spec[**where** *x*=*l* − *l*′]]
using *f*(*3*)
by *auto*
then have *l* ∈ *?S*
using ‹*l*′ ∈ *s*›
apply *auto*
apply (*rule-tac x*=*l*′ **in** *exI*)
apply (*rule-tac x*=*l* − *l*′ **in** *exI*)
apply *auto*
done
}
then show *?thesis*
unfolding *closed-sequential-limits* **by** *fast*
qed

lemma *closed-compact-sums*:
fixes *s t* :: *'a*::*real-normed-vector set*
assumes *closed s*
and *compact t*
shows *closed* {*x* + *y* | *x y*. *x* ∈ *s* ∧ *y* ∈ *t*}
proof −
have {*x* + *y* |*x y*. *x* ∈ *t* ∧ *y* ∈ *s*} = {*x* + *y* |*x y*. *x* ∈ *s* ∧ *y* ∈ *t*}
apply *auto*
apply (*rule-tac x*=*y* **in** *exI*)
apply *auto*
apply (*rule-tac x*=*y* **in** *exI*)
apply *auto*
done
then show *?thesis*

using *compact-closed-sums*[*OF assms*(*2*,*1*)] **by** *simp*
qed

lemma *compact-closed-differences*:
fixes *s t* :: *'a*::*real-normed-vector set*
assumes *compact s*
and *closed t*
shows *closed* {*x* − *y* | *x y*. *x* ∈ *s* ∧ *y* ∈ *t*}
proof −
have {*x* + *y* |*x y*. *x* ∈ *s* ∧ *y* ∈ *uminus* ‘ *t*} = {*x* − *y* |*x y*. *x* ∈ *s* ∧ *y* ∈ *t*}
apply *auto*
apply (*rule-tac x*=*xa* **in** *exI*)
apply *auto*
apply (*rule-tac x*=*xa* **in** *exI*)
apply *auto*
done
then show *?thesis*
using *compact-closed-sums*[*OF assms*(*1*) *closed-negations*[*OF assms*(*2*)]] **by** *auto*
qed

lemma *closed-compact-differences*:
fixes *s t* :: *'a*::*real-normed-vector set*
assumes *closed s*
and *compact t*
shows *closed* {*x* − *y* | *x y*. *x* ∈ *s* ∧ *y* ∈ *t*}
proof −
have {*x* + *y* |*x y*. *x* ∈ *s* ∧ *y* ∈ *uminus* ‘ *t*} = {*x* − *y* |*x y*. *x* ∈ *s* ∧ *y* ∈ *t*}
apply *auto*
apply (*rule-tac x*=*xa* **in** *exI*)
apply *auto*
apply (*rule-tac x*=*xa* **in** *exI*)
apply *auto*
done
then show *?thesis*
using *closed-compact-sums*[*OF assms*(*1*) *compact-negations*[*OF assms*(*2*)]] **by** *simp*
qed

lemma *closed-translation*:
fixes *a* :: *'a*::*real-normed-vector*
assumes *closed s*
shows *closed* ((λ*x*. *a* + *x*) ‘ *s*)
proof −
have {*a* + *y* |*y*. *y* ∈ *s*} = (*op* + *a* ‘ *s*) **by** *auto*
then show *?thesis*
using *compact-closed-sums*[*OF compact-sing*[*of a*] *assms*] **by** *auto*
qed

lemma *translation-Compl*:
fixes $a :: 'a$::*ab-group-add*
shows $(\lambda x.\ a + x)\ `\ (-\ t) = -\ ((\lambda x.\ a + x)\ `\ t)$
apply (*auto simp add*: *image-iff*)
apply (*rule-tac* x=$x - a$ **in** *bexI*)
apply *auto*
done

lemma *translation-UNIV*:
fixes $a :: 'a$::*ab-group-add*
shows *range* $(\lambda x.\ a + x) = UNIV$
apply (*auto simp add*: *image-iff*)
apply (*rule-tac* x=$x - a$ **in** *exI*)
apply *auto*
done

lemma *translation-diff*:
fixes $a :: 'a$::*ab-group-add*
shows $(\lambda x.\ a + x)\ `\ (s - t) = ((\lambda x.\ a + x)\ `\ s) - ((\lambda x.\ a + x)\ `\ t)$
by *auto*

lemma *closure-translation*:
fixes $a :: 'a$::*real-normed-vector*
shows *closure* $((\lambda x.\ a + x)\ `\ s) = (\lambda x.\ a + x)\ `\ (closure\ s)$
proof −
have $*$: $op + a\ `\ (-\ s) = -\ op + a\ `\ s$
apply *auto*
unfolding *image-iff*
apply (*rule-tac* x=$x - a$ **in** *bexI*)
apply *auto*
done
show *?thesis*
unfolding *closure-interior translation-Compl*
using *interior-translation*[*of* $a - s$]
unfolding $*$
by *auto*
qed

lemma *frontier-translation*:
fixes $a :: 'a$::*real-normed-vector*
shows *frontier*$((\lambda x.\ a + x)\ `\ s) = (\lambda x.\ a + x)\ `\ (frontier\ s)$
unfolding *frontier-def translation-diff interior-translation closure-translation*
by *auto*

16.29 Separation between points and sets

lemma *separate-point-closed*:
fixes $s :: 'a$::*heine-borel set*
assumes *closed* s **and** $a \notin s$

```
  shows ∃ d>0. ∀ x∈s. d ≤ dist a x
proof (cases s = {})
  case True
  then show ?thesis by(auto intro!: exI[where x=1])
next
  case False
  from assms obtain x where x∈s ∀ y∈s. dist a x ≤ dist a y
    using ‹s ≠ {}› by (blast intro: distance-attains-inf [of s a])
  with ‹x∈s› show ?thesis using dist-pos-lt[of a x] and‹a ∉ s›
    by blast
qed

lemma separate-compact-closed:
  fixes s t :: 'a::heine-borel set
  assumes compact s
    and t: closed t s ∩ t = {}
  shows ∃ d>0. ∀ x∈s. ∀ y∈t. d ≤ dist x y
proof cases
  assume s ≠ {} ∧ t ≠ {}
  then have s ≠ {} t ≠ {} by auto
  let ?inf = λx. infdist x t
  have continuous-on s ?inf
   by (auto intro!: continuous-at-imp-continuous-on continuous-infdist continuous-ident)
  then obtain x where x: x ∈ s ∀ y∈s. ?inf x ≤ ?inf y
    using continuous-attains-inf[OF ‹compact s› ‹s ≠ {}›] by auto
  then have 0 < ?inf x
    using t ‹t ≠ {}› in-closed-iff-infdist-zero by (auto simp: less-le infdist-nonneg)
  moreover have ∀ x'∈s. ∀ y∈t. ?inf x ≤ dist x' y
    using x by (auto intro: order-trans infdist-le)
  ultimately show ?thesis by auto
qed (auto intro!: exI[of - 1])

lemma separate-closed-compact:
  fixes s t :: 'a::heine-borel set
  assumes closed s
    and compact t
    and s ∩ t = {}
  shows ∃ d>0. ∀ x∈s. ∀ y∈t. d ≤ dist x y
proof −
  have ∗: t ∩ s = {}
    using assms(3) by auto
  show ?thesis
    using separate-compact-closed[OF assms(2,1) ∗]
    apply auto
    apply (rule-tac x=d in exI)
    apply auto
    apply (erule-tac x=y in ballE)
    apply (auto simp add: dist-commute)
    done
```

qed

16.30 Closure of halfspaces and hyperplanes

lemma *isCont-open-vimage*:
assumes $\bigwedge x.\ isCont\ f\ x$
and *open s*
shows *open* $(f\ -`\ s)$
proof −
from *assms*(*1*) **have** *continuous-on UNIV f*
unfolding *isCont-def continuous-on-def* **by** *simp*
then have *open* $\{x \in UNIV.\ f\ x \in s\}$
using *open-UNIV* ‹*open s*› **by** (*rule continuous-open-preimage*)
then show *open* $(f\ -`\ s)$
by (*simp add*: *vimage-def*)
qed

lemma *isCont-closed-vimage*:
assumes $\bigwedge x.\ isCont\ f\ x$
and *closed s*
shows *closed* $(f\ -`\ s)$
using *assms* **unfolding** *closed-def vimage-Compl* [*symmetric*]
by (*rule isCont-open-vimage*)

lemma *open-Collect-less*:
fixes *f g* :: *'a*::*t2-space* ⇒ *real*
assumes *f*: $\bigwedge x.\ isCont\ f\ x$
and *g*: $\bigwedge x.\ isCont\ g\ x$
shows *open* $\{x.\ f\ x < g\ x\}$
proof −
have *open* $((\lambda x.\ g\ x - f\ x)\ -`\ \{0<..\})$
using *isCont-diff* [*OF g f*] *open-real-greaterThan*
by (*rule isCont-open-vimage*)
also have $((\lambda x.\ g\ x - f\ x)\ -`\ \{0<..\}) = \{x.\ f\ x < g\ x\}$
by *auto*
finally show *?thesis* **.**
qed

lemma *closed-Collect-le*:
fixes *f g* :: *'a*::*t2-space* ⇒ *real*
assumes *f*: $\bigwedge x.\ isCont\ f\ x$
and *g*: $\bigwedge x.\ isCont\ g\ x$
shows *closed* $\{x.\ f\ x \leq g\ x\}$
proof −
have *closed* $((\lambda x.\ g\ x - f\ x)\ -`\ \{0..\})$
using *isCont-diff* [*OF g f*] *closed-real-atLeast*
by (*rule isCont-closed-vimage*)
also have $((\lambda x.\ g\ x - f\ x)\ -`\ \{0..\}) = \{x.\ f\ x \leq g\ x\}$
by *auto*

finally show *?thesis* **.**
qed

lemma *closed-Collect-eq*:
fixes $f\ g :: 'a{::}t2\text{-}space \Rightarrow 'b{::}t2\text{-}space$
assumes f: $\bigwedge x.\ isCont\ f\ x$
and g: $\bigwedge x.\ isCont\ g\ x$
shows *closed* $\{x.\ f\ x = g\ x\}$
proof −
have *open* $\{(x{::}'b,\ y{::}'b).\ x \neq y\}$
unfolding *open-prod-def* **by** (*auto dest*!: *hausdorff*)
then have *closed* $\{(x{::}'b,\ y{::}'b).\ x = y\}$
unfolding *closed-def split-def Collect-neg-eq* **.**
with *isCont-Pair* [*OF f g*]
have *closed* $((\lambda x.\ (f\ x,\ g\ x)) -` \{(x,\ y).\ x = y\})$
by (*rule isCont-closed-vimage*)
also have $\ldots = \{x.\ f\ x = g\ x\}$ **by** *auto*
finally show *?thesis* **.**
qed

lemma *continuous-on-closed-Collect-le*:
fixes $f\ g :: 'a{::}t2\text{-}space \Rightarrow real$
assumes f: *continuous-on s f* **and** g: *continuous-on s g* **and** s: *closed s*
shows *closed* $\{x \in s.\ f\ x \leq g\ x\}$
proof −
have *closed* $((\lambda x.\ g\ x - f\ x) -` \{0..\} \cap s)$
using *closed-real-atLeast continuous-on-diff* [*OF g f*]
by (*simp add*: *continuous-on-closed-vimage* [*OF s*])
also have $((\lambda x.\ g\ x - f\ x) -` \{0..\} \cap s) = \{x{\in}s.\ f\ x \leq g\ x\}$
by *auto*
finally show *?thesis* **.**
qed

lemma *continuous-at-inner*: *continuous* (*at x*) (*inner a*)
unfolding *continuous-at* **by** (*intro tendsto-intros*)

lemma *closed-halfspace-le*: *closed* $\{x.\ inner\ a\ x \leq b\}$
by (*simp add*: *closed-Collect-le*)

lemma *closed-halfspace-ge*: *closed* $\{x.\ inner\ a\ x \geq b\}$
by (*simp add*: *closed-Collect-le*)

lemma *closed-hyperplane*: *closed* $\{x.\ inner\ a\ x = b\}$
by (*simp add*: *closed-Collect-eq*)

lemma *closed-halfspace-component-le*: *closed* $\{x{::}'a{::}euclidean\text{-}space.\ x{\cdot}i \leq a\}$
by (*simp add*: *closed-Collect-le*)

lemma *closed-halfspace-component-ge*: *closed* $\{x{::}'a{::}euclidean\text{-}space.\ x{\cdot}i \geq a\}$

by (*simp add*: *closed-Collect-le*)

lemma *closed-interval-left*:
fixes $b :: 'a::euclidean\text{-}space$
shows *closed* $\{x::'a.\ \forall i{\in}Basis.\ x{\cdot}i \leq b{\cdot}i\}$
by (*simp add*: *Collect-ball-eq closed-INT closed-Collect-le*)

lemma *closed-interval-right*:
fixes $a :: 'a::euclidean\text{-}space$
shows *closed* $\{x::'a.\ \forall i{\in}Basis.\ a{\cdot}i \leq x{\cdot}i\}$
by (*simp add*: *Collect-ball-eq closed-INT closed-Collect-le*)

lemma *continuous-le-on-closure*:
fixes $a::real$
assumes f: *continuous-on* (*closure s*) f
and x: $x \in closure(s)$
and *xlo*: $\bigwedge x.\ x \in s \Longrightarrow f(x) \leq a$
shows $f(x) \leq a$
using *image-closure-subset* [*OF f*]
using *image-closure-subset* [*OF f*] *closed-halfspace-le* [*of 1*::*real a*] *assms*
by *force*

lemma *continuous-ge-on-closure*:
fixes $a::real$
assumes f: *continuous-on* (*closure s*) f
and x: $x \in closure(s)$
and *xlo*: $\bigwedge x.\ x \in s \Longrightarrow f(x) \geq a$
shows $f(x) \geq a$
using *image-closure-subset* [*OF f*] *closed-halfspace-ge* [*of a 1*::*real*] *assms*
by *force*

Openness of halfspaces.

lemma *open-halfspace-lt*: *open* $\{x.\ inner\ a\ x < b\}$
by (*simp add*: *open-Collect-less*)

lemma *open-halfspace-gt*: *open* $\{x.\ inner\ a\ x > b\}$
by (*simp add*: *open-Collect-less*)

lemma *open-halfspace-component-lt*: *open* $\{x::'a::euclidean\text{-}space.\ x{\cdot}i < a\}$
by (*simp add*: *open-Collect-less*)

lemma *open-halfspace-component-gt*: *open* $\{x::'a::euclidean\text{-}space.\ x{\cdot}i > a\}$
by (*simp add*: *open-Collect-less*)

This gives a simple derivation of limit component bounds.

lemma *Lim-component-le*:
fixes $f :: 'a \Rightarrow 'b::euclidean\text{-}space$
assumes $(f \longrightarrow l)$ *net*
and $\neg$ (*trivial-limit net*)

and *eventually* $(\lambda x.\ f(x)\cdot i \leq b)$ *net*
shows $l\cdot i \leq b$
by (*rule tendsto-le*[*OF assms*(*2*) *tendsto-const tendsto-inner*[*OF assms*(*1*) *tendsto-const*] *assms*(*3*)])

lemma *Lim-component-ge*:
fixes $f :: 'a \Rightarrow 'b$::*euclidean-space*
assumes $(f \longrightarrow l)$ *net*
and $\neg$ (*trivial-limit net*)
and *eventually* $(\lambda x.\ b \leq (f\ x)\cdot i)$ *net*
shows $b \leq l\cdot i$
by (*rule tendsto-le*[*OF assms*(*2*) *tendsto-inner*[*OF assms*(*1*) *tendsto-const*] *tendsto-const assms*(*3*)])

lemma *Lim-component-eq*:
fixes $f :: 'a \Rightarrow 'b$::*euclidean-space*
assumes *net*: $(f \longrightarrow l)$ *net* $\neg$ *trivial-limit net*
and *ev*:*eventually* $(\lambda x.\ f(x)\cdot i = b)$ *net*
shows $l\cdot i = b$
using *ev*[*unfolded order-eq-iff eventually-conj-iff*]
using *Lim-component-ge*[*OF net*, *of b i*]
using *Lim-component-le*[*OF net*, *of i b*]
by *auto*

Limits relative to a union.

lemma *eventually-within-Un*:
eventually P (*at x within* $(s \cup t)$) $\longleftrightarrow$
eventually P (*at x within s*) $\wedge$ *eventually P* (*at x within t*)
unfolding *eventually-at-filter*
by (*auto elim*!: *eventually-rev-mp*)

lemma *Lim-within-union*:
$(f \longrightarrow l)$ (*at x within* $(s \cup t)$) $\longleftrightarrow$
$(f \longrightarrow l)$ (*at x within s*) $\wedge$ $(f \longrightarrow l)$ (*at x within t*)
unfolding *tendsto-def*
by (*auto simp add*: *eventually-within-Un*)

lemma *Lim-topological*:
$(f \longrightarrow l)$ *net* $\longleftrightarrow$
trivial-limit net $\vee$ $(\forall S.$ *open* $S \longrightarrow l \in S \longrightarrow$ *eventually* $(\lambda x.\ f\ x \in S)$ *net*)
unfolding *tendsto-def trivial-limit-eq* **by** *auto*

Continuity relative to a union.

lemma *continuous-on-Un-local*:
⟦*closedin* (*subtopology euclidean* $(s \cup t)$) *s*; *closedin* (*subtopology euclidean* $(s \cup t)$) *t*;
continuous-on s f; *continuous-on t f*⟧
$\Longrightarrow$ *continuous-on* $(s \cup t)$ *f*
unfolding *continuous-on closedin-limpt*

by (*metis Lim-trivial-limit Lim-within-union Un-iff trivial-limit-within*)

lemma *continuous-on-cases-local*:
⟦*closedin* (*subtopology euclidean* (*s* ∪ *t*)) *s*; *closedin* (*subtopology euclidean* (*s* ∪ *t*)) *t*;
continuous-on s f; *continuous-on t g*;
⋀*x*. ⟦*x* ∈ *s* ∧ ~*P x* ∨ *x* ∈ *t* ∧ *P x*⟧ ⟹ *f x* = *g x*⟧
⟹ *continuous-on* (*s* ∪ *t*) (λ*x*. *if P x then f x else g x*)
by (*rule continuous-on-Un-local*) (*auto intro*: *continuous-on-eq*)

lemma *continuous-on-cases-le*:
fixes *h* :: *'a* :: *topological-space* ⇒ *real*
assumes *continuous-on* {*t* ∈ *s*. *h t* ≤ *a*} *f*
and *continuous-on* {*t* ∈ *s*. *a* ≤ *h t*} *g*
and *h*: *continuous-on s h*
and ⋀*t*. ⟦*t* ∈ *s*; *h t* = *a*⟧ ⟹ *f t* = *g t*
shows *continuous-on s* (λ*t*. *if h t* ≤ *a then f*(*t*) *else g*(*t*))
proof −
have *s*: *s* = {*t* ∈ *s*. *h t* ∈ *atMost a*} ∪ {*t* ∈ *s*. *h t* ∈ *atLeast a*}
by *force*
have *1*: *closedin* (*subtopology euclidean s*) {*t* ∈ *s*. *h t* ∈ *atMost a*}
by (*rule continuous-closedin-preimage* [*OF h closed-atMost*])
have *2*: *closedin* (*subtopology euclidean s*) {*t* ∈ *s*. *h t* ∈ *atLeast a*}
by (*rule continuous-closedin-preimage* [*OF h closed-atLeast*])
show *?thesis*
apply (*rule continuous-on-subset* [*of s*, *OF - order-refl*])
apply (*subst s*)
apply (*rule continuous-on-cases-local*)
using *1 2 s assms* **apply** *auto*
done
qed

lemma *continuous-on-cases-1*:
fixes *s* :: *real set*
assumes *continuous-on* {*t* ∈ *s*. *t* ≤ *a*} *f*
and *continuous-on* {*t* ∈ *s*. *a* ≤ *t*} *g*
and *a* ∈ *s* ⟹ *f a* = *g a*
shows *continuous-on s* (λ*t*. *if t* ≤ *a then f*(*t*) *else g*(*t*))
using *assms*
by (*auto simp*: *continuous-on-id intro*: *continuous-on-cases-le* [**where** *h* = *id*, *simplified*])

Some more convenient intermediate-value theorem formulations.

lemma *connected-ivt-hyperplane*:
assumes *connected s*
and *x* ∈ *s*
and *y* ∈ *s*
and *inner a x* ≤ *b*
and *b* ≤ *inner a y*

shows $\exists z \in s.\ inner\ a\ z = b$
proof (*rule ccontr*)
assume *as*:$\neg$ ($\exists z{\in}s.\ inner\ a\ z = b$)
let *?A* = $\{x.\ inner\ a\ x < b\}$
let *?B* = $\{x.\ inner\ a\ x > b\}$
have *open ?A open ?B*
using *open-halfspace-lt* **and** *open-halfspace-gt* **by** *auto*
moreover
have *?A* $\cap$ *?B* = $\{\}$ **by** *auto*
moreover
have *s* $\subseteq$ *?A* $\cup$ *?B* **using** *as* **by** *auto*
ultimately
show *False*
using *assms*(*1*)[*unfolded connected-def not-ex*,
THEN spec[**where** *x*=*?A*], *THEN spec*[**where** *x*=*?B*]]
using *assms*(*2*−*5*)
by *auto*
qed

lemma *connected-ivt-component*:
fixes *x*::*'a*::*euclidean-space*
shows *connected s* $\Longrightarrow$
x $\in$ *s* $\Longrightarrow$ *y* $\in$ *s* $\Longrightarrow$
x$\cdot$*k* $\leq$ *a* $\Longrightarrow$ *a* $\leq$ *y*$\cdot$*k* $\Longrightarrow$ ($\exists z{\in}s.\ \ z{\cdot}k = a$)
using *connected-ivt-hyperplane*[*of s x y k*::*'a a*]
by (*auto simp*: *inner-commute*)

16.31 Intervals

lemma *open-box*[*intro*]: *open* (*box a b*)
proof −
have *open* ($\bigcap i{\in}Basis.\ (op \cdot i) -` \{a \cdot i <..< b \cdot i\}$)
by (*auto intro*!: *continuous-open-vimage continuous-inner continuous-ident continuous-const*)
also have ($\bigcap i{\in}Basis.\ (op \cdot i) -` \{a \cdot i <..< b \cdot i\}$) = *box a b*
by (*auto simp add*: *box-def inner-commute*)
finally show *?thesis* **.**
qed

instance *euclidean-space* $\subseteq$ *second-countable-topology*
proof
def *a* $\equiv$ λf :: *'a* $\Rightarrow$ (*real* $\times$ *real*). $\sum i{\in}Basis.\ fst\ (f\ i) *_R i$
then have *a*: $\bigwedge f.\ (\sum i{\in}Basis.\ fst\ (f\ i) *_R i) = a\ f$
by *simp*
def *b* $\equiv$ λf :: *'a* $\Rightarrow$ (*real* $\times$ *real*). $\sum i{\in}Basis.\ snd\ (f\ i) *_R i$
then have *b*: $\bigwedge f.\ (\sum i{\in}Basis.\ snd\ (f\ i) *_R i) = b\ f$
by *simp*
def *B* $\equiv$ ($\lambda f.\ box\ (a\ f)\ (b\ f)$) ` (*Basis* $\to_E$ ($\mathbb{Q} \times \mathbb{Q}$))

```
  have Ball B open by (simp add: B-def open-box)
  moreover have (∀ A. open A ⟶ (∃ B'⊆B. ⋃ B' = A))
  proof safe
    fix A::'a set
    assume open A
    show ∃ B'⊆B. ⋃ B' = A
      apply (rule exI[of - {b∈B. b ⊆ A}])
      apply (subst (3) open-UNION-box[OF ‹open A›])
      apply (auto simp add: a b B-def)
      done
  qed
  ultimately
  have topological-basis B
    unfolding topological-basis-def by blast
  moreover
  have countable B
    unfolding B-def
   by (intro countable-image countable-PiE finite-Basis countable-SIGMA countable-rat)
  ultimately show ∃ B::'a set set. countable B ∧ open = generate-topology B
    by (blast intro: topological-basis-imp-subbasis)
qed

instance euclidean-space ⊆ polish-space ..

lemma closed-cbox[intro]:
  fixes a b :: 'a::euclidean-space
  shows closed (cbox a b)
proof −
  have closed (⋂ i∈Basis. (λx. x•i) −‘ {a•i .. b•i})
    by (intro closed-INT ballI continuous-closed-vimage allI
     linear-continuous-at closed-real-atLeastAtMost finite-Basis bounded-linear-inner-left)
  also have (⋂ i∈Basis. (λx. x•i) −‘ {a•i .. b•i}) = cbox a b
    by (auto simp add: cbox-def)
  finally show closed (cbox a b) .
qed

lemma interior-cbox [simp]:
  fixes a b :: 'a::euclidean-space
  shows interior (cbox a b) = box a b (is ?L = ?R)
proof(rule subset-antisym)
  show ?R ⊆ ?L
    using box-subset-cbox open-box
    by (rule interior-maximal)
  {
    fix x
    assume x ∈ interior (cbox a b)
    then obtain s where s: open s x ∈ s s ⊆ cbox a b ..
    then obtain e where e>0 and e:∀ x'. dist x' x < e ⟶ x' ∈ cbox a b
      unfolding open-dist and subset-eq by auto
```

```
    {
      fix i :: 'a
      assume i: i ∈ Basis
      have dist (x − (e / 2) *R i) x < e
        and dist (x + (e / 2) *R i) x < e
        unfolding dist-norm
        apply auto
        unfolding norm-minus-cancel
        using norm-Basis[OF i] ‹e>0›
        apply auto
        done
      then have a · i ≤ (x − (e / 2) *R i) · i and (x + (e / 2) *R i) · i ≤ b · i
        using e[THEN spec[where x=x − (e/2) *R i]]
          and e[THEN spec[where x=x + (e/2) *R i]]
        unfolding mem-box
        using i
        by blast+
      then have a · i < x · i and x · i < b · i
        using ‹e>0› i
        by (auto simp: inner-diff-left inner-Basis inner-add-left)
    }
    then have x ∈ box a b
      unfolding mem-box by auto
  }
  then show ?L ⊆ ?R ..
qed

lemma bounded-cbox:
  fixes a :: 'a::euclidean-space
  shows bounded (cbox a b)
proof −
  let ?b = ∑ i∈Basis. |a·i| + |b·i|
  {
    fix x :: 'a
    assume x: ∀ i∈Basis. a · i ≤ x · i ∧ x · i ≤ b · i
    {
      fix i :: 'a
      assume i ∈ Basis
      then have |x·i| ≤ |a·i| + |b·i|
        using x[THEN bspec[where x=i]] by auto
    }
    then have (∑ i∈Basis. |x · i|) ≤ ?b
      apply −
      apply (rule setsum-mono)
      apply auto
      done
    then have norm x ≤ ?b
      using norm-le-l1[of x] by auto
  }
```

```
  then show ?thesis
    unfolding cbox-def bounded-iff by auto
qed

lemma bounded-box [simp]:
  fixes a :: 'a::euclidean-space
  shows bounded (box a b)
  using bounded-cbox[of a b]
  using box-subset-cbox[of a b]
  using bounded-subset[of cbox a b box a b]
  by simp

lemma not-interval-UNIV [simp]:
  fixes a :: 'a::euclidean-space
  shows cbox a b ≠ UNIV box a b ≠ UNIV
  using bounded-box[of a b] bounded-cbox[of a b] by force+

lemma compact-cbox [simp]:
  fixes a :: 'a::euclidean-space
  shows compact (cbox a b)
  using bounded-closed-imp-seq-compact[of cbox a b] using bounded-cbox[of a b]
  by (auto simp: compact-eq-seq-compact-metric)

lemma box-midpoint:
  fixes a :: 'a::euclidean-space
  assumes box a b ≠ {}
  shows ((1/2) *R (a + b)) ∈ box a b
proof −
  {
    fix i :: 'a
    assume i ∈ Basis
    then have a · i < ((1 / 2) *R (a + b)) · i ∧ ((1 / 2) *R (a + b)) · i < b · i
     using assms[unfolded box-ne-empty, THEN bspec[where x=i]] by (auto simp:
inner-add-left)
  }
  then show ?thesis unfolding mem-box by auto
qed

lemma open-cbox-convex:
  fixes x :: 'a::euclidean-space
  assumes x: x ∈ box a b
    and y: y ∈ cbox a b
    and e: 0 < e e ≤ 1
  shows (e *R x + (1 − e) *R y) ∈ box a b
proof −
  {
    fix i :: 'a
    assume i: i ∈ Basis
    have a · i = e * (a · i) + (1 − e) * (a · i)
```

```
    unfolding left-diff-distrib by simp
  also have ... < e * (x • i) + (1 - e) * (y • i)
    apply (rule add-less-le-mono)
    using e unfolding mult-less-cancel-left and mult-le-cancel-left
    apply simp-all
    using x unfolding mem-box using i
    apply simp
    using y unfolding mem-box using i
    apply simp
    done
  finally have a • i < (e *R x + (1 - e) *R y) • i
    unfolding inner-simps by auto
  moreover
  {
    have b • i = e * (b•i) + (1 - e) * (b•i)
      unfolding left-diff-distrib by simp
    also have ... > e * (x • i) + (1 - e) * (y • i)
      apply (rule add-less-le-mono)
      using e unfolding mult-less-cancel-left and mult-le-cancel-left
      apply simp-all
      using x
      unfolding mem-box
      using i
      apply simp
      using y
      unfolding mem-box
      using i
      apply simp
      done
    finally have (e *R x + (1 - e) *R y) • i < b • i
      unfolding inner-simps by auto
  }
  ultimately have a • i < (e *R x + (1 - e) *R y) • i ∧ (e *R x + (1 - e)
*R y) • i < b • i
    by auto
 }
 then show ?thesis
  unfolding mem-box by auto
qed

lemma closure-box:
 fixes a :: 'a::euclidean-space
  assumes box a b ≠ {}
 shows closure (box a b) = cbox a b
proof −
 have ab: a <e b
  using assms by (simp add: eucl-less-def box-ne-empty)
 let ?c = (1 / 2) *R (a + b)
 {
```

```
  fix x
  assume as:x ∈ cbox a b
  def f ≡ λn::nat. x + (inverse (real n + 1)) *R (?c − x)
  {
    fix n
    assume fn: f n <e b ⟶ a <e f n ⟶ f n = x and xc: x ≠ ?c
    have *: 0 < inverse (real n + 1) inverse (real n + 1) ≤ 1
      unfolding inverse-le-1-iff by auto
    have (inverse (real n + 1)) *R ((1 / 2) *R (a + b)) + (1 − inverse (real n
+ 1)) *R x =
      x + (inverse (real n + 1)) *R (((1 / 2) *R (a + b)) − x)
      by (auto simp add: algebra-simps)
    then have f n <e b and a <e f n
      using open-cbox-convex[OF box-midpoint[OF assms] as *]
      unfolding f-def by (auto simp: box-def eucl-less-def)
    then have False
      using fn unfolding f-def using xc by auto
  }
  moreover
  {
    assume ¬ (f ⟶⟶ x) sequentially
    {
      fix e :: real
      assume e > 0
      then have ∃N::nat. inverse (real (N + 1)) < e
        using real-arch-inverse[of e]
        apply (auto simp add: Suc-pred')
        apply (metis Suc-pred' of-nat-Suc)
        done
      then obtain N :: nat where N: inverse (real (N + 1)) < e
        by auto
      have inverse (real n + 1) < e if N ≤ n for n
        by (auto intro!: that le-less-trans [OF - N])
      then have ∃N::nat. ∀n≥N. inverse (real n + 1) < e by auto
    }
    then have ((λn. inverse (real n + 1)) ⟶⟶ 0) sequentially
      unfolding lim-sequentially by(auto simp add: dist-norm)
    then have (f ⟶⟶ x) sequentially
      unfolding f-def
      using tendsto-add[OF tendsto-const, of λn::nat. (inverse (real n + 1)) *R
((1 / 2) *R (a + b) − x) 0 sequentially x]
      using tendsto-scaleR [OF - tendsto-const, of λn::nat. inverse (real n + 1)
0 sequentially ((1 / 2) *R (a + b) − x)]
      by auto
  }
  ultimately have x ∈ closure (box a b)
    using as and box-midpoint[OF assms]
    unfolding closure-def
    unfolding islimpt-sequential
```

```
    by (cases x=?c) (auto simp: in-box-eucl-less)
  }
  then show ?thesis
    using closure-minimal[OF box-subset-cbox, of a b] by blast
qed

lemma bounded-subset-box-symmetric:
  fixes s::('a::euclidean-space) set
  assumes bounded s
  shows ∃ a. s ⊆ box (−a) a
proof −
  obtain b where b>0 and b: ∀ x∈s. norm x ≤ b
    using assms[unfolded bounded-pos] by auto
  def a ≡ (∑ i∈Basis. (b + 1) *R i)::'a
  {
    fix x
    assume x ∈ s
    fix i :: 'a
    assume i: i ∈ Basis
    then have (−a)•i < x•i and x•i < a•i
      using b[THEN bspec[where x=x], OF ‹x∈s›]
      using Basis-le-norm[OF i, of x]
      unfolding inner-simps and a-def
      by auto
  }
  then show ?thesis
    by (auto intro: exI[where x=a] simp add: box-def)
qed

lemma bounded-subset-open-interval:
  fixes s :: ('a::euclidean-space) set
  shows bounded s ⟹ (∃ a b. s ⊆ box a b)
  by (auto dest!: bounded-subset-box-symmetric)

lemma bounded-subset-cbox-symmetric:
  fixes s :: ('a::euclidean-space) set
   assumes bounded s
  shows ∃ a. s ⊆ cbox (−a) a
proof −
  obtain a where s ⊆ box (−a) a
    using bounded-subset-box-symmetric[OF assms] by auto
  then show ?thesis
    using box-subset-cbox[of −a a] by auto
qed

lemma bounded-subset-cbox:
  fixes s :: ('a::euclidean-space) set
  shows bounded s ⟹ ∃ a b. s ⊆ cbox a b
  using bounded-subset-cbox-symmetric[of s] by auto
```

lemma *frontier-cbox*:
fixes *a b* :: *'a*::*euclidean-space*
shows *frontier* (*cbox a b*) = *cbox a b* − *box a b*
unfolding *frontier-def* **unfolding** *interior-cbox* **and** *closure-closed*[*OF closed-cbox*]
..

lemma *frontier-box*:
fixes *a b* :: *'a*::*euclidean-space*
shows *frontier* (*box a b*) = (*if box a b* = {} *then* {} *else cbox a b* − *box a b*)
proof (*cases box a b* = {})
case *True*
then show *?thesis*
using *frontier-empty* **by** *auto*
next
case *False*
then show *?thesis*
unfolding *frontier-def* **and** *closure-box*[*OF False*] **and** *interior-open*[*OF open-box*]
by *auto*
qed

lemma *inter-interval-mixed-eq-empty*:
fixes *a* :: *'a*::*euclidean-space*
assumes *box c d* ≠ {}
shows *box a b* ∩ *cbox c d* = {} ⟷ *box a b* ∩ *box c d* = {}
unfolding *closure-box*[*OF assms*, *symmetric*]
unfolding *open-Int-closure-eq-empty*[*OF open-box*] **..**

lemma *diameter-cbox*:
fixes *a b*::*'a*::*euclidean-space*
shows (∀ *i* ∈ *Basis*. *a* · *i* ≤ *b* · *i*) ⟹ *diameter* (*cbox a b*) = *dist a b*
by (*force simp add*: *diameter-def intro*!: *cSup-eq-maximum setL2-mono*
simp: *euclidean-dist-l2*[**where** *'a*=*'a*] *cbox-def dist-norm*)

lemma *eucl-less-eq-halfspaces*:
fixes *a* :: *'a*::*euclidean-space*
shows {*x*. *x* <*e a*} = (⋂ *i*∈*Basis*. {*x*. *x* · *i* < *a* · *i*})
{*x*. *a* <*e x*} = (⋂ *i*∈*Basis*. {*x*. *a* · *i* < *x* · *i*})
by (*auto simp*: *eucl-less-def*)

lemma *eucl-le-eq-halfspaces*:
fixes *a* :: *'a*::*euclidean-space*
shows {*x*. ∀ *i*∈*Basis*. *x* · *i* ≤ *a* · *i*} = (⋂ *i*∈*Basis*. {*x*. *x* · *i* ≤ *a* · *i*})
{*x*. ∀ *i*∈*Basis*. *a* · *i* ≤ *x* · *i*} = (⋂ *i*∈*Basis*. {*x*. *a* · *i* ≤ *x* · *i*})
by *auto*

lemma *open-Collect-eucl-less*[*simp*, *intro*]:
fixes *a* :: *'a*::*euclidean-space*
shows *open* {*x*. *x* <*e a*}

```
    open {x. a <e x}
 by (auto simp: eucl-less-eq-halfspaces open-halfspace-component-lt open-halfspace-component-gt)

lemma closed-Collect-eucl-le[simp, intro]:
  fixes a :: 'a::euclidean-space
  shows closed {x. ∀i∈Basis. a · i ≤ x · i}
    closed {x. ∀i∈Basis. x · i ≤ a · i}
  unfolding eucl-le-eq-halfspaces
  by (simp-all add: closed-INT closed-Collect-le)

lemma image-affinity-cbox: fixes m::real
  fixes a b c :: 'a::euclidean-space
  shows (λx. m *R x + c) ' cbox a b =
    (if cbox a b = {} then {}
     else (if 0 ≤ m then cbox (m *R a + c) (m *R b + c)
     else cbox (m *R b + c) (m *R a + c)))
proof (cases m = 0)
  case True
  {
    fix x
    assume ∀i∈Basis. x · i ≤ c · i ∀i∈Basis. c · i ≤ x · i
    then have x = c
      apply −
      apply (subst euclidean-eq-iff)
      apply (auto intro: order-antisym)
      done
  }
  moreover have c ∈ cbox (m *R a + c) (m *R b + c)
    unfolding True by (auto simp add: cbox-sing)
  ultimately show ?thesis using True by (auto simp: cbox-def)
next
  case False
  {
    fix y
    assume ∀i∈Basis. a · i ≤ y · i ∀i∈Basis. y · i ≤ b · i m > 0
    then have ∀i∈Basis. (m *R a + c) · i ≤ (m *R y + c) · i and ∀i∈Basis.
(m *R y + c) · i ≤ (m *R b + c) · i
      by (auto simp: inner-distrib)
  }
  moreover
  {
    fix y
    assume ∀i∈Basis. a · i ≤ y · i ∀i∈Basis. y · i ≤ b · i m < 0
    then have ∀i∈Basis. (m *R b + c) · i ≤ (m *R y + c) · i and ∀i∈Basis.
(m *R y + c) · i ≤ (m *R a + c) · i
      by (auto simp add: mult-left-mono-neg inner-distrib)
  }
  moreover
  {
```

fix y
assume $m > 0$ **and** $\forall i{\in}Basis.\ (m *_R a + c) \cdot i \leq y \cdot i$ **and** $\forall i{\in}Basis.\ y \cdot i \leq (m *_R b + c) \cdot i$
then have $y \in (\lambda x.\ m *_R x + c)$ ‘ *cbox a b*
unfolding *image-iff Bex-def mem-box*
apply (*intro exI*[**where** $x{=}(1\ /\ m) *_R (y - c)$])
apply (*auto simp add*: *pos-le-divide-eq pos-divide-le-eq mult.commute inner-distrib inner-diff-left*)
done
}
moreover
{
fix y
assume $\forall i{\in}Basis.\ (m *_R b + c) \cdot i \leq y \cdot i\ \forall i{\in}Basis.\ y \cdot i \leq (m *_R a + c) \cdot i\ m < 0$
then have $y \in (\lambda x.\ m *_R x + c)$ ‘ *cbox a b*
unfolding *image-iff Bex-def mem-box*
apply (*intro exI*[**where** $x{=}(1\ /\ m) *_R (y - c)$])
apply (*auto simp add*: *neg-le-divide-eq neg-divide-le-eq mult.commute inner-distrib inner-diff-left*)
done
}
ultimately show *?thesis* **using** *False* **by** (*auto simp*: *cbox-def*)
qed

lemma *image-smult-cbox*:$(\lambda x.\ m *_R (x{::}\text{-}{::}euclidean\text{-}space))$ ‘ *cbox a b* =
(*if cbox a b* = {} *then* {} *else if* $0 \leq m$ *then cbox* $(m *_R a)$ $(m *_R b)$ *else cbox* $(m *_R b)$ $(m *_R a)$)
using *image-affinity-cbox*[*of m 0 a b*] **by** *auto*

lemma *islimpt-greaterThanLessThan1*:
fixes *a b*::$'a$::{*linorder-topology*, *dense-order*}
assumes $a < b$
shows *a islimpt* $\{a{<}..{<}b\}$
proof (*rule islimptI*)
fix T
assume *open* T $a \in T$
from *open-right*[*OF this* ‹$a < b$›]
obtain c **where** c: $a < c$ $\{a..{<}c\} \subseteq T$ **by** *auto*
with *assms dense*[*of a min c b*]
show $\exists y{\in}\{a{<}..{<}b\}.\ y \in T \wedge y \neq a$
by (*metis atLeastLessThan-iff greaterThanLessThan-iff min-less-iff-conj not-le order.strict-implies-order subset-eq*)
qed

lemma *islimpt-greaterThanLessThan2*:
fixes *a b*::$'a$::{*linorder-topology*, *dense-order*}
assumes $a < b$
shows *b islimpt* $\{a{<}..{<}b\}$

proof (*rule islimptI*)
fix T
assume *open* T $b \in T$
from *open-left*[*OF this* ‹$a < b$›]
obtain c **where** c: $c < b$ $\{c<..b\} \subseteq T$ **by** *auto*
with *assms dense*[*of max a c b*]
show $\exists y \in \{a<..<b\}.\ y \in T \wedge y \neq b$
by (*metis greaterThanAtMost-iff greaterThanLessThan-iff max-less-iff-conj not-le order.strict-implies-order subset-eq*)
qed

lemma *closure-greaterThanLessThan*[*simp*]:
fixes a b::'a::{*linorder-topology*, *dense-order*}
shows $a < b \Longrightarrow$ *closure* $\{a <..< b\} = \{a\ ..\ b\}$ (**is** - $\Longrightarrow$ *?l* = *?r*)
proof
have *?l* $\subseteq$ *closure ?r*
by (*rule closure-mono*) *auto*
thus *closure* $\{a<..<b\} \subseteq \{a..b\}$ **by** *simp*
qed (*auto simp*: *closure-def order.order-iff-strict islimpt-greaterThanLessThan1 islimpt-greaterThanLessThan2*)

lemma *closure-greaterThan*[*simp*]:
fixes a b::'a::{*no-top*, *linorder-topology*, *dense-order*}
shows *closure* $\{a<..\} = \{a..\}$
proof −
from *gt-ex* **obtain** b **where** $a < b$ **by** *auto*
hence $\{a<..\} = \{a<..<b\} \cup \{b..\}$ **by** *auto*
also have *closure* ... $= \{a..\}$ **using** ‹$a < b$› **unfolding** *closure-union*
by *auto*
finally show *?thesis* **.**
qed

lemma *closure-lessThan*[*simp*]:
fixes b::'a::{*no-bot*, *linorder-topology*, *dense-order*}
shows *closure* $\{..<b\} = \{..b\}$
proof −
from *lt-ex* **obtain** a **where** $a < b$ **by** *auto*
hence $\{..<b\} = \{a<..<b\} \cup \{..a\}$ **by** *auto*
also have *closure* ... $= \{..b\}$ **using** ‹$a < b$› **unfolding** *closure-union*
by *auto*
finally show *?thesis* **.**
qed

lemma *closure-atLeastLessThan*[*simp*]:
fixes a b::'a::{*linorder-topology*, *dense-order*}
assumes $a < b$
shows *closure* $\{a\ ..<\ b\} = \{a\ ..\ b\}$
proof −
from *assms* **have** $\{a\ ..<\ b\} = \{a\} \cup \{a <..<\ b\}$ **by** *auto*

also have *closure ...* $= \{a\ ..\ b\}$ **unfolding** *closure-union*
by (*auto simp add*: *assms less-imp-le*)
finally show *?thesis* **.**
qed

lemma *closure-greaterThanAtMost*[*simp*]:
fixes *a b*::*'a*::{*linorder-topology*, *dense-order*}
assumes $a < b$
shows *closure* $\{a <..\ b\} = \{a\ ..\ b\}$
proof −
from *assms* **have** $\{a <..\ b\} = \{b\} \cup \{a <..< b\}$ **by** *auto*
also have *closure ...* $= \{a\ ..\ b\}$ **unfolding** *closure-union*
by (*auto simp add*: *assms less-imp-le*)
finally show *?thesis* **.**
qed

16.32 Homeomorphisms

definition *homeomorphism s t f g* $\longleftrightarrow$
$(\forall x{\in}s.\ (g(f\ x) = x)) \wedge (f\ `\ s = t) \wedge$ *continuous-on s f* $\wedge$
$(\forall y{\in}t.\ (f(g\ y) = y)) \wedge (g\ `\ t = s) \wedge$ *continuous-on t g*

lemma *homeomorphism-translation*:
fixes *a* :: *'a* :: *real-normed-vector*
shows *homeomorphism* $(op + a\ `\ S)\ S\ (op + (-\ a))\ (op + a)$
unfolding *homeomorphism-def* **by** (*auto simp*: *algebra-simps continuous-intros*)

lemma *homeomorphism-symD*: *homeomorphism S t f g* $\Longrightarrow$ *homeomorphism t S g f*
by (*simp add*: *homeomorphism-def*)

lemma *homeomorphism-sym*: *homeomorphism S t f g* = *homeomorphism t S g f*
by (*force simp*: *homeomorphism-def*)

definition *homeomorphic* :: *'a*::*topological-space set* $\Rightarrow$ *'b*::*topological-space set* $\Rightarrow$ *bool*
(**infixr** *homeomorphic 60*)
where *s homeomorphic t* $\equiv (\exists f\ g.$ *homeomorphism s t f g*$)$

lemma *homeomorphic-refl*: *s homeomorphic s*
unfolding *homeomorphic-def homeomorphism-def*
using *continuous-on-id*
apply (*rule-tac x* $= (\lambda x.\ x)$ **in** *exI*)
apply (*rule-tac x* $= (\lambda x.\ x)$ **in** *exI*)
apply *blast*
done

lemma *homeomorphic-sym*: *s homeomorphic t* $\longleftrightarrow$ *t homeomorphic s*
unfolding *homeomorphic-def homeomorphism-def*

by *blast*

lemma *homeomorphic-trans* [*trans*]:
assumes *s homeomorphic t*
and *t homeomorphic u*
shows *s homeomorphic u*
proof −
obtain *f1 g1* **where** *fg1*: ∀ *x*∈*s*. *g1* (*f1 x*) = *x* *f1* ' *s* = *t*
continuous-on s f1 ∀ *y*∈*t*. *f1* (*g1 y*) = *y g1* ' *t* = *s continuous-on t g1*
using *assms*(*1*) **unfolding** *homeomorphic-def homeomorphism-def* **by** *auto*
obtain *f2 g2* **where** *fg2*: ∀ *x*∈*t*. *g2* (*f2 x*) = *x* *f2* ' *t* = *u continuous-on t f2*
∀ *y*∈*u*. *f2* (*g2 y*) = *y g2* ' *u* = *t continuous-on u g2*
using *assms*(*2*) **unfolding** *homeomorphic-def homeomorphism-def* **by** *auto*
{
fix *x*
assume *x*∈*s*
then have (*g1* ∘ *g2*) ((*f2* ∘ *f1*) *x*) = *x*
using *fg1*(*1*)[*THEN bspec*[**where** *x*=*x*]] **and** *fg2*(*1*)[*THEN bspec*[**where** *x*=*f1*
x]] **and** *fg1*(*2*)
by *auto*
}
moreover have (*f2* ∘ *f1*) ' *s* = *u*
using *fg1*(*2*) *fg2*(*2*) **by** *auto*
moreover have *continuous-on s* (*f2* ∘ *f1*)
using *continuous-on-compose*[*OF fg1*(*3*)] **and** *fg2*(*3*) **unfolding** *fg1*(*2*) **by**
auto
moreover
{
fix *y*
assume *y*∈*u*
then have (*f2* ∘ *f1*) ((*g1* ∘ *g2*) *y*) = *y*
using *fg2*(*4*)[*THEN bspec*[**where** *x*=*y*]] **and** *fg1*(*4*)[*THEN bspec*[**where** *x*=*g2*
y]] **and** *fg2*(*5*)
by *auto*
}
moreover have (*g1* ∘ *g2*) ' *u* = *s* **using** *fg1*(*5*) *fg2*(*5*) **by** *auto*
moreover have *continuous-on u* (*g1* ∘ *g2*)
using *continuous-on-compose*[*OF fg2*(*6*)] **and** *fg1*(*6*)
unfolding *fg2*(*5*)
by *auto*
ultimately show *?thesis*
unfolding *homeomorphic-def homeomorphism-def*
apply (*rule-tac x*=*f2* ∘ *f1* **in** *exI*)
apply (*rule-tac x*=*g1* ∘ *g2* **in** *exI*)
apply *auto*
done
qed

lemma *homeomorphic-minimal*:

```
  s homeomorphic t ⟷
    (∃f g. (∀x∈s. f(x) ∈ t ∧ (g(f(x)) = x)) ∧
          (∀y∈t. g(y) ∈ s ∧ (f(g(y)) = y)) ∧
          continuous-on s f ∧ continuous-on t g)
  unfolding homeomorphic-def homeomorphism-def
  apply auto
  apply (rule-tac x=f in exI)
  apply (rule-tac x=g in exI)
  apply auto
  apply (rule-tac x=f in exI)
  apply (rule-tac x=g in exI)
  apply auto
  unfolding image-iff
  apply (erule-tac x=g x in ballE)
  apply (erule-tac x=x in ballE)
  apply auto
  apply (rule-tac x=g x in bexI)
  apply auto
  apply (erule-tac x=f x in ballE)
  apply (erule-tac x=x in ballE)
  apply auto
  apply (rule-tac x=f x in bexI)
  apply auto
  done
```

Relatively weak hypotheses if a set is compact.

```
lemma homeomorphism-compact:
  fixes f :: 'a::topological-space ⇒ 'b::t2-space
  assumes compact s continuous-on s f  f ‘ s = t  inj-on f s
  shows ∃g. homeomorphism s t f g
proof −
  def g ≡ λx. SOME y. y∈s ∧ f y = x
  have g: ∀x∈s. g (f x) = x
    using assms(3) assms(4)[unfolded inj-on-def] unfolding g-def by auto
  {
    fix y
    assume y ∈ t
    then obtain x where x:f x = y x∈s
      using assms(3) by auto
    then have g (f x) = x using g by auto
    then have f (g y) = y unfolding x(1)[symmetric] by auto
  }
  then have g':∀x∈t. f (g x) = x by auto
  moreover
  {
    fix x
    have x∈s ⟹ x ∈ g ‘ t
      using g[THEN bspec[where x=x]]
      unfolding image-iff
```

```
    using assms(3)
    by (auto intro!: bexI[where x=f x])
  moreover
  {
    assume x∈g ‘ t
    then obtain y where y:y∈t g y = x by auto
    then obtain x' where x':x'∈s f x' = y
      using assms(3) by auto
    then have x ∈ s
      unfolding g-def
      using someI2[of λb. b∈s ∧ f b = y x' λx. x∈s]
      unfolding y(2)[symmetric] and g-def
      by auto
  }
  ultimately have x∈s ⟷ x ∈ g ‘ t ..
 }
 then have g ‘ t = s by auto
 ultimately show ?thesis
  unfolding homeomorphism-def homeomorphic-def
  apply (rule-tac x=g in exI)
  using g and assms(3) and continuous-on-inv[OF assms(2,1), of g, unfolded
assms(3)] and assms(2)
  apply auto
  done
qed

lemma homeomorphic-compact:
 fixes f :: 'a::topological-space ⇒ 'b::t2-space
 shows compact s ⟹ continuous-on s f ⟹ (f ‘ s = t) ⟹ inj-on f s ⟹ s
homeomorphic t
 unfolding homeomorphic-def by (metis homeomorphism-compact)
```

Preservation of topological properties.

```
lemma homeomorphic-compactness: s homeomorphic t ⟹ (compact s ⟷ com-
pact t)
 unfolding homeomorphic-def homeomorphism-def
 by (metis compact-continuous-image)
```

Results on translation, scaling etc.

```
lemma homeomorphic-scaling:
 fixes s :: 'a::real-normed-vector set
 assumes c ≠ 0
 shows s homeomorphic ((λx. c *R x) ‘ s)
 unfolding homeomorphic-minimal
 apply (rule-tac x=λx. c *R x in exI)
 apply (rule-tac x=λx. (1 / c) *R x in exI)
 using assms
 apply (auto simp add: continuous-intros)
 done
```

lemma *homeomorphic-translation*:
fixes $s :: 'a$::*real-normed-vector set*
shows s *homeomorphic* $((\lambda x.\ a + x)\ `\ s)$
unfolding *homeomorphic-minimal*
apply (*rule-tac* $x{=}\lambda x.\ a + x$ **in** *exI*)
apply (*rule-tac* $x{=}\lambda x.\ -a + x$ **in** *exI*)
using *continuous-on-add* [*OF continuous-on-const continuous-on-id*, *of s a*]
continuous-on-add [*OF continuous-on-const continuous-on-id*, *of plus a ` s − a*]
apply *auto*
done

lemma *homeomorphic-affinity*:
fixes $s :: 'a$::*real-normed-vector set*
assumes $c \neq 0$
shows s *homeomorphic* $((\lambda x.\ a + c *_R x)\ `\ s)$
proof −
have $*$: $op + a\ `\ op *_R\ c\ `\ s = (\lambda x.\ a + c *_R x)\ `\ s$ **by** *auto*
show *?thesis*
using *homeomorphic-trans*
using *homeomorphic-scaling*[*OF assms*, *of s*]
using *homeomorphic-translation*[*of* $(\lambda x.\ c *_R x)\ `\ s\ a$]
unfolding $*$
by *auto*
qed

lemma *homeomorphic-balls*:
fixes $a\ b :: 'a$::*real-normed-vector*
assumes $0 < d\ \ 0 < e$
shows (*ball a d*) *homeomorphic* (*ball b e*) (**is** *?th*)
and (*cball a d*) *homeomorphic* (*cball b e*) (**is** *?cth*)
proof −
show *?th* **unfolding** *homeomorphic-minimal*
apply(*rule-tac* $x{=}\lambda x.\ b + (e/d) *_R (x - a)$ **in** *exI*)
apply(*rule-tac* $x{=}\lambda x.\ a + (d/e) *_R (x - b)$ **in** *exI*)
using *assms*
apply (*auto intro*!: *continuous-intros*
simp: *dist-commute dist-norm pos-divide-less-eq mult-strict-left-mono*)
done
show *?cth* **unfolding** *homeomorphic-minimal*
apply(*rule-tac* $x{=}\lambda x.\ b + (e/d) *_R (x - a)$ **in** *exI*)
apply(*rule-tac* $x{=}\lambda x.\ a + (d/e) *_R (x - b)$ **in** *exI*)
using *assms*
apply (*auto intro*!: *continuous-intros*
simp: *dist-commute dist-norm pos-divide-le-eq mult-strict-left-mono*)
done
qed

"Isometry" (up to constant bounds) of injective linear map etc.

lemma *cauchy-isometric*:
assumes *e*: $e > 0$
and *s*: *subspace s*
and *f*: *bounded-linear f*
and *normf*: $\forall\, x{\in}s.\ norm\ (f\ x) \geq e * norm\ x$
and *xs*: $\forall\, n.\ x\ n \in s$
and *cf*: *Cauchy* $(f \circ x)$
shows *Cauchy x*
proof −
interpret *f*: *bounded-linear f* **by** *fact*
{
fix *d* :: *real*
assume $d > 0$
then obtain *N* **where** $N{:}\forall\, n{\geq}N.\ norm\ (f\ (x\ n) - f\ (x\ N)) < e * d$
using *cf*[*unfolded cauchy o-def dist-norm*, *THEN spec*[**where** $x{=}e{*}d$]] *e*
by *auto*
{
fix *n*
assume $n{\geq}N$
have $e * norm\ (x\ n - x\ N) \leq norm\ (f\ (x\ n - x\ N))$
using *subspace-sub*[*OF s*, *of x n x N*]
using *xs*[*THEN spec*[**where** $x{=}N$]] **and** *xs*[*THEN spec*[**where** $x{=}n$]]
using *normf*[*THEN bspec*[**where** $x{=}x\ n - x\ N$]]
by *auto*
also have $norm\ (f\ (x\ n - x\ N)) < e * d$
using ‹$N \leq n$› *N* **unfolding** *f.diff*[*symmetric*] **by** *auto*
finally have $norm\ (x\ n - x\ N) < d$ **using** ‹$e{>}0$› **by** *simp*
}
then have $\exists\, N.\ \forall\, n{\geq}N.\ norm\ (x\ n - x\ N) < d$ **by** *auto*
}
then show *?thesis* **unfolding** *cauchy* **and** *dist-norm* **by** *auto*
qed

lemma *complete-isometric-image*:
assumes $0 < e$
and *s*: *subspace s*
and *f*: *bounded-linear f*
and *normf*: $\forall\, x{\in}s.\ norm(f\ x) \geq e * norm(x)$
and *cs*: *complete s*
shows *complete* $(f\ `\ s)$
proof −
{
fix *g*
assume *as*:$\forall\, n{::}nat.\ g\ n \in f\ `\ s$ **and** *cfg*:*Cauchy g*
then obtain *x* **where** $\forall\, n.\ x\ n \in s \wedge g\ n = f\ (x\ n)$
using *choice*[*of* $\lambda\ n\ xa.\ xa \in s \wedge g\ n = f\ xa$]
by *auto*
then have $x{:}\forall\, n.\ x\ n \in s\ \ \forall\, n.\ g\ n = f\ (x\ n)$
by *auto*

```
    then have f ∘ x = g
      unfolding fun-eq-iff
      by auto
    then obtain l where l∈s and l:(x ⟶ l) sequentially
      using cs[unfolded complete-def, THEN spec[where x=x]]
      using cauchy-isometric[OF ‹0 < e› s f normf] and cfg and x(1)
      by auto
    then have ∃ l∈f ‘ s. (g ⟶ l) sequentially
      using linear-continuous-at[OF f, unfolded continuous-at-sequentially, THEN
spec[where x=x], of l]
      unfolding ‹f ∘ x = g›
      by auto
  }
  then show ?thesis
    unfolding complete-def by auto
qed

lemma injective-imp-isometric:
  fixes f :: 'a::euclidean-space ⇒ 'b::euclidean-space
  assumes s: closed s subspace s
    and f: bounded-linear f ∀ x∈s. f x = 0 ⟶ x = 0
  shows ∃ e>0. ∀ x∈s. norm (f x) ≥ e * norm x
proof (cases s ⊆ {0::'a})
  case True
  {
    fix x
    assume x ∈ s
    then have x = 0 using True by auto
    then have norm x ≤ norm (f x) by auto
  }
  then show ?thesis by (auto intro!: exI[where x=1])
next
  interpret f: bounded-linear f by fact
  case False
  then obtain a where a: a ≠ 0 a ∈ s
    by auto
  from False have s ≠ {}
    by auto
  let ?S = {f x| x. (x ∈ s ∧ norm x = norm a)}
  let ?S' = {x::'a. x∈s ∧ norm x = norm a}
  let ?S'' = {x::'a. norm x = norm a}

  have ?S'' = frontier(cball 0 (norm a))
    by (simp add: sphere-def dist-norm)
  then have compact ?S'' by (metis compact-cball compact-frontier)
  moreover have ?S' = s ∩ ?S'' by auto
  ultimately have compact ?S'
    using closed-Int-compact[of s ?S''] using s(1) by auto
  moreover have ∗:f ‘ ?S' = ?S by auto
```

ultimately have *compact ?S*
using *compact-continuous-image*[*OF linear-continuous-on*[*OF f*(*1*)], *of ?S'*] **by** *auto*
then have *closed ?S* **using** *compact-imp-closed* **by** *auto*
moreover have *?S* ≠ {} **using** *a* **by** *auto*
ultimately obtain *b'* **where** *b'*∈*?S* ∀ *y*∈*?S*. *norm b'* ≤ *norm y*
using *distance-attains-inf*[*of ?S 0*] **unfolding** *dist-0-norm* **by** *auto*
then obtain *b* **where** *b*∈*s*
and *ba*: *norm b* = *norm a*
and *b*: ∀ *x*∈{*x* ∈ *s*. *norm x* = *norm a*}. *norm* (*f b*) ≤ *norm* (*f x*)
unfolding ∗[*symmetric*] **unfolding** *image-iff* **by** *auto*

let *?e* = *norm* (*f b*) / *norm b*
have *norm b* > *0* **using** *ba* **and** *a* **and** *norm-ge-zero* **by** *auto*
moreover have *norm* (*f b*) > *0*
using *f*(*2*)[*THEN bspec*[**where** *x*=*b*], *OF* ‹*b*∈*s*›]
using ‹*norm b* >*0*›
unfolding *zero-less-norm-iff*
by *auto*
ultimately have *0* < *norm* (*f b*) / *norm b* **by** *simp*
moreover
{
fix *x*
assume *x*∈*s*
then have *norm* (*f b*) / *norm b* ∗ *norm x* ≤ *norm* (*f x*)
proof (*cases x*=*0*)
case *True*
then show *norm* (*f b*) / *norm b* ∗ *norm x* ≤ *norm* (*f x*) **by** *auto*
next
case *False*
then have ∗: *0* < *norm a* / *norm x*
using ‹*a*≠*0*›
unfolding *zero-less-norm-iff*[*symmetric*] **by** *simp*
have ∀ *c*. ∀ *x*∈*s*. *c* ∗*R* *x* ∈ *s*
using *s*[*unfolded subspace-def*] **by** *auto*
then have (*norm a* / *norm x*) ∗*R* *x* ∈ {*x* ∈ *s*. *norm x* = *norm a*}
using ‹*x*∈*s*› **and** ‹*x*≠*0*› **by** *auto*
then show *norm* (*f b*) / *norm b* ∗ *norm x* ≤ *norm* (*f x*)
using *b*[*THEN bspec*[**where** *x*=(*norm a* / *norm x*) ∗*R* *x*]]
unfolding *f.scaleR* **and** *ba* **using** ‹*x*≠*0*› ‹*a*≠*0*›
by (*auto simp add*: *mult.commute pos-le-divide-eq pos-divide-le-eq*)
qed
}
ultimately show *?thesis* **by** *auto*
qed

lemma *closed-injective-image-subspace*:
fixes *f* :: *'a*::*euclidean-space* ⇒ *'b*::*euclidean-space*
assumes *subspace s bounded-linear f* ∀ *x*∈*s*. *f x* = *0* ⟶ *x* = *0 closed s*

shows *closed*(*f ‘ s*)
proof −
obtain *e* **where** *e* > *0* **and** *e*: ∀ *x*∈*s*. *e* ∗ *norm x* ≤ *norm* (*f x*)
using *injective-imp-isometric*[*OF assms*(*4*,*1*,*2*,*3*)] **by** *auto*
show *?thesis*
using *complete-isometric-image*[*OF* ‹*e*>*0*› *assms*(*1*,*2*) *e*] **and** *assms*(*4*)
unfolding *complete-eq-closed*[*symmetric*] **by** *auto*
qed

16.33 Some properties of a canonical subspace

lemma *subspace-substandard*:
subspace {*x*::′*a*::*euclidean-space*. (∀ *i*∈*Basis*. *P i* ⟶ *x*•*i* = *0*)}
unfolding *subspace-def* **by** (*auto simp*: *inner-add-left*)

lemma *closed-substandard*:
closed {*x*::′*a*::*euclidean-space*. ∀ *i*∈*Basis*. *P i* −−> *x*•*i* = *0*} (**is** *closed ?A*)
proof −
let *?D* = {*i*∈*Basis*. *P i*}
have *closed* (⋂ *i*∈*?D*. {*x*::′*a*. *x*•*i* = *0*})
by (*simp add*: *closed-INT closed-Collect-eq*)
also have (⋂ *i*∈*?D*. {*x*::′*a*. *x*•*i* = *0*}) = *?A*
by *auto*
finally show *closed ?A* **.**
qed

lemma *dim-substandard*:
assumes *d*: *d* ⊆ *Basis*
shows *dim* {*x*::′*a*::*euclidean-space*. ∀ *i*∈*Basis*. *i* ∉ *d* ⟶ *x*•*i* = *0*} = *card d* (**is** *dim ?A* = -)
proof (*rule dim-unique*)
show *d* ⊆ *?A*
using *d* **by** (*auto simp*: *inner-Basis*)
show *independent d*
using *independent-mono* [*OF independent-Basis d*] **.**
show *?A* ⊆ *span d*
proof (*clarify*)
fix *x* **assume** *x*: ∀ *i*∈*Basis*. *i* ∉ *d* ⟶ *x* • *i* = *0*
have *finite d*
using *finite-subset* [*OF d finite-Basis*] **.**
then have (∑ *i*∈*d*. (*x* • *i*) ∗*R i*) ∈ *span d*
by (*simp add*: *span-setsum span-clauses*)
also have (∑ *i*∈*d*. (*x* • *i*) ∗*R i*) = (∑ *i*∈*Basis*. (*x* • *i*) ∗*R i*)
by (*rule setsum.mono-neutral-cong-left* [*OF finite-Basis d*]) (*auto simp add*: *x*)
finally show *x* ∈ *span d*
unfolding *euclidean-representation* **.**
qed
qed *simp*

Hence closure and completeness of all subspaces.

lemma *ex-card*:
assumes $n \leq$ *card A*
shows $\exists\, S \subseteq A.$ *card* $S = n$
proof *cases*
assume *finite A*
from *ex-bij-betw-nat-finite*[*OF this*] **obtain** f **where** f: *bij-betw f* $\{0..<$*card A*$\}$ *A* **..**
moreover from f ‹$n \leq$ *card A*› **have** $\{..< n\} \subseteq \{..<$ *card A*$\}$ *inj-on f* $\{..< n\}$
by (*auto simp*: *bij-betw-def intro*: *subset-inj-on*)
ultimately have f ‘ $\{..< n\} \subseteq A$ *card* $(f$ ‘ $\{..< n\}) = n$
by (*auto simp*: *bij-betw-def card-image*)
then show *?thesis* **by** *blast*
next
assume ¬ *finite A*
with ‹$n \leq$ *card A*› **show** *?thesis* **by** *force*
qed

lemma *closed-subspace*:
fixes s :: *'a*::*euclidean-space set*
assumes *subspace s*
shows *closed s*
proof −
have *dim s* $\leq$ *card* (*Basis* :: *'a set*)
using *dim-subset-UNIV* **by** *auto*
with *ex-card*[*OF this*] **obtain** d :: *'a set* **where** t: *card d* $=$ *dim s* **and** d: $d \subseteq$ *Basis*
by *auto*
let *?t* $= \{x$::*'a*. $\forall\, i \in$*Basis*. $i \notin d \longrightarrow x \cdot i = 0\}$
have $\exists f.$ *linear f* $\wedge$ *f* ‘ $\{x$::*'a*. $\forall\, i \in$*Basis*. $i \notin d \longrightarrow x \cdot i = 0\} = s \wedge$
inj-on f $\{x$::*'a*. $\forall\, i \in$*Basis*. $i \notin d \longrightarrow x \cdot i = 0\}$
using *dim-substandard*[*of d*] *t d assms*
by (*intro subspace-isomorphism*[*OF subspace-substandard*[*of* $\lambda i.\ i \notin d$]]) (*auto simp*: *inner-Basis*)
then obtain f **where** f:
linear f
f ‘ $\{x.\ \forall\, i \in$*Basis*. $i \notin d \longrightarrow x \cdot i = 0\} = s$
inj-on f $\{x.\ \forall\, i \in$*Basis*. $i \notin d \longrightarrow x \cdot i = 0\}$
by *blast*
interpret f: *bounded-linear f*
using f **unfolding** *linear-conv-bounded-linear* **by** *auto*
{
fix x
have $x \in$ *?t* $\Longrightarrow$ *f x* $= 0 \Longrightarrow x = 0$
using *f.zero d f*(*3*)[*THEN inj-onD*, *of x 0*] **by** *auto*
}
moreover have *closed ?t* **using** *closed-substandard* **.**
moreover have *subspace ?t* **using** *subspace-substandard* **.**
ultimately show *?thesis*

using *closed-injective-image-subspace*[*of ?t f*]
unfolding *f*(*2*) **using** *f*(*1*) **unfolding** *linear-conv-bounded-linear* **by** *auto*
qed

lemma *complete-subspace*:
fixes *s* :: ('*a*::*euclidean-space*) *set*
shows *subspace s* ⟹ *complete s*
using *complete-eq-closed closed-subspace* **by** *auto*

lemma *dim-closure*:
fixes *s* :: ('*a*::*euclidean-space*) *set*
shows *dim*(*closure s*) = *dim s* (**is** *?dc* = *?d*)
proof −
have *?dc* ≤ *?d* **using** *closure-minimal*[*OF span-inc*, *of s*]
using *closed-subspace*[*OF subspace-span*, *of s*]
using *dim-subset*[*of closure s span s*]
unfolding *dim-span*
by *auto*
then show *?thesis* **using** *dim-subset*[*OF closure-subset*, *of s*]
by *auto*
qed

16.34 Affine transformations of intervals

lemma *real-affinity-le*:
0 < (*m*::'*a*::*linordered-field*) ⟹ (*m* ∗ *x* + *c* ≤ *y* ⟷ *x* ≤ *inverse*(*m*) ∗ *y* + −(*c* / *m*))
by (*simp add*: *field-simps*)

lemma *real-le-affinity*:
0 < (*m*::'*a*::*linordered-field*) ⟹ (*y* ≤ *m* ∗ *x* + *c* ⟷ *inverse*(*m*) ∗ *y* + −(*c* / *m*) ≤ *x*)
by (*simp add*: *field-simps*)

lemma *real-affinity-lt*:
0 < (*m*::'*a*::*linordered-field*) ⟹ (*m* ∗ *x* + *c* < *y* ⟷ *x* < *inverse*(*m*) ∗ *y* + −(*c* / *m*))
by (*simp add*: *field-simps*)

lemma *real-lt-affinity*:
0 < (*m*::'*a*::*linordered-field*) ⟹ (*y* < *m* ∗ *x* + *c* ⟷ *inverse*(*m*) ∗ *y* + −(*c* / *m*) < *x*)
by (*simp add*: *field-simps*)

lemma *real-affinity-eq*:
(*m*::'*a*::*linordered-field*) ≠ *0* ⟹ (*m* ∗ *x* + *c* = *y* ⟷ *x* = *inverse*(*m*) ∗ *y* + −(*c* / *m*))
by (*simp add*: *field-simps*)

lemma *real-eq-affinity*:
$(m::'a::linordered\text{-}field) \neq 0 \Longrightarrow (y = m * x + c \longleftrightarrow inverse(m) * y + -(c / m) = x)$
by (*simp add*: *field-simps*)

16.35 Banach fixed point theorem (not really topological...)

theorem *banach-fix*:
assumes *s*: *complete s* $s \neq \{\}$
and *c*: $0 \leq c$ $c < 1$
and *f*: $(f \text{ ` } s) \subseteq s$
and *lipschitz*: $\forall x{\in}s.\ \forall y{\in}s.\ dist\ (f\ x)\ (f\ y) \leq c * dist\ x\ y$
shows $\exists!x{\in}s.\ f\ x = x$
proof −
have $1 - c > 0$ **using** *c* **by** *auto*

from *s*(*2*) **obtain** *z0* **where** $z0 \in s$ **by** *auto*
def $z \equiv \lambda n.\ (f\ \hat{}\hat{}\ n)\ z0$
{
fix *n* :: *nat*
have $z\ n \in s$ **unfolding** *z-def*
proof (*induct n*)
case *0*
then show *?case* **using** ‹*z0* ∈ *s*› **by** *auto*
next
case *Suc*
then show *?case* **using** *f* **by** *auto* **qed**
} **note** *z-in-s* = *this*

def $d \equiv dist\ (z\ 0)\ (z\ 1)$

have *fzn*:$\bigwedge n.\ f\ (z\ n) = z\ (Suc\ n)$ **unfolding** *z-def* **by** *auto*
{
fix *n* :: *nat*
have $dist\ (z\ n)\ (z\ (Suc\ n)) \leq (c\ \hat{}\ n) * d$
proof (*induct n*)
case *0*
then show *?case*
unfolding *d-def* **by** *auto*
next
case (*Suc m*)
then have $c * dist\ (z\ m)\ (z\ (Suc\ m)) \leq c\ \hat{}\ Suc\ m * d$
using ‹*0* ≤ *c*›
using *mult-left-mono*[*of dist* (*z m*) (*z* (*Suc m*)) *c* ^ *m* * *d c*]
by *auto*
then show *?case*
using *lipschitz*[*THEN bspec*[**where** *x*=*z m*], *OF z-in-s*, *THEN bspec*[**where** *x*=*z* (*Suc m*)], *OF z-in-s*]
unfolding *fzn* **and** *mult-le-cancel-left*

```
    by auto
  qed
} note cf-z = this

{
  fix n m :: nat
  have (1 - c) * dist (z m) (z (m+n)) ≤ (c ^ m) * d * (1 - c ^ n)
  proof (induct n)
    case 0
    show ?case by auto
  next
    case (Suc k)
    have (1 - c) * dist (z m) (z (m + Suc k)) ≤
      (1 - c) * (dist (z m) (z (m + k)) + dist (z (m + k)) (z (Suc (m + k))))
      using dist-triangle and c by (auto simp add: dist-triangle)
    also have ... ≤ (1 - c) * (dist (z m) (z (m + k)) + c ^ (m + k) * d)
      using cf-z[of m + k] and c by auto
    also have ... ≤ c ^ m * d * (1 - c ^ k) + (1 - c) * c ^ (m + k) * d
      using Suc by (auto simp add: field-simps)
    also have ... = (c ^ m) * (d * (1 - c ^ k) + (1 - c) * c ^ k * d)
      unfolding power-add by (auto simp add: field-simps)
    also have ... ≤ (c ^ m) * d * (1 - c ^ Suc k)
      using c by (auto simp add: field-simps)
    finally show ?case by auto
  qed
} note cf-z2 = this
{
  fix e :: real
  assume e > 0
  then have ∃ N. ∀ m n. N ≤ m ∧ N ≤ n ⟶ dist (z m) (z n) < e
  proof (cases d = 0)
    case True
    have *: ⋀x. ((1 - c) * x ≤ 0) = (x ≤ 0) using ‹1 - c > 0›
      by (metis mult-zero-left mult.commute real-mult-le-cancel-iff1)
    from True have ⋀n. z n = z0 using cf-z2[of 0] and c unfolding z-def
      by (simp add: *)
    then show ?thesis using ‹e>0› by auto
  next
    case False
    then have d>0 unfolding d-def using zero-le-dist[of z 0 z 1]
      by (metis False d-def less-le)
    hence 0 < e * (1 - c) / d
      using ‹e>0› and ‹1-c>0› by auto
    then obtain N where N:c ^ N < e * (1 - c) / d
      using real-arch-pow-inv[of e * (1 - c) / d c] and c by auto
    {
      fix m n::nat
      assume m>n and as:m≥N n≥N
      have *:c ^ n ≤ c ^ N using ‹n≥N› and c
```

```
      using power-decreasing[OF ‹n≥N›, of c] by auto
    have 1 − c ^ (m − n) > 0
      using c and power-strict-mono[of c 1 m − n] using ‹m>n› by auto
    hence **: d * (1 − c ^ (m − n)) / (1 − c) > 0
      using ‹d>0› ‹0 < 1 − c› by auto

    have dist (z m) (z n) ≤ c ^ n * d * (1 − c ^ (m − n)) / (1 − c)
      using cf-z2[of n m − n] and ‹m>n›
      unfolding pos-le-divide-eq[OF ‹1−c>0›]
      by (auto simp add: mult.commute dist-commute)
    also have ... ≤ c ^ N * d * (1 − c ^ (m − n)) / (1 − c)
      using mult-right-mono[OF * order-less-imp-le[OF **]]
      unfolding mult.assoc by auto
    also have ... < (e * (1 − c) / d) * d * (1 − c ^ (m − n)) / (1 − c)
      using mult-strict-right-mono[OF N **] unfolding mult.assoc by auto
    also have ... = e * (1 − c ^ (m − n))
      using c and ‹d>0› and ‹1 − c > 0› by auto
    also have ... ≤ e using c and ‹1 − c ^ (m − n) > 0› and ‹e>0›
      using mult-right-le-one-le[of e 1 − c ^ (m − n)] by auto
    finally have  dist (z m) (z n) < e by auto
   } note * = this
   {
    fix m n :: nat
    assume as: N ≤ m N ≤ n
    then have dist (z n) (z m) < e
    proof (cases n = m)
     case True
     then show ?thesis using ‹e>0› by auto
    next
     case False
     then show ?thesis using as and *[of n m] *[of m n]
       unfolding nat-neq-iff by (auto simp add: dist-commute)
    qed
   }
   then show ?thesis by auto
  qed
 }
 then have Cauchy z
  unfolding cauchy-def by auto
 then obtain x where x∈s and x:(z ⟶ x) sequentially
   using s(1)[unfolded compact-def complete-def, THEN spec[where x=z]] and
z-in-s by auto

 def e ≡ dist (f x) x
 have e = 0
 proof (rule ccontr)
  assume e ≠ 0
  then have e > 0
   unfolding e-def using zero-le-dist[of f x x]
```

```
    by (metis dist-eq-0-iff dist-nz e-def)
   then obtain N where N:∀ n≥N. dist (z n) x < e / 2
    using x[unfolded lim-sequentially, THEN spec[where x=e/2]] by auto
   then have N':dist (z N) x < e / 2 by auto

   have *: c * dist (z N) x ≤ dist (z N) x
    unfolding mult-le-cancel-right2
    using zero-le-dist[of z N x] and c
    by (metis dist-eq-0-iff dist-nz order-less-asym less-le)
   have dist (f (z N)) (f x) ≤ c * dist (z N) x
    using lipschitz[THEN bspec[where x=z N], THEN bspec[where x=x]]
    using z-in-s[of N] ‹x∈s›
    using c
    by auto
   also have ... < e / 2
    using N' and c using * by auto
   finally show False
    unfolding fzn
    using N[THEN spec[where x=Suc N]] and dist-triangle-half-r[of z (Suc N)
f x e x]
    unfolding e-def
    by auto
  qed
  then have f x = x unfolding e-def by auto
  moreover
  {
   fix y
   assume f y = y y∈s
   then have dist x y ≤ c * dist x y
    using lipschitz[THEN bspec[where x=x], THEN bspec[where x=y]]
    using ‹x∈s› and ‹f x = x›
    by auto
   then have dist x y = 0
    unfolding mult-le-cancel-right1
    using c and zero-le-dist[of x y]
    by auto
   then have y = x by auto
  }
  ultimately show ?thesis using ‹x∈s› by blast+
qed
```

16.36 Edelstein fixed point theorem

```
theorem edelstein-fix:
  fixes s :: 'a::metric-space set
  assumes s: compact s s ≠ {}
    and gs: (g ' s) ⊆ s
    and dist: ∀ x∈s. ∀ y∈s. x ≠ y ⟶ dist (g x) (g y) < dist x y
  shows ∃!x∈s. g x = x
```

proof −
let *?D* = (λ*x*. (*x*, *x*)) ‘ *s*
have *D*: *compact ?D ?D* ≠ {}
by (*rule compact-continuous-image*)
(*auto intro*!: *s continuous-Pair continuous-ident simp*: *continuous-on-eq-continuous-within*)

have ⋀*x y e*. *x* ∈ *s* ⟹ *y* ∈ *s* ⟹ 0 < *e* ⟹ *dist y x* < *e* ⟹ *dist* (*g y*) (*g x*)
< *e*
using *dist* **by** *fastforce*
then have *continuous-on s g*
unfolding *continuous-on-iff* **by** *auto*
then have *cont*: *continuous-on ?D* (λ*x*. *dist* ((*g* ∘ *fst*) *x*) (*snd x*))
unfolding *continuous-on-eq-continuous-within*
by (*intro continuous-dist ballI continuous-within-compose*)
(*auto intro*!: *continuous-fst continuous-snd continuous-ident simp*: *image-image*)

obtain *a* **where** *a* ∈ *s* **and** *le*: ⋀*x*. *x* ∈ *s* ⟹ *dist* (*g a*) *a* ≤ *dist* (*g x*) *x*
using *continuous-attains-inf*[*OF D cont*] **by** *auto*

have *g a* = *a*
proof (*rule ccontr*)
assume *g a* ≠ *a*
with ‹*a* ∈ *s*› *gs* **have** *dist* (*g* (*g a*)) (*g a*) < *dist* (*g a*) *a*
by (*intro dist*[*rule-format*]) *auto*
moreover have *dist* (*g a*) *a* ≤ *dist* (*g* (*g a*)) (*g a*)
using ‹*a* ∈ *s*› *gs* **by** (*intro le*) *auto*
ultimately show *False* **by** *auto*
qed
moreover have ⋀*x*. *x* ∈ *s* ⟹ *g x* = *x* ⟹ *x* = *a*
using *dist*[*THEN bspec*[**where** *x*=*a*]] ‹*g a* = *a*› **and** ‹*a*∈*s*› **by** *auto*
ultimately show ∃!*x*∈*s*. *g x* = *x* **using** ‹*a* ∈ *s*› **by** *blast*
qed

lemma *cball-subset-cball-iff*:
fixes *a* :: ′*a* :: *euclidean-space*
shows *cball a r* ⊆ *cball a*′ *r*′ ⟷ *dist a a*′ + *r* ≤ *r*′ ∨ *r* < 0
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
proof (*cases r* < 0)
case *True* **then show** *?rhs* **by** *simp*
next
case *False*
then have [*simp*]: *r* ≥ 0 **by** *simp*
have *norm* (*a* − *a*′) + *r* ≤ *r*′
proof (*cases a* = *a*′)
case *True* **then show** *?thesis*

```
        using subsetD [where c = a + r *R (SOME i. i ∈ Basis), OF ‹?lhs›]
subsetD [where c = a, OF ‹?lhs›]
       by (force simp add: SOME-Basis dist-norm)
    next
      case False
      have norm (a' − (a + (r / norm (a − a')) *R (a − a'))) = norm (a' − a
− (r / norm (a − a')) *R (a − a'))
        by (simp add: algebra-simps)
      also have ... = norm ((−1 − (r / norm (a − a'))) *R (a − a'))
        by (simp add: algebra-simps)
      also have ... = |− norm (a − a') − r|
        using ‹a ≠ a'› by (simp add: abs-mult-pos field-simps)
      finally have [simp]: norm (a' − (a + (r / norm (a − a')) *R (a − a'))) =
|norm (a − a') + r| by linarith
      show ?thesis
        using subsetD [where c = a' + (1 + r / norm(a − a')) *R (a − a'), OF
‹?lhs›] ‹a ≠ a'›
        by (simp add: dist-norm scaleR-add-left)
    qed
    then show ?rhs by (simp add: dist-norm)
  qed
next
  assume ?rhs then show ?lhs
    apply (auto simp: ball-def dist-norm)
   apply (metis add.commute add-le-cancel-right dist-norm dist-triangle3 order-trans)
    done
qed

lemma cball-subset-ball-iff:
  fixes a :: 'a :: euclidean-space
  shows cball a r ⊆ ball a' r' ⟷ dist a a' + r < r' ∨ r < 0
        (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
  proof (cases r < 0)
    case True then show ?rhs by simp
  next
    case False
    then have [simp]: r ≥ 0 by simp
    have norm (a − a') + r < r'
    proof (cases a = a')
      case True then show ?thesis
        using subsetD [where c = a + r *R (SOME i. i ∈ Basis), OF ‹?lhs›]
subsetD [where c = a, OF ‹?lhs›]
       by (force simp add: SOME-Basis dist-norm)
    next
      case False
      { assume norm (a − a') + r ≥ r'
```

```
      then have |r' − norm (a − a')| ≤ r
        apply (simp split: abs-split)
        by (metis ‹0 ≤ r› ‹?lhs› centre-in-cball dist-commute dist-norm less-asym
mem-ball subset-eq)
      then have False
      using subsetD [where c = a + (r' / norm(a − a') − 1) *R (a − a'), OF
‹?lhs›] ‹a ≠ a'›
      apply (simp add: dist-norm field-simps)
      apply (simp add: diff-divide-distrib scaleR-left-diff-distrib)
      done
    }
    then show ?thesis by force
  qed
  then show ?rhs by (simp add: dist-norm)
 qed
next
 assume ?rhs then show ?lhs
  apply (auto simp: ball-def dist-norm )
 apply (metis add.commute add-le-cancel-right dist-norm dist-triangle3 le-less-trans)
  done
qed

lemma ball-subset-cball-iff:
 fixes a :: 'a :: euclidean-space
 shows ball a r ⊆ cball a' r' ⟷ dist a a' + r ≤ r' ∨ r ≤ 0
      (is ?lhs = ?rhs)
proof (cases r ≤ 0)
 case True then show ?thesis
  using dist-not-less-zero less-le-trans by force
next
 case False show ?thesis
 proof
  assume ?lhs
  then have (cball a r ⊆ cball a' r')
   by (metis False closed-cball closure-ball closure-closed closure-mono not-less)
  then show ?rhs
   using False cball-subset-cball-iff by fastforce
 next
  assume ?rhs with False show ?lhs
   using ball-subset-cball cball-subset-cball-iff by blast
 qed
qed

lemma ball-subset-ball-iff:
 fixes a :: 'a :: euclidean-space
 shows ball a r ⊆ ball a' r' ⟷ dist a a' + r ≤ r' ∨ r ≤ 0
      (is ?lhs = ?rhs)
proof (cases r ≤ 0)
 case True then show ?thesis
```

using *dist-not-less-zero less-le-trans* **by** *force*
next
case *False* **show** *?thesis*
proof
assume *?lhs*
then have $0 < r'$
by (*metis* (*no-types*) *False* ‹*?lhs*› *centre-in-ball dist-norm le-less-trans mem-ball norm-ge-zero not-less set-mp*)
then have (*cball a r* ⊆ *cball a' r'*)
by (*metis False*‹*?lhs*› *closure-ball closure-mono not-less*)
then show *?rhs*
using *False cball-subset-cball-iff* **by** *fastforce*
next
assume *?rhs* **then show** *?lhs*
apply (*auto simp*: *ball-def*)
apply (*metis add.commute add-le-cancel-right dist-commute dist-triangle-lt not-le order-trans*)
using *dist-not-less-zero order.strict-trans2* **apply** *blast*
done
qed
qed

lemma *ball-eq-ball-iff*:
fixes *x* :: *'a* :: *euclidean-space*
shows *ball x d* = *ball y e* ⟷ *d* ≤ *0* ∧ *e* ≤ *0* ∨ *x*=*y* ∧ *d*=*e*
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
proof (*cases d* ≤ *0* ∨ *e* ≤ *0*)
case *True*
with ‹*?lhs*› **show** *?rhs*
by *safe* (*simp-all only*: *ball-eq-empty* [*of y e*, *symmetric*] *ball-eq-empty* [*of x d*, *symmetric*])
next
case *False*
with ‹*?lhs*› **show** *?rhs*
apply (*auto simp add*: *set-eq-subset ball-subset-ball-iff dist-norm norm-minus-commute algebra-simps*)
apply (*metis add-le-same-cancel1 le-add-same-cancel1 norm-ge-zero norm-pths*(*2*) *order-trans*)
apply (*metis add-increasing2 add-le-imp-le-right eq-iff norm-ge-zero*)
done
qed
next
assume *?rhs* **then show** *?lhs*
by (*auto simp add*: *set-eq-subset ball-subset-ball-iff*)
qed

lemma *cball-eq-cball-iff*:
fixes $x :: 'a :: euclidean\text{-}space$
shows $cball\ x\ d = cball\ y\ e \longleftrightarrow d < 0 \wedge e < 0 \vee x{=}y \wedge d{=}e$
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
proof (*cases* $d < 0 \vee e < 0$)
case *True*
with ‹*?lhs*› **show** *?rhs*
by *safe* (*simp-all only*: *cball-eq-empty* [*of y e*, *symmetric*] *cball-eq-empty* [*of x d*, *symmetric*])
next
case *False*
with ‹*?lhs*› **show** *?rhs*
apply (*auto simp add*: *set-eq-subset cball-subset-cball-iff dist-norm norm-minus-commute algebra-simps*)
apply (*metis add-le-same-cancel1 le-add-same-cancel1 norm-ge-zero norm-pths*(*2*) *order-trans*)
apply (*metis add-increasing2 add-le-imp-le-right eq-iff norm-ge-zero*)
done
qed
next
assume *?rhs* **then show** *?lhs*
by (*auto simp add*: *set-eq-subset cball-subset-cball-iff*)
qed

lemma *ball-eq-cball-iff*:
fixes $x :: 'a :: euclidean\text{-}space$
shows $ball\ x\ d = cball\ y\ e \longleftrightarrow d \leq 0 \wedge e < 0$ (**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
apply (*auto simp add*: *set-eq-subset ball-subset-cball-iff cball-subset-ball-iff algebra-simps*)
apply (*metis add-increasing2 add-le-cancel-right add-less-same-cancel1 dist-not-less-zero less-le-trans zero-le-dist*)
apply (*metis add-less-same-cancel1 dist-not-less-zero less-le-trans not-le*)
using ‹*?lhs*› *ball-eq-empty cball-eq-empty* **apply** *blast+*
done
next
assume *?rhs* **then show** *?lhs* **by** *auto*
qed

lemma *cball-eq-ball-iff*:
fixes $x :: 'a :: euclidean\text{-}space$
shows $cball\ x\ d = ball\ y\ e \longleftrightarrow d < 0 \wedge e \leq 0$
using *ball-eq-cball-iff* **by** *blast*

no-notation
eucl-less (**infix** $<e$ *50*)

end

17 Convexity in real vector spaces

theory *Convex*
imports *Product-Vector*
begin

17.1 Convexity

definition *convex* :: *'a::real-vector set* ⇒ *bool*
where *convex s* ⟷ (∀ *x*∈*s*. ∀ *y*∈*s*. ∀ *u*≥*0*. ∀ *v*≥*0*. *u* + *v* = *1* ⟶ *u* *$_R$ *x* + *v* *$_R$ *y* ∈ *s*)

lemma *convexI*:
assumes ⋀*x y u v*. *x* ∈ *s* ⟹ *y* ∈ *s* ⟹ *0* ≤ *u* ⟹ *0* ≤ *v* ⟹ *u* + *v* = *1* ⟹ *u* *$_R$ *x* + *v* *$_R$ *y* ∈ *s*
shows *convex s*
using *assms* **unfolding** *convex-def* **by** *fast*

lemma *convexD*:
assumes *convex s* **and** *x* ∈ *s* **and** *y* ∈ *s* **and** *0* ≤ *u* **and** *0* ≤ *v* **and** *u* + *v* = *1*
shows *u* *$_R$ *x* + *v* *$_R$ *y* ∈ *s*
using *assms* **unfolding** *convex-def* **by** *fast*

lemma *convex-alt*:
convex s ⟷ (∀ *x*∈*s*. ∀ *y*∈*s*. ∀ *u*. *0* ≤ *u* ∧ *u* ≤ *1* ⟶ ((*1* − *u*) *$_R$ *x* + *u* *$_R$ *y*) ∈ *s*)
(**is** - ⟷ *?alt*)
proof
assume *alt*[*rule-format*]: *?alt*
{
fix *x y* **and** *u v* :: *real*
assume *mem*: *x* ∈ *s y* ∈ *s*
assume *0* ≤ *u 0* ≤ *v*
moreover
assume *u* + *v* = *1*
then have *u* = *1* − *v* **by** *auto*
ultimately have *u* *$_R$ *x* + *v* *$_R$ *y* ∈ *s*
using *alt*[*OF mem*] **by** *auto*
}
then show *convex s*
unfolding *convex-def* **by** *auto*
qed (*auto simp*: *convex-def*)

lemma *convexD-alt*:

assumes *convex s a* ∈ *s b* ∈ *s* 0 ≤ *u u* ≤ *1*
shows ((*1* − *u*) *∗R a* + *u ∗R b*) ∈ *s*
using *assms* **unfolding** *convex-alt* **by** *auto*

lemma *mem-convex-alt*:
assumes *convex S x* ∈ *S y* ∈ *S u* ≥ *0 v* ≥ *0 u* + *v* > *0*
shows ((*u*/(*u*+*v*)) *∗R x* + (*v*/(*u*+*v*)) *∗R y*) ∈ *S*
apply (*rule convexD*)
using *assms*
apply (*simp-all add*: *zero-le-divide-iff add-divide-distrib* [*symmetric*])
done

lemma *convex-empty*[*intro*,*simp*]: *convex* {}
unfolding *convex-def* **by** *simp*

lemma *convex-singleton*[*intro*,*simp*]: *convex* {*a*}
unfolding *convex-def* **by** (*auto simp*: *scaleR-left-distrib*[*symmetric*])

lemma *convex-UNIV*[*intro*,*simp*]: *convex UNIV*
unfolding *convex-def* **by** *auto*

lemma *convex-Inter*: (∀ *s*∈*f*. *convex s*) ⟹ *convex*(⋂*f*)
unfolding *convex-def* **by** *auto*

lemma *convex-Int*: *convex s* ⟹ *convex t* ⟹ *convex* (*s* ∩ *t*)
unfolding *convex-def* **by** *auto*

lemma *convex-INT*: ∀ *i*∈*A*. *convex* (*B i*) ⟹ *convex* (⋂ *i*∈*A*. *B i*)
unfolding *convex-def* **by** *auto*

lemma *convex-Times*: *convex s* ⟹ *convex t* ⟹ *convex* (*s* × *t*)
unfolding *convex-def* **by** *auto*

lemma *convex-halfspace-le*: *convex* {*x*. *inner a x* ≤ *b*}
unfolding *convex-def*
by (*auto simp*: *inner-add intro*!: *convex-bound-le*)

lemma *convex-halfspace-ge*: *convex* {*x*. *inner a x* ≥ *b*}
proof −
have ∗: {*x*. *inner a x* ≥ *b*} = {*x*. *inner* (−*a*) *x* ≤ −*b*}
by *auto*
show *?thesis*
unfolding ∗ **using** *convex-halfspace-le*[*of* −*a* −*b*] **by** *auto*
qed

lemma *convex-hyperplane*: *convex* {*x*. *inner a x* = *b*}
proof −
have ∗: {*x*. *inner a x* = *b*} = {*x*. *inner a x* ≤ *b*} ∩ {*x*. *inner a x* ≥ *b*}
by *auto*

show *?thesis* **using** *convex-halfspace-le convex-halfspace-ge*
by (*auto intro*!: *convex-Int simp*: ∗)
qed

lemma *convex-halfspace-lt*: *convex* {*x*. *inner a x* < *b*}
unfolding *convex-def*
by (*auto simp*: *convex-bound-lt inner-add*)

lemma *convex-halfspace-gt*: *convex* {*x*. *inner a x* > *b*}
using *convex-halfspace-lt*[*of* −*a* −*b*] **by** *auto*

lemma *convex-real-interval* [*iff*]:
fixes *a b* :: *real*
shows *convex* {*a*..} **and** *convex* {..*b*}
and *convex* {*a*<..} **and** *convex* {..<*b*}
and *convex* {*a*..*b*} **and** *convex* {*a*<..*b*}
and *convex* {*a*..<*b*} **and** *convex* {*a*<..<*b*}
proof −
have {*a*..} = {*x*. *a* ≤ *inner 1 x*}
by *auto*
then show *1*: *convex* {*a*..}
by (*simp only*: *convex-halfspace-ge*)
have {..*b*} = {*x*. *inner 1 x* ≤ *b*}
by *auto*
then show *2*: *convex* {..*b*}
by (*simp only*: *convex-halfspace-le*)
have {*a*<..} = {*x*. *a* < *inner 1 x*}
by *auto*
then show *3*: *convex* {*a*<..}
by (*simp only*: *convex-halfspace-gt*)
have {..<*b*} = {*x*. *inner 1 x* < *b*}
by *auto*
then show *4*: *convex* {..<*b*}
by (*simp only*: *convex-halfspace-lt*)
have {*a*..*b*} = {*a*..} ∩ {..*b*}
by *auto*
then show *convex* {*a*..*b*}
by (*simp only*: *convex-Int 1 2*)
have {*a*<..*b*} = {*a*<..} ∩ {..*b*}
by *auto*
then show *convex* {*a*<..*b*}
by (*simp only*: *convex-Int 3 2*)
have {*a*..<*b*} = {*a*..} ∩ {..<*b*}
by *auto*
then show *convex* {*a*..<*b*}
by (*simp only*: *convex-Int 1 4*)
have {*a*<..<*b*} = {*a*<..} ∩ {..<*b*}
by *auto*
then show *convex* {*a*<..<*b*}

by (*simp only*: *convex-Int 3 4*)
qed

lemma *convex-Reals*: *convex* $\mathbb{R}$
by (*simp add*: *convex-def scaleR-conv-of-real*)

17.2 Explicit expressions for convexity in terms of arbitrary sums

lemma *convex-setsum*:
fixes C :: $'a$::*real-vector set*
assumes *finite s*
and *convex C*
and $(\sum\ i \in s.\ a\ i) = 1$
assumes $\bigwedge i.\ i \in s \Longrightarrow a\ i \geq 0$
and $\bigwedge i.\ i \in s \Longrightarrow y\ i \in C$
shows $(\sum\ j \in s.\ a\ j *_R y\ j) \in C$
using *assms*(*1*,*3*,*4*,*5*)
proof (*induct arbitrary*: *a set*: *finite*)
case *empty*
then show *?case* **by** *simp*
next
case (*insert i s*) **note** *IH* = *this*(*3*)
have *a i* + *setsum a s* = *1*
and $0 \leq a\ i$
and $\forall j{\in}s.\ 0 \leq a\ j$
and $y\ i \in C$
and $\forall j{\in}s.\ y\ j \in C$
using *insert.hyps*(*1*,*2*) *insert.prems* **by** *simp-all*
then have $0 \leq$ *setsum a s*
by (*simp add*: *setsum-nonneg*)
have $a\ i *_R y\ i + (\sum j{\in}s.\ a\ j *_R y\ j) \in C$
proof (*cases*)
assume *z*: *setsum a s* = *0*
with ‹*a i* + *setsum a s* = *1*› **have** *a i* = *1*
by *simp*
from *setsum-nonneg-0* [*OF* ‹*finite s*› - *z*] ‹$\forall j{\in}s.\ 0 \leq a\ j$› **have** $\forall j{\in}s.\ a\ j = 0$
by *simp*
show *?thesis* **using** ‹*a i* = *1*› **and** ‹$\forall j{\in}s.\ a\ j = 0$› **and** ‹$y\ i \in C$›
by *simp*
next
assume *nz*: *setsum a s* ≠ *0*
with ‹$0 \leq$ *setsum a s*› **have** $0 <$ *setsum a s*
by *simp*
then have $(\sum j{\in}s.\ (a\ j\ /\ \textit{setsum}\ a\ s) *_R y\ j) \in C$
using ‹$\forall j{\in}s.\ 0 \leq a\ j$› **and** ‹$\forall j{\in}s.\ y\ j \in C$›
by (*simp add*: *IH setsum-divide-distrib* [*symmetric*])
from ‹*convex C*› **and** ‹$y\ i \in C$› **and** *this* **and** ‹$0 \leq a\ i$›
and ‹$0 \leq$ *setsum a s*› **and** ‹*a i* + *setsum a s* = *1*›

```
    have a i *R y i + setsum a s *R (∑j∈s. (a j / setsum a s) *R y j) ∈ C
      by (rule convexD)
    then show ?thesis
      by (simp add: scaleR-setsum-right nz)
  qed
  then show ?case using ‹finite s› and ‹i ∉ s›
    by simp
qed

lemma convex:
  convex s ⟷ (∀(k::nat) u x. (∀i. 1≤i ∧ i≤k ⟶ 0 ≤ u i ∧ x i ∈s) ∧ (setsum
u {1..k} = 1)
      ⟶ setsum (λi. u i *R x i) {1..k} ∈ s)
proof safe
  fix k :: nat
  fix u :: nat ⇒ real
  fix x
  assume convex s
    ∀i. 1 ≤ i ∧ i ≤ k ⟶ 0 ≤ u i ∧ x i ∈ s
    setsum u {1..k} = 1
  with convex-setsum[of {1 .. k} s] show (∑j∈{1 .. k}. u j *R x j) ∈ s
    by auto
next
  assume *: ∀k u x. (∀ i :: nat. 1 ≤ i ∧ i ≤ k ⟶ 0 ≤ u i ∧ x i ∈ s) ∧ setsum
u {1..k} = 1
    ⟶ (∑i = 1..k. u i *R (x i :: 'a)) ∈ s
  {
    fix μ :: real
    fix x y :: 'a
    assume xy: x ∈ s y ∈ s
    assume mu: μ ≥ 0 μ ≤ 1
    let ?u = λi. if (i :: nat) = 1 then μ else 1 − μ
    let ?x = λi. if (i :: nat) = 1 then x else y
    have {1 :: nat .. 2} ∩ − {x. x = 1} = {2}
      by auto
    then have card: card ({1 :: nat .. 2} ∩ − {x. x = 1}) = 1
      by simp
    then have setsum ?u {1 .. 2} = 1
      using setsum.If-cases[of {(1 :: nat) .. 2} λ x. x = 1 λ x. μ λ x. 1 − μ]
      by auto
    with *[rule-format, of 2 ?u ?x] have s: (∑j ∈ {1..2}. ?u j *R ?x j) ∈ s
      using mu xy by auto
    have grarr: (∑j ∈ {Suc (Suc 0)..2}. ?u j *R ?x j) = (1 − μ) *R y
      using setsum-head-Suc[of Suc (Suc 0) 2 λ j. (1 − μ) *R y] by auto
    from setsum-head-Suc[of Suc 0 2 λ j. ?u j *R ?x j, simplified this]
    have (∑j ∈ {1..2}. ?u j *R ?x j) = μ *R x + (1 − μ) *R y
      by auto
    then have (1 − μ) *R y + μ *R x ∈ s
      using s by (auto simp: add.commute)
```

```
  }
  then show convex s
    unfolding convex-alt by auto
qed


lemma convex-explicit:
  fixes s :: 'a::real-vector set
  shows convex s ⟷
    (∀ t u. finite t ∧ t ⊆ s ∧ (∀ x∈t. 0 ≤ u x) ∧ setsum u t = 1 ⟶ setsum (λx.
u x *R x) t ∈ s)
proof safe
  fix t
  fix u :: 'a ⇒ real
  assume convex s
    and finite t
    and t ⊆ s ∀ x∈t. 0 ≤ u x setsum u t = 1
  then show (∑ x∈t. u x *R x) ∈ s
    using convex-setsum[of t s u λ x. x] by auto
next
  assume *: ∀ t. ∀ u. finite t ∧ t ⊆ s ∧ (∀ x∈t. 0 ≤ u x) ∧
    setsum u t = 1 ⟶ (∑ x∈t. u x *R x) ∈ s
  show convex s
    unfolding convex-alt
  proof safe
    fix x y
    fix μ :: real
    assume **: x ∈ s y ∈ s 0 ≤ μ μ ≤ 1
    show (1 − μ) *R x + μ *R y ∈ s
    proof (cases x = y)
      case False
      then show ?thesis
        using *[rule-format, of {x, y} λ z. if z = x then 1 − μ else μ] **
          by auto
    next
      case True
      then show ?thesis
        using *[rule-format, of {x, y} λ z. 1] **
          by (auto simp: field-simps real-vector.scale-left-diff-distrib)
    qed
  qed
qed

lemma convex-finite:
  assumes finite s
  shows convex s ⟷ (∀ u. (∀ x∈s. 0 ≤ u x) ∧ setsum u s = 1 ⟶ setsum (λx.
u x *R x) s ∈ s)
  unfolding convex-explicit
proof safe
```

fix *t u*
assume *sum*: $\forall u.\ (\forall x{\in}s.\ 0 \le u\ x) \wedge \mathit{setsum}\ u\ s = 1 \longrightarrow (\sum x{\in}s.\ u\ x *_R x) \in s$
and *as*: *finite t* $t \subseteq s$ $\forall x{\in}t.\ 0 \le u\ x$ *setsum u t* $= (1{::}\mathit{real})$
have $*$: $s \cap t = t$
using *as*(*2*) **by** *auto*
have *if-distrib-arg*: $\bigwedge P\ f\ g\ x.\ (\mathit{if}\ P\ \mathit{then}\ f\ \mathit{else}\ g)\ x = (\mathit{if}\ P\ \mathit{then}\ f\ x\ \mathit{else}\ g\ x)$
by *simp*
show $(\sum x{\in}t.\ u\ x *_R x) \in s$
using *sum*[*THEN spec*[**where** $x{=}\lambda x.\ \mathit{if}\ x{\in}t\ \mathit{then}\ u\ x\ \mathit{else}\ 0$]] *as* $*$
by (*auto simp*: *assms setsum.If-cases if-distrib if-distrib-arg*)
qed (*erule-tac* $x{=}s$ **in** *allE*, *erule-tac* $x{=}u$ **in** *allE*, *auto*)

17.3 Functions that are convex on a set

definition *convex-on* :: *'a*::*real-vector set* $\Rightarrow$ (*'a* $\Rightarrow$ *real*) $\Rightarrow$ *bool*
where *convex-on s f* $\longleftrightarrow$
$(\forall x{\in}s.\ \forall y{\in}s.\ \forall u{\ge}0.\ \forall v{\ge}0.\ u + v = 1 \longrightarrow f\ (u *_R x + v *_R y) \le u * f\ x + v * f\ y)$

lemma *convex-onI* [*intro?*]:
assumes $\bigwedge t\ x\ y.\ t > 0 \Longrightarrow t < 1 \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow$
$f\ ((1 - t) *_R x + t *_R y) \le (1 - t) * f\ x + t * f\ y$
shows *convex-on A f*
unfolding *convex-on-def*
proof *clarify*
fix *x y u v* **assume** *A*: $x \in A$ $y \in A$ $(u{::}\mathit{real}) \ge 0$ $v \ge 0$ $u + v = 1$
from *A*(*5*) **have** [*simp*]: $v = 1 - u$ **by** (*simp add*: *algebra-simps*)
from *A*(*1*−*4*) **show** $f\ (u *_R x + v *_R y) \le u * f\ x + v * f\ y$ **using** *assms*[*of u y x*]
by (*cases* $u = 0 \vee u = 1$) (*auto simp*: *algebra-simps*)
qed

lemma *convex-on-linorderI* [*intro?*]:
fixes *A* :: (*'a*::{*linorder*,*real-vector*}) *set*
assumes $\bigwedge t\ x\ y.\ t > 0 \Longrightarrow t < 1 \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow x < y \Longrightarrow$
$f\ ((1 - t) *_R x + t *_R y) \le (1 - t) * f\ x + t * f\ y$
shows *convex-on A f*
proof
fix *t x y* **assume** *A*: $x \in A$ $y \in A$ $(t{::}\mathit{real}) > 0$ $t < 1$
with *assms*[*of t x y*] *assms*[*of* $1 - t$ *y x*]
show $f\ ((1 - t) *_R x + t *_R y) \le (1 - t) * f\ x + t * f\ y$
by (*cases x y rule*: *linorder-cases*) (*auto simp*: *algebra-simps*)
qed

lemma *convex-onD*:
assumes *convex-on A f*
shows $\bigwedge t\ x\ y.\ t \ge 0 \Longrightarrow t \le 1 \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow$
$f\ ((1 - t) *_R x + t *_R y) \le (1 - t) * f\ x + t * f\ y$

using *assms* **unfolding** *convex-on-def* **by** *auto*

lemma *convex-onD-Icc*:
assumes *convex-on* $\{x..y\}$ f $x \leq (y :: - :: \{real\text{-}vector,preorder\})$
shows $\bigwedge t.\ t \geq 0 \Longrightarrow t \leq 1 \Longrightarrow$
$f\ ((1 - t) *_R x + t *_R y) \leq (1 - t) * f\ x + t * f\ y$
using *assms*(*2*) **by** (*intro convex-onD*[*OF assms*(*1*)]) *simp-all*

lemma *convex-on-subset*: *convex-on* t $f \Longrightarrow s \subseteq t \Longrightarrow$ *convex-on* s f
unfolding *convex-on-def* **by** *auto*

lemma *convex-on-add* [*intro*]:
assumes *convex-on* s f
and *convex-on* s g
shows *convex-on* s $(\lambda x.\ f\ x + g\ x)$
proof −
{
fix x y
assume $x \in s$ $y \in s$
moreover
fix u $v :: real$
assume $0 \leq u$ $0 \leq v$ $u + v = 1$
ultimately
have $f\ (u *_R x + v *_R y) + g\ (u *_R x + v *_R y) \leq (u * f\ x + v * f\ y) + (u * g\ x + v * g\ y)$
using *assms* **unfolding** *convex-on-def* **by** (*auto simp*: *add-mono*)
then have $f\ (u *_R x + v *_R y) + g\ (u *_R x + v *_R y) \leq u * (f\ x + g\ x) + v * (f\ y + g\ y)$
by (*simp add*: *field-simps*)
}
then show *?thesis*
unfolding *convex-on-def* **by** *auto*
qed

lemma *convex-on-cmul* [*intro*]:
fixes $c :: real$
assumes $0 \leq c$
and *convex-on* s f
shows *convex-on* s $(\lambda x.\ c * f\ x)$
proof −
have $*$: $\bigwedge u\ c\ fx\ v\ fy :: real.\ u * (c * fx) + v * (c * fy) = c * (u * fx + v * fy)$
by (*simp add*: *field-simps*)
show *?thesis* **using** *assms*(*2*) **and** *mult-left-mono* [*OF - assms*(*1*)]
unfolding *convex-on-def* **and** $*$ **by** *auto*
qed

lemma *convex-lower*:
assumes *convex-on* s f
and $x \in s$

```
    and y ∈ s
    and 0 ≤ u
    and 0 ≤ v
    and u + v = 1
  shows f (u *R x + v *R y) ≤ max (f x) (f y)
proof −
  let ?m = max (f x) (f y)
  have u * f x + v * f y ≤ u * max (f x) (f y) + v * max (f x) (f y)
    using assms(4,5) by (auto simp: mult-left-mono add-mono)
  also have ... = max (f x) (f y)
    using assms(6) by (simp add: distrib-right [symmetric])
  finally show ?thesis
    using assms unfolding convex-on-def by fastforce
qed

lemma convex-on-dist [intro]:
  fixes s :: 'a::real-normed-vector set
  shows convex-on s (λx. dist a x)
proof (auto simp: convex-on-def dist-norm)
  fix x y
  assume x ∈ s y ∈ s
  fix u v :: real
  assume 0 ≤ u
  assume 0 ≤ v
  assume u + v = 1
  have a = u *R a + v *R a
    unfolding scaleR-left-distrib[symmetric] and ‹u + v = 1› by simp
  then have *: a − (u *R x + v *R y) = (u *R (a − x)) + (v *R (a − y))
    by (auto simp: algebra-simps)
  show norm (a − (u *R x + v *R y)) ≤ u * norm (a − x) + v * norm (a − y)
    unfolding * using norm-triangle-ineq[of u *R (a − x) v *R (a − y)]
    using ‹0 ≤ u› ‹0 ≤ v› by auto
qed
```

17.4 Arithmetic operations on sets preserve convexity

```
lemma convex-linear-image:
  assumes linear f
    and convex s
  shows convex (f ‘ s)
proof −
  interpret f: linear f by fact
  from ‹convex s› show convex (f ‘ s)
    by (simp add: convex-def f.scaleR [symmetric] f.add [symmetric])
qed

lemma convex-linear-vimage:
  assumes linear f
    and convex s
```

shows *convex* (*f* −‘ *s*)
proof −
interpret *f*: *linear f* **by** *fact*
from ‹*convex s*› **show** *convex* (*f* −‘ *s*)
by (*simp add*: *convex-def f.add f.scaleR*)
qed

lemma *convex-scaling*:
assumes *convex s*
shows *convex* ((λ*x*. *c* ∗*R* *x*) ‘ *s*)
proof −
have *linear* (λ*x*. *c* ∗*R* *x*)
by (*simp add*: *linearI scaleR-add-right*)
then show *?thesis*
using ‹*convex s*› **by** (*rule convex-linear-image*)
qed

lemma *convex-scaled*:
assumes *convex s*
shows *convex* ((λ*x*. *x* ∗*R* *c*) ‘ *s*)
proof −
have *linear* (λ*x*. *x* ∗*R* *c*)
by (*simp add*: *linearI scaleR-add-left*)
then show *?thesis*
using ‹*convex s*› **by** (*rule convex-linear-image*)
qed

lemma *convex-negations*:
assumes *convex s*
shows *convex* ((λ*x*. − *x*) ‘ *s*)
proof −
have *linear* (λ*x*. − *x*)
by (*simp add*: *linearI*)
then show *?thesis*
using ‹*convex s*› **by** (*rule convex-linear-image*)
qed

lemma *convex-sums*:
assumes *convex s*
and *convex t*
shows *convex* {*x* + *y*| *x y*. *x* ∈ *s* ∧ *y* ∈ *t*}
proof −
have *linear* (λ(*x*, *y*). *x* + *y*)
by (*auto intro*: *linearI simp*: *scaleR-add-right*)
with *assms* **have** *convex* ((λ(*x*, *y*). *x* + *y*) ‘ (*s* × *t*))
by (*intro convex-linear-image convex-Times*)
also have ((λ(*x*, *y*). *x* + *y*) ‘ (*s* × *t*)) = {*x* + *y*| *x y*. *x* ∈ *s* ∧ *y* ∈ *t*}
by *auto*
finally show *?thesis* **.**

qed

lemma *convex-differences*:
assumes *convex s convex t*
shows *convex* $\{x - y |\ x\ y.\ x \in s \wedge y \in t\}$
proof −
have $\{x - y |\ x\ y.\ x \in s \wedge y \in t\} = \{x + y\ | x\ y.\ x \in s \wedge y \in uminus\ `\ t\}$
by (*auto simp*: *diff-conv-add-uminus simp del*: *add-uminus-conv-diff*)
then show *?thesis*
using *convex-sums*[*OF assms*(*1*) *convex-negations*[*OF assms*(*2*)]] **by** *auto*
qed

lemma *convex-translation*:
assumes *convex s*
shows *convex* $((\lambda x.\ a + x)\ `\ s)$
proof −
have $\{a + y\ | y.\ y \in s\} = (\lambda x.\ a + x)\ `\ s$
by *auto*
then show *?thesis*
using *convex-sums*[*OF convex-singleton*[*of a*] *assms*] **by** *auto*
qed

lemma *convex-affinity*:
assumes *convex s*
shows *convex* $((\lambda x.\ a + c *_R x)\ `\ s)$
proof −
have $(\lambda x.\ a + c *_R x)\ `\ s = op + a\ `\ op *_R c\ `\ s$
by *auto*
then show *?thesis*
using *convex-translation*[*OF convex-scaling*[*OF assms*], *of a c*] **by** *auto*
qed

lemma *pos-is-convex*: *convex* $\{0 :: real <..\}$
unfolding *convex-alt*
proof *safe*
fix $y\ x\ \mu :: real$
assume $*$: $y > 0\ x > 0\ \mu \geq 0\ \mu \leq 1$
{
assume $\mu = 0$
then have $\mu *_R x + (1 - \mu) *_R y = y$ **by** *simp*
then have $\mu *_R x + (1 - \mu) *_R y > 0$ **using** $*$ **by** *simp*
}
moreover
{
assume $\mu = 1$
then have $\mu *_R x + (1 - \mu) *_R y > 0$ **using** $*$ **by** *simp*
}
moreover
{

```
    assume μ ≠ 1 μ ≠ 0
    then have μ > 0 (1 − μ) > 0 using ∗ by auto
    then have μ ∗R x + (1 − μ) ∗R y > 0 using ∗
      by (auto simp: add-pos-pos)
  }
  ultimately show (1 − μ) ∗R y + μ ∗R x > 0
    using assms by fastforce
qed

lemma convex-on-setsum:
  fixes a :: 'a ⇒ real
    and y :: 'a ⇒ 'b::real-vector
    and f :: 'b ⇒ real
  assumes finite s s ≠ {}
    and convex-on C f
    and convex C
    and (∑ i ∈ s. a i) = 1
    and ⋀i. i ∈ s ⟹ a i ≥ 0
    and ⋀i. i ∈ s ⟹ y i ∈ C
  shows f (∑ i ∈ s. a i ∗R y i) ≤ (∑ i ∈ s. a i ∗ f (y i))
  using assms
proof (induct s arbitrary: a rule: finite-ne-induct)
  case (singleton i)
  then have ai: a i = 1 by auto
  then show ?case by auto
next
  case (insert i s)
  then have convex-on C f by simp
  from this[unfolded convex-on-def, rule-format]
  have conv: ⋀x y μ. x ∈ C ⟹ y ∈ C ⟹ 0 ≤ μ ⟹ μ ≤ 1 ⟹
      f (μ ∗R x + (1 − μ) ∗R y) ≤ μ ∗ f x + (1 − μ) ∗ f y
    by simp
  show ?case
  proof (cases a i = 1)
    case True
    then have (∑ j ∈ s. a j) = 0
      using insert by auto
    then have ⋀j. j ∈ s ⟹ a j = 0
      using insert by (fastforce simp: setsum-nonneg-eq-0-iff)
    then show ?thesis
      using insert by auto
  next
    case False
    from insert have yai: y i ∈ C a i ≥ 0
      by auto
    have fis: finite (insert i s)
      using insert by auto
    then have ai1: a i ≤ 1
      using setsum-nonneg-leq-bound[of insert i s a] insert by simp
```

then have $a\ i < 1$
using *False* **by** *auto*
then have *i0*: $1 - a\ i > 0$
by *auto*
let $?a = \lambda j.\ a\ j\ /\ (1 - a\ i)$
have *a-nonneg*: $?a\ j \geq 0$ **if** $j \in s$ **for** j
using *i0 insert that* **by** *fastforce*
have $(\sum\ j \in insert\ i\ s.\ a\ j) = 1$
using *insert* **by** *auto*
then have $(\sum\ j \in s.\ a\ j) = 1 - a\ i$
using *setsum.insert insert* **by** *fastforce*
then have $(\sum\ j \in s.\ a\ j)\ /\ (1 - a\ i) = 1$
using *i0* **by** *auto*
then have *a1*: $(\sum\ j \in s.\ ?a\ j) = 1$
unfolding *setsum-divide-distrib* **by** *simp*
have *convex C* **using** *insert* **by** *auto*
then have *asum*: $(\sum\ j \in s.\ ?a\ j *_R y\ j) \in C$
using *insert convex-setsum*[*OF* ‹*finite s*›
‹*convex C*› *a1 a-nonneg*] **by** *auto*
have *asum-le*: $f\ (\sum\ j \in s.\ ?a\ j *_R y\ j) \leq (\sum\ j \in s.\ ?a\ j * f\ (y\ j))$
using *a-nonneg a1 insert* **by** *blast*
have $f\ (\sum\ j \in insert\ i\ s.\ a\ j *_R y\ j) = f\ ((\sum\ j \in s.\ a\ j *_R y\ j) + a\ i *_R y\ i)$
using *setsum.insert*[*of s i* $\lambda\ j.\ a\ j *_R y\ j$, *OF* ‹*finite s*› ‹$i \notin s$›] *insert*
by (*auto simp only*: *add.commute*)
also have $\ldots = f\ (((1 - a\ i) * inverse\ (1 - a\ i)) *_R (\sum\ j \in s.\ a\ j *_R y\ j)$
$+\ a\ i *_R y\ i)$
using *i0* **by** *auto*
also have $\ldots = f\ ((1 - a\ i) *_R (\sum\ j \in s.\ (a\ j * inverse\ (1 - a\ i)) *_R y\ j)$
$+\ a\ i *_R y\ i)$
using *scaleR-right.setsum*[*of inverse* $(1 - a\ i)$ $\lambda\ j.\ a\ j *_R y\ j\ s$, *symmetric*]
by (*auto simp*: *algebra-simps*)
also have $\ldots = f\ ((1 - a\ i) *_R (\sum\ j \in s.\ ?a\ j *_R y\ j) + a\ i *_R y\ i)$
by (*auto simp*: *divide-inverse*)
also have $\ldots \leq (1 - a\ i) *_R f\ ((\sum\ j \in s.\ ?a\ j *_R y\ j)) + a\ i * f\ (y\ i)$
using *conv*[*of y i* $(\sum\ j \in s.\ ?a\ j *_R y\ j)$ *a i*, *OF yai*(*1*) *asum yai*(*2*) *ai1*]
by (*auto simp*: *add.commute*)
also have $\ldots \leq (1 - a\ i) * (\sum\ j \in s.\ ?a\ j * f\ (y\ j)) + a\ i * f\ (y\ i)$
using *add-right-mono*[*OF mult-left-mono*[*of - - 1 − a i*,
OF asum-le less-imp-le[*OF i0*]], *of a i* $*\ f\ (y\ i)$] **by** *simp*
also have $\ldots = (\sum\ j \in s.\ (1 - a\ i) * ?a\ j * f\ (y\ j)) + a\ i * f\ (y\ i)$
unfolding *setsum-right-distrib*[*of* $1 - a\ i\ \lambda\ j.\ ?a\ j * f\ (y\ j)$] **using** *i0* **by**
auto
also have $\ldots = (\sum\ j \in s.\ a\ j * f\ (y\ j)) + a\ i * f\ (y\ i)$
using *i0* **by** *auto*
also have $\ldots = (\sum\ j \in insert\ i\ s.\ a\ j * f\ (y\ j))$
using *insert* **by** *auto*
finally show *?thesis*
by *simp*
qed

qed

lemma *convex-on-alt*:
fixes C :: *'a*::*real-vector set*
assumes *convex* C
shows *convex-on* $C\ f \longleftrightarrow$
$(\forall x \in C.\ \forall\ y \in C.\ \forall\ \mu :: real.\ \mu \geq 0 \wedge \mu \leq 1 \longrightarrow$
$f\ (\mu *_R x + (1 - \mu) *_R y) \leq \mu * f\ x + (1 - \mu) * f\ y)$
proof *safe*
fix $x\ y$
fix μ :: *real*
assume $*$: *convex-on* $C\ f\ x \in C\ y \in C\ 0 \leq \mu\ \mu \leq 1$
from *this*[*unfolded convex-on-def*, *rule-format*]
have $\bigwedge u\ v.\ 0 \leq u \Longrightarrow 0 \leq v \Longrightarrow u + v = 1 \Longrightarrow f\ (u *_R x + v *_R y) \leq u * f\ x + v * f\ y$
by *auto*
from *this*[*of* $\mu\ 1 - \mu$, *simplified*] $*$
show $f\ (\mu *_R x + (1 - \mu) *_R y) \leq \mu * f\ x + (1 - \mu) * f\ y$
by *auto*
next
assume $*$: $\forall x \in C.\ \forall y \in C.\ \forall \mu.\ 0 \leq \mu \wedge \mu \leq 1 \longrightarrow$
$f\ (\mu *_R x + (1 - \mu) *_R y) \leq \mu * f\ x + (1 - \mu) * f\ y$
{
fix $x\ y$
fix $u\ v$ :: *real*
assume $**$: $x \in C\ y \in C\ u \geq 0\ v \geq 0\ u + v = 1$
then have[*simp*]: $1 - u = v$ **by** *auto*
from $*$[*rule-format*, *of* $x\ y\ u$]
have $f\ (u *_R x + v *_R y) \leq u * f\ x + v * f\ y$
using $**$ **by** *auto*
}
then show *convex-on* $C\ f$
unfolding *convex-on-def* **by** *auto*
qed

lemma *convex-on-diff*:
fixes f :: *real* $\Rightarrow$ *real*
assumes f: *convex-on* $I\ f$
and I: $x \in I\ y \in I$
and t: $x < t\ t < y$
shows $(f\ x - f\ t) / (x - t) \leq (f\ x - f\ y) / (x - y)$
and $(f\ x - f\ y) / (x - y) \leq (f\ t - f\ y) / (t - y)$
proof $-$
def $a \equiv (t - y) / (x - y)$
with t **have** $0 \leq a\ 0 \leq 1 - a$
by (*auto simp*: *field-simps*)
with f ‹$x \in I$› ‹$y \in I$› **have** *cvx*: $f\ (a * x + (1 - a) * y) \leq a * f\ x + (1 - a) * f\ y$
by (*auto simp*: *convex-on-def*)

```
  have a * x + (1 - a) * y = a * (x - y) + y
    by (simp add: field-simps)
  also have ... = t
    unfolding a-def using ‹x < t› ‹t < y› by simp
  finally have f t ≤ a * f x + (1 - a) * f y
    using cvx by simp
  also have ... = a * (f x - f y) + f y
    by (simp add: field-simps)
  finally have f t - f y ≤ a * (f x - f y)
    by simp
  with t show (f x - f t) / (x - t) ≤ (f x - f y) / (x - y)
    by (simp add: le-divide-eq divide-le-eq field-simps a-def)
  with t show (f x - f y) / (x - y) ≤ (f t - f y) / (t - y)
    by (simp add: le-divide-eq divide-le-eq field-simps)
qed

lemma pos-convex-function:
  fixes f :: real ⇒ real
  assumes convex C
    and leq: ⋀x y. x ∈ C ⟹ y ∈ C ⟹ f' x * (y - x) ≤ f y - f x
  shows convex-on C f
  unfolding convex-on-alt[OF assms(1)]
  using assms
proof safe
  fix x y μ :: real
  let ?x = μ *R x + (1 - μ) *R y
  assume *: convex C x ∈ C y ∈ C μ ≥ 0 μ ≤ 1
  then have 1 - μ ≥ 0 by auto
  then have xpos: ?x ∈ C
    using * unfolding convex-alt by fastforce
  have geq: μ * (f x - f ?x) + (1 - μ) * (f y - f ?x) ≥
      μ * f' ?x * (x - ?x) + (1 - μ) * f' ?x * (y - ?x)
    using add-mono[OF mult-left-mono[OF leq[OF xpos *(2)] ‹μ ≥ 0›]
      mult-left-mono[OF leq[OF xpos *(3)] ‹1 - μ ≥ 0›]]
    by auto
  then have μ * f x + (1 - μ) * f y - f ?x ≥ 0
    by (auto simp: field-simps)
  then show f (μ *R x + (1 - μ) *R y) ≤ μ * f x + (1 - μ) * f y
    using convex-on-alt by auto
qed

lemma atMostAtLeast-subset-convex:
  fixes C :: real set
  assumes convex C
    and x ∈ C y ∈ C x < y
  shows {x .. y} ⊆ C
proof safe
  fix z assume z: z ∈ {x .. y}
  have less: z ∈ C if *: x < z z < y
```

```
  proof −
    let ?μ = (y − z) / (y − x)
    have 0 ≤ ?μ ?μ ≤ 1
      using assms ∗ by (auto simp: field-simps)
    then have comb: ?μ ∗ x + (1 − ?μ) ∗ y ∈ C
      using assms iffD1[OF convex-alt, rule-format, of C y x ?μ]
      by (simp add: algebra-simps)
    have ?μ ∗ x + (1 − ?μ) ∗ y = (y − z) ∗ x / (y − x) + (1 − (y − z) / (y −
x)) ∗ y
      by (auto simp: field-simps)
    also have ... = ((y − z) ∗ x + (y − x − (y − z)) ∗ y) / (y − x)
      using assms unfolding add-divide-distrib by (auto simp: field-simps)
    also have ... = z
      using assms by (auto simp: field-simps)
    finally show ?thesis
      using comb by auto
  qed
  show z ∈ C using z less assms
    unfolding atLeastAtMost-iff le-less by auto
qed

lemma f''-imp-f':
  fixes f :: real ⇒ real
  assumes convex C
    and f': ⋀x. x ∈ C ⟹ DERIV f x :> (f' x)
    and f'': ⋀x. x ∈ C ⟹ DERIV f' x :> (f'' x)
    and pos: ⋀x. x ∈ C ⟹ f'' x ≥ 0
    and x ∈ C y ∈ C
  shows f' x ∗ (y − x) ≤ f y − f x
  using assms
proof −
  {
    fix x y :: real
    assume ∗: x ∈ C y ∈ C y > x
    then have ge: y − x > 0 y − x ≥ 0
      by auto
    from ∗ have le: x − y < 0 x − y ≤ 0
      by auto
    then obtain z1 where z1: z1 > x z1 < y f y − f x = (y − x) ∗ f' z1
      using subsetD[OF atMostAtLeast-subset-convex[OF ‹convex C› ‹x ∈ C› ‹y ∈
C› ‹x < y›],
      THEN f', THEN MVT2[OF ‹x < y›, rule-format, unfolded atLeastAtMost-iff[symmetric]]]
      by auto
    then have z1 ∈ C
      using atMostAtLeast-subset-convex ‹convex C› ‹x ∈ C› ‹y ∈ C› ‹x < y›
      by fastforce
    from z1 have z1': f x − f y = (x − y) ∗ f' z1
      by (simp add: field-simps)
    obtain z2 where z2: z2 > x z2 < z1 f' z1 − f' x = (z1 − x) ∗ f'' z2
```

using *subsetD*[*OF atMostAtLeast-subset-convex*[*OF* ‹*convex C*› ‹*x* ∈ *C*› ‹*z1* ∈ *C*› ‹*x* < *z1*›],
THEN f'', *THEN MVT2*[*OF* ‹*x* < *z1*›, *rule-format*, *unfolded atLeastAtMost-iff*[*symmetric*]]] *z1*
by *auto*
obtain *z3* **where** *z3*: *z3* > *z1* *z3* < *y* *f'* *y* − *f'* *z1* = (*y* − *z1*) ∗ *f''* *z3*
using *subsetD*[*OF atMostAtLeast-subset-convex*[*OF* ‹*convex C*› ‹*z1* ∈ *C*› ‹*y* ∈ *C*› ‹*z1* < *y*›],
THEN f'', *THEN MVT2*[*OF* ‹*z1* < *y*›, *rule-format*, *unfolded atLeastAtMost-iff*[*symmetric*]]] *z1*
by *auto*
have *f'* *y* − (*f* *x* − *f* *y*) / (*x* − *y*) = *f'* *y* − *f'* *z1*
using ∗ *z1'* **by** *auto*
also have ... = (*y* − *z1*) ∗ *f''* *z3*
using *z3* **by** *auto*
finally have *cool'*: *f'* *y* − (*f* *x* − *f* *y*) / (*x* − *y*) = (*y* − *z1*) ∗ *f''* *z3*
by *simp*
have *A'*: *y* − *z1* ≥ *0*
using *z1* **by** *auto*
have *z3* ∈ *C*
using *z3* ∗ *atMostAtLeast-subset-convex* ‹*convex C*› ‹*x* ∈ *C*› ‹*z1* ∈ *C*› ‹*x* < *z1*›
by *fastforce*
then have *B'*: *f''* *z3* ≥ *0*
using *assms* **by** *auto*
from *A' B'* **have** (*y* − *z1*) ∗ *f''* *z3* ≥ *0*
by *auto*
from *cool' this* **have** *f'* *y* − (*f* *x* − *f* *y*) / (*x* − *y*) ≥ *0*
by *auto*
from *mult-right-mono-neg*[*OF this le*(*2*)]
have *f'* *y* ∗ (*x* − *y*) − (*f* *x* − *f* *y*) / (*x* − *y*) ∗ (*x* − *y*) ≤ *0* ∗ (*x* − *y*)
by (*simp add*: *algebra-simps*)
then have *f'* *y* ∗ (*x* − *y*) − (*f* *x* − *f* *y*) ≤ *0*
using *le* **by** *auto*
then have *res*: *f'* *y* ∗ (*x* − *y*) ≤ *f* *x* − *f* *y*
by *auto*
have (*f* *y* − *f* *x*) / (*y* − *x*) − *f'* *x* = *f'* *z1* − *f'* *x*
using ∗ *z1* **by** *auto*
also have ... = (*z1* − *x*) ∗ *f''* *z2*
using *z2* **by** *auto*
finally have *cool*: (*f* *y* − *f* *x*) / (*y* − *x*) − *f'* *x* = (*z1* − *x*) ∗ *f''* *z2*
by *simp*
have *A*: *z1* − *x* ≥ *0*
using *z1* **by** *auto*
have *z2* ∈ *C*
using *z2* *z1* ∗ *atMostAtLeast-subset-convex* ‹*convex C*› ‹*z1* ∈ *C*› ‹*y* ∈ *C*› ‹*z1* < *y*›
by *fastforce*
then have *B*: *f''* *z2* ≥ *0*

```
    using assms by auto
  from A B have (z1 − x) ∗ f'' z2 ≥ 0
    by auto
  with cool have (f y − f x) / (y − x) − f' x ≥ 0
    by auto
  from mult-right-mono[OF this ge(2)]
  have (f y − f x) / (y − x) ∗ (y − x) − f' x ∗ (y − x) ≥ 0 ∗ (y − x)
    by (simp add: algebra-simps)
  then have f y − f x − f' x ∗ (y − x) ≥ 0
    using ge by auto
  then have f y − f x ≥ f' x ∗ (y − x) f' y ∗ (x − y) ≤ f x − f y
    using res by auto
 } note less-imp = this
 {
  fix x y :: real
  assume x ∈ C y ∈ C x ≠ y
  then havef y − f x ≥ f' x ∗ (y − x)
  unfolding neq-iff using less-imp by auto
 }
 moreover
 {
  fix x y :: real
  assume x ∈ C y ∈ C x = y
  then have f y − f x ≥ f' x ∗ (y − x) by auto
 }
 ultimately show ?thesis using assms by blast
qed

lemma f''-ge0-imp-convex:
 fixes f :: real ⇒ real
 assumes conv: convex C
   and f': ⋀x. x ∈ C ⟹ DERIV f x :> (f' x)
   and f'': ⋀x. x ∈ C ⟹ DERIV f' x :> (f'' x)
   and pos: ⋀x. x ∈ C ⟹ f'' x ≥ 0
 shows convex-on C f
 using f''-imp-f'[OF conv f' f'' pos] assms pos-convex-function
 by fastforce

lemma minus-log-convex:
 fixes b :: real
 assumes b > 1
 shows convex-on {0 <..} (λ x. − log b x)
proof −
 have ⋀z. z > 0 ⟹ DERIV (log b) z :> 1 / (ln b ∗ z)
   using DERIV-log by auto
 then have f': ⋀z. z > 0 ⟹ DERIV (λ z. − log b z) z :> − 1 / (ln b ∗ z)
   by (auto simp: DERIV-minus)
 have ⋀z :: real. z > 0 ⟹ DERIV inverse z :> − (inverse z ^ Suc (Suc 0))
   using less-imp-neq[THEN not-sym, THEN DERIV-inverse] by auto
```

from *this*[*THEN DERIV-cmult*, *of* - − *1* / *ln b*]
have ⋀*z* :: *real*. *z* > *0* ⟹
DERIV (λ *z*. (− *1* / *ln b*) ∗ *inverse z*) *z* :> (− *1* / *ln b*) ∗ (− (*inverse z* ^ *Suc* (*Suc 0*)))
by *auto*
then have *f''0*: ⋀*z*::*real*. *z* > *0* ⟹
DERIV (λ *z*. − *1* / (*ln b* ∗ *z*)) *z* :> *1* / (*ln b* ∗ *z* ∗ *z*)
unfolding *inverse-eq-divide* **by** (*auto simp*: *mult.assoc*)
have *f''-ge0*: ⋀*z*::*real*. *z* > *0* ⟹ *1* / (*ln b* ∗ *z* ∗ *z*) ≥ *0*
using ‹*b* > *1*› **by** (*auto intro*!: *less-imp-le*)
from *f''-ge0-imp-convex*[*OF pos-is-convex*,
unfolded greaterThan-iff, *OF f' f''0 f''-ge0*]
show *?thesis* **by** *auto*
qed

17.5 Convexity of real functions

lemma *convex-on-realI*:
assumes *connected A*
assumes ⋀*x*. *x* ∈ *A* ⟹ (*f has-real-derivative f' x*) (*at x*)
assumes ⋀*x y*. *x* ∈ *A* ⟹ *y* ∈ *A* ⟹ *x* ≤ *y* ⟹ *f' x* ≤ *f' y*
shows *convex-on A f*
proof (*rule convex-on-linorderI*)
fix *t x y* :: *real*
assume *t*: *t* > *0* *t* < *1* **and** *xy*: *x* ∈ *A* *y* ∈ *A* *x* < *y*
def *z* ≡ (*1* − *t*) ∗ *x* + *t* ∗ *y*
with ‹*connected A*› **and** *xy* **have** *ivl*: {*x*..*y*} ⊆ *A* **using** *connected-contains-Icc*
by *blast*

from *xy t* **have** *xz*: *z* > *x* **by** (*simp add*: *z-def algebra-simps*)
have *y* − *z* = (*1* − *t*) ∗ (*y* − *x*) **by** (*simp add*: *z-def algebra-simps*)
also from *xy t* **have** *...* > *0* **by** (*intro mult-pos-pos*) *simp-all*
finally have *yz*: *z* < *y* **by** *simp*

from *assms xz yz ivl t* **have** ∃ξ. ξ > *x* ∧ ξ < *z* ∧ *f z* − *f x* = (*z* − *x*) ∗ *f'* ξ
by (*intro MVT2*) (*auto intro*!: *assms*(*2*))
then obtain ξ **where** ξ: ξ > *x* ξ < *z* *f'* ξ = (*f z* − *f x*) / (*z* − *x*) **by** *auto*
from *assms xz yz ivl t* **have** ∃η. η > *z* ∧ η < *y* ∧ *f y* − *f z* = (*y* − *z*) ∗ *f'* η
by (*intro MVT2*) (*auto intro*!: *assms*(*2*))
then obtain η **where** η: η > *z* η < *y* *f'* η = (*f y* − *f z*) / (*y* − *z*) **by** *auto*

from η(*3*) **have** (*f y* − *f z*) / (*y* − *z*) = *f'* η **..**
also from ξ η *ivl* **have** ξ ∈ *A* η ∈ *A* **by** *auto*
with ξ η **have** *f'* η ≥ *f'* ξ **by** (*intro assms*(*3*)) *auto*
also from ξ(*3*) **have** *f'* ξ = (*f z* − *f x*) / (*z* − *x*) **.**
finally have (*f y* − *f z*) ∗ (*z* − *x*) ≥ (*f z* − *f x*) ∗ (*y* − *z*)
using *xz yz* **by** (*simp add*: *field-simps*)
also have *z* − *x* = *t* ∗ (*y* − *x*) **by** (*simp add*: *z-def algebra-simps*)
also have *y* − *z* = (*1* − *t*) ∗ (*y* − *x*) **by** (*simp add*: *z-def algebra-simps*)

finally have $(f\ y - f\ z) * t \geq (f\ z - f\ x) * (1 - t)$ **using** *xy* **by** *simp*
thus $(1 - t) * f\ x + t * f\ y \geq f\ ((1 - t) *_R x + t *_R y)$
by (*simp add*: *z-def algebra-simps*)
qed

lemma *convex-on-inverse*:
assumes $A \subseteq \{0<..\}$
shows *convex-on A* (*inverse* :: *real* ⇒ *real*)
proof (*rule convex-on-subset*[*OF - assms*], *intro convex-on-realI*[*of - - λx. −inverse* (*x^2*)])
fix *u v* :: *real* **assume** $u \in \{0<..\}$ $v \in \{0<..\}$ $u \leq v$
with *assms* **show** $-inverse\ (u\hat{}2) \leq -inverse\ (v\hat{}2)$
by (*intro le-imp-neg-le le-imp-inverse-le power-mono*) (*simp-all*)
qed (*insert assms*, *auto intro*!: *derivative-eq-intros simp*: *divide-simps power2-eq-square*)

lemma *convex-onD-Icc′*:
assumes *convex-on* $\{x..y\}$ *f c* ∈ $\{x..y\}$
defines $d \equiv y - x$
shows $f\ c \leq (f\ y - f\ x) / d * (c - x) + f\ x$
proof (*cases y x rule*: *linorder-cases*)
assume *less*: $x < y$
hence *d*: $d > 0$ **by** (*simp add*: *d-def*)
from *assms*(*2*) *less* **have** *A*: $0 \leq (c - x) / d$ $(c - x) / d \leq 1$
by (*simp-all add*: *d-def divide-simps*)
have $f\ c = f\ (x + (c - x) * 1)$ **by** *simp*
also from *less* **have** $1 = ((y - x) / d)$ **by** (*simp add*: *d-def*)
also from *d* **have** $x + (c - x) * \ldots = (1 - (c - x) / d) *_R x + ((c - x) / d) *_R y$
by (*simp add*: *field-simps*)
also have $f\ \ldots \leq (1 - (c - x) / d) * f\ x + (c - x) / d * f\ y$ **using** *assms less*
by (*intro convex-onD-Icc*) *simp-all*
also from *d* **have** $\ldots = (f\ y - f\ x) / d * (c - x) + f\ x$ **by** (*simp add*: *field-simps*)
finally show *?thesis* **.**
qed (*insert assms*(*2*), *simp-all*)

lemma *convex-onD-Icc′′*:
assumes *convex-on* $\{x..y\}$ *f c* ∈ $\{x..y\}$
defines $d \equiv y - x$
shows $f\ c \leq (f\ x - f\ y) / d * (y - c) + f\ y$
proof (*cases y x rule*: *linorder-cases*)
assume *less*: $x < y$
hence *d*: $d > 0$ **by** (*simp add*: *d-def*)
from *assms*(*2*) *less* **have** *A*: $0 \leq (y - c) / d$ $(y - c) / d \leq 1$
by (*simp-all add*: *d-def divide-simps*)
have $f\ c = f\ (y - (y - c) * 1)$ **by** *simp*
also from *less* **have** $1 = ((y - x) / d)$ **by** (*simp add*: *d-def*)
also from *d* **have** $y - (y - c) * \ldots = (1 - (1 - (y - c) / d)) *_R x + (1 - (y - c) / d) *_R y$
by (*simp add*: *field-simps*)

also have $f \ldots \leq (1 - (1 - (y - c) / d)) * f\ x + (1 - (y - c) / d) * f\ y$
using *assms less*
by (*intro convex-onD-Icc*) (*simp-all add*: *field-simps*)
also from d **have** $\ldots = (f\ x - f\ y) / d * (y - c) + f\ y$ **by** (*simp add*: *field-simps*)
finally show *?thesis* **.**
qed (*insert assms*(*2*), *simp-all*)

end

18 Algebraic operations on sets

theory *Set-Algebras*
imports *Main*
begin

This library lifts operations like addition and multiplication to sets. It was designed to support asymptotic calculations. See the comments at the top of theory *BigO*.

instantiation *set* :: (*plus*) *plus*
begin

definition *plus-set* :: $'a$::*plus set* $\Rightarrow$ $'a$ *set* $\Rightarrow$ $'a$ *set* **where**
set-plus-def: $A + B = \{c.\ \exists\, a{\in}A.\ \exists\, b{\in}B.\ c = a + b\}$

instance ..

end

instantiation *set* :: (*times*) *times*
begin

definition *times-set* :: $'a$::*times set* $\Rightarrow$ $'a$ *set* $\Rightarrow$ $'a$ *set* **where**
set-times-def: $A * B = \{c.\ \exists\, a{\in}A.\ \exists\, b{\in}B.\ c = a * b\}$

instance ..

end

instantiation *set* :: (*zero*) *zero*
begin

definition
set-zero[*simp*]: $(0::'a$::*zero set*$) = \{0\}$

instance ..

end

instantiation *set* :: (*one*) *one*
begin

definition
set-one[*simp*]: (*1*::'*a*::*one set*) = {*1*}

instance ..

end

definition *elt-set-plus* :: '*a*::*plus* ⇒ '*a set* ⇒ '*a set* (**infixl** +*o* *70*) **where**
a +*o* *B* = {*c*. ∃ *b*∈*B*. *c* = *a* + *b*}

definition *elt-set-times* :: '*a*::*times* ⇒ '*a set* ⇒ '*a set* (**infixl** ∗*o* *80*) **where**
a ∗*o* *B* = {*c*. ∃ *b*∈*B*. *c* = *a* ∗ *b*}

abbreviation (*input*) *elt-set-eq* :: '*a* ⇒ '*a set* ⇒ *bool* (**infix** =*o* *50*) **where**
x =*o* *A* ≡ *x* ∈ *A*

instance *set* :: (*semigroup-add*) *semigroup-add*
by *standard* (*force simp add*: *set-plus-def add.assoc*)

instance *set* :: (*ab-semigroup-add*) *ab-semigroup-add*
by *standard* (*force simp add*: *set-plus-def add.commute*)

instance *set* :: (*monoid-add*) *monoid-add*
by *standard* (*simp-all add*: *set-plus-def*)

instance *set* :: (*comm-monoid-add*) *comm-monoid-add*
by *standard* (*simp-all add*: *set-plus-def*)

instance *set* :: (*semigroup-mult*) *semigroup-mult*
by *standard* (*force simp add*: *set-times-def mult.assoc*)

instance *set* :: (*ab-semigroup-mult*) *ab-semigroup-mult*
by *standard* (*force simp add*: *set-times-def mult.commute*)

instance *set* :: (*monoid-mult*) *monoid-mult*
by *standard* (*simp-all add*: *set-times-def*)

instance *set* :: (*comm-monoid-mult*) *comm-monoid-mult*
by *standard* (*simp-all add*: *set-times-def*)

lemma *set-plus-intro* [*intro*]: *a* ∈ *C* ⟹ *b* ∈ *D* ⟹ *a* + *b* ∈ *C* + *D*
by (*auto simp add*: *set-plus-def*)

lemma *set-plus-elim*:
assumes *x* ∈ *A* + *B*

obtains a b **where** $x = a + b$ **and** $a \in A$ **and** $b \in B$
using *assms* **unfolding** *set-plus-def* **by** *fast*

lemma *set-plus-intro2* [*intro*]: $b \in C \Longrightarrow a + b \in a +o C$
by (*auto simp add*: *elt-set-plus-def*)

lemma *set-plus-rearrange*:
$((a::'a::comm\text{-}monoid\text{-}add) +o C) + (b +o D) = (a + b) +o (C + D)$
apply (*auto simp add*: *elt-set-plus-def set-plus-def ac-simps*)
apply (*rule-tac* $x = ba + bb$ **in** *exI*)
apply (*auto simp add*: *ac-simps*)
apply (*rule-tac* $x = aa + a$ **in** *exI*)
apply (*auto simp add*: *ac-simps*)
done

lemma *set-plus-rearrange2*: $(a::'a::semigroup\text{-}add) +o (b +o C) = (a + b) +o C$
by (*auto simp add*: *elt-set-plus-def add.assoc*)

lemma *set-plus-rearrange3*: $((a::'a::semigroup\text{-}add) +o B) + C = a +o (B + C)$
apply (*auto simp add*: *elt-set-plus-def set-plus-def*)
apply (*blast intro*: *ac-simps*)
apply (*rule-tac* $x = a + aa$ **in** *exI*)
apply (*rule conjI*)
apply (*rule-tac* $x = aa$ **in** *bexI*)
apply *auto*
apply (*rule-tac* $x = ba$ **in** *bexI*)
apply (*auto simp add*: *ac-simps*)
done

theorem *set-plus-rearrange4*: $C + ((a::'a::comm\text{-}monoid\text{-}add) +o D) = a +o (C + D)$
apply (*auto simp add*: *elt-set-plus-def set-plus-def ac-simps*)
apply (*rule-tac* $x = aa + ba$ **in** *exI*)
apply (*auto simp add*: *ac-simps*)
done

lemmas *set-plus-rearranges* = *set-plus-rearrange set-plus-rearrange2*
set-plus-rearrange3 set-plus-rearrange4

lemma *set-plus-mono* [*intro*!]: $C \subseteq D \Longrightarrow a +o C \subseteq a +o D$
by (*auto simp add*: *elt-set-plus-def*)

lemma *set-plus-mono2* [*intro*]: $(C::'a::plus\ set) \subseteq D \Longrightarrow E \subseteq F \Longrightarrow C + E \subseteq D + F$
by (*auto simp add*: *set-plus-def*)

lemma *set-plus-mono3* [*intro*]: $a \in C \Longrightarrow a +o D \subseteq C + D$
by (*auto simp add*: *elt-set-plus-def set-plus-def*)

lemma *set-plus-mono4* [*intro*]: $(a::'a::comm\text{-}monoid\text{-}add) \in C \Longrightarrow a +o D \subseteq D + C$
by (*auto simp add*: *elt-set-plus-def set-plus-def ac-simps*)

lemma *set-plus-mono5*: $a \in C \Longrightarrow B \subseteq D \Longrightarrow a +o B \subseteq C + D$
apply (*subgoal-tac* $a +o B \subseteq a +o D$)
apply (*erule order-trans*)
apply (*erule set-plus-mono3*)
apply (*erule set-plus-mono*)
done

lemma *set-plus-mono-b*: $C \subseteq D \Longrightarrow x \in a +o C \Longrightarrow x \in a +o D$
apply (*frule set-plus-mono*)
apply *auto*
done

lemma *set-plus-mono2-b*: $C \subseteq D \Longrightarrow E \subseteq F \Longrightarrow x \in C + E \Longrightarrow x \in D + F$
apply (*frule set-plus-mono2*)
prefer *2*
apply *force*
apply *assumption*
done

lemma *set-plus-mono3-b*: $a \in C \Longrightarrow x \in a +o D \Longrightarrow x \in C + D$
apply (*frule set-plus-mono3*)
apply *auto*
done

lemma *set-plus-mono4-b*: $(a::'a::comm\text{-}monoid\text{-}add) : C \Longrightarrow x \in a +o D \Longrightarrow x \in D + C$
apply (*frule set-plus-mono4*)
apply *auto*
done

lemma *set-zero-plus* [*simp*]: $(0::'a::comm\text{-}monoid\text{-}add) +o C = C$
by (*auto simp add*: *elt-set-plus-def*)

lemma *set-zero-plus2*: $(0::'a::comm\text{-}monoid\text{-}add) \in A \Longrightarrow B \subseteq A + B$
apply (*auto simp add*: *set-plus-def*)
apply (*rule-tac* $x = 0$ **in** *bexI*)
apply (*rule-tac* $x = x$ **in** *bexI*)
apply (*auto simp add*: *ac-simps*)
done

lemma *set-plus-imp-minus*: $(a::'a::ab\text{-}group\text{-}add) : b +o C \Longrightarrow (a - b) \in C$
by (*auto simp add*: *elt-set-plus-def ac-simps*)

lemma *set-minus-imp-plus*: $(a::'a::ab\text{-}group\text{-}add) - b : C \Longrightarrow a \in b +o C$
apply (*auto simp add*: *elt-set-plus-def ac-simps*)

apply (*subgoal-tac* $a = (a + - b) + b$)
apply (*rule bexI*, *assumption*)
apply (*auto simp add*: *ac-simps*)
done

lemma *set-minus-plus*: $(a::'a::ab\text{-}group\text{-}add) - b \in C \longleftrightarrow a \in b +o C$
by (*rule iffI*, *rule set-minus-imp-plus*, *assumption*, *rule set-plus-imp-minus*)

lemma *set-times-intro* [*intro*]: $a \in C \Longrightarrow b \in D \Longrightarrow a * b \in C * D$
by (*auto simp add*: *set-times-def*)

lemma *set-times-elim*:
assumes $x \in A * B$
obtains $a\ b$ **where** $x = a * b$ **and** $a \in A$ **and** $b \in B$
using *assms* **unfolding** *set-times-def* **by** *fast*

lemma *set-times-intro2* [*intro*!]: $b \in C \Longrightarrow a * b \in a *o C$
by (*auto simp add*: *elt-set-times-def*)

lemma *set-times-rearrange*:
$((a::'a::comm\text{-}monoid\text{-}mult) *o C) * (b *o D) = (a * b) *o (C * D)$
apply (*auto simp add*: *elt-set-times-def set-times-def*)
apply (*rule-tac* $x = ba * bb$ **in** *exI*)
apply (*auto simp add*: *ac-simps*)
apply (*rule-tac* $x = aa * a$ **in** *exI*)
apply (*auto simp add*: *ac-simps*)
done

lemma *set-times-rearrange2*:
$(a::'a::semigroup\text{-}mult) *o (b *o C) = (a * b) *o C$
by (*auto simp add*: *elt-set-times-def mult.assoc*)

lemma *set-times-rearrange3*:
$((a::'a::semigroup\text{-}mult) *o B) * C = a *o (B * C)$
apply (*auto simp add*: *elt-set-times-def set-times-def*)
apply (*blast intro*: *ac-simps*)
apply (*rule-tac* $x = a * aa$ **in** *exI*)
apply (*rule conjI*)
apply (*rule-tac* $x = aa$ **in** *bexI*)
apply *auto*
apply (*rule-tac* $x = ba$ **in** *bexI*)
apply (*auto simp add*: *ac-simps*)
done

theorem *set-times-rearrange4*:
$C * ((a::'a::comm\text{-}monoid\text{-}mult) *o D) = a *o (C * D)$
apply (*auto simp add*: *elt-set-times-def set-times-def ac-simps*)
apply (*rule-tac* $x = aa * ba$ **in** *exI*)
apply (*auto simp add*: *ac-simps*)

done

lemmas *set-times-rearranges = set-times-rearrange set-times-rearrange2 set-times-rearrange3 set-times-rearrange4*

lemma *set-times-mono* [*intro*]: $C \subseteq D \Longrightarrow a *o C \subseteq a *o D$
by (*auto simp add*: *elt-set-times-def*)

lemma *set-times-mono2* [*intro*]: $(C::'a::times\ set) \subseteq D \Longrightarrow E \subseteq F \Longrightarrow C * E \subseteq D * F$
by (*auto simp add*: *set-times-def*)

lemma *set-times-mono3* [*intro*]: $a \in C \Longrightarrow a *o D \subseteq C * D$
by (*auto simp add*: *elt-set-times-def set-times-def*)

lemma *set-times-mono4* [*intro*]: $(a::'a::comm\text{-}monoid\text{-}mult) : C \Longrightarrow a *o D \subseteq D * C$
by (*auto simp add*: *elt-set-times-def set-times-def ac-simps*)

lemma *set-times-mono5*: $a \in C \Longrightarrow B \subseteq D \Longrightarrow a *o B \subseteq C * D$
apply (*subgoal-tac* $a *o B \subseteq a *o D$)
apply (*erule order-trans*)
apply (*erule set-times-mono3*)
apply (*erule set-times-mono*)
done

lemma *set-times-mono-b*: $C \subseteq D \Longrightarrow x \in a *o C \Longrightarrow x \in a *o D$
apply (*frule set-times-mono*)
apply *auto*
done

lemma *set-times-mono2-b*: $C \subseteq D \Longrightarrow E \subseteq F \Longrightarrow x \in C * E \Longrightarrow x \in D * F$
apply (*frule set-times-mono2*)
prefer *2*
apply *force*
apply *assumption*
done

lemma *set-times-mono3-b*: $a \in C \Longrightarrow x \in a *o D \Longrightarrow x \in C * D$
apply (*frule set-times-mono3*)
apply *auto*
done

lemma *set-times-mono4-b*: $(a::'a::comm\text{-}monoid\text{-}mult) \in C \Longrightarrow x \in a *o D \Longrightarrow x \in D * C$
apply (*frule set-times-mono4*)
apply *auto*
done

lemma *set-one-times* [*simp*]: $(1::'a::comm\text{-}monoid\text{-}mult) *o C = C$
by (*auto simp add*: *elt-set-times-def*)

lemma *set-times-plus-distrib*:
$(a::'a::semiring) *o (b +o C) = (a * b) +o (a *o C)$
by (*auto simp add*: *elt-set-plus-def elt-set-times-def ring-distribs*)

lemma *set-times-plus-distrib2*:
$(a::'a::semiring) *o (B + C) = (a *o B) + (a *o C)$
apply (*auto simp add*: *set-plus-def elt-set-times-def ring-distribs*)
apply *blast*
apply (*rule-tac* $x = b + bb$ **in** *exI*)
apply (*auto simp add*: *ring-distribs*)
done

lemma *set-times-plus-distrib3*: $((a::'a::semiring) +o C) * D \subseteq a *o D + C * D$
apply (*auto simp add*:
elt-set-plus-def elt-set-times-def set-times-def
set-plus-def ring-distribs)
apply *auto*
done

lemmas *set-times-plus-distribs* =
set-times-plus-distrib
set-times-plus-distrib2

lemma *set-neg-intro*: $(a::'a::ring\text{-}1) \in (-1) *o C \Longrightarrow -a \in C$
by (*auto simp add*: *elt-set-times-def*)

lemma *set-neg-intro2*: $(a::'a::ring\text{-}1) \in C \Longrightarrow -a \in (-1) *o C$
by (*auto simp add*: *elt-set-times-def*)

lemma *set-plus-image*: $S + T = (\lambda(x, y).\ x + y)\ `\ (S \times T)$
unfolding *set-plus-def* **by** (*fastforce simp*: *image-iff*)

lemma *set-times-image*: $S * T = (\lambda(x, y).\ x * y)\ `\ (S \times T)$
unfolding *set-times-def* **by** (*fastforce simp*: *image-iff*)

lemma *finite-set-plus*: *finite* $s \Longrightarrow$ *finite* $t \Longrightarrow$ *finite* $(s + t)$
unfolding *set-plus-image* **by** *simp*

lemma *finite-set-times*: *finite* $s \Longrightarrow$ *finite* $t \Longrightarrow$ *finite* $(s * t)$
unfolding *set-times-image* **by** *simp*

lemma *set-setsum-alt*:
assumes *fin*: *finite* I
shows *setsum* $S\ I = \{setsum\ s\ I\ |s.\ \forall i{\in}I.\ s\ i \in S\ i\}$
(**is** - = *?setsum* I)
using *fin*

proof *induct*
case *empty*
then show *?case* **by** *simp*
next
case (*insert x F*)
have *setsum S* (*insert x F*) = *S x* + *?setsum F*
using *insert.hyps* **by** *auto*
also have ... = {*s x* + *setsum s F* |*s*. ∀ *i*∈*insert x F*. *s i* ∈ *S i*}
unfolding *set-plus-def*
proof *safe*
fix *y s*
assume *y* ∈ *S x* ∀ *i*∈*F*. *s i* ∈ *S i*
then show ∃ *s'*. *y* + *setsum s F* = *s' x* + *setsum s' F* ∧ (∀ *i*∈*insert x F*. *s' i* ∈ *S i*)
using *insert.hyps*
by (*intro exI*[*of* - λ*i*. *if i* ∈ *F then s i else y*]) (*auto simp add*: *set-plus-def*)
qed *auto*
finally show *?case*
using *insert.hyps* **by** *auto*
qed

lemma *setsum-set-cond-linear*:
fixes *f* :: *'a*::*comm-monoid-add set* ⇒ *'b*::*comm-monoid-add set*
assumes [*intro*!]: ⋀*A B*. *P A* ⟹ *P B* ⟹ *P* (*A* + *B*) *P* {*0*}
and *f*: ⋀*A B*. *P A* ⟹ *P B* ⟹ *f* (*A* + *B*) = *f A* + *f B f* {*0*} = {*0*}
assumes *all*: ⋀*i*. *i* ∈ *I* ⟹ *P* (*S i*)
shows *f* (*setsum S I*) = *setsum* (*f* ∘ *S*) *I*
proof (*cases finite I*)
case *True*
from *this all* **show** *?thesis*
proof *induct*
case *empty*
then show *?case* **by** (*auto intro*!: *f*)
next
case (*insert x F*)
from ‹*finite F*› ‹⋀*i*. *i* ∈ *insert x F* ⟹ *P* (*S i*)› **have** *P* (*setsum S F*)
by *induct auto*
with *insert* **show** *?case*
by (*simp*, *subst f*) *auto*
qed
next
case *False*
then show *?thesis* **by** (*auto intro*!: *f*)
qed

lemma *setsum-set-linear*:
fixes *f* :: *'a*::*comm-monoid-add set* ⇒ *'b*::*comm-monoid-add set*
assumes ⋀*A B*. *f*(*A*) + *f*(*B*) = *f*(*A* + *B*) *f* {*0*} = {*0*}
shows *f* (*setsum S I*) = *setsum* (*f* ∘ *S*) *I*

using *setsum-set-cond-linear*[*of* $\lambda x.$ *True f I S*] *assms* **by** *auto*

lemma *set-times-Un-distrib*:
$A * (B \cup C) = A * B \cup A * C$
$(A \cup B) * C = A * C \cup B * C$
by (*auto simp*: *set-times-def*)

lemma *set-times-UNION-distrib*:
$A * UNION\ I\ M = (\bigcup i{\in}I.\ A * M\ i)$
$UNION\ I\ M * A = (\bigcup i{\in}I.\ M\ i * A)$
by (*auto simp*: *set-times-def*)

end

19 Convex sets, functions and related things.

theory *Convex-Euclidean-Space*
imports
Topology-Euclidean-Space
~~/*src*/*HOL*/*Library*/*Convex*
~~/*src*/*HOL*/*Library*/*Set-Algebras*
begin

lemma *independent-injective-on-span-image*:
assumes *iS*: *independent S*
and *lf*: *linear f*
and *fi*: *inj-on f* (*span S*)
shows *independent* (*f* ' *S*)
proof −
{
fix *a*
assume *a*: $a \in S\ f\ a \in span\ (f\ `\ S - \{f\ a\})$
have *eq*: $f\ `\ S - \{f\ a\} = f\ `\ (S - \{a\})$
using *fi a span-inc* **by** (*auto simp add*: *inj-on-def*)
from *a* **have** $f\ a \in f\ `\ span\ (S - \{a\})$
unfolding *eq span-linear-image* [*OF lf*, *of S* − {*a*}] **by** *blast*
moreover have $span\ (S - \{a\}) \subseteq span\ S$
using *span-mono*[*of S* − {*a*} *S*] **by** *auto*
ultimately have $a \in span\ (S - \{a\})$
using *fi a span-inc* **by** (*auto simp add*: *inj-on-def*)
with *a*(*1*) *iS* **have** *False*
by (*simp add*: *dependent-def*)
}
then show *?thesis*
unfolding *dependent-def* **by** *blast*
qed

lemma *dim-image-eq*:
fixes *f* :: *'n*::*euclidean-space* ⇒ *'m*::*euclidean-space*

assumes *lf*: *linear f*
and *fi*: *inj-on f* (*span S*)
shows *dim* (*f ' S*) = *dim* (*S*::*'n*::*euclidean-space set*)
proof −
obtain *B* **where** *B*: *B* ⊆ *S independent B S* ⊆ *span B card B* = *dim S*
using *basis-exists*[*of S*] **by** *auto*
then have *span S* = *span B*
using *span-mono*[*of B S*] *span-mono*[*of S span B*] *span-span*[*of B*] **by** *auto*
then have *independent* (*f ' B*)
using *independent-injective-on-span-image*[*of B f*] *B assms* **by** *auto*
moreover have *card* (*f ' B*) = *card B*
using *assms card-image*[*of f B*] *subset-inj-on*[*of f span S B*] *B span-inc* **by** *auto*
moreover have (*f ' B*) ⊆ (*f ' S*)
using *B* **by** *auto*
ultimately have *dim* (*f ' S*) ≥ *dim S*
using *independent-card-le-dim*[*of f ' B f ' S*] *B* **by** *auto*
then show *?thesis*
using *dim-image-le*[*of f S*] *assms* **by** *auto*
qed

lemma *linear-injective-on-subspace-0*:
assumes *lf*: *linear f*
and *subspace S*
shows *inj-on f S* ⟷ (∀ *x* ∈ *S*. *f x* = *0* ⟶ *x* = *0*)
proof −
have *inj-on f S* ⟷ (∀ *x* ∈ *S*. ∀ *y* ∈ *S*. *f x* = *f y* ⟶ *x* = *y*)
by (*simp add*: *inj-on-def*)
also have ... ⟷ (∀ *x* ∈ *S*. ∀ *y* ∈ *S*. *f x* − *f y* = *0* ⟶ *x* − *y* = *0*)
by *simp*
also have ... ⟷ (∀ *x* ∈ *S*. ∀ *y* ∈ *S*. *f* (*x* − *y*) = *0* ⟶ *x* − *y* = *0*)
by (*simp add*: *linear-sub*[*OF lf*])
also have ... ⟷ (∀ *x* ∈ *S*. *f x* = *0* ⟶ *x* = *0*)
using ‹*subspace S*› *subspace-def*[*of S*] *subspace-sub*[*of S*] **by** *auto*
finally show *?thesis* **.**
qed

lemma *subspace-Inter*: ∀ *s* ∈ *f*. *subspace s* ⟹ *subspace* (⋂ *f*)
unfolding *subspace-def* **by** *auto*

lemma *span-eq*[*simp*]: *span s* = *s* ⟷ *subspace s*
unfolding *span-def* **by** (*rule hull-eq*) (*rule subspace-Inter*)

lemma *substdbasis-expansion-unique*:
assumes *d*: *d* ⊆ *Basis*
shows (∑ *i*∈*d*. *f i* ∗$_R$ *i*) = (*x*::*'a*::*euclidean-space*) ⟷
(∀ *i*∈*Basis*. (*i* ∈ *d* ⟶ *f i* = *x* ∙ *i*) ∧ (*i* ∉ *d* ⟶ *x* ∙ *i* = *0*))
proof −
have ∗: ⋀*x a b P*. *x* ∗ (*if P then a else b*) = (*if P then x* ∗ *a else x* ∗ *b*)
by *auto*

have $**$: *finite d*
by (*auto intro*: *finite-subset*[*OF assms*])
have $***$: $\bigwedge i.\ i \in Basis \Longrightarrow (\sum i{\in}d.\ f\ i *_R\ i) \cdot i = (\sum x{\in}d.\ if\ x = i\ then\ f\ x\ else\ 0)$
using *d*
by (*auto intro*!: *setsum.cong simp*: *inner-Basis inner-setsum-left*)
show *?thesis*
unfolding *euclidean-eq-iff*[**where** $'a{=}'a$] **by** (*auto simp*: *setsum.delta*[*OF* $**$] $***$)
qed

lemma *independent-substdbasis*: $d \subseteq Basis \Longrightarrow independent\ d$
by (*rule independent-mono*[*OF independent-Basis*])

lemma *dim-cball*:
assumes $e > 0$
shows $dim\ (cball\ (0 :: 'n{::}euclidean\text{-}space)\ e) = DIM('n)$
proof −
{
fix $x :: 'n{::}euclidean\text{-}space$
def $y \equiv (e\ /\ norm\ x) *_R\ x$
then have $y \in cball\ 0\ e$
using *assms* **by** *auto*
moreover have $*$: $x = (norm\ x\ /\ e) *_R\ y$
using *y-def assms* **by** *simp*
moreover from $*$ **have** $x = (norm\ x/e) *_R\ y$
by *auto*
ultimately have $x \in span\ (cball\ 0\ e)$
using *span-mul*[*of y cball 0 e norm x*/*e*] *span-inc*[*of cball 0 e*]
by (*simp add*: *span-superset*)
}
then have $span\ (cball\ 0\ e) = (UNIV :: 'n{::}euclidean\text{-}space\ set)$
by *auto*
then show *?thesis*
using *dim-span*[*of cball* $(0 :: 'n{::}euclidean\text{-}space)\ e$] **by** (*auto simp add*: *dim-UNIV*)
qed

lemma *indep-card-eq-dim-span*:
fixes $B :: 'n{::}euclidean\text{-}space\ set$
assumes *independent B*
shows $finite\ B \wedge card\ B = dim\ (span\ B)$
using *assms basis-card-eq-dim*[*of B span B*] *span-inc* **by** *auto*

lemma *setsum-not-0*: $setsum\ f\ A \neq 0 \Longrightarrow \exists\, a \in A.\ f\ a \neq 0$
by (*rule ccontr*) *auto*

lemma *subset-translation-eq* [*simp*]:
fixes $a :: 'a{::}real\text{-}vector$ **shows** $op + a\ `\ s \subseteq op + a\ `\ t \longleftrightarrow s \subseteq t$
by *auto*

lemma *translate-inj-on*:
fixes $A :: 'a::ab\text{-}group\text{-}add\ set$
shows *inj-on* $(\lambda x.\ a + x)\ A$
unfolding *inj-on-def* **by** *auto*

lemma *translation-assoc*:
fixes $a\ b :: 'a::ab\text{-}group\text{-}add$
shows $(\lambda x.\ b + x)\ `\ ((\lambda x.\ a + x)\ `\ S) = (\lambda x.\ (a + b) + x)\ `\ S$
by *auto*

lemma *translation-invert*:
fixes $a :: 'a::ab\text{-}group\text{-}add$
assumes $(\lambda x.\ a + x)\ `\ A = (\lambda x.\ a + x)\ `\ B$
shows $A = B$
proof −
have $(\lambda x.\ -a + x)\ `\ ((\lambda x.\ a + x)\ `\ A) = (\lambda x.\ -\ a + x)\ `\ ((\lambda x.\ a + x)\ `\ B)$
using *assms* **by** *auto*
then show *?thesis*
using *translation-assoc*[*of* $-a\ a\ A$] *translation-assoc*[*of* $-a\ a\ B$] **by** *auto*
qed

lemma *translation-galois*:
fixes $a :: 'a::ab\text{-}group\text{-}add$
shows $T = ((\lambda x.\ a + x)\ `\ S) \longleftrightarrow S = ((\lambda x.\ (-\ a) + x)\ `\ T)$
using *translation-assoc*[*of* $-a\ a\ S$]
apply *auto*
using *translation-assoc*[*of* $a\ -a\ T$]
apply *auto*
done

lemma *convex-translation-eq* [*simp*]: *convex* $((\lambda x.\ a + x)\ `\ s) \longleftrightarrow$ *convex* s
by (*metis convex-translation translation-galois*)

lemma *translation-inverse-subset*:
assumes $((\lambda x.\ -\ a + x)\ `\ V) \leq (S :: 'n::ab\text{-}group\text{-}add\ set)$
shows $V \leq ((\lambda x.\ a + x)\ `\ S)$
proof −
{
fix x
assume $x \in V$
then have $x-a \in S$ **using** *assms* **by** *auto*
then have $x \in \{a + v\ |v.\ v \in S\}$
apply *auto*
apply (*rule exI*[*of* - $x-a$])
apply *simp*
done
then have $x \in ((\lambda x.\ a+x)\ `\ S)$ **by** *auto*
}

then show *?thesis* **by** *auto*
qed

lemma *convex-linear-image-eq* [*simp*]:
fixes $f :: 'a::real\text{-}vector \Rightarrow 'b::real\text{-}vector$
shows $[\![linear\ f;\ inj\ f]\!] \Longrightarrow convex\ (f\ `\ s) \longleftrightarrow convex\ s$
by (*metis* (*no-types*) *convex-linear-image convex-linear-vimage inj-vimage-image-eq*)

lemma *basis-to-basis-subspace-isomorphism*:
assumes *s*: *subspace* ($S::('n::euclidean\text{-}space)\ set$)
and *t*: *subspace* ($T::('m::euclidean\text{-}space)\ set$)
and *d*: $dim\ S = dim\ T$
and *B*: $B \subseteq S\ independent\ B\ S \subseteq span\ B\ card\ B = dim\ S$
and *C*: $C \subseteq T\ independent\ C\ T \subseteq span\ C\ card\ C = dim\ T$
shows $\exists f.\ linear\ f \wedge f\ `\ B = C \wedge f\ `\ S = T \wedge inj\text{-}on\ f\ S$
proof −
from *B independent-bound* **have** *fB*: *finite B*
by *blast*
from *C independent-bound* **have** *fC*: *finite C*
by *blast*
from *B*(*4*) *C*(*4*) *card-le-inj*[*of B C*] *d* **obtain** *f* **where**
f: $f\ `\ B \subseteq C\ inj\text{-}on\ f\ B$ **using** ‹*finite B*› ‹*finite C*› **by** *auto*
from *linear-independent-extend*[*OF B*(*2*)] **obtain** *g* **where**
g: $linear\ g\ \forall x \in B.\ g\ x = f\ x$ **by** *blast*
from *inj-on-iff-eq-card*[*OF fB*, *of f*] *f*(*2*)
have $card\ (f\ `\ B) = card\ B$ **by** *simp*
with *B*(*4*) *C*(*4*) **have** *ceq*: $card\ (f\ `\ B) = card\ C$ **using** *d*
by *simp*
have $g\ `\ B = f\ `\ B$ **using** *g*(*2*)
by (*auto simp add*: *image-iff*)
also have $\ldots = C$ **using** *card-subset-eq*[*OF fC f*(*1*) *ceq*] **.**
finally have *gBC*: $g\ `\ B = C$ **.**
have *gi*: *inj-on g B* **using** *f*(*2*) *g*(*2*)
by (*auto simp add*: *inj-on-def*)
note *g0* = *linear-indep-image-lemma*[*OF g*(*1*) *fB*, *unfolded gBC*, *OF C*(*2*) *gi*]
{
fix *x y*
assume *x*: $x \in S$ **and** *y*: $y \in S$ **and** *gxy*: $g\ x = g\ y$
from *B*(*3*) *x y* **have** *x'*: $x \in span\ B$ **and** *y'*: $y \in span\ B$
by *blast*+
from *gxy* **have** *th0*: $g\ (x - y) = 0$
by (*simp add*: *linear-sub*[*OF g*(*1*)])
have *th1*: $x - y \in span\ B$ **using** $x'\ y'$
by (*metis span-sub*)
have $x = y$ **using** *g0*[*OF th1 th0*] **by** *simp*
}
then have *giS*: *inj-on g S* **unfolding** *inj-on-def* **by** *blast*
from *span-subspace*[*OF B*(*1*,*3*) *s*]
have $g\ `\ S = span\ (g\ `\ B)$

```
    by (simp add: span-linear-image[OF g(1)])
  also have ... = span C
    unfolding gBC ..
  also have ... = T
    using span-subspace[OF C(1,3) t] .
  finally have gS: g ‘ S = T .
  from g(1) gS giS gBC show ?thesis
    by blast
qed

lemma closure-bounded-linear-image-subset:
  assumes f: bounded-linear f
  shows f ‘ closure S ⊆ closure (f ‘ S)
  using linear-continuous-on [OF f] closed-closure closure-subset
  by (rule image-closure-subset)

lemma closure-linear-image-subset:
  fixes f :: 'm::euclidean-space ⇒ 'n::real-normed-vector
  assumes linear f
  shows f ‘ (closure S) ⊆ closure (f ‘ S)
  using assms unfolding linear-conv-bounded-linear
  by (rule closure-bounded-linear-image-subset)

lemma closed-injective-linear-image:
    fixes f :: 'a::euclidean-space ⇒ 'b::euclidean-space
    assumes S: closed S and f: linear f inj f
    shows closed (f ‘ S)
proof −
  obtain g where g: linear g g ∘ f = id
    using linear-injective-left-inverse [OF f] by blast
  then have confg: continuous-on (range f) g
    using linear-continuous-on linear-conv-bounded-linear by blast
  have [simp]: g ‘ f ‘ S = S
    using g by (simp add: image-comp)
  have cgf: closed (g ‘ f ‘ S)
    by (simp add: ‹g ∘ f = id› S image-comp)
  have [simp]: {x ∈ range f. g x ∈ S} = f ‘ S
    using g by (simp add: o-def id-def image-def) metis
  show ?thesis
    apply (rule closedin-closed-trans [of range f])
    apply (rule continuous-closedin-preimage [OF confg cgf, simplified])
    apply (rule closed-injective-image-subspace)
    using f
    apply (auto simp: linear-linear linear-injective-0)
    done
qed

lemma closed-injective-linear-image-eq:
    fixes f :: 'a::euclidean-space ⇒ 'b::euclidean-space
```

assumes *f*: *linear f inj f*
shows (*closed*(*image f s*) ⟷ *closed s*)
by (*metis closed-injective-linear-image closure-eq closure-linear-image-subset closure-subset-eq f*(*1*) *f*(*2*) *inj-image-subset-iff*)

lemma *closure-injective-linear-image*:
fixes *f* :: *'a*::*euclidean-space* ⇒ *'b*::*euclidean-space*
shows ⟦*linear f*; *inj f*⟧ ⟹ *f* ‘ (*closure S*) = *closure* (*f* ‘ *S*)
apply (*rule subset-antisym*)
apply (*simp add*: *closure-linear-image-subset*)
by (*simp add*: *closure-minimal closed-injective-linear-image closure-subset image-mono*)

lemma *closure-bounded-linear-image*:
fixes *f* :: *'a*::*euclidean-space* ⇒ *'b*::*euclidean-space*
shows ⟦*linear f*; *bounded S*⟧ ⟹ *f* ‘ (*closure S*) = *closure* (*f* ‘ *S*)
apply (*rule subset-antisym*, *simp add*: *closure-linear-image-subset*)
apply (*rule closure-minimal*, *simp add*: *closure-subset image-mono*)
by (*meson bounded-closure closed-closure compact-continuous-image compact-eq-bounded-closed linear-continuous-on linear-conv-bounded-linear*)

lemma *closure-scaleR*:
fixes *S* :: *'a*::*real-normed-vector set*
shows (*op* ∗*R c*) ‘ (*closure S*) = *closure* ((*op* ∗*R c*) ‘ *S*)
proof
show (*op* ∗*R c*) ‘ (*closure S*) ⊆ *closure* ((*op* ∗*R c*) ‘ *S*)
using *bounded-linear-scaleR-right*
by (*rule closure-bounded-linear-image-subset*)
show *closure* ((*op* ∗*R c*) ‘ *S*) ⊆ (*op* ∗*R c*) ‘ (*closure S*)
by (*intro closure-minimal image-mono closure-subset closed-scaling closed-closure*)
qed

lemma *fst-linear*: *linear fst*
unfolding *linear-iff* **by** (*simp add*: *algebra-simps*)

lemma *snd-linear*: *linear snd*
unfolding *linear-iff* **by** (*simp add*: *algebra-simps*)

lemma *fst-snd-linear*: *linear* (λ(*x*,*y*). *x* + *y*)
unfolding *linear-iff* **by** (*simp add*: *algebra-simps*)

lemma *scaleR-2*:
fixes *x* :: *'a*::*real-vector*
shows *scaleR 2 x* = *x* + *x*
unfolding *one-add-one* [*symmetric*] *scaleR-left-distrib* **by** *simp*

lemma *scaleR-half-double* [*simp*]:
fixes *a* :: *'a*::*real-normed-vector*
shows (*1* / *2*) ∗*R* (*a* + *a*) = *a*
proof −

have $\bigwedge r.\ r *_R (a + a) = (r * 2) *_R a$
by (*metis scaleR-2 scaleR-scaleR*)
then show *?thesis*
by *simp*
qed

lemma *vector-choose-size*:
assumes $0 \leq c$
obtains $x :: 'a::\{real\text{-}normed\text{-}vector, perfect\text{-}space\}$ **where** $norm\ x = c$
proof −
obtain $a::'a$ **where** $a \neq 0$
using *UNIV-not-singleton UNIV-eq-I set-zero singletonI* **by** *fastforce*
then show *?thesis*
by (*rule-tac x=scaleR* (*c / norm a*) *a* **in** *that*) (*simp add*: *assms*)
qed

lemma *vector-choose-dist*:
assumes $0 \leq c$
obtains $y :: 'a::\{real\text{-}normed\text{-}vector, perfect\text{-}space\}$ **where** $dist\ x\ y = c$
by (*metis add-diff-cancel-left' assms dist-commute dist-norm vector-choose-size*)

lemma *sphere-eq-empty* [*simp*]:
fixes $a :: 'a::\{real\text{-}normed\text{-}vector, perfect\text{-}space\}$
shows $sphere\ a\ r = \{\} \longleftrightarrow r < 0$
by (*auto simp*: *sphere-def dist-norm*) (*metis dist-norm le-less-linear vector-choose-dist*)

lemma *setsum-delta-notmem*:
assumes $x \notin s$
shows $setsum\ (\lambda y.\ if\ (y = x)\ then\ P\ x\ else\ Q\ y)\ s = setsum\ Q\ s$
and $setsum\ (\lambda y.\ if\ (x = y)\ then\ P\ x\ else\ Q\ y)\ s = setsum\ Q\ s$
and $setsum\ (\lambda y.\ if\ (y = x)\ then\ P\ y\ else\ Q\ y)\ s = setsum\ Q\ s$
and $setsum\ (\lambda y.\ if\ (x = y)\ then\ P\ y\ else\ Q\ y)\ s = setsum\ Q\ s$
apply (*rule-tac* [!] *setsum.cong*)
using *assms*
apply *auto*
done

lemma *setsum-delta''*:
fixes $s::'a::real\text{-}vector\ set$
assumes *finite s*
shows $(\sum x \in s.\ (if\ y = x\ then\ f\ x\ else\ 0) *_R x) = (if\ y \in s\ then\ (f\ y) *_R y\ else\ 0)$
proof −
have $*$: $\bigwedge x\ y.\ (if\ y = x\ then\ f\ x\ else\ (0::real)) *_R x = (if\ x=y\ then\ (f\ x) *_R x\ else\ 0)$
by *auto*
show *?thesis*
unfolding $*$ **using** *setsum.delta*[*OF assms*, *of y* $\lambda x.\ f\ x *_R x$] **by** *auto*
qed

lemma *if-smult*: (*if P then x else* (*y*::*real*)) $*_R$ *v* = (*if P then x* $*_R$ *v else y* $*_R$ *v*)
by (*fact if-distrib*)

lemma *dist-triangle-eq*:
fixes *x y z* :: *'a*::*real-inner*
shows *dist x z* = *dist x y* + *dist y z* ⟷
norm (*x* − *y*) $*_R$ (*y* − *z*) = *norm* (*y* − *z*) $*_R$ (*x* − *y*)
proof −
have ∗: *x* − *y* + (*y* − *z*) = *x* − *z* **by** *auto*
show *?thesis* **unfolding** *dist-norm norm-triangle-eq*[*of x* − *y y* − *z*, *unfolded* ∗]
by (*auto simp add*:*norm-minus-commute*)
qed

lemma *norm-minus-eqI*: *x* = − *y* ⟹ *norm x* = *norm y* **by** *auto*

lemma *Min-grI*:
assumes *finite A A* ≠ {} ∀ *a*∈*A*. *x* < *a*
shows *x* < *Min A*
unfolding *Min-gr-iff*[*OF assms*(*1*,*2*)] **using** *assms*(*3*) **by** *auto*

lemma *norm-lt*: *norm x* < *norm y* ⟷ *inner x x* < *inner y y*
unfolding *norm-eq-sqrt-inner* **by** *simp*

lemma *norm-le*: *norm x* ≤ *norm y* ⟷ *inner x x* ≤ *inner y y*
unfolding *norm-eq-sqrt-inner* **by** *simp*

19.1 Affine set and affine hull

definition *affine* :: *'a*::*real-vector set* ⇒ *bool*
where *affine s* ⟷ (∀ *x*∈*s*. ∀ *y*∈*s*. ∀ *u v*. *u* + *v* = *1* ⟶ *u* $*_R$ *x* + *v* $*_R$ *y* ∈ *s*)

lemma *affine-alt*: *affine s* ⟷ (∀ *x*∈*s*. ∀ *y*∈*s*. ∀ *u*::*real*. (*1* − *u*) $*_R$ *x* + *u* $*_R$ *y* ∈ *s*)
unfolding *affine-def* **by** (*metis eq-diff-eq'*)

lemma *affine-empty* [*iff*]: *affine* {}
unfolding *affine-def* **by** *auto*

lemma *affine-sing* [*iff*]: *affine* {*x*}
unfolding *affine-alt* **by** (*auto simp add*: *scaleR-left-distrib* [*symmetric*])

lemma *affine-UNIV* [*iff*]: *affine UNIV*
unfolding *affine-def* **by** *auto*

lemma *affine-Inter*[*intro*]: (∀ *s*∈*f*. *affine s*) ⟹ *affine* (⋂ *f*)
unfolding *affine-def* **by** *auto*

lemma *affine-Int*[*intro*]: *affine s* ⟹ *affine t* ⟹ *affine* (*s* ∩ *t*)
unfolding *affine-def* **by** *auto*

lemma *affine-affine-hull* [*simp*]: *affine*(*affine hull s*)
unfolding *hull-def*
using *affine-Inter*[*of* {*t*. *affine t* ∧ *s* ⊆ *t*}] **by** *auto*

lemma *affine-hull-eq*[*simp*]: (*affine hull s* = *s*) ⟷ *affine s*
by (*metis affine-affine-hull hull-same*)

lemma *affine-hyperplane*: *affine* {*x*. *a* ∙ *x* = *b*}
by (*simp add*: *affine-def algebra-simps*) (*metis distrib-right mult.left-neutral*)

19.1.1 Some explicit formulations (from Lars Schewe)

lemma *affine*:
fixes *V*::'*a*::*real-vector set*
shows *affine V* ⟷
(∀ *s u*. *finite s* ∧ *s* ≠ {} ∧ *s* ⊆ *V* ∧ *setsum u s* = *1* ⟶ (*setsum* (λ*x*. (*u x*) *∗R x*)) *s* ∈ *V*)
unfolding *affine-def*
apply *rule*
apply(*rule*, *rule*, *rule*)
apply(*erule conjE*)+
defer
apply (*rule*, *rule*, *rule*, *rule*, *rule*)
proof −
fix *x y u v*
assume *as*: *x* ∈ *V y* ∈ *V u* + *v* = (*1*::*real*)
∀ *s u*. *finite s* ∧ *s* ≠ {} ∧ *s* ⊆ *V* ∧ *setsum u s* = *1* ⟶ (∑ *x*∈*s*. *u x ∗R x*) ∈ *V*
then show *u ∗R x* + *v ∗R y* ∈ *V*
apply (*cases x* = *y*)
using *as*(*4*)[*THEN spec*[**where** *x*={*x*,*y*}], *THEN spec*[**where** *x*=λ*w*. *if w* = *x then u else v*]]
and *as*(*1*−*3*)
apply (*auto simp add*: *scaleR-left-distrib*[*symmetric*])
done
next
fix *s u*
assume *as*: ∀ *x*∈*V*. ∀ *y*∈*V*. ∀ *u v*. *u* + *v* = *1* ⟶ *u ∗R x* + *v ∗R y* ∈ *V*
finite s s ≠ {} *s* ⊆ *V setsum u s* = (*1*::*real*)
def *n* ≡ *card s*
have *card s* = *0* ∨ *card s* = *1* ∨ *card s* = *2* ∨ *card s* > *2* **by** *auto*
then show (∑ *x*∈*s*. *u x ∗R x*) ∈ *V*
proof (*auto simp only*: *disjE*)
assume *card s* = *2*
then have *card s* = *Suc* (*Suc 0*)
by *auto*
then obtain *a b* **where** *s* = {*a*, *b*}
unfolding *card-Suc-eq* **by** *auto*
then show *?thesis*

```
      using as(1)[THEN bspec[where x=a], THEN bspec[where x=b]] using
as(4,5)
    by (auto simp add: setsum-clauses(2))
  next
    assume card s > 2
    then show ?thesis using as and n-def
    proof (induct n arbitrary: u s)
      case 0
      then show ?case by auto
    next
      case (Suc n)
      fix s :: 'a set and u :: 'a ⇒ real
      assume IA:
        ⋀u s. ⟦2 < card s; ∀x∈V. ∀y∈V. ∀u v. u + v = 1 ⟶ u *R x + v *R y
∈ V; finite s;
          s ≠ {}; s ⊆ V; setsum u s = 1; n = card s ⟧ ⟹ (∑x∈s. u x *R x) ∈ V
        and as:
          Suc n = card s 2 < card s ∀x∈V. ∀y∈V. ∀u v. u + v = 1 ⟶ u *R x
+ v *R y ∈ V
           finite s s ≠ {} s ⊆ V setsum u s = 1
      have ∃x∈s. u x ≠ 1
      proof (rule ccontr)
        assume ¬ ?thesis
        then have setsum u s = real-of-nat (card s)
          unfolding card-eq-setsum by auto
        then show False
          using as(7) and ‹card s > 2›
              by (metis One-nat-def less-Suc0 Zero-not-Suc of-nat-1 of-nat-eq-iff
numeral-2-eq-2)
      qed
      then obtain x where x:x ∈ s u x ≠ 1 by auto

      have c: card (s − {x}) = card s − 1
        apply (rule card-Diff-singleton)
        using ‹x∈s› as(4)
        apply auto
        done
      have ∗: s = insert x (s − {x}) finite (s − {x})
        using ‹x∈s› and as(4) by auto
      have ∗∗: setsum u (s − {x}) = 1 − u x
        using setsum-clauses(2)[OF ∗(2), of u x, unfolded ∗(1)[symmetric] as(7)]
by auto
      have ∗∗∗: inverse (1 − u x) ∗ setsum u (s − {x}) = 1
        unfolding ∗∗ using ‹u x ≠ 1› by auto
      have (∑xa∈s − {x}. (inverse (1 − u x) ∗ u xa) *R xa) ∈ V
      proof (cases card (s − {x}) > 2)
        case True
        then have s − {x} ≠ {} card (s − {x}) = n
          unfolding c and as(1)[symmetric]
```

```
      proof (rule-tac ccontr)
        assume ¬ s − {x} ≠ {}
        then have card (s − {x}) = 0 unfolding card-0-eq[OF ∗(2)] by simp
        then show False using True by auto
      qed auto
      then show ?thesis
        apply (rule-tac IA[of s − {x} λy. (inverse (1 − u x) ∗ u y)])
        unfolding setsum-right-distrib[symmetric]
        using as and ∗∗∗ and True
        apply auto
        done
    next
      case False
      then have card (s − {x}) = Suc (Suc 0)
        using as(2) and c by auto
      then obtain a b where (s − {x}) = {a, b} a≠b
        unfolding card-Suc-eq by auto
      then show ?thesis
        using as(3)[THEN bspec[where x=a], THEN bspec[where x=b]]
        using ∗∗∗ ∗(2) and ‹s ⊆ V›
        unfolding setsum-right-distrib
        by (auto simp add: setsum-clauses(2))
    qed
    then have u x + (1 − u x) = 1 ⟹
      u x ∗R x + (1 − u x) ∗R ((∑ xa∈s − {x}. u xa ∗R xa) /R (1 − u x)) ∈
V
      apply −
      apply (rule as(3)[rule-format])
      unfolding  Real-Vector-Spaces.scaleR-right.setsum
      using x(1) as(6)
      apply auto
      done
    then show (∑ x∈s. u x ∗R x) ∈ V
      unfolding scaleR-scaleR[symmetric] and scaleR-right.setsum [symmetric]
      apply (subst ∗)
      unfolding setsum-clauses(2)[OF ∗(2)]
      using ‹u x ≠ 1›
      apply auto
      done
  qed
 next
  assume card s = 1
  then obtain a where s={a}
    by (auto simp add: card-Suc-eq)
  then show ?thesis
    using as(4,5) by simp
 qed (insert ‹s≠{}› ‹finite s›, auto)
qed
```

lemma *affine-hull-explicit*:
affine hull p =
{*y*. ∃ *s u*. *finite s* ∧ *s* ≠ {} ∧ *s* ⊆ *p* ∧ *setsum u s* = *1* ∧ *setsum* (λ*v*. (*u v*) *$*_R$ *v*) *s* = *y*}
apply (*rule hull-unique*)
apply (*subst subset-eq*)
prefer *3*
apply *rule*
unfolding *mem-Collect-eq*
apply (*erule exE*)+
apply (*erule conjE*)+
prefer *2*
apply *rule*
proof −
fix *x*
assume *x*∈*p*
then show ∃ *s u*. *finite s* ∧ *s* ≠ {} ∧ *s* ⊆ *p* ∧ *setsum u s* = *1* ∧ (∑ *v*∈*s*. *u v* *$*_R$ *v*) = *x*
apply (*rule-tac x*={*x*} **in** *exI*)
apply (*rule-tac x*=λ*x*. *1* **in** *exI*)
apply *auto*
done
next
fix *t x s u*
assume *as*: *p* ⊆ *t affine t finite s s* ≠ {}
s ⊆ *p setsum u s* = *1* (∑ *v*∈*s*. *u v* *$*_R$ *v*) = *x*
then show *x* ∈ *t*
using *as*(*2*)[*unfolded affine*, *THEN spec*[**where** *x*=*s*], *THEN spec*[**where** *x*=*u*]]
by *auto*
next
show *affine* {*y*. ∃ *s u*. *finite s* ∧ *s* ≠ {} ∧ *s* ⊆ *p* ∧ *setsum u s* = *1* ∧ (∑ *v*∈*s*. *u v* *$*_R$ *v*) = *y*}
unfolding *affine-def*
apply (*rule*, *rule*, *rule*, *rule*, *rule*)
unfolding *mem-Collect-eq*
proof −
fix *u v* :: *real*
assume *uv*: *u* + *v* = *1*
fix *x*
assume ∃ *s u*. *finite s* ∧ *s* ≠ {} ∧ *s* ⊆ *p* ∧ *setsum u s* = *1* ∧ (∑ *v*∈*s*. *u v* *$*_R$ *v*) = *x*
then obtain *sx ux* **where**
x: *finite sx sx* ≠ {} *sx* ⊆ *p setsum ux sx* = *1* (∑ *v*∈*sx*. *ux v* *$*_R$ *v*) = *x*
by *auto*
fix *y*
assume ∃ *s u*. *finite s* ∧ *s* ≠ {} ∧ *s* ⊆ *p* ∧ *setsum u s* = *1* ∧ (∑ *v*∈*s*. *u v* *$*_R$ *v*) = *y*
then obtain *sy uy* **where**
y: *finite sy sy* ≠ {} *sy* ⊆ *p setsum uy sy* = *1* (∑ *v*∈*sy*. *uy v* *$*_R$ *v*) = *y* **by**

auto
  have *xy*: *finite* (*sx* ∪ *sy*)
    using *x*(*1*) *y*(*1*) **by** *auto*
  have ∗∗: (*sx* ∪ *sy*) ∩ *sx* = *sx* (*sx* ∪ *sy*) ∩ *sy* = *sy*
    by *auto*
  show ∃ *s ua*. *finite s* ∧ *s* ≠ {} ∧ *s* ⊆ *p* ∧
      setsum ua s = *1* ∧ (∑ *v*∈*s*. *ua v* ∗*R v*) = *u* ∗*R x* + *v* ∗*R y*
    apply (*rule-tac x*=*sx* ∪ *sy* **in** *exI*)
    apply (*rule-tac x*=λ*a*. (*if a*∈*sx then u* ∗ *ux a else 0*) + (*if a*∈*sy then v* ∗ *uy a else 0*) **in** *exI*)
    unfolding *scaleR-left-distrib setsum.distrib if-smult scaleR-zero-left*
      ∗∗ *setsum.inter-restrict*[*OF xy*, *symmetric*]
     unfolding *scaleR-scaleR*[*symmetric*] *Real-Vector-Spaces.scaleR-right.setsum*[*symmetric*]
      and *setsum-right-distrib*[*symmetric*]
    unfolding *x y*
    using *x*(*1*−*3*) *y*(*1*−*3*) *uv*
    apply *simp*
    done
 qed
qed

lemma *affine-hull-finite*:
 assumes *finite s*
 shows *affine hull s* = {*y*. ∃ *u*. *setsum u s* = *1* ∧ *setsum* (λ*v*. *u v* ∗*R v*) *s* = *y*}
 unfolding *affine-hull-explicit* **and** *set-eq-iff* **and** *mem-Collect-eq*
 apply (*rule*, *rule*)
 apply (*erule exE*)+
 apply (*erule conjE*)+
 defer
 apply (*erule exE*)
 apply (*erule conjE*)
proof −
 fix *x u*
 assume *setsum u s* = *1* (∑ *v*∈*s*. *u v* ∗*R v*) = *x*
 then show ∃ *sa u*. *finite sa* ∧
     ¬ (∀ *x*. (*x* ∈ *sa*) = (*x* ∈ {})) ∧ *sa* ⊆ *s* ∧ *setsum u sa* = *1* ∧ (∑ *v*∈*sa*. *u v* ∗*R v*) = *x*
   apply (*rule-tac x*=*s* **in** *exI*, *rule-tac x*=*u* **in** *exI*)
   using *assms*
   apply *auto*
   done
next
 fix *x t u*
 assume *t* ⊆ *s*
 then have ∗: *s* ∩ *t* = *t*
   by *auto*
 assume *finite t* ¬ (∀ *x*. (*x* ∈ *t*) = (*x* ∈ {})) *setsum u t* = *1* (∑ *v*∈*t*. *u v* ∗*R v*) = *x*

then show $\exists u.\ setsum\ u\ s = 1 \wedge (\sum v{\in}s.\ u\ v *_R v) = x$
apply (*rule-tac x=*$\lambda x.$ *if* $x{\in}t$ *then u x else 0* **in** *exI*)
unfolding *if-smult scaleR-zero-left* **and** *setsum.inter-restrict*[*OF assms*, *symmetric*] **and** $*$
apply *auto*
done
qed

19.1.2 Stepping theorems and hence small special cases

lemma *affine-hull-empty*[*simp*]: *affine hull* $\{\} = \{\}$
by (*rule hull-unique*) *auto*

lemma *affine-hull-finite-step*:
fixes $y :: 'a{::}real\text{-}vector$
shows
$(\exists u.\ setsum\ u\ \{\} = w \wedge setsum\ (\lambda x.\ u\ x *_R x)\ \{\} = y) \longleftrightarrow w = 0 \wedge y = 0$ (**is** *?th1*)
and
finite s $\Longrightarrow$
$(\exists u.\ setsum\ u\ (insert\ a\ s) = w \wedge setsum\ (\lambda x.\ u\ x *_R x)\ (insert\ a\ s) = y) \longleftrightarrow$
$(\exists v\ u.\ setsum\ u\ s = w - v \wedge setsum\ (\lambda x.\ u\ x *_R x)\ s = y - v *_R a)$ (**is** *?lhs* $=$ *?rhs*)
proof −
show *?th1* **by** *simp*
assume *fin*: *finite s*
show *?lhs* $=$ *?rhs*
proof
assume *?lhs*
then obtain *u* **where** *u*: $setsum\ u\ (insert\ a\ s) = w \wedge (\sum x{\in}insert\ a\ s.\ u\ x *_R x) = y$
by *auto*
show *?rhs*
proof (*cases* $a \in s$)
case *True*
then have $*$: $insert\ a\ s = s$ **by** *auto*
show *?thesis*
using *u*[*unfolded* $*$]
apply(*rule-tac x=0* **in** *exI*)
apply *auto*
done
next
case *False*
then show *?thesis*
apply (*rule-tac x=u a* **in** *exI*)
using *u* **and** *fin*
apply *auto*
done

```
    qed
  next
    assume ?rhs
    then obtain v u where vu: setsum u s = w − v  (∑x∈s. u x *R x) = y − v
*R a
      by auto
    have *: ⋀x M. (if x = a then v else M) *R x = (if x = a then v *R x else M
*R x)
      by auto
    show ?lhs
    proof (cases a ∈ s)
      case True
      then show ?thesis
        apply (rule-tac x=λx. (if x=a then v else 0) + u x in exI)
        unfolding setsum-clauses(2)[OF fin]
        apply simp
        unfolding scaleR-left-distrib and setsum.distrib
        unfolding vu and * and scaleR-zero-left
        apply (auto simp add: setsum.delta[OF fin])
        done
    next
      case False
      then have **:
        ⋀x. x ∈ s ⟹ u x = (if x = a then v else u x)
        ⋀x. x ∈ s ⟹ u x *R x = (if x = a then v *R x else u x *R x) by auto
      from False show ?thesis
        apply (rule-tac x=λx. if x=a then v else u x in exI)
        unfolding setsum-clauses(2)[OF fin] and * using vu
       using setsum.cong [of s - λx. u x *R x λx. if x = a then v *R x else u x *R
x, OF - **(2)]
        using setsum.cong [of s - u λx. if x = a then v else u x, OF - **(1)]
        apply auto
        done
    qed
  qed
qed

lemma affine-hull-2:
  fixes a b :: 'a::real-vector
  shows affine hull {a,b} = {u *R a + v *R b| u v. (u + v = 1)}
  (is ?lhs = ?rhs)
proof −
  have *:
    ⋀x y z. z = x − y ⟷ y + z = (x::real)
    ⋀x y z. z = x − y ⟷ y + z = (x::'a) by auto
  have ?lhs = {y. ∃u. setsum u {a, b} = 1 ∧ (∑v∈{a, b}. u v *R v) = y}
    using affine-hull-finite[of {a,b}] by auto
  also have ... = {y. ∃v u. u b = 1 − v ∧ u b *R b = y − v *R a}
    by (simp add: affine-hull-finite-step(2)[of {b} a])
```

also have ... = *?rhs* **unfolding** $*$ **by** *auto*
finally show *?thesis* **by** *auto*
qed

lemma *affine-hull-3*:
fixes *a b c* :: *'a*::*real-vector*
shows *affine hull* {*a*,*b*,*c*} = { $u *_R a + v *_R b + w *_R c$| $u\ v\ w.\ u + v + w = 1$}
proof −
have $*$:
$\bigwedge x\ y\ z.\ z = x - y \longleftrightarrow y + z = (x::real)$
$\bigwedge x\ y\ z.\ z = x - y \longleftrightarrow y + z = (x::'a)$ **by** *auto*
show *?thesis*
apply (*simp add*: *affine-hull-finite affine-hull-finite-step*)
unfolding $*$
apply *auto*
apply (*rule-tac x*=*v* **in** *exI*)
apply (*rule-tac x*=*va* **in** *exI*)
apply *auto*
apply (*rule-tac x*=*u* **in** *exI*)
apply *force*
done
qed

lemma *mem-affine*:
assumes *affine S x* ∈ *S y* ∈ *S u* + *v* = *1*
shows $u *_R x + v *_R y \in S$
using *assms affine-def*[*of S*] **by** *auto*

lemma *mem-affine-3*:
assumes *affine S x* ∈ *S y* ∈ *S z* ∈ *S u* + *v* + *w* = *1*
shows $u *_R x + v *_R y + w *_R z \in S$
proof −
have $u *_R x + v *_R y + w *_R z \in$ *affine hull* {*x*, *y*, *z*}
using *affine-hull-3*[*of x y z*] *assms* **by** *auto*
moreover
have *affine hull* {*x*, *y*, *z*} ⊆ *affine hull S*
using *hull-mono*[*of* {*x*, *y*, *z*} *S*] *assms* **by** *auto*
moreover
have *affine hull S* = *S*
using *assms affine-hull-eq*[*of S*] **by** *auto*
ultimately show *?thesis* **by** *auto*
qed

lemma *mem-affine-3-minus*:
assumes *affine S x* ∈ *S y* ∈ *S z* ∈ *S*
shows $x + v *_R (y-z) \in S$
using *mem-affine-3*[*of S x y z 1 v* −*v*] *assms*
by (*simp add*: *algebra-simps*)

corollary *mem-affine-3-minus2*:
  $[\![$*affine* S; $x \in S$; $y \in S$; $z \in S]\!] \Longrightarrow x - v *_R (y-z) \in S$
 by (*metis add-uminus-conv-diff mem-affine-3-minus real-vector.scale-minus-left*)

19.1.3 Some relations between affine hull and subspaces

lemma *affine-hull-insert-subset-span*:
 affine hull (*insert a s*) $\subseteq \{a + v| \ v \ . \ v \in$ *span* $\{x - a \mid x \ . \ x \in s\}\}$
 unfolding *subset-eq Ball-def*
 unfolding *affine-hull-explicit span-explicit mem-Collect-eq*
 apply (*rule*, *rule*)
 apply (*erule exE*)+
 apply (*erule conjE*)+
proof −
 fix *x t u*
 assume *as*: *finite t* $t \neq \{\}$ $t \subseteq$ *insert a s setsum u t* $= 1$ $(\sum v \in t. \ u \ v *_R \ v) = x$
 have $(\lambda x. \ x - a)$ ‘ $(t - \{a\}) \subseteq \{x - a \ |x. \ x \in s\}$
  using *as*(*3*) **by** *auto*
 then show $\exists v. \ x = a + v \wedge (\exists S \ u.$ *finite* $S \wedge S \subseteq \{x - a \ |x. \ x \in s\} \wedge (\sum v \in S.$
$u \ v *_R \ v) = v)$
  apply (*rule-tac x*=$x - a$ **in** *exI*)
  apply (*rule conjI*, *simp*)
  apply (*rule-tac x*=$(\lambda x. \ x - a)$ ‘ $(t - \{a\})$ **in** *exI*)
  apply (*rule-tac x*=$\lambda x. \ u \ (x + a)$ **in** *exI*)
  apply (*rule conjI*) **using** *as*(*1*) **apply** *simp*
  apply (*erule conjI*)
  using *as*(*1*)
  apply (*simp add*: *setsum.reindex*[*unfolded inj-on-def*] *scaleR-right-diff-distrib*
  setsum-subtractf scaleR-left.setsum[*symmetric*] *setsum-diff1 scaleR-left-diff-distrib*)
  unfolding *as*
  apply *simp*
  done
qed

lemma *affine-hull-insert-span*:
 assumes $a \notin s$
 shows *affine hull* (*insert a s*) $= \{a + v \mid v \ . \ v \in$ *span* $\{x - a \mid x. \ \ x \in s\}\}$
 apply (*rule*, *rule affine-hull-insert-subset-span*)
 unfolding *subset-eq Ball-def*
 unfolding *affine-hull-explicit* **and** *mem-Collect-eq*
proof (*rule*, *rule*, *erule exE*, *erule conjE*)
 fix *y v*
 assume $y = a + v$ $v \in$ *span* $\{x - a \ |x. \ x \in s\}$
 then obtain *t u* **where** *obt*: *finite t* $t \subseteq \{x - a \ |x. \ x \in s\}$ $a + (\sum v \in t. \ u \ v *_R$
$v) = y$
  unfolding *span-explicit* **by** *auto*
 def $f \equiv (\lambda x. \ x + a)$ ‘ t
 have *f*: *finite f* $f \subseteq s$ $(\sum v \in f. \ u \ (v - a) *_R \ (v - a)) = y - a$

unfolding *f-def* **using** *obt* **by** (*auto simp add*: *setsum.reindex*[*unfolded inj-on-def*])
have ∗: $f \cap \{a\} = \{\}$ $f \cap - \{a\} = f$
using *f*(*2*) *assms* **by** *auto*
show ∃ *sa u*. *finite sa* ∧ *sa* ≠ {} ∧ *sa* ⊆ *insert a s* ∧ *setsum u sa* = *1* ∧ ($\sum$ *v*∈*sa*. *u v* ∗$_R$ *v*) = *y*
apply (*rule-tac x* = *insert a f* **in** *exI*)
apply (*rule-tac x* = λ*x*. *if x*=*a then 1* − *setsum* (λ*x*. *u* (*x* − *a*)) *f else u* (*x* − *a*) **in** *exI*)
using *assms* **and** *f*
unfolding *setsum-clauses*(*2*)[*OF f*(*1*)] **and** *if-smult*
unfolding *setsum.If-cases*[*OF f*(*1*), *of* λ*x*. *x* = *a*]
apply (*auto simp add*: *setsum-subtractf scaleR-left.setsum algebra-simps* ∗)
done
qed

lemma *affine-hull-span*:
assumes *a* ∈ *s*
shows *affine hull s* = {*a* + *v* | *v*. *v* ∈ *span* {*x* − *a* | *x*. *x* ∈ *s* − {*a*}}}
using *affine-hull-insert-span*[*of a s* − {*a*}, *unfolded insert-Diff*[*OF assms*]] **by** *auto*

19.1.4 Parallel affine sets

definition *affine-parallel* :: '*a*::*real-vector set* ⇒ '*a*::*real-vector set* ⇒ *bool*
where *affine-parallel S T* ⟷ (∃ *a*. *T* = (λ*x*. *a* + *x*) ' *S*)

lemma *affine-parallel-expl-aux*:
fixes *S T* :: '*a*::*real-vector set*
assumes ∀ *x*. *x* ∈ *S* ⟷ *a* + *x* ∈ *T*
shows *T* = (λ*x*. *a* + *x*) ' *S*
proof −
{
fix *x*
assume *x* ∈ *T*
then have (− *a*) + *x* ∈ *S*
using *assms* **by** *auto*
then have *x* ∈ ((λ*x*. *a* + *x*) ' *S*)
using *imageI*[*of* −*a*+*x S* (λ*x*. *a*+*x*)] **by** *auto*
}
moreover have *T* ≥ (λ*x*. *a* + *x*) ' *S*
using *assms* **by** *auto*
ultimately show *?thesis* **by** *auto*
qed

lemma *affine-parallel-expl*: *affine-parallel S T* ⟷ (∃ *a*. ∀ *x*. *x* ∈ *S* ⟷ *a* + *x* ∈ *T*)
unfolding *affine-parallel-def*
using *affine-parallel-expl-aux*[*of S* - *T*] **by** *auto*

lemma *affine-parallel-reflex*: *affine-parallel S S*
unfolding *affine-parallel-def*
apply (*rule exI*[*of - 0*])
apply *auto*
done

lemma *affine-parallel-commut*:
assumes *affine-parallel A B*
shows *affine-parallel B A*
proof −
from *assms* **obtain** *a* **where** *B*: $B = (\lambda x.\ a + x)$ ‘ *A*
unfolding *affine-parallel-def* **by** *auto*
have [*simp*]: $(\lambda x.\ x - a) = plus\ (-\ a)$ **by** (*simp add*: *fun-eq-iff*)
from *B* **show** *?thesis*
using *translation-galois* [*of B a A*]
unfolding *affine-parallel-def* **by** *auto*
qed

lemma *affine-parallel-assoc*:
assumes *affine-parallel A B*
and *affine-parallel B C*
shows *affine-parallel A C*
proof −
from *assms* **obtain** *ab* **where** $B = (\lambda x.\ ab + x)$ ‘ *A*
unfolding *affine-parallel-def* **by** *auto*
moreover
from *assms* **obtain** *bc* **where** $C = (\lambda x.\ bc + x)$ ‘ *B*
unfolding *affine-parallel-def* **by** *auto*
ultimately show *?thesis*
using *translation-assoc*[*of bc ab A*] **unfolding** *affine-parallel-def* **by** *auto*
qed

lemma *affine-translation-aux*:
fixes *a* :: *'a*::*real-vector*
assumes *affine* $((\lambda x.\ a + x)$ ‘ $S)$
shows *affine S*
proof −
{
fix *x y u v*
assume *xy*: $x \in S\ y \in S\ (u :: real) + v = 1$
then have $(a + x) \in ((\lambda x.\ a + x)$ ‘ $S)\ (a + y) \in ((\lambda x.\ a + x)$ ‘ $S)$
by *auto*
then have *h1*: $u *_R\ (a + x) + v *_R (a + y) \in (\lambda x.\ a + x)$ ‘ *S*
using *xy assms* **unfolding** *affine-def* **by** *auto*
have $u *_R (a + x) + v *_R (a + y) = (u + v) *_R a + (u *_R x + v *_R y)$
by (*simp add*: *algebra-simps*)
also have $\ldots = a + (u *_R x + v *_R y)$
using ‹$u + v = 1$› **by** *auto*
ultimately have $a + (u *_R x + v *_R y) \in (\lambda x.\ a + x)$ ‘ *S*

```
    using h1 by auto
  then have u *R x + v *R y : S by auto
 }
 then show ?thesis unfolding affine-def by auto
qed

lemma affine-translation:
 fixes a :: 'a::real-vector
 shows affine S ⟷ affine ((λx. a + x) ` S)
proof −
 have affine S ⟹ affine ((λx. a + x) ` S)
  using affine-translation-aux[of −a ((λx. a + x) ` S)]
  using translation-assoc[of −a a S] by auto
 then show ?thesis using affine-translation-aux by auto
qed

lemma parallel-is-affine:
 fixes S T :: 'a::real-vector set
 assumes affine S affine-parallel S T
 shows affine T
proof −
 from assms obtain a where T = (λx. a + x) ` S
  unfolding affine-parallel-def by auto
 then show ?thesis
  using affine-translation assms by auto
qed

lemma subspace-imp-affine: subspace s ⟹ affine s
 unfolding subspace-def affine-def by auto
```

19.1.5 Subspace parallel to an affine set

```
lemma subspace-affine: subspace S ⟷ affine S ∧ 0 ∈ S
proof −
 have h0: subspace S ⟹ affine S ∧ 0 ∈ S
  using subspace-imp-affine[of S] subspace-0 by auto
 {
  assume assm: affine S ∧ 0 ∈ S
  {
   fix c :: real
   fix x
   assume x: x ∈ S
   have c *R x = (1−c) *R 0 + c *R x by auto
   moreover
   have (1 − c) *R 0 + c *R x ∈ S
    using affine-alt[of S] assm x by auto
   ultimately have c *R x ∈ S by auto
  }
  then have h1: ∀ c. ∀ x ∈ S. c *R x ∈ S by auto
```

```
    {
      fix x y
      assume xy: x ∈ S y ∈ S
      def u == (1 :: real)/2
      have (1/2) *R (x+y) = (1/2) *R (x+y)
        by auto
      moreover
      have (1/2) *R (x+y)=(1/2) *R x + (1-(1/2)) *R y
        by (simp add: algebra-simps)
      moreover
      have (1 - u) *R x + u *R y ∈ S
        using affine-alt[of S] assm xy by auto
      ultimately
      have (1/2) *R (x+y) ∈ S
        using u-def by auto
      moreover
      have x + y = 2 *R ((1/2) *R (x+y))
        by auto
      ultimately
      have x + y ∈ S
        using h1[rule-format, of (1/2) *R (x+y) 2] by auto
    }
    then have ∀x ∈ S. ∀y ∈ S. x + y ∈ S
      by auto
    then have subspace S
      using h1 assm unfolding subspace-def by auto
  }
  then show ?thesis using h0 by metis
qed

lemma affine-diffs-subspace:
  assumes affine S a ∈ S
  shows subspace ((λx. (-a)+x) ` S)
proof -
  have [simp]: (λx. x - a) = plus (- a) by (simp add: fun-eq-iff)
  have affine ((λx. (-a)+x) ` S)
    using affine-translation assms by auto
  moreover have 0 : ((λx. (-a)+x) ` S)
    using assms exI[of (λx. x∈S ∧ -a+x = 0) a] by auto
  ultimately show ?thesis using subspace-affine by auto
qed

lemma parallel-subspace-explicit:
  assumes affine S
    and a ∈ S
  assumes L ≡ {y. ∃x ∈ S. (-a) + x = y}
  shows subspace L ∧ affine-parallel S L
proof -
```

from *assms* **have** $L = plus\ (-\ a)\ `\ S$ **by** *auto*
then have *par*: *affine-parallel S L*
unfolding *affine-parallel-def* **..**
then have *affine L* **using** *assms parallel-is-affine* **by** *auto*
moreover have $0 \in L$
using *assms* **by** *auto*
ultimately show *?thesis*
using *subspace-affine par* **by** *auto*
qed

lemma *parallel-subspace-aux*:
assumes *subspace A*
and *subspace B*
and *affine-parallel A B*
shows $A \supseteq B$
proof −
from *assms* **obtain** *a* **where** *a*: $\forall x.\ x \in A \longleftrightarrow a + x \in B$
using *affine-parallel-expl*[*of A B*] **by** *auto*
then have $-a \in A$
using *assms subspace-0*[*of B*] **by** *auto*
then have $a \in A$
using *assms subspace-neg*[*of A* $-a$] **by** *auto*
then show *?thesis*
using *assms a* **unfolding** *subspace-def* **by** *auto*
qed

lemma *parallel-subspace*:
assumes *subspace A*
and *subspace B*
and *affine-parallel A B*
shows $A = B$
proof
show $A \supseteq B$
using *assms parallel-subspace-aux* **by** *auto*
show $A \subseteq B$
using *assms parallel-subspace-aux*[*of B A*] *affine-parallel-commut* **by** *auto*
qed

lemma *affine-parallel-subspace*:
assumes *affine S* $S \neq \{\}$
shows $\exists!L.$ *subspace L* $\wedge$ *affine-parallel S L*
proof −
have *ex*: $\exists\, L.$ *subspace L* $\wedge$ *affine-parallel S L*
using *assms parallel-subspace-explicit* **by** *auto*
{
fix *L1 L2*
assume *ass*: *subspace L1* $\wedge$ *affine-parallel S L1 subspace L2* $\wedge$ *affine-parallel S L2*
then have *affine-parallel L1 L2*

using *affine-parallel-commut*[*of S L1*] *affine-parallel-assoc*[*of L1 S L2*] **by** *auto*
then have *L1 = L2*
using *ass parallel-subspace* **by** *auto*
}
then show *?thesis* **using** *ex* **by** *auto*
qed

19.2 Cones

definition *cone* :: *'a*::*real-vector set* ⇒ *bool*
where *cone s* ⟷ (∀ *x*∈*s*. ∀ *c*≥*0*. *c* ∗$_R$ *x* ∈ *s*)

lemma *cone-empty*[*intro*, *simp*]: *cone* {}
unfolding *cone-def* **by** *auto*

lemma *cone-univ*[*intro*, *simp*]: *cone UNIV*
unfolding *cone-def* **by** *auto*

lemma *cone-Inter*[*intro*]: ∀ *s*∈*f*. *cone s* ⟹ *cone* (⋂ *f*)
unfolding *cone-def* **by** *auto*

19.2.1 Conic hull

lemma *cone-cone-hull*: *cone* (*cone hull s*)
unfolding *hull-def* **by** *auto*

lemma *cone-hull-eq*: *cone hull s* = *s* ⟷ *cone s*
apply (*rule hull-eq*)
using *cone-Inter*
unfolding *subset-eq*
apply *auto*
done

lemma *mem-cone*:
assumes *cone S x* ∈ *S c* ≥ *0*
shows *c* ∗$_R$ *x* : *S*
using *assms cone-def*[*of S*] **by** *auto*

lemma *cone-contains-0*:
assumes *cone S*
shows *S* ≠ {} ⟷ *0* ∈ *S*
proof −
{
assume *S* ≠ {}
then obtain *a* **where** *a* ∈ *S* **by** *auto*
then have *0* ∈ *S*
using *assms mem-cone*[*of S a 0*] **by** *auto*
}
then show *?thesis* **by** *auto*

qed

lemma *cone-0*: *cone* $\{0\}$
unfolding *cone-def* **by** *auto*

lemma *cone-Union*[*intro*]: $(\forall s{\in}f.\ cone\ s) \longrightarrow cone\ (\bigcup f)$
unfolding *cone-def* **by** *blast*

lemma *cone-iff*:
assumes $S \neq \{\}$
shows $cone\ S \longleftrightarrow 0 \in S \wedge (\forall c.\ c > 0 \longrightarrow (op *_R\ c)\ `\ S = S)$
proof −
{
assume *cone S*
{
fix $c :: real$
assume $c > 0$
{
fix x
assume $x \in S$
then have $x \in (op *_R\ c)\ `\ S$
unfolding *image-def*
using ‹*cone S*› ‹$c{>}0$› *mem-cone*[*of S x 1/c*]
exI[*of* $(\lambda t.\ t \in S \wedge x = c *_R t)\ (1\ /\ c) *_R x$]
by *auto*
}
moreover
{
fix x
assume $x \in (op *_R\ c)\ `\ S$
then have $x \in S$
using ‹*cone S*› ‹$c > 0$›
unfolding *cone-def image-def* ‹$c > 0$› **by** *auto*
}
ultimately have $(op *_R\ c)\ `\ S = S$ **by** *auto*
}
then have $0 \in S \wedge (\forall c.\ c > 0 \longrightarrow (op *_R\ c)\ `\ S = S)$
using ‹*cone S*› *cone-contains-0*[*of S*] *assms* **by** *auto*
}
moreover
{
assume *a*: $0 \in S \wedge (\forall c.\ c > 0 \longrightarrow (op *_R\ c)\ `\ S = S)$
{
fix x
assume $x \in S$
fix $c1 :: real$
assume $c1 \geq 0$
then have $c1 = 0 \vee c1 > 0$ **by** *auto*
then have $c1 *_R x \in S$ **using** *a* ‹$x \in S$› **by** *auto*

```
    }
    then have cone S unfolding cone-def by auto
  }
  ultimately show ?thesis by blast
qed

lemma cone-hull-empty: cone hull {} = {}
  by (metis cone-empty cone-hull-eq)

lemma cone-hull-empty-iff: S = {} ⟷ cone hull S = {}
  by (metis bot-least cone-hull-empty hull-subset xtrans(5))

lemma cone-hull-contains-0: S ≠ {} ⟷ 0 ∈ cone hull S
  using cone-cone-hull[of S] cone-contains-0[of cone hull S] cone-hull-empty-iff[of
S]
  by auto

lemma mem-cone-hull:
  assumes x ∈ S c ≥ 0
  shows c *R x ∈ cone hull S
  by (metis assms cone-cone-hull hull-inc mem-cone)

lemma cone-hull-expl: cone hull S = {c *R x | c x. c ≥ 0 ∧ x ∈ S}
  (is ?lhs = ?rhs)
proof −
  {
    fix x
    assume x ∈ ?rhs
    then obtain cx :: real and xx where x: x = cx *R xx cx ≥ 0 xx ∈ S
      by auto
    fix c :: real
    assume c: c ≥ 0
    then have c *R x = (c * cx) *R xx
      using x by (simp add: algebra-simps)
    moreover
    have c * cx ≥ 0 using c x by auto
    ultimately
    have c *R x ∈ ?rhs using x by auto
  }
  then have cone ?rhs
    unfolding cone-def by auto
  then have ?rhs ∈ Collect cone
    unfolding mem-Collect-eq by auto
  {
    fix x
    assume x ∈ S
    then have 1 *R x ∈ ?rhs
      apply auto
      apply (rule-tac x = 1 in exI)
```

```
    apply auto
    done
  then have x ∈ ?rhs by auto
 }
 then have S ⊆ ?rhs by auto
 then have ?lhs ⊆ ?rhs
  using ‹?rhs ∈ Collect cone› hull-minimal[of S ?rhs cone] by auto
 moreover
 {
  fix x
  assume x ∈ ?rhs
  then obtain cx :: real and xx where x: x = cx *R xx cx ≥ 0 xx ∈ S
   by auto
  then have xx ∈ cone hull S
   using hull-subset[of S] by auto
  then have x ∈ ?lhs
   using x cone-cone-hull[of S] cone-def[of cone hull S] by auto
 }
 ultimately show ?thesis by auto
qed

lemma cone-closure:
 fixes S :: 'a::real-normed-vector set
 assumes cone S
 shows cone (closure S)
proof (cases S = {})
 case True
 then show ?thesis by auto
next
 case False
 then have 0 ∈ S ∧ (∀ c. c > 0 ⟶ op *R c ' S = S)
  using cone-iff[of S] assms by auto
 then have 0 ∈ closure S ∧ (∀ c. c > 0 ⟶ op *R c ' closure S = closure S)
  using closure-subset by (auto simp add: closure-scaleR)
 then show ?thesis
  using False cone-iff[of closure S] by auto
qed
```

19.3 Affine dependence and consequential theorems (from Lars Schewe)

```
definition affine-dependent :: 'a::real-vector set ⇒ bool
 where affine-dependent s ⟷ (∃ x∈s. x ∈ affine hull (s − {x}))

lemma affine-dependent-explicit:
 affine-dependent p ⟷
  (∃ s u. finite s ∧ s ⊆ p ∧ setsum u s = 0 ∧
    (∃ v∈s. u v ≠ 0) ∧ setsum (λv. u v *R v) s = 0)
 unfolding affine-dependent-def affine-hull-explicit mem-Collect-eq
```

apply *rule*
apply (*erule bexE*, *erule exE*, *erule exE*)
apply (*erule conjE*)+
defer
apply (*erule exE*, *erule exE*)
apply (*erule conjE*)+
apply (*erule bexE*)
proof −
fix *x s u*
assume *as*: $x \in p$ *finite s* $s \neq \{\}$ $s \subseteq p - \{x\}$ *setsum u s* $= 1$ $(\sum v{\in}s.\ u\ v *_R v) = x$
have $x \notin s$ **using** *as*(*1*,*4*) **by** *auto*
show $\exists s\ u.$ *finite* $s \wedge s \subseteq p \wedge$ *setsum u s* $= 0 \wedge (\exists v{\in}s.\ u\ v \neq 0) \wedge (\sum v{\in}s.\ u\ v *_R v) = 0$
apply (*rule-tac x*=*insert x s* **in** *exI*, *rule-tac x*=$\lambda v.$ *if* $v = x$ *then* -1 *else u v* **in** *exI*)
unfolding *if-smult* **and** *setsum-clauses*(*2*)[*OF as*(*2*)] **and** *setsum-delta-notmem*[*OF* ‹$x \notin s$›] **and** *as*
using *as*
apply *auto*
done
next
fix *s u v*
assume *as*: *finite s* $s \subseteq p$ *setsum u s* $= 0$ $(\sum v{\in}s.\ u\ v *_R v) = 0$ $v \in s$ $u\ v \neq 0$
have $s \neq \{v\}$
using *as*(*3*,*6*) **by** *auto*
then show $\exists x{\in}p.\ \exists s\ u.$ *finite* $s \wedge s \neq \{\} \wedge s \subseteq p - \{x\} \wedge$ *setsum u s* $= 1 \wedge (\sum v{\in}s.\ u\ v *_R v) = x$
apply (*rule-tac x*=*v* **in** *bexI*)
apply (*rule-tac x*=*s* $-$ $\{v\}$ **in** *exI*)
apply (*rule-tac x*=$\lambda x.$ $-$ (*1* / *u v*) $*$ *u x* **in** *exI*)
unfolding *scaleR-scaleR*[*symmetric*] **and** *scaleR-right.setsum* [*symmetric*]
unfolding *setsum-right-distrib*[*symmetric*] **and** *setsum-diff1*[*OF as*(*1*)]
using *as*
apply *auto*
done
qed

lemma *affine-dependent-explicit-finite*:
fixes *s* :: *'a*::*real-vector set*
assumes *finite s*
shows *affine-dependent s* $\longleftrightarrow$
$(\exists u.$ *setsum u s* $= 0 \wedge (\exists v{\in}s.\ u\ v \neq 0) \wedge$ *setsum* $(\lambda v.\ u\ v *_R v)\ s = 0)$
(**is** *?lhs* $=$ *?rhs*)
proof
have $*$: $\bigwedge vt\ u\ v.$ $(\textit{if } vt \textit{ then } u\ v \textit{ else } 0) *_R v = (\textit{if } vt \textit{ then } (u\ v) *_R v \textit{ else } 0{::}'a)$
by *auto*
assume *?lhs*
then obtain *t u v* **where**

finite t t ⊆ s setsum u t = 0 v∈t u v ≠ 0 $(\sum v{\in}t.\ u\ v *_R v) = 0$
unfolding *affine-dependent-explicit* **by** *auto*
then show *?rhs*
apply (*rule-tac x=λx. if x∈t then u x else 0* **in** *exI*)
apply *auto* **unfolding** ∗ **and** *setsum.inter-restrict*[*OF assms*, *symmetric*]
unfolding *Int-absorb1*[*OF* ‹*t⊆s*›]
apply *auto*
done
next
assume *?rhs*
then obtain *u v* **where** *setsum u s = 0 v∈s u v ≠ 0* $(\sum v{\in}s.\ u\ v *_R v) = 0$
by *auto*
then show *?lhs* **unfolding** *affine-dependent-explicit*
using *assms* **by** *auto*
qed

19.4 Connectedness of convex sets

lemma *connectedD*:
connected S ⟹ open A ⟹ open B ⟹ S ⊆ A ∪ B ⟹ A ∩ B ∩ S = {} ⟹ A ∩ S = {} ∨ B ∩ S = {}
by (*rule Topological-Spaces.topological-space-class.connectedD*)

lemma *convex-connected*:
fixes *s* :: *'a*::*real-normed-vector set*
assumes *convex s*
shows *connected s*
proof (*rule connectedI*)
fix *A B*
assume *open A open B A ∩ B ∩ s = {} s ⊆ A ∪ B*
moreover
assume *A ∩ s ≠ {} B ∩ s ≠ {}*
then obtain *a b* **where** *a*: *a ∈ A a ∈ s* **and** *b*: *b ∈ B b ∈ s* **by** *auto*
def *f ≡ λu. u ∗R a + (1 − u) ∗R b*
then have *continuous-on {0 .. 1} f*
by (*auto intro*!: *continuous-intros*)
then have *connected* (*f ‘ {0 .. 1}*)
by (*auto intro*!: *connected-continuous-image*)
note *connectedD*[*OF this*, *of A B*]
moreover have *a ∈ A ∩ f ‘ {0 .. 1}*
using *a* **by** (*auto intro*!: *image-eqI*[*of - - 1*] *simp*: *f-def*)
moreover have *b ∈ B ∩ f ‘ {0 .. 1}*
using *b* **by** (*auto intro*!: *image-eqI*[*of - - 0*] *simp*: *f-def*)
moreover have *f ‘ {0 .. 1} ⊆ s*
using ‹*convex s*› *a b* **unfolding** *convex-def f-def* **by** *auto*
ultimately show *False* **by** *auto*
qed

corollary *connected-UNIV*[*intro*]: *connected* (*UNIV* :: *'a*::*real-normed-vector set*)

by(*simp add*: *convex-connected*)

proposition *clopen*:
fixes $s :: 'a :: real\text{-}normed\text{-}vector\ set$
shows $closed\ s \wedge open\ s \longleftrightarrow s = \{\} \vee s = UNIV$
apply (*rule iffI*)
apply (*rule connected-UNIV* [*unfolded connected-clopen*, *rule-format*])
apply (*force simp add*: *open-openin closed-closedin*, *force*)
done

corollary *compact-open*:
fixes $s :: 'a :: euclidean\text{-}space\ set$
shows $compact\ s \wedge open\ s \longleftrightarrow s = \{\}$
by (*auto simp*: *compact-eq-bounded-closed clopen*)

corollary *finite-imp-not-open*:
fixes $S :: 'a::\{real\text{-}normed\text{-}vector, perfect\text{-}space\}\ set$
shows $[\![finite\ S; open\ S]\!] \Longrightarrow S=\{\}$
using *clopen* [*of S*] *finite-imp-closed not-bounded-UNIV* **by** *blast*

Balls, being convex, are connected.

lemma *convex-prod*:
assumes $\bigwedge i.\ i \in Basis \Longrightarrow convex\ \{x.\ P\ i\ x\}$
shows $convex\ \{x.\ \forall i{\in}Basis.\ P\ i\ (x{\cdot}i)\}$
using *assms* **unfolding** *convex-def*
by (*auto simp*: *inner-add-left*)

lemma *convex-positive-orthant*: $convex\ \{x::'a::euclidean\text{-}space.\ (\forall i{\in}Basis.\ 0 \le x{\cdot}i)\}$
by (*rule convex-prod*) (*simp add*: *atLeast-def*[*symmetric*] *convex-real-interval*)

lemma *convex-local-global-minimum*:
fixes $s :: 'a::real\text{-}normed\text{-}vector\ set$
assumes $e > 0$
and $convex\text{-}on\ s\ f$
and $ball\ x\ e \subseteq s$
and $\forall y{\in}ball\ x\ e.\ f\ x \le f\ y$
shows $\forall y{\in}s.\ f\ x \le f\ y$
proof (*rule ccontr*)
have $x \in s$ **using** *assms*(*1*,*3*) **by** *auto*
assume $\neg$ *?thesis*
then obtain y **where** $y{\in}s$ **and** y: $f\ x > f\ y$ **by** *auto*
then have xy: $0 < dist\ x\ y$ **by** *auto*
then obtain u **where** $0 < u\ u \le 1$ **and** u: $u < e\ /\ dist\ x\ y$
using *real-lbound-gt-zero*[*of 1 e / dist x y*] *xy* ‹*e*>*0*› **by** *auto*
then have $f\ ((1-u) *_R x + u *_R y) \le (1-u) * f\ x + u * f\ y$
using ‹*x*∈*s*› ‹*y*∈*s*›
using *assms*(*2*)[*unfolded convex-on-def*,
THEN bspec[**where** *x*=*x*], *THEN bspec*[**where** *x*=*y*], *THEN spec*[**where**

```
x=1−u]]
    by auto
  moreover
  have ∗: x − ((1 − u) ∗R x + u ∗R y) = u ∗R (x − y)
    by (simp add: algebra-simps)
  have (1 − u) ∗R x + u ∗R y ∈ ball x e
    unfolding mem-ball dist-norm
    unfolding ∗ and norm-scaleR and abs-of-pos[OF ‹0<u›]
    unfolding dist-norm[symmetric]
    using u
    unfolding pos-less-divide-eq[OF xy]
    by auto
  then have f x ≤ f ((1 − u) ∗R x + u ∗R y)
    using assms(4) by auto
  ultimately show False
    using mult-strict-left-mono[OF y ‹u>0›]
    unfolding left-diff-distrib
    by auto
qed

lemma convex-ball [iff]:
  fixes x :: 'a::real-normed-vector
  shows convex (ball x e)
proof (auto simp add: convex-def)
  fix y z
  assume yz: dist x y < e dist x z < e
  fix u v :: real
  assume uv: 0 ≤ u 0 ≤ v u + v = 1
  have dist x (u ∗R y + v ∗R z) ≤ u ∗ dist x y + v ∗ dist x z
    using uv yz
    using convex-on-dist [of ball x e x, unfolded convex-on-def,
      THEN bspec[where x=y], THEN bspec[where x=z]]
    by auto
  then show dist x (u ∗R y + v ∗R z) < e
    using convex-bound-lt[OF yz uv] by auto
qed

lemma convex-cball [iff]:
  fixes x :: 'a::real-normed-vector
  shows convex (cball x e)
proof −
  {
    fix y z
    assume yz: dist x y ≤ e dist x z ≤ e
    fix u v :: real
    assume uv: 0 ≤ u 0 ≤ v u + v = 1
    have dist x (u ∗R y + v ∗R z) ≤ u ∗ dist x y + v ∗ dist x z
      using uv yz
      using convex-on-dist [of cball x e x, unfolded convex-on-def,
```

```
      THEN bspec[where x=y], THEN bspec[where x=z]]
    by auto
  then have dist x (u *R y + v *R z) ≤ e
    using convex-bound-le[OF yz uv] by auto
 }
 then show ?thesis by (auto simp add: convex-def Ball-def)
qed

lemma connected-ball [iff]:
 fixes x :: 'a::real-normed-vector
 shows connected (ball x e)
 using convex-connected convex-ball by auto

lemma connected-cball [iff]:
 fixes x :: 'a::real-normed-vector
 shows connected (cball x e)
 using convex-connected convex-cball by auto
```

19.5 Convex hull

```
lemma convex-convex-hull [iff]: convex (convex hull s)
 unfolding hull-def
 using convex-Inter[of {t. convex t ∧ s ⊆ t}]
 by auto

lemma convex-hull-eq: convex hull s = s ⟷ convex s
 by (metis convex-convex-hull hull-same)

lemma bounded-convex-hull:
 fixes s :: 'a::real-normed-vector set
 assumes bounded s
 shows bounded (convex hull s)
proof −
 from assms obtain B where B: ∀ x∈s. norm x ≤ B
   unfolding bounded-iff by auto
 show ?thesis
   apply (rule bounded-subset[OF bounded-cball, of - 0 B])
   unfolding subset-hull[of convex, OF convex-cball]
   unfolding subset-eq mem-cball dist-norm using B
   apply auto
   done
qed

lemma finite-imp-bounded-convex-hull:
 fixes s :: 'a::real-normed-vector set
 shows finite s ⟹ bounded (convex hull s)
 using bounded-convex-hull finite-imp-bounded
 by auto
```

19.5.1 Convex hull is "preserved" by a linear function

lemma *convex-hull-linear-image*:
assumes *f*: *linear f*
shows *f* ‘ (*convex hull s*) = *convex hull* (*f* ‘ *s*)
proof
show *convex hull* (*f* ‘ *s*) ⊆ *f* ‘ (*convex hull s*)
by (*intro hull-minimal image-mono hull-subset convex-linear-image assms convex-convex-hull*)
show *f* ‘ (*convex hull s*) ⊆ *convex hull* (*f* ‘ *s*)
proof (*unfold image-subset-iff-subset-vimage*, *rule hull-minimal*)
show *s* ⊆ *f* −‘ (*convex hull* (*f* ‘ *s*))
by (*fast intro*: *hull-inc*)
show *convex* (*f* −‘ (*convex hull* (*f* ‘ *s*)))
by (*intro convex-linear-vimage* [*OF f*] *convex-convex-hull*)
qed
qed

lemma *in-convex-hull-linear-image*:
assumes *linear f*
and *x* ∈ *convex hull s*
shows *f x* ∈ *convex hull* (*f* ‘ *s*)
using *convex-hull-linear-image*[*OF assms*(*1*)] *assms*(*2*) **by** *auto*

lemma *convex-hull-Times*:
convex hull (*s* × *t*) = (*convex hull s*) × (*convex hull t*)
proof
show *convex hull* (*s* × *t*) ⊆ (*convex hull s*) × (*convex hull t*)
by (*intro hull-minimal Sigma-mono hull-subset convex-Times convex-convex-hull*)
have ∀ *x*∈*convex hull s*. ∀ *y*∈*convex hull t*. (*x*, *y*) ∈ *convex hull* (*s* × *t*)
proof (*intro hull-induct*)
fix *x y* **assume** *x* ∈ *s* **and** *y* ∈ *t*
then show (*x*, *y*) ∈ *convex hull* (*s* × *t*)
by (*simp add*: *hull-inc*)
next
fix *x* **let** *?S* = ((λ*y*. (*0*, *y*)) −‘ (λ*p*. (− *x*, *0*) + *p*) ‘ (*convex hull s* × *t*))
have *convex ?S*
by (*intro convex-linear-vimage convex-translation convex-convex-hull*,
simp add: *linear-iff*)
also have *?S* = {*y*. (*x*, *y*) ∈ *convex hull* (*s* × *t*)}
by (*auto simp add*: *image-def Bex-def*)
finally show *convex* {*y*. (*x*, *y*) ∈ *convex hull* (*s* × *t*)} **.**
next
show *convex* {*x*. ∀ *y*∈*convex hull t*. (*x*, *y*) ∈ *convex hull* (*s* × *t*)}
proof (*unfold Collect-ball-eq*, *rule convex-INT* [*rule-format*])
fix *y* **let** *?S* = ((λ*x*. (*x*, *0*)) −‘ (λ*p*. (*0*, − *y*) + *p*) ‘ (*convex hull s* × *t*))
have *convex ?S*
by (*intro convex-linear-vimage convex-translation convex-convex-hull*,
simp add: *linear-iff*)
also have *?S* = {*x*. (*x*, *y*) ∈ *convex hull* (*s* × *t*)}
by (*auto simp add*: *image-def Bex-def*)

```
    finally show convex {x. (x, y) ∈ convex hull (s × t)} .
   qed
  qed
  then show (convex hull s) × (convex hull t) ⊆ convex hull (s × t)
   unfolding subset-eq split-paired-Ball-Sigma .
qed
```

19.5.2 Stepping theorems for convex hulls of finite sets

```
lemma convex-hull-empty[simp]: convex hull {} = {}
  by (rule hull-unique) auto

lemma convex-hull-singleton[simp]: convex hull {a} = {a}
  by (rule hull-unique) auto

lemma convex-hull-insert:
  fixes s :: 'a::real-vector set
  assumes s ≠ {}
  shows convex hull (insert a s) =
    {x. ∃u≥0. ∃v≥0. ∃b. (u + v = 1) ∧ b ∈ (convex hull s) ∧ (x = u *R a + v
*R b)}
  (is - = ?hull)
  apply (rule, rule hull-minimal, rule)
  unfolding insert-iff
  prefer 3
  apply rule
proof −
  fix x
  assume x: x = a ∨ x ∈ s
  then show x ∈ ?hull
    apply rule
    unfolding mem-Collect-eq
    apply (rule-tac x=1 in exI)
    defer
    apply (rule-tac x=0 in exI)
    using assms hull-subset[of s convex]
    apply auto
    done
next
  fix x
  assume x ∈ ?hull
  then obtain u v b where obt: u≥0 v≥0 u + v = 1 b ∈ convex hull s x = u *R
a + v *R b
    by auto
  have a ∈ convex hull insert a s b ∈ convex hull insert a s
    using hull-mono[of s insert a s convex] hull-mono[of {a} insert a s convex]
and obt(4)
    by auto
  then show x ∈ convex hull insert a s
```

```
    unfolding obt(5) using obt(1−3)
    by (rule convexD [OF convex-convex-hull])
next
  show convex ?hull
  proof (rule convexI)
    fix x y u v
    assume as: (0::real) ≤ u 0 ≤ v u + v = 1 x∈?hull y∈?hull
    from as(4) obtain u1 v1 b1 where
      obt1: u1≥0 v1≥0 u1 + v1 = 1 b1 ∈ convex hull s x = u1 *R a + v1 *R b1
      by auto
    from as(5) obtain u2 v2 b2 where
      obt2: u2≥0 v2≥0 u2 + v2 = 1 b2 ∈ convex hull s y = u2 *R a + v2 *R b2
      by auto
    have *: ⋀(x::'a) s1 s2. x − s1 *R x − s2 *R x = ((1::real) − (s1 + s2)) *R x
      by (auto simp add: algebra-simps)
    have **: ∃ b ∈ convex hull s. u *R x + v *R y =
      (u * u1) *R a + (v * u2) *R a + (b − (u * u1) *R b − (v * u2) *R b)
    proof (cases u * v1 + v * v2 = 0)
      case True
      have *: ⋀(x::'a) s1 s2. x − s1 *R x − s2 *R x = ((1::real) − (s1 + s2))
*R x
        by (auto simp add: algebra-simps)
      from True have ***: u * v1 = 0 v * v2 = 0
          using mult-nonneg-nonneg[OF ‹u≥0› ‹v1≥0›] mult-nonneg-nonneg[OF
‹v≥0› ‹v2≥0›]
        by arith+
      then have u * u1 + v * u2 = 1
        using as(3) obt1(3) obt2(3) by auto
      then show ?thesis
        unfolding obt1(5) obt2(5) *
        using assms hull-subset[of s convex]
        by (auto simp add: *** scaleR-right-distrib)
    next
      case False
      have 1 − (u * u1 + v * u2) = (u + v) − (u * u1 + v * u2)
        using as(3) obt1(3) obt2(3) by (auto simp add: field-simps)
      also have ... = u * (v1 + u1 − u1) + v * (v2 + u2 − u2)
        using as(3) obt1(3) obt2(3) by (auto simp add: field-simps)
      also have ... = u * v1 + v * v2
        by simp
      finally have **:1 − (u * u1 + v * u2) = u * v1 + v * v2 by auto
      have 0 ≤ u * v1 + v * v2 0 ≤ u * v1 0 ≤ u * v1 + v * v2 0 ≤ v * v2
        using as(1,2) obt1(1,2) obt2(1,2) by auto
      then show ?thesis
        unfolding obt1(5) obt2(5)
        unfolding * and **
        using False
        apply (rule-tac
          x = ((u * v1) / (u * v1 + v * v2)) *R b1 + ((v * v2) / (u * v1 + v *
```

```
v2)) *R b2 in bexI)
      defer
      apply (rule convexD [OF convex-convex-hull])
      using obt1(4) obt2(4)
      unfolding add-divide-distrib[symmetric] and zero-le-divide-iff
      apply (auto simp add: scaleR-left-distrib scaleR-right-distrib)
      done
  qed
  have u1: u1 ≤ 1
    unfolding obt1(3)[symmetric] and not-le using obt1(2) by auto
  have u2: u2 ≤ 1
    unfolding obt2(3)[symmetric] and not-le using obt2(2) by auto
  have u1 * u + u2 * v ≤ max u1 u2 * u + max u1 u2 * v
    apply (rule add-mono)
    apply (rule-tac [!] mult-right-mono)
    using as(1,2) obt1(1,2) obt2(1,2)
    apply auto
    done
  also have ... ≤ 1
    unfolding distrib-left[symmetric] and as(3) using u1 u2 by auto
  finally show u *R x + v *R y ∈ ?hull
    unfolding mem-Collect-eq
    apply (rule-tac x=u * u1 + v * u2 in exI)
    apply (rule conjI)
    defer
    apply (rule-tac x=1 - u * u1 - v * u2 in exI)
    unfolding Bex-def
    using as(1,2) obt1(1,2) obt2(1,2) **
    apply (auto simp add: algebra-simps)
    done
 qed
qed
```

19.5.3 Explicit expression for convex hull

```
lemma convex-hull-indexed:
 fixes s :: 'a::real-vector set
 shows convex hull s =
   {y. ∃k u x.
     (∀i∈{1::nat .. k}. 0 ≤ u i ∧ x i ∈ s) ∧
     (setsum u {1..k} = 1) ∧ (setsum (λi. u i *R x i) {1..k} = y)}
 (is ?xyz = ?hull)
 apply (rule hull-unique)
 apply rule
 defer
 apply (rule convexI)
proof -
 fix x
 assume x∈s
```

then show *x ∈ ?hull*
unfolding *mem-Collect-eq*
apply (*rule-tac x=1* **in** *exI*, *rule-tac x=λx. 1* **in** *exI*)
apply *auto*
done
next
fix *t*
assume *as*: *s ⊆ t convex t*
show *?hull ⊆ t*
apply *rule*
unfolding *mem-Collect-eq*
apply (*elim exE conjE*)
proof −
fix *x k u y*
assume *assm*:
∀ i∈{1::nat..k}. 0 ≤ u i ∧ y i ∈ s
setsum u {1..k} = 1 (∑ i = 1..k. u i ∗R y i) = x
show *x∈t*
unfolding *assm(3)* [*symmetric*]
apply (*rule as(2)*[*unfolded convex*, *rule-format*])
using *assm(1,2) as(1)* **apply** *auto*
done
qed
next
fix *x y u v*
assume *uv*: *0 ≤ u 0 ≤ v u + v = (1::real)*
assume *xy*: *x ∈ ?hull y ∈ ?hull*
from *xy* **obtain** *k1 u1 x1* **where**
x: *∀ i∈{1::nat..k1}. 0≤u1 i ∧ x1 i ∈ s setsum u1 {Suc 0..k1} = 1 (∑ i = Suc 0..k1. u1 i ∗R x1 i) = x*
by *auto*
from *xy* **obtain** *k2 u2 x2* **where**
y: *∀ i∈{1::nat..k2}. 0≤u2 i ∧ x2 i ∈ s setsum u2 {Suc 0..k2} = 1 (∑ i = Suc 0..k2. u2 i ∗R x2 i) = y*
by *auto*
have ∗: *⋀P (x1::'a) x2 s1 s2 i.*
(if P i then s1 else s2) ∗R (if P i then x1 else x2) = (if P i then s1 ∗R x1 else s2 ∗R x2)
{1..k1 + k2} ∩ {1..k1} = {1..k1} {1..k1 + k2} ∩ − {1..k1} = (λi. i + k1) ‘ {1..k2}
prefer *3*
apply (*rule*, *rule*)
unfolding *image-iff*
apply (*rule-tac x = x − k1* **in** *bexI*)
apply (*auto simp add*: *not-le*)
done
have *inj*: *inj-on (λi. i + k1) {1..k2}*
unfolding *inj-on-def* **by** *auto*
show *u ∗R x + v ∗R y ∈ ?hull*

```
    apply rule
    apply (rule-tac x=k1 + k2 in exI)
    apply (rule-tac x=λi. if i ∈ {1..k1} then u * u1 i else v * u2 (i − k1) in exI)
    apply (rule-tac x=λi. if i ∈ {1..k1} then x1 i else x2 (i − k1) in exI)
    apply (rule, rule)
    defer
    apply rule
    unfolding * and setsum.If-cases[OF finite-atLeastAtMost[of 1 k1 + k2]] and
      setsum.reindex[OF inj] and o-def Collect-mem-eq
   unfolding scaleR-scaleR[symmetric] scaleR-right.setsum [symmetric] setsum-right-distrib[symmetric]
  proof −
    fix i
    assume i: i ∈ {1..k1+k2}
    show 0 ≤ (if i ∈ {1..k1} then u * u1 i else v * u2 (i − k1)) ∧
      (if i ∈ {1..k1} then x1 i else x2 (i − k1)) ∈ s
    proof (cases i∈{1..k1})
      case True
      then show ?thesis
        using uv(1) x(1)[THEN bspec[where x=i]] by auto
    next
      case False
      def j ≡ i − k1
      from i False have j ∈ {1..k2}
        unfolding j-def by auto
      then show ?thesis
        using False uv(2) y(1)[THEN bspec[where x=j]]
        by (auto simp: j-def[symmetric])
    qed
  qed (auto simp add: not-le x(2,3) y(2,3) uv(3))
qed

lemma convex-hull-finite:
  fixes s :: 'a::real-vector set
  assumes finite s
  shows convex hull s = {y. ∃ u. (∀ x∈s. 0 ≤ u x) ∧
    setsum u s = 1 ∧ setsum (λx. u x *R x) s = y}
  (is ?HULL = ?set)
proof (rule hull-unique, auto simp add: convex-def[of ?set])
  fix x
  assume x ∈ s
  then show ∃ u. (∀ x∈s. 0 ≤ u x) ∧ setsum u s = 1 ∧ (∑ x∈s. u x *R x) = x
    apply (rule-tac x=λy. if x=y then 1 else 0 in exI)
    apply auto
    unfolding setsum.delta'[OF assms] and setsum-delta''[OF assms]
    apply auto
    done
next
  fix u v :: real
  assume uv: 0 ≤ u 0 ≤ v u + v = 1
```

fix *ux* **assume** *ux*: $\forall x{\in}s.\ 0 \leq ux\ x$ *setsum* $ux\ s = (1{::}real)$
fix *uy* **assume** *uy*: $\forall x{\in}s.\ 0 \leq uy\ x$ *setsum* $uy\ s = (1{::}real)$
{
fix *x*
assume $x{\in}s$
then have $0 \leq u * ux\ x + v * uy\ x$
using *ux*(*1*)[*THEN bspec*[**where** *x*=*x*]] *uy*(*1*)[*THEN bspec*[**where** *x*=*x*]] **and** *uv*(*1*,*2*)
by *auto*
}
moreover
have $(\sum x{\in}s.\ u * ux\ x + v * uy\ x) = 1$
unfolding *setsum.distrib* **and** *setsum-right-distrib*[*symmetric*] **and** *ux*(*2*) *uy*(*2*)
using *uv*(*3*) **by** *auto*
moreover
have $(\sum x{\in}s.\ (u * ux\ x + v * uy\ x) *_R x) = u *_R (\sum x{\in}s.\ ux\ x *_R x) + v *_R (\sum x{\in}s.\ uy\ x *_R x)$
unfolding *scaleR-left-distrib* **and** *setsum.distrib* **and** *scaleR-scaleR*[*symmetric*] **and** *scaleR-right.setsum* [*symmetric*]
by *auto*
ultimately
show $\exists uc.\ (\forall x{\in}s.\ 0 \leq uc\ x) \wedge$ *setsum* $uc\ s = 1 \wedge (\sum x{\in}s.\ uc\ x *_R x) = u *_R (\sum x{\in}s.\ ux\ x *_R x) + v *_R (\sum x{\in}s.\ uy\ x *_R x)$
apply (*rule-tac* $x{=}\lambda x.\ u * ux\ x + v * uy\ x$ **in** *exI*)
apply *auto*
done
next
fix *t*
assume *t*: $s \subseteq t$ *convex* *t*
fix *u*
assume *u*: $\forall x{\in}s.\ 0 \leq u\ x$ *setsum* $u\ s = (1{::}real)$
then show $(\sum x{\in}s.\ u\ x *_R x) \in t$
using *t*(*2*)[*unfolded convex-explicit*, *THEN spec*[**where** *x*=*s*], *THEN spec*[**where** *x*=*u*]]
using *assms* **and** *t*(*1*) **by** *auto*
qed

19.5.4 Another formulation from Lars Schewe

lemma *convex-hull-explicit*:
fixes *p* :: *'a*::*real-vector set*
shows *convex hull p* =
$\{y.\ \exists s\ u.$ *finite* $s \wedge s \subseteq p \wedge (\forall x{\in}s.\ 0 \leq u\ x) \wedge$ *setsum* $u\ s = 1 \wedge$ *setsum* $(\lambda v.\ u\ v *_R v)\ s = y\}$
(**is** *?lhs* = *?rhs*)
proof −
{
fix *x*
assume $x{\in}?lhs$

```
  then obtain k u y where
      obt: ∀i∈{1::nat..k}. 0 ≤ u i ∧ y i ∈ p setsum u {1..k} = 1 (∑ i = 1..k.
u i *R y i) = x
    unfolding convex-hull-indexed by auto

  have fin: finite {1..k} by auto
  have fin': ⋀v. finite {i ∈ {1..k}. y i = v} by auto
  {
    fix j
    assume j∈{1..k}
    then have y j ∈ p 0 ≤ setsum u {i. Suc 0 ≤ i ∧ i ≤ k ∧ y i = y j}
      using obt(1)[THEN bspec[where x=j]] and obt(2)
      apply simp
      apply (rule setsum-nonneg)
      using obt(1)
      apply auto
      done
  }
  moreover
  have (∑ v∈y ' {1..k}. setsum u {i ∈ {1..k}. y i = v}) = 1
    unfolding setsum-image-gen[OF fin, symmetric] using obt(2) by auto
  moreover have (∑ v∈y ' {1..k}. setsum u {i ∈ {1..k}. y i = v} *R v) = x
    using setsum-image-gen[OF fin, of λi. u i *R y i y, symmetric]
    unfolding scaleR-left.setsum using obt(3) by auto
  ultimately
  have ∃ s u. finite s ∧ s ⊆ p ∧ (∀ x∈s. 0 ≤ u x) ∧ setsum u s = 1 ∧ (∑ v∈s.
u v *R v) = x
    apply (rule-tac x=y ' {1..k} in exI)
    apply (rule-tac x=λv. setsum u {i∈{1..k}. y i = v} in exI)
    apply auto
    done
  then have x∈?rhs by auto
 }
 moreover
 {
  fix y
  assume y∈?rhs
  then obtain s u where
    obt: finite s s ⊆ p ∀ x∈s. 0 ≤ u x setsum u s = 1 (∑ v∈s. u v *R v) = y
    by auto

  obtain f where f: inj-on f {1..card s} f ' {1..card s} = s
    using ex-bij-betw-nat-finite-1[OF obt(1)] unfolding bij-betw-def by auto

  {
    fix i :: nat
    assume i∈{1..card s}
    then have f i ∈ s
      apply (subst f(2)[symmetric])
```

```
      apply auto
      done
    then have 0 ≤ u (f i) f i ∈ p using obt(2,3) by auto
  }
  moreover have ∗: finite {1..card s} by auto
  {
    fix y
    assume y∈s
    then obtain i where i∈{1..card s} f i = y
      using f using image-iff[of y f {1..card s}]
      by auto
    then have {x. Suc 0 ≤ x ∧ x ≤ card s ∧ f x = y} = {i}
      apply auto
      using f(1)[unfolded inj-on-def]
      apply(erule-tac x=x in ballE)
      apply auto
      done
    then have card {x. Suc 0 ≤ x ∧ x ≤ card s ∧ f x = y} = 1 by auto
    then have (∑ x∈{x ∈ {1..card s}. f x = y}. u (f x)) = u y
        (∑ x∈{x ∈ {1..card s}. f x = y}. u (f x) ∗R f x) = u y ∗R y
      by (auto simp add: setsum-constant-scaleR)
  }
  then have (∑ x = 1..card s. u (f x)) = 1 (∑ i = 1..card s. u (f i) ∗R f i) = y
    unfolding setsum-image-gen[OF ∗(1), of λx. u (f x) ∗R f x f]
      and setsum-image-gen[OF ∗(1), of λx. u (f x) f]
    unfolding f
    using setsum.cong [of s s λy. (∑ x∈{x ∈ {1..card s}. f x = y}. u (f x) ∗R f
x) λv. u v ∗R v]
    using setsum.cong [of s s λy. (∑ x∈{x ∈ {1..card s}. f x = y}. u (f x)) u]
    unfolding obt(4,5)
    by auto
  ultimately
  have ∃ k u x. (∀ i∈{1..k}. 0 ≤ u i ∧ x i ∈ p) ∧ setsum u {1..k} = 1 ∧
      (∑ i::nat = 1..k. u i ∗R x i) = y
    apply (rule-tac x=card s in exI)
    apply (rule-tac x=u ∘ f in exI)
    apply (rule-tac x=f in exI)
    apply fastforce
    done
  then have y ∈ ?lhs
    unfolding convex-hull-indexed by auto
 }
 ultimately show ?thesis
  unfolding set-eq-iff by blast
qed
```

19.5.5 A stepping theorem for that expansion

lemma *convex-hull-finite-step*:

fixes s :: $'a$::*real-vector set*
assumes *finite* s
shows
$(\exists u.\ (\forall x{\in}\mathit{insert}\ a\ s.\ 0 \le u\ x) \wedge \mathit{setsum}\ u\ (\mathit{insert}\ a\ s) = w \wedge \mathit{setsum}\ (\lambda x.\ u\ x *_R x)\ (\mathit{insert}\ a\ s) = y)$
$\longleftrightarrow (\exists v{\ge}0.\ \exists u.\ (\forall x{\in}s.\ 0 \le u\ x) \wedge \mathit{setsum}\ u\ s = w - v \wedge \mathit{setsum}\ (\lambda x.\ u\ x *_R x)\ s = y - v *_R a)$
(**is** *?lhs = ?rhs*)
proof (*rule*, *case-tac*[!] $a{\in}s$)
assume $a \in s$
then have $*$: *insert a s = s* **by** *auto*
assume *?lhs*
then show *?rhs*
unfolding $*$
apply (*rule-tac x=0* **in** *exI*)
apply *auto*
done
next
assume *?lhs*
then obtain u **where**
u: $\forall x{\in}\mathit{insert}\ a\ s.\ 0 \le u\ x\ \mathit{setsum}\ u\ (\mathit{insert}\ a\ s) = w\ (\sum x{\in}\mathit{insert}\ a\ s.\ u\ x *_R x) = y$
by *auto*
assume $a \notin s$
then show *?rhs*
apply (*rule-tac x=u a* **in** *exI*)
using *u*(*1*)[*THEN bspec*[**where** *x=a*]]
apply *simp*
apply (*rule-tac x=u* **in** *exI*)
using *u*[*unfolded setsum-clauses*(*2*)[*OF assms*]] **and** ‹$a{\notin}s$›
apply *auto*
done
next
assume $a \in s$
then have $*$: *insert a s = s* **by** *auto*
have *fin*: *finite* (*insert a s*) **using** *assms* **by** *auto*
assume *?rhs*
then obtain v u **where** *uv*: $v{\ge}0$ $\forall x{\in}s.\ 0 \le u\ x\ \mathit{setsum}\ u\ s = w - v\ (\sum x{\in}s.\ u\ x *_R x) = y - v *_R a$
by *auto*
show *?lhs*
apply (*rule-tac x* = $\lambda x.$ (*if a = x then v else 0*) + *u x* **in** *exI*)
unfolding *scaleR-left-distrib* **and** *setsum.distrib* **and** *setsum-delta''*[*OF fin*] **and** *setsum.delta'*[*OF fin*]
unfolding *setsum-clauses*(*2*)[*OF assms*]
using *uv* **and** *uv*(*2*)[*THEN bspec*[**where** *x=a*]] **and** ‹$a{\in}s$›
apply *auto*
done
next

assume *?rhs*
then obtain *v u* **where**
uv: $v \geq 0$ $\forall x \in s.\ 0 \leq u\ x$ *setsum u s* $= w - v$ $(\sum x \in s.\ u\ x *_R x) = y - v *_R a$
by *auto*
moreover
assume $a \notin s$
moreover
have $(\sum x \in s.$ *if* $a = x$ *then* v *else* $u\ x) =$ *setsum u s*
and $(\sum x \in s.$ $(\textit{if}\ a = x\ \textit{then}\ v\ \textit{else}\ u\ x) *_R x) = (\sum x \in s.\ u\ x *_R x)$
apply (*rule-tac setsum.cong*) **apply** *rule*
defer
apply (*rule-tac setsum.cong*) **apply** *rule*
using ‹$a \notin s$›
apply *auto*
done
ultimately show *?lhs*
apply (*rule-tac* $x = \lambda x.$ *if* $a = x$ *then* v *else* $u\ x$ **in** *exI*)
unfolding *setsum-clauses*(*2*)[*OF assms*]
apply *auto*
done
qed

19.5.6 Hence some special cases

lemma *convex-hull-2*:
convex hull $\{a,b\} = \{u *_R a + v *_R b \mid u\ v.\ 0 \leq u \wedge 0 \leq v \wedge u + v = 1\}$
proof −
have $*$: $\bigwedge u.$ $(\forall x \in \{a, b\}.\ 0 \leq u\ x) \longleftrightarrow 0 \leq u\ a \wedge 0 \leq u\ b$
by *auto*
have $**$: *finite* $\{b\}$ **by** *auto*
show *?thesis*
apply (*simp add*: *convex-hull-finite*)
unfolding *convex-hull-finite-step*[*OF* $**$, *of a 1*, *unfolded* $*$ *conj-assoc*]
apply *auto*
apply (*rule-tac* $x = v$ **in** *exI*)
apply (*rule-tac* $x = 1 - v$ **in** *exI*)
apply *simp*
apply (*rule-tac* $x = u$ **in** *exI*)
apply *simp*
apply (*rule-tac* $x = \lambda x.\ v$ **in** *exI*)
apply *simp*
done
qed

lemma *convex-hull-2-alt*: *convex hull* $\{a,b\} = \{a + u *_R (b - a) \mid u.\ \ 0 \leq u \wedge u \leq 1\}$
unfolding *convex-hull-2*
proof (*rule Collect-cong*)
have $*$: $\bigwedge x\ y$ ::*real*. $x + y = 1 \longleftrightarrow x = 1 - y$

```
    by auto
  fix x
  show (∃ v u. x = v *R a + u *R b ∧ 0 ≤ v ∧ 0 ≤ u ∧ v + u = 1) ⟷
    (∃ u. x = a + u *R (b − a) ∧ 0 ≤ u ∧ u ≤ 1)
    unfolding *
    apply auto
    apply (rule-tac[!] x=u in exI)
    apply (auto simp add: algebra-simps)
    done
qed

lemma convex-hull-3:
  convex hull {a,b,c} = { u *R a + v *R b + w *R c | u v w. 0 ≤ u ∧ 0 ≤ v ∧
0 ≤ w ∧ u + v + w = 1}
proof −
  have fin: finite {a,b,c} finite {b,c} finite {c}
    by auto
  have *: ⋀x y z ::real. x + y + z = 1 ⟷ x = 1 − y − z
    by (auto simp add: field-simps)
  show ?thesis
    unfolding convex-hull-finite[OF fin(1)] and convex-hull-finite-step[OF fin(2)]
and *
    unfolding convex-hull-finite-step[OF fin(3)]
    apply (rule Collect-cong)
    apply simp
    apply auto
    apply (rule-tac x=va in exI)
    apply (rule-tac x=u c in exI)
    apply simp
    apply (rule-tac x=1 − v − w in exI)
    apply simp
    apply (rule-tac x=v in exI)
    apply simp
    apply (rule-tac x=λx. w in exI)
    apply simp
    done
qed

lemma convex-hull-3-alt:
  convex hull {a,b,c} = {a + u *R (b − a) + v *R (c − a) | u v.  0 ≤ u ∧ 0 ≤
v ∧ u + v ≤ 1}
proof −
  have *: ⋀x y z ::real. x + y + z = 1 ⟷ x = 1 − y − z
    by auto
  show ?thesis
    unfolding convex-hull-3
    apply (auto simp add: *)
    apply (rule-tac x=v in exI)
    apply (rule-tac x=w in exI)
```

apply (*simp add*: *algebra-simps*)
apply (*rule-tac x*=*u* **in** *exI*)
apply (*rule-tac x*=*v* **in** *exI*)
apply (*simp add*: *algebra-simps*)
done
qed

19.6 Relations among closure notions and corresponding hulls

lemma *affine-imp-convex*: *affine s* $\Longrightarrow$ *convex s*
unfolding *affine-def convex-def* **by** *auto*

lemma *subspace-imp-convex*: *subspace s* $\Longrightarrow$ *convex s*
using *subspace-imp-affine affine-imp-convex* **by** *auto*

lemma *affine-hull-subset-span*: (*affine hull s*) $\subseteq$ (*span s*)
by (*metis hull-minimal span-inc subspace-imp-affine subspace-span*)

lemma *convex-hull-subset-span*: (*convex hull s*) $\subseteq$ (*span s*)
by (*metis hull-minimal span-inc subspace-imp-convex subspace-span*)

lemma *convex-hull-subset-affine-hull*: (*convex hull s*) $\subseteq$ (*affine hull s*)
by (*metis affine-affine-hull affine-imp-convex hull-minimal hull-subset*)

lemma *affine-dependent-imp-dependent*: *affine-dependent s* $\Longrightarrow$ *dependent s*
unfolding *affine-dependent-def dependent-def*
using *affine-hull-subset-span* **by** *auto*

lemma *dependent-imp-affine-dependent*:
assumes *dependent* $\{x - a| \ x \ . \ x \in s\}$
and $a \notin s$
shows *affine-dependent* (*insert a s*)
proof −
from *assms*(*1*)[*unfolded dependent-explicit*] **obtain** *S u v*
where *obt*: *finite S* $S \subseteq \{x - a \ |x. \ x \in s\}$ $v \in S$ *u v* $\neq$ *0* $(\sum v \in S. \ u \ v *_R \ v)$
= *0*
by *auto*
def $t \equiv (\lambda x. \ x + a) \ ` \ S$

have *inj*: *inj-on* $(\lambda x. \ x + a)$ *S*
unfolding *inj-on-def* **by** *auto*
have *0* $\notin$ *S*
using *obt*(*2*) *assms*(*2*) **unfolding** *subset-eq* **by** *auto*
have *fin*: *finite t* **and** $t \subseteq s$
unfolding *t-def* **using** *obt*(*1*,*2*) **by** *auto*
then have *finite* (*insert a t*) **and** *insert a t* $\subseteq$ *insert a s*
by *auto*
moreover have $*$: $\bigwedge P \ Q. \ (\sum x \in t. \ (if \ x = a \ then \ P \ x \ else \ Q \ x)) = (\sum x \in t. \ Q$

x)
apply (*rule setsum.cong*)
using ‹*a*∉*s*› ‹*t*⊆*s*›
apply *auto*
done
have (∑ *x*∈*insert a t*. *if x* = *a then* − (∑ *x*∈*t*. *u* (*x* − *a*)) *else u* (*x* − *a*)) = *0*
unfolding *setsum-clauses*(*2*)[*OF fin*]
using ‹*a*∉*s*› ‹*t*⊆*s*›
apply *auto*
unfolding ∗
apply *auto*
done
moreover have ∃ *v*∈*insert a t*. (*if v* = *a then* − (∑ *x*∈*t*. *u* (*x* − *a*)) *else u* (*v* − *a*)) ≠ *0*
apply (*rule-tac x*=*v* + *a* **in** *bexI*)
using *obt*(*3*,*4*) **and** ‹*0*∉*S*›
unfolding *t-def*
apply *auto*
done
moreover have ∗: ⋀*P Q*. (∑ *x*∈*t*. (*if x* = *a then P x else Q x*) ∗*R x*) = (∑ *x*∈*t*. *Q x* ∗*R x*)
apply (*rule setsum.cong*)
using ‹*a*∉*s*› ‹*t*⊆*s*›
apply *auto*
done
have (∑ *x*∈*t*. *u* (*x* − *a*)) ∗*R a* = (∑ *v*∈*t*. *u* (*v* − *a*) ∗*R v*)
unfolding *scaleR-left.setsum*
unfolding *t-def* **and** *setsum.reindex*[*OF inj*] **and** *o-def*
using *obt*(*5*)
by (*auto simp add*: *setsum.distrib scaleR-right-distrib*)
then have (∑ *v*∈*insert a t*. (*if v* = *a then* − (∑ *x*∈*t*. *u* (*x* − *a*)) *else u* (*v* − *a*)) ∗*R v*) = *0*
unfolding *setsum-clauses*(*2*)[*OF fin*]
using ‹*a*∉*s*› ‹*t*⊆*s*›
by (*auto simp add*: ∗)
ultimately show *?thesis*
unfolding *affine-dependent-explicit*
apply (*rule-tac x*=*insert a t* **in** *exI*)
apply *auto*
done
qed

lemma *convex-cone*:
convex s ∧ *cone s* ⟷ (∀ *x*∈*s*. ∀ *y*∈*s*. (*x* + *y*) ∈ *s*) ∧ (∀ *x*∈*s*. ∀ *c*≥*0*. (*c* ∗*R x*) ∈ *s*)
(**is** *?lhs* = *?rhs*)
proof −
{
fix *x y*

```
    assume x∈s y∈s and ?lhs
    then have 2 *R x ∈s 2 *R y ∈ s
      unfolding cone-def by auto
    then have x + y ∈ s
      using ‹?lhs›[unfolded convex-def, THEN conjunct1]
      apply (erule-tac x=2*R x in ballE)
      apply (erule-tac x=2*R y in ballE)
      apply (erule-tac x=1/2 in allE)
      apply simp
      apply (erule-tac x=1/2 in allE)
      apply auto
      done
  }
  then show ?thesis
    unfolding convex-def cone-def by blast
qed

lemma affine-dependent-biggerset:
  fixes s :: 'a::euclidean-space set
  assumes finite s card s ≥ DIM('a) + 2
  shows affine-dependent s
proof −
  have s ≠ {} using assms by auto
  then obtain a where a∈s by auto
  have ∗: {x − a |x. x ∈ s − {a}} = (λx. x − a) ‘ (s − {a})
    by auto
  have card {x − a |x. x ∈ s − {a}} = card (s − {a})
    unfolding ∗
    apply (rule card-image)
    unfolding inj-on-def
    apply auto
    done
  also have ... > DIM('a) using assms(2)
    unfolding card-Diff-singleton[OF assms(1) ‹a∈s›] by auto
  finally show ?thesis
    apply (subst insert-Diff[OF ‹a∈s›, symmetric])
    apply (rule dependent-imp-affine-dependent)
    apply (rule dependent-biggerset)
    apply auto
    done
qed

lemma affine-dependent-biggerset-general:
  assumes finite (s :: 'a::euclidean-space set)
    and card s ≥ dim s + 2
  shows affine-dependent s
proof −
  from assms(2) have s ≠ {} by auto
  then obtain a where a∈s by auto
```

```
  have ∗: {x − a |x. x ∈ s − {a}} = (λx. x − a) ‘ (s − {a})
    by auto
  have ∗∗: card {x − a |x. x ∈ s − {a}} = card (s − {a})
    unfolding ∗
    apply (rule card-image)
    unfolding inj-on-def
    apply auto
    done
  have dim {x − a |x. x ∈ s − {a}} ≤ dim s
    apply (rule subset-le-dim)
    unfolding subset-eq
    using ‹a∈s›
    apply (auto simp add:span-superset span-sub)
    done
  also have ... < dim s + 1 by auto
  also have ... ≤ card (s − {a})
    using assms
    using card-Diff-singleton[OF assms(1) ‹a∈s›]
    by auto
  finally show ?thesis
    apply (subst insert-Diff[OF ‹a∈s›, symmetric])
    apply (rule dependent-imp-affine-dependent)
    apply (rule dependent-biggerset-general)
    unfolding ∗∗
    apply auto
    done
qed
```

19.7 Some Properties of Affine Dependent Sets

```
lemma affine-independent-empty: ¬ affine-dependent {}
  by (simp add: affine-dependent-def)

lemma affine-independent-sing: ¬ affine-dependent {a}
  by (simp add: affine-dependent-def)

lemma affine-hull-translation: affine hull ((λx. a + x) ‘ S) = (λx. a + x) ‘
(affine hull S)
proof −
  have affine ((λx. a + x) ‘ (affine hull S))
    using affine-translation affine-affine-hull by blast
  moreover have (λx. a + x) ‘ S ⊆ (λx. a + x) ‘ (affine hull S)
    using hull-subset[of S] by auto
  ultimately have h1: affine hull ((λx. a + x) ‘ S) ⊆ (λx. a + x) ‘ (affine hull
S)
    by (metis hull-minimal)
  have affine((λx. −a + x) ‘ (affine hull ((λx. a + x) ‘ S)))
    using affine-translation affine-affine-hull by blast
  moreover have (λx. −a + x) ‘ (λx. a + x) ‘ S ⊆ (λx. −a + x) ‘ (affine hull
```

$((\lambda x.\ a + x)\ `\ S))$
using *hull-subset*[*of* $(\lambda x.\ a + x)\ `\ S$] **by** *auto*
moreover have $S = (\lambda x.\ -a + x)\ `\ (\lambda x.\ a + x)\ `\ S$
using *translation-assoc*[*of* $-a\ a$] **by** *auto*
ultimately have $(\lambda x.\ -a + x)\ `\ (\mathit{affine\ hull}\ ((\lambda x.\ a + x)\ `\ S)) >= (\mathit{affine\ hull}\ S)$
by (*metis hull-minimal*)
then have $\mathit{affine\ hull}\ ((\lambda x.\ a + x)\ `\ S) >= (\lambda x.\ a + x)\ `\ (\mathit{affine\ hull}\ S)$
by *auto*
then show *?thesis* **using** *h1* **by** *auto*
qed

lemma *affine-dependent-translation*:
assumes *affine-dependent S*
shows *affine-dependent* $((\lambda x.\ a + x)\ `\ S)$
proof −
obtain *x* **where** *x*: $x \in S \wedge x \in \mathit{affine\ hull}\ (S - \{x\})$
using *assms affine-dependent-def* **by** *auto*
have $op + a\ `\ (S - \{x\}) = op + a\ `\ S - \{a + x\}$
by *auto*
then have $a + x \in \mathit{affine\ hull}\ ((\lambda x.\ a + x)\ `\ S - \{a + x\})$
using *affine-hull-translation*[*of a S* − $\{x\}$] *x* **by** *auto*
moreover have $a + x \in (\lambda x.\ a + x)\ `\ S$
using *x* **by** *auto*
ultimately show *?thesis*
unfolding *affine-dependent-def* **by** *auto*
qed

lemma *affine-dependent-translation-eq*:
affine-dependent S $\longleftrightarrow$ *affine-dependent* $((\lambda x.\ a + x)\ `\ S)$
proof −
{
assume *affine-dependent* $((\lambda x.\ a + x)\ `\ S)$
then have *affine-dependent S*
using *affine-dependent-translation*[*of* $((\lambda x.\ a + x)\ `\ S)\ -a$] *translation-assoc*[*of* $-a\ a$]
by *auto*
}
then show *?thesis*
using *affine-dependent-translation* **by** *auto*
qed

lemma *affine-hull-0-dependent*:
assumes $0 \in \mathit{affine\ hull}\ S$
shows *dependent S*
proof −
obtain *s u* **where** *s-u*: $\mathit{finite}\ s \wedge s \neq \{\} \wedge s \subseteq S \wedge \mathit{setsum}\ u\ s = 1 \wedge (\sum v \in s.\ u\ v *_R v) = 0$
using *assms affine-hull-explicit*[*of S*] **by** *auto*

```
  then have ∃v∈s. u v ≠ 0
    using setsum-not-0[of u s] by auto
  then have finite s ∧ s ⊆ S ∧ (∃v∈s. u v ≠ 0 ∧ (∑v∈s. u v *R v) = 0)
    using s-u by auto
  then show ?thesis
    unfolding dependent-explicit[of S] by auto
qed

lemma affine-dependent-imp-dependent2:
  assumes affine-dependent (insert 0 S)
  shows dependent S
proof −
  obtain x where x: x ∈ insert 0 S ∧ x ∈ affine hull (insert 0 S − {x})
    using affine-dependent-def[of (insert 0 S)] assms by blast
  then have x ∈ span (insert 0 S − {x})
    using affine-hull-subset-span by auto
  moreover have span (insert 0 S − {x}) = span (S − {x})
    using insert-Diff-if[of 0 S {x}] span-insert-0[of S−{x}] by auto
  ultimately have x ∈ span (S − {x}) by auto
  then have x ≠ 0 ⟹ dependent S
    using x dependent-def by auto
  moreover
  {
    assume x = 0
    then have 0 ∈ affine hull S
      using x hull-mono[of S − {0} S] by auto
    then have dependent S
      using affine-hull-0-dependent by auto
  }
  ultimately show ?thesis by auto
qed

lemma affine-dependent-iff-dependent:
  assumes a ∉ S
  shows affine-dependent (insert a S) ⟷ dependent ((λx. −a + x) ' S)
proof −
  have (op + (− a) ' S) = {x − a| x . x : S} by auto
  then show ?thesis
    using affine-dependent-translation-eq[of (insert a S) −a]
      affine-dependent-imp-dependent2 assms
      dependent-imp-affine-dependent[of a S]
    by (auto simp del: uminus-add-conv-diff)
qed

lemma affine-dependent-iff-dependent2:
  assumes a ∈ S
  shows affine-dependent S ⟷ dependent ((λx. −a + x) ' (S−{a}))
proof −
  have insert a (S − {a}) = S
```

```
    using assms by auto
  then show ?thesis
    using assms affine-dependent-iff-dependent[of a S−{a}] by auto
qed

lemma affine-hull-insert-span-gen:
  affine hull (insert a s) = (λx. a + x) ‘ span ((λx. − a + x) ‘ s)
proof −
  have h1: {x − a |x. x ∈ s} = ((λx. −a+x) ‘ s)
    by auto
  {
    assume a ∉ s
    then have ?thesis
      using affine-hull-insert-span[of a s] h1 by auto
  }
  moreover
  {
    assume a1: a ∈ s
    have ∃x. x ∈ s ∧ −a+x=0
      apply (rule exI[of - a])
      using a1
      apply auto
      done
    then have insert 0 ((λx. −a+x) ‘ (s − {a})) = (λx. −a+x) ‘ s
      by auto
    then have span ((λx. −a+x) ‘ (s − {a}))=span ((λx. −a+x) ‘ s)
    using span-insert-0[of op + (− a) ‘ (s − {a})] by (auto simp del: uminus-add-conv-diff)
    moreover have {x − a |x. x ∈ (s − {a})} = ((λx. −a+x) ‘ (s − {a}))
      by auto
    moreover have insert a (s − {a}) = insert a s
      using assms by auto
    ultimately have ?thesis
      using assms affine-hull-insert-span[of a s−{a}] by auto
  }
  ultimately show ?thesis by auto
qed

lemma affine-hull-span2:
  assumes a ∈ s
  shows affine hull s = (λx. a+x) ‘ span ((λx. −a+x) ‘ (s−{a}))
  using affine-hull-insert-span-gen[of a s − {a}, unfolded insert-Diff[OF assms]]
  by auto

lemma affine-hull-span-gen:
  assumes a ∈ affine hull s
  shows affine hull s = (λx. a+x) ‘ span ((λx. −a+x) ‘ s)
proof −
  have affine hull (insert a s) = affine hull s
    using hull-redundant[of a affine s] assms by auto
```

then show *?thesis*
using *affine-hull-insert-span-gen*[*of a s*] **by** *auto*
qed

lemma *affine-hull-span-0*:
assumes *0* ∈ *affine hull S*
shows *affine hull S* = *span S*
using *affine-hull-span-gen*[*of 0 S*] *assms* **by** *auto*

lemma *extend-to-affine-basis*:
fixes *S V* :: *'n*::*euclidean-space set*
assumes ¬ *affine-dependent S S* ⊆ *V S* ≠ {}
shows ∃ *T*. ¬ *affine-dependent T* ∧ *S* ⊆ *T* ∧ *T* ⊆ *V* ∧ *affine hull T* = *affine hull V*
proof −
obtain *a* **where** *a*: *a* ∈ *S*
using *assms* **by** *auto*
then have *h0*: *independent* ((λ*x*. −*a* + *x*) ' (*S*−{*a*}))
using *affine-dependent-iff-dependent2 assms* **by** *auto*
then obtain *B* **where** *B*:
(λ*x*. −*a*+*x*) ' (*S* − {*a*}) ⊆ *B* ∧ *B* ⊆ (λ*x*. −*a*+*x*) ' *V* ∧ *independent B* ∧ (λ*x*. −*a*+*x*) ' *V* ⊆ *span B*
using *maximal-independent-subset-extend*[*of* (λ*x*. −*a*+*x*) ' (*S*−{*a*}) (λ*x*. −*a* + *x*) ' *V*] *assms*
by *blast*
def *T* ≡ (λ*x*. *a*+*x*) ' *insert 0 B*
then have *T* = *insert a* ((λ*x*. *a*+*x*) ' *B*)
by *auto*
then have *affine hull T* = (λ*x*. *a*+*x*) ' *span B*
using *affine-hull-insert-span-gen*[*of a* ((λ*x*. *a*+*x*) ' *B*)] *translation-assoc*[*of* −*a a B*]
by *auto*
then have *V* ⊆ *affine hull T*
using *B assms translation-inverse-subset*[*of a V span B*]
by *auto*
moreover have *T* ⊆ *V*
using *T-def B a assms* **by** *auto*
ultimately have *affine hull T* = *affine hull V*
by (*metis Int-absorb1 Int-absorb2 hull-hull hull-mono*)
moreover have *S* ⊆ *T*
using *T-def B translation-inverse-subset*[*of a S*−{*a*} *B*]
by *auto*
moreover have ¬ *affine-dependent T*
using *T-def affine-dependent-translation-eq*[*of insert 0 B*]
affine-dependent-imp-dependent2 B
by *auto*
ultimately show *?thesis* **using** ‹*T* ⊆ *V*› **by** *auto*
qed

lemma *affine-basis-exists*:
fixes $V :: 'n::euclidean\text{-}space\ set$
shows $\exists B.\ B \subseteq V \wedge \neg\ affine\text{-}dependent\ B \wedge affine\ hull\ V = affine\ hull\ B$
proof (*cases* $V = \{\}$)
case *True*
then show *?thesis*
using *affine-independent-empty* **by** *auto*
next
case *False*
then obtain x **where** $x \in V$ **by** *auto*
then show *?thesis*
using *affine-dependent-def*[*of* $\{x\}$] *extend-to-affine-basis*[*of* $\{x\}$ V]
by *auto*
qed

19.8 Affine Dimension of a Set

definition *aff-dim* :: ($'a::euclidean\text{-}space$) *set* ⇒ *int*
where *aff-dim* $V =$
(*SOME* d :: *int*.
$\exists B.\ affine\ hull\ B = affine\ hull\ V \wedge \neg\ affine\text{-}dependent\ B \wedge of\text{-}nat\ (card\ B) = d + 1$)

lemma *aff-dim-basis-exists*:
fixes V :: ($'n::euclidean\text{-}space$) *set*
shows $\exists B.\ affine\ hull\ B = affine\ hull\ V \wedge \neg\ affine\text{-}dependent\ B \wedge of\text{-}nat\ (card\ B) = aff\text{-}dim\ V + 1$
proof −
obtain B **where** $\neg\ affine\text{-}dependent\ B \wedge affine\ hull\ B = affine\ hull\ V$
using *affine-basis-exists*[*of* V] **by** *auto*
then show *?thesis*
unfolding *aff-dim-def*
some-eq-ex[*of* $\lambda d.\ \exists B.\ affine\ hull\ B = affine\ hull\ V \wedge \neg\ affine\text{-}dependent\ B \wedge of\text{-}nat\ (card\ B) = d + 1$]
apply *auto*
apply (*rule exI*[*of* - *int* (*card* B) − (*1* :: *int*)])
apply (*rule exI*[*of* - B])
apply *auto*
done
qed

lemma *affine-hull-nonempty*: $S \neq \{\} \longleftrightarrow affine\ hull\ S \neq \{\}$
proof −
have $S = \{\} \Longrightarrow affine\ hull\ S = \{\}$
using *affine-hull-empty* **by** *auto*
moreover have $affine\ hull\ S = \{\} \Longrightarrow S = \{\}$
unfolding *hull-def* **by** *auto*
ultimately show *?thesis* **by** *blast*

qed

lemma *aff-dim-parallel-subspace-aux*:
fixes $B :: 'n::euclidean\text{-}space\ set$
assumes $\neg$ *affine-dependent* B $a \in B$
shows *finite* $B \wedge ((card\ B) - 1 = dim\ (span\ ((\lambda x.\ -a+x)\ `\ (B-\{a\}))))$
proof −
have *independent* $((\lambda x.\ -a + x)\ `\ (B-\{a\}))$
using *affine-dependent-iff-dependent2 assms* **by** *auto*
then have *fin*: *dim* $(span\ ((\lambda x.\ -a+x)\ `\ (B-\{a\}))) = card\ ((\lambda x.\ -a + x)\ `\ (B-\{a\}))$
finite $((\lambda x.\ -a + x)\ `\ (B - \{a\}))$
using *indep-card-eq-dim-span*[*of* $(\lambda x.\ -a+x)\ `\ (B-\{a\})$] **by** *auto*
show *?thesis*
proof (*cases* $(\lambda x.\ -a + x)\ `\ (B - \{a\}) = \{\}$)
case *True*
have $B = insert\ a\ ((\lambda x.\ a + x)\ `\ (\lambda x.\ -a + x)\ `\ (B - \{a\}))$
using *translation-assoc*[*of* $a\ -a\ (B - \{a\})$] *assms* **by** *auto*
then have $B = \{a\}$ **using** *True* **by** *auto*
then show *?thesis* **using** *assms fin* **by** *auto*
next
case *False*
then have $card\ ((\lambda x.\ -a + x)\ `\ (B - \{a\})) > 0$
using *fin* **by** *auto*
moreover have *h1*: $card\ ((\lambda x.\ -a + x)\ `\ (B-\{a\})) = card\ (B-\{a\})$
apply (*rule card-image*)
using *translate-inj-on*
apply (*auto simp del*: *uminus-add-conv-diff*)
done
ultimately have $card\ (B-\{a\}) > 0$ **by** *auto*
then have $*$: *finite* $(B - \{a\})$
using *card-gt-0-iff*[*of* $(B - \{a\})$] **by** *auto*
then have $card\ (B - \{a\}) = card\ B - 1$
using *card-Diff-singleton assms* **by** *auto*
with $*$ **show** *?thesis* **using** *fin h1* **by** *auto*
qed
qed

lemma *aff-dim-parallel-subspace*:
fixes $V\ L :: 'n::euclidean\text{-}space\ set$
assumes $V \neq \{\}$
and *subspace* L
and *affine-parallel* $(affine\ hull\ V)\ L$
shows *aff-dim* $V = int\ (dim\ L)$
proof −
obtain B **where**
B: *affine hull* B = *affine hull* $V \wedge \neg$ *affine-dependent* $B \wedge int\ (card\ B) =$ *aff-dim* $V + 1$
using *aff-dim-basis-exists* **by** *auto*

```
  then have B ≠ {}
    using assms B affine-hull-nonempty[of V] affine-hull-nonempty[of B]
    by auto
  then obtain a where a: a ∈ B by auto
  def Lb ≡ span ((λx. −a+x) ‘ (B−{a}))
  moreover have affine-parallel (affine hull B) Lb
    using Lb-def B assms affine-hull-span2[of a B] a
      affine-parallel-commut[of Lb (affine hull B)]
    unfolding affine-parallel-def
    by auto
  moreover have subspace Lb
    using Lb-def subspace-span by auto
  moreover have affine hull B ≠ {}
    using assms B affine-hull-nonempty[of V] by auto
  ultimately have L = Lb
    using assms affine-parallel-subspace[of affine hull B] affine-affine-hull[of B] B
    by auto
  then have dim L = dim Lb
    by auto
  moreover have card B − 1 = dim Lb and finite B
    using Lb-def aff-dim-parallel-subspace-aux a B by auto
  ultimately show ?thesis
    using B ‹B ≠ {}› card-gt-0-iff[of B] by auto
qed

lemma aff-independent-finite:
  fixes B :: 'n::euclidean-space set
  assumes ¬ affine-dependent B
  shows finite B
proof −
  {
    assume B ≠ {}
    then obtain a where a ∈ B by auto
    then have ?thesis
      using aff-dim-parallel-subspace-aux assms by auto
  }
  then show ?thesis by auto
qed

lemma independent-finite:
  fixes B :: 'n::euclidean-space set
  assumes independent B
  shows finite B
  using affine-dependent-imp-dependent[of B] aff-independent-finite[of B] assms
  by auto

lemma subspace-dim-equal:
  assumes subspace (S :: ('n::euclidean-space) set)
    and subspace T
```

and $S \subseteq T$
and $dim\ S \geq dim\ T$
shows $S = T$
proof −
obtain B **where** B: $B \leq S$ *independent* $B \wedge S \subseteq span\ B$ *card* $B = dim\ S$
using *basis-exists*[*of* S] **by** *auto*
then have $span\ B \subseteq S$
using *span-mono*[*of* $B\ S$] *span-eq*[*of* S] *assms* **by** *metis*
then have $span\ B = S$
using B **by** *auto*
have $dim\ S = dim\ T$
using *assms dim-subset*[*of* $S\ T$] **by** *auto*
then have $T \subseteq span\ B$
using *card-eq-dim*[*of* $B\ T$] B *independent-finite assms* **by** *auto*
then show *?thesis*
using *assms* ‹*span* $B = S$› **by** *auto*
qed

lemma *span-substd-basis*:
assumes d: $d \subseteq Basis$
shows $span\ d = \{x.\ \forall i{\in}Basis.\ i \notin d \longrightarrow x{\cdot}i = 0\}$
(**is** - = *?B*)
proof −
have $d \subseteq$ *?B*
using d **by** (*auto simp*: *inner-Basis*)
moreover have s: *subspace ?B*
using *subspace-substandard*[*of* $\lambda i.\ i \notin d$] **.**
ultimately have $span\ d \subseteq$ *?B*
using *span-mono*[*of* d *?B*] *span-eq*[*of ?B*] **by** *blast*
moreover have $*$: $card\ d \leq dim\ (span\ d)$
using *independent-card-le-dim*[*of* d *span* d] *independent-substdbasis*[*OF assms*] *span-inc*[*of* d]
by *auto*
moreover from $*$ **have** dim *?B* $\leq dim\ (span\ d)$
using *dim-substandard*[*OF assms*] **by** *auto*
ultimately show *?thesis*
using s *subspace-dim-equal*[*of span* d *?B*] *subspace-span*[*of* d] **by** *auto*
qed

lemma *basis-to-substdbasis-subspace-isomorphism*:
fixes B :: *'a*::*euclidean-space set*
assumes *independent* B
shows $\exists f\ d{::}'a\ set.\ card\ d = card\ B \wedge linear\ f \wedge f\ `\ B = d \wedge$
$f\ `\ span\ B = \{x.\ \forall i{\in}Basis.\ i \notin d \longrightarrow x \cdot i = 0\} \wedge inj\text{-}on\ f\ (span\ B) \wedge d \subseteq Basis$
proof −
have B: *card* $B = dim\ B$
using *dim-unique*[*of* $B\ B$ *card* B] *assms span-inc*[*of* B] **by** *auto*
have $dim\ B \leq card\ (Basis :: 'a\ set)$

using *dim-subset-UNIV*[*of B*] **by** *simp*
from *ex-card*[*OF this*] **obtain** *d* :: *'a set* **where** *d*: *d* ⊆ *Basis* **and** *t*: *card d* = *dim B*
by *auto*
let *?t* = {*x*::*'a*::*euclidean-space*. ∀ *i*∈*Basis*. *i* ∉ *d* ⟶ *x*•*i* = *0*}
have ∃*f*. *linear f* ∧ *f* ' *B* = *d* ∧ *f* ' *span B* = *?t* ∧ *inj-on f* (*span B*)
apply (*rule basis-to-basis-subspace-isomorphism*[*of span B ?t B d*])
apply (*rule subspace-span*)
apply (*rule subspace-substandard*)
defer
apply (*rule span-inc*)
apply (*rule assms*)
defer
unfolding *dim-span*[*of B*]
apply(*rule B*)
unfolding *span-substd-basis*[*OF d*, *symmetric*]
apply (*rule span-inc*)
apply (*rule independent-substdbasis*[*OF d*])
apply *rule*
apply *assumption*
unfolding *t*[*symmetric*] *span-substd-basis*[*OF d*] *dim-substandard*[*OF d*]
apply *auto*
done
with *t* ‹*card B* = *dim B*› *d* **show** *?thesis* **by** *auto*
qed

lemma *aff-dim-empty*:
fixes *S* :: *'n*::*euclidean-space set*
shows *S* = {} ⟷ *aff-dim S* = *−1*
proof −
obtain *B* **where** ∗: *affine hull B* = *affine hull S*
and ¬ *affine-dependent B*
and *int* (*card B*) = *aff-dim S* + *1*
using *aff-dim-basis-exists* **by** *auto*
moreover
from ∗ **have** *S* = {} ⟷ *B* = {}
using *affine-hull-nonempty*[*of B*] *affine-hull-nonempty*[*of S*] **by** *auto*
ultimately show *?thesis*
using *aff-independent-finite*[*of B*] *card-gt-0-iff*[*of B*] **by** *auto*
qed

lemma *aff-dim-empty-eq* [*simp*]: *aff-dim* ({}::*'a*::*euclidean-space set*) = *−1*
by (*simp add*: *aff-dim-empty* [*symmetric*])

lemma *aff-dim-affine-hull*: *aff-dim* (*affine hull S*) = *aff-dim S*
unfolding *aff-dim-def* **using** *hull-hull*[*of - S*] **by** *auto*

lemma *aff-dim-affine-hull2*:
assumes *affine hull S* = *affine hull T*

shows *aff-dim S = aff-dim T*
unfolding *aff-dim-def* **using** *assms* **by** *auto*

lemma *aff-dim-unique*:
fixes *B V* :: *'n*::*euclidean-space set*
assumes *affine hull B = affine hull V* ∧ ¬ *affine-dependent B*
shows *of-nat* (*card B*) = *aff-dim V + 1*
proof (*cases B* = {})
case *True*
then have *V* = {}
using *affine-hull-nonempty*[*of V*] *affine-hull-nonempty*[*of B*] *assms*
by *auto*
then have *aff-dim V* = (−*1*::*int*)
using *aff-dim-empty* **by** *auto*
then show *?thesis*
using ‹*B* = {}› **by** *auto*
next
case *False*
then obtain *a* **where** *a*: *a* ∈ *B* **by** *auto*
def *Lb* ≡ *span* ((λ*x*. −*a*+*x*) ‘ (*B*−{*a*}))
have *affine-parallel* (*affine hull B*) *Lb*
using *Lb-def affine-hull-span2*[*of a B*] *a*
affine-parallel-commut[*of Lb* (*affine hull B*)]
unfolding *affine-parallel-def* **by** *auto*
moreover have *subspace Lb*
using *Lb-def subspace-span* **by** *auto*
ultimately have *aff-dim B* = *int*(*dim Lb*)
using *aff-dim-parallel-subspace*[*of B Lb*] ‹*B* ≠ {}› **by** *auto*
moreover have (*card B*) − *1* = *dim Lb finite B*
using *Lb-def aff-dim-parallel-subspace-aux a assms* **by** *auto*
ultimately have *of-nat* (*card B*) = *aff-dim B + 1*
using ‹*B* ≠ {}› *card-gt-0-iff*[*of B*] **by** *auto*
then show *?thesis*
using *aff-dim-affine-hull2 assms* **by** *auto*
qed

lemma *aff-dim-affine-independent*:
fixes *B* :: *'n*::*euclidean-space set*
assumes ¬ *affine-dependent B*
shows *of-nat* (*card B*) = *aff-dim B + 1*
using *aff-dim-unique*[*of B B*] *assms* **by** *auto*

lemma *affine-independent-iff-card*:
fixes *s* :: *'a*::*euclidean-space set*
shows ~ *affine-dependent s* ⟷ *finite s* ∧ *aff-dim s* = *int*(*card s*) − *1*
apply (*rule iffI*)
apply (*simp add*: *aff-dim-affine-independent aff-independent-finite*)
by (*metis affine-basis-exists* [*of s*] *aff-dim-unique card-subset-eq diff-add-cancel of-nat-eq-iff*)

lemma *aff-dim-sing* [*simp*]:
fixes $a :: 'n::euclidean\text{-}space$
shows *aff-dim* $\{a\} = 0$
using *aff-dim-affine-independent*[*of* $\{a\}$] *affine-independent-sing* **by** *auto*

lemma *aff-dim-inner-basis-exists*:
fixes $V :: ('n::euclidean\text{-}space)\ set$
shows $\exists B.\ B \subseteq V \wedge$ *affine hull* $B =$ *affine hull* $V \wedge$
$\neg$ *affine-dependent* $B \wedge$ *of-nat* (*card* B) = *aff-dim* $V + 1$
proof −
obtain B **where** B: $\neg$ *affine-dependent* B $B \subseteq V$ *affine hull* $B =$ *affine hull* V
using *affine-basis-exists*[*of* V] **by** *auto*
then have *of-nat*(*card* B) = *aff-dim* $V+1$ **using** *aff-dim-unique* **by** *auto*
with B **show** *?thesis* **by** *auto*
qed

lemma *aff-dim-le-card*:
fixes $V :: 'n::euclidean\text{-}space\ set$
assumes *finite* V
shows *aff-dim* $V \leq$ *of-nat* (*card* V) − 1
proof −
obtain B **where** B: $B \subseteq V$ *of-nat* (*card* B) = *aff-dim* $V + 1$
using *aff-dim-inner-basis-exists*[*of* V] **by** *auto*
then have *card* $B \leq$ *card* V
using *assms card-mono* **by** *auto*
with B **show** *?thesis* **by** *auto*
qed

lemma *aff-dim-parallel-eq*:
fixes $S\ T :: 'n::euclidean\text{-}space\ set$
assumes *affine-parallel* (*affine hull* S) (*affine hull* T)
shows *aff-dim* $S =$ *aff-dim* T
proof −
{
assume $T \neq \{\}$ $S \neq \{\}$
then obtain L **where** L: *subspace* $L \wedge$ *affine-parallel* (*affine hull* T) L
using *affine-parallel-subspace*[*of affine hull* T]
affine-affine-hull[*of* T] *affine-hull-nonempty*
by *auto*
then have *aff-dim* $T =$ *int* (*dim* L)
using *aff-dim-parallel-subspace* ‹$T \neq \{\}$› **by** *auto*
moreover have $*$: *subspace* $L \wedge$ *affine-parallel* (*affine hull* S) L
using L *affine-parallel-assoc*[*of affine hull* S *affine hull* T L] *assms* **by** *auto*
moreover from $*$ **have** *aff-dim* $S =$ *int* (*dim* L)
using *aff-dim-parallel-subspace* ‹$S \neq \{\}$› **by** *auto*
ultimately have *?thesis* **by** *auto*
}
moreover

```
  {
    assume S = {}
    then have S = {} and T = {}
      using assms affine-hull-nonempty
      unfolding affine-parallel-def
      by auto
    then have ?thesis using aff-dim-empty by auto
  }
  moreover
  {
    assume T = {}
    then have S = {} and T = {}
      using assms affine-hull-nonempty
      unfolding affine-parallel-def
      by auto
    then have ?thesis
      using aff-dim-empty by auto
  }
  ultimately show ?thesis by blast
qed

lemma aff-dim-translation-eq:
  fixes a :: 'n::euclidean-space
  shows aff-dim ((λx. a + x) ` S) = aff-dim S
proof −
  have affine-parallel (affine hull S) (affine hull ((λx. a + x) ` S))
    unfolding affine-parallel-def
    apply (rule exI[of - a])
    using affine-hull-translation[of a S]
    apply auto
    done
  then show ?thesis
    using aff-dim-parallel-eq[of S (λx. a + x) ` S] by auto
qed

lemma aff-dim-affine:
  fixes S L :: 'n::euclidean-space set
  assumes S ≠ {}
    and affine S
    and subspace L
    and affine-parallel S L
  shows aff-dim S = int (dim L)
proof −
  have ∗: affine hull S = S
    using assms affine-hull-eq[of S] by auto
  then have affine-parallel (affine hull S) L
    using assms by (simp add: ∗)
  then show ?thesis
    using assms aff-dim-parallel-subspace[of S L] by blast
```

qed

lemma *dim-affine-hull*:
fixes $S :: 'n::euclidean\text{-}space\ set$
shows $dim\ (affine\ hull\ S) = dim\ S$
proof −
have $dim\ (affine\ hull\ S) \geq dim\ S$
using *dim-subset* **by** *auto*
moreover have $dim\ (span\ S) \geq dim\ (affine\ hull\ S)$
using *dim-subset affine-hull-subset-span* **by** *blast*
moreover have $dim\ (span\ S) = dim\ S$
using *dim-span* **by** *auto*
ultimately show *?thesis* **by** *auto*
qed

lemma *aff-dim-subspace*:
fixes $S :: 'n::euclidean\text{-}space\ set$
assumes $S \neq \{\}$
and *subspace S*
shows *aff-dim S* = *int* (*dim S*)
using *aff-dim-affine*[*of S S*] *assms subspace-imp-affine*[*of S*] *affine-parallel-reflex*[*of S*]
by *auto*

lemma *aff-dim-zero*:
fixes $S :: 'n::euclidean\text{-}space\ set$
assumes $0 \in affine\ hull\ S$
shows *aff-dim S* = *int* (*dim S*)
proof −
have *subspace* (*affine hull S*)
using *subspace-affine*[*of affine hull S*] *affine-affine-hull assms*
by *auto*
then have *aff-dim* (*affine hull S*) = *int* (*dim* (*affine hull S*))
using *assms aff-dim-subspace*[*of affine hull S*] **by** *auto*
then show *?thesis*
using *aff-dim-affine-hull*[*of S*] *dim-affine-hull*[*of S*]
by *auto*
qed

lemma *aff-dim-univ*: *aff-dim* (*UNIV* :: $'n::euclidean\text{-}space\ set$) = $int(DIM('n))$
using *aff-dim-subspace*[*of* (*UNIV* :: $'n::euclidean\text{-}space\ set$)]
dim-UNIV[**where** $'a='n::euclidean\text{-}space$]
by *auto*

lemma *aff-dim-geq*:
fixes $V :: 'n::euclidean\text{-}space\ set$
shows *aff-dim V* ≥ −*1*
proof −
obtain *B* **where** *affine hull B* = *affine hull V*

and ¬ *affine-dependent B*
and *int* (*card B*) = *aff-dim V* + *1*
using *aff-dim-basis-exists* **by** *auto*
then show *?thesis* **by** *auto*
qed

lemma *independent-card-le-aff-dim*:
fixes *B* :: *'n*::*euclidean-space set*
assumes *B* ⊆ *V*
assumes ¬ *affine-dependent B*
shows *int* (*card B*) ≤ *aff-dim V* + *1*
proof (*cases B* = {})
case *True*
then have −*1* ≤ *aff-dim V*
using *aff-dim-geq* **by** *auto*
with *True* **show** *?thesis* **by** *auto*
next
case *False*
then obtain *T* **where** *T*: ¬ *affine-dependent T* ∧ *B* ⊆ *T* ∧ *T* ⊆ *V* ∧ *affine hull T* = *affine hull V*
using *assms extend-to-affine-basis*[*of B V*] **by** *auto*
then have *of-nat* (*card T*) = *aff-dim V* + *1*
using *aff-dim-unique* **by** *auto*
then show *?thesis*
using *T card-mono*[*of T B*] *aff-independent-finite*[*of T*] **by** *auto*
qed

lemma *aff-dim-subset*:
fixes *S T* :: *'n*::*euclidean-space set*
assumes *S* ⊆ *T*
shows *aff-dim S* ≤ *aff-dim T*
proof −
obtain *B* **where** *B*: ¬ *affine-dependent B B* ⊆ *S affine hull B* = *affine hull S*
of-nat (*card B*) = *aff-dim S* + *1*
using *aff-dim-inner-basis-exists*[*of S*] **by** *auto*
then have *int* (*card B*) ≤ *aff-dim T* + *1*
using *assms independent-card-le-aff-dim*[*of B T*] **by** *auto*
with *B* **show** *?thesis* **by** *auto*
qed

lemma *aff-dim-subset-univ*:
fixes *S* :: *'n*::*euclidean-space set*
shows *aff-dim S* ≤ *int* (*DIM*(*'n*))
proof −
have *aff-dim* (*UNIV* :: *'n*::*euclidean-space set*) = *int*(*DIM*(*'n*))
using *aff-dim-univ* **by** *auto*
then show *aff-dim* (*S*:: *'n*::*euclidean-space set*) ≤ *int*(*DIM*(*'n*))
using *assms aff-dim-subset*[*of S* (*UNIV* :: (*'n*::*euclidean-space*) *set*)] *subset-UNIV*
by *auto*

qed

lemma *affine-dim-equal*:
fixes S :: *'n::euclidean-space set*
assumes *affine* S *affine* T $S \neq \{\}$ $S \subseteq T$ *aff-dim* S = *aff-dim* T
shows $S = T$
proof −
obtain a **where** $a \in S$ **using** *assms* **by** *auto*
then have $a \in T$ **using** *assms* **by** *auto*
def $LS \equiv \{y.\ \exists x \in S.\ (-a) + x = y\}$
then have *ls*: *subspace LS affine-parallel S LS*
using *assms parallel-subspace-explicit*[*of S a LS*] ‹$a \in S$› **by** *auto*
then have *h1*: *int*(*dim LS*) = *aff-dim S*
using *assms aff-dim-affine*[*of S LS*] **by** *auto*
have $T \neq \{\}$ **using** *assms* **by** *auto*
def $LT \equiv \{y.\ \exists x \in T.\ (-a) + x = y\}$
then have *lt*: *subspace LT* ∧ *affine-parallel T LT*
using *assms parallel-subspace-explicit*[*of T a LT*] ‹$a \in T$› **by** *auto*
then have *int*(*dim LT*) = *aff-dim T*
using *assms aff-dim-affine*[*of T LT*] ‹$T \neq \{\}$› **by** *auto*
then have *dim LS* = *dim LT*
using *h1 assms* **by** *auto*
moreover have $LS \leq LT$
using *LS-def LT-def assms* **by** *auto*
ultimately have $LS = LT$
using *subspace-dim-equal*[*of LS LT*] *ls lt* **by** *auto*
moreover have $S = \{x.\ \exists y \in LS.\ a{+}y{=}x\}$
using *LS-def* **by** *auto*
moreover have $T = \{x.\ \exists y \in LT.\ a{+}y{=}x\}$
using *LT-def* **by** *auto*
ultimately show *?thesis* **by** *auto*
qed

lemma *affine-hull-univ*:
fixes S :: *'n::euclidean-space set*
assumes *aff-dim* S = *int*(*DIM*('*n*))
shows *affine hull* S = (*UNIV* :: ('*n*::*euclidean-space*) *set*)
proof −
have $S \neq \{\}$
using *assms aff-dim-empty*[*of S*] **by** *auto*
have *h0*: $S \subseteq$ *affine hull S*
using *hull-subset*[*of S* -] **by** *auto*
have *h1*: *aff-dim* (*UNIV* :: ('*n*::*euclidean-space*) *set*) = *aff-dim S*
using *aff-dim-univ assms* **by** *auto*
then have *h2*: *aff-dim* (*affine hull S*) ≤ *aff-dim* (*UNIV* :: ('*n*::*euclidean-space*) *set*)
using *aff-dim-subset-univ*[*of affine hull S*] *assms h0* **by** *auto*
have *h3*: *aff-dim S* ≤ *aff-dim* (*affine hull S*)
using *h0 aff-dim-subset*[*of S affine hull S*] *assms* **by** *auto*

then have *h4*: *aff-dim* (*affine hull S*) = *aff-dim* (*UNIV* :: (*'n*::*euclidean-space*) *set*)
using *h0 h1 h2* **by** *auto*
then show *?thesis*
using *affine-dim-equal*[*of affine hull S* (*UNIV* :: (*'n*::*euclidean-space*) *set*)]
affine-affine-hull[*of S*] *affine-UNIV assms h4 h0* ‹*S* ≠ {}›
by *auto*
qed

lemma *aff-dim-convex-hull*:
fixes *S* :: *'n*::*euclidean-space set*
shows *aff-dim* (*convex hull S*) = *aff-dim S*
using *aff-dim-affine-hull*[*of S*] *convex-hull-subset-affine-hull*[*of S*]
hull-subset[*of S convex*] *aff-dim-subset*[*of S convex hull S*]
aff-dim-subset[*of convex hull S affine hull S*]
by *auto*

lemma *aff-dim-cball*:
fixes *a* :: *'n*::*euclidean-space*
assumes *e* > *0*
shows *aff-dim* (*cball a e*) = *int* (*DIM*(*'n*))
proof −
have (λ*x*. *a* + *x*) ‘ (*cball 0 e*) ⊆ *cball a e*
unfolding *cball-def dist-norm* **by** *auto*
then have *aff-dim* (*cball* (*0* :: *'n*::*euclidean-space*) *e*) ≤ *aff-dim* (*cball a e*)
using *aff-dim-translation-eq*[*of a cball 0 e*]
aff-dim-subset[*of op* + *a* ‘ *cball 0 e cball a e*]
by *auto*
moreover have *aff-dim* (*cball* (*0* :: *'n*::*euclidean-space*) *e*) = *int* (*DIM*(*'n*))
using *hull-inc*[*of* (*0* :: *'n*::*euclidean-space*) *cball 0 e*]
centre-in-cball[*of* (*0* :: *'n*::*euclidean-space*)] *assms*
by (*simp add*: *dim-cball*[*of e*] *aff-dim-zero*[*of cball 0 e*])
ultimately show *?thesis*
using *aff-dim-subset-univ*[*of cball a e*] **by** *auto*
qed

lemma *aff-dim-open*:
fixes *S* :: *'n*::*euclidean-space set*
assumes *open S*
and *S* ≠ {}
shows *aff-dim S* = *int* (*DIM*(*'n*))
proof −
obtain *x* **where** *x* ∈ *S*
using *assms* **by** *auto*
then obtain *e* **where** *e*: *e* > *0 cball x e* ⊆ *S*
using *open-contains-cball*[*of S*] *assms* **by** *auto*
then have *aff-dim* (*cball x e*) ≤ *aff-dim S*
using *aff-dim-subset* **by** *auto*
with *e* **show** *?thesis*

using *aff-dim-cball*[*of e x*] *aff-dim-subset-univ*[*of S*] **by** *auto*
qed

lemma *low-dim-interior*:
fixes *S* :: *'n*::*euclidean-space set*
assumes ¬ *aff-dim S* = *int* (*DIM*(*'n*))
shows *interior S* = {}
proof −
have *aff-dim*(*interior S*) ≤ *aff-dim S*
using *interior-subset aff-dim-subset*[*of interior S S*] **by** *auto*
then show *?thesis*
using *aff-dim-open*[*of interior S*] *aff-dim-subset-univ*[*of S*] *assms* **by** *auto*
qed

corollary *empty-interior-lowdim*:
fixes *S* :: *'n*::*euclidean-space set*
shows *dim S* < *DIM* (*'n*) ⟹ *interior S* = {}
by (*metis low-dim-interior affine-hull-univ dim-affine-hull less-not-refl dim-UNIV*)

19.9 Caratheodory's theorem.

lemma *convex-hull-caratheodory-aff-dim*:
fixes *p* :: (*'a*::*euclidean-space*) *set*
shows *convex hull p* =
{*y*. ∃ *s u*. *finite s* ∧ *s* ⊆ *p* ∧ *card s* ≤ *aff-dim p* + *1* ∧
(∀ *x*∈*s*. *0* ≤ *u x*) ∧ *setsum u s* = *1* ∧ *setsum* (λ*v*. *u v* ∗*R v*) *s* = *y*}
unfolding *convex-hull-explicit set-eq-iff mem-Collect-eq*
proof (*intro allI iffI*)
fix *y*
let *?P* = λ*n*. ∃ *s u*. *finite s* ∧ *card s* = *n* ∧ *s* ⊆ *p* ∧ (∀ *x*∈*s*. *0* ≤ *u x*) ∧
setsum u s = *1* ∧ (∑ *v*∈*s*. *u v* ∗*R v*) = *y*
assume ∃ *s u*. *finite s* ∧ *s* ⊆ *p* ∧ (∀ *x*∈*s*. *0* ≤ *u x*) ∧ *setsum u s* = *1* ∧ (∑ *v*∈*s*. *u v* ∗*R v*) = *y*
then obtain *N* **where** *?P N* **by** *auto*
then have ∃ *n*≤*N*. (∀ *k*<*n*. ¬ *?P k*) ∧ *?P n*
apply (*rule-tac ex-least-nat-le*)
apply *auto*
done
then obtain *n* **where** *?P n* **and** *smallest*: ∀ *k*<*n*. ¬ *?P k*
by *blast*
then obtain *s u* **where** *obt*: *finite s card s* = *n s*⊆*p* ∀ *x*∈*s*. *0* ≤ *u x*
setsum u s = *1* (∑ *v*∈*s*. *u v* ∗*R v*) = *y* **by** *auto*

have *card s* ≤ *aff-dim p* + *1*
proof (*rule ccontr*, *simp only*: *not-le*)
assume *aff-dim p* + *1* < *card s*
then have *affine-dependent s*
using *affine-dependent-biggerset*[*OF obt*(*1*)] *independent-card-le-aff-dim not-less obt*(*3*)

```
    by blast
  then obtain w v where wv: setsum w s = 0 v∈s w v ≠ 0 (∑ v∈s. w v *R v)
= 0
    using affine-dependent-explicit-finite[OF obt(1)] by auto
  def i ≡ (λv. (u v) / (− w v)) ‘ {v∈s. w v < 0}
  def t ≡ Min i
  have ∃ x∈s. w x < 0
  proof (rule ccontr, simp add: not-less)
    assume as:∀ x∈s. 0 ≤ w x
    then have setsum w (s − {v}) ≥ 0
      apply (rule-tac setsum-nonneg)
      apply auto
      done
    then have setsum w s > 0
      unfolding setsum.remove[OF obt(1) ‹v∈s›]
      using as[THEN bspec[where x=v]] ‹v∈s› ‹w v ≠ 0› by auto
    then show False using wv(1) by auto
  qed
  then have i ≠ {} unfolding i-def by auto
  then have t ≥ 0
    using Min-ge-iff[of i 0 ] and obt(1)
    unfolding t-def i-def
    using obt(4)[unfolded le-less]
    by (auto simp: divide-le-0-iff)
  have t: ∀ v∈s. u v + t * w v ≥ 0
  proof
    fix v
    assume v ∈ s
    then have v: 0 ≤ u v
      using obt(4)[THEN bspec[where x=v]] by auto
    show 0 ≤ u v + t * w v
    proof (cases w v < 0)
      case False
      thus ?thesis using v ‹t≥0› by auto
    next
      case True
      then have t ≤ u v / (− w v)
        using ‹v∈s› unfolding t-def i-def
        apply (rule-tac Min-le)
        using obt(1) apply auto
        done
      then show ?thesis
        unfolding real-0-le-add-iff
        using pos-le-divide-eq[OF True[unfolded neg-0-less-iff-less[symmetric]]]
        by auto
    qed
  qed
  obtain a where a ∈ s and t = (λv. (u v) / (− w v)) a and w a < 0
    using Min-in[OF - ‹i≠{}›] and obt(1) unfolding i-def t-def by auto
```

then have *a*: $a \in s\ u\ a + t * w\ a = 0$ **by** *auto*
have $*$: $\bigwedge f.\ setsum\ f\ (s - \{a\}) = setsum\ f\ s - ((f\ a)::'b::ab\text{-}group\text{-}add)$
unfolding *setsum.remove*[*OF obt*(*1*) ‹$a \in s$›] **by** *auto*
have $(\sum v \in s.\ u\ v + t * w\ v) = 1$
unfolding *setsum.distrib wv*(*1*) *setsum-right-distrib*[*symmetric*] *obt*(*5*) **by** *auto*
moreover have $(\sum v \in s.\ u\ v *_R v + (t * w\ v) *_R v) - (u\ a *_R a + (t * w\ a) *_R a) = y$
unfolding *setsum.distrib obt*(*6*) *scaleR-scaleR*[*symmetric*] *scaleR-right.setsum* [*symmetric*] *wv*(*4*)
using *a*(*2*) [*THEN eq-neg-iff-add-eq-0* [*THEN iffD2*]] **by** *simp*
ultimately have *?P* $(n - 1)$
apply (*rule-tac x*=$(s - \{a\})$ **in** *exI*)
apply (*rule-tac x*=$\lambda v.\ u\ v + t * w\ v$ **in** *exI*)
using *obt*(*1*−*3*) **and** *t* **and** *a*
apply (*auto simp add*: $*$ *scaleR-left-distrib*)
done
then show *False*
using *smallest*[*THEN spec*[**where** *x*=$n - 1$]] **by** *auto*
qed
then show $\exists s\ u.\ finite\ s \wedge s \subseteq p \wedge card\ s \leq aff\text{-}dim\ p + 1 \wedge (\forall x \in s.\ 0 \leq u\ x) \wedge setsum\ u\ s = 1 \wedge (\sum v \in s.\ u\ v *_R v) = y$
using *obt* **by** *auto*
qed *auto*

lemma *caratheodory-aff-dim*:
fixes *p* :: ('*a*::*euclidean-space*) *set*
shows *convex hull p* = $\{x.\ \exists s.\ finite\ s \wedge s \subseteq p \wedge card\ s \leq aff\text{-}dim\ p + 1 \wedge x \in convex\ hull\ s\}$
(**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⊆ *?rhs*
apply (*subst convex-hull-caratheodory-aff-dim*)
apply *clarify*
apply (*rule-tac x*=*s* **in** *exI*)
apply (*simp add*: *hull-subset convex-explicit* [*THEN iffD1*, *OF convex-convex-hull*])
done
next
show *?rhs* ⊆ *?lhs*
using *hull-mono* **by** *blast*
qed

lemma *convex-hull-caratheodory*:
fixes *p* :: ('*a*::*euclidean-space*) *set*
shows *convex hull p* =
$\{y.\ \exists s\ u.\ finite\ s \wedge s \subseteq p \wedge card\ s \leq DIM('a) + 1 \wedge (\forall x \in s.\ 0 \leq u\ x) \wedge setsum\ u\ s = 1 \wedge setsum\ (\lambda v.\ u\ v *_R v)\ s = y\}$
(**is** *?lhs* = *?rhs*)
proof (*intro set-eqI iffI*)

fix x
assume $x \in$ *?lhs* **then show** $x \in$ *?rhs*
apply (*simp only*: *convex-hull-caratheodory-aff-dim Set.mem-Collect-eq*)
apply (*erule ex-forward*)+
using *aff-dim-subset-univ* [*of p*]
apply *simp*
done
next
fix x
assume $x \in$ *?rhs* **then show** $x \in$ *?lhs*
by (*auto simp add*: *convex-hull-explicit*)
qed

theorem *caratheodory*:
convex hull p =
$\{x::'a::$*euclidean-space*$.\ \exists\, s.$ *finite* $s \wedge s \subseteq p \wedge$
card $s \leq$ *DIM*$('a) + 1 \wedge x \in$ *convex hull* $s\}$
proof *safe*
fix x
assume $x \in$ *convex hull p*
then obtain $s\ u$ **where** *finite* s $s \subseteq p$ *card* $s \leq$ *DIM*$('a) + 1$
$\forall x{\in}s.\ 0 \leq u\ x$ *setsum* $u\ s = 1\ (\sum v{\in}s.\ u\ v *_R v) = x$
unfolding *convex-hull-caratheodory* **by** *auto*
then show $\exists\, s.$ *finite* $s \wedge s \subseteq p \wedge$ *card* $s \leq$ *DIM*$('a) + 1 \wedge x \in$ *convex hull* s
apply (*rule-tac x*=s **in** *exI*)
using *hull-subset*[*of s convex*]
using *convex-convex-hull*[*unfolded convex-explicit*, *of s*,
THEN spec[**where** x=s], *THEN spec*[**where** x=u]]
apply *auto*
done
next
fix $x\ s$
assume *finite* s $s \subseteq p$ *card* $s \leq$ *DIM*$('a) + 1\ x \in$ *convex hull* s
then show $x \in$ *convex hull p*
using *hull-mono*[*OF* ‹$s \subseteq p$›] **by** *auto*
qed

19.10 Relative interior of a set

definition *rel-interior* $S =$
$\{x.\ \exists\, T.$ *openin* (*subtopology euclidean* (*affine hull* S)) $T \wedge x \in T \wedge T \subseteq S\}$

lemma *rel-interior*:
rel-interior $S = \{x \in S.\ \exists\, T.$ *open* $T \wedge x \in T \wedge T \cap$ *affine hull* $S \subseteq S\}$
unfolding *rel-interior-def*[*of S*] *openin-open*[*of affine hull S*]
apply *auto*
proof −
fix $x\ T$
assume $*$: $x \in S$ *open* $T\ x \in T\ T \cap$ *affine hull* $S \subseteq S$

then have $**$: $x \in T \cap$ *affine hull* S
using *hull-inc* **by** *auto*
show $\exists$ *Tb*. ($\exists$ *Ta*. *open Ta* $\wedge$ *Tb* = *affine hull S* $\cap$ *Ta*) $\wedge$ $x \in$ *Tb* $\wedge$ *Tb* $\subseteq S$
apply (*rule-tac x* = *T* $\cap$ (*affine hull S*) **in** *exI*)
using $*$ $**$
apply *auto*
done
qed

lemma *mem-rel-interior*: $x \in$ *rel-interior S* $\longleftrightarrow$ ($\exists$ *T*. *open T* $\wedge$ $x \in T \cap S \wedge T \cap$ *affine hull S* $\subseteq S$)
by (*auto simp add*: *rel-interior*)

lemma *mem-rel-interior-ball*:
$x \in$ *rel-interior S* $\longleftrightarrow$ $x \in S \wedge$ ($\exists$ *e*. *e* > *0* $\wedge$ *ball x e* $\cap$ *affine hull S* $\subseteq S$)
apply (*simp add*: *rel-interior*, *safe*)
apply (*force simp add*: *open-contains-ball*)
apply (*rule-tac x* = *ball x e* **in** *exI*)
apply *simp*
done

lemma *rel-interior-ball*:
rel-interior S = {$x \in S$. $\exists$ *e*. *e* > *0* $\wedge$ *ball x e* $\cap$ *affine hull S* $\subseteq S$}
using *mem-rel-interior-ball* [*of* - *S*] **by** *auto*

lemma *mem-rel-interior-cball*:
$x \in$ *rel-interior S* $\longleftrightarrow$ $x \in S \wedge$ ($\exists$ *e*. *e* > *0* $\wedge$ *cball x e* $\cap$ *affine hull S* $\subseteq S$)
apply (*simp add*: *rel-interior*, *safe*)
apply (*force simp add*: *open-contains-cball*)
apply (*rule-tac x* = *ball x e* **in** *exI*)
apply (*simp add*: *subset-trans* [*OF ball-subset-cball*])
apply *auto*
done

lemma *rel-interior-cball*:
rel-interior S = {$x \in S$. $\exists$ *e*. *e* > *0* $\wedge$ *cball x e* $\cap$ *affine hull S* $\subseteq S$}
using *mem-rel-interior-cball* [*of* - *S*] **by** *auto*

lemma *rel-interior-empty* [*simp*]: *rel-interior* {} = {}
by (*auto simp add*: *rel-interior-def*)

lemma *affine-hull-sing* [*simp*]: *affine hull* {*a* :: *'n*::*euclidean-space*} = {*a*}
by (*metis affine-hull-eq affine-sing*)

lemma *rel-interior-sing* [*simp*]: *rel-interior* {*a* :: *'n*::*euclidean-space*} = {*a*}
unfolding *rel-interior-ball affine-hull-sing*
apply *auto*
apply (*rule-tac x* = *1* :: *real* **in** *exI*)
apply *simp*

done

lemma *subset-rel-interior*:
fixes S T :: *'n::euclidean-space set*
assumes $S \subseteq T$
and *affine hull* S = *affine hull* T
shows *rel-interior* $S \subseteq$ *rel-interior* T
using *assms* **by** (*auto simp add*: *rel-interior-def*)

lemma *rel-interior-subset*: *rel-interior* $S \subseteq S$
by (*auto simp add*: *rel-interior-def*)

lemma *rel-interior-subset-closure*: *rel-interior* $S \subseteq$ *closure* S
using *rel-interior-subset* **by** (*auto simp add*: *closure-def*)

lemma *interior-subset-rel-interior*: *interior* $S \subseteq$ *rel-interior* S
by (*auto simp add*: *rel-interior interior-def*)

lemma *interior-rel-interior*:
fixes S :: *'n::euclidean-space set*
assumes *aff-dim* S = *int*(*DIM*(*'n*))
shows *rel-interior* S = *interior* S
proof −
have *affine hull* S = *UNIV*
using *assms affine-hull-univ*[*of* S] **by** *auto*
then show *?thesis*
unfolding *rel-interior interior-def* **by** *auto*
qed

lemma *rel-interior-interior*:
fixes S :: *'n::euclidean-space set*
assumes *affine hull* S = *UNIV*
shows *rel-interior* S = *interior* S
using *assms* **unfolding** *rel-interior interior-def* **by** *auto*

lemma *rel-interior-open*:
fixes S :: *'n::euclidean-space set*
assumes *open* S
shows *rel-interior* S = S
by (*metis assms interior-eq interior-subset-rel-interior rel-interior-subset set-eq-subset*)

lemma *interior-ball* [*simp*]: *interior* (*ball* x e) = *ball* x e
by (*simp add*: *interior-open*)

lemma *interior-rel-interior-gen*:
fixes S :: *'n::euclidean-space set*
shows *interior* S = (*if aff-dim* S = *int*(*DIM*(*'n*)) *then rel-interior* S *else* {})
by (*metis interior-rel-interior low-dim-interior*)

lemma *rel-interior-univ*:
fixes S :: *'n::euclidean-space set*
shows *rel-interior (affine hull S) = affine hull S*
proof −
have ∗: *rel-interior (affine hull S) ⊆ affine hull S*
using *rel-interior-subset* **by** *auto*
{
fix x
assume x: $x \in$ *affine hull S*
def $e \equiv$ *1::real*
then have $e > 0$ *ball x e ∩ affine hull (affine hull S) ⊆ affine hull S*
using *hull-hull[of - S]* **by** *auto*
then have $x \in$ *rel-interior (affine hull S)*
using *x rel-interior-ball[of affine hull S]* **by** *auto*
}
then show *?thesis* **using** ∗ **by** *auto*
qed

lemma *rel-interior-univ2*: *rel-interior (UNIV :: ('n::euclidean-space) set) = UNIV*
by (*metis open-UNIV rel-interior-open*)

lemma *rel-interior-convex-shrink*:
fixes S :: *'a::euclidean-space set*
assumes *convex S*
and $c \in$ *rel-interior S*
and $x \in S$
and $0 < e$
and $e \leq 1$
shows $x - e *_R (x - c) \in$ *rel-interior S*
proof −
obtain d **where** $d > 0$ **and** d: *ball c d ∩ affine hull S ⊆ S*
using *assms(2)* **unfolding** *mem-rel-interior-ball* **by** *auto*
{
fix y
assume *as*: *dist (x − e ∗R (x − c)) y < e ∗ d y ∈ affine hull S*
have ∗: $y = (1 - (1 - e)) *_R ((1 / e) *_R y - ((1 - e) / e) *_R x) + (1 - e) *_R x$
using ‹$e > 0$› **by** (*auto simp add*: *scaleR-left-diff-distrib scaleR-right-diff-distrib*)
have $x \in$ *affine hull S*
using *assms hull-subset[of S]* **by** *auto*
moreover have $1 / e + - ((1 - e) / e) = 1$
using ‹$e > 0$› *left-diff-distrib[of 1 (1−e) 1/e]* **by** *auto*
ultimately have ∗∗: $(1 / e) *_R y - ((1 - e) / e) *_R x \in$ *affine hull S*
using *as affine-affine-hull[of S] mem-affine[of affine hull S y x (1 / e) −((1 − e) / e)]*
by (*simp add*: *algebra-simps*)
have *dist c ((1 / e) ∗R y − ((1 − e) / e) ∗R x) = |1/e| ∗ norm (e ∗R c − y + (1 − e) ∗R x)*
unfolding *dist-norm norm-scaleR[symmetric]*

```
    apply (rule arg-cong[where f=norm])
    using ‹e > 0›
   apply (auto simp add: euclidean-eq-iff[where 'a='a] field-simps inner-simps)
    done
  also have ... = |1/e| * norm (x - e *R (x - c) - y)
    by (auto intro!:arg-cong[where f=norm] simp add: algebra-simps)
  also have ... < d
    using as[unfolded dist-norm] and ‹e > 0›
    by (auto simp add:pos-divide-less-eq[OF ‹e > 0›] mult.commute)
  finally have y ∈ S
    apply (subst *)
    apply (rule assms(1)[unfolded convex-alt,rule-format])
    apply (rule d[unfolded subset-eq,rule-format])
    unfolding mem-ball
    using assms(3-5) **
    apply auto
    done
 }
 then have ball (x - e *R (x - c)) (e*d) ∩ affine hull S ⊆ S
  by auto
 moreover have e * d > 0
  using ‹e > 0› ‹d > 0› by simp
 moreover have c: c ∈ S
  using assms rel-interior-subset by auto
 moreover from c have x - e *R (x - c) ∈ S
  using convexD-alt[of S x c e]
  apply (simp add: algebra-simps)
  using assms
  apply auto
  done
 ultimately show ?thesis
  using mem-rel-interior-ball[of x - e *R (x - c) S] ‹e > 0› by auto
qed

lemma interior-real-semiline:
 fixes a :: real
 shows interior {a..} = {a<..}
proof -
 {
  fix y
  assume a < y
  then have y ∈ interior {a..}
   apply (simp add: mem-interior)
   apply (rule-tac x=(y-a) in exI)
   apply (auto simp add: dist-norm)
   done
 }
 moreover
 {
```

```
    fix y
    assume y ∈ interior {a..}
    then obtain e where e: e > 0 cball y e ⊆ {a..}
      using mem-interior-cball[of y {a..}] by auto
    moreover from e have y − e ∈ cball y e
      by (auto simp add: cball-def dist-norm)
    ultimately have a ≤ y − e by blast
    then have a < y using e by auto
  }
  ultimately show ?thesis by auto
qed

lemma continuous-ge-on-Ioo:
  assumes continuous-on {c..d} g ⋀x. x ∈ {c<..<d} ⟹ g x ≥ a c < d x ∈
{c..d}
  shows g (x::real) ≥ (a::real)
proof−
 from assms(3) have {c..d} = closure {c<..<d} by (rule closure-greaterThanLessThan[symmetric])
  also from assms(2) have {c<..<d} ⊆ (g −‘ {a..} ∩ {c..d}) by auto
  hence closure {c<..<d} ⊆ closure (g −‘ {a..} ∩ {c..d}) by (rule closure-mono)
  also from assms(1) have closed (g −‘ {a..} ∩ {c..d})
    by (auto simp: continuous-on-closed-vimage)
  hence closure (g −‘ {a..} ∩ {c..d}) = g −‘ {a..} ∩ {c..d} by simp
  finally show ?thesis using ‹x ∈ {c..d}› by auto
qed

lemma interior-real-semiline′:
  fixes a :: real
  shows interior {..a} = {..<a}
proof −
  {
    fix y
    assume a > y
    then have y ∈ interior {..a}
      apply (simp add: mem-interior)
      apply (rule-tac x=(a−y) in exI)
      apply (auto simp add: dist-norm)
      done
  }
  moreover
  {
    fix y
    assume y ∈ interior {..a}
    then obtain e where e: e > 0 cball y e ⊆ {..a}
      using mem-interior-cball[of y {..a}] by auto
    moreover from e have y + e ∈ cball y e
      by (auto simp add: cball-def dist-norm)
    ultimately have a ≥ y + e by auto
    then have a > y using e by auto
```

```
  }
  ultimately show ?thesis by auto
qed

lemma interior-atLeastAtMost-real: interior {a..b} = {a<..<b :: real}
proof−
  have {a..b} = {a..} ∩ {..b} by auto
  also have interior ... = {a<..} ∩ {..<b}
    by (simp add: interior-real-semiline interior-real-semiline′)
  also have ... = {a<..<b} by auto
  finally show ?thesis .
qed

lemma frontier-real-Iic:
  fixes a :: real
  shows frontier {..a} = {a}
  unfolding frontier-def by (auto simp add: interior-real-semiline′)

lemma rel-interior-real-box:
  fixes a b :: real
  assumes a < b
  shows rel-interior {a .. b} = {a <..< b}
proof −
  have box a b ≠ {}
    using assms
    unfolding set-eq-iff
    by (auto intro!: exI[of - (a + b) / 2] simp: box-def)
  then show ?thesis
    using interior-rel-interior-gen[of cbox a b, symmetric]
   by (simp split: if-split-asm del: box-real add: box-real[symmetric] interior-cbox)
qed

lemma rel-interior-real-semiline:
  fixes a :: real
  shows rel-interior {a..} = {a<..}
proof −
  have ∗: {a<..} ≠ {}
    unfolding set-eq-iff by (auto intro!: exI[of - a + 1])
 then show ?thesis using interior-real-semiline interior-rel-interior-gen[of {a..}]
    by (auto split: if-split-asm)
qed
```

19.10.1 Relative open sets

```
definition rel-open S ⟷ rel-interior S = S

lemma rel-open: rel-open S ⟷ openin (subtopology euclidean (affine hull S)) S
  unfolding rel-open-def rel-interior-def
  apply auto
```

using *openin-subopen*[*of subtopology euclidean* (*affine hull S*) *S*]
apply *auto*
done

lemma *opein-rel-interior*: *openin* (*subtopology euclidean* (*affine hull S*)) (*rel-interior S*)
apply (*simp add*: *rel-interior-def*)
apply (*subst openin-subopen*)
apply *blast*
done

lemma *affine-rel-open*:
fixes *S* :: *'n*::*euclidean-space set*
assumes *affine S*
shows *rel-open S*
unfolding *rel-open-def*
using *assms rel-interior-univ*[*of S*] *affine-hull-eq*[*of S*]
by *metis*

lemma *affine-closed*:
fixes *S* :: *'n*::*euclidean-space set*
assumes *affine S*
shows *closed S*
proof −
{
assume *S* ≠ {}
then obtain *L* **where** *L*: *subspace L affine-parallel S L*
using *assms affine-parallel-subspace*[*of S*] **by** *auto*
then obtain *a* **where** *a*: *S* = (*op* + *a* ' *L*)
using *affine-parallel-def*[*of L S*] *affine-parallel-commut* **by** *auto*
from *L* **have** *closed L* **using** *closed-subspace* **by** *auto*
then have *closed S*
using *closed-translation a* **by** *auto*
}
then show *?thesis* **by** *auto*
qed

lemma *closure-affine-hull*:
fixes *S* :: *'n*::*euclidean-space set*
shows *closure S* ⊆ *affine hull S*
by (*intro closure-minimal hull-subset affine-closed affine-affine-hull*)

lemma *closure-same-affine-hull* [*simp*]:
fixes *S* :: *'n*::*euclidean-space set*
shows *affine hull* (*closure S*) = *affine hull S*
proof −
have *affine hull* (*closure S*) ⊆ *affine hull S*
using *hull-mono*[*of closure S affine hull S affine*]
closure-affine-hull[*of S*] *hull-hull*[*of affine S*]

```
    by auto
  moreover have affine hull (closure S) ⊇ affine hull S
    using hull-mono[of S closure S affine] closure-subset by auto
  ultimately show ?thesis by auto
qed

lemma closure-aff-dim:
  fixes S :: 'n::euclidean-space set
  shows aff-dim (closure S) = aff-dim S
proof −
  have aff-dim S ≤ aff-dim (closure S)
    using aff-dim-subset closure-subset by auto
  moreover have aff-dim (closure S) ≤ aff-dim (affine hull S)
    using aff-dim-subset closure-affine-hull by auto
  moreover have aff-dim (affine hull S) = aff-dim S
    using aff-dim-affine-hull by auto
  ultimately show ?thesis by auto
qed

lemma rel-interior-closure-convex-shrink:
  fixes S :: -::euclidean-space set
  assumes convex S
    and c ∈ rel-interior S
    and x ∈ closure S
    and e > 0
    and e ≤ 1
  shows x − e *R (x − c) ∈ rel-interior S
proof −
  obtain d where d > 0 and d: ball c d ∩ affine hull S ⊆ S
    using assms(2) unfolding mem-rel-interior-ball by auto
  have ∃ y ∈ S. norm (y − x) * (1 − e) < e * d
  proof (cases x ∈ S)
    case True
    then show ?thesis using ‹e > 0› ‹d > 0›
      apply (rule-tac bexI[where x=x])
      apply (auto)
      done
  next
    case False
    then have x: x islimpt S
      using assms(3)[unfolded closure-def] by auto
    show ?thesis
    proof (cases e = 1)
      case True
      obtain y where y ∈ S y ≠ x dist y x < 1
        using x[unfolded islimpt-approachable,THEN spec[where x=1]] by auto
      then show ?thesis
        apply (rule-tac x=y in bexI)
        unfolding True
```

```
      using ‹d > 0›
      apply auto
      done
   next
    case False
    then have 0 < e * d / (1 - e) and *: 1 - e > 0
     using ‹e ≤ 1› ‹e > 0› ‹d > 0› by (auto)
    then obtain y where y ∈ S y ≠ x dist y x < e * d / (1 - e)
     using x[unfolded islimpt-approachable,THEN spec[where x=e*d / (1 - e)]]
by auto
    then show ?thesis
     apply (rule-tac x=y in bexI)
     unfolding dist-norm
     using pos-less-divide-eq[OF *]
     apply auto
     done
   qed
  qed
  then obtain y where y ∈ S and y: norm (y - x) * (1 - e) < e * d
   by auto
  def z ≡ c + ((1 - e) / e) *R (x - y)
  have *: x - e *R (x - c) = y - e *R (y - z)
   unfolding z-def using ‹e > 0›
   by (auto simp add: scaleR-right-diff-distrib scaleR-right-distrib scaleR-left-diff-distrib)
  have zball: z ∈ ball c d
   using mem-ball z-def dist-norm[of c]
   using y and assms(4,5)
   by (auto simp add:field-simps norm-minus-commute)
  have x ∈ affine hull S
   using closure-affine-hull assms by auto
  moreover have y ∈ affine hull S
   using ‹y ∈ S› hull-subset[of S] by auto
  moreover have c ∈ affine hull S
   using assms rel-interior-subset hull-subset[of S] by auto
  ultimately have z ∈ affine hull S
   using z-def affine-affine-hull[of S]
     mem-affine-3-minus [of affine hull S c x y (1 - e) / e]
     assms
   by (auto simp add: field-simps)
  then have z ∈ S using d zball by auto
  obtain d1 where d1 > 0 and d1: ball z d1 ≤ ball c d
   using zball open-ball[of c d] openE[of ball c d z] by auto
  then have ball z d1 ∩ affine hull S ⊆ ball c d ∩ affine hull S
   by auto
  then have ball z d1 ∩ affine hull S ⊆ S
   using d by auto
  then have z ∈ rel-interior S
   using mem-rel-interior-ball using ‹d1 > 0› ‹z ∈ S› by auto
  then have y - e *R (y - z) ∈ rel-interior S
```

using *rel-interior-convex-shrink*[*of S z y e*] *assms* ‹*y* ∈ *S*› **by** *auto*
then show *?thesis* **using** ∗ **by** *auto*
qed

lemma *rel-interior-eq*:
rel-interior s = *s* ⟷ *openin*(*subtopology euclidean* (*affine hull s*)) *s*
using *rel-open rel-open-def* **by** *blast*

lemma *rel-interior-openin*:
openin(*subtopology euclidean* (*affine hull s*)) *s* ⟹ *rel-interior s* = *s*
by (*simp add*: *rel-interior-eq*)

19.10.2 Relative interior preserves under linear transformations

lemma *rel-interior-translation-aux*:
fixes *a* :: *'n*::*euclidean-space*
shows ((λ*x*. *a* + *x*) ‘ *rel-interior S*) ⊆ *rel-interior* ((λ*x*. *a* + *x*) ‘ *S*)
proof −
{
fix *x*
assume *x*: *x* ∈ *rel-interior S*
then obtain *T* **where** *open T x* ∈ *T* ∩ *S T* ∩ *affine hull S* ⊆ *S*
using *mem-rel-interior*[*of x S*] **by** *auto*
then have *open* ((λ*x*. *a* + *x*) ‘ *T*)
and *a* + *x* ∈ ((λ*x*. *a* + *x*) ‘ *T*) ∩ ((λ*x*. *a* + *x*) ‘ *S*)
and ((λ*x*. *a* + *x*) ‘ *T*) ∩ *affine hull* ((λ*x*. *a* + *x*) ‘ *S*) ⊆ (λ*x*. *a* + *x*) ‘ *S*
using *affine-hull-translation*[*of a S*] *open-translation*[*of T a*] *x* **by** *auto*
then have *a* + *x* ∈ *rel-interior* ((λ*x*. *a* + *x*) ‘ *S*)
using *mem-rel-interior*[*of a*+*x* ((λ*x*. *a* + *x*) ‘ *S*)] **by** *auto*
}
then show *?thesis* **by** *auto*
qed

lemma *rel-interior-translation*:
fixes *a* :: *'n*::*euclidean-space*
shows *rel-interior* ((λ*x*. *a* + *x*) ‘ *S*) = (λ*x*. *a* + *x*) ‘ *rel-interior S*
proof −
have (λ*x*. (−*a*) + *x*) ‘ *rel-interior* ((λ*x*. *a* + *x*) ‘ *S*) ⊆ *rel-interior S*
using *rel-interior-translation-aux*[*of* −*a* (λ*x*. *a* + *x*) ‘ *S*]
translation-assoc[*of* −*a a*]
by *auto*
then have ((λ*x*. *a* + *x*) ‘ *rel-interior S*) ⊇ *rel-interior* ((λ*x*. *a* + *x*) ‘ *S*)
using *translation-inverse-subset*[*of a rel-interior* (*op* + *a* ‘ *S*) *rel-interior S*]
by *auto*
then show *?thesis*
using *rel-interior-translation-aux*[*of a S*] **by** *auto*
qed

lemma *affine-hull-linear-image*:
assumes *bounded-linear f*
shows *f ' (affine hull s) = affine hull f ' s*
apply *rule*
unfolding *subset-eq ball-simps*
apply (*rule-tac*[!] *hull-induct*, *rule hull-inc*)
prefer *3*
apply (*erule imageE*)
apply (*rule-tac x=xa* **in** *image-eqI*)
apply *assumption*
apply (*rule hull-subset*[*unfolded subset-eq*, *rule-format*])
apply *assumption*
proof −
interpret *f*: *bounded-linear f* **by** *fact*
show *affine* {*x*. *f x* ∈ *affine hull f ' s*}
unfolding *affine-def*
by (*auto simp add*: *f.scaleR f.add affine-affine-hull*[*unfolded affine-def*, *rule-format*])
show *affine* {*x*. *x* ∈ *f ' (affine hull s)*}
using *affine-affine-hull*[*unfolded affine-def*, *of s*]
unfolding *affine-def* **by** (*auto simp add*: *f.scaleR* [*symmetric*] *f.add* [*symmetric*])
qed *auto*

lemma *rel-interior-injective-on-span-linear-image*:
fixes *f* :: *'m*::*euclidean-space* ⇒ *'n*::*euclidean-space*
and *S* :: *'m*::*euclidean-space set*
assumes *bounded-linear f*
and *inj-on f* (*span S*)
shows *rel-interior* (*f ' S*) = *f ' (rel-interior S)*
proof −
{
fix *z*
assume *z*: *z* ∈ *rel-interior* (*f ' S*)
then have *z* ∈ *f ' S*
using *rel-interior-subset*[*of f ' S*] **by** *auto*
then obtain *x* **where** *x*: *x* ∈ *S f x* = *z* **by** *auto*
obtain *e2* **where** *e2*: *e2* > *0 cball z e2* ∩ *affine hull* (*f ' S*) ⊆ (*f ' S*)
using *z rel-interior-cball*[*of f ' S*] **by** *auto*
obtain *K* **where** *K*: *K* > *0* ⋀*x*. *norm* (*f x*) ≤ *norm x* ∗ *K*
using *assms Real-Vector-Spaces.bounded-linear.pos-bounded*[*of f*] **by** *auto*
def *e1* ≡ *1* / *K*
then have *e1*: *e1* > *0* ⋀*x*. *e1* ∗ *norm* (*f x*) ≤ *norm x*
using *K pos-le-divide-eq*[*of e1*] **by** *auto*
def *e* ≡ *e1* ∗ *e2*
then have *e* > *0* **using** *e1 e2* **by** *auto*
{
fix *y*
assume *y*: *y* ∈ *cball x e* ∩ *affine hull S*
then have *h1*: *f y* ∈ *affine hull* (*f ' S*)

```
      using affine-hull-linear-image[of f S] assms by auto
    from y have norm (x−y) ≤ e1 * e2
      using cball-def[of x e] dist-norm[of x y] e-def by auto
    moreover have f x − f y = f (x − y)
      using assms linear-sub[of f x y] linear-conv-bounded-linear[of f] by auto
    moreover have e1 * norm (f (x−y)) ≤ norm (x − y)
      using e1 by auto
    ultimately have e1 * norm ((f x)−(f y)) ≤ e1 * e2
      by auto
    then have f y ∈ cball z e2
      using cball-def[of f x e2] dist-norm[of f x f y] e1 x by auto
    then have f y ∈ f ‘ S
      using y e2 h1 by auto
    then have y ∈ S
      using assms y hull-subset[of S] affine-hull-subset-span
        inj-on-image-mem-iff [OF ‹inj-on f (span S)›]
      by (metis Int-iff span-inc subsetCE)
  }
  then have z ∈ f ‘ (rel-interior S)
    using mem-rel-interior-cball[of x S] ‹e > 0› x by auto
}
moreover
{
  fix x
  assume x: x ∈ rel-interior S
  then obtain e2 where e2: e2 > 0 cball x e2 ∩ affine hull S ⊆ S
    using rel-interior-cball[of S] by auto
  have x ∈ S using x rel-interior-subset by auto
  then have *: f x ∈ f ‘ S by auto
  have ∀ x∈span S. f x = 0 ⟶ x = 0
    using assms subspace-span linear-conv-bounded-linear[of f]
      linear-injective-on-subspace-0[of f span S]
    by auto
  then obtain e1 where e1: e1 > 0 ∀ x ∈ span S. e1 * norm x ≤ norm (f x)
    using assms injective-imp-isometric[of span S f]
      subspace-span[of S] closed-subspace[of span S]
    by auto
  def e ≡ e1 * e2
  hence e > 0 using e1 e2 by auto
  {
    fix y
    assume y: y ∈ cball (f x) e ∩ affine hull (f ‘ S)
    then have y ∈ f ‘ (affine hull S)
      using affine-hull-linear-image[of f S] assms by auto
    then obtain xy where xy: xy ∈ affine hull S f xy = y by auto
    with y have norm (f x − f xy) ≤ e1 * e2
      using cball-def[of f x e] dist-norm[of f x y] e-def by auto
    moreover have f x − f xy = f (x − xy)
      using assms linear-sub[of f x xy] linear-conv-bounded-linear[of f] by auto
```

```
    moreover have *: x - xy ∈ span S
      using subspace-sub[of span S x xy] subspace-span ‹x ∈ S› xy
        affine-hull-subset-span[of S] span-inc
      by auto
    moreover from * have e1 * norm (x - xy) ≤ norm (f (x - xy))
      using e1 by auto
    ultimately have e1 * norm (x - xy) ≤ e1 * e2
      by auto
    then have xy ∈ cball x e2
      using cball-def[of x e2] dist-norm[of x xy] e1 by auto
    then have y ∈ f ' S
      using xy e2 by auto
  }
  then have f x ∈ rel-interior (f ' S)
    using mem-rel-interior-cball[of (f x) (f ' S)] * ‹e > 0› by auto
 }
 ultimately show ?thesis by auto
qed

lemma rel-interior-injective-linear-image:
 fixes f :: 'm::euclidean-space ⇒ 'n::euclidean-space
 assumes bounded-linear f
  and inj f
 shows rel-interior (f ' S) = f ' (rel-interior S)
 using assms rel-interior-injective-on-span-linear-image[of f S]
  subset-inj-on[of f UNIV span S]
 by auto
```

19.11 Some Properties of subset of standard basis

```
lemma affine-hull-substd-basis:
 assumes d ⊆ Basis
 shows affine hull (insert 0 d) = {x::'a::euclidean-space. ∀ i∈Basis. i ∉ d ⟶ x•i
= 0}
 (is affine hull (insert 0 ?A) = ?B)
proof -
 have *: ⋀A. op + (0::'a) ' A = A ⋀A. op + (- (0::'a)) ' A = A
  by auto
 show ?thesis
  unfolding affine-hull-insert-span-gen span-substd-basis[OF assms,symmetric] *
..
qed

lemma affine-hull-convex-hull [simp]: affine hull (convex hull S) = affine hull S
 by (metis Int-absorb1 Int-absorb2 convex-hull-subset-affine-hull hull-hull hull-mono
hull-subset)
```

19.12 Openness and compactness are preserved by convex hull operation.

lemma *open-convex-hull*[*intro*]:
fixes $s :: 'a$::*real-normed-vector set*
assumes *open s*
shows *open* (*convex hull s*)
unfolding *open-contains-cball convex-hull-explicit*
unfolding *mem-Collect-eq ball-simps*(*8*)
proof (*rule*, *rule*)
fix *a*
assume $\exists$ *sa u. finite sa* $\wedge$ *sa* $\subseteq$ *s* $\wedge$ ($\forall$ *x*$\in$*sa. 0* $\leq$ *u x*) $\wedge$ *setsum u sa = 1* $\wedge$ ($\sum$ *v*$\in$*sa. u v* $*_R$ *v*) = *a*
then obtain *t u* **where** *obt*: *finite t t*$\subseteq$*s* $\forall$ *x*$\in$*t. 0* $\leq$ *u x setsum u t = 1* ($\sum$ *v*$\in$*t. u v* $*_R$ *v*) = *a*
by *auto*

from *assms*[*unfolded open-contains-cball*] **obtain** *b*
where *b*: $\forall$ *x*$\in$*s. 0 < b x* $\wedge$ *cball x* (*b x*) $\subseteq$ *s*
using *bchoice*[*of s* λ*x e. e > 0* $\wedge$ *cball x e* $\subseteq$ *s*] **by** *auto*
have *b* ‘ *t* $\neq$ {}
using *obt* **by** *auto*
def *i* $\equiv$ *b* ‘ *t*

show $\exists$ *e > 0*.
cball a e $\subseteq$ {*y*. $\exists$ *sa u. finite sa* $\wedge$ *sa* $\subseteq$ *s* $\wedge$ ($\forall$ *x*$\in$*sa. 0* $\leq$ *u x*) $\wedge$ *setsum u sa = 1* $\wedge$ ($\sum$ *v*$\in$*sa. u v* $*_R$ *v*) = *y*}
apply (*rule-tac x = Min i* **in** *exI*)
unfolding *subset-eq*
apply *rule*
defer
apply *rule*
unfolding *mem-Collect-eq*
proof −
show *0 < Min i*
unfolding *i-def* **and** *Min-gr-iff*[*OF finite-imageI*[*OF obt*(*1*)] ‹*b* ‘ *t*≠{}›]
using *b*
apply *simp*
apply *rule*
apply (*erule-tac x*=*x* **in** *ballE*)
using ‹*t*⊆*s*›
apply *auto*
done
next
fix *y*
assume *y* $\in$ *cball a* (*Min i*)
then have *y*: *norm* (*a* − *y*) $\leq$ *Min i*
unfolding *dist-norm*[*symmetric*] **by** *auto*
{
fix *x*

```
      assume x ∈ t
      then have Min i ≤ b x
        unfolding i-def
        apply (rule-tac Min-le)
        using obt(1)
        apply auto
        done
      then have x + (y − a) ∈ cball x (b x)
        using y unfolding mem-cball dist-norm by auto
      moreover from ‹x∈t› have x ∈ s
        using obt(2) by auto
      ultimately have x + (y − a) ∈ s
        using y and b[THEN bspec[where x=x]] unfolding subset-eq by fast
    }
    moreover
    have ∗: inj-on (λv. v + (y − a)) t
      unfolding inj-on-def by auto
    have (∑ v∈(λv. v + (y − a)) ‘ t. u (v − (y − a))) = 1
      unfolding setsum.reindex[OF ∗] o-def using obt(4) by auto
    moreover have (∑ v∈(λv. v + (y − a)) ‘ t. u (v − (y − a)) ∗R v) = y
      unfolding setsum.reindex[OF ∗] o-def using obt(4,5)
      by (simp add: setsum.distrib setsum-subtractf scaleR-left.setsum[symmetric]
scaleR-right-distrib)
    ultimately
    show ∃ sa u. finite sa ∧ (∀ x∈sa. x ∈ s) ∧ (∀ x∈sa. 0 ≤ u x) ∧ setsum u sa =
1 ∧ (∑ v∈sa. u v ∗R v) = y
      apply (rule-tac x=(λv. v + (y − a)) ‘ t in exI)
      apply (rule-tac x=λv. u (v − (y − a)) in exI)
      using obt(1, 3)
      apply auto
      done
  qed
qed

lemma compact-convex-combinations:
  fixes s t :: 'a::real-normed-vector set
  assumes compact s compact t
  shows compact { (1 − u) ∗R x + u ∗R y | x y u. 0 ≤ u ∧ u ≤ 1 ∧ x ∈ s ∧ y
∈ t}
proof −
  let ?X = {0..1} × s × t
  let ?h = (λz. (1 − fst z) ∗R fst (snd z) + fst z ∗R snd (snd z))
  have ∗: { (1 − u) ∗R x + u ∗R y | x y u. 0 ≤ u ∧ u ≤ 1 ∧ x ∈ s ∧ y ∈ t} =
?h ‘ ?X
    apply (rule set-eqI)
    unfolding image-iff mem-Collect-eq
    apply rule
    apply auto
    apply (rule-tac x=u in rev-bexI)
```

```
    apply simp
    apply (erule rev-bexI)
    apply (erule rev-bexI)
    apply simp
    apply auto
    done
  have continuous-on ?X (λz. (1 − fst z) *R fst (snd z) + fst z *R snd (snd z))
    unfolding continuous-on by (rule ballI) (intro tendsto-intros)
  then show ?thesis
    unfolding ∗
    apply (rule compact-continuous-image)
    apply (intro compact-Times compact-Icc assms)
    done
qed

lemma finite-imp-compact-convex-hull:
  fixes s :: 'a::real-normed-vector set
  assumes finite s
  shows compact (convex hull s)
proof (cases s = {})
  case True
  then show ?thesis by simp
next
  case False
  with assms show ?thesis
  proof (induct rule: finite-ne-induct)
    case (singleton x)
    show ?case by simp
  next
    case (insert x A)
    let ?f = λ(u, y::'a). u *R x + (1 − u) *R y
    let ?T = {0..1::real} × (convex hull A)
    have continuous-on ?T ?f
      unfolding split-def continuous-on by (intro ballI tendsto-intros)
    moreover have compact ?T
      by (intro compact-Times compact-Icc insert)
    ultimately have compact (?f ‘ ?T)
      by (rule compact-continuous-image)
    also have ?f ‘ ?T = convex hull (insert x A)
      unfolding convex-hull-insert [OF ‹A ≠ {}›]
      apply safe
      apply (rule-tac x=a in exI, simp)
      apply (rule-tac x=1 − a in exI, simp)
      apply fast
      apply (rule-tac x=(u, b) in image-eqI, simp-all)
      done
    finally show compact (convex hull (insert x A)) .
  qed
qed
```

lemma *compact-convex-hull*:
fixes $s :: 'a::euclidean\text{-}space\ set$
assumes *compact s*
shows *compact* (*convex hull s*)
proof (*cases* $s = \{\}$)
case *True*
then show *?thesis* **using** *compact-empty* **by** *simp*
next
case *False*
then obtain *w* **where** $w \in s$ **by** *auto*
show *?thesis*
unfolding *caratheodory*[*of s*]
proof (*induct* (*DIM*('*a*) + *1*))
case *0*
have $*$: $\{x.\exists\, sa.\ finite\ sa \wedge sa \subseteq s \wedge card\ sa \leq 0 \wedge x \in convex\ hull\ sa\} = \{\}$
using *compact-empty* **by** *auto*
from *0* **show** *?case* **unfolding** $*$ **by** *simp*
next
case (*Suc n*)
show *?case*
proof (*cases n* = *0*)
case *True*
have $\{x.\ \exists\, t.\ finite\ t \wedge t \subseteq s \wedge card\ t \leq Suc\ n \wedge x \in convex\ hull\ t\} = s$
unfolding *set-eq-iff* **and** *mem-Collect-eq*
proof (*rule*, *rule*)
fix *x*
assume $\exists\, t.\ finite\ t \wedge t \subseteq s \wedge card\ t \leq Suc\ n \wedge x \in convex\ hull\ t$
then obtain *t* **where** *t*: *finite t t* $\subseteq$ *s card t* $\leq$ *Suc n x* $\in$ *convex hull t*
by *auto*
show $x \in s$
proof (*cases card t* = *0*)
case *True*
then show *?thesis*
using *t*(*4*) **unfolding** *card-0-eq*[*OF t*(*1*)] **by** *simp*
next
case *False*
then have *card t* = *Suc 0* **using** *t*(*3*) ‹*n*=*0*› **by** *auto*
then obtain *a* **where** $t = \{a\}$ **unfolding** *card-Suc-eq* **by** *auto*
then show *?thesis* **using** *t*(*2*,*4*) **by** *simp*
qed
next
fix *x* **assume** $x \in s$
then show $\exists\, t.\ finite\ t \wedge t \subseteq s \wedge card\ t \leq Suc\ n \wedge x \in convex\ hull\ t$
apply (*rule-tac x*=$\{x\}$ **in** *exI*)
unfolding *convex-hull-singleton*
apply *auto*
done
qed

```
    then show ?thesis using assms by simp
  next
    case False
    have {x. ∃t. finite t ∧ t ⊆ s ∧ card t ≤ Suc n ∧ x ∈ convex hull t} =
      {(1 − u) *R x + u *R y | x y u.
        0 ≤ u ∧ u ≤ 1 ∧ x ∈ s ∧ y ∈ {x. ∃t. finite t ∧ t ⊆ s ∧ card t ≤ n ∧ x
∈ convex hull t}}
      unfolding set-eq-iff and mem-Collect-eq
    proof (rule, rule)
      fix x
      assume ∃u v c. x = (1 − c) *R u + c *R v ∧
        0 ≤ c ∧ c ≤ 1 ∧ u ∈ s ∧ (∃t. finite t ∧ t ⊆ s ∧ card t ≤ n ∧ v ∈ convex
hull t)
      then obtain u v c t where obt: x = (1 − c) *R u + c *R v
        0 ≤ c ∧ c ≤ 1 u ∈ s finite t t ⊆ s card t ≤ n  v ∈ convex hull t
        by auto
      moreover have (1 − c) *R u + c *R v ∈ convex hull insert u t
        apply (rule convexD-alt)
       using obt(2) and convex-convex-hull and hull-subset[of insert u t convex]
        using obt(7) and hull-mono[of t insert u t]
        apply auto
        done
      ultimately show ∃t. finite t ∧ t ⊆ s ∧ card t ≤ Suc n ∧ x ∈ convex hull t
        apply (rule-tac x=insert u t in exI)
        apply (auto simp add: card-insert-if)
        done
    next
      fix x
      assume ∃t. finite t ∧ t ⊆ s ∧ card t ≤ Suc n ∧ x ∈ convex hull t
      then obtain t where t: finite t t ⊆ s card t ≤ Suc n x ∈ convex hull t
        by auto
      show ∃u v c. x = (1 − c) *R u + c *R v ∧
        0 ≤ c ∧ c ≤ 1 ∧ u ∈ s ∧ (∃t. finite t ∧ t ⊆ s ∧ card t ≤ n ∧ v ∈ convex
hull t)
      proof (cases card t = Suc n)
        case False
        then have card t ≤ n using t(3) by auto
        then show ?thesis
          apply (rule-tac x=w in exI, rule-tac x=x in exI, rule-tac x=1 in exI)
          using ‹w∈s› and t
          apply (auto intro!: exI[where x=t])
          done
      next
        case True
        then obtain a u where au: t = insert a u a∉u
          apply (drule-tac card-eq-SucD)
          apply auto
          done
        show ?thesis
```

```
      proof (cases u = {})
        case True
        then have x = a using t(4)[unfolded au] by auto
        show ?thesis unfolding ‹x = a›
          apply (rule-tac x=a in exI)
          apply (rule-tac x=a in exI)
          apply (rule-tac x=1 in exI)
          using t and ‹n ≠ 0›
          unfolding au
          apply (auto intro!: exI[where x={a}])
          done
      next
        case False
        obtain ux vx b where obt: ux≥0 vx≥0 ux + vx = 1
          b ∈ convex hull u x = ux *R a + vx *R b
          using t(4)[unfolded au convex-hull-insert[OF False]]
          by auto
        have *: 1 - vx = ux using obt(3) by auto
        show ?thesis
          apply (rule-tac x=a in exI)
          apply (rule-tac x=b in exI)
          apply (rule-tac x=vx in exI)
          using obt and t(1-3)
          unfolding au and * using card-insert-disjoint[OF - au(2)]
          apply (auto intro!: exI[where x=u])
          done
      qed
    qed
   qed
   then show ?thesis
    using compact-convex-combinations[OF assms Suc] by simp
  qed
 qed
qed
```

19.13 Extremal points of a simplex are some vertices.

```
lemma dist-increases-online:
 fixes a b d :: 'a::real-inner
 assumes d ≠ 0
 shows dist a (b + d) > dist a b ∨ dist a (b - d) > dist a b
proof (cases inner a d - inner b d > 0)
 case True
 then have 0 < inner d d + (inner a d * 2 - inner b d * 2)
  apply (rule-tac add-pos-pos)
  using assms
  apply auto
  done
 then show ?thesis
```

apply (*rule-tac disjI2*)
unfolding *dist-norm* **and** *norm-eq-sqrt-inner* **and** *real-sqrt-less-iff*
apply (*simp add*: *algebra-simps inner-commute*)
done
next
case *False*
then have *0 < inner d d + (inner b d * 2 − inner a d * 2)*
apply (*rule-tac add-pos-nonneg*)
using *assms*
apply *auto*
done
then show *?thesis*
apply (*rule-tac disjI1*)
unfolding *dist-norm* **and** *norm-eq-sqrt-inner* **and** *real-sqrt-less-iff*
apply (*simp add*: *algebra-simps inner-commute*)
done
qed

lemma *norm-increases-online*:
fixes *d* :: *'a*::*real-inner*
shows *d ≠ 0 ⟹ norm (a + d) > norm a ∨ norm(a − d) > norm a*
using *dist-increases-online*[*of d a 0*] **unfolding** *dist-norm* **by** *auto*

lemma *simplex-furthest-lt*:
fixes *s* :: *'a*::*real-inner set*
assumes *finite s*
shows *∀ x ∈ convex hull s. x ∉ s ⟶ (∃ y ∈ convex hull s. norm (x − a) < norm(y − a))*
using *assms*
proof *induct*
fix *x s*
assume *as*: *finite s x∉s ∀ x∈convex hull s. x ∉ s ⟶ (∃ y∈convex hull s. norm (x − a) < norm (y − a))*
show *∀ xa∈convex hull insert x s. xa ∉ insert x s ⟶*
(∃ y∈convex hull insert x s. norm (xa − a) < norm (y − a))
proof (*rule*, *rule*, *cases s = {}*)
case *False*
fix *y*
assume *y*: *y ∈ convex hull insert x s y ∉ insert x s*
obtain *u v b* **where** *obt*: *u≥0 v≥0 u + v = 1 b ∈ convex hull s y = u *R x + v *R b*
using *y*(*1*)[*unfolded convex-hull-insert*[*OF False*]] **by** *auto*
show *∃ z∈convex hull insert x s. norm (y − a) < norm (z − a)*
proof (*cases y ∈ convex hull s*)
case *True*
then obtain *z* **where** *z ∈ convex hull s norm (y − a) < norm (z − a)*
using *as*(*3*)[*THEN bspec*[**where** *x=y*]] **and** *y*(*2*) **by** *auto*
then show *?thesis*
apply (*rule-tac x=z* **in** *bexI*)

```
    unfolding convex-hull-insert[OF False]
    apply auto
    done
  next
    case False
    show ?thesis
    using obt(3)
    proof (cases u = 0, case-tac[!] v = 0)
    assume u = 0 v ≠ 0
    then have y = b using obt by auto
    then show ?thesis using False and obt(4) by auto
    next
    assume u ≠ 0 v = 0
    then have y = x using obt by auto
    then show ?thesis using y(2) by auto
    next
    assume u ≠ 0 v ≠ 0
    then obtain w where w: w>0 w<u w<v
    using real-lbound-gt-zero[of u v] and obt(1,2) by auto
    have x ≠ b
    proof
    assume x = b
    then have y = b unfolding obt(5)
    using obt(3) by (auto simp add: scaleR-left-distrib[symmetric])
    then show False using obt(4) and False by simp
    qed
    then have *: w *R (x − b) ≠ 0 using w(1) by auto
    show ?thesis
    using dist-increases-online[OF *, of a y]
    proof (elim disjE)
    assume dist a y < dist a (y + w *R (x − b))
    then have norm (y − a) < norm ((u + w) *R x + (v − w) *R b − a)
    unfolding dist-commute[of a]
    unfolding dist-norm obt(5)
    by (simp add: algebra-simps)
    moreover have (u + w) *R x + (v − w) *R b ∈ convex hull insert x s
    unfolding convex-hull-insert[OF ‹s≠{}›] and mem-Collect-eq
    apply (rule-tac x=u + w in exI)
    apply rule
    defer
    apply (rule-tac x=v − w in exI)
    using ‹u ≥ 0› and w and obt(3,4)
    apply auto
    done
    ultimately show ?thesis by auto
    next
    assume dist a y < dist a (y − w *R (x − b))
    then have norm (y − a) < norm ((u − w) *R x + (v + w) *R b − a)
    unfolding dist-commute[of a]
```

```
          unfolding dist-norm obt(5)
          by (simp add: algebra-simps)
        moreover have (u - w) *R x + (v + w) *R b ∈ convex hull insert x s
          unfolding convex-hull-insert[OF ‹s≠{}›] and mem-Collect-eq
          apply (rule-tac x=u - w in exI)
          apply rule
          defer
          apply (rule-tac x=v + w in exI)
          using ‹u ≥ 0› and w and obt(3,4)
          apply auto
          done
        ultimately show ?thesis by auto
      qed
    qed auto
  qed
qed (auto simp add: assms)

lemma simplex-furthest-le:
  fixes s :: 'a::real-inner set
  assumes finite s
    and s ≠ {}
  shows ∃y∈s. ∀x∈ convex hull s. norm (x - a) ≤ norm (y - a)
proof -
  have convex hull s ≠ {}
    using hull-subset[of s convex] and assms(2) by auto
  then obtain x where x: x ∈ convex hull s ∀y∈convex hull s. norm (y - a) ≤
norm (x - a)
    using distance-attains-sup[OF finite-imp-compact-convex-hull[OF assms(1)], of
a]
    unfolding dist-commute[of a]
    unfolding dist-norm
    by auto
  show ?thesis
  proof (cases x ∈ s)
    case False
    then obtain y where y ∈ convex hull s norm (x - a) < norm (y - a)
      using simplex-furthest-lt[OF assms(1), THEN bspec[where x=x]] and x(1)
      by auto
    then show ?thesis
      using x(2)[THEN bspec[where x=y]] by auto
  next
    case True
    with x show ?thesis by auto
  qed
qed

lemma simplex-furthest-le-exists:
  fixes s :: ('a::real-inner) set
```

shows *finite s* $\Longrightarrow \forall x{\in}(\mathit{convex\ hull\ s}).\ \exists y{\in}s.\ \mathit{norm}\ (x - a) \leq \mathit{norm}\ (y - a)$
using *simplex-furthest-le*[*of s*] **by** (*cases s* = {}) *auto*

lemma *simplex-extremal-le*:
fixes *s* :: *'a*::*real-inner set*
assumes *finite s*
and *s* ≠ {}
shows ∃ *u*∈*s*. ∃ *v*∈*s*. ∀ *x*∈*convex hull s*. ∀ *y* ∈ *convex hull s*. *norm* (*x* − *y*) ≤ *norm* (*u* − *v*)
proof −
have *convex hull s* ≠ {}
using *hull-subset*[*of s convex*] **and** *assms*(*2*) **by** *auto*
then obtain *u v* **where** *obt*: *u* ∈ *convex hull s v* ∈ *convex hull s*
∀ *x*∈*convex hull s*. ∀ *y*∈*convex hull s*. *norm* (*x* − *y*) ≤ *norm* (*u* − *v*)
using *compact-sup-maxdistance*[*OF finite-imp-compact-convex-hull*[*OF assms*(*1*)]]
by (*auto simp*: *dist-norm*)
then show *?thesis*
proof (*cases u*∉*s* ∨ *v*∉*s*, *elim disjE*)
assume *u* ∉ *s*
then obtain *y* **where** *y* ∈ *convex hull s norm* (*u* − *v*) < *norm* (*y* − *v*)
using *simplex-furthest-lt*[*OF assms*(*1*), *THEN bspec*[**where** *x*=*u*]] **and** *obt*(*1*)
by *auto*
then show *?thesis*
using *obt*(*3*)[*THEN bspec*[**where** *x*=*y*], *THEN bspec*[**where** *x*=*v*]] **and** *obt*(*2*)
by *auto*
next
assume *v* ∉ *s*
then obtain *y* **where** *y* ∈ *convex hull s norm* (*v* − *u*) < *norm* (*y* − *u*)
using *simplex-furthest-lt*[*OF assms*(*1*), *THEN bspec*[**where** *x*=*v*]] **and** *obt*(*2*)
by *auto*
then show *?thesis*
using *obt*(*3*)[*THEN bspec*[**where** *x*=*u*], *THEN bspec*[**where** *x*=*y*]] **and** *obt*(*1*)
by (*auto simp add*: *norm-minus-commute*)
qed *auto*
qed

lemma *simplex-extremal-le-exists*:
fixes *s* :: *'a*::*real-inner set*
shows *finite s* $\Longrightarrow$ *x* ∈ *convex hull s* $\Longrightarrow$ *y* ∈ *convex hull s* $\Longrightarrow$
∃ *u*∈*s*. ∃ *v*∈*s*. *norm* (*x* − *y*) ≤ *norm* (*u* − *v*)
using *convex-hull-empty simplex-extremal-le*[*of s*]
by(*cases s* = {}) *auto*

19.14 Closest point of a convex set is unique, with a continuous projection.

definition *closest-point* :: *'a*::{*real-inner*,*heine-borel*} *set* ⇒ *'a* ⇒ *'a*
where *closest-point s a* = (*SOME x*. *x* ∈ *s* ∧ (∀ *y*∈*s*. *dist a x* ≤ *dist a y*))

lemma *closest-point-exists*:
assumes *closed s*
and $s \neq \{\}$
shows *closest-point s a* $\in s$
and $\forall y{\in}s.$ *dist a* (*closest-point s a*) $\leq$ *dist a y*
unfolding *closest-point-def*
apply(*rule-tac*[!] *someI2-ex*)
apply (*auto intro*: *distance-attains-inf*[*OF assms*(*1*,*2*), *of a*])
done

lemma *closest-point-in-set*: *closed s* $\Longrightarrow$ $s \neq \{\}$ $\Longrightarrow$ *closest-point s a* $\in s$
by (*meson closest-point-exists*)

lemma *closest-point-le*: *closed s* $\Longrightarrow$ $x \in s$ $\Longrightarrow$ *dist a* (*closest-point s a*) $\leq$ *dist a x*
using *closest-point-exists*[*of s*] **by** *auto*

lemma *closest-point-self*:
assumes $x \in s$
shows *closest-point s x* $= x$
unfolding *closest-point-def*
apply (*rule some1-equality*, *rule ex1I*[*of* - *x*])
using *assms*
apply *auto*
done

lemma *closest-point-refl*: *closed s* $\Longrightarrow$ $s \neq \{\}$ $\Longrightarrow$ *closest-point s x* $= x \longleftrightarrow x \in s$
using *closest-point-in-set*[*of s x*] *closest-point-self*[*of x s*]
by *auto*

lemma *closer-points-lemma*:
assumes *inner y z* > 0
shows $\exists u{>}0.\ \forall v{>}0.\ v \leq u \longrightarrow$ *norm*$(v *_R z - y) <$ *norm y*
proof −
have *z*: *inner z z* > 0
unfolding *inner-gt-zero-iff* **using** *assms* **by** *auto*
then show *?thesis*
using *assms*
apply (*rule-tac x* $=$ *inner y z* / *inner z z* **in** *exI*)
apply *rule*
defer
proof *rule*+
fix *v*
assume $0 < v$ **and** $v \leq$ *inner y z* / *inner z z*
then show *norm* $(v *_R z - y) <$ *norm y*
unfolding *norm-lt* **using** *z* **and** *assms*
by (*simp add*: *field-simps inner-diff inner-commute mult-strict-left-mono*[*OF* - ‹*0*<*v*›])

qed *auto*
qed

lemma *closer-point-lemma*:
assumes *inner* $(y - x)$ $(z - x) > 0$
shows $\exists u{>}0.\ u \leq 1 \wedge$ *dist* $(x + u *_R (z - x))$ $y <$ *dist* x y
proof −
obtain u **where** $u > 0$
and u: $\forall v{>}0.\ v \leq u \longrightarrow$ *norm* $(v *_R (z - x) - (y - x)) <$ *norm* $(y - x)$
using *closer-points-lemma*[*OF assms*] **by** *auto*
show *?thesis*
apply (*rule-tac x*=*min u 1* **in** *exI*)
using u[*THEN spec*[**where** x=*min u 1*]] **and** ‹$u > 0$›
unfolding *dist-norm* **by** (*auto simp add*: *norm-minus-commute field-simps*)
qed

lemma *any-closest-point-dot*:
assumes *convex s closed s* $x \in s$ $y \in s$ $\forall z{\in}s.$ *dist a x* $\leq$ *dist a z*
shows *inner* $(a - x)$ $(y - x) \leq 0$
proof (*rule ccontr*)
assume ¬ *?thesis*
then obtain u **where** u: $u{>}0$ $u{\leq}1$ *dist* $(x + u *_R (y - x))$ $a <$ *dist x a*
using *closer-point-lemma*[*of a x y*] **by** *auto*
let *?z* $= (1 - u) *_R x + u *_R y$
have *?z* $\in s$
using *convexD-alt*[*OF assms*(*1*,*3*,*4*), *of u*] **using** u **by** *auto*
then show *False*
using *assms*(*5*)[*THEN bspec*[**where** x=*?z*]] **and** u(*3*)
by (*auto simp add*: *dist-commute algebra-simps*)
qed

lemma *any-closest-point-unique*:
fixes x :: *'a*::*real-inner*
assumes *convex s closed s* $x \in s$ $y \in s$
$\forall z{\in}s.$ *dist a x* $\leq$ *dist a z* $\forall z{\in}s.$ *dist a y* $\leq$ *dist a z*
shows $x = y$
using *any-closest-point-dot*[*OF assms*(*1*−*4*,*5*)] **and** *any-closest-point-dot*[*OF assms*(*1*−*2*,*4*,*3*,*6*)]
unfolding *norm-pths*(*1*) **and** *norm-le-square*
by (*auto simp add*: *algebra-simps*)

lemma *closest-point-unique*:
assumes *convex s closed s* $x \in s$ $\forall z{\in}s.$ *dist a x* $\leq$ *dist a z*
shows $x =$ *closest-point s a*
using *any-closest-point-unique*[*OF assms*(*1*−*3*) - *assms*(*4*), *of closest-point s a*]
using *closest-point-exists*[*OF assms*(*2*)] **and** *assms*(*3*) **by** *auto*

lemma *closest-point-dot*:
assumes *convex s closed s* $x \in s$

shows *inner* (*a* − *closest-point s a*) (*x* − *closest-point s a*) ≤ *0*
apply (*rule any-closest-point-dot*[*OF assms*(*1*,*2*) *- assms*(*3*)])
using *closest-point-exists*[*OF assms*(*2*)] **and** *assms*(*3*)
apply *auto*
done

lemma *closest-point-lt*:
assumes *convex s closed s x* ∈ *s x* ≠ *closest-point s a*
shows *dist a* (*closest-point s a*) < *dist a x*
apply (*rule ccontr*)
apply (*rule-tac notE*[*OF assms*(*4*)])
apply (*rule closest-point-unique*[*OF assms*(*1*−*3*), *of a*])
using *closest-point-le*[*OF assms*(*2*), *of - a*]
apply *fastforce*
done

lemma *closest-point-lipschitz*:
assumes *convex s*
and *closed s s* ≠ {}
shows *dist* (*closest-point s x*) (*closest-point s y*) ≤ *dist x y*
proof −
have *inner* (*x* − *closest-point s x*) (*closest-point s y* − *closest-point s x*) ≤ *0*
and *inner* (*y* − *closest-point s y*) (*closest-point s x* − *closest-point s y*) ≤ *0*
apply (*rule-tac*[!] *any-closest-point-dot*[*OF assms*(*1*−*2*)])
using *closest-point-exists*[*OF assms*(*2*−*3*)]
apply *auto*
done
then show *?thesis* **unfolding** *dist-norm* **and** *norm-le*
using *inner-ge-zero*[*of* (*x* − *closest-point s x*) − (*y* − *closest-point s y*)]
by (*simp add*: *inner-add inner-diff inner-commute*)
qed

lemma *continuous-at-closest-point*:
assumes *convex s*
and *closed s*
and *s* ≠ {}
shows *continuous* (*at x*) (*closest-point s*)
unfolding *continuous-at-eps-delta*
using *le-less-trans*[*OF closest-point-lipschitz*[*OF assms*]] **by** *auto*

lemma *continuous-on-closest-point*:
assumes *convex s*
and *closed s*
and *s* ≠ {}
shows *continuous-on t* (*closest-point s*)
by (*metis continuous-at-imp-continuous-on continuous-at-closest-point*[*OF assms*])

19.14.1 Various point-to-set separating/supporting hyperplane theorems.

lemma *supporting-hyperplane-closed-point*:
fixes $z :: 'a::\{real\text{-}inner,heine\text{-}borel\}$
assumes *convex s*
and *closed s*
and $s \neq \{\}$
and $z \notin s$
shows $\exists\, a\ b.\ \exists\, y{\in}s.\ inner\ a\ z < b \wedge inner\ a\ y = b \wedge (\forall\, x{\in}s.\ inner\ a\ x \geq b)$
proof −
obtain *y* **where** $y \in s$ **and** y: $\forall\, x{\in}s.\ dist\ z\ y \leq dist\ z\ x$
by (*metis distance-attains-inf*[*OF assms*(2−3)])
show *?thesis*
apply (*rule-tac x*=*y* − *z* **in** *exI*)
apply (*rule-tac x*=*inner* (*y* − *z*) *y* **in** *exI*)
apply (*rule-tac x*=*y* **in** *bexI*)
apply *rule*
defer
apply *rule*
defer
apply *rule*
apply (*rule ccontr*)
using ‹*y* ∈ *s*›
proof −
show *inner* (*y* − *z*) *z* < *inner* (*y* − *z*) *y*
apply (*subst diff-gt-0-iff-gt* [*symmetric*])
unfolding *inner-diff-right*[*symmetric*] **and** *inner-gt-zero-iff*
using ‹*y*∈*s*› ‹*z*∉*s*›
apply *auto*
done
next
fix *x*
assume *x* ∈ *s*
have ∗: ∀ *u*. *0* ≤ *u* ∧ *u* ≤ *1* ⟶ *dist z y* ≤ *dist z* ((*1* − *u*) ∗$_R$ *y* + *u* ∗$_R$ *x*)
using *assms*(*1*)[*unfolded convex-alt*] **and** *y* **and** ‹*x*∈*s*› **and** ‹*y*∈*s*› **by** *auto*
assume ¬ *inner* (*y* − *z*) *y* ≤ *inner* (*y* − *z*) *x*
then obtain *v* **where** *v* > *0* *v* ≤ *1* *dist* (*y* + *v* ∗$_R$ (*x* − *y*)) *z* < *dist y z*
using *closer-point-lemma*[*of z y x*] **by** (*auto simp add*: *inner-diff*)
then show *False*
using ∗[*THEN spec*[**where** *x*=*v*]] **by** (*auto simp add*: *dist-commute algebra-simps*)
qed *auto*
qed

lemma *separating-hyperplane-closed-point*:
fixes $z :: 'a::\{real\text{-}inner,heine\text{-}borel\}$
assumes *convex s*
and *closed s*
and $z \notin s$
shows $\exists\, a\ b.\ inner\ a\ z < b \wedge (\forall\, x{\in}s.\ inner\ a\ x > b)$

proof (*cases* $s = \{\}$)
case *True*
then show *?thesis*
apply (*rule-tac* $x{=}-z$ **in** *exI*)
apply (*rule-tac* $x{=}1$ **in** *exI*)
using *less-le-trans*[*OF - inner-ge-zero*[*of z*]]
apply *auto*
done
next
case *False*
obtain y **where** $y \in s$ **and** y: $\forall x{\in}s.\ dist\ z\ y \leq dist\ z\ x$
by (*metis distance-attains-inf*[*OF assms*(2) *False*])
show *?thesis*
apply (*rule-tac* $x{=}y - z$ **in** *exI*)
apply (*rule-tac* $x{=}inner\ (y - z)\ z + (norm\ (y - z))^2\ /\ 2$ **in** *exI*)
apply *rule*
defer
apply *rule*
proof −
fix x
assume $x \in s$
have $\neg\ 0 < inner\ (z - y)\ (x - y)$
apply (*rule notI*)
apply (*drule closer-point-lemma*)
proof −
assume $\exists u{>}0.\ u \leq 1 \wedge dist\ (y + u *_R (x - y))\ z < dist\ y\ z$
then obtain u **where** $u > 0\ u \leq 1\ dist\ (y + u *_R (x - y))\ z < dist\ y\ z$
by *auto*
then show *False* **using** y[*THEN bspec*[**where** $x{=}y + u *_R (x - y)$]]
using *assms*(1)[*unfolded convex-alt*, *THEN bspec*[**where** $x{=}y$]]
using ‹$x{\in}s$› ‹$y{\in}s$› **by** (*auto simp add*: *dist-commute algebra-simps*)
qed
moreover have $0 < (norm\ (y - z))^2$
using ‹$y{\in}s$› ‹$z{\notin}s$› **by** *auto*
then have $0 < inner\ (y - z)\ (y - z)$
unfolding *power2-norm-eq-inner* **by** *simp*
ultimately show $inner\ (y - z)\ z + (norm\ (y - z))^2\ /\ 2 < inner\ (y - z)\ x$
unfolding *power2-norm-eq-inner* **and** *not-less*
by (*auto simp add*: *field-simps inner-commute inner-diff*)
qed (*insert* ‹$y{\in}s$› ‹$z{\notin}s$›, *auto*)
qed

lemma *separating-hyperplane-closed-0*:
assumes *convex* (s::('a::*euclidean-space*) *set*)
and *closed s*
and $0 \notin s$
shows $\exists a\ b.\ a \neq 0 \wedge 0 < b \wedge (\forall x{\in}s.\ inner\ a\ x > b)$
proof (*cases* $s = \{\}$)
case *True*

```
  have norm ((SOME i. i∈Basis)::'a) = 1 (SOME i. i∈Basis) ≠ (0::'a)
    defer
    apply (subst norm-le-zero-iff[symmetric])
    apply (auto simp: SOME-Basis)
    done
  then show ?thesis
    apply (rule-tac x=SOME i. i∈Basis in exI)
    apply (rule-tac x=1 in exI)
    using True using DIM-positive[where 'a='a]
    apply auto
    done
next
  case False
  then show ?thesis
    using False using separating-hyperplane-closed-point[OF assms]
    apply (elim exE)
    unfolding inner-zero-right
    apply (rule-tac x=a in exI)
    apply (rule-tac x=b in exI)
    apply auto
    done
qed
```

19.14.2 Now set-to-set for closed/compact sets

```
lemma separating-hyperplane-closed-compact:
  fixes s :: 'a::euclidean-space set
  assumes convex s
    and closed s
    and convex t
    and compact t
    and t ≠ {}
    and s ∩ t = {}
  shows ∃ a b. (∀ x∈s. inner a x < b) ∧ (∀ x∈t. inner a x > b)
proof (cases s = {})
  case True
  obtain b where b: b > 0 ∀ x∈t. norm x ≤ b
    using compact-imp-bounded[OF assms(4)] unfolding bounded-pos by auto
  obtain z :: 'a where z: norm z = b + 1
    using vector-choose-size[of b + 1] and b(1) by auto
  then have z ∉ t using b(2)[THEN bspec[where x=z]] by auto
  then obtain a b where ab: inner a z < b ∀ x∈t. b < inner a x
    using separating-hyperplane-closed-point[OF assms(3) compact-imp-closed[OF
assms(4)], of z]
    by auto
  then show ?thesis
    using True by auto
next
  case False
```

```
  then obtain y where y ∈ s by auto
  obtain a b where 0 < b ∀x∈{x − y |x y. x ∈ s ∧ y ∈ t}. b < inner a x
   using separating-hyperplane-closed-point[OF convex-differences[OF assms(1,3)],
of 0]
    using closed-compact-differences[OF assms(2,4)]
    using assms(6) by auto blast
  then have ab: ∀x∈s. ∀y∈t. b + inner a y < inner a x
    apply −
    apply rule
    apply rule
    apply (erule-tac x=x − y in ballE)
    apply (auto simp add: inner-diff)
    done
  def k ≡ SUP x:t. a · x
  show ?thesis
    apply (rule-tac x=−a in exI)
    apply (rule-tac x=−(k + b / 2) in exI)
    apply (intro conjI ballI)
    unfolding inner-minus-left and neg-less-iff-less
  proof −
    fix x assume x ∈ t
    then have inner a x − b / 2 < k
      unfolding k-def
    proof (subst less-cSUP-iff)
      show t ≠ {} by fact
      show bdd-above (op · a ' t)
        using ab[rule-format, of y] ‹y ∈ s›
         by (intro bdd-aboveI2[where M=inner a y − b]) (auto simp: field-simps
intro: less-imp-le)
    qed (auto intro!: bexI[of - x] ‹0<b›)
    then show inner a x < k + b / 2
      by auto
  next
    fix x
    assume x ∈ s
    then have k ≤ inner a x − b
      unfolding k-def
      apply (rule-tac cSUP-least)
      using assms(5)
      using ab[THEN bspec[where x=x]]
      apply auto
      done
    then show k + b / 2 < inner a x
      using ‹0 < b› by auto
  qed
qed

lemma separating-hyperplane-compact-closed:
  fixes s :: 'a::euclidean-space set
```

assumes *convex s*
and *compact s*
and $s \neq \{\}$
and *convex t*
and *closed t*
and $s \cap t = \{\}$
shows $\exists\, a\ b.\ (\forall\, x{\in}s.\ inner\ a\ x < b) \wedge (\forall\, x{\in}t.\ inner\ a\ x > b)$
proof −
obtain *a b* **where** $(\forall\, x{\in}t.\ inner\ a\ x < b) \wedge (\forall\, x{\in}s.\ b < inner\ a\ x)$
using *separating-hyperplane-closed-compact*[*OF assms*(*4*−*5*,*1*−*2*,*3*)] **and** *assms*(*6*)
by *auto*
then show *?thesis*
apply (*rule-tac x*=−*a* **in** *exI*)
apply (*rule-tac x*=−*b* **in** *exI*)
apply *auto*
done
qed

19.14.3 General case without assuming closure and getting non-strict separation

lemma *separating-hyperplane-set-0*:
assumes *convex s* (*0*::*'a*::*euclidean-space*) $\notin$ *s*
shows $\exists\, a.\ a \neq 0 \wedge (\forall\, x{\in}s.\ 0 \leq inner\ a\ x)$
proof −
let *?k* = $\lambda c.\ \{x::'a.\ 0 \leq inner\ c\ x\}$
have ∗: *frontier* (*cball 0 1*) ∩ $\bigcap f \neq \{\}$ **if** *as*: $f \subseteq$ *?k* ' *s finite f* **for** *f*
proof −
obtain *c* **where** *c*: f = *?k* ' *c c* ⊆ *s finite c*
using *finite-subset-image*[*OF as*(*2*,*1*)] **by** *auto*
then obtain *a b* **where** *ab*: $a \neq 0\ 0 < b\ \forall\, x{\in}convex\ hull\ c.\ b < inner\ a\ x$
using *separating-hyperplane-closed-0*[*OF convex-convex-hull*, *of c*]
using *finite-imp-compact-convex-hull*[*OF c*(*3*), *THEN compact-imp-closed*]
and *assms*(*2*)
using *subset-hull*[*of convex*, *OF assms*(*1*), *symmetric*, *of c*]
by *force*
then have $\exists\, x.\ norm\ x = 1 \wedge (\forall\, y{\in}c.\ 0 \leq inner\ y\ x)$
apply (*rule-tac x* = *inverse*(*norm a*) $*_R$ *a* **in** *exI*)
using *hull-subset*[*of c convex*]
unfolding *subset-eq* **and** *inner-scaleR*
by (*auto simp add*: *inner-commute del*: *ballE elim*!: *ballE*)
then show *frontier* (*cball 0 1*) ∩ $\bigcap f \neq \{\}$
unfolding *c*(*1*) *frontier-cball sphere-def dist-norm* **by** *auto*
qed
have *frontier* (*cball 0 1*) ∩ ($\bigcap$ (*?k* ' *s*)) ≠ {}
apply (*rule compact-imp-fip*)
apply (*rule compact-frontier*[*OF compact-cball*])
using ∗ *closed-halfspace-ge*
by *auto*

then obtain x **where** $norm\ x = 1\ \forall y{\in}s.\ x{\in}?k\ y$
unfolding *frontier-cball dist-norm sphere-def* **by** *auto*
then show *?thesis*
by (*metis inner-commute mem-Collect-eq norm-eq-zero zero-neq-one*)
qed

lemma *separating-hyperplane-sets*:
fixes $s\ t :: 'a{::}euclidean\text{-}space\ set$
assumes *convex s*
and *convex t*
and $s \neq \{\}$
and $t \neq \{\}$
and $s \cap t = \{\}$
shows $\exists a\ b.\ a \neq 0 \wedge (\forall x{\in}s.\ inner\ a\ x \leq b) \wedge (\forall x{\in}t.\ inner\ a\ x \geq b)$
proof −
from *separating-hyperplane-set-0*[*OF convex-differences*[*OF assms*(*2*,*1*)]]
obtain a **where** $a \neq 0\ \forall x{\in}\{x - y\ |x\ y.\ x \in t \wedge y \in s\}.\ 0 \leq inner\ a\ x$
using *assms*(*3*−*5*) **by** *fastforce*
then have $*$: $\bigwedge x\ y.\ x \in t \Longrightarrow y \in s \Longrightarrow inner\ a\ y \leq inner\ a\ x$
by (*force simp add*: *inner-diff*)
then have *bdd*: *bdd-above* $((op \cdot a)`s)$
using ‹$t \neq \{\}$› **by** (*auto intro*: *bdd-aboveI2*[*OF* $*$])
show *?thesis*
using ‹$a{\neq}0$›
by (*intro exI*[*of - a*] *exI*[*of - SUP x*:*s. a* · *x*])
(*auto intro*!: *cSUP-upper bdd cSUP-least* ‹$a \neq 0$› ‹$s \neq \{\}$› $*$)
qed

19.15 More convexity generalities

lemma *convex-closure* [*intro*,*simp*]:
fixes $s :: 'a{::}real\text{-}normed\text{-}vector\ set$
assumes *convex s*
shows *convex* (*closure s*)
apply (*rule convexI*)
apply (*unfold closure-sequential*, *elim exE*)
apply (*rule-tac* $x{=}\lambda n.\ u *_R xa\ n + v *_R xb\ n$ **in** *exI*)
apply (*rule*,*rule*)
apply (*rule convexD* [*OF assms*])
apply (*auto del*: *tendsto-const intro*!: *tendsto-intros*)
done

lemma *convex-interior* [*intro*,*simp*]:
fixes $s :: 'a{::}real\text{-}normed\text{-}vector\ set$
assumes *convex s*
shows *convex* (*interior s*)
unfolding *convex-alt Ball-def mem-interior*
apply (*rule*,*rule*,*rule*,*rule*,*rule*,*rule*)
apply (*elim exE conjE*)

proof −
fix *x y u*
assume *u*: *0* ≤ *u u* ≤ (*1*::*real*)
fix *e d*
assume *ed*: *ball x e* ⊆ *s ball y d* ⊆ *s 0*<*d 0*<*e*
show ∃ *e*>*0*. *ball* ((*1* − *u*) ∗*R x* + *u* ∗*R y*) *e* ⊆ *s*
apply (*rule-tac x*=*min d e* **in** *exI*)
apply *rule*
unfolding *subset-eq*
defer
apply *rule*
proof −
fix *z*
assume *z* ∈ *ball* ((*1* − *u*) ∗*R x* + *u* ∗*R y*) (*min d e*)
then have (*1*− *u*) ∗*R* (*z* − *u* ∗*R* (*y* − *x*)) + *u* ∗*R* (*z* + (*1* − *u*) ∗*R* (*y* − *x*))
∈ *s*
apply (*rule-tac assms*[*unfolded convex-alt*, *rule-format*])
using *ed*(*1*,*2*) **and** *u*
unfolding *subset-eq mem-ball Ball-def dist-norm*
apply (*auto simp add*: *algebra-simps*)
done
then show *z* ∈ *s*
using *u* **by** (*auto simp add*: *algebra-simps*)
qed(*insert u ed*(*3*−*4*), *auto*)
qed

lemma *convex-hull-eq-empty*[*simp*]: *convex hull s* = {} ⟷ *s* = {}
using *hull-subset*[*of s convex*] *convex-hull-empty* **by** *auto*

19.16 Moving and scaling convex hulls.

lemma *convex-hull-set-plus*:
convex hull (*s* + *t*) = *convex hull s* + *convex hull t*
unfolding *set-plus-image*
apply (*subst convex-hull-linear-image* [*symmetric*])
apply (*simp add*: *linear-iff scaleR-right-distrib*)
apply (*simp add*: *convex-hull-Times*)
done

lemma *translation-eq-singleton-plus*: (λ*x*. *a* + *x*) ‘ *t* = {*a*} + *t*
unfolding *set-plus-def* **by** *auto*

lemma *convex-hull-translation*:
convex hull ((λ*x*. *a* + *x*) ‘ *s*) = (λ*x*. *a* + *x*) ‘ (*convex hull s*)
unfolding *translation-eq-singleton-plus*
by (*simp only*: *convex-hull-set-plus convex-hull-singleton*)

lemma *convex-hull-scaling*:
convex hull ((λ*x*. *c* ∗*R x*) ‘ *s*) = (λ*x*. *c* ∗*R x*) ‘ (*convex hull s*)

using *linear-scaleR* **by** (*rule convex-hull-linear-image* [*symmetric*])

lemma *convex-hull-affinity*:
convex hull ((λ*x*. *a* + *c* ∗*R* *x*) ‘ *s*) = (λ*x*. *a* + *c* ∗*R* *x*) ‘ (*convex hull s*)
by(*simp only*: *image-image*[*symmetric*] *convex-hull-scaling convex-hull-translation*)

19.17 Convexity of cone hulls

lemma *convex-cone-hull*:
assumes *convex S*
shows *convex* (*cone hull S*)
proof (*rule convexI*)
fix *x y*
assume *xy*: *x* ∈ *cone hull S y* ∈ *cone hull S*
then have *S* ≠ {}
using *cone-hull-empty-iff*[*of S*] **by** *auto*
fix *u v* :: *real*
assume *uv*: *u* ≥ *0 v* ≥ *0 u* + *v* = *1*
then have ∗: *u* ∗*R* *x* ∈ *cone hull S v* ∗*R* *y* ∈ *cone hull S*
using *cone-cone-hull*[*of S*] *xy cone-def*[*of cone hull S*] **by** *auto*
from ∗ **obtain** *cx* :: *real* **and** *xx* **where** *x*: *u* ∗*R* *x* = *cx* ∗*R* *xx cx* ≥ *0 xx* ∈ *S*
using *cone-hull-expl*[*of S*] **by** *auto*
from ∗ **obtain** *cy* :: *real* **and** *yy* **where** *y*: *v* ∗*R* *y* = *cy* ∗*R* *yy cy* ≥ *0 yy* ∈ *S*
using *cone-hull-expl*[*of S*] **by** *auto*
{
assume *cx* + *cy* ≤ *0*
then have *u* ∗*R* *x* = *0* **and** *v* ∗*R* *y* = *0*
using *x y* **by** *auto*
then have *u* ∗*R* *x* + *v* ∗*R* *y* = *0*
by *auto*
then have *u* ∗*R* *x* + *v* ∗*R* *y* ∈ *cone hull S*
using *cone-hull-contains-0*[*of S*] ‹*S* ≠ {}› **by** *auto*
}
moreover
{
assume *cx* + *cy* > *0*
then have (*cx* / (*cx* + *cy*)) ∗*R* *xx* + (*cy* / (*cx* + *cy*)) ∗*R* *yy* ∈ *S*
using *assms mem-convex-alt*[*of S xx yy cx cy*] *x y* **by** *auto*
then have *cx* ∗*R* *xx* + *cy* ∗*R* *yy* ∈ *cone hull S*
using *mem-cone-hull*[*of* (*cx*/(*cx*+*cy*)) ∗*R* *xx* + (*cy*/(*cx*+*cy*)) ∗*R* *yy S cx*+*cy*]
‹*cx*+*cy*>*0*›
by (*auto simp add*: *scaleR-right-distrib*)
then have *u* ∗*R* *x* + *v* ∗*R* *y* ∈ *cone hull S*
using *x y* **by** *auto*
}
moreover have *cx* + *cy* ≤ *0* ∨ *cx* + *cy* > *0* **by** *auto*
ultimately show *u* ∗*R* *x* + *v* ∗*R* *y* ∈ *cone hull S* **by** *blast*
qed

lemma *cone-convex-hull*:
assumes *cone S*
shows *cone* (*convex hull S*)
proof (*cases S* = {})
case *True*
then show *?thesis* **by** *auto*
next
case *False*
then have ∗: *0* ∈ *S* ∧ (∀ *c*. *c* > *0* ⟶ *op* ∗*R* *c* ‘ *S* = *S*)
using *cone-iff*[*of S*] *assms* **by** *auto*
{
fix *c* :: *real*
assume *c* > *0*
then have *op* ∗*R* *c* ‘ (*convex hull S*) = *convex hull* (*op* ∗*R* *c* ‘ *S*)
using *convex-hull-scaling*[*of* - *S*] **by** *auto*
also have ... = *convex hull S*
using ∗ ‹*c* > *0*› **by** *auto*
finally have *op* ∗*R* *c* ‘ (*convex hull S*) = *convex hull S*
by *auto*
}
then have *0* ∈ *convex hull S* ⋀*c*. *c* > *0* ⟹ (*op* ∗*R* *c* ‘ (*convex hull S*)) = (*convex hull S*)
using ∗ *hull-subset*[*of S convex*] **by** *auto*
then show *?thesis*
using ‹*S* ≠ {}› *cone-iff*[*of convex hull S*] **by** *auto*
qed

19.18 Convex set as intersection of halfspaces

lemma *convex-halfspace-intersection*:
fixes *s* :: (*'a*::*euclidean-space*) *set*
assumes *closed s convex s*
shows *s* = ⋂{*h*. *s* ⊆ *h* ∧ (∃ *a b*. *h* = {*x*. *inner a x* ≤ *b*})}
apply (*rule set-eqI*)
apply *rule*
unfolding *Inter-iff Ball-def mem-Collect-eq*
apply (*rule*,*rule*,*erule conjE*)
proof −
fix *x*
assume ∀ *xa*. *s* ⊆ *xa* ∧ (∃ *a b*. *xa* = {*x*. *inner a x* ≤ *b*}) ⟶ *x* ∈ *xa*
then have ∀ *a b*. *s* ⊆ {*x*. *inner a x* ≤ *b*} ⟶ *x* ∈ {*x*. *inner a x* ≤ *b*}
by *blast*
then show *x* ∈ *s*
apply (*rule-tac ccontr*)
apply (*drule separating-hyperplane-closed-point*[*OF assms*(*2*,*1*)])
apply (*erule exE*)+
apply (*erule-tac x*=−*a* **in** *allE*)
apply (*erule-tac x*=−*b* **in** *allE*)
apply *auto*

done
qed *auto*

19.19 Radon's theorem (from Lars Schewe)

lemma *radon-ex-lemma*:
assumes *finite c affine-dependent c*
shows $\exists u.\ setsum\ u\ c = 0 \wedge (\exists v{\in}c.\ u\ v \neq 0) \wedge setsum\ (\lambda v.\ u\ v *_R v)\ c = 0$
proof −
from *assms*(2)[*unfolded affine-dependent-explicit*]
obtain *s u* **where**
finite s s $\subseteq$ *c setsum u s* = 0 $\exists v{\in}s.\ u\ v \neq 0$ $(\sum v{\in}s.\ u\ v *_R v) = 0$
by *blast*
then show *?thesis*
apply (*rule-tac x*=$\lambda v.$ *if v*$\in$*s then u v else 0* **in** *exI*)
unfolding *if-smult scaleR-zero-left* **and** *setsum.inter-restrict*[*OF assms*(1), *symmetric*]
apply (*auto simp add*: *Int-absorb1*)
done
qed

lemma *radon-s-lemma*:
assumes *finite s*
and *setsum f s* = (0::*real*)
shows *setsum f* {*x*$\in$*s*. 0 < *f x*} = − *setsum f* {*x*$\in$*s*. *f x* < 0}
proof −
have ∗: $\bigwedge x.$ (*if f x* < 0 *then f x else* 0) + (*if* 0 < *f x then f x else* 0) = *f x*
by *auto*
show *?thesis*
unfolding *add-eq-0-iff*[*symmetric*] **and** *setsum.inter-filter*[*OF assms*(1)]
and *setsum.distrib*[*symmetric*] **and** ∗
using *assms*(2)
by *assumption*
qed

lemma *radon-v-lemma*:
assumes *finite s*
and *setsum f s* = 0
and $\forall x.$ *g x* = (0::*real*) ⟶ *f x* = (0::'*a*::*euclidean-space*)
shows (*setsum f* {*x*$\in$*s*. 0 < *g x*}) = − *setsum f* {*x*$\in$*s*. *g x* < 0}
proof −
have ∗: $\bigwedge x.$ (*if* 0 < *g x then f x else* 0) + (*if g x* < 0 *then f x else* 0) = *f x*
using *assms*(3) **by** *auto*
show *?thesis*
unfolding *eq-neg-iff-add-eq-0* **and** *setsum.inter-filter*[*OF assms*(1)]
and *setsum.distrib*[*symmetric*] **and** ∗
using *assms*(2)
apply *assumption*
done

qed

lemma *radon-partition*:
assumes *finite c affine-dependent c*
shows $\exists m\ p.\ m \cap p = \{\} \wedge m \cup p = c \wedge (\textit{convex hull } m) \cap (\textit{convex hull } p) \neq \{\}$
proof −
obtain *u v* **where** *uv*: *setsum u c = 0 v∈c u v ≠ 0* $(\sum v{\in}c.\ u\ v *_R v) = 0$
using *radon-ex-lemma*[*OF assms*] **by** *auto*
have *fin*: *finite* $\{x \in c.\ 0 < u\ x\}$ *finite* $\{x \in c.\ 0 > u\ x\}$
using *assms*(*1*) **by** *auto*
def $z \equiv \textit{inverse}\ (\textit{setsum}\ u\ \{x{\in}c.\ u\ x > 0\}) *_R \textit{setsum}\ (\lambda x.\ u\ x *_R x)\ \{x{\in}c.\ u\ x > 0\}$
have $\textit{setsum}\ u\ \{x \in c.\ 0 < u\ x\} \neq 0$
proof (*cases u v ≥ 0*)
case *False*
then have *u v < 0* **by** *auto*
then show *?thesis*
proof (*cases* $\exists w{\in}\{x \in c.\ 0 < u\ x\}.\ u\ w > 0$)
case *True*
then show *?thesis*
using *setsum-nonneg-eq-0-iff*[*of - u*, *OF fin*(*1*)] **by** *auto*
next
case *False*
then have $\textit{setsum}\ u\ c \leq \textit{setsum}\ (\lambda x.\ \textit{if } x{=}v \textit{ then } u\ v \textit{ else } 0)\ c$
apply (*rule-tac setsum-mono*)
apply *auto*
done
then show *?thesis*
unfolding *setsum.delta*[*OF assms*(*1*)] **using** *uv*(*2*) **and** ‹*u v < 0*› **and** *uv*(*1*) **by** *auto*
qed
qed (*insert setsum-nonneg-eq-0-iff*[*of - u*, *OF fin*(*1*)] *uv*(*2−3*), *auto*)

then have ∗: $\textit{setsum}\ u\ \{x{\in}c.\ u\ x > 0\} > 0$
unfolding *less-le*
apply (*rule-tac conjI*)
apply (*rule-tac setsum-nonneg*)
apply *auto*
done
moreover have $\textit{setsum}\ u\ (\{x \in c.\ 0 < u\ x\} \cup \{x \in c.\ u\ x < 0\}) = \textit{setsum}\ u\ c$
$(\sum x{\in}\{x \in c.\ 0 < u\ x\} \cup \{x \in c.\ u\ x < 0\}.\ u\ x *_R x) = (\sum x{\in}c.\ u\ x *_R x)$
using *assms*(*1*)
apply (*rule-tac*[!] *setsum.mono-neutral-left*)
apply *auto*
done
then have $\textit{setsum}\ u\ \{x \in c.\ 0 < u\ x\} = -\ \textit{setsum}\ u\ \{x \in c.\ 0 > u\ x\}$
$(\sum x{\in}\{x \in c.\ 0 < u\ x\}.\ u\ x *_R x) = -\ (\sum x{\in}\{x \in c.\ 0 > u\ x\}.\ u\ x *_R x)$
unfolding *eq-neg-iff-add-eq-0*

using *uv(1,4)*
by (*auto simp add*: *setsum.union-inter-neutral*[*OF fin*, *symmetric*])
moreover have $\forall x{\in}\{v \in c.\ u\ v < 0\}.\ 0 \leq inverse\ (setsum\ u\ \{x \in c.\ 0 < u\ x\}) * - u\ x$
apply *rule*
apply (*rule mult-nonneg-nonneg*)
using $*$
apply *auto*
done
ultimately have $z \in convex\ hull\ \{v \in c.\ u\ v \leq 0\}$
unfolding *convex-hull-explicit mem-Collect-eq*
apply (*rule-tac* $x{=}\{v \in c.\ u\ v < 0\}$ **in** *exI*)
apply (*rule-tac* $x{=}\lambda y.\ inverse\ (setsum\ u\ \{x{\in}c.\ u\ x > 0\}) * - u\ y$ **in** *exI*)
using *assms(1)* **unfolding** *scaleR-scaleR*[*symmetric*] *scaleR-right.setsum* [*symmetric*] **and** *z-def*
apply (*auto simp add*: *setsum-negf setsum-right-distrib*[*symmetric*])
done
moreover have $\forall x{\in}\{v \in c.\ 0 < u\ v\}.\ 0 \leq inverse\ (setsum\ u\ \{x \in c.\ 0 < u\ x\}) * u\ x$
apply *rule*
apply (*rule mult-nonneg-nonneg*)
using $*$
apply *auto*
done
then have $z \in convex\ hull\ \{v \in c.\ u\ v > 0\}$
unfolding *convex-hull-explicit mem-Collect-eq*
apply (*rule-tac* $x{=}\{v \in c.\ 0 < u\ v\}$ **in** *exI*)
apply (*rule-tac* $x{=}\lambda y.\ inverse\ (setsum\ u\ \{x{\in}c.\ u\ x > 0\}) * u\ y$ **in** *exI*)
using *assms(1)*
unfolding *scaleR-scaleR*[*symmetric*] *scaleR-right.setsum* [*symmetric*] **and** *z-def*
using $*$
apply (*auto simp add*: *setsum-negf setsum-right-distrib*[*symmetric*])
done
ultimately show *?thesis*
apply (*rule-tac* $x{=}\{v{\in}c.\ u\ v \leq 0\}$ **in** *exI*)
apply (*rule-tac* $x{=}\{v{\in}c.\ u\ v > 0\}$ **in** *exI*)
apply *auto*
done
qed

lemma *radon*:
assumes *affine-dependent c*
obtains *m p* **where** $m \subseteq c\ p \subseteq c\ m \cap p = \{\}\ (convex\ hull\ m) \cap (convex\ hull\ p) \neq \{\}$
proof −
from *assms*[*unfolded affine-dependent-explicit*]
obtain *s u* **where**
finite s $s \subseteq c$ *setsum u s* $= 0\ \exists v{\in}s.\ u\ v \neq 0\ (\sum v{\in}s.\ u\ v *_R v) = 0$
by *blast*

```
  then have ∗: finite s affine-dependent s and s: s ⊆ c
    unfolding affine-dependent-explicit by auto
  from radon-partition[OF ∗]
  obtain m p where m ∩ p = {} m ∪ p = s convex hull m ∩ convex hull p ≠ {}
    by blast
  then show ?thesis
    apply (rule-tac that[of p m])
    using s
    apply auto
    done
qed
```

19.20 Helly's theorem

```
lemma helly-induct:
  fixes f :: 'a::euclidean-space set set
  assumes card f = n
    and n ≥ DIM('a) + 1
    and ∀ s∈f. convex s ∀ t⊆f. card t = DIM('a) + 1 ⟶ ⋂ t ≠ {}
  shows ⋂ f ≠ {}
  using assms
proof (induct n arbitrary: f)
  case 0
  then show ?case by auto
next
  case (Suc n)
  have finite f
    using ‹card f = Suc n› by (auto intro: card-ge-0-finite)
  show ⋂ f ≠ {}
    apply (cases n = DIM('a))
    apply (rule Suc(5)[rule-format])
    unfolding ‹card f = Suc n›
  proof −
    assume ng: n ≠ DIM('a)
    then have ∃ X. ∀ s∈f. X s ∈ ⋂ (f − {s})
      apply (rule-tac bchoice)
      unfolding ex-in-conv
      apply (rule, rule Suc(1)[rule-format])
      unfolding card-Diff-singleton-if[OF ‹finite f›] ‹card f = Suc n›
      defer
      defer
      apply (rule Suc(4)[rule-format])
      defer
      apply (rule Suc(5)[rule-format])
      using Suc(3) ‹finite f›
      apply auto
      done
    then obtain X where X: ∀ s∈f. X s ∈ ⋂ (f − {s}) by auto
    show ?thesis
```

```
  proof (cases inj-on X f)
    case False
    then obtain s t where st: s≠t s∈f t∈f X s = X t
      unfolding inj-on-def by auto
    then have ∗: ⋂f = ⋂(f − {s}) ∩ ⋂(f − {t}) by auto
    show ?thesis
      unfolding ∗
      unfolding ex-in-conv[symmetric]
      apply (rule-tac x=X s in exI)
      apply rule
      apply (rule X[rule-format])
      using X st
      apply auto
      done
  next
    case True
    then obtain m p where mp: m ∩ p = {} m ∪ p = X ‘ f convex hull m ∩
convex hull p ≠ {}
      using radon-partition[of X ‘ f] and affine-dependent-biggerset[of X ‘ f]
      unfolding card-image[OF True] and ‹card f = Suc n›
      using Suc(3) ‹finite f› and ng
      by auto
    have m ⊆ X ‘ f p ⊆ X ‘ f
      using mp(2) by auto
    then obtain g h where gh:m = X ‘ g p = X ‘ h g ⊆ f h ⊆ f
      unfolding subset-image-iff by auto
    then have f ∪ (g ∪ h) = f by auto
    then have f: f = g ∪ h
      using inj-on-Un-image-eq-iff[of X f g ∪ h] and True
      unfolding mp(2)[unfolded image-Un[symmetric] gh]
      by auto
    have ∗: g ∩ h = {}
      using mp(1)
      unfolding gh
      using inj-on-image-Int[OF True gh(3,4)]
      by auto
    have convex hull (X ‘ h) ⊆ ⋂g convex hull (X ‘ g) ⊆ ⋂h
      apply (rule-tac [!] hull-minimal)
      using Suc gh(3−4)
      unfolding subset-eq
      apply (rule-tac [2] convex-Inter, rule-tac [4] convex-Inter)
      apply rule
      prefer 3
      apply rule
    proof −
      fix x
      assume x ∈ X ‘ g
      then obtain y where y ∈ g x = X y
        unfolding image-iff ..
```

```
      then show x ∈ ⋂h
        using X[THEN bspec[where x=y]] using * f by auto
    next
      fix x
      assume x ∈ X ' h
      then obtain y where y ∈ h x = X y
        unfolding image-iff ..
      then show x ∈ ⋂g
        using X[THEN bspec[where x=y]] using * f by auto
    qed auto
    then show ?thesis
      unfolding f using mp(3)[unfolded gh] by blast
  qed
 qed auto
qed

lemma helly:
 fixes f :: 'a::euclidean-space set set
 assumes card f ≥ DIM('a) + 1 ∀s∈f. convex s
   and ∀t⊆f. card t = DIM('a) + 1 ⟶ ⋂t ≠ {}
 shows ⋂f ≠ {}
 apply (rule helly-induct)
 using assms
 apply auto
 done
```

19.21 Homeomorphism of all convex compact sets with nonempty interior

```
lemma compact-frontier-line-lemma:
 fixes s :: 'a::euclidean-space set
 assumes compact s
   and 0 ∈ s
   and x ≠ 0
 obtains u where 0 ≤ u and (u *R x) ∈ frontier s ∀v>u. (v *R x) ∉ s
proof −
 obtain b where b: b > 0 ∀x∈s. norm x ≤ b
   using compact-imp-bounded[OF assms(1), unfolded bounded-pos] by auto
 let ?A = {y. ∃u. 0 ≤ u ∧ u ≤ b / norm(x) ∧ (y = u *R x)}
 have A: ?A = (λu. u *R x) ' {0 .. b / norm x}
   by auto
 have *: ⋀x A B. x∈A ⟹ x∈B ⟹ A∩B ≠ {} by blast
 have compact ?A
   unfolding A
   apply (rule compact-continuous-image)
   apply (rule continuous-at-imp-continuous-on)
   apply rule
   apply (intro continuous-intros)
   apply (rule compact-Icc)
```

```
    done
  moreover have {y. ∃ u≥0. u ≤ b / norm x ∧ y = u *R x} ∩ s ≠ {}
    apply(rule *[OF - assms(2)])
    unfolding mem-Collect-eq
    using ‹b > 0› assms(3)
    apply auto
    done
  ultimately obtain u y where obt: u≥0 u ≤ b / norm x y = u *R x
    y ∈ ?A y ∈ s ∀ z∈?A ∩ s. dist 0 z ≤ dist 0 y
     using distance-attains-sup[OF compact-Int[OF - assms(1), of ?A], of 0] by
blast
  have norm x > 0
    using assms(3)[unfolded zero-less-norm-iff[symmetric]] by auto
  {
    fix v
    assume as: v > u v *R x ∈ s
    then have v ≤ b / norm x
      using b(2)[rule-format, OF as(2)]
      using ‹u≥0›
      unfolding pos-le-divide-eq[OF ‹norm x > 0›]
      by auto
    then have norm (v *R x) ≤ norm y
      apply (rule-tac obt(6)[rule-format, unfolded dist-0-norm])
      apply (rule IntI)
      defer
      apply (rule as(2))
      unfolding mem-Collect-eq
      apply (rule-tac x=v in exI)
      using as(1) ‹u≥0›
      apply (auto simp add: field-simps)
      done
    then have False
      unfolding obt(3) using ‹u≥0› ‹norm x > 0› ‹v > u›
      by (auto simp add:field-simps)
  } note u-max = this

  have u *R x ∈ frontier s
    unfolding frontier-straddle
    apply (rule,rule,rule)
    apply (rule-tac x=u *R x in bexI)
    unfolding obt(3)[symmetric]
    prefer 3
    apply (rule-tac x=(u + (e / 2) / norm x) *R x in exI)
    apply (rule, rule)
  proof −
    fix e
    assume e > 0 and as: (u + e / 2 / norm x) *R x ∈ s
    then have u + e / 2 / norm x > u
      using ‹norm x > 0› by (auto simp del:zero-less-norm-iff)
```

```
    then show False using u-max[OF - as] by auto
  qed (insert ‹y∈s›, auto simp add: dist-norm scaleR-left-distrib obt(3))
  then show ?thesis by(metis that[of u] u-max obt(1))
qed

lemma starlike-compact-projective:
  assumes compact s
    and cball (0::'a::euclidean-space) 1 ⊆ s
    and ∀x∈s. ∀u. 0 ≤ u ∧ u < 1 ⟶ u *R x ∈ s − frontier s
  shows s homeomorphic (cball (0::'a::euclidean-space) 1)
proof −
  have fs: frontier s ⊆ s
    apply (rule frontier-subset-closed)
    using compact-imp-closed[OF assms(1)]
    apply simp
    done
  def pi ≡ λx::'a. inverse (norm x) *R x
  have 0 ∉ frontier s
    unfolding frontier-straddle
    apply (rule notI)
    apply (erule-tac x=1 in allE)
    using assms(2)[unfolded subset-eq Ball-def mem-cball]
    apply auto
    done
  have injpi: ⋀x y. pi x = pi y ∧ norm x = norm y ⟷ x = y
    unfolding pi-def by auto

  have contpi: continuous-on (UNIV − {0}) pi
    apply (rule continuous-at-imp-continuous-on)
    apply rule unfolding pi-def
    apply (intro continuous-intros)
    apply simp
    done
  def sphere ≡ {x::'a. norm x = 1}
  have pi: ⋀x. x ≠ 0 ⟹ pi x ∈ sphere ⋀x u. u>0 ⟹ pi (u *R x) = pi x
    unfolding pi-def sphere-def by auto

  have 0 ∈ s
    using assms(2) and centre-in-cball[of 0 1] by auto
  have front-smul: ∀x∈frontier s. ∀u≥0. u *R x ∈ s ⟷ u ≤ 1
  proof (rule,rule,rule)
    fix x and u :: real
    assume x: x ∈ frontier s and 0 ≤ u
    then have x ≠ 0
      using ‹0 ∉ frontier s› by auto
    obtain v where v: 0 ≤ v v *R x ∈ frontier s ∀w>v. w *R x ∉ s
      using compact-frontier-line-lemma[OF assms(1) ‹0∈s› ‹x≠0›] by auto
    have v = 1
      apply (rule ccontr)
```

```
      unfolding neq-iff
      apply (erule disjE)
    proof −
      assume v < 1
      then show False
          using v(3)[THEN spec[where x=1]] using x fs by (simp add: pth-1
subset-iff)
    next
      assume v > 1
      then show False
     using assms(3)[THEN bspec[where x=v *R x], THEN spec[where x=inverse
v]]
        using v and x and fs
        unfolding inverse-less-1-iff by auto
    qed
    show u *R x ∈ s ⟷ u ≤ 1
      apply rule
      using v(3)[unfolded ‹v=1›, THEN spec[where x=u]]
    proof −
      assume u ≤ 1
      then show u *R x ∈ s
      apply (cases u = 1)
        using assms(3)[THEN bspec[where x=x], THEN spec[where x=u]]
        using ‹0≤u› and x and fs
        by auto
    qed auto
  qed

  have ∃ surf. homeomorphism (frontier s) sphere pi surf
    apply (rule homeomorphism-compact)
    apply (rule compact-frontier[OF assms(1)])
    apply (rule continuous-on-subset[OF contpi])
    defer
    apply (rule set-eqI)
    apply rule
    unfolding inj-on-def
    prefer 3
    apply(rule,rule,rule)
  proof −
    fix x
    assume x ∈ pi ‘ frontier s
    then obtain y where y ∈ frontier s x = pi y by auto
    then show x ∈ sphere
      using pi(1)[of y] and ‹0 ∉ frontier s› by auto
  next
    fix x
    assume x ∈ sphere
    then have norm x = 1 x ≠ 0
      unfolding sphere-def by auto
```

then obtain u **where** $0 \leq u$ $u *_R x \in$ *frontier* s $\forall v{>}u.$ $v *_R x \notin s$
using *compact-frontier-line-lemma*[*OF assms*(*1*) ‹*0*∈*s*›, *of x*] **by** *auto*
then show $x \in$ *pi* ‘ *frontier s*
unfolding *image-iff le-less pi-def*
apply (*rule-tac x*=$u *_R x$ **in** *bexI*)
using ‹*norm x* = *1*› ‹*0* ∉ *frontier s*›
apply *auto*
done
next
fix x y
assume *as*: $x \in$ *frontier* s $y \in$ *frontier* s *pi* $x =$ *pi* y
then have *xys*: $x \in s$ $y \in s$
using *fs* **by** *auto*
from *as*(*1*,*2*) **have** *nor*: *norm* $x \neq 0$ *norm* $y \neq 0$
using ‹*0*∉*frontier s*› **by** *auto*
from *nor* **have** *x*: $x =$ *norm* $x *_R$ ((*inverse* (*norm* y)) $*_R$ y)
unfolding *as*(*3*)[*unfolded pi-def*, *symmetric*] **by** *auto*
from *nor* **have** *y*: $y =$ *norm* $y *_R$ ((*inverse* (*norm* x)) $*_R$ x)
unfolding *as*(*3*)[*unfolded pi-def*] **by** *auto*
have *0* ≤ *norm* y ∗ *inverse* (*norm* x) **and** *0* ≤ *norm* x ∗ *inverse* (*norm* y)
using *nor*
apply *auto*
done
then have *norm* $x =$ *norm* y
apply −
apply (*rule ccontr*)
unfolding *neq-iff*
using x y **and** *front-smul*[*THEN bspec*, *OF as*(*1*), *THEN spec*[**where** *x*=*norm* y ∗ (*inverse* (*norm* x))]]
using *front-smul*[*THEN bspec*, *OF as*(*2*), *THEN spec*[**where** *x*=*norm* x ∗ (*inverse* (*norm* y))]]
using *xys nor*
apply (*auto simp add*: *field-simps*)
done
then show $x = y$
apply (*subst injpi*[*symmetric*])
using *as*(*3*)
apply *auto*
done
qed (*insert* ‹*0* ∉ *frontier s*›, *auto*)
then obtain *surf* **where**
surf: ∀ *x*∈*frontier s*. *surf* (*pi x*) = *x* *pi* ‘ *frontier s* = *sphere continuous-on* (*frontier s*) *pi*
∀ *y*∈*sphere*. *pi* (*surf y*) = *y surf* ‘ *sphere* = *frontier s continuous-on sphere surf*
unfolding *homeomorphism-def* **by** *auto*

have *cont-surfpi*: *continuous-on* (*UNIV* − {*0*}) (*surf* ∘ *pi*)
apply (*rule continuous-on-compose*)
apply (*rule contpi*)

```
  apply (rule continuous-on-subset[of sphere])
  apply (rule surf(6))
  using pi(1)
  apply auto
  done

{
  fix x
  assume as: x ∈ cball (0::'a) 1
  have norm x *R surf (pi x) ∈ s
  proof (cases x=0 ∨ norm x = 1)
    case False
    then have pi x ∈ sphere norm x < 1
      using pi(1)[of x] as by(auto simp add: dist-norm)
    then show ?thesis
      apply (rule-tac assms(3)[rule-format, THEN DiffD1])
      apply (rule-tac fs[unfolded subset-eq, rule-format])
      unfolding surf(5)[symmetric]
      apply auto
      done
  next
    case True
    then show ?thesis
      apply rule
      defer
      unfolding pi-def
      apply (rule fs[unfolded subset-eq, rule-format])
      unfolding surf(5)[unfolded sphere-def, symmetric]
      using ‹0∈s›
      apply auto
      done
  qed
} note hom = this

{
  fix x
  assume x ∈ s
  then have x ∈ (λx. norm x *R surf (pi x)) ‘ cball 0 1
  proof (cases x = 0)
    case True
    show ?thesis
      unfolding image-iff True
      apply (rule-tac x=0 in bexI)
      apply auto
      done
  next
    let ?a = inverse (norm (surf (pi x)))
    case False
    then have invn: inverse (norm x) ≠ 0 by auto
```

```
    from False have pix: pi x∈sphere using pi(1) by auto
    then have pi (surf (pi x)) = pi x
      apply (rule-tac surf(4)[rule-format])
      apply assumption
      done
    then have **: norm x *R (?a *R surf (pi x)) = x
      apply (rule-tac scaleR-left-imp-eq[OF invn])
      unfolding pi-def
      using invn
      apply auto
      done
    then have *: ?a * norm x > 0 and ?a > 0 ?a ≠ 0
      using surf(5) ‹0∉frontier s›
      apply -
      apply (rule mult-pos-pos)
      using False[unfolded zero-less-norm-iff[symmetric]]
      apply auto
      done
    have norm (surf (pi x)) ≠ 0
      using ** False by auto
    then have norm x = norm ((?a * norm x) *R surf (pi x))
      unfolding norm-scaleR abs-mult abs-norm-cancel abs-of-pos[OF ‹?a > 0›]
by auto
    moreover have pi x = pi ((inverse (norm (surf (pi x))) * norm x) *R surf
(pi x))
      unfolding pi(2)[OF *] surf(4)[rule-format, OF pix] ..
    moreover have surf (pi x) ∈ frontier s
      using surf(5) pix by auto
    then have dist 0 (inverse (norm (surf (pi x))) *R x) ≤ 1
      unfolding dist-norm
      using ** and *
       using front-smul[THEN bspec[where x=surf (pi x)], THEN spec[where
x=norm x * ?a]]
      using False ‹x∈s›
      by (auto simp add: field-simps)
    ultimately show ?thesis
      unfolding image-iff
      apply (rule-tac x=inverse (norm (surf(pi x))) *R x in bexI)
      apply (subst injpi[symmetric])
      unfolding abs-mult abs-norm-cancel abs-of-pos[OF ‹?a > 0›]
      unfolding pi(2)[OF ‹?a > 0›]
      apply auto
      done
  qed
 } note hom2 = this

 show ?thesis
  apply (subst homeomorphic-sym)
  apply (rule homeomorphic-compact[where f=λx. norm x *R surf (pi x)])
```

```
    apply (rule compact-cball)
    defer
    apply (rule set-eqI)
    apply rule
    apply (erule imageE)
    apply (drule hom)
    prefer 4
    apply (rule continuous-at-imp-continuous-on)
    apply rule
    apply (rule-tac [3] hom2)
  proof −
    fix x :: 'a
    assume as: x ∈ cball 0 1
    then show continuous (at x) (λx. norm x *R surf (pi x))
    proof (cases x = 0)
      case False
      then show ?thesis
        apply (intro continuous-intros)
        using cont-surfpi
       unfolding continuous-on-eq-continuous-at[OF open-delete[OF open-UNIV]]
o-def
        apply auto
        done
    next
      case True
      obtain B where B: ∀x∈s. norm x ≤ B
        using compact-imp-bounded[OF assms(1)] unfolding bounded-iff by auto
      then have B > 0
        using assms(2)
        unfolding subset-eq
        apply (erule-tac x=SOME i. i∈Basis in ballE)
        defer
        apply (erule-tac x=SOME i. i∈Basis in ballE)
        unfolding Ball-def mem-cball dist-norm
        using DIM-positive[where 'a='a]
        apply (auto simp: SOME-Basis)
        done
      show ?thesis
        unfolding True continuous-at Lim-at
        apply(rule,rule)
        apply(rule-tac x=e / B in exI)
        apply rule
        apply (rule divide-pos-pos)
        prefer 3
        apply(rule,rule,erule conjE)
          unfolding norm-zero scaleR-zero-left dist-norm diff-0-right norm-scaleR
abs-norm-cancel
      proof −
        fix e and x :: 'a
```

```
      assume as: norm x < e / B 0 < norm x e > 0
      then have surf (pi x) ∈ frontier s
        using pi(1)[of x] unfolding surf(5)[symmetric] by auto
      then have norm (surf (pi x)) ≤ B
        using B fs by auto
      then have norm x * norm (surf (pi x)) ≤ norm x * B
        using as(2) by auto
      also have ... < e / B * B
        apply (rule mult-strict-right-mono)
        using as(1) ‹B>0›
        apply auto
        done
      also have ... = e using ‹B > 0› by auto
      finally show norm x * norm (surf (pi x)) < e .
    qed (insert ‹B>0›, auto)
  qed
 next
  {
    fix x
    assume as: surf (pi x) = 0
    have x = 0
    proof (rule ccontr)
      assume x ≠ 0
      then have pi x ∈ sphere
        using pi(1) by auto
      then have surf (pi x) ∈ frontier s
        using surf(5) by auto
      then show False
        using ‹0∉frontier s› unfolding as by simp
    qed
  } note surf-0 = this
  show inj-on (λx. norm x *R surf (pi x)) (cball 0 1)
    unfolding inj-on-def
  proof (rule,rule,rule)
    fix x y
    assume as: x ∈ cball 0 1 y ∈ cball 0 1 norm x *R surf (pi x) = norm y *R
surf (pi y)
    then show x = y
    proof (cases x=0 ∨ y=0)
      case True
      then show ?thesis
        using as by (auto elim: surf-0)
    next
      case False
      then have pi (surf (pi x)) = pi (surf (pi y))
        using as(3)
        using pi(2)[of norm x surf (pi x)] pi(2)[of norm y surf (pi y)]
        by auto
      moreover have pi x ∈ sphere pi y ∈ sphere
```

```
        using pi(1) False by auto
      ultimately have *: pi x = pi y
        using surf(4)[THEN bspec[where x=pi x]] surf(4)[THEN bspec[where
x=pi y]]
        by auto
      moreover have norm x = norm y
        using as(3)[unfolded *] using False
        by (auto dest:surf-0)
      ultimately show ?thesis
        using injpi by auto
    qed
  qed
 qed auto
qed

lemma homeomorphic-convex-compact-lemma:
 fixes s :: 'a::euclidean-space set
 assumes convex s
   and compact s
   and cball 0 1 ⊆ s
 shows s homeomorphic (cball (0::'a) 1)
proof (rule starlike-compact-projective[OF assms(2−3)], clarify)
 fix x u
 assume x ∈ s and 0 ≤ u and u < (1::real)
 have open (ball (u *R x) (1 − u))
   by (rule open-ball)
 moreover have u *R x ∈ ball (u *R x) (1 − u)
   unfolding centre-in-ball using ‹u < 1› by simp
 moreover have ball (u *R x) (1 − u) ⊆ s
 proof
   fix y
   assume y ∈ ball (u *R x) (1 − u)
   then have dist (u *R x) y < 1 − u
     unfolding mem-ball .
   with ‹u < 1› have inverse (1 − u) *R (y − u *R x) ∈ cball 0 1
     by (simp add: dist-norm inverse-eq-divide norm-minus-commute)
   with assms(3) have inverse (1 − u) *R (y − u *R x) ∈ s ..
   with assms(1) have (1 − u) *R ((y − u *R x) /R (1 − u)) + u *R x ∈ s
     using ‹x ∈ s› ‹0 ≤ u› ‹u < 1› [THEN less-imp-le] by (rule convexD-alt)
   then show y ∈ s using ‹u < 1›
     by simp
 qed
 ultimately have u *R x ∈ interior s ..
 then show u *R x ∈ s − frontier s
   using frontier-def and interior-subset by auto
qed

lemma homeomorphic-convex-compact-cball:
 fixes e :: real
```

and $s :: 'a::euclidean\text{-}space\ set$
assumes *convex s*
and *compact s*
and $interior\ s \neq \{\}$
and $e > 0$
shows $s\ homeomorphic\ (cball\ (b::'a)\ e)$
proof −
obtain a **where** $a \in interior\ s$
using *assms*(3) **by** *auto*
then obtain d **where** $d > 0$ **and** d: $cball\ a\ d \subseteq s$
unfolding *mem-interior-cball* **by** *auto*
let $?d = inverse\ d$ **and** $?n = 0::'a$
have $cball\ ?n\ 1 \subseteq (\lambda x.\ inverse\ d *_R (x - a))\ `\ s$
apply *rule*
apply (*rule-tac* $x{=}d *_R x + a$ **in** *image-eqI*)
defer
apply (*rule d*[*unfolded subset-eq*, *rule-format*])
using ‹$d > 0$›
unfolding *mem-cball dist-norm*
apply (*auto simp add*: *mult-right-le-one-le*)
done
then have $(\lambda x.\ inverse\ d *_R (x - a))\ `\ s\ homeomorphic\ cball\ ?n\ 1$
using *homeomorphic-convex-compact-lemma*[*of* $(\lambda x.\ ?d *_R -a + ?d *_R x)\ `\ s$,
OF convex-affinity compact-affinity]
using *assms*(1,2)
by (*auto simp add*: *scaleR-right-diff-distrib*)
then show *?thesis*
apply (*rule-tac homeomorphic-trans*[*OF - homeomorphic-balls*(2)[*of 1 - ?n*]])
apply (*rule homeomorphic-trans*[*OF homeomorphic-affinity*[*of ?d s ?d* $*_R -a$]])
using ‹$d{>}0$› ‹$e{>}0$›
apply (*auto simp add*: *scaleR-right-diff-distrib*)
done
qed

lemma *homeomorphic-convex-compact*:
fixes $s :: 'a::euclidean\text{-}space\ set$
and $t :: 'a\ set$
assumes *convex s compact s* $interior\ s \neq \{\}$
and *convex t compact t* $interior\ t \neq \{\}$
shows *s homeomorphic t*
using *assms*
by (*meson zero-less-one homeomorphic-trans homeomorphic-convex-compact-cball homeomorphic-sym*)

19.22 Epigraphs of convex functions

definition $epigraph\ s\ (f :: - \Rightarrow real) = \{xy.\ fst\ xy \in s \wedge f\ (fst\ xy) \leq snd\ xy\}$

lemma *mem-epigraph*: $(x, y) \in epigraph\ s\ f \longleftrightarrow x \in s \wedge f\ x \leq y$

unfolding *epigraph-def* **by** *auto*

lemma *convex-epigraph*: *convex* (*epigraph s f*) ⟷ *convex-on s f* ∧ *convex s*
unfolding *convex-def convex-on-def*
unfolding *Ball-def split-paired-All epigraph-def*
unfolding *mem-Collect-eq fst-conv snd-conv fst-add snd-add fst-scaleR snd-scaleR Ball-def*[*symmetric*]
apply *safe*
defer
apply (*erule-tac x*=*x* **in** *allE*)
apply (*erule-tac x*=*f x* **in** *allE*)
apply *safe*
apply (*erule-tac x*=*xa* **in** *allE*)
apply (*erule-tac x*=*f xa* **in** *allE*)
prefer *3*
apply (*rule-tac y*=*u* ∗ *f a* + *v* ∗ *f aa* **in** *order-trans*)
defer
apply (*auto intro*!:*mult-left-mono add-mono*)
done

lemma *convex-epigraphI*: *convex-on s f* ⟹ *convex s* ⟹ *convex* (*epigraph s f*)
unfolding *convex-epigraph* **by** *auto*

lemma *convex-epigraph-convex*: *convex s* ⟹ *convex-on s f* ⟷ *convex*(*epigraph s f*)
by (*simp add*: *convex-epigraph*)

19.22.1 Use this to derive general bound property of convex function

lemma *convex-on*:
assumes *convex s*
shows *convex-on s f* ⟷
(∀ *k u x*. (∀ *i*∈{*1*..*k*::*nat*}. *0* ≤ *u i* ∧ *x i* ∈ *s*) ∧ *setsum u* {*1*..*k*} = *1* ⟶
f (*setsum* (λ*i*. *u i* ∗$_R$ *x i*) {*1*..*k*}) ≤ *setsum* (λ*i*. *u i* ∗ *f*(*x i*)) {*1*..*k*})
unfolding *convex-epigraph-convex*[*OF assms*] *convex epigraph-def Ball-def mem-Collect-eq*
unfolding *fst-setsum snd-setsum fst-scaleR snd-scaleR*
apply *safe*
apply (*drule-tac x*=*k* **in** *spec*)
apply (*drule-tac x*=*u* **in** *spec*)
apply (*drule-tac x*=λ*i*. (*x i*, *f* (*x i*)) **in** *spec*)
apply *simp*
using *assms*[*unfolded convex*]
apply *simp*
apply (*rule-tac y*=∑ *i* = *1*..*k*. *u i* ∗ *f* (*fst* (*x i*)) **in** *order-trans*)
defer
apply (*rule setsum-mono*)
apply (*erule-tac x*=*i* **in** *allE*)
unfolding *real-scaleR-def*

apply (*rule mult-left-mono*)
using *assms*[*unfolded convex*]
apply *auto*
done

19.23 Convexity of general and special intervals

lemma *is-interval-convex*:
fixes $s :: 'a$::*euclidean-space set*
assumes *is-interval s*
shows *convex s*
proof (*rule convexI*)
fix *x y* **and** *u v* :: *real*
assume *as*: $x \in s$ $y \in s$ $0 \leq u$ $0 \leq v$ $u + v = 1$
then have $*$: $u = 1 - v$ $1 - v \geq 0$ **and** $**$: $v = 1 - u$ $1 - u \geq 0$
by *auto*
{
fix *a b*
assume $\neg$ $b \leq u * a + v * b$
then have $u * a < (1 - v) * b$
unfolding *not-le* **using** *as*(*4*) **by** (*auto simp add*: *field-simps*)
then have $a < b$
unfolding $*$ **using** *as*(*4*) $*$(*2*)
apply (*rule-tac mult-left-less-imp-less*[*of* $1 - v$])
apply (*auto simp add*: *field-simps*)
done
then have $a \leq u * a + v * b$
unfolding $*$ **using** *as*(*4*)
by (*auto simp add*: *field-simps intro*!:*mult-right-mono*)
}
moreover
{
fix *a b*
assume $\neg$ $u * a + v * b \leq a$
then have $v * b > (1 - u) * a$
unfolding *not-le* **using** *as*(*4*) **by** (*auto simp add*: *field-simps*)
then have $a < b$
unfolding $*$ **using** *as*(*4*)
apply (*rule-tac mult-left-less-imp-less*)
apply (*auto simp add*: *field-simps*)
done
then have $u * a + v * b \leq b$
unfolding $**$
using $**$(*2*) *as*(*3*)
by (*auto simp add*: *field-simps intro*!:*mult-right-mono*)
}
ultimately show $u *_R x + v *_R y \in s$
apply $-$
apply (*rule assms*[*unfolded is-interval-def*, *rule-format*, *OF as*(*1*,*2*)])

```
    using as(3-) DIM-positive[where 'a='a]
    apply (auto simp: inner-simps)
    done
qed

lemma is-interval-connected:
  fixes s :: 'a::euclidean-space set
  shows is-interval s ⟹ connected s
  using is-interval-convex convex-connected by auto

lemma convex-box [simp]: convex (cbox a b) convex (box a (b::'a::euclidean-space))
  apply (rule-tac[!] is-interval-convex)+
  using is-interval-box is-interval-cbox
  apply auto
  done
```

19.24 On *real*, *is-interval*, *convex* and *connected* are all equivalent.

```
lemma is-interval-1:
  is-interval (s::real set) ⟷ (∀ a∈s. ∀ b∈s. ∀ x. a ≤ x ∧ x ≤ b ⟶ x ∈ s)
  unfolding is-interval-def by auto

lemma is-interval-connected-1:
  fixes s :: real set
  shows is-interval s ⟷ connected s
  apply rule
  apply (rule is-interval-connected, assumption)
  unfolding is-interval-1
  apply rule
  apply rule
  apply rule
  apply rule
  apply (erule conjE)
  apply (rule ccontr)
proof -
  fix a b x
  assume as: connected s a ∈ s b ∈ s a ≤ x x ≤ b x ∉ s
  then have *: a < x x < b
    unfolding not-le [symmetric] by auto
  let ?halfl = {..<x}
  let ?halfr = {x<..}
  {
    fix y
    assume y ∈ s
    with ‹x ∉ s› have x ≠ y by auto
    then have y ∈ ?halfr ∪ ?halfl by auto
  }
  moreover have a ∈ ?halfl b ∈ ?halfr using * by auto
```

```
  then have ?halfl ∩ s ≠ {} ?halfr ∩ s ≠ {}
    using as(2−3) by auto
  ultimately show False
    apply (rule-tac notE[OF as(1)[unfolded connected-def]])
    apply (rule-tac x = ?halfl in exI)
    apply (rule-tac x = ?halfr in exI)
    apply rule
    apply (rule open-lessThan)
    apply rule
    apply (rule open-greaterThan)
    apply auto
    done
qed

lemma is-interval-convex-1:
  fixes s :: real set
  shows is-interval s ⟷ convex s
  by (metis is-interval-convex convex-connected is-interval-connected-1)

lemma connected-convex-1:
  fixes s :: real set
  shows connected s ⟷ convex s
  by (metis is-interval-convex convex-connected is-interval-connected-1)

lemma connected-convex-1-gen:
  fixes s :: 'a :: euclidean-space set
  assumes DIM('a) = 1
  shows connected s ⟷ convex s
proof −
  obtain f:: 'a ⇒ real where linf: linear f and inj f
    using subspace-isomorphism [where 'a = 'a and 'b = real]
    by (metis DIM-real dim-UNIV subspace-UNIV assms)
  then have f −` (f ` s) = s
    by (simp add: inj-vimage-image-eq)
  then show ?thesis
   by (metis connected-convex-1 convex-linear-vimage linf convex-connected connected-linear-image)
qed
```

19.25 Another intermediate value theorem formulation

```
lemma ivt-increasing-component-on-1:
  fixes f :: real ⇒ 'a::euclidean-space
  assumes a ≤ b
    and continuous-on {a..b} f
    and (f a)•k ≤ y y ≤ (f b)•k
  shows ∃x∈{a..b}. (f x)•k = y
proof −
  have f a ∈ f ` cbox a b f b ∈ f ` cbox a b
    apply (rule-tac[!] imageI)
```

using *assms(1)*
apply *auto*
done
then show *?thesis*
using *connected-ivt-component*[*of f ‘ cbox a b f a f b k y*]
by (*simp add*: *Topology-Euclidean-Space.connected-continuous-image assms*)
qed

lemma *ivt-increasing-component-1*:
fixes $f :: real \Rightarrow 'a::euclidean\text{-}space$
shows $a \le b \Longrightarrow \forall x{\in}\{a..b\}.\ continuous\ (at\ x)\ f \Longrightarrow$
$f\ a{\cdot}k \le y \Longrightarrow y \le f\ b{\cdot}k \Longrightarrow \exists x{\in}\{a..b\}.\ (f\ x){\cdot}k = y$
by (*rule ivt-increasing-component-on-1*) (*auto simp add*: *continuous-at-imp-continuous-on*)

lemma *ivt-decreasing-component-on-1*:
fixes $f :: real \Rightarrow 'a::euclidean\text{-}space$
assumes $a \le b$
and *continuous-on* $\{a..b\}$ f
and $(f\ b){\cdot}k \le y$
and $y \le (f\ a){\cdot}k$
shows $\exists x{\in}\{a..b\}.\ (f\ x){\cdot}k = y$
apply (*subst neg-equal-iff-equal*[*symmetric*])
using *ivt-increasing-component-on-1*[*of a b* $\lambda x.\ -\ f\ x\ k\ -\ y$]
using *assms* **using** *continuous-on-minus*
apply *auto*
done

lemma *ivt-decreasing-component-1*:
fixes $f :: real \Rightarrow 'a::euclidean\text{-}space$
shows $a \le b \Longrightarrow \forall x{\in}\{a..b\}.\ continuous\ (at\ x)\ f \Longrightarrow$
$f\ b{\cdot}k \le y \Longrightarrow y \le f\ a{\cdot}k \Longrightarrow \exists x{\in}\{a..b\}.\ (f\ x){\cdot}k = y$
by (*rule ivt-decreasing-component-on-1*) (*auto simp*: *continuous-at-imp-continuous-on*)

19.26 A bound within a convex hull, and so an interval

lemma *convex-on-convex-hull-bound*:
assumes *convex-on* (*convex hull s*) f
and $\forall x{\in}s.\ f\ x \le b$
shows $\forall x{\in}\ convex\ hull\ s.\ f\ x \le b$
proof
fix x
assume $x \in convex\ hull\ s$
then obtain $k\ u\ v$ **where**
obt: $\forall i{\in}\{1..k::nat\}.\ 0 \le u\ i \wedge v\ i \in s\ setsum\ u\ \{1..k\} = 1\ (\sum i = 1..k.\ u\ i$
$*_R\ v\ i) = x$
unfolding *convex-hull-indexed mem-Collect-eq* **by** *auto*
have $(\sum i = 1..k.\ u\ i * f\ (v\ i)) \le b$
using *setsum-mono*[*of* $\{1..k\}$ $\lambda i.\ u\ i * f\ (v\ i)$ $\lambda i.\ u\ i * b$]
unfolding *setsum-left-distrib*[*symmetric*] *obt*(*2*) *mult-1*

apply (*drule-tac meta-mp*)
apply (*rule mult-left-mono*)
using *assms*(*2*) *obt*(*1*)
apply *auto*
done
then show $f\ x \leq b$
using *assms*(*1*)[*unfolded convex-on*[*OF convex-convex-hull*], *rule-format*, *of k u v*]
unfolding *obt*(*2*−*3*)
using *obt*(*1*) **and** *hull-subset*[*unfolded subset-eq*, *rule-format*, *of - s*]
by *auto*
qed

lemma *inner-setsum-Basis*[*simp*]: $i \in Basis \Longrightarrow (\sum Basis) \cdot i = 1$
by (*simp add*: *inner-setsum-left setsum.If-cases inner-Basis*)

lemma *convex-set-plus*:
assumes *convex s* **and** *convex t* **shows** *convex* $(s + t)$
proof −
have *convex* $\{x + y \mid x\ y.\ x \in s \wedge y \in t\}$
using *assms* **by** (*rule convex-sums*)
moreover have $\{x + y \mid x\ y.\ x \in s \wedge y \in t\} = s + t$
unfolding *set-plus-def* **by** *auto*
finally show *convex* $(s + t)$ **.**
qed

lemma *convex-set-setsum*:
assumes $\bigwedge i.\ i \in A \Longrightarrow$ *convex* $(B\ i)$
shows *convex* $(\sum i{\in}A.\ B\ i)$
proof (*cases finite A*)
case *True* **then show** *?thesis* **using** *assms*
by *induct* (*auto simp*: *convex-set-plus*)
qed *auto*

lemma *finite-set-setsum*:
assumes *finite A* **and** $\forall\, i{\in}A.$ *finite* $(B\ i)$ **shows** *finite* $(\sum i{\in}A.\ B\ i)$
using *assms* **by** (*induct set*: *finite*, *simp*, *simp add*: *finite-set-plus*)

lemma *set-setsum-eq*:
finite A $\Longrightarrow (\sum i{\in}A.\ B\ i) = \{\sum i{\in}A.\ f\ i \mid f.\ \forall\, i{\in}A.\ f\ i \in B\ i\}$
apply (*induct set*: *finite*)
apply *simp*
apply *simp*
apply (*safe elim*!: *set-plus-elim*)
apply (*rule-tac x*=*fun-upd f x a* **in** *exI*)
apply *simp*
apply (*rule-tac f*=$\lambda x.\ a + x$ **in** *arg-cong*)
apply (*rule setsum.cong* [*OF refl*])
apply *clarsimp*

apply *fast*
done

lemma *box-eq-set-setsum-Basis*:
shows $\{x.\ \forall i{\in}Basis.\ x{\cdot}i \in B\ i\} = (\sum i{\in}Basis.\ image\ (\lambda x.\ x *_R i)\ (B\ i))$
apply (*subst set-setsum-eq* [*OF finite-Basis*])
apply *safe*
apply (*fast intro*: *euclidean-representation* [*symmetric*])
apply (*subst inner-setsum-left*)
apply (*subgoal-tac* $(\sum x{\in}Basis.\ f\ x \cdot i) = f\ i \cdot i$)
apply (*drule* (*1*) *bspec*)
apply *clarsimp*
apply (*frule setsum.remove* [*OF finite-Basis*])
apply (*erule trans*)
apply *simp*
apply (*rule setsum.neutral*)
apply *clarsimp*
apply (*frule-tac* $x{=}i$ **in** *bspec*, *assumption*)
apply (*drule-tac* $x{=}x$ **in** *bspec*, *assumption*)
apply *clarsimp*
apply (*cut-tac* $u{=}x$ **and** $v{=}i$ **in** *inner-Basis*, *assumption*+)
apply (*rule ccontr*)
apply *simp*
done

lemma *convex-hull-set-setsum*:
convex hull $(\sum i{\in}A.\ B\ i) = (\sum i{\in}A.\ convex\ hull\ (B\ i))$
proof (*cases finite A*)
assume *finite A* **then show** *?thesis*
by (*induct set*: *finite*, *simp*, *simp add*: *convex-hull-set-plus*)
qed *simp*

lemma *convex-hull-eq-real-cbox*:
fixes $x\ y$ :: *real* **assumes** $x \le y$
shows *convex hull* $\{x, y\} = cbox\ x\ y$
proof (*rule hull-unique*)
show $\{x, y\} \subseteq cbox\ x\ y$ **using** ‹$x \le y$› **by** *auto*
show *convex* (*cbox x y*)
by (*rule convex-box*)
next
fix *s* **assume** $\{x, y\} \subseteq s$ **and** *convex s*
then show $cbox\ x\ y \subseteq s$
unfolding *is-interval-convex-1* [*symmetric*] *is-interval-def Basis-real-def*
by − (*clarify*, *simp* (*no-asm-use*), *fast*)
qed

lemma *unit-interval-convex-hull*:
cbox (*0*::*'a*::*euclidean-space*) *One* = *convex hull* $\{x.\ \forall i{\in}Basis.\ (x{\cdot}i = 0) \vee (x{\cdot}i = 1)\}$

(**is** *?int = convex hull ?points*)
proof −
have *One*[*simp*]: ⋀*i*. *i* ∈ *Basis* ⟹ *One* ⋅ *i* = *1*
by (*simp add*: *inner-setsum-left setsum.If-cases inner-Basis*)
have *?int* = {*x*. ∀ *i*∈*Basis*. *x* ⋅ *i* ∈ *cbox 0 1*}
by (*auto simp*: *cbox-def*)
also have ... = (∑ *i*∈*Basis*. (λ*x*. *x* ∗$_R$ *i*) ‘ *cbox 0 1*)
by (*simp only*: *box-eq-set-setsum-Basis*)
also have ... = (∑ *i*∈*Basis*. (λ*x*. *x* ∗$_R$ *i*) ‘ (*convex hull* {*0*, *1*}))
by (*simp only*: *convex-hull-eq-real-cbox zero-le-one*)
also have ... = (∑ *i*∈*Basis*. *convex hull* ((λ*x*. *x* ∗$_R$ *i*) ‘ {*0*, *1*}))
by (*simp only*: *convex-hull-linear-image linear-scaleR-left*)
also have ... = *convex hull* (∑ *i*∈*Basis*. (λ*x*. *x* ∗$_R$ *i*) ‘ {*0*, *1*})
by (*simp only*: *convex-hull-set-setsum*)
also have ... = *convex hull* {*x*. ∀ *i*∈*Basis*. *x*⋅*i* ∈ {*0*, *1*}}
by (*simp only*: *box-eq-set-setsum-Basis*)
also have *convex hull* {*x*. ∀ *i*∈*Basis*. *x*⋅*i* ∈ {*0*, *1*}} = *convex hull ?points*
by *simp*
finally show *?thesis* **.**
qed

And this is a finite set of vertices.

lemma *unit-cube-convex-hull*:
obtains *s* :: *'a*::*euclidean-space set*
where *finite s* **and** *cbox 0* (∑ *Basis*) = *convex hull s*
apply (*rule that*[*of* {*x*::*'a*. ∀ *i*∈*Basis*. *x*⋅*i*=*0* ∨ *x*⋅*i*=*1*}])
apply (*rule finite-subset*[*of* - (λ*s*. (∑ *i*∈*Basis*. (*if i*∈*s then 1 else 0*) ∗$_R$ *i*)::*'a*) ‘ *Pow Basis*])
prefer *3*
apply (*rule unit-interval-convex-hull*)
apply *rule*
unfolding *mem-Collect-eq*
proof −
fix *x* :: *'a*
assume *as*: ∀ *i*∈*Basis*. *x* ⋅ *i* = *0* ∨ *x* ⋅ *i* = *1*
show *x* ∈ (λ*s*. ∑ *i*∈*Basis*. (*if i*∈*s then 1 else 0*) ∗$_R$ *i*) ‘ *Pow Basis*
apply (*rule image-eqI*[**where** *x*={*i*. *i*∈*Basis* ∧ *x*⋅*i* = *1*}])
using *as*
apply (*subst euclidean-eq-iff*)
apply *auto*
done
qed *auto*

Hence any cube (could do any nonempty interval).

lemma *cube-convex-hull*:
assumes *d* > *0*
obtains *s* :: *'a*::*euclidean-space set* **where**
finite s **and** *cbox* (*x* − (∑ *i*∈*Basis*. *d*∗$_R$*i*)) (*x* + (∑ *i*∈*Basis*. *d*∗$_R$*i*)) = *convex hull s*

proof −
let *?d* = (∑ *i*∈*Basis*. *d*∗$_R$*i*)::′*a*
have ∗: *cbox* (*x* − *?d*) (*x* + *?d*) = (λ*y*. *x* − *?d* + (*2* ∗ *d*) ∗$_R$ *y*) ‘ *cbox 0* (∑ *Basis*)
apply (*rule set-eqI*, *rule*)
unfolding *image-iff*
defer
apply (*erule bexE*)
proof −
fix *y*
assume *as*: *y*∈*cbox* (*x* − *?d*) (*x* + *?d*)
then have *inverse* (*2* ∗ *d*) ∗$_R$ (*y* − (*x* − *?d*)) ∈ *cbox 0* (∑ *Basis*)
using *assms* **by** (*simp add*: *mem-box field-simps inner-simps*)
with ‹*0* < *d*› **show** ∃ *z*∈*cbox 0* (∑ *Basis*). *y* = *x* − *?d* + (*2* ∗ *d*) ∗$_R$ *z*
by (*intro bexI*[*of* - *inverse* (*2* ∗ *d*) ∗$_R$ (*y* − (*x* − *?d*))]) *auto*
next
fix *y z*
assume *as*: *z*∈*cbox 0* (∑ *Basis*) *y* = *x* − *?d* + (*2*∗*d*) ∗$_R$ *z*
have ⋀*i*. *i*∈*Basis* ⟹ *0* ≤ *d* ∗ (*z* ∙ *i*) ∧ *d* ∗ (*z* ∙ *i*) ≤ *d*
using *assms as*(*1*)[*unfolded mem-box*]
apply (*erule-tac x*=*i* **in** *ballE*)
apply *rule*
prefer *2*
apply (*rule mult-right-le-one-le*)
using *assms*
apply *auto*
done
then show *y* ∈ *cbox* (*x* − *?d*) (*x* + *?d*)
unfolding *as*(*2*) *mem-box*
apply −
apply *rule*
using *as*(*1*)[*unfolded mem-box*]
apply (*erule-tac x*=*i* **in** *ballE*)
using *assms*
apply (*auto simp*: *inner-simps*)
done
qed
obtain *s* **where** *finite s cbox 0* (∑ *Basis*::′*a*) = *convex hull s*
using *unit-cube-convex-hull* **by** *auto*
then show *?thesis*
apply (*rule-tac that*[*of* (λ*y*. *x* − *?d* + (*2* ∗ *d*) ∗$_R$ *y*)‘ *s*])
unfolding ∗ **and** *convex-hull-affinity*
apply *auto*
done
qed

19.27 Bounded convex function on open set is continuous

lemma *convex-on-bounded-continuous*:

```
  fixes s :: ('a::real-normed-vector) set
  assumes open s
    and convex-on s f
    and ∀ x∈s. |f x| ≤ b
  shows continuous-on s f
  apply (rule continuous-at-imp-continuous-on)
  unfolding continuous-at-real-range
proof (rule,rule,rule)
  fix x and e :: real
  assume x ∈ s e > 0
  def B ≡ |b| + 1
  have B: 0 < B ⋀x. x∈s ⟹ |f x| ≤ B
    unfolding B-def
    defer
    apply (drule assms(3)[rule-format])
    apply auto
    done
  obtain k where k > 0 and k: cball x k ⊆ s
    using assms(1)[unfolded open-contains-cball, THEN bspec[where x=x]]
    using ‹x∈s› by auto
  show ∃ d>0. ∀ x'. norm (x' − x) < d ⟶ |f x' − f x| < e
    apply (rule-tac x=min (k / 2) (e / (2 * B) * k) in exI)
    apply rule
    defer
  proof (rule, rule)
    fix y
    assume as: norm (y − x) < min (k / 2) (e / (2 * B) * k)
    show |f y − f x| < e
    proof (cases y = x)
      case False
      def t ≡ k / norm (y − x)
      have 2 < t 0<t
        unfolding t-def using as False and ‹k>0›
        by (auto simp add:field-simps)
      have y ∈ s
        apply (rule k[unfolded subset-eq,rule-format])
        unfolding mem-cball dist-norm
        apply (rule order-trans[of - 2 * norm (x − y)])
        using as
        by (auto simp add: field-simps norm-minus-commute)
      {
        def w ≡ x + t *R (y − x)
        have w ∈ s
          unfolding w-def
          apply (rule k[unfolded subset-eq,rule-format])
          unfolding mem-cball dist-norm
          unfolding t-def
          using ‹k>0›
          apply auto
```

```
      done
    have (1 / t) *R x + - x + ((t - 1) / t) *R x = (1 / t - 1 + (t - 1) /
t) *R x
      by (auto simp add: algebra-simps)
    also have ... = 0
      using ‹t > 0› by (auto simp add:field-simps)
    finally have w: (1 / t) *R w + ((t - 1) / t) *R x = y
      unfolding w-def using False and ‹t > 0›
      by (auto simp add: algebra-simps)
    have  2 * B < e * t
      unfolding t-def using ‹0 < e› ‹0 < k› ‹B > 0› and as and False
      by (auto simp add:field-simps)
    then have (f w - f x) / t < e
      using B(2)[OF ‹w∈s›] and B(2)[OF ‹x∈s›]
      using ‹t > 0› by (auto simp add:field-simps)
    then have th1: f y - f x < e
      apply -
      apply (rule le-less-trans)
      defer
      apply assumption
      using assms(2)[unfolded convex-on-def,rule-format,of w x 1/t (t - 1)/t,
unfolded w]
      using ‹0 < t› ‹2 < t› and ‹x ∈ s› ‹w ∈ s›
      by (auto simp add:field-simps)
  }
  moreover
  {
    def w ≡ x - t *R (y - x)
    have w ∈ s
      unfolding w-def
      apply (rule k[unfolded subset-eq,rule-format])
      unfolding mem-cball dist-norm
      unfolding t-def
      using ‹k > 0›
      apply auto
      done
    have (1 / (1 + t)) *R x + (t / (1 + t)) *R x = (1 / (1 + t) + t / (1 +
t)) *R x
      by (auto simp add: algebra-simps)
    also have ... = x
      using ‹t > 0› by (auto simp add:field-simps)
    finally have w: (1 / (1+t)) *R w + (t / (1 + t)) *R y = x
      unfolding w-def using False and ‹t > 0›
      by (auto simp add: algebra-simps)
    have 2 * B < e * t
      unfolding t-def
      using ‹0 < e› ‹0 < k› ‹B > 0› and as and False
      by (auto simp add:field-simps)
    then have *: (f w - f y) / t < e
```

```
        using B(2)[OF ‹w∈s›] and B(2)[OF ‹y∈s›]
        using ‹t > 0›
        by (auto simp add:field-simps)
      have f x ≤ 1 / (1 + t) * f w + (t / (1 + t)) * f y
        using assms(2)[unfolded convex-on-def,rule-format,of w y 1/(1+t) t /
(1+t),unfolded w]
        using ‹0 < t› ‹2 < t› and ‹y ∈ s› ‹w ∈ s›
        by (auto simp add:field-simps)
      also have ... = (f w + t * f y) / (1 + t)
        using ‹t > 0› by (auto simp add: divide-simps)
      also have ... < e + f y
        using ‹t > 0› * ‹e > 0› by (auto simp add: field-simps)
      finally have f x − f y < e by auto
    }
    ultimately show ?thesis by auto
  qed (insert ‹0<e›, auto)
 qed (insert ‹0<e› ‹0<k› ‹0<B›, auto simp: field-simps)
qed
```

19.28 Upper bound on a ball implies upper and lower bounds

```
lemma convex-bounds-lemma:
 fixes x :: 'a::real-normed-vector
 assumes convex-on (cball x e) f
   and ∀ y ∈ cball x e. f y ≤ b
 shows ∀ y ∈ cball x e. |f y| ≤ b + 2 * |f x|
 apply rule
proof (cases 0 ≤ e)
 case True
 fix y
 assume y: y ∈ cball x e
 def z ≡ 2 *R x − y
 have *: x − (2 *R x − y) = y − x
   by (simp add: scaleR-2)
 have z: z ∈ cball x e
  using y unfolding z-def mem-cball dist-norm * by (auto simp add: norm-minus-commute)
 have (1 / 2) *R y + (1 / 2) *R z = x
   unfolding z-def by (auto simp add: algebra-simps)
 then show |f y| ≤ b + 2 * |f x|
   using assms(1)[unfolded convex-on-def,rule-format, OF y z, of 1/2 1/2]
   using assms(2)[rule-format,OF y] assms(2)[rule-format,OF z]
   by (auto simp add:field-simps)
next
 case False
 fix y
 assume y ∈ cball x e
 then have dist x y < 0
   using False unfolding mem-cball not-le by (auto simp del: dist-not-less-zero)
 then show |f y| ≤ b + 2 * |f x|
```

using *zero-le-dist*[*of x y*] **by** *auto*
qed

19.28.1 Hence a convex function on an open set is continuous

lemma *real-of-nat-ge-one-iff*: $1 \leq real\ (n::nat) \longleftrightarrow 1 \leq n$
by *auto*

lemma *convex-on-continuous*:
assumes *open* (*s*::('*a*::*euclidean-space*) *set*) *convex-on s f*
shows *continuous-on s f*
unfolding *continuous-on-eq-continuous-at*[*OF assms*(*1*)]
proof
note *dimge1* = *DIM-positive*[**where** '*a*='*a*]
fix *x*
assume $x \in s$
then obtain *e* **where** *e*: *cball x e* $\subseteq$ *s e* > *0*
using *assms*(*1*) **unfolding** *open-contains-cball* **by** *auto*
def $d \equiv e\ /\ real\ DIM('a)$
have *0* < *d*
unfolding *d-def* **using** ‹*e* > *0*› *dimge1* **by** *auto*
let *?d* = ($\sum i{\in}Basis.\ d *_R i$)::'*a*
obtain *c*
where *c*: *finite c*
and *c1*: *convex hull c* $\subseteq$ *cball x e*
and *c2*: *cball x d* $\subseteq$ *convex hull c*
proof
def $c \equiv \sum i{\in}Basis.\ (\lambda a.\ a *_R i)$ ` {*x*•*i* − *d*, *x*•*i* + *d*}
show *finite c*
unfolding *c-def* **by** (*simp add*: *finite-set-setsum*)
have *1*: *convex hull c* = {*a*. ∀ *i*∈*Basis*. *a* • *i* ∈ *cbox* (*x* • *i* − *d*) (*x* • *i* + *d*)}
unfolding *box-eq-set-setsum-Basis*
unfolding *c-def convex-hull-set-setsum*
apply (*subst convex-hull-linear-image* [*symmetric*])
apply (*simp add*: *linear-iff scaleR-add-left*)
apply (*rule setsum.cong* [*OF refl*])
apply (*rule image-cong* [*OF* - *refl*])
apply (*rule convex-hull-eq-real-cbox*)
apply (*cut-tac* ‹*0* < *d*›, *simp*)
done
then have *2*: *convex hull c* = {*a*. ∀ *i*∈*Basis*. *a* • *i* ∈ *cball* (*x* • *i*) *d*}
by (*simp add*: *dist-norm abs-le-iff algebra-simps*)
show *cball x d* $\subseteq$ *convex hull c*
unfolding *2*
apply *clarsimp*
apply (*simp only*: *dist-norm*)
apply (*subst inner-diff-left* [*symmetric*])
apply *simp*
apply (*erule* (*1*) *order-trans* [*OF Basis-le-norm*])

```
    done
  have e': e = (∑ (i::'a)∈Basis. d)
    by (simp add: d-def DIM-positive)
  show convex hull c ⊆ cball x e
    unfolding 2
    apply clarsimp
    apply (subst euclidean-dist-l2)
    apply (rule order-trans [OF setL2-le-setsum])
    apply (rule zero-le-dist)
    unfolding e'
    apply (rule setsum-mono)
    apply simp
    done
qed
def k ≡ Max (f ` c)
have convex-on (convex hull c) f
  apply(rule convex-on-subset[OF assms(2)])
  apply(rule subset-trans[OF - e(1)])
  apply(rule c1)
  done
then have k: ∀ y∈convex hull c. f y ≤ k
  apply (rule-tac convex-on-convex-hull-bound)
  apply assumption
  unfolding k-def
  apply (rule, rule Max-ge)
  using c(1)
  apply auto
  done
have d ≤ e
  unfolding d-def
  apply (rule mult-imp-div-pos-le)
  using ‹e > 0›
  unfolding mult-le-cancel-left1
  apply (auto simp: real-of-nat-ge-one-iff Suc-le-eq DIM-positive)
  done
then have dsube: cball x d ⊆ cball x e
  by (rule subset-cball)
have conv: convex-on (cball x d) f
  apply (rule convex-on-subset)
  apply (rule convex-on-subset[OF assms(2)])
  apply (rule e(1))
  apply (rule dsube)
  done
then have ∀ y∈cball x d. |f y| ≤ k + 2 * |f x|
  apply (rule convex-bounds-lemma)
  apply (rule ballI)
  apply (rule k [rule-format])
  apply (erule rev-subsetD)
  apply (rule c2)
```

done
then have *continuous-on* (*ball x d*) *f*
apply (*rule-tac convex-on-bounded-continuous*)
apply (*rule open-ball*, *rule convex-on-subset*[*OF conv*])
apply (*rule ball-subset-cball*)
apply *force*
done
then show *continuous* (*at x*) *f*
unfolding *continuous-on-eq-continuous-at*[*OF open-ball*]
using ‹*d* > *0*› **by** *auto*
qed

19.29 Line segments, Starlike Sets, etc.

definition *midpoint* :: *'a*::*real-vector* ⇒ *'a* ⇒ *'a*
where *midpoint a b* = (*inverse* (*2*::*real*)) *∗R* (*a* + *b*)

definition *closed-segment* :: *'a*::*real-vector* ⇒ *'a* ⇒ *'a set*
where *closed-segment a b* = {(*1* − *u*) *∗R a* + *u ∗R b* | *u*::*real*. *0* ≤ *u* ∧ *u* ≤ *1*}

definition *open-segment* :: *'a*::*real-vector* ⇒ *'a* ⇒ *'a set* **where**
open-segment a b ≡ *closed-segment a b* − {*a*,*b*}

lemmas *segment* = *open-segment-def closed-segment-def*

lemma *in-segment*:
x ∈ *closed-segment a b* ⟷ (∃ *u*. *0* ≤ *u* ∧ *u* ≤ *1* ∧ *x* = (*1* − *u*) *∗R a* + *u ∗R b*)
x ∈ *open-segment a b* ⟷ *a* ≠ *b* ∧ (∃ *u*. *0* < *u* ∧ *u* < *1* ∧ *x* = (*1* − *u*) *∗R a* + *u ∗R b*)
using *less-eq-real-def* **by** (*auto simp*: *segment algebra-simps*)

definition *between* = (λ(*a*,*b*) *x*. *x* ∈ *closed-segment a b*)

definition *starlike s* ⟷ (∃ *a*∈*s*. ∀ *x*∈*s*. *closed-segment a x* ⊆ *s*)

lemma *starlike-UNIV* [*simp*]: *starlike UNIV*
by (*simp add*: *starlike-def*)

lemma *midpoint-refl*: *midpoint x x* = *x*
unfolding *midpoint-def*
unfolding *scaleR-right-distrib*
unfolding *scaleR-left-distrib*[*symmetric*]
by *auto*

lemma *midpoint-sym*: *midpoint a b* = *midpoint b a*
unfolding *midpoint-def* **by** (*auto simp add*: *scaleR-right-distrib*)

lemma *midpoint-eq-iff*: *midpoint a b* = *c* ⟷ *a* + *b* = *c* + *c*

proof −
have *midpoint a b = c ⟷ scaleR 2 (midpoint a b) = scaleR 2 c*
by *simp*
then show *?thesis*
unfolding *midpoint-def scaleR-2* [*symmetric*] **by** *simp*
qed

lemma *dist-midpoint*:
fixes *a b* :: *'a*::*real-normed-vector* **shows**
dist a (*midpoint a b*) = (*dist a b*) / *2* (**is** *?t1*)
dist b (*midpoint a b*) = (*dist a b*) / *2* (**is** *?t2*)
dist (*midpoint a b*) *a* = (*dist a b*) / *2* (**is** *?t3*)
dist (*midpoint a b*) *b* = (*dist a b*) / *2* (**is** *?t4*)
proof −
have ∗: ⋀*x y*::*'a*. *2* ∗*R x* = − *y* ⟹ *norm x* = (*norm y*) / *2*
unfolding *equation-minus-iff* **by** *auto*
have ∗∗: ⋀*x y*::*'a*. *2* ∗*R x* = *y* ⟹ *norm x* = (*norm y*) / *2*
by *auto*
note *scaleR-right-distrib* [*simp*]
show *?t1*
unfolding *midpoint-def dist-norm*
apply (*rule* ∗∗)
apply (*simp add*: *scaleR-right-diff-distrib*)
apply (*simp add*: *scaleR-2*)
done
show *?t2*
unfolding *midpoint-def dist-norm*
apply (*rule* ∗)
apply (*simp add*: *scaleR-right-diff-distrib*)
apply (*simp add*: *scaleR-2*)
done
show *?t3*
unfolding *midpoint-def dist-norm*
apply (*rule* ∗)
apply (*simp add*: *scaleR-right-diff-distrib*)
apply (*simp add*: *scaleR-2*)
done
show *?t4*
unfolding *midpoint-def dist-norm*
apply (*rule* ∗∗)
apply (*simp add*: *scaleR-right-diff-distrib*)
apply (*simp add*: *scaleR-2*)
done
qed

lemma *midpoint-eq-endpoint*:
midpoint a b = *a* ⟷ *a* = *b*
midpoint a b = *b* ⟷ *a* = *b*
unfolding *midpoint-eq-iff* **by** *auto*

lemma *convex-contains-segment*:
convex s $\longleftrightarrow$ $(\forall\, a{\in}s.\ \forall\, b{\in}s.$ *closed-segment a b* $\subseteq$ *s*)
unfolding *convex-alt closed-segment-def* **by** *auto*

lemma *closed-segment-subset*: $[\![x \in s;\ y \in s;$ *convex s*$]\!] \Longrightarrow$ *closed-segment x y* $\subseteq$ *s*
by (*simp add*: *convex-contains-segment*)

lemma *closed-segment-subset-convex-hull*:
$[\![x \in$ *convex hull s*; $y \in$ *convex hull s*$]\!] \Longrightarrow$ *closed-segment x y* $\subseteq$ *convex hull s*
using *convex-contains-segment* **by** *blast*

lemma *convex-imp-starlike*:
convex s $\Longrightarrow$ *s* $\neq$ {} $\Longrightarrow$ *starlike s*
unfolding *convex-contains-segment starlike-def* **by** *auto*

lemma *segment-convex-hull*:
closed-segment a b = *convex hull* {*a*,*b*}
proof −
have ∗: $\bigwedge x.$ {*x*} $\neq$ {} **by** *auto*
show *?thesis*
unfolding *segment convex-hull-insert*[*OF* ∗] *convex-hull-singleton*
by (*safe*; *rule-tac x=1 − u* **in** *exI*; *force*)
qed

lemma *open-closed-segment*: *u* $\in$ *open-segment w z* $\Longrightarrow$ *u* $\in$ *closed-segment w z*
by (*auto simp add*: *closed-segment-def open-segment-def*)

lemma *segment-open-subset-closed*:
open-segment a b $\subseteq$ *closed-segment a b*
by (*auto simp*: *closed-segment-def open-segment-def*)

lemma *bounded-closed-segment*:
fixes *a* :: *'a*::*euclidean-space* **shows** *bounded* (*closed-segment a b*)
by (*simp add*: *segment-convex-hull compact-convex-hull compact-imp-bounded*)

lemma *bounded-open-segment*:
fixes *a* :: *'a*::*euclidean-space* **shows** *bounded* (*open-segment a b*)
by (*rule bounded-subset* [*OF bounded-closed-segment segment-open-subset-closed*])

lemmas *bounded-segment* = *bounded-closed-segment open-closed-segment*

lemma *ends-in-segment* [*iff*]: *a* $\in$ *closed-segment a b b* $\in$ *closed-segment a b*
unfolding *segment-convex-hull*
by (*auto intro*!: *hull-subset*[*unfolded subset-eq*, *rule-format*])

lemma *segment-furthest-le*:
fixes *a b x y* :: *'a*::*euclidean-space*

assumes $x \in$ *closed-segment a b*
shows *norm* $(y - x) \leq$ *norm* $(y - a) \vee$ *norm* $(y - x) \leq$ *norm* $(y - b)$
proof −
obtain z **where** $z \in \{a, b\}$ *norm* $(x - y) \leq$ *norm* $(z - y)$
using *simplex-furthest-le*[*of* $\{a, b\}$ y]
using *assms*[*unfolded segment-convex-hull*]
by *auto*
then show *?thesis*
by (*auto simp add*:*norm-minus-commute*)
qed

lemma *closed-segment-commute*: *closed-segment a b* = *closed-segment b a*
proof −
have $\{a, b\} = \{b, a\}$ **by** *auto*
thus *?thesis*
by (*simp add*: *segment-convex-hull*)
qed

lemma *segment-bound1*:
assumes $x \in$ *closed-segment a b*
shows *norm* $(x - a) \leq$ *norm* $(b - a)$
proof −
obtain u **where** $x = (1 - u) *_R a + u *_R b$ $0 \leq u$ $u \leq 1$
using *assms* **by** (*auto simp add*: *closed-segment-def*)
then show *norm* $(x - a) \leq$ *norm* $(b - a)$
apply *clarify*
apply (*auto simp*: *algebra-simps*)
apply (*simp add*: *scaleR-diff-right* [*symmetric*] *mult-left-le-one-le*)
done
qed

lemma *segment-bound*:
assumes $x \in$ *closed-segment a b*
shows *norm* $(x - a) \leq$ *norm* $(b - a)$ *norm* $(x - b) \leq$ *norm* $(b - a)$
apply (*simp add*: *assms segment-bound1*)
by (*metis assms closed-segment-commute dist-commute dist-norm segment-bound1*)

lemma *open-segment-commute*: *open-segment a b* = *open-segment b a*
proof −
have $\{a, b\} = \{b, a\}$ **by** *auto*
thus *?thesis*
by (*simp add*: *closed-segment-commute open-segment-def*)
qed

lemma *closed-segment-idem* [*simp*]: *closed-segment a a* = $\{a\}$
unfolding *segment* **by** (*auto simp add*: *algebra-simps*)

lemma *open-segment-idem* [*simp*]: *open-segment a a* = $\{\}$
by (*simp add*: *open-segment-def*)

lemma *closed-segment-eq-open*: *closed-segment a b = open-segment a b ∪ {a,b}*
using *open-segment-def* **by** *auto*

lemma *closed-segment-eq-real-ivl*:
fixes *a b*::*real*
shows *closed-segment a b = (if a ≤ b then {a .. b} else {b .. a})*
proof −
have *b ≤ a ⟹ closed-segment b a = {b .. a}*
and *a ≤ b ⟹ closed-segment a b = {a .. b}*
by (*auto simp*: *convex-hull-eq-real-cbox segment-convex-hull*)
thus *?thesis*
by (*auto simp*: *closed-segment-commute*)
qed

lemma *closed-segment-real-eq*:
fixes *u*::*real* **shows** *closed-segment u v = (λx. (v − u) ∗ x + u) ‘ {0..1}*
by (*simp add*: *add.commute* [*of u*] *image-affinity-atLeastAtMost* [**where** *c*=*u*] *closed-segment-eq-real-ivl*)

19.29.1 More lemmas, especially for working with the underlying formula

lemma *segment-eq-compose*:
fixes *a* :: *'a* :: *real-vector*
shows *(λu. (1 − u) ∗R a + u ∗R b) = (λx. a + x) o (λu. u ∗R (b − a))*
by (*simp add*: *o-def algebra-simps*)

lemma *segment-degen-1*:
fixes *a* :: *'a* :: *real-vector*
shows *(1 − u) ∗R a + u ∗R b = b ⟷ a=b ∨ u=1*
proof −
{ assume *(1 − u) ∗R a + u ∗R b = b*
then have *(1 − u) ∗R a = (1 − u) ∗R b*
by (*simp add*: *algebra-simps*)
then have *a=b ∨ u=1*
by *simp*
} then show *?thesis*
by (*auto simp*: *algebra-simps*)
qed

lemma *segment-degen-0*:
fixes *a* :: *'a* :: *real-vector*
shows *(1 − u) ∗R a + u ∗R b = a ⟷ a=b ∨ u=0*
using *segment-degen-1* [*of 1−u b a*]
by (*auto simp*: *algebra-simps*)

lemma *closed-segment-image-interval*:
closed-segment a b = (λu. (1 − u) ∗R a + u ∗R b) ‘ {0..1}

by (*auto simp*: *set-eq-iff image-iff closed-segment-def*)

lemma *open-segment-image-interval*:
open-segment a b = (*if a=b then* {} *else* (λ*u*. (1 − *u*) ∗$_R$ *a* + *u* ∗$_R$ *b*) ‘ {0<..<1})
by (*auto simp*: *open-segment-def closed-segment-def segment-degen-0 segment-degen-1*)

lemmas *segment-image-interval* = *closed-segment-image-interval open-segment-image-interval*

lemma *open-segment-bound1*:
assumes *x* ∈ *open-segment a b*
shows *norm* (*x* − *a*) < *norm* (*b* − *a*)
proof −
obtain *u* **where** *x* = (1 − *u*) ∗$_R$ *a* + *u* ∗$_R$ *b* 0 < *u u* < 1 *a* ≠ *b*
using *assms* **by** (*auto simp add*: *open-segment-image-interval split*: *if-split-asm*)
then show *norm* (*x* − *a*) < *norm* (*b* − *a*)
apply *clarify*
apply (*auto simp*: *algebra-simps*)
apply (*simp add*: *scaleR-diff-right* [*symmetric*])
done
qed

lemma *compact-segment* [*simp*]:
fixes *a* :: ′*a*::*real-normed-vector*
shows *compact* (*closed-segment a b*)
by (*auto simp*: *segment-image-interval intro*!: *compact-continuous-image continuous-intros*)

lemma *closed-segment* [*simp*]:
fixes *a* :: ′*a*::*real-normed-vector*
shows *closed* (*closed-segment a b*)
by (*simp add*: *compact-imp-closed*)

lemma *closure-closed-segment* [*simp*]:
fixes *a* :: ′*a*::*real-normed-vector*
shows *closure*(*closed-segment a b*) = *closed-segment a b*
by *simp*

lemma *open-segment-bound*:
assumes *x* ∈ *open-segment a b*
shows *norm* (*x* − *a*) < *norm* (*b* − *a*) *norm* (*x* − *b*) < *norm* (*b* − *a*)
apply (*simp add*: *assms open-segment-bound1*)
by (*metis assms norm-minus-commute open-segment-bound1 open-segment-commute*)

lemma *closure-open-segment* [*simp*]:
fixes *a* :: ′*a*::*euclidean-space*
shows *closure*(*open-segment a b*) = (*if a* = *b then* {} *else closed-segment a b*)
proof −
have *closure* ((λ*u*. *u* ∗$_R$ (*b* − *a*)) ‘ {0<..<1}) = (λ*u*. *u* ∗$_R$ (*b* − *a*)) ‘ *closure* {0<..<1} **if** *a* ≠ *b*

apply (*rule closure-injective-linear-image* [*symmetric*])
apply (*simp add*:)
using *that* **by** (*simp add*: *inj-on-def*)
then show *?thesis*
by (*simp add*: *segment-image-interval segment-eq-compose closure-greaterThanLessThan* [*symmetric*]
closure-translation image-comp [*symmetric*] *del*: *closure-greaterThanLessThan*)
qed

lemma *closed-open-segment-iff* [*simp*]:
fixes *a* :: *'a*::*euclidean-space* **shows** *closed*(*open-segment a b*) ⟷ *a* = *b*
by (*metis open-segment-def DiffE closure-eq closure-open-segment ends-in-segment*(*1*) *insert-iff segment-image-interval*(*2*))

lemma *compact-open-segment-iff* [*simp*]:
fixes *a* :: *'a*::*euclidean-space* **shows** *compact*(*open-segment a b*) ⟷ *a* = *b*
by (*simp add*: *bounded-open-segment compact-eq-bounded-closed*)

lemma *convex-closed-segment* [*iff*]: *convex* (*closed-segment a b*)
unfolding *segment-convex-hull* **by**(*rule convex-convex-hull*)

lemma *convex-open-segment* [*iff*]: *convex*(*open-segment a b*)
proof −
have *convex* ((λ*u*. *u* *$_R$ (*b*−*a*)) ‘ {*0*<..<*1*})
by (*rule convex-linear-image*) *auto*
then show *?thesis*
apply (*simp add*: *open-segment-image-interval segment-eq-compose*)
by (*metis image-comp convex-translation*)
qed

lemmas *convex-segment* = *convex-closed-segment convex-open-segment*

lemma *connected-segment* [*iff*]:
fixes *x* :: *'a* :: *real-normed-vector*
shows *connected* (*closed-segment x y*)
by (*simp add*: *convex-connected*)

lemma *affine-hull-closed-segment* [*simp*]:
affine hull (*closed-segment a b*) = *affine hull* {*a*,*b*}
by (*simp add*: *segment-convex-hull*)

lemma *affine-hull-open-segment* [*simp*]:
fixes *a* :: *'a*::*euclidean-space*
shows *affine hull* (*open-segment a b*) = (*if a* = *b then* {} *else affine hull* {*a*,*b*})
by (*metis affine-hull-convex-hull affine-hull-empty closure-open-segment closure-same-affine-hull segment-convex-hull*)

lemma *rel-interior-closure-convex-segment*:
fixes *S* :: -::*euclidean-space set*

assumes *convex S a* ∈ *rel-interior S b* ∈ *closure S*
shows *open-segment a b* ⊆ *rel-interior S*
proof
fix *x*
have [*simp*]: (*1* − *u*) *∗R a* + *u ∗R b* = *b* − (*1* − *u*) *∗R* (*b* − *a*) **for** *u*
by (*simp add*: *algebra-simps*)
assume *x* ∈ *open-segment a b*
then show *x* ∈ *rel-interior S*
unfolding *closed-segment-def open-segment-def* **using** *assms*
by (*auto intro*: *rel-interior-closure-convex-shrink*)
qed

19.30 More results about segments

lemma *dist-half-times2*:
fixes *a* :: *'a* :: *real-normed-vector*
shows *dist* ((*1* / *2*) *∗R* (*a* + *b*)) *x ∗ 2* = *dist* (*a*+*b*) (*2 ∗R x*)
proof −
have *norm* ((*1* / *2*) *∗R* (*a* + *b*) − *x*) ∗ *2* = *norm* (*2 ∗R* ((*1* / *2*) *∗R* (*a* + *b*) − *x*))
by *simp*
also have ... = *norm* ((*a* + *b*) − *2 ∗R x*)
by (*simp add*: *real-vector.scale-right-diff-distrib*)
finally show *?thesis*
by (*simp only*: *dist-norm*)
qed

lemma *closed-segment-as-ball*:
closed-segment a b = *affine hull* {*a*,*b*} ∩ *cball*(*inverse 2 ∗R* (*a* + *b*))(*norm*(*b* − *a*) / *2*)
proof (*cases b* = *a*)
case *True* **then show** *?thesis* **by** (*auto simp*: *hull-inc*)
next
case *False*
then have ∗: ((∃ *u v*. *x* = *u ∗R a* + *v ∗R b* ∧ *u* + *v* = *1*) ∧
dist ((*1* / *2*) *∗R* (*a* + *b*)) *x ∗ 2* ≤ *norm* (*b* − *a*)) =
(∃ *u*. *x* = (*1* − *u*) *∗R a* + *u ∗R b* ∧ *0* ≤ *u* ∧ *u* ≤ *1*) **for** *x*
proof −
have ((∃ *u v*. *x* = *u ∗R a* + *v ∗R b* ∧ *u* + *v* = *1*) ∧
dist ((*1* / *2*) *∗R* (*a* + *b*)) *x ∗ 2* ≤ *norm* (*b* − *a*)) =
((∃ *u*. *x* = (*1* − *u*) *∗R a* + *u ∗R b*) ∧
dist ((*1* / *2*) *∗R* (*a* + *b*)) *x ∗ 2* ≤ *norm* (*b* − *a*))
unfolding *eq-diff-eq* [*symmetric*] **by** *simp*
also have ... = (∃ *u*. *x* = (*1* − *u*) *∗R a* + *u ∗R b* ∧
norm ((*a*+*b*) − (*2 ∗R x*)) ≤ *norm* (*b* − *a*))
by (*simp add*: *dist-half-times2*) (*simp add*: *dist-norm*)
also have ... = (∃ *u*. *x* = (*1* − *u*) *∗R a* + *u ∗R b* ∧
norm ((*a*+*b*) − (*2 ∗R* ((*1* − *u*) *∗R a* + *u ∗R b*))) ≤ *norm* (*b* − *a*))
by *auto*

```
    also have ... = (∃u. x = (1 - u) *R a + u *R b ∧
              norm ((1 - u * 2) *R (b - a)) ≤ norm (b - a))
      by (simp add: algebra-simps scaleR-2)
    also have ... = (∃u. x = (1 - u) *R a + u *R b ∧
                        |1 - u * 2| * norm (b - a) ≤ norm (b - a))
      by simp
    also have ... = (∃u. x = (1 - u) *R a + u *R b ∧ |1 - u * 2| ≤ 1)
      by (simp add: mult-le-cancel-right2 False)
    also have ... = (∃u. x = (1 - u) *R a + u *R b ∧ 0 ≤ u ∧ u ≤ 1)
      by auto
    finally show ?thesis .
  qed
  show ?thesis
    by (simp add: affine-hull-2 Set.set-eq-iff closed-segment-def *)
qed

lemma open-segment-as-ball:
    open-segment a b =
     affine hull {a,b} ∩ ball(inverse 2 *R (a + b))(norm(b - a) / 2)
proof (cases b = a)
  case True then show ?thesis by (auto simp: hull-inc)
next
  case False
  then have *: ((∃u v. x = u *R a + v *R b ∧ u + v = 1) ∧
                dist ((1 / 2) *R (a + b)) x * 2 < norm (b - a)) =
               (∃u. x = (1 - u) *R a + u *R b ∧ 0 < u ∧ u < 1) for x
  proof -
    have ((∃u v. x = u *R a + v *R b ∧ u + v = 1) ∧
                dist ((1 / 2) *R (a + b)) x * 2 < norm (b - a)) =
         ((∃u. x = (1 - u) *R a + u *R b) ∧
                dist ((1 / 2) *R (a + b)) x * 2 < norm (b - a))
      unfolding eq-diff-eq [symmetric] by simp
    also have ... = (∃u. x = (1 - u) *R a + u *R b ∧
                       norm ((a+b) - (2 *R x)) < norm (b - a))
      by (simp add: dist-half-times2) (simp add: dist-norm)
    also have ... = (∃u. x = (1 - u) *R a + u *R b ∧
          norm ((a+b) - (2 *R ((1 - u) *R a + u *R b))) < norm (b - a))
      by auto
    also have ... = (∃u. x = (1 - u) *R a + u *R b ∧
              norm ((1 - u * 2) *R (b - a)) < norm (b - a))
      by (simp add: algebra-simps scaleR-2)
    also have ... = (∃u. x = (1 - u) *R a + u *R b ∧
                        |1 - u * 2| * norm (b - a) < norm (b - a))
      by simp
    also have ... = (∃u. x = (1 - u) *R a + u *R b ∧ |1 - u * 2| < 1)
      by (simp add: mult-le-cancel-right2 False)
    also have ... = (∃u. x = (1 - u) *R a + u *R b ∧ 0 < u ∧ u < 1)
      by auto
    finally show ?thesis .
```

qed
show *?thesis*
using *False* **by** (*force simp*: *affine-hull-2 Set.set-eq-iff open-segment-image-interval*
∗)
qed

lemmas *segment-as-ball* = *closed-segment-as-ball open-segment-as-ball*

lemma *closed-segment-neq-empty* [*simp*]: *closed-segment a b* ≠ {}
by *auto*

lemma *open-segment-eq-empty* [*simp*]: *open-segment a b* = {} ⟷ *a* = *b*
proof −
{ assume *a1*: *open-segment a b* = {}
have {} ≠ {*0*::*real*<..<*1*}
by *simp*
then have *a* = *b*
using *a1 open-segment-image-interval* **by** *fastforce*
} then show *?thesis* **by** *auto*
qed

lemma *open-segment-eq-empty′* [*simp*]: {} = *open-segment a b* ⟷ *a* = *b*
using *open-segment-eq-empty* **by** *blast*

lemmas *segment-eq-empty* = *closed-segment-neq-empty open-segment-eq-empty*

lemma *inj-segment*:
fixes *a* :: *′a* :: *real-vector*
assumes *a* ≠ *b*
shows *inj-on* (*λu*. (*1* − *u*) *∗R a* + *u ∗R b*) *I*
proof
fix *x y*
assume (*1* − *x*) *∗R a* + *x ∗R b* = (*1* − *y*) *∗R a* + *y ∗R b*
then have *x ∗R* (*b* − *a*) = *y ∗R* (*b* − *a*)
by (*simp add*: *algebra-simps*)
with *assms* **show** *x* = *y*
by (*simp add*: *real-vector.scale-right-imp-eq*)
qed

lemma *finite-closed-segment* [*simp*]: *finite*(*closed-segment a b*) ⟷ *a* = *b*
apply *auto*
apply (*rule ccontr*)
apply (*simp add*: *segment-image-interval*)
using *infinite-Icc* [*OF zero-less-one*] *finite-imageD* [*OF - inj-segment*] **apply**
blast
done

lemma *finite-open-segment* [*simp*]: *finite*(*open-segment a b*) ⟷ *a* = *b*
by (*auto simp*: *open-segment-def*)

lemmas *finite-segment = finite-closed-segment finite-open-segment*

lemma *closed-segment-eq-sing*: *closed-segment a b = {c} ⟷ a = c ∧ b = c*
by *auto*

lemma *open-segment-eq-sing*: *open-segment a b ≠ {c}*
by (*metis finite-insert finite-open-segment insert-not-empty open-segment-image-interval*)

lemmas *segment-eq-sing = closed-segment-eq-sing open-segment-eq-sing*

lemma *subset-closed-segment*:
closed-segment a b ⊆ closed-segment c d ⟷
a ∈ closed-segment c d ∧ b ∈ closed-segment c d
by *auto* (*meson contra-subsetD convex-closed-segment convex-contains-segment*)

lemma *subset-co-segment*:
closed-segment a b ⊆ open-segment c d ⟷
a ∈ open-segment c d ∧ b ∈ open-segment c d
using *closed-segment-subset* **by** *blast*

lemma *subset-open-segment*:
fixes *a* :: *'a*::*euclidean-space*
shows *open-segment a b ⊆ open-segment c d ⟷*
a = b ∨ a ∈ closed-segment c d ∧ b ∈ closed-segment c d
(**is** *?lhs = ?rhs*)
proof (*cases a = b*)
case *True* **then show** *?thesis* **by** *simp*
next
case *False* **show** *?thesis*
proof
assume *rhs*: *?rhs*
with ‹*a ≠ b*› **have** *c ≠ d*
using *closed-segment-idem singleton-iff* **by** *auto*
have ∃ *uc*. (*1 − u*) *∗R* ((*1 − ua*) *∗R c* + *ua ∗R d*) + *u ∗R* ((*1 − ub*) *∗R c* + *ub ∗R d*) =
(*1 − uc*) *∗R c* + *uc ∗R d* ∧ *0 < uc* ∧ *uc < 1*
if *neq*: (*1 − ua*) *∗R c* + *ua ∗R d ≠* (*1 − ub*) *∗R c* + *ub ∗R d c ≠ d*
and *a* = (*1 − ua*) *∗R c* + *ua ∗R d b* = (*1 − ub*) *∗R c* + *ub ∗R d*
and *u*: *0 < u u < 1* **and** *uab*: *0 ≤ ua ua ≤ 1 0 ≤ ub ub ≤ 1*
for *u ua ub*
proof −
have *ua ≠ ub*
using *neq* **by** *auto*
moreover have (*u − 1*) *∗ ua ≤ 0* **using** *u uab*
by (*simp add*: *mult-nonpos-nonneg*)
ultimately have *lt*: (*u − 1*) *∗ ua < u ∗ ub* **using** *u uab*
by (*metis antisym-conv diff-ge-0-iff-ge le-less-trans mult-eq-0-iff mult-le-0-iff not-less*)

have $p * ua + q * ub < p+q$ **if** p: $0 < p$ **and** q: $0 < q$ **for** $p\ q$
proof −
have ¬ $p \leq 0$ ¬ $q \leq 0$
using $p\ q$ *not-less* **by** *blast+*
then show *?thesis*
by (*metis* ‹$ua \neq ub$› *add-less-cancel-left add-less-cancel-right add-mono-thms-linordered-field*(5)
less-eq-real-def mult-cancel-left1 mult-less-cancel-left2 uab(2) *uab*(4))
qed
then have $(1 - u) * ua + u * ub < 1$ **using** u ‹$ua \neq ub$›
by (*metis diff-add-cancel diff-gt-0-iff-gt*)
with *lt* **show** *?thesis*
by (*rule-tac* x=$ua + u$*(ub−ua) **in** *exI*) (*simp add*: *algebra-simps*)
qed
with *rhs* ‹$a \neq b$› ‹$c \neq d$› **show** *?lhs*
unfolding *open-segment-image-interval closed-segment-def*
by (*fastforce simp add*:)
next
assume *lhs*: *?lhs*
with ‹$a \neq b$› **have** $c \neq d$
by (*meson finite-open-segment rev-finite-subset*)
have *closure* (*open-segment a b*) ⊆ *closure* (*open-segment c d*)
using *lhs closure-mono* **by** *blast*
then have *closed-segment a b* ⊆ *closed-segment c d*
by (*simp add*: ‹$a \neq b$› ‹$c \neq d$›)
then show *?rhs*
by (*force simp*: ‹$a \neq b$›)
qed
qed

lemma *subset-oc-segment*:
fixes a :: $'a$::*euclidean-space*
shows *open-segment a b* ⊆ *closed-segment c d* ⟷
$a = b \vee a \in$ *closed-segment c d* $\wedge$ $b \in$ *closed-segment c d*
apply (*simp add*: *subset-open-segment* [*symmetric*])
apply (*rule iffI*)
apply (*metis closure-closed-segment closure-mono closure-open-segment subset-closed-segment subset-open-segment*)
apply (*meson dual-order.trans segment-open-subset-closed*)
done

lemmas *subset-segment* = *subset-closed-segment subset-co-segment subset-oc-segment subset-open-segment*

19.31 Betweenness

lemma *between-mem-segment*: *between* (a,b) x ⟷ $x \in$ *closed-segment a b*
unfolding *between-def* **by** *auto*

lemma *between*: *between* (a, b) (x::$'a$::*euclidean-space*) ⟷ *dist a b* = (*dist a x*)

```
+ (dist x b)
proof (cases a = b)
  case True
  then show ?thesis
    unfolding between-def split-conv
    by (auto simp add: dist-commute)
next
  case False
  then have Fal: norm (a - b) ≠ 0 and Fal2: norm (a - b) > 0
    by auto
  have *: ⋀u. a - ((1 - u) *R a + u *R b) = u *R (a - b)
    by (auto simp add: algebra-simps)
  show ?thesis
    unfolding between-def split-conv closed-segment-def mem-Collect-eq
    apply rule
    apply (elim exE conjE)
    apply (subst dist-triangle-eq)
  proof -
    fix u
    assume as: x = (1 - u) *R a + u *R b 0 ≤ u u ≤ 1
    then have *: a - x = u *R (a - b) x - b = (1 - u) *R (a - b)
      unfolding as(1) by (auto simp add:algebra-simps)
    show norm (a - x) *R (x - b) = norm (x - b) *R (a - x)
      unfolding norm-minus-commute[of x a] * using as(2,3)
      by (auto simp add: field-simps)
  next
    assume as: dist a b = dist a x + dist x b
    have norm (a - x) / norm (a - b) ≤ 1
      using Fal2 unfolding as[unfolded dist-norm] norm-ge-zero by auto
    then show ∃ u. x = (1 - u) *R a + u *R b ∧ 0 ≤ u ∧ u ≤ 1
      apply (rule-tac x=dist a x / dist a b in exI)
      unfolding dist-norm
      apply (subst euclidean-eq-iff)
      apply rule
      defer
      apply rule
      prefer 3
      apply rule
    proof -
      fix i :: 'a
      assume i: i ∈ Basis
      have ((1 - norm (a - x) / norm (a - b)) *R a + (norm (a - x) / norm
(a - b)) *R b) • i =
        ((norm (a - b) - norm (a - x)) * (a • i) + norm (a - x) * (b • i)) /
norm (a - b)
        using Fal by (auto simp add: field-simps inner-simps)
      also have ... = x•i
        apply (rule divide-eq-imp[OF Fal])
        unfolding as[unfolded dist-norm]
```

using *as*[*unfolded dist-triangle-eq*]
apply −
apply (*subst* (*asm*) *euclidean-eq-iff*)
using *i*
apply (*erule-tac x*=*i* **in** *ballE*)
apply (*auto simp add*: *field-simps inner-simps*)
done
finally show $x \cdot i =$
$((1 - norm\ (a - x)\ /\ norm\ (a - b)) *_R a + (norm\ (a - x)\ /\ norm\ (a - b)) *_R b) \cdot i$
by *auto*
qed (*insert Fal2*, *auto*)
qed
qed

lemma *between-midpoint*:
fixes *a* :: *'a*::*euclidean-space*
shows *between* (*a*,*b*) (*midpoint a b*) (**is** *?t1*)
and *between* (*b*,*a*) (*midpoint a b*) (**is** *?t2*)
proof −
have ∗: $\bigwedge x\ y\ z.\ x = (1/2::real) *_R z \Longrightarrow y = (1/2) *_R z \Longrightarrow norm\ z = norm\ x + norm\ y$
by *auto*
show *?t1 ?t2*
unfolding *between midpoint-def dist-norm*
apply(*rule-tac*[!] ∗)
unfolding *euclidean-eq-iff*[**where** *'a*=*'a*]
apply (*auto simp add*: *field-simps inner-simps*)
done
qed

lemma *between-mem-convex-hull*:
between (*a*,*b*) *x* ⟷ *x* ∈ *convex hull* {*a*,*b*}
unfolding *between-mem-segment segment-convex-hull* **..**

19.32 Shrinking towards the interior of a convex set

lemma *mem-interior-convex-shrink*:
fixes *s* :: *'a*::*euclidean-space set*
assumes *convex s*
and *c* ∈ *interior s*
and *x* ∈ *s*
and *0* < *e*
and *e* ≤ *1*
shows $x - e *_R (x - c) \in interior\ s$
proof −
obtain *d* **where** *d* > *0* **and** *d*: *ball c d* ⊆ *s*
using *assms*(*2*) **unfolding** *mem-interior* **by** *auto*
show *?thesis*

```
    unfolding mem-interior
    apply (rule-tac x=e*d in exI)
    apply rule
    defer
    unfolding subset-eq Ball-def mem-ball
  proof (rule, rule)
    fix y
    assume as: dist (x - e *R (x - c)) y < e * d
    have *: y = (1 - (1 - e)) *R ((1 / e) *R y - ((1 - e) / e) *R x) + (1 -
e) *R x
      using ‹e > 0› by (auto simp add: scaleR-left-diff-distrib scaleR-right-diff-distrib)
    have dist c ((1 / e) *R y - ((1 - e) / e) *R x) = |1/e| * norm (e *R c - y
+ (1 - e) *R x)
      unfolding dist-norm
      unfolding norm-scaleR[symmetric]
      apply (rule arg-cong[where f=norm])
      using ‹e > 0›
      by (auto simp add: euclidean-eq-iff[where 'a='a] field-simps inner-simps)
    also have ... = |1/e| * norm (x - e *R (x - c) - y)
      by (auto intro!:arg-cong[where f=norm] simp add: algebra-simps)
    also have ... < d
      using as[unfolded dist-norm] and ‹e > 0›
      by (auto simp add:pos-divide-less-eq[OF ‹e > 0›] mult.commute)
    finally show y ∈ s
      apply (subst *)
      apply (rule assms(1)[unfolded convex-alt,rule-format])
      apply (rule d[unfolded subset-eq,rule-format])
      unfolding mem-ball
      using assms(3-5)
      apply auto
      done
  qed (insert ‹e>0› ‹d>0›, auto)
qed

lemma mem-interior-closure-convex-shrink:
  fixes s :: 'a::euclidean-space set
  assumes convex s
    and c ∈ interior s
    and x ∈ closure s
    and 0 < e
    and e ≤ 1
  shows x - e *R (x - c) ∈ interior s
proof -
  obtain d where d > 0 and d: ball c d ⊆ s
    using assms(2) unfolding mem-interior by auto
  have ∃y∈s. norm (y - x) * (1 - e) < e * d
  proof (cases x ∈ s)
    case True
    then show ?thesis
```

```
      using ‹e > 0› ‹d > 0›
      apply (rule-tac bexI[where x=x])
      apply (auto)
      done
  next
    case False
    then have x: x islimpt s
      using assms(3)[unfolded closure-def] by auto
    show ?thesis
    proof (cases e = 1)
      case True
      obtain y where y ∈ s y ≠ x dist y x < 1
        using x[unfolded islimpt-approachable,THEN spec[where x=1]] by auto
      then show ?thesis
        apply (rule-tac x=y in bexI)
        unfolding True
        using ‹d > 0›
        apply auto
        done
    next
      case False
      then have 0 < e * d / (1 - e) and *: 1 - e > 0
        using ‹e ≤ 1› ‹e > 0› ‹d > 0› by auto
      then obtain y where y ∈ s y ≠ x dist y x < e * d / (1 - e)
       using x[unfolded islimpt-approachable,THEN spec[where x=e*d / (1 - e)]]
by auto
      then show ?thesis
        apply (rule-tac x=y in bexI)
        unfolding dist-norm
        using pos-less-divide-eq[OF *]
        apply auto
        done
    qed
  qed
  then obtain y where y ∈ s and y: norm (y - x) * (1 - e) < e * d
    by auto
  def z ≡ c + ((1 - e) / e) *R (x - y)
  have *: x - e *R (x - c) = y - e *R (y - z)
    unfolding z-def using ‹e > 0›
   by (auto simp add: scaleR-right-diff-distrib scaleR-right-distrib scaleR-left-diff-distrib)
  have z ∈ interior s
    apply (rule interior-mono[OF d,unfolded subset-eq,rule-format])
    unfolding interior-open[OF open-ball] mem-ball z-def dist-norm using y and
assms(4,5)
    apply (auto simp add:field-simps norm-minus-commute)
    done
  then show ?thesis
    unfolding *
    apply -
```

apply (*rule mem-interior-convex-shrink*)
using *assms*(*1*,*4*−*5*) ‹*y*∈*s*›
apply *auto*
done
qed

19.33 Some obvious but surprisingly hard simplex lemmas

lemma *simplex*:
assumes *finite s*
and *0* ∉ *s*
shows *convex hull* (*insert 0 s*) =
{*y*. (∃ *u*. (∀ *x*∈*s*. *0* ≤ *u x*) ∧ *setsum u s* ≤ *1* ∧ *setsum* (λ*x*. *u x* ∗$_R$ *x*) *s* = *y*)}
unfolding *convex-hull-finite*[*OF finite.insertI*[*OF assms*(*1*)]]
apply (*rule set-eqI*, *rule*)
unfolding *mem-Collect-eq*
apply (*erule-tac*[!] *exE*)
apply (*erule-tac*[!] *conjE*)+
unfolding *setsum-clauses*(*2*)[*OF assms*(*1*)]
apply (*rule-tac x*=*u* **in** *exI*)
defer
apply (*rule-tac x*=λ*x*. *if x* = *0 then 1* − *setsum u s else u x* **in** *exI*)
using *assms*(*2*)
unfolding *if-smult* **and** *setsum-delta-notmem*[*OF assms*(*2*)]
apply *auto*
done

lemma *substd-simplex*:
assumes *d*: *d* ⊆ *Basis*
shows *convex hull* (*insert 0 d*) =
{*x*. (∀ *i*∈*Basis*. *0* ≤ *x*•*i*) ∧ (∑ *i*∈*d*. *x*•*i*) ≤ *1* ∧ (∀ *i*∈*Basis*. *i* ∉ *d* ⟶ *x*•*i* = *0*)}
(**is** *convex hull* (*insert 0 ?p*) = *?s*)
proof −
let *?D* = *d*
have *0* ∉ *?p*
using *assms* **by** (*auto simp*: *image-def*)
from *d* **have** *finite d*
by (*blast intro*: *finite-subset finite-Basis*)
show *?thesis*
unfolding *simplex*[*OF* ‹*finite d*› ‹*0* ∉ *?p*›]
apply (*rule set-eqI*)
unfolding *mem-Collect-eq*
apply *rule*
apply (*elim exE conjE*)
apply (*erule-tac*[*2*] *conjE*)+
proof −
fix *x* :: *'a*::*euclidean-space*
fix *u*

```
    assume as: ∀x∈?D. 0 ≤ u x setsum u ?D ≤ 1 (∑x∈?D. u x *R x) = x
    have *: ∀i∈Basis. i:d ⟶ u i = x•i
      and (∀i∈Basis. i ∉ d ⟶ x • i = 0)
      using as(3)
      unfolding substdbasis-expansion-unique[OF assms]
      by auto
    then have **: setsum u ?D = setsum (op • x) ?D
      apply −
      apply (rule setsum.cong)
      using assms
      apply auto
      done
    have (∀i∈Basis. 0 ≤ x•i) ∧ setsum (op • x) ?D ≤ 1
    proof (rule,rule)
      fix i :: 'a
      assume i: i ∈ Basis
      have i ∈ d ⟹ 0 ≤ x•i
        unfolding *[rule-format,OF i,symmetric]
        apply (rule-tac as(1)[rule-format])
        apply auto
        done
      moreover have i ∉ d ⟹ 0 ≤ x•i
        using ‹(∀i∈Basis. i ∉ d ⟶ x • i = 0)›[rule-format, OF i] by auto
      ultimately show 0 ≤ x•i by auto
    qed (insert as(2)[unfolded **], auto)
    then show (∀i∈Basis. 0 ≤ x•i) ∧ setsum (op • x) ?D ≤ 1 ∧ (∀i∈Basis. i ∉
d ⟶ x • i = 0)
      using ‹(∀i∈Basis. i ∉ d ⟶ x • i = 0)› by auto
  next
    fix x :: 'a::euclidean-space
    assume as: ∀i∈Basis. 0 ≤ x • i setsum (op • x) ?D ≤ 1 (∀i∈Basis. i ∉ d
⟶ x • i = 0)
    show ∃u. (∀x∈?D. 0 ≤ u x) ∧ setsum u ?D ≤ 1 ∧ (∑x∈?D. u x *R x) = x
      using as d
      unfolding substdbasis-expansion-unique[OF assms]
      apply (rule-tac x=inner x in exI)
      apply auto
      done
  qed
qed

lemma std-simplex:
  convex hull (insert 0 Basis) =
    {x::'a::euclidean-space. (∀i∈Basis. 0 ≤ x•i) ∧ setsum (λi. x•i) Basis ≤ 1}
  using substd-simplex[of Basis] by auto

lemma interior-std-simplex:
  interior (convex hull (insert 0 Basis)) =
    {x::'a::euclidean-space. (∀i∈Basis. 0 < x•i) ∧ setsum (λi. x•i) Basis < 1}
```

```
  apply (rule set-eqI)
  unfolding mem-interior std-simplex
  unfolding subset-eq mem-Collect-eq Ball-def mem-ball
  unfolding Ball-def[symmetric]
  apply rule
  apply (elim exE conjE)
  defer
  apply (erule conjE)
proof −
  fix x :: 'a
  fix e
  assume e > 0 and as: ∀ xa. dist x xa < e ⟶ (∀ x∈Basis. 0 ≤ xa • x) ∧ setsum
(op • xa) Basis ≤ 1
  show (∀ xa∈Basis. 0 < x • xa) ∧ setsum (op • x) Basis < 1
    apply safe
  proof −
    fix i :: 'a
    assume i: i ∈ Basis
    then show 0 < x • i
      using as[THEN spec[where x=x − (e / 2) *R i]] and ‹e > 0›
      unfolding dist-norm
      by (auto elim!: ballE[where x=i] simp: inner-simps)
  next
    have **: dist x (x + (e / 2) *R (SOME i. i∈Basis)) < e using ‹e > 0›
      unfolding dist-norm
      by (auto intro!: mult-strict-left-mono simp: SOME-Basis)
    have ⋀i. i ∈ Basis ⟹ (x + (e / 2) *R (SOME i. i∈Basis)) • i =
      x•i + (if i = (SOME i. i∈Basis) then e/2 else 0)
      by (auto simp: SOME-Basis inner-Basis inner-simps)
    then have *: setsum (op • (x + (e / 2) *R (SOME i. i∈Basis))) Basis =
      setsum (λi. x•i + (if (SOME i. i∈Basis) = i then e/2 else 0)) Basis
      apply (rule-tac setsum.cong)
      apply auto
      done
   have setsum (op • x) Basis < setsum (op • (x + (e / 2) *R (SOME i. i∈Basis)))
Basis
      unfolding * setsum.distrib
      using ‹e > 0› DIM-positive[where 'a='a]
      apply (subst setsum.delta')
      apply (auto simp: SOME-Basis)
      done
    also have ... ≤ 1
      using **
      apply (drule-tac as[rule-format])
      apply auto
      done
    finally show setsum (op • x) Basis < 1 by auto
  qed
next
```

fix $x :: 'a$
assume *as*: $\forall i{\in}Basis.\ 0 < x \cdot i$ *setsum* $(op \cdot x)$ *Basis* < 1
obtain $a :: 'b$ **where** $a \in UNIV$ **using** *UNIV-witness* **..**
let *?d* $= (1 - setsum\ (op \cdot x)\ Basis) / real\ (DIM('a))$
have *Min* $((op \cdot x)\ `\ Basis) > 0$
apply (*rule Min-grI*)
using *as*(*1*)
apply *auto*
done
moreover have *?d* > 0
using *as*(*2*) **by** (*auto simp*: *Suc-le-eq DIM-positive*)
ultimately show $\exists e{>}0.\ \forall y.\ dist\ x\ y < e \longrightarrow (\forall i{\in}Basis.\ 0 \le y \cdot i) \wedge setsum\ (op \cdot y)\ Basis \le 1$
apply (*rule-tac x*=*min* (*Min* $((op \cdot x)\ `\ Basis))$ *D* **for** *D* **in** *exI*)
apply *rule*
defer
apply (*rule*, *rule*)
proof −
fix *y*
assume *y*: *dist x y* $<$ *min* (*Min* $(op \cdot x\ `\ Basis))$ *?d*
have *setsum* $(op \cdot y)$ *Basis* $\le$ *setsum* $(\lambda i.\ x{\cdot}i + ?d)$ *Basis*
proof (*rule setsum-mono*)
fix $i :: 'a$
assume *i*: $i \in Basis$
then have $|y{\cdot}i - x{\cdot}i| <$ *?d*
apply −
apply (*rule le-less-trans*)
using *Basis-le-norm*[*OF i*, *of y* − *x*]
using *y*[*unfolded min-less-iff-conj dist-norm*, *THEN conjunct2*]
apply (*auto simp add*: *norm-minus-commute inner-diff-left*)
done
then show $y \cdot i \le x \cdot i$ + *?d* **by** *auto*
qed
also have $\ldots \le 1$
unfolding *setsum.distrib setsum-constant*
by (*auto simp add*: *Suc-le-eq*)
finally show $(\forall i{\in}Basis.\ 0 \le y \cdot i) \wedge setsum\ (op \cdot y)\ Basis \le 1$
proof *safe*
fix $i :: 'a$
assume *i*: $i \in Basis$
have *norm* $(x - y) < x{\cdot}i$
apply (*rule less-le-trans*)
apply (*rule y*[*unfolded min-less-iff-conj dist-norm*, *THEN conjunct1*])
using *i*
apply *auto*
done
then show $0 \le y{\cdot}i$
using *Basis-le-norm*[*OF i*, *of x* − *y*] **and** *as*(*1*)[*rule-format*, *OF i*]
by (*auto simp*: *inner-simps*)

```
    qed
  qed auto
qed

lemma interior-std-simplex-nonempty:
  obtains a :: 'a::euclidean-space where
    a ∈ interior(convex hull (insert 0 Basis))
proof −
  let ?D = Basis :: 'a set
  let ?a = setsum (λb::'a. inverse (2 ∗ real DIM('a)) ∗R b) Basis
  {
    fix i :: 'a
    assume i: i ∈ Basis
    have ?a • i = inverse (2 ∗ real DIM('a))
      by (rule trans[of - setsum (λj. if i = j then inverse (2 ∗ real DIM('a)) else
0) ?D])
        (simp-all add: setsum.If-cases i) }
  note ∗∗ = this
  show ?thesis
    apply (rule that[of ?a])
    unfolding interior-std-simplex mem-Collect-eq
  proof safe
    fix i :: 'a
    assume i: i ∈ Basis
    show 0 < ?a • i
      unfolding ∗∗[OF i] by (auto simp add: Suc-le-eq DIM-positive)
  next
    have setsum (op • ?a) ?D = setsum (λi. inverse (2 ∗ real DIM('a))) ?D
      apply (rule setsum.cong)
      apply rule
      apply auto
      done
    also have ... < 1
      unfolding setsum-constant divide-inverse[symmetric]
      by (auto simp add: field-simps)
    finally show setsum (op • ?a) ?D < 1 by auto
  qed
qed

lemma rel-interior-substd-simplex:
  assumes d: d ⊆ Basis
  shows rel-interior (convex hull (insert 0 d)) =
    {x::'a::euclidean-space. (∀ i∈d. 0 < x•i) ∧ (∑ i∈d. x•i) < 1 ∧ (∀ i∈Basis. i ∉
d ⟶ x•i = 0)}
  (is rel-interior (convex hull (insert 0 ?p)) = ?s)
proof −
  have finite d
    apply (rule finite-subset)
    using assms
```

```
    apply auto
    done
  show ?thesis
  proof (cases d = {})
    case True
    then show ?thesis
      using rel-interior-sing using euclidean-eq-iff[of - 0] by auto
  next
    case False
    have h0: affine hull (convex hull (insert 0 ?p)) =
      {x::'a::euclidean-space. (∀ i∈Basis. i ∉ d ⟶ x•i = 0)}
      using affine-hull-convex-hull affine-hull-substd-basis assms by auto
    have aux: ⋀x::'a. ∀ i∈Basis. (∀ i∈d. 0 ≤ x•i) ∧ (∀ i∈Basis. i ∉ d ⟶ x•i =
0) ⟶ 0 ≤ x•i
      by auto
    {
      fix x :: 'a::euclidean-space
      assume x: x ∈ rel-interior (convex hull (insert 0 ?p))
      then obtain e where e0: e > 0 and
        ball x e ∩ {xa. (∀ i∈Basis. i ∉ d ⟶ xa•i = 0)} ⊆ convex hull (insert 0 ?p)
        using mem-rel-interior-ball[of x convex hull (insert 0 ?p)] h0 by auto
      then have as: ∀ xa. dist x xa < e ∧ (∀ i∈Basis. i ∉ d ⟶ xa•i = 0) ⟶
        (∀ i∈d. 0 ≤ xa • i) ∧ setsum (op • xa) d ≤ 1
        unfolding ball-def unfolding substd-simplex[OF assms] using assms by
auto
      have x0: (∀ i∈Basis. i ∉ d ⟶ x•i = 0)
        using x rel-interior-subset  substd-simplex[OF assms] by auto
      have (∀ i∈d. 0 < x • i) ∧ setsum (op • x) d < 1 ∧ (∀ i∈Basis. i ∉ d ⟶
x•i = 0)
        apply rule
        apply rule
      proof -
        fix i :: 'a
        assume i ∈ d
        then have ∀ ia∈d. 0 ≤ (x - (e / 2) *R i) • ia
          apply -
          apply (rule as[rule-format,THEN conjunct1])
          unfolding dist-norm
          using d ‹e > 0› x0
          apply (auto simp: inner-simps inner-Basis)
          done
        then show 0 < x • i
          apply (erule-tac x=i in ballE)
          using ‹e > 0› ‹i ∈ d› d
          apply (auto simp: inner-simps inner-Basis)
          done
      next
        obtain a where a: a ∈ d
          using ‹d ≠ {}› by auto
```

```
      then have **: dist x (x + (e / 2) *R a) < e
        using ‹e > 0› norm-Basis[of a] d
        unfolding dist-norm
        by auto
      have ⋀i. i ∈ Basis ⟹ (x + (e / 2) *R a) • i = x•i + (if i = a then e/2 else 0)
        using a d by (auto simp: inner-simps inner-Basis)
      then have *: setsum (op • (x + (e / 2) *R a)) d =
        setsum (λi. x•i + (if a = i then e/2 else 0)) d
        using d by (intro setsum.cong) auto
      have a ∈ Basis
        using ‹a ∈ d› d by auto
      then have h1: (∀ i∈Basis. i ∉ d ⟶ (x + (e / 2) *R a) • i = 0)
        using x0 d ‹a∈d› by (auto simp add: inner-add-left inner-Basis)
      have setsum (op • x) d < setsum (op • (x + (e / 2) *R a)) d
        unfolding * setsum.distrib
        using ‹e > 0› ‹a ∈ d›
        using ‹finite d›
        by (auto simp add: setsum.delta')
      also have ... ≤ 1
        using ** h1 as[rule-format, of x + (e / 2) *R a]
        by auto
      finally show setsum (op • x) d < 1 ∧ (∀ i∈Basis. i ∉ d ⟶ x•i = 0)
        using x0 by auto
    qed
  }
  moreover
  {
    fix x :: 'a::euclidean-space
    assume as: x ∈ ?s
    have ∀ i. 0 < x•i ∨ 0 = x•i ⟶ 0 ≤ x•i
      by auto
    moreover have ∀ i. i ∈ d ∨ i ∉ d by auto
    ultimately
    have ∀ i. (∀ i∈d. 0 < x•i) ∧ (∀ i. i ∉ d ⟶ x•i = 0) ⟶ 0 ≤ x•i
      by metis
    then have h2: x ∈ convex hull (insert 0 ?p)
      using as assms
      unfolding substd-simplex[OF assms] by fastforce
    obtain a where a: a ∈ d
      using ‹d ≠ {}› by auto
    let ?d = (1 - setsum (op • x) d) / real (card d)
    have 0 < card d using ‹d ≠ {}› ‹finite d›
      by (simp add: card-gt-0-iff)
    have Min ((op • x) ` d) > 0
      using as ‹d ≠ {}› ‹finite d› by (simp add: Min-grI)
    moreover have ?d > 0 using as using ‹0 < card d› by auto
    ultimately have h3: min (Min ((op • x) ` d)) ?d > 0
      by auto
```

```
  have x ∈ rel-interior (convex hull (insert 0 ?p))
    unfolding rel-interior-ball mem-Collect-eq h0
    apply (rule,rule h2)
    unfolding substd-simplex[OF assms]
    apply (rule-tac x=min (Min ((op · x) ` d)) ?d in exI)
    apply (rule, rule h3)
    apply safe
    unfolding mem-ball
  proof −
    fix y :: 'a
    assume y: dist x y < min (Min (op · x ` d)) ?d
    assume y2: ∀ i∈Basis. i ∉ d ⟶ y·i = 0
    have setsum (op · y) d ≤ setsum (λi. x·i + ?d) d
    proof (rule setsum-mono)
      fix i
      assume i ∈ d
      with d have i: i ∈ Basis
        by auto
      have |y·i − x·i| < ?d
        apply (rule le-less-trans)
        using Basis-le-norm[OF i, of y − x]
        using y[unfolded min-less-iff-conj dist-norm, THEN conjunct2]
        apply (auto simp add: norm-minus-commute inner-simps)
        done
      then show y · i ≤ x · i + ?d by auto
    qed
    also have ... ≤ 1
      unfolding setsum.distrib setsum-constant  using ‹0 < card d›
      by auto
    finally show setsum (op · y) d ≤ 1 .

    fix i :: 'a
    assume i: i ∈ Basis
    then show 0 ≤ y·i
    proof (cases i∈d)
      case True
      have norm (x − y) < x·i
        using y[unfolded min-less-iff-conj dist-norm, THEN conjunct1]
        using Min-gr-iff[of op · x ` d norm (x − y)] ‹0 < card d› ‹i:d›
        by (simp add: card-gt-0-iff)
      then show 0 ≤ y·i
        using Basis-le-norm[OF i, of x − y] and as(1)[rule-format]
        by (auto simp: inner-simps)
    qed (insert y2, auto)
  qed
}
ultimately have
  ⋀x. x ∈ rel-interior (convex hull insert 0 d) ⟷
```

```
      x ∈ {x. (∀i∈d. 0 < x · i) ∧ setsum (op · x) d < 1 ∧ (∀i∈Basis. i ∉ d
⟶ x · i = 0)}
    by blast
  then show ?thesis by (rule set-eqI)
 qed
qed

lemma rel-interior-substd-simplex-nonempty:
 assumes d ≠ {}
   and d ⊆ Basis
 obtains a :: 'a::euclidean-space
   where a ∈ rel-interior (convex hull (insert 0 d))
proof −
 let ?D = d
 let ?a = setsum (λb::'a::euclidean-space. inverse (2 * real (card d)) *R b) ?D
 have finite d
   apply (rule finite-subset)
   using assms(2)
   apply auto
   done
 then have d1: 0 < real (card d)
   using ‹d ≠ {}› by auto
 {
   fix i
   assume i ∈ d
   have ?a · i = inverse (2 * real (card d))
     apply (rule trans[of - setsum (λj. if i = j then inverse (2 * real (card d))
else 0) ?D])
     unfolding inner-setsum-left
     apply (rule setsum.cong)
     using ‹i ∈ d› ‹finite d› setsum.delta'[of d i (λk. inverse (2 * real (card d)))]
       d1 assms(2)
     by (auto simp: inner-Basis set-rev-mp[OF - assms(2)])
 }
 note ** = this
 show ?thesis
   apply (rule that[of ?a])
   unfolding rel-interior-substd-simplex[OF assms(2)] mem-Collect-eq
 proof safe
   fix i
   assume i ∈ d
   have 0 < inverse (2 * real (card d))
     using d1 by auto
   also have ... = ?a · i using **[of i] ‹i ∈ d›
     by auto
   finally show 0 < ?a · i by auto
 next
   have setsum (op · ?a) ?D = setsum (λi. inverse (2 * real (card d))) ?D
     by (rule setsum.cong) (rule refl, rule **)
```

```
    also have ... < 1
      unfolding setsum-constant divide-real-def[symmetric]
      by (auto simp add: field-simps)
    finally show setsum (op · ?a) ?D < 1 by auto
  next
    fix i
    assume i ∈ Basis and i ∉ d
    have ?a ∈ span d
    proof (rule span-setsum[of d (λb. b /R (2 * real (card d))) d])
      {
        fix x :: 'a::euclidean-space
        assume x ∈ d
        then have x ∈ span d
          using span-superset[of - d] by auto
        then have x /R (2 * real (card d)) ∈ span d
          using span-mul[of x d (inverse (real (card d)) / 2)] by auto
      }
      then show ∀x∈d. x /R (2 * real (card d)) ∈ span d
        by auto
    qed
    then show ?a · i = 0
      using ‹i ∉ d› unfolding span-substd-basis[OF assms(2)] using ‹i ∈ Basis›
by auto
  qed
qed
```

19.34 Relative interior of convex set

```
lemma rel-interior-convex-nonempty-aux:
  fixes S :: 'n::euclidean-space set
  assumes convex S
    and 0 ∈ S
  shows rel-interior S ≠ {}
proof (cases S = {0})
  case True
  then show ?thesis using rel-interior-sing by auto
next
  case False
  obtain B where B: independent B ∧ B ≤ S ∧ S ≤ span B ∧ card B = dim S
    using basis-exists[of S] by auto
  then have B ≠ {}
    using B assms ‹S ≠ {0}› span-empty by auto
  have insert 0 B ≤ span B
    using subspace-span[of B] subspace-0[of span B] span-inc by auto
  then have span (insert 0 B) ≤ span B
    using span-span[of B] span-mono[of insert 0 B span B] by blast
  then have convex hull insert 0 B ≤ span B
    using convex-hull-subset-span[of insert 0 B] by auto
  then have span (convex hull insert 0 B) ≤ span B
```

using *span-span*[*of B*] *span-mono*[*of convex hull insert 0 B span B*] **by** *blast*
then have ∗: *span* (*convex hull insert 0 B*) = *span B*
using *span-mono*[*of B convex hull insert 0 B*] *hull-subset*[*of insert 0 B*] **by** *auto*
then have *span* (*convex hull insert 0 B*) = *span S*
using *B span-mono*[*of B S*] *span-mono*[*of S span B*] *span-span*[*of B*] **by** *auto*
moreover have *0* ∈ *affine hull* (*convex hull insert 0 B*)
using *hull-subset*[*of convex hull insert 0 B*] *hull-subset*[*of insert 0 B*] **by** *auto*
ultimately have ∗∗: *affine hull* (*convex hull insert 0 B*) = *affine hull S*
using *affine-hull-span-0*[*of convex hull insert 0 B*] *affine-hull-span-0*[*of S*]
assms hull-subset[*of S*]
by *auto*
obtain *d* **and** *f* :: *'n* ⇒ *'n* **where**
fd: *card d* = *card B linear f f ' B* = *d*
f ' span B = {*x*. ∀ *i*∈*Basis*. *i* ∉ *d* ⟶ *x* ∙ *i* = (*0*::*real*)} ∧ *inj-on f* (*span B*)
and *d*: *d* ⊆ *Basis*
using *basis-to-substdbasis-subspace-isomorphism*[*of B*,*OF -] B* **by** *auto*
then have *bounded-linear f*
using *linear-conv-bounded-linear* **by** *auto*
have *d* ≠ {}
using *fd B* ‹*B* ≠ {}› **by** *auto*
have *insert 0 d* = *f '* (*insert 0 B*)
using *fd linear-0* **by** *auto*
then have (*convex hull* (*insert 0 d*)) = *f '* (*convex hull* (*insert 0 B*))
using *convex-hull-linear-image*[*of f* (*insert 0 d*)]
convex-hull-linear-image[*of f* (*insert 0 B*)] ‹*linear f*›
by *auto*
moreover have *rel-interior* (*f '* (*convex hull insert 0 B*)) =
f ' rel-interior (*convex hull insert 0 B*)
apply (*rule rel-interior-injective-on-span-linear-image*[*of f* (*convex hull insert 0 B*)])
using ‹*bounded-linear f*› *fd* ∗
apply *auto*
done
ultimately have *rel-interior* (*convex hull insert 0 B*) ≠ {}
using *rel-interior-substd-simplex-nonempty*[*OF* ‹*d* ≠ {}› *d*]
apply *auto*
apply *blast*
done
moreover have *convex hull* (*insert 0 B*) ⊆ *S*
using *B assms hull-mono*[*of insert 0 B S convex*] *convex-hull-eq*
by *auto*
ultimately show *?thesis*
using *subset-rel-interior*[*of convex hull insert 0 B S*] ∗∗ **by** *auto*
qed

lemma *rel-interior-eq-empty*:
fixes *S* :: *'n*::*euclidean-space set*
assumes *convex S*
shows *rel-interior S* = {} ⟷ *S* = {}

proof −
{
assume $S \neq \{\}$
then obtain a **where** $a \in S$ **by** *auto*
then have $0 \in op + (-a)$ ` S
using *assms exI*[*of* $(\lambda x.\ x \in S \wedge -\ a + x = 0)$ a] **by** *auto*
then have *rel-interior* $(op + (-a)$ ` $S) \neq \{\}$
using *rel-interior-convex-nonempty-aux*[*of* $op + (-a)$ ` S]
convex-translation[*of* S $-a$] *assms*
by *auto*
then have *rel-interior* $S \neq \{\}$
using *rel-interior-translation* **by** *auto*
}
then show *?thesis*
using *rel-interior-empty* **by** *auto*
qed

lemma *convex-rel-interior*:
fixes S :: *'n*::*euclidean-space set*
assumes *convex* S
shows *convex* (*rel-interior* S)
proof −
{
fix x y **and** u :: *real*
assume *assm*: $x \in$ *rel-interior* S $y \in$ *rel-interior* S $0 \leq u$ $u \leq 1$
then have $x \in S$
using *rel-interior-subset* **by** *auto*
have $x - u *_R (x - y) \in$ *rel-interior* S
proof (*cases* $0 = u$)
case *False*
then have $0 < u$ **using** *assm* **by** *auto*
then show *?thesis*
using *assm rel-interior-convex-shrink*[*of* S y x u] *assms* ‹$x \in S$› **by** *auto*
next
case *True*
then show *?thesis* **using** *assm* **by** *auto*
qed
then have $(1 - u) *_R x + u *_R y \in$ *rel-interior* S
by (*simp add*: *algebra-simps*)
}
then show *?thesis*
unfolding *convex-alt* **by** *auto*
qed

lemma *convex-closure-rel-interior*:
fixes S :: *'n*::*euclidean-space set*
assumes *convex* S
shows *closure* (*rel-interior* S) = *closure* S
proof −

```
have h1: closure (rel-interior S) ≤ closure S
  using closure-mono[of rel-interior S S] rel-interior-subset[of S] by auto
show ?thesis
proof (cases S = {})
  case False
  then obtain a where a: a ∈ rel-interior S
    using rel-interior-eq-empty assms by auto
  { fix x
    assume x: x ∈ closure S
    {
      assume x = a
      then have x ∈ closure (rel-interior S)
        using a unfolding closure-def by auto
    }
    moreover
    {
      assume x ≠ a
      {
        fix e :: real
        assume e > 0
        def e1 ≡ min 1 (e/norm (x − a))
        then have e1: e1 > 0 e1 ≤ 1 e1 * norm (x − a) ≤ e
          using ‹x ≠ a› ‹e > 0› le-divide-eq[of e1 e norm (x − a)]
          by simp-all
        then have *: x − e1 *R (x − a) : rel-interior S
          using rel-interior-closure-convex-shrink[of S a x e1] assms x a e1-def
          by auto
        have ∃ y. y ∈ rel-interior S ∧ y ≠ x ∧ dist y x ≤ e
          apply (rule-tac x=x − e1 *R (x − a) in exI)
          using * e1 dist-norm[of x − e1 *R (x − a) x] ‹x ≠ a›
          apply simp
          done
      }
      then have x islimpt rel-interior S
        unfolding islimpt-approachable-le by auto
      then have x ∈ closure(rel-interior S)
        unfolding closure-def by auto
    }
    ultimately have x ∈ closure(rel-interior S) by auto
  }
  then show ?thesis using h1 by auto
next
  case True
  then have rel-interior S = {}
    using rel-interior-empty by auto
  then have closure (rel-interior S) = {}
    using closure-empty by auto
  with True show ?thesis by auto
qed
```

qed

lemma *rel-interior-same-affine-hull*:
fixes S :: *'n::euclidean-space set*
assumes *convex S*
shows *affine hull* (*rel-interior S*) = *affine hull S*
by (*metis assms closure-same-affine-hull convex-closure-rel-interior*)

lemma *rel-interior-aff-dim*:
fixes S :: *'n::euclidean-space set*
assumes *convex S*
shows *aff-dim* (*rel-interior S*) = *aff-dim S*
by (*metis aff-dim-affine-hull2 assms rel-interior-same-affine-hull*)

lemma *rel-interior-rel-interior*:
fixes S :: *'n::euclidean-space set*
assumes *convex S*
shows *rel-interior* (*rel-interior S*) = *rel-interior S*
proof −
have *openin* (*subtopology euclidean* (*affine hull* (*rel-interior S*))) (*rel-interior S*)
using *opein-rel-interior*[*of S*] *rel-interior-same-affine-hull*[*of S*] *assms* **by** *auto*
then show *?thesis*
using *rel-interior-def* **by** *auto*
qed

lemma *rel-interior-rel-open*:
fixes S :: *'n::euclidean-space set*
assumes *convex S*
shows *rel-open* (*rel-interior S*)
unfolding *rel-open-def* **using** *rel-interior-rel-interior assms* **by** *auto*

lemma *convex-rel-interior-closure-aux*:
fixes x y z :: *'n::euclidean-space*
assumes $0 < a$ $0 < b$ $(a + b) *_R z = a *_R x + b *_R y$
obtains e **where** $0 < e$ $e \leq 1$ $z = y - e *_R (y - x)$
proof −
def $e \equiv a / (a + b)$
have $z = (1 / (a + b)) *_R ((a + b) *_R z)$
apply *auto*
using *assms*
apply *simp*
done
also have ... = $(1 / (a + b)) *_R (a *_R x + b *_R y)$
using *assms scaleR-cancel-left*[*of* $1/(a+b)$ $(a + b) *_R z$ $a *_R x + b *_R y$]
by *auto*
also have ... = $y - e *_R (y-x)$
using *e-def*
apply (*simp add*: *algebra-simps*)
using *scaleR-left-distrib*[*of* $a/(a+b)$ $b/(a+b)$ y] *assms add-divide-distrib*[*of a b*

```
a+b]
    apply auto
    done
  finally have z = y − e *R (y−x)
    by auto
  moreover have e > 0 using e-def assms by auto
  moreover have e ≤ 1 using e-def assms by auto
  ultimately show ?thesis using that[of e] by auto
qed

lemma convex-rel-interior-closure:
  fixes S :: 'n::euclidean-space set
  assumes convex S
  shows rel-interior (closure S) = rel-interior S
proof (cases S = {})
  case True
  then show ?thesis
    using assms rel-interior-eq-empty by auto
next
  case False
  have rel-interior (closure S) ⊇ rel-interior S
    using subset-rel-interior[of S closure S] closure-same-affine-hull closure-subset
    by auto
  moreover
  {
    fix z
    assume z: z ∈ rel-interior (closure S)
    obtain x where x: x ∈ rel-interior S
      using ‹S ≠ {}› assms rel-interior-eq-empty by auto
    have z ∈ rel-interior S
    proof (cases x = z)
      case True
      then show ?thesis using x by auto
    next
      case False
      obtain e where e: e > 0 cball z e ∩ affine hull closure S ≤ closure S
        using z rel-interior-cball[of closure S] by auto
      hence *: 0 < e/norm(z−x) using e False by auto
      def y ≡ z + (e/norm(z−x)) *R (z−x)
      have yball: y ∈ cball z e
        using mem-cball y-def dist-norm[of z y] e by auto
      have x ∈ affine hull closure S
        using x rel-interior-subset-closure hull-inc[of x closure S] by blast
      moreover have z ∈ affine hull closure S
        using z rel-interior-subset hull-subset[of closure S] by blast
      ultimately have y ∈ affine hull closure S
        using y-def affine-affine-hull[of closure S]
          mem-affine-3-minus [of affine hull closure S z z x e/norm(z−x)] by auto
      then have y ∈ closure S using e yball by auto
```

```
    have (1 + (e/norm(z−x))) *R z = (e/norm(z−x)) *R x + y
      using y-def by (simp add: algebra-simps)
    then obtain e1 where 0 < e1 e1 ≤ 1 z = y − e1 *R (y − x)
      using * convex-rel-interior-closure-aux[of e / norm (z − x) 1 z x y]
      by (auto simp add: algebra-simps)
    then show ?thesis
      using rel-interior-closure-convex-shrink assms x ‹y ∈ closure S›
      by auto
  qed
 }
 ultimately show ?thesis by auto
qed

lemma convex-interior-closure:
 fixes S :: 'n::euclidean-space set
 assumes convex S
 shows interior (closure S) = interior S
 using closure-aff-dim[of S] interior-rel-interior-gen[of S]
  interior-rel-interior-gen[of closure S]
  convex-rel-interior-closure[of S] assms
 by auto

lemma closure-eq-rel-interior-eq:
 fixes S1 S2 :: 'n::euclidean-space set
 assumes convex S1
  and convex S2
 shows closure S1 = closure S2 ⟷ rel-interior S1 = rel-interior S2
 by (metis convex-rel-interior-closure convex-closure-rel-interior assms)

lemma closure-eq-between:
 fixes S1 S2 :: 'n::euclidean-space set
 assumes convex S1
  and convex S2
 shows closure S1 = closure S2 ⟷ rel-interior S1 ≤ S2 ∧ S2 ⊆ closure S1
 (is ?A ⟷ ?B)
proof
 assume ?A
 then show ?B
  by (metis assms closure-subset convex-rel-interior-closure rel-interior-subset)
next
 assume ?B
 then have closure S1 ⊆ closure S2
  by (metis assms(1) convex-closure-rel-interior closure-mono)
 moreover from ‹?B› have closure S1 ⊇ closure S2
  by (metis closed-closure closure-minimal)
 ultimately show ?A ..
qed

lemma open-inter-closure-rel-interior:
```

fixes S A :: $'n$::*euclidean-space set*
assumes *convex* S
and *open* A
shows $A \cap$ *closure* $S = \{\} \longleftrightarrow A \cap$ *rel-interior* $S = \{\}$
by (*metis assms convex-closure-rel-interior open-Int-closure-eq-empty*)

lemma *rel-interior-open-segment*:
fixes a :: $'a$ :: *euclidean-space*
shows *rel-interior*(*open-segment* a b) = *open-segment* a b
proof (*cases* $a = b$)
case *True* **then show** *?thesis* **by** *auto*
next
case *False* **then show** *?thesis*
apply (*simp add*: *rel-interior-eq openin-open*)
apply (*rule-tac x=ball* (*inverse 2* $*_R$ (a + b)) (*norm*(b − a) / *2*) **in** *exI*)
apply (*simp add*: *open-segment-as-ball*)
done
qed

lemma *rel-interior-closed-segment*:
fixes a :: $'a$ :: *euclidean-space*
shows *rel-interior*(*closed-segment* a b) =
(*if* $a = b$ *then* {a} *else open-segment* a b)
proof (*cases* $a = b$)
case *True* **then show** *?thesis* **by** *auto*
next
case *False* **then show** *?thesis*
by *simp*
(*metis closure-open-segment convex-open-segment convex-rel-interior-closure*
rel-interior-open-segment)
qed

lemmas *rel-interior-segment* = *rel-interior-closed-segment rel-interior-open-segment*

lemma *starlike-convex-tweak-boundary-points*:
fixes S :: $'a$::*euclidean-space set*
assumes *convex* S $S \neq \{\}$ **and** *ST*: *rel-interior* $S \subseteq T$ **and** *TS*: $T \subseteq$ *closure* S
shows *starlike* T
proof −
have *rel-interior* $S \neq \{\}$
by (*simp add*: *assms rel-interior-eq-empty*)
then obtain a **where** a: $a \in$ *rel-interior* S **by** *blast*
with *ST* **have** $a \in T$ **by** *blast*
have $*$: $\bigwedge x.\ x \in T \Longrightarrow$ *open-segment* a $x \subseteq$ *rel-interior* S
apply (*rule rel-interior-closure-convex-segment* [*OF* ‹*convex S*› a])
using *assms* **by** *blast*
show *?thesis*
unfolding *starlike-def*
apply (*rule bexI* [*OF* - ‹$a \in T$›])

apply (*simp add*: *closed-segment-eq-open*)
apply (*intro conjI ballI a* ‹*a* ∈ *T*› *rel-interior-closure-convex-segment* [*OF* ‹*convex S*› *a*])
apply (*simp add*: *order-trans* [*OF* ∗ *ST*])
done
qed

19.35 The relative frontier of a set

definition *rel-frontier S* = *closure S* − *rel-interior S*

lemma *closed-affine-hull*:
fixes *S* :: *'n*::*euclidean-space set*
shows *closed* (*affine hull S*)
by (*metis affine-affine-hull affine-closed*)

lemma *closed-rel-frontier*:
fixes *S* :: *'n*::*euclidean-space set*
shows *closed* (*rel-frontier S*)
proof −
have ∗: *closedin* (*subtopology euclidean* (*affine hull S*)) (*closure S* − *rel-interior S*)
apply (*rule closedin-diff*[*of subtopology euclidean* (*affine hull S*)*closure S rel-interior S*])
using *closed-closedin-trans*[*of affine hull S closure S*] *closed-affine-hull*[*of S*] *closure-affine-hull*[*of S*] *opein-rel-interior*[*of S*]
apply *auto*
done
show *?thesis*
apply (*rule closedin-closed-trans*[*of affine hull S rel-frontier S*])
unfolding *rel-frontier-def*
using ∗ *closed-affine-hull*
apply *auto*
done
qed

lemma *convex-rel-frontier-aff-dim*:
fixes *S1 S2* :: *'n*::*euclidean-space set*
assumes *convex S1*
and *convex S2*
and *S2* ≠ {}
and *S1* ≤ *rel-frontier S2*
shows *aff-dim S1* < *aff-dim S2*
proof −
have *S1* ⊆ *closure S2*
using *assms* **unfolding** *rel-frontier-def* **by** *auto*
then have ∗: *affine hull S1* ⊆ *affine hull S2*
using *hull-mono*[*of S1 closure S2*] *closure-same-affine-hull*[*of S2*] **by** *blast*

```
  then have aff-dim S1 ≤ aff-dim S2
    using ∗ aff-dim-affine-hull[of S1] aff-dim-affine-hull[of S2]
      aff-dim-subset[of affine hull S1 affine hull S2]
    by auto
  moreover
  {
    assume eq: aff-dim S1 = aff-dim S2
    then have S1 ≠ {}
      using aff-dim-empty[of S1] aff-dim-empty[of S2] ‹S2 ≠ {}› by auto
    have ∗∗: affine hull S1 = affine hull S2
      apply (rule affine-dim-equal)
      using ∗ affine-affine-hull
      apply auto
      using ‹S1 ≠ {}› hull-subset[of S1]
      apply auto
      using eq aff-dim-affine-hull[of S1] aff-dim-affine-hull[of S2]
      apply auto
      done
    obtain a where a: a ∈ rel-interior S1
      using ‹S1 ≠ {}› rel-interior-eq-empty assms by auto
    obtain T where T: open T a ∈ T ∩ S1 T ∩ affine hull S1 ⊆ S1
      using mem-rel-interior[of a S1] a by auto
    then have a ∈ T ∩ closure S2
      using a assms unfolding rel-frontier-def by auto
    then obtain b where b: b ∈ T ∩ rel-interior S2
      using open-inter-closure-rel-interior[of S2 T] assms T by auto
    then have b ∈ affine hull S1
      using rel-interior-subset hull-subset[of S2] ∗∗ by auto
    then have b ∈ S1
      using T b by auto
    then have False
      using b assms unfolding rel-frontier-def by auto
  }
  ultimately show ?thesis
    using less-le by auto
qed


lemma convex-rel-interior-if:
  fixes S :: 'n::euclidean-space set
  assumes convex S
    and z ∈ rel-interior S
  shows ∀ x∈affine hull S. ∃ m. m > 1 ∧ (∀ e. e > 1 ∧ e ≤ m ⟶ (1 − e) ∗R x
+ e ∗R z ∈ S)
proof −
  obtain e1 where e1: e1 > 0 ∧ cball z e1 ∩ affine hull S ⊆ S
    using mem-rel-interior-cball[of z S] assms by auto
  {
    fix x
```

```
  assume x: x ∈ affine hull S
  {
   assume x ≠ z
   def m ≡ 1 + e1/norm(x−z)
   hence m > 1 using e1 ‹x ≠ z› by auto
   {
    fix e
    assume e: e > 1 ∧ e ≤ m
    have z ∈ affine hull S
     using assms rel-interior-subset hull-subset[of S] by auto
    then have ∗: (1 − e)∗R x + e ∗R z ∈ affine hull S
     using mem-affine[of affine hull S x z (1−e) e] affine-affine-hull[of S] x
     by auto
    have norm (z + e ∗R x − (x + e ∗R z)) = norm ((e − 1) ∗R (x − z))
     by (simp add: algebra-simps)
    also have ... = (e − 1) ∗ norm (x−z)
     using norm-scaleR e by auto
    also have ... ≤ (m − 1) ∗ norm (x − z)
     using e mult-right-mono[of - - norm(x−z)] by auto
    also have ... = (e1 / norm (x − z)) ∗ norm (x − z)
     using m-def by auto
    also have ... = e1
     using ‹x ≠ z› e1 by simp
    finally have ∗∗: norm (z + e ∗R x − (x + e ∗R z)) ≤ e1
     by auto
    have (1 − e)∗R x+ e ∗R z ∈ cball z e1
     using m-def ∗∗
     unfolding cball-def dist-norm
     by (auto simp add: algebra-simps)
    then have (1 − e) ∗R x+ e ∗R z ∈ S
     using e ∗ e1 by auto
   }
   then have ∃ m. m > 1 ∧ (∀ e. e > 1 ∧ e ≤ m ⟶ (1 − e) ∗R x + e ∗R z
∈ S )
    using ‹m> 1 › by auto
  }
  moreover
  {
   assume x = z
   def m ≡ 1 + e1
   then have m > 1
    using e1 by auto
   {
    fix e
    assume e: e > 1 ∧ e ≤ m
    then have (1 − e) ∗R x + e ∗R z ∈ S
     using e1 x ‹x = z› by (auto simp add: algebra-simps)
    then have (1 − e) ∗R x + e ∗R z ∈ S
     using e by auto
```

```
    }
    then have ∃ m. m > 1 ∧ (∀ e. e > 1 ∧ e ≤ m ⟶ (1 − e) *R x + e *R z
∈ S)
      using ‹m > 1› by auto
   }
   ultimately have ∃ m. m > 1 ∧ (∀ e. e > 1 ∧ e ≤ m ⟶ (1 − e) *R x + e
*R z ∈ S )
    by blast
 }
 then show ?thesis by auto
qed

lemma convex-rel-interior-if2:
 fixes S :: 'n::euclidean-space set
 assumes convex S
 assumes z ∈ rel-interior S
 shows ∀ x∈affine hull S. ∃ e. e > 1 ∧ (1 − e)*R x + e *R z ∈ S
 using convex-rel-interior-if[of S z] assms by auto

lemma convex-rel-interior-only-if:
 fixes S :: 'n::euclidean-space set
 assumes convex S
   and S ≠ {}
 assumes ∀ x∈S. ∃ e. e > 1 ∧ (1 − e) *R x + e *R z ∈ S
 shows z ∈ rel-interior S
proof −
 obtain x where x: x ∈ rel-interior S
   using rel-interior-eq-empty assms by auto
 then have x ∈ S
   using rel-interior-subset by auto
 then obtain e where e: e > 1 ∧ (1 − e) *R x + e *R z ∈ S
   using assms by auto
 def y ≡ (1 − e) *R x + e *R z
 then have y ∈ S using e by auto
 def e1 ≡ 1/e
 then have 0 < e1 ∧ e1 < 1 using e by auto
 then have z =y − (1 − e1) *R (y − x)
   using e1-def y-def by (auto simp add: algebra-simps)
 then show ?thesis
   using rel-interior-convex-shrink[of S x y 1−e1] ‹0 < e1 ∧ e1 < 1› ‹y ∈ S› x
assms
   by auto
qed

lemma convex-rel-interior-iff:
 fixes S :: 'n::euclidean-space set
 assumes convex S
   and S ≠ {}
 shows z ∈ rel-interior S ⟷ (∀ x∈S. ∃ e. e > 1 ∧ (1 − e) *R x + e *R z ∈ S)
```

using *assms hull-subset*[*of S affine*]
convex-rel-interior-if[*of S z*] *convex-rel-interior-only-if*[*of S z*]
by *auto*

lemma *convex-rel-interior-iff2*:
fixes $S :: 'n::euclidean\text{-}space\ set$
assumes *convex S*
and $S \neq \{\}$
shows $z \in rel\text{-}interior\ S \longleftrightarrow (\forall x{\in}affine\ hull\ S.\ \exists e.\ e > 1 \wedge (1 - e) *_R x + e *_R z \in S)$
using *assms hull-subset*[*of S*] *convex-rel-interior-if2*[*of S z*] *convex-rel-interior-only-if*[*of S z*]
by *auto*

lemma *convex-interior-iff*:
fixes $S :: 'n::euclidean\text{-}space\ set$
assumes *convex S*
shows $z \in interior\ S \longleftrightarrow (\forall x.\ \exists e.\ e > 0 \wedge z + e *_R x \in S)$
proof (*cases aff-dim S = int DIM*('*n*))
case *False*
{
assume $z \in interior\ S$
then have *False*
using *False interior-rel-interior-gen*[*of S*] **by** *auto*
}
moreover
{
assume *r*: $\forall x.\ \exists e.\ e > 0 \wedge z + e *_R x \in S$
{
fix *x*
obtain *e1* **where** *e1*: $e1 > 0 \wedge z + e1 *_R (x - z) \in S$
using *r* **by** *auto*
obtain *e2* **where** *e2*: $e2 > 0 \wedge z + e2 *_R (z - x) \in S$
using *r* **by** *auto*
def $x1 \equiv z + e1 *_R (x - z)$
then have *x1*: $x1 \in affine\ hull\ S$
using *e1 hull-subset*[*of S*] **by** *auto*
def $x2 \equiv z + e2 *_R (z - x)$
then have *x2*: $x2 \in affine\ hull\ S$
using *e2 hull-subset*[*of S*] **by** *auto*
have $*$: $e1/(e1+e2) + e2/(e1+e2) = 1$
using *add-divide-distrib*[*of e1 e2 e1+e2*] *e1 e2* **by** *simp*
then have $z = (e2/(e1+e2)) *_R x1 + (e1/(e1+e2)) *_R x2$
using *x1-def x2-def*
apply (*auto simp add*: *algebra-simps*)
using *scaleR-left-distrib*[*of e1/(e1+e2) e2/(e1+e2) z*]
apply *auto*
done
then have *z*: $z \in affine\ hull\ S$

```
      using mem-affine[of affine hull S x1 x2 e2/(e1+e2) e1/(e1+e2)]
        x1 x2 affine-affine-hull[of S] *
      by auto
    have x1 − x2 = (e1 + e2) *R (x − z)
      using x1-def x2-def by (auto simp add: algebra-simps)
    then have x = z+(1/(e1+e2)) *R (x1−x2)
      using e1 e2 by simp
    then have x ∈ affine hull S
      using mem-affine-3-minus[of affine hull S z x1 x2 1/(e1+e2)]
        x1 x2 z affine-affine-hull[of S]
      by auto
  }
  then have affine hull S = UNIV
    by auto
  then have aff-dim S = int DIM('n)
    using aff-dim-affine-hull[of S] by (simp add: aff-dim-univ)
  then have False
    using False by auto
 }
 ultimately show ?thesis by auto
next
 case True
 then have S ≠ {}
  using aff-dim-empty[of S] by auto
 have *: affine hull S = UNIV
  using True affine-hull-univ by auto
 {
  assume z ∈ interior S
  then have z ∈ rel-interior S
    using True interior-rel-interior-gen[of S] by auto
  then have **: ∀ x. ∃ e. e > 1 ∧ (1 − e) *R x + e *R z ∈ S
    using convex-rel-interior-iff2[of S z] assms ‹S ≠ {}› * by auto
  fix x
  obtain e1 where e1: e1 > 1 (1 − e1) *R (z − x) + e1 *R z ∈ S
    using **[rule-format, of z−x] by auto
  def e ≡ e1 − 1
  then have (1 − e1) *R (z − x) + e1 *R z = z + e *R x
    by (simp add: algebra-simps)
  then have e > 0 z + e *R x ∈ S
    using e1 e-def by auto
  then have ∃ e. e > 0 ∧ z + e *R x ∈ S
    by auto
 }
 moreover
 {
  assume r: ∀ x. ∃ e. e > 0 ∧ z + e *R x ∈ S
  {
   fix x
   obtain e1 where e1: e1 > 0 z + e1 *R (z − x) ∈ S
```

```
      using r[rule-format, of z−x] by auto
    def e ≡ e1 + 1
    then have z + e1 *R (z − x) = (1 − e) *R x + e *R z
      by (simp add: algebra-simps)
    then have e > 1 (1 − e)*R x + e *R z ∈ S
      using e1 e-def by auto
    then have ∃ e. e > 1 ∧ (1 − e) *R x + e *R z ∈ S by auto
  }
  then have z ∈ rel-interior S
    using convex-rel-interior-iff2[of S z] assms ‹S ≠ {}› by auto
  then have z ∈ interior S
    using True interior-rel-interior-gen[of S] by auto
 }
 ultimately show ?thesis by auto
qed
```

19.35.1 Relative interior and closure under common operations

```
lemma rel-interior-inter-aux: ⋂{rel-interior S |S. S : I} ⊆ ⋂I
proof −
 {
  fix y
  assume y ∈ ⋂{rel-interior S |S. S : I}
  then have y: ∀ S ∈ I. y ∈ rel-interior S
    by auto
  {
   fix S
   assume S ∈ I
   then have y ∈ S
     using rel-interior-subset y by auto
  }
  then have y ∈ ⋂I by auto
 }
 then show ?thesis by auto
qed

lemma closure-inter: closure (⋂I) ≤ ⋂{closure S |S. S ∈ I}
proof −
 {
  fix y
  assume y ∈ ⋂I
  then have y: ∀ S ∈ I. y ∈ S by auto
  {
   fix S
   assume S ∈ I
   then have y ∈ closure S
     using closure-subset y by auto
  }
  then have y ∈ ⋂{closure S |S. S ∈ I}
```

```
    by auto
  }
  then have ⋂I ⊆ ⋂{closure S |S. S ∈ I}
    by auto
  moreover have closed (⋂{closure S |S. S ∈ I})
    unfolding closed-Inter closed-closure by auto
  ultimately show ?thesis using closure-hull[of ⋂I]
    hull-minimal[of ⋂I ⋂{closure S |S. S ∈ I} closed] by auto
qed

lemma convex-closure-rel-interior-inter:
  assumes ∀S∈I. convex (S :: 'n::euclidean-space set)
    and ⋂{rel-interior S |S. S ∈ I} ≠ {}
  shows ⋂{closure S |S. S ∈ I} ≤ closure (⋂{rel-interior S |S. S ∈ I})
proof −
  obtain x where x: ∀S∈I. x ∈ rel-interior S
    using assms by auto
  {
    fix y
    assume y ∈ ⋂{closure S |S. S ∈ I}
    then have y: ∀S ∈ I. y ∈ closure S
      by auto
    {
      assume y = x
      then have y ∈ closure (⋂{rel-interior S |S. S ∈ I})
        using x closure-subset[of ⋂{rel-interior S |S. S ∈ I}] by auto
    }
    moreover
    {
      assume y ≠ x
      { fix e :: real
        assume e: e > 0
        def e1 ≡ min 1 (e/norm (y − x))
        then have e1: e1 > 0 e1 ≤ 1 e1 * norm (y − x) ≤ e
          using ‹y ≠ x› ‹e > 0› le-divide-eq[of e1 e norm (y − x)]
          by simp-all
        def z ≡ y − e1 *R (y − x)
        {
          fix S
          assume S ∈ I
          then have z ∈ rel-interior S
            using rel-interior-closure-convex-shrink[of S x y e1] assms x y e1 z-def
            by auto
        }
        then have *: z ∈ ⋂{rel-interior S |S. S ∈ I}
          by auto
        have ∃z. z ∈ ⋂{rel-interior S |S. S ∈ I} ∧ z ≠ y ∧ dist z y ≤ e
          apply (rule-tac x=z in exI)
          using ‹y ≠ x› z-def * e1 e dist-norm[of z y]
```

```
      apply simp
      done
    }
    then have y islimpt ⋂{rel-interior S |S. S ∈ I}
      unfolding islimpt-approachable-le by blast
    then have y ∈ closure (⋂{rel-interior S |S. S ∈ I})
      unfolding closure-def by auto
  }
  ultimately have y ∈ closure (⋂{rel-interior S |S. S ∈ I})
    by auto
 }
 then show ?thesis by auto
qed

lemma convex-closure-inter:
 assumes ∀S∈I. convex (S :: 'n::euclidean-space set)
  and ⋂{rel-interior S |S. S ∈ I} ≠ {}
 shows closure (⋂I) = ⋂{closure S |S. S ∈ I}
proof −
 have ⋂{closure S |S. S ∈ I} ≤ closure (⋂{rel-interior S |S. S ∈ I})
  using convex-closure-rel-interior-inter assms by auto
 moreover
 have closure (⋂{rel-interior S |S. S ∈ I}) ≤ closure (⋂I)
  using rel-interior-inter-aux closure-mono[of ⋂{rel-interior S |S. S ∈ I} ⋂I]
  by auto
 ultimately show ?thesis
  using closure-inter[of I] by auto
qed

lemma convex-inter-rel-interior-same-closure:
 assumes ∀S∈I. convex (S :: 'n::euclidean-space set)
  and ⋂{rel-interior S |S. S ∈ I} ≠ {}
 shows closure (⋂{rel-interior S |S. S ∈ I}) = closure (⋂I)
proof −
 have ⋂{closure S |S. S ∈ I} ≤ closure (⋂{rel-interior S |S. S ∈ I})
  using convex-closure-rel-interior-inter assms by auto
 moreover
 have closure (⋂{rel-interior S |S. S ∈ I}) ≤ closure (⋂I)
  using rel-interior-inter-aux closure-mono[of ⋂{rel-interior S |S. S ∈ I} ⋂I]
  by auto
 ultimately show ?thesis
  using closure-inter[of I] by auto
qed

lemma convex-rel-interior-inter:
 assumes ∀S∈I. convex (S :: 'n::euclidean-space set)
  and ⋂{rel-interior S |S. S ∈ I} ≠ {}
 shows rel-interior (⋂I) ⊆ ⋂{rel-interior S |S. S ∈ I}
proof −
```

```
  have convex (⋂ I)
    using assms convex-Inter by auto
  moreover
  have convex (⋂ {rel-interior S |S. S ∈ I})
    apply (rule convex-Inter)
    using assms convex-rel-interior
    apply auto
    done
  ultimately
  have rel-interior (⋂ {rel-interior S |S. S ∈ I}) = rel-interior (⋂ I)
    using convex-inter-rel-interior-same-closure assms
      closure-eq-rel-interior-eq[of ⋂ {rel-interior S |S. S ∈ I} ⋂ I]
    by blast
  then show ?thesis
    using rel-interior-subset[of ⋂ {rel-interior S |S. S ∈ I}] by auto
qed

lemma convex-rel-interior-finite-inter:
  assumes ∀ S∈I. convex (S :: 'n::euclidean-space set)
    and ⋂ {rel-interior S |S. S ∈ I} ≠ {}
    and finite I
  shows rel-interior (⋂ I) = ⋂ {rel-interior S |S. S ∈ I}
proof −
  have ⋂ I ≠ {}
    using assms rel-interior-inter-aux[of I] by auto
  have convex (⋂ I)
    using convex-Inter assms by auto
  show ?thesis
  proof (cases I = {})
    case True
    then show ?thesis
      using Inter-empty rel-interior-univ2 by auto
  next
    case False
    {
      fix z
      assume z: z ∈ ⋂ {rel-interior S |S. S ∈ I}
      {
        fix x
        assume x: x ∈ ⋂ I
        {
          fix S
          assume S: S ∈ I
          then have z ∈ rel-interior S x ∈ S
            using z x by auto
          then have ∃ m. m > 1 ∧ (∀ e. e > 1 ∧ e ≤ m ⟶ (1 − e)*R x + e *R
z ∈ S)
            using convex-rel-interior-if[of S z] S assms hull-subset[of S] by auto
        }
```

```
      then obtain mS where
        mS: ∀ S∈I. mS S > 1 ∧ (∀ e. e > 1 ∧ e ≤ mS S ⟶ (1 − e) *R x + e
*R z ∈ S) by metis
      def e ≡ Min (mS ‘ I)
      then have e ∈ mS ‘ I using assms ‹I ≠ {}› by simp
      then have e > 1 using mS by auto
      moreover have ∀ S∈I. e ≤ mS S
        using e-def assms by auto
      ultimately have ∃ e > 1. (1 − e) *R x + e *R z ∈ ⋂ I
        using mS by auto
    }
    then have z ∈ rel-interior (⋂ I)
      using convex-rel-interior-iff[of ⋂ I z] ‹⋂ I ≠ {}› ‹convex (⋂ I)› by auto
  }
  then show ?thesis
    using convex-rel-interior-inter[of I] assms by auto
 qed
qed

lemma convex-closure-inter-two:
 fixes S T :: 'n::euclidean-space set
 assumes convex S
   and convex T
 assumes rel-interior S ∩ rel-interior T ≠ {}
 shows closure (S ∩ T) = closure S ∩ closure T
 using convex-closure-inter[of {S,T}] assms by auto

lemma convex-rel-interior-inter-two:
 fixes S T :: 'n::euclidean-space set
 assumes convex S
   and convex T
   and rel-interior S ∩ rel-interior T ≠ {}
 shows rel-interior (S ∩ T) = rel-interior S ∩ rel-interior T
 using convex-rel-interior-finite-inter[of {S,T}] assms by auto

lemma convex-affine-closure-inter:
 fixes S T :: 'n::euclidean-space set
 assumes convex S
   and affine T
   and rel-interior S ∩ T ≠ {}
 shows closure (S ∩ T) = closure S ∩ T
proof −
 have affine hull T = T
   using assms by auto
 then have rel-interior T = T
   using rel-interior-univ[of T] by metis
 moreover have closure T = T
   using assms affine-closed[of T] by auto
 ultimately show ?thesis
```

using *convex-closure-inter-two*[*of S T*] *assms affine-imp-convex* **by** *auto*
qed

lemma *connected-component-1-gen*:
fixes $S :: 'a :: euclidean\text{-}space\ set$
assumes $DIM('a) = 1$
shows $connected\text{-}component\ S\ a\ b \longleftrightarrow closed\text{-}segment\ a\ b \subseteq S$
unfolding *connected-component-def*
by (*metis* (*no-types*, *lifting*) *assms subsetD subsetI convex-contains-segment convex-segment*(*1*)
ends-in-segment connected-convex-1-gen)

lemma *connected-component-1*:
fixes $S :: real\ set$
shows $connected\text{-}component\ S\ a\ b \longleftrightarrow closed\text{-}segment\ a\ b \subseteq S$
by (*simp add*: *connected-component-1-gen*)

lemma *convex-affine-rel-interior-inter*:
fixes $S\ T :: 'n::euclidean\text{-}space\ set$
assumes *convex S*
and *affine T*
and $rel\text{-}interior\ S \cap T \neq \{\}$
shows $rel\text{-}interior\ (S \cap T) = rel\text{-}interior\ S \cap T$
proof −
have $affine\ hull\ T = T$
using *assms* **by** *auto*
then have $rel\text{-}interior\ T = T$
using *rel-interior-univ*[*of T*] **by** *metis*
moreover have $closure\ T = T$
using *assms affine-closed*[*of T*] **by** *auto*
ultimately show *?thesis*
using *convex-rel-interior-inter-two*[*of S T*] *assms affine-imp-convex* **by** *auto*
qed

lemma *subset-rel-interior-convex*:
fixes $S\ T :: 'n::euclidean\text{-}space\ set$
assumes *convex S*
and *convex T*
and $S \leq closure\ T$
and $\neg\ S \subseteq rel\text{-}frontier\ T$
shows $rel\text{-}interior\ S \subseteq rel\text{-}interior\ T$
proof −
have ∗: $S \cap closure\ T = S$
using *assms* **by** *auto*
have $\neg\ rel\text{-}interior\ S \subseteq rel\text{-}frontier\ T$
using *closure-mono*[*of rel-interior S rel-frontier T*] *closed-rel-frontier*[*of T*]
closure-closed[*of S*] *convex-closure-rel-interior*[*of S*] *closure-subset*[*of S*] *assms*
by *auto*
then have $rel\text{-}interior\ S \cap rel\text{-}interior\ (closure\ T) \neq \{\}$
using *assms rel-frontier-def*[*of T*] *rel-interior-subset convex-rel-interior-closure*[*of*

```
T]
    by auto
  then have rel-interior S ∩ rel-interior T = rel-interior (S ∩ closure T)
    using assms convex-closure convex-rel-interior-inter-two[of S closure T]
      convex-rel-interior-closure[of T]
    by auto
  also have ... = rel-interior S
    using ∗ by auto
  finally show ?thesis
    by auto
qed

lemma rel-interior-convex-linear-image:
  fixes f :: 'm::euclidean-space ⇒ 'n::euclidean-space
  assumes linear f
    and convex S
  shows f ‘ (rel-interior S) = rel-interior (f ‘ S)
proof (cases S = {})
  case True
  then show ?thesis
    using assms rel-interior-empty rel-interior-eq-empty by auto
next
  case False
  have ∗: f ‘ (rel-interior S) ⊆ f ‘ S
    unfolding image-mono using rel-interior-subset by auto
  have f ‘ S ⊆ f ‘ (closure S)
    unfolding image-mono using closure-subset by auto
  also have ... = f ‘ (closure (rel-interior S))
    using convex-closure-rel-interior assms by auto
  also have ... ⊆ closure (f ‘ (rel-interior S))
    using closure-linear-image-subset assms by auto
  finally have closure (f ‘ S) = closure (f ‘ rel-interior S)
    using closure-mono[of f ‘ S closure (f ‘ rel-interior S)] closure-closure
      closure-mono[of f ‘ rel-interior S f ‘ S] ∗
    by auto
  then have rel-interior (f ‘ S) = rel-interior (f ‘ rel-interior S)
    using assms convex-rel-interior
      linear-conv-bounded-linear[of f] convex-linear-image[of - S]
      convex-linear-image[of - rel-interior S]
      closure-eq-rel-interior-eq[of f ‘ S f ‘ rel-interior S]
    by auto
  then have rel-interior (f ‘ S) ⊆ f ‘ rel-interior S
    using rel-interior-subset by auto
  moreover
  {
    fix z
    assume z ∈ f ‘ rel-interior S
    then obtain z1 where z1: z1 ∈ rel-interior S f z1 = z by auto
    {
```

```
    fix x
    assume x ∈ f ‘ S
    then obtain x1 where x1: x1 ∈ S f x1 = x by auto
    then obtain e where e: e > 1 (1 − e) *R x1 + e *R z1 : S
      using convex-rel-interior-iff[of S z1] ‹convex S› x1 z1 by auto
    moreover have f ((1 − e) *R x1 + e *R z1) = (1 − e) *R x + e *R z
      using x1 z1 ‹linear f› by (simp add: linear-add-cmul)
    ultimately have (1 − e) *R x + e *R z : f ‘ S
      using imageI[of (1 − e) *R x1 + e *R z1 S f] by auto
    then have ∃ e. e > 1 ∧ (1 − e) *R x + e *R z : f ‘ S
      using e by auto
  }
  then have z ∈ rel-interior (f ‘ S)
    using convex-rel-interior-iff[of f ‘ S z] ‹convex S›
     ‹linear f› ‹S ≠ {}› convex-linear-image[of f S]  linear-conv-bounded-linear[of
f]
    by auto
 }
 ultimately show ?thesis by auto
qed

lemma rel-interior-convex-linear-preimage:
 fixes f :: 'm::euclidean-space ⇒ 'n::euclidean-space
 assumes linear f
   and convex S
   and f −‘ (rel-interior S) ≠ {}
 shows rel-interior (f −‘ S) = f −‘ (rel-interior S)
proof −
 have S ≠ {}
   using assms rel-interior-empty by auto
 have nonemp: f −‘ S ≠ {}
   by (metis assms(3) rel-interior-subset subset-empty vimage-mono)
 then have S ∩ (range f) ≠ {}
   by auto
 have conv: convex (f −‘ S)
   using convex-linear-vimage assms by auto
 then have convex (S ∩ range f)
   by (metis assms(1) assms(2) convex-Int subspace-UNIV subspace-imp-convex
subspace-linear-image)
 {
   fix z
   assume z ∈ f −‘ (rel-interior S)
   then have z: f z : rel-interior S
     by auto
   {
     fix x
     assume x ∈ f −‘ S
     then have f x ∈ S by auto
     then obtain e where e: e > 1 (1 − e) *R f x + e *R f z ∈ S
```

```
      using convex-rel-interior-iff[of S f z] z assms ‹S ≠ {}› by auto
    moreover have (1 − e) *R f x + e *R f z = f ((1 − e) *R x + e *R z)
      using ‹linear f› by (simp add: linear-iff)
    ultimately have ∃e. e > 1 ∧ (1 − e) *R x + e *R z ∈ f −‘ S
      using e by auto
  }
  then have z ∈ rel-interior (f −‘ S)
    using convex-rel-interior-iff[of f −‘ S z] conv nonemp by auto
 }
 moreover
 {
  fix z
  assume z: z ∈ rel-interior (f −‘ S)
  {
   fix x
   assume x ∈ S ∩ range f
   then obtain y where y: f y = x y ∈ f −‘ S by auto
   then obtain e where e: e > 1 (1 − e) *R y + e *R z ∈ f −‘ S
    using convex-rel-interior-iff[of f −‘ S z] z conv by auto
   moreover have (1 − e) *R x + e *R f z = f ((1 − e) *R y + e *R z)
    using ‹linear f› y by (simp add: linear-iff)
   ultimately have ∃e. e > 1 ∧ (1 − e) *R x + e *R f z ∈ S ∩ range f
    using e by auto
  }
  then have f z ∈ rel-interior (S ∩ range f)
   using ‹convex (S ∩ (range f))› ‹S ∩ range f ≠ {}›
    convex-rel-interior-iff[of S ∩ (range f) f z]
   by auto
  moreover have affine (range f)
   by (metis assms(1) subspace-UNIV subspace-imp-affine subspace-linear-image)
  ultimately have f z ∈ rel-interior S
   using convex-affine-rel-interior-inter[of S range f] assms by auto
  then have z ∈ f −‘ (rel-interior S)
   by auto
 }
 ultimately show ?thesis by auto
qed

lemma rel-interior-direct-sum:
 fixes S :: 'n::euclidean-space set
  and T :: 'm::euclidean-space set
 assumes convex S
  and convex T
 shows rel-interior (S × T) = rel-interior S × rel-interior T
proof −
 { assume S = {}
  then have ?thesis
   by auto
 }
```

```
  moreover
  { assume T = {}
    then have ?thesis
      by auto
  }
  moreover
  {
    assume S ≠ {} T ≠ {}
    then have ri: rel-interior S ≠ {} rel-interior T ≠ {}
      using rel-interior-eq-empty assms by auto
    then have fst -` rel-interior S ≠ {}
      using fst-vimage-eq-Times[of rel-interior S] by auto
    then have rel-interior ((fst :: 'n * 'm ⇒ 'n) -` S) = fst -` rel-interior S
       using fst-linear ‹convex S› rel-interior-convex-linear-preimage[of fst S] by
auto
    then have s: rel-interior (S × (UNIV :: 'm set)) = rel-interior S × UNIV
      by (simp add: fst-vimage-eq-Times)
    from ri have snd -` rel-interior T ≠ {}
      using snd-vimage-eq-Times[of rel-interior T] by auto
    then have rel-interior ((snd :: 'n * 'm ⇒ 'm) -` T) = snd -` rel-interior T
      using snd-linear ‹convex T› rel-interior-convex-linear-preimage[of snd T] by
auto
    then have t: rel-interior ((UNIV :: 'n set) × T) = UNIV × rel-interior T
      by (simp add: snd-vimage-eq-Times)
    from s t have *: rel-interior (S × (UNIV :: 'm set)) ∩ rel-interior ((UNIV ::
'n set) × T) =
      rel-interior S × rel-interior T by auto
    have S × T = S × (UNIV :: 'm set) ∩ (UNIV :: 'n set) × T
      by auto
     then have rel-interior (S × T) = rel-interior ((S × (UNIV :: 'm set)) ∩
((UNIV :: 'n set) × T))
      by auto
    also have ... = rel-interior (S × (UNIV :: 'm set)) ∩ rel-interior ((UNIV ::
'n set) × T)
       apply (subst convex-rel-interior-inter-two[of S × (UNIV :: 'm set) (UNIV
:: 'n set) × T])
       using * ri assms convex-Times
       apply auto
       done
    finally have ?thesis using * by auto
  }
  ultimately show ?thesis by blast
qed

lemma rel-interior-scaleR:
  fixes S :: 'n::euclidean-space set
  assumes c ≠ 0
  shows (op *R c) ` (rel-interior S) = rel-interior ((op *R c) ` S)
  using rel-interior-injective-linear-image[of (op *R c) S]
```

linear-conv-bounded-linear[*of op* $*_R$ *c*] *linear-scaleR injective-scaleR*[*of c*] *assms*
by *auto*

lemma *rel-interior-convex-scaleR*:
fixes *S* :: *'n*::*euclidean-space set*
assumes *convex S*
shows (*op* $*_R$ *c*) ' (*rel-interior S*) = *rel-interior* ((*op* $*_R$ *c*) ' *S*)
by (*metis assms linear-scaleR rel-interior-convex-linear-image*)

lemma *convex-rel-open-scaleR*:
fixes *S* :: *'n*::*euclidean-space set*
assumes *convex S*
and *rel-open S*
shows *convex* ((*op* $*_R$ *c*) ' *S*) ∧ *rel-open* ((*op* $*_R$ *c*) ' *S*)
by (*metis assms convex-scaling rel-interior-convex-scaleR rel-open-def*)

lemma *convex-rel-open-finite-inter*:
assumes ∀ *S*∈*I*. *convex* (*S* :: *'n*::*euclidean-space set*) ∧ *rel-open S*
and *finite I*
shows *convex* (⋂ *I*) ∧ *rel-open* (⋂ *I*)
proof (*cases* ⋂{*rel-interior S* |*S*. *S* ∈ *I*} = {})
case *True*
then have ⋂ *I* = {}
using *assms* **unfolding** *rel-open-def* **by** *auto*
then show *?thesis*
unfolding *rel-open-def* **using** *rel-interior-empty* **by** *auto*
next
case *False*
then have *rel-open* (⋂ *I*)
using *assms* **unfolding** *rel-open-def*
using *convex-rel-interior-finite-inter*[*of I*]
by *auto*
then show *?thesis*
using *convex-Inter assms* **by** *auto*
qed

lemma *convex-rel-open-linear-image*:
fixes *f* :: *'m*::*euclidean-space* ⇒ *'n*::*euclidean-space*
assumes *linear f*
and *convex S*
and *rel-open S*
shows *convex* (*f* ' *S*) ∧ *rel-open* (*f* ' *S*)
by (*metis assms convex-linear-image rel-interior-convex-linear-image rel-open-def*)

lemma *convex-rel-open-linear-preimage*:
fixes *f* :: *'m*::*euclidean-space* ⇒ *'n*::*euclidean-space*
assumes *linear f*
and *convex S*
and *rel-open S*

```
  shows convex (f -` S) ∧ rel-open (f -` S)
proof (cases f -` (rel-interior S) = {})
  case True
  then have f -` S = {}
    using assms unfolding rel-open-def by auto
  then show ?thesis
    unfolding rel-open-def using rel-interior-empty by auto
next
  case False
  then have rel-open (f -` S)
    using assms unfolding rel-open-def
    using rel-interior-convex-linear-preimage[of f S]
    by auto
  then show ?thesis
    using convex-linear-vimage assms
    by auto
qed

lemma rel-interior-projection:
  fixes S :: ('m::euclidean-space × 'n::euclidean-space) set
    and f :: 'm::euclidean-space ⇒ 'n::euclidean-space set
  assumes convex S
    and f = (λy. {z. (y, z) ∈ S})
  shows (y, z) ∈ rel-interior S ⟷ (y ∈ rel-interior {y. (f y ≠ {})} ∧ z ∈
rel-interior (f y))
proof -
  {
    fix y
    assume y ∈ {y. f y ≠ {}}
    then obtain z where (y, z) ∈ S
      using assms by auto
    then have ∃ x. x ∈ S ∧ y = fst x
      apply (rule-tac x=(y, z) in exI)
      apply auto
      done
    then obtain x where x ∈ S y = fst x
      by blast
    then have y ∈ fst ` S
      unfolding image-def by auto
  }
  then have fst ` S = {y. f y ≠ {}}
    unfolding fst-def using assms by auto
  then have h1: fst ` rel-interior S = rel-interior {y. f y ≠ {}}
    using rel-interior-convex-linear-image[of fst S] assms fst-linear by auto
  {
    fix y
    assume y ∈ rel-interior {y. f y ≠ {}}
    then have y ∈ fst ` rel-interior S
      using h1 by auto
```

```
    then have ∗: rel-interior S ∩ fst −‘ {y} ≠ {}
      by auto
    moreover have aff: affine (fst −‘ {y})
      unfolding affine-alt by (simp add: algebra-simps)
    ultimately have ∗∗: rel-interior (S ∩ fst −‘ {y}) = rel-interior S ∩ fst −‘
{y}
      using convex-affine-rel-interior-inter[of S fst −‘ {y}] assms by auto
    have conv: convex (S ∩ fst −‘ {y})
      using convex-Int assms aff affine-imp-convex by auto
    {
      fix x
      assume x ∈ f y
      then have (y, x) ∈ S ∩ (fst −‘ {y})
        using assms by auto
      moreover have x = snd (y, x) by auto
      ultimately have x ∈ snd ‘ (S ∩ fst −‘ {y})
        by blast
    }
    then have snd ‘ (S ∩ fst −‘ {y}) = f y
      using assms by auto
    then have ∗∗∗: rel-interior (f y) = snd ‘ rel-interior (S ∩ fst −‘ {y})
      using rel-interior-convex-linear-image[of snd S ∩ fst −‘ {y}] snd-linear conv
      by auto
    {
      fix z
      assume z ∈ rel-interior (f y)
      then have z ∈ snd ‘ rel-interior (S ∩ fst −‘ {y})
        using ∗∗∗ by auto
      moreover have {y} = fst ‘ rel-interior (S ∩ fst −‘ {y})
        using ∗ ∗∗ rel-interior-subset by auto
      ultimately have (y, z) ∈ rel-interior (S ∩ fst −‘ {y})
        by force
      then have (y,z) ∈ rel-interior S
        using ∗∗ by auto
    }
    moreover
    {
      fix z
      assume (y, z) ∈ rel-interior S
      then have (y, z) ∈ rel-interior (S ∩ fst −‘ {y})
        using ∗∗ by auto
      then have z ∈ snd ‘ rel-interior (S ∩ fst −‘ {y})
        by (metis Range-iff snd-eq-Range)
      then have z ∈ rel-interior (f y)
        using ∗∗∗ by auto
    }
    ultimately have ⋀z. (y, z) ∈ rel-interior S ⟷ z ∈ rel-interior (f y)
      by auto
  }
```

```
  then have h2: ⋀y z. y ∈ rel-interior {t. f t ≠ {}} ⟹
    (y, z) ∈ rel-interior S ⟷ z ∈ rel-interior (f y)
    by auto
  {
    fix y z
    assume asm: (y, z) ∈ rel-interior S
    then have y ∈ fst ` rel-interior S
      by (metis Domain-iff fst-eq-Domain)
    then have y ∈ rel-interior {t. f t ≠ {}}
      using h1 by auto
    then have y ∈ rel-interior {t. f t ≠ {}} and (z : rel-interior (f y))
      using h2 asm by auto
  }
  then show ?thesis using h2 by blast
qed
```

19.35.2 Relative interior of convex cone

```
lemma cone-rel-interior:
  fixes S :: 'm::euclidean-space set
  assumes cone S
  shows cone ({0} ∪ rel-interior S)
proof (cases S = {})
  case True
  then show ?thesis
    by (simp add: rel-interior-empty cone-0)
next
  case False
  then have ∗: 0 ∈ S ∧ (∀ c. c > 0 ⟶ op ∗R c ` S = S)
    using cone-iff[of S] assms by auto
  then have ∗: 0 ∈ ({0} ∪ rel-interior S)
    and ∀ c. c > 0 ⟶ op ∗R c ` ({0} ∪ rel-interior S) = ({0} ∪ rel-interior S)
    by (auto simp add: rel-interior-scaleR)
  then show ?thesis
    using cone-iff[of {0} ∪ rel-interior S] by auto
qed

lemma rel-interior-convex-cone-aux:
  fixes S :: 'm::euclidean-space set
  assumes convex S
  shows (c, x) ∈ rel-interior (cone hull ({(1 :: real)} × S)) ⟷
    c > 0 ∧ x ∈ ((op ∗R c) ` (rel-interior S))
proof (cases S = {})
  case True
  then show ?thesis
    by (simp add: rel-interior-empty cone-hull-empty)
next
  case False
  then obtain s where s ∈ S by auto
```

```
  have conv: convex ({(1 :: real)} × S)
    using convex-Times[of {(1 :: real)} S] assms convex-singleton[of 1 :: real]
    by auto
  def f ≡ λy. {z. (y, z) ∈ cone hull ({1 :: real} × S)}
  then have ∗: (c, x) ∈ rel-interior (cone hull ({(1 :: real)} × S)) =
    (c ∈ rel-interior {y. f y ≠ {}} ∧ x ∈ rel-interior (f c))
    apply (subst rel-interior-projection[of cone hull ({(1 :: real)} × S) f c x])
    using convex-cone-hull[of {(1 :: real)} × S] conv
    apply auto
    done
  {
    fix y :: real
    assume y ≥ 0
    then have y ∗R (1,s) ∈ cone hull ({1 :: real} × S)
      using cone-hull-expl[of {(1 :: real)} × S] ‹s ∈ S› by auto
    then have f y ≠ {}
      using f-def by auto
  }
  then have {y. f y ≠ {}} = {0..}
    using f-def cone-hull-expl[of {1 :: real} × S] by auto
  then have ∗∗: rel-interior {y. f y ≠ {}} = {0<..}
    using rel-interior-real-semiline by auto
  {
    fix c :: real
    assume c > 0
    then have f c = (op ∗R c ' S)
      using f-def cone-hull-expl[of {1 :: real} × S] by auto
    then have rel-interior (f c) = op ∗R c ' rel-interior S
      using rel-interior-convex-scaleR[of S c] assms by auto
  }
  then show ?thesis using ∗ ∗∗ by auto
qed

lemma rel-interior-convex-cone:
  fixes S :: 'm::euclidean-space set
  assumes convex S
  shows rel-interior (cone hull ({1 :: real} × S)) =
    {(c, c ∗R x) | c x. c > 0 ∧ x ∈ rel-interior S}
  (is ?lhs = ?rhs)
proof −
  {
    fix z
    assume z ∈ ?lhs
    have ∗: z = (fst z, snd z)
      by auto
    have z ∈ ?rhs
      using rel-interior-convex-cone-aux[of S fst z snd z] assms ‹z ∈ ?lhs›
      apply auto
      apply (rule-tac x = fst z in exI)
```

```
    apply (rule-tac x = x in exI)
    using ∗
    apply auto
    done
  }
  moreover
  {
    fix z
    assume z ∈ ?rhs
    then have z ∈ ?lhs
      using rel-interior-convex-cone-aux[of S fst z snd z] assms
      by auto
  }
  ultimately show ?thesis by blast
qed

lemma convex-hull-finite-union:
  assumes finite I
  assumes ∀ i∈I. convex (S i) ∧ (S i) ≠ {}
  shows convex hull (⋃(S ‘ I)) =
    {setsum (λi. c i ∗R s i) I | c s. (∀ i∈I. c i ≥ 0) ∧ setsum c I = 1 ∧ (∀ i∈I. s
i ∈ S i)}
  (is ?lhs = ?rhs)
proof −
  have ?lhs ⊇ ?rhs
  proof
    fix x
    assume x : ?rhs
    then obtain c s where ∗: setsum (λi. c i ∗R s i) I = x setsum c I = 1
      (∀ i∈I. c i ≥ 0) ∧ (∀ i∈I. s i ∈ S i) by auto
    then have ∀ i∈I. s i ∈ convex hull (⋃(S ‘ I))
      using hull-subset[of ⋃(S ‘ I) convex] by auto
    then show x ∈ ?lhs
      unfolding ∗(1)[symmetric]
      apply (subst convex-setsum[of I convex hull ⋃(S ‘ I) c s])
      using ∗ assms convex-convex-hull
      apply auto
      done
  qed

  {
    fix i
    assume i ∈ I
    with assms have ∃ p. p ∈ S i by auto
  }
  then obtain p where p: ∀ i∈I. p i ∈ S i by metis

  {
    fix i
```

```
    assume i ∈ I
    {
      fix x
      assume x ∈ S i
      def c ≡ λj. if j = i then 1::real else 0
      then have ∗: setsum c I = 1
        using ⟨finite I⟩ ⟨i ∈ I⟩ setsum.delta[of I i λj::'a. 1::real]
        by auto
      def s ≡ λj. if j = i then x else p j
      then have ∀j. c j ∗R s j = (if j = i then x else 0)
        using c-def by (auto simp add: algebra-simps)
      then have x = setsum (λi. c i ∗R s i) I
        using s-def c-def ⟨finite I⟩ ⟨i ∈ I⟩ setsum.delta[of I i λj::'a. x]
        by auto
      then have x ∈ ?rhs
        apply auto
        apply (rule-tac x = c in exI)
        apply (rule-tac x = s in exI)
        using ∗ c-def s-def p ⟨x ∈ S i⟩
        apply auto
        done
    }
    then have ?rhs ⊇ S i by auto
  }
  then have ∗: ?rhs ⊇ ⋃(S ' I) by auto

  {
    fix u v :: real
    assume uv: u ≥ 0 ∧ v ≥ 0 ∧ u + v = 1
    fix x y
    assume xy: x ∈ ?rhs ∧ y ∈ ?rhs
    from xy obtain c s where
      xc: x = setsum (λi. c i ∗R s i) I ∧ (∀i∈I. c i ≥ 0) ∧ setsum c I = 1 ∧
(∀i∈I. s i ∈ S i)
      by auto
    from xy obtain d t where
      yc: y = setsum (λi. d i ∗R t i) I ∧ (∀i∈I. d i ≥ 0) ∧ setsum d I = 1 ∧
(∀i∈I. t i ∈ S i)
      by auto
    def e ≡ λi. u ∗ c i + v ∗ d i
    have ge0: ∀i∈I. e i ≥ 0
      using e-def xc yc uv by simp
    have setsum (λi. u ∗ c i) I = u ∗ setsum c I
      by (simp add: setsum-right-distrib)
    moreover have setsum (λi. v ∗ d i) I = v ∗ setsum d I
      by (simp add: setsum-right-distrib)
    ultimately have sum1: setsum e I = 1
      using e-def xc yc uv by (simp add: setsum.distrib)
   def q ≡ λi. if e i = 0 then p i else (u ∗ c i / e i) ∗R s i + (v ∗ d i / e i) ∗R t i
```

```
  {
   fix i
   assume i: i ∈ I
   have q i ∈ S i
   proof (cases e i = 0)
    case True
    then show ?thesis using i p q-def by auto
   next
    case False
    then show ?thesis
     using mem-convex-alt[of S i s i t i u * (c i) v * (d i)]
      mult-nonneg-nonneg[of u c i] mult-nonneg-nonneg[of v d i]
      assms q-def e-def i False xc yc uv
     by (auto simp del: mult-nonneg-nonneg)
   qed
  }
  then have qs: ∀ i∈I. q i ∈ S i by auto
  {
   fix i
   assume i: i ∈ I
   have (u * c i) *R s i + (v * d i) *R t i = e i *R q i
   proof (cases e i = 0)
    case True
    have ge: u * (c i) ≥ 0 ∧ v * d i ≥ 0
     using xc yc uv i by simp
    moreover from ge have u * c i ≤ 0 ∧ v * d i ≤ 0
     using True e-def i by simp
    ultimately have u * c i = 0 ∧ v * d i = 0 by auto
    with True show ?thesis by auto
   next
    case False
    then have (u * (c i)/(e i))*R (s i)+(v * (d i)/(e i))*R (t i) = q i
     using q-def by auto
    then have e i *R ((u * (c i)/(e i))*R (s i)+(v * (d i)/(e i))*R (t i))
          = (e i) *R (q i) by auto
    with False show ?thesis by (simp add: algebra-simps)
   qed
  }
  then have *: ∀ i∈I. (u * c i) *R s i + (v * d i) *R t i = e i *R q i
   by auto
  have u *R x + v *R y = setsum (λi. (u * c i) *R s i + (v * d i) *R t i) I
   using xc yc by (simp add: algebra-simps scaleR-right.setsum setsum.distrib)
  also have ... = setsum (λi. e i *R q i) I
   using * by auto
  finally have u *R x + v *R y = setsum (λi. (e i) *R (q i)) I
   by auto
  then have u *R x + v *R y ∈ ?rhs
   using ge0 sum1 qs by auto
 }
```

then have *convex ?rhs* **unfolding** *convex-def* **by** *auto*
then show *?thesis*
using ‹*?lhs* ⊇ *?rhs*› ∗ *hull-minimal*[*of* ⋃(*S* ‘ *I*) *?rhs convex*]
by *blast*
qed

lemma *convex-hull-union-two*:
fixes *S T* :: *'m*::*euclidean-space set*
assumes *convex S*
and *S* ≠ {}
and *convex T*
and *T* ≠ {}
shows *convex hull* (*S* ∪ *T*) =
{*u* ∗*R* *s* + *v* ∗*R* *t* | *u v s t*. *u* ≥ *0* ∧ *v* ≥ *0* ∧ *u* + *v* = *1* ∧ *s* ∈ *S* ∧ *t* ∈ *T*}
(**is** *?lhs* = *?rhs*)
proof
def *I* ≡ {*1*::*nat*, *2*}
def *s* ≡ λ*i*. *if i* = (*1*::*nat*) *then S else T*
have ⋃(*s* ‘ *I*) = *S* ∪ *T*
using *s-def I-def* **by** *auto*
then have *convex hull* (⋃(*s* ‘ *I*)) = *convex hull* (*S* ∪ *T*)
by *auto*
moreover have *convex hull* ⋃(*s* ‘ *I*) =
{∑ *i*∈*I*. *c i* ∗*R* *sa i* | *c sa*. (∀ *i*∈*I*. *0* ≤ *c i*) ∧ *setsum c I* = *1* ∧ (∀ *i*∈*I*. *sa i* ∈ *s i*)}
apply (*subst convex-hull-finite-union*[*of I s*])
using *assms s-def I-def*
apply *auto*
done
moreover have
{∑ *i*∈*I*. *c i* ∗*R* *sa i* | *c sa*. (∀ *i*∈*I*. *0* ≤ *c i*) ∧ *setsum c I* = *1* ∧ (∀ *i*∈*I*. *sa i* ∈ *s i*)} ≤ *?rhs*
using *s-def I-def* **by** *auto*
ultimately show *?lhs* ⊆ *?rhs* **by** *auto*
{
fix *x*
assume *x* ∈ *?rhs*
then obtain *u v s t* **where** ∗: *x* = *u* ∗*R* *s* + *v* ∗*R* *t* ∧ *u* ≥ *0* ∧ *v* ≥ *0* ∧ *u* + *v* = *1* ∧ *s* ∈ *S* ∧ *t* ∈ *T*
by *auto*
then have *x* ∈ *convex hull* {*s*, *t*}
using *convex-hull-2*[*of s t*] **by** *auto*
then have *x* ∈ *convex hull* (*S* ∪ *T*)
using ∗ *hull-mono*[*of* {*s*, *t*} *S* ∪ *T*] **by** *auto*
}
then show *?lhs* ⊇ *?rhs* **by** *blast*
qed

19.36 Convexity on direct sums

lemma *closure-sum*:
fixes *S T* :: *'a*::*real-normed-vector set*
shows *closure S* + *closure T* ⊆ *closure* (*S* + *T*)
unfolding *set-plus-image closure-Times* [*symmetric*] *split-def*
by (*intro closure-bounded-linear-image-subset bounded-linear-add bounded-linear-fst bounded-linear-snd*)

lemma *rel-interior-sum*:
fixes *S T* :: *'n*::*euclidean-space set*
assumes *convex S* **and** *convex T*
shows *rel-interior* (*S* + *T*) = *rel-interior S* + *rel-interior T*
proof −
have *rel-interior S* + *rel-interior T* = (λ(*x*,*y*). *x* + *y*) ' (*rel-interior S* × *rel-interior T*)
by (*simp add*: *set-plus-image*)
also have ... = (λ(*x*,*y*). *x* + *y*) ' *rel-interior* (*S* × *T*)
using *rel-interior-direct-sum assms* **by** *auto*
also have ... = *rel-interior* (*S* + *T*)
using *fst-snd-linear convex-Times assms*
rel-interior-convex-linear-image[*of* (λ(*x*,*y*). *x* + *y*) *S* × *T*]
by (*auto simp add*: *set-plus-image*)
finally show *?thesis* **..**
qed

lemma *rel-interior-sum-gen*:
fixes *S* :: *'a* ⇒ *'n*::*euclidean-space set*
assumes ∀ *i*∈*I*. *convex* (*S i*)
shows *rel-interior* (*setsum S I*) = *setsum* (λ*i*. *rel-interior* (*S i*)) *I*
apply (*subst setsum-set-cond-linear*[*of convex*])
using *rel-interior-sum rel-interior-sing*[*of 0*] *assms*
apply (*auto simp add*: *convex-set-plus*)
done

lemma *convex-rel-open-direct-sum*:
fixes *S T* :: *'n*::*euclidean-space set*
assumes *convex S* **and** *rel-open S* **and** *convex T* **and** *rel-open T*
shows *convex* (*S* × *T*) ∧ *rel-open* (*S* × *T*)
by (*metis assms convex-Times rel-interior-direct-sum rel-open-def*)

lemma *convex-rel-open-sum*:
fixes *S T* :: *'n*::*euclidean-space set*
assumes *convex S* **and** *rel-open S* **and** *convex T*

```
    and rel-open T
  shows convex (S + T) ∧ rel-open (S + T)
  by (metis assms convex-set-plus rel-interior-sum rel-open-def)

lemma convex-hull-finite-union-cones:
  assumes finite I
    and I ≠ {}
  assumes ∀ i∈I. convex (S i) ∧ cone (S i) ∧ S i ≠ {}
  shows convex hull (⋃(S ' I)) = setsum S I
  (is ?lhs = ?rhs)
proof −
  {
    fix x
    assume x ∈ ?lhs
    then obtain c xs where
      x: x = setsum (λi. c i *R xs i) I ∧ (∀ i∈I. c i ≥ 0) ∧ setsum c I = 1 ∧
(∀ i∈I. xs i ∈ S i)
      using convex-hull-finite-union[of I S] assms by auto
    def s ≡ λi. c i *R xs i
    {
      fix i
      assume i ∈ I
      then have s i ∈ S i
        using s-def x assms mem-cone[of S i xs i c i] by auto
    }
    then have ∀ i∈I. s i ∈ S i by auto
    moreover have x = setsum s I using x s-def by auto
    ultimately have x ∈ ?rhs
      using set-setsum-alt[of I S] assms by auto
  }
  moreover
  {
    fix x
    assume x ∈ ?rhs
    then obtain s where x: x = setsum s I ∧ (∀ i∈I. s i ∈ S i)
      using set-setsum-alt[of I S] assms by auto
    def xs ≡ λi. of-nat(card I) *R s i
    then have x = setsum (λi. ((1 :: real) / of-nat(card I)) *R xs i) I
      using x assms by auto
    moreover have ∀ i∈I. xs i ∈ S i
      using x xs-def assms by (simp add: cone-def)
    moreover have ∀ i∈I. (1 :: real) / of-nat (card I) ≥ 0
      by auto
    moreover have setsum (λi. (1 :: real) / of-nat (card I)) I = 1
      using assms by auto
    ultimately have x ∈ ?lhs
      apply (subst convex-hull-finite-union[of I S])
      using assms
      apply blast
```

```
    using assms
    apply blast
    apply rule
    apply (rule-tac x = (λi. (1 :: real) / of-nat (card I)) in exI)
    apply auto
    done
  }
  ultimately show ?thesis by auto
qed

lemma convex-hull-union-cones-two:
  fixes S T :: 'm::euclidean-space set
  assumes convex S
    and cone S
    and S ≠ {}
  assumes convex T
    and cone T
    and T ≠ {}
  shows convex hull (S ∪ T) = S + T
proof −
  def I ≡ {1::nat, 2}
  def A ≡ (λi. if i = (1::nat) then S else T)
  have ⋃(A ‘ I) = S ∪ T
    using A-def I-def by auto
  then have convex hull (⋃(A ‘ I)) = convex hull (S ∪ T)
    by auto
  moreover have convex hull ⋃(A ‘ I) = setsum A I
    apply (subst convex-hull-finite-union-cones[of I A])
    using assms A-def I-def
    apply auto
    done
  moreover have setsum A I = S + T
    using A-def I-def
    unfolding set-plus-def
    apply auto
    unfolding set-plus-def
    apply auto
    done
  ultimately show ?thesis by auto
qed

lemma rel-interior-convex-hull-union:
  fixes S :: 'a ⇒ 'n::euclidean-space set
  assumes finite I
    and ∀ i∈I. convex (S i) ∧ S i ≠ {}
  shows rel-interior (convex hull (⋃(S ‘ I))) =
    {setsum (λi. c i *R s i) I | c s. (∀ i∈I. c i > 0) ∧ setsum c I = 1 ∧
      (∀ i∈I. s i ∈ rel-interior(S i))}
  (is ?lhs = ?rhs)
```

proof (*cases* $I = \{\}$)
case *True*
then show *?thesis*
using *convex-hull-empty rel-interior-empty* **by** *auto*
next
case *False*
def *C0* $\equiv$ *convex hull* $(\bigcup (S \text{ ` } I))$
have $\forall\, i{\in}I.\ C0 \geq S\ i$
unfolding *C0-def* **using** *hull-subset*[*of* $\bigcup (S \text{ ` } I)$] **by** *auto*
def *K0* $\equiv$ *cone hull* $(\{1 :: real\} \times C0)$
def $K \equiv \lambda i.$ *cone hull* $(\{1 :: real\} \times S\ i)$
have $\forall\, i{\in}I.\ K\ i \neq \{\}$
unfolding *K-def* **using** *assms*
by (*simp add*: *cone-hull-empty-iff*[*symmetric*])
{
fix i
assume $i \in I$
then have *convex* $(K\ i)$
unfolding *K-def*
apply (*subst convex-cone-hull*)
apply (*subst convex-Times*)
using *assms*
apply *auto*
done
}
then have *convK*: $\forall\, i{\in}I.$ *convex* $(K\ i)$
by *auto*
{
fix i
assume $i \in I$
then have $K0 \supseteq K\ i$
unfolding *K0-def K-def*
apply (*subst hull-mono*)
using $\langle \forall\, i{\in}I.\ C0 \geq S\ i \rangle$
apply *auto*
done
}
then have $K0 \supseteq \bigcup (K \text{ ` } I)$ **by** *auto*
moreover have *convex K0*
unfolding *K0-def*
apply (*subst convex-cone-hull*)
apply (*subst convex-Times*)
unfolding *C0-def*
using *convex-convex-hull*
apply *auto*
done
ultimately have *geq*: $K0 \supseteq$ *convex hull* $(\bigcup (K \text{ ` } I))$
using *hull-minimal*[*of - K0 convex*] **by** *blast*
have $\forall\, i{\in}I.\ K\ i \supseteq \{1 :: real\} \times S\ i$

```
  using K-def by (simp add: hull-subset)
 then have ⋃(K ‘ I) ⊇ {1 :: real} × ⋃(S ‘ I)
  by auto
 then have convex hull ⋃(K ‘ I) ⊇ convex hull ({1 :: real} × ⋃(S ‘ I))
  by (simp add: hull-mono)
 then have convex hull ⋃(K ‘ I) ⊇ {1 :: real} × C0
  unfolding C0-def
  using convex-hull-Times[of {(1 :: real)} ⋃(S ‘ I)] convex-hull-singleton
  by auto
 moreover have cone (convex hull (⋃(K ‘ I)))
  apply (subst cone-convex-hull)
  using cone-Union[of K ‘ I]
  apply auto
  unfolding K-def
  using cone-cone-hull
  apply auto
  done
 ultimately have convex hull (⋃(K ‘ I)) ⊇ K0
  unfolding K0-def
  using hull-minimal[of - convex hull (⋃(K ‘ I)) cone]
  by blast
 then have K0 = convex hull (⋃(K ‘ I))
  using geq by auto
 also have ... = setsum K I
  apply (subst convex-hull-finite-union-cones[of I K])
  using assms
  apply blast
  using False
  apply blast
  unfolding K-def
  apply rule
  apply (subst convex-cone-hull)
  apply (subst convex-Times)
  using assms cone-cone-hull ⟨∀ i∈I. K i ≠ {}⟩ K-def
  apply auto
  done
 finally have K0 = setsum K I by auto
 then have ∗: rel-interior K0 = setsum (λi. (rel-interior (K i))) I
  using rel-interior-sum-gen[of I K] convK by auto
 {
  fix x
  assume x ∈ ?lhs
  then have (1::real, x) ∈ rel-interior K0
  using K0-def C0-def rel-interior-convex-cone-aux[of C0 1::real x] convex-convex-hull
   by auto
  then obtain k where k: (1::real, x) = setsum k I ∧ (∀ i∈I. k i ∈ rel-interior
(K i))
   using ⟨finite I⟩ ∗ set-setsum-alt[of I λi. rel-interior (K i)] by auto
  {
```

```
    fix i
    assume i ∈ I
    then have convex (S i) ∧ k i ∈ rel-interior (cone hull {1} × S i)
      using k K-def assms by auto
    then have ∃ ci si. k i = (ci, ci *R si) ∧ 0 < ci ∧ si ∈ rel-interior (S i)
      using rel-interior-convex-cone[of S i] by auto
  }
  then obtain c s where
    cs: ∀ i∈I. k i = (c i, c i *R s i) ∧ 0 < c i ∧ s i ∈ rel-interior (S i)
    by metis
  then have x = (∑ i∈I. c i *R s i) ∧ setsum c I = 1
    using k by (simp add: setsum-prod)
  then have x ∈ ?rhs
    using k
    apply auto
    apply (rule-tac x = c in exI)
    apply (rule-tac x = s in exI)
    using cs
    apply auto
    done
 }
 moreover
 {
  fix x
  assume x ∈ ?rhs
  then obtain c s where cs: x = setsum (λi. c i *R s i) I ∧
      (∀ i∈I. c i > 0) ∧ setsum c I = 1 ∧ (∀ i∈I. s i ∈ rel-interior (S i))
    by auto
  def k ≡ λi. (c i, c i *R s i)
  {
    fix i assume i:I
    then have k i ∈ rel-interior (K i)
      using k-def K-def assms cs rel-interior-convex-cone[of S i]
      by auto
  }
  then have (1::real, x) ∈ rel-interior K0
    using K0-def * set-setsum-alt[of I (λi. rel-interior (K i))] assms k-def cs
    apply auto
    apply (rule-tac x = k in exI)
    apply (simp add: setsum-prod)
    done
  then have x ∈ ?lhs
    using K0-def C0-def rel-interior-convex-cone-aux[of C0 1 x]
    by (auto simp add: convex-convex-hull)
 }
 ultimately show ?thesis by blast
qed
```

lemma *convex-le-Inf-differential*:
fixes $f :: real \Rightarrow real$
assumes *convex-on I f*
and $x \in interior\ I$
and $y \in I$
shows $f\ y \geq f\ x + Inf\ ((\lambda t.\ (f\ x - f\ t)\ /\ (x - t))\ `\ (\{x<..\} \cap I)) * (y - x)$
(**is** - $\geq$ - + *Inf* (*?F x*) $*$ $(y - x)$)
proof (*cases rule*: *linorder-cases*)
assume $x < y$
moreover
have *open* (*interior I*) **by** *auto*
from *openE*[*OF this* ‹$x \in interior\ I$›]
obtain *e* **where** *e*: $0 < e$ *ball x e* $\subseteq$ *interior I* **.**
moreover def $t \equiv min\ (x + e\ /\ 2)\ ((x + y)\ /\ 2)$
ultimately have $x < t\ t < y\ t \in ball\ x\ e$
by (*auto simp*: *dist-real-def field-simps split*: *split-min*)
with ‹$x \in interior\ I$› *e interior-subset*[*of I*] **have** $t \in I\ x \in I$ **by** *auto*

have *open* (*interior I*) **by** *auto*
from *openE*[*OF this* ‹$x \in interior\ I$›]
obtain *e* **where** $0 < e$ *ball x e* $\subseteq$ *interior I* **.**
moreover def $K \equiv x - e\ /\ 2$
with ‹$0 < e$›**have** $K \in ball\ x\ e\ K < x$
by (*auto simp*: *dist-real-def*)
ultimately have $K \in I\ K < x\ x \in I$
using *interior-subset*[*of I*] ‹$x \in interior\ I$› **by** *auto*

have *Inf* (*?F x*) $\leq (f\ x - f\ y)\ /\ (x - y)$
proof (*intro bdd-belowI cInf-lower2*)
show $(f\ x - f\ t)\ /\ (x - t) \in$ *?F x*
using ‹$t \in I$› ‹$x < t$› **by** *auto*
show $(f\ x - f\ t)\ /\ (x - t) \leq (f\ x - f\ y)\ /\ (x - y)$
using ‹*convex-on I f*› ‹$x \in I$› ‹$y \in I$› ‹$x < t$› ‹$t < y$›
by (*rule convex-on-diff*)
next
fix *y*
assume $y \in$ *?F x*
with *order-trans*[*OF convex-on-diff*[*OF* ‹*convex-on I f*› ‹$K \in I$› - ‹$K < x$› -]]
show $(f\ K - f\ x)\ /\ (K - x) \leq y$ **by** *auto*
qed
then show *?thesis*
using ‹$x < y$› **by** (*simp add*: *field-simps*)
next
assume $y < x$
moreover
have *open* (*interior I*) **by** *auto*
from *openE*[*OF this* ‹$x \in interior\ I$›]
obtain *e* **where** *e*: $0 < e$ *ball x e* $\subseteq$ *interior I* **.**
moreover def $t \equiv x + e\ /\ 2$

ultimately have $x < t\ t \in ball\ x\ e$
by (*auto simp*: *dist-real-def field-simps*)
with ‹$x \in interior\ I$› e *interior-subset*[*of I*] **have** $t \in I\ x \in I$ **by** *auto*

have $(f\ x - f\ y) / (x - y) \leq Inf\ (?F\ x)$
proof (*rule cInf-greatest*)
have $(f\ x - f\ y) / (x - y) = (f\ y - f\ x) / (y - x)$
using ‹$y < x$› **by** (*auto simp*: *field-simps*)
also
fix z
assume $z \in ?F\ x$
with *order-trans*[*OF convex-on-diff*[*OF* ‹*convex-on I f*› ‹$y \in I$› - ‹$y < x$›]]
have $(f\ y - f\ x) / (y - x) \leq z$
by *auto*
finally show $(f\ x - f\ y) / (x - y) \leq z$ **.**
next
have *open* (*interior I*) **by** *auto*
from *openE*[*OF this* ‹$x \in interior\ I$›]
obtain e **where** e: $0 < e\ ball\ x\ e \subseteq interior\ I$ **.**
then have $x + e / 2 \in ball\ x\ e$
by (*auto simp*: *dist-real-def*)
with e *interior-subset*[*of I*] **have** $x + e / 2 \in \{x<..\} \cap I$
by *auto*
then show $?F\ x \neq \{\}$
by *blast*
qed
then show *?thesis*
using ‹$y < x$› **by** (*simp add*: *field-simps*)
qed *simp*

19.37 Explicit formulas for interior and relative interior of convex hull

lemma *interior-atLeastAtMost* [*simp*]:
fixes a::*real* **shows** $interior\ \{a..b\} = \{a<..<b\}$
using *interior-cbox* [*of a b*] **by** *auto*

lemma *interior-atLeastLessThan* [*simp*]:
fixes a::*real* **shows** $interior\ \{a..<b\} = \{a<..<b\}$
by (*metis atLeastLessThan-def greaterThanLessThan-def interior-atLeastAtMost interior-Int interior-interior interior-real-semiline*)

lemma *interior-lessThanAtMost* [*simp*]:
fixes a::*real* **shows** $interior\ \{a<..b\} = \{a<..<b\}$
by (*metis atLeastAtMost-def greaterThanAtMost-def interior-atLeastAtMost interior-Int interior-interior interior-real-semiline*)

lemma *at-within-closed-interval*:
fixes x::*real*

shows $a < x \Longrightarrow x < b \Longrightarrow (at\ x\ within\ \{a..b\}) = at\ x$
by (*metis at-within-interior greaterThanLessThan-iff interior-atLeastAtMost*)

lemma *affine-independent-convex-affine-hull*:
fixes $s :: 'a::euclidean\text{-}space\ set$
assumes $\sim affine\text{-}dependent\ s\ t \subseteq s$
shows $convex\ hull\ t = affine\ hull\ t \cap convex\ hull\ s$
proof −
have *fin*: *finite s finite t* **using** *assms aff-independent-finite finite-subset* **by** *auto*
{ fix *u v x*
assume *uv*: $setsum\ u\ t = 1\ \forall x{\in}s.\ 0 \leq v\ x\ setsum\ v\ s = 1$
$(\sum x{\in}s.\ v\ x *_R x) = (\sum v{\in}t.\ u\ v *_R v)\ x \in t$
then have *s*: $s = (s - t) \cup t$ — split into separate cases
using *assms* **by** *auto*
have [*simp*]: $(\sum x{\in}t.\ v\ x *_R x) + (\sum x{\in}s - t.\ v\ x *_R x) = (\sum x{\in}t.\ u\ x *_R x)$
$setsum\ v\ t + setsum\ v\ (s - t) = 1$
using *uv fin s*
by (*auto simp*: *setsum.union-disjoint* [*symmetric*] *Un-commute*)
have $(\sum x{\in}s.\ if\ x \in t\ then\ v\ x - u\ x\ else\ v\ x) = 0$
$(\sum x{\in}s.\ (if\ x \in t\ then\ v\ x - u\ x\ else\ v\ x) *_R x) = 0$
using *uv fin*
by (*subst s*, *subst setsum.union-disjoint*, *auto simp*: *algebra-simps setsum-subtractf*)+
} note [*simp*] = *this*
have $convex\ hull\ t \subseteq affine\ hull\ t$
using *convex-hull-subset-affine-hull* **by** *blast*
moreover have $convex\ hull\ t \subseteq convex\ hull\ s$
using *assms hull-mono* **by** *blast*
moreover have $affine\ hull\ t \cap convex\ hull\ s \subseteq convex\ hull\ t$
using *assms*
apply (*simp add*: *convex-hull-finite affine-hull-finite fin affine-dependent-explicit*)
apply (*drule-tac x*=*s* **in** *spec*)
apply (*auto simp*: *fin*)
apply (*rule-tac x*=*u* **in** *exI*)
apply (*rename-tac v*)
apply (*drule-tac x*=$\lambda x.\ if\ x \in t\ then\ v\ x - u\ x\ else\ v\ x$ **in** *spec*)
apply (*force*)+
done
ultimately show *?thesis*
by *blast*
qed

lemma *affine-independent-span-eq*:
fixes $s :: 'a::euclidean\text{-}space\ set$
assumes $\sim affine\text{-}dependent\ s\ card\ s = Suc\ (DIM\ ('a))$
shows $affine\ hull\ s = UNIV$
proof (*cases* $s = \{\}$)
case *True* **then show** *?thesis*
using *assms* **by** *simp*

next
case *False*
then obtain *a t* **where** *t*: $a \notin t$ $s = insert\ a\ t$
by *blast*
then have *fin*: *finite t* **using** *assms*
by (*metis finite-insert aff-independent-finite*)
show *?thesis*
using *assms t fin*
apply (*simp add*: *affine-dependent-iff-dependent affine-hull-insert-span-gen*)
apply (*rule subset-antisym*)
apply *force*
apply (*rule Fun.vimage-subsetD*)
apply (*metis add.commute diff-add-cancel surj-def*)
apply (*rule card-ge-dim-independent*)
apply (*auto simp*: *card-image inj-on-def dim-subset-UNIV*)
done
qed

lemma *affine-independent-span-gt*:
fixes *s* :: *'a*::*euclidean-space set*
assumes *ind*: ~ *affine-dependent s* **and** *dim*: $DIM\ ('a) < card\ s$
shows *affine hull s* = *UNIV*
apply (*rule affine-independent-span-eq* [*OF ind*])
apply (*rule antisym*)
using *assms*
apply *auto*
apply (*metis add-2-eq-Suc' not-less-eq-eq affine-dependent-biggerset aff-independent-finite*)
done

lemma *empty-interior-affine-hull*:
fixes *s* :: *'a*::*euclidean-space set*
assumes *finite s* **and** *dim*: $card\ s \leq DIM\ ('a)$
shows $interior(affine\ hull\ s) = \{\}$
using *assms*
apply (*induct s rule*: *finite-induct*)
apply (*simp-all add*: *affine-dependent-iff-dependent affine-hull-insert-span-gen interior-translation*)
apply (*rule empty-interior-lowdim*)
apply (*simp add*: *affine-dependent-iff-dependent affine-hull-insert-span-gen*)
apply (*metis Suc-le-lessD not-less order-trans card-image-le finite-imageI dim-le-card*)
done

lemma *empty-interior-convex-hull*:
fixes *s* :: *'a*::*euclidean-space set*
assumes *finite s* **and** *dim*: $card\ s \leq DIM\ ('a)$
shows $interior(convex\ hull\ s) = \{\}$
by (*metis Diff-empty Diff-eq-empty-iff convex-hull-subset-affine-hull*
interior-mono empty-interior-affine-hull [*OF assms*])

lemma *explicit-subset-rel-interior-convex-hull*:
fixes s :: *'a::euclidean-space set*
shows *finite s*
$\Longrightarrow \{y.\ \exists u.\ (\forall x \in s.\ 0 < u\ x \wedge u\ x < 1) \wedge setsum\ u\ s = 1 \wedge setsum\ (\lambda x.\ u\ x *_R x)\ s = y\}$
$\subseteq$ *rel-interior* (*convex hull s*)
by (*force simp add*: *rel-interior-convex-hull-union* [**where** $S{=}\lambda x.\ \{x\}$ **and** $I{=}s$, *simplified*])

lemma *explicit-subset-rel-interior-convex-hull-minimal*:
fixes s :: *'a::euclidean-space set*
shows *finite s*
$\Longrightarrow \{y.\ \exists u.\ (\forall x \in s.\ 0 < u\ x) \wedge setsum\ u\ s = 1 \wedge setsum\ (\lambda x.\ u\ x *_R x)\ s = y\}$
$\subseteq$ *rel-interior* (*convex hull s*)
by (*force simp add*: *rel-interior-convex-hull-union* [**where** $S{=}\lambda x.\ \{x\}$ **and** $I{=}s$, *simplified*])

lemma *rel-interior-convex-hull-explicit*:
fixes s :: *'a::euclidean-space set*
assumes ~ *affine-dependent s*
shows *rel-interior*(*convex hull s*) =
$\{y.\ \exists u.\ (\forall x \in s.\ 0 < u\ x) \wedge setsum\ u\ s = 1 \wedge setsum\ (\lambda x.\ u\ x *_R x)\ s = y\}$
(**is** *?lhs* = *?rhs*)
proof
show *?rhs* $\leq$ *?lhs*
by (*simp add*: *aff-independent-finite explicit-subset-rel-interior-convex-hull-minimal assms*)
next
show *?lhs* $\leq$ *?rhs*
proof (*cases* $\exists a.\ s = \{a\}$)
case *True* **then show** *?lhs* $\leq$ *?rhs*
by *force*
next
case *False*
have *fs*: *finite s*
using *assms* **by** (*simp add*: *aff-independent-finite*)
{ fix *a b* **and** *d*::*real*
assume *ab*: $a \in s\ b \in s\ a \neq b$
then have *s*: $s = (s - \{a,b\}) \cup \{a,b\}$ — split into separate cases
by *auto*
have $(\sum x{\in}s.\ \textit{if } x = a \textit{ then } -d \textit{ else if } x = b \textit{ then } d \textit{ else } 0) = 0$
$(\sum x{\in}s.\ (\textit{if } x = a \textit{ then } -d \textit{ else if } x = b \textit{ then } d \textit{ else } 0) *_R x) = d *_R b - d *_R a$
using *ab fs*
by (*subst s*, *subst setsum.union-disjoint*, *auto*)+
} note [*simp*] = *this*
{ fix *y*

```
    assume y: y ∈ convex hull s y ∉ ?rhs
    { fix u T a
      assume ua: ∀x∈s. 0 ≤ u x setsum u s = 1 ¬ 0 < u a a ∈ s
        and yT: y = (∑x∈s. u x *R x) y ∈ T open T
        and sb: T ∩ affine hull s ⊆ {w. ∃u. (∀x∈s. 0 ≤ u x) ∧ setsum u s = 1
∧ (∑x∈s. u x *R x) = w}
      have ua0: u a = 0
        using ua by auto
      obtain b where b: b∈s a ≠ b
        using ua False by auto
      obtain e where e: 0 < e ball (∑x∈s. u x *R x) e ⊆ T
        using yT by (auto elim: openE)
      with b obtain d where d: 0 < d norm(d *R (a−b)) < e
        by (auto intro: that [of e / 2 / norm(a−b)])
      have (∑x∈s. u x *R x) ∈ affine hull s
        using yT y by (metis affine-hull-convex-hull hull-redundant-eq)
      then have (∑x∈s. u x *R x) − d *R (a − b) ∈ affine hull s
        using ua b by (auto simp: hull-inc intro: mem-affine-3-minus2)
      then have y − d *R (a − b) ∈ T ∩ affine hull s
        using d e yT by auto
      then obtain v where ∀x∈s. 0 ≤ v x
                          setsum v s = 1
                          (∑x∈s. v x *R x) = (∑x∈s. u x *R x) − d *R (a − b)
        using subsetD [OF sb] yT
        by auto
      then have False
        using assms
        apply (simp add: affine-dependent-explicit-finite fs)
         apply (drule-tac x=λx. (v x − u x) − (if x = a then −d else if x = b
then d else 0) in spec)
        using ua b d
        apply (auto simp: algebra-simps setsum-subtractf setsum.distrib)
        done
    } note * = this
    have y ∉ rel-interior (convex hull s)
      using y
      apply (simp add: mem-rel-interior affine-hull-convex-hull)
      apply (auto simp: convex-hull-finite [OF fs])
      apply (drule-tac x=u in spec)
      apply (auto intro: *)
      done
  } with rel-interior-subset show ?lhs ≤ ?rhs
    by blast
 qed
qed

lemma interior-convex-hull-explicit-minimal:
 fixes s :: 'a::euclidean-space set
 shows
```

```
  ~ affine-dependent s
      ==> interior(convex hull s) =
          (if card(s) ≤ DIM('a) then {}
           else {y. ∃u. (∀x ∈ s. 0 < u x) ∧ setsum u s = 1 ∧ (∑x∈s. u x *R
x) = y})
  apply (simp add: aff-independent-finite empty-interior-convex-hull, clarify)
  apply (rule trans [of - rel-interior(convex hull s)])
 apply (simp add: affine-hull-convex-hull affine-independent-span-gt rel-interior-interior)
  by (simp add: rel-interior-convex-hull-explicit)

lemma interior-convex-hull-explicit:
  fixes s :: 'a::euclidean-space set
  assumes ~ affine-dependent s
  shows
   interior(convex hull s) =
           (if card(s) ≤ DIM('a) then {}
           else {y. ∃u. (∀x ∈ s. 0 < u x ∧ u x < 1) ∧ setsum u s = 1 ∧ (∑x∈s.
u x *R x) = y})
proof −
  { fix u :: 'a ⇒ real and a
    assume card Basis < card s and u: ⋀x. x∈s ⟹ 0 < u x setsum u s = 1 and
a: a ∈ s
    then have cs: Suc 0 < card s
      by (metis DIM-positive less-trans-Suc)
    obtain b where b: b ∈ s a ≠ b
    proof (cases s ≤ {a})
      case True
      then show thesis
        using cs subset-singletonD by fastforce
    next
      case False
      then show thesis
      by (blast intro: that)
    qed
    have u a + u b ≤ setsum u {a,b}
      using a b by simp
    also have ... ≤ setsum u s
      apply (rule Groups-Big.setsum-mono2)
      using a b u
      apply (auto simp: less-imp-le aff-independent-finite assms)
      done
    finally have u a < 1
      using ‹b ∈ s› u by fastforce
  } note [simp] = this
  show ?thesis
    using assms
    apply (auto simp: interior-convex-hull-explicit-minimal)
    apply (rule-tac x=u in exI)
    apply (auto simp: not-le)
```

```
    done
qed

lemma interior-closed-segment-ge2:
  fixes a :: 'a::euclidean-space
  assumes 2 ≤ DIM('a)
    shows  interior(closed-segment a b) = {}
using assms unfolding segment-convex-hull
proof −
  have card {a, b} ≤ DIM('a)
    using assms
    by (simp add: card-insert-if linear not-less-eq-eq numeral-2-eq-2)
  then show interior (convex hull {a, b}) = {}
    by (metis empty-interior-convex-hull finite.insertI finite.emptyI)
qed

lemma interior-open-segment:
  fixes a :: 'a::euclidean-space
  shows  interior(open-segment a b) =
                (if 2 ≤ DIM('a) then {} else open-segment a b)
proof (simp add: not-le, intro conjI impI)
  assume 2 ≤ DIM('a)
  then show interior (open-segment a b) = {}
    apply (simp add: segment-convex-hull open-segment-def)
   apply (metis Diff-subset interior-mono segment-convex-hull subset-empty interior-closed-segment-ge2)
    done
next
  assume le2: DIM('a) < 2
  show interior (open-segment a b) = open-segment a b
  proof (cases a = b)
    case True then show ?thesis by auto
  next
    case False
    with le2 have affine hull (open-segment a b) = UNIV
      apply simp
      apply (rule affine-independent-span-gt)
      apply (simp-all add: affine-dependent-def insert-Diff-if)
      done
    then show interior (open-segment a b) = open-segment a b
      using rel-interior-interior rel-interior-open-segment by blast
  qed
qed

lemma interior-closed-segment:
  fixes a :: 'a::euclidean-space
  shows interior(closed-segment a b) =
                (if 2 ≤ DIM('a) then {} else open-segment a b)
proof (cases a = b)
  case True then show ?thesis by simp
```

next
case *False*
then have *closure* (*open-segment a b*) = *closed-segment a b*
by *simp*
then show *?thesis*
by (*metis* (*no-types*) *convex-interior-closure convex-open-segment interior-open-segment*)
qed

lemmas *interior-segment* = *interior-closed-segment interior-open-segment*

lemma *closed-segment-eq* [*simp*]:
fixes *a* :: *'a*::*euclidean-space*
shows *closed-segment a b* = *closed-segment c d* ⟷ {*a*,*b*} = {*c*,*d*}
proof
assume *abcd*: *closed-segment a b* = *closed-segment c d*
show {*a*,*b*} = {*c*,*d*}
proof (*cases a*=*b* ∨ *c*=*d*)
case *True* **with** *abcd* **show** *?thesis* **by** *force*
next
case *False*
then have *neq*: *a* ≠ *b* ∧ *c* ≠ *d* **by** *force*
have ∗: *closed-segment c d* − {*a*, *b*} = *rel-interior* (*closed-segment c d*)
using *neq abcd* **by** (*metis* (*no-types*) *open-segment-def rel-interior-closed-segment*)
have *b* ∈ {*c*, *d*}
proof −
have *insert b* (*closed-segment c d*) = *closed-segment c d*
using *abcd* **by** *blast*
then show *?thesis*
by (*metis DiffD2 Diff-insert2 False* ∗ *insertI1 insert-Diff-if open-segment-def rel-interior-closed-segment*)
qed
moreover have *a* ∈ {*c*, *d*}
by (*metis Diff-iff False* ∗ *abcd ends-in-segment*(*1*) *insertI1 open-segment-def rel-interior-closed-segment*)
ultimately show {*a*, *b*} = {*c*, *d*}
using *neq* **by** *fastforce*
qed
next
assume {*a*,*b*} = {*c*,*d*}
then show *closed-segment a b* = *closed-segment c d*
by (*simp add*: *segment-convex-hull*)
qed

lemma *closed-open-segment-eq* [*simp*]:
fixes *a* :: *'a*::*euclidean-space*
shows *closed-segment a b* ≠ *open-segment c d*
by (*metis DiffE closed-segment-neq-empty closure-closed-segment closure-open-segment ends-in-segment*(*1*) *insertI1 open-segment-def*)

lemma *open-closed-segment-eq* [*simp*]:
fixes *a* :: *'a*::*euclidean-space*
shows *open-segment a b* ≠ *closed-segment c d*
using *closed-open-segment-eq* **by** *blast*

lemma *open-segment-eq* [*simp*]:
fixes *a* :: *'a*::*euclidean-space*
shows *open-segment a b* = *open-segment c d* ⟷ *a* = *b* ∧ *c* = *d* ∨ {*a*,*b*} = {*c*,*d*}
(**is** *?lhs* = *?rhs*)
proof
assume *abcd*: *?lhs*
show *?rhs*
proof (*cases a*=*b* ∨ *c*=*d*)
case *True* **with** *abcd* **show** *?thesis*
using *finite-open-segment* **by** *fastforce*
next
case *False*
then have *a2*: *a* ≠ *b* ∧ *c* ≠ *d* **by** *force*
with *abcd* **show** *?rhs*
unfolding *open-segment-def*
by (*metis* (*no-types*) *abcd closed-segment-eq closure-open-segment*)
qed
next
assume *?rhs*
then show *?lhs*
by (*metis Diff-cancel convex-hull-singleton insert-absorb2 open-segment-def segment-convex-hull*)
qed

19.38 Similar results for closure and (relative or absolute) frontier.

lemma *closure-convex-hull* [*simp*]:
fixes *s* :: *'a*::*euclidean-space set*
shows *compact s* ==> *closure*(*convex hull s*) = *convex hull s*
by (*simp add*: *compact-imp-closed compact-convex-hull*)

lemma *rel-frontier-convex-hull-explicit*:
fixes *s* :: *'a*::*euclidean-space set*
assumes ~ *affine-dependent s*
shows *rel-frontier*(*convex hull s*) =
{*y*. ∃ *u*. (∀ *x* ∈ *s*. *0* ≤ *u x*) ∧ (∃ *x* ∈ *s*. *u x* = *0*) ∧ *setsum u s* = *1* ∧ *setsum* (λ*x*. *u x* ∗*R x*) *s* = *y*}
proof −
have *fs*: *finite s*
using *assms* **by** (*simp add*: *aff-independent-finite*)
show *?thesis*
apply (*simp add*: *rel-frontier-def finite-imp-compact rel-interior-convex-hull-explicit*

assms fs)
apply (*auto simp*: *convex-hull-finite fs*)
apply (*drule-tac x*=*u* **in** *spec*)
apply (*rule-tac x*=*u* **in** *exI*)
apply *force*
apply (*rename-tac v*)
apply (*rule notE* [*OF assms*])
apply (*simp add*: *affine-dependent-explicit*)
apply (*rule-tac x*=*s* **in** *exI*)
apply (*auto simp*: *fs*)
apply (*rule-tac x* = $\lambda x.\ u\ x - v\ x$ **in** *exI*)
apply (*force simp*: *setsum-subtractf scaleR-diff-left*)
done
qed

lemma *frontier-convex-hull-explicit*:
fixes *s* :: *'a*::*euclidean-space set*
assumes ~ *affine-dependent s*
shows *frontier*(*convex hull s*) =
$\{y.\ \exists u.\ (\forall x \in s.\ 0 \le u\ x) \wedge (DIM\ ('a) < card\ s \longrightarrow (\exists x \in s.\ u\ x = 0))$
$\wedge$
$setsum\ u\ s = 1 \wedge setsum\ (\lambda x.\ u\ x *_R x)\ s = y\}$
proof −
have *fs*: *finite s*
using *assms* **by** (*simp add*: *aff-independent-finite*)
show *?thesis*
proof (*cases DIM* (*'a*) < *card s*)
case *True*
with *assms fs* **show** *?thesis*
by (*simp add*: *rel-frontier-def frontier-def rel-frontier-convex-hull-explicit* [*symmetric*]
interior-convex-hull-explicit-minimal rel-interior-convex-hull-explicit)
next
case *False*
then have *card s* ≤ *DIM* (*'a*)
by *linarith*
then show *?thesis*
using *assms fs*
apply (*simp add*: *frontier-def interior-convex-hull-explicit finite-imp-compact*)
apply (*simp add*: *convex-hull-finite*)
done
qed
qed

lemma *rel-frontier-convex-hull-cases*:
fixes *s* :: *'a*::*euclidean-space set*
assumes ~ *affine-dependent s*
shows *rel-frontier*(*convex hull s*) = $\bigcup\{convex\ hull\ (s - \{x\})\ |x.\ x \in s\}$
proof −

have *fs*: *finite s*
using *assms* **by** (*simp add*: *aff-independent-finite*)
{ fix *u a*
have $\forall x{\in}s.\ 0 \leq u\ x \Longrightarrow a \in s \Longrightarrow u\ a = 0 \Longrightarrow setsum\ u\ s = 1 \Longrightarrow$
$\exists x\ v.\ x \in s \wedge$
$(\forall x{\in}s - \{x\}.\ 0 \leq v\ x) \wedge$
$setsum\ v\ (s - \{x\}) = 1 \wedge (\sum x{\in}s - \{x\}.\ v\ x *_R x) = (\sum x{\in}s.$
$u\ x *_R x)$
apply (*rule-tac x*=*a* **in** *exI*)
apply (*rule-tac x*=*u* **in** *exI*)
apply (*simp add*: *Groups-Big.setsum-diff1 fs*)
done }
moreover
{ fix *a u*
have $a \in s \Longrightarrow \forall x{\in}s - \{a\}.\ 0 \leq u\ x \Longrightarrow setsum\ u\ (s - \{a\}) = 1 \Longrightarrow$
$\exists v.\ (\forall x{\in}s.\ 0 \leq v\ x) \wedge$
$(\exists x{\in}s.\ v\ x = 0) \wedge setsum\ v\ s = 1 \wedge (\sum x{\in}s.\ v\ x *_R x) = (\sum x{\in}s$
$- \{a\}.\ u\ x *_R x)$
apply (*rule-tac x*=$\lambda x.$ *if x = a then 0 else u x* **in** *exI*)
apply (*auto simp*: *setsum.If-cases Diff-eq if-smult fs*)
done }
ultimately show *?thesis*
using *assms*
apply (*simp add*: *rel-frontier-convex-hull-explicit*)
apply (*simp add*: *convex-hull-finite fs Union-SetCompr-eq*, *auto*)
done
qed

lemma *frontier-convex-hull-eq-rel-frontier*:
fixes *s* :: *'a*::*euclidean-space set*
assumes ~ *affine-dependent s*
shows *frontier*(*convex hull s*) =
(*if card s* ≤ *DIM* (*'a*) *then convex hull s else rel-frontier*(*convex hull s*))
using *assms*
unfolding *rel-frontier-def frontier-def*
by (*simp add*: *affine-independent-span-gt rel-interior-interior*
finite-imp-compact empty-interior-convex-hull aff-independent-finite)

lemma *frontier-convex-hull-cases*:
fixes *s* :: *'a*::*euclidean-space set*
assumes ~ *affine-dependent s*
shows *frontier*(*convex hull s*) =
(*if card s* ≤ *DIM* (*'a*) *then convex hull s else* $\bigcup$ {*convex hull* (*s* − {*x*}) |*x*. *x* ∈ *s*})
by (*simp add*: *assms frontier-convex-hull-eq-rel-frontier rel-frontier-convex-hull-cases*)

lemma *in-frontier-convex-hull*:
fixes *s* :: *'a*::*euclidean-space set*
assumes *finite s card s* ≤ *Suc* (*DIM* (*'a*)) *x* ∈ *s*

shows $x \in \mathit{frontier}(\mathit{convex\ hull}\ s)$
proof (*cases affine-dependent s*)
case *True*
with *assms* **show** *?thesis*
apply (*auto simp*: *affine-dependent-def frontier-def finite-imp-compact hull-inc*)
by (*metis card.insert-remove convex-hull-subset-affine-hull empty-interior-affine-hull finite-Diff hull-redundant insert-Diff insert-Diff-single insert-not-empty interior-mono not-less-eq-eq subset-empty*)
next
case *False*
{ assume *card s* = *Suc* (*card Basis*)
then have *cs*: *Suc 0* < *card s*
by (*simp add*: *DIM-positive*)
with *subset-singletonD* **have** $\exists\, y \in s.\ y \neq x$
by (*cases s* $\leq$ $\{x\}$) *fastforce*+
} note [*dest*!] = *this*
show *?thesis* **using** *assms*
unfolding *frontier-convex-hull-cases* [*OF False*] *Union-SetCompr-eq*
by (*auto simp*: *le-Suc-eq hull-inc*)
qed

lemma *not-in-interior-convex-hull*:
fixes *s* :: *'a*::*euclidean-space set*
assumes *finite s card s* $\leq$ *Suc* (*DIM* (*'a*)) *x* $\in$ *s*
shows $x \notin \mathit{interior}(\mathit{convex\ hull}\ s)$
using *in-frontier-convex-hull* [*OF assms*]
by (*metis Diff-iff frontier-def*)

lemma *interior-convex-hull-eq-empty*:
fixes *s* :: *'a*::*euclidean-space set*
assumes *card s* = *Suc* (*DIM* (*'a*))
shows $\mathit{interior}(\mathit{convex\ hull}\ s) = \{\} \longleftrightarrow \mathit{affine\text{-}dependent}\ s$
proof −
{ fix *a b*
assume *ab*: *a* $\in$ *interior* (*convex hull s*) *b* $\in$ *s b* $\in$ *affine hull* (*s* − {*b*})
then have *interior*(*affine hull s*) = {} **using** *assms*
by (*metis DIM-positive One-nat-def Suc-mono card.remove card-infinite empty-interior-affine-hull eq-iff hull-redundant insert-Diff not-less zero-le-one*)
then have *False* **using** *ab*
by (*metis convex-hull-subset-affine-hull equals0D interior-mono subset-eq*)
} then
show *?thesis*
using *assms*
apply *auto*
apply (*metis UNIV-I affine-hull-convex-hull affine-hull-empty affine-independent-span-eq convex-convex-hull empty-iff rel-interior-interior rel-interior-same-affine-hull*)
apply (*auto simp*: *affine-dependent-def*)
done
qed

19.39 Coplanarity, and collinearity in terms of affine hull

definition *coplanar* **where**
  coplanar s ≡ ∃ *u v w*. *s* ⊆ *affine hull* {*u*,*v*,*w*}

lemma *collinear-affine-hull*:
  collinear s ⟷ (∃ *u v*. *s* ⊆ *affine hull* {*u*,*v*})
proof (*cases s*={})
  case *True* **then show** *?thesis*
    by *simp*
next
  case *False*
  then obtain *x* **where** *x*: *x* ∈ *s* **by** *auto*
  { fix *u*
    assume ∗: ⋀*x y*. ⟦*x*∈*s*; *y*∈*s*⟧ ⟹ ∃ *c*. *x* − *y* = *c* ∗*R u*
    have ∃ *u v*. *s* ⊆ {*a* ∗*R u* + *b* ∗*R v* |*a b*. *a* + *b* = *1*}
      apply (*rule-tac x*=*x* **in** *exI*)
      apply (*rule-tac x*=*x*+*u* **in** *exI*, *clarify*)
      apply (*erule exE* [*OF* ∗ [*OF x*]])
      apply (*rename-tac c*)
      apply (*rule-tac x*=*1*+*c* **in** *exI*)
      apply (*rule-tac x*=−*c* **in** *exI*)
      apply (*simp add*: *algebra-simps*)
      done
  } moreover
  { fix *u v x y*
    assume ∗: *s* ⊆ {*a* ∗*R u* + *b* ∗*R v* |*a b*. *a* + *b* = *1*}
    have *x*∈*s* ⟹ *y*∈*s* ⟹ ∃ *c*. *x* − *y* = *c* ∗*R* (*v*−*u*)
      apply (*drule subsetD* [*OF* ∗])+
      apply *simp*
      apply *clarify*
      apply (*rename-tac r1 r2*)
      apply (*rule-tac x*=*r1*−*r2* **in** *exI*)
      apply (*simp add*: *algebra-simps*)
      apply (*metis scaleR-left.add*)
      done
  } ultimately
  show *?thesis*
  unfolding *collinear-def affine-hull-2*
    by *blast*
qed

lemma *collinear-closed-segment* [*simp*]: *collinear* (*closed-segment a b*)
by (*metis affine-hull-convex-hull collinear-affine-hull hull-subset segment-convex-hull*)

lemma *collinear-open-segment* [*simp*]: *collinear* (*open-segment a b*)
  unfolding *open-segment-def*
  by (*metis convex-hull-subset-affine-hull segment-convex-hull dual-order.trans*
    convex-hull-subset-affine-hull Diff-subset collinear-affine-hull)

lemma *subset-continuous-image-segment-1*:
fixes *f* :: *'a*::*euclidean-space* ⇒ *real*
assumes *continuous-on* (*closed-segment a b*) *f*
shows *closed-segment* (*f a*) (*f b*) ⊆ *image f* (*closed-segment a b*)
by (*metis connected-segment convex-contains-segment ends-in-segment imageI*
is-interval-connected-1 is-interval-convex connected-continuous-image [*OF assms*])

lemma *collinear-imp-coplanar*:
collinear s ==> *coplanar s*
by (*metis collinear-affine-hull coplanar-def insert-absorb2*)

lemma *collinear-small*:
assumes *finite s card s* ≤ *2*
shows *collinear s*
proof −
have *card s* = *0* ∨ *card s* = *1* ∨ *card s* = *2*
using *assms* **by** *linarith*
then show *?thesis* **using** *assms*
using *card-eq-SucD*
by *auto* (*metis collinear-2 numeral-2-eq-2*)
qed

lemma *coplanar-small*:
assumes *finite s card s* ≤ *3*
shows *coplanar s*
proof −
have *card s* ≤ *2* ∨ *card s* = *Suc* (*Suc* (*Suc 0*))
using *assms* **by** *linarith*
then show *?thesis* **using** *assms*
apply *safe*
apply (*simp add*: *collinear-small collinear-imp-coplanar*)
apply (*safe dest*!: *card-eq-SucD*)
apply (*auto simp*: *coplanar-def*)
apply (*metis hull-subset insert-subset*)
done
qed

lemma *coplanar-empty*: *coplanar* {}
by (*simp add*: *coplanar-small*)

lemma *coplanar-sing*: *coplanar* {*a*}
by (*simp add*: *coplanar-small*)

lemma *coplanar-2*: *coplanar* {*a*,*b*}
by (*auto simp*: *card-insert-if coplanar-small*)

lemma *coplanar-3*: *coplanar* {*a*,*b*,*c*}
by (*auto simp*: *card-insert-if coplanar-small*)

lemma *collinear-affine-hull-collinear*: *collinear*(*affine hull s*) ⟷ *collinear s*
unfolding *collinear-affine-hull*
by (*metis affine-affine-hull subset-hull hull-hull hull-mono*)

lemma *coplanar-affine-hull-coplanar*: *coplanar*(*affine hull s*) ⟷ *coplanar s*
unfolding *coplanar-def*
by (*metis affine-affine-hull subset-hull hull-hull hull-mono*)

lemma *coplanar-linear-image*:
fixes *f* :: *'a*::*euclidean-space* ⇒ *'b*::*real-normed-vector*
assumes *coplanar s linear f* **shows** *coplanar*(*f ' s*)
proof −
{ fix *u v w*
assume *s* ⊆ *affine hull* {*u*, *v*, *w*}
then have *f ' s* ⊆ *f '* (*affine hull* {*u*, *v*, *w*})
by (*simp add*: *image-mono*)
then have *f ' s* ⊆ *affine hull* (*f '* {*u*, *v*, *w*})
by (*metis assms*(*2*) *linear-conv-bounded-linear affine-hull-linear-image*)
} then
show *?thesis*
by *auto* (*meson assms*(*1*) *coplanar-def*)
qed

lemma *coplanar-translation-imp*: *coplanar s* ⟹ *coplanar* ((λ*x*. *a* + *x*) *' s*)
unfolding *coplanar-def*
apply *clarify*
apply (*rule-tac x*=*u*+*a* **in** *exI*)
apply (*rule-tac x*=*v*+*a* **in** *exI*)
apply (*rule-tac x*=*w*+*a* **in** *exI*)
using *affine-hull-translation* [*of a* {*u*,*v*,*w*} **for** *u v w*]
apply (*force simp*: *add.commute*)
done

lemma *coplanar-translation-eq*: *coplanar*((λ*x*. *a* + *x*) *' s*) ⟷ *coplanar s*
by (*metis* (*no-types*) *coplanar-translation-imp translation-galois*)

lemma *coplanar-linear-image-eq*:
fixes *f* :: *'a*::*euclidean-space* ⇒ *'b*::*euclidean-space*
assumes *linear f inj f* **shows** *coplanar*(*f ' s*) = *coplanar s*
proof
assume *coplanar s*
then show *coplanar* (*f ' s*)
unfolding *coplanar-def*
using *affine-hull-linear-image* [*of f* {*u*,*v*,*w*} **for** *u v w*] *assms*
by (*meson coplanar-def coplanar-linear-image*)
next
obtain *g* **where** *g*: *linear g g* ∘ *f* = *id*
using *linear-injective-left-inverse* [*OF assms*]

by *blast*
assume *coplanar* (*f ' s*)
then obtain *u v w* **where** *f ' s* ⊆ *affine hull* {*u*, *v*, *w*}
by (*auto simp*: *coplanar-def*)
then have *g ' f ' s* ⊆ *g '* (*affine hull* {*u*, *v*, *w*})
by *blast*
then have *s* ⊆ *g '* (*affine hull* {*u*, *v*, *w*})
using *g* **by** (*simp add*: *Fun.image-comp*)
then show *coplanar s*
unfolding *coplanar-def*
using *affine-hull-linear-image* [*of g* {*u*,*v*,*w*} **for** *u v w*] ‹*linear g*› *linear-conv-bounded-linear*
by *fastforce*
qed

lemma *coplanar-subset*: ⟦*coplanar t*; *s* ⊆ *t*⟧ ⟹ *coplanar s*
by (*meson coplanar-def order-trans*)

lemma *affine-hull-3-imp-collinear*: *c* ∈ *affine hull* {*a*,*b*} ⟹ *collinear* {*a*,*b*,*c*}
by (*metis collinear-2 collinear-affine-hull-collinear hull-redundant insert-commute*)

lemma *collinear-3-imp-in-affine-hull*: ⟦*collinear* {*a*,*b*,*c*}; *a* ≠ *b*⟧ ⟹ *c* ∈ *affine hull* {*a*,*b*}
unfolding *collinear-def*
apply *clarify*
apply (*frule-tac x*=*b* **in** *bspec*, *blast*, *drule-tac x*=*a* **in** *bspec*, *blast*, *erule exE*)
apply (*drule-tac x*=*c* **in** *bspec*, *blast*, *drule-tac x*=*a* **in** *bspec*, *blast*, *erule exE*)
apply (*rename-tac y x*)
apply (*simp add*: *affine-hull-2*)
apply (*rule-tac x*=*1* − *x*/*y* **in** *exI*)
apply (*simp add*: *algebra-simps*)
done

lemma *collinear-3-affine-hull*:
assumes *a* ≠ *b*
shows *collinear* {*a*,*b*,*c*} ⟷ *c* ∈ *affine hull* {*a*,*b*}
using *affine-hull-3-imp-collinear assms collinear-3-imp-in-affine-hull* **by** *blast*

lemma *collinear-3-eq-affine-dependent*:
collinear{*a*,*b*,*c*} ⟷ *a* = *b* ∨ *a* = *c* ∨ *b* = *c* ∨ *affine-dependent* {*a*,*b*,*c*}
apply (*case-tac a*=*b*, *simp*)
apply (*case-tac a*=*c*)
apply (*simp add*: *insert-commute*)
apply (*case-tac b*=*c*)
apply (*simp add*: *insert-commute*)
apply (*auto simp*: *affine-dependent-def collinear-3-affine-hull insert-Diff-if*)
apply (*metis collinear-3-affine-hull insert-commute*)+
done

lemma *affine-dependent-imp-collinear-3*:
affine-dependent {*a*,*b*,*c*} ⟹ *collinear*{*a*,*b*,*c*}
by (*simp add*: *collinear-3-eq-affine-dependent*)

lemma *collinear-3*: *NO-MATCH 0 x* ⟹ *collinear* {*x*,*y*,*z*} ⟷ *collinear* {*0*, *x*−*y*, *z*−*y*}
by (*auto simp add*: *collinear-def*)

thm *affine-hull-nonempty*
corollary *affine-hull-eq-empty* [*simp*]: *affine hull S* = {} ⟷ *S* = {}
using *affine-hull-nonempty* **by** *blast*

lemma *affine-hull-2-alt*:
fixes *a b* :: *'a*::*real-vector*
shows *affine hull* {*a*,*b*} = *range* (λ*u*. *a* + *u* *$_R$ (*b* − *a*))
apply (*simp add*: *affine-hull-2*, *safe*)
apply (*rule-tac x*=*v* **in** *image-eqI*)
apply (*simp add*: *algebra-simps*)
apply (*metis scaleR-add-left scaleR-one*, *simp*)
apply (*rule-tac x*=*1*−*u* **in** *exI*)
apply (*simp add*: *algebra-simps*)
done

lemma *interior-convex-hull-3-minimal*:
fixes *a* :: *'a*::*euclidean-space*
shows ⟦~ *collinear*{*a*,*b*,*c*}; *DIM*(*'a*) = *2*⟧
⟹ *interior*(*convex hull* {*a*,*b*,*c*}) =
{*v*. ∃*x y z*. *0* < *x* ∧ *0* < *y* ∧ *0* < *z* ∧ *x* + *y* + *z* = *1* ∧
x *$_R$ *a* + *y* *$_R$ *b* + *z* *$_R$ *c* = *v*}
apply (*simp add*: *collinear-3-eq-affine-dependent interior-convex-hull-explicit-minimal*, *safe*)
apply (*rule-tac x*=*u a* **in** *exI*, *simp*)
apply (*rule-tac x*=*u b* **in** *exI*, *simp*)
apply (*rule-tac x*=*u c* **in** *exI*, *simp*)
apply (*rename-tac uu x y z*)
apply (*rule-tac x*=λ*r*. (*if r*=*a then x else if r*=*b then y else if r*=*c then z else 0*)
in *exI*)
apply *simp*
done

19.40 The infimum of the distance between two sets

definition *setdist* :: *'a*::*metric-space set* ⇒ *'a set* ⇒ *real* **where**
setdist s t ≡
(*if s* = {} ∨ *t* = {} *then 0*
else Inf {*dist x y*| *x y*. *x* ∈ *s* ∧ *y* ∈ *t*})

lemma *setdist-empty1* [*simp*]: *setdist* {} *t* = *0*

by (*simp add*: *setdist-def*)

lemma *setdist-empty2* [*simp*]: *setdist t* {} = *0*
by (*simp add*: *setdist-def*)

lemma *setdist-pos-le*: *0* ≤ *setdist s t*
by (*auto simp*: *setdist-def ex-in-conv* [*symmetric*] *intro*: *cInf-greatest*)

lemma *le-setdistI*:
assumes *s* ≠ {} *t* ≠ {} ⋀*x y*. ⟦*x* ∈ *s*; *y* ∈ *t*⟧ ⟹ *d* ≤ *dist x y*
shows *d* ≤ *setdist s t*
using *assms*
by (*auto simp*: *setdist-def Set.ex-in-conv* [*symmetric*] *intro*: *cInf-greatest*)

lemma *setdist-le-dist*: ⟦*x* ∈ *s*; *y* ∈ *t*⟧ ⟹ *setdist s t* ≤ *dist x y*
unfolding *setdist-def*
by (*auto intro*!: *bdd-belowI* [**where** *m*=*0*] *cInf-lower*)

lemma *le-setdist-iff*:
d ≤ *setdist s t* ⟷
(∀ *x* ∈ *s*. ∀ *y* ∈ *t*. *d* ≤ *dist x y*) ∧ (*s* = {} ∨ *t* = {} ⟶ *d* ≤ *0*)
apply (*cases s* = {} ∨ *t* = {})
apply (*force simp add*: *setdist-def*)
apply (*intro iffI conjI*)
using *setdist-le-dist* **apply** *fastforce*
apply (*auto simp*: *intro*: *le-setdistI*)
done

lemma *setdist-ltE*:
assumes *setdist s t* < *b s* ≠ {} *t* ≠ {}
obtains *x y* **where** *x* ∈ *s y* ∈ *t dist x y* < *b*
using *assms*
by (*auto simp*: *not-le* [*symmetric*] *le-setdist-iff*)

lemma *setdist-refl*: *setdist s s* = *0*
apply (*cases s* = {})
apply (*force simp add*: *setdist-def*)
apply (*rule antisym* [*OF - setdist-pos-le*])
apply (*metis all-not-in-conv dist-self setdist-le-dist*)
done

lemma *setdist-sym*: *setdist s t* = *setdist t s*
by (*force simp*: *setdist-def dist-commute intro*!: *arg-cong* [**where** *f*=*Inf*])

lemma *setdist-triangle*: *setdist s t* ≤ *setdist s* {*a*} + *setdist* {*a*} *t*
proof (*cases s* = {} ∨ *t* = {})
case *True* **then show** *?thesis*
using *setdist-pos-le* **by** *fastforce*
next

case *False*
have $\bigwedge x.\ x \in s \Longrightarrow$ *setdist s t − dist x a* $\leq$ *setdist* $\{a\}$ *t*
apply (*rule le-setdistI*, *blast*)
using *False* **apply** (*fastforce intro*: *le-setdistI*)
apply (*simp add*: *algebra-simps*)
apply (*metis dist-commute dist-triangle3 order-trans* [*OF setdist-le-dist*])
done
then have *setdist s t − setdist* $\{a\}$ *t* $\leq$ *setdist s* $\{a\}$
using *False* **by** (*fastforce intro*: *le-setdistI*)
then show *?thesis*
by (*simp add*: *algebra-simps*)
qed

lemma *setdist-singletons* [*simp*]: *setdist* $\{x\}$ $\{y\}$ = *dist x y*
by (*simp add*: *setdist-def*)

lemma *setdist-Lipschitz*: |*setdist* $\{x\}$ *s − setdist* $\{y\}$ *s*| $\leq$ *dist x y*
apply (*subst setdist-singletons* [*symmetric*])
by (*metis abs-diff-le-iff diff-le-eq setdist-triangle setdist-sym*)

lemma *continuous-at-setdist*: *continuous* (*at x*) ($\lambda y.$ (*setdist* $\{y\}$ *s*))
by (*force simp*: *continuous-at-eps-delta dist-real-def intro*: *le-less-trans* [*OF setdist-Lipschitz*])

lemma *continuous-on-setdist*: *continuous-on t* ($\lambda y.$ (*setdist* $\{y\}$ *s*))
by (*metis continuous-at-setdist continuous-at-imp-continuous-on*)

lemma *uniformly-continuous-on-setdist*: *uniformly-continuous-on t* ($\lambda y.$ (*setdist* $\{y\}$ *s*))
by (*force simp*: *uniformly-continuous-on-def dist-real-def intro*: *le-less-trans* [*OF setdist-Lipschitz*])

lemma *setdist-subset-right*: $[\![t \neq \{\};\ t \subseteq u]\!] \Longrightarrow$ *setdist s u* $\leq$ *setdist s t*
apply (*cases s* = $\{\} \vee$ *u* = $\{\}$, *force*)
apply (*auto simp*: *setdist-def intro*!: *bdd-belowI* [**where** *m*=*0*] *cInf-superset-mono*)
done

lemma *setdist-subset-left*: $[\![s \neq \{\};\ s \subseteq t]\!] \Longrightarrow$ *setdist t u* $\leq$ *setdist s u*
by (*metis setdist-subset-right setdist-sym*)

lemma *setdist-closure-1* [*simp*]: *setdist* (*closure s*) *t* = *setdist s t*
proof (*cases s* = $\{\} \vee$ *t* = $\{\}$)
case *True* **then show** *?thesis* **by** *force*
next
case *False*
{ fix *y*
assume *y* $\in$ *t*
have *continuous-on* (*closure s*) ($\lambda a.$ *dist a y*)
by (*auto simp*: *continuous-intros dist-norm*)
then have $*$: $\bigwedge x.\ x \in$ *closure s* $\Longrightarrow$ *setdist s t* $\leq$ *dist x y*

```
    apply (rule continuous-ge-on-closure)
    apply assumption
    apply (blast intro: setdist-le-dist ‹y ∈ t› )
    done
  } note ∗ = this
  show ?thesis
   apply (rule antisym)
    using False closure-subset apply (blast intro: setdist-subset-left)
   using False ∗
   apply (force simp add: closure-eq-empty intro!: le-setdistI)
   done
qed

lemma setdist-closure-2 [simp]: setdist t (closure s) = setdist t s
by (metis setdist-closure-1 setdist-sym)

lemma setdist-compact-closed:
  fixes s :: 'a::euclidean-space set
  assumes s: compact s and t: closed t
      and s ≠ {} t ≠ {}
    shows ∃x ∈ s. ∃y ∈ t. dist x y = setdist s t
proof −
  have {x − y |x y. x ∈ s ∧ y ∈ t} ≠ {}
   using assms by blast
  then
  have ∃x ∈ s. ∃y ∈ t. dist x y ≤ setdist s t
    apply (rule distance-attains-inf [where a=0, OF compact-closed-differences
[OF s t]])
   apply (simp add: dist-norm le-setdist-iff)
   apply blast
   done
  then show ?thesis
   by (blast intro!: antisym [OF - setdist-le-dist] )
qed

lemma setdist-closed-compact:
  fixes s :: 'a::euclidean-space set
  assumes s: closed s and t: compact t
      and s ≠ {} t ≠ {}
    shows ∃x ∈ s. ∃y ∈ t. dist x y = setdist s t
  using setdist-compact-closed [OF t s ‹t ≠ {}› ‹s ≠ {}›]
  by (metis dist-commute setdist-sym)

lemma setdist-eq-0I: ⟦x ∈ s; x ∈ t⟧ ⟹ setdist s t = 0
  by (metis antisym dist-self setdist-le-dist setdist-pos-le)

lemma setdist-eq-0-compact-closed:
  fixes s :: 'a::euclidean-space set
  assumes s: compact s and t: closed t
```

shows *setdist s t = 0 ⟷ s = {} ∨ t = {} ∨ s ∩ t ≠ {}*
apply (*cases s = {} ∨ t = {}*, *force*)
using *setdist-compact-closed* [*OF s t*]
apply (*force intro*: *setdist-eq-0I*)
done

corollary *setdist-gt-0-compact-closed*:
fixes *s* :: *'a*::*euclidean-space set*
assumes *s*: *compact s* **and** *t*: *closed t*
shows *setdist s t > 0 ⟷ (s ≠ {} ∧ t ≠ {} ∧ s ∩ t = {})*
using *setdist-pos-le* [*of s t*] *setdist-eq-0-compact-closed* [*OF assms*]
by *linarith*

lemma *setdist-eq-0-closed-compact*:
fixes *s* :: *'a*::*euclidean-space set*
assumes *s*: *closed s* **and** *t*: *compact t*
shows *setdist s t = 0 ⟷ s = {} ∨ t = {} ∨ s ∩ t ≠ {}*
using *setdist-eq-0-compact-closed* [*OF t s*]
by (*metis Int-commute setdist-sym*)

lemma *setdist-eq-0-bounded*:
fixes *s* :: *'a*::*euclidean-space set*
assumes *bounded s ∨ bounded t*
shows *setdist s t = 0 ⟷ s = {} ∨ t = {} ∨ closure s ∩ closure t ≠ {}*
apply (*cases s = {} ∨ t = {}*, *force*)
using *setdist-eq-0-compact-closed* [*of closure s closure t*]
setdist-eq-0-closed-compact [*of closure s closure t*] *assms*
apply (*force simp add*: *bounded-closure compact-eq-bounded-closed*)
done

lemma *setdist-unique*:
⟦*a ∈ s*; *b ∈ t*; ⋀*x y*. *x ∈ s ∧ y ∈ t ==> dist a b ≤ dist x y*⟧
⟹ setdist s t = dist a b
by (*force simp add*: *setdist-le-dist le-setdist-iff intro*: *antisym*)

lemma *setdist-closest-point*:
⟦*closed s*; *s ≠ {}*⟧ *⟹ setdist {a} s = dist a* (*closest-point s a*)
apply (*rule setdist-unique*)
using *closest-point-le*
apply (*auto simp*: *closest-point-in-set*)
done

lemma *setdist-eq-0-sing-1* [*simp*]:
fixes *s* :: *'a*::*euclidean-space set*
shows *setdist {x} s = 0 ⟷ s = {} ∨ x ∈ closure s*
by (*auto simp*: *setdist-eq-0-bounded*)

lemma *setdist-eq-0-sing-2* [*simp*]:
fixes *s* :: *'a*::*euclidean-space set*

shows *setdist s {x} = 0 ⟷ s = {} ∨ x ∈ closure s*
by (*auto simp*: *setdist-eq-0-bounded*)

lemma *setdist-sing-in-set*:
fixes *s* :: *'a*::*euclidean-space set*
shows *x ∈ s ⟹ setdist {x} s = 0*
using *closure-subset* **by** *force*

lemma *setdist-le-sing*: *x ∈ s ==> setdist s t ≤ setdist {x} t*
using *setdist-subset-left* **by** *auto*

end

20 Continuous paths and path-connected sets

theory *Path-Connected*
imports *Convex-Euclidean-Space*
begin

20.1 Paths and Arcs

definition *path* :: (*real ⇒ 'a*::*topological-space*) ⇒ *bool*
where *path g ⟷ continuous-on {0..1} g*

definition *pathstart* :: (*real ⇒ 'a*::*topological-space*) ⇒ *'a*
where *pathstart g = g 0*

definition *pathfinish* :: (*real ⇒ 'a*::*topological-space*) ⇒ *'a*
where *pathfinish g = g 1*

definition *path-image* :: (*real ⇒ 'a*::*topological-space*) ⇒ *'a set*
where *path-image g = g ' {0 .. 1}*

definition *reversepath* :: (*real ⇒ 'a*::*topological-space*) ⇒ *real ⇒ 'a*
where *reversepath g = (λx. g(1 − x))*

definition *joinpaths* :: (*real ⇒ 'a*::*topological-space*) ⇒ (*real ⇒ 'a*) ⇒ *real ⇒ 'a*
(**infixr** *+++ 75*)
where *g1 +++ g2 = (λx. if x ≤ 1/2 then g1 (2 ∗ x) else g2 (2 ∗ x − 1))*

definition *simple-path* :: (*real ⇒ 'a*::*topological-space*) ⇒ *bool*
where *simple-path g ⟷*
path g ∧ (∀ x∈{0..1}. ∀ y∈{0..1}. g x = g y ⟶ x = y ∨ x = 0 ∧ y = 1 ∨ x = 1 ∧ y = 0)

definition *arc* :: (*real ⇒ 'a* :: *topological-space*) ⇒ *bool*
where *arc g ⟷ path g ∧ inj-on g {0..1}*

20.2 Invariance theorems

lemma *path-eq*: *path p* $\Longrightarrow$ $(\bigwedge t.\ t \in \{0..1\} \Longrightarrow p\ t = q\ t) \Longrightarrow$ *path q*
using *continuous-on-eq path-def* **by** *blast*

lemma *path-continuous-image*: *path g* $\Longrightarrow$ *continuous-on* (*path-image g*) *f* $\Longrightarrow$ *path*(*f o g*)
unfolding *path-def path-image-def*
using *continuous-on-compose* **by** *blast*

lemma *path-translation-eq*:
fixes *g* :: *real* $\Rightarrow$ *'a* :: *real-normed-vector*
shows *path*$((\lambda x.\ a + x)\ o\ g) =$ *path g*
proof −
have *g*: $g = (\lambda x.\ -a + x)\ o\ ((\lambda x.\ a + x)\ o\ g)$
by (*rule ext*) *simp*
show *?thesis*
unfolding *path-def*
apply *safe*
apply (*subst g*)
apply (*rule continuous-on-compose*)
apply (*auto intro*: *continuous-intros*)
done
qed

lemma *path-linear-image-eq*:
fixes *f* :: *'a*::*euclidean-space* $\Rightarrow$ *'b*::*euclidean-space*
assumes *linear f inj f*
shows *path*(*f o g*) = *path g*
proof −
from *linear-injective-left-inverse* [*OF assms*]
obtain *h* **where** *h*: *linear h h* $\circ$ *f* = *id*
by *blast*
then have *g*: $g = h\ o\ (f\ o\ g)$
by (*metis comp-assoc id-comp*)
show *?thesis*
unfolding *path-def*
using *h assms*
by (*metis g continuous-on-compose linear-continuous-on linear-conv-bounded-linear*)
qed

lemma *pathstart-translation*: *pathstart*$((\lambda x.\ a + x)\ o\ g) = a +$ *pathstart g*
by (*simp add*: *pathstart-def*)

lemma *pathstart-linear-image-eq*: *linear f* $\Longrightarrow$ *pathstart*(*f o g*) = *f*(*pathstart g*)
by (*simp add*: *pathstart-def*)

lemma *pathfinish-translation*: *pathfinish*$((\lambda x.\ a + x)\ o\ g) = a +$ *pathfinish g*
by (*simp add*: *pathfinish-def*)

lemma *pathfinish-linear-image*: *linear f* $\Longrightarrow$ *pathfinish*(*f o g*) = *f*(*pathfinish g*)
by (*simp add*: *pathfinish-def*)

lemma *path-image-translation*: *path-image*((λ*x*. *a* + *x*) *o g*) = (λ*x*. *a* + *x*) ‘ (*path-image g*)
by (*simp add*: *image-comp path-image-def*)

lemma *path-image-linear-image*: *linear f* $\Longrightarrow$ *path-image*(*f o g*) = *f* ‘ (*path-image g*)
by (*simp add*: *image-comp path-image-def*)

lemma *reversepath-translation*: *reversepath*((λ*x*. *a* + *x*) *o g*) = (λ*x*. *a* + *x*) *o reversepath g*
by (*rule ext*) (*simp add*: *reversepath-def*)

lemma *reversepath-linear-image*: *linear f* $\Longrightarrow$ *reversepath*(*f o g*) = *f o reversepath g*
by (*rule ext*) (*simp add*: *reversepath-def*)

lemma *joinpaths-translation*:
((λ*x*. *a* + *x*) *o g1*) +++ ((λ*x*. *a* + *x*) *o g2*) = (λ*x*. *a* + *x*) *o* (*g1* +++ *g2*)
by (*rule ext*) (*simp add*: *joinpaths-def*)

lemma *joinpaths-linear-image*: *linear f* $\Longrightarrow$ (*f o g1*) +++ (*f o g2*) = *f o* (*g1* +++ *g2*)
by (*rule ext*) (*simp add*: *joinpaths-def*)

lemma *simple-path-translation-eq*:
fixes *g* :: *real* ⇒ *'a*::*euclidean-space*
shows *simple-path*((λ*x*. *a* + *x*) *o g*) = *simple-path g*
by (*simp add*: *simple-path-def path-translation-eq*)

lemma *simple-path-linear-image-eq*:
fixes *f* :: *'a*::*euclidean-space* ⇒ *'b*::*euclidean-space*
assumes *linear f inj f*
shows *simple-path*(*f o g*) = *simple-path g*
using *assms inj-on-eq-iff* [*of f*]
by (*auto simp*: *path-linear-image-eq simple-path-def path-translation-eq*)

lemma *arc-translation-eq*:
fixes *g* :: *real* ⇒ *'a*::*euclidean-space*
shows *arc*((λ*x*. *a* + *x*) *o g*) = *arc g*
by (*auto simp*: *arc-def inj-on-def path-translation-eq*)

lemma *arc-linear-image-eq*:
fixes *f* :: *'a*::*euclidean-space* ⇒ *'b*::*euclidean-space*
assumes *linear f inj f*
shows *arc*(*f o g*) = *arc g*
using *assms inj-on-eq-iff* [*of f*]

by (*auto simp*: *arc-def inj-on-def path-linear-image-eq*)

20.3 Basic lemmas about paths

lemma *arc-imp-simple-path*: *arc g* $\Longrightarrow$ *simple-path g*
by (*simp add*: *arc-def inj-on-def simple-path-def*)

lemma *arc-imp-path*: *arc g* $\Longrightarrow$ *path g*
using *arc-def* **by** *blast*

lemma *simple-path-imp-path*: *simple-path g* $\Longrightarrow$ *path g*
using *simple-path-def* **by** *blast*

lemma *simple-path-cases*: *simple-path g* $\Longrightarrow$ *arc g* $\vee$ *pathfinish g* = *pathstart g*
unfolding *simple-path-def arc-def inj-on-def pathfinish-def pathstart-def*
by (*force*)

lemma *simple-path-imp-arc*: *simple-path g* $\Longrightarrow$ *pathfinish g* $\neq$ *pathstart g* $\Longrightarrow$ *arc g*
using *simple-path-cases* **by** *auto*

lemma *arc-distinct-ends*: *arc g* $\Longrightarrow$ *pathfinish g* $\neq$ *pathstart g*
unfolding *arc-def inj-on-def pathfinish-def pathstart-def*
by *fastforce*

lemma *arc-simple-path*: *arc g* $\longleftrightarrow$ *simple-path g* $\wedge$ *pathfinish g* $\neq$ *pathstart g*
using *arc-distinct-ends arc-imp-simple-path simple-path-cases* **by** *blast*

lemma *simple-path-eq-arc*: *pathfinish g* $\neq$ *pathstart g* $\Longrightarrow$ (*simple-path g* = *arc g*)
by (*simp add*: *arc-simple-path*)

lemma *path-image-nonempty* [*simp*]: *path-image g* $\neq$ {}
unfolding *path-image-def image-is-empty box-eq-empty*
by *auto*

lemma *pathstart-in-path-image*[*intro*]: *pathstart g* $\in$ *path-image g*
unfolding *pathstart-def path-image-def*
by *auto*

lemma *pathfinish-in-path-image*[*intro*]: *pathfinish g* $\in$ *path-image g*
unfolding *pathfinish-def path-image-def*
by *auto*

lemma *connected-path-image*[*intro*]: *path g* $\Longrightarrow$ *connected* (*path-image g*)
unfolding *path-def path-image-def*
using *connected-continuous-image connected-Icc* **by** *blast*

lemma *compact-path-image*[*intro*]: *path g* $\Longrightarrow$ *compact* (*path-image g*)
unfolding *path-def path-image-def*

using *compact-continuous-image connected-Icc* **by** *blast*

lemma *reversepath-reversepath*[*simp*]: *reversepath* (*reversepath g*) = *g*
unfolding *reversepath-def*
by *auto*

lemma *pathstart-reversepath*[*simp*]: *pathstart* (*reversepath g*) = *pathfinish g*
unfolding *pathstart-def reversepath-def pathfinish-def*
by *auto*

lemma *pathfinish-reversepath*[*simp*]: *pathfinish* (*reversepath g*) = *pathstart g*
unfolding *pathstart-def reversepath-def pathfinish-def*
by *auto*

lemma *pathstart-join*[*simp*]: *pathstart* (*g1* +++ *g2*) = *pathstart g1*
unfolding *pathstart-def joinpaths-def pathfinish-def*
by *auto*

lemma *pathfinish-join*[*simp*]: *pathfinish* (*g1* +++ *g2*) = *pathfinish g2*
unfolding *pathstart-def joinpaths-def pathfinish-def*
by *auto*

lemma *path-image-reversepath*[*simp*]: *path-image* (*reversepath g*) = *path-image g*
proof −
have ∗: ⋀*g*. *path-image* (*reversepath g*) ⊆ *path-image g*
unfolding *path-image-def subset-eq reversepath-def Ball-def image-iff*
by *force*
show *?thesis*
using ∗[*of g*] ∗[*of reversepath g*]
unfolding *reversepath-reversepath*
by *auto*
qed

lemma *path-reversepath* [*simp*]: *path* (*reversepath g*) ⟷ *path g*
proof −
have ∗: ⋀*g*. *path g* ⟹ *path* (*reversepath g*)
unfolding *path-def reversepath-def*
apply (*rule continuous-on-compose*[*unfolded o-def*, *of* - λ*x*. *1* − *x*])
apply (*intro continuous-intros*)
apply (*rule continuous-on-subset*[*of* {*0..1*}])
apply *assumption*
apply *auto*
done
show *?thesis*
using ∗[*of reversepath g*] ∗[*of g*]
unfolding *reversepath-reversepath*
by (*rule iffI*)
qed

lemma *arc-reversepath*:
assumes *arc g* **shows** *arc*(*reversepath g*)
proof −
have *injg*: *inj-on g* {*0..1*}
using *assms*
by (*simp add*: *arc-def*)
have ∗∗: ⋀*x y*::*real*. *1*−*x* = *1*−*y* ⟹ *x* = *y*
by *simp*
show *?thesis*
apply (*auto simp*: *arc-def inj-on-def path-reversepath*)
apply (*simp add*: *arc-imp-path assms*)
apply (*rule* ∗∗)
apply (*rule inj-onD* [*OF injg*])
apply (*auto simp*: *reversepath-def*)
done
qed

lemma *simple-path-reversepath*: *simple-path g* ⟹ *simple-path* (*reversepath g*)
apply (*simp add*: *simple-path-def*)
apply (*force simp*: *reversepath-def*)
done

lemmas *reversepath-simps* =
path-reversepath path-image-reversepath pathstart-reversepath pathfinish-reversepath

lemma *path-join*[*simp*]:
assumes *pathfinish g1* = *pathstart g2*
shows *path* (*g1* +++ *g2*) ⟷ *path g1* ∧ *path g2*
unfolding *path-def pathfinish-def pathstart-def*
proof *safe*
assume *cont*: *continuous-on* {*0..1*} (*g1* +++ *g2*)
have *g1*: *continuous-on* {*0..1*} *g1* ⟷ *continuous-on* {*0..1*} ((*g1* +++ *g2*) ∘ (λ*x*. *x* / *2*))
by (*intro continuous-on-cong refl*) (*auto simp*: *joinpaths-def*)
have *g2*: *continuous-on* {*0..1*} *g2* ⟷ *continuous-on* {*0..1*} ((*g1* +++ *g2*) ∘ (λ*x*. *x* / *2* + *1*/*2*))
using *assms*
by (*intro continuous-on-cong refl*) (*auto simp*: *joinpaths-def pathfinish-def pathstart-def*)
show *continuous-on* {*0..1*} *g1* **and** *continuous-on* {*0..1*} *g2*
unfolding *g1 g2*
by (*auto intro*!: *continuous-intros continuous-on-subset*[*OF cont*] *simp del*: *o-apply*)
next
assume *g1g2*: *continuous-on* {*0..1*} *g1 continuous-on* {*0..1*} *g2*
have *01*: {*0* .. *1*} = {*0..1*/*2*} ∪ {*1*/*2* .. *1*::*real*}
by *auto*
{
fix *x* :: *real*

```
    assume 0 ≤ x and x ≤ 1
    then have x ∈ (λx. x * 2) ` {0..1 / 2}
      by (intro image-eqI[where x=x/2]) auto
  }
  note 1 = this
  {
    fix x :: real
    assume 0 ≤ x and x ≤ 1
    then have x ∈ (λx. x * 2 − 1) ` {1 / 2..1}
      by (intro image-eqI[where x=x/2 + 1/2]) auto
  }
  note 2 = this
  show continuous-on {0..1} (g1 +++ g2)
    using assms
    unfolding joinpaths-def 01
   apply (intro continuous-on-cases closed-atLeastAtMost g1g2[THEN continuous-on-compose2]
continuous-intros)
    apply (auto simp: field-simps pathfinish-def pathstart-def intro!: 1 2)
    done
qed
```

21 Path Images

```
lemma bounded-path-image: path g ⟹ bounded(path-image g)
  by (simp add: compact-imp-bounded compact-path-image)

lemma closed-path-image:
  fixes g :: real ⇒ 'a::t2-space
  shows path g ⟹ closed(path-image g)
  by (metis compact-path-image compact-imp-closed)

lemma connected-simple-path-image: simple-path g ⟹ connected(path-image g)
  by (metis connected-path-image simple-path-imp-path)

lemma compact-simple-path-image: simple-path g ⟹ compact(path-image g)
  by (metis compact-path-image simple-path-imp-path)

lemma bounded-simple-path-image: simple-path g ⟹ bounded(path-image g)
  by (metis bounded-path-image simple-path-imp-path)

lemma closed-simple-path-image:
  fixes g :: real ⇒ 'a::t2-space
  shows simple-path g ⟹ closed(path-image g)
  by (metis closed-path-image simple-path-imp-path)

lemma connected-arc-image: arc g ⟹ connected(path-image g)
  by (metis connected-path-image arc-imp-path)

lemma compact-arc-image: arc g ⟹ compact(path-image g)
```

by (*metis compact-path-image arc-imp-path*)

lemma *bounded-arc-image*: *arc g* $\Longrightarrow$ *bounded*(*path-image g*)
by (*metis bounded-path-image arc-imp-path*)

lemma *closed-arc-image*:
fixes *g* :: *real* $\Rightarrow$ *'a*::*t2-space*
shows *arc g* $\Longrightarrow$ *closed*(*path-image g*)
by (*metis closed-path-image arc-imp-path*)

lemma *path-image-join-subset*: *path-image* (*g1 +++ g2*) $\subseteq$ *path-image g1* $\cup$ *path-image g2*
unfolding *path-image-def joinpaths-def*
by *auto*

lemma *subset-path-image-join*:
assumes *path-image g1* $\subseteq$ *s*
and *path-image g2* $\subseteq$ *s*
shows *path-image* (*g1 +++ g2*) $\subseteq$ *s*
using *path-image-join-subset*[*of g1 g2*] **and** *assms*
by *auto*

lemma *path-image-join*:
pathfinish g1 = *pathstart g2* $\Longrightarrow$ *path-image*(*g1 +++ g2*) = *path-image g1* $\cup$ *path-image g2*
apply (*rule subset-antisym* [*OF path-image-join-subset*])
apply (*auto simp*: *pathfinish-def pathstart-def path-image-def joinpaths-def image-def*)
apply (*drule sym*)
apply (*rule-tac x=xa/2* **in** *bexI*, *auto*)
apply (*rule ccontr*)
apply (*drule-tac x=(xa+1)/2* **in** *bspec*)
apply (*auto simp*: *field-simps*)
apply (*drule-tac x=1/2* **in** *bspec*, *auto*)
done

lemma *not-in-path-image-join*:
assumes *x* $\notin$ *path-image g1*
and *x* $\notin$ *path-image g2*
shows *x* $\notin$ *path-image* (*g1 +++ g2*)
using *assms* **and** *path-image-join-subset*[*of g1 g2*]
by *auto*

lemma *pathstart-compose*: *pathstart*(*f o p*) = *f*(*pathstart p*)
by (*simp add*: *pathstart-def*)

lemma *pathfinish-compose*: *pathfinish*(*f o p*) = *f*(*pathfinish p*)
by (*simp add*: *pathfinish-def*)

lemma *path-image-compose*: *path-image* (*f o p*) = *f* ' (*path-image p*)

by (*simp add*: *image-comp path-image-def*)

lemma *path-compose-join*: *f o* (*p* +++ *q*) = (*f o p*) +++ (*f o q*)
by (*rule ext*) (*simp add*: *joinpaths-def*)

lemma *path-compose-reversepath*: *f o reversepath p* = *reversepath*(*f o p*)
by (*rule ext*) (*simp add*: *reversepath-def*)

lemma *joinpaths-eq*:
(⋀*t*. *t* ∈ {*0..1*} ⟹ *p t* = *p*′ *t*) ⟹
(⋀*t*. *t* ∈ {*0..1*} ⟹ *q t* = *q*′ *t*)
⟹ *t* ∈ {*0..1*} ⟹ (*p* +++ *q*) *t* = (*p*′ +++ *q*′) *t*
by (*auto simp*: *joinpaths-def*)

lemma *simple-path-inj-on*: *simple-path g* ⟹ *inj-on g* {*0*<*..*<*1*}
by (*auto simp*: *simple-path-def path-image-def inj-on-def less-eq-real-def Ball-def*)

21.1 Simple paths with the endpoints removed

lemma *simple-path-endless*:
simple-path c ⟹ *path-image c* − {*pathstart c*,*pathfinish c*} = *c* ‘ {*0*<*..*<*1*}
apply (*auto simp*: *simple-path-def path-image-def pathstart-def pathfinish-def Ball-def Bex-def image-def*)
apply (*metis eq-iff le-less-linear*)
apply (*metis leD linear*)
using *less-eq-real-def zero-le-one* **apply** *blast*
using *less-eq-real-def zero-le-one* **apply** *blast*
done

lemma *connected-simple-path-endless*:
simple-path c ⟹ *connected*(*path-image c* − {*pathstart c*,*pathfinish c*})
apply (*simp add*: *simple-path-endless*)
apply (*rule connected-continuous-image*)
apply (*meson continuous-on-subset greaterThanLessThan-subseteq-atLeastAtMost-iff le-numeral-extra*(*3*) *le-numeral-extra*(*4*) *path-def simple-path-imp-path*)
by *auto*

lemma *nonempty-simple-path-endless*:
simple-path c ⟹ *path-image c* − {*pathstart c*,*pathfinish c*} ≠ {}
by (*simp add*: *simple-path-endless*)

21.2 The operations on paths

lemma *path-image-subset-reversepath*: *path-image*(*reversepath g*) ≤ *path-image g*
by (*auto simp*: *path-image-def reversepath-def*)

lemma *path-imp-reversepath*: *path g* ⟹ *path*(*reversepath g*)
apply (*auto simp*: *path-def reversepath-def*)
using *continuous-on-compose* [*of* {*0..1*} λ*x*. *1* − *x g*]
apply (*auto simp*: *continuous-on-op-minus*)

done

lemma *half-bounded-equal*: $1 \leq x * 2 \Longrightarrow x * 2 \leq 1 \longleftrightarrow x = (1/2::real)$
by *simp*

lemma *continuous-on-joinpaths*:
assumes *continuous-on* $\{0..1\}$ *g1 continuous-on* $\{0..1\}$ *g2 pathfinish g1 = pathstart g2*
shows *continuous-on* $\{0..1\}$ $(g1 +++ g2)$
proof −
have $*$: $\{0..1::real\} = \{0..1/2\} \cup \{1/2..1\}$
by *auto*
have *gg*: *g2 0 = g1 1*
by (*metis assms*(3) *pathfinish-def pathstart-def*)
have *1*: *continuous-on* $\{0..1/2\}$ $(g1 +++ g2)$
apply (*rule continuous-on-eq* [*of - g1 o* $(\lambda x.\ 2*x)$])
apply (*rule continuous-intros* | *simp add*: *joinpaths-def assms*)+
done
have *continuous-on* $\{1/2..1\}$ $(g2\ o\ (\lambda x.\ 2*x-1))$
apply (*rule continuous-on-subset* [*of* $\{1/2..1\}$])
apply (*rule continuous-intros* | *simp add*: *image-affinity-atLeastAtMost-diff assms*)+
done
then have *2*: *continuous-on* $\{1/2..1\}$ $(g1 +++ g2)$
apply (*rule continuous-on-eq* [*of* $\{1/2..1\}$ *g2 o* $(\lambda x.\ 2*x-1)$])
apply (*rule assms continuous-intros* | *simp add*: *joinpaths-def mult.commute half-bounded-equal gg*)+
done
show *?thesis*
apply (*subst* $*$)
apply (*rule continuous-on-closed-Un*)
using *1 2*
apply *auto*
done
qed

lemma *path-join-imp*: $[\![path\ g1; path\ g2; pathfinish\ g1 = pathstart\ g2]\!] \Longrightarrow path(g1 +++ g2)$
by (*simp add*: *path-join*)

lemma *simple-path-join-loop*:
assumes *arc g1 arc g2*
pathfinish g1 = pathstart g2 pathfinish g2 = pathstart g1
path-image g1 $\cap$ *path-image g2* $\subseteq$ $\{pathstart\ g1, pathstart\ g2\}$
shows *simple-path*$(g1 +++ g2)$
proof −
have *injg1*: *inj-on g1* $\{0..1\}$
using *assms*
by (*simp add*: *arc-def*)

```
 have injg2: inj-on g2 {0..1}
  using assms
  by (simp add: arc-def)
 have g12: g1 1 = g2 0
  and g21: g2 1 = g1 0
  and sb:  g1 ` {0..1} ∩ g2 ` {0..1} ⊆ {g1 0, g2 0}
  using assms
  by (simp-all add: arc-def pathfinish-def pathstart-def path-image-def)
 { fix x and y::real
  assume xyI: x = 1 ⟶ y ≠ 0
    and xy: x ≤ 1 0 ≤ y  y * 2 ≤ 1 ¬ x * 2 ≤ 1 g2 (2 * x − 1) = g1 (2 * y)
  have g1im: g1 (2 * y) ∈ g1 ` {0..1} ∩ g2 ` {0..1}
   using xy
   apply simp
   apply (rule-tac x=2 * x − 1 in image-eqI, auto)
   done
  have False
   using subsetD [OF sb g1im] xy
   apply auto
   apply (drule inj-onD [OF injg1])
   using g21 [symmetric] xyI
   apply (auto dest: inj-onD [OF injg2])
   done
 } note ∗ = this
 { fix x and y::real
  assume xy: y ≤ 1 0 ≤ x ¬ y * 2 ≤ 1 x * 2 ≤ 1 g1 (2 * x) = g2 (2 * y − 1)
  have g1im: g1 (2 * x) ∈ g1 ` {0..1} ∩ g2 ` {0..1}
   using xy
   apply simp
   apply (rule-tac x=2 * x in image-eqI, auto)
   done
  have x = 0 ∧ y = 1
   using subsetD [OF sb g1im] xy
   apply auto
   apply (force dest: inj-onD [OF injg1])
   using  g21 [symmetric]
   apply (auto dest: inj-onD [OF injg2])
   done
 } note ∗∗ = this
 show ?thesis
  using assms
  apply (simp add: arc-def simple-path-def path-join, clarify)
  apply (simp add: joinpaths-def split: if-split-asm)
  apply (force dest: inj-onD [OF injg1])
  apply (metis ∗)
  apply (metis ∗∗)
  apply (force dest: inj-onD [OF injg2])
  done
qed
```

lemma *arc-join*:
assumes *arc g1 arc g2*
pathfinish g1 = pathstart g2
path-image g1 ∩ path-image g2 ⊆ {pathstart g2}
shows *arc(g1 +++ g2)*
proof −
have *injg1*: *inj-on g1 {0..1}*
using *assms*
by (*simp add*: *arc-def*)
have *injg2*: *inj-on g2 {0..1}*
using *assms*
by (*simp add*: *arc-def*)
have *g11*: *g1 1 = g2 0*
and *sb*: *g1 ' {0..1} ∩ g2 ' {0..1} ⊆ {g2 0}*
using *assms*
by (*simp-all add*: *arc-def pathfinish-def pathstart-def path-image-def*)
{ fix *x* **and** *y*::*real*
assume *xy*: *x ≤ 1 0 ≤ y y * 2 ≤ 1 ¬ x * 2 ≤ 1 g2 (2 * x − 1) = g1 (2 * y)*
have *g1im*: *g1 (2 * y) ∈ g1 ' {0..1} ∩ g2 ' {0..1}*
using *xy*
apply *simp*
apply (*rule-tac x=2 * x − 1* **in** *image-eqI*, *auto*)
done
have *False*
using *subsetD* [*OF sb g1im*] *xy*
by (*auto dest*: *inj-onD* [*OF injg2*])
} note ∗ = *this*
show *?thesis*
apply (*simp add*: *arc-def inj-on-def*)
apply (*clarsimp simp add*: *arc-imp-path assms path-join*)
apply (*simp add*: *joinpaths-def split*: *if-split-asm*)
apply (*force dest*: *inj-onD* [*OF injg1*])
apply (*metis* ∗)
apply (*metis* ∗)
apply (*force dest*: *inj-onD* [*OF injg2*])
done
qed

lemma *reversepath-joinpaths*:
pathfinish g1 = pathstart g2 ⟹ reversepath(g1 +++ g2) = reversepath g2 +++ reversepath g1
unfolding *reversepath-def pathfinish-def pathstart-def joinpaths-def*
by (*rule ext*) (*auto simp*: *mult.commute*)

21.3 Some reversed and "if and only if" versions of joining theorems

lemma *path-join-path-ends*:

fixes *g1* :: *real* ⇒ *'a*::*metric-space*
assumes *path*(*g1* +++ *g2*) *path g2*
shows *pathfinish g1* = *pathstart g2*
proof (*rule ccontr*)
def *e* ≡ *dist* (*g1 1*) (*g2 0*)
assume *Neg*: *pathfinish g1* ≠ *pathstart g2*
then have *0* < *dist* (*pathfinish g1*) (*pathstart g2*)
by *auto*
then have *e* > *0*
by (*metis e-def pathfinish-def pathstart-def*)
then obtain *d1* **where** *d1* > *0*
and *d1*: ⋀*x'*. ⟦*x'*∈{*0..1*}; *norm x'* < *d1*⟧ ⟹ *dist* (*g2 x'*) (*g2 0*) < *e*/*2*
using *assms*(*2*) **unfolding** *path-def continuous-on-iff*
apply (*drule-tac x*=*0* **in** *bspec*, *simp*)
by (*metis half-gt-zero-iff norm-conv-dist*)
obtain *d2* **where** *d2* > *0*
and *d2*: ⋀*x'*. ⟦*x'*∈{*0..1*}; *dist x'* (*1*/*2*) < *d2*⟧
⟹ *dist* ((*g1* +++ *g2*) *x'*) (*g1 1*) < *e*/*2*
using *assms*(*1*) ‹*e* > *0*› **unfolding** *path-def continuous-on-iff*
apply (*drule-tac x*=*1*/*2* **in** *bspec*, *simp*)
apply (*drule-tac x*=*e*/*2* **in** *spec*)
apply (*force simp*: *joinpaths-def*)
done
have *int01-1*: *min* (*1*/*2*) (*min d1 d2*) / *2* ∈ {*0..1*}
using ‹*d1* > *0*› ‹*d2* > *0*› **by** (*simp add*: *min-def*)
have *dist1*: *norm* (*min* (*1* / *2*) (*min d1 d2*) / *2*) < *d1*
using ‹*d1* > *0*› ‹*d2* > *0*› **by** (*simp add*: *min-def dist-norm*)
have *int01-2*: *1*/*2* + *min* (*1*/*2*) (*min d1 d2*) / *4* ∈ {*0..1*}
using ‹*d1* > *0*› ‹*d2* > *0*› **by** (*simp add*: *min-def*)
have *dist2*: *dist* (*1* / *2* + *min* (*1* / *2*) (*min d1 d2*) / *4*) (*1* / *2*) < *d2*
using ‹*d1* > *0*› ‹*d2* > *0*› **by** (*simp add*: *min-def dist-norm*)
have [*simp*]: ~ *min* (*1* / *2*) (*min d1 d2*) ≤ *0*
using ‹*d1* > *0*› ‹*d2* > *0*› **by** (*simp add*: *min-def*)
have *dist* (*g2* (*min* (*1* / *2*) (*min d1 d2*) / *2*)) (*g1 1*) < *e*/*2*
dist (*g2* (*min* (*1* / *2*) (*min d1 d2*) / *2*)) (*g2 0*) < *e*/*2*
using *d1* [*OF int01-1 dist1*] *d2* [*OF int01-2 dist2*] **by** (*simp-all add*: *joinpaths-def*)
then have *dist* (*g1 1*) (*g2 0*) < *e*/*2* + *e*/*2*
using *dist-triangle-half-r e-def* **by** *blast*
then show *False*
by (*simp add*: *e-def* [*symmetric*])
qed

lemma *path-join-eq* [*simp*]:
fixes *g1* :: *real* ⇒ *'a*::*metric-space*
assumes *path g1 path g2*
shows *path*(*g1* +++ *g2*) ⟷ *pathfinish g1* = *pathstart g2*
using *assms* **by** (*metis path-join-path-ends path-join-imp*)

lemma *simple-path-joinE*:

assumes *simple-path*(*g1 +++ g2*) **and** *pathfinish g1 = pathstart g2*
obtains *arc g1 arc g2*
path-image g1 ∩ path-image g2 ⊆ {pathstart g1, pathstart g2}
proof −
have ∗: ⋀*x y*. ⟦*0 ≤ x*; *x ≤ 1*; *0 ≤ y*; *y ≤ 1*; (*g1 +++ g2*) *x* = (*g1 +++ g2*) *y*⟧
⟹ *x = y ∨ x = 0 ∧ y = 1 ∨ x = 1 ∧ y = 0*
using *assms* **by** (*simp add*: *simple-path-def*)
have *path g1*
using *assms path-join simple-path-imp-path* **by** *blast*
moreover have *inj-on g1 {0..1}*
proof (*clarsimp simp*: *inj-on-def*)
fix *x y*
assume *g1 x = g1 y 0 ≤ x x ≤ 1 0 ≤ y y ≤ 1*
then show *x = y*
using ∗ [*of x/2 y/2*] **by** (*simp add*: *joinpaths-def split-ifs*)
qed
ultimately have *arc g1*
using *assms* **by** (*simp add*: *arc-def*)
have [*simp*]: *g2 0 = g1 1*
using *assms* **by** (*metis pathfinish-def pathstart-def*)
have *path g2*
using *assms path-join simple-path-imp-path* **by** *blast*
moreover have *inj-on g2 {0..1}*
proof (*clarsimp simp*: *inj-on-def*)
fix *x y*
assume *g2 x = g2 y 0 ≤ x x ≤ 1 0 ≤ y y ≤ 1*
then show *x = y*
using ∗ [*of* (*x + 1*) */ 2* (*y + 1*) */ 2*]
by (*force simp*: *joinpaths-def split-ifs divide-simps*)
qed
ultimately have *arc g2*
using *assms* **by** (*simp add*: *arc-def*)
have *g2 y = g1 0 ∨ g2 y = g1 1*
if *g1 x = g2 y 0 ≤ x x ≤ 1 0 ≤ y y ≤ 1* **for** *x y*
using ∗ [*of x / 2* (*y + 1*) */ 2*] *that*
by (*auto simp*: *joinpaths-def split-ifs divide-simps*)
then have *path-image g1 ∩ path-image g2 ⊆ {pathstart g1, pathstart g2}*
by (*fastforce simp*: *pathstart-def pathfinish-def path-image-def*)
with ‹*arc g1*› ‹*arc g2*› **show** *?thesis* **using** *that* **by** *blast*
qed

lemma *simple-path-join-loop-eq*:
assumes *pathfinish g2 = pathstart g1 pathfinish g1 = pathstart g2*
shows *simple-path*(*g1 +++ g2*) ⟷
arc g1 ∧ arc g2 ∧ path-image g1 ∩ path-image g2 ⊆ {pathstart g1, pathstart g2}
by (*metis assms simple-path-joinE simple-path-join-loop*)

lemma *arc-join-eq*:
assumes *pathfinish g1 = pathstart g2*
shows *arc(g1 +++ g2) ⟷*
arc g1 ∧ arc g2 ∧ path-image g1 ∩ path-image g2 ⊆ {pathstart g2}
(**is** *?lhs = ?rhs*)
proof
assume *?lhs*
then have *simple-path(g1 +++ g2)* **by** (*rule arc-imp-simple-path*)
then have ∗: ⋀*x y*. ⟦*0 ≤ x*; *x ≤ 1*; *0 ≤ y*; *y ≤ 1*; (*g1 +++ g2*) *x* = (*g1 +++ g2*) *y*⟧
⟹ x = y ∨ x = 0 ∧ y = 1 ∨ x = 1 ∧ y = 0
using *assms* **by** (*simp add*: *simple-path-def*)
have *False* **if** *g1 0 = g2 u 0 ≤ u u ≤ 1* **for** *u*
using ∗ [*of 0* (*u + 1*) */ 2*] *that assms arc-distinct-ends* [*OF* ‹*?lhs*›]
by (*auto simp*: *joinpaths-def pathstart-def pathfinish-def split-ifs divide-simps*)
then have *n1*: ~ (*pathstart g1 ∈ path-image g2*)
unfolding *pathstart-def path-image-def*
using *atLeastAtMost-iff* **by** *blast*
show *?rhs* **using** ‹*?lhs*›
apply (*rule simple-path-joinE* [*OF arc-imp-simple-path assms*])
using *n1* **by** *force*
next
assume *?rhs* **then show** *?lhs*
using *assms*
by (*fastforce simp*: *pathfinish-def pathstart-def intro*!: *arc-join*)
qed

lemma *arc-join-eq-alt*:
pathfinish g1 = pathstart g2
⟹ (*arc(g1 +++ g2*) *⟷*
arc g1 ∧ arc g2 ∧
path-image g1 ∩ path-image g2 = {pathstart g2})
using *pathfinish-in-path-image* **by** (*fastforce simp*: *arc-join-eq*)

21.4 The joining of paths is associative

lemma *path-assoc*:
⟦*pathfinish p = pathstart q*; *pathfinish q = pathstart r*⟧
⟹ path(*p +++* (*q +++ r*)) *⟷ path*((*p +++ q*) *+++ r*)
by *simp*

lemma *simple-path-assoc*:
assumes *pathfinish p = pathstart q pathfinish q = pathstart r*
shows *simple-path* (*p +++* (*q +++ r*)) *⟷ simple-path* ((*p +++ q*) *+++ r*)
proof (*cases pathstart p = pathfinish r*)
case *True* **show** *?thesis*
proof
assume *simple-path* (*p +++ q +++ r*)
with *assms True* **show** *simple-path* ((*p +++ q*) *+++ r*)

by (*fastforce simp add*: *simple-path-join-loop-eq arc-join-eq path-image-join dest*: *arc-distinct-ends* [*of r*])
next
assume *0*: *simple-path* ((*p* +++ *q*) +++ *r*)
with *assms True* **have** *q*: *pathfinish r* ∉ *path-image q*
using *arc-distinct-ends*
by (*fastforce simp add*: *simple-path-join-loop-eq arc-join-eq path-image-join*)
have *pathstart r* ∉ *path-image p*
using *assms*
by (*metis 0 IntI arc-distinct-ends arc-join-eq-alt empty-iff insert-iff pathfinish-in-path-image pathfinish-join simple-path-joinE*)
with *assms 0 q True* **show** *simple-path* (*p* +++ *q* +++ *r*)
by (*auto simp*: *simple-path-join-loop-eq arc-join-eq path-image-join dest*!: *subsetD* [*OF - IntI*])
qed
next
case *False*
{ fix *x* :: *'a*
assume *a*: *path-image p* ∩ *path-image q* ⊆ {*pathstart q*}
(*path-image p* ∪ *path-image q*) ∩ *path-image r* ⊆ {*pathstart r*}
x ∈ *path-image p x* ∈ *path-image r*
have *pathstart r* ∈ *path-image q*
by (*metis assms*(*2*) *pathfinish-in-path-image*)
with *a* **have** *x* = *pathstart q*
by *blast*
}
with *False assms* **show** *?thesis*
by (*auto simp*: *simple-path-eq-arc simple-path-join-loop-eq arc-join-eq path-image-join*)
qed

lemma *arc-assoc*:
⟦*pathfinish p* = *pathstart q*; *pathfinish q* = *pathstart r*⟧
⟹ *arc*(*p* +++ (*q* +++ *r*)) ⟷ *arc*((*p* +++ *q*) +++ *r*)
by (*simp add*: *arc-simple-path simple-path-assoc*)

21.4.1 Symmetry and loops

lemma *path-sym*:
⟦*pathfinish p* = *pathstart q*; *pathfinish q* = *pathstart p*⟧ ⟹ *path*(*p* +++ *q*) ⟷ *path*(*q* +++ *p*)
by *auto*

lemma *simple-path-sym*:
⟦*pathfinish p* = *pathstart q*; *pathfinish q* = *pathstart p*⟧
⟹ *simple-path*(*p* +++ *q*) ⟷ *simple-path*(*q* +++ *p*)
by (*metis* (*full-types*) *inf-commute insert-commute simple-path-joinE simple-path-join-loop*)

lemma *path-image-sym*:
⟦*pathfinish p* = *pathstart q*; *pathfinish q* = *pathstart p*⟧

$\Longrightarrow$ *path-image*$(p$ *+++* $q) =$ *path-image*$(q$ *+++* $p)$
by (*simp add*: *path-image-join sup-commute*)

22 Choosing a subpath of an existing path

definition *subpath* :: *real* $\Rightarrow$ *real* $\Rightarrow$ (*real* $\Rightarrow$ $'a$) $\Rightarrow$ *real* $\Rightarrow$ $'a$::*real-normed-vector*
where *subpath a b g* $\equiv \lambda x.\ g((b - a) * x + a)$

lemma *path-image-subpath-gen*:
fixes *g* :: - $\Rightarrow$ $'a$::*real-normed-vector*
shows *path-image*(*subpath u v g*) = *g* ‘ (*closed-segment u v*)
apply (*simp add*: *closed-segment-real-eq path-image-def subpath-def*)
apply (*subst o-def* [*of g*, *symmetric*])
apply (*simp add*: *image-comp* [*symmetric*])
done

lemma *path-image-subpath*:
fixes *g* :: *real* $\Rightarrow$ $'a$::*real-normed-vector*
shows *path-image*(*subpath u v g*) = (*if u* $\leq$ *v then g* ‘ $\{u..v\}$ *else g* ‘ $\{v..u\}$)
by (*simp add*: *path-image-subpath-gen closed-segment-eq-real-ivl*)

lemma *path-subpath* [*simp*]:
fixes *g* :: *real* $\Rightarrow$ $'a$::*real-normed-vector*
assumes *path g u* $\in \{0..1\}$ *v* $\in \{0..1\}$
shows *path*(*subpath u v g*)
proof −
have *continuous-on* $\{0..1\}$ (*g o* $(\lambda x.\ ((v-u) * x+ u))$)
apply (*rule continuous-intros* | *simp*)+
apply (*simp add*: *image-affinity-atLeastAtMost* [**where** *c*=*u*])
using *assms*
apply (*auto simp*: *path-def continuous-on-subset*)
done
then show *?thesis*
by (*simp add*: *path-def subpath-def*)
qed

lemma *pathstart-subpath* [*simp*]: *pathstart*(*subpath u v g*) = *g*(*u*)
by (*simp add*: *pathstart-def subpath-def*)

lemma *pathfinish-subpath* [*simp*]: *pathfinish*(*subpath u v g*) = *g*(*v*)
by (*simp add*: *pathfinish-def subpath-def*)

lemma *subpath-trivial* [*simp*]: *subpath 0 1 g* = *g*
by (*simp add*: *subpath-def*)

lemma *subpath-reversepath*: *subpath 1 0 g* = *reversepath g*
by (*simp add*: *reversepath-def subpath-def*)

lemma *reversepath-subpath*: *reversepath*(*subpath u v g*) = *subpath v u g*

by (*simp add*: *reversepath-def subpath-def algebra-simps*)

lemma *subpath-translation*: *subpath u v* ((λ*x*. *a* + *x*) *o g*) = (λ*x*. *a* + *x*) *o subpath u v g*
by (*rule ext*) (*simp add*: *subpath-def*)

lemma *subpath-linear-image*: *linear f* ⟹ *subpath u v* (*f o g*) = *f o subpath u v g*
by (*rule ext*) (*simp add*: *subpath-def*)

lemma *affine-ineq*:
fixes *x* :: *'a*::*linordered-idom*
assumes *x* ≤ *1 v* ≤ *u*
shows *v* + *x* ∗ *u* ≤ *u* + *x* ∗ *v*
proof −
have (*1*−*x*)∗(*u*−*v*) ≥ *0*
using *assms* **by** *auto*
then show *?thesis*
by (*simp add*: *algebra-simps*)
qed

lemma *sum-le-prod1*:
fixes *a*::*real* **shows** ⟦*a* ≤ *1*; *b* ≤ *1*⟧ ⟹ *a* + *b* ≤ *1* + *a* ∗ *b*
by (*metis add.commute affine-ineq less-eq-real-def mult.right-neutral*)

lemma *simple-path-subpath-eq*:
simple-path(*subpath u v g*) ⟷
path(*subpath u v g*) ∧ *u*≠*v* ∧
(∀ *x y*. *x* ∈ *closed-segment u v* ∧ *y* ∈ *closed-segment u v* ∧ *g x* = *g y*
⟶ *x* = *y* ∨ *x* = *u* ∧ *y* = *v* ∨ *x* = *v* ∧ *y* = *u*)
(**is** *?lhs* = *?rhs*)
proof (*rule iffI*)
assume *?lhs*
then have *p*: *path* (λ*x*. *g* ((*v* − *u*) ∗ *x* + *u*))
and *sim*: (⋀*x y*. ⟦*x*∈{*0*..*1*}; *y*∈{*0*..*1*}; *g* ((*v* − *u*) ∗ *x* + *u*) = *g* ((*v* − *u*) ∗ *y* + *u*)⟧
⟹ *x* = *y* ∨ *x* = *0* ∧ *y* = *1* ∨ *x* = *1* ∧ *y* = *0*)
by (*auto simp*: *simple-path-def subpath-def*)
{ fix *x y*
assume *x* ∈ *closed-segment u v y* ∈ *closed-segment u v g x* = *g y*
then have *x* = *y* ∨ *x* = *u* ∧ *y* = *v* ∨ *x* = *v* ∧ *y* = *u*
using *sim* [*of* (*x*−*u*)/(*v*−*u*) (*y*−*u*)/(*v*−*u*)] *p*
by (*auto simp*: *closed-segment-real-eq image-affinity-atLeastAtMost divide-simps split*: *if-split-asm*)
} moreover
have *path*(*subpath u v g*) ∧ *u*≠*v*
using *sim* [*of 1*/*3 2*/*3*] *p*
by (*auto simp*: *subpath-def*)
ultimately show *?rhs*
by *metis*

next
assume *?rhs*
then
have *d1*: $\bigwedge x\ y.\ [\![g\ x = g\ y;\ u \le x;\ x \le v;\ u \le y;\ y \le v]\!] \Longrightarrow x = y \vee x = u \wedge y = v \vee x = v \wedge y = u$
and *d2*: $\bigwedge x\ y.\ [\![g\ x = g\ y;\ v \le x;\ x \le u;\ v \le y;\ y \le u]\!] \Longrightarrow x = y \vee x = u \wedge y = v \vee x = v \wedge y = u$
and *ne*: $u < v \vee v < u$
and *psp*: *path* (*subpath u v g*)
by (*auto simp*: *closed-segment-real-eq image-affinity-atLeastAtMost*)
have [*simp*]: $\bigwedge x.\ u + x * v = v + x * u \longleftrightarrow u{=}v \vee x{=}1$
by *algebra*
show *?lhs* **using** *psp ne*
unfolding *simple-path-def subpath-def*
by (*fastforce simp add*: *algebra-simps affine-ineq mult-left-mono crossproduct-eq dest*: *d1 d2*)
qed

lemma *arc-subpath-eq*:
arc(*subpath u v g*) $\longleftrightarrow$ *path*(*subpath u v g*) $\wedge$ $u{\neq}v$ $\wedge$ *inj-on g* (*closed-segment u v*)
(**is** *?lhs = ?rhs*)
proof (*rule iffI*)
assume *?lhs*
then have *p*: *path* ($\lambda x.\ g\ ((v - u) * x + u)$)
and *sim*: $(\bigwedge x\ y.\ [\![x{\in}\{0..1\};\ y{\in}\{0..1\};\ g\ ((v - u) * x + u) = g\ ((v - u) * y + u)]\!] \Longrightarrow x = y)$
by (*auto simp*: *arc-def inj-on-def subpath-def*)
{ fix *x y*
assume $x \in$ *closed-segment u v* $y \in$ *closed-segment u v* $g\ x = g\ y$
then have $x = y$
using *sim* [*of* $(x{-}u)/(v{-}u)$ $(y{-}u)/(v{-}u)$] *p*
by (*force simp add*: *inj-on-def closed-segment-real-eq image-affinity-atLeastAtMost divide-simps split*: *if-split-asm*)
} moreover
have *path*(*subpath u v g*) $\wedge$ $u{\neq}v$
using *sim* [*of 1/3 2/3*] *p*
by (*auto simp*: *subpath-def*)
ultimately show *?rhs*
unfolding *inj-on-def*
by *metis*
next
assume *?rhs*
then
have *d1*: $\bigwedge x\ y.\ [\![g\ x = g\ y;\ u \le x;\ x \le v;\ u \le y;\ y \le v]\!] \Longrightarrow x = y$
and *d2*: $\bigwedge x\ y.\ [\![g\ x = g\ y;\ v \le x;\ x \le u;\ v \le y;\ y \le u]\!] \Longrightarrow x = y$
and *ne*: $u < v \vee v < u$

and *psp*: *path* (*subpath u v g*)
by (*auto simp*: *inj-on-def closed-segment-real-eq image-affinity-atLeastAtMost*)
show *?lhs* **using** *psp ne*
unfolding *arc-def subpath-def inj-on-def*
by (*auto simp*: *algebra-simps affine-ineq mult-left-mono crossproduct-eq dest*: *d1 d2*)
qed

lemma *simple-path-subpath*:
assumes *simple-path g u* ∈ {*0..1*} *v* ∈ {*0..1*} *u* ≠ *v*
shows *simple-path*(*subpath u v g*)
using *assms*
apply (*simp add*: *simple-path-subpath-eq simple-path-imp-path*)
apply (*simp add*: *simple-path-def closed-segment-real-eq image-affinity-atLeastAtMost*, *fastforce*)
done

lemma *arc-simple-path-subpath*:
⟦*simple-path g*; *u* ∈ {*0..1*}; *v* ∈ {*0..1*}; *g u* ≠ *g v*⟧ ⟹ *arc*(*subpath u v g*)
by (*force intro*: *simple-path-subpath simple-path-imp-arc*)

lemma *arc-subpath-arc*:
⟦*arc g*; *u* ∈ {*0..1*}; *v* ∈ {*0..1*}; *u* ≠ *v*⟧ ⟹ *arc*(*subpath u v g*)
by (*meson arc-def arc-imp-simple-path arc-simple-path-subpath inj-onD*)

lemma *arc-simple-path-subpath-interior*:
⟦*simple-path g*; *u* ∈ {*0..1*}; *v* ∈ {*0..1*}; *u* ≠ *v*; |*u*−*v*| < *1*⟧ ⟹ *arc*(*subpath u v g*)
apply (*rule arc-simple-path-subpath*)
apply (*force simp*: *simple-path-def*)+
done

lemma *path-image-subpath-subset*:
⟦*path g*; *u* ∈ {*0..1*}; *v* ∈ {*0..1*}⟧ ⟹ *path-image*(*subpath u v g*) ⊆ *path-image g*
apply (*simp add*: *closed-segment-real-eq image-affinity-atLeastAtMost path-image-subpath*)
apply (*auto simp*: *path-image-def*)
done

lemma *join-subpaths-middle*: *subpath* (*0*) ((*1* / *2*)) *p* +++ *subpath* ((*1* / *2*)) *1 p* = *p*
by (*rule ext*) (*simp add*: *joinpaths-def subpath-def divide-simps*)

22.1 There is a subpath to the frontier

lemma *subpath-to-frontier-explicit*:
fixes *S* :: *'a*::*metric-space set*
assumes *g*: *path g* **and** *pathfinish g* ∉ *S*
obtains *u* **where** *0* ≤ *u u* ≤ *1*

$\bigwedge x.\ 0 \le x \wedge x < u \Longrightarrow g\ x \in \mathit{interior}\ S$
$(g\ u \notin \mathit{interior}\ S)\ (u = 0 \vee g\ u \in \mathit{closure}\ S)$
proof −
have *gcon*: *continuous-on* $\{0..1\}$ *g* **using** *g* **by** (*simp add*: *path-def*)
then have *com*: *compact* $(\{0..1\} \cap \{u.\ g\ u \in \mathit{closure}\ (-S)\})$
apply (*simp add*: *Int-commute* [*of* $\{0..1\}$] *compact-eq-bounded-closed closed-vimage-Int* [*unfolded vimage-def*])
using *compact-eq-bounded-closed* **apply** *fastforce*
done
have $1 \in \{u.\ g\ u \in \mathit{closure}\ (-S)\}$
using *assms* **by** (*simp add*: *pathfinish-def closure-def*)
then have *dis*: $\{0..1\} \cap \{u.\ g\ u \in \mathit{closure}\ (-S)\} \neq \{\}$
using *atLeastAtMost-iff zero-le-one* **by** *blast*
then obtain *u* **where** $0 \le u\ u \le 1$ **and** *gu*: $g\ u \in \mathit{closure}\ (-S)$
and *umin*: $\bigwedge t.\ [\![0 \le t;\ t \le 1;\ g\ t \in \mathit{closure}\ (-S)]\!] \Longrightarrow u \le t$
using *compact-attains-inf* [*OF com dis*] **by** *fastforce*
then have *umin'*: $\bigwedge t.\ [\![0 \le t;\ t \le 1;\ t < u]\!] \Longrightarrow\ g\ t \in S$
using *closure-def* **by** *fastforce*
{ assume $u \neq 0$
then have $u > 0$ **using** ‹$0 \le u$› **by** *auto*
{ fix *e*::*real* **assume** $e > 0$
obtain *d* **where** $d{>}0$ **and** *d*: $\bigwedge x'.\ [\![x' \in \{0..1\};\ \mathit{dist}\ x'\ u \le d]\!] \Longrightarrow \mathit{dist}\ (g\ x')\ (g\ u) < e$
using *continuous-onE* [*OF gcon* - ‹$e > 0$›] ‹$0 \le$ -› ‹- ≤ 1› *atLeastAtMost-iff* **by** *auto*
have ∗: $\mathit{dist}\ (\mathit{max}\ 0\ (u - d\ /\ 2))\ u \le d$
using ‹$0 \le u$› ‹$u \le 1$› ‹$d > 0$› **by** (*simp add*: *dist-real-def*)
have $\exists y{\in}S.\ \mathit{dist}\ y\ (g\ u) < e$
using ‹$0 < u$› ‹$u \le 1$› ‹$d > 0$›
by (*force intro*: *d* [*OF* - ∗] *umin'*)
}
then have $g\ u \in \mathit{closure}\ S$
by (*simp add*: *frontier-def closure-approachable*)
}
then show *?thesis*
apply (*rule-tac u*=*u* **in** *that*)
apply (*auto simp*: ‹$0 \le u$› ‹$u \le 1$› *gu interior-closure umin*)
using ‹- ≤ 1› *interior-closure umin* **apply** *fastforce*
done
qed

lemma *subpath-to-frontier-strong*:
assumes *g*: *path g* **and** *pathfinish* $g \notin S$
obtains *u* **where** $0 \le u\ u \le 1\ g\ u \notin \mathit{interior}\ S$
$u = 0 \vee (\forall x.\ 0 \le x \wedge x < 1 \longrightarrow \mathit{subpath}\ 0\ u\ g\ x \in \mathit{interior}\ S) \wedge\ g\ u \in \mathit{closure}\ S$
proof −
obtain *u* **where** $0 \le u\ u \le 1$
and *gxin*: $\bigwedge x.\ 0 \le x \wedge x < u \Longrightarrow g\ x \in \mathit{interior}\ S$

```
          and gunot: (g u ∉ interior S) and u0: (u = 0 ∨ g u ∈ closure S)
    using subpath-to-frontier-explicit [OF assms] by blast
  show ?thesis
    apply (rule that [OF ‹0 ≤ u› ‹u ≤ 1›])
    apply (simp add: gunot)
    using ‹0 ≤ u› u0 by (force simp: subpath-def gxin)
qed

lemma subpath-to-frontier:
    assumes g: path g and g0: pathstart g ∈ closure S and g1: pathfinish g ∉ S
    obtains u where 0 ≤ u u ≤ 1 g u ∈ frontier S (path-image(subpath 0 u g) −
{g u}) ⊆ interior S
proof −
  obtain u where 0 ≤ u u ≤ 1
          and notin: g u ∉ interior S
          and disj: u = 0 ∨
                    (∀ x. 0 ≤ x ∧ x < 1 ⟶ subpath 0 u g x ∈ interior S) ∧ g u
∈ closure S
    using subpath-to-frontier-strong [OF g g1] by blast
  show ?thesis
    apply (rule that [OF ‹0 ≤ u› ‹u ≤ 1›])
    apply (metis DiffI disj frontier-def g0 notin pathstart-def)
    using ‹0 ≤ u› g0 disj
    apply (simp add: path-image-subpath-gen)
   apply (auto simp: closed-segment-eq-real-ivl pathstart-def pathfinish-def subpath-def)
    apply (rename-tac y)
    apply (drule-tac x=y/u in spec)
    apply (auto split: if-split-asm)
    done
qed

lemma exists-path-subpath-to-frontier:
    fixes S :: 'a::real-normed-vector set
    assumes path g pathstart g ∈ closure S pathfinish g ∉ S
    obtains h where path h pathstart h = pathstart g path-image h ⊆ path-image
g
                    path-image h − {pathfinish h} ⊆ interior S
                    pathfinish h ∈ frontier S
proof −
  obtain u where u: 0 ≤ u u ≤ 1 g u ∈ frontier S (path-image(subpath 0 u g) −
{g u}) ⊆ interior S
    using subpath-to-frontier [OF assms] by blast
  show ?thesis
    apply (rule that [of subpath 0 u g])
    using assms u
    apply (simp-all add: path-image-subpath)
    apply (simp add: pathstart-def)
    apply (force simp: closed-segment-eq-real-ivl path-image-def)
    done
```

qed

lemma *exists-path-subpath-to-frontier-closed*:
fixes S :: *'a::real-normed-vector set*
assumes S: *closed S* **and** g: *path g* **and** *g0*: *pathstart g* $\in S$ **and** *g1*: *pathfinish g* $\notin S$
obtains h **where** *path h pathstart h = pathstart g path-image h* $\subseteq$ *path-image g* $\cap$ *S*
pathfinish h $\in$ *frontier S*
proof −
obtain h **where** h: *path h pathstart h = pathstart g path-image h* $\subseteq$ *path-image g*
path-image h − {*pathfinish h*} $\subseteq$ *interior S*
pathfinish h $\in$ *frontier S*
using *exists-path-subpath-to-frontier* [*OF g - g1*] *closure-closed* [*OF S*] *g0* **by** *auto*
show *?thesis*
apply (*rule that* [*OF* ‹*path h*›])
using *assms h*
apply *auto*
apply (*metis Diff-single-insert frontier-subset-eq insert-iff interior-subset subset-iff*)
done
qed

22.2 Reparametrizing a closed curve to start at some chosen point

definition *shiftpath* :: *real* ⇒ (*real* ⇒ *'a::topological-space*) ⇒ *real* ⇒ *'a*
where *shiftpath a f* = (λ*x*. *if* (*a* + *x*) ≤ *1 then f* (*a* + *x*) *else f* (*a* + *x* − *1*))

lemma *pathstart-shiftpath*: *a* ≤ *1* ⟹ *pathstart* (*shiftpath a g*) = *g a*
unfolding *pathstart-def shiftpath-def* **by** *auto*

lemma *pathfinish-shiftpath*:
assumes *0* ≤ *a*
and *pathfinish g* = *pathstart g*
shows *pathfinish* (*shiftpath a g*) = *g a*
using *assms*
unfolding *pathstart-def pathfinish-def shiftpath-def*
by *auto*

lemma *endpoints-shiftpath*:
assumes *pathfinish g* = *pathstart g*
and *a* ∈ {*0* .. *1*}
shows *pathfinish* (*shiftpath a g*) = *g a*
and *pathstart* (*shiftpath a g*) = *g a*
using *assms*
by (*auto intro*!: *pathfinish-shiftpath pathstart-shiftpath*)

lemma *closed-shiftpath*:
assumes *pathfinish g = pathstart g*
and *a ∈ {0..1}*
shows *pathfinish (shiftpath a g) = pathstart (shiftpath a g)*
using *endpoints-shiftpath[OF assms]*
by *auto*

lemma *path-shiftpath*:
assumes *path g*
and *pathfinish g = pathstart g*
and *a ∈ {0..1}*
shows *path (shiftpath a g)*
proof −
have ∗: *{0 .. 1} = {0 .. 1−a} ∪ {1−a .. 1}*
using *assms(3)* **by** *auto*
have ∗∗: *⋀x. x + a = 1 ⟹ g (x + a − 1) = g (x + a)*
using *assms(2)[unfolded pathfinish-def pathstart-def]*
by *auto*
show *?thesis*
unfolding *path-def shiftpath-def* ∗
apply *(rule continuous-on-closed-Un)*
apply *(rule closed-real-atLeastAtMost)+*
apply *(rule continuous-on-eq[of - g ∘ (λx. a + x)])*
prefer *3*
apply *(rule continuous-on-eq[of - g ∘ (λx. a − 1 + x)])*
prefer *3*
apply *(rule continuous-intros)+*
prefer *2*
apply *(rule continuous-intros)+*
apply *(rule-tac[1−2] continuous-on-subset[OF assms(1)[unfolded path-def]])*
using *assms(3)* **and** ∗∗
apply *auto*
apply *(auto simp add: field-simps)*
done
qed

lemma *shiftpath-shiftpath*:
assumes *pathfinish g = pathstart g*
and *a ∈ {0..1}*
and *x ∈ {0..1}*
shows *shiftpath (1 − a) (shiftpath a g) x = g x*
using *assms*
unfolding *pathfinish-def pathstart-def shiftpath-def*
by *auto*

lemma *path-image-shiftpath*:
assumes *a ∈ {0..1}*
and *pathfinish g = pathstart g*
shows *path-image (shiftpath a g) = path-image g*

proof −
{ fix *x*
assume *as*: *g 1 = g 0 x* ∈ {*0..1*::*real*} ∀ *y*∈{*0..1*} ∩ {*x*. ¬ *a + x* ≤ *1*}. *g x* ≠ *g* (*a + y − 1*)
then have ∃ *y*∈{*0..1*} ∩ {*x*. *a + x* ≤ *1*}. *g x = g* (*a + y*)
proof (*cases a* ≤ *x*)
case *False*
then show *?thesis*
apply (*rule-tac x=1 + x − a* **in** *bexI*)
using *as*(*1*,*2*) **and** *as*(*3*)[*THEN bspec*[**where** *x=1 + x − a*]] **and** *assms*(*1*)
apply (*auto simp add*: *field-simps atomize-not*)
done
next
case *True*
then show *?thesis*
using *as*(*1−2*) **and** *assms*(*1*)
apply (*rule-tac x=x − a* **in** *bexI*)
apply (*auto simp add*: *field-simps*)
done
qed
}
then show *?thesis*
using *assms*
unfolding *shiftpath-def path-image-def pathfinish-def pathstart-def*
by (*auto simp add*: *image-iff*)
qed

22.3 Special case of straight-line paths

definition *linepath* :: *'a*::*real-normed-vector* ⇒ *'a* ⇒ *real* ⇒ *'a*
where *linepath a b* = (λ*x*. (*1 − x*) *∗R a + x ∗R b*)

lemma *pathstart-linepath*[*simp*]: *pathstart* (*linepath a b*) = *a*
unfolding *pathstart-def linepath-def*
by *auto*

lemma *pathfinish-linepath*[*simp*]: *pathfinish* (*linepath a b*) = *b*
unfolding *pathfinish-def linepath-def*
by *auto*

lemma *continuous-linepath-at*[*intro*]: *continuous* (*at x*) (*linepath a b*)
unfolding *linepath-def*
by (*intro continuous-intros*)

lemma *continuous-on-linepath* [*intro*,*continuous-intros*]: *continuous-on s* (*linepath a b*)
using *continuous-linepath-at*
by (*auto intro*!: *continuous-at-imp-continuous-on*)

lemma *path-linepath*[*iff*]: *path* (*linepath a b*)
unfolding *path-def*
by (*rule continuous-on-linepath*)

lemma *path-image-linepath*[*simp*]: *path-image* (*linepath a b*) = *closed-segment a b*
unfolding *path-image-def segment linepath-def*
by *auto*

lemma *reversepath-linepath*[*simp*]: *reversepath* (*linepath a b*) = *linepath b a*
unfolding *reversepath-def linepath-def*
by *auto*

lemma *linepath-0* [*simp*]: *linepath 0 b x* = *x* $*_R$ *b*
by (*simp add*: *linepath-def*)

lemma *arc-linepath*:
assumes *a* ≠ *b* **shows** [*simp*]: *arc* (*linepath a b*)
proof −
{
fix *x y* :: *real*
assume *x* $*_R$ *b* + *y* $*_R$ *a* = *x* $*_R$ *a* + *y* $*_R$ *b*
then have (*x* − *y*) $*_R$ *a* = (*x* − *y*) $*_R$ *b*
by (*simp add*: *algebra-simps*)
with *assms* **have** *x* = *y*
by *simp*
}
then show *?thesis*
unfolding *arc-def inj-on-def*
by (*simp add*: *path-linepath*) (*force simp*: *algebra-simps linepath-def*)
qed

lemma *simple-path-linepath*[*intro*]: *a* ≠ *b* ⟹ *simple-path* (*linepath a b*)
by (*simp add*: *arc-imp-simple-path arc-linepath*)

lemma *linepath-trivial* [*simp*]: *linepath a a x* = *a*
by (*simp add*: *linepath-def real-vector.scale-left-diff-distrib*)

lemma *subpath-refl*: *subpath a a g* = *linepath* (*g a*) (*g a*)
by (*simp add*: *subpath-def linepath-def algebra-simps*)

lemma *linepath-of-real*: (*linepath* (*of-real a*) (*of-real b*) *x*) = *of-real* ((*1* − *x*)∗*a* + *x*∗*b*)
by (*simp add*: *scaleR-conv-of-real linepath-def*)

lemma *of-real-linepath*: *of-real* (*linepath a b x*) = *linepath* (*of-real a*) (*of-real b*) *x*
by (*metis linepath-of-real mult.right-neutral of-real-def real-scaleR-def*)

22.4 Segments via convex hulls

lemma *segments-subset-convex-hull*:
closed-segment a b ⊆ (*convex hull* {*a*,*b*,*c*})
closed-segment a c ⊆ (*convex hull* {*a*,*b*,*c*})
closed-segment b c ⊆ (*convex hull* {*a*,*b*,*c*})
closed-segment b a ⊆ (*convex hull* {*a*,*b*,*c*})
closed-segment c a ⊆ (*convex hull* {*a*,*b*,*c*})
closed-segment c b ⊆ (*convex hull* {*a*,*b*,*c*})
by (*auto simp*: *segment-convex-hull linepath-of-real elim*!: *rev-subsetD* [*OF - hull-mono*])

lemma *midpoints-in-convex-hull*:
assumes *x* ∈ *convex hull s y* ∈ *convex hull s*
shows *midpoint x y* ∈ *convex hull s*
proof −
have (*1* − *inverse*(*2*)) ∗$_R$ *x* + *inverse*(*2*) ∗$_R$ *y* ∈ *convex hull s*
apply (*rule convexD-alt*)
using *assms*
apply (*auto simp*: *convex-convex-hull*)
done
then show *?thesis*
by (*simp add*: *midpoint-def algebra-simps*)
qed

lemma *convex-hull-subset*:
s ⊆ *convex hull t* ⟹ *convex hull s* ⊆ *convex hull t*
by (*simp add*: *convex-convex-hull subset-hull*)

lemma *not-in-interior-convex-hull-3*:
fixes *a* :: *complex*
shows *a* ∉ *interior*(*convex hull* {*a*,*b*,*c*})
b ∉ *interior*(*convex hull* {*a*,*b*,*c*})
c ∉ *interior*(*convex hull* {*a*,*b*,*c*})
by (*auto simp*: *card-insert-le-m1 not-in-interior-convex-hull*)

lemma *midpoint-in-closed-segment* [*simp*]: *midpoint a b* ∈ *closed-segment a b*
using *midpoints-in-convex-hull segment-convex-hull* **by** *blast*

lemma *midpoint-in-open-segment* [*simp*]: *midpoint a b* ∈ *open-segment a b* ⟷ *a* ≠ *b*
by (*simp add*: *midpoint-eq-endpoint*(*1*) *midpoint-eq-endpoint*(*2*) *open-segment-def*)

22.5 Bounding a point away from a path

lemma *not-on-path-ball*:
fixes *g* :: *real* ⇒ *'a*::*heine-borel*
assumes *path g*
and *z* ∉ *path-image g*
shows ∃ *e* > *0*. *ball z e* ∩ *path-image g* = {}
proof −

obtain *a* **where** $a \in$ *path-image g* $\forall\, y \in$ *path-image g*. *dist z a* $\leq$ *dist z y*
apply (*rule distance-attains-inf*[*OF - path-image-nonempty*, *of g z*])
using *compact-path-image*[*THEN compact-imp-closed*, *OF assms*(*1*)] **by** *auto*
then show *?thesis*
apply (*rule-tac x*=*dist z a* **in** *exI*)
using *assms*(*2*)
apply (*auto intro*!: *dist-pos-lt*)
done
qed

lemma *not-on-path-cball*:
fixes *g* :: *real* $\Rightarrow$ *'a*::*heine-borel*
assumes *path g*
and $z \notin$ *path-image g*
shows $\exists\, e{>}0$. *cball z e* $\cap$ (*path-image g*) = {}
proof −
obtain *e* **where** *ball z e* $\cap$ *path-image g* = {} $e > 0$
using *not-on-path-ball*[*OF assms*] **by** *auto*
moreover have *cball z* (*e*/*2*) $\subseteq$ *ball z e*
using ‹$e > 0$› **by** *auto*
ultimately show *?thesis*
apply (*rule-tac x*=*e*/*2* **in** *exI*)
apply *auto*
done
qed

23 Path component, considered as a "joinability" relation (from Tom Hales)

definition *path-component s x y* $\longleftrightarrow$
($\exists\, g$. *path g* $\wedge$ *path-image g* $\subseteq$ *s* $\wedge$ *pathstart g* = *x* $\wedge$ *pathfinish g* = *y*)

abbreviation
path-component-set s x $\equiv$ *Collect* (*path-component s x*)

lemmas *path-defs* = *path-def pathstart-def pathfinish-def path-image-def path-component-def*

lemma *path-component-mem*:
assumes *path-component s x y*
shows $x \in s$ **and** $y \in s$
using *assms*
unfolding *path-defs*
by *auto*

lemma *path-component-refl*:
assumes $x \in s$
shows *path-component s x x*
unfolding *path-defs*

apply (*rule-tac x*=λ*u*. *x* **in** *exI*)
using *assms*
apply (*auto intro*!: *continuous-intros*)
done

lemma *path-component-refl-eq*: *path-component s x x* ⟷ *x* ∈ *s*
by (*auto intro*!: *path-component-mem path-component-refl*)

lemma *path-component-sym*: *path-component s x y* ⟹ *path-component s y x*
using *assms*
unfolding *path-component-def*
apply (*erule exE*)
apply (*rule-tac x*=*reversepath g* **in** *exI*)
apply *auto*
done

lemma *path-component-trans*:
assumes *path-component s x y* **and** *path-component s y z*
shows *path-component s x z*
using *assms*
unfolding *path-component-def*
apply (*elim exE*)
apply (*rule-tac x*=*g* +++ *ga* **in** *exI*)
apply (*auto simp add*: *path-image-join*)
done

lemma *path-component-of-subset*: *s* ⊆ *t* ⟹ *path-component s x y* ⟹ *path-component t x y*
unfolding *path-component-def* **by** *auto*

lemma *path-connected-linepath*:
fixes *s* :: *'a*::*real-normed-vector set*
shows *closed-segment a b* ⊆ *s* ⟹ *path-component s a b*
apply (*simp add*: *path-component-def*)
apply (*rule-tac x*=*linepath a b* **in** *exI*, *auto*)
done

23.0.1 Path components as sets

lemma *path-component-set*:
path-component-set s x =
{*y*. (∃ *g*. *path g* ∧ *path-image g* ⊆ *s* ∧ *pathstart g* = *x* ∧ *pathfinish g* = *y*)}
by (*auto simp*: *path-component-def*)

lemma *path-component-subset*: *path-component-set s x* ⊆ *s*
by (*auto simp add*: *path-component-mem*(*2*))

lemma *path-component-eq-empty*: *path-component-set s x* = {} ⟷ *x* ∉ *s*
using *path-component-mem path-component-refl-eq*

by *fastforce*

lemma *path-component-mono*:
$s \subseteq t \Longrightarrow (\textit{path-component-set}\ s\ x) \subseteq (\textit{path-component-set}\ t\ x)$
by (*simp add*: *Collect-mono path-component-of-subset*)

lemma *path-component-eq*:
$y \in \textit{path-component-set}\ s\ x \Longrightarrow \textit{path-component-set}\ s\ y = \textit{path-component-set}\ s\ x$
by (*metis* (*no-types*, *lifting*) *Collect-cong mem-Collect-eq path-component-sym path-component-trans*)

23.1 Path connectedness of a space

definition *path-connected* $s \longleftrightarrow$
$(\forall x{\in}s.\ \forall y{\in}s.\ \exists g.\ \textit{path}\ g \wedge \textit{path-image}\ g \subseteq s \wedge \textit{pathstart}\ g = x \wedge \textit{pathfinish}\ g = y)$

lemma *path-connected-component*: *path-connected* $s \longleftrightarrow (\forall x{\in}s.\ \forall y{\in}s.\ \textit{path-component}\ s\ x\ y)$
unfolding *path-connected-def path-component-def* **by** *auto*

lemma *path-connected-component-set*: *path-connected* $s \longleftrightarrow (\forall x{\in}s.\ \textit{path-component-set}\ s\ x = s)$
unfolding *path-connected-component path-component-subset*
using *path-component-mem* **by** *blast*

lemma *path-component-maximal*:
$[\![x \in t;\ \textit{path-connected}\ t;\ t \subseteq s]\!] \Longrightarrow t \subseteq (\textit{path-component-set}\ s\ x)$
by (*metis path-component-mono path-connected-component-set*)

lemma *convex-imp-path-connected*:
fixes $s :: 'a{::}\textit{real-normed-vector set}$
assumes *convex s*
shows *path-connected s*
unfolding *path-connected-def*
apply *rule*
apply *rule*
apply (*rule-tac* $x = \textit{linepath}\ x\ y$ **in** *exI*)
unfolding *path-image-linepath*
using *assms* [*unfolded convex-contains-segment*]
apply *auto*
done

lemma *path-connected-UNIV* [*iff*]: *path-connected* $(\textit{UNIV} :: 'a{::}\textit{real-normed-vector set})$
by (*simp add*: *convex-imp-path-connected*)

lemma *path-component-UNIV*: $\textit{path-component-set}\ \textit{UNIV}\ x = (\textit{UNIV} :: 'a{::}\textit{real-normed-vector set})$

using *path-connected-component-set* **by** *auto*

lemma *path-connected-imp-connected*:
assumes *path-connected s*
shows *connected s*
unfolding *connected-def not-ex*
apply *rule*
apply *rule*
apply (*rule ccontr*)
unfolding *not-not*
apply (*elim conjE*)
proof −
fix *e1 e2*
assume *as*: *open e1 open e2 s* ⊆ *e1* ∪ *e2 e1* ∩ *e2* ∩ *s* = {} *e1* ∩ *s* ≠ {} *e2* ∩ *s* ≠ {}
then obtain *x1 x2* **where** *obt*:*x1* ∈ *e1* ∩ *s x2* ∈ *e2* ∩ *s*
by *auto*
then obtain *g* **where** *g*: *path g path-image g* ⊆ *s pathstart g* = *x1 pathfinish g* = *x2*
using *assms*[*unfolded path-connected-def*,*rule-format*,*of x1 x2*] **by** *auto*
have ∗: *connected* {*0..1*::*real*}
by (*auto intro*!: *convex-connected convex-real-interval*)
have {*0..1*} ⊆ {*x* ∈ {*0..1*}. *g x* ∈ *e1*} ∪ {*x* ∈ {*0..1*}. *g x* ∈ *e2*}
using *as*(*3*) *g*(*2*)[*unfolded path-defs*] **by** *blast*
moreover have {*x* ∈ {*0..1*}. *g x* ∈ *e1*} ∩ {*x* ∈ {*0..1*}. *g x* ∈ *e2*} = {}
using *as*(*4*) *g*(*2*)[*unfolded path-defs*]
unfolding *subset-eq*
by *auto*
moreover have {*x* ∈ {*0..1*}. *g x* ∈ *e1*} ≠ {} ∧ {*x* ∈ {*0..1*}. *g x* ∈ *e2*} ≠ {}
using *g*(*3*,*4*)[*unfolded path-defs*]
using *obt*
by (*simp add*: *ex-in-conv* [*symmetric*], *metis zero-le-one order-refl*)
ultimately show *False*
using ∗[*unfolded connected-local not-ex*, *rule-format*,
of {*x*∈{*0..1*}. *g x* ∈ *e1*} {*x*∈{*0..1*}. *g x* ∈ *e2*}]
using *continuous-openin-preimage*[*OF g*(*1*)[*unfolded path-def*] *as*(*1*)]
using *continuous-openin-preimage*[*OF g*(*1*)[*unfolded path-def*] *as*(*2*)]
by *auto*
qed

lemma *open-path-component*:
fixes *s* :: *'a*::*real-normed-vector set*
assumes *open s*
shows *open* (*path-component-set s x*)
unfolding *open-contains-ball*
proof
fix *y*
assume *as*: *y* ∈ *path-component-set s x*
then have *y* ∈ *s*

```
    apply −
    apply (rule path-component-mem(2))
    unfolding mem-Collect-eq
    apply auto
    done
  then obtain e where e: e > 0 ball y e ⊆ s
    using assms[unfolded open-contains-ball]
    by auto
  show ∃ e > 0. ball y e ⊆ path-component-set s x
    apply (rule-tac x=e in exI)
    apply (rule,rule ‹e>0›)
    apply rule
    unfolding mem-ball mem-Collect-eq
  proof −
    fix z
    assume dist y z < e
    then show path-component s x z
      apply (rule-tac path-component-trans[of - - y])
      defer
      apply (rule path-component-of-subset[OF e(2)])
    apply (rule convex-imp-path-connected[OF convex-ball, unfolded path-connected-component,
rule-format])
      using ‹e > 0› as
      apply auto
      done
  qed
qed

lemma open-non-path-component:
  fixes s :: 'a::real-normed-vector set
  assumes open s
  shows open (s − path-component-set s x)
  unfolding open-contains-ball
proof
  fix y
  assume as: y ∈ s − path-component-set s x
  then obtain e where e: e > 0 ball y e ⊆ s
    using assms [unfolded open-contains-ball]
    by auto
  show ∃ e>0. ball y e ⊆ s − path-component-set s x
    apply (rule-tac x=e in exI)
    apply rule
    apply (rule ‹e>0›)
    apply rule
    apply rule
    defer
  proof (rule ccontr)
    fix z
    assume z ∈ ball y e ¬ z ∉ path-component-set s x
```

```
    then have y ∈ path-component-set s x
      unfolding not-not mem-Collect-eq using ‹e>0›
      apply −
      apply (rule path-component-trans, assumption)
      apply (rule path-component-of-subset[OF e(2)])
    apply (rule convex-imp-path-connected[OF convex-ball, unfolded path-connected-component,
rule-format])
      apply auto
      done
    then show False
      using as by auto
  qed (insert e(2), auto)
qed

lemma connected-open-path-connected:
  fixes s :: 'a::real-normed-vector set
  assumes open s
    and connected s
  shows path-connected s
  unfolding path-connected-component-set
proof (rule, rule, rule path-component-subset, rule)
  fix x y
  assume x ∈ s and y ∈ s
  show y ∈ path-component-set s x
  proof (rule ccontr)
    assume ¬ ?thesis
    moreover have path-component-set s x ∩ s ≠ {}
      using ‹x ∈ s› path-component-eq-empty path-component-subset[of s x]
      by auto
    ultimately
    show False
    using ‹y ∈ s› open-non-path-component[OF assms(1)] open-path-component[OF
assms(1)]
      using assms(2)[unfolded connected-def not-ex, rule-format,
        of path-component-set s x s − path-component-set s x]
      by auto
  qed
qed

lemma path-connected-continuous-image:
  assumes continuous-on s f
    and path-connected s
  shows path-connected (f ‘ s)
  unfolding path-connected-def
proof (rule, rule)
  fix x' y'
  assume x' ∈ f ‘ s y' ∈ f ‘ s
  then obtain x y where x: x ∈ s and y: y ∈ s and x': x' = f x and y': y' = f
y
```

by *auto*
from *x y* **obtain** *g* **where** *path g* ∧ *path-image g* ⊆ *s* ∧ *pathstart g* = *x* ∧ *pathfinish g* = *y*
using *assms*(*2*)[*unfolded path-connected-def*] **by** *fast*
then show ∃ *g*. *path g* ∧ *path-image g* ⊆ *f* ' *s* ∧ *pathstart g* = *x'* ∧ *pathfinish g* = *y'*
unfolding *x' y'*
apply (*rule-tac x*=*f* ∘ *g* **in** *exI*)
unfolding *path-defs*
apply (*intro conjI continuous-on-compose continuous-on-subset*[*OF assms*(*1*)])
apply *auto*
done
qed

lemma *path-connected-segment*:
fixes *a* :: *'a*::*real-normed-vector*
shows *path-connected* (*closed-segment a b*)
by (*simp add*: *convex-imp-path-connected*)

lemma *path-connected-open-segment*:
fixes *a* :: *'a*::*real-normed-vector*
shows *path-connected* (*open-segment a b*)
by (*simp add*: *convex-imp-path-connected*)

lemma *homeomorphic-path-connectedness*:
s homeomorphic t ⟹ *path-connected s* ⟷ *path-connected t*
unfolding *homeomorphic-def homeomorphism-def* **by** (*metis path-connected-continuous-image*)

lemma *path-connected-empty*: *path-connected* {}
unfolding *path-connected-def* **by** *auto*

lemma *path-connected-singleton*: *path-connected* {*a*}
unfolding *path-connected-def pathstart-def pathfinish-def path-image-def*
apply *clarify*
apply (*rule-tac x*=λ*x*. *a* **in** *exI*)
apply (*simp add*: *image-constant-conv*)
apply (*simp add*: *path-def continuous-on-const*)
done

lemma *path-connected-Un*:
assumes *path-connected s*
and *path-connected t*
and *s* ∩ *t* ≠ {}
shows *path-connected* (*s* ∪ *t*)
unfolding *path-connected-component*
proof (*rule*, *rule*)
fix *x y*
assume *as*: *x* ∈ *s* ∪ *t y* ∈ *s* ∪ *t*
from *assms*(*3*) **obtain** *z* **where** *z* ∈ *s* ∩ *t*

```
    by auto
  then show path-component (s ∪ t) x y
    using as and assms(1−2)[unfolded path-connected-component]
    apply −
    apply (erule-tac[!] UnE)+
    apply (rule-tac[2−3] path-component-trans[of - - z])
   apply (auto simp add:path-component-of-subset [OF Un-upper1] path-component-of-subset[OF
Un-upper2])
    done
qed

lemma path-connected-UNION:
  assumes ⋀i. i ∈ A ⟹ path-connected (S i)
    and ⋀i. i ∈ A ⟹ z ∈ S i
  shows path-connected (⋃i∈A. S i)
  unfolding path-connected-component
proof clarify
  fix x i y j
  assume ∗: i ∈ A x ∈ S i j ∈ A y ∈ S j
  then have path-component (S i) x z and path-component (S j) z y
    using assms by (simp-all add: path-connected-component)
  then have path-component (⋃i∈A. S i) x z and path-component (⋃i∈A. S i)
z y
    using ∗(1,3) by (auto elim!: path-component-of-subset [rotated])
  then show path-component (⋃i∈A. S i) x y
    by (rule path-component-trans)
qed

lemma path-component-path-image-pathstart:
  assumes p: path p and x: x ∈ path-image p
  shows path-component (path-image p) (pathstart p) x
using x
proof (clarsimp simp add: path-image-def)
  fix y
  assume x = p y and y: 0 ≤ y y ≤ 1
  show path-component (p ‘ {0..1}) (pathstart p) (p y)
  proof (cases y=0)
    case True then show ?thesis
      by (simp add: path-component-refl-eq pathstart-def)
  next
    case False have continuous-on {0..1} (p o (op∗y))
      apply (rule continuous-intros)+
      using p [unfolded path-def] y
      apply (auto simp: mult-le-one intro: continuous-on-subset [of - p])
      done
    then have path (λu. p (y ∗ u))
      by (simp add: path-def)
    then show ?thesis
      apply (simp add: path-component-def)
```

```
    apply (rule-tac x = λu. p (y * u) in exI)
    apply (intro conjI)
    using y False
    apply (auto simp: mult-le-one pathstart-def pathfinish-def path-image-def)
    done
  qed
qed

lemma path-connected-path-image: path p ⟹ path-connected(path-image p)
  unfolding path-connected-component
 by (meson path-component-path-image-pathstart path-component-sym path-component-trans)

lemma path-connected-path-component:
   path-connected (path-component-set s x)
proof −
  { fix y z
    assume pa: path-component s x y path-component s x z
    then have pae: path-component-set s x = path-component-set s y
      using path-component-eq by auto
    have yz: path-component s y z
      using pa path-component-sym path-component-trans by blast
    then have ∃ g. path g ∧ path-image g ⊆ path-component-set s x ∧ pathstart g
= y ∧ pathfinish g = z
      apply (simp add: path-component-def, clarify)
      apply (rule-tac x=g in exI)
     by (simp add: pae path-component-maximal path-connected-path-image pathstart-in-path-image)
  }
  then show ?thesis
    by (simp add: path-connected-def)
qed

lemma path-component: path-component s x y ⟷ (∃ t. path-connected t ∧ t ⊆ s
∧ x ∈ t ∧ y ∈ t)
  apply (intro iffI)
   apply (metis path-connected-path-image path-defs(5) pathfinish-in-path-image
pathstart-in-path-image)
  using path-component-of-subset path-connected-component by blast

lemma path-component-path-component [simp]:
   path-component-set (path-component-set s x) x = path-component-set s x
proof (cases x ∈ s)
  case True show ?thesis
    apply (rule subset-antisym)
    apply (simp add: path-component-subset)
   by (simp add: True path-component-maximal path-component-refl path-connected-path-component)
next
  case False then show ?thesis
    by (metis False empty-iff path-component-eq-empty)
qed
```

lemma *path-component-subset-connected-component*:
(*path-component-set s x*) ⊆ (*connected-component-set s x*)
proof (*cases x ∈ s*)
case *True* **show** *?thesis*
apply (*rule connected-component-maximal*)
apply (*auto simp*: *True path-component-subset path-component-refl path-connected-imp-connected path-connected-path-component*)
done
next
case *False* **then show** *?thesis*
using *path-component-eq-empty* **by** *auto*
qed

23.2 Lemmas about path-connectedness

lemma *path-connected-linear-image*:
fixes *f* :: *'a*::*real-normed-vector* ⇒ *'b*::*real-normed-vector*
assumes *path-connected s bounded-linear f*
shows *path-connected*(*f ' s*)
by (*auto simp*: *linear-continuous-on assms path-connected-continuous-image*)

lemma *is-interval-path-connected*: *is-interval s* ⟹ *path-connected s*
by (*simp add*: *convex-imp-path-connected is-interval-convex*)

lemma *linear-homeomorphism-image*:
fixes *f* :: *'a*::*euclidean-space* ⇒ *'b*::*euclidean-space*
assumes *linear f inj f*
obtains *g* **where** *homeomorphism* (*f ' S*) *S g f*
using *linear-injective-left-inverse* [*OF assms*]
apply *clarify*
apply (*rule-tac g*=*g* **in** *that*)
using *assms*
apply (*auto simp*: *homeomorphism-def eq-id-iff* [*symmetric*] *image-comp comp-def linear-conv-bounded-linear linear-continuous-on*)
done

lemma *linear-homeomorphic-image*:
fixes *f* :: *'a*::*euclidean-space* ⇒ *'b*::*euclidean-space*
assumes *linear f inj f*
shows *S homeomorphic f ' S*
by (*meson homeomorphic-def homeomorphic-sym linear-homeomorphism-image* [*OF assms*])

lemma *path-connected-Times*:
assumes *path-connected s path-connected t*
shows *path-connected* (*s* × *t*)
proof (*simp add*: *path-connected-def Sigma-def*, *clarify*)
fix *x1 y1 x2 y2*

assume *x1* ∈ *s y1* ∈ *t x2* ∈ *s y2* ∈ *t*
obtain *g* **where** *path g* **and** *g*: *path-image g* ⊆ *s* **and** *gs*: *pathstart g* = *x1* **and** *gf*: *pathfinish g* = *x2*
using ‹*x1* ∈ *s*› ‹*x2* ∈ *s*› *assms* **by** (*force simp*: *path-connected-def*)
obtain *h* **where** *path h* **and** *h*: *path-image h* ⊆ *t* **and** *hs*: *pathstart h* = *y1* **and** *hf*: *pathfinish h* = *y2*
using ‹*y1* ∈ *t*› ‹*y2* ∈ *t*› *assms* **by** (*force simp*: *path-connected-def*)
have *path* (λ*z*. (*x1*, *h z*))
using ‹*path h*›
apply (*simp add*: *path-def*)
apply (*rule continuous-on-compose2* [**where** *f* = *h*])
apply (*rule continuous-intros* | *force*)+
done
moreover have *path* (λ*z*. (*g z*, *y2*))
using ‹*path g*›
apply (*simp add*: *path-def*)
apply (*rule continuous-on-compose2* [**where** *f* = *g*])
apply (*rule continuous-intros* | *force*)+
done
ultimately have *1*: *path* ((λ*z*. (*x1*, *h z*)) +++ (λ*z*. (*g z*, *y2*)))
by (*metis hf gs path-join-imp pathstart-def pathfinish-def*)
have *path-image* ((λ*z*. (*x1*, *h z*)) +++ (λ*z*. (*g z*, *y2*))) ⊆ *path-image* (λ*z*. (*x1*, *h z*)) ∪ *path-image* (λ*z*. (*g z*, *y2*))
by (*rule Path-Connected.path-image-join-subset*)
also have ... ⊆ (⋃*x*∈*s*. ⋃*x1*∈*t*. {(*x*, *x1*)})
using *g h* ‹*x1* ∈ *s*› ‹*y2* ∈ *t*› **by** (*force simp*: *path-image-def*)
finally have *2*: *path-image* ((λ*z*. (*x1*, *h z*)) +++ (λ*z*. (*g z*, *y2*))) ⊆ (⋃*x*∈*s*. ⋃*x1*∈*t*. {(*x*, *x1*)}) **.**
show ∃*g*. *path g* ∧ *path-image g* ⊆ (⋃*x*∈*s*. ⋃*x1*∈*t*. {(*x*, *x1*)}) ∧
pathstart g = (*x1*, *y1*) ∧ *pathfinish g* = (*x2*, *y2*)
apply (*intro exI conjI*)
apply (*rule 1*)
apply (*rule 2*)
apply (*metis hs pathstart-def pathstart-join*)
by (*metis gf pathfinish-def pathfinish-join*)
qed

lemma *is-interval-path-connected-1*:
fixes *s* :: *real set*
shows *is-interval s* ⟷ *path-connected s*
using *is-interval-connected-1 is-interval-path-connected path-connected-imp-connected*
by *blast*

lemma *Union-path-component* [*simp*]:
Union {*path-component-set S x* |*x*. *x* ∈ *S*} = *S*
apply (*rule subset-antisym*)
using *path-component-subset* **apply** *force*
using *path-component-refl* **by** *auto*

lemma *path-component-disjoint*:
disjnt (*path-component-set S a*) (*path-component-set S b*) ⟷
(*a* ∉ *path-component-set S b*)
apply (*auto simp*: *disjnt-def*)
using *path-component-eq* **apply** *fastforce*
using *path-component-sym path-component-trans* **by** *blast*

lemma *path-component-eq-eq*:
path-component S x = *path-component S y* ⟷
(*x* ∉ *S*) ∧ (*y* ∉ *S*) ∨ *x* ∈ *S* ∧ *y* ∈ *S* ∧ *path-component S x y*
apply (*rule iffI*, *metis* (*no-types*) *path-component-mem*(*1*) *path-component-refl*)
apply (*erule disjE*, *metis Collect-empty-eq-bot path-component-eq-empty*)
apply (*rule ext*)
apply (*metis path-component-trans path-component-sym*)
done

lemma *path-component-unique*:
assumes *x* ∈ *c c* ⊆ *S path-connected c*
⋀*c'*. ⟦*x* ∈ *c'*; *c'* ⊆ *S*; *path-connected c'*⟧ ⟹ *c'* ⊆ *c*
shows *path-component-set S x* = *c*
apply (*rule subset-antisym*)
using *assms*
apply (*metis mem-Collect-eq subsetCE path-component-eq-eq path-component-subset*
path-connected-path-component)
by (*simp add*: *assms path-component-maximal*)

lemma *path-component-intermediate-subset*:
path-component-set u a ⊆ *t* ∧ *t* ⊆ *u*
⟹ *path-component-set t a* = *path-component-set u a*
by (*metis* (*no-types*) *path-component-mono path-component-path-component subset-antisym*)

lemma *complement-path-component-Union*:
fixes *x* :: *'a* :: *topological-space*
shows *S* − *path-component-set S x* =
⋃({*path-component-set S y*| *y*. *y* ∈ *S*} − {*path-component-set S x*})
proof −
have ∗: (⋀*x*. *x* ∈ *S* − {*a*} ⟹ *disjnt a x*) ⟹ ⋃*S* − *a* = ⋃(*S* − {*a*})
for *a*::*'a set* **and** *S*
by (*auto simp*: *disjnt-def*)
have ⋀*y*. *y* ∈ {*path-component-set S x* |*x*. *x* ∈ *S*} − {*path-component-set S x*}
⟹ *disjnt* (*path-component-set S x*) *y*
using *path-component-disjoint path-component-eq* **by** *fastforce*
then have ⋃{*path-component-set S x* |*x*. *x* ∈ *S*} − *path-component-set S x* =
⋃({*path-component-set S y* |*y*. *y* ∈ *S*} − {*path-component-set S x*})
by (*meson* ∗)
then show *?thesis* **by** *simp*
qed

23.3 Sphere is path-connected

lemma *path-connected-punctured-universe*:
assumes $2 \leq DIM('a::euclidean\text{-}space)$
shows *path-connected* $(- \{a::'a\})$
proof −
let *?A* $= \{x::'a.\ \exists i{\in}Basis.\ x \cdot i < a \cdot i\}$
let *?B* $= \{x::'a.\ \exists i{\in}Basis.\ a \cdot i < x \cdot i\}$

have *A*: *path-connected ?A*
unfolding *Collect-bex-eq*
proof (*rule path-connected-UNION*)
fix $i :: 'a$
assume $i \in Basis$
then show $(\sum i{\in}Basis.\ (a \cdot i - 1) *_R i) \in \{x::'a.\ x \cdot i < a \cdot i\}$
by *simp*
show *path-connected* $\{x.\ x \cdot i < a \cdot i\}$
using *convex-imp-path-connected* [*OF convex-halfspace-lt*, *of i a* $\cdot$ *i*]
by (*simp add*: *inner-commute*)
qed
have *B*: *path-connected ?B*
unfolding *Collect-bex-eq*
proof (*rule path-connected-UNION*)
fix $i :: 'a$
assume $i \in Basis$
then show $(\sum i{\in}Basis.\ (a \cdot i + 1) *_R i) \in \{x::'a.\ a \cdot i < x \cdot i\}$
by *simp*
show *path-connected* $\{x.\ a \cdot i < x \cdot i\}$
using *convex-imp-path-connected* [*OF convex-halfspace-gt*, *of a* $\cdot$ *i i*]
by (*simp add*: *inner-commute*)
qed
obtain $S :: 'a\ set$ **where** $S \subseteq Basis$ **and** $card\ S = Suc\ (Suc\ 0)$
using *ex-card*[*OF assms*]
by *auto*
then obtain *b0 b1* $:: 'a$ **where** $b0 \in Basis$ **and** $b1 \in Basis$ **and** $b0 \neq b1$
unfolding *card-Suc-eq* **by** *auto*
then have $a + b0 - b1 \in ?A \cap ?B$
by (*auto simp*: *inner-simps inner-Basis*)
then have $?A \cap ?B \neq \{\}$
by *fast*
with *A B* **have** *path-connected* $(?A \cup ?B)$
by (*rule path-connected-Un*)
also have $?A \cup ?B = \{x.\ \exists i{\in}Basis.\ x \cdot i \neq a \cdot i\}$
unfolding *neq-iff bex-disj-distrib Collect-disj-eq* **..**
also have $\ldots = \{x.\ x \neq a\}$
unfolding *euclidean-eq-iff* [**where** $'a='a$]
by (*simp add*: *Bex-def*)
also have $\ldots = - \{a\}$
by *auto*
finally show *?thesis* **.**

qed

lemma *path-connected-sphere*:
assumes *2* ≤ *DIM*('a::*euclidean-space*)
shows *path-connected* {x::'a. *norm* (x − a) = r}
proof (*rule linorder-cases* [*of r 0*])
assume r < *0*
then have {x::'a. *norm*(x − a) = r} = {}
by *auto*
then show *?thesis*
using *path-connected-empty* **by** *simp*
next
assume r = *0*
then show *?thesis*
using *path-connected-singleton* **by** *simp*
next
assume r: *0* < r
have ∗: {x::'a. *norm*(x − a) = r} = (λx. a + r ∗$_R$ x) ‘ {x. *norm* x = *1*}
apply (*rule set-eqI*)
apply *rule*
unfolding *image-iff*
apply (*rule-tac* x=(*1*/r) ∗$_R$ (x − a) **in** *bexI*)
unfolding *mem-Collect-eq norm-scaleR*
using r
apply (*auto simp add*: *scaleR-right-diff-distrib*)
done
have ∗∗: {x::'a. *norm* x = *1*} = (λx. (*1*/*norm* x) ∗$_R$ x) ‘ (− {*0*})
apply (*rule set-eqI*)
apply *rule*
unfolding *image-iff*
apply (*rule-tac* x=x **in** *bexI*)
unfolding *mem-Collect-eq*
apply (*auto split*: *if-split-asm*)
done
have *continuous-on* (− {*0*}) (λx::'a. *1* / *norm* x)
by (*auto intro*!: *continuous-intros*)
then show *?thesis*
unfolding ∗ ∗∗
using *path-connected-punctured-universe*[*OF assms*]
by (*auto intro*!: *path-connected-continuous-image continuous-intros*)
qed

corollary *connected-sphere*: *2* ≤ *DIM*('a::*euclidean-space*) ⟹ *connected* {x::'a. *norm* (x − a) = r}
using *path-connected-sphere path-connected-imp-connected*
by *auto*

corollary *path-connected-complement-bounded-convex*:
fixes s :: 'a :: *euclidean-space set*

assumes *bounded s convex s* **and** *2*: *2* ≤ *DIM*('*a*)
shows *path-connected* (− *s*)
proof (*cases s*={})
case *True* **then show** *?thesis*
using *convex-imp-path-connected* **by** *auto*
next
case *False*
then obtain *a* **where** *a* ∈ *s* **by** *auto*
{ fix *x y* **assume** *x* ∉ *s y* ∉ *s*
then have *x* ≠ *a y* ≠ *a* **using** ‹*a* ∈ *s*› **by** *auto*
then have *bxy*: *bounded*(*insert x* (*insert y s*))
by (*simp add*: ‹*bounded s*›)
then obtain *B*::*real* **where** *B*: *0* < *B* **and** *Bx*: *norm* (*a* − *x*) < *B* **and** *By*: *norm* (*a* − *y*) < *B*
and *s* ⊆ *ball a B*
using *bounded-subset-ballD* [*OF bxy*, *of a*] **by** (*auto simp*: *dist-norm*)
def *C* == *B* / *norm*(*x* − *a*)
{ fix *u*
assume *u*: (*1* − *u*) *∗R x* + *u ∗R* (*a* + *C ∗R* (*x* − *a*)) ∈ *s* **and** *0* ≤ *u u* ≤ *1*
have *CC*: *1* ≤ *1* + (*C* − *1*) ∗ *u*
using ‹*x* ≠ *a*› ‹*0* ≤ *u*›
apply (*simp add*: *C-def divide-simps norm-minus-commute*)
using *Bx* **by** *auto*
have ∗: ⋀*v*. (*1* − *u*) *∗R x* + *u ∗R* (*a* + *v ∗R* (*x* − *a*)) = *a* + (*1* + (*v* − *1*) ∗ *u*) *∗R* (*x* − *a*)
by (*simp add*: *algebra-simps*)
have *a* + ((*1* / (*1* + *C* ∗ *u* − *u*)) *∗R x* + ((*u* / (*1* + *C* ∗ *u* − *u*)) *∗R a* + (*C* ∗ *u* / (*1* + *C* ∗ *u* − *u*)) *∗R x*)) =
(*1* + (*u* / (*1* + *C* ∗ *u* − *u*))) *∗R a* + ((*1* / (*1* + *C* ∗ *u* − *u*)) + (*C* ∗ *u* / (*1* + *C* ∗ *u* − *u*))) *∗R x*
by (*simp add*: *algebra-simps*)
also have ... = (*1* + (*u* / (*1* + *C* ∗ *u* − *u*))) *∗R a* + (*1* + (*u* / (*1* + *C* ∗ *u* − *u*))) *∗R x*
using *CC* **by** (*simp add*: *field-simps*)
also have ... = *x* + (*1* + (*u* / (*1* + *C* ∗ *u* − *u*))) *∗R a* + (*u* / (*1* + *C* ∗ *u* − *u*)) *∗R x*
by (*simp add*: *algebra-simps*)
also have ... = *x* + ((*1* / (*1* + *C* ∗ *u* − *u*)) *∗R a* +
((*u* / (*1* + *C* ∗ *u* − *u*)) *∗R x* + (*C* ∗ *u* / (*1* + *C* ∗ *u* − *u*)) *∗R a*))
using *CC* **by** (*simp add*: *field-simps*) (*simp add*: *add-divide-distrib scaleR-add-left*)
finally have *xeq*: (*1* − *1* / (*1* + (*C* − *1*) ∗ *u*)) *∗R a* + (*1* / (*1* + (*C* − *1*) ∗ *u*)) *∗R* (*a* + (*1* + (*C* − *1*) ∗ *u*) *∗R* (*x* − *a*)) = *x*
by (*simp add*: *algebra-simps*)
have *False*
using ‹*convex s*›
apply (*simp add*: *convex-alt*)
apply (*drule-tac x*=*a* **in** *bspec*)
apply (*rule* ‹*a* ∈ *s*›)
apply (*drule-tac x*=*a* + (*1* + (*C* − *1*) ∗ *u*) *∗R* (*x* − *a*) **in** *bspec*)

```
      using u apply (simp add: *)
     apply (drule-tac x=1 / (1 + (C - 1) * u) in spec)
     using ‹x ≠ a› ‹x ∉ s› ‹0 ≤ u› CC
     apply (auto simp: xeq)
     done
  }
  then have pcx: path-component (- s) x (a + C *R (x - a))
    by (force simp: closed-segment-def intro!: path-connected-linepath)
  def D == B / norm(y - a)  — massive duplication with the proof above
  { fix u
    assume u: (1 - u) *R y + u *R (a + D *R (y - a)) ∈ s and 0 ≤ u u ≤ 1
    have DD: 1 ≤ 1 + (D - 1) * u
      using ‹y ≠ a› ‹0 ≤ u›
      apply (simp add: D-def divide-simps norm-minus-commute)
      using By by auto
    have *: ⋀v. (1 - u) *R y + u *R (a + v *R (y - a)) = a + (1 + (v - 1)
* u) *R (y - a)
      by (simp add: algebra-simps)
    have a + ((1 / (1 + D * u - u)) *R y + ((u / (1 + D * u - u)) *R a +
(D * u / (1 + D * u - u)) *R y)) =
          (1 + (u / (1 + D * u - u))) *R a + ((1 / (1 + D * u - u)) + (D *
u / (1 + D * u - u))) *R y
      by (simp add: algebra-simps)
    also have ... = (1 + (u / (1 + D * u - u))) *R a + (1 + (u / (1 + D * u
- u))) *R y
      using DD by (simp add: field-simps)
    also have ... = y + (1 + (u / (1 + D * u - u))) *R a + (u / (1 + D * u
- u)) *R y
      by (simp add: algebra-simps)
    also have ... = y + ((1 / (1 + D * u - u)) *R a +
          ((u / (1 + D * u - u)) *R y + (D * u / (1 + D * u - u)) *R a))
    using DD by (simp add: field-simps) (simp add: add-divide-distrib scaleR-add-left)
    finally have xeq: (1 - 1 / (1 + (D - 1) * u)) *R a + (1 / (1 + (D - 1)
* u)) *R (a + (1 + (D - 1) * u) *R (y - a)) = y
      by (simp add: algebra-simps)
    have False
      using ‹convex s›
      apply (simp add: convex-alt)
      apply (drule-tac x=a in bspec)
       apply (rule ‹a ∈ s›)
      apply (drule-tac x=a + (1 + (D - 1) * u) *R (y - a) in bspec)
       using u apply (simp add: *)
      apply (drule-tac x=1 / (1 + (D - 1) * u) in spec)
      using ‹y ≠ a› ‹y ∉ s› ‹0 ≤ u› DD
      apply (auto simp: xeq)
      done
  }
  then have pdy: path-component (- s) y (a + D *R (y - a))
    by (force simp: closed-segment-def intro!: path-connected-linepath)
```

have *pyx*: *path-component* $(-\ s)$ $(a + D *_R (y - a))$ $(a + C *_R (x - a))$
apply (*rule path-component-of-subset* [*of* $\{x.\ norm(x - a) = B\}$])
using ‹$s \subseteq$ *ball a B*›
apply (*force simp*: *ball-def dist-norm norm-minus-commute*)
apply (*rule path-connected-sphere* [*OF 2*, *of a B*, *simplified path-connected-component*, *rule-format*])
using ‹$x \neq a$› **using** ‹$y \neq a$› *B* **apply** (*auto simp*: *C-def D-def*)
done
have *path-component* $(-\ s)$ *x y*
by (*metis path-component-trans path-component-sym pcx pdy pyx*)
}
then show *?thesis*
by (*auto simp*: *path-connected-component*)
qed

lemma *connected-complement-bounded-convex*:
fixes *s* :: $'a$ :: *euclidean-space set*
assumes *bounded s convex s 2* $\leq$ *DIM*($'a$)
shows *connected* $(-\ s)$
using *path-connected-complement-bounded-convex* [*OF assms*] *path-connected-imp-connected*
by *blast*

lemma *connected-diff-ball*:
fixes *s* :: $'a$ :: *euclidean-space set*
assumes *connected s cball a r* $\subseteq$ *s 2* $\leq$ *DIM*($'a$)
shows *connected* $(s - ball\ a\ r)$
apply (*rule connected-diff-open-from-closed* [*OF ball-subset-cball*])
using *assms connected-sphere*
apply (*auto simp*: *cball-diff-eq-sphere dist-norm*)
done

proposition *connected-open-delete*:
assumes *open S connected S* **and** *2*: *2* $\leq$ *DIM*($'N$::*euclidean-space*)
shows *connected*$(S - \{a::'N\})$
proof (*cases a* $\in$ *S*)
case *True*
with ‹*open S*› **obtain** ε **where** $\varepsilon > 0$ **and** ε: *cball a* $\varepsilon \subseteq S$
using *open-contains-cball-eq* **by** *blast*
have *dist a* $(a + \varepsilon *_R (SOME\ i.\ i \in Basis)) = \varepsilon$
by (*simp add*: *dist-norm SOME-Basis* ‹$0 < \varepsilon$› *less-imp-le*)
with ε **have** $\bigcap \{S - ball\ a\ r\ |r.\ 0 < r \wedge r < \varepsilon\} \subseteq \{\} \Longrightarrow$ *False*
apply (*drule-tac c*=*a + scaleR* (ε) $((SOME\ i.\ i \in Basis))$ **in** *subsetD*)
by *auto*
then have *nonemp*: $(\bigcap \{S - ball\ a\ r\ |r.\ 0 < r \wedge r < \varepsilon\}) = \{\} \Longrightarrow$ *False*
by *auto*
have *con*: $\bigwedge r.\ r < \varepsilon \Longrightarrow$ *connected* $(S - ball\ a\ r)$
using ε **by** (*force intro*: *connected-diff-ball* [*OF* ‹*connected S*› *- 2*])
have $x \in \bigcup \{S - ball\ a\ r\ |r.\ 0 < r \wedge r < \varepsilon\}$ **if** $x \in S - \{a\}$ **for** *x*

apply (*rule UnionI* [*of* $S - \mathit{ball}\ a\ (\mathit{min}\ \varepsilon\ (\mathit{dist}\ a\ x)\ /\ 2)$])
using *that* ‹$0 < \varepsilon$› **apply** (*simp-all add*:)
apply (*rule-tac x*=*min* ε (*dist a x*) / 2 **in** *exI*)
apply *auto*
done
then have $S - \{a\} = \bigcup \{S - \mathit{ball}\ a\ r \mid r.\ 0 < r \wedge r < \varepsilon\}$
by *auto*
then show *?thesis*
by (*auto intro*: *connected-Union con dest*!: *nonemp*)
next
case *False* **then show** *?thesis*
by (*simp add*: ‹*connected S*›)
qed

corollary *path-connected-open-delete*:
assumes *open S connected S* **and** 2: $2 \le \mathit{DIM}('N{::}\mathit{euclidean\text{-}space})$
shows *path-connected*$(S - \{a{::}'N\})$
by (*simp add*: *assms connected-open-delete connected-open-path-connected open-delete*)

corollary *path-connected-punctured-ball*:
$2 \le \mathit{DIM}('N{::}\mathit{euclidean\text{-}space}) \Longrightarrow \mathit{path\text{-}connected}(\mathit{ball}\ a\ r - \{a{::}'N\})$
by (*simp add*: *path-connected-open-delete*)

lemma *connected-punctured-ball*:
$2 \le \mathit{DIM}('N{::}\mathit{euclidean\text{-}space}) \Longrightarrow \mathit{connected}(\mathit{ball}\ a\ r - \{a{::}'N\})$
by (*simp add*: *connected-open-delete*)

23.4 Relations between components and path components

lemma *open-connected-component*:
fixes $s :: 'a{::}\mathit{real\text{-}normed\text{-}vector\ set}$
shows *open s* $\Longrightarrow$ *open* (*connected-component-set s x*)
apply (*simp add*: *open-contains-ball*, *clarify*)
apply (*rename-tac y*)
apply (*drule-tac x*=*y* **in** *bspec*)
apply (*simp add*: *connected-component-in*, *clarify*)
apply (*rule-tac x*=*e* **in** *exI*)
by (*metis mem-Collect-eq connected-component-eq connected-component-maximal centre-in-ball connected-ball*)

corollary *open-components*:
fixes $s :: 'a{::}\mathit{real\text{-}normed\text{-}vector\ set}$
shows ⟦*open u*; *s* ∈ *components u*⟧ $\Longrightarrow$ *open s*
by (*simp add*: *components-iff*) (*metis open-connected-component*)

lemma *in-closure-connected-component*:
fixes $s :: 'a{::}\mathit{real\text{-}normed\text{-}vector\ set}$
assumes *x*: $x \in s$ **and** *s*: *open s*
shows $x \in \mathit{closure}$ (*connected-component-set s y*) $\longleftrightarrow$ $x \in$ *connected-component-set*

s y
proof −
{ assume *x* ∈ *closure* (*connected-component-set s y*)
moreover have *x* ∈ *connected-component-set s x*
using *x* **by** *simp*
ultimately have *x* ∈ *connected-component-set s y*
using *s* **by** (*meson Compl-disjoint closure-iff-nhds-not-empty connected-component-disjoint disjoint-eq-subset-Compl open-connected-component*)
}
then show *?thesis*
by (*auto simp*: *closure-def*)
qed

23.5 Existence of unbounded components

lemma *cobounded-unbounded-component*:
fixes *s* :: *'a* :: *euclidean-space set*
assumes *bounded* (−*s*)
shows ∃*x*. *x* ∈ *s* ∧ ~ *bounded* (*connected-component-set s x*)
proof −
obtain *i*::*'a* **where** *i*: *i* ∈ *Basis*
using *nonempty-Basis* **by** *blast*
obtain *B* **where** *B*: *B*>*0* −*s* ⊆ *ball 0 B*
using *bounded-subset-ballD* [*OF assms*, *of 0*] **by** *auto*
then have ∗: ⋀*x*. *B* ≤ *norm x* ⟹ *x* ∈ *s*
by (*force simp add*: *ball-def dist-norm*)
have *unbounded-inner*: ~ *bounded* {*x*. *inner i x* ≥ *B*}
apply (*auto simp*: *bounded-def dist-norm*)
apply (*rule-tac x*=*x* + (*max B e* + *1* + |*i* ⋅ *x*|) ∗*R* *i* **in** *exI*)
apply *simp*
using *i*
apply (*auto simp*: *algebra-simps*)
done
have ∗∗: {*x*. *B* ≤ *i* ⋅ *x*} ⊆ *connected-component-set s* (*B* ∗*R* *i*)
apply (*rule connected-component-maximal*)
apply (*auto simp*: *i intro*: *convex-connected convex-halfspace-ge* [*of B*])
apply (*rule* ∗)
apply (*rule order-trans* [*OF - Basis-le-norm* [*OF i*]])
by (*simp add*: *inner-commute*)
have *B* ∗*R* *i* ∈ *s*
by (*rule* ∗) (*simp add*: *norm-Basis* [*OF i*])
then show *?thesis*
apply (*rule-tac x*=*B* ∗*R* *i* **in** *exI*, *clarify*)
apply (*frule bounded-subset* [*of - {x. B ≤ i ⋅ x}*, *OF - ∗∗*])
using *unbounded-inner* **apply** *blast*
done
qed

lemma *cobounded-unique-unbounded-component*:

fixes $s :: 'a :: euclidean\text{-}space\ set$
assumes bs: $bounded\ (-s)$ **and** $2 \leq DIM('a)$
and bo: $\sim bounded(connected\text{-}component\text{-}set\ s\ x)$
$\sim bounded(connected\text{-}component\text{-}set\ s\ y)$
shows $connected\text{-}component\text{-}set\ s\ x = connected\text{-}component\text{-}set\ s\ y$
proof −
obtain $i::'a$ **where** i: $i \in Basis$
using $nonempty\text{-}Basis$ **by** $blast$
obtain B **where** B: $B>0\ -s \subseteq ball\ 0\ B$
using $bounded\text{-}subset\text{-}ballD\ [OF\ bs,\ of\ 0]$ **by** $auto$
then have $*$: $\bigwedge x.\ B \leq norm\ x \Longrightarrow x \in s$
by $(force\ simp\ add$: $ball\text{-}def\ dist\text{-}norm)$
have ccb: $connected\ (-\ ball\ 0\ B :: 'a\ set)$
using $assms$ **by** $(auto\ intro$: $connected\text{-}complement\text{-}bounded\text{-}convex)$
obtain x' **where** x': $connected\text{-}component\ s\ x\ x'\ norm\ x' > B$
using $bo\ [unfolded\ bounded\text{-}def\ dist\text{-}norm,\ simplified,\ rule\text{-}format]$
by $(metis\ diff\text{-}zero\ norm\text{-}minus\text{-}commute\ not\text{-}less)$
obtain y' **where** y': $connected\text{-}component\ s\ y\ y'\ norm\ y' > B$
using $bo\ [unfolded\ bounded\text{-}def\ dist\text{-}norm,\ simplified,\ rule\text{-}format]$
by $(metis\ diff\text{-}zero\ norm\text{-}minus\text{-}commute\ not\text{-}less)$
have $x'y'$: $connected\text{-}component\ s\ x'\ y'$
apply $(simp\ add$: $connected\text{-}component\text{-}def)$
apply $(rule\text{-}tac\ x{=}-\ ball\ 0\ B$ **in** $exI)$
using $x'\ y'$
apply $(auto\ simp$: $ccb\ dist\text{-}norm\ *)$
done
show $?thesis$
apply $(rule\ connected\text{-}component\text{-}eq)$
using $x'\ y'\ x'y'$
by $(metis\ (no\text{-}types,\ lifting)\ connected\text{-}component\text{-}eq\text{-}empty\ connected\text{-}component\text{-}eq\text{-}eq\ connected\text{-}component\text{-}idemp\ connected\text{-}component\text{-}in)$
qed

lemma $cobounded\text{-}unbounded\text{-}components$:
fixes $s :: 'a :: euclidean\text{-}space\ set$
shows $bounded\ (-s) \Longrightarrow \exists c.\ c \in components\ s \wedge \sim bounded\ c$
by $(metis\ cobounded\text{-}unbounded\text{-}component\ components\text{-}def\ imageI)$

lemma $cobounded\text{-}unique\text{-}unbounded\text{-}components$:
fixes $s :: 'a :: euclidean\text{-}space\ set$
shows $[\![bounded\ (-\ s);\ c \in components\ s;\ \neg\ bounded\ c;\ c' \in components\ s;\ \neg\ bounded\ c';\ 2 \leq DIM('a)]\!] \Longrightarrow c' = c$
unfolding $components\text{-}iff$
by $(metis\ cobounded\text{-}unique\text{-}unbounded\text{-}component)$

lemma $cobounded\text{-}has\text{-}bounded\text{-}component$:
fixes $s :: 'a :: euclidean\text{-}space\ set$
shows $[\![bounded\ (-\ s);\ \sim connected\ s;\ 2 \leq DIM('a)]\!] \Longrightarrow \exists c.\ c \in components\ s \wedge bounded\ c$

by (*meson cobounded-unique-unbounded-components connected-eq-connected-components-eq*)

24 The "inside" and "outside" of a set

The inside comprises the points in a bounded connected component of the set's complement. The outside comprises the points in unbounded connected component of the complement.

definition *inside* **where**
inside s ≡ {*x*. (*x* ∉ *s*) ∧ *bounded*(*connected-component-set* (− *s*) *x*)}

definition *outside* **where**
outside s ≡ −*s* ∩ {*x*. ~ *bounded*(*connected-component-set* (− *s*) *x*)}

lemma *outside*: *outside s* = {*x*. ~ *bounded*(*connected-component-set* (− *s*) *x*)}
by (*auto simp*: *outside-def*) (*metis Compl-iff bounded-empty connected-component-eq-empty*)

lemma *inside-no-overlap* [*simp*]: *inside s* ∩ *s* = {}
by (*auto simp*: *inside-def*)

lemma *outside-no-overlap* [*simp*]:
outside s ∩ *s* = {}
by (*auto simp*: *outside-def*)

lemma *inside-inter-outside* [*simp*]: *inside s* ∩ *outside s* = {}
by (*auto simp*: *inside-def outside-def*)

lemma *inside-union-outside* [*simp*]: *inside s* ∪ *outside s* = (− *s*)
by (*auto simp*: *inside-def outside-def*)

lemma *inside-eq-outside*:
inside s = *outside s* ⟷ *s* = *UNIV*
by (*auto simp*: *inside-def outside-def*)

lemma *inside-outside*: *inside s* = (− (*s* ∪ *outside s*))
by (*force simp add*: *inside-def outside*)

lemma *outside-inside*: *outside s* = (− (*s* ∪ *inside s*))
by (*auto simp*: *inside-outside*) (*metis IntI equals0D outside-no-overlap*)

lemma *union-with-inside*: *s* ∪ *inside s* = − *outside s*
by (*auto simp*: *inside-outside*) (*simp add*: *outside-inside*)

lemma *union-with-outside*: *s* ∪ *outside s* = − *inside s*
by (*simp add*: *inside-outside*)

lemma *outside-mono*: *s* ⊆ *t* ⟹ *outside t* ⊆ *outside s*
by (*auto simp*: *outside bounded-subset connected-component-mono*)

lemma *inside-mono*: $s \subseteq t \Longrightarrow$ *inside* $s - t \subseteq$ *inside* t
by (*auto simp*: *inside-def bounded-subset connected-component-mono*)

lemma *segment-bound-lemma*:
fixes u::*real*
assumes $x \geq B$ $y \geq B$ $0 \leq u$ $u \leq 1$
shows $(1 - u) * x + u * y \geq B$
proof −
obtain *dx dy* **where** $dx \geq 0$ $dy \geq 0$ $x = B + dx$ $y = B + dy$
using *assms* **by** *auto* (*metis add.commute diff-add-cancel*)
with ‹$0 \leq u$› ‹$u \leq 1$› **show** *?thesis*
by (*simp add*: *add-increasing2 mult-left-le field-simps*)
qed

lemma *cobounded-outside*:
fixes s :: $'a$ :: *real-normed-vector set*
assumes *bounded s* **shows** *bounded* (− *outside s*)
proof −
obtain B **where** B: $B>0$ $s \subseteq$ *ball 0 B*
using *bounded-subset-ballD* [*OF assms*, *of 0*] **by** *auto*
{ fix x::$'a$ **and** C::*real*
assume *Bno*: $B \leq$ *norm x* **and** C: $0 < C$
have $\exists y$. *connected-component* (− *s*) *x y* ∧ *norm y* > *C*
proof (*cases* $x = 0$)
case *True* **with** *B Bno* **show** *?thesis* **by** *force*
next
case *False* **with** *B C* **show** *?thesis*
apply (*rule-tac* x=((*B*+*C*)/*norm x*) $*_R$ *x* **in** *exI*)
apply (*simp add*: *connected-component-def*)
apply (*rule-tac* x=*closed-segment x* (((*B*+*C*)/*norm x*) $*_R$ *x*) **in** *exI*)
apply *simp*
apply (*rule-tac* y=− *ball 0 B* **in** *order-trans*)
prefer *2* **apply** *force*
apply (*simp add*: *closed-segment-def ball-def dist-norm*, *clarify*)
apply (*simp add*: *real-vector-class.scaleR-add-left* [*symmetric*] *divide-simps*)
using *segment-bound-lemma* [*of B norm x B*+*C*] *Bno*
by (*meson le-add-same-cancel1 less-eq-real-def not-le*)
qed
}
then show *?thesis*
apply (*simp add*: *outside-def assms*)
apply (*rule bounded-subset* [*OF bounded-ball* [*of 0 B*]])
apply (*force simp add*: *dist-norm not-less bounded-pos*)
done
qed

lemma *unbounded-outside*:
fixes s :: $'a$::{*real-normed-vector*, *perfect-space*} *set*
shows *bounded s* $\Longrightarrow$ ~ *bounded*(*outside s*)

using *cobounded-imp-unbounded cobounded-outside* **by** *blast*

lemma *bounded-inside*:
fixes *s* :: *'a*::{*real-normed-vector*, *perfect-space*} *set*
shows *bounded s* ⟹ *bounded*(*inside s*)
by (*simp add*: *bounded-Int cobounded-outside inside-outside*)

lemma *connected-outside*:
fixes *s* :: *'a*::*euclidean-space set*
assumes *bounded s 2* ≤ *DIM*(*'a*)
shows *connected*(*outside s*)
apply (*simp add*: *connected-iff-connected-component*, *clarify*)
apply (*simp add*: *outside*)
apply (*rule-tac s*=*connected-component-set* (− *s*) *x* **in** *connected-component-of-subset*)
apply (*metis* (*no-types*) *assms cobounded-unbounded-component cobounded-unique-unbounded-component connected-component-eq-eq connected-component-idemp double-complement mem-Collect-eq*)
apply *clarify*
apply (*metis connected-component-eq-eq connected-component-in*)
done

lemma *outside-connected-component-lt*:
outside s = {*x*. ∀ *B*. ∃ *y*. *B* < *norm*(*y*) ∧ *connected-component* (− *s*) *x y*}
apply (*auto simp*: *outside bounded-def dist-norm*)
apply (*metis diff-0 norm-minus-cancel not-less*)
by (*metis less-diff-eq norm-minus-commute norm-triangle-ineq2 order.trans pinf*(*6*))

lemma *outside-connected-component-le*:
outside s =
{*x*. ∀ *B*. ∃ *y*. *B* ≤ *norm*(*y*) ∧
connected-component (− *s*) *x y*}
apply (*simp add*: *outside-connected-component-lt*)
apply (*simp add*: *Set.set-eq-iff*)
by (*meson gt-ex leD le-less-linear less-imp-le order.trans*)

lemma *not-outside-connected-component-lt*:
fixes *s* :: *'a*::*euclidean-space set*
assumes *s*: *bounded s* **and** *2* ≤ *DIM*(*'a*)
shows − (*outside s*) = {*x*. ∀ *B*. ∃ *y*. *B* < *norm*(*y*) ∧ ~ (*connected-component* (− *s*) *x y*)}
proof −
obtain *B*::*real* **where** *B*: *0* < *B* **and** *Bno*: ⋀*x*. *x* ∈ *s* ⟹ *norm x* ≤ *B*
using *s* [*simplified bounded-pos*] **by** *auto*
{ fix *y*::*'a* **and** *z*::*'a*
assume *yz*: *B* < *norm z B* < *norm y*
have *connected-component* (− *cball 0 B*) *y z*
apply (*rule connected-componentI* [*OF - subset-refl*])
apply (*rule connected-complement-bounded-convex*)
using *assms yz*
by (*auto simp*: *dist-norm*)

then have *connected-component* $(-s)$ *y z*
apply (*rule connected-component-of-subset*)
apply (*metis Bno Compl-anti-mono mem-cball-0 subset-iff*)
done
} note *cyz* = *this*
show *?thesis*
apply (*auto simp*: *outside*)
apply (*metis Compl-iff bounded-iff cobounded-imp-unbounded mem-Collect-eq not-le*)
apply (*simp add*: *bounded-pos*)
by (*metis B connected-component-trans cyz not-le*)
qed

lemma *not-outside-connected-component-le*:
fixes *s* :: *'a*::*euclidean-space set*
assumes *s*: *bounded s* $2 \leq DIM('a)$
shows $-$ (*outside s*) = $\{x.\ \forall B.\ \exists y.\ B \leq norm(y) \wedge \sim$ (*connected-component* $(-s)\ x\ y)\}$
apply (*auto intro*: *less-imp-le simp*: *not-outside-connected-component-lt* [*OF assms*])
by (*meson gt-ex less-le-trans*)

lemma *inside-connected-component-lt*:
fixes *s* :: *'a*::*euclidean-space set*
assumes *s*: *bounded s* $2 \leq DIM('a)$
shows *inside s* = $\{x.\ (x \notin s) \wedge (\forall B.\ \exists y.\ B < norm(y) \wedge \sim$(*connected-component* $(-s)\ x\ y))\}$
by (*auto simp*: *inside-outside not-outside-connected-component-lt* [*OF assms*])

lemma *inside-connected-component-le*:
fixes *s* :: *'a*::*euclidean-space set*
assumes *s*: *bounded s* $2 \leq DIM('a)$
shows *inside s* = $\{x.\ (x \notin s) \wedge (\forall B.\ \exists y.\ B \leq norm(y) \wedge \sim$(*connected-component* $(-s)\ x\ y))\}$
by (*auto simp*: *inside-outside not-outside-connected-component-le* [*OF assms*])

lemma *inside-subset*:
assumes *connected u* **and** $\sim$*bounded u* **and** $t \cup u = -s$
shows *inside s* $\subseteq t$
apply (*auto simp*: *inside-def*)
by (*metis bounded-subset* [*of connected-component-set* $(-s)$ -] *connected-component-maximal Compl-iff Un-iff assms subsetI*)

lemma *frontier-interiors*: *frontier s* $= -$ *interior*$(s) -$ *interior*$(-s)$
by (*simp add*: *Int-commute frontier-def interior-closure*)

lemma *frontier-interior-subset*: *frontier*(*interior S*) $\subseteq$ *frontier S*
by (*simp add*: *Diff-mono frontier-interiors interior-mono interior-subset*)

lemma *connected-Int-frontier*:

```
    ⟦connected s; s ∩ t ≠ {}; s − t ≠ {}⟧ ⟹ (s ∩ frontier t ≠ {})
  apply (simp add: frontier-interiors connected-openin, safe)
  apply (drule-tac x=s ∩ interior t in spec, safe)
   apply (drule-tac [2] x=s ∩ interior (−t) in spec)
    apply (auto simp: disjoint-eq-subset-Compl dest: interior-subset [THEN subsetD])
  done

lemma frontier-not-empty:
  fixes S :: 'a :: real-normed-vector set
  shows ⟦S ≠ {}; S ≠ UNIV⟧ ⟹ frontier S ≠ {}
    using connected-Int-frontier [of UNIV S] by auto

lemma frontier-eq-empty:
  fixes S :: 'a :: real-normed-vector set
  shows frontier S = {} ⟷ S = {} ∨ S = UNIV
using frontier-UNIV frontier-empty frontier-not-empty by blast

lemma frontier-of-connected-component-subset:
  fixes S :: 'a::real-normed-vector set
  shows frontier(connected-component-set S x) ⊆ frontier S
proof −
  { fix y
    assume y1: y ∈ closure (connected-component-set S x)
       and y2: y ∉ interior (connected-component-set S x)
    have 1: y ∈ closure S
      using y1 closure-mono connected-component-subset by blast
    have z ∈ interior (connected-component-set S x)
          if 0 < e ball y e ⊆ interior S dist y z < e for e z
    proof −
      have ball y e ⊆ connected-component-set S y
        apply (rule connected-component-maximal)
        using that interior-subset mem-ball apply auto
        done
      then show ?thesis
        using y1 apply (simp add: closure-approachable open-contains-ball-eq [OF
open-interior])
      by (metis (no-types, hide-lams) connected-component-eq-eq connected-component-in
subsetD
                    dist-commute mem-Collect-eq mem-ball mem-interior ‹0 < e›
y2)
    qed
    then have 2: y ∉ interior S
      using y2 by (force simp: open-contains-ball-eq [OF open-interior])
    note 1 2
  }
  then show ?thesis by (auto simp: frontier-def)
qed
```

lemma *frontier-Union-subset-closure*:
fixes $F :: 'a$::*real-normed-vector set set*
shows *frontier*$(\bigcup F) \subseteq$ *closure*$(\bigcup t \in F.$ *frontier* $t)$
proof −
have $\exists y{\in}F.\ \exists y{\in}$*frontier* $y.$ *dist* $y\ x < e$
if $T \in F\ y \in T$ *dist* $y\ x < e$
$x \notin$ *interior* $(\bigcup F)\ 0 < e$ **for** $x\ y\ e\ T$
proof (*cases* $x \in T$)
case *True* **with** *that* **show** *?thesis*
by (*metis Diff-iff Sup-upper closure-subset contra-subsetD dist-self frontier-def interior-mono*)
next
case *False*
have *1*: *closed-segment* $x\ y \cap T \neq \{\}$ **using** ‹$y \in T$› **by** *blast*
have *2*: *closed-segment* $x\ y - T \neq \{\}$
using *False* **by** *blast*
obtain c **where** $c \in$ *closed-segment* $x\ y\ c \in$ *frontier* T
using *False connected-Int-frontier* [*OF connected-segment 1 2*] **by** *auto*
then show *?thesis*
proof −
have *norm* $(y - x) < e$
by (*metis dist-norm* ‹*dist* $y\ x < e$›)
moreover have *norm* $(c - x) \leq$ *norm* $(y - x)$
by (*simp add*: ‹$c \in$ *closed-segment* $x\ y$› *segment-bound*(*1*))
ultimately have *norm* $(c - x) < e$
by *linarith*
then show *?thesis*
by (*metis* (*no-types*) ‹$c \in$ *frontier* T› *dist-norm that*(*1*))
qed
qed
then show *?thesis*
by (*fastforce simp add*: *frontier-def closure-approachable*)
qed

lemma *frontier-Union-subset*:
fixes $F :: 'a$::*real-normed-vector set set*
shows *finite* $F \Longrightarrow$ *frontier*$(\bigcup F) \subseteq (\bigcup t \in F.$ *frontier* $t)$
by (*rule order-trans* [*OF frontier-Union-subset-closure*])
(*auto simp*: *closure-subset-eq*)

lemma *connected-component-UNIV* [*simp*]:
fixes $x :: 'a$::*real-normed-vector*
shows *connected-component-set UNIV* $x =$ *UNIV*
using *connected-iff-eq-connected-component-set* [*of UNIV*::$'a$ *set*] *connected-UNIV*
by *auto*

lemma *connected-component-eq-UNIV*:
fixes $x :: 'a$::*real-normed-vector*
shows *connected-component-set* $s\ x =$ *UNIV* $\longleftrightarrow$ $s =$ *UNIV*

using *connected-component-in connected-component-UNIV* **by** *blast*

lemma *components-univ* [*simp*]: *components UNIV* = {*UNIV* :: *'a*::*real-normed-vector set*}
by (*auto simp*: *components-eq-sing-iff*)

lemma *interior-inside-frontier*:
fixes *s* :: *'a*::*real-normed-vector set*
assumes *bounded s*
shows *interior s* ⊆ *inside* (*frontier s*)
proof −
{ fix *x y*
assume *x*: *x* ∈ *interior s* **and** *y*: *y* ∉ *s*
and *cc*: *connected-component* (− *frontier s*) *x y*
have *connected-component-set* (− *frontier s*) *x* ∩ *frontier s* ≠ {}
apply (*rule connected-Int-frontier*, *simp*)
apply (*metis IntI cc connected-component-in connected-component-refl empty-iff interiorE mem-Collect-eq set-rev-mp x*)
using *y cc*
by *blast*
then have *bounded* (*connected-component-set* (− *frontier s*) *x*)
using *connected-component-in* **by** *auto*
}
then show *?thesis*
apply (*auto simp*: *inside-def frontier-def*)
apply (*rule classical*)
apply (*rule bounded-subset* [*OF assms*], *blast*)
done
qed

lemma *inside-empty* [*simp*]: *inside* {} = ({} :: *'a* :: {*real-normed-vector*, *perfect-space*} *set*)
by (*simp add*: *inside-def connected-component-UNIV*)

lemma *outside-empty* [*simp*]: *outside* {} = (*UNIV* :: *'a* :: {*real-normed-vector*, *perfect-space*} *set*)
using *inside-empty inside-union-outside* **by** *blast*

lemma *inside-same-component*:
⟦*connected-component* (− *s*) *x y*; *x* ∈ *inside s*⟧ ⟹ *y* ∈ *inside s*
using *connected-component-eq connected-component-in*
by (*fastforce simp add*: *inside-def*)

lemma *outside-same-component*:
⟦*connected-component* (− *s*) *x y*; *x* ∈ *outside s*⟧ ⟹ *y* ∈ *outside s*
using *connected-component-eq connected-component-in*
by (*fastforce simp add*: *outside-def*)

lemma *convex-in-outside*:

fixes $s :: 'a :: \{real\text{-}normed\text{-}vector, perfect\text{-}space\}$ *set*
assumes *s*: *convex s* **and** *z*: $z \notin s$
shows $z \in outside\ s$
proof (*cases s*={})
case *True* **then show** *?thesis* **by** *simp*
next
case *False* **then obtain** *a* **where** $a \in s$ **by** *blast*
with *z* **have** *zna*: $z \neq a$ **by** *auto*
{ assume *bounded* (*connected-component-set* (− *s*) *z*)
with *bounded-pos-less* **obtain** *B* **where** *B*>*0* **and** *B*: $\bigwedge x.$ *connected-component* (− *s*) *z x* $\Longrightarrow$ *norm x* < *B*
by (*metis mem-Collect-eq*)
def *C* ≡ ((*B* + *1* + *norm z*) / *norm* (*z*−*a*))
have *C* > *0*
using ‹*0* < *B*› *zna* **by** (*simp add*: *C-def divide-simps add-strict-increasing*)
have |*norm* (*z* + *C* $*_R$ (*z*−*a*)) − *norm* (*C* $*_R$ (*z*−*a*))| ≤ *norm z*
by (*metis add-diff-cancel norm-triangle-ineq3*)
moreover have *norm* (*C* $*_R$ (*z*−*a*)) > *norm z* + *B*
using *zna* ‹*B*>*0*› **by** (*simp add*: *C-def le-max-iff-disj field-simps*)
ultimately have *C*: *norm* (*z* + *C* $*_R$ (*z*−*a*)) > *B* **by** *linarith*
{ fix *u*::*real*
assume *u*: *0*≤*u u*≤*1* **and** *ins*: (*1* − *u*) $*_R$ *z* + *u* $*_R$ (*z* + *C* $*_R$ (*z* − *a*)) ∈ *s*
then have *Cpos*: *1* + *u* ∗ *C* > *0*
by (*meson* ‹*0* < *C*› *add-pos-nonneg less-eq-real-def zero-le-mult-iff zero-less-one*)
then have ∗: (*1* / (*1* + *u* ∗ *C*)) $*_R$ *z* + (*u* ∗ *C* / (*1* + *u* ∗ *C*)) $*_R$ *z* = *z*
by (*simp add*: *scaleR-add-left* [*symmetric*] *divide-simps*)
then have *False*
using *convexD-alt* [*OF s* ‹*a* ∈ *s*› *ins*, *of 1*/(*u*∗*C* + *1*)] ‹*C*>*0*› ‹*z* ∉ *s*› *Cpos u*
by (*simp add*: ∗ *divide-simps algebra-simps*)
} note *contra* = *this*
have *connected-component* (− *s*) *z* (*z* + *C* $*_R$ (*z*−*a*))
apply (*rule connected-componentI* [*OF connected-segment* [*of z z* + *C* $*_R$ (*z*−*a*)]])
apply (*simp add*: *closed-segment-def*)
using *contra*
apply *auto*
done
then have *False*
using *zna B* [*of z* + *C* $*_R$ (*z*−*a*)] *C*
by (*auto simp*: *divide-simps max-mult-distrib-right*)
}
then show *?thesis*
by (*auto simp*: *outside-def z*)
qed

lemma *outside-convex*:
fixes $s :: 'a :: \{real\text{-}normed\text{-}vector, perfect\text{-}space\}$ *set*
assumes *convex s*

shows *outside s = − s*
by (*metis ComplD assms convex-in-outside equalityI inside-union-outside subsetI sup.cobounded2*)

lemma *inside-convex*:
fixes *s* :: *'a* :: {*real-normed-vector*, *perfect-space*} *set*
shows *convex s ⟹ inside s = {}*
by (*simp add*: *inside-outside outside-convex*)

lemma *outside-subset-convex*:
fixes *s* :: *'a* :: {*real-normed-vector*, *perfect-space*} *set*
shows ⟦*convex t*; *s ⊆ t*⟧ *⟹ − t ⊆ outside s*
using *outside-convex outside-mono* **by** *blast*

lemma *outside-frontier-misses-closure*:
fixes *s* :: *'a*::*real-normed-vector set*
assumes *bounded s*
shows *outside*(*frontier s*) *⊆ − closure s*
unfolding *outside-inside Lattices.boolean-algebra-class.compl-le-compl-iff*
proof −
{ assume *interior s ⊆ inside* (*frontier s*)
hence *interior s ∪ inside* (*frontier s*) = *inside* (*frontier s*)
by (*simp add*: *subset-Un-eq*)
then have *closure s ⊆ frontier s ∪ inside* (*frontier s*)
using *frontier-def* **by** *auto*
}
then show *closure s ⊆ frontier s ∪ inside* (*frontier s*)
using *interior-inside-frontier* [*OF assms*] **by** *blast*
qed

lemma *outside-frontier-eq-complement-closure*:
fixes *s* :: *'a* :: {*real-normed-vector*, *perfect-space*} *set*
assumes *bounded s convex s*
shows *outside*(*frontier s*) = *− closure s*
by (*metis Diff-subset assms convex-closure frontier-def outside-frontier-misses-closure outside-subset-convex subset-antisym*)

lemma *inside-frontier-eq-interior*:
fixes *s* :: *'a* :: {*real-normed-vector*, *perfect-space*} *set*
shows ⟦*bounded s*; *convex s*⟧ *⟹ inside*(*frontier s*) = *interior s*
apply (*simp add*: *inside-outside outside-frontier-eq-complement-closure*)
using *closure-subset interior-subset*
apply (*auto simp add*: *frontier-def*)
done

lemma *open-inside*:
fixes *s* :: *'a*::*real-normed-vector set*
assumes *closed s*
shows *open* (*inside s*)

proof −
{ fix x **assume** x: $x \in$ *inside* s
have *open* (*connected-component-set* (− s) x)
using *assms open-connected-component* **by** *blast*
then obtain e **where** e: $e>0$ **and** e: $\bigwedge y$. *dist* $y\ x < e \longrightarrow$ *connected-component* (− s) $x\ y$
using *dist-not-less-zero*
apply (*simp add*: *open-dist*)
by (*metis* (*no-types*, *lifting*) *Compl-iff connected-component-refl-eq inside-def mem-Collect-eq* x)
then have $\exists\, e>0$. *ball* $x\ e \subseteq$ *inside* s
by (*metis e dist-commute inside-same-component mem-ball subsetI* x)
}
then show *?thesis*
by (*simp add*: *open-contains-ball*)
qed

lemma *open-outside*:
fixes s :: *'a*::*real-normed-vector set*
assumes *closed* s
shows *open* (*outside* s)
proof −
{ fix x **assume** x: $x \in$ *outside* s
have *open* (*connected-component-set* (− s) x)
using *assms open-connected-component* **by** *blast*
then obtain e **where** e: $e>0$ **and** e: $\bigwedge y$. *dist* $y\ x < e \longrightarrow$ *connected-component* (− s) $x\ y$
using *dist-not-less-zero*
apply (*simp add*: *open-dist*)
by (*metis Int-iff outside-def connected-component-refl-eq* x)
then have $\exists\, e>0$. *ball* $x\ e \subseteq$ *outside* s
by (*metis e dist-commute outside-same-component mem-ball subsetI* x)
}
then show *?thesis*
by (*simp add*: *open-contains-ball*)
qed

lemma *closure-inside-subset*:
fixes s :: *'a*::*real-normed-vector set*
assumes *closed* s
shows *closure*(*inside* s) $\subseteq$ $s \cup$ *inside* s
by (*metis assms closure-minimal open-closed open-outside sup.cobounded2 union-with-inside*)

lemma *frontier-inside-subset*:
fixes s :: *'a*::*real-normed-vector set*
assumes *closed* s
shows *frontier*(*inside* s) $\subseteq$ s
proof −
have *closure* (*inside* s) ∩ − *inside* s = *closure* (*inside* s) − *interior* (*inside* s)

by (*metis* (*no-types*) *Diff-Compl assms closure-closed interior-closure open-closed open-inside*)
moreover have − *inside s* ∩ − *outside s* = *s*
by (*metis* (*no-types*) *compl-sup double-compl inside-union-outside*)
moreover have *closure* (*inside s*) ⊆ − *outside s*
by (*metis* (*no-types*) *assms closure-inside-subset union-with-inside*)
ultimately have *closure* (*inside s*) − *interior* (*inside s*) ⊆ *s*
by *blast*
then show *?thesis*
by (*simp add*: *frontier-def open-inside interior-open*)
qed

lemma *closure-outside-subset*:
fixes *s* :: *'a*::*real-normed-vector set*
assumes *closed s*
shows *closure*(*outside s*) ⊆ *s* ∪ *outside s*
apply (*rule closure-minimal*, *simp*)
by (*metis assms closed-open inside-outside open-inside*)

lemma *frontier-outside-subset*:
fixes *s* :: *'a*::*real-normed-vector set*
assumes *closed s*
shows *frontier*(*outside s*) ⊆ *s*
apply (*simp add*: *frontier-def open-outside interior-open*)
by (*metis Diff-subset-conv assms closure-outside-subset interior-eq open-outside sup.commute*)

lemma *inside-complement-unbounded-connected-empty*:
⟦*connected* (− *s*); ¬ *bounded* (− *s*)⟧ ⟹ *inside s* = {}
apply (*simp add*: *inside-def*)
by (*meson Compl-iff bounded-subset connected-component-maximal order-refl*)

lemma *inside-bounded-complement-connected-empty*:
fixes *s* :: *'a*::{*real-normed-vector*, *perfect-space*} *set*
shows ⟦*connected* (− *s*); *bounded s*⟧ ⟹ *inside s* = {}
by (*metis inside-complement-unbounded-connected-empty cobounded-imp-unbounded*)

lemma *inside-inside*:
assumes *s* ⊆ *inside t*
shows *inside s* − *t* ⊆ *inside t*
unfolding *inside-def*
proof *clarify*
fix *x*
assume *x*: *x* ∉ *t x* ∉ *s* **and** *bo*: *bounded* (*connected-component-set* (− *s*) *x*)
show *bounded* (*connected-component-set* (− *t*) *x*)
proof (*cases s* ∩ *connected-component-set* (− *t*) *x* = {})
case *True* **show** *?thesis*
apply (*rule bounded-subset* [*OF bo*])
apply (*rule connected-component-maximal*)

```
    using x True apply auto
    done
 next
  case False then show ?thesis
    using assms [unfolded inside-def] x
    apply (simp add: disjoint-iff-not-equal, clarify)
    apply (drule subsetD, assumption, auto)
    by (metis (no-types, hide-lams) ComplI connected-component-eq-eq)
 qed
qed

lemma inside-inside-subset: inside(inside s) ⊆ s
 using inside-inside union-with-outside by fastforce

lemma inside-outside-intersect-connected:
    ⟦connected t; inside s ∩ t ≠ {}; outside s ∩ t ≠ {}⟧ ⟹ s ∩ t ≠ {}
 apply (simp add: inside-def outside-def ex-in-conv [symmetric] disjoint-eq-subset-Compl,
clarify)
 by (metis (no-types, hide-lams) Compl-anti-mono connected-component-eq connected-component-maximal
contra-subsetD double-compl)

lemma outside-bounded-nonempty:
 fixes s :: 'a :: {real-normed-vector, perfect-space} set
  assumes bounded s shows outside s ≠ {}
 by (metis (no-types, lifting) Collect-empty-eq Collect-mem-eq Compl-eq-Diff-UNIV
Diff-cancel
          Diff-disjoint UNIV-I assms ball-eq-empty bounded-diff cobounded-outside
convex-ball
                double-complement order-refl outside-convex outside-def)

lemma outside-compact-in-open:
   fixes s :: 'a :: {real-normed-vector,perfect-space} set
   assumes s: compact s and t: open t and s ⊆ t t ≠ {}
     shows outside s ∩ t ≠ {}
proof −
 have outside s ≠ {}
   by (simp add: compact-imp-bounded outside-bounded-nonempty s)
 with assms obtain a b where a: a ∈ outside s and b: b ∈ t by auto
 show ?thesis
 proof (cases a ∈ t)
   case True with a show ?thesis by blast
 next
   case False
     have front: frontier t ⊆ − s
       using ‹s ⊆ t› frontier-disjoint-eq t by auto
     { fix γ
      assume path γ and pimg-sbs: path-image γ − {pathfinish γ} ⊆ interior (−
t)
          and pf: pathfinish γ ∈ frontier t and ps: pathstart γ = a
```

```
      def c ≡ pathfinish γ
      have c ∈ −s unfolding c-def using front pf by blast
      moreover have open (−s) using s compact-imp-closed by blast
      ultimately obtain ε::real where ε > 0 and ε: cball c ε ⊆ −s
        using open-contains-cball[of −s] s by blast
      then obtain d where d ∈ t and d: dist d c < ε
        using closure-approachable [of c t] pf unfolding c-def
        by (metis Diff-iff frontier-def)
      then have d ∈ −s using ε
     using dist-commute by (metis contra-subsetD mem-cball not-le not-less-iff-gr-or-eq)
      have pimg-sbs-cos: path-image γ ⊆ −s
        using pimg-sbs apply (auto simp: path-image-def)
        apply (drule subsetD)
        using ‹c ∈ − s› ‹s ⊆ t› interior-subset apply (auto simp: c-def)
        done
      have closed-segment c d ≤ cball c ε
        apply (simp add: segment-convex-hull)
        apply (rule hull-minimal)
        using ‹ε > 0› d apply (auto simp: dist-commute)
        done
      with ε have closed-segment c d ⊆ −s by blast
      moreover have con-gcd: connected (path-image γ ∪ closed-segment c d)
        by (rule connected-Un) (auto simp: c-def ‹path γ› connected-path-image)
      ultimately have connected-component (− s) a d
        unfolding connected-component-def using pimg-sbs-cos ps by blast
      then have outside s ∩ t ≠ {}
       using outside-same-component [OF - a]  by (metis IntI ‹d ∈ t› empty-iff)
    } note ∗ = this
    have pal: pathstart (linepath a b) ∈ closure (− t)
      by (auto simp: False closure-def)
    show ?thesis
     by (rule exists-path-subpath-to-frontier [OF path-linepath pal - ∗]) (auto simp:
b)
  qed
qed

lemma inside-inside-compact-connected:
    fixes s :: 'a :: euclidean-space set
    assumes s: closed s and t: compact t and connected t s ⊆ inside t
      shows inside s ⊆ inside t
proof (cases inside t = {})
  case True with assms show ?thesis by auto
next
  case False
  consider DIM('a) = 1 | DIM('a) ≥ 2
    using antisym not-less-eq-eq by fastforce
  then show ?thesis
  proof cases
    case 1 then show ?thesis
```

using *connected-convex-1-gen assms False inside-convex* **by** *blast*
next
case *2*
have *coms*: *compact s*
using *assms* **apply** (*simp add*: *s compact-eq-bounded-closed*)
by (*meson bounded-inside bounded-subset compact-imp-bounded*)
then have *bst*: *bounded* (*s* ∪ *t*)
by (*simp add*: *compact-imp-bounded t*)
then obtain *r* **where** *0* < *r* **and** *r*: *s* ∪ *t* ⊆ *ball 0 r*
using *bounded-subset-ballD* **by** *blast*
have *outst*: *outside s* ∩ *outside t* ≠ {}
proof −
have − *ball 0 r* ⊆ *outside s*
apply (*rule outside-subset-convex*)
using *r* **by** *auto*
moreover have − *ball 0 r* ⊆ *outside t*
apply (*rule outside-subset-convex*)
using *r* **by** *auto*
ultimately show *?thesis*
by (*metis Compl-subset-Compl-iff Int-subset-iff bounded-ball inf.orderE outside-bounded-nonempty outside-no-overlap*)
qed
have *s* ∩ *t* = {} **using** *assms*
by (*metis disjoint-iff-not-equal inside-no-overlap subsetCE*)
moreover have *outside s* ∩ *inside t* ≠ {}
by (*meson False assms*(*4*) *compact-eq-bounded-closed coms open-inside outside-compact-in-open t*)
ultimately have *inside s* ∩ *t* = {}
using *inside-outside-intersect-connected* [*OF* ‹*connected t*›, *of s*]
by (*metis 2 compact-eq-bounded-closed coms connected-outside inf.commute inside-outside-intersect-connected outst*)
then show *?thesis*
using *inside-inside* [*OF* ‹*s* ⊆ *inside t*›] **by** *blast*
qed
qed

lemma *connected-with-inside*:
fixes *s* :: *'a* :: *real-normed-vector set*
assumes *s*: *closed s* **and** *cons*: *connected s*
shows *connected*(*s* ∪ *inside s*)
proof (*cases s* ∪ *inside s* = *UNIV*)
case *True* **with** *assms* **show** *?thesis* **by** *auto*
next
case *False*
then obtain *b* **where** *b*: *b* ∉ *s b* ∉ *inside s* **by** *blast*
have ∗: ∃ *y t*. *y* ∈ *s* ∧ *connected t* ∧ *a* ∈ *t* ∧ *y* ∈ *t* ∧ *t* ⊆ (*s* ∪ *inside s*) **if** *a* ∈ (*s* ∪ *inside s*) **for** *a*
using *that* **proof**
assume *a* ∈ *s* **then show** *?thesis*

```
      apply (rule-tac x=a in exI)
      apply (rule-tac x={a} in exI)
      apply (simp add:)
      done
  next
    assume a: a ∈ inside s
    show ?thesis
     apply (rule exists-path-subpath-to-frontier [OF path-linepath [of a b], of inside
s])
      using a apply (simp add: closure-def)
      apply (simp add: b)
      apply (rule-tac x=pathfinish h in exI)
      apply (rule-tac x=path-image h in exI)
      apply (simp add: pathfinish-in-path-image connected-path-image, auto)
      using frontier-inside-subset s apply fastforce
    by (metis (no-types, lifting) frontier-inside-subset insertE insert-Diff interior-eq
open-inside pathfinish-in-path-image s subsetCE)
  qed
  show ?thesis
    apply (simp add: connected-iff-connected-component)
    apply (simp add: connected-component-def)
    apply (clarify dest!: *)
    apply (rename-tac u u' t t')
    apply (rule-tac x=(s ∪ t ∪ t') in exI)
    apply (auto simp: intro!: connected-Un cons)
    done
qed
```

The proof is virtually the same as that above.

```
lemma connected-with-outside:
    fixes s :: 'a :: real-normed-vector set
    assumes s: closed s and cons: connected s
      shows connected(s ∪ outside s)
proof (cases s ∪ outside s = UNIV)
  case True with assms show ?thesis by auto
next
  case False
  then obtain b where b: b ∉ s b ∉ outside s by blast
  have *: ∃ y t. y ∈ s ∧ connected t ∧ a ∈ t ∧ y ∈ t ∧ t ⊆ (s ∪ outside s) if a ∈
(s ∪ outside s) for a
  using that proof
    assume a ∈ s then show ?thesis
      apply (rule-tac x=a in exI)
      apply (rule-tac x={a} in exI)
      apply (simp add:)
      done
  next
    assume a: a ∈ outside s
    show ?thesis
```

apply (*rule exists-path-subpath-to-frontier* [*OF path-linepath* [*of a b*], *of outside s*])
using *a* **apply** (*simp add*: *closure-def*)
apply (*simp add*: *b*)
apply (*rule-tac x=pathfinish h* **in** *exI*)
apply (*rule-tac x=path-image h* **in** *exI*)
apply (*simp add*: *pathfinish-in-path-image connected-path-image*, *auto*)
using *frontier-outside-subset s* **apply** *fastforce*
by (*metis* (*no-types*, *lifting*) *frontier-outside-subset insertE insert-Diff interior-eq open-outside pathfinish-in-path-image s subsetCE*)
qed
show *?thesis*
apply (*simp add*: *connected-iff-connected-component*)
apply (*simp add*: *connected-component-def*)
apply (*clarify dest*!: ∗)
apply (*rename-tac u u′ t t′*)
apply (*rule-tac x=*(*s* ∪ *t* ∪ *t′*) **in** *exI*)
apply (*auto simp*: *intro*!: *connected-Un cons*)
done
qed

lemma *inside-inside-eq-empty* [*simp*]:
fixes *s* :: *′a* :: {*real-normed-vector*, *perfect-space*} *set*
assumes *s*: *closed s* **and** *cons*: *connected s*
shows *inside* (*inside s*) = {}
by (*metis* (*no-types*) *unbounded-outside connected-with-outside* [*OF assms*] *bounded-Un inside-complement-unbounded-connected-empty unbounded-outside union-with-outside*)

lemma *inside-in-components*:
inside s ∈ *components* (− *s*) ⟷ *connected*(*inside s*) ∧ *inside s* ≠ {}
apply (*simp add*: *in-components-maximal*)
apply (*auto intro*: *inside-same-component connected-componentI*)
apply (*metis IntI empty-iff inside-no-overlap*)
done

The proof is virtually the same as that above.

lemma *outside-in-components*:
outside s ∈ *components* (− *s*) ⟷ *connected*(*outside s*) ∧ *outside s* ≠ {}
apply (*simp add*: *in-components-maximal*)
apply (*auto intro*: *outside-same-component connected-componentI*)
apply (*metis IntI empty-iff outside-no-overlap*)
done

lemma *bounded-unique-outside*:
fixes *s* :: *′a* :: *euclidean-space set*
shows ⟦*bounded s*; *DIM*(*′a*) ≥ *2*⟧ ⟹ (*c* ∈ *components* (− *s*) ∧ ~*bounded c* ⟷ *c* = *outside s*)
apply (*rule iffI*)
apply (*metis cobounded-unique-unbounded-components connected-outside double-compl*

outside-bounded-nonempty outside-in-components unbounded-outside)
by (*simp add*: *connected-outside outside-bounded-nonempty outside-in-components unbounded-outside*)

24.1 Condition for an open map's image to contain a ball

lemma *ball-subset-open-map-image*:
fixes $f :: 'a{::}heine\text{-}borel \Rightarrow 'b :: \{real\text{-}normed\text{-}vector, heine\text{-}borel\}$
assumes *contf*: *continuous-on* (*closure S*) *f*
and *oint*: *open* (*f* ' *interior S*)
and *le-no*: $\bigwedge z.\ z \in frontier\ S \Longrightarrow r \leq norm(f\ z - f\ a)$
and *bounded S a* $\in S\ 0 < r$
shows *ball* (*f a*) *r* $\subseteq$ *f* ' *S*
proof (*cases f* ' *S* = *UNIV*)
case *True* **then show** *?thesis* **by** *simp*
next
case *False*
obtain *w* **where** *w*: $w \in frontier\ (f\ `\ S)$
and *dw-le*: $\bigwedge y.\ y \in frontier\ (f\ `\ S) \Longrightarrow norm\ (f\ a - w) \leq norm\ (f\ a - y)$
apply (*rule distance-attains-inf* [*of frontier*(*f* ' *S*) *f a*])
using ‹*a* ∈ *S*› **by** (*auto simp*: *frontier-eq-empty dist-norm False*)
then obtain ξ **where** ξ: $\bigwedge n.\ \xi\ n \in f\ `\ S$ **and** *tendsw*: $\xi \longrightarrow w$
by (*metis Diff-iff frontier-def closure-sequential*)
then have $\bigwedge n.\ \exists x \in S.\ \xi\ n = f\ x$ **by** *force*
then obtain *z* **where** *zs*: $\bigwedge n.\ z\ n \in S$ **and** *fz*: $\bigwedge n.\ \xi\ n = f\ (z\ n)$
by *metis*
then obtain *y K* **where** *y*: *y* ∈ *closure S* **and** *subseq K* **and** *Klim*: $(z \circ K) \longrightarrow y$
using ‹*bounded S*›
apply (*simp add*: *compact-closure* [*symmetric*] *compact-def*)
apply (*drule-tac x*=*z* **in** *spec*)
using *closure-subset* **apply** *force*
done
then have *ftendsw*: $((\lambda n.\ f\ (z\ n)) \circ K) \longrightarrow w$
by (*metis LIMSEQ-subseq-LIMSEQ fun.map-cong0 fz tendsw*)
have *zKs*: $\bigwedge n.\ (z\ o\ K)\ n \in S$ **by** (*simp add*: *zs*)
have $f \circ z = \xi\ \ (\lambda n.\ f\ (z\ n)) = \xi$
using *fz* **by** *auto*
moreover then have $(\xi \circ K) \longrightarrow f\ y$
by (*metis* (*no-types*) *Klim zKs y contf comp-assoc continuous-on-closure-sequentially*)
ultimately have *wy*: $w = f\ y$ **using** *fz LIMSEQ-unique ftendsw* **by** *auto*
have *rle*: $r \leq norm\ (f\ y - f\ a)$
apply (*rule le-no*)
using *w wy oint*
by (*force simp*: *imageI image-mono interiorI interior-subset frontier-def y*)
have $**$: $(\sim(b \cap (-\ S) = \{\}) \wedge \sim(b - (-\ S) = \{\}) \Longrightarrow (b \cap f \neq \{\})) \Longrightarrow (b \cap S \neq \{\}) \Longrightarrow b \cap f = \{\} \Longrightarrow b \subseteq S$ **for** *b f* **and** *S* :: *'b set*

by *blast*
show *?thesis*
apply (*rule* ∗∗)
apply (*rule connected-Int-frontier* [**where** $t = f`S$, *OF connected-ball*])
using ‹$a \in S$› ‹$0 < r$›
apply (*auto simp*: *disjoint-iff-not-equal dist-norm*)
by (*metis dw-le norm-minus-commute not-less order-trans rle wy*)
qed

25 Homotopy of maps p,q : X=¿Y with property P of all intermediate maps.

We often just want to require that it fixes some subset, but to take in the case of a loop homotopy, it's convenient to have a general property P.

definition *homotopic-with* ::
[('*a*::*topological-space* ⇒ '*b*::*topological-space*) ⇒ *bool*, '*a set*, '*b set*, '*a* ⇒ '*b*, '*a* ⇒ '*b*] ⇒ *bool*
where
homotopic-with P X Y p q ≡
(∃ *h*:: *real* × '*a* ⇒ '*b*.
continuous-on ({*0..1*} × *X*) *h* ∧
h ` ({*0..1*} × *X*) ⊆ *Y* ∧
(∀ *x*. *h*(*0*, *x*) = *p x*) ∧
(∀ *x*. *h*(*1*, *x*) = *q x*) ∧
(∀ *t* ∈ {*0..1*}. *P*(λ*x*. *h*(*t*, *x*))))

We often want to just localize the ending function equality or whatever.

proposition *homotopic-with*:
fixes *X* :: '*a*::*topological-space set* **and** *Y* :: '*b*::*topological-space set*
assumes ⋀*h k*. (⋀*x*. *x* ∈ *X* ⟹ *h x* = *k x*) ⟹ (*P h* ⟷ *P k*)
shows *homotopic-with P X Y p q* ⟷
(∃ *h* :: *real* × '*a* ⇒ '*b*.
continuous-on ({*0..1*} × *X*) *h* ∧
h ` ({*0..1*} × *X*) ⊆ *Y* ∧
(∀ *x* ∈ *X*. *h*(*0*,*x*) = *p x*) ∧
(∀ *x* ∈ *X*. *h*(*1*,*x*) = *q x*) ∧
(∀ *t* ∈ {*0..1*}. *P*(λ*x*. *h*(*t*, *x*))))
unfolding *homotopic-with-def*
apply (*rule iffI*, *blast*, *clarify*)
apply (*rule-tac x*=λ(*u*,*v*). *if v* ∈ *X then h*(*u*,*v*) *else if u* = *0 then p v else q v* **in** *exI*)
apply (*auto simp*:)
apply (*force elim*: *continuous-on-eq*)
apply (*drule-tac x*=*t* **in** *bspec*, *force*)
apply (*subst assms*; *simp*)
done

proposition *homotopic-with-eq*:
  assumes *h*: *homotopic-with P X Y f g*
    and *f'*: $\bigwedge x.\ x \in X \Longrightarrow f'\ x = f\ x$
    and *g'*: $\bigwedge x.\ x \in X \Longrightarrow g'\ x = g\ x$
    and *P*: $(\bigwedge h\ k.\ (\bigwedge x.\ x \in X \Longrightarrow h\ x = k\ x) \Longrightarrow (P\ h \longleftrightarrow P\ k))$
  shows *homotopic-with P X Y f' g'*
 using *h* **unfolding** *homotopic-with-def*
 apply *safe*
 apply (*rule-tac x*=$\lambda(u,v)$. *if v* $\in$ *X then h*(u,v) *else if u = 0 then f' v else g' v*
in *exI*)
 apply (*simp add*: *f' g'*, *safe*)
 apply (*fastforce intro*: *continuous-on-eq*)
 apply *fastforce*
 apply (*subst P*; *fastforce*)
 done

proposition *homotopic-with-equal*:
  assumes *contf*: *continuous-on X f* **and** *fXY*: *f* ' *X* $\subseteq$ *Y*
    and *gf*: $\bigwedge x.\ x \in X \Longrightarrow g\ x = f\ x$
    and *P*: *P f P g*
  shows *homotopic-with P X Y f g*
 unfolding *homotopic-with-def*
 apply (*rule-tac x*=$\lambda(u,v)$. *if u = 1 then g v else f v* **in** *exI*)
 using *assms*
 apply (*intro conjI*)
 apply (*rule continuous-on-eq* [**where** *f* = *f o snd*])
 apply (*rule continuous-intros* | *force*)+
 apply *clarify*
 apply (*case-tac t=1*; *force*)
 done

lemma *image-Pair-const*: $(\lambda x.\ (x,\ c))$ ' $A = A \times \{c\}$
 by (*auto simp*:)

lemma *homotopic-constant-maps*:
  homotopic-with $(\lambda x.\ True)$ *s t* $(\lambda x.\ a)$ $(\lambda x.\ b) \longleftrightarrow s = \{\} \vee$ *path-component t a b*
proof (*cases s* $= \{\} \vee$ *t* $= \{\}$)
 case *True* **with** *continuous-on-const* **show** *?thesis*
  by (*auto simp*: *homotopic-with path-component-def*)
next
 case *False*
 then obtain *c* **where** *c* $\in$ *s* **by** *blast*
 show *?thesis*
 proof
  assume *homotopic-with* $(\lambda x.\ True)$ *s t* $(\lambda x.\ a)$ $(\lambda x.\ b)$
  then obtain *h* :: *real* $\times$ *'a* $\Rightarrow$ *'b*
    where *conth*: *continuous-on* $(\{0..1\} \times s)$ *h*

```
        and h: h ' ({0..1} × s) ⊆ t (∀x∈s. h (0, x) = a) (∀x∈s. h (1, x) = b)
      by (auto simp: homotopic-with)
    have continuous-on {0..1} (h ∘ (λt. (t, c)))
      apply (rule continuous-intros conth | simp add: image-Pair-const)+
      apply (blast intro: ‹c ∈ s› continuous-on-subset [OF conth] )
      done
    with ‹c ∈ s› h show s = {} ∨ path-component t a b
      apply (simp-all add: homotopic-with path-component-def)
      apply (auto simp:)
      apply (drule-tac x=h o (λt. (t, c)) in spec)
      apply (auto simp: pathstart-def pathfinish-def path-image-def path-def)
      done
  next
    assume s = {} ∨ path-component t a b
    with False show homotopic-with (λx. True) s t (λx. a) (λx. b)
    apply (clarsimp simp: homotopic-with path-component-def pathstart-def pathfinish-def
path-image-def path-def)
      apply (rule-tac x=g o fst in exI)
      apply (rule conjI continuous-intros | force)+
      done
  qed
qed
```

25.1 Trivial properties.

```
lemma homotopic-with-imp-property: homotopic-with P X Y f g ⟹ P f ∧ P g
  unfolding homotopic-with-def Ball-def
  apply clarify
  apply (frule-tac x=0 in spec)
  apply (drule-tac x=1 in spec)
  apply (auto simp:)
  done

lemma continuous-on-o-Pair: ⟦continuous-on (T × X) h; t ∈ T⟧ ⟹ continuous-on
X (h o Pair t)
  by (fast intro: continuous-intros elim!: continuous-on-subset)

lemma homotopic-with-imp-continuous:
    assumes homotopic-with P X Y f g
    shows continuous-on X f ∧ continuous-on X g
proof −
  obtain h :: real × 'a ⇒ 'b
    where conth: continuous-on ({0..1} × X) h
      and h: ∀x. h (0, x) = f x ∀x. h (1, x) = g x
    using assms by (auto simp: homotopic-with-def)
  have ∗: t ∈ {0..1} ⟹ continuous-on X (h o (λx. (t,x))) for t
    by (rule continuous-intros continuous-on-subset [OF conth] | force)+
  show ?thesis
    using h ∗[of 0] ∗[of 1] by auto
```

qed

proposition *homotopic-with-imp-subset1*:
homotopic-with P X Y f g $\Longrightarrow$ *f ' X* $\subseteq$ *Y*
by (*simp add*: *homotopic-with-def image-subset-iff*) (*metis atLeastAtMost-iff order-refl zero-le-one*)

proposition *homotopic-with-imp-subset2*:
homotopic-with P X Y f g $\Longrightarrow$ *g ' X* $\subseteq$ *Y*
by (*simp add*: *homotopic-with-def image-subset-iff*) (*metis atLeastAtMost-iff order-refl zero-le-one*)

proposition *homotopic-with-mono*:
assumes *hom*: *homotopic-with P X Y f g*
and *Q*: $\bigwedge h$. $[\![$*continuous-on X h*; *image h X* $\subseteq$ *Y* $\wedge$ *P h*$]\!] \Longrightarrow$ *Q h*
shows *homotopic-with Q X Y f g*
using *hom*
apply (*simp add*: *homotopic-with-def*)
apply (*erule ex-forward*)
apply (*force simp*: *intro*!: *Q dest*: *continuous-on-o-Pair*)
done

proposition *homotopic-with-subset-left*:
$[\![$*homotopic-with P X Y f g*; *Z* $\subseteq$ *X*$]\!] \Longrightarrow$ *homotopic-with P Z Y f g*
apply (*simp add*: *homotopic-with-def*)
apply (*fast elim*!: *continuous-on-subset ex-forward*)
done

proposition *homotopic-with-subset-right*:
$[\![$*homotopic-with P X Y f g*; *Y* $\subseteq$ *Z*$]\!] \Longrightarrow$ *homotopic-with P X Z f g*
apply (*simp add*: *homotopic-with-def*)
apply (*fast elim*!: *continuous-on-subset ex-forward*)
done

proposition *homotopic-with-compose-continuous-right*:
$[\![$*homotopic-with* $(\lambda f.\ p\ (f \circ h))$ *X Y f g*; *continuous-on W h*; *h ' W* $\subseteq$ *X*$]\!]$
$\Longrightarrow$ *homotopic-with p W Y* $(f\ o\ h)$ $(g\ o\ h)$
apply (*clarsimp simp add*: *homotopic-with-def*)
apply (*rename-tac k*)
apply (*rule-tac x*=*k o* $(\lambda y.\ (fst\ y,\ h\ (snd\ y)))$ **in** *exI*)
apply (*rule conjI continuous-intros continuous-on-compose* [**where** *f*=*snd* **and** *g*=*h*, *unfolded o-def*] | *simp*)+
apply (*erule continuous-on-subset*)
apply (*fastforce simp*: *o-def*)+
done

proposition *homotopic-compose-continuous-right*:
$[\![$*homotopic-with* $(\lambda f.\ True)$ *X Y f g*; *continuous-on W h*; *h ' W* $\subseteq$ *X*$]\!]$
$\Longrightarrow$ *homotopic-with* $(\lambda f.\ True)$ *W Y* $(f\ o\ h)$ $(g\ o\ h)$

using *homotopic-with-compose-continuous-right* **by** *fastforce*

proposition *homotopic-with-compose-continuous-left*:
⟦*homotopic-with* (λ*f*. *p* (*h* ∘ *f*)) *X Y f g*; *continuous-on Y h*; *h* ‘ *Y* ⊆ *Z*⟧
⟹ *homotopic-with p X Z* (*h o f*) (*h o g*)
apply (*clarsimp simp add*: *homotopic-with-def*)
apply (*rename-tac k*)
apply (*rule-tac x*=*h o k* **in** *exI*)
apply (*rule conjI continuous-intros continuous-on-compose* [**where** *f*=*snd* **and** *g*=*h*, *unfolded o-def*] | *simp*)+
apply (*erule continuous-on-subset*)
apply (*fastforce simp*: *o-def*)+
done

proposition *homotopic-compose-continuous-left*:
⟦*homotopic-with* (λ-. *True*) *X Y f g*;
continuous-on Y h; *h* ‘ *Y* ⊆ *Z*⟧
⟹ *homotopic-with* (λ*f*. *True*) *X Z* (*h o f*) (*h o g*)
using *homotopic-with-compose-continuous-left* **by** *fastforce*

proposition *homotopic-with-Pair*:
assumes *hom*: *homotopic-with p s t f g homotopic-with p′ s′ t′ f′ g′*
and *q*: ⋀*f g*. ⟦*p f*; *p′ g*⟧ ⟹ *q*(λ(*x*,*y*). (*f x*, *g y*))
shows *homotopic-with q* (*s* × *s′*) (*t* × *t′*)
(λ(*x*,*y*). (*f x*, *f′ y*)) (λ(*x*,*y*). (*g x*, *g′ y*))
using *hom*
apply (*clarsimp simp add*: *homotopic-with-def*)
apply (*rename-tac k k′*)
apply (*rule-tac x*=λ*z*. ((*k o* (λ*x*. (*fst x*, *fst* (*snd x*)))) *z*, (*k′ o* (λ*x*. (*fst x*, *snd* (*snd x*)))) *z*) **in** *exI*)
apply (*rule conjI continuous-intros* | *erule continuous-on-subset* | *clarsimp*)+
apply (*auto intro*!: *q* [*unfolded case-prod-unfold*])
done

lemma *homotopic-on-empty* [*simp*]: *homotopic-with* (λ*x*. *True*) {} *t f g*
by (*metis continuous-on-def empty-iff homotopic-with-equal image-subset-iff*)

Homotopy with P is an equivalence relation (on continuous functions mapping X into Y that satisfy P, though this only affects reflexivity.

proposition *homotopic-with-refl*:
homotopic-with P X Y f f ⟷ *continuous-on X f* ∧ *image f X* ⊆ *Y* ∧ *P f*
apply (*rule iffI*)
using *homotopic-with-imp-continuous homotopic-with-imp-property homotopic-with-imp-subset2*
apply *blast*
apply (*simp add*: *homotopic-with-def*)
apply (*rule-tac x*=*f o snd* **in** *exI*)
apply (*rule conjI continuous-intros* | *force*)+
done

lemma *homotopic-with-symD*:
fixes $X :: 'a$::*real-normed-vector set*
assumes *homotopic-with P X Y f g*
shows *homotopic-with P X Y g f*
using *assms*
apply (*clarsimp simp add*: *homotopic-with-def*)
apply (*rename-tac h*)
apply (*rule-tac x*=*h o* (λy. (*1 − fst y*, *snd y*)) **in** *exI*)
apply (*rule conjI continuous-intros* | *erule continuous-on-subset* | *force simp add*: *image-subset-iff*)+
done

proposition *homotopic-with-sym*:
fixes $X :: 'a$::*real-normed-vector set*
shows *homotopic-with P X Y f g* $\longleftrightarrow$ *homotopic-with P X Y g f*
using *homotopic-with-symD* **by** *blast*

lemma *split-01*: $\{0..1::real\} = \{0..1/2\} \cup \{1/2..1\}$
by *force*

lemma *split-01-prod*: $\{0..1::real\} \times X = (\{0..1/2\} \times X) \cup (\{1/2..1\} \times X)$
by *force*

proposition *homotopic-with-trans*:
fixes $X :: 'a$::*real-normed-vector set*
assumes *homotopic-with P X Y f g* **and** *homotopic-with P X Y g h*
shows *homotopic-with P X Y f h*
proof −
have *clo1*: *closedin* (*subtopology euclidean* ($\{0..1/2\} \times X \cup \{1/2..1\} \times X$)) ($\{0..1/2::real\} \times X$)
apply (*simp add*: *closedin-closed split-01-prod* [*symmetric*])
apply (*rule-tac x*=$\{0..1/2\} \times$ *UNIV* **in** *exI*)
apply (*force simp add*: *closed-Times*)
done
have *clo2*: *closedin* (*subtopology euclidean* ($\{0..1/2\} \times X \cup \{1/2..1\} \times X$)) ($\{1/2..1::real\} \times X$)
apply (*simp add*: *closedin-closed split-01-prod* [*symmetric*])
apply (*rule-tac x*=$\{1/2..1\} \times$ *UNIV* **in** *exI*)
apply (*force simp add*: *closed-Times*)
done
{ fix *k1 k2*:: *real* × $'a \Rightarrow 'b$
assume *cont*: *continuous-on* ($\{0..1\} \times X$) *k1 continuous-on* ($\{0..1\} \times X$) *k2*
and *Y*: *k1* ‘ ($\{0..1\} \times X$) ⊆ *Y k2* ‘ ($\{0..1\} \times X$) ⊆ *Y*
and *geq*: ∀*x*. *k1* (*1*, *x*) = *g x* ∀*x*. *k2* (*0*, *x*) = *g x*
and *k12*: ∀*x*. *k1* (*0*, *x*) = *f x* ∀*x*. *k2* (*1*, *x*) = *h x*
and *P*: ∀*t*∈$\{0..1\}$. *P* (λx. *k1* (*t*, *x*)) ∀*t*∈$\{0..1\}$. *P* (λx. *k2* (*t*, *x*))
def $k \equiv \lambda y$. *if fst y* ≤ *1 / 2 then* (*k1 o* (λx. (*2* $*_R$ *fst x*, *snd x*))) *y*
else (*k2 o* (λx. (*2* $*_R$ *fst x* −*1*, *snd x*))) *y*
have *keq*: *k1* (*2* ∗ *u*, *v*) = *k2* (*2* ∗ *u* − *1*, *v*) **if** *u* = *1/2* **for** *u v*

by (*simp add*: *geq that*)
have *continuous-on* ({*0..1*} × *X*) *k*
using *cont*
apply (*simp add*: *split-01-prod k-def*)
apply (*rule clo1 clo2 continuous-on-cases-local continuous-intros* | *erule continuous-on-subset* | *simp add*: *linear image-subset-iff*)+
apply (*force simp add*: *keq*)
done
moreover have *k* ‘ ({*0..1*} × *X*) ⊆ *Y*
using *Y* **by** (*force simp add*: *k-def*)
moreover have ∀ *x*. *k* (*0*, *x*) = *f x*
by (*simp add*: *k-def k12*)
moreover have (∀ *x*. *k* (*1*, *x*) = *h x*)
by (*simp add*: *k-def k12*)
moreover have ∀ *t*∈{*0..1*}. *P* (λ*x*. *k* (*t*, *x*))
using *P*
apply (*clarsimp simp add*: *k-def*)
apply (*case-tac t* ≤ *1/2*)
apply (*auto simp*:)
done
ultimately have ∗: ∃ *k* :: *real* × *'a* ⇒ *'b*.
continuous-on ({*0..1*} × *X*) *k* ∧ *k* ‘ ({*0..1*} × *X*) ⊆ *Y* ∧
(∀ *x*. *k* (*0*, *x*) = *f x*) ∧ (∀ *x*. *k* (*1*, *x*) = *h x*) ∧ (∀ *t*∈{*0..1*}. *P* (λ*x*. *k* (*t*, *x*)))
by *blast*
} note ∗ = *this*
show *?thesis*
using *assms* **by** (*auto intro*: ∗ *simp add*: *homotopic-with-def*)
qed

proposition *homotopic-compose*:
fixes *s* :: *'a*::*real-normed-vector set*
shows ⟦*homotopic-with* (λ*x*. *True*) *s t f f'*; *homotopic-with* (λ*x*. *True*) *t u g g'*⟧
⟹ *homotopic-with* (λ*x*. *True*) *s u* (*g o f*) (*g' o f'*)
apply (*rule homotopic-with-trans* [**where** *g* = *g o f'*])
apply (*metis homotopic-compose-continuous-left homotopic-with-imp-continuous homotopic-with-imp-subset1*)
by (*metis homotopic-compose-continuous-right homotopic-with-imp-continuous homotopic-with-imp-subset2*)

25.2 Homotopy of paths, maintaining the same endpoints.

definition *homotopic-paths* :: [*'a set*, *real* ⇒ *'a*, *real* ⇒ *'a*::*topological-space*] ⇒ *bool*
where
homotopic-paths s p q ≡
homotopic-with (λ*r*. *pathstart r* = *pathstart p* ∧ *pathfinish r* = *pathfinish p*) {*0..1*} *s p q*

lemma *homotopic-paths*:
homotopic-paths s p q $\longleftrightarrow$
$(\exists h.$ *continuous-on* $(\{0..1\} \times \{0..1\})$ *h* $\wedge$
h ' $(\{0..1\} \times \{0..1\}) \subseteq s \wedge$
$(\forall x \in \{0..1\}.$ *h*(0,*x*) = *p x*) $\wedge$
$(\forall x \in \{0..1\}.$ *h*(1,*x*) = *q x*) $\wedge$
$(\forall t \in \{0..1::real\}.$ *pathstart*(*h o Pair t*) = *pathstart p* $\wedge$
pathfinish(*h o Pair t*) = *pathfinish p*))
by (*auto simp*: *homotopic-paths-def homotopic-with pathstart-def pathfinish-def*)

proposition *homotopic-paths-imp-pathstart*:
homotopic-paths s p q $\Longrightarrow$ *pathstart p* = *pathstart q*
by (*metis* (*mono-tags*, *lifting*) *homotopic-paths-def homotopic-with-imp-property*)

proposition *homotopic-paths-imp-pathfinish*:
homotopic-paths s p q $\Longrightarrow$ *pathfinish p* = *pathfinish q*
by (*metis* (*mono-tags*, *lifting*) *homotopic-paths-def homotopic-with-imp-property*)

lemma *homotopic-paths-imp-path*:
homotopic-paths s p q $\Longrightarrow$ *path p* $\wedge$ *path q*
using *homotopic-paths-def homotopic-with-imp-continuous path-def* **by** *blast*

lemma *homotopic-paths-imp-subset*:
homotopic-paths s p q $\Longrightarrow$ *path-image p* $\subseteq$ *s* $\wedge$ *path-image q* $\subseteq$ *s*
by (*simp add*: *homotopic-paths-def homotopic-with-imp-subset1 homotopic-with-imp-subset2 path-image-def*)

proposition *homotopic-paths-refl* [*simp*]: *homotopic-paths s p p* $\longleftrightarrow$ *path p* $\wedge$ *path-image p* $\subseteq$ *s*
by (*simp add*: *homotopic-paths-def homotopic-with-refl path-def path-image-def*)

proposition *homotopic-paths-sym*: *homotopic-paths s p q* $\Longrightarrow$ *homotopic-paths s q p*
by (*metis* (*mono-tags*) *homotopic-paths-def homotopic-paths-imp-pathfinish homotopic-paths-imp-pathstart homotopic-with-symD*)

proposition *homotopic-paths-sym-eq*: *homotopic-paths s p q* $\longleftrightarrow$ *homotopic-paths s q p*
by (*metis homotopic-paths-sym*)

proposition *homotopic-paths-trans* [*trans*]:
$[\![$*homotopic-paths s p q*; *homotopic-paths s q r*$]\!] \Longrightarrow$ *homotopic-paths s p r*
apply (*simp add*: *homotopic-paths-def*)
apply (*rule homotopic-with-trans*, *assumption*)
by (*metis* (*mono-tags*, *lifting*) *homotopic-with-imp-property homotopic-with-mono*)

proposition *homotopic-paths-eq*:
$[\![$*path p*; *path-image p* $\subseteq$ *s*; $\bigwedge t.$ *t* $\in \{0..1\} \Longrightarrow$ *p t* = *q t*$]\!] \Longrightarrow$ *homotopic-paths*

s p q
apply (*simp add*: *homotopic-paths-def*)
apply (*rule homotopic-with-eq*)
apply (*auto simp*: *path-def homotopic-with-refl pathstart-def pathfinish-def path-image-def elim*: *continuous-on-eq*)
done

proposition *homotopic-paths-reparametrize*:
assumes *path p*
and *pips*: *path-image p* $\subseteq$ *s*
and *contf*: *continuous-on* $\{0..1\}$ *f*
and *f01*:*f* ' $\{0..1\} \subseteq \{0..1\}$
and [*simp*]: $f(0) = 0\ f(1) = 1$
and *q*: $\bigwedge t.\ t \in \{0..1\} \Longrightarrow q(t) = p(f\ t)$
shows *homotopic-paths s p q*
proof −
have *contp*: *continuous-on* $\{0..1\}$ *p*
by (*metis* ‹*path p*› *path-def*)
then have *continuous-on* $\{0..1\}$ (*p o f*)
using *contf continuous-on-compose continuous-on-subset f01* **by** *blast*
then have *path q*
by (*simp add*: *path-def*) (*metis q continuous-on-cong*)
have *piqs*: *path-image q* $\subseteq$ *s*
by (*metis* (*no-types*, *hide-lams*) *pips f01 image-subset-iff path-image-def q*)
have *fb0*: $\bigwedge a\ b.\ [\![0 \le a;\ a \le 1;\ 0 \le b;\ b \le 1]\!] \Longrightarrow 0 \le (1 - a) * f\ b + a * b$
using *f01* **by** *force*
have *fb1*: $[\![0 \le a;\ a \le 1;\ 0 \le b;\ b \le 1]\!] \Longrightarrow (1 - a) * f\ b + a * b \le 1$ **for** *a b*
using *f01* [*THEN subsetD*, *of f b*] **by** (*simp add*: *convex-bound-le*)
have *homotopic-paths s q p*
proof (*rule homotopic-paths-trans*)
show *homotopic-paths s q* (*p* ∘ *f*)
using *q* **by** (*force intro*: *homotopic-paths-eq* [*OF* ‹*path q*› *piqs*])
next
show *homotopic-paths s* (*p* ∘ *f*) *p*
apply (*simp add*: *homotopic-paths-def homotopic-with-def*)
apply (*rule-tac x*=*p o* (λ*y*. (1 − (*fst y*)) $*_R$ ((*f o snd*) *y*) + (*fst y*) $*_R$ *snd y*) **in** *exI*)
apply (*rule conjI contf continuous-intros continuous-on-subset* [*OF contp*] | *simp*)+
using *pips* [*unfolded path-image-def*]
apply (*auto simp*: *fb0 fb1 pathstart-def pathfinish-def*)
done
qed
then show *?thesis*
by (*simp add*: *homotopic-paths-sym*)
qed

lemma *homotopic-paths-subset*: $[\![$*homotopic-paths s p q*; *s* $\subseteq$ *t*$]\!] \Longrightarrow$ *homotopic-paths t p q*

using *homotopic-paths-def homotopic-with-subset-right* **by** *blast*

A slightly ad-hoc but useful lemma in constructing homotopies.

lemma *homotopic-join-lemma*:
fixes $q :: [real,real] \Rightarrow 'a::topological\text{-}space$
assumes *p*: *continuous-on* $(\{0..1\} \times \{0..1\})$ $(\lambda y.\ p\ (fst\ y)\ (snd\ y))$
and *q*: *continuous-on* $(\{0..1\} \times \{0..1\})$ $(\lambda y.\ q\ (fst\ y)\ (snd\ y))$
and *pf*: $\bigwedge t.\ t \in \{0..1\} \Longrightarrow pathfinish(p\ t) = pathstart(q\ t)$
shows *continuous-on* $(\{0..1\} \times \{0..1\})$ $(\lambda y.\ (p(fst\ y) +\!\!+\!\!+ q(fst\ y))\ (snd\ y))$
proof −
have *1*: $(\lambda y.\ p\ (fst\ y)\ (2 * snd\ y)) = (\lambda y.\ p\ (fst\ y)\ (snd\ y))\ o\ (\lambda y.\ (fst\ y,\ 2 * snd\ y))$
by (*rule ext*) (*simp*)
have *2*: $(\lambda y.\ q\ (fst\ y)\ (2 * snd\ y - 1)) = (\lambda y.\ q\ (fst\ y)\ (snd\ y))\ o\ (\lambda y.\ (fst\ y,\ 2 * snd\ y - 1))$
by (*rule ext*) (*simp*)
show *?thesis*
apply (*simp add*: *joinpaths-def*)
apply (*rule continuous-on-cases-le*)
apply (*simp-all only*: *1 2*)
apply (*rule continuous-intros continuous-on-subset* [*OF p*] *continuous-on-subset* [*OF q*] | *force*)+
using *pf*
apply (*auto simp*: *mult.commute pathstart-def pathfinish-def*)
done
qed

Congruence properties of homotopy w.r.t. path-combining operations.

lemma *homotopic-paths-reversepath-D*:
assumes *homotopic-paths s p q*
shows *homotopic-paths s* (*reversepath p*) (*reversepath q*)
using *assms*
apply (*simp add*: *homotopic-paths-def homotopic-with-def*, *clarify*)
apply (*rule-tac x*=*h o* $(\lambda x.\ (fst\ x,\ 1 - snd\ x))$ **in** *exI*)
apply (*rule conjI continuous-intros*)+
apply (*auto simp*: *reversepath-def pathstart-def pathfinish-def elim*!: *continuous-on-subset*)
done

proposition *homotopic-paths-reversepath*:
homotopic-paths s (*reversepath p*) (*reversepath q*) $\longleftrightarrow$ *homotopic-paths s p q*
using *homotopic-paths-reversepath-D* **by** *force*

proposition *homotopic-paths-join*:
$[\![$*homotopic-paths s p p'*; *homotopic-paths s q q'*; *pathfinish p* = *pathstart q*$]\!] \Longrightarrow$ *homotopic-paths s* (*p* +++ *q*) (*p'* +++ *q'*)
apply (*simp add*: *homotopic-paths-def homotopic-with-def*, *clarify*)
apply (*rename-tac k1 k2*)
apply (*rule-tac x*=$(\lambda y.\ ((k1\ o\ Pair\ (fst\ y)) +\!\!+\!\!+ (k2\ o\ Pair\ (fst\ y)))\ (snd\ y))$

in *exI*)
apply (*rule conjI continuous-intros homotopic-join-lemma*)+
apply (*auto simp*: *joinpaths-def pathstart-def pathfinish-def path-image-def*)
done

proposition *homotopic-paths-continuous-image*:
⟦*homotopic-paths s f g*; *continuous-on s h*; *h ' s ⊆ t*⟧ ⟹ *homotopic-paths t* (*h o f*) (*h o g*)
unfolding *homotopic-paths-def*
apply (*rule homotopic-with-compose-continuous-left* [*of* - - - *s*])
apply (*auto simp*: *pathstart-def pathfinish-def elim*!: *homotopic-with-mono*)
done

25.3 Group properties for homotopy of paths

So taking equivalence classes under homotopy would give the fundamental group

proposition *homotopic-paths-rid*:
⟦*path p*; *path-image p ⊆ s*⟧ ⟹ *homotopic-paths s* (*p +++ linepath* (*pathfinish p*) (*pathfinish p*)) *p*
apply (*subst homotopic-paths-sym*)
apply (*rule homotopic-paths-reparametrize* [**where** *f* = λ*t*. *if t ≤ 1 / 2 then 2 *$_R$ t else 1*])
apply (*simp-all del*: *le-divide-eq-numeral1*)
apply (*subst split-01*)
apply (*rule continuous-on-cases continuous-intros* | *force simp*: *pathfinish-def joinpaths-def*)+
done

proposition *homotopic-paths-lid*:
⟦*path p*; *path-image p ⊆ s*⟧ ⟹ *homotopic-paths s* (*linepath* (*pathstart p*) (*pathstart p*) *+++ p*) *p*
using *homotopic-paths-rid* [*of reversepath p s*]
by (*metis homotopic-paths-reversepath path-image-reversepath path-reversepath pathfinish-linepath*
pathfinish-reversepath reversepath-joinpaths reversepath-linepath)

proposition *homotopic-paths-assoc*:
⟦*path p*; *path-image p ⊆ s*; *path q*; *path-image q ⊆ s*; *path r*; *path-image r ⊆ s*; *pathfinish p = pathstart q*;
pathfinish q = pathstart r⟧
⟹ *homotopic-paths s* (*p +++* (*q +++ r*)) ((*p +++ q*) *+++ r*)
apply (*subst homotopic-paths-sym*)
apply (*rule homotopic-paths-reparametrize*
[**where** *f* = λ*t*. *if t ≤ 1 / 2 then inverse 2 *$_R$ t*
else if t ≤ 3 / 4 then t − (1 / 4)
*else 2 *$_R$ *t − 1*])
apply (*simp-all del*: *le-divide-eq-numeral1*)
apply (*simp add*: *subset-path-image-join*)

apply (*rule continuous-on-cases-1 continuous-intros*)+
apply (*auto simp*: *joinpaths-def*)
done

proposition *homotopic-paths-rinv*:
assumes *path p path-image p* ⊆ *s*
shows *homotopic-paths s* (*p* +++ *reversepath p*) (*linepath* (*pathstart p*) (*pathstart p*))
proof −
have *continuous-on* ({*0..1*} × {*0..1*}) (λ*x*. (*subpath 0* (*fst x*) *p* +++ *reversepath* (*subpath 0* (*fst x*) *p*)) (*snd x*))
using *assms*
apply (*simp add*: *joinpaths-def subpath-def reversepath-def path-def del*: *le-divide-eq-numeral1*)
apply (*rule continuous-on-cases-le*)
apply (*rule-tac* [*2*] *continuous-on-compose* [*of - - p*, *unfolded o-def*])
apply (*rule continuous-on-compose* [*of - - p*, *unfolded o-def*])
apply (*auto intro*!: *continuous-intros simp del*: *eq-divide-eq-numeral1*)
apply (*force elim*!: *continuous-on-subset simp add*: *mult-le-one*)+
done
then show *?thesis*
using *assms*
apply (*subst homotopic-paths-sym-eq*)
unfolding *homotopic-paths-def homotopic-with-def*
apply (*rule-tac x*=(λ*y*. (*subpath 0* (*fst y*) *p* +++ *reversepath*(*subpath 0* (*fst y*) *p*)) (*snd y*)) **in** *exI*)
apply (*simp add*: *path-defs joinpaths-def subpath-def reversepath-def*)
apply (*force simp*: *mult-le-one*)
done
qed

proposition *homotopic-paths-linv*:
assumes *path p path-image p* ⊆ *s*
shows *homotopic-paths s* (*reversepath p* +++ *p*) (*linepath* (*pathfinish p*) (*pathfinish p*))
using *homotopic-paths-rinv* [*of reversepath p s*] *assms* **by** *simp*

25.4 Homotopy of loops without requiring preservation of endpoints.

definition *homotopic-loops* :: *'a*::*topological-space set* ⇒ (*real* ⇒ *'a*) ⇒ (*real* ⇒ *'a*) ⇒ *bool* **where**
homotopic-loops s p q ≡
homotopic-with (λ*r*. *pathfinish r* = *pathstart r*) {*0..1*} *s p q*

lemma *homotopic-loops*:
homotopic-loops s p q ⟷
(∃*h*. *continuous-on* ({*0..1*::*real*} × {*0..1*}) *h* ∧
image h ({*0..1*} × {*0..1*}) ⊆ *s* ∧
(∀*x* ∈ {*0..1*}. *h*(*0*,*x*) = *p x*) ∧

$(\forall x \in \{0..1\}.\ h(1,x) = q\ x) \wedge$
$(\forall t \in \{0..1\}.\ pathfinish(h\ o\ Pair\ t) = pathstart(h\ o\ Pair\ t)))$
by (*simp add*: *homotopic-loops-def pathstart-def pathfinish-def homotopic-with*)

proposition *homotopic-loops-imp-loop*:
$homotopic\text{-}loops\ s\ p\ q \Longrightarrow pathfinish\ p = pathstart\ p \wedge pathfinish\ q = pathstart\ q$
using *homotopic-with-imp-property homotopic-loops-def* **by** *blast*

proposition *homotopic-loops-imp-path*:
$homotopic\text{-}loops\ s\ p\ q \Longrightarrow path\ p \wedge path\ q$
unfolding *homotopic-loops-def path-def*
using *homotopic-with-imp-continuous* **by** *blast*

proposition *homotopic-loops-imp-subset*:
$homotopic\text{-}loops\ s\ p\ q \Longrightarrow path\text{-}image\ p \subseteq s \wedge path\text{-}image\ q \subseteq s$
unfolding *homotopic-loops-def path-image-def*
by (*metis homotopic-with-imp-subset1 homotopic-with-imp-subset2*)

proposition *homotopic-loops-refl*:
$homotopic\text{-}loops\ s\ p\ p \longleftrightarrow$
$path\ p \wedge path\text{-}image\ p \subseteq s \wedge pathfinish\ p = pathstart\ p$
by (*simp add*: *homotopic-loops-def homotopic-with-refl path-image-def path-def*)

proposition *homotopic-loops-sym*: $homotopic\text{-}loops\ s\ p\ q \Longrightarrow homotopic\text{-}loops\ s\ q\ p$
by (*simp add*: *homotopic-loops-def homotopic-with-sym*)

proposition *homotopic-loops-sym-eq*: $homotopic\text{-}loops\ s\ p\ q \longleftrightarrow homotopic\text{-}loops\ s\ q\ p$
by (*metis homotopic-loops-sym*)

proposition *homotopic-loops-trans*:
$[\![homotopic\text{-}loops\ s\ p\ q;\ homotopic\text{-}loops\ s\ q\ r]\!] \Longrightarrow homotopic\text{-}loops\ s\ p\ r$
unfolding *homotopic-loops-def* **by** (*blast intro*: *homotopic-with-trans*)

proposition *homotopic-loops-subset*:
$[\![homotopic\text{-}loops\ s\ p\ q;\ s \subseteq t]\!] \Longrightarrow homotopic\text{-}loops\ t\ p\ q$
by (*simp add*: *homotopic-loops-def homotopic-with-subset-right*)

proposition *homotopic-loops-eq*:
$[\![path\ p;\ path\text{-}image\ p \subseteq s;\ pathfinish\ p = pathstart\ p;\ \bigwedge t.\ t \in \{0..1\} \Longrightarrow p(t) = q(t)]\!]$
$\Longrightarrow homotopic\text{-}loops\ s\ p\ q$
unfolding *homotopic-loops-def*
apply (*rule homotopic-with-eq*)
apply (*rule homotopic-with-refl* [**where** $f = p$, *THEN iffD2*])
apply (*simp-all add*: *path-image-def path-def pathstart-def pathfinish-def*)
done

proposition *homotopic-loops-continuous-image*:
⟦*homotopic-loops s f g*; *continuous-on s h*; *h ' s ⊆ t*⟧ ⟹ *homotopic-loops t* (*h ∘ f*) (*h ∘ g*)
unfolding *homotopic-loops-def*
apply (*rule homotopic-with-compose-continuous-left*)
apply (*erule homotopic-with-mono*)
by (*simp add*: *pathfinish-def pathstart-def*)

25.5 Relations between the two variants of homotopy

proposition *homotopic-paths-imp-homotopic-loops*:
⟦*homotopic-paths s p q*; *pathfinish p = pathstart p*; *pathfinish q = pathstart p*⟧
⟹ *homotopic-loops s p q*
by (*auto simp*: *homotopic-paths-def homotopic-loops-def intro*: *homotopic-with-mono*)

proposition *homotopic-loops-imp-homotopic-paths-null*:
assumes *homotopic-loops s p* (*linepath a a*)
shows *homotopic-paths s p* (*linepath* (*pathstart p*) (*pathstart p*))
proof −
have *path p* **by** (*metis assms homotopic-loops-imp-path*)
have *ploop*: *pathfinish p = pathstart p* **by** (*metis assms homotopic-loops-imp-loop*)
have *pip*: *path-image p ⊆ s* **by** (*metis assms homotopic-loops-imp-subset*)
obtain *h* **where** *conth*: *continuous-on* ({*0..1*::*real*} × {*0..1*}) *h*
and *hs*: *h ' ({0..1} × {0..1}) ⊆ s*
and [*simp*]: ⋀*x*. *x ∈ {0..1}* ⟹ *h*(*0*,*x*) = *p x*
and [*simp*]: ⋀*x*. *x ∈ {0..1}* ⟹ *h*(*1*,*x*) = *a*
and *ends*: ⋀*t*. *t ∈ {0..1}* ⟹ *pathfinish* (*h ∘ Pair t*) = *pathstart* (*h ∘ Pair t*)
using *assms* **by** (*auto simp*: *homotopic-loops homotopic-with*)
have *conth0*: *path* (λ*u*. *h* (*u*, *0*))
unfolding *path-def*
apply (*rule continuous-on-compose* [*of - - h*, *unfolded o-def*])
apply (*force intro*: *continuous-intros continuous-on-subset* [*OF conth*])+
done
have *pih0*: *path-image* (λ*u*. *h* (*u*, *0*)) ⊆ *s*
using *hs* **by** (*force simp*: *path-image-def*)
have *c1*: *continuous-on* ({*0..1*} × {*0..1*}) (λ*x*. *h* (*fst x* ∗ *snd x*, *0*))
apply (*rule continuous-on-compose* [*of - - h*, *unfolded o-def*])
apply (*force simp*: *mult-le-one intro*: *continuous-intros continuous-on-subset* [*OF conth*])+
done
have *c2*: *continuous-on* ({*0..1*} × {*0..1*}) (λ*x*. *h* (*fst x* − *fst x* ∗ *snd x*, *0*))
apply (*rule continuous-on-compose* [*of - - h*, *unfolded o-def*])
apply (*force simp*: *mult-left-le mult-le-one intro*: *continuous-intros continuous-on-subset* [*OF conth*])+
apply (*rule continuous-on-subset* [*OF conth*])
apply (*auto simp*: *algebra-simps add-increasing2 mult-left-le*)
done

```
 have [simp]: ⋀t. ⟦0 ≤ t ∧ t ≤ 1⟧ ⟹ h (t, 1) = h (t, 0)
   using ends by (simp add: pathfinish-def pathstart-def)
 have adhoc-le: c * 4 ≤ 1 + c * (d * 4) if ¬ d * 4 ≤ 3 0 ≤ c c ≤ 1 for c d::real
 proof −
   have c * 3 ≤ c * (d * 4) using that less-eq-real-def by auto
   with ‹c ≤ 1› show ?thesis by fastforce
 qed
 have ∗: ⋀p x. (path p ∧ path(reversepath p)) ∧
               (path-image p ⊆ s ∧ path-image(reversepath p) ⊆ s) ∧
               (pathfinish p = pathstart(linepath a a +++ reversepath p) ∧
                pathstart(reversepath p) = a) ∧ pathstart p = x
                 ⟹ homotopic-paths s (p +++ linepath a a +++ reversepath p)
(linepath x x)
   by (metis homotopic-paths-lid homotopic-paths-join
         homotopic-paths-trans homotopic-paths-sym homotopic-paths-rinv)
 have 1: homotopic-paths s p (p +++ linepath (pathfinish p) (pathfinish p))
   using ‹path p› homotopic-paths-rid homotopic-paths-sym pip by blast
  moreover have homotopic-paths s (p +++ linepath (pathfinish p) (pathfinish
p))
                                  (linepath (pathstart p) (pathstart p) +++ p +++
linepath (pathfinish p) (pathfinish p))
   apply (rule homotopic-paths-sym)
   using homotopic-paths-lid [of p +++ linepath (pathfinish p) (pathfinish p) s]
  by (metis 1 homotopic-paths-imp-path homotopic-paths-imp-pathstart homotopic-paths-imp-subset)
 moreover have homotopic-paths s (linepath (pathstart p) (pathstart p) +++ p
+++ linepath (pathfinish p) (pathfinish p))
                                ((λu. h (u, 0)) +++ linepath a a +++ reversepath
(λu. h (u, 0)))
   apply (simp add: homotopic-paths-def homotopic-with-def)
   apply (rule-tac x=λy. (subpath 0 (fst y) (λu. h (u, 0)) +++ (λu. h (Pair (fst
y) u)) +++ subpath (fst y) 0 (λu. h (u, 0))) (snd y) in exI)
   apply (simp add: subpath-reversepath)
   apply (intro conjI homotopic-join-lemma)
   using ploop
   apply (simp-all add: path-defs joinpaths-def o-def subpath-def conth c1 c2)
   apply (force simp: algebra-simps mult-le-one mult-left-le intro: hs [THEN sub-
setD] adhoc-le)
   done
  moreover have homotopic-paths s ((λu. h (u, 0)) +++ linepath a a +++
reversepath (λu. h (u, 0)))
                              (linepath (pathstart p) (pathstart p))
   apply (rule ∗)
   apply (simp add: pih0 pathstart-def pathfinish-def conth0)
   apply (simp add: reversepath-def joinpaths-def)
   done
 ultimately show ?thesis
   by (blast intro: homotopic-paths-trans)
qed
```

proposition *homotopic-loops-conjugate*:
fixes s :: *'a*::*real-normed-vector set*
assumes *path p path q* **and** *pip*: *path-image p* $\subseteq$ s **and** *piq*: *path-image q* $\subseteq$ s
and *papp*: *pathfinish p* = *pathstart q* **and** *qloop*: *pathfinish q* = *pathstart q*
shows *homotopic-loops s* (*p* +++ *q* +++ *reversepath p*) *q*
proof −
have *contp*: *continuous-on* {*0..1*} *p* **using** ‹*path p*› [*unfolded path-def*] **by** *blast*
have *contq*: *continuous-on* {*0..1*} *q* **using** ‹*path q*› [*unfolded path-def*] **by** *blast*
have *c1*: *continuous-on* ({*0..1*} × {*0..1*}) (λ*x*. *p* ((*1* − *fst x*) ∗ *snd x* + *fst x*))
apply (*rule continuous-on-compose* [*of - - p*, *unfolded o-def*])
apply (*force simp*: *mult-le-one intro*!: *continuous-intros*)
apply (*rule continuous-on-subset* [*OF contp*])
apply (*auto simp*: *algebra-simps add-increasing2 mult-right-le-one-le sum-le-prod1*)
done
have *c2*: *continuous-on* ({*0..1*} × {*0..1*}) (λ*x*. *p* ((*fst x* − *1*) ∗ *snd x* + *1*))
apply (*rule continuous-on-compose* [*of - - p*, *unfolded o-def*])
apply (*force simp*: *mult-le-one intro*!: *continuous-intros*)
apply (*rule continuous-on-subset* [*OF contp*])
apply (*auto simp*: *algebra-simps add-increasing2 mult-left-le-one-le*)
done
have *ps1*: ⋀*a b*. ⟦*b* ∗ *2* ≤ *1*; *0* ≤ *b*; *0* ≤ *a*; *a* ≤ *1*⟧ ⟹ *p* ((*1* − *a*) ∗ (*2* ∗ *b*) + *a*) ∈ *s*
using *sum-le-prod1*
by (*force simp*: *algebra-simps add-increasing2 mult-left-le intro*: *pip* [*unfolded path-image-def*, *THEN subsetD*])
have *ps2*: ⋀*a b*. ⟦¬ *4* ∗ *b* ≤ *3*; *b* ≤ *1*; *0* ≤ *a*; *a* ≤ *1*⟧ ⟹ *p* ((*a* − *1*) ∗ (*4* ∗ *b* − *3*) + *1*) ∈ *s*
apply (*rule pip* [*unfolded path-image-def*, *THEN subsetD*])
apply (*rule image-eqI*, *blast*)
apply (*simp add*: *algebra-simps*)
by (*metis add-mono-thms-linordered-semiring*(*1*) *affine-ineq linear mult.commute mult.left-neutral mult-right-mono not-le*
add.commute zero-le-numeral)
have *qs*: ⋀*a b*. ⟦*4* ∗ *b* ≤ *3*; ¬ *b* ∗ *2* ≤ *1*⟧ ⟹ *q* (*4* ∗ *b* − *2*) ∈ *s*
using *path-image-def piq* **by** *fastforce*
have *homotopic-loops s* (*p* +++ *q* +++ *reversepath p*)
(*linepath* (*pathstart q*) (*pathstart q*) +++ *q* +++ *linepath* (*pathstart q*) (*pathstart q*))
apply (*simp add*: *homotopic-loops-def homotopic-with-def*)
apply (*rule-tac x*=(λ*y*. (*subpath* (*fst y*) *1 p* +++ *q* +++ *subpath 1* (*fst y*) *p*) (*snd y*)) **in** *exI*)
apply (*simp add*: *subpath-refl subpath-reversepath*)
apply (*intro conjI homotopic-join-lemma*)
using *papp qloop*
apply (*simp-all add*: *path-defs joinpaths-def o-def subpath-def c1 c2*)
apply (*force simp*: *contq intro*: *continuous-on-compose* [*of - - q*, *unfolded o-def*] *continuous-on-id continuous-on-snd*)
apply (*auto simp*: *ps1 ps2 qs*)
done

moreover have *homotopic-loops s* (*linepath* (*pathstart q*) (*pathstart q*) +++ *q* +++ *linepath* (*pathstart q*) (*pathstart q*)) *q*
proof −
have *homotopic-paths s* (*linepath* (*pathfinish q*) (*pathfinish q*) +++ *q*) *q*
using ‹*path q*› *homotopic-paths-lid qloop piq* **by** *auto*
hence *1*: ⋀*f*. *homotopic-paths s f q* ∨ ¬ *homotopic-paths s f* (*linepath* (*pathfinish q*) (*pathfinish q*) +++ *q*)
using *homotopic-paths-trans* **by** *blast*
hence *homotopic-paths s* (*linepath* (*pathfinish q*) (*pathfinish q*) +++ *q* +++ *linepath* (*pathfinish q*) (*pathfinish q*)) *q*
proof −
have *homotopic-paths s* (*q* +++ *linepath* (*pathfinish q*) (*pathfinish q*)) *q*
by (*simp add*: ‹*path q*› *homotopic-paths-rid piq*)
thus *?thesis*
by (*metis* (*no-types*) *1* ‹*path q*› *homotopic-paths-join homotopic-paths-rinv homotopic-paths-sym*
homotopic-paths-trans qloop pathfinish-linepath piq)
qed
thus *?thesis*
by (*metis* (*no-types*) *qloop homotopic-loops-sym homotopic-paths-imp-homotopic-loops homotopic-paths-imp-pathfinish homotopic-paths-sym*)
qed
ultimately show *?thesis*
by (*blast intro*: *homotopic-loops-trans*)
qed

25.6 Homotopy of "nearby" function, paths and loops.

lemma *homotopic-with-linear*:
fixes *f g* :: - ⇒ *'b*::*real-normed-vector*
assumes *contf*: *continuous-on s f*
and *contg*:*continuous-on s g*
and *sub*: ⋀*x*. *x* ∈ *s* ⟹ *closed-segment* (*f x*) (*g x*) ⊆ *t*
shows *homotopic-with* (λ*z*. *True*) *s t f g*
apply (*simp add*: *homotopic-with-def*)
apply (*rule-tac x*=λ*y*. ((*1* − (*fst y*)) ∗$_R$ *f*(*snd y*) + (*fst y*) ∗$_R$ *g*(*snd y*)) **in** *exI*)
apply (*intro conjI*)
apply (*rule subset-refl continuous-intros continuous-on-subset* [*OF contf*] *continuous-on-compose2* [**where** *g*=*f*]
continuous-on-subset [*OF contg*] *continuous-on-compose2* [**where** *g*=*g*]| *simp*)+
using *sub closed-segment-def* **apply** *fastforce*+
done

lemma *homotopic-paths-linear*:
fixes *g h* :: *real* ⇒ *'a*::*real-normed-vector*
assumes *path g path h pathstart h* = *pathstart g pathfinish h* = *pathfinish g*
⋀*t x*. *t* ∈ {*0..1*} ⟹ *closed-segment* (*g t*) (*h t*) ⊆ *s*
shows *homotopic-paths s g h*

using *assms*
unfolding *path-def*
apply (*simp add*: *closed-segment-def pathstart-def pathfinish-def homotopic-paths-def homotopic-with-def*)
apply (*rule-tac x=λy.* ((*1* − (*fst y*)) *∗R g*(*snd y*) + (*fst y*) *∗R h*(*snd y*)) **in** *exI*)
apply (*intro conjI subsetI continuous-intros*)
apply (*fastforce intro*: *continuous-intros continuous-on-compose2* [**where** *g=g*] *continuous-on-compose2* [**where** *g=h*])+
done

lemma *homotopic-loops-linear*:
fixes *g h* :: *real* ⇒ *'a*::*real-normed-vector*
assumes *path g path h pathfinish g* = *pathstart g pathfinish h* = *pathstart h*
⋀*t x*. *t* ∈ {*0..1*} ⟹ *closed-segment* (*g t*) (*h t*) ⊆ *s*
shows *homotopic-loops s g h*
using *assms*
unfolding *path-def*
apply (*simp add*: *pathstart-def pathfinish-def homotopic-loops-def homotopic-with-def*)
apply (*rule-tac x=λy.* ((*1* − (*fst y*)) *∗R g*(*snd y*) + (*fst y*) *∗R h*(*snd y*)) **in** *exI*)
apply (*auto intro*!: *continuous-intros intro*: *continuous-on-compose2* [**where** *g=g*] *continuous-on-compose2* [**where** *g=h*])
apply (*force simp*: *closed-segment-def*)
done

lemma *homotopic-paths-nearby-explicit*:
assumes *path g path h pathstart h* = *pathstart g pathfinish h* = *pathfinish g*
and *no*: ⋀*t x*. ⟦*t* ∈ {*0..1*}; *x* ∉ *s*⟧ ⟹ *norm*(*h t* − *g t*) < *norm*(*g t* − *x*)
shows *homotopic-paths s g h*
apply (*rule homotopic-paths-linear* [*OF assms*(*1−4*)])
by (*metis no segment-bound*(*1*) *subsetI norm-minus-commute not-le*)

lemma *homotopic-loops-nearby-explicit*:
assumes *path g path h pathfinish g* = *pathstart g pathfinish h* = *pathstart h*
and *no*: ⋀*t x*. ⟦*t* ∈ {*0..1*}; *x* ∉ *s*⟧ ⟹ *norm*(*h t* − *g t*) < *norm*(*g t* − *x*)
shows *homotopic-loops s g h*
apply (*rule homotopic-loops-linear* [*OF assms*(*1−4*)])
by (*metis no segment-bound*(*1*) *subsetI norm-minus-commute not-le*)

lemma *homotopic-nearby-paths*:
fixes *g h* :: *real* ⇒ *'a*::*euclidean-space*
assumes *path g open s path-image g* ⊆ *s*
shows ∃ *e*. *0* < *e* ∧
(∀ *h*. *path h* ∧
pathstart h = *pathstart g* ∧ *pathfinish h* = *pathfinish g* ∧
(∀ *t* ∈ {*0..1*}. *norm*(*h t* − *g t*) < *e*) ⟶ *homotopic-paths s g h*)
proof −
obtain *e* **where** *e* > *0* **and** *e*: ⋀*x y*. *x* ∈ *path-image g* ⟹ *y* ∈ − *s* ⟹ *e* ≤ *dist x y*
using *separate-compact-closed* [*of path-image g* − *s*] *assms* **by** *force*

show *?thesis*
apply (*intro exI conjI*)
using *e* [*unfolded dist-norm*]
apply (*auto simp*: *intro*!: *homotopic-paths-nearby-explicit assms* ‹*e* > *0*›)
by (*metis atLeastAtMost-iff imageI le-less-trans not-le path-image-def*)
qed

lemma *homotopic-nearby-loops*:
fixes *g h* :: *real* ⇒ *'a*::*euclidean-space*
assumes *path g open s path-image g* ⊆ *s pathfinish g* = *pathstart g*
shows ∃*e*. *0* < *e* ∧
(∀*h*. *path h* ∧ *pathfinish h* = *pathstart h* ∧
(∀*t* ∈ {*0*..*1*}. *norm*(*h t* − *g t*) < *e*) ⟶ *homotopic-loops s g h*)
proof −
obtain *e* **where** *e* > *0* **and** *e*: ⋀*x y*. *x* ∈ *path-image g* ⟹ *y* ∈ − *s* ⟹ *e* ≤ *dist x y*
using *separate-compact-closed* [*of path-image g* −*s*] *assms* **by** *force*
show *?thesis*
apply (*intro exI conjI*)
using *e* [*unfolded dist-norm*]
apply (*auto simp*: *intro*!: *homotopic-loops-nearby-explicit assms* ‹*e* > *0*›)
by (*metis atLeastAtMost-iff imageI le-less-trans not-le path-image-def*)
qed

25.7 Homotopy and subpaths

lemma *homotopic-join-subpaths1*:
assumes *path g* **and** *pag*: *path-image g* ⊆ *s*
and *u*: *u* ∈ {*0*..*1*} **and** *v*: *v* ∈ {*0*..*1*} **and** *w*: *w* ∈ {*0*..*1*} *u* ≤ *v v* ≤ *w*
shows *homotopic-paths s* (*subpath u v g* +++ *subpath v w g*) (*subpath u w g*)
proof −
have *1*: *t* ∗ *2* ≤ *1* ⟹ *u* + *t* ∗ (*v* ∗ *2*) ≤ *v* + *t* ∗ (*u* ∗ *2*) **for** *t*
using *affine-ineq* ‹*u* ≤ *v*› **by** *fastforce*
have *2*: *t* ∗ *2* > *1* ⟹ *u* + (*2*∗*t* − *1*) ∗ *v* ≤ *v* + (*2*∗*t* − *1*) ∗ *w* **for** *t*
by (*metis add-mono-thms-linordered-semiring*(*1*) *diff-gt-0-iff-gt less-eq-real-def mult.commute mult-right-mono* ‹*u* ≤ *v*› ‹*v* ≤ *w*›)
have *t2*: ⋀*t*::*real*. *t*∗*2* = *1* ⟹ *t* = *1*/*2* **by** *auto*
show *?thesis*
apply (*rule homotopic-paths-subset* [*OF - pag*])
using *assms*
apply (*cases w* = *u*)
using *homotopic-paths-rinv* [*of subpath u v g path-image g*]
apply (*force simp*: *closed-segment-eq-real-ivl image-mono path-image-def subpath-refl*)
apply (*rule homotopic-paths-sym*)
apply (*rule homotopic-paths-reparametrize*
[**where** *f* = λ*t*. *if t* ≤ *1* / *2*
then inverse((*w* − *u*)) ∗*R* (*2* ∗ (*v* − *u*)) ∗*R t*
else inverse((*w* − *u*)) ∗*R* ((*v* − *u*) + (*w* − *v*) ∗*R* (*2* ∗*R t* − *1*))])

using ‹*path g*› *path-subpath u w* **apply** *blast*
using ‹*path g*› *path-image-subpath-subset u w*(*1*) **apply** *blast*
apply *simp-all*
apply (*subst split-01*)
apply (*rule continuous-on-cases continuous-intros* | *force simp*: *pathfinish-def joinpaths-def*)+
apply (*simp-all add*: *field-simps not-le*)
apply (*force dest*!: *t2*)
apply (*force simp*: *algebra-simps mult-left-mono affine-ineq dest*!: *1 2*)
apply (*simp add*: *joinpaths-def subpath-def*)
apply (*force simp*: *algebra-simps*)
done
qed

lemma *homotopic-join-subpaths2*:
assumes *homotopic-paths s* (*subpath u v g* +++ *subpath v w g*) (*subpath u w g*)
shows *homotopic-paths s* (*subpath w v g* +++ *subpath v u g*) (*subpath w u g*)
by (*metis assms homotopic-paths-reversepath-D pathfinish-subpath pathstart-subpath reversepath-joinpaths reversepath-subpath*)

lemma *homotopic-join-subpaths3*:
assumes *hom*: *homotopic-paths s* (*subpath u v g* +++ *subpath v w g*) (*subpath u w g*)
and *path g* **and** *pag*: *path-image g* ⊆ *s*
and *u*: *u* ∈ {*0..1*} **and** *v*: *v* ∈ {*0..1*} **and** *w*: *w* ∈ {*0..1*}
shows *homotopic-paths s* (*subpath v w g* +++ *subpath w u g*) (*subpath v u g*)
proof −
have *homotopic-paths s* (*subpath u w g* +++ *subpath w v g*) ((*subpath u v g* +++ *subpath v w g*) +++ *subpath w v g*)
apply (*rule homotopic-paths-join*)
using *hom homotopic-paths-sym-eq* **apply** *blast*
apply (*metis* ‹*path g*› *homotopic-paths-eq pag path-image-subpath-subset path-subpath subset-trans v w*)
apply (*simp add*:)
done
also have *homotopic-paths s* ((*subpath u v g* +++ *subpath v w g*) +++ *subpath w v g*) (*subpath u v g* +++ *subpath v w g* +++ *subpath w v g*)
apply (*rule homotopic-paths-sym* [*OF homotopic-paths-assoc*])
using *assms* **by** (*simp-all add*: *path-image-subpath-subset* [*THEN order-trans*])
also have *homotopic-paths s* (*subpath u v g* +++ *subpath v w g* +++ *subpath w v g*)
(*subpath u v g* +++ *linepath* (*pathfinish* (*subpath u v g*)) (*pathfinish* (*subpath u v g*)))
apply (*rule homotopic-paths-join*)
apply (*metis* ‹*path g*› *homotopic-paths-eq order.trans pag path-image-subpath-subset path-subpath u v*)
apply (*metis* (*no-types*, *lifting*) ‹*path g*› *homotopic-paths-linv order-trans pag path-image-subpath-subset path-subpath pathfinish-subpath reversepath-subpath v w*)
apply (*simp add*:)

done
also have *homotopic-paths s* (*subpath u v g +++ linepath* (*pathfinish* (*subpath u v g*)) (*pathfinish* (*subpath u v g*))) (*subpath u v g*)
apply (*rule homotopic-paths-rid*)
using ‹*path g*› *path-subpath u v* **apply** *blast*
apply (*meson* ‹*path g*› *order.trans pag path-image-subpath-subset u v*)
done
finally have *homotopic-paths s* (*subpath u w g +++ subpath w v g*) (*subpath u v g*) **.**
then show *?thesis*
using *homotopic-join-subpaths2* **by** *blast*
qed

proposition *homotopic-join-subpaths*:
⟦*path g*; *path-image g* ⊆ *s*; *u* ∈ {*0..1*}; *v* ∈ {*0..1*}; *w* ∈ {*0..1*}⟧
⟹ *homotopic-paths s* (*subpath u v g +++ subpath v w g*) (*subpath u w g*)
apply (*rule le-cases3* [*of u v w*])
using *homotopic-join-subpaths1 homotopic-join-subpaths2 homotopic-join-subpaths3*
by *metis+*

Relating homotopy of trivial loops to path-connectedness.

lemma *path-component-imp-homotopic-points*:
path-component S a b ⟹ *homotopic-loops S* (*linepath a a*) (*linepath b b*)
apply (*simp add*: *path-component-def homotopic-loops-def homotopic-with-def pathstart-def pathfinish-def path-image-def path-def*, *clarify*)
apply (*rule-tac x=g o fst* **in** *exI*)
apply (*intro conjI continuous-intros continuous-on-compose*)+
apply (*auto elim*!: *continuous-on-subset*)
done

lemma *homotopic-loops-imp-path-component-value*:
⟦*homotopic-loops S p q*; *0* ≤ *t*; *t* ≤ *1*⟧
⟹ *path-component S* (*p t*) (*q t*)
apply (*simp add*: *path-component-def homotopic-loops-def homotopic-with-def pathstart-def pathfinish-def path-image-def path-def*, *clarify*)
apply (*rule-tac x=h o* (λ*u*. (*u*, *t*)) **in** *exI*)
apply (*intro conjI continuous-intros continuous-on-compose*)+
apply (*auto elim*!: *continuous-on-subset*)
done

lemma *homotopic-points-eq-path-component*:
homotopic-loops S (*linepath a a*) (*linepath b b*) ⟷
path-component S a b
by (*auto simp*: *path-component-imp-homotopic-points*
dest: *homotopic-loops-imp-path-component-value* [**where** *t=1*])

lemma *path-connected-eq-homotopic-points*:
path-connected S ⟷
(∀ *a b*. *a* ∈ *S* ∧ *b* ∈ *S* ⟶ *homotopic-loops S* (*linepath a a*) (*linepath b b*))

by (*auto simp*: *path-connected-def path-component-def homotopic-points-eq-path-component*)

25.8 Simply connected sets

defined as "all loops are homotopic (as loops)

definition *simply-connected* **where**
simply-connected S ≡
∀ *p q*. *path p* ∧ *pathfinish p* = *pathstart p* ∧ *path-image p* ⊆ *S* ∧
path q ∧ *pathfinish q* = *pathstart q* ∧ *path-image q* ⊆ *S*
⟶ *homotopic-loops S p q*

lemma *simply-connected-empty* [*iff*]: *simply-connected* {}
by (*simp add*: *simply-connected-def*)

lemma *simply-connected-imp-path-connected*:
fixes *S* :: -::*real-normed-vector set*
shows *simply-connected S* ⟹ *path-connected S*
by (*simp add*: *simply-connected-def path-connected-eq-homotopic-points*)

lemma *simply-connected-imp-connected*:
fixes *S* :: -::*real-normed-vector set*
shows *simply-connected S* ⟹ *connected S*
by (*simp add*: *path-connected-imp-connected simply-connected-imp-path-connected*)

lemma *simply-connected-eq-contractible-loop-any*:
fixes *S* :: -::*real-normed-vector set*
shows *simply-connected S* ⟷
(∀ *p a*. *path p* ∧ *path-image p* ⊆ *S* ∧
pathfinish p = *pathstart p* ∧ *a* ∈ *S*
⟶ *homotopic-loops S p* (*linepath a a*))
apply (*simp add*: *simply-connected-def*)
apply (*rule iffI*, *force*, *clarify*)
apply (*rule-tac q* = *linepath* (*pathstart p*) (*pathstart p*) **in** *homotopic-loops-trans*)
apply (*fastforce simp add*:)
using *homotopic-loops-sym* **apply** *blast*
done

lemma *simply-connected-eq-contractible-loop-some*:
fixes *S* :: -::*real-normed-vector set*
shows *simply-connected S* ⟷
path-connected S ∧
(∀ *p*. *path p* ∧ *path-image p* ⊆ *S* ∧ *pathfinish p* = *pathstart p*
⟶ (∃ *a*. *a* ∈ *S* ∧ *homotopic-loops S p* (*linepath a a*)))
apply (*rule iffI*)
apply (*fastforce simp*: *simply-connected-imp-path-connected simply-connected-eq-contractible-loop-any*)
apply (*clarsimp simp add*: *simply-connected-eq-contractible-loop-any*)
apply (*drule-tac x*=*p* **in** *spec*)
using *homotopic-loops-trans path-connected-eq-homotopic-points*
apply *blast*

done

lemma *simply-connected-eq-contractible-loop-all*:
fixes S :: -::*real-normed-vector set*
shows *simply-connected* $S \longleftrightarrow$
$S = \{\} \vee$
$(\exists\, a \in S.\ \forall\, p.\ path\ p \wedge path\text{-}image\ p \subseteq S \wedge pathfinish\ p = pathstart\ p$
$\longrightarrow homotopic\text{-}loops\ S\ p\ (linepath\ a\ a))$
(**is** *?lhs* = *?rhs*)
proof (*cases* $S = \{\}$)
case *True* **then show** *?thesis* **by** *force*
next
case *False*
then obtain a **where** $a \in S$ **by** *blast*
show *?thesis*
proof
assume *simply-connected S*
then show *?rhs*
using ‹$a \in S$› ‹*simply-connected S*› *simply-connected-eq-contractible-loop-any*
by *blast*
next
assume *?rhs*
then show *simply-connected S*
apply (*simp add*: *simply-connected-eq-contractible-loop-any False*)
by (*meson homotopic-loops-refl homotopic-loops-sym homotopic-loops-trans*
path-component-imp-homotopic-points path-component-refl)
qed
qed

lemma *simply-connected-eq-contractible-path*:
fixes S :: -::*real-normed-vector set*
shows *simply-connected* $S \longleftrightarrow$
path-connected $S \wedge$
$(\forall\, p.\ path\ p \wedge path\text{-}image\ p \subseteq S \wedge pathfinish\ p = pathstart\ p$
$\longrightarrow homotopic\text{-}paths\ S\ p\ (linepath\ (pathstart\ p)\ (pathstart\ p)))$
apply (*rule iffI*)
apply (*simp add*: *simply-connected-imp-path-connected*)
apply (*metis simply-connected-eq-contractible-loop-some homotopic-loops-imp-homotopic-paths-null*)
by (*meson homotopic-paths-imp-homotopic-loops pathfinish-linepath pathstart-in-path-image*

simply-connected-eq-contractible-loop-some subset-iff)

lemma *simply-connected-eq-homotopic-paths*:
fixes S :: -::*real-normed-vector set*
shows *simply-connected* $S \longleftrightarrow$
path-connected $S \wedge$
$(\forall\, p\ q.\ path\ p \wedge path\text{-}image\ p \subseteq S \wedge$
$path\ q \wedge path\text{-}image\ q \subseteq S \wedge$
$pathstart\ q = pathstart\ p \wedge pathfinish\ q = pathfinish\ p$

```
            ⟶ homotopic-paths S p q)
       (is ?lhs = ?rhs)
proof
  assume ?lhs
  then have pc: path-connected S
       and ∗: ⋀p. ⟦path p; path-image p ⊆ S;
                   pathfinish p = pathstart p⟧
                  ⟹ homotopic-paths S p (linepath (pathstart p) (pathstart p))
    by (auto simp: simply-connected-eq-contractible-path)
  have homotopic-paths S p q
       if path p path-image p ⊆ S path q
          path-image q ⊆ S pathstart q = pathstart p
          pathfinish q = pathfinish p for p q
  proof −
    have homotopic-paths S p (p +++ linepath (pathfinish p) (pathfinish p))
      by (simp add: homotopic-paths-rid homotopic-paths-sym that)
    also have homotopic-paths S (p +++ linepath (pathfinish p) (pathfinish p))
                                (p +++ reversepath q +++ q)
      using that
       by (metis homotopic-paths-join homotopic-paths-linv homotopic-paths-refl
homotopic-paths-sym-eq pathstart-linepath)
    also have homotopic-paths S (p +++ reversepath q +++ q)
                                ((p +++ reversepath q) +++ q)
      by (simp add: that homotopic-paths-assoc)
    also have homotopic-paths S ((p +++ reversepath q) +++ q)
                                (linepath (pathstart q) (pathstart q) +++ q)
      using ∗ [of p +++ reversepath q] that
      by (simp add: homotopic-paths-join path-image-join)
    also have homotopic-paths S (linepath (pathstart q) (pathstart q) +++ q) q
      using that homotopic-paths-lid by blast
    finally show ?thesis .
  qed
  then show ?rhs
    by (blast intro: pc ∗)
next
  assume ?rhs
  then show ?lhs
    by (force simp: simply-connected-eq-contractible-path)
qed

proposition simply-connected-Times:
  fixes S :: 'a::real-normed-vector set and T :: 'b::real-normed-vector set
  assumes S: simply-connected S and T: simply-connected T
    shows simply-connected(S × T)
proof −
  have homotopic-loops (S × T) p (linepath (a, b) (a, b))
       if path p path-image p ⊆ S × T p 1 = p 0 a ∈ S b ∈ T
       for p a b
  proof −
```

```
    have path (fst ∘ p)
      apply (rule Path-Connected.path-continuous-image [OF ‹path p›])
      apply (rule continuous-intros)+
      done
    moreover have path-image (fst ∘ p) ⊆ S
      using that apply (simp add: path-image-def) by force
    ultimately have p1: homotopic-loops S (fst o p) (linepath a a)
      using S that
      apply (simp add: simply-connected-eq-contractible-loop-any)
      apply (drule-tac x=fst o p in spec)
      apply (drule-tac x=a in spec)
      apply (auto simp: pathstart-def pathfinish-def)
      done
    have path (snd ∘ p)
      apply (rule Path-Connected.path-continuous-image [OF ‹path p›])
      apply (rule continuous-intros)+
      done
    moreover have path-image (snd ∘ p) ⊆ T
      using that apply (simp add: path-image-def) by force
    ultimately have p2: homotopic-loops T (snd o p) (linepath b b)
      using T that
      apply (simp add: simply-connected-eq-contractible-loop-any)
      apply (drule-tac x=snd o p in spec)
      apply (drule-tac x=b in spec)
      apply (auto simp: pathstart-def pathfinish-def)
      done
    show ?thesis
      using p1 p2
      apply (simp add: homotopic-loops, clarify)
      apply (rename-tac h k)
      apply (rule-tac x=λz. Pair (h z) (k z) in exI)
      apply (intro conjI continuous-intros | assumption)+
      apply (auto simp: pathstart-def pathfinish-def)
      done
  qed
  with assms show ?thesis
   by (simp add: simply-connected-eq-contractible-loop-any pathfinish-def pathstart-def)
qed
```

25.9 Contractible sets

```
definition contractible where
 contractible S ≡ ∃ a. homotopic-with (λx. True) S S id (λx. a)

proposition contractible-imp-simply-connected:
  fixes S :: -::real-normed-vector set
  assumes contractible S shows simply-connected S
proof (cases S = {})
  case True then show ?thesis by force
```

next
case *False*
obtain *a* **where** *a*: *homotopic-with* (λ*x*. *True*) *S S id* (λ*x*. *a*)
using *assms* **by** (*force simp add*: *contractible-def*)
then have *a* ∈ *S*
by (*metis False homotopic-constant-maps homotopic-with-symD homotopic-with-trans path-component-mem*(*2*))
show *?thesis*
apply (*simp add*: *simply-connected-eq-contractible-loop-all False*)
apply (*rule bexI* [*OF* - ‹*a* ∈ *S*›])
using *a* **apply** (*simp add*: *homotopic-loops-def homotopic-with-def path-def path-image-def pathfinish-def pathstart-def*)
apply *clarify*
apply (*rule-tac x*=(*h o* (λ*y*. (*fst y*, (*p* ∘ *snd*) *y*))) **in** *exI*)
apply (*intro conjI continuous-on-compose continuous-intros*)
apply (*erule continuous-on-subset* | *force*)+
done
qed

corollary *contractible-imp-connected*:
fixes *S* :: -::*real-normed-vector set*
shows *contractible S* ⟹ *connected S*
by (*simp add*: *contractible-imp-simply-connected simply-connected-imp-connected*)

lemma *contractible-imp-path-connected*:
fixes *S* :: -::*real-normed-vector set*
shows *contractible S* ⟹ *path-connected S*
by (*simp add*: *contractible-imp-simply-connected simply-connected-imp-path-connected*)

lemma *nullhomotopic-through-contractible*:
fixes *S* :: -::*topological-space set*
assumes *f*: *continuous-on S f f* ‘ *S* ⊆ *T*
and *g*: *continuous-on T g g* ‘ *T* ⊆ *U*
and *T*: *contractible T*
obtains *c* **where** *homotopic-with* (λ*h*. *True*) *S U* (*g o f*) (λ*x*. *c*)
proof −
obtain *b* **where** *b*: *homotopic-with* (λ*x*. *True*) *T T id* (λ*x*. *b*)
using *assms* **by** (*force simp add*: *contractible-def*)
have *homotopic-with* (λ*f*. *True*) *T U* (*g* ∘ *id*) (*g* ∘ (λ*x*. *b*))
by (*rule homotopic-compose-continuous-left* [*OF b g*])
then have *homotopic-with* (λ*f*. *True*) *S U* (*g* ∘ *id* ∘ *f*) (*g* ∘ (λ*x*. *b*) ∘ *f*)
by (*rule homotopic-compose-continuous-right* [*OF* - *f*])
then show *?thesis*
by (*simp add*: *comp-def that*)
qed

lemma *nullhomotopic-into-contractible*:
assumes *f*: *continuous-on S f f* ‘ *S* ⊆ *T*
and *T*: *contractible T*

```
    obtains c where homotopic-with (λh. True) S T f (λx. c)
apply (rule nullhomotopic-through-contractible [OF f, of id T])
using assms
apply (auto simp: continuous-on-id)
done

lemma nullhomotopic-from-contractible:
  assumes f: continuous-on S f f ‘ S ⊆ T
      and S: contractible S
    obtains c where homotopic-with (λh. True) S T f (λx. c)
apply (rule nullhomotopic-through-contractible [OF continuous-on-id - f S, of S])
using assms
apply (auto simp: comp-def)
done

lemma homotopic-through-contractible:
  fixes S :: -::real-normed-vector set
  assumes continuous-on S f1 f1 ‘ S ⊆ T
          continuous-on T g1 g1 ‘ T ⊆ U
          continuous-on S f2 f2 ‘ S ⊆ T
          continuous-on T g2 g2 ‘ T ⊆ U
          contractible T path-connected U
   shows homotopic-with (λh. True) S U (g1 o f1) (g2 o f2)
proof −
  obtain c1 where c1: homotopic-with (λh. True) S U (g1 o f1) (λx. c1)
    apply (rule nullhomotopic-through-contractible [of S f1 T g1 U])
    using assms apply (auto simp: )
    done
  obtain c2 where c2: homotopic-with (λh. True) S U (g2 o f2) (λx. c2)
    apply (rule nullhomotopic-through-contractible [of S f2 T g2 U])
    using assms apply (auto simp: )
    done
  have ∗: S = {} ∨ (∃ t. path-connected t ∧ t ⊆ U ∧ c2 ∈ t ∧ c1 ∈ t)
  proof (cases S = {})
    case True then show ?thesis by force
  next
    case False
    with c1 c2 have c1 ∈ U c2 ∈ U
      using homotopic-with-imp-subset2 all-not-in-conv image-subset-iff by blast+
    with ‹path-connected U› show ?thesis by blast
  qed
  show ?thesis
    apply (rule homotopic-with-trans [OF c1])
    apply (rule homotopic-with-symD)
    apply (rule homotopic-with-trans [OF c2])
    apply (simp add: path-component homotopic-constant-maps ∗)
    done
qed
```

lemma *homotopic-into-contractible*:
fixes S :: *'a::real-normed-vector set* **and** T:: *'b::real-normed-vector set*
assumes f: *continuous-on S f f ' S ⊆ T*
and g: *continuous-on S g g ' S ⊆ T*
and T: *contractible T*
shows *homotopic-with* (λ*h*. *True*) *S T f g*
using *homotopic-through-contractible* [*of S f T id T g id*]
by (*simp add*: *assms contractible-imp-path-connected continuous-on-id*)

lemma *homotopic-from-contractible*:
fixes S :: *'a::real-normed-vector set* **and** T:: *'b::real-normed-vector set*
assumes f: *continuous-on S f f ' S ⊆ T*
and g: *continuous-on S g g ' S ⊆ T*
and *contractible S path-connected T*
shows *homotopic-with* (λ*h*. *True*) *S T f g*
using *homotopic-through-contractible* [*of S id S f T id g*]
by (*simp add*: *assms contractible-imp-path-connected continuous-on-id*)

lemma *starlike-imp-contractible-gen*:
fixes S :: *'a::real-normed-vector set*
assumes S: *starlike S*
and P: $\bigwedge a\ T.\ [\![a \in S;\ 0 \le T;\ T \le 1]\!] \Longrightarrow P(\lambda x.\ (1 - T) *_R x + T *_R a)$
obtains *a* **where** *homotopic-with P S S* (λ*x*. *x*) (λ*x*. *a*)
proof −
obtain *a* **where** $a \in S$ **and** a: $\bigwedge x.\ x \in S \Longrightarrow$ *closed-segment a x ⊆ S*
using *S* **by** (*auto simp add*: *starlike-def*)
have (λ*y*. (*1* − *fst y*) $*_R$ *snd y* + *fst y* $*_R$ *a*) ' ({*0..1*} × *S*) ⊆ *S*
apply *clarify*
apply (*erule a* [*unfolded closed-segment-def*, *THEN subsetD*])
apply (*simp add*:)
apply (*metis add-diff-cancel-right' diff-ge-0-iff-ge le-add-diff-inverse pth-c*(*1*))
done
then show *?thesis*
apply (*rule-tac a*=*a* **in** *that*)
using ‹*a* ∈ *S*›
apply (*simp add*: *homotopic-with-def*)
apply (*rule-tac x*=λ*y*. (*1* − (*fst y*)) $*_R$ *snd y* + (*fst y*) $*_R$ *a* **in** *exI*)
apply (*intro conjI ballI continuous-on-compose continuous-intros*)
apply (*simp-all add*: *P*)
done
qed

lemma *starlike-imp-contractible*:
fixes S :: *'a::real-normed-vector set*
shows *starlike S* ⟹ *contractible S*
using *starlike-imp-contractible-gen contractible-def* **by** (*fastforce simp*: *id-def*)

lemma *contractible-UNIV*: *contractible* (*UNIV* :: *'a::real-normed-vector set*)
by (*simp add*: *starlike-imp-contractible*)

lemma *starlike-imp-simply-connected*:
fixes S :: *'a::real-normed-vector set*
shows *starlike* $S \Longrightarrow$ *simply-connected* S
by (*simp add*: *contractible-imp-simply-connected starlike-imp-contractible*)

lemma *convex-imp-simply-connected*:
fixes S :: *'a::real-normed-vector set*
shows *convex* $S \Longrightarrow$ *simply-connected* S
using *convex-imp-starlike starlike-imp-simply-connected* **by** *blast*

lemma *starlike-imp-path-connected*:
fixes S :: *'a::real-normed-vector set*
shows *starlike* $S \Longrightarrow$ *path-connected* S
by (*simp add*: *simply-connected-imp-path-connected starlike-imp-simply-connected*)

lemma *starlike-imp-connected*:
fixes S :: *'a::real-normed-vector set*
shows *starlike* $S \Longrightarrow$ *connected* S
by (*simp add*: *path-connected-imp-connected starlike-imp-path-connected*)

lemma *is-interval-simply-connected-1*:
fixes S :: *real set*
shows *is-interval* $S \longleftrightarrow$ *simply-connected* S
using *convex-imp-simply-connected is-interval-convex-1 is-interval-path-connected-1*
simply-connected-imp-path-connected **by** *auto*

lemma *contractible-empty*: *contractible* {}
by (*simp add*: *continuous-on-empty contractible-def homotopic-with*)

lemma *contractible-convex-tweak-boundary-points*:
fixes S :: *'a::euclidean-space set*
assumes *convex* S **and** *TS*: *rel-interior* $S \subseteq T$ $T \subseteq$ *closure* S
shows *contractible* T
proof (*cases* $S = \{\}$)
case *True*
with *assms* **show** *?thesis*
by (*simp add*: *contractible-empty subsetCE*)
next
case *False*
show *?thesis*
apply (*rule starlike-imp-contractible*)
apply (*rule starlike-convex-tweak-boundary-points* [*OF* ‹*convex S*› *False TS*])
done
qed

lemma *convex-imp-contractible*:
fixes S :: *'a::real-normed-vector set*
shows *convex* $S \Longrightarrow$ *contractible* S

using *contractible-empty convex-imp-starlike starlike-imp-contractible* **by** *auto*

lemma *contractible-sing*:
fixes *a* :: *'a*::*real-normed-vector*
shows *contractible* {*a*}
by (*rule convex-imp-contractible* [*OF convex-singleton*])

lemma *is-interval-contractible-1*:
fixes *S* :: *real set*
shows *is-interval S* ⟷ *contractible S*
using *contractible-imp-simply-connected convex-imp-contractible is-interval-convex-1*

is-interval-simply-connected-1 **by** *auto*

lemma *contractible-times*:
fixes *S* :: *'a*::*euclidean-space set* **and** *T* :: *'b*::*euclidean-space set*
assumes *S*: *contractible S* **and** *T*: *contractible T*
shows *contractible* (*S* × *T*)
proof −
obtain *a h* **where** *conth*: *continuous-on* ({*0..1*} × *S*) *h*
and *hsub*: *h* ‘ ({*0..1*} × *S*) ⊆ *S*
and [*simp*]: ⋀*x*. *x* ∈ *S* ⟹ *h* (*0*, *x*) = *x*
and [*simp*]: ⋀*x*. *x* ∈ *S* ⟹ *h* (*1*::*real*, *x*) = *a*
using *S* **by** (*auto simp add*: *contractible-def homotopic-with*)
obtain *b k* **where** *contk*: *continuous-on* ({*0..1*} × *T*) *k*
and *ksub*: *k* ‘ ({*0..1*} × *T*) ⊆ *T*
and [*simp*]: ⋀*x*. *x* ∈ *T* ⟹ *k* (*0*, *x*) = *x*
and [*simp*]: ⋀*x*. *x* ∈ *T* ⟹ *k* (*1*::*real*, *x*) = *b*
using *T* **by** (*auto simp add*: *contractible-def homotopic-with*)
show *?thesis*
apply (*simp add*: *contractible-def homotopic-with*)
apply (*rule exI* [**where** *x*=*a*])
apply (*rule exI* [**where** *x*=*b*])
apply (*rule exI* [**where** *x* = λ*z*. (*h* (*fst z*, *fst*(*snd z*)), *k* (*fst z*, *snd*(*snd z*)))])
apply (*intro conjI ballI continuous-intros continuous-on-compose2* [*OF conth*]
continuous-on-compose2 [*OF contk*])
using *hsub ksub*
apply (*auto simp*:)
done
qed

lemma *homotopy-dominated-contractibility*:
fixes *S* :: *'a*::*real-normed-vector set* **and** *T* :: *'b*::*real-normed-vector set*
assumes *S*: *contractible S*
and *f*: *continuous-on S f image f S* ⊆ *T*
and *g*: *continuous-on T g image g T* ⊆ *S*
and *hom*: *homotopic-with* (λ*x*. *True*) *T T* (*f o g*) *id*
shows *contractible T*
proof −

obtain *b* **where** *homotopic-with* (λ*h*. *True*) *S T f* (λ*x*. *b*)
using *nullhomotopic-from-contractible* [*OF f S*] **.**
then have *homg*: *homotopic-with* (λ*x*. *True*) *T T* ((λ*x*. *b*) ∘ *g*) (*f* ∘ *g*)
by (*rule homotopic-with-compose-continuous-right* [*OF homotopic-with-symD g*])
show *?thesis*
apply (*simp add*: *contractible-def*)
apply (*rule exI* [**where** *x* = *b*])
apply (*rule homotopic-with-symD*)
apply (*rule homotopic-with-trans* [*OF* - *hom*])
using *homg* **apply** (*simp add*: *o-def*)
done
qed

25.10 Local versions of topological properties in general

definition *locally* :: ('*a*::*topological-space set* ⇒ *bool*) ⇒ '*a set* ⇒ *bool*
where
locally P S ≡
∀ *w x*. *openin* (*subtopology euclidean S*) *w* ∧ *x* ∈ *w*
⟶ (∃ *u v*. *openin* (*subtopology euclidean S*) *u* ∧ *P v* ∧
x ∈ *u* ∧ *u* ⊆ *v* ∧ *v* ⊆ *w*)

lemma *locallyI*:
assumes ⋀*w x*. ⟦*openin* (*subtopology euclidean S*) *w*; *x* ∈ *w*⟧
⟹ ∃ *u v*. *openin* (*subtopology euclidean S*) *u* ∧ *P v* ∧
x ∈ *u* ∧ *u* ⊆ *v* ∧ *v* ⊆ *w*
shows *locally P S*
using *assms* **by** (*force simp*: *locally-def*)

lemma *locallyE*:
assumes *locally P S openin* (*subtopology euclidean S*) *w x* ∈ *w*
obtains *u v* **where** *openin* (*subtopology euclidean S*) *u*
P v x ∈ *u u* ⊆ *v v* ⊆ *w*
using *assms* **by** (*force simp*: *locally-def*)

lemma *locally-mono*:
assumes *locally P S* ⋀*t*. *P t* ⟹ *Q t*
shows *locally Q S*
by (*metis assms locally-def*)

lemma *locally-open-subset*:
assumes *locally P S openin* (*subtopology euclidean S*) *t*
shows *locally P t*
using *assms*
apply (*simp add*: *locally-def*)
apply (*erule all-forward*)+
apply (*rule impI*)
apply (*erule impCE*)

using *openin-trans* **apply** *blast*
apply (*erule ex-forward*)
by (*metis* (*no-types*, *hide-lams*) *Int-absorb1 Int-lower1 Int-subset-iff openin-open openin-subtopology-Int-subset*)

lemma *locally-diff-closed*:
  ⟦*locally P S*; *closedin* (*subtopology euclidean S*) *t*⟧ ⟹ *locally P* (*S* − *t*)
 using *locally-open-subset closedin-def* **by** *fastforce*

lemma *locally-empty* [*iff*]: *locally P* {}
 by (*simp add*: *locally-def openin-subtopology*)

lemma *locally-singleton* [*iff*]:
 fixes *a* :: *'a*::*metric-space*
 shows *locally P* {*a*} ⟷ *P* {*a*}
apply (*simp add*: *locally-def openin-euclidean-subtopology-iff subset-singleton-iff conj-disj-distribR cong*: *conj-cong*)
using *zero-less-one* **by** *blast*

lemma *locally-iff*:
  locally P S ⟷
  (∀ *T x*. *open T* ∧ *x* ∈ *S* ∩ *T* ⟶ (∃ *U*. *open U* ∧ (∃ *v*. *P v* ∧ *x* ∈ *S* ∩ *U* ∧ *S* ∩ *U* ⊆ *v* ∧ *v* ⊆ *S* ∩ *T*)))
apply (*simp add*: *le-inf-iff locally-def openin-open*, *safe*)
apply (*metis IntE IntI le-inf-iff*)
apply (*metis IntI Int-subset-iff*)
done

lemma *locally-Int*:
 assumes *S*: *locally P S* **and** *t*: *locally P t*
  and *P*: ⋀*S t*. *P S* ∧ *P t* ⟹ *P*(*S* ∩ *t*)
 shows *locally P* (*S* ∩ *t*)
using *S t* **unfolding** *locally-iff*
apply *clarify*
apply (*drule-tac x*=*T* **in** *spec*)+
apply (*drule-tac x*=*x* **in** *spec*)+
apply *clarsimp*
apply (*rename-tac U1 U2 V1 V2*)
apply (*rule-tac x*=*U1* ∩ *U2* **in** *exI*)
apply (*simp add*: *open-Int*)
apply (*rule-tac x*=*V1* ∩ *V2* **in** *exI*)
apply (*auto intro*: *P*)
done

proposition *homeomorphism-locally-imp*:
 fixes *S* :: *'a*::*metric-space set* **and** *t* :: *'b*::*t2-space set*
 assumes *S*: *locally P S* **and** *hom*: *homeomorphism S t f g*
  and *Q*: ⋀*S t*. ⟦*P S*; *homeomorphism S t f g*⟧ ⟹ *Q t*

shows *locally Q t*
proof (*clarsimp simp*: *locally-def*)
fix *w y*
assume *y* ∈ *w* **and** *openin* (*subtopology euclidean t*) *w*
then obtain *T* **where** *T*: *open T w* = *t* ∩ *T*
by (*force simp*: *openin-open*)
then have *w* ⊆ *t* **by** *auto*
have *f*: ⋀*x*. *x* ∈ *S* ⟹ *g*(*f x*) = *x f* ‘ *S* = *t continuous-on S f*
and *g*: ⋀*y*. *y* ∈ *t* ⟹ *f*(*g y*) = *y g* ‘ *t* = *S continuous-on t g*
using *hom* **by** (*auto simp*: *homeomorphism-def*)
have *gw*: *g* ‘ *w* = *S* ∩ {*x*. *f x* ∈ *w*}
using ‹*w* ⊆ *t*›
apply *auto*
using ‹*g* ‘ *t* = *S*› ‹*w* ⊆ *t*› **apply** *blast*
using *g* ‹*w* ⊆ *t*› **apply** *auto*[*1*]
by (*simp add*: *f rev-image-eqI*)
have *o*: *openin* (*subtopology euclidean S*) (*g* ‘ *w*)
proof −
have *continuous-on S f*
using *f*(*3*) **by** *blast*
then show *openin* (*subtopology euclidean S*) (*g* ‘ *w*)
by (*simp add*: *gw Collect-conj-eq* ‹*openin* (*subtopology euclidean t*) *w*› *continuous-on-open f*(*2*))
qed
then obtain *u v*
where *osu*: *openin* (*subtopology euclidean S*) *u* **and** *uv*: *P v g y* ∈ *u u* ⊆ *v v* ⊆ *g* ‘ *w*
using *S* [*unfolded locally-def*, *rule-format*, *of g* ‘ *w g y*] ‹*y* ∈ *w*› **by** *force*
have *v* ⊆ *S* **using** *uv* **by** (*simp add*: *gw*)
have *fv*: *f* ‘ *v* = *t* ∩ {*x*. *g x* ∈ *v*}
using ‹*f* ‘ *S* = *t*› *f* ‹*v* ⊆ *S*› **by** *auto*
have *f* ‘ *v* ⊆ *w*
using *uv* **using** *Int-lower2 gw image-subsetI mem-Collect-eq subset-iff* **by** *auto*
have *contvf*: *continuous-on v f*
using ‹*v* ⊆ *S*› *continuous-on-subset f*(*3*) **by** *blast*
have *contvg*: *continuous-on* (*f* ‘ *v*) *g*
using ‹*f* ‘ *v* ⊆ *w*› ‹*w* ⊆ *t*› *continuous-on-subset g*(*3*) **by** *blast*
have *homv*: *homeomorphism v* (*f* ‘ *v*) *f g*
using ‹*v* ⊆ *S*› ‹*w* ⊆ *t*› *f*
apply (*simp add*: *homeomorphism-def contvf contvg*, *auto*)
by (*metis f*(*1*) *rev-image-eqI rev-subsetD*)
have *1*: *openin* (*subtopology euclidean t*) {*x* ∈ *t*. *g x* ∈ *u*}
apply (*rule continuous-on-open* [*THEN iffD1*, *rule-format*])
apply (*rule* ‹*continuous-on t g*›)
using ‹*g* ‘ *t* = *S*› **apply** (*simp add*: *osu*)
done
have *2*: ∃*v*. *Q v* ∧ *y* ∈ {*x* ∈ *t*. *g x* ∈ *u*} ∧ {*x* ∈ *t*. *g x* ∈ *u*} ⊆ *v* ∧ *v* ⊆ *w*
apply (*rule-tac x*=*f* ‘ *v* **in** *exI*)
apply (*intro conjI Q* [*OF* ‹*P v*› *homv*])

```
    using ‹w ⊆ t› ‹y ∈ w›  ‹f ‘ v ⊆ w›  uv apply (auto simp: fv)
    done
  show ∃u. openin (subtopology euclidean t) u ∧
          (∃v. Q v ∧ y ∈ u ∧ u ⊆ v ∧ v ⊆ w)
    by (meson 1 2)
qed

lemma homeomorphism-locally:
  fixes f:: 'a::metric-space ⇒ 'b::metric-space
  assumes hom: homeomorphism S t f g
      and eq: ⋀S t. homeomorphism S t f g ⟹ (P S ⟷ Q t)
    shows locally P S ⟷ locally Q t
apply (rule iffI)
apply (erule homeomorphism-locally-imp [OF - hom])
apply (simp add: eq)
apply (erule homeomorphism-locally-imp)
using eq homeomorphism-sym homeomorphism-symD [OF hom] apply blast+
done

lemma locally-translation:
  fixes P :: 'a :: real-normed-vector set ⇒ bool
  shows
   (⋀S. P (image (λx. a + x) S) ⟷ P S)
       ⟹ locally P (image (λx. a + x) S) ⟷ locally P S
apply (rule homeomorphism-locally [OF homeomorphism-translation])
apply (simp add: homeomorphism-def)
by metis

lemma locally-injective-linear-image:
  fixes f :: 'a::euclidean-space ⇒ 'b::euclidean-space
  assumes f: linear f inj f and iff: ⋀S. P (f ‘ S) ⟷ Q S
    shows locally P (f ‘ S) ⟷ locally Q S
apply (rule linear-homeomorphism-image [OF f])
apply (rule-tac f=g and g = f in homeomorphism-locally, assumption)
by (metis iff homeomorphism-def)

lemma locally-open-map-image:
  fixes f :: 'a::real-normed-vector ⇒ 'b::real-normed-vector
  assumes P: locally P S
      and f: continuous-on S f
      and oo: ⋀t. openin (subtopology euclidean S) t
                ⟹ openin (subtopology euclidean (f ‘ S)) (f ‘ t)
      and Q: ⋀t. ⟦t ⊆ S; P t⟧ ⟹ Q(f ‘ t)
    shows locally Q (f ‘ S)
proof (clarsimp simp add: locally-def)
  fix w y
  assume oiw: openin (subtopology euclidean (f ‘ S)) w and y ∈ w
  then have w ⊆ f ‘ S by (simp add: openin-euclidean-subtopology-iff)
  have oivf: openin (subtopology euclidean S) {x ∈ S. f x ∈ w}
```

by (*rule continuous-on-open* [*THEN iffD1*, *rule-format*, *OF f oiw*])
then obtain x **where** $x \in S\ f\ x = y$
using ‹$w \subseteq f$ ‘ S› ‹$y \in w$› **by** *blast*
then obtain $u\ v$
where *openin* (*subtopology euclidean S*) $u\ P\ v\ x \in u\ u \subseteq v\ v \subseteq \{x \in S.\ f\ x \in w\}$
using P [*unfolded locally-def*, *rule-format*, *of* $\{x.\ x \in S \wedge f\ x \in w\}\ x$] *oivf* ‹$y \in w$›
by *auto*
then show $\exists u.$ *openin* (*subtopology euclidean* (f ‘ S)) $u \wedge$
$(\exists v.\ Q\ v \wedge y \in u \wedge u \subseteq v \wedge v \subseteq w)$
apply (*rule-tac x=f ‘ u* **in** *exI*)
apply (*rule conjI*, *blast intro!*: *oo*)
apply (*rule-tac x=f ‘ v* **in** *exI*)
apply (*force simp*: ‹$f\ x = y$› *rev-image-eqI intro*: Q)
done
qed

25.11 Basic properties of local compactness

lemma *locally-compact*:
fixes s :: *'a* :: *metric-space set*
shows
locally compact s $\longleftrightarrow$
$(\forall x \in s.\ \exists u\ v.\ x \in u \wedge u \subseteq v \wedge v \subseteq s \wedge$
openin (*subtopology euclidean s*) $u \wedge$ *compact* v)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
apply *clarify*
apply (*erule-tac* $w = s \cap$ *ball x 1* **in** *locallyE*)
by *auto*
next
assume r [*rule-format*]: *?rhs*
have $*$: $\exists u\ v.$
openin (*subtopology euclidean s*) $u \wedge$
compact $v \wedge x \in u \wedge u \subseteq v \wedge v \subseteq s \cap T$
if *open* $T\ x \in s\ x \in T$ **for** $x\ T$
proof −
obtain $u\ v$ **where** *uv*: $x \in u\ u \subseteq v\ v \subseteq s$ *compact* v *openin* (*subtopology euclidean s*) u
using r [*OF* ‹$x \in s$›] **by** *auto*
obtain e **where** $e{>}0$ **and** e: *cball* $x\ e \subseteq T$
using *open-contains-cball* ‹*open* T› ‹$x \in T$› **by** *blast*
show *?thesis*
apply (*rule-tac x=*($s \cap$ *ball x e*) $\cap u$ **in** *exI*)
apply (*rule-tac x=cball x e* $\cap v$ **in** *exI*)
using *that* ‹$e > 0$› e *uv*

```
    apply auto
    done
  qed
  show ?lhs
    apply (rule locallyI)
    apply (subst (asm) openin-open)
    apply (blast intro: *)
    done
qed

lemma locally-compactE:
  fixes s :: 'a :: metric-space set
  assumes locally compact s
  obtains u v where ⋀x. x ∈ s ⟹ x ∈ u x ∧ u x ⊆ v x ∧ v x ⊆ s ∧
                          openin (subtopology euclidean s) (u x) ∧ compact (v x)
using assms
unfolding locally-compact by metis

lemma locally-compact-alt:
  fixes s :: 'a :: heine-borel set
  shows locally compact s ⟷
        (∀ x ∈ s. ∃ u. x ∈ u ∧
                  openin (subtopology euclidean s) u ∧ compact(closure u) ∧ closure
u ⊆ s)
apply (simp add: locally-compact)
apply (intro ball-cong ex-cong refl iffI)
apply (metis bounded-subset closure-eq closure-mono compact-eq-bounded-closed
dual-order.trans)
by (meson closure-subset compact-closure)

lemma locally-compact-Int-cball:
  fixes s :: 'a :: heine-borel set
  shows locally compact s ⟷ (∀ x ∈ s. ∃ e. 0 < e ∧ closed(cball x e ∩ s))
       (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
    apply (simp add: locally-compact openin-contains-cball)
    apply (clarify | assumption | drule bspec)+
     by (metis (no-types, lifting)  compact-cball compact-imp-closed compact-Int
inf.absorb-iff2 inf.orderE inf-sup-aci(2))
next
  assume ?rhs
  then show ?lhs
    apply (simp add: locally-compact openin-contains-cball)
    apply (clarify | assumption | drule bspec)+
    apply (rule-tac x=ball x e ∩ s in exI, simp)
    apply (rule-tac x=cball x e ∩ s in exI)
    using compact-eq-bounded-closed
```

```
    apply auto
    apply (metis open-ball le-infI1 mem-ball open-contains-cball-eq)
    done
qed

lemma locally-compact-compact:
  fixes s :: 'a :: heine-borel set
  shows locally compact s ⟷
        (∀ k. k ⊆ s ∧ compact k
            ⟶ (∃ u v. k ⊆ u ∧ u ⊆ v ∧ v ⊆ s ∧
                      openin (subtopology euclidean s) u ∧ compact v))
       (is ?lhs = ?rhs)
proof
  assume ?lhs
  then obtain u v where
    uv: ⋀x. x ∈ s ⟹ x ∈ u x ∧ u x ⊆ v x ∧ v x ⊆ s ∧
                     openin (subtopology euclidean s) (u x) ∧ compact (v x)
    by (metis locally-compactE)
  have ∗: ∃ u v. k ⊆ u ∧ u ⊆ v ∧ v ⊆ s ∧ openin (subtopology euclidean s) u ∧
compact v
         if k ⊆ s compact k for k
  proof −
    have ⋀C. (∀ c∈C. openin (subtopology euclidean k) c) ∧ k ⊆ ⋃ C ⟹
                ∃ D⊆C. finite D ∧ k ⊆ ⋃ D
      using that by (simp add: compact-eq-openin-cover)
    moreover have ∀ c ∈ (λx. k ∩ u x) ‘ k. openin (subtopology euclidean k) c
     using that by clarify (metis subsetD inf.absorb-iff2 openin-subset openin-subtopology-Int-subset
topspace-euclidean-subtopology uv)
    moreover have k ⊆ ⋃ ((λx. k ∩ u x) ‘ k)
      using that by clarsimp (meson subsetCE uv)
    ultimately obtain D where D ⊆ (λx. k ∩ u x) ‘ k finite D k ⊆ ⋃ D
      by metis
    then obtain T where T: T ⊆ k finite T k ⊆ ⋃ ((λx. k ∩ u x) ‘ T)
      by (metis finite-subset-image)
    have Tuv: UNION T u ⊆ UNION T v
      using T that by (force simp: dest!: uv)
    show ?thesis
      apply (rule-tac x=⋃ (u ‘ T) in exI)
      apply (rule-tac x=⋃ (v ‘ T) in exI)
      apply (simp add: Tuv)
      using T that
      apply (auto simp: dest!: uv)
      done
  qed
  show ?rhs
    by (blast intro: ∗)
next
  assume ?rhs
  then show ?lhs
```

```
    apply (clarsimp simp add: locally-compact)
    apply (drule-tac x={x} in spec, simp)
    done
qed

lemma open-imp-locally-compact:
  fixes s :: 'a :: heine-borel set
  assumes open s
    shows locally compact s
proof −
  have ∗: ∃ u v. x ∈ u ∧ u ⊆ v ∧ v ⊆ s ∧ openin (subtopology euclidean s) u ∧
compact v
        if x ∈ s for x
  proof −
    obtain e where e>0 and e: cball x e ⊆ s
      using open-contains-cball assms ‹x ∈ s› by blast
    have ope: openin (subtopology euclidean s) (ball x e)
      by (meson e open-ball ball-subset-cball dual-order.trans open-subset)
    show ?thesis
      apply (rule-tac x=ball x e in exI)
      apply (rule-tac x=cball x e in exI)
      using ‹e > 0› e apply (auto simp: ope)
      done
  qed
  show ?thesis
    unfolding locally-compact
    by (blast intro: ∗)
qed

lemma closed-imp-locally-compact:
  fixes s :: 'a :: heine-borel set
  assumes closed s
    shows locally compact s
proof −
  have ∗: ∃ u v. x ∈ u ∧ u ⊆ v ∧ v ⊆ s ∧
                 openin (subtopology euclidean s) u ∧ compact v
        if x ∈ s for x
  proof −
    show ?thesis
      apply (rule-tac x = s ∩ ball x 1 in exI)
      apply (rule-tac x = s ∩ cball x 1 in exI)
      using ‹x ∈ s› assms apply auto
      done
  qed
  show ?thesis
    unfolding locally-compact
    by (blast intro: ∗)
qed
```

lemma *locally-compact-UNIV*: *locally compact* (*UNIV* :: *'a* :: *heine-borel set*)
by (*simp add*: *closed-imp-locally-compact*)

lemma *locally-compact-Int*:
fixes *s* :: *'a* :: *t2-space set*
shows ⟦*locally compact s*; *locally compact t*⟧ ⟹ *locally compact* (*s* ∩ *t*)
by (*simp add*: *compact-Int locally-Int*)

lemma *locally-compact-closedin*:
fixes *s* :: *'a* :: *heine-borel set*
shows ⟦*closedin* (*subtopology euclidean s*) *t*; *locally compact s*⟧
⟹ *locally compact t*
unfolding *closedin-closed*
using *closed-imp-locally-compact locally-compact-Int* **by** *blast*

lemma *locally-compact-delete*:
fixes *s* :: *'a* :: *t1-space set*
shows *locally compact s* ⟹ *locally compact* (*s* − {*a*})
by (*auto simp*: *openin-delete locally-open-subset*)

lemma *locally-closed*:
fixes *s* :: *'a* :: *heine-borel set*
shows *locally closed s* ⟷ *locally compact s*
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
apply (*simp only*: *locally-def*)
apply (*erule all-forward imp-forward asm-rl exE*)+
apply (*rule-tac x* = *u* ∩ *ball x 1* **in** *exI*)
apply (*rule-tac x* = *v* ∩ *cball x 1* **in** *exI*)
apply (*force intro*: *openin-trans*)
done
next
assume *?rhs* **then show** *?lhs*
using *compact-eq-bounded-closed locally-mono* **by** *blast*
qed

25.12 Important special cases of local connectedness and path connectedness

lemma *locally-connected-1*:
assumes
⋀*v x*. ⟦*openin* (*subtopology euclidean S*) *v*; *x* ∈ *v*⟧
⟹ ∃*u*. *openin* (*subtopology euclidean S*) *u* ∧
connected u ∧ *x* ∈ *u* ∧ *u* ⊆ *v*
shows *locally connected S*
apply (*clarsimp simp add*: *locally-def*)
apply (*drule assms*; *blast*)

done

lemma *locally-connected-2*:
assumes *locally connected S*
openin (*subtopology euclidean S*) *t*
$x \in t$
shows *openin* (*subtopology euclidean S*) (*connected-component-set t x*)
proof −
{ fix $y :: 'a$
let *?SS* = *subtopology euclidean S*
assume *1*: *openin ?SS t*
$\forall\, w\ x.$ *openin ?SS w* $\wedge$ $x \in w \longrightarrow (\exists\, u.$ *openin ?SS u* $\wedge$ $(\exists\, v.$ *connected v* $\wedge$ $x \in u \wedge u \subseteq v \wedge v \subseteq w))$
and *connected-component t x y*
then have $y \in t$ **and** *y*: $y \in$ *connected-component-set t x*
using *connected-component-subset* **by** *blast+*
obtain *F* **where**
$\forall\, x\ y.$ $(\exists\, w.$ *openin ?SS w* $\wedge$ $(\exists\, u.$ *connected u* $\wedge$ $x \in w \wedge w \subseteq u \wedge u \subseteq y))$ = (*openin ?SS* (*F x y*) $\wedge$ $(\exists\, u.$ *connected u* $\wedge$ $x \in F\ x\ y \wedge F\ x\ y \subseteq u \wedge u \subseteq y))$
by *moura*
then obtain *G* **where**
$\forall\, a\ A.$ $(\exists\, U.$ *openin ?SS U* $\wedge$ $(\exists\, V.$ *connected V* $\wedge$ $a \in U \wedge U \subseteq V \wedge V \subseteq A))$ = (*openin ?SS* (*F a A*) $\wedge$ *connected* (*G a A*) $\wedge$ $a \in F\ a\ A \wedge F\ a\ A \subseteq G\ a\ A \wedge G\ a\ A \subseteq A$)
by *moura*
then have $*$: *openin ?SS* (*F y t*) $\wedge$ *connected* (*G y t*) $\wedge$ $y \in F\ y\ t \wedge F\ y\ t \subseteq G\ y\ t \wedge G\ y\ t \subseteq t$
using *1* ‹$y \in t$› **by** *presburger*
have *G y t* $\subseteq$ *connected-component-set t y*
by (*metis* (*no-types*) $*$ *connected-component-eq-self connected-component-mono contra-subsetD*)
then have $\exists\, A.$ *openin ?SS A* $\wedge$ $y \in A \wedge A \subseteq$ *connected-component-set t x*
by (*metis* (*no-types*) $*$ *connected-component-eq dual-order.trans y*)
}
then show *?thesis*
using *assms openin-subopen* **by** (*force simp*: *locally-def*)
qed

lemma *locally-connected-3*:
assumes $\bigwedge t\ x.$ $[\![$*openin* (*subtopology euclidean S*) *t*; $x \in t]\!]$
$\Longrightarrow$ *openin* (*subtopology euclidean S*)
(*connected-component-set t x*)
openin (*subtopology euclidean S*) *v* $x \in v$
shows $\exists\, u.$ *openin* (*subtopology euclidean S*) *u* $\wedge$ *connected u* $\wedge$ $x \in u \wedge u \subseteq v$
using *assms connected-component-subset* **by** *fastforce*

lemma *locally-connected*:
locally connected S $\longleftrightarrow$
$(\forall\, v\ x.$ *openin* (*subtopology euclidean S*) *v* $\wedge$ $x \in v$

$\longrightarrow$ ($\exists\, u.$ *openin* (*subtopology euclidean S*) $u \wedge$ *connected* $u \wedge x \in u \wedge u \subseteq v$))
by (*metis locally-connected-1 locally-connected-2 locally-connected-3*)

lemma *locally-connected-open-connected-component*:
locally connected S $\longleftrightarrow$
($\forall\, t\ x.$ *openin* (*subtopology euclidean S*) $t \wedge x \in t$
$\longrightarrow$ *openin* (*subtopology euclidean S*) (*connected-component-set t x*))
by (*metis locally-connected-1 locally-connected-2 locally-connected-3*)

lemma *locally-path-connected-1*:
assumes
$\bigwedge v\ x.$ $[\![$*openin* (*subtopology euclidean S*) v; $x \in v]\!]$
$\Longrightarrow \exists\, u.$ *openin* (*subtopology euclidean S*) $u \wedge$ *path-connected* $u \wedge x \in u \wedge u \subseteq v$
shows *locally path-connected S*
apply (*clarsimp simp add*: *locally-def*)
apply (*drule assms*; *blast*)
done

lemma *locally-path-connected-2*:
assumes *locally path-connected S*
openin (*subtopology euclidean S*) *t*
$x \in t$
shows *openin* (*subtopology euclidean S*) (*path-component-set t x*)
proof −
{ fix $y :: 'a$
let *?SS* = *subtopology euclidean S*
assume *1*: *openin ?SS t*
$\forall\, w\ x.$ *openin ?SS* $w \wedge x \in w \longrightarrow (\exists\, u.$ *openin ?SS* $u \wedge (\exists\, v.$ *path-connected* $v \wedge x \in u \wedge u \subseteq v \wedge v \subseteq w))$
and *path-component t x y*
then have $y \in t$ **and** *y*: $y \in$ *path-component-set t x*
using *path-component-mem*(*2*) **by** *blast+*
obtain *F* **where**
$\forall\, x\ y.$ $(\exists\, w.$ *openin ?SS* $w \wedge (\exists\, u.$ *path-connected* $u \wedge x \in w \wedge w \subseteq u \wedge u \subseteq y)) = ($*openin ?SS* $(F\ x\ y) \wedge (\exists\, u.$ *path-connected* $u \wedge x \in F\ x\ y \wedge F\ x\ y \subseteq u \wedge u \subseteq y))$
by *moura*
then obtain *G* **where**
$\forall\, a\ A.$ $(\exists\, U.$ *openin ?SS* $U \wedge (\exists\, V.$ *path-connected* $V \wedge a \in U \wedge U \subseteq V \wedge V \subseteq A)) = ($*openin ?SS* $(F\ a\ A) \wedge$ *path-connected* $(G\ a\ A) \wedge a \in F\ a\ A \wedge F\ a\ A \subseteq G\ a\ A \wedge G\ a\ A \subseteq A)$
by *moura*
then have $*$: *openin ?SS* $(F\ y\ t) \wedge$ *path-connected* $(G\ y\ t) \wedge y \in F\ y\ t \wedge F\ y\ t \subseteq G\ y\ t \wedge G\ y\ t \subseteq t$
using *1* ‹$y \in t$› **by** *presburger*
have $G\ y\ t \subseteq$ *path-component-set t y*
using $*$ *path-component-maximal set-rev-mp* **by** *blast*

then have $\exists A.$ *openin ?SS A* $\wedge$ $y \in A \wedge A \subseteq$ *path-component-set t x*
by (*metis* $*$ ‹*G y t* $\subseteq$ *path-component-set t y*› *dual-order.trans path-component-eq y*)
}
then show *?thesis*
using *assms openin-subopen* **by** (*force simp*: *locally-def*)
qed

lemma *locally-path-connected-3*:
assumes $\bigwedge t\ x.$ ⟦*openin* (*subtopology euclidean S*) *t*; $x \in t$⟧
$\Longrightarrow$ *openin* (*subtopology euclidean S*) (*path-component-set t x*)
openin (*subtopology euclidean S*) *v* $x \in v$
shows $\exists u.$ *openin* (*subtopology euclidean S*) *u* $\wedge$ *path-connected u* $\wedge$ $x \in u \wedge u \subseteq v$
proof −
have *path-component v x x*
by (*meson assms*(*3*) *path-component-refl*)
then show *?thesis*
by (*metis assms*(*1*) *assms*(*2*) *assms*(*3*) *mem-Collect-eq path-component-subset path-connected-path-component*)
qed

proposition *locally-path-connected*:
locally path-connected S $\longleftrightarrow$
($\forall v\ x.$ *openin* (*subtopology euclidean S*) *v* $\wedge$ $x \in v$
$\longrightarrow$ ($\exists u.$ *openin* (*subtopology euclidean S*) *u* $\wedge$ *path-connected u* $\wedge$ $x \in u \wedge u \subseteq v$))
by (*metis locally-path-connected-1 locally-path-connected-2 locally-path-connected-3*)

proposition *locally-path-connected-open-path-connected-component*:
locally path-connected S $\longleftrightarrow$
($\forall t\ x.$ *openin* (*subtopology euclidean S*) *t* $\wedge$ $x \in t$
$\longrightarrow$ *openin* (*subtopology euclidean S*) (*path-component-set t x*))
by (*metis locally-path-connected-1 locally-path-connected-2 locally-path-connected-3*)

lemma *locally-connected-open-component*:
locally connected S $\longleftrightarrow$
($\forall t\ c.$ *openin* (*subtopology euclidean S*) *t* $\wedge$ $c \in$ *components t*
$\longrightarrow$ *openin* (*subtopology euclidean S*) *c*)
by (*metis components-iff locally-connected-open-connected-component*)

proposition *locally-connected-im-kleinen*:
locally connected S $\longleftrightarrow$
($\forall v\ x.$ *openin* (*subtopology euclidean S*) *v* $\wedge$ $x \in v$
$\longrightarrow$ ($\exists u.$ *openin* (*subtopology euclidean S*) *u* $\wedge$
$x \in u \wedge u \subseteq v \wedge$
($\forall y.$ $y \in u \longrightarrow$ ($\exists c.$ *connected c* $\wedge$ $c \subseteq v \wedge x \in c \wedge y \in c$))))
(**is** *?lhs* = *?rhs*)
proof

```
  assume ?lhs
  then show ?rhs
    by (fastforce simp add: locally-connected)
next
  assume ?rhs
  have *: ∃ T. openin (subtopology euclidean S) T ∧ x ∈ T ∧ T ⊆ c
       if openin (subtopology euclidean S) t and c: c ∈ components t and x ∈ c
for t c x
  proof −
    from that ‹?rhs› [rule-format, of t x]
    obtain u where u:
      openin (subtopology euclidean S) u ∧ x ∈ u ∧ u ⊆ t ∧
      (∀ y. y ∈ u ⟶ (∃ c. connected c ∧ c ⊆ t ∧ x ∈ c ∧ y ∈ c))
      by auto (meson subsetD in-components-subset)
    obtain F :: 'a set ⇒ 'a set ⇒ 'a where
      ∀ x y. (∃ z. z ∈ x ∧ y = connected-component-set x z) = (F x y ∈ x ∧ y =
connected-component-set x (F x y))
      by moura
    then have F: F t c ∈ t ∧ c = connected-component-set t (F t c)
      by (meson components-iff c)
    obtain G :: 'a set ⇒ 'a set ⇒ 'a where
        G: ∀ x y. (∃ z. z ∈ y ∧ z ∉ x) = (G x y ∈ y ∧ G x y ∉ x)
      by moura
     have G c u ∉ u ∨ G c u ∈ c
    using F by (metis (full-types) u connected-componentI connected-component-eq
mem-Collect-eq that(3))
    then show ?thesis
      using G u by auto
  qed
  show ?lhs
    apply (clarsimp simp add: locally-connected-open-component)
    apply (subst openin-subopen)
    apply (blast intro: *)
    done
qed

proposition locally-path-connected-im-kleinen:
  locally path-connected S ⟷
   (∀ v x. openin (subtopology euclidean S) v ∧ x ∈ v
       ⟶ (∃ u. openin (subtopology euclidean S) u ∧
               x ∈ u ∧ u ⊆ v ∧
               (∀ y. y ∈ u ⟶ (∃ p. path p ∧ path-image p ⊆ v ∧
                                   pathstart p = x ∧ pathfinish p = y))))
   (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
    apply (simp add: locally-path-connected path-connected-def)
    apply (erule all-forward ex-forward imp-forward conjE | simp)+
```

```
    by (meson dual-order.trans)
next
  assume ?rhs
  have ∗: ∃ T. openin (subtopology euclidean S) T ∧
              x ∈ T ∧ T ⊆ path-component-set u z
      if openin (subtopology euclidean S) u and z ∈ u and c: path-component u z
x for u z x
  proof −
    have x ∈ u
      by (meson c path-component-mem(2))
    with that ‹?rhs› [rule-format, of u x]
    obtain U where U:
      openin (subtopology euclidean S) U ∧ x ∈ U ∧ U ⊆ u ∧
        (∀ y. y ∈ U ⟶ (∃ p. path p ∧ path-image p ⊆ u ∧ pathstart p = x ∧
pathfinish p = y))
       by blast
    show ?thesis
      apply (rule-tac x=U in exI)
      apply (auto simp: U)
      apply (metis U c path-component-trans path-component-def)
      done
  qed
  show ?lhs
   apply (clarsimp simp add: locally-path-connected-open-path-connected-component)
    apply (subst openin-subopen)
    apply (blast intro: ∗)
    done
qed

lemma locally-path-connected-imp-locally-connected:
  locally path-connected S ⟹ locally connected S
using locally-mono path-connected-imp-connected by blast

lemma locally-connected-components:
  ⟦locally connected S; c ∈ components S⟧ ⟹ locally connected c
by (meson locally-connected-open-component locally-open-subset openin-subtopology-self)

lemma locally-path-connected-components:
  ⟦locally path-connected S; c ∈ components S⟧ ⟹ locally path-connected c
by (meson locally-connected-open-component locally-open-subset locally-path-connected-imp-locally-connected
openin-subtopology-self)

lemma locally-path-connected-connected-component:
  locally path-connected S ⟹ locally path-connected (connected-component-set S
x)
by (metis components-iff connected-component-eq-empty locally-empty locally-path-connected-components)

lemma open-imp-locally-path-connected:
  fixes S :: 'a :: real-normed-vector set
```

shows *open S ⟹ locally path-connected S*
apply (*rule locally-mono* [*of convex*])
apply (*simp-all add*: *locally-def openin-open-eq convex-imp-path-connected*)
apply (*meson Topology-Euclidean-Space.open-ball centre-in-ball convex-ball openE order-trans*)
done

lemma *open-imp-locally-connected*:
fixes *S* :: *'a* :: *real-normed-vector set*
shows *open S ⟹ locally connected S*
by (*simp add*: *locally-path-connected-imp-locally-connected open-imp-locally-path-connected*)

lemma *locally-path-connected-UNIV*: *locally path-connected* (*UNIV*::*'a* :: *real-normed-vector set*)
by (*simp add*: *open-imp-locally-path-connected*)

lemma *locally-connected-UNIV*: *locally connected* (*UNIV*::*'a* :: *real-normed-vector set*)
by (*simp add*: *open-imp-locally-connected*)

lemma *openin-connected-component-locally-connected*:
locally connected S
⟹ openin (*subtopology euclidean S*) (*connected-component-set S x*)
apply (*simp add*: *locally-connected-open-connected-component*)
by (*metis connected-component-eq-empty connected-component-subset open-empty open-subset openin-subtopology-self*)

lemma *openin-components-locally-connected*:
⟦*locally connected S*; *c ∈ components S*⟧ *⟹ openin* (*subtopology euclidean S*) *c*
using *locally-connected-open-component openin-subtopology-self* **by** *blast*

lemma *openin-path-component-locally-path-connected*:
locally path-connected S
⟹ openin (*subtopology euclidean S*) (*path-component-set S x*)
by (*metis* (*no-types*) *empty-iff locally-path-connected-2 openin-subopen openin-subtopology-self path-component-eq-empty*)

lemma *closedin-path-component-locally-path-connected*:
locally path-connected S
⟹ closedin (*subtopology euclidean S*) (*path-component-set S x*)
apply (*simp add*: *closedin-def path-component-subset complement-path-component-Union*)
apply (*rule openin-Union*)
using *openin-path-component-locally-path-connected* **by** *auto*

lemma *convex-imp-locally-path-connected*:
fixes *S* :: *'a*:: *real-normed-vector set*
shows *convex S ⟹ locally path-connected S*
apply (*clarsimp simp add*: *locally-path-connected*)
apply (*subst* (*asm*) *openin-open*)

apply *clarify*
apply (*erule* (*1*) *Topology-Euclidean-Space.openE*)
apply (*rule-tac* $x = S \cap$ *ball* x e **in** *exI*)
apply (*force simp*: *convex-Int convex-imp-path-connected*)
done

25.13 Retracts, in a general sense, preserve (co)homotopic triviality)

locale *Retracts* =
  fixes s h t k
  assumes *conth*: *continuous-on* s h
    and *imh*: h ‘ $s = t$
    and *contk*: *continuous-on* t k
    and *imk*: k ‘ $t \subseteq s$
    and *idhk*: $\bigwedge y.\ y \in t \Longrightarrow h(k\ y) = y$

begin

lemma *homotopically-trivial-retraction-gen*:
  assumes P: $\bigwedge f.\ [\![$*continuous-on* u f; f ‘ $u \subseteq t$; Q $f]\!] \Longrightarrow P(k\ o\ f)$
    and Q: $\bigwedge f.\ [\![$*continuous-on* u f; f ‘ $u \subseteq s$; P $f]\!] \Longrightarrow Q(h\ o\ f)$
    and *Qeq*: $\bigwedge h\ k.\ (\bigwedge x.\ x \in u \Longrightarrow h\ x = k\ x) \Longrightarrow Q\ h = Q\ k$
    and *hom*: $\bigwedge f\ g.\ [\![$*continuous-on* u f; f ‘ $u \subseteq s$; P f;
        continuous-on u g; g ‘ $u \subseteq s$; P $g]\!]$
        $\Longrightarrow$ *homotopic-with* P u s f g
    and *contf*: *continuous-on* u f **and** *imf*: f ‘ $u \subseteq t$ **and** *Qf*: Q f
    and *contg*: *continuous-on* u g **and** *img*: g ‘ $u \subseteq t$ **and** *Qg*: Q g
  shows *homotopic-with* Q u t f g
proof −
  have *feq*: $\bigwedge x.\ x \in u \Longrightarrow (h \circ (k \circ f))\ x = f\ x$ **using** *idhk imf* **by** *auto*
  have *geq*: $\bigwedge x.\ x \in u \Longrightarrow (h \circ (k \circ g))\ x = g\ x$ **using** *idhk img* **by** *auto*
  have *continuous-on* u $(k \circ f)$
    using *contf continuous-on-compose continuous-on-subset contk imf* **by** *blast*
  moreover have $(k \circ f)$ ‘ $u \subseteq s$
    using *imf imk* **by** *fastforce*
  moreover have P $(k \circ f)$
    by (*simp add*: P *Qf contf imf*)
  moreover have *continuous-on* u $(k \circ g)$
    using *contg continuous-on-compose continuous-on-subset contk img* **by** *blast*
  moreover have $(k \circ g)$ ‘ $u \subseteq s$
    using *img imk* **by** *fastforce*
  moreover have P $(k \circ g)$
    by (*simp add*: P *Qg contg img*)
  ultimately have *homotopic-with* P u s $(k \circ f)$ $(k \circ g)$
    by (*rule hom*)
  then have *homotopic-with* Q u t $(h \circ (k \circ f))$ $(h \circ (k \circ g))$
   apply (*rule homotopic-with-compose-continuous-left* [*OF homotopic-with-mono*])
    using Q **by** (*auto simp*: *conth imh*)

then show *?thesis*
apply (*rule homotopic-with-eq*)
apply (*metis feq*)
apply (*metis geq*)
apply (*metis Qeq*)
done
qed

lemma *homotopically-trivial-retraction-null-gen*:
assumes *P*: $\bigwedge f.\ [\![\mathit{continuous\text{-}on}\ u\ f;\ f\ `\ u \subseteq t;\ Q\ f]\!] \Longrightarrow P(k\ o\ f)$
and *Q*: $\bigwedge f.\ [\![\mathit{continuous\text{-}on}\ u\ f;\ f\ `\ u \subseteq s;\ P\ f]\!] \Longrightarrow Q(h\ o\ f)$
and *Qeq*: $\bigwedge h\ k.\ (\bigwedge x.\ x \in u \Longrightarrow h\ x = k\ x) \Longrightarrow Q\ h = Q\ k$
and *hom*: $\bigwedge f.\ [\![\mathit{continuous\text{-}on}\ u\ f;\ f\ `\ u \subseteq s;\ P\ f]\!]$
$\Longrightarrow \exists c.\ \mathit{homotopic\text{-}with}\ P\ u\ s\ f\ (\lambda x.\ c)$
and *contf*: *continuous-on u f* **and** *imf*:$f\ `\ u \subseteq t$ **and** *Qf*: *Q f*
obtains *c* **where** *homotopic-with Q u t f* $(\lambda x.\ c)$
proof −
have *feq*: $\bigwedge x.\ x \in u \Longrightarrow (h \circ (k \circ f))\ x = f\ x$ **using** *idhk imf* **by** *auto*
have *continuous-on u* $(k \circ f)$
using *contf continuous-on-compose continuous-on-subset contk imf* **by** *blast*
moreover have $(k \circ f)\ `\ u \subseteq s$
using *imf imk* **by** *fastforce*
moreover have *P* $(k \circ f)$
by (*simp add*: *P Qf contf imf*)
ultimately obtain *c* **where** *homotopic-with P u s* $(k \circ f)$ $(\lambda x.\ c)$
by (*metis hom*)
then have *homotopic-with Q u t* $(h \circ (k \circ f))$ $(h\ o\ (\lambda x.\ c))$
apply (*rule homotopic-with-compose-continuous-left* [*OF homotopic-with-mono*])
using *Q* **by** (*auto simp*: *conth imh*)
then show *?thesis*
apply (*rule-tac c* = *h c* **in** *that*)
apply (*erule homotopic-with-eq*)
apply (*metis feq*, *simp*)
apply (*metis Qeq*)
done
qed

lemma *cohomotopically-trivial-retraction-gen*:
assumes *P*: $\bigwedge f.\ [\![\mathit{continuous\text{-}on}\ t\ f;\ f\ `\ t \subseteq u;\ Q\ f]\!] \Longrightarrow P(f\ o\ h)$
and *Q*: $\bigwedge f.\ [\![\mathit{continuous\text{-}on}\ s\ f;\ f\ `\ s \subseteq u;\ P\ f]\!] \Longrightarrow Q(f\ o\ k)$
and *Qeq*: $\bigwedge h\ k.\ (\bigwedge x.\ x \in t \Longrightarrow h\ x = k\ x) \Longrightarrow Q\ h = Q\ k$
and *hom*: $\bigwedge f\ g.\ [\![\mathit{continuous\text{-}on}\ s\ f;\ f\ `\ s \subseteq u;\ P\ f;$
continuous-on s g; $g\ `\ s \subseteq u$; *P g*$]\!]$
$\Longrightarrow \mathit{homotopic\text{-}with}\ P\ s\ u\ f\ g$
and *contf*: *continuous-on t f* **and** *imf*: $f\ `\ t \subseteq u$ **and** *Qf*: *Q f*
and *contg*: *continuous-on t g* **and** *img*: $g\ `\ t \subseteq u$ **and** *Qg*: *Q g*
shows *homotopic-with Q t u f g*
proof −
have *feq*: $\bigwedge x.\ x \in t \Longrightarrow (f \circ h \circ k)\ x = f\ x$ **using** *idhk imf* **by** *auto*

```
  have geq: ⋀x. x ∈ t ⟹ (g ∘ h ∘ k) x = g x using idhk img by auto
  have continuous-on s (f ∘ h)
    using contf conth continuous-on-compose imh by blast
  moreover have (f ∘ h) ‘ s ⊆ u
    using imf imh by fastforce
  moreover have P (f ∘ h)
    by (simp add: P Qf contf imf)
  moreover have continuous-on s (g o h)
    using contg continuous-on-compose continuous-on-subset conth imh by blast
  moreover have (g ∘ h) ‘ s ⊆ u
    using img imh by fastforce
  moreover have P (g ∘ h)
    by (simp add: P Qg contg img)
  ultimately have homotopic-with P s u (f o h) (g ∘ h)
    by (rule hom)
  then have homotopic-with Q t u (f o h o k) (g ∘ h o k)
   apply (rule homotopic-with-compose-continuous-right [OF homotopic-with-mono])
    using Q by (auto simp: contk imk)
  then show ?thesis
    apply (rule homotopic-with-eq)
    apply (metis feq)
    apply (metis geq)
    apply (metis Qeq)
    done
qed

lemma cohomotopically-trivial-retraction-null-gen:
  assumes P: ⋀f. ⟦continuous-on t f; f ‘ t ⊆ u; Q f⟧ ⟹ P(f o h)
      and Q: ⋀f. ⟦continuous-on s f; f ‘ s ⊆ u; P f⟧ ⟹ Q(f o k)
      and Qeq: ⋀h k. (⋀x. x ∈ t ⟹ h x = k x) ⟹ Q h = Q k
      and hom: ⋀f g. ⟦continuous-on s f; f ‘ s ⊆ u; P f⟧
                     ⟹ ∃c. homotopic-with P s u f (λx. c)
      and contf: continuous-on t f and imf: f ‘ t ⊆ u and Qf: Q f
  obtains c where homotopic-with Q t u f (λx. c)
proof −
  have feq: ⋀x. x ∈ t ⟹ (f ∘ h ∘ k) x = f x using idhk imf by auto
  have continuous-on s (f ∘ h)
    using contf conth continuous-on-compose imh by blast
  moreover have (f ∘ h) ‘ s ⊆ u
    using imf imh by fastforce
  moreover have P (f ∘ h)
    by (simp add: P Qf contf imf)
  ultimately obtain c where homotopic-with P s u (f o h) (λx. c)
    by (metis hom)
  then have homotopic-with Q t u (f o h o k) ((λx. c) o k)
   apply (rule homotopic-with-compose-continuous-right [OF homotopic-with-mono])
    using Q by (auto simp: contk imk)
  then show ?thesis
    apply (rule-tac c = c in that)
```

apply (*erule homotopic-with-eq*)
apply (*metis feq*, *simp*)
apply (*metis Qeq*)
done
qed

end

lemma *simply-connected-retraction-gen*:
shows ⟦*simply-connected S*; *continuous-on S h*; *h ' S = T*;
continuous-on T k; *k ' T ⊆ S*; ⋀*y*. *y ∈ T ⟹ h*(*k y*) = *y*⟧
⟹ *simply-connected T*
apply (*simp add*: *simply-connected-def path-def path-image-def homotopic-loops-def*, *clarify*)
apply (*rule Retracts.homotopically-trivial-retraction-gen*
[*of S h - k - λp*. *pathfinish p = pathstart p λp*. *pathfinish p = pathstart p*])
apply (*simp-all add*: *Retracts-def pathfinish-def pathstart-def*)
done

lemma *homeomorphic-simply-connected*:
⟦*S homeomorphic T*; *simply-connected S*⟧ ⟹ *simply-connected T*
by (*auto simp*: *homeomorphic-def homeomorphism-def intro*: *simply-connected-retraction-gen*)

lemma *homeomorphic-simply-connected-eq*:
S homeomorphic T ⟹ (*simply-connected S* ⟷ *simply-connected T*)
by (*metis homeomorphic-simply-connected homeomorphic-sym*)

end

26 Results connected with topological dimension.

theory *Brouwer-Fixpoint*
imports *Path-Connected*
begin

lemma *bij-betw-singleton-eq*:
assumes *f*: *bij-betw f A B* **and** *g*: *bij-betw g A B* **and** *a*: *a ∈ A*
assumes *eq*: (⋀*x*. *x ∈ A ⟹ x ≠ a ⟹ f x = g x*)
shows *f a = g a*
proof −
have *f ' (A − {a}) = g ' (A − {a})*
by (*intro image-cong*) (*simp-all add*: *eq*)
then have *B − {f a} = B − {g a}*
using *f g a* **by** (*auto simp*: *bij-betw-def inj-on-image-set-diff set-eq-iff Diff-subset*)
moreover have *f a ∈ B g a ∈ B*
using *f g a* **by** (*auto simp*: *bij-betw-def*)
ultimately show *?thesis*
by *auto*
qed

lemma *swap-image*:
Fun.swap i j f ‘ A = (if i ∈ A then (if j ∈ A then f ‘ A else f ‘ ((A − {i}) ∪ {j}))
else (if j ∈ A then f ‘ ((A − {j}) ∪ {i}) else f ‘ A))
apply (*auto simp*: *Fun.swap-def image-iff*)
apply *metis*
apply (*metis member-remove remove-def*)
apply (*metis member-remove remove-def*)
done

lemma *swap-apply1*: *Fun.swap x y f x = f y*
by (*simp add*: *Fun.swap-def*)

lemma *swap-apply2*: *Fun.swap x y f y = f x*
by (*simp add*: *Fun.swap-def*)

lemma *lessThan-empty-iff*: *{..< n::nat} = {} ⟷ n = 0*
by *auto*

lemma *Zero-notin-Suc*: *0 ∉ Suc ‘ A*
by *auto*

lemma *atMost-Suc-eq-insert-0*: *{.. Suc n} = insert 0 (Suc ‘ {.. n})*
apply *auto*
apply (*case-tac x*)
apply *auto*
done

lemma *setsum-union-disjoint'*:
assumes *finite A*
and *finite B*
and *A ∩ B = {}*
and *A ∪ B = C*
shows *setsum g C = setsum g A + setsum g B*
using *setsum.union-disjoint*[*OF assms*(*1−3*)] **and** *assms*(*4*) **by** *auto*

lemma *pointwise-minimal-pointwise-maximal*:
fixes *s :: (nat ⇒ nat) set*
assumes *finite s*
and *s ≠ {}*
and *∀ x∈s. ∀ y∈s. x ≤ y ∨ y ≤ x*
shows *∃ a∈s. ∀ x∈s. a ≤ x*
and *∃ a∈s. ∀ x∈s. x ≤ a*
using *assms*
proof (*induct s rule*: *finite-ne-induct*)
case (*insert b s*)
assume ∗: *∀ x∈insert b s. ∀ y∈insert b s. x ≤ y ∨ y ≤ x*
moreover then obtain *u l* **where** *l ∈ s ∀ b∈s. l ≤ b u ∈ s ∀ b∈s. b ≤ u*

using *insert* **by** *auto*
ultimately show $\exists\, a{\in}insert\ b\ s.\ \forall\, x{\in}insert\ b\ s.\ a \leq x\ \exists\, a{\in}insert\ b\ s.\ \forall\, x{\in}insert\ b\ s.\ x \leq a$
using $*[rule\text{-}format,\ of\ b\ u]$ $*[rule\text{-}format,\ of\ b\ l]$ **by** (*metis insert-iff order.trans*)+
qed *auto*

lemma *brouwer-compactness-lemma*:
fixes $f :: 'a{::}metric\text{-}space \Rightarrow 'b{::}real\text{-}normed\text{-}vector$
assumes *compact s*
and *continuous-on s f*
and $\neg\ (\exists\, x{\in}s.\ f\ x = 0)$
obtains d **where** $0 < d$ **and** $\forall\, x{\in}s.\ d \leq norm\ (f\ x)$
proof (*cases* $s = \{\}$)
case *True*
show *thesis*
by (*rule that* [*of 1*]) (*auto simp*: *True*)
next
case *False*
have *continuous-on s* $(norm \circ f)$
by (*rule continuous-intros continuous-on-norm assms*(*2*))+
with *False* **obtain** x **where** x: $x \in s$ $\forall\, y{\in}s.\ (norm \circ f)\ x \leq (norm \circ f)\ y$
using *continuous-attains-inf*[*OF assms*(*1*), *of norm* $\circ$ *f*]
unfolding *o-def*
by *auto*
have $(norm \circ f)\ x > 0$
using *assms*(*3*) **and** $x(1)$
by *auto*
then show *?thesis*
by (*rule that*) (*insert x*(*2*), *auto simp*: *o-def*)
qed

lemma *kuhn-labelling-lemma*:
fixes $P\ Q :: 'a{::}euclidean\text{-}space \Rightarrow bool$
assumes $\forall\, x.\ P\ x \longrightarrow P\ (f\ x)$
and $\forall\, x.\ P\ x \longrightarrow (\forall\, i{\in}Basis.\ Q\ i \longrightarrow 0 \leq x{\cdot}i \wedge x{\cdot}i \leq 1)$
shows $\exists\, l.\ (\forall\, x.\forall\, i{\in}Basis.\ l\ x\ i \leq (1{::}nat)) \wedge$
$(\forall\, x.\forall\, i{\in}Basis.\ P\ x \wedge Q\ i \wedge (x{\cdot}i = 0) \longrightarrow (l\ x\ i = 0)) \wedge$
$(\forall\, x.\forall\, i{\in}Basis.\ P\ x \wedge Q\ i \wedge (x{\cdot}i = 1) \longrightarrow (l\ x\ i = 1)) \wedge$
$(\forall\, x.\forall\, i{\in}Basis.\ P\ x \wedge Q\ i \wedge (l\ x\ i = 0) \longrightarrow x{\cdot}i \leq f\ x{\cdot}i) \wedge$
$(\forall\, x.\forall\, i{\in}Basis.\ P\ x \wedge Q\ i \wedge (l\ x\ i = 1) \longrightarrow f\ x{\cdot}i \leq x{\cdot}i)$
proof −
{ fix $x\ i$
let $?R = \lambda y.\ (P\ x \wedge Q\ i \wedge x \cdot i = 0 \longrightarrow y = (0{::}nat)) \wedge$
$(P\ x \wedge Q\ i \wedge x \cdot i = 1 \longrightarrow y = 1) \wedge$
$(P\ x \wedge Q\ i \wedge y = 0 \longrightarrow x \cdot i \leq f\ x \cdot i) \wedge$
$(P\ x \wedge Q\ i \wedge y = 1 \longrightarrow f\ x \cdot i \leq x \cdot i)$
{ assume $P\ x\ Q\ i\ i \in Basis$ **with** *assms* **have** $0 \leq f\ x \cdot i \wedge f\ x \cdot i \leq 1$ **by** *auto* **}**

then have $i \in Basis \Longrightarrow ?R\ 0 \vee ?R\ 1$ **by** *auto* **}**
then show *?thesis*
unfolding *all-conj-distrib*[*symmetric*] *Ball-def*
by (*subst choice-iff*[*symmetric*])+ *blast*
qed

26.1 The key "counting" observation, somewhat abstracted.

lemma *kuhn-counting-lemma*:
fixes *bnd compo compo' face S F*
defines $nF\ s == card\ \{f{\in}F.\ face\ f\ s \wedge compo'\ f\}$
assumes [*simp*, *intro*]: *finite F* — faces **and** [*simp*, *intro*]: *finite S* — simplices
and $\bigwedge f.\ f \in F \Longrightarrow bnd\ f \Longrightarrow card\ \{s{\in}S.\ face\ f\ s\} = 1$
and $\bigwedge f.\ f \in F \Longrightarrow \neg\ bnd\ f \Longrightarrow card\ \{s{\in}S.\ face\ f\ s\} = 2$
and $\bigwedge s.\ s \in S \Longrightarrow compo\ s \Longrightarrow nF\ s = 1$
and $\bigwedge s.\ s \in S \Longrightarrow \neg\ compo\ s \Longrightarrow nF\ s = 0 \vee nF\ s = 2$
and $odd\ (card\ \{f{\in}F.\ compo'\ f \wedge bnd\ f\})$
shows $odd\ (card\ \{s{\in}S.\ compo\ s\})$
proof −
have $(\sum s \mid s \in S \wedge \neg\ compo\ s.\ nF\ s) + (\sum s \mid s \in S \wedge compo\ s.\ nF\ s) = (\sum s{\in}S.\ nF\ s)$
by (*subst setsum.union-disjoint*[*symmetric*]) (*auto intro*!: *setsum.cong*)
also have $\ldots = (\sum s{\in}S.\ card\ \{f \in \{f{\in}F.\ compo'\ f \wedge bnd\ f\}.\ face\ f\ s\}) + (\sum s{\in}S.\ card\ \{f \in \{f{\in}F.\ compo'\ f \wedge \neg\ bnd\ f\}.\ face\ f\ s\})$
unfolding *setsum.distrib*[*symmetric*]
by (*subst card-Un-disjoint*[*symmetric*])
(*auto simp*: *nF-def intro*!: *setsum.cong arg-cong*[**where** *f*=*card*])
also have $\ldots = 1 * card\ \{f{\in}F.\ compo'\ f \wedge bnd\ f\} + 2 * card\ \{f{\in}F.\ compo'\ f \wedge \neg\ bnd\ f\}$
using *assms*(*4*,*5*) **by** (*fastforce intro*!: *arg-cong2*[**where** *f*=*op +*] *setsum-multicount*)
finally have $odd\ ((\sum s \mid s \in S \wedge \neg\ compo\ s.\ nF\ s) + card\ \{s{\in}S.\ compo\ s\})$
using *assms*(*6*,*8*) **by** *simp*
moreover have $(\sum s \mid s \in S \wedge \neg\ compo\ s.\ nF\ s) = (\sum s \mid s \in S \wedge \neg\ compo\ s \wedge nF\ s = 0.\ nF\ s) + (\sum s \mid s \in S \wedge \neg\ compo\ s \wedge nF\ s = 2.\ nF\ s)$
using *assms*(*7*) **by** (*subst setsum.union-disjoint*[*symmetric*]) (*fastforce intro*!: *setsum.cong*)+
ultimately show *?thesis*
by *auto*
qed

26.2 The odd/even result for faces of complete vertices, generalized.

lemma *kuhn-complete-lemma*:
assumes [*simp*]: *finite simplices*
and *face*: $\bigwedge f\ s.\ face\ f\ s \longleftrightarrow (\exists a{\in}s.\ f = s - \{a\})$
and *card-s*[*simp*]: $\bigwedge s.\ s \in simplices \Longrightarrow card\ s = n + 2$
and *rl-bd*: $\bigwedge s.\ s \in simplices \Longrightarrow rl\ `\ s \subseteq \{..Suc\ n\}$

and *bnd*: $\bigwedge f\ s.\ s \in simplices \Longrightarrow face\ f\ s \Longrightarrow bnd\ f \Longrightarrow card\ \{s{\in}simplices.\ face\ f\ s\} = 1$
and *nbnd*: $\bigwedge f\ s.\ s \in simplices \Longrightarrow face\ f\ s \Longrightarrow \neg\ bnd\ f \Longrightarrow card\ \{s{\in}simplices.\ face\ f\ s\} = 2$
and *odd-card*: $odd\ (card\ \{f.\ (\exists\, s{\in}simplices.\ face\ f\ s) \wedge rl\ `\ f = \{..n\} \wedge bnd\ f\})$
shows $odd\ (card\ \{s{\in}simplices.\ (rl\ `\ s = \{..Suc\ n\})\})$
proof (*rule kuhn-counting-lemma*)
have *finite-s*[*simp*]: $\bigwedge s.\ s \in simplices \Longrightarrow finite\ s$
by (*metis add-is-0 zero-neq-numeral card-infinite assms*(*3*))

let *?F* = $\{f.\ \exists\, s{\in}simplices.\ face\ f\ s\}$
have *F-eq*: *?F* = $(\bigcup s{\in}simplices.\ \bigcup a{\in}s.\ \{s - \{a\}\})$
by (*auto simp*: *face*)
show *finite ?F*
using ‹*finite simplices*› **unfolding** *F-eq* **by** *auto*

show $card\ \{s \in simplices.\ face\ f\ s\} = 1$ **if** $f \in ?F\ bnd\ f$ **for** *f*
using *bnd that* **by** *auto*

show $card\ \{s \in simplices.\ face\ f\ s\} = 2$ **if** $f \in ?F\ \neg\ bnd\ f$ **for** *f*
using *nbnd that* **by** *auto*

show $odd\ (card\ \{f \in \{f.\ \exists\, s{\in}simplices.\ face\ f\ s\}.\ rl\ `\ f = \{..n\} \wedge bnd\ f\})$
using *odd-card* **by** *simp*

fix *s* **assume** *s*[*simp*]: $s \in simplices$
let *?S* = $\{f \in \{f.\ \exists\, s{\in}simplices.\ face\ f\ s\}.\ face\ f\ s \wedge rl\ `\ f = \{..n\}\}$
have *?S* = $(\lambda a.\ s - \{a\})\ `\ \{a{\in}s.\ rl\ `\ (s - \{a\}) = \{..n\}\}$
using *s* **by** (*fastforce simp*: *face*)
then have *card-S*: *card ?S* = *card* $\{a{\in}s.\ rl\ `\ (s - \{a\}) = \{..n\}\}$
by (*auto intro*!: *card-image inj-onI*)

{ assume *rl*: $rl\ `\ s = \{..Suc\ n\}$
then have *inj-rl*: *inj-on rl s*
by (*intro eq-card-imp-inj-on*) *auto*
moreover obtain *a* **where** $rl\ a = Suc\ n\ a \in s$
by (*metis atMost-iff image-iff le-Suc-eq rl*)
ultimately have *n*: $\{..n\} = rl\ `\ (s - \{a\})$
by (*auto simp add*: *inj-on-image-set-diff Diff-subset rl*)
have $\{a{\in}s.\ rl\ `\ (s - \{a\}) = \{..n\}\} = \{a\}$
using *inj-rl* ‹*a* ∈ *s*› **by** (*auto simp add*: *n inj-on-image-eq-iff*[*OF inj-rl*] *Diff-subset*)
then show *card ?S* = *1*
unfolding *card-S* **by** *simp* **}**

{ assume *rl*: $rl\ `\ s \neq \{..Suc\ n\}$
show *card ?S* = *0* ∨ *card ?S* = *2*
proof *cases*
assume ∗: $\{..n\} \subseteq rl\ `\ s$

```
    with rl rl-bd[OF s] have rl-s: rl ‘ s = {..n}
      by (auto simp add: atMost-Suc subset-insert-iff split: if-split-asm)
    then have ¬ inj-on rl s
      by (intro pigeonhole) simp
    then obtain a b where ab: a ∈ s b ∈ s rl a = rl b a ≠ b
      by (auto simp: inj-on-def)
    then have eq: rl ‘ (s − {a}) = rl ‘ s
      by auto
    with ab have inj: inj-on rl (s − {a})
      by (intro eq-card-imp-inj-on) (auto simp add: rl-s card-Diff-singleton-if)

    { fix x assume x ∈ s x ∉ {a, b}
      then have rl ‘ s − {rl x} = rl ‘ ((s − {a}) − {x})
        by (auto simp: eq Diff-subset inj-on-image-set-diff[OF inj])
      also have ... = rl ‘ (s − {x})
        using ab ‹x ∉ {a, b}› by auto
      also assume ... = rl ‘ s
      finally have False
        using ‹x∈s› by auto }
    moreover
    { fix x assume x ∈ {a, b} with ab have x ∈ s ∧ rl ‘ (s − {x}) = rl ‘ s
        by (simp add: set-eq-iff image-iff Bex-def) metis }
    ultimately have {a∈s. rl ‘ (s − {a}) = {..n}} = {a, b}
      unfolding rl-s[symmetric] by fastforce
    with ‹a ≠ b› show card ?S = 0 ∨ card ?S = 2
      unfolding card-S by simp
  next
    assume ¬ {..n} ⊆ rl ‘ s
    then have ⋀x. rl ‘ (s − {x}) ≠ {..n}
      by auto
    then show card ?S = 0 ∨ card ?S = 2
      unfolding card-S by simp
  qed }
qed fact

locale kuhn-simplex =
  fixes p n and base upd and s :: (nat ⇒ nat) set
  assumes base: base ∈ {..< n} → {..< p}
  assumes base-out: ⋀i. n ≤ i ⟹ base i = p
  assumes upd: bij-betw upd {..< n} {..< n}
  assumes s-pre: s = (λi j. if j ∈ upd‘{..< i} then Suc (base j) else base j) ‘ {..
n}
begin

definition enum i j = (if j ∈ upd‘{..< i} then Suc (base j) else base j)

lemma s-eq: s = enum ‘ {.. n}
  unfolding s-pre enum-def[abs-def] ..
```

lemma *upd-space*: $i < n \Longrightarrow upd\ i < n$
using *upd* **by** (*auto dest*!: *bij-betwE*)

lemma *s-space*: $s \subseteq \{..< n\} \to \{..\ p\}$
proof −
{ fix i **assume** $i \leq n$ **then have** $enum\ i \in \{..< n\} \to \{..\ p\}$
proof (*induct i*)
case *0* **then show** *?case*
using *base* **by** (*auto simp*: *Pi-iff less-imp-le enum-def*)
next
case (*Suc i*) **with** *base* **show** *?case*
by (*auto simp*: *Pi-iff Suc-le-eq less-imp-le enum-def intro*: *upd-space*)
qed }
then show *?thesis*
by (*auto simp*: *s-eq*)
qed

lemma *inj-upd*: *inj-on upd* $\{..< n\}$
using *upd* **by** (*simp add*: *bij-betw-def*)

lemma *inj-enum*: *inj-on enum* $\{..\ n\}$
proof −
{ fix $x\ y :: nat$ **assume** $x \neq y\ x \leq n\ y \leq n$
with *upd* **have** *upd* ‘ $\{..< x\} \neq$ *upd* ‘ $\{..< y\}$
by (*subst inj-on-image-eq-iff* [**where** C=$\{..< n\}$]) (*auto simp*: *bij-betw-def*)
then have $enum\ x \neq enum\ y$
by (*auto simp add*: *enum-def fun-eq-iff*) **}**
then show *?thesis*
by (*auto simp*: *inj-on-def*)
qed

lemma *enum-0*: $enum\ 0 = base$
by (*simp add*: *enum-def* [*abs-def*])

lemma *base-in-s*: $base \in s$
unfolding *s-eq* **by** (*subst enum-0*[*symmetric*]) *auto*

lemma *enum-in*: $i \leq n \Longrightarrow enum\ i \in s$
unfolding *s-eq* **by** *auto*

lemma *one-step*:
assumes *a*: $a \in s\ j < n$
assumes $*$: $\bigwedge a'.\ a' \in s \Longrightarrow a' \neq a \Longrightarrow a'\ j = p'$
shows $a\ j \neq p'$
proof
assume $a\ j = p'$
with $*\ a$ **have** $\bigwedge a'.\ a' \in s \Longrightarrow a'\ j = p'$
by *auto*
then have $\bigwedge i.\ i \leq n \Longrightarrow enum\ i\ j = p'$

unfolding *s-eq* **by** *auto*
from *this*[*of 0*] *this*[*of n*] **have** $j \notin upd\ `\ \{..< n\}$
by (*auto simp*: *enum-def fun-eq-iff split*: *if-split-asm*)
with *upd* ‹$j < n$› **show** *False*
by (*auto simp*: *bij-betw-def*)
qed

lemma *upd-inj*: $i < n \Longrightarrow j < n \Longrightarrow upd\ i = upd\ j \longleftrightarrow i = j$
using *upd* **by** (*auto simp*: *bij-betw-def inj-on-eq-iff*)

lemma *upd-surj*: $upd\ `\ \{..< n\} = \{..< n\}$
using *upd* **by** (*auto simp*: *bij-betw-def*)

lemma *in-upd-image*: $A \subseteq \{..< n\} \Longrightarrow i < n \Longrightarrow upd\ i \in upd\ `\ A \longleftrightarrow i \in A$
using *inj-on-image-mem-iff*[*of upd* $\{..< n\}$] *upd*
by (*auto simp*: *bij-betw-def*)

lemma *enum-inj*: $i \leq n \Longrightarrow j \leq n \Longrightarrow enum\ i = enum\ j \longleftrightarrow i = j$
using *inj-enum* **by** (*auto simp*: *inj-on-eq-iff*)

lemma *in-enum-image*: $A \subseteq \{..\ n\} \Longrightarrow i \leq n \Longrightarrow enum\ i \in enum\ `\ A \longleftrightarrow i \in A$
using *inj-on-image-mem-iff*[*OF inj-enum*] **by** *auto*

lemma *enum-mono*: $i \leq n \Longrightarrow j \leq n \Longrightarrow enum\ i \leq enum\ j \longleftrightarrow i \leq j$
by (*auto simp*: *enum-def le-fun-def in-upd-image Ball-def*[*symmetric*])

lemma *enum-strict-mono*: $i \leq n \Longrightarrow j \leq n \Longrightarrow enum\ i < enum\ j \longleftrightarrow i < j$
using *enum-mono*[*of i j*] *enum-inj*[*of i j*] **by** (*auto simp add*: *le-less*)

lemma *chain*: $a \in s \Longrightarrow b \in s \Longrightarrow a \leq b \vee b \leq a$
by (*auto simp*: *s-eq enum-mono*)

lemma *less*: $a \in s \Longrightarrow b \in s \Longrightarrow a\ i < b\ i \Longrightarrow a < b$
using *chain*[*of a b*] **by** (*auto simp*: *less-fun-def le-fun-def not-le*[*symmetric*])

lemma *enum-0-bot*: $a \in s \Longrightarrow a = enum\ 0 \longleftrightarrow (\forall\, a' \in s.\ a \leq a')$
unfolding *s-eq* **by** (*auto simp*: *enum-mono Ball-def*)

lemma *enum-n-top*: $a \in s \Longrightarrow a = enum\ n \longleftrightarrow (\forall\, a' \in s.\ a' \leq a)$
unfolding *s-eq* **by** (*auto simp*: *enum-mono Ball-def*)

lemma *enum-Suc*: $i < n \Longrightarrow enum\ (Suc\ i) = (enum\ i)(upd\ i := Suc\ (enum\ i\ (upd\ i)))$
by (*auto simp*: *fun-eq-iff enum-def upd-inj*)

lemma *enum-eq-p*: $i \leq n \Longrightarrow n \leq j \Longrightarrow enum\ i\ j = p$
by (*induct i*) (*auto simp*: *enum-Suc enum-0 base-out upd-space not-less*[*symmetric*])

lemma *out-eq-p*: $a \in s \Longrightarrow n \leq j \Longrightarrow a\ j = p$

unfolding *s-eq* **by** (*auto simp add*: *enum-eq-p*)

lemma *s-le-p*: $a \in s \Longrightarrow a\ j \leq p$
using *out-eq-p*[*of a j*] *s-space* **by** (*cases* $j < n$) *auto*

lemma *le-Suc-base*: $a \in s \Longrightarrow a\ j \leq Suc\ (base\ j)$
unfolding *s-eq* **by** (*auto simp*: *enum-def*)

lemma *base-le*: $a \in s \Longrightarrow base\ j \leq a\ j$
unfolding *s-eq* **by** (*auto simp*: *enum-def*)

lemma *enum-le-p*: $i \leq n \Longrightarrow j < n \Longrightarrow enum\ i\ j \leq p$
using *enum-in*[*of i*] *s-space* **by** *auto*

lemma *enum-less*: $a \in s \Longrightarrow i < n \Longrightarrow enum\ i < a \longleftrightarrow enum\ (Suc\ i) \leq a$
unfolding *s-eq* **by** (*auto simp*: *enum-strict-mono enum-mono*)

lemma *ksimplex-0*:
$n = 0 \Longrightarrow s = \{(\lambda x.\ p)\}$
using *s-eq enum-def base-out* **by** *auto*

lemma *replace-0*:
assumes $j < n\ a \in s$ **and** p: $\forall x \in s - \{a\}.\ x\ j = 0$ **and** $x \in s$
shows $x \leq a$
proof *cases*
assume $x \neq a$
have $a\ j \neq 0$
using *assms* **by** (*intro one-step*[**where** $a{=}a$]) *auto*
with *less*[*OF* ‹$x \in s$› ‹$a \in s$›, *of j*] *p*[*rule-format*, *of x*] ‹$x \in s$› ‹$x \neq a$›
show *?thesis*
by *auto*
qed *simp*

lemma *replace-1*:
assumes $j < n\ a \in s$ **and** p: $\forall x \in s - \{a\}.\ x\ j = p$ **and** $x \in s$
shows $a \leq x$
proof *cases*
assume $x \neq a$
have $a\ j \neq p$
using *assms* **by** (*intro one-step*[**where** $a{=}a$]) *auto*
with *enum-le-p*[*of - j*] ‹$j < n$› ‹$a \in s$›
have $a\ j < p$
by (*auto simp*: *less-le s-eq*)
with *less*[*OF* ‹$a \in s$› ‹$x \in s$›, *of j*] *p*[*rule-format*, *of x*] ‹$x \in s$› ‹$x \neq a$›
show *?thesis*
by *auto*
qed *simp*

end

locale *kuhn-simplex-pair* = *s*: *kuhn-simplex p n b-s u-s s* + *t*: *kuhn-simplex p n b-t u-t t*
  for *p n b-s u-s s b-t u-t t*
begin

lemma *enum-eq*:
  assumes *l*: $i \leq l\ l \leq j$ **and** $j + d \leq n$
  assumes *eq*: *s.enum* ‘ $\{i .. j\}$ = *t.enum* ‘ $\{i + d .. j + d\}$
  shows *s.enum l* = *t.enum* $(l + d)$
using *l* **proof** (*induct l rule*: *dec-induct*)
  case *base*
  then have *s*: *s.enum i* ∈ *t.enum* ‘ $\{i + d .. j + d\}$ **and** *t*: *t.enum* $(i + d)$ ∈ *s.enum* ‘ $\{i .. j\}$
    using *eq* **by** *auto*
  from *t* ‹$i \leq j$› ‹$j + d \leq n$› **have** *s.enum i* ≤ *t.enum* $(i + d)$
    by (*auto simp*: *s.enum-mono*)
  moreover from *s* ‹$i \leq j$› ‹$j + d \leq n$› **have** *t.enum* $(i + d)$ ≤ *s.enum i*
    by (*auto simp*: *t.enum-mono*)
  ultimately show *?case*
    by *auto*
next
  case (*step l*)
  moreover from *step.prems* ‹$j + d \leq n$› **have**
      s.enum l < *s.enum* (*Suc l*)
      t.enum $(l + d)$ < *t.enum* (*Suc l* + *d*)
    by (*simp-all add*: *s.enum-strict-mono t.enum-strict-mono*)
  moreover have
      s.enum (*Suc l*) ∈ *t.enum* ‘ $\{i + d .. j + d\}$
      t.enum (*Suc l* + *d*) ∈ *s.enum* ‘ $\{i .. j\}$
    using *step* ‹$j + d \leq n$› *eq* **by** (*auto simp*: *s.enum-inj t.enum-inj*)
  ultimately have *s.enum* (*Suc l*) = *t.enum* (*Suc* $(l + d)$)
    using ‹$j + d \leq n$›
    by (*intro antisym s.enum-less*[*THEN iffD1*] *t.enum-less*[*THEN iffD1*])
      (*auto intro*!: *s.enum-in t.enum-in*)
  then show *?case* **by** *simp*
qed

lemma *ksimplex-eq-bot*:
  assumes *a*: $a \in s\ \bigwedge a'.\ a' \in s \Longrightarrow a \leq a'$
  assumes *b*: $b \in t\ \bigwedge b'.\ b' \in t \Longrightarrow b \leq b'$
  assumes *eq*: $s - \{a\} = t - \{b\}$
  shows $s = t$
proof *cases*
  assume *n* = *0* **with** *s.ksimplex-0 t.ksimplex-0* **show** *?thesis* **by** *simp*
next
  assume $n \neq 0$
  have *s.enum 0* = (*s.enum* (*Suc 0*)) (*u-s 0* := *s.enum* (*Suc 0*) (*u-s 0*) − *1*)
      t.enum 0 = (*t.enum* (*Suc 0*)) (*u-t 0* := *t.enum* (*Suc 0*) (*u-t 0*) − *1*)

```
    using ‹n ≠ 0› by (simp-all add: s.enum-Suc t.enum-Suc)
  moreover have e0: a = s.enum 0 b = t.enum 0
    using a b by (simp-all add: s.enum-0-bot t.enum-0-bot)
  moreover
  { fix j assume 0 < j j ≤ n
    moreover have s - {a} = s.enum ` {Suc 0 .. n} t - {b} = t.enum ` {Suc 0
.. n}
      unfolding s.s-eq t.s-eq e0 by (auto simp: s.enum-inj t.enum-inj)
    ultimately have s.enum j = t.enum j
      using enum-eq[of 1 j n 0] eq by auto }
  note enum-eq = this
  then have s.enum (Suc 0) = t.enum (Suc 0)
    using ‹n ≠ 0› by auto
  moreover
  { fix j assume Suc j < n
    with enum-eq[of Suc j] enum-eq[of Suc (Suc j)]
    have u-s (Suc j) = u-t (Suc j)
      using s.enum-Suc[of Suc j] t.enum-Suc[of Suc j]
      by (auto simp: fun-eq-iff split: if-split-asm) }
  then have ⋀j. 0 < j ⟹ j < n ⟹ u-s j = u-t j
    by (auto simp: gr0-conv-Suc)
  with ‹n ≠ 0› have u-t 0 = u-s 0
    by (intro bij-betw-singleton-eq[OF t.upd s.upd, of 0]) auto
  ultimately have a = b
    by simp
  with assms show s = t
    by auto
qed

lemma ksimplex-eq-top:
  assumes a: a ∈ s ⋀a'. a' ∈ s ⟹ a' ≤ a
  assumes b: b ∈ t ⋀b'. b' ∈ t ⟹ b' ≤ b
  assumes eq: s - {a} = t - {b}
  shows s = t
proof (cases n)
  assume n = 0 with s.ksimplex-0 t.ksimplex-0 show ?thesis by simp
next
  case (Suc n')
  have s.enum n = (s.enum n') (u-s n' := Suc (s.enum n' (u-s n')))
       t.enum n = (t.enum n') (u-t n' := Suc (t.enum n' (u-t n')))
    using Suc by (simp-all add: s.enum-Suc t.enum-Suc)
  moreover have en: a = s.enum n b = t.enum n
    using a b by (simp-all add: s.enum-n-top t.enum-n-top)
  moreover
  { fix j assume j < n
    moreover have s - {a} = s.enum ` {0 .. n'} t - {b} = t.enum ` {0 .. n'}
      unfolding s.s-eq t.s-eq en by (auto simp: s.enum-inj t.enum-inj Suc)
    ultimately have s.enum j = t.enum j
      using enum-eq[of 0 j n' 0] eq Suc by auto }
```

```
  note enum-eq = this
  then have s.enum n' = t.enum n'
    using Suc by auto
  moreover
  { fix j assume j < n'
    with enum-eq[of j] enum-eq[of Suc j]
    have u-s j = u-t j
      using s.enum-Suc[of j] t.enum-Suc[of j]
      by (auto simp: Suc fun-eq-iff split: if-split-asm) }
  then have ⋀j. j < n' ⟹ u-s j = u-t j
    by (auto simp: gr0-conv-Suc)
  then have u-t n' = u-s n'
    by (intro bij-betw-singleton-eq[OF t.upd s.upd, of n']) (auto simp: Suc)
  ultimately have a = b
    by simp
  with assms show s = t
    by auto
qed

end

inductive ksimplex for p n :: nat where
  ksimplex: kuhn-simplex p n base upd s ⟹ ksimplex p n s

lemma finite-ksimplexes: finite {s. ksimplex p n s}
proof (rule finite-subset)
  { fix a s assume ksimplex p n s a ∈ s
    then obtain b u where kuhn-simplex p n b u s by (auto elim: ksimplex.cases)
    then interpret kuhn-simplex p n b u s .
    from s-space ‹a ∈ s› out-eq-p[OF ‹a ∈ s›]
    have a ∈ (λf x. if n ≤ x then p else f x) ` ({..< n} →_E {.. p})
      by (auto simp: image-iff subset-eq Pi-iff split: if-split-asm
            intro!: bexI[of - restrict a {..< n}]) }
  then show {s. ksimplex p n s} ⊆ Pow ((λf x. if n ≤ x then p else f x) ` ({..<
n} →_E {.. p}))
    by auto
qed (simp add: finite-PiE)

lemma ksimplex-card:
  assumes ksimplex p n s shows card s = Suc n
using assms proof cases
  case (ksimplex u b)
  then interpret kuhn-simplex p n u b s .
  show ?thesis
    by (simp add: card-image s-eq inj-enum)
qed

lemma simplex-top-face:
  assumes 0 < p ∀x∈s'. x n = p
```

shows *ksimplex p n s'* $\longleftrightarrow$ $(\exists\, s\ a.\ ksimplex\ p\ (Suc\ n)\ s \wedge a \in s \wedge s' = s - \{a\})$
using *assms*
proof *safe*
fix *s a* **assume** *ksimplex p (Suc n) s* **and** *a*: $a \in s$ **and** *na*: $\forall\, x{\in}s - \{a\}.\ x\ n = p$
then show *ksimplex p n* $(s - \{a\})$
proof *cases*
case (*ksimplex base upd*)
then interpret *kuhn-simplex p Suc n base upd s* **.**

have $a\ n < p$
using *one-step*[*of a n p*] *na* ‹$a{\in}s$› *s-space* **by** (*auto simp*: *less-le*)
then have $a = enum\ 0$
using ‹$a \in s$› *na* **by** (*subst enum-0-bot*) (*auto simp*: *le-less intro*!: *less*[*of a - n*])
then have *s-eq*: $s - \{a\} = enum\ `\ Suc\ `\ \{..\ n\}$
using *s-eq* **by** (*simp add*: *atMost-Suc-eq-insert-0 insert-ident Zero-notin-Suc in-enum-image subset-eq*)
then have $enum\ 1 \in s - \{a\}$
by *auto*
then have $upd\ 0 = n$
using ‹$a\ n < p$› ‹$a = enum\ 0$› *na*[*rule-format*, *of enum 1*]
by (*auto simp*: *fun-eq-iff enum-Suc split*: *if-split-asm*)
then have *bij-betw upd* $(Suc\ `\ \{..<n\})\ \{..<n\}$
using *upd*
by (*subst notIn-Un-bij-betw3*[**where** *b=0*])
(*auto simp*: *lessThan-Suc*[*symmetric*] *lessThan-Suc-eq-insert-0*)
then have *bij-betw* $(upd{\circ}Suc)\ \{..{<}n\}\ \{..{<}n\}$
by (*rule bij-betw-trans*[*rotated*]) (*auto simp*: *bij-betw-def*)

have $a\ n = p - 1$
using *enum-Suc*[*of 0*] *na*[*rule-format*, *OF* ‹$enum\ 1 \in s - \{a\}$›] ‹$a = enum\ 0$› **by** (*auto simp*: ‹$upd\ 0 = n$›)

show *?thesis*
proof (*rule ksimplex.intros*, *standard*)
show *bij-betw* $(upd{\circ}Suc)\ \{..<n\}\ \{..<n\}$ **by** *fact*
show $base(n := p) \in \{..{<}n\} \to \{..{<}p\} \bigwedge i.\ n{\le}i \Longrightarrow (base(n := p))\ i = p$
using *base base-out* **by** (*auto simp*: *Pi-iff*)

have $\bigwedge i.\ Suc\ `\ \{..<i\} = \{..<Suc\ i\} - \{0\}$
by (*auto simp*: *image-iff Ball-def*) *arith*
then have *upd-Suc*: $\bigwedge i.\ i \le n \Longrightarrow (upd{\circ}Suc)\ `\ \{..<i\} = upd\ `\ \{..<Suc\ i\} - \{n\}$
using ‹$upd\ 0 = n$› *upd-inj*
by (*auto simp add*: *image-comp*[*symmetric*] *inj-on-image-set-diff*[*OF inj-upd*])
have *n-in-upd*: $\bigwedge i.\ n \in upd\ `\ \{..<Suc\ i\}$
using ‹$upd\ 0 = n$› **by** *auto*

```
      def f' ≡ λi j. if j ∈ (upd∘Suc)‘{..< i} then Suc ((base(n := p)) j) else
(base(n := p)) j
    { fix x i assume i[arith]: i ≤ n then have enum (Suc i) x = f' i x
      unfolding f'-def enum-def using ‹a n < p› ‹a = enum 0› ‹upd 0 = n› ‹a
n = p − 1›
       by (simp add: upd-Suc enum-0 n-in-upd) }
    then show s − {a} = f' ‘ {.. n}
     unfolding s-eq image-comp by (intro image-cong) auto
  qed
 qed
next
 assume ksimplex p n s' and ∗: ∀x∈s'. x n = p
 then show ∃ s a. ksimplex p (Suc n) s ∧ a ∈ s ∧ s' = s − {a}
 proof cases
  case (ksimplex base upd)
  then interpret kuhn-simplex p n base upd s' .
  def b ≡ base (n := p − 1)
  def u ≡ λi. case i of 0 ⇒ n | Suc i ⇒ upd i

  have ksimplex p (Suc n) (s' ∪ {b})
  proof (rule ksimplex.intros, standard)
   show b ∈ {..<Suc n} → {..<p}
    using base ‹0 < p› unfolding lessThan-Suc b-def by (auto simp: PiE-iff)
   show ⋀i. Suc n ≤ i ⟹ b i = p
    using base-out by (auto simp: b-def)

   have bij-betw u (Suc ‘ {..< n} ∪ {0}) ({..<n} ∪ {u 0})
    using upd
     by (intro notIn-Un-bij-betw) (auto simp: u-def bij-betw-def image-comp
comp-def inj-on-def)
   then show bij-betw u {..<Suc n} {..<Suc n}
    by (simp add: u-def lessThan-Suc[symmetric] lessThan-Suc-eq-insert-0)

   def f' ≡ λi j. if j ∈ u‘{..< i} then Suc (b j) else b j

   have u-eq: ⋀i. i ≤ n ⟹ u ‘ {..< Suc i} = upd ‘ {..< i} ∪ { n }
    by (auto simp: u-def image-iff upd-inj Ball-def split: nat.split) arith

   { fix x have x ≤ n ⟹ n ∉ upd ‘ {..<x}
     using upd-space by (simp add: image-iff neq-iff) }
   note n-not-upd = this

   have ∗: f' ‘ {.. Suc n} = f' ‘ (Suc ‘ {.. n} ∪ {0})
    unfolding atMost-Suc-eq-insert-0 by simp
   also have ... = (f' ∘ Suc) ‘ {.. n} ∪ {b}
    by (auto simp: f'-def)
   also have (f' ∘ Suc) ‘ {.. n} = s'
    using ‹0 < p› base-out[of n]
    unfolding s-eq enum-def[abs-def] f'-def[abs-def] upd-space
```

```
      by (intro image-cong) (simp-all add: u-eq b-def fun-eq-iff n-not-upd)
    finally show s' ∪ {b} = f' ‘ {.. Suc n} ..
   qed
   moreover have b ∉ s'
    using ∗ ‹0 < p› by (auto simp: b-def)
   ultimately show ?thesis by auto
  qed
qed

lemma ksimplex-replace-0:
  assumes s: ksimplex p n s and a: a ∈ s
  assumes j: j < n and p: ∀ x∈s − {a}. x j = 0
  shows card {s'. ksimplex p n s' ∧ (∃ b∈s'. s' − {b} = s − {a})} = 1
  using s
proof cases
  case (ksimplex b-s u-s)

  { fix t b assume ksimplex p n t
    then obtain b-t u-t where kuhn-simplex p n b-t u-t t
      by (auto elim: ksimplex.cases)
    interpret kuhn-simplex-pair p n b-s u-s s b-t u-t t
      by intro-locales fact+

    assume b: b ∈ t t − {b} = s − {a}
    with a j p s.replace-0[of - a] t.replace-0[of - b] have s = t
      by (intro ksimplex-eq-top[of a b]) auto }
  then have {s'. ksimplex p n s' ∧ (∃ b∈s'. s' − {b} = s − {a})} = {s}
    using s ‹a ∈ s› by auto
  then show ?thesis
    by simp
qed

lemma ksimplex-replace-1:
  assumes s: ksimplex p n s and a: a ∈ s
  assumes j: j < n and p: ∀ x∈s − {a}. x j = p
  shows card {s'. ksimplex p n s' ∧ (∃ b∈s'. s' − {b} = s − {a})} = 1
  using s
proof cases
  case (ksimplex b-s u-s)

  { fix t b assume ksimplex p n t
    then obtain b-t u-t where kuhn-simplex p n b-t u-t t
      by (auto elim: ksimplex.cases)
    interpret kuhn-simplex-pair p n b-s u-s s b-t u-t t
      by intro-locales fact+

    assume b: b ∈ t t − {b} = s − {a}
    with a j p s.replace-1[of - a] t.replace-1[of - b] have s = t
      by (intro ksimplex-eq-bot[of a b]) auto }
```

```
  then have {s'. ksimplex p n s' ∧ (∃ b∈s'. s' − {b} = s − {a})} = {s}
    using s ‹a ∈ s› by auto
  then show ?thesis
    by simp
qed

lemma card-2-exists: card s = 2 ⟷ (∃ x∈s. ∃ y∈s. x ≠ y ∧ (∀ z∈s. z = x ∨ z
= y))
  by (auto simp add: card-Suc-eq eval-nat-numeral)

lemma ksimplex-replace-2:
  assumes s: ksimplex p n s and a ∈ s and n ≠ 0
    and lb: ∀ j<n. ∃ x∈s − {a}. x j ≠ 0
    and ub: ∀ j<n. ∃ x∈s − {a}. x j ≠ p
  shows card {s'. ksimplex p n s' ∧ (∃ b∈s'. s' − {b} = s − {a})} = 2
  using s
proof cases
  case (ksimplex base upd)
  then interpret kuhn-simplex p n base upd s .

  from ‹a ∈ s› obtain i where i ≤ n a = enum i
    unfolding s-eq by auto

  from ‹i ≤ n› have i = 0 ∨ i = n ∨ (0 < i ∧ i < n)
    by linarith
  then have ∃!s'. s' ≠ s ∧ ksimplex p n s' ∧ (∃ b∈s'. s − {a} = s'− {b})
  proof (elim disjE conjE)
    assume i = 0
    def rot ≡ λi. if i + 1 = n then 0 else i + 1
    let ?upd = upd ∘ rot

    have rot: bij-betw rot {..< n} {..< n}
      by (auto simp: bij-betw-def inj-on-def image-iff Ball-def rot-def)
        arith+
    from rot upd have bij-betw ?upd {..<n} {..<n}
      by (rule bij-betw-trans)

    def f' ≡ λi j. if j ∈ ?upd‘{..< i} then Suc (enum (Suc 0) j) else enum (Suc
0) j

    interpret b: kuhn-simplex p n enum (Suc 0) upd ∘ rot f' ‘ {.. n}
    proof
      from ‹a = enum i› ub ‹n ≠ 0› ‹i = 0›
      obtain i' where i' ≤ n enum i' ≠ enum 0 enum i' (upd 0) ≠ p
        unfolding s-eq by (auto intro: upd-space simp: enum-inj)
      then have enum 1 ≤ enum i' enum i' (upd 0) < p
        using enum-le-p[of i' upd 0] by (auto simp add: enum-inj enum-mono
upd-space)
      then have enum 1 (upd 0) < p
```

by (*auto simp add*: *le-fun-def intro*: *le-less-trans*)
then show *enum* (*Suc 0*) ∈ {..<*n*} → {..<*p*}
using *base* ‹*n* ≠ *0*› **by** (*auto simp add*: *enum-0 enum-Suc PiE-iff extensional-def upd-space*)

{ fix *i* **assume** *n* ≤ *i* **then show** *enum* (*Suc 0*) *i* = *p*
using ‹*n* ≠ *0*› **by** (*auto simp*: *enum-eq-p*) **}**
show *bij-betw ?upd* {..<*n*} {..<*n*} **by** *fact*
qed (*simp add*: *f'-def*)
have *ks-f'*: *ksimplex p n* (*f'* ‘ {.. *n*})
by *rule unfold-locales*

have *b-enum*: *b.enum* = *f'* **unfolding** *f'-def b.enum-def*[*abs-def*] **..**
with *b.inj-enum* **have** *inj-f'*: *inj-on f'* {.. *n*} **by** *simp*

have [*simp*]: ⋀*j*. *j* < *n* ⟹ *rot* ‘ {..< *j*} = {*0* <..< *Suc j*}
by (*auto simp*: *rot-def image-iff Ball-def*)
arith

{ fix *j* **assume** *j*: *j* < *n*
from *j* ‹*n* ≠ *0*› **have** *f' j* = *enum* (*Suc j*)
by (*auto simp add*: *f'-def enum-def upd-inj in-upd-image image-comp*[*symmetric*] *fun-eq-iff*) **}**
note *f'-eq-enum* = *this*
then have *enum* ‘ *Suc* ‘ {..< *n*} = *f'* ‘ {..< *n*}
by (*force simp*: *enum-inj*)
also have *Suc* ‘ {..< *n*} = {.. *n*} − {*0*}
by (*auto simp*: *image-iff Ball-def*) *arith*
also have {..< *n*} = {.. *n*} − {*n*}
by *auto*
finally have *eq*: *s* − {*a*} = *f'* ‘ {.. *n*} − {*f' n*}
unfolding *s-eq* ‹*a* = *enum i*› ‹*i* = *0*›
by (*simp add*: *Diff-subset inj-on-image-set-diff*[*OF inj-enum*] *inj-on-image-set-diff*[*OF inj-f'*])

have *enum 0* < *f' 0*
using ‹*n* ≠ *0*› **by** (*simp add*: *enum-strict-mono f'-eq-enum*)
also have ... < *f' n*
using ‹*n* ≠ *0*› *b.enum-strict-mono*[*of 0 n*] **unfolding** *b-enum* **by** *simp*
finally have *a* ≠ *f' n*
using ‹*a* = *enum i*› ‹*i* = *0*› **by** *auto*

{ fix *t c* **assume** *ksimplex p n t c* ∈ *t* **and** *eq-sma*: *s* − {*a*} = *t* − {*c*}
obtain *b u* **where** *kuhn-simplex p n b u t*
using ‹*ksimplex p n t*› **by** (*auto elim*: *ksimplex.cases*)
then interpret *t*: *kuhn-simplex p n b u t* **.**

{ fix *x* **assume** *x* ∈ *s x* ≠ *a*
then have *x* (*upd 0*) = *enum* (*Suc 0*) (*upd 0*)

```
        by (auto simp: ‹a = enum i› ‹i = 0› s-eq enum-def enum-inj) }
    then have eq-upd0: ∀ x∈t−{c}. x (upd 0) = enum (Suc 0) (upd 0)
      unfolding eq-sma[symmetric] by auto
    then have c (upd 0) ≠ enum (Suc 0) (upd 0)
      using ‹n ≠ 0› by (intro t.one-step[OF ‹c∈t› ]) (auto simp: upd-space)
    then have c (upd 0) < enum (Suc 0) (upd 0) ∨ c (upd 0) > enum (Suc 0)
(upd 0)
      by auto
    then have t = s ∨ t = f' ‘ {..n}
    proof (elim disjE conjE)
      assume *: c (upd 0) < enum (Suc 0) (upd 0)
      interpret st: kuhn-simplex-pair p n base upd s b u t ..
      { fix x assume x ∈ t with * ‹c∈t› eq-upd0[rule-format, of x] have c ≤ x
          by (auto simp: le-less intro!: t.less[of - - upd 0]) }
      note top = this
      have s = t
        using ‹a = enum i› ‹i = 0› ‹c ∈ t›
        by (intro st.ksimplex-eq-bot[OF - - - - eq-sma])
           (auto simp: s-eq enum-mono t.s-eq t.enum-mono top)
      then show ?thesis by simp
    next
      assume *: c (upd 0) > enum (Suc 0) (upd 0)
      interpret st: kuhn-simplex-pair p n enum (Suc 0) upd ∘ rot f' ‘ {.. n} b u
t ..
      have eq: f' ‘ {..n} − {f' n} = t − {c}
        using eq-sma eq by simp
      { fix x assume x ∈ t with * ‹c∈t› eq-upd0[rule-format, of x] have x ≤ c
          by (auto simp: le-less intro!: t.less[of - - upd 0]) }
      note top = this
      have f' ‘ {..n} = t
        using ‹a = enum i› ‹i = 0› ‹c ∈ t›
        by (intro st.ksimplex-eq-top[OF - - - - eq])
           (auto simp: b.s-eq b.enum-mono t.s-eq t.enum-mono b-enum[symmetric]
top)
      then show ?thesis by simp
    qed }
  with ks-f' eq ‹a ≠ f' n› ‹n ≠ 0› show ?thesis
    apply (intro ex1I[of - f' ‘ {.. n}])
    apply auto []
    apply metis
    done
next
  assume i = n
  from ‹n ≠ 0› obtain n' where n': n = Suc n'
    by (cases n) auto

  def rot ≡ λi. case i of 0 ⇒ n' | Suc i ⇒ i
  let ?upd = upd ∘ rot
```

have *rot*: *bij-betw rot* {..< *n*} {..< *n*}
by (*auto simp*: *bij-betw-def inj-on-def image-iff Bex-def rot-def n′ split*: *nat.splits*)
arith
from *rot upd* **have** *bij-betw ?upd* {..<*n*} {..<*n*}
by (*rule bij-betw-trans*)

def *b* ≡ *base* (*upd n′* := *base* (*upd n′*) − *1*)
def *f′* ≡ λ*i j*. *if j* ∈ *?upd*‘{..< *i*} *then Suc* (*b j*) *else b j*

interpret *b*: *kuhn-simplex p n b upd* ∘ *rot f′* ‘ {.. *n*}
proof
{ fix *i* **assume** *n* ≤ *i* **then show** *b i* = *p*
using *base-out*[*of i*] *upd-space*[*of n′*] **by** (*auto simp*: *b-def n′*) **}**
show *b* ∈ {..<*n*} → {..<*p*}
using *base* ‹*n* ≠ *0*› *upd-space*[*of n′*]
by (*auto simp*: *b-def PiE-def Pi-iff Ball-def upd-space extensional-def n′*)

show *bij-betw ?upd* {..<*n*} {..<*n*} **by** *fact*
qed (*simp add*: *f′-def*)
have *f′*: *b.enum* = *f′* **unfolding** *f′-def b.enum-def*[*abs-def*] **..**
have *ks-f′*: *ksimplex p n* (*b.enum* ‘ {.. *n*})
unfolding *f′* **by** *rule unfold-locales*

have *0* < *n*
using ‹*n* ≠ *0*› **by** *auto*

{ from ‹*a* = *enum i*› ‹*n* ≠ *0*› ‹*i* = *n*› *lb upd-space*[*of n′*]
obtain *i′* **where** *i′* ≤ *n enum i′* ≠ *enum n 0* < *enum i′* (*upd n′*)
unfolding *s-eq* **by** (*auto simp*: *enum-inj n′*)
moreover have *enum i′* (*upd n′*) = *base* (*upd n′*)
unfolding *enum-def* **using** ‹*i′* ≤ *n*› ‹*enum i′* ≠ *enum n*› **by** (*auto simp*: *n′ upd-inj enum-inj*)
ultimately have *0* < *base* (*upd n′*)
by *auto* **}**
then have *benum1*: *b.enum* (*Suc 0*) = *base*
unfolding *b.enum-Suc*[*OF* ‹*0*<*n*›] *b.enum-0* **by** (*auto simp*: *b-def rot-def*)

have [*simp*]: ⋀*j*. *Suc j* < *n* ⟹ *rot* ‘ {..< *Suc j*} = {*n′*} ∪ {..< *j*}
by (*auto simp*: *rot-def image-iff Ball-def split*: *nat.splits*)
have *rot-simps*: ⋀*j*. *rot* (*Suc j*) = *j rot 0* = *n′*
by (*simp-all add*: *rot-def*)

{ fix *j* **assume** *j*: *Suc j* ≤ *n* **then have** *b.enum* (*Suc j*) = *enum j*
by (*induct j*) (*auto simp add*: *benum1 enum-0 b.enum-Suc enum-Suc rot-simps*) **}**
note *b-enum-eq-enum* = *this*
then have *enum* ‘ {..< *n*} = *b.enum* ‘ *Suc* ‘ {..< *n*}
by (*auto simp add*: *image-comp intro*!: *image-cong*)

```
  also have Suc ` {..< n} = {.. n} - {0}
    by (auto simp: image-iff Ball-def) arith
  also have {..< n} = {.. n} - {n}
    by auto
  finally have eq: s - {a} = b.enum ` {.. n} - {b.enum 0}
    unfolding s-eq ‹a = enum i› ‹i = n›
    using inj-on-image-set-diff[OF inj-enum Diff-subset, of {n}]
          inj-on-image-set-diff[OF b.inj-enum Diff-subset, of {0}]
    by (simp add: comp-def )

  have b.enum 0 ≤ b.enum n
    by (simp add: b.enum-mono)
  also have b.enum n < enum n
    using ‹n ≠ 0› by (simp add: enum-strict-mono b-enum-eq-enum n')
  finally have a ≠ b.enum 0
    using ‹a = enum i› ‹i = n› by auto

  { fix t c assume ksimplex p n t c ∈ t and eq-sma: s - {a} = t - {c}
    obtain b' u where kuhn-simplex p n b' u t
      using ‹ksimplex p n t› by (auto elim: ksimplex.cases)
    then interpret t: kuhn-simplex p n b' u t .

    { fix x assume x ∈ s x ≠ a
      then have x (upd n') = enum n' (upd n')
            by (auto simp: ‹a = enum i› n' ‹i = n› s-eq enum-def enum-inj
in-upd-image) }
    then have eq-upd0: ∀ x∈t-{c}. x (upd n') = enum n' (upd n')
      unfolding eq-sma[symmetric] by auto
    then have c (upd n') ≠ enum n' (upd n')
    using ‹n ≠ 0› by (intro t.one-step[OF ‹c∈t› ]) (auto simp: n' upd-space[unfolded
n'])
    then have c (upd n') < enum n' (upd n') ∨ c (upd n') > enum n' (upd n')
      by auto
    then have t = s ∨ t = b.enum ` {..n}
    proof (elim disjE conjE)
      assume *: c (upd n') > enum n' (upd n')
      interpret st: kuhn-simplex-pair p n base upd s b' u t ..
      { fix x assume x ∈ t with * ‹c∈t› eq-upd0[rule-format, of x] have x ≤ c
        by (auto simp: le-less intro!: t.less[of - - upd n']) }
      note top = this
      have s = t
        using ‹a = enum i› ‹i = n› ‹c ∈ t›
        by (intro st.ksimplex-eq-top[OF - - - - eq-sma])
           (auto simp: s-eq enum-mono t.s-eq t.enum-mono top)
      then show ?thesis by simp
    next
      assume *: c (upd n') < enum n' (upd n')
      interpret st: kuhn-simplex-pair p n b upd ∘ rot f' ` {.. n} b' u t ..
      have eq: f' ` {..n} - {b.enum 0} = t - {c}
```

```
      using eq-sma eq f' by simp
    { fix x assume x ∈ t with * ‹c∈t› eq-upd0[rule-format, of x] have c ≤ x
      by (auto simp: le-less intro!: t.less[of - - upd n']) }
    note bot = this
    have f' ` {..n} = t
     using ‹a = enum i› ‹i = n› ‹c ∈ t›
     by (intro st.ksimplex-eq-bot[OF - - - - eq])
       (auto simp: b.s-eq b.enum-mono t.s-eq t.enum-mono bot)
    with f' show ?thesis by simp
   qed }
  with ks-f' eq ‹a ≠ b.enum 0› ‹n ≠ 0› show ?thesis
   apply (intro ex1I[of - b.enum ` {.. n}])
   apply auto []
   apply metis
   done
next
  assume i: 0 < i i < n
  def i' ≡ i − 1
  with i have Suc i' < n
   by simp
  with i have Suc-i': Suc i' = i
   by (simp add: i'-def)

  let ?upd = Fun.swap i' i upd
  from i upd have bij-betw ?upd {..< n} {..< n}
   by (subst bij-betw-swap-iff) (auto simp: i'-def)

  def f' ≡ λi j. if j ∈ ?upd`{..< i} then Suc (base j) else base j
  interpret b: kuhn-simplex p n base ?upd f' ` {.. n}
  proof
   show base ∈ {..<n} → {..<p} by fact
   { fix i assume n ≤ i then show base i = p by fact }
   show bij-betw ?upd {..<n} {..<n} by fact
  qed (simp add: f'-def)
  have f': b.enum = f' unfolding f'-def b.enum-def[abs-def] ..
  have ks-f': ksimplex p n (b.enum ` {.. n})
   unfolding f' by rule unfold-locales

  have {i} ⊆ {..n}
   using i by auto
  { fix j assume j ≤ n
   moreover have j < i ∨ i = j ∨ i < j by arith
   moreover note i
   ultimately have enum j = b.enum j ⟷ j ≠ i
    unfolding enum-def[abs-def] b.enum-def[abs-def]
    by (auto simp add: fun-eq-iff swap-image i'-def
                in-upd-image inj-on-image-set-diff[OF inj-upd]) }
  note enum-eq-benum = this
  then have enum ` ({.. n} − {i}) = b.enum ` ({.. n} − {i})
```

by (*intro image-cong*) *auto*
then have *eq*: $s - \{a\} = b.enum\ `\ \{..\ n\} - \{b.enum\ i\}$
unfolding *s-eq* ‹$a = enum\ i$›
using *inj-on-image-set-diff*[*OF inj-enum Diff-subset* ‹$\{i\} \subseteq \{..n\}$›]
inj-on-image-set-diff[*OF b.inj-enum Diff-subset* ‹$\{i\} \subseteq \{..n\}$›]
by (*simp add*: *comp-def*)

have $a \neq b.enum\ i$
using ‹$a = enum\ i$› *enum-eq-benum i* **by** *auto*

{ fix *t c* **assume** *ksimplex p n t c* ∈ *t* **and** *eq-sma*: $s - \{a\} = t - \{c\}$
obtain $b'\ u$ **where** *kuhn-simplex p n b' u t*
using ‹*ksimplex p n t*› **by** (*auto elim*: *ksimplex.cases*)
then interpret *t*: *kuhn-simplex p n b' u t* **.**
have $enum\ i' \in s - \{a\}\ enum\ (i + 1) \in s - \{a\}$
using ‹$a = enum\ i$› *i enum-in* **by** (*auto simp*: *enum-inj i'-def*)
then obtain *l k* **where**
l: $t.enum\ l = enum\ i'\ l \leq n\ t.enum\ l \neq c$ **and**
k: $t.enum\ k = enum\ (i + 1)\ k \leq n\ t.enum\ k \neq c$
unfolding *eq-sma* **by** (*auto simp*: *t.s-eq*)
with *i* **have** $t.enum\ l < t.enum\ k$
by (*simp add*: *enum-strict-mono i'-def*)
with ‹$l \leq n$› ‹$k \leq n$› **have** $l < k$
by (*simp add*: *t.enum-strict-mono*)
{ assume $Suc\ l = k$
have $enum\ (Suc\ (Suc\ i')) = t.enum\ (Suc\ l)$
using *i* **by** (*simp add*: *k* ‹$Suc\ l = k$› *i'-def*)
then have *False*
using ‹$l < k$› ‹$k \leq n$› ‹$Suc\ i' < n$›
by (*auto simp*: *t.enum-Suc enum-Suc l upd-inj fun-eq-iff split*: *if-split-asm*)
(*metis Suc-lessD n-not-Suc-n upd-inj*) **}**
with ‹$l < k$› **have** $Suc\ l < k$
by *arith*
have *c-eq*: $c = t.enum\ (Suc\ l)$
proof (*rule ccontr*)
assume $c \neq t.enum\ (Suc\ l)$
then have $t.enum\ (Suc\ l) \in s - \{a\}$
using ‹$l < k$› ‹$k \leq n$› **by** (*simp add*: *t.s-eq eq-sma*)
then obtain *j* **where** $t.enum\ (Suc\ l) = enum\ j\ j \leq n\ enum\ j \neq enum\ i$
unfolding *s-eq* ‹$a = enum\ i$› **by** *auto*
with *i* **have** $t.enum\ (Suc\ l) \leq t.enum\ l \vee t.enum\ k \leq t.enum\ (Suc\ l)$
by (*auto simp add*: *i'-def enum-mono enum-inj l k*)
with ‹$Suc\ l < k$› ‹$k \leq n$› **show** *False*
by (*simp add*: *t.enum-mono*)
qed

{ have $t.enum\ (Suc\ (Suc\ l)) \in s - \{a\}$
unfolding *eq-sma c-eq t.s-eq* **using** ‹$Suc\ l < k$› ‹$k \leq n$› **by** (*auto simp*: *t.enum-inj*)

```
      then obtain j where eq: t.enum (Suc (Suc l)) = enum j and j ≤ n j ≠ i
        by (auto simp: s-eq ‹a = enum i›)
      moreover have enum i' < t.enum (Suc (Suc l))
        unfolding l(1)[symmetric] using ‹Suc l < k› ‹k ≤ n› by (auto simp:
t.enum-strict-mono)
      ultimately have i' < j
        using i by (simp add: enum-strict-mono i'-def)
      with ‹j ≠ i› ‹j ≤ n› have t.enum k ≤ t.enum (Suc (Suc l))
        unfolding i'-def by (simp add: enum-mono k eq)
      then have k ≤ Suc (Suc l)
        using ‹k ≤ n› ‹Suc l < k› by (simp add: t.enum-mono) }
    with ‹Suc l < k› have Suc (Suc l) = k by simp
    then have enum (Suc (Suc i')) = t.enum (Suc (Suc l))
      using i by (simp add: k i'-def)
     also have ... = (enum i') (u l := Suc (enum i' (u l)), u (Suc l) := Suc
(enum i' (u (Suc l))))
      using ‹Suc l < k› ‹k ≤ n› by (simp add: t.enum-Suc l t.upd-inj)
    finally have (u l = upd i' ∧ u (Suc l) = upd (Suc i')) ∨
      (u l = upd (Suc i') ∧ u (Suc l) = upd i')
      using ‹Suc i' < n› by (auto simp: enum-Suc fun-eq-iff split: if-split-asm)

    then have t = s ∨ t = b.enum ' {..n}
    proof (elim disjE conjE)
      assume u: u l = upd i'
      have c = t.enum (Suc l) unfolding c-eq ..
      also have t.enum (Suc l) = enum (Suc i')
       using u ‹l < k› ‹k ≤ n› ‹Suc i' < n› by (simp add: enum-Suc t.enum-Suc
l)
      also have ... = a
        using ‹a = enum i› i by (simp add: i'-def)
      finally show ?thesis
        using eq-sma ‹a ∈ s› ‹c ∈ t› by auto
    next
      assume u: u l = upd (Suc i')
      def B ≡ b.enum ' {..n}
      have b.enum i' = enum i'
        using enum-eq-benum[of i'] i by (auto simp add: i'-def gr0-conv-Suc)
      have c = t.enum (Suc l) unfolding c-eq ..
      also have t.enum (Suc l) = b.enum (Suc i')
        using u ‹l < k› ‹k ≤ n› ‹Suc i' < n›
       by (simp-all add: enum-Suc t.enum-Suc l b.enum-Suc ‹b.enum i' = enum
i'› swap-apply1)
          (simp add: Suc-i')
      also have ... = b.enum i
        using i by (simp add: i'-def)
      finally have c = b.enum i .
      then have t - {c} = B - {c} c ∈ B
        unfolding eq-sma[symmetric] eq B-def using i by auto
      with ‹c ∈ t› have t = B
```

```
        by auto
      then show ?thesis
        by (simp add: B-def)
    qed }
  with ks-f' eq ‹a ≠ b.enum i› ‹n ≠ 0› ‹i ≤ n› show ?thesis
    apply (intro ex1I[of - b.enum ` {.. n}])
    apply auto []
    apply metis
    done
 qed
 then show ?thesis
  using s ‹a ∈ s› by (simp add: card-2-exists Ex1-def) metis
qed
```

Hence another step towards concreteness.

```
lemma kuhn-simplex-lemma:
 assumes ∀ s. ksimplex p (Suc n) s ⟶ rl ` s ⊆ {.. Suc n}
  and odd (card {f. ∃ s a. ksimplex p (Suc n) s ∧ a ∈ s ∧ (f = s - {a}) ∧
    rl ` f = {..n} ∧ ((∃ j≤n. ∀ x∈f. x j = 0) ∨ (∃ j≤n. ∀ x∈f. x j = p))})
 shows odd (card {s. ksimplex p (Suc n) s ∧ rl ` s = {..Suc n}})
proof (rule kuhn-complete-lemma[OF finite-ksimplexes refl, unfolded mem-Collect-eq,
   where bnd=λf. (∃ j∈{..n}. ∀ x∈f. x j = 0) ∨ (∃ j∈{..n}. ∀ x∈f. x j = p)],
   safe del: notI)

 have *: ⋀x y. x = y ⟹ odd (card x) ⟹ odd (card y)
  by auto
 show odd (card {f. (∃ s∈{s. ksimplex p (Suc n) s}. ∃ a∈s. f = s - {a}) ∧
  rl ` f = {..n} ∧ ((∃ j∈{..n}. ∀ x∈f. x j = 0) ∨ (∃ j∈{..n}. ∀ x∈f. x j = p))})
  apply (rule *[OF - assms(2)])
  apply (auto simp: atLeast0AtMost)
  done

next

 fix s assume s: ksimplex p (Suc n) s
 then show card s = n + 2
  by (simp add: ksimplex-card)

 fix a assume a: a ∈ s then show rl a ≤ Suc n
  using assms(1) s by (auto simp: subset-eq)

 let ?S = {t. ksimplex p (Suc n) t ∧ (∃ b∈t. s - {a} = t - {b})}
 { fix j assume j: j ≤ n ∀ x∈s - {a}. x j = 0
  with s a show card ?S = 1
   using ksimplex-replace-0[of p n + 1 s a j]
   by (subst eq-commute) simp }

 { fix j assume j: j ≤ n ∀ x∈s - {a}. x j = p
  with s a show card ?S = 1
```

using *ksimplex-replace-1*[*of p n + 1 s a j*]
by (*subst eq-commute*) *simp* **}**

{ assume *card ?S ≠ 2* ¬ (∃*j*∈{..*n*}. ∀*x*∈*s* − {*a*}. *x j* = *p*)
with *s a* **show** ∃*j*∈{..*n*}. ∀*x*∈*s* − {*a*}. *x j* = *0*
using *ksimplex-replace-2*[*of p n + 1 s a*]
by (*subst* (*asm*) *eq-commute*) *auto* **}**
qed

26.3 Reduced labelling

definition *reduced* :: *nat* ⇒ (*nat* ⇒ *nat*) ⇒ *nat* **where** *reduced n x* = (*LEAST k*. *k* = *n* ∨ *x k* ≠ *0*)

lemma *reduced-labelling*:
shows *reduced n x* ≤ *n*
and ∀*i*<*reduced n x*. *x i* = *0*
and *reduced n x* = *n* ∨ *x* (*reduced n x*) ≠ *0*
proof −
show *reduced n x* ≤ *n*
unfolding *reduced-def* **by** (*rule LeastI2-wellorder*[**where** *a*=*n*]) *auto*
show ∀*i*<*reduced n x*. *x i* = *0*
unfolding *reduced-def* **by** (*rule LeastI2-wellorder*[**where** *a*=*n*]) *fastforce*+
show *reduced n x* = *n* ∨ *x* (*reduced n x*) ≠ *0*
unfolding *reduced-def* **by** (*rule LeastI2-wellorder*[**where** *a*=*n*]) *fastforce*+
qed

lemma *reduced-labelling-unique*:
r ≤ *n* ⟹ ∀*i*<*r*. *x i* = *0* ⟹ *r* = *n* ∨ *x r* ≠ *0* ⟹ *reduced n x* = *r*
unfolding *reduced-def* **by** (*rule LeastI2-wellorder*[**where** *a*=*n*]) (*metis le-less not-le*)+

lemma *reduced-labelling-zero*: *j* < *n* ⟹ *x j* = *0* ⟹ *reduced n x* ≠ *j*
using *reduced-labelling*[*of n x*] **by** *auto*

lemma *reduce-labelling-zero*[*simp*]: *reduced 0 x* = *0*
by (*rule reduced-labelling-unique*) *auto*

lemma *reduced-labelling-nonzero*: *j* < *n* ⟹ *x j* ≠ *0* ⟹ *reduced n x* ≤ *j*
using *reduced-labelling*[*of n x*] **by** (*elim allE*[**where** *x*=*j*]) *auto*

lemma *reduced-labelling-Suc*: *reduced* (*Suc n*) *x* ≠ *Suc n* ⟹ *reduced* (*Suc n*) *x* = *reduced n x*
using *reduced-labelling*[*of Suc n x*]
by (*intro reduced-labelling-unique*[*symmetric*]) *auto*

lemma *complete-face-top*:
assumes ∀*x*∈*f*. ∀*j*≤*n*. *x j* = *0* ⟶ *lab x j* = *0*
and ∀*x*∈*f*. ∀*j*≤*n*. *x j* = *p* ⟶ *lab x j* = *1*

```
    and eq: (reduced (Suc n) ∘ lab) ‘ f = {..n}
  shows ((∃j≤n. ∀x∈f. x j = 0) ∨ (∃j≤n. ∀x∈f. x j = p)) ⟷ (∀x∈f. x n =
p)
proof (safe del: disjCI)
  fix x j assume j: j ≤ n ∀x∈f. x j = 0
  { fix x assume x ∈ f with assms j have reduced (Suc n) (lab x) ≠ j
      by (intro reduced-labelling-zero) auto }
  moreover have j ∈ (reduced (Suc n) ∘ lab) ‘ f
    using j eq by auto
  ultimately show x n = p
    by force
next
  fix x j assume j: j ≤ n ∀x∈f. x j = p and x: x ∈ f
  have j = n
  proof (rule ccontr)
    assume ¬ ?thesis
    { fix x assume x ∈ f
      with assms j have reduced (Suc n) (lab x) ≤ j
        by (intro reduced-labelling-nonzero) auto
      then have reduced (Suc n) (lab x) ≠ n
        using ‹j ≠ n› ‹j ≤ n› by simp }
    moreover
    have n ∈ (reduced (Suc n) ∘ lab) ‘ f
      using eq by auto
    ultimately show False
      by force
  qed
  moreover have j ∈ (reduced (Suc n) ∘ lab) ‘ f
    using j eq by auto
  ultimately show x n = p
    using j x by auto
qed auto
```

Hence we get just about the nice induction.

```
lemma kuhn-induction:
  assumes 0 < p
    and lab-0: ∀x. ∀j≤n. (∀j. x j ≤ p) ∧ x j = 0 ⟶ lab x j = 0
    and lab-1: ∀x. ∀j≤n. (∀j. x j ≤ p) ∧ x j = p ⟶ lab x j = 1
    and odd: odd (card {s. ksimplex p n s ∧ (reduced n∘lab) ‘ s = {..n}})
  shows odd (card {s. ksimplex p (Suc n) s ∧ (reduced (Suc n)∘lab) ‘ s = {..Suc
n}})
proof −
  let ?rl = reduced (Suc n) ∘ lab and ?ext = λf v. ∃j≤n. ∀x∈f. x j = v
  let ?ext = λs. (∃j≤n. ∀x∈s. x j = 0) ∨ (∃j≤n. ∀x∈s. x j = p)
  have ∀s. ksimplex p (Suc n) s ⟶ ?rl ‘ s ⊆ {..Suc n}
    by (simp add: reduced-labelling subset-eq)
  moreover
  have {s. ksimplex p n s ∧ (reduced n ∘ lab) ‘ s = {..n}} =
        {f. ∃s a. ksimplex p (Suc n) s ∧ a ∈ s ∧ f = s − {a} ∧ ?rl ‘ f = {..n} ∧
```

```
?ext f}
 proof (intro set-eqI, safe del: disjCI equalityI disjE)
   fix s assume s: ksimplex p n s and rl: (reduced n ∘ lab) ‘ s = {..n}
  from s obtain u b where kuhn-simplex p n u b s by (auto elim: ksimplex.cases)
   then interpret kuhn-simplex p n u b s .
   have all-eq-p: ∀ x∈s. x n = p
     by (auto simp: out-eq-p)
   moreover
   { fix x assume x ∈ s
     with lab-1[rule-format, of n x] all-eq-p s-le-p[of x]
     have ?rl x ≤ n
       by (auto intro!: reduced-labelling-nonzero)
     then have ?rl x = reduced n (lab x)
       by (auto intro!: reduced-labelling-Suc) }
   then have ?rl ‘ s = {..n}
     using rl by (simp cong: image-cong)
   moreover
   obtain t a where ksimplex p (Suc n) t a ∈ t s = t − {a}
     using s unfolding simplex-top-face[OF ‹0 < p› all-eq-p] by auto
   ultimately
   show ∃ t a. ksimplex p (Suc n) t ∧ a ∈ t ∧ s = t − {a} ∧ ?rl ‘ s = {..n} ∧
?ext s
     by auto
 next
   fix x s a assume s: ksimplex p (Suc n) s and rl: ?rl ‘ (s − {a}) = {.. n}
     and a: a ∈ s and ?ext (s − {a})
    from s obtain u b where kuhn-simplex p (Suc n) u b s by (auto elim:
ksimplex.cases)
   then interpret kuhn-simplex p Suc n u b s .
   have all-eq-p: ∀ x∈s. x (Suc n) = p
     by (auto simp: out-eq-p)

   { fix x assume x ∈ s − {a}
     then have ?rl x ∈ ?rl ‘ (s − {a})
       by auto
     then have ?rl x ≤ n
       unfolding rl by auto
     then have ?rl x = reduced n (lab x)
       by (auto intro!: reduced-labelling-Suc) }
   then show rl′: (reduced n∘lab) ‘ (s − {a}) = {..n}
     unfolding rl[symmetric] by (intro image-cong) auto

   from ‹?ext (s − {a})›
   have all-eq-p: ∀ x∈s − {a}. x n = p
   proof (elim disjE exE conjE)
     fix j assume j ≤ n ∀ x∈s − {a}. x j = 0
     with lab-0[rule-format, of j] all-eq-p s-le-p
     have ⋀x. x ∈ s − {a} ⟹ reduced (Suc n) (lab x) ≠ j
       by (intro reduced-labelling-zero) auto
```

```
    moreover have j ∈ ?rl ‘ (s − {a})
      using ‹j ≤ n› unfolding rl by auto
    ultimately show ?thesis
      by force
  next
    fix j assume j ≤ n and eq-p: ∀x∈s − {a}. x j = p
    show ?thesis
    proof cases
      assume j = n with eq-p show ?thesis by simp
    next
      assume j ≠ n
      { fix x assume x: x ∈ s − {a}
        have reduced n (lab x) ≤ j
        proof (rule reduced-labelling-nonzero)
          show lab x j ≠ 0
            using lab-1[rule-format, of j x] x s-le-p[of x] eq-p ‹j ≤ n› by auto
          show j < n
            using ‹j ≤ n› ‹j ≠ n› by simp
        qed
        then have reduced n (lab x) ≠ n
          using ‹j ≤ n› ‹j ≠ n› by simp }
      moreover have n ∈ (reduced n∘lab) ‘ (s − {a})
        unfolding rl' by auto
      ultimately show ?thesis
        by force
    qed
  qed
  show ksimplex p n (s − {a})
    unfolding simplex-top-face[OF ‹0 < p› all-eq-p] using s a by auto
qed
ultimately show ?thesis
  using assms by (intro kuhn-simplex-lemma) auto
qed
```

And so we get the final combinatorial result.

```
lemma ksimplex-0: ksimplex p 0 s ⟷ s = {(λx. p)}
proof
  assume ksimplex p 0 s then show s = {(λx. p)}
    by (blast dest: kuhn-simplex.ksimplex-0 elim: ksimplex.cases)
next
  assume s: s = {(λx. p)}
  show ksimplex p 0 s
  proof (intro ksimplex, unfold-locales)
    show (λ-. p) ∈ {..<0::nat} → {..<p} by auto
    show bij-betw id {..<0} {..<0}
      by simp
  qed (auto simp: s)
qed
```

lemma *kuhn-combinatorial*:
assumes $0 < p$
and $\forall x\ j.\ (\forall j.\ x\ j \leq p) \wedge j < n \wedge x\ j = 0 \longrightarrow lab\ x\ j = 0$
and $\forall x\ j.\ (\forall j.\ x\ j \leq p) \wedge j < n \wedge x\ j = p \longrightarrow lab\ x\ j = 1$
shows *odd* (*card* $\{s.\ ksimplex\ p\ n\ s \wedge (reduced\ n \circ lab)\ `\ s = \{..n\}\}$)
(**is** *odd* (*card* (*?M n*)))
using *assms*
proof (*induct n*)
case *0* **then show** *?case*
by (*simp add*: *ksimplex-0 cong*: *conj-cong*)
next
case (*Suc n*)
then have *odd* (*card* (*?M n*))
by *force*
with *Suc* **show** *?case*
using *kuhn-induction*[*of p n*] **by** (*auto simp*: *comp-def*)
qed

lemma *kuhn-lemma*:
fixes *n p* :: *nat*
assumes $0 < p$
and $\forall x.\ (\forall i{<}n.\ x\ i \leq p) \longrightarrow (\forall i{<}n.\ label\ x\ i = (0{::}nat) \vee label\ x\ i = 1)$
and $\forall x.\ (\forall i{<}n.\ x\ i \leq p) \longrightarrow (\forall i{<}n.\ x\ i = 0 \longrightarrow label\ x\ i = 0)$
and $\forall x.\ (\forall i{<}n.\ x\ i \leq p) \longrightarrow (\forall i{<}n.\ x\ i = p \longrightarrow label\ x\ i = 1)$
obtains *q* **where** $\forall i{<}n.\ q\ i < p$
and $\forall i{<}n.\ \exists r\ s.\ (\forall j{<}n.\ q\ j \leq r\ j \wedge r\ j \leq q\ j + 1) \wedge (\forall j{<}n.\ q\ j \leq s\ j \wedge s\ j \leq q\ j + 1) \wedge label\ r\ i \neq label\ s\ i$
proof −
let *?rl* = *reduced n* ∘ *label*
let *?A* = $\{s.\ ksimplex\ p\ n\ s \wedge ?rl\ `\ s = \{..n\}\}$
have *odd* (*card ?A*)
using *assms* **by** (*intro kuhn-combinatorial*[*of p n label*]) *auto*
then have *?A* ≠ {}
by *fastforce*
then obtain *s b u* **where** *kuhn-simplex p n b u s* **and** *rl*: *?rl* ` *s* = {..*n*}
by (*auto elim*: *ksimplex.cases*)
interpret *kuhn-simplex p n b u s* **by** *fact*

show *?thesis*
proof (*intro that*[*of b*] *allI impI*)
fix *i*
assume $i < n$
then show $b\ i < p$
using *base* **by** *auto*
next
fix *i*
assume $i < n$
then have $i \in \{..\ n\}$ *Suc* $i \in \{..\ n\}$
by *auto*

then obtain *u v* **where** *u*: $u \in s$ *Suc* $i = ?rl\ u$ **and** *v*: $v \in s\ i = ?rl\ v$
unfolding *rl*[*symmetric*] **by** *blast*

have *label u i* $\neq$ *label v i*
using *reduced-labelling* [*of n label u*] *reduced-labelling* [*of n label v*]
u(2)[*symmetric*] *v*(2)[*symmetric*] ‹*i* < *n*›
by *auto*
moreover
have $b\ j \leq u\ j\ u\ j \leq b\ j + 1\ b\ j \leq v\ j\ v\ j \leq b\ j + 1$ **if** $j < n$ **for** *j*
using *that base-le*[*OF* ‹*u*∈*s*›] *le-Suc-base*[*OF* ‹*u*∈*s*›] *base-le*[*OF* ‹*v*∈*s*›] *le-Suc-base*[*OF* ‹*v*∈*s*›]
by *auto*
ultimately show $\exists r\ s.\ (\forall j{<}n.\ b\ j \leq r\ j \wedge r\ j \leq b\ j + 1) \wedge$
$(\forall j{<}n.\ b\ j \leq s\ j \wedge s\ j \leq b\ j + 1) \wedge$ *label r i* $\neq$ *label s i*
by *blast*
qed
qed

26.4 The main result for the unit cube

lemma *kuhn-labelling-lemma'*:
assumes $(\forall x{::}nat{\Rightarrow}real.\ P\ x \longrightarrow P\ (f\ x))$
and $\forall x.\ P\ x \longrightarrow (\forall i{::}nat.\ Q\ i \longrightarrow 0 \leq x\ i \wedge x\ i \leq 1)$
shows $\exists l.\ (\forall x\ i.\ l\ x\ i \leq (1{::}nat)) \wedge$
$(\forall x\ i.\ P\ x \wedge Q\ i \wedge x\ i = 0 \longrightarrow l\ x\ i = 0) \wedge$
$(\forall x\ i.\ P\ x \wedge Q\ i \wedge x\ i = 1 \longrightarrow l\ x\ i = 1) \wedge$
$(\forall x\ i.\ P\ x \wedge Q\ i \wedge l\ x\ i = 0 \longrightarrow x\ i \leq f\ x\ i) \wedge$
$(\forall x\ i.\ P\ x \wedge Q\ i \wedge l\ x\ i = 1 \longrightarrow f\ x\ i \leq x\ i)$
proof −
have *and-forall-thm*: $\bigwedge P\ Q.\ (\forall x.\ P\ x) \wedge (\forall x.\ Q\ x) \longleftrightarrow (\forall x.\ P\ x \wedge Q\ x)$
by *auto*
have ∗: $\forall x\ y{::}real.\ 0 \leq x \wedge x \leq 1 \wedge 0 \leq y \wedge y \leq 1 \longrightarrow x \neq 1 \wedge x \leq y \vee x \neq 0 \wedge y \leq x$
by *auto*
show *?thesis*
unfolding *and-forall-thm*
apply (*subst choice-iff*[*symmetric*])+
apply *rule*
apply *rule*
proof −
fix *x x'*
let $?R = \lambda y{::}nat.$
$(P\ x \wedge Q\ x' \wedge x\ x' = 0 \longrightarrow y = 0) \wedge$
$(P\ x \wedge Q\ x' \wedge x\ x' = 1 \longrightarrow y = 1) \wedge$
$(P\ x \wedge Q\ x' \wedge y = 0 \longrightarrow x\ x' \leq (f\ x)\ x') \wedge$
$(P\ x \wedge Q\ x' \wedge y = 1 \longrightarrow (f\ x)\ x' \leq x\ x')$
have $0 \leq f\ x\ x' \wedge f\ x\ x' \leq 1$ **if** $P\ x\ Q\ x'$
using *assms*(2)[*rule-format*,*of f x x'*] *that*
apply (*drule-tac assms*(1)[*rule-format*])

```
    apply auto
    done
  then have ?R 0 ∨ ?R 1
    by auto
  then show ∃ y≤1. ?R y
    by auto
 qed
qed

definition unit-cube :: 'a::euclidean-space set
 where unit-cube = {x. ∀ i∈Basis. 0 ≤ x • i ∧ x • i ≤ 1}

lemma mem-unit-cube: x ∈ unit-cube ⟷ (∀ i∈Basis. 0 ≤ x • i ∧ x • i ≤ 1)
 unfolding unit-cube-def by simp

lemma bounded-unit-cube: bounded unit-cube
 unfolding bounded-def
proof (intro exI ballI)
 fix y :: 'a assume y: y ∈ unit-cube
 have dist 0 y = norm y by (rule dist-0-norm)
 also have ... = norm (∑ i∈Basis. (y • i) *R i) unfolding euclidean-representation
..
 also have ... ≤ (∑ i∈Basis. norm ((y • i) *R i)) by (rule norm-setsum)
 also have ... ≤ (∑ i::'a∈Basis. 1)
   by (rule setsum-mono, simp add: y [unfolded mem-unit-cube])
 finally show dist 0 y ≤ (∑ i::'a∈Basis. 1) .
qed

lemma closed-unit-cube: closed unit-cube
 unfolding unit-cube-def Collect-ball-eq Collect-conj-eq
 by (rule closed-INT, auto intro!: closed-Collect-le)

lemma compact-unit-cube: compact unit-cube (is compact ?C)
 unfolding compact-eq-seq-compact-metric
 using bounded-unit-cube closed-unit-cube
 by (rule bounded-closed-imp-seq-compact)

lemma brouwer-cube:
 fixes f :: 'a::euclidean-space ⇒ 'a
 assumes continuous-on unit-cube f
   and f ` unit-cube ⊆ unit-cube
 shows ∃ x∈unit-cube. f x = x
proof (rule ccontr)
 def n ≡ DIM('a)
 have n: 1 ≤ n 0 < n n ≠ 0
   unfolding n-def by (auto simp add: Suc-le-eq DIM-positive)
 assume ¬ ?thesis
 then have ∗: ¬ (∃ x∈unit-cube. f x − x = 0)
   by auto
```

obtain d **where**
d: $d > 0$ $\bigwedge x.\ x \in$ *unit-cube* $\Longrightarrow d \leq$ *norm* $(f\ x - x)$
apply (*rule brouwer-compactness-lemma*[*OF compact-unit-cube* - $*$])
apply (*rule continuous-intros assms*)+
apply *blast*
done
have $*$: $\forall x.\ x \in$ *unit-cube* $\longrightarrow f\ x \in$ *unit-cube*
$\forall x.\ x \in$ (*unit-cube*::$'a$ *set*) $\longrightarrow (\forall i{\in}Basis.\ True \longrightarrow 0 \leq x \cdot i \wedge x \cdot i \leq 1)$
using *assms*(*2*)[*unfolded image-subset-iff Ball-def*]
unfolding *mem-unit-cube*
by *auto*
obtain *label* :: $'a \Rightarrow 'a \Rightarrow$ *nat* **where**
$\forall x.\ \forall i{\in}Basis.\ label\ x\ i \leq 1$
$\forall x.\ \forall i{\in}Basis.\ x \in$ *unit-cube* $\wedge\ True \wedge x \cdot i = 0 \longrightarrow label\ x\ i = 0$
$\forall x.\ \forall i{\in}Basis.\ x \in$ *unit-cube* $\wedge\ True \wedge x \cdot i = 1 \longrightarrow label\ x\ i = 1$
$\forall x.\ \forall i{\in}Basis.\ x \in$ *unit-cube* $\wedge\ True \wedge label\ x\ i = 0 \longrightarrow x \cdot i \leq f\ x \cdot i$
$\forall x.\ \forall i{\in}Basis.\ x \in$ *unit-cube* $\wedge\ True \wedge label\ x\ i = 1 \longrightarrow f\ x \cdot i \leq x \cdot i$
using *kuhn-labelling-lemma*[*OF* $*$] **by** *blast*
note *label* = *this* [*rule-format*]
have *lem1*: $\forall x{\in}$*unit-cube*. $\forall y{\in}$*unit-cube*. $\forall i{\in}Basis.\ label\ x\ i \neq label\ y\ i \longrightarrow$
$|f\ x \cdot i - x \cdot i| \leq$ *norm* $(f\ y - f\ x)$ + *norm* $(y - x)$
proof *safe*
fix $x\ y$ :: $'a$
assume x: $x \in$ *unit-cube*
assume y: $y \in$ *unit-cube*
fix i
assume i: *label* $x\ i \neq$ *label* $y\ i\ i \in Basis$
have $*$: $\bigwedge x\ y\ fx\ fy$ :: *real*. $x \leq fx \wedge fy \leq y \vee fx \leq x \wedge y \leq fy \Longrightarrow$
$|fx - x| \leq |fy - fx| + |y - x|$ **by** *auto*
have $|(f\ x - x) \cdot i| \leq |(f\ y - f\ x){\cdot}i| + |(y - x){\cdot}i|$
unfolding *inner-simps*
apply (*rule* $*$)
apply (*cases label* $x\ i = 0$)
apply (*rule disjI1*)
apply *rule*
prefer *3*
apply (*rule disjI2*)
apply *rule*
proof −
assume *lx*: *label* $x\ i = 0$
then have *ly*: *label* $y\ i = 1$
using i *label*(*1*)[*of* $i\ y$]
by *auto*
show $x \cdot i \leq f\ x \cdot i$
apply (*rule label*(*4*)[*rule-format*])
using $x\ y$ *lx* i(*2*)
apply *auto*
done
show $f\ y \cdot i \leq y \cdot i$

```
      apply (rule label(5)[rule-format])
      using x y ly i(2)
      apply auto
      done
  next
    assume label x i ≠ 0
    then have l: label x i = 1 label y i = 0
      using i label(1)[of i x] label(1)[of i y]
      by auto
    show f x · i ≤ x · i
      apply (rule label(5)[rule-format])
      using x y l i(2)
      apply auto
      done
    show y · i ≤ f y · i
      apply (rule label(4)[rule-format])
      using x y l i(2)
      apply auto
      done
  qed
  also have ... ≤ norm (f y − f x) + norm (y − x)
    apply (rule add-mono)
    apply (rule Basis-le-norm[OF i(2)])+
    done
  finally show |f x · i − x · i| ≤ norm (f y − f x) + norm (y − x)
    unfolding inner-simps .
qed
have ∃ e>0. ∀ x∈unit-cube. ∀ y∈unit-cube. ∀ z∈unit-cube. ∀ i∈Basis.
  norm (x − z) < e ∧ norm (y − z) < e ∧ label x i ≠ label y i ⟶
    |(f(z) − z)·i| < d / (real n)
proof −
  have d': d / real n / 8 > 0
    using d(1) by (simp add: n-def DIM-positive)
  have ∗: uniformly-continuous-on unit-cube f
    by (rule compact-uniformly-continuous[OF assms(1) compact-unit-cube])
  obtain e where e:
      e > 0
      ⋀x x'. x ∈ unit-cube ⟹
        x' ∈ unit-cube ⟹
        norm (x' − x) < e ⟹
        norm (f x' − f x) < d / real n / 8
    using ∗[unfolded uniformly-continuous-on-def,rule-format,OF d']
    unfolding dist-norm
    by blast
  show ?thesis
    apply (rule-tac x=min (e/2) (d/real n/8) in exI)
    apply safe
  proof −
    show 0 < min (e / 2) (d / real n / 8)
```

```
  using d' e by auto
fix x y z i
assume as:
  x ∈ unit-cube y ∈ unit-cube z ∈ unit-cube
  norm (x − z) < min (e / 2) (d / real n / 8)
  norm (y − z) < min (e / 2) (d / real n / 8)
  label x i ≠ label y i
assume i: i ∈ Basis
have ∗: ⋀z fz x fx n1 n2 n3 n4 d4 d :: real. |fx − x| ≤ n1 + n2 ⟹
  |fx − fz| ≤ n3 ⟹ |x − z| ≤ n4 ⟹
  n1 < d4 ⟹ n2 < 2 ∗ d4 ⟹ n3 < d4 ⟹ n4 < d4 ⟹
  (8 ∗ d4 = d) ⟹ |fz − z| < d
  by auto
show |(f z − z) · i| < d / real n
  unfolding inner-simps
proof (rule ∗)
  show |f x · i − x · i| ≤ norm (f y −f x) + norm (y − x)
    apply (rule lem1[rule-format])
    using as i
    apply auto
    done
  show |f x · i − f z · i| ≤ norm (f x − f z) |x · i − z · i| ≤ norm (x − z)
    unfolding inner-diff-left[symmetric]
    by (rule Basis-le-norm[OF i])+
  have tria: norm (y − x) ≤ norm (y − z) + norm (x − z)
    using dist-triangle[of y x z, unfolded dist-norm]
    unfolding norm-minus-commute
    by auto
  also have ... < e / 2 + e / 2
    apply (rule add-strict-mono)
    using as(4,5)
    apply auto
    done
  finally show norm (f y − f x) < d / real n / 8
    apply −
    apply (rule e(2))
    using as
    apply auto
    done
  have norm (y − z) + norm (x − z) < d / real n / 8 + d / real n / 8
    apply (rule add-strict-mono)
    using as
    apply auto
    done
  then show norm (y − x) < 2 ∗ (d / real n / 8)
    using tria
    by auto
  show norm (f x − f z) < d / real n / 8
    apply (rule e(2))
```

```
      using as e(1)
      apply auto
      done
    qed (insert as, auto)
  qed
qed
then
obtain e where e:
  e > 0
  ⋀x y z i. x ∈ unit-cube ⟹
    y ∈ unit-cube ⟹
    z ∈ unit-cube ⟹
    i ∈ Basis ⟹
    norm (x − z) < e ∧ norm (y − z) < e ∧ label x i ≠ label y i ⟹
    |(f z − z) • i| < d / real n
  by blast
obtain p :: nat where p: 1 + real n / e ≤ real p
  using real-arch-simple ..
have 1 + real n / e > 0
  using e(1) n by (simp add: add-pos-pos)
then have p > 0
  using p by auto

obtain b :: nat ⇒ 'a where b: bij-betw b {..< n} Basis
  by atomize-elim (auto simp: n-def intro!: finite-same-card-bij)
def b' ≡ inv-into {..< n} b
then have b': bij-betw b' Basis {..< n}
  using bij-betw-inv-into[OF b] by auto
then have b'-Basis: ⋀i. i ∈ Basis ⟹ b' i ∈ {..< n}
  unfolding bij-betw-def by (auto simp: set-eq-iff)
have bb'[simp]:⋀i. i ∈ Basis ⟹ b (b' i) = i
  unfolding b'-def
  using b
  by (auto simp: f-inv-into-f bij-betw-def)
have b'b[simp]:⋀i. i < n ⟹ b' (b i) = i
  unfolding b'-def
  using b
  by (auto simp: inv-into-f-eq bij-betw-def)
have ∗: ⋀x :: nat. x = 0 ∨ x = 1 ⟷ x ≤ 1
  by auto
have b'': ⋀j. j < n ⟹ b j ∈ Basis
  using b unfolding bij-betw-def by auto
have q1: 0 < p ∀x. (∀i<n. x i ≤ p) ⟶
  (∀i<n. (label (∑i∈Basis. (real (x (b' i)) / real p) ∗R i) ∘ b) i = 0 ∨
        (label (∑i∈Basis. (real (x (b' i)) / real p) ∗R i) ∘ b) i = 1)
  unfolding ∗
  using ‹p > 0› ‹n > 0›
  using label(1)[OF b'']
  by auto
```

{ fix $x :: nat \Rightarrow nat$ **and** i **assume** $\forall\, i{<}n.\ x\ i \leq p\ i < n\ x\ i = p \vee x\ i = 0$
then have $(\sum i{\in}Basis.\ (real\ (x\ (b'\ i))\ /\ real\ p) *_R\ i) \in (unit\text{-}cube{::}'a\ set)$
using *b'-Basis*
by (*auto simp add*: *mem-unit-cube inner-simps bij-betw-def zero-le-divide-iff divide-le-eq-1*) **}**
note *cube = this*
have *q2*: $\forall\, x.\ (\forall\, i{<}n.\ x\ i \leq p) \longrightarrow (\forall\, i{<}n.\ x\ i = 0 \longrightarrow$
$(label\ (\sum i{\in}Basis.\ (real\ (x\ (b'\ i))\ /\ real\ p) *_R\ i) \circ b)\ i = 0)$
unfolding *o-def* **using** *cube* ‹$p > 0$› **by** (*intro allI impI label*(*2*)) (*auto simp add*: *b''*)
have *q3*: $\forall\, x.\ (\forall\, i{<}n.\ x\ i \leq p) \longrightarrow (\forall\, i{<}n.\ x\ i = p \longrightarrow$
$(label\ (\sum i{\in}Basis.\ (real\ (x\ (b'\ i))\ /\ real\ p) *_R\ i) \circ b)\ i = 1)$
using *cube* ‹$p > 0$› **unfolding** *o-def* **by** (*intro allI impI label*(*3*)) (*auto simp add*: *b''*)
obtain q **where** q:
$\forall\, i{<}n.\ q\ i < p$
$\forall\, i{<}n.$
$\exists\, r\ s.\ (\forall\, j{<}n.\ q\ j \leq r\ j \wedge r\ j \leq q\ j + 1) \wedge$
$(\forall\, j{<}n.\ q\ j \leq s\ j \wedge s\ j \leq q\ j + 1) \wedge$
$(label\ (\sum i{\in}Basis.\ (real\ (r\ (b'\ i))\ /\ real\ p) *_R\ i) \circ b)\ i \neq$
$(label\ (\sum i{\in}Basis.\ (real\ (s\ (b'\ i))\ /\ real\ p) *_R\ i) \circ b)\ i$
by (*rule kuhn-lemma*[*OF q1 q2 q3*])
def $z \equiv (\sum i{\in}Basis.\ (real\ (q\ (b'\ i))\ /\ real\ p) *_R\ i){::}'a$
have $\exists\, i{\in}Basis.\ d\ /\ real\ n \leq |(f\ z - z){\cdot}i|$
proof (*rule ccontr*)
have $\forall\, i{\in}Basis.\ q\ (b'\ i) \in \{0..p\}$
using *q*(*1*) *b'*
by (*auto intro*: *less-imp-le simp*: *bij-betw-def*)
then have $z \in unit\text{-}cube$
unfolding *z-def mem-unit-cube*
using *b'-Basis*
by (*auto simp add*: *bij-betw-def zero-le-divide-iff divide-le-eq-1*)
then have *d-fz-z*: $d \leq norm\ (f\ z - z)$
by (*rule d*)
assume $\neg$ *?thesis*
then have *as*: $\forall\, i{\in}Basis.\ |f\ z \cdot i - z \cdot i| < d\ /\ real\ n$
using ‹$n > 0$›
by (*auto simp add*: *not-le inner-diff*)
have $norm\ (f\ z - z) \leq (\sum i{\in}Basis.\ |f\ z \cdot i - z \cdot i|)$
unfolding *inner-diff-left*[*symmetric*]
by (*rule norm-le-l1*)
also have $\ldots < (\sum (i{::}'a) \in Basis.\ d\ /\ real\ n)$
apply (*rule setsum-strict-mono*)
using *as*
apply *auto*
done
also have $\ldots = d$
using *DIM-positive*[**where** *'a='a*]
by (*auto simp*: *n-def*)

finally show *False*
using *d-fz-z* **by** *auto*
qed
then obtain i **where** i: $i \in Basis\ d\ /\ real\ n \leq |(f\ z - z) \cdot i|$ **..**
have $*$: $b'\ i < n$
using i **and** b'[*unfolded bij-betw-def*]
by *auto*
obtain $r\ s$ **where** *rs*:
$\bigwedge j.\ j < n \Longrightarrow q\ j \leq r\ j \wedge r\ j \leq q\ j + 1$
$\bigwedge j.\ j < n \Longrightarrow q\ j \leq s\ j \wedge s\ j \leq q\ j + 1$
$(label\ (\sum i{\in}Basis.\ (real\ (r\ (b'\ i))\ /\ real\ p)\ *_R\ i) \circ b)\ (b'\ i) \neq$
$(label\ (\sum i{\in}Basis.\ (real\ (s\ (b'\ i))\ /\ real\ p)\ *_R\ i) \circ b)\ (b'\ i)$
using $q(2)$[*rule-format*,*OF* $*$] **by** *blast*
have b'-*im*: $\bigwedge i.\ i \in Basis \Longrightarrow\ b'\ i < n$
using b' **unfolding** *bij-betw-def* **by** *auto*
def $r' \equiv (\sum i{\in}Basis.\ (real\ (r\ (b'\ i))\ /\ real\ p)\ *_R\ i)::'a$
have $\bigwedge i.\ i \in Basis \Longrightarrow r\ (b'\ i) \leq p$
apply (*rule order-trans*)
apply (*rule rs*(1)[*OF* b'-*im*,*THEN conjunct2*])
using $q(1)$[*rule-format*,*OF* b'-*im*]
apply (*auto simp add*: *Suc-le-eq*)
done
then have $r' \in unit\text{-}cube$
unfolding r'-*def mem-unit-cube*
using b'-*Basis*
by (*auto simp add*: *bij-betw-def zero-le-divide-iff divide-le-eq-1*)
def $s' \equiv (\sum i{\in}Basis.\ (real\ (s\ (b'\ i))\ /\ real\ p)\ *_R\ i)::'a$
have $\bigwedge i.\ i \in Basis \Longrightarrow s\ (b'\ i) \leq p$
apply (*rule order-trans*)
apply (*rule rs*(2)[*OF* b'-*im*, *THEN conjunct2*])
using $q(1)$[*rule-format*,*OF* b'-*im*]
apply (*auto simp add*: *Suc-le-eq*)
done
then have $s' \in unit\text{-}cube$
unfolding s'-*def mem-unit-cube*
using b'-*Basis*
by (*auto simp add*: *bij-betw-def zero-le-divide-iff divide-le-eq-1*)
have $z \in unit\text{-}cube$
unfolding *z-def mem-unit-cube*
using b'-*Basis* $q(1)$[*rule-format*,*OF* b'-*im*] ‹$p > 0$›
by (*auto simp add*: *bij-betw-def zero-le-divide-iff divide-le-eq-1 less-imp-le*)
have $*$: $\bigwedge x.\ 1 + real\ x = real\ (Suc\ x)$
by *auto*
{
have $(\sum i{\in}Basis.\ |real\ (r\ (b'\ i)) - real\ (q\ (b'\ i))|) \leq (\sum (i::'a){\in}Basis.\ 1)$
apply (*rule setsum-mono*)
using *rs*(1)[*OF* b'-*im*]
apply (*auto simp add*:$*$ *field-simps simp del*: *of-nat-Suc*)
done

```
  also have ... < e * real p
    using p ‹e > 0› ‹p > 0›
    by (auto simp add: field-simps n-def)
  finally have (∑ i∈Basis. |real (r (b' i)) − real (q (b' i))|) < e * real p .
 }
 moreover
 {
  have (∑ i∈Basis. |real (s (b' i)) − real (q (b' i))|) ≤ (∑ (i::'a)∈Basis. 1)
    apply (rule setsum-mono)
    using rs(2)[OF b'-im]
    apply (auto simp add:* field-simps simp del: of-nat-Suc)
    done
  also have ... < e * real p
    using p ‹e > 0› ‹p > 0›
    by (auto simp add: field-simps n-def)
  finally have (∑ i∈Basis. |real (s (b' i)) − real (q (b' i))|) < e * real p .
 }
 ultimately
 have norm (r' − z) < e and norm (s' − z) < e
  unfolding r'-def s'-def z-def
  using ‹p > 0›
  apply (rule-tac[!] le-less-trans[OF norm-le-l1])
 apply (auto simp add: field-simps setsum-divide-distrib[symmetric] inner-diff-left)
  done
 then have |(f z − z) · i| < d / real n
  using rs(3) i
  unfolding r'-def[symmetric] s'-def[symmetric] o-def bb'
  by (intro e(2)[OF ‹r'∈unit-cube› ‹s'∈unit-cube› ‹z∈unit-cube›]) auto
 then show False
  using i by auto
qed
```

26.5 Retractions

definition *retraction s t r* ⟷ *t* ⊆ *s* ∧ *continuous-on s r* ∧ *r* ‘ *s* ⊆ *t* ∧ (∀ *x*∈*t*. *r x* = *x*)

definition *retract-of* (**infixl** *retract'-of 50*)
where (*t retract-of s*) ⟷ (∃ *r*. *retraction s t r*)

lemma *retraction-idempotent*: *retraction s t r* ⟹ *x* ∈ *s* ⟹ *r* (*r x*) = *r x*
unfolding *retraction-def* **by** *auto*

26.6 Preservation of fixpoints under (more general notion of) retraction

lemma *invertible-fixpoint-property*:
fixes *s* :: *'a*::*euclidean-space set*
and *t* :: *'b*::*euclidean-space set*

```
  assumes continuous-on t i
    and i ‘ t ⊆ s
    and continuous-on s r
    and r ‘ s ⊆ t
    and ∀y∈t. r (i y) = y
    and ∀f. continuous-on s f ∧ f ‘ s ⊆ s ⟶ (∃x∈s. f x = x)
    and continuous-on t g
    and g ‘ t ⊆ t
  obtains y where y ∈ t and g y = y
proof −
  have ∃x∈s. (i ∘ g ∘ r) x = x
    apply (rule assms(6)[rule-format])
    apply rule
    apply (rule continuous-on-compose assms)+
    apply ((rule continuous-on-subset)?, rule assms)+
    using assms(2,4,8)
    apply auto
    apply blast
    done
  then obtain x where x: x ∈ s (i ∘ g ∘ r) x = x ..
  then have ∗: g (r x) ∈ t
    using assms(4,8) by auto
  have r ((i ∘ g ∘ r) x) = r x
    using x by auto
  then show ?thesis
    apply (rule-tac that[of r x])
    using x
    unfolding o-def
    unfolding assms(5)[rule-format,OF ∗]
    using assms(4)
    apply auto
    done
qed

lemma homeomorphic-fixpoint-property:
  fixes s :: 'a::euclidean-space set
    and t :: 'b::euclidean-space set
  assumes s homeomorphic t
  shows (∀f. continuous-on s f ∧ f ‘ s ⊆ s ⟶ (∃x∈s. f x = x)) ⟷
    (∀g. continuous-on t g ∧ g ‘ t ⊆ t ⟶ (∃y∈t. g y = y))
proof −
  obtain r i where
      ∀x∈s. i (r x) = x
      r ‘ s = t
      continuous-on s r
      ∀y∈t. r (i y) = y
      i ‘ t = s
      continuous-on t i
    using assms
```

unfolding *homeomorphic-def homeomorphism-def*
by *blast*
then show *?thesis*
apply −
apply *rule*
apply (*rule-tac*[!] *allI impI*)+
apply (*rule-tac g*=*g* **in** *invertible-fixpoint-property*[*of t i s r*])
prefer *10*
apply (*rule-tac g*=*f* **in** *invertible-fixpoint-property*[*of s r t i*])
apply *auto*
done
qed

lemma *retract-fixpoint-property*:
fixes *f* :: *'a*::*euclidean-space* ⇒ *'b*::*euclidean-space*
and *s* :: *'a set*
assumes *t retract-of s*
and ∀*f*. *continuous-on s f* ∧ *f* ' *s* ⊆ *s* ⟶ (∃ *x*∈*s*. *f x* = *x*)
and *continuous-on t g*
and *g* ' *t* ⊆ *t*
obtains *y* **where** *y* ∈ *t* **and** *g y* = *y*
proof −
obtain *h* **where** *retraction s t h*
using *assms*(*1*) **unfolding** *retract-of-def* **..**
then show *?thesis*
unfolding *retraction-def*
apply −
apply (*rule invertible-fixpoint-property*[*OF continuous-on-id - - - - assms*(*2*), *of t h g*])
prefer *7*
apply (*rule-tac y* = *y* **in** *that*)
using *assms*
apply *auto*
done
qed

26.7 The Brouwer theorem for any set with nonempty interior

lemma *convex-unit-cube*: *convex unit-cube*
apply (*rule is-interval-convex*)
apply (*clarsimp simp add*: *is-interval-def mem-unit-cube*)
apply (*drule* (*1*) *bspec*)+
apply *auto*
done

lemma *brouwer-weak*:
fixes *f* :: *'a*::*euclidean-space* ⇒ *'a*
assumes *compact s*

```
    and convex s
    and interior s ≠ {}
    and continuous-on s f
    and f ‘ s ⊆ s
  obtains x where x ∈ s and f x = x
proof −
  let ?U = unit-cube :: 'a set
  have ∑ Basis /R 2 ∈ interior ?U
  proof (rule interiorI)
    let ?I = (⋂ i∈Basis. {x::'a. 0 < x • i} ∩ {x. x • i < 1})
    show open ?I
      by (intro open-INT finite-Basis ballI open-Int, auto intro: open-Collect-less)
    show ∑ Basis /R 2 ∈ ?I
      by simp
    show ?I ⊆ unit-cube
      unfolding unit-cube-def by force
  qed
  then have ∗: interior ?U ≠ {} by fast
  have ∗: ?U homeomorphic s
    using homeomorphic-convex-compact[OF convex-unit-cube compact-unit-cube ∗
assms(2,1,3)] .
  have ∀f. continuous-on ?U f ∧ f ‘ ?U ⊆ ?U ⟶
    (∃ x∈?U. f x = x)
    using brouwer-cube by auto
  then show ?thesis
    unfolding homeomorphic-fixpoint-property[OF ∗]
    using assms
    by (auto simp: intro: that)
qed
```

And in particular for a closed ball.

```
lemma brouwer-ball:
  fixes f :: 'a::euclidean-space ⇒ 'a
  assumes e > 0
    and continuous-on (cball a e) f
    and f ‘ cball a e ⊆ cball a e
  obtains x where x ∈ cball a e and f x = x
  using brouwer-weak[OF compact-cball convex-cball, of a e f]
  unfolding interior-cball ball-eq-empty
  using assms by auto
```

Still more general form; could derive this directly without using the rather involved *HOMEOMORPHIC-CONVEX-COMPACT* theorem, just using a scaling and translation to put the set inside the unit cube.

```
lemma brouwer:
  fixes f :: 'a::euclidean-space ⇒ 'a
  assumes compact s
    and convex s
    and s ≠ {}
```

```
    and continuous-on s f
    and f ' s ⊆ s
  obtains x where x ∈ s and f x = x
proof −
  have ∃ e>0. s ⊆ cball 0 e
    using compact-imp-bounded[OF assms(1)]
    unfolding bounded-pos
    apply (erule-tac exE)
    apply (rule-tac x=b in exI)
    apply (auto simp add: dist-norm)
    done
  then obtain e where e: e > 0 s ⊆ cball 0 e
    by blast
  have ∃ x∈ cball 0 e. (f ∘ closest-point s) x = x
    apply (rule-tac brouwer-ball[OF e(1), of 0 f ∘ closest-point s])
    apply (rule continuous-on-compose )
     apply (rule continuous-on-closest-point[OF assms(2) compact-imp-closed[OF
assms(1)] assms(3)])
    apply (rule continuous-on-subset[OF assms(4)])
   apply (insert closest-point-in-set[OF compact-imp-closed[OF assms(1)] assms(3)])
    using assms(5)[unfolded subset-eq]
    using e(2)[unfolded subset-eq mem-cball]
    apply (auto simp add: dist-norm)
    done
  then obtain x where x: x ∈ cball 0 e (f ∘ closest-point s) x = x ..
  have ∗: closest-point s x = x
    apply (rule closest-point-self)
     apply (rule assms(5)[unfolded subset-eq,THEN bspec[where x=x], unfolded
image-iff])
    apply (rule-tac x=closest-point s x in bexI)
    using x
    unfolding o-def
     using closest-point-in-set[OF compact-imp-closed[OF assms(1)] assms(3), of
x]
    apply auto
    done
  show thesis
    apply (rule-tac x=closest-point s x in that)
    unfolding x(2)[unfolded o-def]
   apply (rule closest-point-in-set[OF compact-imp-closed[OF assms(1)] assms(3)])
    using ∗
    apply auto
    done
qed
```

So we get the no-retraction theorem.

```
lemma no-retraction-cball:
  fixes a :: 'a::euclidean-space
  assumes e > 0
```

shows ¬ (*frontier* (*cball a e*) *retract-of* (*cball a e*))
proof
assume ∗: *frontier* (*cball a e*) *retract-of* (*cball a e*)
have ∗∗: ⋀*xa*. *a* − (*2* ∗$_R$ *a* − *xa*) = − (*a* − *xa*)
using *scaleR-left-distrib*[*of 1 1 a*] **by** *auto*
obtain *x* **where** *x*:
x ∈ {*x*. *norm* (*a* − *x*) = *e*}
2 ∗$_R$ *a* − *x* = *x*
apply (*rule retract-fixpoint-property*[*OF* ∗, *of* λ*x*. *scaleR 2 a* − *x*])
apply (*blast intro*: *brouwer-ball*[*OF assms*])
apply (*intro continuous-intros*)
unfolding *frontier-cball subset-eq Ball-def image-iff dist-norm sphere-def*
apply (*auto simp add*: ∗∗ *norm-minus-commute*)
done
then have *scaleR 2 a* = *scaleR 1 x* + *scaleR 1 x*
by (*auto simp add*: *algebra-simps*)
then have *a* = *x*
unfolding *scaleR-left-distrib*[*symmetric*]
by *auto*
then show *False*
using *x assms* **by** *auto*
qed

26.8 Retractions

lemma *retraction*:
retraction s t r ⟷
t ⊆ *s* ∧ *continuous-on s r* ∧ *r* ‘ *s* = *t* ∧ (∀ *x* ∈ *t*. *r x* = *x*)
by (*force simp*: *retraction-def*)

lemma *retract-of-imp-extensible*:
assumes *s retract-of t* **and** *continuous-on s f* **and** *f* ‘ *s* ⊆ *u*
obtains *g* **where** *continuous-on t g g* ‘ *t* ⊆ *u* ⋀*x*. *x* ∈ *s* ⟹ *g x* = *f x*
using *assms*
apply (*clarsimp simp add*: *retract-of-def retraction*)
apply (*rule-tac g* = *f o r* **in** *that*)
apply (*auto simp*: *continuous-on-compose2*)
done

lemma *idempotent-imp-retraction*:
assumes *continuous-on s f* **and** *f* ‘ *s* ⊆ *s* **and** ⋀*x*. *x* ∈ *s* ⟹ *f*(*f x*) = *f x*
shows *retraction s* (*f* ‘ *s*) *f*
by (*simp add*: *assms retraction*)

lemma *retraction-subset*:
assumes *retraction s t r* **and** *t* ⊆ *s*′ **and** *s*′ ⊆ *s*
shows *retraction s*′ *t r*
apply (*simp add*: *retraction-def*)
by (*metis assms continuous-on-subset image-mono retraction*)

lemma *retract-of-subset*:
  assumes *t retract-of s* **and** $t \subseteq s'$ **and** $s' \subseteq s$
    shows *t retract-of s'*
by (*meson assms retract-of-def retraction-subset*)

lemma *retraction-refl* [*simp*]: *retraction s s* ($\lambda x.\ x$)
by (*simp add*: *continuous-on-id retraction*)

lemma *retract-of-refl* [*iff*]: *s retract-of s*
  using *continuous-on-id retract-of-def retraction-def* **by** *fastforce*

lemma *retract-of-imp-subset*:
   s retract-of t $\Longrightarrow$ $s \subseteq t$
by (*simp add*: *retract-of-def retraction-def*)

lemma *retract-of-empty* [*simp*]:
     ({} *retract-of s*) $\longleftrightarrow$ $s = \{\}$ (*s retract-of* {}) $\longleftrightarrow$ $s = \{\}$
by (*auto simp*: *retract-of-def retraction-def*)

lemma *retract-of-singleton* [*iff*]: ({x} *retract-of s*) $\longleftrightarrow$ $x \in s$
  using *continuous-on-const*
  by (*auto simp*: *retract-of-def retraction-def*)

lemma *retraction-comp*:
   ⟦*retraction s t f*; *retraction t u g*⟧
       $\Longrightarrow$ *retraction s u* (*g o f*)
apply (*auto simp*: *retraction-def intro*: *continuous-on-compose2*)
by *blast*

lemma *retract-of-trans*:
  assumes *s retract-of t* **and** *t retract-of u*
    shows *s retract-of u*
using *assms* **by** (*auto simp*: *retract-of-def intro*: *retraction-comp*)

lemma *closedin-retract*:
  fixes $s :: 'a :: real\text{-}normed\text{-}vector\ set$
  assumes *s retract-of t*
    shows *closedin* (*subtopology euclidean t*) *s*
proof −
  obtain *r* **where** $s \subseteq t$ *continuous-on t r r* ‘ $t \subseteq s$ $\bigwedge x.\ x \in s \Longrightarrow r\ x = x$
    using *assms* **by** (*auto simp*: *retract-of-def retraction-def*)
  then have *s*: $s = \{x \in t.\ (norm(r\ x - x)) = 0\}$ **by** *auto*
  show *?thesis*
    apply (*subst s*)
    apply (*rule continuous-closedin-preimage-constant*)
   by (*simp add*: ‹*continuous-on t r*› *continuous-on-diff continuous-on-id continuous-on-norm*)
qed

lemma *retract-of-contractible*:
assumes *contractible t s retract-of t*
shows *contractible s*
using *assms*
apply (*clarsimp simp add*: *retract-of-def contractible-def retraction-def homotopic-with*)
apply (*rule-tac x*=*r a* **in** *exI*)
apply (*rule-tac x*=*r o h* **in** *exI*)
apply (*intro conjI continuous-intros continuous-on-compose*)
apply (*erule continuous-on-subset* | *force*)+
done

lemma *retract-of-compact*:
⟦*compact t*; *s retract-of t*⟧ ⟹ *compact s*
by (*metis compact-continuous-image retract-of-def retraction*)

lemma *retract-of-closed*:
fixes *s* :: *'a* :: *real-normed-vector set*
shows ⟦*closed t*; *s retract-of t*⟧ ⟹ *closed s*
by (*metis closedin-retract closedin-closed-eq*)

lemma *retract-of-connected*:
⟦*connected t*; *s retract-of t*⟧ ⟹ *connected s*
by (*metis Topological-Spaces.connected-continuous-image retract-of-def retraction*)

lemma *retract-of-path-connected*:
⟦*path-connected t*; *s retract-of t*⟧ ⟹ *path-connected s*
by (*metis path-connected-continuous-image retract-of-def retraction*)

lemma *retract-of-simply-connected*:
⟦*simply-connected t*; *s retract-of t*⟧ ⟹ *simply-connected s*
apply (*simp add*: *retract-of-def retraction-def*, *clarify*)
apply (*rule simply-connected-retraction-gen*)
apply (*force simp*: *continuous-on-id elim*!: *continuous-on-subset*)+
done

lemma *retract-of-homotopically-trivial*:
assumes *ts*: *t retract-of s*
and *hom*: ⋀*f g*. ⟦*continuous-on u f*; *f ' u* ⊆ *s*;
continuous-on u g; *g ' u* ⊆ *s*⟧
⟹ *homotopic-with* (λ*x*. *True*) *u s f g*
and *continuous-on u f f ' u* ⊆ *t*
and *continuous-on u g g ' u* ⊆ *t*
shows *homotopic-with* (λ*x*. *True*) *u t f g*
proof −
obtain *r* **where** *r ' s* ⊆ *s continuous-on s r* ∀*x*∈*s*. *r* (*r x*) = *r x t* = *r ' s*
using *ts* **by** (*auto simp*: *retract-of-def retraction*)
then obtain *k* **where** *Retracts s r t k*
unfolding *Retracts-def*

by (*metis continuous-on-subset dual-order.trans image-iff image-mono*)
then show *?thesis*
apply (*rule Retracts.homotopically-trivial-retraction-gen*)
using *assms*
apply (*force simp*: *hom*)+
done
qed

lemma *retract-of-homotopically-trivial-null*:
assumes *ts*: *t retract-of s*
and *hom*: $\bigwedge f.\ [\![$*continuous-on u f*; *f ‘ u* $\subseteq$ *s*$]\!] \Longrightarrow \exists c.$ *homotopic-with* $(\lambda x.\ True)$ *u s f* $(\lambda x.\ c)$
and *continuous-on u f f ‘ u* $\subseteq$ *t*
obtains *c* **where** *homotopic-with* $(\lambda x.\ True)$ *u t f* $(\lambda x.\ c)$
proof −
obtain *r* **where** *r ‘ s* $\subseteq$ *s continuous-on s r* $\forall x{\in}s.\ r\ (r\ x) = r\ x\ t = r\ ‘\ s$
using *ts* **by** (*auto simp*: *retract-of-def retraction*)
then obtain *k* **where** *Retracts s r t k*
unfolding *Retracts-def*
by (*metis continuous-on-subset dual-order.trans image-iff image-mono*)
then show *?thesis*
apply (*rule Retracts.homotopically-trivial-retraction-null-gen*)
apply (*rule TrueI refl assms that* | *assumption*)+
done
qed

lemma *retraction-imp-quotient-map*:
retraction s t r
$\Longrightarrow$ *u* $\subseteq$ *t*
$\Longrightarrow$ (*openin* (*subtopology euclidean s*) $\{x.\ x \in s \wedge r\ x \in u\} \longleftrightarrow$
openin (*subtopology euclidean t*) *u*)
apply (*clarsimp simp add*: *retraction*)
apply (*rule continuous-right-inverse-imp-quotient-map* [**where** *g*=*r*])
apply (*auto simp*: *elim*: *continuous-on-subset*)
done

lemma *retract-of-locally-compact*:
fixes *s* :: *'a* :: {*heine-borel*,*real-normed-vector*} *set*
shows $[\![$ *locally compact s*; *t retract-of s*$]\!] \Longrightarrow$ *locally compact t*
by (*metis locally-compact-closedin closedin-retract*)

lemma *retract-of-times*:
$[\![$*s retract-of s'*; *t retract-of t'*$]\!] \Longrightarrow$ (*s* × *t*) *retract-of* (*s'* × *t'*)
apply (*simp add*: *retract-of-def retraction-def Sigma-mono*, *clarify*)
apply (*rename-tac f g*)
apply (*rule-tac x*=$\lambda z.$ ((*f o fst*) *z*, (*g o snd*) *z*) **in** *exI*)
apply (*rule conjI continuous-intros* | *erule continuous-on-subset* | *force*)+
done

lemma *homotopic-into-retract*:
  ⟦*f* ‘ *s* ⊆ *t*; *g* ‘ *s* ⊆ *t*; *t retract-of u*;
    homotopic-with (λ*x*. *True*) *s u f g*⟧
    ⟹ *homotopic-with* (λ*x*. *True*) *s t f g*
apply (*subst* (*asm*) *homotopic-with-def*)
apply (*simp add*: *homotopic-with retract-of-def retraction-def*, *clarify*)
apply (*rule-tac x*=*r o h* **in** *exI*)
apply (*rule conjI continuous-intros* | *erule continuous-on-subset* | *force simp*: *image-subset-iff*)+
done

end

27 A generic phantom type

theory *Phantom-Type*
imports *Main*
begin

datatype ('*a*, '*b*) *phantom* = *phantom* (*of-phantom*: '*b*)

lemma *type-definition-phantom'*: *type-definition of-phantom phantom UNIV*
by(*unfold-locales*) *simp-all*

lemma *phantom-comp-of-phantom* [*simp*]: *phantom* ∘ *of-phantom* = *id*
  and *of-phantom-comp-phantom* [*simp*]: *of-phantom* ∘ *phantom* = *id*
by(*simp-all add*: *o-def id-def*)

syntax -*Phantom* :: *type* ⇒ *logic* ((*1Phantom*/(*1* '(-'))))
translations
  Phantom('*t*) => *CONST phantom* :: - ⇒ ('*t*, -) *phantom*

typed-print-translation ‹
  let
    fun phantom-tr' ctxt (*Type* (@{*type-name fun*}, [-, *Type* (@{*type-name phantom*}, [*T*, -])])) *ts* =
          list-comb
            (*Syntax.const* @{*syntax-const -Phantom*} $ *Syntax-Phases.term-of-typ ctxt T*, *ts*)
      | *phantom-tr' - - - = raise Match*;
  in [(@{*const-syntax phantom*}, *phantom-tr'*)] *end*
›

lemma *of-phantom-inject* [*simp*]:
  of-phantom x = *of-phantom y* ⟷ *x* = *y*
by(*cases x y rule*: *phantom.exhaust*[*case-product phantom.exhaust*]) *simp*

end

28 Cardinality of types

theory *Cardinality*
imports *Phantom-Type*
begin

28.1 Preliminary lemmas

lemma (**in** *type-definition*) *univ*:
  $UNIV = Abs\ `\ A$
proof
  show $Abs\ `\ A \subseteq UNIV$ **by** (*rule subset-UNIV*)
  show $UNIV \subseteq Abs\ `\ A$
  proof
    fix $x :: 'b$
    have $x = Abs\ (Rep\ x)$ **by** (*rule Rep-inverse* [*symmetric*])
    moreover have $Rep\ x \in A$ **by** (*rule Rep*)
    ultimately show $x \in Abs\ `\ A$ **by** (*rule image-eqI*)
  qed
qed

lemma (**in** *type-definition*) *card*: $card\ (UNIV :: 'b\ set) = card\ A$
  by (*simp add*: *univ card-image inj-on-def Abs-inject*)

lemma *finite-range-Some*: *finite* (*range* (*Some* :: $'a \Rightarrow 'a\ option$)) = *finite* (*UNIV* :: $'a\ set$)
by(*auto dest*: *finite-imageD intro*: *inj-Some*)

lemma *infinite-literal*: $\neg$ *finite* (*UNIV* :: *String.literal set*)
proof −
  have *inj STR* **by**(*auto intro*: *injI*)
  thus *?thesis*
   by(*auto simp add*: *type-definition.univ*[*OF type-definition-literal*] *infinite-UNIV-listI dest*: *finite-imageD*)
qed

28.2 Cardinalities of types

syntax *-type-card* :: *type* => *nat* ((*1CARD*/(*1'*(*-'*))))

translations $CARD('t) => CONST\ card\ (CONST\ UNIV :: 't\ set)$

print-translation ‹
  let
    fun card-univ-tr' ctxt [*Const* (@{*const-syntax UNIV*}, *Type* (-, [*T*]))] =
      Syntax.const @{*syntax-const -type-card*} $ *Syntax-Phases.term-of-typ ctxt T*
  in [(@{*const-syntax card*}, *card-univ-tr'*)] *end*
›

lemma *card-prod* [*simp*]: $CARD('a \times 'b) = CARD('a) * CARD('b)$

unfolding *UNIV-Times-UNIV* [*symmetric*] **by** (*simp only*: *card-cartesian-product*)

lemma *card-UNIV-sum*: $CARD('a + 'b) = (if\ CARD('a) \neq 0 \wedge CARD('b) \neq 0\ then\ CARD('a) + CARD('b)\ else\ 0)$
unfolding *UNIV-Plus-UNIV*[*symmetric*]
by(*auto simp add*: *card-eq-0-iff card-Plus simp del*: *UNIV-Plus-UNIV*)

lemma *card-sum* [*simp*]: $CARD('a + 'b) = CARD('a::finite) + CARD('b::finite)$
by(*simp add*: *card-UNIV-sum*)

lemma *card-UNIV-option*: $CARD('a\ option) = (if\ CARD('a) = 0\ then\ 0\ else\ CARD('a) + 1)$
proof −
have $(None :: 'a\ option) \notin range\ Some$ **by** *clarsimp*
thus *?thesis*
by (*simp add*: *UNIV-option-conv card-eq-0-iff finite-range-Some card-image*)
qed

lemma *card-option* [*simp*]: $CARD('a\ option) = Suc\ CARD('a::finite)$
by(*simp add*: *card-UNIV-option*)

lemma *card-UNIV-set*: $CARD('a\ set) = (if\ CARD('a) = 0\ then\ 0\ else\ 2\ \hat{}\ CARD('a))$
by(*simp add*: *Pow-UNIV*[*symmetric*] *card-eq-0-iff card-Pow del*: *Pow-UNIV*)

lemma *card-set* [*simp*]: $CARD('a\ set) = 2\ \hat{}\ CARD('a::finite)$
by(*simp add*: *card-UNIV-set*)

lemma *card-nat* [*simp*]: $CARD(nat) = 0$
by (*simp add*: *card-eq-0-iff*)

lemma *card-fun*: $CARD('a \Rightarrow 'b) = (if\ CARD('a) \neq 0 \wedge CARD('b) \neq 0 \vee CARD('b) = 1\ then\ CARD('b)\ \hat{}\ CARD('a)\ else\ 0)$
proof −
{ **assume** $0 < CARD('a)$ **and** $0 < CARD('b)$
hence *fina*: *finite* $(UNIV :: 'a\ set)$ **and** *finb*: *finite* $(UNIV :: 'b\ set)$
by(*simp-all only*: *card-ge-0-finite*)
from *finite-distinct-list*[*OF finb*] **obtain** *bs*
where *bs*: $set\ bs = (UNIV :: 'b\ set)$ **and** *distb*: *distinct bs* **by** *blast*
from *finite-distinct-list*[*OF fina*] **obtain** *as*
where *as*: $set\ as = (UNIV :: 'a\ set)$ **and** *dista*: *distinct as* **by** *blast*
have *cb*: $CARD('b) = length\ bs$
unfolding *bs*[*symmetric*] *distinct-card*[*OF distb*] **..**
have *ca*: $CARD('a) = length\ as$
unfolding *as*[*symmetric*] *distinct-card*[*OF dista*] **..**
let *?xs* = $map\ (\lambda ys.\ the \circ map\text{-}of\ (zip\ as\ ys))\ (List.n\text{-}lists\ (length\ as)\ bs)$
have *UNIV* = *set ?xs*
proof(*rule UNIV-eq-I*)
fix $f :: 'a \Rightarrow 'b$
from *as* **have** $f = the \circ map\text{-}of\ (zip\ as\ (map\ f\ as))$

```
      by(auto simp add: map-of-zip-map)
    thus f ∈ set ?xs using bs by(auto simp add: set-n-lists)
  qed
  moreover have distinct ?xs unfolding distinct-map
  proof(intro conjI distinct-n-lists distb inj-onI)
    fix xs ys :: 'b list
    assume xs: xs ∈ set (List.n-lists (length as) bs)
      and ys: ys ∈ set (List.n-lists (length as) bs)
      and eq: the ∘ map-of (zip as xs) = the ∘ map-of (zip as ys)
    from xs ys have [simp]: length xs = length as length ys = length as
      by(simp-all add: length-n-lists-elem)
    have map-of (zip as xs) = map-of (zip as ys)
    proof
      fix x
      from as bs have ∃ y. map-of (zip as xs) x = Some y ∃ y. map-of (zip as
ys) x = Some y
        by(simp-all add: map-of-zip-is-Some[symmetric])
      with eq show map-of (zip as xs) x = map-of (zip as ys) x
        by(auto dest: fun-cong[where x=x])
    qed
    with dista show xs = ys by(simp add: map-of-zip-inject)
  qed
  hence card (set ?xs) = length ?xs by(simp only: distinct-card)
  moreover have length ?xs = length bs ^ length as by(simp add: length-n-lists)
  ultimately have CARD('a ⇒ 'b) = CARD('b) ^ CARD('a) using cb ca by
simp }
 moreover {
  assume cb: CARD('b) = 1
  then obtain b where b: UNIV = {b :: 'b} by(auto simp add: card-Suc-eq)
  have eq: UNIV = {λx :: 'a. b ::'b}
  proof(rule UNIV-eq-I)
    fix x :: 'a ⇒ 'b
    { fix y
      have x y ∈ UNIV ..
      hence x y = b unfolding b by simp }
    thus x ∈ {λx. b} by(auto)
  qed
  have CARD('a ⇒ 'b) = 1 unfolding eq by simp }
 ultimately show ?thesis
  by(auto simp del: One-nat-def)(auto simp add: card-eq-0-iff dest: finite-fun-UNIVD2
finite-fun-UNIVD1)
qed

corollary finite-UNIV-fun:
 finite (UNIV :: ('a ⇒ 'b) set) ⟷
  finite (UNIV :: 'a set) ∧ finite (UNIV :: 'b set) ∨ CARD('b) = 1
 (is ?lhs ⟷ ?rhs)
proof −
 have ?lhs ⟷ CARD('a ⇒ 'b) > 0 by(simp add: card-gt-0-iff)
```

also have ... $\longleftrightarrow$ $CARD('a) > 0 \wedge CARD('b) > 0 \vee CARD('b) = 1$
by(*simp add*: *card-fun*)
also have ... = *?rhs* **by**(*simp add*: *card-gt-0-iff*)
finally show *?thesis* **.**
qed

lemma *card-literal*: $CARD(String.literal) = 0$
by(*simp add*: *card-eq-0-iff infinite-literal*)

28.3 Classes with at least 1 and 2

Class finite already captures "at least 1"

lemma *zero-less-card-finite* [*simp*]: $0 < CARD('a::finite)$
unfolding *neq0-conv* [*symmetric*] **by** *simp*

lemma *one-le-card-finite* [*simp*]: $Suc\ 0 \leq CARD('a::finite)$
by (*simp add*: *less-Suc-eq-le* [*symmetric*])

Class for cardinality "at least 2"

class *card2* = *finite* +
assumes *two-le-card*: $2 \leq CARD('a)$

lemma *one-less-card*: $Suc\ 0 < CARD('a::card2)$
using *two-le-card* [**where** $'a='a$] **by** *simp*

lemma *one-less-int-card*: $1 < int\ CARD('a::card2)$
using *one-less-card* [**where** $'a='a$] **by** *simp*

28.4 A type class for deciding finiteness of types

type-synonym $'a$ *finite-UNIV* = $('a, bool)$ *phantom*

class *finite-UNIV* =
fixes *finite-UNIV* :: $('a, bool)$ *phantom*
assumes *finite-UNIV*: *finite-UNIV* = *Phantom*$('a)$ (*finite* (*UNIV* :: $'a$ *set*))

lemma *finite-UNIV-code* [*code-unfold*]:
finite (*UNIV* :: $'a$ :: *finite-UNIV set*)
$\longleftrightarrow$ *of-phantom* (*finite-UNIV* :: $'a$ *finite-UNIV*)
by(*simp add*: *finite-UNIV*)

28.5 A type class for computing the cardinality of types

definition *is-list-UNIV* :: $'a$ *list* $\Rightarrow$ *bool*
where *is-list-UNIV xs* = (*let c* = *CARD*$('a)$ *in if c* = 0 *then False else size* (*remdups xs*) = *c*)

lemma *is-list-UNIV-iff*: *is-list-UNIV xs* $\longleftrightarrow$ *set xs* = *UNIV*
by(*auto simp add*: *is-list-UNIV-def Let-def card-eq-0-iff List.card-set*[*symmetric*]

dest: *subst*[**where** *P*=*finite*, *OF* - *finite-set*] *card-eq-UNIV-imp-eq-UNIV*)

type-synonym *'a card-UNIV* = (*'a*, *nat*) *phantom*

class *card-UNIV* = *finite-UNIV* +
fixes *card-UNIV* :: *'a card-UNIV*
assumes *card-UNIV*: *card-UNIV* = *Phantom*(*'a*) *CARD*(*'a*)

28.6 Instantiations for *card-UNIV*

instantiation *nat* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*nat*) *False*
definition *card-UNIV* = *Phantom*(*nat*) *0*
instance by *intro-classes* (*simp-all add*: *finite-UNIV-nat-def card-UNIV-nat-def*)
end

instantiation *int* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*int*) *False*
definition *card-UNIV* = *Phantom*(*int*) *0*
instance by *intro-classes* (*simp-all add*: *card-UNIV-int-def finite-UNIV-int-def infinite-UNIV-int*)
end

instantiation *natural* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*natural*) *False*
definition *card-UNIV* = *Phantom*(*natural*) *0*
instance
by *standard*
(*auto simp add*: *finite-UNIV-natural-def card-UNIV-natural-def card-eq-0-iff*
type-definition.univ [*OF type-definition-natural*] *natural-eq-iff*
dest!: *finite-imageD intro*: *inj-onI*)
end

instantiation *integer* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*integer*) *False*
definition *card-UNIV* = *Phantom*(*integer*) *0*
instance
by *standard*
(*auto simp add*: *finite-UNIV-integer-def card-UNIV-integer-def card-eq-0-iff*
type-definition.univ [*OF type-definition-integer*] *infinite-UNIV-int*
dest!: *finite-imageD intro*: *inj-onI*)
end

instantiation *list* :: (*type*) *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*'a list*) *False*
definition *card-UNIV* = *Phantom*(*'a list*) *0*
instance by *intro-classes* (*simp-all add*: *card-UNIV-list-def finite-UNIV-list-def infinite-UNIV-listI*)
end

instantiation *unit* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*unit*) *True*
definition *card-UNIV* = *Phantom*(*unit*) *1*
instance by *intro-classes* (*simp-all add*: *card-UNIV-unit-def finite-UNIV-unit-def*)
end

instantiation *bool* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*bool*) *True*
definition *card-UNIV* = *Phantom*(*bool*) *2*
instance by(*intro-classes*)(*simp-all add*: *card-UNIV-bool-def finite-UNIV-bool-def*)
end

instantiation *char* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*char*) *True*
definition *card-UNIV* = *Phantom*(*char*) *256*
instance by *intro-classes* (*simp-all add*: *card-UNIV-char-def card-UNIV-char finite-UNIV-char-def*)
end

instantiation *prod* :: (*finite-UNIV*, *finite-UNIV*) *finite-UNIV* **begin**
definition *finite-UNIV* = *Phantom*('*a* × '*b*)
(*of-phantom* (*finite-UNIV* :: '*a finite-UNIV*) ∧ *of-phantom* (*finite-UNIV* :: '*b finite-UNIV*))
instance by *intro-classes* (*simp add*: *finite-UNIV-prod-def finite-UNIV finite-prod*)
end

instantiation *prod* :: (*card-UNIV*, *card-UNIV*) *card-UNIV* **begin**
definition *card-UNIV* = *Phantom*('*a* × '*b*)
(*of-phantom* (*card-UNIV* :: '*a card-UNIV*) ∗ *of-phantom* (*card-UNIV* :: '*b card-UNIV*))
instance by *intro-classes* (*simp add*: *card-UNIV-prod-def card-UNIV*)
end

instantiation *sum* :: (*finite-UNIV*, *finite-UNIV*) *finite-UNIV* **begin**
definition *finite-UNIV* = *Phantom*('*a* + '*b*)
(*of-phantom* (*finite-UNIV* :: '*a finite-UNIV*) ∧ *of-phantom* (*finite-UNIV* :: '*b finite-UNIV*))
instance
by *intro-classes* (*simp add*: *UNIV-Plus-UNIV*[*symmetric*] *finite-UNIV-sum-def finite-UNIV del*: *UNIV-Plus-UNIV*)
end

instantiation *sum* :: (*card-UNIV*, *card-UNIV*) *card-UNIV* **begin**
definition *card-UNIV* = *Phantom*('*a* + '*b*)
(*let ca* = *of-phantom* (*card-UNIV* :: '*a card-UNIV*);
cb = *of-phantom* (*card-UNIV* :: '*b card-UNIV*)
in if ca ≠ *0* ∧ *cb* ≠ *0 then ca* + *cb else 0*)
instance by *intro-classes* (*auto simp add*: *card-UNIV-sum-def card-UNIV card-UNIV-sum*)
end

instantiation *fun* :: (*finite-UNIV*, *card-UNIV*) *finite-UNIV* **begin**
definition *finite-UNIV* = *Phantom*('*a* ⇒ '*b*)
(*let cb* = *of-phantom* (*card-UNIV* :: '*b card-UNIV*)
in cb = *1* ∨ *of-phantom* (*finite-UNIV* :: '*a finite-UNIV*) ∧ *cb* ≠ *0*)
instance
by *intro-classes* (*auto simp add*: *finite-UNIV-fun-def Let-def card-UNIV finite-UNIV finite-UNIV-fun card-gt-0-iff*)
end

instantiation *fun* :: (*card-UNIV*, *card-UNIV*) *card-UNIV* **begin**
definition *card-UNIV* = *Phantom*('*a* ⇒ '*b*)
(*let ca* = *of-phantom* (*card-UNIV* :: '*a card-UNIV*);
cb = *of-phantom* (*card-UNIV* :: '*b card-UNIV*)
in if ca ≠ *0* ∧ *cb* ≠ *0* ∨ *cb* = *1 then cb* ^ *ca else 0*)
instance by *intro-classes* (*simp add*: *card-UNIV-fun-def card-UNIV Let-def card-fun*)
end

instantiation *option* :: (*finite-UNIV*) *finite-UNIV* **begin**
definition *finite-UNIV* = *Phantom*('*a option*) (*of-phantom* (*finite-UNIV* :: '*a finite-UNIV*))
instance by *intro-classes* (*simp add*: *finite-UNIV-option-def finite-UNIV*)
end

instantiation *option* :: (*card-UNIV*) *card-UNIV* **begin**
definition *card-UNIV* = *Phantom*('*a option*)
(*let c* = *of-phantom* (*card-UNIV* :: '*a card-UNIV*) *in if c* ≠ *0 then Suc c else 0*)
instance by *intro-classes* (*simp add*: *card-UNIV-option-def card-UNIV card-UNIV-option*)
end

instantiation *String.literal* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*String.literal*) *False*
definition *card-UNIV* = *Phantom*(*String.literal*) *0*
instance
by *intro-classes* (*simp-all add*: *card-UNIV-literal-def finite-UNIV-literal-def infinite-literal card-literal*)
end

instantiation *set* :: (*finite-UNIV*) *finite-UNIV* **begin**
definition *finite-UNIV* = *Phantom*('*a set*) (*of-phantom* (*finite-UNIV* :: '*a finite-UNIV*))
instance by *intro-classes* (*simp add*: *finite-UNIV-set-def finite-UNIV Finite-Set.finite-set*)
end

instantiation *set* :: (*card-UNIV*) *card-UNIV* **begin**
definition *card-UNIV* = *Phantom*('*a set*)
(*let c* = *of-phantom* (*card-UNIV* :: '*a card-UNIV*) *in if c* = *0 then 0 else 2* ^ *c*)
instance by *intro-classes* (*simp add*: *card-UNIV-set-def card-UNIV-set card-UNIV*)
end

lemma *UNIV-finite-1*: *UNIV* = *set* [*finite-1*.a_1]

by(*auto intro*: *finite-1.exhaust*)

lemma *UNIV-finite-2*: *UNIV* = *set* [*finite-2*.a_1, *finite-2*.a_2]
by(*auto intro*: *finite-2.exhaust*)

lemma *UNIV-finite-3*: *UNIV* = *set* [*finite-3*.a_1, *finite-3*.a_2, *finite-3*.a_3]
by(*auto intro*: *finite-3.exhaust*)

lemma *UNIV-finite-4*: *UNIV* = *set* [*finite-4*.a_1, *finite-4*.a_2, *finite-4*.a_3, *finite-4*.a_4]
by(*auto intro*: *finite-4.exhaust*)

lemma *UNIV-finite-5*:
UNIV = *set* [*finite-5*.a_1, *finite-5*.a_2, *finite-5*.a_3, *finite-5*.a_4, *finite-5*.a_5]
by(*auto intro*: *finite-5.exhaust*)

instantiation *Enum.finite-1* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*Enum.finite-1*) *True*
definition *card-UNIV* = *Phantom*(*Enum.finite-1*) *1*
instance
by *intro-classes* (*simp-all add*: *UNIV-finite-1 card-UNIV-finite-1-def finite-UNIV-finite-1-def*)
end

instantiation *Enum.finite-2* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*Enum.finite-2*) *True*
definition *card-UNIV* = *Phantom*(*Enum.finite-2*) *2*
instance
by *intro-classes* (*simp-all add*: *UNIV-finite-2 card-UNIV-finite-2-def finite-UNIV-finite-2-def*)
end

instantiation *Enum.finite-3* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*Enum.finite-3*) *True*
definition *card-UNIV* = *Phantom*(*Enum.finite-3*) *3*
instance
by *intro-classes* (*simp-all add*: *UNIV-finite-3 card-UNIV-finite-3-def finite-UNIV-finite-3-def*)
end

instantiation *Enum.finite-4* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*Enum.finite-4*) *True*
definition *card-UNIV* = *Phantom*(*Enum.finite-4*) *4*
instance
by *intro-classes* (*simp-all add*: *UNIV-finite-4 card-UNIV-finite-4-def finite-UNIV-finite-4-def*)
end

instantiation *Enum.finite-5* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*Enum.finite-5*) *True*
definition *card-UNIV* = *Phantom*(*Enum.finite-5*) *5*
instance
by *intro-classes* (*simp-all add*: *UNIV-finite-5 card-UNIV-finite-5-def finite-UNIV-finite-5-def*)
end

28.7 Code setup for sets

Implement *CARD('a)* via *card-UNIV-class.card-UNIV* and provide implementations for *finite*, *card*, *op* $\subseteq$, and *op* =if the calling context already provides *finite-UNIV* and *card-UNIV* instances. If we implemented the latter always via *card-UNIV-class.card-UNIV*, we would require instances of essentially all element types, i.e., a lot of instantiation proofs and – at run time – possibly slow dictionary constructions.

context
begin

qualified definition *card-UNIV'* :: *'a card-UNIV*
where [*code del*]: *card-UNIV'* = *Phantom('a) CARD('a)*

lemma *CARD-code* [*code-unfold*]:
CARD('a) = *of-phantom* (*card-UNIV'* :: *'a card-UNIV*)
by(*simp add*: *card-UNIV'-def*)

lemma *card-UNIV'-code* [*code*]:
card-UNIV' = *card-UNIV*
by(*simp add*: *card-UNIV card-UNIV'-def*)

end

lemma *card-Compl*:
finite A $\Longrightarrow$ *card* (− *A*) = *card* (*UNIV* :: *'a set*) − *card* (*A* :: *'a set*)
by (*metis Compl-eq-Diff-UNIV card-Diff-subset top-greatest*)

context fixes *xs* :: *'a* :: *finite-UNIV list*
begin

qualified definition *finite'* :: *'a set* ⇒ *bool*
where [*simp*, *code del*, *code-abbrev*]: *finite'* = *finite*

lemma *finite'-code* [*code*]:
finite' (*set xs*) $\longleftrightarrow$ *True*
finite' (*List.coset xs*) $\longleftrightarrow$ *of-phantom* (*finite-UNIV* :: *'a finite-UNIV*)
by(*simp-all add*: *card-gt-0-iff finite-UNIV*)

end

context fixes *xs* :: *'a* :: *card-UNIV list*
begin

qualified definition *card'* :: *'a set* ⇒ *nat*
where [*simp*, *code del*, *code-abbrev*]: *card'* = *card*

lemma *card'-code* [*code*]:

card′ (set xs) = length (remdups xs)
card′ (List.coset xs) = of-phantom (card-UNIV :: 'a card-UNIV) − length (remdups xs)
by(*simp-all add*: *List.card-set card-Compl card-UNIV*)

qualified definition *subset′ :: 'a set ⇒ 'a set ⇒ bool*
where [*simp*, *code del*, *code-abbrev*]: *subset′ = op ⊆*

lemma *subset′-code* [*code*]:
subset′ A (List.coset ys) ⟷ (∀ y ∈ set ys. y ∉ A)
subset′ (set ys) B ⟷ (∀ y ∈ set ys. y ∈ B)
subset′ (List.coset xs) (set ys) ⟷ (let n = CARD('a) in n > 0 ∧ card(set (xs @ ys)) = n)
by(*auto simp add*: *Let-def card-gt-0-iff dest*: *card-eq-UNIV-imp-eq-UNIV intro*: *arg-cong*[**where** *f=card*])
(*metis finite-compl finite-set rev-finite-subset*)

qualified definition *eq-set :: 'a set ⇒ 'a set ⇒ bool*
where [*simp*, *code del*, *code-abbrev*]: *eq-set = op =*

lemma *eq-set-code* [*code*]:
fixes *ys*
defines *rhs ≡*
let n = CARD('a)
in if n = 0 then False else
let xs′ = remdups xs; *ys′ = remdups ys*
in length xs′ + length ys′ = n ∧ (∀ x ∈ set xs′. x ∉ set ys′) ∧ (∀ y ∈ set ys′. y ∉ set xs′)
shows *eq-set (List.coset xs) (set ys) ⟷ rhs*
and *eq-set (set ys) (List.coset xs) ⟷ rhs*
and *eq-set (set xs) (set ys) ⟷ (∀ x ∈ set xs. x ∈ set ys) ∧ (∀ y ∈ set ys. y ∈ set xs)*
and *eq-set (List.coset xs) (List.coset ys) ⟷ (∀ x ∈ set xs. x ∈ set ys) ∧ (∀ y ∈ set ys. y ∈ set xs)*
proof *goal-cases*
{
case *1*
show *?case* (**is** *?lhs ⟷ ?rhs*)
proof
show *?rhs* **if** *?lhs*
using *that*
by (*auto simp add*: *rhs-def Let-def List.card-set*[*symmetric*]
card-Un-Int[**where** *A=set xs* **and** *B=− set xs*] *card-UNIV*
Compl-partition card-gt-0-iff dest: *sym*)(*metis finite-compl finite-set*)
show *?lhs* **if** *?rhs*
proof −
have ⟦ *∀ y∈set xs. y ∉ set ys*; *∀ x∈set ys. x ∉ set xs* ⟧ *⟹ set xs ∩ set ys = {}* **by** *blast*

```
      with that show ?thesis
       by (auto simp add: rhs-def Let-def List.card-set[symmetric]
         card-UNIV card-gt-0-iff card-Un-Int[where A=set xs and B=set ys]
         dest: card-eq-UNIV-imp-eq-UNIV split: if-split-asm)
    qed
  qed
 }
 moreover
 case 2
 ultimately show ?case unfolding eq-set-def by blast
next
 case 3
 show ?case unfolding eq-set-def List.coset-def by blast
next
 case 4
 show ?case unfolding eq-set-def List.coset-def by blast
qed

end
```

Provide more informative exceptions than Match for non-rewritten cases. If generated code raises one these exceptions, then a code equation calls the mentioned operator for an element type that is not an instance of *card-UNIV* and is therefore not implemented via *card-UNIV-class.card-UNIV*. Constrain the element type with sort *card-UNIV* to change this.

```
lemma card-coset-error [code]:
  card (List.coset xs) =
   Code.abort (STR ''card (List.coset -) requires type class instance card-UNIV'')
    (λ-. card (List.coset xs))
by(simp)

lemma coset-subseteq-set-code [code]:
  List.coset xs ⊆ set ys ⟷
  (if xs = [] ∧ ys = [] then False
   else Code.abort
   (STR ''subset-eq (List.coset -) (List.set -) requires type class instance card-UNIV'')
    (λ-. List.coset xs ⊆ set ys))
by simp

notepad begin — test code setup
have List.coset [True] = set [False] ∧
     List.coset [] ⊆ List.set [True, False] ∧
     finite (List.coset [True])
 by eval
end

end
```

29 Numeral Syntax for Types

theory *Numeral-Type*
imports *Cardinality*
begin

29.1 Numeral Types

typedef *num0* = *UNIV* :: *nat set* **..**
typedef *num1* = *UNIV* :: *unit set* **..**

typedef *'a bit0* = {*0* ..< *2* * *int CARD*('a::*finite*)}
proof
  show *0* ∈ {*0* ..< *2* * *int CARD*('a)}
    by *simp*
qed

typedef *'a bit1* = {*0* ..< *1* + *2* * *int CARD*('a::*finite*)}
proof
  show *0* ∈ {*0* ..< *1* + *2* * *int CARD*('a)}
    by *simp*
qed

lemma *card-num0* [*simp*]: *CARD* (*num0*) = *0*
  unfolding *type-definition.card* [*OF type-definition-num0*]
  by *simp*

lemma *infinite-num0*: ¬ *finite* (*UNIV* :: *num0 set*)
  using *card-num0*[*unfolded card-eq-0-iff*]
  by *simp*

lemma *card-num1* [*simp*]: *CARD*(*num1*) = *1*
  unfolding *type-definition.card* [*OF type-definition-num1*]
  by (*simp only*: *card-UNIV-unit*)

lemma *card-bit0* [*simp*]: *CARD*('a *bit0*) = *2* * *CARD*('a::*finite*)
  unfolding *type-definition.card* [*OF type-definition-bit0*]
  by *simp*

lemma *card-bit1* [*simp*]: *CARD*('a *bit1*) = *Suc* (*2* * *CARD*('a::*finite*))
  unfolding *type-definition.card* [*OF type-definition-bit1*]
  by *simp*

instance *num1* :: *finite*
proof
  show *finite* (*UNIV*::*num1 set*)
    unfolding *type-definition.univ* [*OF type-definition-num1*]
    using *finite* **by** (*rule finite-imageI*)
qed

instance *bit0* :: (*finite*) *card2*
proof
  show *finite* (*UNIV*::'*a bit0 set*)
    unfolding *type-definition.univ* [*OF type-definition-bit0*]
    by *simp*
  show *2* ≤ *CARD*('*a bit0*)
    by *simp*
qed

instance *bit1* :: (*finite*) *card2*
proof
  show *finite* (*UNIV*::'*a bit1 set*)
    unfolding *type-definition.univ* [*OF type-definition-bit1*]
    by *simp*
  show *2* ≤ *CARD*('*a bit1*)
    by *simp*
qed

29.2 Locales for for modular arithmetic subtypes

locale *mod-type* =
  fixes *n* :: *int*
  and *Rep* :: '*a*::{*zero*,*one*,*plus*,*times*,*uminus*,*minus*} ⇒ *int*
  and *Abs* :: *int* ⇒ '*a*::{*zero*,*one*,*plus*,*times*,*uminus*,*minus*}
  assumes *type*: *type-definition Rep Abs* {*0*..<*n*}
  and *size1*: *1* < *n*
  and *zero-def*: *0* = *Abs 0*
  and *one-def*: *1* = *Abs 1*
  and *add-def*: *x* + *y* = *Abs* ((*Rep x* + *Rep y*) *mod n*)
  and *mult-def*: *x* * *y* = *Abs* ((*Rep x* * *Rep y*) *mod n*)
  and *diff-def*: *x* − *y* = *Abs* ((*Rep x* − *Rep y*) *mod n*)
  and *minus-def*: − *x* = *Abs* ((− *Rep x*) *mod n*)
begin

lemma *size0*: *0* < *n*
using *size1* **by** *simp*

lemmas *definitions* =
  zero-def one-def add-def mult-def minus-def diff-def

lemma *Rep-less-n*: *Rep x* < *n*
by (*rule type-definition.Rep* [*OF type*, *simplified*, *THEN conjunct2*])

lemma *Rep-le-n*: *Rep x* ≤ *n*
by (*rule Rep-less-n* [*THEN order-less-imp-le*])

lemma *Rep-inject-sym*: *x* = *y* ⟷ *Rep x* = *Rep y*
by (*rule type-definition.Rep-inject* [*OF type*, *symmetric*])

lemma *Rep-inverse*: *Abs* (*Rep x*) = *x*
by (*rule type-definition.Rep-inverse* [*OF type*])

lemma *Abs-inverse*: *m* ∈ {*0*..<*n*} ⟹ *Rep* (*Abs m*) = *m*
by (*rule type-definition.Abs-inverse* [*OF type*])

lemma *Rep-Abs-mod*: *Rep* (*Abs* (*m mod n*)) = *m mod n*
by (*simp add*: *Abs-inverse pos-mod-conj* [*OF size0*])

lemma *Rep-Abs-0*: *Rep* (*Abs 0*) = *0*
by (*simp add*: *Abs-inverse size0*)

lemma *Rep-0*: *Rep 0* = *0*
by (*simp add*: *zero-def Rep-Abs-0*)

lemma *Rep-Abs-1*: *Rep* (*Abs 1*) = *1*
by (*simp add*: *Abs-inverse size1*)

lemma *Rep-1*: *Rep 1* = *1*
by (*simp add*: *one-def Rep-Abs-1*)

lemma *Rep-mod*: *Rep x mod n* = *Rep x*
apply (*rule-tac x*=*x* **in** *type-definition.Abs-cases* [*OF type*])
apply (*simp add*: *type-definition.Abs-inverse* [*OF type*])
apply (*simp add*: *mod-pos-pos-trivial*)
done

lemmas *Rep-simps* =
Rep-inject-sym Rep-inverse Rep-Abs-mod Rep-mod Rep-Abs-0 Rep-Abs-1

lemma *comm-ring-1*: *OFCLASS*('*a*, *comm-ring-1-class*)
apply (*intro-classes*, *unfold definitions*)
apply (*simp-all add*: *Rep-simps zmod-simps field-simps*)
done

end

locale *mod-ring* = *mod-type n Rep Abs*
for *n* :: *int*
and *Rep* :: '*a*::{*comm-ring-1*} ⇒ *int*
and *Abs* :: *int* ⇒ '*a*::{*comm-ring-1*}
begin

lemma *of-nat-eq*: *of-nat k* = *Abs* (*int k mod n*)
apply (*induct k*)
apply (*simp add*: *zero-def*)
apply (*simp add*: *Rep-simps add-def one-def zmod-simps ac-simps*)
done

lemma *of-int-eq*: *of-int z = Abs (z mod n)*
apply (*cases z rule*: *int-diff-cases*)
apply (*simp add*: *Rep-simps of-nat-eq diff-def zmod-simps*)
done

lemma *Rep-numeral*:
Rep (numeral w) = numeral w mod n
using *of-int-eq* [*of numeral w*]
by (*simp add*: *Rep-inject-sym Rep-Abs-mod*)

lemma *iszero-numeral*:
iszero (numeral w::'a) ⟷ numeral w mod n = 0
by (*simp add*: *Rep-inject-sym Rep-numeral Rep-0 iszero-def*)

lemma *cases*:
assumes *1*: $\bigwedge z.\ [\![(x::'a) = \textit{of-int } z;\ 0 \leq z;\ z < n]\!] \Longrightarrow P$
shows *P*
apply (*cases x rule*: *type-definition.Abs-cases* [*OF type*])
apply (*rule-tac z=y* **in** *1*)
apply (*simp-all add*: *of-int-eq mod-pos-pos-trivial*)
done

lemma *induct*:
$(\bigwedge z.\ [\![0 \leq z;\ z < n]\!] \Longrightarrow P\ (\textit{of-int } z)) \Longrightarrow P\ (x::'a)$
by (*cases x rule*: *cases*) *simp*

end

29.3 Ring class instances

Unfortunately *ring-1* instance is not possible for *num1*, since 0 and 1 are not distinct.

instantiation *num1* :: {*comm-ring*,*comm-monoid-mult*,*numeral*}
begin

lemma *num1-eq-iff*: *(x::num1) = (y::num1) ⟷ True*
by (*induct x*, *induct y*) *simp*

instance
by *standard* (*simp-all add*: *num1-eq-iff*)

end

instantiation
bit0 **and** *bit1* :: (*finite*) {*zero*,*one*,*plus*,*times*,*uminus*,*minus*}
begin

definition *Abs-bit0'* :: *int ⇒ 'a bit0* **where**
Abs-bit0' x = Abs-bit0 (x mod int CARD('a bit0))

definition *Abs-bit1'* :: *int* ⇒ *'a bit1* **where**
Abs-bit1' x = *Abs-bit1* (*x mod int CARD*(*'a bit1*))

definition *0* = *Abs-bit0 0*
definition *1* = *Abs-bit0 1*
definition *x* + *y* = *Abs-bit0'* (*Rep-bit0 x* + *Rep-bit0 y*)
definition *x* ∗ *y* = *Abs-bit0'* (*Rep-bit0 x* ∗ *Rep-bit0 y*)
definition *x* − *y* = *Abs-bit0'* (*Rep-bit0 x* − *Rep-bit0 y*)
definition − *x* = *Abs-bit0'* (− *Rep-bit0 x*)

definition *0* = *Abs-bit1 0*
definition *1* = *Abs-bit1 1*
definition *x* + *y* = *Abs-bit1'* (*Rep-bit1 x* + *Rep-bit1 y*)
definition *x* ∗ *y* = *Abs-bit1'* (*Rep-bit1 x* ∗ *Rep-bit1 y*)
definition *x* − *y* = *Abs-bit1'* (*Rep-bit1 x* − *Rep-bit1 y*)
definition − *x* = *Abs-bit1'* (− *Rep-bit1 x*)

instance ..

end

interpretation *bit0*:
mod-type int CARD(*'a*::*finite bit0*)
Rep-bit0 :: *'a*::*finite bit0* ⇒ *int*
Abs-bit0 :: *int* ⇒ *'a*::*finite bit0*
apply (*rule mod-type.intro*)
apply (*simp add*: *of-nat-mult type-definition-bit0*)
apply (*rule one-less-int-card*)
apply (*rule zero-bit0-def*)
apply (*rule one-bit0-def*)
apply (*rule plus-bit0-def* [*unfolded Abs-bit0'-def*])
apply (*rule times-bit0-def* [*unfolded Abs-bit0'-def*])
apply (*rule minus-bit0-def* [*unfolded Abs-bit0'-def*])
apply (*rule uminus-bit0-def* [*unfolded Abs-bit0'-def*])
done

interpretation *bit1*:
mod-type int CARD(*'a*::*finite bit1*)
Rep-bit1 :: *'a*::*finite bit1* ⇒ *int*
Abs-bit1 :: *int* ⇒ *'a*::*finite bit1*
apply (*rule mod-type.intro*)
apply (*simp add*: *of-nat-mult type-definition-bit1*)
apply (*rule one-less-int-card*)
apply (*rule zero-bit1-def*)
apply (*rule one-bit1-def*)
apply (*rule plus-bit1-def* [*unfolded Abs-bit1'-def*])
apply (*rule times-bit1-def* [*unfolded Abs-bit1'-def*])
apply (*rule minus-bit1-def* [*unfolded Abs-bit1'-def*])

apply (*rule uminus-bit1-def* [*unfolded Abs-bit1'-def*])
done

instance *bit0* :: (*finite*) *comm-ring-1*
by (*rule bit0.comm-ring-1*)

instance *bit1* :: (*finite*) *comm-ring-1*
by (*rule bit1.comm-ring-1*)

interpretation *bit0*:
mod-ring int CARD('a::finite bit0)
Rep-bit0 :: *'a::finite bit0* ⇒ *int*
Abs-bit0 :: *int* ⇒ *'a::finite bit0*
..

interpretation *bit1*:
mod-ring int CARD('a::finite bit1)
Rep-bit1 :: *'a::finite bit1* ⇒ *int*
Abs-bit1 :: *int* ⇒ *'a::finite bit1*
..

Set up cases, induction, and arithmetic

lemmas *bit0-cases* [*case-names of-int*, *cases type*: *bit0*] = *bit0.cases*
lemmas *bit1-cases* [*case-names of-int*, *cases type*: *bit1*] = *bit1.cases*

lemmas *bit0-induct* [*case-names of-int*, *induct type*: *bit0*] = *bit0.induct*
lemmas *bit1-induct* [*case-names of-int*, *induct type*: *bit1*] = *bit1.induct*

lemmas *bit0-iszero-numeral* [*simp*] = *bit0.iszero-numeral*
lemmas *bit1-iszero-numeral* [*simp*] = *bit1.iszero-numeral*

lemmas [*simp*] = *eq-numeral-iff-iszero* [**where** *'a='a bit0*] **for** *dummy* :: *'a::finite*
lemmas [*simp*] = *eq-numeral-iff-iszero* [**where** *'a='a bit1*] **for** *dummy* :: *'a::finite*

29.4 Order instances

instantiation *bit0* **and** *bit1* :: (*finite*) *linorder* **begin**
definition $a < b \longleftrightarrow$ *Rep-bit0 a* $<$ *Rep-bit0 b*
definition $a \leq b \longleftrightarrow$ *Rep-bit0 a* $\leq$ *Rep-bit0 b*
definition $a < b \longleftrightarrow$ *Rep-bit1 a* $<$ *Rep-bit1 b*
definition $a \leq b \longleftrightarrow$ *Rep-bit1 a* $\leq$ *Rep-bit1 b*

instance
by(*intro-classes*)
(*auto simp add*: *less-eq-bit0-def less-bit0-def less-eq-bit1-def less-bit1-def Rep-bit0-inject Rep-bit1-inject*)
end

lemma (**in** *preorder*) *tranclp-less*: *op* $<^{++}$ = *op* $<$

by(*auto simp add*: *fun-eq-iff intro*: *less-trans elim*: *tranclp.induct*)

instance *bit0* **and** *bit1* :: (*finite*) *wellorder*
proof −
  have *wf* {(*x* :: *'a bit0*, *y*). *x* < *y*}
    by(*auto simp add*: *trancl-def tranclp-less intro*!: *finite-acyclic-wf acyclicI*)
  thus *OFCLASS*(*'a bit0*, *wellorder-class*)
    by(*rule wf-wellorderI*) *intro-classes*
next
  have *wf* {(*x* :: *'a bit1*, *y*). *x* < *y*}
    by(*auto simp add*: *trancl-def tranclp-less intro*!: *finite-acyclic-wf acyclicI*)
  thus *OFCLASS*(*'a bit1*, *wellorder-class*)
    by(*rule wf-wellorderI*) *intro-classes*
qed

29.5 Code setup and type classes for code generation

Code setup for *num0* and *num1*

definition *Num0* :: *num0* **where** *Num0* = *Abs-num0 0*
code-datatype *Num0*

instantiation *num0* :: *equal* **begin**
definition *equal-num0* :: *num0* ⇒ *num0* ⇒ *bool*
  where *equal-num0* = *op* =
instance by *intro-classes* (*simp add*: *equal-num0-def*)
end

lemma *equal-num0-code* [*code*]:
  equal-class.equal Num0 Num0 = *True*
by(*rule equal-refl*)

code-datatype *1* :: *num1*

instantiation *num1* :: *equal* **begin**
definition *equal-num1* :: *num1* ⇒ *num1* ⇒ *bool*
  where *equal-num1* = *op* =
instance by *intro-classes* (*simp add*: *equal-num1-def*)
end

lemma *equal-num1-code* [*code*]:
  equal-class.equal (*1* :: *num1*) *1* = *True*
by(*rule equal-refl*)

instantiation *num1* :: *enum* **begin**
definition *enum-class.enum* = [*1* :: *num1*]
definition *enum-class.enum-all P* = *P* (*1* :: *num1*)
definition *enum-class.enum-ex P* = *P* (*1* :: *num1*)
instance
  by *intro-classes*

(*auto simp add*: *enum-num1-def enum-all-num1-def enum-ex-num1-def num1-eq-iff Ball-def*,
(*metis* (*full-types*) *num1-eq-iff*)+)
end

instantiation *num0* **and** *num1* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*num0*) *False*
definition *card-UNIV* = *Phantom*(*num0*) *0*
definition *finite-UNIV* = *Phantom*(*num1*) *True*
definition *card-UNIV* = *Phantom*(*num1*) *1*
instance
by *intro-classes*
(*simp-all add*: *finite-UNIV-num0-def card-UNIV-num0-def infinite-num0 finite-UNIV-num1-def card-UNIV-num1-def*)
end

Code setup for *'a bit0* and *'a bit1*

declare
bit0.Rep-inverse[*code abstype*]
bit0.Rep-0[*code abstract*]
bit0.Rep-1[*code abstract*]

lemma *Abs-bit0'-code* [*code abstract*]:
Rep-bit0 (*Abs-bit0'* *x* :: *'a* :: *finite bit0*) = *x mod int* (*CARD*(*'a bit0*))
by(*auto simp add*: *Abs-bit0'-def intro*!: *Abs-bit0-inverse*)

lemma *inj-on-Abs-bit0*:
inj-on (*Abs-bit0* :: *int* ⇒ *'a bit0*) {*0..<2* ∗ *int CARD*(*'a* :: *finite*)}
by(*auto intro*: *inj-onI simp add*: *Abs-bit0-inject*)

declare
bit1.Rep-inverse[*code abstype*]
bit1.Rep-0[*code abstract*]
bit1.Rep-1[*code abstract*]

lemma *Abs-bit1'-code* [*code abstract*]:
Rep-bit1 (*Abs-bit1'* *x* :: *'a* :: *finite bit1*) = *x mod int* (*CARD*(*'a bit1*))
by(*auto simp add*: *Abs-bit1'-def intro*!: *Abs-bit1-inverse*)

lemma *inj-on-Abs-bit1*:
inj-on (*Abs-bit1* :: *int* ⇒ *'a bit1*) {*0..<1* + *2* ∗ *int CARD*(*'a* :: *finite*)}
by(*auto intro*: *inj-onI simp add*: *Abs-bit1-inject*)

instantiation *bit0* **and** *bit1* :: (*finite*) *equal* **begin**

definition *equal-class.equal x y* ⟷ *Rep-bit0 x* = *Rep-bit0 y*
definition *equal-class.equal x y* ⟷ *Rep-bit1 x* = *Rep-bit1 y*

instance

by *intro-classes* (*simp-all add*: *equal-bit0-def equal-bit1-def Rep-bit0-inject Rep-bit1-inject*)

end

instantiation *bit0* :: (*finite*) *enum* **begin**
definition (*enum-class.enum* :: *'a bit0 list*) = *map* (*Abs-bit0'* ∘ *int*) (*upt 0* (*CARD*('*a bit0*')))
definition *enum-class.enum-all P* = (∀ *b* :: *'a bit0* ∈ *set enum-class.enum*. *P b*)
definition *enum-class.enum-ex P* = (∃ *b* :: *'a bit0* ∈ *set enum-class.enum*. *P b*)

instance
proof(*intro-classes*)
show *distinct* (*enum-class.enum* :: *'a bit0 list*)
by (*simp add*: *enum-bit0-def distinct-map inj-on-def Abs-bit0'-def Abs-bit0-inject mod-pos-pos-trivial*)

show *univ-eq*: (*UNIV* :: *'a bit0 set*) = *set enum-class.enum*
unfolding *enum-bit0-def type-definition.Abs-image*[*OF type-definition-bit0*, *symmetric*]
by(*simp add*: *image-comp* [*symmetric*] *inj-on-Abs-bit0 card-image image-int-atLeastLessThan*)
(*auto intro*!: *image-cong*[*OF refl*] *simp add*: *Abs-bit0'-def mod-pos-pos-trivial*)

fix *P* :: *'a bit0* ⇒ *bool*
show *enum-class.enum-all P* = *Ball UNIV P*
and *enum-class.enum-ex P* = *Bex UNIV P*
by(*simp-all add*: *enum-all-bit0-def enum-ex-bit0-def univ-eq*)
qed

end

instantiation *bit1* :: (*finite*) *enum* **begin**
definition (*enum-class.enum* :: *'a bit1 list*) = *map* (*Abs-bit1'* ∘ *int*) (*upt 0* (*CARD*('*a bit1*')))
definition *enum-class.enum-all P* = (∀ *b* :: *'a bit1* ∈ *set enum-class.enum*. *P b*)
definition *enum-class.enum-ex P* = (∃ *b* :: *'a bit1* ∈ *set enum-class.enum*. *P b*)

instance
proof(*intro-classes*)
show *distinct* (*enum-class.enum* :: *'a bit1 list*)
by(*simp only*: *Abs-bit1'-def zmod-int*[*symmetric*] *enum-bit1-def distinct-map Suc-eq-plus1 card-bit1 o-apply inj-on-def*)
(*clarsimp simp add*: *Abs-bit1-inject*)

show *univ-eq*: (*UNIV* :: *'a bit1 set*) = *set enum-class.enum*
unfolding *enum-bit1-def type-definition.Abs-image*[*OF type-definition-bit1*, *symmetric*]
by(*simp add*: *image-comp* [*symmetric*] *inj-on-Abs-bit1 card-image image-int-atLeastLessThan*)
(*auto intro*!: *image-cong*[*OF refl*] *simp add*: *Abs-bit1'-def mod-pos-pos-trivial*)

```
  fix P :: 'a bit1 ⇒ bool
  show enum-class.enum-all P = Ball UNIV P
    and enum-class.enum-ex P = Bex UNIV P
    by(simp-all add: enum-all-bit1-def enum-ex-bit1-def univ-eq)
qed

end

instantiation bit0 and bit1 :: (finite) finite-UNIV begin
definition finite-UNIV = Phantom('a bit0) True
definition finite-UNIV = Phantom('a bit1) True
instance by intro-classes (simp-all add: finite-UNIV-bit0-def finite-UNIV-bit1-def)
end

instantiation bit0 and bit1 :: ({finite,card-UNIV}) card-UNIV begin
definition card-UNIV = Phantom('a bit0) (2 * of-phantom (card-UNIV :: 'a
card-UNIV))
definition card-UNIV = Phantom('a bit1) (1 + 2 * of-phantom (card-UNIV ::
'a card-UNIV))
instance by intro-classes (simp-all add: card-UNIV-bit0-def card-UNIV-bit1-def
card-UNIV)
end
```

29.6 Syntax

```
syntax
  -NumeralType :: num-token => type  (-)
  -NumeralType0 :: type  (0)
  -NumeralType1 :: type  (1)

translations
  (type) 1 == (type) num1
  (type) 0 == (type) num0

parse-translation ‹
  let
    fun mk-bintype n =
      let
        fun mk-bit 0 = Syntax.const @{type-syntax bit0}
          | mk-bit 1 = Syntax.const @{type-syntax bit1};
        fun bin-of n =
          if n = 1 then Syntax.const @{type-syntax num1}
          else if n = 0 then Syntax.const @{type-syntax num0}
          else if n = ~1 then raise TERM (negative type numeral, [])
          else
            let val (q, r) = Integer.div-mod n 2;
            in mk-bit r $ bin-of q end;
      in bin-of n end;
```

```
    fun numeral-tr [Free (str, -)] = mk-bintype (the (Int.fromString str))
      | numeral-tr ts = raise TERM (numeral-tr, ts);

  in [(@{syntax-const -NumeralType}, K numeral-tr)] end;
⟩

print-translation ⟨
  let
    fun int-of [] = 0
      | int-of (b :: bs) = b + 2 * int-of bs;

    fun bin-of (Const (@{type-syntax num0}, -)) = []
      | bin-of (Const (@{type-syntax num1}, -)) = [1]
      | bin-of (Const (@{type-syntax bit0}, -) $ bs) = 0 :: bin-of bs
      | bin-of (Const (@{type-syntax bit1}, -) $ bs) = 1 :: bin-of bs
      | bin-of t = raise TERM (bin-of, [t]);

    fun bit-tr' b [t] =
          let
            val rev-digs = b :: bin-of t handle TERM - => raise Match
            val i = int-of rev-digs;
            val num = string-of-int (abs i);
          in
            Syntax.const @{syntax-const -NumeralType} $ Syntax.free num
          end
      | bit-tr' b - = raise Match;
  in
   [(@{type-syntax bit0}, K (bit-tr' 0)),
    (@{type-syntax bit1}, K (bit-tr' 1))]
  end;
⟩
```

29.7 Examples

lemma *CARD(0) = 0* **by** *simp*
lemma *CARD(17) = 17* **by** *simp*
lemma *8 * 11 ^ 3 - 6 = (2::5)* **by** *simp*

end

30 Definition of finite Cartesian product types.

theory *Finite-Cartesian-Product*
imports
  Euclidean-Space
  L2-Norm
  ~~/src/HOL/Library/Numeral-Type
begin

30.1 Finite Cartesian products, with indexing and lambdas.

typedef $('a, 'b)\ vec = UNIV :: (('b::finite) \Rightarrow 'a)\ set$
  morphisms *vec-nth vec-lambda* **..**

notation
  vec-nth (**infixl** $ *90*) **and**
  vec-lambda (**binder** χ *10*)

syntax *-finite-vec* :: *type* ⇒ *type* ⇒ *type* ((- ^/ -) [*15*, *16*] *15*)

parse-translation ‹
  let
    fun vec t u = Syntax.const @{*type-syntax vec*} $ *t* $ *u*;
    fun finite-vec-tr [*t*, *u*] =
      (*case Term-Position.strip-positions u of*
        v as Free (*x*, -) =>
          if Lexicon.is-tid x then
            vec t (*Syntax.const* @{*syntax-const -ofsort*} $ *v* $
              Syntax.const @{*class-syntax finite*})
          else vec t u
      | - => *vec t u*)
  in
    [(@{*syntax-const -finite-vec*}, *K finite-vec-tr*)]
  end
›

lemma *vec-eq-iff*: $(x = y) \longleftrightarrow (\forall i.\ x\$i = y\$i)$
  by (*simp add*: *vec-nth-inject* [*symmetric*] *fun-eq-iff*)

lemma *vec-lambda-beta* [*simp*]: *vec-lambda g* $ *i* = *g i*
  by (*simp add*: *vec-lambda-inverse*)

lemma *vec-lambda-unique*: $(\forall i.\ f\$i = g\ i) \longleftrightarrow vec\text{-}lambda\ g = f$
  by (*auto simp add*: *vec-eq-iff*)

lemma *vec-lambda-eta*: $(\chi\ i.\ (g\$i)) = g$
  by (*simp add*: *vec-eq-iff*)

30.2 Group operations and class instances

instantiation *vec* :: (*zero*, *finite*) *zero*
begin
  definition $0 \equiv (\chi\ i.\ 0)$
  instance **..**
end

instantiation *vec* :: (*plus*, *finite*) *plus*

begin
definition $op + \equiv (\lambda\ x\ y.\ (\chi\ i.\ x\$i + y\$i))$
instance ..
end

instantiation *vec* :: (*minus*, *finite*) *minus*
begin
definition $op - \equiv (\lambda\ x\ y.\ (\chi\ i.\ x\$i - y\$i))$
instance ..
end

instantiation *vec* :: (*uminus*, *finite*) *uminus*
begin
definition $uminus \equiv (\lambda\ x.\ (\chi\ i.\ -\ (x\$i)))$
instance ..
end

lemma *zero-index* [*simp*]: $0\ \$\ i = 0$
unfolding *zero-vec-def* **by** *simp*

lemma *vector-add-component* [*simp*]: $(x + y)\$i = x\$i + y\$i$
unfolding *plus-vec-def* **by** *simp*

lemma *vector-minus-component* [*simp*]: $(x - y)\$i = x\$i - y\$i$
unfolding *minus-vec-def* **by** *simp*

lemma *vector-uminus-component* [*simp*]: $(-\ x)\$i = -\ (x\$i)$
unfolding *uminus-vec-def* **by** *simp*

instance *vec* :: (*semigroup-add*, *finite*) *semigroup-add*
by *standard* (*simp add*: *vec-eq-iff add.assoc*)

instance *vec* :: (*ab-semigroup-add*, *finite*) *ab-semigroup-add*
by *standard* (*simp add*: *vec-eq-iff add.commute*)

instance *vec* :: (*monoid-add*, *finite*) *monoid-add*
by *standard* (*simp-all add*: *vec-eq-iff*)

instance *vec* :: (*comm-monoid-add*, *finite*) *comm-monoid-add*
by *standard* (*simp add*: *vec-eq-iff*)

instance *vec* :: (*cancel-semigroup-add*, *finite*) *cancel-semigroup-add*
by *standard* (*simp-all add*: *vec-eq-iff*)

instance *vec* :: (*cancel-ab-semigroup-add*, *finite*) *cancel-ab-semigroup-add*
by *standard* (*simp-all add*: *vec-eq-iff diff-diff-eq*)

instance *vec* :: (*cancel-comm-monoid-add*, *finite*) *cancel-comm-monoid-add* **..**

instance *vec* :: (*group-add*, *finite*) *group-add*
by *standard* (*simp-all add*: *vec-eq-iff*)

instance *vec* :: (*ab-group-add*, *finite*) *ab-group-add*
by *standard* (*simp-all add*: *vec-eq-iff*)

30.3 Real vector space

instantiation *vec* :: (*real-vector*, *finite*) *real-vector*
begin

definition *scaleR* ≡ (λ *r x*. (χ *i*. *scaleR r* (*x*$*i*)))

lemma *vector-scaleR-component* [*simp*]: (*scaleR r x*)$*i* = *scaleR r* (*x*$*i*)
unfolding *scaleR-vec-def* **by** *simp*

instance
by *standard* (*simp-all add*: *vec-eq-iff scaleR-left-distrib scaleR-right-distrib*)

end

30.4 Topological space

instantiation *vec* :: (*topological-space*, *finite*) *topological-space*
begin

definition [*code del*]:
open (*S* :: ('*a* ^ '*b*) *set*) ⟷
(∀ *x*∈*S*. ∃ *A*. (∀ *i*. *open* (*A i*) ∧ *x*$*i* ∈ *A i*) ∧
(∀ *y*. (∀ *i*. *y*$*i* ∈ *A i*) ⟶ *y* ∈ *S*))

instance proof
show *open* (*UNIV* :: ('*a* ^ '*b*) *set*)
unfolding *open-vec-def* **by** *auto*
next
fix *S T* :: ('*a* ^ '*b*) *set*
assume *open S open T* **thus** *open* (*S* ∩ *T*)
unfolding *open-vec-def*
apply *clarify*
apply (*drule* (*1*) *bspec*)+
apply (*clarify*, *rename-tac Sa Ta*)
apply (*rule-tac x*=λ*i*. *Sa i* ∩ *Ta i* **in** *exI*)
apply (*simp add*: *open-Int*)
done
next
fix *K* :: ('*a* ^ '*b*) *set set*
assume ∀ *S*∈*K*. *open S* **thus** *open* (⋃ *K*)
unfolding *open-vec-def*
apply *clarify*
apply (*drule* (*1*) *bspec*)

apply (*drule* (*1*) *bspec*)
apply *clarify*
apply (*rule-tac x=A* **in** *exI*)
apply *fast*
done
qed

end

lemma *open-vector-box*: $\forall i.\ open\ (S\ i) \Longrightarrow open\ \{x.\ \forall i.\ x\ \$\ i \in S\ i\}$
unfolding *open-vec-def* **by** *auto*

lemma *open-vimage-vec-nth*: $open\ S \Longrightarrow open\ ((\lambda x.\ x\ \$\ i) -` S)$
unfolding *open-vec-def*
apply *clarify*
apply (*rule-tac x=*$\lambda k.\ if\ k = i\ then\ S\ else\ UNIV$ **in** *exI*, *simp*)
done

lemma *closed-vimage-vec-nth*: $closed\ S \Longrightarrow closed\ ((\lambda x.\ x\ \$\ i) -` S)$
unfolding *closed-open vimage-Compl* [*symmetric*]
by (*rule open-vimage-vec-nth*)

lemma *closed-vector-box*: $\forall i.\ closed\ (S\ i) \Longrightarrow closed\ \{x.\ \forall i.\ x\ \$\ i \in S\ i\}$
proof −
have $\{x.\ \forall i.\ x\ \$\ i \in S\ i\} = (\bigcap i.\ (\lambda x.\ x\ \$\ i) -` S\ i)$ **by** *auto*
thus $\forall i.\ closed\ (S\ i) \Longrightarrow closed\ \{x.\ \forall i.\ x\ \$\ i \in S\ i\}$
by (*simp add*: *closed-INT closed-vimage-vec-nth*)
qed

lemma *tendsto-vec-nth* [*tendsto-intros*]:
assumes $((\lambda x.\ f\ x) \longrightarrow a)\ net$
shows $((\lambda x.\ f\ x\ \$\ i) \longrightarrow a\ \$\ i)\ net$
proof (*rule topological-tendstoI*)
fix *S* **assume** *open S a* $\$\ i \in S$
then have $open\ ((\lambda y.\ y\ \$\ i) -` S)\ a \in ((\lambda y.\ y\ \$\ i) -` S)$
by (*simp-all add*: *open-vimage-vec-nth*)
with *assms* **have** *eventually* $(\lambda x.\ f\ x \in (\lambda y.\ y\ \$\ i) -` S)\ net$
by (*rule topological-tendstoD*)
then show *eventually* $(\lambda x.\ f\ x\ \$\ i \in S)\ net$
by *simp*
qed

lemma *isCont-vec-nth* [*simp*]: $isCont\ f\ a \Longrightarrow isCont\ (\lambda x.\ f\ x\ \$\ i)\ a$
unfolding *isCont-def* **by** (*rule tendsto-vec-nth*)

lemma *vec-tendstoI*:
assumes $\bigwedge i.\ ((\lambda x.\ f\ x\ \$\ i) \longrightarrow a\ \$\ i)\ net$
shows $((\lambda x.\ f\ x) \longrightarrow a)\ net$
proof (*rule topological-tendstoI*)

fix S **assume** *open* S **and** $a \in S$
then obtain A **where** A: $\bigwedge i.$ *open* $(A\ i)$ $\bigwedge i.\ a\ \$\ i \in A\ i$
and S: $\bigwedge y.\ \forall i.\ y\ \$\ i \in A\ i \Longrightarrow y \in S$
unfolding *open-vec-def* **by** *metis*
have $\bigwedge i.$ *eventually* $(\lambda x.\ f\ x\ \$\ i \in A\ i)$ *net*
using *assms* A **by** (*rule topological-tendstoD*)
hence *eventually* $(\lambda x.\ \forall i.\ f\ x\ \$\ i \in A\ i)$ *net*
by (*rule eventually-all-finite*)
thus *eventually* $(\lambda x.\ f\ x \in S)$ *net*
by (*rule eventually-mono*, *simp add*: S)
qed

lemma *tendsto-vec-lambda* [*tendsto-intros*]:
assumes $\bigwedge i.$ $((\lambda x.\ f\ x\ i) \longrightarrow a\ i)$ *net*
shows $((\lambda x.\ \chi\ i.\ f\ x\ i) \longrightarrow (\chi\ i.\ a\ i))$ *net*
using *assms* **by** (*simp add*: *vec-tendstoI*)

lemma *open-image-vec-nth*: **assumes** *open* S **shows** *open* $((\lambda x.\ x\ \$\ i)\ `\ S)$
proof (*rule openI*)
fix a **assume** $a \in (\lambda x.\ x\ \$\ i)\ `\ S$
then obtain z **where** $a = z\ \$\ i$ **and** $z \in S$ **..**
then obtain A **where** A: $\forall i.$ *open* $(A\ i) \wedge z\ \$\ i \in A\ i$
and S: $\forall y.\ (\forall i.\ y\ \$\ i \in A\ i) \longrightarrow y \in S$
using ‹*open S*› **unfolding** *open-vec-def* **by** *auto*
hence $A\ i \subseteq (\lambda x.\ x\ \$\ i)\ `\ S$
by (*clarsimp*, *rule-tac* $x{=}\chi\ j.$ *if* $j = i$ *then* x *else* $z\ \$\ j$ **in** *image-eqI*, *simp-all*)
hence *open* $(A\ i) \wedge a \in A\ i \wedge A\ i \subseteq (\lambda x.\ x\ \$\ i)\ `\ S$
using A ‹$a = z\ \$\ i$› **by** *simp*
then show $\exists T.$ *open* $T \wedge a \in T \wedge T \subseteq (\lambda x.\ x\ \$\ i)\ `\ S$ **by** $-$ (*rule exI*)
qed

instance *vec* :: (*perfect-space*, *finite*) *perfect-space*
proof
fix x :: $'a\ \hat{}\ 'b$ **show** $\neg$ *open* $\{x\}$
proof
assume *open* $\{x\}$
hence $\forall i.$ *open* $((\lambda x.\ x\ \$\ i)\ `\ \{x\})$ **by** (*fast intro*: *open-image-vec-nth*)
hence $\forall i.$ *open* $\{x\ \$\ i\}$ **by** *simp*
thus *False* **by** (*simp add*: *not-open-singleton*)
qed
qed

30.5 Metric space

instantiation *vec* :: (*metric-space*, *finite*) *dist*
begin

definition

dist x y = setL2 (λi. dist (x$i) (y$i)) UNIV

instance ..
end

instantiation *vec* :: (*metric-space*, *finite*) *uniformity-dist*
begin

definition [*code del*]:
(*uniformity* :: (('*a*, '*b*) *vec* × ('*a*, '*b*) *vec*) *filter*) =
(*INF e*:{*0* <..}. *principal* {(*x*, *y*). *dist x y* < *e*})

instance
by *standard* (*rule uniformity-vec-def*)
end

declare *uniformity-Abort*[**where** '*a*='*a* :: *metric-space* ^ '*b* :: *finite*, *code*]

instantiation *vec* :: (*metric-space*, *finite*) *metric-space*
begin

lemma *dist-vec-nth-le*: *dist* (*x* $ *i*) (*y* $ *i*) ≤ *dist x y*
unfolding *dist-vec-def* **by** (*rule member-le-setL2*) *simp-all*

instance proof
fix *x y* :: '*a* ^ '*b*
show *dist x y* = *0* ⟷ *x* = *y*
unfolding *dist-vec-def*
by (*simp add*: *setL2-eq-0-iff vec-eq-iff*)
next
fix *x y z* :: '*a* ^ '*b*
show *dist x y* ≤ *dist x z* + *dist y z*
unfolding *dist-vec-def*
apply (*rule order-trans* [*OF* - *setL2-triangle-ineq*])
apply (*simp add*: *setL2-mono dist-triangle2*)
done
next
fix *S* :: ('*a* ^ '*b*) *set*
have ∗: *open S* ⟷ (∀ *x*∈*S*. ∃ *e*>*0*. ∀ *y*. *dist y x* < *e* ⟶ *y* ∈ *S*)
proof
assume *open S* **show** ∀ *x*∈*S*. ∃ *e*>*0*. ∀ *y*. *dist y x* < *e* ⟶ *y* ∈ *S*
proof
fix *x* **assume** *x* ∈ *S*
obtain *A* **where** *A*: ∀ *i*. *open* (*A i*) ∀ *i*. *x* $ *i* ∈ *A i*
and *S*: ∀ *y*. (∀ *i*. *y* $ *i* ∈ *A i*) ⟶ *y* ∈ *S*
using ‹*open S*› **and** ‹*x* ∈ *S*› **unfolding** *open-vec-def* **by** *metis*
have ∀ *i*∈*UNIV*. ∃ *r*>*0*. ∀ *y*. *dist y* (*x* $ *i*) < *r* ⟶ *y* ∈ *A i*
using *A* **unfolding** *open-dist* **by** *simp*
hence ∃ *r*. ∀ *i*∈*UNIV*. *0* < *r i* ∧ (∀ *y*. *dist y* (*x* $ *i*) < *r i* ⟶ *y* ∈ *A i*)

```
      by (rule finite-set-choice [OF finite])
    then obtain r where r1: ∀ i. 0 < r i
      and r2: ∀ i y. dist y (x $ i) < r i ⟶ y ∈ A i by fast
    have 0 < Min (range r) ∧ (∀ y. dist y x < Min (range r) ⟶ y ∈ S)
      by (simp add: r1 r2 S le-less-trans [OF dist-vec-nth-le])
    thus ∃ e>0. ∀ y. dist y x < e ⟶ y ∈ S ..
  qed
 next
  assume ∗: ∀ x∈S. ∃ e>0. ∀ y. dist y x < e ⟶ y ∈ S show open S
  proof (unfold open-vec-def, rule)
    fix x assume x ∈ S
    then obtain e where 0 < e and S: ∀ y. dist y x < e ⟶ y ∈ S
      using ∗ by fast
    def r ≡ λi::'b. e / sqrt (of-nat CARD('b))
    from ‹0 < e› have r: ∀ i. 0 < r i
      unfolding r-def by simp-all
    from ‹0 < e› have e: e = setL2 r UNIV
      unfolding r-def by (simp add: setL2-constant)
    def A ≡ λi. {y. dist (x $ i) y < r i}
    have ∀ i. open (A i) ∧ x $ i ∈ A i
      unfolding A-def by (simp add: open-ball r)
    moreover have ∀ y. (∀ i. y $ i ∈ A i) ⟶ y ∈ S
      by (simp add: A-def S dist-vec-def e setL2-strict-mono dist-commute)
    ultimately show ∃ A. (∀ i. open (A i) ∧ x $ i ∈ A i) ∧
      (∀ y. (∀ i. y $ i ∈ A i) ⟶ y ∈ S) by metis
  qed
 qed
 show open S = (∀ x∈S. ∀ F (x', y) in uniformity. x' = x ⟶ y ∈ S)
  unfolding ∗ eventually-uniformity-metric
  by (simp del: split-paired-All add: dist-vec-def dist-commute)
qed

end

lemma Cauchy-vec-nth:
 Cauchy (λn. X n) ⟹ Cauchy (λn. X n $ i)
 unfolding Cauchy-def by (fast intro: le-less-trans [OF dist-vec-nth-le])

lemma vec-CauchyI:
 fixes X :: nat ⇒ 'a::metric-space ^ 'n
 assumes X: ⋀i. Cauchy (λn. X n $ i)
 shows Cauchy (λn. X n)
proof (rule metric-CauchyI)
 fix r :: real assume 0 < r
 hence 0 < r / of-nat CARD('n) (is 0 < ?s) by simp
 def N ≡ λi. LEAST N. ∀ m≥N. ∀ n≥N. dist (X m $ i) (X n $ i) < ?s
 def M ≡ Max (range N)
 have ⋀i. ∃ N. ∀ m≥N. ∀ n≥N. dist (X m $ i) (X n $ i) < ?s
  using X ‹0 < ?s› by (rule metric-CauchyD)
```

```
  hence ⋀i. ∀ m≥N i. ∀ n≥N i. dist (X m $ i) (X n $ i) < ?s
    unfolding N-def by (rule LeastI-ex)
  hence M: ⋀i. ∀ m≥M. ∀ n≥M. dist (X m $ i) (X n $ i) < ?s
    unfolding M-def by simp
  {
    fix m n :: nat
    assume M ≤ m M ≤ n
    have dist (X m) (X n) = setL2 (λi. dist (X m $ i) (X n $ i)) UNIV
      unfolding dist-vec-def ..
    also have ... ≤ setsum (λi. dist (X m $ i) (X n $ i)) UNIV
      by (rule setL2-le-setsum [OF zero-le-dist])
    also have ... < setsum (λi::'n. ?s) UNIV
      by (rule setsum-strict-mono, simp-all add: M ‹M ≤ m› ‹M ≤ n›)
    also have ... = r
      by simp
    finally have dist (X m) (X n) < r .
  }
  hence ∀ m≥M. ∀ n≥M. dist (X m) (X n) < r
    by simp
  then show ∃ M. ∀ m≥M. ∀ n≥M. dist (X m) (X n) < r ..
qed

instance vec :: (complete-space, finite) complete-space
proof
  fix X :: nat ⇒ 'a ^ 'b assume Cauchy X
  have ⋀i. (λn. X n $ i) ⟶ lim (λn. X n $ i)
    using Cauchy-vec-nth [OF ‹Cauchy X›]
    by (simp add: Cauchy-convergent-iff convergent-LIMSEQ-iff)
  hence X ⟶ vec-lambda (λi. lim (λn. X n $ i))
    by (simp add: vec-tendstoI)
  then show convergent X
    by (rule convergentI)
qed
```

30.6 Normed vector space

```
instantiation vec :: (real-normed-vector, finite) real-normed-vector
begin

definition norm x = setL2 (λi. norm (x$i)) UNIV

definition sgn (x::'a^'b) = scaleR (inverse (norm x)) x

instance proof
  fix a :: real and x y :: 'a ^ 'b
  show norm x = 0 ⟷ x = 0
    unfolding norm-vec-def
    by (simp add: setL2-eq-0-iff vec-eq-iff)
  show norm (x + y) ≤ norm x + norm y
```

unfolding *norm-vec-def*
apply (*rule order-trans* [*OF - setL2-triangle-ineq*])
apply (*simp add*: *setL2-mono norm-triangle-ineq*)
done
show *norm* (*scaleR a x*) = |*a*| ∗ *norm x*
unfolding *norm-vec-def*
by (*simp add*: *setL2-right-distrib*)
show *sgn x* = *scaleR* (*inverse* (*norm x*)) *x*
by (*rule sgn-vec-def*)
show *dist x y* = *norm* (*x* − *y*)
unfolding *dist-vec-def norm-vec-def*
by (*simp add*: *dist-norm*)
qed

end

lemma *norm-nth-le*: *norm* (*x* $ *i*) ≤ *norm x*
unfolding *norm-vec-def*
by (*rule member-le-setL2*) *simp-all*

lemma *bounded-linear-vec-nth*: *bounded-linear* (λ*x*. *x* $ *i*)
apply *standard*
apply (*rule vector-add-component*)
apply (*rule vector-scaleR-component*)
apply (*rule-tac x=1* **in** *exI*, *simp add*: *norm-nth-le*)
done

instance *vec* :: (*banach*, *finite*) *banach* **..**

30.7 Inner product space

instantiation *vec* :: (*real-inner*, *finite*) *real-inner*
begin

definition *inner x y* = *setsum* (λ*i*. *inner* (*x*$*i*) (*y*$*i*)) *UNIV*

instance proof
fix *r* :: *real* **and** *x y z* :: *'a* ^ *'b*
show *inner x y* = *inner y x*
unfolding *inner-vec-def*
by (*simp add*: *inner-commute*)
show *inner* (*x* + *y*) *z* = *inner x z* + *inner y z*
unfolding *inner-vec-def*
by (*simp add*: *inner-add-left setsum.distrib*)
show *inner* (*scaleR r x*) *y* = *r* ∗ *inner x y*
unfolding *inner-vec-def*
by (*simp add*: *setsum-right-distrib*)
show *0* ≤ *inner x x*
unfolding *inner-vec-def*

by (*simp add*: *setsum-nonneg*)
show *inner x x = 0 ⟷ x = 0*
unfolding *inner-vec-def*
by (*simp add*: *vec-eq-iff setsum-nonneg-eq-0-iff*)
show *norm x = sqrt* (*inner x x*)
unfolding *inner-vec-def norm-vec-def setL2-def*
by (*simp add*: *power2-norm-eq-inner*)
qed

end

30.8 Euclidean space

Vectors pointing along a single axis.

definition *axis k x* = (*χ i. if i = k then x else 0*)

lemma *axis-nth* [*simp*]: *axis i x $ i = x*
unfolding *axis-def* **by** *simp*

lemma *axis-eq-axis*: *axis i x = axis j y ⟷ x = y ∧ i = j ∨ x = 0 ∧ y = 0*
unfolding *axis-def vec-eq-iff* **by** *auto*

lemma *inner-axis-axis*:
inner (*axis i x*) (*axis j y*) = (*if i = j then inner x y else 0*)
unfolding *inner-vec-def*
apply (*cases i = j*)
apply *clarsimp*
apply (*subst setsum.remove* [*of - j*], *simp-all*)
apply (*rule setsum.neutral*, *simp add*: *axis-def*)
apply (*rule setsum.neutral*, *simp add*: *axis-def*)
done

lemma *setsum-single*:
assumes *finite A* **and** *k ∈ A* **and** *f k = y*
assumes *⋀i. i ∈ A ⟹ i ≠ k ⟹ f i = 0*
shows (*∑ i∈A. f i*) = *y*
apply (*subst setsum.remove* [*OF assms*(*1*,*2*)])
apply (*simp add*: *setsum.neutral assms*(*3*,*4*))
done

lemma *inner-axis*: *inner x* (*axis i y*) = *inner* (*x $ i*) *y*
unfolding *inner-vec-def*
apply (*rule-tac k=i* **in** *setsum-single*)
apply *simp-all*
apply (*simp add*: *axis-def*)
done

instantiation *vec* :: (*euclidean-space*, *finite*) *euclidean-space*
begin

definition *Basis* = $(\bigcup i.\ \bigcup u{\in}Basis.\ \{axis\ i\ u\})$

instance proof
show $(Basis :: ('a\ \hat{}\ 'b)\ set) \neq \{\}$
unfolding *Basis-vec-def* **by** *simp*
next
show *finite* $(Basis :: ('a\ \hat{}\ 'b)\ set)$
unfolding *Basis-vec-def* **by** *simp*
next
fix $u\ v :: 'a\ \hat{}\ 'b$
assume $u \in Basis$ **and** $v \in Basis$
thus $inner\ u\ v = (if\ u = v\ then\ 1\ else\ 0)$
unfolding *Basis-vec-def*
by (*auto simp add*: *inner-axis-axis axis-eq-axis inner-Basis*)
next
fix $x :: 'a\ \hat{}\ 'b$
show $(\forall u{\in}Basis.\ inner\ x\ u = 0) \longleftrightarrow x = 0$
unfolding *Basis-vec-def*
by (*simp add*: *inner-axis euclidean-all-zero-iff vec-eq-iff*)
qed

lemma *DIM-cart*[*simp*]: $DIM('a\hat{}'b) = CARD('b) * DIM('a)$
apply (*simp add*: *Basis-vec-def*)
apply (*subst card-UN-disjoint*)
apply *simp*
apply *simp*
apply (*auto simp*: *axis-eq-axis*) [*1*]
apply (*subst card-UN-disjoint*)
apply (*auto simp*: *axis-eq-axis*)
done

end

lemma *cart-eq-inner-axis*: $a\ \$\ i = inner\ a\ (axis\ i\ 1)$
by (*simp add*: *inner-axis*)

lemma *axis-in-Basis*: $a \in Basis \Longrightarrow axis\ i\ a \in Basis$
by (*auto simp add*: *Basis-vec-def axis-eq-axis*)

end

31 Operator Norm

theory *Operator-Norm*
imports *Complex-Main*
begin

This formulation yields zero if $'a$ is the trivial vector space.

definition *onorm* :: (*'a*::*real-normed-vector* ⇒ *'b*::*real-normed-vector*) ⇒ *real*
where *onorm f* = (*SUP x*. *norm* (*f x*) / *norm x*)

lemma *onorm-bound*:
assumes *0* ≤ *b* **and** ⋀*x*. *norm* (*f x*) ≤ *b* ∗ *norm x*
shows *onorm f* ≤ *b*
unfolding *onorm-def*
proof (*rule cSUP-least*)
fix *x*
show *norm* (*f x*) / *norm x* ≤ *b*
using *assms* **by** (*cases x* = *0*) (*simp-all add*: *pos-divide-le-eq*)
qed *simp*

In non-trivial vector spaces, the first assumption is redundant.

lemma *onorm-le*:
fixes *f* :: *'a*::{*real-normed-vector*, *perfect-space*} ⇒ *'b*::*real-normed-vector*
assumes ⋀*x*. *norm* (*f x*) ≤ *b* ∗ *norm x*
shows *onorm f* ≤ *b*
proof (*rule onorm-bound* [*OF* - *assms*])
have {*0*::*'a*} ≠ *UNIV* **by** (*metis not-open-singleton open-UNIV*)
then obtain *a* :: *'a* **where** *a* ≠ *0* **by** *fast*
have *0* ≤ *b* ∗ *norm a*
by (*rule order-trans* [*OF norm-ge-zero assms*])
with ‹*a* ≠ *0*› **show** *0* ≤ *b*
by (*simp add*: *zero-le-mult-iff*)
qed

lemma *le-onorm*:
assumes *bounded-linear f*
shows *norm* (*f x*) / *norm x* ≤ *onorm f*
proof −
interpret *f*: *bounded-linear f* **by** *fact*
obtain *b* **where** *0* ≤ *b* **and** ∀ *x*. *norm* (*f x*) ≤ *norm x* ∗ *b*
using *f.nonneg-bounded* **by** *auto*
then have ∀ *x*. *norm* (*f x*) / *norm x* ≤ *b*
by (*clarify*, *case-tac x* = *0*,
simp-all add: *f.zero pos-divide-le-eq mult.commute*)
then have *bdd-above* (*range* (λ*x*. *norm* (*f x*) / *norm x*))
unfolding *bdd-above-def* **by** *fast*
with *UNIV-I* **show** *?thesis*
unfolding *onorm-def* **by** (*rule cSUP-upper*)
qed

lemma *onorm*:
assumes *bounded-linear f*
shows *norm* (*f x*) ≤ *onorm f* ∗ *norm x*
proof −
interpret *f*: *bounded-linear f* **by** *fact*
show *?thesis*

proof (*cases*)
assume $x = 0$
then show *?thesis* **by** (*simp add*: $f.zero$)
next
assume $x \neq 0$
have $norm\ (f\ x)\ /\ norm\ x \leq onorm\ f$
by (*rule le-onorm* [*OF assms*])
then show $norm\ (f\ x) \leq onorm\ f * norm\ x$
by (*simp add*: *pos-divide-le-eq* ‹$x \neq 0$›)
qed
qed

lemma *onorm-pos-le*:
assumes f: *bounded-linear f*
shows $0 \leq onorm\ f$
using *le-onorm* [*OF f*, **where** $x{=}0$] **by** *simp*

lemma *onorm-zero*: $onorm\ (\lambda x.\ 0) = 0$
proof (*rule order-antisym*)
show $onorm\ (\lambda x.\ 0) \leq 0$
by (*simp add*: *onorm-bound*)
show $0 \leq onorm\ (\lambda x.\ 0)$
using *bounded-linear-zero* **by** (*rule onorm-pos-le*)
qed

lemma *onorm-eq-0*:
assumes f: *bounded-linear f*
shows $onorm\ f = 0 \longleftrightarrow (\forall x.\ f\ x = 0)$
using *onorm* [*OF f*] **by** (*auto simp*: *fun-eq-iff* [*symmetric*] *onorm-zero*)

lemma *onorm-pos-lt*:
assumes f: *bounded-linear f*
shows $0 < onorm\ f \longleftrightarrow \neg\ (\forall x.\ f\ x = 0)$
by (*simp add*: *less-le onorm-pos-le* [*OF f*] *onorm-eq-0* [*OF f*])

lemma *onorm-id-le*: $onorm\ (\lambda x.\ x) \leq 1$
by (*rule onorm-bound*) *simp-all*

lemma *onorm-id*: $onorm\ (\lambda x.\ x{::}'a{::}\{real\text{-}normed\text{-}vector, perfect\text{-}space\}) = 1$
proof (*rule antisym*[*OF onorm-id-le*])
have $\{0{::}'a\} \neq UNIV$ **by** (*metis not-open-singleton open-UNIV*)
then obtain $x :: 'a$ **where** $x \neq 0$ **by** *fast*
hence $1 \leq norm\ x\ /\ norm\ x$
by *simp*
also have $\ldots \leq onorm\ (\lambda x{::}'a.\ x)$
by (*rule le-onorm*) (*rule bounded-linear-ident*)
finally show $1 \leq onorm\ (\lambda x{::}'a.\ x)$ **.**
qed

lemma *onorm-compose*:
assumes *f*: *bounded-linear f*
assumes *g*: *bounded-linear g*
shows *onorm* (*f* ∘ *g*) ≤ *onorm f* ∗ *onorm g*
proof (*rule onorm-bound*)
show *0* ≤ *onorm f* ∗ *onorm g*
by (*intro mult-nonneg-nonneg onorm-pos-le f g*)
next
fix *x*
have *norm* (*f* (*g x*)) ≤ *onorm f* ∗ *norm* (*g x*)
by (*rule onorm* [*OF f*])
also have *onorm f* ∗ *norm* (*g x*) ≤ *onorm f* ∗ (*onorm g* ∗ *norm x*)
by (*rule mult-left-mono* [*OF onorm* [*OF g*] *onorm-pos-le* [*OF f*]])
finally show *norm* ((*f* ∘ *g*) *x*) ≤ *onorm f* ∗ *onorm g* ∗ *norm x*
by (*simp add*: *mult.assoc*)
qed

lemma *onorm-scaleR-lemma*:
assumes *f*: *bounded-linear f*
shows *onorm* (λ*x*. *r* ∗*R* *f x*) ≤ |*r*| ∗ *onorm f*
proof (*rule onorm-bound*)
show *0* ≤ |*r*| ∗ *onorm f*
by (*intro mult-nonneg-nonneg onorm-pos-le abs-ge-zero f*)
next
fix *x*
have |*r*| ∗ *norm* (*f x*) ≤ |*r*| ∗ (*onorm f* ∗ *norm x*)
by (*intro mult-left-mono onorm abs-ge-zero f*)
then show *norm* (*r* ∗*R* *f x*) ≤ |*r*| ∗ *onorm f* ∗ *norm x*
by (*simp only*: *norm-scaleR mult.assoc*)
qed

lemma *onorm-scaleR*:
assumes *f*: *bounded-linear f*
shows *onorm* (λ*x*. *r* ∗*R* *f x*) = |*r*| ∗ *onorm f*
proof (*cases r* = *0*)
assume *r* ≠ *0*
show *?thesis*
proof (*rule order-antisym*)
show *onorm* (λ*x*. *r* ∗*R* *f x*) ≤ |*r*| ∗ *onorm f*
using *f* **by** (*rule onorm-scaleR-lemma*)
next
have *bounded-linear* (λ*x*. *r* ∗*R* *f x*)
using *bounded-linear-scaleR-right f* **by** (*rule bounded-linear-compose*)
then have *onorm* (λ*x*. *inverse r* ∗*R* *r* ∗*R* *f x*) ≤ |*inverse r*| ∗ *onorm* (λ*x*. *r* ∗*R* *f x*)
by (*rule onorm-scaleR-lemma*)
with ‹*r* ≠ *0*› **show** |*r*| ∗ *onorm f* ≤ *onorm* (λ*x*. *r* ∗*R* *f x*)
by (*simp add*: *inverse-eq-divide pos-le-divide-eq mult.commute*)
qed

qed (*simp add*: *onorm-zero*)

lemma *onorm-scaleR-left-lemma*:
assumes *r*: *bounded-linear r*
shows *onorm* (λ*x*. *r x* $*_R$ *f*) ≤ *onorm r* * *norm f*
proof (*rule onorm-bound*)
fix *x*
have *norm* (*r x* $*_R$ *f*) = *norm* (*r x*) * *norm f*
by *simp*
also have ... ≤ *onorm r* * *norm x* * *norm f*
by (*intro mult-right-mono onorm r norm-ge-zero*)
finally show *norm* (*r x* $*_R$ *f*) ≤ *onorm r* * *norm f* * *norm x*
by (*simp add*: *ac-simps*)
qed (*intro mult-nonneg-nonneg norm-ge-zero onorm-pos-le r*)

lemma *onorm-scaleR-left*:
assumes *f*: *bounded-linear r*
shows *onorm* (λ*x*. *r x* $*_R$ *f*) = *onorm r* * *norm f*
proof (*cases f* = *0*)
assume *f* ≠ *0*
show *?thesis*
proof (*rule order-antisym*)
show *onorm* (λ*x*. *r x* $*_R$ *f*) ≤ *onorm r* * *norm f*
using *f* **by** (*rule onorm-scaleR-left-lemma*)
next
have *bl1*: *bounded-linear* (λ*x*. *r x* $*_R$ *f*)
by (*metis bounded-linear-scaleR-const f*)
have *bounded-linear* (λ*x*. *r x* * *norm f*)
by (*metis bounded-linear-mult-const f*)
from *onorm-scaleR-left-lemma*[*OF this*, *of inverse* (*norm f*)]
have *onorm r* ≤ *onorm* (λ*x*. *r x* * *norm f*) * *inverse* (*norm f*)
using ‹*f* ≠ *0*›
by (*simp add*: *inverse-eq-divide*)
also have *onorm* (λ*x*. *r x* * *norm f*) ≤ *onorm* (λ*x*. *r x* $*_R$ *f*)
by (*rule onorm-bound*)
(*auto simp*: *abs-mult bl1 onorm-pos-le intro*!: *order-trans*[*OF* - *onorm*])
finally show *onorm r* * *norm f* ≤ *onorm* (λ*x*. *r x* $*_R$ *f*)
using ‹*f* ≠ *0*›
by (*simp add*: *inverse-eq-divide pos-le-divide-eq mult.commute*)
qed
qed (*simp add*: *onorm-zero*)

lemma *onorm-neg*:
shows *onorm* (λ*x*. − *f x*) = *onorm f*
unfolding *onorm-def* **by** *simp*

lemma *onorm-triangle*:
assumes *f*: *bounded-linear f*
assumes *g*: *bounded-linear g*

shows *onorm* $(\lambda x.\ f\ x + g\ x) \leq$ *onorm* $f +$ *onorm* g
proof (*rule onorm-bound*)
show $0 \leq$ *onorm* $f +$ *onorm* g
by (*intro add-nonneg-nonneg onorm-pos-le f g*)
next
fix x
have *norm* $(f\ x + g\ x) \leq$ *norm* $(f\ x) +$ *norm* $(g\ x)$
by (*rule norm-triangle-ineq*)
also have *norm* $(f\ x) +$ *norm* $(g\ x) \leq$ *onorm* $f *$ *norm* $x +$ *onorm* $g *$ *norm* x
by (*intro add-mono onorm f g*)
finally show *norm* $(f\ x + g\ x) \leq$ (*onorm* $f +$ *onorm* $g) *$ *norm* x
by (*simp only*: *distrib-right*)
qed

lemma *onorm-triangle-le*:
assumes *bounded-linear f*
assumes *bounded-linear g*
assumes *onorm* $f +$ *onorm* $g \leq e$
shows *onorm* $(\lambda x.\ f\ x + g\ x) \leq e$
using *assms* **by** (*rule onorm-triangle* [*THEN order-trans*])

lemma *onorm-triangle-lt*:
assumes *bounded-linear f*
assumes *bounded-linear g*
assumes *onorm* $f +$ *onorm* $g < e$
shows *onorm* $(\lambda x.\ f\ x + g\ x) < e$
using *assms* **by** (*rule onorm-triangle* [*THEN order-le-less-trans*])

end

32 Countable Complete Lattices

theory *Countable-Complete-Lattices*
imports *Main Countable-Set*
begin

lemma *UNIV-nat-eq*: *UNIV* $=$ *insert 0* (*range Suc*)
by (*metis UNIV-eq-I nat.nchotomy insertCI rangeI*)

class *countable-complete-lattice* $=$ *lattice* $+$ *Inf* $+$ *Sup* $+$ *bot* $+$ *top* $+$
assumes *ccInf-lower*: *countable* $A \Longrightarrow x \in A \Longrightarrow$ *Inf* $A \leq x$
assumes *ccInf-greatest*: *countable* $A \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow z \leq x) \Longrightarrow z \leq$ *Inf* A
assumes *ccSup-upper*: *countable* $A \Longrightarrow x \in A \Longrightarrow x \leq$ *Sup* A
assumes *ccSup-least*: *countable* $A \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow x \leq z) \Longrightarrow$ *Sup* $A \leq z$
assumes *ccInf-empty* [*simp*]: *Inf* $\{\} =$ *top*
assumes *ccSup-empty* [*simp*]: *Sup* $\{\} =$ *bot*
begin

subclass *bounded-lattice*

proof
fix *a*
show *bot* $\leq$ *a* **by** (*auto intro*: *ccSup-least simp only*: *ccSup-empty* [*symmetric*])
show *a* $\leq$ *top* **by** (*auto intro*: *ccInf-greatest simp only*: *ccInf-empty* [*symmetric*])
qed

lemma *ccINF-lower*: *countable A* $\Longrightarrow$ *i* $\in$ *A* $\Longrightarrow$ (*INF i* :*A*. *f i*) $\leq$ *f i*
using *ccInf-lower* [*of f* ' *A*] **by** *simp*

lemma *ccINF-greatest*: *countable A* $\Longrightarrow$ ($\bigwedge$*i*. *i* $\in$ *A* $\Longrightarrow$ *u* $\leq$ *f i*) $\Longrightarrow$ *u* $\leq$ (*INF i* :*A*. *f i*)
using *ccInf-greatest* [*of f* ' *A*] **by** *auto*

lemma *ccSUP-upper*: *countable A* $\Longrightarrow$ *i* $\in$ *A* $\Longrightarrow$ *f i* $\leq$ (*SUP i* :*A*. *f i*)
using *ccSup-upper* [*of f* ' *A*] **by** *simp*

lemma *ccSUP-least*: *countable A* $\Longrightarrow$ ($\bigwedge$*i*. *i* $\in$ *A* $\Longrightarrow$ *f i* $\leq$ *u*) $\Longrightarrow$ (*SUP i* :*A*. *f i*) $\leq$ *u*
using *ccSup-least* [*of f* ' *A*] **by** *auto*

lemma *ccInf-lower2*: *countable A* $\Longrightarrow$ *u* $\in$ *A* $\Longrightarrow$ *u* $\leq$ *v* $\Longrightarrow$ *Inf A* $\leq$ *v*
using *ccInf-lower* [*of A u*] **by** *auto*

lemma *ccINF-lower2*: *countable A* $\Longrightarrow$ *i* $\in$ *A* $\Longrightarrow$ *f i* $\leq$ *u* $\Longrightarrow$ (*INF i* :*A*. *f i*) $\leq$ *u*
using *ccINF-lower* [*of A i f*] **by** *auto*

lemma *ccSup-upper2*: *countable A* $\Longrightarrow$ *u* $\in$ *A* $\Longrightarrow$ *v* $\leq$ *u* $\Longrightarrow$ *v* $\leq$ *Sup A*
using *ccSup-upper* [*of A u*] **by** *auto*

lemma *ccSUP-upper2*: *countable A* $\Longrightarrow$ *i* $\in$ *A* $\Longrightarrow$ *u* $\leq$ *f i* $\Longrightarrow$ *u* $\leq$ (*SUP i* :*A*. *f i*)
using *ccSUP-upper* [*of A i f*] **by** *auto*

lemma *le-ccInf-iff*: *countable A* $\Longrightarrow$ *b* $\leq$ *Inf A* $\longleftrightarrow$ ($\forall$ *a*$\in$*A*. *b* $\leq$ *a*)
by (*auto intro*: *ccInf-greatest dest*: *ccInf-lower*)

lemma *le-ccINF-iff*: *countable A* $\Longrightarrow$ *u* $\leq$ (*INF i* :*A*. *f i*) $\longleftrightarrow$ ($\forall$ *i*$\in$*A*. *u* $\leq$ *f i*)
using *le-ccInf-iff* [*of f* ' *A*] **by** *simp*

lemma *ccSup-le-iff*: *countable A* $\Longrightarrow$ *Sup A* $\leq$ *b* $\longleftrightarrow$ ($\forall$ *a*$\in$*A*. *a* $\leq$ *b*)
by (*auto intro*: *ccSup-least dest*: *ccSup-upper*)

lemma *ccSUP-le-iff*: *countable A* $\Longrightarrow$ (*SUP i* :*A*. *f i*) $\leq$ *u* $\longleftrightarrow$ ($\forall$ *i*$\in$*A*. *f i* $\leq$ *u*)
using *ccSup-le-iff* [*of f* ' *A*] **by** *simp*

lemma *ccInf-insert* [*simp*]: *countable A* $\Longrightarrow$ *Inf* (*insert a A*) = *inf a* (*Inf A*)
by (*force intro*: *le-infI le-infI1 le-infI2 antisym ccInf-greatest ccInf-lower*)

lemma *ccINF-insert* [*simp*]: *countable A* $\Longrightarrow$ (*INF x*:*insert a A*. *f x*) = *inf* (*f a*)

(*INFIMUM A f*)
unfolding *image-insert* **by** *simp*

lemma *ccSup-insert* [*simp*]: *countable A* $\Longrightarrow$ *Sup* (*insert a A*) = *sup a* (*Sup A*)
by (*force intro*: *le-supI le-supI1 le-supI2 antisym ccSup-least ccSup-upper*)

lemma *ccSUP-insert* [*simp*]: *countable A* $\Longrightarrow$ (*SUP x*:*insert a A*. *f x*) = *sup* (*f a*) (*SUPREMUM A f*)
unfolding *image-insert* **by** *simp*

lemma *ccINF-empty* [*simp*]: (*INF x*:{}. *f x*) = *top*
unfolding *image-empty* **by** *simp*

lemma *ccSUP-empty* [*simp*]: (*SUP x*:{}. *f x*) = *bot*
unfolding *image-empty* **by** *simp*

lemma *ccInf-superset-mono*: *countable A* $\Longrightarrow$ *B* $\subseteq$ *A* $\Longrightarrow$ *Inf A* $\leq$ *Inf B*
by (*auto intro*: *ccInf-greatest ccInf-lower countable-subset*)

lemma *ccSup-subset-mono*: *countable B* $\Longrightarrow$ *A* $\subseteq$ *B* $\Longrightarrow$ *Sup A* $\leq$ *Sup B*
by (*auto intro*: *ccSup-least ccSup-upper countable-subset*)

lemma *ccInf-mono*:
assumes [*intro*]: *countable B countable A*
assumes $\bigwedge b.\ b \in B \Longrightarrow \exists\, a{\in}A.\ a \leq b$
shows *Inf A* $\leq$ *Inf B*
proof (*rule ccInf-greatest*)
fix *b* **assume** *b* $\in$ *B*
with *assms* **obtain** *a* **where** *a* $\in$ *A* **and** *a* $\leq$ *b* **by** *blast*
from ‹*a* $\in$ *A*› **have** *Inf A* $\leq$ *a* **by** (*rule ccInf-lower*[*rotated*]) *auto*
with ‹*a* $\leq$ *b*› **show** *Inf A* $\leq$ *b* **by** *auto*
qed *auto*

lemma *ccINF-mono*:
countable A $\Longrightarrow$ *countable B* $\Longrightarrow$ ($\bigwedge m.\ m \in B \Longrightarrow \exists\, n{\in}A.\ f\ n \leq g\ m$) $\Longrightarrow$ (*INF n*:*A*. *f n*) $\leq$ (*INF n*:*B*. *g n*)
using *ccInf-mono* [*of g* ‘ *B f* ‘ *A*] **by** *auto*

lemma *ccSup-mono*:
assumes [*intro*]: *countable B countable A*
assumes $\bigwedge a.\ a \in A \Longrightarrow \exists\, b{\in}B.\ a \leq b$
shows *Sup A* $\leq$ *Sup B*
proof (*rule ccSup-least*)
fix *a* **assume** *a* $\in$ *A*
with *assms* **obtain** *b* **where** *b* $\in$ *B* **and** *a* $\leq$ *b* **by** *blast*
from ‹*b* $\in$ *B*› **have** *b* $\leq$ *Sup B* **by** (*rule ccSup-upper*[*rotated*]) *auto*
with ‹*a* $\leq$ *b*› **show** *a* $\leq$ *Sup B* **by** *auto*
qed *auto*

lemma *ccSUP-mono*:
countable A $\Longrightarrow$ *countable B* $\Longrightarrow$ $(\bigwedge n.\ n \in A \Longrightarrow \exists\, m{\in}B.\ f\ n \leq g\ m) \Longrightarrow (SUP\ n{:}A.\ f\ n) \leq (SUP\ n{:}B.\ g\ n)$
using *ccSup-mono* [*of g ' B f ' A*] **by** *auto*

lemma *ccINF-superset-mono*:
countable A $\Longrightarrow$ $B \subseteq A \Longrightarrow (\bigwedge x.\ x \in B \Longrightarrow f\ x \leq g\ x) \Longrightarrow (INF\ x{:}A.\ f\ x) \leq (INF\ x{:}B.\ g\ x)$
by (*blast intro*: *ccINF-mono countable-subset dest*: *subsetD*)

lemma *ccSUP-subset-mono*:
countable B $\Longrightarrow$ $A \subseteq B \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow f\ x \leq g\ x) \Longrightarrow (SUP\ x{:}A.\ f\ x) \leq (SUP\ x{:}B.\ g\ x)$
by (*blast intro*: *ccSUP-mono countable-subset dest*: *subsetD*)

lemma *less-eq-ccInf-inter*: *countable A* $\Longrightarrow$ *countable B* $\Longrightarrow$ *sup* (*Inf A*) (*Inf B*) $\leq$ *Inf* $(A \cap B)$
by (*auto intro*: *ccInf-greatest ccInf-lower*)

lemma *ccSup-inter-less-eq*: *countable A* $\Longrightarrow$ *countable B* $\Longrightarrow$ *Sup* $(A \cap B) \leq$ *inf* (*Sup A*) (*Sup B*)
by (*auto intro*: *ccSup-least ccSup-upper*)

lemma *ccInf-union-distrib*: *countable A* $\Longrightarrow$ *countable B* $\Longrightarrow$ *Inf* $(A \cup B) =$ *inf* (*Inf A*) (*Inf B*)
by (*rule antisym*) (*auto intro*: *ccInf-greatest ccInf-lower le-infI1 le-infI2*)

lemma *ccINF-union*:
countable A $\Longrightarrow$ *countable B* $\Longrightarrow$ $(INF\ i{:}A \cup B.\ M\ i) =$ *inf* $(INF\ i{:}A.\ M\ i)$ $(INF\ i{:}B.\ M\ i)$
by (*auto intro*!: *antisym ccINF-mono intro*: *le-infI1 le-infI2 ccINF-greatest ccINF-lower*)

lemma *ccSup-union-distrib*: *countable A* $\Longrightarrow$ *countable B* $\Longrightarrow$ *Sup* $(A \cup B) =$ *sup* (*Sup A*) (*Sup B*)
by (*rule antisym*) (*auto intro*: *ccSup-least ccSup-upper le-supI1 le-supI2*)

lemma *ccSUP-union*:
countable A $\Longrightarrow$ *countable B* $\Longrightarrow$ $(SUP\ i{:}A \cup B.\ M\ i) =$ *sup* $(SUP\ i{:}A.\ M\ i)$ $(SUP\ i{:}B.\ M\ i)$
by (*auto intro*!: *antisym ccSUP-mono intro*: *le-supI1 le-supI2 ccSUP-least ccSUP-upper*)

lemma *ccINF-inf-distrib*: *countable A* $\Longrightarrow$ *inf* $(INF\ a{:}A.\ f\ a)$ $(INF\ a{:}A.\ g\ a) =$ $(INF\ a{:}A.\ inf\ (f\ a)\ (g\ a))$
by (*rule antisym*) (*rule ccINF-greatest*, *auto intro*: *le-infI1 le-infI2 ccINF-lower ccINF-mono*)

lemma *ccSUP-sup-distrib*: *countable A* $\Longrightarrow$ *sup* $(SUP\ a{:}A.\ f\ a)$ $(SUP\ a{:}A.\ g\ a) =$ $(SUP\ a{:}A.\ sup\ (f\ a)\ (g\ a))$

by (*rule antisym*[*rotated*]) (*rule ccSUP-least*, *auto intro*: *le-supI1 le-supI2 ccSUP-upper ccSUP-mono*)

lemma *ccINF-const* [*simp*]: $A \neq \{\} \Longrightarrow (INF\ i\ :A.\ f) = f$
unfolding *image-constant-conv* **by** *auto*

lemma *ccSUP-const* [*simp*]: $A \neq \{\} \Longrightarrow (SUP\ i\ :A.\ f) = f$
unfolding *image-constant-conv* **by** *auto*

lemma *ccINF-top* [*simp*]: $(INF\ x{:}A.\ top) = top$
by (*cases* $A = \{\}$) *simp-all*

lemma *ccSUP-bot* [*simp*]: $(SUP\ x{:}A.\ bot) = bot$
by (*cases* $A = \{\}$) *simp-all*

lemma *ccINF-commute*: *countable* $A \Longrightarrow$ *countable* $B \Longrightarrow (INF\ i{:}A.\ INF\ j{:}B.\ f\ i\ j) = (INF\ j{:}B.\ INF\ i{:}A.\ f\ i\ j)$
by (*iprover intro*: *ccINF-lower ccINF-greatest order-trans antisym*)

lemma *ccSUP-commute*: *countable* $A \Longrightarrow$ *countable* $B \Longrightarrow (SUP\ i{:}A.\ SUP\ j{:}B.\ f\ i\ j) = (SUP\ j{:}B.\ SUP\ i{:}A.\ f\ i\ j)$
by (*iprover intro*: *ccSUP-upper ccSUP-least order-trans antisym*)

end

context
fixes $a :: 'a{::}\{$*countable-complete-lattice*, *linorder*$\}$
begin

lemma *less-ccSup-iff*: *countable* $S \Longrightarrow a < Sup\ S \longleftrightarrow (\exists\, x{\in}S.\ a < x)$
unfolding *not-le* [*symmetric*] **by** (*subst ccSup-le-iff*) *auto*

lemma *less-ccSUP-iff*: *countable* $A \Longrightarrow a < (SUP\ i{:}A.\ f\ i) \longleftrightarrow (\exists\, x{\in}A.\ a < f\ x)$
using *less-ccSup-iff* [*of f ' A*] **by** *simp*

lemma *ccInf-less-iff*: *countable* $S \Longrightarrow Inf\ S < a \longleftrightarrow (\exists\, x{\in}S.\ x < a)$
unfolding *not-le* [*symmetric*] **by** (*subst le-ccInf-iff*) *auto*

lemma *ccINF-less-iff*: *countable* $A \Longrightarrow (INF\ i{:}A.\ f\ i) < a \longleftrightarrow (\exists\, x{\in}A.\ f\ x < a)$
using *ccInf-less-iff* [*of f ' A*] **by** *simp*

end

class *countable-complete-distrib-lattice* = *countable-complete-lattice* +
assumes *sup-ccInf*: *countable* $B \Longrightarrow sup\ a\ (Inf\ B) = (INF\ b{:}B.\ sup\ a\ b)$
assumes *inf-ccSup*: *countable* $B \Longrightarrow inf\ a\ (Sup\ B) = (SUP\ b{:}B.\ inf\ a\ b)$
begin

lemma *sup-ccINF*:

countable B $\Longrightarrow$ *sup a* (*INF b*:*B*. *f b*) = (*INF b*:*B*. *sup a* (*f b*))
by (*simp only*: *sup-ccInf image-image countable-image*)

lemma *inf-ccSUP*:
countable B $\Longrightarrow$ *inf a* (*SUP b*:*B*. *f b*) = (*SUP b*:*B*. *inf a* (*f b*))
by (*simp only*: *inf-ccSup image-image countable-image*)

subclass *distrib-lattice*
proof
fix *a b c*
from *sup-ccInf*[*of* {*b*, *c*} *a*] **have** *sup a* (*Inf* {*b*, *c*}) = (*INF d*:{*b*, *c*}. *sup a d*)
by *simp*
then show *sup a* (*inf b c*) = *inf* (*sup a b*) (*sup a c*)
by *simp*
qed

lemma *ccInf-sup*:
countable B $\Longrightarrow$ *sup* (*Inf B*) *a* = (*INF b*:*B*. *sup b a*)
by (*simp add*: *sup-ccInf sup-commute*)

lemma *ccSup-inf*:
countable B $\Longrightarrow$ *inf* (*Sup B*) *a* = (*SUP b*:*B*. *inf b a*)
by (*simp add*: *inf-ccSup inf-commute*)

lemma *ccINF-sup*:
countable B $\Longrightarrow$ *sup* (*INF b*:*B*. *f b*) *a* = (*INF b*:*B*. *sup* (*f b*) *a*)
by (*simp add*: *sup-ccINF sup-commute*)

lemma *ccSUP-inf*:
countable B $\Longrightarrow$ *inf* (*SUP b*:*B*. *f b*) *a* = (*SUP b*:*B*. *inf* (*f b*) *a*)
by (*simp add*: *inf-ccSUP inf-commute*)

lemma *ccINF-sup-distrib2*:
countable A $\Longrightarrow$ *countable B* $\Longrightarrow$ *sup* (*INF a*:*A*. *f a*) (*INF b*:*B*. *g b*) = (*INF a*:*A*. *INF b*:*B*. *sup* (*f a*) (*g b*))
by (*subst ccINF-commute*) (*simp-all add*: *sup-ccINF ccINF-sup*)

lemma *ccSUP-inf-distrib2*:
countable A $\Longrightarrow$ *countable B* $\Longrightarrow$ *inf* (*SUP a*:*A*. *f a*) (*SUP b*:*B*. *g b*) = (*SUP a*:*A*. *SUP b*:*B*. *inf* (*f a*) (*g b*))
by (*subst ccSUP-commute*) (*simp-all add*: *inf-ccSUP ccSUP-inf*)

context
fixes *f* :: *'a* $\Rightarrow$ *'b*::*countable-complete-lattice*
assumes *mono f*
begin

lemma *mono-ccInf*:
countable A $\Longrightarrow$ *f* (*Inf A*) $\leq$ (*INF x*:*A*. *f x*)

using ‹*mono f*›
by (*auto intro*!: *countable-complete-lattice-class.ccINF-greatest intro*: *ccInf-lower dest*: *monoD*)

lemma *mono-ccSup*:
countable A ⟹ (*SUP x*:*A*. *f x*) ≤ *f* (*Sup A*)
using ‹*mono f*› **by** (*auto intro*: *countable-complete-lattice-class.ccSUP-least ccSup-upper dest*: *monoD*)

lemma *mono-ccINF*:
countable I ⟹ *f* (*INF i* : *I*. *A i*) ≤ (*INF x* : *I*. *f* (*A x*))
by (*intro countable-complete-lattice-class.ccINF-greatest monoD*[*OF* ‹*mono f*›] *ccINF-lower*)

lemma *mono-ccSUP*:
countable I ⟹ (*SUP x* : *I*. *f* (*A x*)) ≤ *f* (*SUP i* : *I*. *A i*)
by (*intro countable-complete-lattice-class.ccSUP-least monoD*[*OF* ‹*mono f*›] *ccSUP-upper*)

end

end

32.0.1 Instances of countable complete lattices

instance *fun* :: (*type*, *countable-complete-lattice*) *countable-complete-lattice*
by *standard*
(*auto simp*: *le-fun-def intro*!: *ccSUP-upper ccSUP-least ccINF-lower ccINF-greatest*)

subclass (**in** *complete-lattice*) *countable-complete-lattice*
by *standard* (*auto intro*: *Sup-upper Sup-least Inf-lower Inf-greatest*)

subclass (**in** *complete-distrib-lattice*) *countable-complete-distrib-lattice*
by *standard* (*auto intro*: *sup-Inf inf-Sup*)

end

33 Continuity and iterations

theory *Order-Continuity*
imports *Complex-Main Countable-Complete-Lattices*
begin

lemma *SUP-nat-binary*:
(*SUP n*::*nat*. *if n* = *0 then A else B*) = (*sup A B*::*'a*::*countable-complete-lattice*)
apply (*auto intro*!: *antisym ccSUP-least*)
apply (*rule ccSUP-upper2*[**where** *i*=*0*])
apply *simp-all*

apply (*rule ccSUP-upper2*[**where** *i=1*])
apply *simp-all*
done

lemma *INF-nat-binary*:
(*INF n*::*nat*. *if n = 0 then A else B*) = (*inf A B*::*'a*::*countable-complete-lattice*)
apply (*auto intro*!: *antisym ccINF-greatest*)
apply (*rule ccINF-lower2*[**where** *i=0*])
apply *simp-all*
apply (*rule ccINF-lower2*[**where** *i=1*])
apply *simp-all*
done

The name *continuous* is already taken in *Complex-Main*, so we use *sup-continuous* and *inf-continuous*. These names appear sometimes in literature and have the advantage that these names are duals.

named-theorems *order-continuous-intros*

33.1 Continuity for complete lattices

definition
sup-continuous :: (*'a*::*countable-complete-lattice* ⇒ *'b*::*countable-complete-lattice*) ⇒ *bool*
where
sup-continuous F ⟷ (∀ *M*::*nat* ⇒ *'a*. *mono M* ⟶ *F* (*SUP i*. *M i*) = (*SUP i*. *F* (*M i*)))

lemma *sup-continuousD*: *sup-continuous F* ⟹ *mono M* ⟹ *F* (*SUP i*::*nat*. *M i*) = (*SUP i*. *F* (*M i*))
by (*auto simp*: *sup-continuous-def*)

lemma *sup-continuous-mono*:
assumes [*simp*]: *sup-continuous F* **shows** *mono F*
proof
fix *A B* :: *'a* **assume** [*simp*]: *A* ≤ *B*
have *F B* = *F* (*SUP n*::*nat*. *if n = 0 then A else B*)
by (*simp add*: *sup-absorb2 SUP-nat-binary*)
also have ... = (*SUP n*::*nat*. *if n = 0 then F A else F B*)
by (*auto simp*: *sup-continuousD mono-def intro*!: *SUP-cong*)
finally show *F A* ≤ *F B*
by (*simp add*: *SUP-nat-binary le-iff-sup*)
qed

lemma [*order-continuous-intros*]:
shows *sup-continuous-const*: *sup-continuous* (λ*x*. *c*)
and *sup-continuous-id*: *sup-continuous* (λ*x*. *x*)
and *sup-continuous-apply*: *sup-continuous* (λ*f*. *f x*)
and *sup-continuous-fun*: (⋀*s*. *sup-continuous* (λ*x*. *P x s*)) ⟹ *sup-continuous P*

and *sup-continuous-If*: *sup-continuous F* $\Longrightarrow$ *sup-continuous G* $\Longrightarrow$ *sup-continuous* (*λf*. *if C then F f else G f*)
by (*auto simp*: *sup-continuous-def*)

lemma *sup-continuous-compose*:
assumes *f*: *sup-continuous f* **and** *g*: *sup-continuous g*
shows *sup-continuous* (*λx*. *f* (*g x*))
unfolding *sup-continuous-def*
proof *safe*
fix *M* :: *nat* ⇒ *'c* **assume** *mono M*
moreover then have *mono* (*λi*. *g* (*M i*))
using *sup-continuous-mono*[*OF g*] **by** (*auto simp*: *mono-def*)
ultimately show *f* (*g* (*SUPREMUM UNIV M*)) = (*SUP i*. *f* (*g* (*M i*)))
by (*auto simp*: *sup-continuous-def g*[*THEN sup-continuousD*] *f*[*THEN sup-continuousD*])
qed

lemma *sup-continuous-sup*[*order-continuous-intros*]:
sup-continuous f $\Longrightarrow$ *sup-continuous g* $\Longrightarrow$ *sup-continuous* (*λx*. *sup* (*f x*) (*g x*))
by (*simp add*: *sup-continuous-def ccSUP-sup-distrib*)

lemma *sup-continuous-inf*[*order-continuous-intros*]:
fixes *P Q* :: *'a* :: *countable-complete-lattice* ⇒ *'b* :: *countable-complete-distrib-lattice*
assumes *P*: *sup-continuous P* **and** *Q*: *sup-continuous Q*
shows *sup-continuous* (*λx*. *inf* (*P x*) (*Q x*))
unfolding *sup-continuous-def*
proof (*safe intro*!: *antisym*)
fix *M* :: *nat* ⇒ *'a* **assume** *M*: *incseq M*
have *inf* (*P* (*SUP i*. *M i*)) (*Q* (*SUP i*. *M i*)) ≤ (*SUP j i*. *inf* (*P* (*M i*)) (*Q* (*M j*)))
by (*simp add*: *sup-continuousD*[*OF P M*] *sup-continuousD*[*OF Q M*] *inf-ccSUP ccSUP-inf*)
also have ... ≤ (*SUP i*. *inf* (*P* (*M i*)) (*Q* (*M i*)))
proof (*intro ccSUP-least*)
fix *i j* **from** *M assms*[*THEN sup-continuous-mono*] **show** *inf* (*P* (*M i*)) (*Q* (*M j*)) ≤ (*SUP i*. *inf* (*P* (*M i*)) (*Q* (*M i*)))
by (*intro ccSUP-upper2*[*of - sup i j*] *inf-mono*) (*auto simp*: *mono-def*)
qed *auto*
finally show *inf* (*P* (*SUP i*. *M i*)) (*Q* (*SUP i*. *M i*)) ≤ (*SUP i*. *inf* (*P* (*M i*)) (*Q* (*M i*))) **.**

show (*SUP i*. *inf* (*P* (*M i*)) (*Q* (*M i*))) ≤ *inf* (*P* (*SUP i*. *M i*)) (*Q* (*SUP i*. *M i*))
unfolding *sup-continuousD*[*OF P M*] *sup-continuousD*[*OF Q M*] **by** (*intro ccSUP-least inf-mono ccSUP-upper*) *auto*
qed

lemma *sup-continuous-and*[*order-continuous-intros*]:
sup-continuous P $\Longrightarrow$ *sup-continuous Q* $\Longrightarrow$ *sup-continuous* (*λx*. *P x* ∧ *Q x*)
using *sup-continuous-inf*[*of P Q*] **by** *simp*

lemma *sup-continuous-or*[*order-continuous-intros*]:
  sup-continuous P ⟹ *sup-continuous Q* ⟹ *sup-continuous* (λ*x*. *P x* ∨ *Q x*)
  by (*auto simp*: *sup-continuous-def*)

lemma *sup-continuous-lfp*:
  assumes *sup-continuous F* **shows** *lfp F* = (*SUP i*. (*F* ^^ *i*) *bot*) (**is** *lfp F* = *?U*)
proof (*rule antisym*)
  note *mono* = *sup-continuous-mono*[*OF* ‹*sup-continuous F*›]
  show *?U* ≤ *lfp F*
  proof (*rule SUP-least*)
    fix *i* **show** (*F* ^^ *i*) *bot* ≤ *lfp F*
    proof (*induct i*)
      case (*Suc i*)
      have (*F* ^^ *Suc i*) *bot* = *F* ((*F* ^^ *i*) *bot*) **by** *simp*
      also have ... ≤ *F* (*lfp F*) **by** (*rule monoD*[*OF mono Suc*])
      also have ... = *lfp F* **by** (*simp add*: *lfp-unfold*[*OF mono*, *symmetric*])
      finally show *?case* **.**
    qed *simp*
  qed
  show *lfp F* ≤ *?U*
  proof (*rule lfp-lowerbound*)
    have *mono* (λ*i*::*nat*. (*F* ^^ *i*) *bot*)
    proof −
      { fix *i*::*nat* **have** (*F* ^^ *i*) *bot* ≤ (*F* ^^ (*Suc i*)) *bot*
        proof (*induct i*)
          case *0* **show** *?case* **by** *simp*
        next
          case *Suc* **thus** *?case* **using** *monoD*[*OF mono Suc*] **by** *auto*
        qed **}**
      thus *?thesis* **by** (*auto simp add*: *mono-iff-le-Suc*)
    qed
    hence *F ?U* = (*SUP i*. (*F* ^^ *Suc i*) *bot*)
      using ‹*sup-continuous F*› **by** (*simp add*: *sup-continuous-def*)
    also have ... ≤ *?U*
      by (*fast intro*: *SUP-least SUP-upper*)
    finally show *F ?U* ≤ *?U* **.**
  qed
qed

lemma *lfp-transfer-bounded*:
  assumes *P*: *P bot* ⋀*x*. *P x* ⟹ *P* (*f x*) ⋀*M*. (⋀*i*. *P* (*M i*)) ⟹ *P* (*SUP i*::*nat*. *M i*)
  assumes α: ⋀*M*. *mono M* ⟹ (⋀*i*::*nat*. *P* (*M i*)) ⟹ α (*SUP i*. *M i*) = (*SUP i*. α (*M i*))
  assumes *f*: *sup-continuous f* **and** *g*: *sup-continuous g*
  assumes [*simp*]: ⋀*x*. *P x* ⟹ *x* ≤ *lfp f* ⟹ α (*f x*) = *g* (α *x*)
  assumes *g-bound*: ⋀*x*. α *bot* ≤ *g x*
  shows α (*lfp f*) = *lfp g*

proof (*rule antisym*)
note *mono-g* = *sup-continuous-mono*[*OF g*]
note *mono-f* = *sup-continuous-mono*[*OF f*]
have *lfp-bound*: α *bot* $\leq$ *lfp g*
by (*subst lfp-unfold*[*OF mono-g*]) (*rule g-bound*)

have *P-pow*: *P* ((*f* ^^ *i*) *bot*) **for** *i*
by (*induction i*) (*auto intro*!: *P*)
have *incseq-pow*: *mono* (λ*i*. (*f* ^^ *i*) *bot*)
unfolding *mono-iff-le-Suc*
proof
fix *i* **show** (*f* ^^ *i*) *bot* $\leq$ (*f* ^^ (*Suc i*)) *bot*
proof (*induct i*)
case *Suc* **thus** *?case* **using** *monoD*[*OF sup-continuous-mono*[*OF f*] *Suc*] **by** *auto*
qed (*simp add*: *le-fun-def*)
qed
have *P-lfp*: *P* (*lfp f*)
using *P-pow* **unfolding** *sup-continuous-lfp*[*OF f*] **by** (*auto intro*!: *P*)

have *iter-le-lfp*: (*f* ^^ *n*) *bot* $\leq$ *lfp f* **for** *n*
apply (*induction n*)
apply *simp*
apply (*subst lfp-unfold*[*OF mono-f*])
apply (*auto intro*!: *monoD*[*OF mono-f*])
done

have α (*lfp f*) = (*SUP i*. α ((*f*^^*i*) *bot*))
unfolding *sup-continuous-lfp*[*OF f*] **using** *incseq-pow P-pow* **by** (*rule* α)
also have ... $\leq$ *lfp g*
proof (*rule SUP-least*)
fix *i* **show** α ((*f*^^*i*) *bot*) $\leq$ *lfp g*
proof (*induction i*)
case (*Suc n*) **then show** *?case*
by (*subst lfp-unfold*[*OF mono-g*]) (*simp add*: *monoD*[*OF mono-g*] *P-pow iter-le-lfp*)
qed (*simp add*: *lfp-bound*)
qed
finally show α (*lfp f*) $\leq$ *lfp g* **.**

show *lfp g* $\leq$ α (*lfp f*)
proof (*induction rule*: *lfp-ordinal-induct*[*OF mono-g*])
case (*1 S*) **then show** *?case*
by (*subst lfp-unfold*[*OF sup-continuous-mono*[*OF f*]])
(*simp add*: *monoD*[*OF mono-g*] *P-lfp*)
qed (*auto intro*: *Sup-least*)
qed

lemma *lfp-transfer*:

sup-continuous $\alpha \Longrightarrow$ *sup-continuous* $f \Longrightarrow$ *sup-continuous* $g \Longrightarrow$
$(\bigwedge x.\ \alpha\ bot \leq g\ x) \Longrightarrow (\bigwedge x.\ x \leq lfp\ f \Longrightarrow \alpha\ (f\ x) = g\ (\alpha\ x)) \Longrightarrow \alpha\ (lfp\ f) = lfp\ g$
by (*rule lfp-transfer-bounded*[**where** P=*top*]) (*auto dest*: *sup-continuousD*)

definition
inf-continuous :: ('*a*::*countable-complete-lattice* ⇒ '*b*::*countable-complete-lattice*) ⇒ *bool*
where
inf-continuous $F \longleftrightarrow (\forall M{::}nat \Rightarrow {'a}.\ antimono\ M \longrightarrow F\ (INF\ i.\ M\ i) = (INF\ i.\ F\ (M\ i)))$

lemma *inf-continuousD*: *inf-continuous* $F \Longrightarrow$ *antimono* $M \Longrightarrow F\ (INF\ i{::}nat.\ M\ i) = (INF\ i.\ F\ (M\ i))$
by (*auto simp*: *inf-continuous-def*)

lemma *inf-continuous-mono*:
assumes [*simp*]: *inf-continuous* F **shows** *mono* F
proof
fix $A\ B :: {'a}$ **assume** [*simp*]: $A \leq B$
have $F\ A = F\ (INF\ n{::}nat.\ if\ n = 0\ then\ B\ else\ A)$
by (*simp add*: *inf-absorb2 INF-nat-binary*)
also have $\ldots = (INF\ n{::}nat.\ if\ n = 0\ then\ F\ B\ else\ F\ A)$
by (*auto simp*: *inf-continuousD antimono-def intro*!: *INF-cong*)
finally show $F\ A \leq F\ B$
by (*simp add*: *INF-nat-binary le-iff-inf inf-commute*)
qed

lemma [*order-continuous-intros*]:
shows *inf-continuous-const*: *inf-continuous* $(\lambda x.\ c)$
and *inf-continuous-id*: *inf-continuous* $(\lambda x.\ x)$
and *inf-continuous-apply*: *inf-continuous* $(\lambda f.\ f\ x)$
and *inf-continuous-fun*: $(\bigwedge s.$ *inf-continuous* $(\lambda x.\ P\ x\ s)) \Longrightarrow$ *inf-continuous* P
and *inf-continuous-If*: *inf-continuous* $F \Longrightarrow$ *inf-continuous* $G \Longrightarrow$ *inf-continuous* $(\lambda f.\ if\ C\ then\ F\ f\ else\ G\ f)$
by (*auto simp*: *inf-continuous-def*)

lemma *inf-continuous-inf*[*order-continuous-intros*]:
inf-continuous $f \Longrightarrow$ *inf-continuous* $g \Longrightarrow$ *inf-continuous* $(\lambda x.\ inf\ (f\ x)\ (g\ x))$
by (*simp add*: *inf-continuous-def ccINF-inf-distrib*)

lemma *inf-continuous-sup*[*order-continuous-intros*]:
fixes $P\ Q :: {'a} ::$ *countable-complete-lattice* ⇒ $'b ::$ *countable-complete-distrib-lattice*
assumes P: *inf-continuous* P **and** Q: *inf-continuous* Q
shows *inf-continuous* $(\lambda x.\ sup\ (P\ x)\ (Q\ x))$
unfolding *inf-continuous-def*
proof (*safe intro*!: *antisym*)
fix $M :: nat \Rightarrow {'a}$ **assume** M: *decseq* M
show $sup\ (P\ (INF\ i.\ M\ i))\ (Q\ (INF\ i.\ M\ i)) \leq (INF\ i.\ sup\ (P\ (M\ i))\ (Q\ (M$

i)))
unfolding *inf-continuousD*[*OF P M*] *inf-continuousD*[*OF Q M*] **by** (*intro ccINF-greatest sup-mono ccINF-lower*) *auto*

have (*INF i. sup* (*P* (*M i*)) (*Q* (*M i*))) ≤ (*INF j i. sup* (*P* (*M i*)) (*Q* (*M j*)))
proof (*intro ccINF-greatest*)
fix *i j* **from** *M assms*[*THEN inf-continuous-mono*] **show** *sup* (*P* (*M i*)) (*Q* (*M j*)) ≥ (*INF i. sup* (*P* (*M i*)) (*Q* (*M i*)))
by (*intro ccINF-lower2*[*of - sup i j*] *sup-mono*) (*auto simp*: *mono-def antimono-def*)
qed *auto*
also have ... ≤ *sup* (*P* (*INF i. M i*)) (*Q* (*INF i. M i*))
by (*simp add*: *inf-continuousD*[*OF P M*] *inf-continuousD*[*OF Q M*] *ccINF-sup sup-ccINF*)
finally show *sup* (*P* (*INF i. M i*)) (*Q* (*INF i. M i*)) ≥ (*INF i. sup* (*P* (*M i*)) (*Q* (*M i*))) **.**
qed

lemma *inf-continuous-and*[*order-continuous-intros*]:
inf-continuous P ⟹ *inf-continuous Q* ⟹ *inf-continuous* (λ*x. P x* ∧ *Q x*)
using *inf-continuous-inf*[*of P Q*] **by** *simp*

lemma *inf-continuous-or*[*order-continuous-intros*]:
inf-continuous P ⟹ *inf-continuous Q* ⟹ *inf-continuous* (λ*x. P x* ∨ *Q x*)
using *inf-continuous-sup*[*of P Q*] **by** *simp*

lemma *inf-continuous-compose*:
assumes *f*: *inf-continuous f* **and** *g*: *inf-continuous g*
shows *inf-continuous* (λ*x. f* (*g x*))
unfolding *inf-continuous-def*
proof *safe*
fix *M* :: *nat* ⇒ *'c* **assume** *antimono M*
moreover then have *antimono* (λ*i. g* (*M i*))
using *inf-continuous-mono*[*OF g*] **by** (*auto simp*: *mono-def antimono-def*)
ultimately show *f* (*g* (*INFIMUM UNIV M*)) = (*INF i. f* (*g* (*M i*)))
by (*auto simp*: *inf-continuous-def g*[*THEN inf-continuousD*] *f*[*THEN inf-continuousD*])
qed

lemma *inf-continuous-gfp*:
assumes *inf-continuous F* **shows** *gfp F* = (*INF i.* (*F* ^^ *i*) *top*) (**is** *gfp F* = *?U*)
proof (*rule antisym*)
note *mono* = *inf-continuous-mono*[*OF* ‹*inf-continuous F*›]
show *gfp F* ≤ *?U*
proof (*rule INF-greatest*)
fix *i* **show** *gfp F* ≤ (*F* ^^ *i*) *top*
proof (*induct i*)
case (*Suc i*)
have *gfp F* = *F* (*gfp F*) **by** (*simp add*: *gfp-unfold*[*OF mono*, *symmetric*])
also have ... ≤ *F* ((*F* ^^ *i*) *top*) **by** (*rule monoD*[*OF mono Suc*])
also have ... = (*F* ^^ *Suc i*) *top* **by** *simp*

```
    finally show ?case .
  qed simp
 qed
 show ?U ≤ gfp F
 proof (rule gfp-upperbound)
  have ∗: antimono (λi::nat. (F ^^ i) top)
  proof −
   { fix i::nat have (F ^^ Suc i) top ≤ (F ^^ i) top
    proof (induct i)
     case 0 show ?case by simp
    next
     case Suc thus ?case using monoD[OF mono Suc] by auto
    qed }
   thus ?thesis by (auto simp add: antimono-iff-le-Suc)
  qed
  have ?U ≤ (INF i. (F ^^ Suc i) top)
   by (fast intro: INF-greatest INF-lower)
  also have ... ≤ F ?U
   by (simp add: inf-continuousD ⟨inf-continuous F⟩ ∗)
  finally show ?U ≤ F ?U .
 qed
qed

lemma gfp-transfer:
 assumes α: inf-continuous α and f: inf-continuous f and g: inf-continuous g
 assumes [simp]: α top = top ⋀x. α (f x) = g (α x)
 shows α (gfp f) = gfp g
proof −
 have α (gfp f) = (INF i. α ((f^^i) top))
  unfolding inf-continuous-gfp[OF f] by (intro f α inf-continuousD antimono-funpow
inf-continuous-mono)
 moreover have α ((f^^i) top) = (g^^i) top for i
  by (induction i; simp)
 ultimately show ?thesis
  unfolding inf-continuous-gfp[OF g] by simp
qed

lemma gfp-transfer-bounded:
 assumes P: P (f top) ⋀x. P x ⟹ P (f x) ⋀M. antimono M ⟹ (⋀i. P (M
i)) ⟹ P (INF i::nat. M i)
 assumes α: ⋀M. antimono M ⟹ (⋀i::nat. P (M i)) ⟹ α (INF i. M i) =
(INF i. α (M i))
 assumes f: inf-continuous f and g: inf-continuous g
 assumes [simp]: ⋀x. P x ⟹ α (f x) = g (α x)
 assumes g-bound: ⋀x. g x ≤ α (f top)
 shows α (gfp f) = gfp g
proof (rule antisym)
 note mono-g = inf-continuous-mono[OF g]
```

```
  have P-pow: P ((f ^^ i) (f top)) for i
    by (induction i) (auto intro!: P)

  have antimono-pow: antimono (λi. (f ^^ i) top)
    unfolding antimono-iff-le-Suc
  proof
    fix i show (f ^^ Suc i) top ≤ (f ^^ i) top
    proof (induct i)
      case Suc thus ?case using monoD[OF inf-continuous-mono[OF f] Suc] by
auto
    qed (simp add: le-fun-def)
  qed
  have antimono-pow2: antimono (λi. (f ^^ i) (f top))
  proof
    show x ≤ y ⟹ (f ^^ y) (f top) ≤ (f ^^ x) (f top) for x y
      using antimono-pow[THEN antimonoD, of Suc x Suc y]
      unfolding funpow-Suc-right by simp
  qed

  have gfp-f: gfp f = (INF i. (f ^^ i) (f top))
    unfolding inf-continuous-gfp[OF f]
  proof (rule INF-eq)
    show ∃j∈UNIV. (f ^^ j) (f top) ≤ (f ^^ i) top for i
      by (intro bexI[of - i − 1]) (auto simp: diff-Suc funpow-Suc-right simp del:
funpow.simps(2) split: nat.split)
    show ∃j∈UNIV. (f ^^ j) top ≤ (f ^^ i) (f top) for i
      by (intro bexI[of - Suc i]) (auto simp: funpow-Suc-right simp del: fun-
pow.simps(2))
  qed

  have P-lfp: P (gfp f)
    unfolding gfp-f by (auto intro!: P P-pow antimono-pow2)

  have α (gfp f) = (INF i. α ((f^^i) (f top)))
    unfolding gfp-f by (rule α) (auto intro!: P-pow antimono-pow2)
  also have ... ≥ gfp g
  proof (rule INF-greatest)
    fix i show gfp g ≤ α ((f^^i) (f top))
    proof (induction i)
      case (Suc n) then show ?case
        by (subst gfp-unfold[OF mono-g]) (simp add: monoD[OF mono-g] P-pow)
    next
      case 0
      have gfp g ≤ α (f top)
        by (subst gfp-unfold[OF mono-g]) (rule g-bound)
      then show ?case
        by simp
    qed
  qed
```

finally show *gfp g* ≤ α (*gfp f*) **.**

show α (*gfp f*) ≤ *gfp g*
proof (*induction rule*: *gfp-ordinal-induct*[*OF mono-g*])
case (*1 S*) **then show** *?case*
by (*subst gfp-unfold*[*OF inf-continuous-mono*[*OF f*]])
(*simp add*: *monoD*[*OF mono-g*] *P-lfp*)
qed (*auto intro*: *Inf-greatest*)
qed

33.1.1 Least fixed points in countable complete lattices

definition (**in** *countable-complete-lattice*) *cclfp* :: ('*a* ⇒ '*a*) ⇒ '*a*
where *cclfp f* = (*SUP i*. (*f* ^^ *i*) *bot*)

lemma *cclfp-unfold*:
assumes *sup-continuous F* **shows** *cclfp F* = *F* (*cclfp F*)
proof −
have *cclfp F* = (*SUP i*. *F* ((*F* ^^ *i*) *bot*))
unfolding *cclfp-def* **by** (*subst UNIV-nat-eq*) *auto*
also have ... = *F* (*cclfp F*)
unfolding *cclfp-def*
by (*intro sup-continuousD*[*symmetric*] *assms mono-funpow sup-continuous-mono*)
finally show *?thesis* **.**
qed

lemma *cclfp-lowerbound*: **assumes** *f*: *mono f* **and** *A*: *f A* ≤ *A* **shows** *cclfp f* ≤ *A*
unfolding *cclfp-def*
proof (*intro ccSUP-least*)
fix *i* **show** (*f* ^^ *i*) *bot* ≤ *A*
proof (*induction i*)
case (*Suc i*) **from** *monoD*[*OF f this*] *A* **show** *?case*
by *auto*
qed *simp*
qed *simp*

lemma *cclfp-transfer*:
assumes *sup-continuous* α *mono f*
assumes α *bot* = *bot* ⋀*x*. α (*f x*) = *g* (α *x*)
shows α (*cclfp f*) = *cclfp g*
proof −
have α (*cclfp f*) = (*SUP i*. α ((*f* ^^ *i*) *bot*))
unfolding *cclfp-def* **by** (*intro sup-continuousD assms mono-funpow sup-continuous-mono*)
moreover have α ((*f* ^^ *i*) *bot*) = (*g* ^^ *i*) *bot* **for** *i*
by (*induction i*) (*simp-all add*: *assms*)
ultimately show *?thesis*
by (*simp add*: *cclfp-def*)
qed

end

34 Extended natural numbers (i.e. with infinity)

theory *Extended-Nat*
imports *Main Countable Order-Continuity*
begin

class *infinity* =
  fixes *infinity* :: *'a* (∞)

context
  fixes *f* :: *nat* ⇒ *'a*::{*canonically-ordered-monoid-add*, *linorder-topology*, *complete-linorder*}
begin

lemma *sums-SUP*[*simp*, *intro*]: *f sums* (*SUP n.* $\sum$ *i*<*n. f i*)
  unfolding *sums-def* **by** (*intro LIMSEQ-SUP monoI setsum-mono2 zero-le*) *auto*

lemma *suminf-eq-SUP*: *suminf f* = (*SUP n.* $\sum$ *i*<*n. f i*)
  using *sums-SUP* **by** (*rule sums-unique*[*symmetric*])

end

34.1 Type definition

We extend the standard natural numbers by a special value indicating infinity.

typedef *enat* = *UNIV* :: *nat option set* **..**

TODO: introduce enat as coinductive datatype, enat is just *of-nat*

definition *enat* :: *nat* ⇒ *enat* **where**
  enat n = *Abs-enat* (*Some n*)

instantiation *enat* :: *infinity*
begin

definition ∞ = *Abs-enat None*
instance **..**

end

instance *enat* :: *countable*
proof
  show ∃ *to-nat*::*enat* ⇒ *nat. inj to-nat*
    by (*rule exI*[*of - to-nat* ∘ *Rep-enat*]) (*simp add*: *inj-on-def Rep-enat-inject*)
qed

old-rep-datatype *enat* ∞ :: *enat*
proof −
  fix *P i* **assume** $\bigwedge j$. *P* (*enat j*) *P* ∞
  then show *P i*
  proof *induct*
    case (*Abs-enat y*) **then show** *?case*
      by (*cases y rule*: *option.exhaust*)
        (*auto simp*: *enat-def infinity-enat-def*)
  qed
qed (*auto simp add*: *enat-def infinity-enat-def Abs-enat-inject*)

declare [[*coercion enat*::*nat*⇒*enat*]]

lemmas *enat2-cases* = *enat.exhaust*[*case-product enat.exhaust*]
lemmas *enat3-cases* = *enat.exhaust*[*case-product enat.exhaust enat.exhaust*]

lemma *not-infinity-eq* [*iff*]: ($x \neq \infty$) = ($\exists\, i.\ x$ = *enat i*)
  by (*cases x*) *auto*

lemma *not-enat-eq* [*iff*]: ($\forall\, y.\ x \neq$ *enat y*) = ($x = \infty$)
  by (*cases x*) *auto*

lemma *enat-ex-split*: ($\exists\, c$::*enat*. *P c*) $\longleftrightarrow$ *P* ∞ $\vee$ ($\exists\, c$::*nat*. *P c*)
  by (*metis enat.exhaust*)

primrec *the-enat* :: *enat* ⇒ *nat*
  where *the-enat* (*enat n*) = *n*

34.2 Constructors and numbers

instantiation *enat* :: *zero-neq-one*
begin

definition
  0 = *enat 0*

definition
  1 = *enat 1*

instance
  proof qed (*simp add*: *zero-enat-def one-enat-def*)

end

definition *eSuc* :: *enat* ⇒ *enat* **where**
  eSuc i = (*case i of enat n* ⇒ *enat* (*Suc n*) | ∞ ⇒ ∞)

lemma *enat-0* [*code-post*]: *enat 0* = *0*
  by (*simp add*: *zero-enat-def*)

lemma *enat-1* [*code-post*]: *enat 1 = 1*
by (*simp add*: *one-enat-def*)

lemma *enat-0-iff*: *enat x = 0 ⟷ x = 0 0 = enat x ⟷ x = 0*
by (*auto simp add*: *zero-enat-def*)

lemma *enat-1-iff*: *enat x = 1 ⟷ x = 1 1 = enat x ⟷ x = 1*
by (*auto simp add*: *one-enat-def*)

lemma *one-eSuc*: *1 = eSuc 0*
by (*simp add*: *zero-enat-def one-enat-def eSuc-def*)

lemma *infinity-ne-i0* [*simp*]: *(∞::enat) ≠ 0*
by (*simp add*: *zero-enat-def*)

lemma *i0-ne-infinity* [*simp*]: *0 ≠ (∞::enat)*
by (*simp add*: *zero-enat-def*)

lemma *zero-one-enat-neq*:
¬ 0 = (1::enat)
¬ 1 = (0::enat)
unfolding *zero-enat-def one-enat-def* **by** *simp-all*

lemma *infinity-ne-i1* [*simp*]: *(∞::enat) ≠ 1*
by (*simp add*: *one-enat-def*)

lemma *i1-ne-infinity* [*simp*]: *1 ≠ (∞::enat)*
by (*simp add*: *one-enat-def*)

lemma *eSuc-enat*: *eSuc (enat n) = enat (Suc n)*
by (*simp add*: *eSuc-def*)

lemma *eSuc-infinity* [*simp*]: *eSuc ∞ = ∞*
by (*simp add*: *eSuc-def*)

lemma *eSuc-ne-0* [*simp*]: *eSuc n ≠ 0*
by (*simp add*: *eSuc-def zero-enat-def split*: *enat.splits*)

lemma *zero-ne-eSuc* [*simp*]: *0 ≠ eSuc n*
by (*rule eSuc-ne-0* [*symmetric*])

lemma *eSuc-inject* [*simp*]: *eSuc m = eSuc n ⟷ m = n*
by (*simp add*: *eSuc-def split*: *enat.splits*)

lemma *eSuc-enat-iff*: *eSuc x = enat y ⟷ (∃ n. y = Suc n ∧ x = enat n)*
by (*cases y*) (*auto simp*: *enat-0 eSuc-enat*[*symmetric*])

lemma *enat-eSuc-iff*: *enat y = eSuc x ⟷ (∃ n. y = Suc n ∧ enat n = x)*

by (*cases y*) (*auto simp*: *enat-0 eSuc-enat*[*symmetric*])

34.3 Addition

instantiation *enat* :: *comm-monoid-add*
begin

definition [*nitpick-simp*]:
$m + n = (case\ m\ of\ \infty \Rightarrow \infty \mid enat\ m \Rightarrow (case\ n\ of\ \infty \Rightarrow \infty \mid enat\ n \Rightarrow enat\ (m + n)))$

lemma *plus-enat-simps* [*simp*, *code*]:
fixes q :: *enat*
shows $enat\ m + enat\ n = enat\ (m + n)$
and $\infty + q = \infty$
and $q + \infty = \infty$
by (*simp-all add*: *plus-enat-def split*: *enat.splits*)

instance
proof
fix $n\ m\ q$ :: *enat*
show $n + m + q = n + (m + q)$
by (*cases n m q rule*: *enat3-cases*) *auto*
show $n + m = m + n$
by (*cases n m rule*: *enat2-cases*) *auto*
show $0 + n = n$
by (*cases n*) (*simp-all add*: *zero-enat-def*)
qed

end

lemma *eSuc-plus-1*:
$eSuc\ n = n + 1$
by (*cases n*) (*simp-all add*: *eSuc-enat one-enat-def*)

lemma *plus-1-eSuc*:
$1 + q = eSuc\ q$
$q + 1 = eSuc\ q$
by (*simp-all add*: *eSuc-plus-1 ac-simps*)

lemma *iadd-Suc*: $eSuc\ m + n = eSuc\ (m + n)$
by (*simp-all add*: *eSuc-plus-1 ac-simps*)

lemma *iadd-Suc-right*: $m + eSuc\ n = eSuc\ (m + n)$
by (*simp only*: *add.commute*[*of m*] *iadd-Suc*)

34.4 Multiplication

instantiation *enat* :: {*comm-semiring-1*, *semiring-no-zero-divisors*}
begin

definition *times-enat-def* [*nitpick-simp*]:
m ∗ *n* = (*case m of* ∞ ⇒ *if n* = *0 then 0 else* ∞ | *enat m* ⇒
(*case n of* ∞ ⇒ *if m* = *0 then 0 else* ∞ | *enat n* ⇒ *enat* (*m* ∗ *n*)))

lemma *times-enat-simps* [*simp*, *code*]:
enat m ∗ *enat n* = *enat* (*m* ∗ *n*)
∞ ∗ ∞ = (∞::*enat*)
∞ ∗ *enat n* = (*if n* = *0 then 0 else* ∞)
enat m ∗ ∞ = (*if m* = *0 then 0 else* ∞)
unfolding *times-enat-def zero-enat-def*
by (*simp-all split*: *enat.split*)

instance
proof
fix *a b c* :: *enat*
show (*a* ∗ *b*) ∗ *c* = *a* ∗ (*b* ∗ *c*)
unfolding *times-enat-def zero-enat-def*
by (*simp split*: *enat.split*)
show *comm*: *a* ∗ *b* = *b* ∗ *a*
unfolding *times-enat-def zero-enat-def*
by (*simp split*: *enat.split*)
show *1* ∗ *a* = *a*
unfolding *times-enat-def zero-enat-def one-enat-def*
by (*simp split*: *enat.split*)
show *distr*: (*a* + *b*) ∗ *c* = *a* ∗ *c* + *b* ∗ *c*
unfolding *times-enat-def zero-enat-def*
by (*simp split*: *enat.split add*: *distrib-right*)
show *0* ∗ *a* = *0*
unfolding *times-enat-def zero-enat-def*
by (*simp split*: *enat.split*)
show *a* ∗ *0* = *0*
unfolding *times-enat-def zero-enat-def*
by (*simp split*: *enat.split*)
show *a* ∗ (*b* + *c*) = *a* ∗ *b* + *a* ∗ *c*
by (*cases a b c rule*: *enat3-cases*) (*auto simp*: *times-enat-def zero-enat-def distrib-left*)
show *a* ≠ *0* ⟹ *b* ≠ *0* ⟹ *a* ∗ *b* ≠ *0*
by (*cases a b rule*: *enat2-cases*) (*auto simp*: *times-enat-def zero-enat-def*)
qed

end

lemma *mult-eSuc*: *eSuc m* ∗ *n* = *n* + *m* ∗ *n*
unfolding *eSuc-plus-1* **by** (*simp add*: *algebra-simps*)

lemma *mult-eSuc-right*: *m* ∗ *eSuc n* = *m* + *m* ∗ *n*
unfolding *eSuc-plus-1* **by** (*simp add*: *algebra-simps*)

lemma *of-nat-eq-enat*: *of-nat n = enat n*
apply (*induct n*)
apply (*simp add*: *enat-0*)
apply (*simp add*: *plus-1-eSuc eSuc-enat*)
done

instance *enat* :: *semiring-char-0*
proof
have *inj enat* **by** (*rule injI*) *simp*
then show *inj* (λ*n*. *of-nat n* :: *enat*) **by** (*simp add*: *of-nat-eq-enat*)
qed

lemma *imult-is-infinity*: ((*a*::*enat*) ∗ *b* = ∞) = (*a* = ∞ ∧ *b* ≠ *0* ∨ *b* = ∞ ∧ *a* ≠ *0*)
by (*auto simp add*: *times-enat-def zero-enat-def split*: *enat.split*)

34.5 Numerals

lemma *numeral-eq-enat*:
numeral k = enat (*numeral k*)
using *of-nat-eq-enat* [*of numeral k*] **by** *simp*

lemma *enat-numeral* [*code-abbrev*]:
enat (*numeral k*) = *numeral k*
using *numeral-eq-enat* **..**

lemma *infinity-ne-numeral* [*simp*]: (∞::*enat*) ≠ *numeral k*
by (*simp add*: *numeral-eq-enat*)

lemma *numeral-ne-infinity* [*simp*]: *numeral k* ≠ (∞::*enat*)
by (*simp add*: *numeral-eq-enat*)

lemma *eSuc-numeral* [*simp*]: *eSuc* (*numeral k*) = *numeral* (*k* + *Num.One*)
by (*simp only*: *eSuc-plus-1 numeral-plus-one*)

34.6 Subtraction

instantiation *enat* :: *minus*
begin

definition *diff-enat-def*:
a − *b* = (*case a of* (*enat x*) ⇒ (*case b of* (*enat y*) ⇒ *enat* (*x* − *y*) | ∞ ⇒ *0*)
| ∞ ⇒ ∞)

instance ..

end

lemma *idiff-enat-enat* [*simp*, *code*]: *enat a* − *enat b* = *enat* (*a* − *b*)
by (*simp add*: *diff-enat-def*)

lemma *idiff-infinity* [*simp*, *code*]: $\infty - n = (\infty::enat)$
by (*simp add*: *diff-enat-def*)

lemma *idiff-infinity-right* [*simp*, *code*]: *enat a* $- \infty = 0$
by (*simp add*: *diff-enat-def*)

lemma *idiff-0* [*simp*]: $(0::enat) - n = 0$
by (*cases n*, *simp-all add*: *zero-enat-def*)

lemmas *idiff-enat-0* [*simp*] = *idiff-0* [*unfolded zero-enat-def*]

lemma *idiff-0-right* [*simp*]: $(n::enat) - 0 = n$
by (*cases n*) (*simp-all add*: *zero-enat-def*)

lemmas *idiff-enat-0-right* [*simp*] = *idiff-0-right* [*unfolded zero-enat-def*]

lemma *idiff-self* [*simp*]: $n \neq \infty \Longrightarrow (n::enat) - n = 0$
by (*auto simp*: *zero-enat-def*)

lemma *eSuc-minus-eSuc* [*simp*]: *eSuc n* $-$ *eSuc m* $= n - m$
by (*simp add*: *eSuc-def split*: *enat.split*)

lemma *eSuc-minus-1* [*simp*]: *eSuc n* $- 1 = n$
by (*simp add*: *one-enat-def eSuc-enat*[*symmetric*] *zero-enat-def*[*symmetric*])

34.7 Ordering

instantiation *enat* :: *linordered-ab-semigroup-add*
begin

definition [*nitpick-simp*]:
$m \leq n = (case\ n\ of\ enat\ n1 \Rightarrow (case\ m\ of\ enat\ m1 \Rightarrow m1 \leq n1 \mid \infty \Rightarrow False) \mid \infty \Rightarrow True)$

definition [*nitpick-simp*]:
$m < n = (case\ m\ of\ enat\ m1 \Rightarrow (case\ n\ of\ enat\ n1 \Rightarrow m1 < n1 \mid \infty \Rightarrow True) \mid \infty \Rightarrow False)$

lemma *enat-ord-simps* [*simp*]:
enat m $\leq$ *enat n* $\longleftrightarrow m \leq n$
enat m $<$ *enat n* $\longleftrightarrow m < n$
$q \leq (\infty::enat)$
$q < (\infty::enat) \longleftrightarrow q \neq \infty$
$(\infty::enat) \leq q \longleftrightarrow q = \infty$
$(\infty::enat) < q \longleftrightarrow False$
by (*simp-all add*: *less-eq-enat-def less-enat-def split*: *enat.splits*)

lemma *numeral-le-enat-iff* [*simp*]:

shows *numeral m* ≤ *enat n* ⟷ *numeral m* ≤ *n*
by (*auto simp*: *numeral-eq-enat*)

lemma *numeral-less-enat-iff* [*simp*]:
shows *numeral m* < *enat n* ⟷ *numeral m* < *n*
by (*auto simp*: *numeral-eq-enat*)

lemma *enat-ord-code* [*code*]:
enat m ≤ *enat n* ⟷ *m* ≤ *n*
enat m < *enat n* ⟷ *m* < *n*
q ≤ (∞::*enat*) ⟷ *True*
enat m < ∞ ⟷ *True*
∞ ≤ *enat n* ⟷ *False*
(∞::*enat*) < *q* ⟷ *False*
by *simp-all*

instance
by *standard* (*auto simp add*: *less-eq-enat-def less-enat-def plus-enat-def split*: *enat.splits*)

end

instance *enat* :: *dioid*
proof
fix *a b* :: *enat* **show** (*a* ≤ *b*) = (∃ *c*. *b* = *a* + *c*)
by (*cases a b rule*: *enat2-cases*) (*auto simp*: *le-iff-add enat-ex-split*)
qed

instance *enat* :: {*linordered-nonzero-semiring*, *strict-ordered-comm-monoid-add*}
proof
fix *a b c* :: *enat*
show *a* ≤ *b* ⟹ *0* ≤ *c* ⟹*c* ∗ *a* ≤ *c* ∗ *b*
unfolding *times-enat-def less-eq-enat-def zero-enat-def*
by (*simp split*: *enat.splits*)
show *a* < *b* ⟹ *c* < *d* ⟹ *a* + *c* < *b* + *d* **for** *a b c d* :: *enat*
by (*cases a b c d rule*: *enat2-cases*[*case-product enat2-cases*]) *auto*
qed (*simp add*: *zero-enat-def one-enat-def*)

lemma *enat-ord-number* [*simp*]:
(*numeral m* :: *enat*) ≤ *numeral n* ⟷ (*numeral m* :: *nat*) ≤ *numeral n*
(*numeral m* :: *enat*) < *numeral n* ⟷ (*numeral m* :: *nat*) < *numeral n*
by (*simp-all add*: *numeral-eq-enat*)

lemma *infinity-ileE* [*elim*!]: ∞ ≤ *enat m* ⟹ *R*
by (*simp add*: *zero-enat-def less-eq-enat-def split*: *enat.splits*)

lemma *infinity-ilessE* [*elim*!]: ∞ < *enat m* ⟹ *R*

by *simp*

lemma *eSuc-ile-mono* [*simp*]: $eSuc\ n \leq eSuc\ m \longleftrightarrow n \leq m$
by (*simp add*: *eSuc-def less-eq-enat-def split*: *enat.splits*)

lemma *eSuc-mono* [*simp*]: $eSuc\ n < eSuc\ m \longleftrightarrow n < m$
by (*simp add*: *eSuc-def less-enat-def split*: *enat.splits*)

lemma *ile-eSuc* [*simp*]: $n \leq eSuc\ n$
by (*simp add*: *eSuc-def less-eq-enat-def split*: *enat.splits*)

lemma *not-eSuc-ilei0* [*simp*]: $\neg\ eSuc\ n \leq 0$
by (*simp add*: *zero-enat-def eSuc-def less-eq-enat-def split*: *enat.splits*)

lemma *i0-iless-eSuc* [*simp*]: $0 < eSuc\ n$
by (*simp add*: *zero-enat-def eSuc-def less-enat-def split*: *enat.splits*)

lemma *iless-eSuc0*[*simp*]: $(n < eSuc\ 0) = (n = 0)$
by (*simp add*: *zero-enat-def eSuc-def less-enat-def split*: *enat.split*)

lemma *ileI1*: $m < n \Longrightarrow eSuc\ m \leq n$
by (*simp add*: *eSuc-def less-eq-enat-def less-enat-def split*: *enat.splits*)

lemma *Suc-ile-eq*: $enat\ (Suc\ m) \leq n \longleftrightarrow enat\ m < n$
by (*cases n*) *auto*

lemma *iless-Suc-eq* [*simp*]: $enat\ m < eSuc\ n \longleftrightarrow enat\ m \leq n$
by (*auto simp add*: *eSuc-def less-enat-def split*: *enat.splits*)

lemma *imult-infinity*: $(0::enat) < n \Longrightarrow \infty * n = \infty$
by (*simp add*: *zero-enat-def less-enat-def split*: *enat.splits*)

lemma *imult-infinity-right*: $(0::enat) < n \Longrightarrow n * \infty = \infty$
by (*simp add*: *zero-enat-def less-enat-def split*: *enat.splits*)

lemma *enat-0-less-mult-iff*: $(0 < (m::enat) * n) = (0 < m \wedge 0 < n)$
by (*simp only*: *zero-less-iff-neq-zero mult-eq-0-iff*, *simp*)

lemma *mono-eSuc*: *mono eSuc*
by (*simp add*: *mono-def*)

lemma *min-enat-simps* [*simp*]:
$min\ (enat\ m)\ (enat\ n) = enat\ (min\ m\ n)$
$min\ q\ 0 = 0$
$min\ 0\ q = 0$
$min\ q\ (\infty::enat) = q$
$min\ (\infty::enat)\ q = q$
by (*auto simp add*: *min-def*)

lemma *max-enat-simps* [*simp*]:
max (*enat m*) (*enat n*) = *enat* (*max m n*)
max q 0 = *q*
max 0 q = *q*
max q ∞ = (∞::*enat*)
max ∞ *q* = (∞::*enat*)
by (*simp-all add*: *max-def*)

lemma *enat-ile*: *n* ≤ *enat m* ⟹ ∃ *k*. *n* = *enat k*
by (*cases n*) *simp-all*

lemma *enat-iless*: *n* < *enat m* ⟹ ∃ *k*. *n* = *enat k*
by (*cases n*) *simp-all*

lemma *iadd-le-enat-iff*:
x + *y* ≤ *enat n* ⟷ (∃ *y' x'*. *x* = *enat x'* ∧ *y* = *enat y'* ∧ *x'* + *y'* ≤ *n*)
by(*cases x y rule*: *enat.exhaust*[*case-product enat.exhaust*]) *simp-all*

lemma *chain-incr*: ∀ *i*. ∃ *j*. *Y i* < *Y j* ⟹ ∃ *j*. *enat k* < *Y j*
apply (*induct-tac k*)
apply (*simp* (*no-asm*) *only*: *enat-0*)
apply (*fast intro*: *le-less-trans* [*OF zero-le*])
apply (*erule exE*)
apply (*drule spec*)
apply (*erule exE*)
apply (*drule ileI1*)
apply (*rule eSuc-enat* [*THEN subst*])
apply (*rule exI*)
apply (*erule* (*1*) *le-less-trans*)
done

lemma *eSuc-max*: *eSuc* (*max x y*) = *max* (*eSuc x*) (*eSuc y*)
by (*simp add*: *eSuc-def split*: *enat.split*)

lemma *eSuc-Max*:
assumes *finite A A* ≠ {}
shows *eSuc* (*Max A*) = *Max* (*eSuc* ‘ *A*)
using *assms* **proof** *induction*
case (*insert x A*)
thus *?case* **by**(*cases A* = {})(*simp-all add*: *eSuc-max*)
qed *simp*

instantiation *enat* :: {*order-bot*, *order-top*}
begin

definition *bot-enat* :: *enat* **where** *bot-enat* = *0*
definition *top-enat* :: *enat* **where** *top-enat* = ∞

instance

by *standard* (*simp-all add*: *bot-enat-def top-enat-def*)

end

lemma *finite-enat-bounded*:
assumes *le-fin*: $\bigwedge y.\ y \in A \Longrightarrow y \leq \mathit{enat}\ n$
shows *finite A*
proof (*rule finite-subset*)
show *finite* (*enat* ' {..*n*}) **by** *blast*
have $A \subseteq \{..\mathit{enat}\ n\}$ **using** *le-fin* **by** *fastforce*
also have $\ldots \subseteq \mathit{enat}$ ' {..*n*}
apply (*rule subsetI*)
subgoal for *x* **by** (*cases x*) *auto*
done
finally show $A \subseteq \mathit{enat}$ ' {..*n*} **.**
qed

34.8 Cancellation simprocs

lemma *enat-add-left-cancel*: $a + b = a + c \longleftrightarrow a = (\infty::\mathit{enat}) \vee b = c$
unfolding *plus-enat-def* **by** (*simp split*: *enat.split*)

lemma *enat-add-left-cancel-le*: $a + b \leq a + c \longleftrightarrow a = (\infty::\mathit{enat}) \vee b \leq c$
unfolding *plus-enat-def* **by** (*simp split*: *enat.split*)

lemma *enat-add-left-cancel-less*: $a + b < a + c \longleftrightarrow a \neq (\infty::\mathit{enat}) \wedge b < c$
unfolding *plus-enat-def* **by** (*simp split*: *enat.split*)

ML ‹
```
structure Cancel-Enat-Common =
struct
  (* copied from src/HOL/Tools/nat-numeral-simprocs.ML *)
  fun find-first-t - - []         = raise TERM(find-first-t, [])
    | find-first-t past u (t::terms) =
          if u aconv t then (rev past @ terms)
          else find-first-t (t::past) u terms

  fun dest-summing (Const (@{const-name Groups.plus}, -) $ t $ u, ts) =
        dest-summing (t, dest-summing (u, ts))
    | dest-summing (t, ts) = t :: ts

  val mk-sum = Arith-Data.long-mk-sum
  fun dest-sum t = dest-summing (t, [])
  val find-first = find-first-t []
  val trans-tac = Numeral-Simprocs.trans-tac
  val norm-ss =
    simpset-of (put-simpset HOL-basic-ss @{context}
      addsimps @{thms ac-simps add-0-left add-0-right})
  fun norm-tac ctxt = ALLGOALS (simp-tac (put-simpset norm-ss ctxt))
```

```
  fun simplify-meta-eq ctxt cancel-th th =
    Arith-Data.simplify-meta-eq [] ctxt
      ([th, cancel-th] MRS trans)
  fun mk-eq (a, b) = HOLogic.mk-Trueprop (HOLogic.mk-eq (a, b))
end

structure Eq-Enat-Cancel = ExtractCommonTermFun
(open Cancel-Enat-Common
  val mk-bal = HOLogic.mk-eq
  val dest-bal = HOLogic.dest-bin @{const-name HOL.eq} @{typ enat}
  fun simp-conv - - = SOME @{thm enat-add-left-cancel}
)

structure Le-Enat-Cancel = ExtractCommonTermFun
(open Cancel-Enat-Common
  val mk-bal = HOLogic.mk-binrel @{const-name Orderings.less-eq}
  val dest-bal = HOLogic.dest-bin @{const-name Orderings.less-eq} @{typ enat}
  fun simp-conv - - = SOME @{thm enat-add-left-cancel-le}
)

structure Less-Enat-Cancel = ExtractCommonTermFun
(open Cancel-Enat-Common
  val mk-bal = HOLogic.mk-binrel @{const-name Orderings.less}
  val dest-bal = HOLogic.dest-bin @{const-name Orderings.less} @{typ enat}
  fun simp-conv - - = SOME @{thm enat-add-left-cancel-less}
)
›

simproc-setup enat-eq-cancel
  ((l::enat) + m = n | (l::enat) = m + n) =
  ‹fn phi => fn ctxt => fn ct => Eq-Enat-Cancel.proc ctxt (Thm.term-of ct)›

simproc-setup enat-le-cancel
  ((l::enat) + m ≤ n | (l::enat) ≤ m + n) =
  ‹fn phi => fn ctxt => fn ct => Le-Enat-Cancel.proc ctxt (Thm.term-of ct)›

simproc-setup enat-less-cancel
  ((l::enat) + m < n | (l::enat) < m + n) =
  ‹fn phi => fn ctxt => fn ct => Less-Enat-Cancel.proc ctxt (Thm.term-of ct)›
```

TODO: add regression tests for these simprocs

TODO: add simprocs for combining and cancelling numerals

34.9 Well-ordering

```
lemma less-enatE:
  [| n < enat m; !!k. n = enat k ==> k < m ==> P |] ==> P
by (induct n) auto
```

lemma *less-infinityE*:
[| $n < \infty$; !!k. n = *enat* k ==> P |] ==> P
by (*induct n*) *auto*

lemma *enat-less-induct*:
assumes *prem*: !!n. $\forall m$::*enat*. $m < n$ --> $P\ m$ ==> $P\ n$ **shows** $P\ n$
proof −
have *P-enat*: !!k. P (*enat* k)
apply (*rule nat-less-induct*)
apply (*rule prem*, *clarify*)
apply (*erule less-enatE*, *simp*)
done
show *?thesis*
proof (*induct n*)
fix *nat*
show P (*enat nat*) **by** (*rule P-enat*)
next
show $P\ \infty$
apply (*rule prem*, *clarify*)
apply (*erule less-infinityE*)
apply (*simp add*: *P-enat*)
done
qed
qed

instance *enat* :: *wellorder*
proof
fix P **and** n
assume *hyp*: ($\bigwedge n$::*enat*. ($\bigwedge m$::*enat*. $m < n \Longrightarrow P\ m$) $\Longrightarrow P\ n$)
show $P\ n$ **by** (*blast intro*: *enat-less-induct hyp*)
qed

34.10 Complete Lattice

instantiation *enat* :: *complete-lattice*
begin

definition *inf-enat* :: *enat* ⇒ *enat* ⇒ *enat* **where**
inf-enat = *min*

definition *sup-enat* :: *enat* ⇒ *enat* ⇒ *enat* **where**
sup-enat = *max*

definition *Inf-enat* :: *enat set* ⇒ *enat* **where**
Inf-enat A = (*if A* = {} *then* ∞ *else* (*LEAST x*. $x \in A$))

definition *Sup-enat* :: *enat set* ⇒ *enat* **where**
Sup-enat A = (*if A* = {} *then 0 else if finite A then Max A else* ∞)
instance

proof
fix x :: *enat* **and** A :: *enat set*
{ assume $x \in A$ **then show** *Inf* $A \leq x$
unfolding *Inf-enat-def* **by** (*auto intro*: *Least-le*) **}**
{ assume $\bigwedge y.\ y \in A \Longrightarrow x \leq y$ **then show** $x \leq$ *Inf* A
unfolding *Inf-enat-def*
by (*cases* $A = \{\}$) (*auto intro*: *LeastI2-ex*) **}**
{ assume $x \in A$ **then show** $x \leq$ *Sup* A
unfolding *Sup-enat-def* **by** (*cases finite* A) *auto* **}**
{ assume $\bigwedge y.\ y \in A \Longrightarrow y \leq x$ **then show** *Sup* $A \leq x$
unfolding *Sup-enat-def* **using** *finite-enat-bounded* **by** *auto* **}**
qed (*simp-all add*:
inf-enat-def sup-enat-def bot-enat-def top-enat-def Inf-enat-def Sup-enat-def)
end

instance *enat* :: *complete-linorder* **..**

lemma *eSuc-Sup*: $A \neq \{\} \Longrightarrow$ *eSuc* (*Sup* A) = *Sup* (*eSuc* ‘ A)
by(*auto simp add*: *Sup-enat-def eSuc-Max inj-on-def dest*: *finite-imageD*)

lemma *sup-continuous-eSuc*: *sup-continuous* $f \Longrightarrow$ *sup-continuous* ($\lambda x.$ *eSuc* (f x))
using *eSuc-Sup*[*of* - ‘ *UNIV*] **by** (*auto simp*: *sup-continuous-def*)

34.11 Traditional theorem names

lemmas *enat-defs* = *zero-enat-def one-enat-def eSuc-def*
plus-enat-def less-eq-enat-def less-enat-def

lemma *iadd-is-0*: (m + n = (0::*enat*)) = (m = 0 ∧ n = 0)
by (*rule add-eq-0-iff-both-eq-0*)

lemma *i0-lb* : (0::*enat*) ≤ n
by (*rule zero-le*)

lemma *ile0-eq*: n ≤ (0::*enat*) ⟷ n = 0
by (*rule le-zero-eq*)

lemma *not-iless0*: ¬ n < (0::*enat*)
by (*rule not-less-zero*)

lemma *i0-less*[*simp*]: (0::*enat*) < n ⟷ $n \neq 0$
by (*rule zero-less-iff-neq-zero*)

lemma *imult-is-0*: ((m::*enat*) * n = 0) = (m = 0 ∨ n = 0)
by (*rule mult-eq-0-iff*)

end

35 Liminf and Limsup on conditionally complete lattices

theory *Liminf-Limsup*
imports *Complex-Main*
begin

lemma (**in** *conditionally-complete-linorder*) *le-cSup-iff*:
  assumes $A \neq \{\}$ *bdd-above A*
  shows $x \leq Sup\ A \longleftrightarrow (\forall\, y{<}x.\ \exists\, a{\in}A.\ y < a)$
proof *safe*
  fix *y* **assume** $x \leq Sup\ A\ y < x$
  then have $y < Sup\ A$ **by** *auto*
  then show $\exists\, a{\in}A.\ y < a$
    unfolding *less-cSup-iff*[*OF assms*] **.**
qed (*auto elim*!: *allE*[*of* - *Sup A*] *simp add*: *not-le*[*symmetric*] *cSup-upper assms*)

lemma (**in** *conditionally-complete-linorder*) *le-cSUP-iff*:
  $A \neq \{\} \Longrightarrow$ *bdd-above* (*f*`*A*) $\Longrightarrow x \leq SUPREMUM\ A\ f \longleftrightarrow (\forall\, y{<}x.\ \exists\, i{\in}A.\ y < f\ i)$
  using *le-cSup-iff* [*of f* ` *A*] **by** *simp*

lemma *le-cSup-iff-less*:
  fixes $x :: 'a :: \{$*conditionally-complete-linorder*, *dense-linorder*$\}$
  shows $A \neq \{\} \Longrightarrow$ *bdd-above* (*f*`*A*) $\Longrightarrow x \leq (SUP\ i{:}A.\ f\ i) \longleftrightarrow (\forall\, y{<}x.\ \exists\, i{\in}A.\ y \leq f\ i)$
  by (*simp add*: *le-cSUP-iff*)
    (*blast intro*: *less-imp-le less-trans less-le-trans dest*: *dense*)

lemma *le-Sup-iff-less*:
  fixes $x :: 'a :: \{$*complete-linorder*, *dense-linorder*$\}$
  shows $x \leq (SUP\ i{:}A.\ f\ i) \longleftrightarrow (\forall\, y{<}x.\ \exists\, i{\in}A.\ y \leq f\ i)$ (**is** *?lhs* = *?rhs*)
  unfolding *le-SUP-iff*
  by (*blast intro*: *less-imp-le less-trans less-le-trans dest*: *dense*)

lemma (**in** *conditionally-complete-linorder*) *cInf-le-iff*:
  assumes $A \neq \{\}$ *bdd-below A*
  shows $Inf\ A \leq x \longleftrightarrow (\forall\, y{>}x.\ \exists\, a{\in}A.\ y > a)$
proof *safe*
  fix *y* **assume** $x \geq Inf\ A\ y > x$
  then have $y > Inf\ A$ **by** *auto*
  then show $\exists\, a{\in}A.\ y > a$
    unfolding *cInf-less-iff*[*OF assms*] **.**
qed (*auto elim*!: *allE*[*of* - *Inf A*] *simp add*: *not-le*[*symmetric*] *cInf-lower assms*)

lemma (**in** *conditionally-complete-linorder*) *cINF-le-iff*:
  $A \neq \{\} \Longrightarrow$ *bdd-below* (*f*`*A*) $\Longrightarrow INFIMUM\ A\ f \leq x \longleftrightarrow (\forall\, y{>}x.\ \exists\, i{\in}A.\ y > f\ i)$
  using *cInf-le-iff* [*of f* ` *A*] **by** *simp*

lemma *cInf-le-iff-less*:
fixes $x :: 'a :: \{conditionally\text{-}complete\text{-}linorder, dense\text{-}linorder\}$
shows $A \neq \{\} \Longrightarrow bdd\text{-}below\ (f`A) \Longrightarrow (INF\ i{:}A.\ f\ i) \leq x \longleftrightarrow (\forall\, y{>}x.\ \exists\, i{\in}A.\ f\ i \leq y)$
by (*simp add*: *cINF-le-iff*)
(*blast intro*: *less-imp-le less-trans le-less-trans dest*: *dense*)

lemma *Inf-le-iff-less*:
fixes $x :: 'a :: \{complete\text{-}linorder, dense\text{-}linorder\}$
shows $(INF\ i{:}A.\ f\ i) \leq x \longleftrightarrow (\forall\, y{>}x.\ \exists\, i{\in}A.\ f\ i \leq y)$
unfolding *INF-le-iff*
by (*blast intro*: *less-imp-le less-trans le-less-trans dest*: *dense*)

lemma *SUP-pair*:
fixes $f :: - \Rightarrow - \Rightarrow - :: complete\text{-}lattice$
shows $(SUP\ i : A.\ SUP\ j : B.\ f\ i\ j) = (SUP\ p : A \times B.\ f\ (fst\ p)\ (snd\ p))$
by (*rule antisym*) (*auto intro*!: *SUP-least SUP-upper2*)

lemma *INF-pair*:
fixes $f :: - \Rightarrow - \Rightarrow - :: complete\text{-}lattice$
shows $(INF\ i : A.\ INF\ j : B.\ f\ i\ j) = (INF\ p : A \times B.\ f\ (fst\ p)\ (snd\ p))$
by (*rule antisym*) (*auto intro*!: *INF-greatest INF-lower2*)

35.0.1 *Liminf* and *Limsup*

definition $Liminf :: 'a\ filter \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'b :: complete\text{-}lattice$ **where**
$Liminf\ F\ f = (SUP\ P{:}\{P.\ eventually\ P\ F\}.\ INF\ x{:}\{x.\ P\ x\}.\ f\ x)$

definition $Limsup :: 'a\ filter \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'b :: complete\text{-}lattice$ **where**
$Limsup\ F\ f = (INF\ P{:}\{P.\ eventually\ P\ F\}.\ SUP\ x{:}\{x.\ P\ x\}.\ f\ x)$

abbreviation $liminf \equiv Liminf\ sequentially$

abbreviation $limsup \equiv Limsup\ sequentially$

lemma *Liminf-eqI*:
$(\bigwedge P.\ eventually\ P\ F \Longrightarrow INFIMUM\ (Collect\ P)\ f \leq x) \Longrightarrow$
$(\bigwedge y.\ (\bigwedge P.\ eventually\ P\ F \Longrightarrow INFIMUM\ (Collect\ P)\ f \leq y) \Longrightarrow x \leq y) \Longrightarrow Liminf\ F\ f = x$
unfolding *Liminf-def* **by** (*auto intro*!: *SUP-eqI*)

lemma *Limsup-eqI*:
$(\bigwedge P.\ eventually\ P\ F \Longrightarrow x \leq SUPREMUM\ (Collect\ P)\ f) \Longrightarrow$
$(\bigwedge y.\ (\bigwedge P.\ eventually\ P\ F \Longrightarrow y \leq SUPREMUM\ (Collect\ P)\ f) \Longrightarrow y \leq x) \Longrightarrow Limsup\ F\ f = x$
unfolding *Limsup-def* **by** (*auto intro*!: *INF-eqI*)

lemma *liminf-SUP-INF*: $liminf\ f = (SUP\ n.\ INF\ m{:}\{n..\}.\ f\ m)$

unfolding *Liminf-def eventually-sequentially*
by (*rule SUP-eq*) (*auto simp*: *atLeast-def intro*!: *INF-mono*)

lemma *limsup-INF-SUP*: *limsup f* = (*INF n*. *SUP m*:{*n*..}. *f m*)
unfolding *Limsup-def eventually-sequentially*
by (*rule INF-eq*) (*auto simp*: *atLeast-def intro*!: *SUP-mono*)

lemma *Limsup-const*:
assumes *ntriv*: ¬ *trivial-limit F*
shows *Limsup F* (λ*x*. *c*) = *c*
proof −
have ∗: ⋀*P*. *Ex P* ⟷ *P* ≠ (λ*x*. *False*) **by** *auto*
have ⋀*P*. *eventually P F* ⟹ (*SUP x* : {*x*. *P x*}. *c*) = *c*
using *ntriv* **by** (*intro SUP-const*) (*auto simp*: *eventually-False* ∗)
then show *?thesis*
unfolding *Limsup-def* **using** *eventually-True*
by (*subst INF-cong*[**where** *D*=λ*x*. *c*])
(*auto intro*!: *INF-const simp del*: *eventually-True*)
qed

lemma *Liminf-const*:
assumes *ntriv*: ¬ *trivial-limit F*
shows *Liminf F* (λ*x*. *c*) = *c*
proof −
have ∗: ⋀*P*. *Ex P* ⟷ *P* ≠ (λ*x*. *False*) **by** *auto*
have ⋀*P*. *eventually P F* ⟹ (*INF x* : {*x*. *P x*}. *c*) = *c*
using *ntriv* **by** (*intro INF-const*) (*auto simp*: *eventually-False* ∗)
then show *?thesis*
unfolding *Liminf-def* **using** *eventually-True*
by (*subst SUP-cong*[**where** *D*=λ*x*. *c*])
(*auto intro*!: *SUP-const simp del*: *eventually-True*)
qed

lemma *Liminf-mono*:
assumes *ev*: *eventually* (λ*x*. *f x* ≤ *g x*) *F*
shows *Liminf F f* ≤ *Liminf F g*
unfolding *Liminf-def*
proof (*safe intro*!: *SUP-mono*)
fix *P* **assume** *eventually P F*
with *ev* **have** *eventually* (λ*x*. *f x* ≤ *g x* ∧ *P x*) *F* (**is** *eventually ?Q F*) **by** (*rule eventually-conj*)
then show ∃ *Q*∈{*P*. *eventually P F*}. *INFIMUM* (*Collect P*) *f* ≤ *INFIMUM* (*Collect Q*) *g*
by (*intro bexI*[*of* - *?Q*]) (*auto intro*!: *INF-mono*)
qed

lemma *Liminf-eq*:
assumes *eventually* (λ*x*. *f x* = *g x*) *F*
shows *Liminf F f* = *Liminf F g*

by (*intro antisym Liminf-mono eventually-mono*[*OF assms*]) *auto*

lemma *Limsup-mono*:
assumes *ev*: *eventually* (λ*x*. *f x* ≤ *g x*) *F*
shows *Limsup F f* ≤ *Limsup F g*
unfolding *Limsup-def*
proof (*safe intro*!: *INF-mono*)
fix *P* **assume** *eventually P F*
with *ev* **have** *eventually* (λ*x*. *f x* ≤ *g x* ∧ *P x*) *F* (**is** *eventually ?Q F*) **by** (*rule eventually-conj*)
then show ∃ *Q*∈{*P*. *eventually P F*}. *SUPREMUM* (*Collect Q*) *f* ≤ *SUPREMUM* (*Collect P*) *g*
by (*intro bexI*[*of - ?Q*]) (*auto intro*!: *SUP-mono*)
qed

lemma *Limsup-eq*:
assumes *eventually* (λ*x*. *f x* = *g x*) *net*
shows *Limsup net f* = *Limsup net g*
by (*intro antisym Limsup-mono eventually-mono*[*OF assms*]) *auto*

lemma *Liminf-le-Limsup*:
assumes *ntriv*: ¬ *trivial-limit F*
shows *Liminf F f* ≤ *Limsup F f*
unfolding *Limsup-def Liminf-def*
apply (*rule SUP-least*)
apply (*rule INF-greatest*)
proof *safe*
fix *P Q* **assume** *eventually P F eventually Q F*
then have *eventually* (λ*x*. *P x* ∧ *Q x*) *F* (**is** *eventually ?C F*) **by** (*rule eventually-conj*)
then have *not-False*: (λ*x*. *P x* ∧ *Q x*) ≠ (λ*x*. *False*)
using *ntriv* **by** (*auto simp add*: *eventually-False*)
have *INFIMUM* (*Collect P*) *f* ≤ *INFIMUM* (*Collect ?C*) *f*
by (*rule INF-mono*) *auto*
also have ... ≤ *SUPREMUM* (*Collect ?C*) *f*
using *not-False* **by** (*intro INF-le-SUP*) *auto*
also have ... ≤ *SUPREMUM* (*Collect Q*) *f*
by (*rule SUP-mono*) *auto*
finally show *INFIMUM* (*Collect P*) *f* ≤ *SUPREMUM* (*Collect Q*) *f* **.**
qed

lemma *Liminf-bounded*:
assumes *ntriv*: ¬ *trivial-limit F*
assumes *le*: *eventually* (λ*n*. *C* ≤ *X n*) *F*
shows *C* ≤ *Liminf F X*
using *Liminf-mono*[*OF le*] *Liminf-const*[*OF ntriv*, *of C*] **by** *simp*

lemma *Limsup-bounded*:
assumes *ntriv*: ¬ *trivial-limit F*
assumes *le*: *eventually* (λ*n*. *X n* ≤ *C*) *F*

shows *Limsup F X* ≤ *C*
using *Limsup-mono*[*OF le*] *Limsup-const*[*OF ntriv*, *of C*] **by** *simp*

lemma *le-Limsup*:
assumes *F*: *F* ≠ *bot* **and** *x*: ∀ *x in F*. *l* ≤ *f x*
shows *l* ≤ *Limsup F f*
proof −
have *l* = *Limsup F* (λ*x*. *l*)
using *F* **by** (*simp add*: *Limsup-const*)
also have ... ≤ *Limsup F f*
by (*intro Limsup-mono x*)
finally show *?thesis* **.**
qed

lemma *le-Liminf-iff*:
fixes *X* :: - ⇒ - :: *complete-linorder*
shows *C* ≤ *Liminf F X* ⟷ (∀ *y*<*C*. *eventually* (λ*x*. *y* < *X x*) *F*)
proof −
have *eventually* (λ*x*. *y* < *X x*) *F*
if *eventually P F y* < *INFIMUM* (*Collect P*) *X* **for** *y P*
using *that* **by** (*auto elim*!: *eventually-mono dest*: *less-INF-D*)
moreover
have ∃ *P*. *eventually P F* ∧ *y* < *INFIMUM* (*Collect P*) *X*
if *y* < *C* **and** *y*: ∀ *y*<*C*. *eventually* (λ*x*. *y* < *X x*) *F* **for** *y P*
proof (*cases* ∃ *z*. *y* < *z* ∧ *z* < *C*)
case *True*
then obtain *z* **where** *z*: *y* < *z* ∧ *z* < *C* **..**
moreover from *z* **have** *z* ≤ *INFIMUM* {*x*. *z* < *X x*} *X*
by (*auto intro*!: *INF-greatest*)
ultimately show *?thesis*
using *y* **by** (*intro exI*[*of* - λ*x*. *z* < *X x*]) *auto*
next
case *False*
then have *C* ≤ *INFIMUM* {*x*. *y* < *X x*} *X*
by (*intro INF-greatest*) *auto*
with ‹*y* < *C*› **show** *?thesis*
using *y* **by** (*intro exI*[*of* - λ*x*. *y* < *X x*]) *auto*
qed
ultimately show *?thesis*
unfolding *Liminf-def le-SUP-iff* **by** *auto*
qed

lemma *Limsup-le-iff*:
fixes *X* :: - ⇒ - :: *complete-linorder*
shows *C* ≥ *Limsup F X* ⟷ (∀ *y*>*C*. *eventually* (λ*x*. *y* > *X x*) *F*)
proof −
{ fix *y P* **assume** *eventually P F y* > *SUPREMUM* (*Collect P*) *X*
then have *eventually* (λ*x*. *y* > *X x*) *F*
by (*auto elim*!: *eventually-mono dest*: *SUP-lessD*) **}**

```
  moreover
  { fix y P assume y > C and y: ∀ y>C. eventually (λx. y > X x) F
    have ∃ P. eventually P F ∧ y > SUPREMUM (Collect P) X
    proof (cases ∃ z. C < z ∧ z < y)
      case True
      then obtain z where z: C < z ∧ z < y ..
      moreover from z have z ≥ SUPREMUM {x. z > X x} X
        by (auto intro!: SUP-least)
      ultimately show ?thesis
        using y by (intro exI[of - λx. z > X x]) auto
    next
      case False
      then have C ≥ SUPREMUM {x. y > X x} X
        by (intro SUP-least) (auto simp: not-less)
      with ‹y > C› show ?thesis
        using y by (intro exI[of - λx. y > X x]) auto
    qed }
  ultimately show ?thesis
    unfolding Limsup-def INF-le-iff by auto
qed

lemma less-LiminfD:
  y < Liminf F (f :: - ⇒ 'a :: complete-linorder) ⟹ eventually (λx. f x > y) F
  using le-Liminf-iff[of Liminf F f F f] by simp

lemma Limsup-lessD:
  y > Limsup F (f :: - ⇒ 'a :: complete-linorder) ⟹ eventually (λx. f x < y) F
  using Limsup-le-iff[of F f Limsup F f] by simp

lemma lim-imp-Liminf:
  fixes f :: 'a ⇒ - :: {complete-linorder,linorder-topology}
  assumes ntriv: ¬ trivial-limit F
  assumes lim: (f ⟶ f0) F
  shows Liminf F f = f0
proof (intro Liminf-eqI)
  fix P assume P: eventually P F
  then have eventually (λx. INFIMUM (Collect P) f ≤ f x) F
    by eventually-elim (auto intro!: INF-lower)
  then show INFIMUM (Collect P) f ≤ f0
    by (rule tendsto-le[OF ntriv lim tendsto-const])
next
  fix y assume upper: ⋀P. eventually P F ⟹ INFIMUM (Collect P) f ≤ y
  show f0 ≤ y
  proof cases
    assume ∃ z. y < z ∧ z < f0
    then obtain z where y < z ∧ z < f0 ..
    moreover have z ≤ INFIMUM {x. z < f x} f
      by (rule INF-greatest) simp
    ultimately show ?thesis
```

```
    using lim[THEN topological-tendstoD, THEN upper, of {z <..}] by auto
  next
    assume discrete: ¬ (∃ z. y < z ∧ z < f0)
    show ?thesis
    proof (rule classical)
      assume ¬ f0 ≤ y
      then have eventually (λx. y < f x) F
        using lim[THEN topological-tendstoD, of {y <..}] by auto
      then have eventually (λx. f0 ≤ f x) F
        using discrete by (auto elim!: eventually-mono)
      then have INFIMUM {x. f0 ≤ f x} f ≤ y
        by (rule upper)
      moreover have f0 ≤ INFIMUM {x. f0 ≤ f x} f
        by (intro INF-greatest) simp
      ultimately show f0 ≤ y by simp
    qed
  qed
qed

lemma lim-imp-Limsup:
  fixes f :: 'a ⇒ - :: {complete-linorder,linorder-topology}
  assumes ntriv: ¬ trivial-limit F
  assumes lim: (f ⟶ f0) F
  shows Limsup F f = f0
proof (intro Limsup-eqI)
  fix P assume P: eventually P F
  then have eventually (λx. f x ≤ SUPREMUM (Collect P) f) F
    by eventually-elim (auto intro!: SUP-upper)
  then show f0 ≤ SUPREMUM (Collect P) f
    by (rule tendsto-le[OF ntriv tendsto-const lim])
next
  fix y assume lower: ⋀P. eventually P F ⟹ y ≤ SUPREMUM (Collect P) f
  show y ≤ f0
  proof (cases ∃ z. f0 < z ∧ z < y)
    case True
    then obtain z where f0 < z ∧ z < y ..
    moreover have SUPREMUM {x. f x < z} f ≤ z
      by (rule SUP-least) simp
    ultimately show ?thesis
      using lim[THEN topological-tendstoD, THEN lower, of {..< z}] by auto
  next
    case False
    show ?thesis
    proof (rule classical)
      assume ¬ y ≤ f0
      then have eventually (λx. f x < y) F
        using lim[THEN topological-tendstoD, of {..< y}] by auto
      then have eventually (λx. f x ≤ f0) F
        using False by (auto elim!: eventually-mono simp: not-less)
```

```
      then have y ≤ SUPREMUM {x. f x ≤ f0} f
        by (rule lower)
      moreover have SUPREMUM {x. f x ≤ f0} f ≤ f0
        by (intro SUP-least) simp
      ultimately show y ≤ f0 by simp
    qed
  qed
qed

lemma Liminf-eq-Limsup:
  fixes f0 :: 'a :: {complete-linorder,linorder-topology}
  assumes ntriv: ¬ trivial-limit F
    and lim: Liminf F f = f0 Limsup F f = f0
  shows (f ⟶ f0) F
proof (rule order-tendstoI)
  fix a assume f0 < a
  with assms have Limsup F f < a by simp
  then obtain P where eventually P F SUPREMUM (Collect P) f < a
    unfolding Limsup-def INF-less-iff by auto
  then show eventually (λx. f x < a) F
    by (auto elim!: eventually-mono dest: SUP-lessD)
next
  fix a assume a < f0
  with assms have a < Liminf F f by simp
  then obtain P where eventually P F a < INFIMUM (Collect P) f
    unfolding Liminf-def less-SUP-iff by auto
  then show eventually (λx. a < f x) F
    by (auto elim!: eventually-mono dest: less-INF-D)
qed

lemma tendsto-iff-Liminf-eq-Limsup:
  fixes f0 :: 'a :: {complete-linorder,linorder-topology}
  shows ¬ trivial-limit F ⟹ (f ⟶ f0) F ⟷ (Liminf F f = f0 ∧ Limsup F
f = f0)
  by (metis Liminf-eq-Limsup lim-imp-Limsup lim-imp-Liminf)

lemma liminf-subseq-mono:
  fixes X :: nat ⇒ 'a :: complete-linorder
  assumes subseq r
  shows liminf X ≤ liminf (X ∘ r)
proof−
  have ⋀n. (INF m:{n..}. X m) ≤ (INF m:{n..}. (X ∘ r) m)
  proof (safe intro!: INF-mono)
    fix n m :: nat assume n ≤ m then show ∃ma∈{n..}. X ma ≤ (X ∘ r) m
      using seq-suble[OF ‹subseq r›, of m] by (intro bexI[of - r m]) auto
  qed
  then show ?thesis by (auto intro!: SUP-mono simp: liminf-SUP-INF comp-def)
qed
```

lemma *limsup-subseq-mono*:
fixes $X :: nat \Rightarrow 'a :: complete\text{-}linorder$
assumes *subseq r*
shows $limsup\ (X \circ r) \leq limsup\ X$
proof−
have $(SUP\ m{:}\{n..\}.\ (X \circ r)\ m) \leq (SUP\ m{:}\{n..\}.\ X\ m)$ **for** n
proof (*safe intro*!: *SUP-mono*)
fix $m :: nat$
assume $n \leq m$
then show $\exists\, ma{\in}\{n..\}.\ (X \circ r)\ m \leq X\ ma$
using *seq-suble*[*OF* ‹*subseq r*›, *of m*] **by** (*intro bexI*[*of - r m*]) *auto*
qed
then show *?thesis*
by (*auto intro*!: *INF-mono simp*: *limsup-INF-SUP comp-def*)
qed

lemma *continuous-on-imp-continuous-within*:
$continuous\text{-}on\ s\ f \Longrightarrow t \subseteq s \Longrightarrow x \in s \Longrightarrow continuous\ (at\ x\ within\ t)\ f$
unfolding *continuous-on-eq-continuous-within*
by (*auto simp*: *continuous-within intro*: *tendsto-within-subset*)

lemma *Liminf-compose-continuous-mono*:
fixes $f :: 'a{::}\{complete\text{-}linorder,\ linorder\text{-}topology\} \Rightarrow 'b{::}\{complete\text{-}linorder,\ linorder\text{-}topology\}$
assumes *c*: *continuous-on UNIV f* **and** *am*: *mono f* **and** *F*: $F \neq bot$
shows $Liminf\ F\ (\lambda n.\ f\ (g\ n)) = f\ (Liminf\ F\ g)$
proof −
{ fix *P* **assume** *eventually P F*
have $\exists\, x.\ P\ x$
proof (*rule ccontr*)
assume $\neg\ (\exists\, x.\ P\ x)$ **then have** $P = (\lambda x.\ False)$
by *auto*
with ‹*eventually P F*› *F* **show** *False*
by *auto*
qed **}**
note $* = this$

have $f\ (Liminf\ F\ g) = (SUP\ P : \{P.\ eventually\ P\ F\}.\ f\ (Inf\ (g\ `\ Collect\ P)))$
unfolding *Liminf-def*
by (*subst continuous-at-Sup-mono*[*OF am continuous-on-imp-continuous-within*[*OF c*]])
(*auto intro*: *eventually-True*)
also have $\ldots = (SUP\ P : \{P.\ eventually\ P\ F\}.\ INFIMUM\ (g\ `\ Collect\ P)\ f)$
by (*intro SUP-cong refl continuous-at-Inf-mono*[*OF am continuous-on-imp-continuous-within*[*OF c*]])
(*auto dest*!: *eventually-happens simp*: *F*)
finally show *?thesis* **by** (*auto simp*: *Liminf-def*)
qed

lemma *Limsup-compose-continuous-mono*:
fixes f :: $'a$::{*complete-linorder*, *linorder-topology*} ⇒ $'b$::{*complete-linorder*, *linorder-topology*}
assumes *c*: *continuous-on UNIV f* **and** *am*: *mono f* **and** *F*: $F \neq bot$
shows *Limsup F* (λ*n*. *f* (*g n*)) = *f* (*Limsup F g*)
proof −
{ fix *P* **assume** *eventually P F*
have ∃ *x*. *P x*
proof (*rule ccontr*)
assume ¬ (∃ *x*. *P x*) **then have** *P* = (λ*x*. *False*)
by *auto*
with ‹*eventually P F*› *F* **show** *False*
by *auto*
qed }
note ∗ = *this*

have *f* (*Limsup F g*) = (*INF P* : {*P*. *eventually P F*}. *f* (*Sup* (*g* ‘ *Collect P*)))
unfolding *Limsup-def*
by (*subst continuous-at-Inf-mono*[*OF am continuous-on-imp-continuous-within*[*OF c*]])
(*auto intro*: *eventually-True*)
also have ... = (*INF P* : {*P*. *eventually P F*}. *SUPREMUM* (*g* ‘ *Collect P*) *f*)
by (*intro INF-cong refl continuous-at-Sup-mono*[*OF am continuous-on-imp-continuous-within*[*OF c*]])
(*auto dest*!: *eventually-happens simp*: *F*)
finally show *?thesis* **by** (*auto simp*: *Limsup-def*)
qed

lemma *Liminf-compose-continuous-antimono*:
fixes *f* :: $'a$::{*complete-linorder*,*linorder-topology*} ⇒ $'b$::{*complete-linorder*,*linorder-topology*}
assumes *c*: *continuous-on UNIV f*
and *am*: *antimono f*
and *F*: $F \neq bot$
shows *Liminf F* (λ*n*. *f* (*g n*)) = *f* (*Limsup F g*)
proof −
have ∗: ∃ *x*. *P x* **if** *eventually P F* **for** *P*
proof (*rule ccontr*)
assume ¬ (∃ *x*. *P x*) **then have** *P* = (λ*x*. *False*)
by *auto*
with ‹*eventually P F*› *F* **show** *False*
by *auto*
qed
have *f* (*Limsup F g*) = (*SUP P* : {*P*. *eventually P F*}. *f* (*Sup* (*g* ‘ *Collect P*)))
unfolding *Limsup-def*
by (*subst continuous-at-Inf-antimono*[*OF am continuous-on-imp-continuous-within*[*OF c*]])
(*auto intro*: *eventually-True*)
also have ... = (*SUP P* : {*P*. *eventually P F*}. *INFIMUM* (*g* ‘ *Collect P*) *f*)
by (*intro SUP-cong refl continuous-at-Sup-antimono*[*OF am continuous-on-imp-continuous-within*[*OF*

c]])
(auto dest!: eventually-happens simp: F)
finally show ?thesis
by (auto simp: Liminf-def)
qed

lemma Limsup-compose-continuous-antimono:
fixes f :: 'a::{complete-linorder, linorder-topology} ⇒ 'b::{complete-linorder, linorder-topology}
assumes c: continuous-on UNIV f **and** am: antimono f **and** F: F ≠ bot
shows Limsup F (λn. f (g n)) = f (Liminf F g)
proof −
{ fix P **assume** eventually P F
have ∃ x. P x
proof (rule ccontr)
assume ¬ (∃ x. P x) **then have** P = (λx. False)
by auto
with ‹eventually P F› F **show** False
by auto
qed }
note ∗ = this

have f (Liminf F g) = (INF P : {P. eventually P F}. f (Inf (g ' Collect P)))
unfolding Liminf-def
by (subst continuous-at-Sup-antimono[OF am continuous-on-imp-continuous-within[OF c]])
(auto intro: eventually-True)
also have ... = (INF P : {P. eventually P F}. SUPREMUM (g ' Collect P) f)
by (intro INF-cong refl continuous-at-Inf-antimono[OF am continuous-on-imp-continuous-within[OF c]])
(auto dest!: eventually-happens simp: F)
finally show ?thesis
by (auto simp: Limsup-def)
qed

35.1 More Limits

lemma convergent-limsup-cl:
fixes X :: nat ⇒ 'a::{complete-linorder,linorder-topology}
shows convergent X ⟹ limsup X = lim X
by (auto simp: convergent-def limI lim-imp-Limsup)

lemma convergent-liminf-cl:
fixes X :: nat ⇒ 'a::{complete-linorder,linorder-topology}
shows convergent X ⟹ liminf X = lim X
by (auto simp: convergent-def limI lim-imp-Liminf)

lemma lim-increasing-cl:
assumes ⋀n m. n ≥ m ⟹ f n ≥ f m

obtains l **where** $f \longrightarrow (l::'a::\{complete\text{-}linorder,linorder\text{-}topology\})$
proof
show $f \longrightarrow (SUP\ n.\ f\ n)$
using *assms*
by (*intro increasing-tendsto*)
(*auto simp*: *SUP-upper eventually-sequentially less-SUP-iff intro*: *less-le-trans*)
qed

lemma *lim-decreasing-cl*:
assumes $\bigwedge n\ m.\ n \geq m \Longrightarrow f\ n \leq f\ m$
obtains l **where** $f \longrightarrow (l::'a::\{complete\text{-}linorder,linorder\text{-}topology\})$
proof
show $f \longrightarrow (INF\ n.\ f\ n)$
using *assms*
by (*intro decreasing-tendsto*)
(*auto simp*: *INF-lower eventually-sequentially INF-less-iff intro*: *le-less-trans*)
qed

lemma *compact-complete-linorder*:
fixes $X :: nat \Rightarrow 'a::\{complete\text{-}linorder,linorder\text{-}topology\}$
shows $\exists l\ r.\ subseq\ r \wedge (X \circ r) \longrightarrow l$
proof −
obtain r **where** *subseq r* **and** *mono*: *monoseq* $(X \circ r)$
using *seq-monosub*[*of X*]
unfolding *comp-def*
by *auto*
then have $(\forall n\ m.\ m \leq n \longrightarrow (X \circ r)\ m \leq (X \circ r)\ n) \vee (\forall n\ m.\ m \leq n \longrightarrow (X \circ r)\ n \leq (X \circ r)\ m)$
by (*auto simp add*: *monoseq-def*)
then obtain l **where** $(X \circ r) \longrightarrow l$
using *lim-increasing-cl*[*of X* ∘ *r*] *lim-decreasing-cl*[*of X* ∘ *r*]
by *auto*
then show *?thesis*
using ‹*subseq r*› **by** *auto*
qed

lemma *tendsto-Limsup*:
fixes $f :: - \Rightarrow 'a :: \{complete\text{-}linorder,linorder\text{-}topology\}$
shows $F \neq bot \Longrightarrow Limsup\ F\ f = Liminf\ F\ f \Longrightarrow (f \longrightarrow Limsup\ F\ f)\ F$
by (*subst tendsto-iff-Liminf-eq-Limsup*) *auto*

lemma *tendsto-Liminf*:
fixes $f :: - \Rightarrow 'a :: \{complete\text{-}linorder,linorder\text{-}topology\}$
shows $F \neq bot \Longrightarrow Limsup\ F\ f = Liminf\ F\ f \Longrightarrow (f \longrightarrow Liminf\ F\ f)\ F$
by (*subst tendsto-iff-Liminf-eq-Limsup*) *auto*

end

36 Extended real number line

theory *Extended-Real*
imports *Complex-Main Extended-Nat Liminf-Limsup*
begin

This should be part of *Extended-Nat* or *Order-Continuity*, but then the AFP-entry *Jinja-Thread* fails, as it does overload certain named from *Complex-Main*.

lemma *incseq-setsumI2*:
fixes $f :: 'i \Rightarrow nat \Rightarrow 'a$::*ordered-comm-monoid-add*
shows $(\bigwedge n.\ n \in A \Longrightarrow mono\ (f\ n)) \Longrightarrow mono\ (\lambda i.\ \sum n{\in}A.\ f\ n\ i)$
unfolding *incseq-def* **by** (*auto intro*: *setsum-mono*)

lemma *incseq-setsumI*:
fixes $f :: nat \Rightarrow 'a$::*ordered-comm-monoid-add*
assumes $\bigwedge i.\ 0 \leq f\ i$
shows *incseq* $(\lambda i.\ setsum\ f\ \{..<i\})$
proof (*intro incseq-SucI*)
fix *n*
have $setsum\ f\ \{..<n\} + 0 \leq setsum\ f\ \{..<n\} + f\ n$
using *assms* **by** (*rule add-left-mono*)
then show $setsum\ f\ \{..<n\} \leq setsum\ f\ \{..<Suc\ n\}$
by *auto*
qed

lemma *continuous-at-left-imp-sup-continuous*:
fixes $f :: 'a$::{*complete-linorder*, *linorder-topology*} $\Rightarrow$ $'b$::{*complete-linorder*, *linorder-topology*}
assumes *mono f* $\bigwedge x.$ *continuous* (*at-left x*) *f*
shows *sup-continuous f*
unfolding *sup-continuous-def*
proof *safe*
fix $M :: nat \Rightarrow 'a$ **assume** *incseq M* **then show** $f\ (SUP\ i.\ M\ i) = (SUP\ i.\ f\ (M\ i))$
using *continuous-at-Sup-mono*[*OF assms*, *of range M*] **by** *simp*
qed

lemma *sup-continuous-at-left*:
fixes $f :: 'a$::{*complete-linorder*, *linorder-topology*, *first-countable-topology*} $\Rightarrow$ $'b$::{*complete-linorder*, *linorder-topology*}
assumes *f*: *sup-continuous f*
shows *continuous* (*at-left x*) *f*
proof *cases*
assume $x = bot$ **then show** *?thesis*
by (*simp add*: *trivial-limit-at-left-bot*)
next
assume *x*: $x \neq bot$
show *?thesis*
unfolding *continuous-within*

proof (*intro tendsto-at-left-sequentially*[*of bot*])
fix $S :: nat \Rightarrow 'a$ **assume** S: *incseq S* **and** S-x: $S \longrightarrow x$
from S-x **have** *x-eq*: $x = (SUP\ i.\ S\ i)$
by (*rule LIMSEQ-unique*) (*intro LIMSEQ-SUP S*)
show $(\lambda n.\ f\ (S\ n)) \longrightarrow f\ x$
unfolding *x-eq sup-continuousD*[*OF f S*]
using *S sup-continuous-mono*[*OF f*] **by** (*intro LIMSEQ-SUP*) (*auto simp*: *mono-def*)
qed (*insert x*, *auto simp*: *bot-less*)
qed

lemma *sup-continuous-iff-at-left*:
fixes $f :: 'a$::{*complete-linorder*, *linorder-topology*, *first-countable-topology*} ⇒ $'b$::{*complete-linorder*, *linorder-topology*}
shows *sup-continuous f* ⟷ $(\forall x.$ *continuous* (*at-left x*) *f*$) \wedge$ *mono f*
using *sup-continuous-at-left*[*of f*] *continuous-at-left-imp-sup-continuous*[*of f*] *sup-continuous-mono*[*of f*] **by** *auto*

lemma *continuous-at-right-imp-inf-continuous*:
fixes $f :: 'a$::{*complete-linorder*, *linorder-topology*} ⇒ $'b$::{*complete-linorder*, *linorder-topology*}
assumes *mono f* $\bigwedge x.$ *continuous* (*at-right x*) *f*
shows *inf-continuous f*
unfolding *inf-continuous-def*
proof *safe*
fix $M :: nat \Rightarrow 'a$ **assume** *decseq M* **then show** *f* $(INF\ i.\ M\ i) = (INF\ i.\ f\ (M\ i))$
using *continuous-at-Inf-mono*[*OF assms*, *of range M*] **by** *simp*
qed

lemma *inf-continuous-at-right*:
fixes $f :: 'a$::{*complete-linorder*, *linorder-topology*, *first-countable-topology*} ⇒ $'b$::{*complete-linorder*, *linorder-topology*}
assumes *f*: *inf-continuous f*
shows *continuous* (*at-right x*) *f*
proof *cases*
assume $x = top$ **then show** *?thesis*
by (*simp add*: *trivial-limit-at-right-top*)
next
assume *x*: $x \neq top$
show *?thesis*
unfolding *continuous-within*
proof (*intro tendsto-at-right-sequentially*[*of - top*])
fix $S :: nat \Rightarrow 'a$ **assume** S: *decseq S* **and** S-x: $S \longrightarrow x$
from S-x **have** *x-eq*: $x = (INF\ i.\ S\ i)$
by (*rule LIMSEQ-unique*) (*intro LIMSEQ-INF S*)
show $(\lambda n.\ f\ (S\ n)) \longrightarrow f\ x$
unfolding *x-eq inf-continuousD*[*OF f S*]
using *S inf-continuous-mono*[*OF f*] **by** (*intro LIMSEQ-INF*) (*auto simp*:

mono-def antimono-def)
qed (*insert x*, *auto simp*: *less-top*)
qed

lemma *inf-continuous-iff-at-right*:
fixes *f* :: *'a*::{*complete-linorder*, *linorder-topology*, *first-countable-topology*} ⇒
'b::{*complete-linorder*, *linorder-topology*}
shows *inf-continuous f* ⟷ (∀ *x*. *continuous* (*at-right x*) *f*) ∧ *mono f*
using *inf-continuous-at-right*[*of f*] *continuous-at-right-imp-inf-continuous*[*of f*]
inf-continuous-mono[*of f*] **by** *auto*

instantiation *enat* :: *linorder-topology*
begin

definition *open-enat* :: *enat set* ⇒ *bool* **where**
open-enat = *generate-topology* (*range lessThan* ∪ *range greaterThan*)

instance
proof qed (*rule open-enat-def*)

end

lemma *open-enat*: *open* {*enat n*}
proof (*cases n*)
case *0*
then have {*enat n*} = {..< *eSuc 0*}
by (*auto simp*: *enat-0*)
then show *?thesis*
by *simp*
next
case (*Suc n'*)
then have {*enat n*} = {*enat n'* <..< *enat* (*Suc n*)}
apply *auto*
apply (*case-tac x*)
apply *auto*
done
then show *?thesis*
by *simp*
qed

lemma *open-enat-iff*:
fixes *A* :: *enat set*
shows *open A* ⟷ (∞ ∈ *A* ⟶ (∃ *n*::*nat*. {*n* <..} ⊆ *A*))
proof *safe*
assume ∞ ∉ *A*
then have *A* = (⋃ *n*∈{*n*. *enat n* ∈ *A*}. {*enat n*})
apply *auto*
apply (*case-tac x*)
apply *auto*

```
    done
  moreover have open ...
    by (auto intro: open-enat)
  ultimately show open A
    by simp
next
  fix n assume {enat n <..} ⊆ A
  then have A = (⋃n∈{n. enat n ∈ A}. {enat n}) ∪ {enat n <..}
    apply auto
    apply (case-tac x)
    apply auto
    done
  moreover have open ...
    by (intro open-Un open-UN ballI open-enat open-greaterThan)
  ultimately show open A
    by simp
next
  assume open A ∞ ∈ A
  then have generate-topology (range lessThan ∪ range greaterThan) A ∞ ∈ A
    unfolding open-enat-def by auto
  then show ∃ n::nat. {n <..} ⊆ A
  proof induction
    case (Int A B)
    then obtain n m where {enat n<..} ⊆ A {enat m<..} ⊆ B
      by auto
    then have {enat (max n m) <..} ⊆ A ∩ B
      by (auto simp add: subset-eq Ball-def max-def enat-ord-code(1)[symmetric]
simp del: enat-ord-code(1))
    then show ?case
      by auto
  next
    case (UN K)
    then obtain k where k ∈ K ∞ ∈ k
      by auto
    with UN.IH[OF this] show ?case
      by auto
  qed auto
qed

lemma nhds-enat: nhds x = (if x = ∞ then INF i. principal {enat i..} else prin-
cipal {x})
proof auto
  show nhds ∞ = (INF i. principal {enat i..})
    unfolding nhds-def
   apply (auto intro!: antisym INF-greatest simp add: open-enat-iff cong: rev-conj-cong)
    apply (auto intro!: INF-lower Ioi-le-Ico) []
    subgoal for x i
      by (auto intro!: INF-lower2[of Suc i] simp: subset-eq Ball-def eSuc-enat
Suc-ile-eq)
```

```
    done
  show nhds (enat i) = principal {enat i} for i
    by (simp add: nhds-discrete-open open-enat)
qed

instance enat :: topological-comm-monoid-add
proof
  have [simp]: enat i ≤ aa ⟹ enat i ≤ aa + ba for aa ba i
    by (rule order-trans[OF - add-mono[of aa aa 0 ba]]) auto
  then have [simp]: enat i ≤ ba ⟹ enat i ≤ aa + ba for aa ba i
    by (metis add.commute)
  fix a b :: enat show ((λx. fst x + snd x) ⟶ a + b) (nhds a ×_F nhds b)
    apply (auto simp: nhds-enat filterlim-INF prod-filter-INF1 prod-filter-INF2
                      filterlim-principal principal-prod-principal eventually-principal)
    subgoal for i
      by (auto intro!: eventually-INF1[of i] simp: eventually-principal)
    subgoal for j i
      by (auto intro!: eventually-INF1[of i] simp: eventually-principal)
    subgoal for j i
      by (auto intro!: eventually-INF1[of i] simp: eventually-principal)
    done
qed
```

For more lemmas about the extended real numbers go to `~~/src/HOL/Multivariate_Analysis/Extended_Real_Limits.thy`

36.1 Definition and basic properties

```
datatype ereal = ereal real | PInfty | MInfty

instantiation ereal :: uminus
begin

fun uminus-ereal where
  - (ereal r) = ereal (- r)
| - PInfty = MInfty
| - MInfty = PInfty

instance ..

end

instantiation ereal :: infinity
begin

definition (∞::ereal) = PInfty
instance ..

end
```

declare [[*coercion ereal* :: *real* ⇒ *ereal*]]

lemma *ereal-uminus-uminus*[*simp*]:
fixes *a* :: *ereal*
shows − (− *a*) = *a*
by (*cases a*) *simp-all*

lemma
shows *PInfty-eq-infinity*[*simp*]: *PInfty* = ∞
and *MInfty-eq-minfinity*[*simp*]: *MInfty* = − ∞
and *MInfty-neq-PInfty*[*simp*]: ∞ ≠ − (∞::*ereal*) − ∞ ≠ (∞::*ereal*)
and *MInfty-neq-ereal*[*simp*]: *ereal r* ≠ − ∞ − ∞ ≠ *ereal r*
and *PInfty-neq-ereal*[*simp*]: *ereal r* ≠ ∞ ∞ ≠ *ereal r*
and *PInfty-cases*[*simp*]: (*case* ∞ *of ereal r* ⇒ *f r* | *PInfty* ⇒ *y* | *MInfty* ⇒ *z*) = *y*
and *MInfty-cases*[*simp*]: (*case* − ∞ *of ereal r* ⇒ *f r* | *PInfty* ⇒ *y* | *MInfty* ⇒ *z*) = *z*
by (*simp-all add*: *infinity-ereal-def*)

declare
PInfty-eq-infinity[*code-post*]
MInfty-eq-minfinity[*code-post*]

lemma [*code-unfold*]:
∞ = *PInfty*
− *PInfty* = *MInfty*
by *simp-all*

lemma *inj-ereal*[*simp*]: *inj-on ereal A*
unfolding *inj-on-def* **by** *auto*

lemma *ereal-cases*[*cases type*: *ereal*]:
obtains (*real*) *r* **where** *x* = *ereal r*
| (*PInf*) *x* = ∞
| (*MInf*) *x* = −∞
using *assms* **by** (*cases x*) *auto*

lemmas *ereal2-cases* = *ereal-cases*[*case-product ereal-cases*]
lemmas *ereal3-cases* = *ereal2-cases*[*case-product ereal-cases*]

lemma *ereal-all-split*: ⋀*P*. (∀ *x*::*ereal*. *P x*) ⟷ *P* ∞ ∧ (∀ *x*. *P* (*ereal x*)) ∧ *P* (−∞)
by (*metis ereal-cases*)

lemma *ereal-ex-split*: ⋀*P*. (∃ *x*::*ereal*. *P x*) ⟷ *P* ∞ ∨ (∃ *x*. *P* (*ereal x*)) ∨ *P* (−∞)
by (*metis ereal-cases*)

lemma *ereal-uminus-eq-iff*[*simp*]:
  fixes *a b* :: *ereal*
  shows $-a = -b \longleftrightarrow a = b$
  by (*cases rule*: *ereal2-cases*[*of a b*]) *simp-all*

function *real-of-ereal* :: *ereal* ⇒ *real* **where**
  real-of-ereal (*ereal r*) = *r*
| *real-of-ereal* ∞ = *0*
| *real-of-ereal* (−∞) = *0*
  by (*auto intro*: *ereal-cases*)
termination by *standard* (*rule wf-empty*)

lemma *real-of-ereal*[*simp*]:
  real-of-ereal (− *x* :: *ereal*) = − (*real-of-ereal x*)
  by (*cases x*) *simp-all*

lemma *range-ereal*[*simp*]: *range ereal* = *UNIV* − {∞, −∞}
proof *safe*
  fix *x*
  assume *x* ∉ *range ereal x* ≠ ∞
  then show *x* = −∞
    by (*cases x*) *auto*
qed *auto*

lemma *ereal-range-uminus*[*simp*]: *range uminus* = (*UNIV*::*ereal set*)
proof *safe*
  fix *x* :: *ereal*
  show *x* ∈ *range uminus*
    by (*intro image-eqI*[*of - - −x*]) *auto*
qed *auto*

instantiation *ereal* :: *abs*
begin

function *abs-ereal* **where**
  |*ereal r*| = *ereal* |*r*|
| |−∞| = (∞::*ereal*)
| |∞| = (∞::*ereal*)
by (*auto intro*: *ereal-cases*)
termination proof qed (*rule wf-empty*)

instance ..

end

lemma *abs-eq-infinity-cases*[*elim*!]:
  fixes *x* :: *ereal*
  assumes |*x*| = ∞
  obtains *x* = ∞ | *x* = −∞

using *assms* **by** (*cases x*) *auto*

lemma *abs-neq-infinity-cases*[*elim*!]:
fixes x :: *ereal*
assumes $|x| \neq \infty$
obtains r **where** $x = ereal\ r$
using *assms* **by** (*cases x*) *auto*

lemma *abs-ereal-uminus*[*simp*]:
fixes x :: *ereal*
shows $|-x| = |x|$
by (*cases x*) *auto*

lemma *ereal-infinity-cases*:
fixes a :: *ereal*
shows $a \neq \infty \Longrightarrow a \neq -\infty \Longrightarrow |a| \neq \infty$
by *auto*

36.1.1 Addition

instantiation *ereal* :: {*one*,*comm-monoid-add*,*zero-neq-one*}
begin

definition $0 = ereal\ 0$
definition $1 = ereal\ 1$

function *plus-ereal* **where**
ereal r + *ereal p* = *ereal* (*r* + *p*)
| ∞ + *a* = (∞::*ereal*)
| *a* + ∞ = (∞::*ereal*)
| *ereal r* + −∞ = − ∞
| −∞ + *ereal p* = −(∞::*ereal*)
| −∞ + −∞ = −(∞::*ereal*)
proof *goal-cases*
case *prems*: (*1 P x*)
then obtain *a b* **where** *x* = (*a*, *b*)
by (*cases x*) *auto*
with *prems* **show** *P*
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*
qed *auto*
termination by *standard* (*rule wf-empty*)

lemma *Infty-neq-0*[*simp*]:
(∞::*ereal*) ≠ *0 0* ≠ (∞::*ereal*)
−(∞::*ereal*) ≠ *0 0* ≠ −(∞::*ereal*)
by (*simp-all add*: *zero-ereal-def*)

lemma *ereal-eq-0*[*simp*]:
ereal r = *0* ⟷ *r* = *0*

```
  0 = ereal r ⟷ r = 0
  unfolding zero-ereal-def by simp-all

lemma ereal-eq-1[simp]:
  ereal r = 1 ⟷ r = 1
  1 = ereal r ⟷ r = 1
  unfolding one-ereal-def by simp-all

instance
proof
  fix a b c :: ereal
  show 0 + a = a
    by (cases a) (simp-all add: zero-ereal-def)
  show a + b = b + a
    by (cases rule: ereal2-cases[of a b]) simp-all
  show a + b + c = a + (b + c)
    by (cases rule: ereal3-cases[of a b c]) simp-all
  show 0 ≠ (1::ereal)
    by (simp add: one-ereal-def zero-ereal-def)
qed

end

lemma ereal-0-plus [simp]: ereal 0 + x = x
  and plus-ereal-0 [simp]: x + ereal 0 = x
by(simp-all add: zero-ereal-def[symmetric])

instance ereal :: numeral ..

lemma real-of-ereal-0[simp]: real-of-ereal (0::ereal) = 0
  unfolding zero-ereal-def by simp

lemma abs-ereal-zero[simp]: |0| = (0::ereal)
  unfolding zero-ereal-def abs-ereal.simps by simp

lemma ereal-uminus-zero[simp]: − 0 = (0::ereal)
  by (simp add: zero-ereal-def)

lemma ereal-uminus-zero-iff[simp]:
  fixes a :: ereal
  shows −a = 0 ⟷ a = 0
  by (cases a) simp-all

lemma ereal-plus-eq-PInfty[simp]:
  fixes a b :: ereal
  shows a + b = ∞ ⟷ a = ∞ ∨ b = ∞
  by (cases rule: ereal2-cases[of a b]) auto

lemma ereal-plus-eq-MInfty[simp]:
```

fixes *a b* :: *ereal*
shows $a + b = -\infty \longleftrightarrow (a = -\infty \vee b = -\infty) \wedge a \neq \infty \wedge b \neq \infty$
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

lemma *ereal-add-cancel-left*:
fixes *a b* :: *ereal*
assumes $a \neq -\infty$
shows $a + b = a + c \longleftrightarrow a = \infty \vee b = c$
using *assms* **by** (*cases rule*: *ereal3-cases*[*of a b c*]) *auto*

lemma *ereal-add-cancel-right*:
fixes *a b* :: *ereal*
assumes $a \neq -\infty$
shows $b + a = c + a \longleftrightarrow a = \infty \vee b = c$
using *assms* **by** (*cases rule*: *ereal3-cases*[*of a b c*]) *auto*

lemma *ereal-real*: *ereal* (*real-of-ereal x*) = (*if* $|x| = \infty$ *then 0 else x*)
by (*cases x*) *simp-all*

lemma *real-of-ereal-add*:
fixes *a b* :: *ereal*
shows *real-of-ereal* (*a* + *b*) =
(*if* ($|a| = \infty$) ∧ ($|b| = \infty$) ∨ ($|a| \neq \infty$) ∧ ($|b| \neq \infty$) *then real-of-ereal a* + *real-of-ereal b else 0*)
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

36.1.2 Linear order on *ereal*

instantiation *ereal* :: *linorder*
begin

function *less-ereal*
where
ereal x < *ereal y* ⟷ *x* < *y*
| (∞::*ereal*) < *a* ⟷ *False*
| *a* < −(∞::*ereal*) ⟷ *False*
| *ereal x* < ∞ ⟷ *True*
| −∞ < *ereal r* ⟷ *True*
| −∞ < (∞::*ereal*) ⟷ *True*
proof *goal-cases*
case *prems*: (*1 P x*)
then obtain *a b* **where** *x* = (*a*,*b*) **by** (*cases x*) *auto*
with *prems* **show** *P* **by** (*cases rule*: *ereal2-cases*[*of a b*]) *auto*
qed *simp-all*
termination by (*relation* {}) *simp*

definition *x* ≤ (*y*::*ereal*) ⟷ *x* < *y* ∨ *x* = *y*

lemma *ereal-infty-less*[*simp*]:

fixes x :: *ereal*
shows $x < \infty \longleftrightarrow (x \neq \infty)$
$-\infty < x \longleftrightarrow (x \neq -\infty)$
by (*cases* x, *simp-all*) (*cases* x, *simp-all*)

lemma *ereal-infty-less-eq*[*simp*]:
fixes x :: *ereal*
shows $\infty \leq x \longleftrightarrow x = \infty$
and $x \leq -\infty \longleftrightarrow x = -\infty$
by (*auto simp add*: *less-eq-ereal-def*)

lemma *ereal-less*[*simp*]:
ereal $r < 0 \longleftrightarrow (r < 0)$
$0 <$ *ereal* $r \longleftrightarrow (0 < r)$
ereal $r < 1 \longleftrightarrow (r < 1)$
$1 <$ *ereal* $r \longleftrightarrow (1 < r)$
$0 < (\infty$::*ereal*$)$
$-(\infty$::*ereal*$) < 0$
by (*simp-all add*: *zero-ereal-def one-ereal-def*)

lemma *ereal-less-eq*[*simp*]:
$x \leq (\infty$::*ereal*$)$
$-(\infty$::*ereal*$) \leq x$
ereal $r \leq$ *ereal* $p \longleftrightarrow r \leq p$
ereal $r \leq 0 \longleftrightarrow r \leq 0$
$0 \leq$ *ereal* $r \longleftrightarrow 0 \leq r$
ereal $r \leq 1 \longleftrightarrow r \leq 1$
$1 \leq$ *ereal* $r \longleftrightarrow 1 \leq r$
by (*auto simp add*: *less-eq-ereal-def zero-ereal-def one-ereal-def*)

lemma *ereal-infty-less-eq2*:
$a \leq b \Longrightarrow a = \infty \Longrightarrow b = (\infty$::*ereal*$)$
$a \leq b \Longrightarrow b = -\infty \Longrightarrow a = -(\infty$::*ereal*$)$
by *simp-all*

instance
proof
fix x y z :: *ereal*
show $x \leq x$
by (*cases* x) *simp-all*
show $x < y \longleftrightarrow x \leq y \wedge \neg\ y \leq x$
by (*cases rule*: *ereal2-cases*[*of* x y]) *auto*
show $x \leq y \vee y \leq x$
by (*cases rule*: *ereal2-cases*[*of* x y]) *auto*
{
assume $x \leq y$ $y \leq x$
then show $x = y$
by (*cases rule*: *ereal2-cases*[*of* x y]) *auto*
}

```
  {
    assume x ≤ y y ≤ z
    then show x ≤ z
      by (cases rule: ereal3-cases[of x y z]) auto
  }
qed

end

lemma ereal-dense2: x < y ⟹ ∃ z. x < ereal z ∧ ereal z < y
  using lt-ex gt-ex dense by (cases x y rule: ereal2-cases) auto

instance ereal :: dense-linorder
  by standard (blast dest: ereal-dense2)

instance ereal :: ordered-comm-monoid-add
proof
  fix a b c :: ereal
  assume a ≤ b
  then show c + a ≤ c + b
    by (cases rule: ereal3-cases[of a b c]) auto
qed

lemma ereal-one-not-less-zero-ereal[simp]: ¬ 1 < (0::ereal)
  by (simp add: zero-ereal-def)

lemma real-of-ereal-positive-mono:
  fixes x y :: ereal
  shows 0 ≤ x ⟹ x ≤ y ⟹ y ≠ ∞ ⟹ real-of-ereal x ≤ real-of-ereal y
  by (cases rule: ereal2-cases[of x y]) auto

lemma ereal-MInfty-lessI[intro, simp]:
  fixes a :: ereal
  shows a ≠ −∞ ⟹ −∞ < a
  by (cases a) auto

lemma ereal-less-PInfty[intro, simp]:
  fixes a :: ereal
  shows a ≠ ∞ ⟹ a < ∞
  by (cases a) auto

lemma ereal-less-ereal-Ex:
  fixes a b :: ereal
  shows x < ereal r ⟷ x = −∞ ∨ (∃ p. p < r ∧ x = ereal p)
  by (cases x) auto

lemma less-PInf-Ex-of-nat: x ≠ ∞ ⟷ (∃ n::nat. x < ereal (real n))
proof (cases x)
  case (real r)
```

then show *?thesis*
using *reals-Archimedean2*[*of r*] **by** *simp*
qed *simp-all*

lemma *ereal-add-mono*:
fixes *a b c d* :: *ereal*
assumes $a \leq b$
and $c \leq d$
shows $a + c \leq b + d$
using *assms*
apply (*cases a*)
apply (*cases rule*: *ereal3-cases*[*of b c d*], *auto*)
apply (*cases rule*: *ereal3-cases*[*of b c d*], *auto*)
done

lemma *ereal-minus-le-minus*[*simp*]:
fixes *a b* :: *ereal*
shows $- a \leq - b \longleftrightarrow b \leq a$
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

lemma *ereal-minus-less-minus*[*simp*]:
fixes *a b* :: *ereal*
shows $- a < - b \longleftrightarrow b < a$
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

lemma *ereal-le-real-iff*:
$x \leq$ *real-of-ereal y* $\longleftrightarrow (|y| \neq \infty \longrightarrow$ *ereal x* $\leq y) \wedge (|y| = \infty \longrightarrow x \leq 0)$
by (*cases y*) *auto*

lemma *real-le-ereal-iff*:
real-of-ereal y $\leq x \longleftrightarrow (|y| \neq \infty \longrightarrow y \leq$ *ereal x*$) \wedge (|y| = \infty \longrightarrow 0 \leq x)$
by (*cases y*) *auto*

lemma *ereal-less-real-iff*:
$x <$ *real-of-ereal y* $\longleftrightarrow (|y| \neq \infty \longrightarrow$ *ereal x* $< y) \wedge (|y| = \infty \longrightarrow x < 0)$
by (*cases y*) *auto*

lemma *real-less-ereal-iff*:
real-of-ereal y $< x \longleftrightarrow (|y| \neq \infty \longrightarrow y <$ *ereal x*$) \wedge (|y| = \infty \longrightarrow 0 < x)$
by (*cases y*) *auto*

lemma *real-of-ereal-pos*:
fixes *x* :: *ereal*
shows $0 \leq x \Longrightarrow 0 \leq$ *real-of-ereal x* **by** (*cases x*) *auto*

lemmas *real-of-ereal-ord-simps* =
ereal-le-real-iff real-le-ereal-iff ereal-less-real-iff real-less-ereal-iff

lemma *abs-ereal-ge0*[*simp*]: $0 \leq x \Longrightarrow |x :: ereal| = x$

by (*cases x*) *auto*

lemma *abs-ereal-less0*[*simp*]: $x < 0 \Longrightarrow |x :: ereal| = -x$
by (*cases x*) *auto*

lemma *abs-ereal-pos*[*simp*]: $0 \leq |x :: ereal|$
by (*cases x*) *auto*

lemma *ereal-abs-leI*:
fixes *x y* :: *ereal*
shows $[\![x \leq y; -x \leq y]\!] \Longrightarrow |x| \leq y$
by(*cases x y rule*: *ereal2-cases*)(*simp-all*)

lemma *real-of-ereal-le-0*[*simp*]: *real-of-ereal* $(x :: ereal) \leq 0 \longleftrightarrow x \leq 0 \vee x = \infty$
by (*cases x*) *auto*

lemma *abs-real-of-ereal*[*simp*]: $|$*real-of-ereal* $(x :: ereal)| =$ *real-of-ereal* $|x|$
by (*cases x*) *auto*

lemma *zero-less-real-of-ereal*:
fixes *x* :: *ereal*
shows $0 <$ *real-of-ereal x* $\longleftrightarrow 0 < x \wedge x \neq \infty$
by (*cases x*) *auto*

lemma *ereal-0-le-uminus-iff*[*simp*]:
fixes *a* :: *ereal*
shows $0 \leq - a \longleftrightarrow a \leq 0$
by (*cases rule*: *ereal2-cases*[*of a*]) *auto*

lemma *ereal-uminus-le-0-iff*[*simp*]:
fixes *a* :: *ereal*
shows $- a \leq 0 \longleftrightarrow 0 \leq a$
by (*cases rule*: *ereal2-cases*[*of a*]) *auto*

lemma *ereal-add-strict-mono*:
fixes *a b c d* :: *ereal*
assumes $a \leq b$
and $0 \leq a$
and $a \neq \infty$
and $c < d$
shows $a + c < b + d$
using *assms*
by (*cases rule*: *ereal3-cases*[*case-product ereal-cases*, *of a b c d*]) *auto*

lemma *ereal-less-add*:
fixes *a b c* :: *ereal*
shows $|a| \neq \infty \Longrightarrow c < b \Longrightarrow a + c < a + b$
by (*cases rule*: *ereal2-cases*[*of b c*]) *auto*

lemma *ereal-add-nonneg-eq-0-iff*:
fixes $a\ b :: ereal$
shows $0 \leq a \Longrightarrow 0 \leq b \Longrightarrow a + b = 0 \longleftrightarrow a = 0 \wedge b = 0$
by (*cases a b rule*: *ereal2-cases*) *auto*

lemma *ereal-uminus-eq-reorder*: $- a = b \longleftrightarrow a = (-b::ereal)$
by *auto*

lemma *ereal-uminus-less-reorder*: $- a < b \longleftrightarrow -b < (a::ereal)$
by (*subst* (*3*) *ereal-uminus-uminus*[*symmetric*]) (*simp only*: *ereal-minus-less-minus*)

lemma *ereal-less-uminus-reorder*: $a < - b \longleftrightarrow b < - (a::ereal)$
by (*subst* (*3*) *ereal-uminus-uminus*[*symmetric*]) (*simp only*: *ereal-minus-less-minus*)

lemma *ereal-uminus-le-reorder*: $- a \leq b \longleftrightarrow -b \leq (a::ereal)$
by (*subst* (*3*) *ereal-uminus-uminus*[*symmetric*]) (*simp only*: *ereal-minus-le-minus*)

lemmas *ereal-uminus-reorder* =
ereal-uminus-eq-reorder ereal-uminus-less-reorder ereal-uminus-le-reorder

lemma *ereal-bot*:
fixes $x :: ereal$
assumes $\bigwedge B.\ x \leq ereal\ B$
shows $x = - \infty$
proof (*cases x*)
case (*real r*)
with *assms*[*of r − 1*] **show** *?thesis*
by *auto*
next
case *PInf*
with *assms*[*of 0*] **show** *?thesis*
by *auto*
next
case *MInf*
then show *?thesis*
by *simp*
qed

lemma *ereal-top*:
fixes $x :: ereal$
assumes $\bigwedge B.\ x \geq ereal\ B$
shows $x = \infty$
proof (*cases x*)
case (*real r*)
with *assms*[*of r + 1*] **show** *?thesis*
by *auto*
next
case *MInf*
with *assms*[*of 0*] **show** *?thesis*

```
    by auto
next
  case PInf
  then show ?thesis
    by simp
qed

lemma
  shows ereal-max[simp]: ereal (max x y) = max (ereal x) (ereal y)
    and ereal-min[simp]: ereal (min x y) = min (ereal x) (ereal y)
  by (simp-all add: min-def max-def)

lemma ereal-max-0: max 0 (ereal r) = ereal (max 0 r)
  by (auto simp: zero-ereal-def)

lemma
  fixes f :: nat ⇒ ereal
  shows ereal-incseq-uminus[simp]: incseq (λx. − f x) ⟷ decseq f
    and ereal-decseq-uminus[simp]: decseq (λx. − f x) ⟷ incseq f
  unfolding decseq-def incseq-def by auto

lemma incseq-ereal: incseq f ⟹ incseq (λx. ereal (f x))
  unfolding incseq-def by auto

lemma ereal-add-nonneg-nonneg[simp]:
  fixes a b :: ereal
  shows 0 ≤ a ⟹ 0 ≤ b ⟹ 0 ≤ a + b
  using add-mono[of 0 a 0 b] by simp

lemma setsum-ereal[simp]: (∑ x∈A. ereal (f x)) = ereal (∑ x∈A. f x)
proof (cases finite A)
  case True
  then show ?thesis by induct auto
next
  case False
  then show ?thesis by simp
qed

lemma setsum-Pinfty:
  fixes f :: 'a ⇒ ereal
  shows (∑ x∈P. f x) = ∞ ⟷ finite P ∧ (∃ i∈P. f i = ∞)
proof safe
  assume ∗: setsum f P = ∞
  show finite P
  proof (rule ccontr)
    assume ¬ finite P
    with ∗ show False
      by auto
  qed
```

```
  show ∃ i∈P. f i = ∞
  proof (rule ccontr)
    assume ¬ ?thesis
    then have ⋀i. i ∈ P ⟹ f i ≠ ∞
      by auto
    with ‹finite P› have setsum f P ≠ ∞
      by induct auto
    with * show False
      by auto
  qed
next
  fix i
  assume finite P and i ∈ P and f i = ∞
  then show setsum f P = ∞
  proof induct
    case (insert x A)
    show ?case using insert by (cases x = i) auto
  qed simp
qed

lemma setsum-Inf:
  fixes f :: 'a ⇒ ereal
  shows |setsum f A| = ∞ ⟷ finite A ∧ (∃ i∈A. |f i| = ∞)
proof
  assume *: |setsum f A| = ∞
  have finite A
    by (rule ccontr) (insert *, auto)
  moreover have ∃ i∈A. |f i| = ∞
  proof (rule ccontr)
    assume ¬ ?thesis
    then have ∀ i∈A. ∃ r. f i = ereal r
      by auto
    from bchoice[OF this] obtain r where ∀ x∈A. f x = ereal (r x) ..
    with * show False
      by auto
  qed
  ultimately show finite A ∧ (∃ i∈A. |f i| = ∞)
    by auto
next
  assume finite A ∧ (∃ i∈A. |f i| = ∞)
  then obtain i where finite A i ∈ A and |f i| = ∞
    by auto
  then show |setsum f A| = ∞
  proof induct
    case (insert j A)
    then show ?case
      by (cases rule: ereal3-cases[of f i f j setsum f A]) auto
  qed simp
qed
```

lemma *setsum-real-of-ereal*:
fixes $f :: 'i \Rightarrow ereal$
assumes $\bigwedge x.\ x \in S \Longrightarrow |f\ x| \neq \infty$
shows $(\sum x{\in}S.\ real\text{-}of\text{-}ereal\ (f\ x)) = real\text{-}of\text{-}ereal\ (setsum\ f\ S)$
proof −
have $\forall x{\in}S.\ \exists r.\ f\ x = ereal\ r$
proof
fix x
assume $x \in S$
from *assms*[*OF this*] **show** $\exists r.\ f\ x = ereal\ r$
by (*cases f x*) *auto*
qed
from *bchoice*[*OF this*] **obtain** r **where** $\forall x{\in}S.\ f\ x = ereal\ (r\ x)$ **..**
then show *?thesis*
by *simp*
qed

lemma *setsum-ereal-0*:
fixes $f :: 'a \Rightarrow ereal$
assumes *finite A*
and $\bigwedge i.\ i \in A \Longrightarrow 0 \leq f\ i$
shows $(\sum x{\in}A.\ f\ x) = 0 \longleftrightarrow (\forall i{\in}A.\ f\ i = 0)$
proof
assume *setsum f A* $= 0$ **with** *assms* **show** $\forall i{\in}A.\ f\ i = 0$
proof (*induction A*)
case (*insert a A*)
then have $f\ a = 0 \wedge (\sum a{\in}A.\ f\ a) = 0$
by (*subst ereal-add-nonneg-eq-0-iff*[*symmetric*]) (*simp-all add*: *setsum-nonneg*)
with *insert* **show** *?case*
by *simp*
qed *simp*
qed *auto*

36.1.3 Multiplication

instantiation *ereal* :: {*comm-monoid-mult*,*sgn*}
begin

function *sgn-ereal* :: *ereal* ⇒ *ereal* **where**
sgn (*ereal r*) = *ereal* (*sgn r*)
| *sgn* (∞::*ereal*) = *1*
| *sgn* (−∞::*ereal*) = *−1*
by (*auto intro*: *ereal-cases*)
termination by *standard* (*rule wf-empty*)

function *times-ereal* **where**
ereal r ∗ *ereal p* = *ereal* (*r* ∗ *p*)
| *ereal r* ∗ ∞ = (*if r* = *0 then 0 else if r* > *0 then* ∞ *else* −∞)

| $\infty * ereal\ r = (if\ r = 0\ then\ 0\ else\ if\ r > 0\ then\ \infty\ else\ -\infty)$
| $ereal\ r * -\infty = (if\ r = 0\ then\ 0\ else\ if\ r > 0\ then\ -\infty\ else\ \infty)$
| $-\infty * ereal\ r = (if\ r = 0\ then\ 0\ else\ if\ r > 0\ then\ -\infty\ else\ \infty)$
| $(\infty::ereal) * \infty = \infty$
| $-(\infty::ereal) * \infty = -\infty$
| $(\infty::ereal) * -\infty = -\infty$
| $-(\infty::ereal) * -\infty = \infty$
proof *goal-cases*
  case *prems*: (*1 P x*)
  then obtain *a b* **where** $x = (a, b)$
    by (*cases x*) *auto*
  with *prems* **show** *P*
    by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*
qed *simp-all*
termination by (*relation* {}) *simp*

instance
proof
  fix *a b c* :: *ereal*
  show *1 * a = a*
    by (*cases a*) (*simp-all add*: *one-ereal-def*)
  show *a * b = b * a*
    by (*cases rule*: *ereal2-cases*[*of a b*]) *simp-all*
  show *a * b * c = a * (b * c)*
    by (*cases rule*: *ereal3-cases*[*of a b c*])
      (*simp-all add*: *zero-ereal-def zero-less-mult-iff*)
qed

end

lemma [*simp*]:
  shows *ereal-1-times*: *ereal 1 * x = x*
  and *times-ereal-1*: *x * ereal 1 = x*
by(*simp-all add*: *one-ereal-def*[*symmetric*])

lemma *one-not-le-zero-ereal*[*simp*]: ¬ (*1* ≤ (*0*::*ereal*))
  by (*simp add*: *one-ereal-def zero-ereal-def*)

lemma *real-ereal-1*[*simp*]: *real-of-ereal* (*1*::*ereal*) = *1*
  unfolding *one-ereal-def* **by** *simp*

lemma *real-of-ereal-le-1*:
  fixes *a* :: *ereal*
  shows *a* ≤ *1* ⟹ *real-of-ereal a* ≤ *1*
  by (*cases a*) (*auto simp*: *one-ereal-def*)

lemma *abs-ereal-one*[*simp*]: |*1*| = (*1*::*ereal*)
  unfolding *one-ereal-def* **by** *simp*

lemma *ereal-mult-zero*[*simp*]:
fixes a :: *ereal*
shows $a * 0 = 0$
by (*cases a*) (*simp-all add*: *zero-ereal-def*)

lemma *ereal-zero-mult*[*simp*]:
fixes a :: *ereal*
shows $0 * a = 0$
by (*cases a*) (*simp-all add*: *zero-ereal-def*)

lemma *ereal-m1-less-0*[*simp*]: $-(1::ereal) < 0$
by (*simp add*: *zero-ereal-def one-ereal-def*)

lemma *ereal-times*[*simp*]:
$1 \neq (\infty::ereal)$ $(\infty::ereal) \neq 1$
$1 \neq -(\infty::ereal)$ $-(\infty::ereal) \neq 1$
by (*auto simp*: *one-ereal-def*)

lemma *ereal-plus-1*[*simp*]:
$1 + ereal\ r = ereal\ (r + 1)$
$ereal\ r + 1 = ereal\ (r + 1)$
$1 + -(\infty::ereal) = -\infty$
$-(\infty::ereal) + 1 = -\infty$
unfolding *one-ereal-def* **by** *auto*

lemma *ereal-zero-times*[*simp*]:
fixes $a\ b$ :: *ereal*
shows $a * b = 0 \longleftrightarrow a = 0 \vee b = 0$
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

lemma *ereal-mult-eq-PInfty*[*simp*]:
$a * b = (\infty::ereal) \longleftrightarrow$
$(a = \infty \wedge b > 0) \vee (a > 0 \wedge b = \infty) \vee (a = -\infty \wedge b < 0) \vee (a < 0 \wedge b = -\infty)$
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

lemma *ereal-mult-eq-MInfty*[*simp*]:
$a * b = -(\infty::ereal) \longleftrightarrow$
$(a = \infty \wedge b < 0) \vee (a < 0 \wedge b = \infty) \vee (a = -\infty \wedge b > 0) \vee (a > 0 \wedge b = -\infty)$
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

lemma *ereal-abs-mult*: $|x * y :: ereal| = |x| * |y|$
by (*cases x y rule*: *ereal2-cases*) (*auto simp*: *abs-mult*)

lemma *ereal-0-less-1*[*simp*]: $0 < (1::ereal)$
by (*simp-all add*: *zero-ereal-def one-ereal-def*)

lemma *ereal-mult-minus-left*[*simp*]:

fixes $a\ b :: ereal$
shows $-a * b = -(a * b)$
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

lemma *ereal-mult-minus-right*[*simp*]:
fixes $a\ b :: ereal$
shows $a * -b = -(a * b)$
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

lemma *ereal-mult-infty*[*simp*]:
$a * (\infty::ereal) = (if\ a = 0\ then\ 0\ else\ if\ 0 < a\ then\ \infty\ else\ -\infty)$
by (*cases a*) *auto*

lemma *ereal-infty-mult*[*simp*]:
$(\infty::ereal) * a = (if\ a = 0\ then\ 0\ else\ if\ 0 < a\ then\ \infty\ else\ -\infty)$
by (*cases a*) *auto*

lemma *ereal-mult-strict-right-mono*:
assumes $a < b$
and $0 < c$
and $c < (\infty::ereal)$
shows $a * c < b * c$
using *assms*
by (*cases rule*: *ereal3-cases*[*of a b c*]) (*auto simp*: *zero-le-mult-iff*)

lemma *ereal-mult-strict-left-mono*:
$a < b \Longrightarrow 0 < c \Longrightarrow c < (\infty::ereal) \Longrightarrow c * a < c * b$
using *ereal-mult-strict-right-mono*
by (*simp add*: *mult.commute*[*of c*])

lemma *ereal-mult-right-mono*:
fixes $a\ b\ c :: ereal$
shows $a \leq b \Longrightarrow 0 \leq c \Longrightarrow a * c \leq b * c$
using *assms*
apply (*cases c = 0*)
apply *simp*
apply (*cases rule*: *ereal3-cases*[*of a b c*])
apply (*auto simp*: *zero-le-mult-iff*)
done

lemma *ereal-mult-left-mono*:
fixes $a\ b\ c :: ereal$
shows $a \leq b \Longrightarrow 0 \leq c \Longrightarrow c * a \leq c * b$
using *ereal-mult-right-mono*
by (*simp add*: *mult.commute*[*of c*])

lemma *zero-less-one-ereal*[*simp*]: $0 \leq (1::ereal)$
by (*simp add*: *one-ereal-def zero-ereal-def*)

lemma *ereal-0-le-mult*[*simp*]: $0 \leq a \Longrightarrow 0 \leq b \Longrightarrow 0 \leq a * (b :: ereal)$
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

lemma *ereal-right-distrib*:
fixes *r a b* :: *ereal*
shows $0 \leq a \Longrightarrow 0 \leq b \Longrightarrow r * (a + b) = r * a + r * b$
by (*cases rule*: *ereal3-cases*[*of r a b*]) (*simp-all add*: *field-simps*)

lemma *ereal-left-distrib*:
fixes *r a b* :: *ereal*
shows $0 \leq a \Longrightarrow 0 \leq b \Longrightarrow (a + b) * r = a * r + b * r$
by (*cases rule*: *ereal3-cases*[*of r a b*]) (*simp-all add*: *field-simps*)

lemma *ereal-mult-le-0-iff*:
fixes *a b* :: *ereal*
shows $a * b \leq 0 \longleftrightarrow (0 \leq a \wedge b \leq 0) \vee (a \leq 0 \wedge 0 \leq b)$
by (*cases rule*: *ereal2-cases*[*of a b*]) (*simp-all add*: *mult-le-0-iff*)

lemma *ereal-zero-le-0-iff*:
fixes *a b* :: *ereal*
shows $0 \leq a * b \longleftrightarrow (0 \leq a \wedge 0 \leq b) \vee (a \leq 0 \wedge b \leq 0)$
by (*cases rule*: *ereal2-cases*[*of a b*]) (*simp-all add*: *zero-le-mult-iff*)

lemma *ereal-mult-less-0-iff*:
fixes *a b* :: *ereal*
shows $a * b < 0 \longleftrightarrow (0 < a \wedge b < 0) \vee (a < 0 \wedge 0 < b)$
by (*cases rule*: *ereal2-cases*[*of a b*]) (*simp-all add*: *mult-less-0-iff*)

lemma *ereal-zero-less-0-iff*:
fixes *a b* :: *ereal*
shows $0 < a * b \longleftrightarrow (0 < a \wedge 0 < b) \vee (a < 0 \wedge b < 0)$
by (*cases rule*: *ereal2-cases*[*of a b*]) (*simp-all add*: *zero-less-mult-iff*)

lemma *ereal-left-mult-cong*:
fixes *a b c* :: *ereal*
shows $c = d \Longrightarrow (d \neq 0 \Longrightarrow a = b) \Longrightarrow a * c = b * d$
by (*cases* $c = 0$) *simp-all*

lemma *ereal-right-mult-cong*:
fixes *a b c* :: *ereal*
shows $c = d \Longrightarrow (d \neq 0 \Longrightarrow a = b) \Longrightarrow c * a = d * b$
by (*cases* $c = 0$) *simp-all*

lemma *ereal-distrib*:
fixes *a b c* :: *ereal*
assumes $a \neq \infty \vee b \neq -\infty$
and $a \neq -\infty \vee b \neq \infty$
and $|c| \neq \infty$
shows $(a + b) * c = a * c + b * c$

using *assms*
by (*cases rule*: *ereal3-cases*[*of a b c*]) (*simp-all add*: *field-simps*)

lemma *numeral-eq-ereal* [*simp*]: *numeral w* = *ereal* (*numeral w*)
apply (*induct w rule*: *num-induct*)
apply (*simp only*: *numeral-One one-ereal-def*)
apply (*simp only*: *numeral-inc ereal-plus-1*)
done

lemma *distrib-left-ereal-nn*:
c ≥ *0* ⟹ (*x* + *y*) ∗ *ereal c* = *x* ∗ *ereal c* + *y* ∗ *ereal c*
by(*cases x y rule*: *ereal2-cases*)(*simp-all add*: *ring-distribs*)

lemma *setsum-ereal-right-distrib*:
fixes *f* :: *'a* ⇒ *ereal*
shows (⋀*i*. *i* ∈ *A* ⟹ *0* ≤ *f i*) ⟹ *r* ∗ *setsum f A* = (∑ *n*∈*A*. *r* ∗ *f n*)
by (*induct A rule*: *infinite-finite-induct*) (*auto simp*: *ereal-right-distrib setsum-nonneg*)

lemma *setsum-ereal-left-distrib*:
(⋀*i*. *i* ∈ *A* ⟹ *0* ≤ *f i*) ⟹ *setsum f A* ∗ *r* = (∑ *n*∈*A*. *f n* ∗ *r* :: *ereal*)
using *setsum-ereal-right-distrib*[*of A f r*] **by** (*simp add*: *mult-ac*)

lemma *setsum-left-distrib-ereal*:
c ≥ *0* ⟹ *setsum f A* ∗ *ereal c* = (∑ *x*∈*A*. *f x* ∗ *c* :: *ereal*)
by(*subst setsum-comp-morphism*[**where** *h*=λ*x*. *x* ∗ *ereal c*, *symmetric*])(*simp-all add*: *distrib-left-ereal-nn*)

lemma *ereal-le-epsilon*:
fixes *x y* :: *ereal*
assumes ∀ *e*. *0* < *e* ⟶ *x* ≤ *y* + *e*
shows *x* ≤ *y*
proof −
{
assume *a*: ∃ *r*. *y* = *ereal r*
then obtain *r* **where** *r-def*: *y* = *ereal r*
by *auto*
{
assume *x* = −∞
then have *?thesis* **by** *auto*
}
moreover
{
assume *x* ≠ −∞
then obtain *p* **where** *p-def*: *x* = *ereal p*
using *a assms*[*rule-format*, *of 1*]
by (*cases x*) *auto*
{
fix *e*
have *0* < *e* ⟶ *p* ≤ *r* + *e*

```
        using assms[rule-format, of ereal e] p-def r-def by auto
      }
      then have p ≤ r
        apply (subst field-le-epsilon)
        apply auto
        done
      then have ?thesis
        using r-def p-def by auto
    }
    ultimately have ?thesis
      by blast
  }
  moreover
  {
    assume y = −∞ | y = ∞
    then have ?thesis
      using assms[rule-format, of 1] by (cases x) auto
  }
  ultimately show ?thesis
    by (cases y) auto
qed

lemma ereal-le-epsilon2:
  fixes x y :: ereal
  assumes ∀ e. 0 < e ⟶ x ≤ y + ereal e
  shows x ≤ y
proof −
  {
    fix e :: ereal
    assume e > 0
    {
      assume e = ∞
      then have x ≤ y + e
        by auto
    }
    moreover
    {
      assume e ≠ ∞
      then obtain r where e = ereal r
        using ‹e > 0› by (cases e) auto
      then have x ≤ y + e
        using assms[rule-format, of r] ‹e>0› by auto
    }
    ultimately have x ≤ y + e
      by blast
  }
  then show ?thesis
    using ereal-le-epsilon by auto
qed
```

lemma *ereal-le-real*:
fixes $x\ y :: ereal$
assumes $\forall z.\ x \leq ereal\ z \longrightarrow y \leq ereal\ z$
shows $y \leq x$
by (*metis assms ereal-bot ereal-cases ereal-infty-less-eq*(*2*) *ereal-less-eq*(*1*) *linorder-le-cases*)

lemma *setprod-ereal-0*:
fixes $f :: 'a \Rightarrow ereal$
shows $(\prod i{\in}A.\ f\ i) = 0 \longleftrightarrow finite\ A \wedge (\exists i{\in}A.\ f\ i = 0)$
proof (*cases finite A*)
case *True*
then show *?thesis* **by** (*induct A*) *auto*
next
case *False*
then show *?thesis* **by** *auto*
qed

lemma *setprod-ereal-pos*:
fixes $f :: 'a \Rightarrow ereal$
assumes *pos*: $\bigwedge i.\ i \in I \Longrightarrow 0 \leq f\ i$
shows $0 \leq (\prod i{\in}I.\ f\ i)$
proof (*cases finite I*)
case *True*
from *this pos* **show** *?thesis*
by *induct auto*
next
case *False*
then show *?thesis* **by** *simp*
qed

lemma *setprod-PInf*:
fixes $f :: 'a \Rightarrow ereal$
assumes $\bigwedge i.\ i \in I \Longrightarrow 0 \leq f\ i$
shows $(\prod i{\in}I.\ f\ i) = \infty \longleftrightarrow finite\ I \wedge (\exists i{\in}I.\ f\ i = \infty) \wedge (\forall i{\in}I.\ f\ i \neq 0)$
proof (*cases finite I*)
case *True*
from *this assms* **show** *?thesis*
proof (*induct I*)
case (*insert i I*)
then have *pos*: $0 \leq f\ i\ 0 \leq setprod\ f\ I$
by (*auto intro*!: *setprod-ereal-pos*)
from *insert* **have** $(\prod j{\in}insert\ i\ I.\ f\ j) = \infty \longleftrightarrow setprod\ f\ I * f\ i = \infty$
by *auto*
also have $\ldots \longleftrightarrow (setprod\ f\ I = \infty \vee f\ i = \infty) \wedge f\ i \neq 0 \wedge setprod\ f\ I \neq 0$
using *setprod-ereal-pos*[*of I f*] *pos*
by (*cases rule*: *ereal2-cases*[*of f i setprod f I*]) *auto*
also have $\ldots \longleftrightarrow finite\ (insert\ i\ I) \wedge (\exists j{\in}insert\ i\ I.\ f\ j = \infty) \wedge (\forall j{\in}insert\ i\ I.\ f\ j \neq 0)$

```
    using insert by (auto simp: setprod-ereal-0)
  finally show ?case .
 qed simp
next
 case False
 then show ?thesis by simp
qed

lemma setprod-ereal: (∏ i∈A. ereal (f i)) = ereal (setprod f A)
proof (cases finite A)
 case True
 then show ?thesis
  by induct (auto simp: one-ereal-def)
next
 case False
 then show ?thesis
  by (simp add: one-ereal-def)
qed
```

36.1.4 Power

```
lemma ereal-power[simp]: (ereal x) ^ n = ereal (x^n)
 by (induct n) (auto simp: one-ereal-def)

lemma ereal-power-PInf[simp]: (∞::ereal) ^ n = (if n = 0 then 1 else ∞)
 by (induct n) (auto simp: one-ereal-def)

lemma ereal-power-uminus[simp]:
 fixes x :: ereal
 shows (− x) ^ n = (if even n then x ^ n else − (x^n))
 by (induct n) (auto simp: one-ereal-def)

lemma ereal-power-numeral[simp]:
 (numeral num :: ereal) ^ n = ereal (numeral num ^ n)
 by (induct n) (auto simp: one-ereal-def)

lemma zero-le-power-ereal[simp]:
 fixes a :: ereal
 assumes 0 ≤ a
 shows 0 ≤ a ^ n
 using assms by (induct n) (auto simp: ereal-zero-le-0-iff)
```

36.1.5 Subtraction

```
lemma ereal-minus-minus-image[simp]:
 fixes S :: ereal set
 shows uminus ‘ uminus ‘ S = S
 by (auto simp: image-iff)

lemma ereal-uminus-lessThan[simp]:
```

fixes a :: *ereal*
shows *uminus* ' $\{..<a\} = \{-a<..\}$
proof −
{
fix x
assume $-a < x$
then have $- x < - (- a)$
by (*simp del*: *ereal-uminus-uminus*)
then have $- x < a$
by *simp*
}
then show *?thesis*
by *force*
qed

lemma *ereal-uminus-greaterThan*[*simp*]: *uminus* ' $\{(a::ereal)<..\} = \{..<-a\}$
by (*metis ereal-uminus-lessThan ereal-uminus-uminus ereal-minus-minus-image*)

instantiation *ereal* :: *minus*
begin

definition $x - y = x + -(y::ereal)$
instance ..

end

lemma *ereal-minus*[*simp*]:
$ereal\ r - ereal\ p = ereal\ (r - p)$
$-\infty - ereal\ r = -\infty$
$ereal\ r - \infty = -\infty$
$(\infty::ereal) - x = \infty$
$-(\infty::ereal) - \infty = -\infty$
$x - -y = x + y$
$x - 0 = x$
$0 - x = -x$
by (*simp-all add*: *minus-ereal-def*)

lemma *ereal-x-minus-x*[*simp*]: $x - x = (if\ |x| = \infty\ then\ \infty\ else\ 0::ereal)$
by (*cases x*) *simp-all*

lemma *ereal-eq-minus-iff*:
fixes $x\ y\ z$ :: *ereal*
shows $x = z - y \longleftrightarrow$
$(|y| \neq \infty \longrightarrow x + y = z) \wedge$
$(y = -\infty \longrightarrow x = \infty) \wedge$
$(y = \infty \longrightarrow z = \infty \longrightarrow x = \infty) \wedge$
$(y = \infty \longrightarrow z \neq \infty \longrightarrow x = -\infty)$
by (*cases rule*: *ereal3-cases*[*of x y z*]) *auto*

lemma *ereal-eq-minus*:
fixes $x\ y\ z$ :: *ereal*
shows $|y| \neq \infty \Longrightarrow x = z - y \longleftrightarrow x + y = z$
by (*auto simp*: *ereal-eq-minus-iff*)

lemma *ereal-less-minus-iff*:
fixes $x\ y\ z$ :: *ereal*
shows $x < z - y \longleftrightarrow$
$(y = \infty \longrightarrow z = \infty \wedge x \neq \infty) \wedge$
$(y = -\infty \longrightarrow x \neq \infty) \wedge$
$(|y| \neq \infty \longrightarrow x + y < z)$
by (*cases rule*: *ereal3-cases*[*of x y z*]) *auto*

lemma *ereal-less-minus*:
fixes $x\ y\ z$ :: *ereal*
shows $|y| \neq \infty \Longrightarrow x < z - y \longleftrightarrow x + y < z$
by (*auto simp*: *ereal-less-minus-iff*)

lemma *ereal-le-minus-iff*:
fixes $x\ y\ z$ :: *ereal*
shows $x \leq z - y \longleftrightarrow (y = \infty \longrightarrow z \neq \infty \longrightarrow x = -\infty) \wedge (|y| \neq \infty \longrightarrow x + y \leq z)$
by (*cases rule*: *ereal3-cases*[*of x y z*]) *auto*

lemma *ereal-le-minus*:
fixes $x\ y\ z$ :: *ereal*
shows $|y| \neq \infty \Longrightarrow x \leq z - y \longleftrightarrow x + y \leq z$
by (*auto simp*: *ereal-le-minus-iff*)

lemma *ereal-minus-less-iff*:
fixes $x\ y\ z$ :: *ereal*
shows $x - y < z \longleftrightarrow y \neq -\infty \wedge (y = \infty \longrightarrow x \neq \infty \wedge z \neq -\infty) \wedge (y \neq \infty \longrightarrow x < z + y)$
by (*cases rule*: *ereal3-cases*[*of x y z*]) *auto*

lemma *ereal-minus-less*:
fixes $x\ y\ z$ :: *ereal*
shows $|y| \neq \infty \Longrightarrow x - y < z \longleftrightarrow x < z + y$
by (*auto simp*: *ereal-minus-less-iff*)

lemma *ereal-minus-le-iff*:
fixes $x\ y\ z$ :: *ereal*
shows $x - y \leq z \longleftrightarrow$
$(y = -\infty \longrightarrow z = \infty) \wedge$
$(y = \infty \longrightarrow x = \infty \longrightarrow z = \infty) \wedge$
$(|y| \neq \infty \longrightarrow x \leq z + y)$
by (*cases rule*: *ereal3-cases*[*of x y z*]) *auto*

lemma *ereal-minus-le*:

fixes $x\ y\ z :: ereal$
shows $|y| \neq \infty \Longrightarrow x - y \leq z \longleftrightarrow x \leq z + y$
by (*auto simp*: *ereal-minus-le-iff*)

lemma *ereal-minus-eq-minus-iff*:
fixes $a\ b\ c :: ereal$
shows $a - b = a - c \longleftrightarrow$
$b = c \vee a = \infty \vee (a = -\infty \wedge b \neq -\infty \wedge c \neq -\infty)$
by (*cases rule*: *ereal3-cases*[*of a b c*]) *auto*

lemma *ereal-add-le-add-iff*:
fixes $a\ b\ c :: ereal$
shows $c + a \leq c + b \longleftrightarrow$
$a \leq b \vee c = \infty \vee (c = -\infty \wedge a \neq \infty \wedge b \neq \infty)$
by (*cases rule*: *ereal3-cases*[*of a b c*]) (*simp-all add*: *field-simps*)

lemma *ereal-add-le-add-iff2*:
fixes $a\ b\ c :: ereal$
shows $a + c \leq b + c \longleftrightarrow a \leq b \vee c = \infty \vee (c = -\infty \wedge a \neq \infty \wedge b \neq \infty)$
by(*cases rule*: *ereal3-cases*[*of a b c*])(*simp-all add*: *field-simps*)

lemma *ereal-mult-le-mult-iff*:
fixes $a\ b\ c :: ereal$
shows $|c| \neq \infty \Longrightarrow c * a \leq c * b \longleftrightarrow (0 < c \longrightarrow a \leq b) \wedge (c < 0 \longrightarrow b \leq a)$
by (*cases rule*: *ereal3-cases*[*of a b c*]) (*simp-all add*: *mult-le-cancel-left*)

lemma *ereal-minus-mono*:
fixes $A\ B\ C\ D :: ereal$ **assumes** $A \leq B\ D \leq C$
shows $A - C \leq B - D$
using *assms*
by (*cases rule*: *ereal3-cases*[*case-product ereal-cases*, *of A B C D*]) *simp-all*

lemma *ereal-mono-minus-cancel*:
fixes $a\ b\ c :: ereal$
shows $c - a \leq c - b \Longrightarrow 0 \leq c \Longrightarrow c < \infty \Longrightarrow b \leq a$
by (*cases a b c rule*: *ereal3-cases*) *auto*

lemma *real-of-ereal-minus*:
fixes $a\ b :: ereal$
shows *real-of-ereal* $(a - b) =$ (*if* $|a| = \infty \vee |b| = \infty$ *then 0 else real-of-ereal a* $-$ *real-of-ereal b*)
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

lemma *real-of-ereal-minus'*: $|x| = \infty \longleftrightarrow |y| = \infty \Longrightarrow$ *real-of-ereal x* $-$ *real-of-ereal y* $=$ *real-of-ereal* $(x - y :: ereal)$
by(*subst real-of-ereal-minus*) *auto*

lemma *ereal-diff-positive*:
fixes $a\ b :: ereal$ **shows** $a \leq b \Longrightarrow 0 \leq b - a$

by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

lemma *ereal-between*:
fixes $x\ e$ :: *ereal*
assumes $|x| \neq \infty$
and $0 < e$
shows $x - e < x$
and $x < x + e$
using *assms*
apply (*cases x*, *cases e*)
apply *auto*
using *assms*
apply (*cases x*, *cases e*)
apply *auto*
done

lemma *ereal-minus-eq-PInfty-iff*:
fixes $x\ y$ :: *ereal*
shows $x - y = \infty \longleftrightarrow y = -\infty \vee x = \infty$
by (*cases x y rule*: *ereal2-cases*) *simp-all*

lemma *ereal-diff-add-eq-diff-diff-swap*:
fixes $x\ y\ z$ :: *ereal*
shows $|y| \neq \infty \Longrightarrow x - (y + z) = x - y - z$
by(*cases x y z rule*: *ereal3-cases*) *simp-all*

lemma *ereal-diff-add-assoc2*:
fixes $x\ y\ z$ :: *ereal*
shows $x + y - z = x - z + y$
by(*cases x y z rule*: *ereal3-cases*) *simp-all*

lemma *ereal-add-uminus-conv-diff*: **fixes** $x\ y\ z$ :: *ereal* **shows** $-x + y = y - x$
by(*cases x y rule*: *ereal2-cases*) *simp-all*

lemma *ereal-minus-diff-eq*:
fixes $x\ y$ :: *ereal*
shows $[\![x = \infty \longrightarrow y \neq \infty; x = -\infty \longrightarrow y \neq -\infty]\!] \Longrightarrow -(x - y) = y - x$
by(*cases x y rule*: *ereal2-cases*) *simp-all*

lemma *ediff-le-self* [*simp*]: $x - y \leq (x$ :: *enat*$)$
by(*cases x y rule*: *enat.exhaust*[*case-product enat.exhaust*]) *simp-all*

36.1.6 Division

instantiation *ereal* :: *inverse*
begin

function *inverse-ereal* **where**
inverse (*ereal r*) = (*if r* = *0 then* ∞ *else ereal* (*inverse r*))

| *inverse* (∞::*ereal*) = *0*
| *inverse* (−∞::*ereal*) = *0*
by (*auto intro*: *ereal-cases*)
termination by (*relation* {}) *simp*

definition *x div y* = *x* ∗ *inverse* (*y* :: *ereal*)

instance ..

end

lemma *real-of-ereal-inverse*[*simp*]:
fixes *a* :: *ereal*
shows *real-of-ereal* (*inverse a*) = *1* / *real-of-ereal a*
by (*cases a*) (*auto simp*: *inverse-eq-divide*)

lemma *ereal-inverse*[*simp*]:
inverse (*0*::*ereal*) = ∞
inverse (*1*::*ereal*) = *1*
by (*simp-all add*: *one-ereal-def zero-ereal-def*)

lemma *ereal-divide*[*simp*]:
ereal r / *ereal p* = (*if p* = *0 then ereal r* ∗ ∞ *else ereal* (*r* / *p*))
unfolding *divide-ereal-def* **by** (*auto simp*: *divide-real-def*)

lemma *ereal-divide-same*[*simp*]:
fixes *x* :: *ereal*
shows *x* / *x* = (*if* |*x*| = ∞ ∨ *x* = *0 then 0 else 1*)
by (*cases x*) (*simp-all add*: *divide-real-def divide-ereal-def one-ereal-def*)

lemma *ereal-inv-inv*[*simp*]:
fixes *x* :: *ereal*
shows *inverse* (*inverse x*) = (*if x* ≠ −∞ *then x else* ∞)
by (*cases x*) *auto*

lemma *ereal-inverse-minus*[*simp*]:
fixes *x* :: *ereal*
shows *inverse* (− *x*) = (*if x* = *0 then* ∞ *else* −*inverse x*)
by (*cases x*) *simp-all*

lemma *ereal-uminus-divide*[*simp*]:
fixes *x y* :: *ereal*
shows − *x* / *y* = − (*x* / *y*)
unfolding *divide-ereal-def* **by** *simp*

lemma *ereal-divide-Infty*[*simp*]:
fixes *x* :: *ereal*
shows *x* / ∞ = *0 x* / −∞ = *0*
unfolding *divide-ereal-def* **by** *simp-all*

lemma *ereal-divide-one*[*simp*]: x / $1 = (x::ereal)$
unfolding *divide-ereal-def* **by** *simp*

lemma *ereal-divide-ereal*[*simp*]: ∞ / *ereal* r = (*if* $0 \leq r$ *then* ∞ *else* $-\infty$)
unfolding *divide-ereal-def* **by** *simp*

lemma *ereal-inverse-nonneg-iff*: $0 \leq$ *inverse* $(x ::$ *ereal*$) \longleftrightarrow 0 \leq x \vee x = -\infty$
by (*cases* x) *auto*

lemma *inverse-ereal-ge0I*: $0 \leq (x ::$ *ereal*$) \Longrightarrow 0 \leq$ *inverse* x
by(*cases* x) *simp-all*

lemma *zero-le-divide-ereal*[*simp*]:
fixes $a ::$ *ereal*
assumes $0 \leq a$
and $0 \leq b$
shows $0 \leq a$ / b
using *assms* **by** (*cases rule*: *ereal2-cases*[*of a b*]) (*auto simp*: *zero-le-divide-iff*)

lemma *ereal-le-divide-pos*:
fixes $x\ y\ z ::$ *ereal*
shows $x > 0 \Longrightarrow x \neq \infty \Longrightarrow y \leq z$ / $x \longleftrightarrow x * y \leq z$
by (*cases rule*: *ereal3-cases*[*of x y z*]) (*auto simp*: *field-simps*)

lemma *ereal-divide-le-pos*:
fixes $x\ y\ z ::$ *ereal*
shows $x > 0 \Longrightarrow x \neq \infty \Longrightarrow z$ / $x \leq y \longleftrightarrow z \leq x * y$
by (*cases rule*: *ereal3-cases*[*of x y z*]) (*auto simp*: *field-simps*)

lemma *ereal-le-divide-neg*:
fixes $x\ y\ z ::$ *ereal*
shows $x < 0 \Longrightarrow x \neq -\infty \Longrightarrow y \leq z$ / $x \longleftrightarrow z \leq x * y$
by (*cases rule*: *ereal3-cases*[*of x y z*]) (*auto simp*: *field-simps*)

lemma *ereal-divide-le-neg*:
fixes $x\ y\ z ::$ *ereal*
shows $x < 0 \Longrightarrow x \neq -\infty \Longrightarrow z$ / $x \leq y \longleftrightarrow x * y \leq z$
by (*cases rule*: *ereal3-cases*[*of x y z*]) (*auto simp*: *field-simps*)

lemma *ereal-inverse-antimono-strict*:
fixes $x\ y ::$ *ereal*
shows $0 \leq x \Longrightarrow x < y \Longrightarrow$ *inverse* $y <$ *inverse* x
by (*cases rule*: *ereal2-cases*[*of x y*]) *auto*

lemma *ereal-inverse-antimono*:
fixes $x\ y ::$ *ereal*
shows $0 \leq x \Longrightarrow x \leq y \Longrightarrow$ *inverse* $y \leq$ *inverse* x
by (*cases rule*: *ereal2-cases*[*of x y*]) *auto*

lemma *inverse-inverse-Pinfty-iff* [*simp*]:
fixes x :: *ereal*
shows *inverse* $x = \infty \longleftrightarrow x = 0$
by (*cases* x) *auto*

lemma *ereal-inverse-eq-0*:
fixes x :: *ereal*
shows *inverse* $x = 0 \longleftrightarrow x = \infty \vee x = -\infty$
by (*cases* x) *auto*

lemma *ereal-0-gt-inverse*:
fixes x :: *ereal*
shows $0 <$ *inverse* $x \longleftrightarrow x \neq \infty \wedge 0 \leq x$
by (*cases* x) *auto*

lemma *ereal-inverse-le-0-iff*:
fixes x :: *ereal*
shows *inverse* $x \leq 0 \longleftrightarrow x < 0 \vee x = \infty$
by(*cases* x) *auto*

lemma *ereal-divide-eq-0-iff*: x / $y = 0 \longleftrightarrow x = 0 \vee |y :: ereal| = \infty$
by(*cases* x y *rule*: *ereal2-cases*) *simp-all*

lemma *ereal-mult-less-right*:
fixes a b c :: *ereal*
assumes $b * a < c * a$
and $0 < a$
and $a < \infty$
shows $b < c$
using *assms*
by (*cases rule*: *ereal3-cases*[*of a b c*])
(*auto split*: *if-split-asm simp*: *zero-less-mult-iff zero-le-mult-iff*)

lemma *ereal-mult-divide*: **fixes** a b :: *ereal* **shows** $0 < b \Longrightarrow b < \infty \Longrightarrow b * (a$ / $b) = a$
by (*cases* a b *rule*: *ereal2-cases*) *auto*

lemma *ereal-power-divide*:
fixes x y :: *ereal*
shows $y \neq 0 \Longrightarrow (x$ / $y)$ ^ $n = x$^n / y^n
by (*cases rule*: *ereal2-cases* [*of x y*])
(*auto simp*: *one-ereal-def zero-ereal-def power-divide zero-le-power-eq*)

lemma *ereal-le-mult-one-interval*:
fixes x y :: *ereal*
assumes y: $y \neq -\infty$
assumes z: $\bigwedge z.\ 0 < z \Longrightarrow z < 1 \Longrightarrow z * x \leq y$
shows $x \leq y$

proof (*cases x*)
case *PInf*
with *z*[*of 1 / 2*] **show** $x \leq y$
by (*simp add*: *one-ereal-def*)
next
case (*real r*)
note $r = this$
show $x \leq y$
proof (*cases y*)
case (*real p*)
note $p = this$
have $r \leq p$
proof (*rule field-le-mult-one-interval*)
fix $z :: real$
assume $0 < z$ **and** $z < 1$
with *z*[*of ereal z*] **show** $z * r \leq p$
using *p r* **by** (*auto simp*: *zero-le-mult-iff one-ereal-def*)
qed
then show $x \leq y$
using *p r* **by** *simp*
qed (*insert y*, *simp-all*)
qed *simp*

lemma *ereal-divide-right-mono*[*simp*]:
fixes *x y z* :: *ereal*
assumes $x \leq y$
and $0 < z$
shows $x / z \leq y / z$
using *assms* **by** (*cases x y z rule*: *ereal3-cases*) (*auto intro*: *divide-right-mono*)

lemma *ereal-divide-left-mono*[*simp*]:
fixes *x y z* :: *ereal*
assumes $y \leq x$
and $0 < z$
and $0 < x * y$
shows $z / x \leq z / y$
using *assms*
by (*cases x y z rule*: *ereal3-cases*)
(*auto intro*: *divide-left-mono simp*: *field-simps zero-less-mult-iff mult-less-0-iff split*: *if-split-asm*)

lemma *ereal-divide-zero-left*[*simp*]:
fixes *a* :: *ereal*
shows $0 / a = 0$
by (*cases a*) (*auto simp*: *zero-ereal-def*)

lemma *ereal-times-divide-eq-left*[*simp*]:
fixes *a b c* :: *ereal*
shows $b / c * a = b * a / c$

by (*cases a b c rule*: *ereal3-cases*) (*auto simp*: *field-simps zero-less-mult-iff mult-less-0-iff*)

lemma *ereal-times-divide-eq*: $a * (b / c :: ereal) = a * b / c$
by (*cases a b c rule*: *ereal3-cases*)
(*auto simp*: *field-simps zero-less-mult-iff*)

lemma *ereal-inverse-real*: $|z| \neq \infty \Longrightarrow z \neq 0 \Longrightarrow ereal\ (inverse\ (real\text{-}of\text{-}ereal\ z))$
$= inverse\ z$
by (*cases z*) *simp-all*

lemma *ereal-inverse-mult*:
$a \neq 0 \Longrightarrow b \neq 0 \Longrightarrow inverse\ (a * (b::ereal)) = inverse\ a * inverse\ b$
by (*cases a*; *cases b*) *auto*

36.2 Complete lattice

instantiation *ereal* :: *lattice*
begin

definition [*simp*]: $sup\ x\ y = (max\ x\ y :: ereal)$
definition [*simp*]: $inf\ x\ y = (min\ x\ y :: ereal)$
instance by *standard simp-all*

end

instantiation *ereal* :: *complete-lattice*
begin

definition $bot = (-\infty::ereal)$
definition $top = (\infty::ereal)$

definition $Sup\ S = (SOME\ x :: ereal.\ (\forall\, y{\in}S.\ y \leq x) \wedge (\forall\, z.\ (\forall\, y{\in}S.\ y \leq z) \longrightarrow x \leq z))$
definition $Inf\ S = (SOME\ x :: ereal.\ (\forall\, y{\in}S.\ x \leq y) \wedge (\forall\, z.\ (\forall\, y{\in}S.\ z \leq y) \longrightarrow z \leq x))$

lemma *ereal-complete-Sup*:
fixes $S :: ereal\ set$
shows $\exists\, x.\ (\forall\, y{\in}S.\ y \leq x) \wedge (\forall\, z.\ (\forall\, y{\in}S.\ y \leq z) \longrightarrow x \leq z)$
proof (*cases* $\exists\, x.\ \forall\, a{\in}S.\ a \leq ereal\ x$)
case *True*
then obtain *y* **where** *y*: $\bigwedge a.\ a{\in}S \Longrightarrow a \leq ereal\ y$
by *auto*
then have $\infty \notin S$
by *force*
show *?thesis*
proof (*cases* $S \neq \{-\infty\} \wedge S \neq \{\}$)
case *True*
with ‹$\infty \notin S$› **obtain** *x* **where** *x*: $x \in S\ |x| \neq \infty$

```
      by auto
    obtain s where s: ∀x∈ereal -` S. x ≤ s ⋀z. (∀x∈ereal -` S. x ≤ z) ⟹ s
≤ z
    proof (atomize-elim, rule complete-real)
      show ∃x. x ∈ ereal -` S
        using x by auto
      show ∃z. ∀x∈ereal -` S. x ≤ z
        by (auto dest: y intro!: exI[of - y])
    qed
    show ?thesis
    proof (safe intro!: exI[of - ereal s])
      fix y
      assume y ∈ S
      with s ‹∞ ∉ S› show y ≤ ereal s
        by (cases y) auto
    next
      fix z
      assume ∀y∈S. y ≤ z
      with ‹S ≠ {-∞} ∧ S ≠ {}› show ereal s ≤ z
        by (cases z) (auto intro!: s)
    qed
  next
    case False
    then show ?thesis
      by (auto intro!: exI[of - -∞])
  qed
next
  case False
  then show ?thesis
    by (fastforce intro!: exI[of - ∞] ereal-top intro: order-trans dest: less-imp-le
simp: not-le)
qed

lemma ereal-complete-uminus-eq:
  fixes S :: ereal set
  shows (∀y∈uminus`S. y ≤ x) ∧ (∀z. (∀y∈uminus`S. y ≤ z) ⟶ x ≤ z)
    ⟷ (∀y∈S. -x ≤ y) ∧ (∀z. (∀y∈S. z ≤ y) ⟶ z ≤ -x)
  by simp (metis ereal-minus-le-minus ereal-uminus-uminus)

lemma ereal-complete-Inf:
  ∃x. (∀y∈S::ereal set. x ≤ y) ∧ (∀z. (∀y∈S. z ≤ y) ⟶ z ≤ x)
  using ereal-complete-Sup[of uminus ` S]
  unfolding ereal-complete-uminus-eq
  by auto

instance
proof
  show Sup {} = (bot::ereal)
    apply (auto simp: bot-ereal-def Sup-ereal-def)
```

apply (*rule some1-equality*)
apply (*metis ereal-bot ereal-less-eq*(*2*))
apply (*metis ereal-less-eq*(*2*))
done
show *Inf* {} = (*top*::*ereal*)
apply (*auto simp*: *top-ereal-def Inf-ereal-def*)
apply (*rule some1-equality*)
apply (*metis ereal-top ereal-less-eq*(*1*))
apply (*metis ereal-less-eq*(*1*))
done
qed (*auto intro*: *someI2-ex ereal-complete-Sup ereal-complete-Inf*
simp: *Sup-ereal-def Inf-ereal-def bot-ereal-def top-ereal-def*)

end

instance *ereal* :: *complete-linorder* **..**

instance *ereal* :: *linear-continuum*
proof
show ∃ *a b*::*ereal*. *a* ≠ *b*
using *zero-neq-one* **by** *blast*
qed

36.2.1 Topological space

instantiation *ereal* :: *linear-continuum-topology*
begin

definition *open-ereal* :: *ereal set* ⇒ *bool* **where**
open-ereal-generated: *open-ereal* = *generate-topology* (*range lessThan* ∪ *range greaterThan*)

instance
by *standard* (*simp add*: *open-ereal-generated*)

end

lemma *continuous-on-ereal*[*continuous-intros*]:
assumes *f*: *continuous-on s f* **shows** *continuous-on s* (λ*x*. *ereal* (*f x*))
by (*rule continuous-on-compose2* [*OF continuous-onI-mono*[*of ereal UNIV*] *f*])
auto

lemma *tendsto-ereal*[*tendsto-intros*, *simp*, *intro*]: (*f* ⟶ *x*) *F* ⟹ ((λ*x*. *ereal* (*f x*)) ⟶ *ereal x*) *F*
using *isCont-tendsto-compose*[*of x ereal f F*] *continuous-on-ereal*[*of UNIV* λ*x*. *x*]
by (*simp add*: *continuous-on-eq-continuous-at*)

lemma *tendsto-uminus-ereal*[*tendsto-intros*, *simp*, *intro*]: (*f* ⟶ *x*) *F* ⟹ ((λ*x*.

$- f\ x$::*ereal*) $\longrightarrow - x)\ F$
apply (*rule tendsto-compose*[**where** *g*=*uminus*])
apply (*auto intro*!: *order-tendstoI simp*: *eventually-at-topological*)
apply (*rule-tac x*={..< $-a$} **in** *exI*)
apply (*auto split*: *ereal.split simp*: *ereal-less-uminus-reorder*) []
apply (*rule-tac x*={$-$ *a* <..} **in** *exI*)
apply (*auto split*: *ereal.split simp*: *ereal-uminus-reorder*) []
done

lemma *at-infty-ereal-eq-at-top*: *at* ∞ = *filtermap ereal at-top*
unfolding *filter-eq-iff eventually-at-filter eventually-at-top-linorder eventually-filtermap top-ereal-def*[*symmetric*]
apply (*subst eventually-nhds-top*[*of 0*])
apply (*auto simp*: *top-ereal-def less-le ereal-all-split ereal-ex-split*)
apply (*metis PInfty-neq-ereal*(*2*) *ereal-less-eq*(*3*) *ereal-top le-cases order-trans*)
done

lemma *ereal-Lim-uminus*: (*f* $\longrightarrow$ *f0*) *net* $\longleftrightarrow$ (($\lambda x.\ -$ *f x*::*ereal*) $\longrightarrow -$ *f0*) *net*
using *tendsto-uminus-ereal*[*of f f0 net*] *tendsto-uminus-ereal*[*of* $\lambda x.\ -$ *f x* $-$ *f0 net*]
by *auto*

lemma *ereal-divide-less-iff*: *0* < (*c*::*ereal*) $\Longrightarrow$ *c* < ∞ $\Longrightarrow$ *a* / *c* < *b* $\longleftrightarrow$ *a* < *b* * *c*
by (*cases a b c rule*: *ereal3-cases*) (*auto simp*: *field-simps*)

lemma *ereal-less-divide-iff*: *0* < (*c*::*ereal*) $\Longrightarrow$ *c* < ∞ $\Longrightarrow$ *a* < *b* / *c* $\longleftrightarrow$ *a* * *c* < *b*
by (*cases a b c rule*: *ereal3-cases*) (*auto simp*: *field-simps*)

lemma *tendsto-cmult-ereal*[*tendsto-intros*, *simp*, *intro*]:
assumes *c*: |*c*| $\neq \infty$ **and** *f*: (*f* $\longrightarrow$ *x*) *F* **shows** (($\lambda x.\ c * f\ x$::*ereal*) $\longrightarrow$ *c* * *x*) *F*
proof $-$
{ fix *c* :: *ereal* **assume** *0* < *c c* < ∞
then have (($\lambda x.\ c * f\ x$::*ereal*) $\longrightarrow$ *c* * *x*) *F*
apply (*intro tendsto-compose*[*OF - f*])
apply (*auto intro*!: *order-tendstoI simp*: *eventually-at-topological*)
apply (*rule-tac x*={*a*/*c* <..} **in** *exI*)
apply (*auto split*: *ereal.split simp*: *ereal-divide-less-iff mult.commute*) []
apply (*rule-tac x*={..< *a*/*c*} **in** *exI*)
apply (*auto split*: *ereal.split simp*: *ereal-less-divide-iff mult.commute*) []
done }
note * = *this*

have ((*0* < *c* $\wedge$ *c* < ∞) $\vee$ ($-\infty$ < *c* $\wedge$ *c* < *0*) $\vee$ *c* = *0*)
using *c* **by** (*cases c*) *auto*
then show *?thesis*

```
  proof (elim disjE conjE)
    assume − ∞ < c c < 0
    then have 0 < − c − c < ∞
      by (auto simp: ereal-uminus-reorder ereal-less-uminus-reorder[of 0])
    then have ((λx. (− c) * f x) ⟶ (− c) * x) F
      by (rule *)
    from tendsto-uminus-ereal[OF this] show ?thesis
      by simp
  qed (auto intro!: *)
qed

lemma tendsto-cmult-ereal-not-0[tendsto-intros, simp, intro]:
  assumes x ≠ 0 and f: (f ⟶ x) F shows ((λx. c * f x::ereal) ⟶ c * x)
F
proof cases
  assume |c| = ∞
  show ?thesis
  proof (rule filterlim-cong[THEN iffD1, OF refl refl - tendsto-const])
    have 0 < x ∨ x < 0
      using ‹x ≠ 0› by (auto simp add: neq-iff)
    then show eventually (λx'. c * x = c * f x') F
    proof
      assume 0 < x from order-tendstoD(1)[OF f this] show ?thesis
        by eventually-elim (insert ‹0<x› ‹|c| = ∞›, auto)
    next
      assume x < 0 from order-tendstoD(2)[OF f this] show ?thesis
        by eventually-elim (insert ‹x<0› ‹|c| = ∞›, auto)
    qed
  qed
qed (rule tendsto-cmult-ereal[OF - f])

lemma tendsto-cadd-ereal[tendsto-intros, simp, intro]:
  assumes c: y ≠ − ∞ x ≠ − ∞ and f: (f ⟶ x) F shows ((λx. f x + y::ereal)
⟶ x + y) F
  apply (intro tendsto-compose[OF - f])
  apply (auto intro!: order-tendstoI simp: eventually-at-topological)
  apply (rule-tac x={a − y <..} in exI)
  apply (auto split: ereal.split simp: ereal-minus-less-iff c) []
  apply (rule-tac x={..< a − y} in exI)
  apply (auto split: ereal.split simp: ereal-less-minus-iff c) []
  done

lemma tendsto-add-left-ereal[tendsto-intros, simp, intro]:
  assumes c: |y| ≠ ∞ and f: (f ⟶ x) F shows ((λx. f x + y::ereal) ⟶ x
+ y) F
  apply (intro tendsto-compose[OF - f])
  apply (auto intro!: order-tendstoI simp: eventually-at-topological)
  apply (rule-tac x={a − y <..} in exI)
  apply (insert c, auto split: ereal.split simp: ereal-minus-less-iff) []
```

apply (*rule-tac x*=$\{..< a - y\}$ **in** *exI*)
apply (*auto split*: *ereal.split simp*: *ereal-less-minus-iff c*) []
done

lemma *continuous-at-ereal*[*continuous-intros*]: *continuous F f* $\Longrightarrow$ *continuous F* (λx. *ereal* (*f x*))
unfolding *continuous-def* **by** *auto*

lemma *ereal-Sup*:
assumes $*$: $|SUP\ a{:}A.\ ereal\ a| \neq \infty$
shows *ereal* (*Sup A*) = (*SUP a*:*A*. *ereal a*)
proof (*rule continuous-at-Sup-mono*)
obtain *r* **where** *r*: *ereal r* = (*SUP a*:*A*. *ereal a*) *A* $\neq$ {}
using $*$ **by** (*force simp*: *bot-ereal-def*)
then show *bdd-above A A* $\neq$ {}
by (*auto intro*!: *SUP-upper bdd-aboveI*[*of* - *r*] *simp add*: *ereal-less-eq*(*3*)[*symmetric*] *simp del*: *ereal-less-eq*)
qed (*auto simp*: *mono-def continuous-at-imp-continuous-at-within continuous-at-ereal*)

lemma *ereal-SUP*: $|SUP\ a{:}A.\ ereal\ (f\ a)| \neq \infty \Longrightarrow$ *ereal* (*SUP a*:*A*. *f a*) = (*SUP a*:*A*. *ereal* (*f a*))
using *ereal-Sup*[*of f*‘*A*] **by** *auto*

lemma *ereal-Inf*:
assumes $*$: $|INF\ a{:}A.\ ereal\ a| \neq \infty$
shows *ereal* (*Inf A*) = (*INF a*:*A*. *ereal a*)
proof (*rule continuous-at-Inf-mono*)
obtain *r* **where** *r*: *ereal r* = (*INF a*:*A*. *ereal a*) *A* $\neq$ {}
using $*$ **by** (*force simp*: *top-ereal-def*)
then show *bdd-below A A* $\neq$ {}
by (*auto intro*!: *INF-lower bdd-belowI*[*of* - *r*] *simp add*: *ereal-less-eq*(*3*)[*symmetric*] *simp del*: *ereal-less-eq*)
qed (*auto simp*: *mono-def continuous-at-imp-continuous-at-within continuous-at-ereal*)

lemma *ereal-Inf'*:
assumes $*$: *bdd-below A A* $\neq$ {}
shows *ereal* (*Inf A*) = (*INF a*:*A*. *ereal a*)
proof (*rule ereal-Inf*)
from $*$ **obtain** *l u* **where** $\bigwedge x.\ x \in A \Longrightarrow l \leq x\ u \in A$
by (*auto simp*: *bdd-below-def*)
then have *l* $\leq$ (*INF x*:*A*. *ereal x*) (*INF x*:*A*. *ereal x*) $\leq$ *u*
by (*auto intro*!: *INF-greatest INF-lower*)
then show $|INF\ a{:}A.\ ereal\ a| \neq \infty$
by *auto*
qed

lemma *ereal-INF*: $|INF\ a{:}A.\ ereal\ (f\ a)| \neq \infty \Longrightarrow$ *ereal* (*INF a*:*A*. *f a*) = (*INF a*:*A*. *ereal* (*f a*))
using *ereal-Inf*[*of f*‘*A*] **by** *auto*

lemma *ereal-Sup-uminus-image-eq*: *Sup* (*uminus* ‘ *S*::*ereal set*) = − *Inf S*
by (*auto intro*!: *SUP-eqI*
simp: *Ball-def*[*symmetric*] *ereal-uminus-le-reorder le-Inf-iff*
intro!: *complete-lattice-class.Inf-lower2*)

lemma *ereal-SUP-uminus-eq*:
fixes *f* :: *'a* ⇒ *ereal*
shows (*SUP x*:*S*. *uminus* (*f x*)) = − (*INF x*:*S*. *f x*)
using *ereal-Sup-uminus-image-eq* [*of f* ‘ *S*] **by** (*simp add*: *comp-def*)

lemma *ereal-inj-on-uminus*[*intro*, *simp*]: *inj-on uminus* (*A* :: *ereal set*)
by (*auto intro*!: *inj-onI*)

lemma *ereal-Inf-uminus-image-eq*: *Inf* (*uminus* ‘ *S*::*ereal set*) = − *Sup S*
using *ereal-Sup-uminus-image-eq*[*of uminus* ‘ *S*] **by** *simp*

lemma *ereal-INF-uminus-eq*:
fixes *f* :: *'a* ⇒ *ereal*
shows (*INF x*:*S*. − *f x*) = − (*SUP x*:*S*. *f x*)
using *ereal-Inf-uminus-image-eq* [*of f* ‘ *S*] **by** (*simp add*: *comp-def*)

lemma *ereal-SUP-uminus*:
fixes *f* :: *'a* ⇒ *ereal*
shows (*SUP i* : *R*. − *f i*) = − (*INF i* : *R*. *f i*)
using *ereal-Sup-uminus-image-eq*[*of f*‘*R*]
by (*simp add*: *image-image*)

lemma *ereal-SUP-not-infty*:
fixes *f* :: - ⇒ *ereal*
shows *A* ≠ {} ⟹ *l* ≠ −∞ ⟹ *u* ≠ ∞ ⟹ ∀ *a*∈*A*. *l* ≤ *f a* ∧ *f a* ≤ *u* ⟹
|*SUPREMUM A f*| ≠ ∞
using *SUP-upper2*[*of* - *A l f*] *SUP-least*[*of A f u*]
by (*cases SUPREMUM A f*) *auto*

lemma *ereal-INF-not-infty*:
fixes *f* :: - ⇒ *ereal*
shows *A* ≠ {} ⟹ *l* ≠ −∞ ⟹ *u* ≠ ∞ ⟹ ∀ *a*∈*A*. *l* ≤ *f a* ∧ *f a* ≤ *u* ⟹
|*INFIMUM A f*| ≠ ∞
using *INF-lower2*[*of* - *A f u*] *INF-greatest*[*of A l f*]
by (*cases INFIMUM A f*) *auto*

lemma *ereal-image-uminus-shift*:
fixes *X Y* :: *ereal set*
shows *uminus* ‘ *X* = *Y* ⟷ *X* = *uminus* ‘ *Y*
proof
assume *uminus* ‘ *X* = *Y*
then have *uminus* ‘ *uminus* ‘ *X* = *uminus* ‘ *Y*
by (*simp add*: *inj-image-eq-iff*)

then show $X = uminus\ `\ Y$
by (*simp add*: *image-image*)
qed (*simp add*: *image-image*)

lemma *Sup-eq-MInfty*:
fixes S :: *ereal set*
shows $Sup\ S = -\infty \longleftrightarrow S = \{\} \vee S = \{-\infty\}$
unfolding *bot-ereal-def*[*symmetric*] **by** *auto*

lemma *Inf-eq-PInfty*:
fixes S :: *ereal set*
shows $Inf\ S = \infty \longleftrightarrow S = \{\} \vee S = \{\infty\}$
using *Sup-eq-MInfty*[*of uminus`S*]
unfolding *ereal-Sup-uminus-image-eq ereal-image-uminus-shift* **by** *simp*

lemma *Inf-eq-MInfty*:
fixes S :: *ereal set*
shows $-\infty \in S \Longrightarrow Inf\ S = -\infty$
unfolding *bot-ereal-def*[*symmetric*] **by** *auto*

lemma *Sup-eq-PInfty*:
fixes S :: *ereal set*
shows $\infty \in S \Longrightarrow Sup\ S = \infty$
unfolding *top-ereal-def*[*symmetric*] **by** *auto*

lemma *not-MInfty-nonneg*[*simp*]: $0 \leq (x::ereal) \Longrightarrow x \neq -\infty$
by *auto*

lemma *Sup-ereal-close*:
fixes e :: *ereal*
assumes $0 < e$
and S: $|Sup\ S| \neq \infty\ S \neq \{\}$
shows $\exists x{\in}S.\ Sup\ S - e < x$
using *assms* **by** (*cases e*) (*auto intro*!: *less-Sup-iff*[*THEN iffD1*])

lemma *Inf-ereal-close*:
fixes e :: *ereal*
assumes $|Inf\ X| \neq \infty$
and $0 < e$
shows $\exists x{\in}X.\ x < Inf\ X + e$
proof (*rule Inf-less-iff*[*THEN iffD1*])
show $Inf\ X < Inf\ X + e$
using *assms* **by** (*cases e*) *auto*
qed

lemma *SUP-PInfty*:
$(\bigwedge n::nat.\ \exists i{\in}A.\ ereal\ (real\ n) \leq f\ i) \Longrightarrow (SUP\ i{:}A.\ f\ i :: ereal) = \infty$
unfolding *top-ereal-def*[*symmetric*] *SUP-eq-top-iff*
by (*metis MInfty-neq-PInfty*(*2*) *PInfty-neq-ereal*(*2*) *less-PInf-Ex-of-nat less-ereal.elims*(*2*)

less-le-trans)

lemma *SUP-nat-Infty*: $(SUP\ i{::}nat.\ ereal\ (real\ i)) = \infty$
by (*rule SUP-PInfty*) *auto*

lemma *SUP-ereal-add-left*:
assumes $I \neq \{\}$ $c \neq -\infty$
shows $(SUP\ i{:}I.\ f\ i + c :: ereal) = (SUP\ i{:}I.\ f\ i) + c$
proof *cases*
assume $(SUP\ i{:}I.\ f\ i) = -\infty$
moreover then have $\bigwedge i.\ i \in I \Longrightarrow f\ i = -\infty$
unfolding *Sup-eq-MInfty* **by** *auto*
ultimately show *?thesis*
by (*cases c*) (*auto simp*: ‹$I \neq \{\}$›)
next
assume $(SUP\ i{:}I.\ f\ i) \neq -\infty$ **then show** *?thesis*
by (*subst continuous-at-Sup-mono*[**where** $f{=}\lambda x.\ x + c$])
(*auto simp*: *continuous-at-imp-continuous-at-within continuous-at mono-def ereal-add-mono* ‹$I \neq \{\}$› ‹$c \neq -\infty$›)
qed

lemma *SUP-ereal-add-right*:
fixes $c :: ereal$
shows $I \neq \{\} \Longrightarrow c \neq -\infty \Longrightarrow (SUP\ i{:}I.\ c + f\ i) = c + (SUP\ i{:}I.\ f\ i)$
using *SUP-ereal-add-left*[*of I c f*] **by** (*simp add*: *add.commute*)

lemma *SUP-ereal-minus-right*:
assumes $I \neq \{\}$ $c \neq -\infty$
shows $(SUP\ i{:}I.\ c - f\ i :: ereal) = c - (INF\ i{:}I.\ f\ i)$
using *SUP-ereal-add-right*[*OF assms*, *of* $\lambda i.\ -f\ i$]
by (*simp add*: *ereal-SUP-uminus minus-ereal-def*)

lemma *SUP-ereal-minus-left*:
assumes $I \neq \{\}$ $c \neq \infty$
shows $(SUP\ i{:}I.\ f\ i - c{::}\ ereal) = (SUP\ i{:}I.\ f\ i) - c$
using *SUP-ereal-add-left*[*OF* ‹$I \neq \{\}$›, *of* $-c\ f$] **by** (*simp add*: ‹$c \neq \infty$› *minus-ereal-def*)

lemma *INF-ereal-minus-right*:
assumes $I \neq \{\}$ **and** $|c| \neq \infty$
shows $(INF\ i{:}I.\ c - f\ i) = c - (SUP\ i{:}I.\ f\ i{::}ereal)$
proof −
{ fix *b* **have** $(-c) + b = -(c - b)$
using ‹$|c| \neq \infty$› **by** (*cases c b rule*: *ereal2-cases*) *auto* **}**
note $* = this$
show *?thesis*
using *SUP-ereal-add-right*[*OF* ‹$I \neq \{\}$›, *of* $-c\ f$] ‹$|c| \neq \infty$›
by (*auto simp add*: $*$ *ereal-SUP-uminus-eq*)
qed

lemma *SUP-ereal-le-addI*:
fixes $f :: 'i \Rightarrow ereal$
assumes $\bigwedge i.\ f\ i + y \leq z$ **and** $y \neq -\infty$
shows $SUPREMUM\ UNIV\ f + y \leq z$
unfolding *SUP-ereal-add-left*[*OF UNIV-not-empty* ‹$y \neq -\infty$›, *symmetric*]
by (*rule SUP-least assms*)+

lemma *SUP-combine*:
fixes $f :: 'a$::*semilattice-sup* $\Rightarrow 'a$::*semilattice-sup* $\Rightarrow 'b$::*complete-lattice*
assumes *mono*: $\bigwedge a\ b\ c\ d.\ a \leq b \Longrightarrow c \leq d \Longrightarrow f\ a\ c \leq f\ b\ d$
shows $(SUP\ i{:}UNIV.\ SUP\ j{:}UNIV.\ f\ i\ j) = (SUP\ i.\ f\ i\ i)$
proof (*rule antisym*)
show $(SUP\ i\ j.\ f\ i\ j) \leq (SUP\ i.\ f\ i\ i)$
by (*rule SUP-least SUP-upper2*[**where** *i*=*sup i j* **for** *i j*] *UNIV-I mono sup-ge1 sup-ge2*)+
show $(SUP\ i.\ f\ i\ i) \leq (SUP\ i\ j.\ f\ i\ j)$
by (*rule SUP-least SUP-upper2 UNIV-I mono order-refl*)+
qed

lemma *SUP-ereal-add*:
fixes $f\ g :: nat \Rightarrow ereal$
assumes *inc*: *incseq f incseq g*
and *pos*: $\bigwedge i.\ f\ i \neq -\infty\ \bigwedge i.\ g\ i \neq -\infty$
shows $(SUP\ i.\ f\ i + g\ i) = SUPREMUM\ UNIV\ f + SUPREMUM\ UNIV\ g$
apply (*subst SUP-ereal-add-left*[*symmetric*, *OF UNIV-not-empty*])
apply (*metis SUP-upper UNIV-I assms*(*4*) *ereal-infty-less-eq*(*2*))
apply (*subst* (*2*) *add.commute*)
apply (*subst SUP-ereal-add-left*[*symmetric*, *OF UNIV-not-empty assms*(*3*)])
apply (*rule SUP-combine*[*symmetric*] *ereal-add-mono inc*[*THEN monoD*] | *assumption*)+
done

lemma *INF-ereal-add*:
fixes $f :: nat \Rightarrow ereal$
assumes *decseq f decseq g*
and *fin*: $\bigwedge i.\ f\ i \neq \infty\ \bigwedge i.\ g\ i \neq \infty$
shows $(INF\ i.\ f\ i + g\ i) = INFIMUM\ UNIV\ f + INFIMUM\ UNIV\ g$
proof −
have *INF-less*: $(INF\ i.\ f\ i) < \infty\ (INF\ i.\ g\ i) < \infty$
using *assms* **unfolding** *INF-less-iff* **by** *auto*
{ fix $a\ b :: ereal$ **assume** $a \neq \infty\ b \neq \infty$
then have $-((-a) + (-b)) = a + b$
by (*cases a b rule*: *ereal2-cases*) *auto* **}**
note $* = this$
have $(INF\ i.\ f\ i + g\ i) = (INF\ i.\ -((-f\ i) + (-g\ i)))$
by (*simp add*: *fin* ∗)
also have $\ldots = INFIMUM\ UNIV\ f + INFIMUM\ UNIV\ g$
unfolding *ereal-INF-uminus-eq*

```
    using assms INF-less
    by (subst SUP-ereal-add) (auto simp: ereal-SUP-uminus fin *)
  finally show ?thesis .
qed

lemma SUP-ereal-add-pos:
  fixes f g :: nat ⇒ ereal
  assumes inc: incseq f incseq g
    and pos: ⋀i. 0 ≤ f i ⋀i. 0 ≤ g i
  shows (SUP i. f i + g i) = SUPREMUM UNIV f + SUPREMUM UNIV g
proof (intro SUP-ereal-add inc)
  fix i
  show f i ≠ −∞ g i ≠ −∞
    using pos[of i] by auto
qed

lemma SUP-ereal-setsum:
  fixes f g :: 'a ⇒ nat ⇒ ereal
  assumes ⋀n. n ∈ A ⟹ incseq (f n)
    and pos: ⋀n i. n ∈ A ⟹ 0 ≤ f n i
  shows (SUP i. ∑ n∈A. f n i) = (∑ n∈A. SUPREMUM UNIV (f n))
proof (cases finite A)
  case True
  then show ?thesis using assms
    by induct (auto simp: incseq-setsumI2 setsum-nonneg SUP-ereal-add-pos)
next
  case False
  then show ?thesis by simp
qed

lemma SUP-ereal-mult-left:
  fixes f :: 'a ⇒ ereal
  assumes I ≠ {}
  assumes f: ⋀i. i ∈ I ⟹ 0 ≤ f i and c: 0 ≤ c
  shows (SUP i:I. c * f i) = c * (SUP i:I. f i)
proof cases
  assume (SUP i: I. f i) = 0
  moreover then have ⋀i. i ∈ I ⟹ f i = 0
    by (metis SUP-upper f antisym)
  ultimately show ?thesis
    by simp
next
  assume (SUP i:I. f i) ≠ 0 then show ?thesis
    by (subst continuous-at-Sup-mono[where f=λx. c * x])
       (auto simp: mono-def continuous-at continuous-at-imp-continuous-at-within
‹I ≠ {}›
            intro!: ereal-mult-left-mono c)
qed
```

lemma *countable-approach*:
fixes x :: *ereal*
assumes $x \neq -\infty$
shows $\exists f.$ *incseq* $f \wedge (\forall i{::}nat.\ f\ i < x) \wedge (f \longrightarrow x)$
proof (*cases* x)
case (*real* r)
moreover have $(\lambda n.\ r - inverse\ (real\ (Suc\ n))) \longrightarrow r - 0$
by (*intro tendsto-intros LIMSEQ-inverse-real-of-nat*)
ultimately show *?thesis*
by (*intro exI*[*of* - $\lambda n.\ x - inverse\ (Suc\ n)$]) (*auto simp*: *incseq-def*)
next
case *PInf* **with** *LIMSEQ-SUP*[*of* $\lambda n{::}nat.\ ereal\ (real\ n)$] **show** *?thesis*
by (*intro exI*[*of* - $\lambda n.\ ereal\ (real\ n)$]) (*auto simp*: *incseq-def SUP-nat-Infty*)
qed (*simp add*: *assms*)

lemma *Sup-countable-SUP*:
assumes $A \neq \{\}$
shows $\exists f{::}nat \Rightarrow ereal.\ incseq\ f \wedge range\ f \subseteq A \wedge Sup\ A = (SUP\ i.\ f\ i)$
proof *cases*
assume $Sup\ A = -\infty$
with ‹$A \neq \{\}$› **have** $A = \{-\infty\}$
by (*auto simp*: *Sup-eq-MInfty*)
then show *?thesis*
by (*auto intro*!: *exI*[*of* - λ-. $-\infty$] *simp*: *bot-ereal-def*)
next
assume $Sup\ A \neq -\infty$
then obtain l **where** *incseq* l **and** l: $\bigwedge i{::}nat.\ l\ i < Sup\ A$ **and** *l-Sup*: $l \longrightarrow Sup\ A$
by (*auto dest*: *countable-approach*)

have $\exists f.\ \forall n.\ (f\ n \in A \wedge l\ n \leq f\ n) \wedge (f\ n \leq f\ (Suc\ n))$
proof (*rule dependent-nat-choice*)
show $\exists x.\ x \in A \wedge l\ 0 \leq x$
using *l*[*of 0*] **by** (*auto simp*: *less-Sup-iff*)
next
fix $x\ n$ **assume** $x \in A \wedge l\ n \leq x$
moreover from *l*[*of Suc n*] **obtain** y **where** $y \in A\ l\ (Suc\ n) < y$
by (*auto simp*: *less-Sup-iff*)
ultimately show $\exists y.\ (y \in A \wedge l\ (Suc\ n) \leq y) \wedge x \leq y$
by (*auto intro*!: *exI*[*of* - *max x y*] *split*: *split-max*)
qed
then guess f **..** **note** $f = this$
then have *range* $f \subseteq A$ *incseq* f
by (*auto simp*: *incseq-Suc-iff*)
moreover
have $(SUP\ i.\ f\ i) = Sup\ A$
proof (*rule tendsto-unique*)
show $f \longrightarrow (SUP\ i.\ f\ i)$
by (*rule LIMSEQ-SUP* ‹*incseq f*›)+

show $f \longrightarrow Sup\ A$
using *l f*
by (*intro tendsto-sandwich*[*OF - - l-Sup tendsto-const*])
(*auto simp*: *Sup-upper*)
qed *simp*
ultimately show *?thesis*
by *auto*
qed

lemma *SUP-countable-SUP*:
$A \neq \{\} \Longrightarrow \exists f{::}nat \Rightarrow ereal.\ range\ f \subseteq g`A \wedge SUPREMUM\ A\ g = SUPREMUM\ UNIV\ f$
using *Sup-countable-SUP* [*of g`A*] **by** *auto*

36.3 Relation to *enat*

definition *ereal-of-enat n* = (*case n of enat n* $\Rightarrow$ *ereal* (*real n*) | $\infty \Rightarrow \infty$)

declare [[*coercion ereal-of-enat* :: *enat* $\Rightarrow$ *ereal*]]
declare [[*coercion* ($\lambda n.$ *ereal* (*real n*)) :: *nat* $\Rightarrow$ *ereal*]]

lemma *ereal-of-enat-simps*[*simp*]:
ereal-of-enat (*enat n*) = *ereal n*
ereal-of-enat $\infty = \infty$
by (*simp-all add*: *ereal-of-enat-def*)

lemma *ereal-of-enat-le-iff*[*simp*]: *ereal-of-enat m* $\leq$ *ereal-of-enat n* $\longleftrightarrow$ *m* $\leq$ *n*
by (*cases m n rule*: *enat2-cases*) *auto*

lemma *ereal-of-enat-less-iff*[*simp*]: *ereal-of-enat m* $<$ *ereal-of-enat n* $\longleftrightarrow$ *m* $<$ *n*
by (*cases m n rule*: *enat2-cases*) *auto*

lemma *numeral-le-ereal-of-enat-iff*[*simp*]: *numeral m* $\leq$ *ereal-of-enat n* $\longleftrightarrow$ *numeral m* $\leq$ *n*
by (*cases n*) (*auto*)

lemma *numeral-less-ereal-of-enat-iff*[*simp*]: *numeral m* $<$ *ereal-of-enat n* $\longleftrightarrow$ *numeral m* $<$ *n*
by (*cases n*) *auto*

lemma *ereal-of-enat-ge-zero-cancel-iff*[*simp*]: *0* $\leq$ *ereal-of-enat n* $\longleftrightarrow$ *0* $\leq$ *n*
by (*cases n*) (*auto simp*: *enat-0*[*symmetric*])

lemma *ereal-of-enat-gt-zero-cancel-iff*[*simp*]: *0* $<$ *ereal-of-enat n* $\longleftrightarrow$ *0* $<$ *n*
by (*cases n*) (*auto simp*: *enat-0*[*symmetric*])

lemma *ereal-of-enat-zero*[*simp*]: *ereal-of-enat 0* = *0*
by (*auto simp*: *enat-0*[*symmetric*])

lemma *ereal-of-enat-inf*[*simp*]: *ereal-of-enat n* = ∞ ⟷ *n* = ∞
by (*cases n*) *auto*

lemma *ereal-of-enat-add*: *ereal-of-enat* (*m* + *n*) = *ereal-of-enat m* + *ereal-of-enat n*
by (*cases m n rule*: *enat2-cases*) *auto*

lemma *ereal-of-enat-sub*:
assumes *n* ≤ *m*
shows *ereal-of-enat* (*m* − *n*) = *ereal-of-enat m* − *ereal-of-enat n*
using *assms* **by** (*cases m n rule*: *enat2-cases*) *auto*

lemma *ereal-of-enat-mult*:
ereal-of-enat (*m* ∗ *n*) = *ereal-of-enat m* ∗ *ereal-of-enat n*
by (*cases m n rule*: *enat2-cases*) *auto*

lemmas *ereal-of-enat-pushin* = *ereal-of-enat-add ereal-of-enat-sub ereal-of-enat-mult*
lemmas *ereal-of-enat-pushout* = *ereal-of-enat-pushin*[*symmetric*]

lemma *ereal-of-enat-nonneg*: *ereal-of-enat n* ≥ *0*
by(*cases n*) *simp-all*

lemma *ereal-of-enat-Sup*:
assumes *A* ≠ {} **shows** *ereal-of-enat* (*Sup A*) = (*SUP a* : *A*. *ereal-of-enat a*)
proof (*intro antisym mono-Sup*)
show *ereal-of-enat* (*Sup A*) ≤ (*SUP a* : *A*. *ereal-of-enat a*)
proof *cases*
assume *finite A*
with ‹*A* ≠ {}› **obtain** *a* **where** *a* ∈ *A ereal-of-enat* (*Sup A*) = *ereal-of-enat a*
using *Max-in*[*of A*] **by** (*auto simp*: *Sup-enat-def simp del*: *Max-in*)
then show *?thesis*
by (*auto intro*: *SUP-upper*)
next
assume ¬ *finite A*
have [*simp*]: (*SUP a* : *A*. *ereal-of-enat a*) = *top*
unfolding *SUP-eq-top-iff*
proof *safe*
fix *x* :: *ereal* **assume** *x* < *top*
then obtain *n* :: *nat* **where** *x* < *n*
using *less-PInf-Ex-of-nat top-ereal-def* **by** *auto*
obtain *a* **where** *a* ∈ *A* − *enat* ‘ {.. *n*}
by (*metis* ‹¬ *finite A*› *all-not-in-conv finite-Diff2 finite-atMost finite-imageI finite.emptyI*)
then have *a* ∈ *A ereal n* ≤ *ereal-of-enat a*
by (*auto simp*: *image-iff Ball-def*)
(*metis enat-iless enat-ord-simps*(*1*) *ereal-of-enat-less-iff ereal-of-enat-simps*(*1*) *less-le not-less*)
with ‹*x* < *n*› **show** ∃ *i*∈*A*. *x* < *ereal-of-enat i*
by (*auto intro*!: *bexI*[*of* - *a*])

```
    qed
    show ?thesis
      by simp
  qed
qed (simp add: mono-def)

lemma ereal-of-enat-SUP:
  A ≠ {} ⟹ ereal-of-enat (SUP a:A. f a) = (SUP a : A. ereal-of-enat (f a))
  using ereal-of-enat-Sup[of f‘A] by auto
```

36.4 Limits on *ereal*

```
lemma open-PInfty: open A ⟹ ∞ ∈ A ⟹ (∃ x. {ereal x<..} ⊆ A)
  unfolding open-ereal-generated
proof (induct rule: generate-topology.induct)
  case (Int A B)
  then obtain x z where ∞ ∈ A ⟹ {ereal x <..} ⊆ A ∞ ∈ B ⟹ {ereal z <..}
⊆ B
    by auto
  with Int show ?case
    by (intro exI[of - max x z]) fastforce
next
  case (Basis S)
  {
    fix x
    have x ≠ ∞ ⟹ ∃ t. x ≤ ereal t
      by (cases x) auto
  }
  moreover note Basis
  ultimately show ?case
    by (auto split: ereal.split)
qed (fastforce simp add: vimage-Union)+

lemma open-MInfty: open A ⟹ −∞ ∈ A ⟹ (∃ x. {..<ereal x} ⊆ A)
  unfolding open-ereal-generated
proof (induct rule: generate-topology.induct)
  case (Int A B)
  then obtain x z where −∞ ∈ A ⟹ {..< ereal x} ⊆ A −∞ ∈ B ⟹ {..< ereal
z} ⊆ B
    by auto
  with Int show ?case
    by (intro exI[of - min x z]) fastforce
next
  case (Basis S)
  {
    fix x
    have x ≠ − ∞ ⟹ ∃ t. ereal t ≤ x
      by (cases x) auto
  }
```

```
  moreover note Basis
  ultimately show ?case
    by (auto split: ereal.split)
qed (fastforce simp add: vimage-Union)+

lemma open-ereal-vimage: open S ⟹ open (ereal −` S)
  by (intro open-vimage continuous-intros)

lemma open-ereal: open S ⟹ open (ereal ` S)
  unfolding open-generated-order[where 'a=real]
proof (induct rule: generate-topology.induct)
  case (Basis S)
  moreover {
    fix x
    have ereal ` {..< x} = { −∞ <..< ereal x }
      apply auto
      apply (case-tac xa)
      apply auto
      done
  }
  moreover {
    fix x
    have ereal ` {x <..} = { ereal x <..< ∞ }
      apply auto
      apply (case-tac xa)
      apply auto
      done
  }
  ultimately show ?case
    by auto
qed (auto simp add: image-Union image-Int)


lemma eventually-finite:
  fixes x :: ereal
  assumes |x| ≠ ∞ (f ⟶ x) F
  shows eventually (λx. |f x| ≠ ∞) F
proof −
  have (f ⟶ ereal (real-of-ereal x)) F
    using assms by (cases x) auto
  then have eventually (λx. f x ∈ ereal ` UNIV) F
    by (rule topological-tendstoD) (auto intro: open-ereal)
  also have (λx. f x ∈ ereal ` UNIV) = (λx. |f x| ≠ ∞)
    by auto
  finally show ?thesis .
qed


lemma open-ereal-def:
```

open A $\longleftrightarrow$ *open* (*ereal* $-$‘ *A*) $\wedge$ ($\infty \in A \longrightarrow (\exists x.\ \{ereal\ x <..\} \subseteq A)) \wedge (-\infty \in A \longrightarrow (\exists x.\ \{..<ereal\ x\} \subseteq A)$)
(**is** *open A* $\longleftrightarrow$ *?rhs*)
proof
assume *open A*
then show *?rhs*
using *open-PInfty open-MInfty open-ereal-vimage* **by** *auto*
next
assume *?rhs*
then obtain *x y* **where** *A*: *open* (*ereal* $-$‘ *A*) $\infty \in A \Longrightarrow \{ereal\ x<..\} \subseteq A$ $-\infty \in A \Longrightarrow \{..<\ ereal\ y\} \subseteq A$
by *auto*
have $*$: $A = ereal$ ‘ (*ereal* $-$‘ *A*) $\cup$ (*if* $\infty \in A$ *then* $\{ereal\ x<..\}$ *else* $\{\}$) $\cup$ (*if* $-\infty \in A$ *then* $\{..<\ ereal\ y\}$ *else* $\{\}$)
using *A*(*2*,*3*) **by** *auto*
from *open-ereal*[*OF A*(*1*)] **show** *open A*
by (*subst* $*$) (*auto simp*: *open-Un*)
qed

lemma *open-PInfty2*:
assumes *open A*
and $\infty \in A$
obtains *x* **where** $\{ereal\ x<..\} \subseteq A$
using *open-PInfty*[*OF assms*] **by** *auto*

lemma *open-MInfty2*:
assumes *open A*
and $-\infty \in A$
obtains *x* **where** $\{..<ereal\ x\} \subseteq A$
using *open-MInfty*[*OF assms*] **by** *auto*

lemma *ereal-openE*:
assumes *open A*
obtains *x y* **where** *open* (*ereal* $-$‘ *A*)
and $\infty \in A \Longrightarrow \{ereal\ x<..\} \subseteq A$
and $-\infty \in A \Longrightarrow \{..<ereal\ y\} \subseteq A$
using *assms open-ereal-def* **by** *auto*

lemmas *open-ereal-lessThan* = *open-lessThan*[**where** *'a*=*ereal*]
lemmas *open-ereal-greaterThan* = *open-greaterThan*[**where** *'a*=*ereal*]
lemmas *ereal-open-greaterThanLessThan* = *open-greaterThanLessThan*[**where** *'a*=*ereal*]
lemmas *closed-ereal-atLeast* = *closed-atLeast*[**where** *'a*=*ereal*]
lemmas *closed-ereal-atMost* = *closed-atMost*[**where** *'a*=*ereal*]
lemmas *closed-ereal-atLeastAtMost* = *closed-atLeastAtMost*[**where** *'a*=*ereal*]
lemmas *closed-ereal-singleton* = *closed-singleton*[**where** *'a*=*ereal*]

lemma *ereal-open-cont-interval*:
fixes *S* :: *ereal set*
assumes *open S*

and $x \in S$
and $|x| \neq \infty$
obtains e **where** $e > 0$ **and** $\{x-e <..< x+e\} \subseteq S$
proof −
from ‹*open S*›
have *open* (*ereal* −‘ *S*)
by (*rule ereal-openE*)
then obtain e **where** $e > 0$ **and** e: $\bigwedge y.$ *dist y* (*real-of-ereal x*) < *e* ⟹ *ereal y* ∈ *S*
using *assms* **unfolding** *open-dist* **by** *force*
show *thesis*
proof (*intro that subsetI*)
show $0 <$ *ereal e*
using ‹$0 < e$› **by** *auto*
fix y
assume $y \in \{x -$ *ereal e*$<..<x +$ *ereal e*$\}$
with *assms* **obtain** t **where** $y =$ *ereal t dist t* (*real-of-ereal x*) < *e*
by (*cases y*) (*auto simp*: *dist-real-def*)
then show $y \in S$
using *e*[*of t*] **by** *auto*
qed
qed

lemma *ereal-open-cont-interval2*:
fixes S :: *ereal set*
assumes *open S*
and $x \in S$
and x: $|x| \neq \infty$
obtains a b **where** $a < x$ **and** $x < b$ **and** $\{a <..< b\} \subseteq S$
proof −
obtain e **where** $0 < e$ $\{x -$ *e*$<..<x +$ *e*$\} \subseteq S$
using *assms* **by** (*rule ereal-open-cont-interval*)
with *that*[*of x − e x + e*] *ereal-between*[*OF x*, *of e*]
show *thesis*
by *auto*
qed

36.4.1 Convergent sequences

lemma *lim-real-of-ereal*[*simp*]:
assumes *lim*: (*f* ⟶ *ereal x*) *net*
shows ((λ*x*. *real-of-ereal* (*f x*)) ⟶ *x*) *net*
proof (*intro topological-tendstoI*)
fix *S*
assume *open S* **and** $x \in S$
then have *S*: *open S ereal x* ∈ *ereal* ‘ *S*
by (*simp-all add*: *inj-image-mem-iff*)
show *eventually* (λ*x*. *real-of-ereal* (*f x*) ∈ *S*) *net*
by (*auto intro*: *eventually-mono* [*OF lim*[*THEN topological-tendstoD*, *OF open-ereal*,

OF S]])
qed

lemma *lim-ereal*[*simp*]: $((\lambda n.\ ereal\ (f\ n)) \longrightarrow ereal\ x)\ net \longleftrightarrow (f \longrightarrow x)\ net$
by (*auto dest*!: *lim-real-of-ereal*)

lemma *convergent-real-imp-convergent-ereal*:
assumes *convergent a*
shows *convergent* $(\lambda n.\ ereal\ (a\ n))$ **and** $lim\ (\lambda n.\ ereal\ (a\ n)) = ereal\ (lim\ a)$
proof −
from *assms* **obtain** *L* **where** *L*: $a \longrightarrow L$ **unfolding** *convergent-def* **..**
hence *lim*: $(\lambda n.\ ereal\ (a\ n)) \longrightarrow ereal\ L$ **using** *lim-ereal* **by** *auto*
thus *convergent* $(\lambda n.\ ereal\ (a\ n))$ **unfolding** *convergent-def* **..**
thus $lim\ (\lambda n.\ ereal\ (a\ n)) = ereal\ (lim\ a)$ **using** *lim L limI* **by** *metis*
qed

lemma *tendsto-PInfty*: $(f \longrightarrow \infty)\ F \longleftrightarrow (\forall r.\ eventually\ (\lambda x.\ ereal\ r < f\ x)\ F)$
proof −
{
fix *l* :: *ereal*
assume $\forall r.\ eventually\ (\lambda x.\ ereal\ r < f\ x)\ F$
from *this*[*THEN spec*, *of real-of-ereal l*] **have** $l \neq \infty \Longrightarrow eventually\ (\lambda x.\ l < f\ x)\ F$
by (*cases l*) (*auto elim*: *eventually-mono*)
}
then show *?thesis*
by (*auto simp*: *order-tendsto-iff*)
qed

lemma *tendsto-PInfty'*: $(f \longrightarrow \infty)\ F = (\forall r{>}c.\ eventually\ (\lambda x.\ ereal\ r < f\ x)\ F)$
proof (*subst tendsto-PInfty*, *intro iffI allI impI*)
assume *A*: $\forall r{>}c.\ eventually\ (\lambda x.\ ereal\ r < f\ x)\ F$
fix *r* :: *real*
from *A* **have** *A*: $eventually\ (\lambda x.\ ereal\ r < f\ x)\ F$ **if** $r > c$ **for** *r* **using** *that* **by** *blast*
show $eventually\ (\lambda x.\ ereal\ r < f\ x)\ F$
proof (*cases* $r > c$)
case *False*
hence *B*: $ereal\ r \leq ereal\ (c + 1)$ **by** *simp*
have $c < c + 1$ **by** *simp*
from *A*[*OF this*] **show** $eventually\ (\lambda x.\ ereal\ r < f\ x)\ F$
by *eventually-elim* (*rule le-less-trans*[*OF B*])
qed (*simp add*: *A*)
qed *simp*

lemma *tendsto-PInfty-eq-at-top*:
$((\lambda z.\ ereal\ (f\ z)) \longrightarrow \infty)\ F \longleftrightarrow (LIM\ z\ F.\ f\ z :> at\text{-}top)$
unfolding *tendsto-PInfty filterlim-at-top-dense* **by** *simp*

lemma *tendsto-MInfty*: $(f \longrightarrow -\infty)\ F \longleftrightarrow (\forall r.\ \mathit{eventually}\ (\lambda x.\ f\ x < \mathit{ereal}\ r)\ F)$
unfolding *tendsto-def*
proof *safe*
fix S :: *ereal set*
assume *open* S $-\infty \in S$
from *open-MInfty*[*OF this*] **obtain** B **where** $\{..<\mathit{ereal}\ B\} \subseteq S$ **..**
moreover
assume $\forall r{::}\mathit{real}.\ \mathit{eventually}\ (\lambda z.\ f\ z < r)\ F$
then have *eventually* $(\lambda z.\ f\ z \in \{..< B\})\ F$
by *auto*
ultimately show *eventually* $(\lambda z.\ f\ z \in S)\ F$
by (*auto elim*!: *eventually-mono*)
next
fix x
assume $\forall S.\ \mathit{open}\ S \longrightarrow -\infty \in S \longrightarrow \mathit{eventually}\ (\lambda x.\ f\ x \in S)\ F$
from *this*[*rule-format*, *of* $\{..< \mathit{ereal}\ x\}$] **show** *eventually* $(\lambda y.\ f\ y < \mathit{ereal}\ x)\ F$
by *auto*
qed

lemma *tendsto-MInfty'*: $(f \longrightarrow -\infty)\ F = (\forall r{<}c.\ \mathit{eventually}\ (\lambda x.\ \mathit{ereal}\ r > f\ x)\ F)$
proof (*subst tendsto-MInfty*, *intro iffI allI impI*)
assume *A*: $\forall r{<}c.\ \mathit{eventually}\ (\lambda x.\ \mathit{ereal}\ r > f\ x)\ F$
fix r :: *real*
from *A* **have** *A*: *eventually* $(\lambda x.\ \mathit{ereal}\ r > f\ x)\ F$ **if** $r < c$ **for** r **using** *that* **by** *blast*
show *eventually* $(\lambda x.\ \mathit{ereal}\ r > f\ x)\ F$
proof (*cases* $r < c$)
case *False*
hence *B*: *ereal* $r \geq \mathit{ereal}\ (c - 1)$ **by** *simp*
have $c > c - 1$ **by** *simp*
from *A*[*OF this*] **show** *eventually* $(\lambda x.\ \mathit{ereal}\ r > f\ x)\ F$
by *eventually-elim* (*erule less-le-trans*[*OF* - *B*])
qed (*simp add*: *A*)
qed *simp*

lemma *Lim-PInfty*: $f \longrightarrow \infty \longleftrightarrow (\forall B.\ \exists N.\ \forall n{\geq}N.\ f\ n \geq \mathit{ereal}\ B)$
unfolding *tendsto-PInfty eventually-sequentially*
proof *safe*
fix r
assume $\forall r.\ \exists N.\ \forall n{\geq}N.\ \mathit{ereal}\ r \leq f\ n$
then obtain N **where** $\forall n{\geq}N.\ \mathit{ereal}\ (r + 1) \leq f\ n$
by *blast*
moreover have *ereal* $r < \mathit{ereal}\ (r + 1)$
by *auto*
ultimately show $\exists N.\ \forall n{\geq}N.\ \mathit{ereal}\ r < f\ n$
by (*blast intro*: *less-le-trans*)

qed (*blast intro*: *less-imp-le*)

lemma *Lim-MInfty*: $f \longrightarrow -\infty \longleftrightarrow (\forall B.\ \exists N.\ \forall n{\geq}N.\ ereal\ B \geq f\ n)$
unfolding *tendsto-MInfty eventually-sequentially*
proof *safe*
fix r
assume $\forall r.\ \exists N.\ \forall n{\geq}N.\ f\ n \leq ereal\ r$
then obtain N **where** $\forall n{\geq}N.\ f\ n \leq ereal\ (r - 1)$
by *blast*
moreover have $ereal\ (r - 1) < ereal\ r$
by *auto*
ultimately show $\exists N.\ \forall n{\geq}N.\ f\ n < ereal\ r$
by (*blast intro*: *le-less-trans*)
qed (*blast intro*: *less-imp-le*)

lemma *Lim-bounded-PInfty*: $f \longrightarrow l \Longrightarrow (\bigwedge n.\ f\ n \leq ereal\ B) \Longrightarrow l \neq \infty$
using *LIMSEQ-le-const2*[*of f l ereal B*] **by** *auto*

lemma *Lim-bounded-MInfty*: $f \longrightarrow l \Longrightarrow (\bigwedge n.\ ereal\ B \leq f\ n) \Longrightarrow l \neq -\infty$
using *LIMSEQ-le-const*[*of f l ereal B*] **by** *auto*

lemma *tendsto-zero-erealI*:
assumes $\bigwedge e.\ e > 0 \Longrightarrow eventually\ (\lambda x.\ |f\ x| < ereal\ e)\ F$
shows $(f \longrightarrow 0)\ F$
proof (*subst filterlim-cong*[*OF refl refl*])
from *assms*[*OF zero-less-one*] **show** *eventually* ($\lambda x.\ f\ x = ereal\ (real\text{-}of\text{-}ereal\ (f\ x)))\ F$
by *eventually-elim* (*auto simp*: *ereal-real*)
hence *eventually* ($\lambda x.\ abs\ (real\text{-}of\text{-}ereal\ (f\ x)) < e)\ F$ **if** $e > 0$ **for** e **using** *assms*[*OF that*]
by *eventually-elim* (*simp add*: *real-less-ereal-iff that*)
hence $((\lambda x.\ real\text{-}of\text{-}ereal\ (f\ x)) \longrightarrow 0)\ F$ **unfolding** *tendsto-iff*
by (*auto simp*: *tendsto-iff dist-real-def*)
thus $((\lambda x.\ ereal\ (real\text{-}of\text{-}ereal\ (f\ x))) \longrightarrow 0)\ F$ **by** (*simp add*: *zero-ereal-def*)
qed

lemma *tendsto-explicit*:
$f \longrightarrow f0 \longleftrightarrow (\forall S.\ open\ S \longrightarrow f0 \in S \longrightarrow (\exists N.\ \forall n{\geq}N.\ f\ n \in S))$
unfolding *tendsto-def eventually-sequentially* **by** *auto*

lemma *Lim-bounded-PInfty2*: $f \longrightarrow l \Longrightarrow \forall n{\geq}N.\ f\ n \leq ereal\ B \Longrightarrow l \neq \infty$
using *LIMSEQ-le-const2*[*of f l ereal B*] **by** *fastforce*

lemma *Lim-bounded-ereal*: $f \longrightarrow (l :: 'a{::}linorder\text{-}topology) \Longrightarrow \forall n{\geq}M.\ f\ n \leq C \Longrightarrow l \leq C$
by (*intro LIMSEQ-le-const2*) *auto*

lemma *Lim-bounded2-ereal*:
assumes *lim*:$f \longrightarrow (l :: 'a{::}linorder\text{-}topology)$

and *ge*: $\forall\, n{\geq}N.\ f\ n \geq C$
shows $l \geq C$
using *ge*
by (*intro tendsto-le*[*OF trivial-limit-sequentially lim tendsto-const*])
(*auto simp*: *eventually-sequentially*)

lemma *real-of-ereal-mult*[*simp*]:
fixes $a\ b$ :: *ereal*
shows *real-of-ereal* $(a * b) =$ *real-of-ereal* $a *$ *real-of-ereal* b
by (*cases rule*: *ereal2-cases*[*of* $a\ b$]) *auto*

lemma *real-of-ereal-eq-0*:
fixes x :: *ereal*
shows *real-of-ereal* $x = 0 \longleftrightarrow x = \infty \vee x = -\infty \vee x = 0$
by (*cases* x) *auto*

lemma *tendsto-ereal-realD*:
fixes f :: $'a \Rightarrow$ *ereal*
assumes $x \neq 0$
and *tendsto*: $((\lambda x.$ *ereal* (*real-of-ereal* $(f\,x))) \longrightarrow x)$ *net*
shows $(f \longrightarrow x)$ *net*
proof (*intro topological-tendstoI*)
fix S
assume S: *open* $S\ x \in S$
with ‹$x \neq 0$›
have *open* $(S - \{0\})\ x \in S - \{0\}$
by *auto*
from *tendsto*[*THEN topological-tendstoD*, *OF this*]
show *eventually* $(\lambda x.\ f\,x \in S)$ *net*
by (*rule eventually-rev-mp*) (*auto simp*: *ereal-real*)
qed

lemma *tendsto-ereal-realI*:
fixes f :: $'a \Rightarrow$ *ereal*
assumes x: $|x| \neq \infty$ **and** *tendsto*: $(f \longrightarrow x)$ *net*
shows $((\lambda x.$ *ereal* (*real-of-ereal* $(f\,x))) \longrightarrow x)$ *net*
proof (*intro topological-tendstoI*)
fix S
assume *open* S **and** $x \in S$
with x **have** *open* $(S - \{\infty, -\infty\})\ x \in S - \{\infty, -\infty\}$
by *auto*
from *tendsto*[*THEN topological-tendstoD*, *OF this*]
show *eventually* $(\lambda x.$ *ereal* (*real-of-ereal* $(f\,x)) \in S)$ *net*
by (*elim eventually-mono*) (*auto simp*: *ereal-real*)
qed

lemma *ereal-mult-cancel-left*:
fixes $a\ b\ c$ :: *ereal*
shows $a * b = a * c \longleftrightarrow (|a| = \infty \wedge 0 < b * c) \vee a = 0 \vee b = c$
by (*cases rule*: *ereal3-cases*[*of* $a\ b\ c$]) (*simp-all add*: *zero-less-mult-iff*)

lemma *tendsto-add-ereal*:
fixes $x\ y$ :: *ereal*
assumes x: $|x| \neq \infty$ **and** y: $|y| \neq \infty$
assumes f: $(f \longrightarrow x)\ F$ **and** g: $(g \longrightarrow y)\ F$
shows $((\lambda x.\ f\ x + g\ x) \longrightarrow x + y)\ F$
proof −
from x **obtain** r **where** x': $x = ereal\ r$ **by** (*cases* x) *auto*
with f **have** $((\lambda i.\ real\text{-}of\text{-}ereal\ (f\ i)) \longrightarrow r)\ F$ **by** *simp*
moreover
from y **obtain** p **where** y': $y = ereal\ p$ **by** (*cases* y) *auto*
with g **have** $((\lambda i.\ real\text{-}of\text{-}ereal\ (g\ i)) \longrightarrow p)\ F$ **by** *simp*
ultimately have $((\lambda i.\ real\text{-}of\text{-}ereal\ (f\ i) + real\text{-}of\text{-}ereal\ (g\ i)) \longrightarrow r + p)\ F$
by (*rule tendsto-add*)
moreover
from *eventually-finite*[*OF x f*] *eventually-finite*[*OF y g*]
have *eventually* $(\lambda x.\ f\ x + g\ x = ereal\ (real\text{-}of\text{-}ereal\ (f\ x) + real\text{-}of\text{-}ereal\ (g\ x)))\ F$
by *eventually-elim auto*
ultimately show *?thesis*
by (*simp add*: $x'\ y'$ *cong*: *filterlim-cong*)
qed

lemma *tendsto-add-ereal-nonneg*:
fixes $x\ y$ :: *ereal*
assumes $x \neq -\infty\ y \neq -\infty\ (f \longrightarrow x)\ F\ (g \longrightarrow y)\ F$
shows $((\lambda x.\ f\ x + g\ x) \longrightarrow x + y)\ F$
proof *cases*
assume $x = \infty \vee y = \infty$
moreover
{ fix y :: *ereal* **and** $f\ g$ :: $'a \Rightarrow ereal$ **assume** $y \neq -\infty\ (f \longrightarrow \infty)\ F\ (g \longrightarrow y)\ F$
then obtain y' **where** $-\infty < y'\ y' < y$
using *dense*[*of* $-\infty\ y$] **by** *auto*
have $((\lambda x.\ f\ x + g\ x) \longrightarrow \infty)\ F$
proof (*rule tendsto-sandwich*)
have $\forall_F\ x\ in\ F.\ y' < g\ x$
using *order-tendstoD*(*1*)[*OF* ‹$(g \longrightarrow y)\ F$› ‹$y' < y$›] **by** *auto*
then show $\forall_F\ x\ in\ F.\ f\ x + y' \leq f\ x + g\ x$
by *eventually-elim* (*auto intro*!: *add-mono*)
show $\forall_F\ n\ in\ F.\ f\ n + g\ n \leq \infty$
by *auto*
show $((\lambda x.\ f\ x + y') \longrightarrow \infty)\ F$
using *tendsto-cadd-ereal*[*of* $y'\ \infty\ f\ F$] ‹$(f \longrightarrow \infty)\ F$› ‹$-\infty < y'$› **by** *auto*
qed }
note *this*[*of* $y\ f\ g$] *this*[*of* $x\ g\ f$]
ultimately show *?thesis*
using *assms* **by** (*auto simp*: *add-ac*)
next

assume $\neg\ (x = \infty \vee y = \infty)$
with *assms tendsto-add-ereal*[*of x y f F g*]
show *?thesis*
by *auto*
qed

lemma *ereal-inj-affinity*:
fixes $m\ t$ :: *ereal*
assumes $|m| \neq \infty$
and $m \neq 0$
and $|t| \neq \infty$
shows *inj-on* $(\lambda x.\ m * x + t)\ A$
using *assms*
by (*cases rule*: *ereal2-cases*[*of m t*])
(*auto intro*!: *inj-onI simp*: *ereal-add-cancel-right ereal-mult-cancel-left*)

lemma *ereal-PInfty-eq-plus*[*simp*]:
fixes $a\ b$ :: *ereal*
shows $\infty = a + b \longleftrightarrow a = \infty \vee b = \infty$
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

lemma *ereal-MInfty-eq-plus*[*simp*]:
fixes $a\ b$ :: *ereal*
shows $-\infty = a + b \longleftrightarrow (a = -\infty \wedge b \neq \infty) \vee (b = -\infty \wedge a \neq \infty)$
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

lemma *ereal-less-divide-pos*:
fixes $x\ y$ :: *ereal*
shows $x > 0 \Longrightarrow x \neq \infty \Longrightarrow y < z\ /\ x \longleftrightarrow x * y < z$
by (*cases rule*: *ereal3-cases*[*of x y z*]) (*auto simp*: *field-simps*)

lemma *ereal-divide-less-pos*:
fixes $x\ y\ z$ :: *ereal*
shows $x > 0 \Longrightarrow x \neq \infty \Longrightarrow y\ /\ x < z \longleftrightarrow y < x * z$
by (*cases rule*: *ereal3-cases*[*of x y z*]) (*auto simp*: *field-simps*)

lemma *ereal-divide-eq*:
fixes $a\ b\ c$ :: *ereal*
shows $b \neq 0 \Longrightarrow |b| \neq \infty \Longrightarrow a\ /\ b = c \longleftrightarrow a = b * c$
by (*cases rule*: *ereal3-cases*[*of a b c*])
(*simp-all add*: *field-simps*)

lemma *ereal-inverse-not-MInfty*[*simp*]: *inverse* $(a$::*ereal*$) \neq -\infty$
by (*cases a*) *auto*

lemma *ereal-mult-m1*[*simp*]: $x * ereal\ (-1) = -x$
by (*cases x*) *auto*

lemma *ereal-real'*:

assumes $|x| \neq \infty$
shows *ereal* (*real-of-ereal* x) $= x$
using *assms* **by** *auto*

lemma *real-ereal-id*: *real-of-ereal* $\circ$ *ereal* $=$ *id*
proof $-$
{
fix x
have (*real-of-ereal o ereal*) $x =$ *id* x
by *auto*
}
then show *?thesis*
using *ext* **by** *blast*
qed

lemma *open-image-ereal*: *open*(*UNIV*$-\{\infty$, $(-\infty ::$ *ereal*$)\}$)
by (*metis range-ereal open-ereal open-UNIV*)

lemma *ereal-le-distrib*:
fixes a b $c ::$ *ereal*
shows $c * (a + b) \leq c * a + c * b$
by (*cases rule*: *ereal3-cases*[*of a b c*])
(*auto simp add*: *field-simps not-le mult-le-0-iff mult-less-0-iff*)

lemma *ereal-pos-distrib*:
fixes a b $c ::$ *ereal*
assumes $0 \leq c$
and $c \neq \infty$
shows $c * (a + b) = c * a + c * b$
using *assms*
by (*cases rule*: *ereal3-cases*[*of a b c*])
(*auto simp add*: *field-simps not-le mult-le-0-iff mult-less-0-iff*)

lemma *ereal-max-mono*: (a::*ereal*) $\leq b \Longrightarrow c \leq d \Longrightarrow$ *max a c* $\leq$ *max b d*
by (*metis sup-ereal-def sup-mono*)

lemma *ereal-max-least*: (a::*ereal*) $\leq x \Longrightarrow c \leq x \Longrightarrow$ *max a c* $\leq x$
by (*metis sup-ereal-def sup-least*)

lemma *ereal-LimI-finite*:
fixes $x ::$ *ereal*
assumes $|x| \neq \infty$
and $\bigwedge r.\ 0 < r \Longrightarrow \exists N.\ \forall n \geq N.\ u\ n < x + r \wedge x < u\ n + r$
shows $u \longrightarrow x$
proof (*rule topological-tendstoI*, *unfold eventually-sequentially*)
obtain *rx* **where** *rx*: $x =$ *ereal rx*
using *assms* **by** (*cases x*) *auto*
fix S
assume *open* S **and** $x \in S$

```
  then have open (ereal −‘ S)
    unfolding open-ereal-def by auto
  with ‹x ∈ S› obtain r where 0 < r and dist: ⋀y. dist y rx < r ⟹ ereal y ∈
S
    unfolding open-dist rx by auto
  then obtain n where
    upper: ⋀N. n ≤ N ⟹ u N < x + ereal r and
    lower: ⋀N. n ≤ N ⟹ x < u N + ereal r
    using assms(2)[of ereal r] by auto
  show ∃ N. ∀ n≥N. u n ∈ S
  proof (safe intro!: exI[of - n])
    fix N
    assume n ≤ N
    from upper[OF this] lower[OF this] assms ‹0 < r›
    have u N ∉ {∞,(−∞)}
      by auto
    then obtain ra where ra-def: (u N) = ereal ra
      by (cases u N) auto
    then have rx < ra + r and ra < rx + r
      using rx assms ‹0 < r› lower[OF ‹n ≤ N›] upper[OF ‹n ≤ N›]
      by auto
    then have dist (real-of-ereal (u N)) rx < r
      using rx ra-def
      by (auto simp: dist-real-def abs-diff-less-iff field-simps)
    from dist[OF this] show u N ∈ S
      using ‹u N  ∉ {∞, −∞}›
      by (auto simp: ereal-real split: if-split-asm)
  qed
qed

lemma tendsto-obtains-N:
  assumes f ⟶ f0
  assumes open S
    and f0 ∈ S
  obtains N where ∀ n≥N. f n ∈ S
  using assms using tendsto-def
  using tendsto-explicit[of f f0] assms by auto

lemma ereal-LimI-finite-iff:
  fixes x :: ereal
  assumes |x| ≠ ∞
  shows u ⟶ x ⟷ (∀ r. 0 < r ⟶ (∃ N. ∀ n≥N. u n < x + r ∧ x < u n
+ r))
  (is ?lhs ⟷ ?rhs)
proof
  assume lim: u ⟶ x
  {
    fix r :: ereal
    assume r > 0
```

```
    then obtain N where ∀ n≥N. u n ∈ {x − r <..< x + r}
      apply (subst tendsto-obtains-N[of u x {x − r <..< x + r}])
      using lim ereal-between[of x r] assms ‹r > 0›
      apply auto
      done
    then have ∃ N. ∀ n≥N. u n < x + r ∧ x < u n + r
     using ereal-minus-less[of r x]
     by (cases r) auto
  }
  then show ?rhs
    by auto
next
  assume ?rhs
  then show u ⟶ x
    using ereal-LimI-finite[of x] assms by auto
qed

lemma ereal-Limsup-uminus:
  fixes f :: 'a ⇒ ereal
  shows Limsup net (λx. − (f x)) = − Liminf net f
  unfolding Limsup-def Liminf-def ereal-SUP-uminus ereal-INF-uminus-eq ..

lemma liminf-bounded-iff:
  fixes x :: nat ⇒ ereal
  shows C ≤ liminf x ⟷ (∀ B<C. ∃ N. ∀ n≥N. B < x n)
  (is ?lhs ⟷ ?rhs)
  unfolding le-Liminf-iff eventually-sequentially ..

lemma Liminf-add-le:
  fixes f g :: - ⇒ ereal
  assumes F: F ≠ bot
  assumes ev: eventually (λx. 0 ≤ f x) F eventually (λx. 0 ≤ g x) F
  shows Liminf F f + Liminf F g ≤ Liminf F (λx. f x + g x)
  unfolding Liminf-def
proof (subst SUP-ereal-add-left[symmetric])
  let ?F = {P. eventually P F}
  let ?INF = λP g. INFIMUM (Collect P) g
  show ?F ≠ {}
    by (auto intro: eventually-True)
  show (SUP P:?F. ?INF P g) ≠ − ∞
    unfolding bot-ereal-def[symmetric] SUP-bot-conv INF-eq-bot-iff
    by (auto intro!: exI[of - 0] ev simp: bot-ereal-def)
  have (SUP P:?F. ?INF P f + (SUP P:?F. ?INF P g)) ≤ (SUP P:?F. (SUP
P':?F. ?INF P f + ?INF P' g))
  proof (safe intro!: SUP-mono bexI[of - λx. P x ∧ 0 ≤ f x for P])
    fix P let ?P' = λx. P x ∧ 0 ≤ f x
    assume eventually P F
    with ev show eventually ?P' F
      by eventually-elim auto
```

```
    have ?INF P f + (SUP P:?F. ?INF P g) ≤ ?INF ?P' f + (SUP P:?F. ?INF
P g)
      by (intro ereal-add-mono INF-mono) auto
    also have ... = (SUP P':?F. ?INF ?P' f + ?INF P' g)
    proof (rule SUP-ereal-add-right[symmetric])
      show INFIMUM {x. P x ∧ 0 ≤ f x} f ≠ − ∞
        unfolding bot-ereal-def[symmetric] INF-eq-bot-iff
        by (auto intro!: exI[of - 0] ev simp: bot-ereal-def)
    qed fact
    finally show ?INF P f + (SUP P:?F. ?INF P g) ≤ (SUP P':?F. ?INF ?P' f
+ ?INF P' g) .
  qed
  also have ... ≤ (SUP P:?F. INF x:Collect P. f x + g x)
  proof (safe intro!: SUP-least)
    fix P Q assume ∗: eventually P F eventually Q F
    show ?INF P f + ?INF Q g ≤ (SUP P:?F. INF x:Collect P. f x + g x)
    proof (rule SUP-upper2)
      show (λx. P x ∧ Q x) ∈ ?F
        using ∗ by (auto simp: eventually-conj)
      show ?INF P f + ?INF Q g ≤ (INF x:{x. P x ∧ Q x}. f x + g x)
        by (intro INF-greatest ereal-add-mono) (auto intro: INF-lower)
    qed
  qed
  finally show (SUP P:?F. ?INF P f + (SUP P:?F. ?INF P g)) ≤ (SUP P:?F.
INF x:Collect P. f x + g x) .
qed

lemma Sup-ereal-mult-right':
  assumes nonempty: Y ≠ {}
  and x: x ≥ 0
  shows (SUP i:Y. f i) ∗ ereal x = (SUP i:Y. f i ∗ ereal x) (is ?lhs = ?rhs)
proof(cases x = 0)
  case True thus ?thesis by(auto simp add: nonempty zero-ereal-def[symmetric])
next
  case False
  show ?thesis
  proof(rule antisym)
    show ?rhs ≤ ?lhs
      by(rule SUP-least)(simp add: ereal-mult-right-mono SUP-upper x)
  next
     have ?lhs / ereal x = (SUP i:Y. f i) ∗ (ereal x / ereal x) by(simp only:
ereal-times-divide-eq)
    also have ... = (SUP i:Y. f i) using False by simp
    also have ... ≤ ?rhs / x
    proof(rule SUP-least)
      fix i
      assume i ∈ Y
      have f i = f i ∗ (ereal x / ereal x) using False by simp
      also have ... = f i ∗ x / x by(simp only: ereal-times-divide-eq)
```

```
    also from ‹i ∈ Y› have f i * x ≤ ?rhs by(rule SUP-upper)
    hence f i * x / x ≤ ?rhs / x using x False by simp
    finally show f i ≤ ?rhs / x .
  qed
  finally have (?lhs / x) * x ≤ (?rhs / x) * x
    by(rule ereal-mult-right-mono)(simp add: x)
  also have ... = ?rhs using False ereal-divide-eq mult.commute by force
  also have (?lhs / x) * x = ?lhs using False ereal-divide-eq mult.commute by
force
  finally show ?lhs ≤ ?rhs .
 qed
qed

lemma Sup-ereal-mult-left':
 ⟦ Y ≠ {}; x ≥ 0 ⟧ ⟹ ereal x * (SUP i:Y. f i) = (SUP i:Y. ereal x * f i)
by(subst (1 2) mult.commute)(rule Sup-ereal-mult-right')

lemma sup-continuous-add[order-continuous-intros]:
 fixes f g :: 'a::complete-lattice ⇒ ereal
 assumes nn: ⋀x. 0 ≤ f x ⋀x. 0 ≤ g x and cont: sup-continuous f sup-continuous
g
 shows sup-continuous (λx. f x + g x)
 unfolding sup-continuous-def
proof safe
 fix M :: nat ⇒ 'a assume incseq M
 then show f (SUP i. M i) + g (SUP i. M i) = (SUP i. f (M i) + g (M i))
   using SUP-ereal-add-pos[of λi. f (M i) λi. g (M i)] nn
     cont[THEN sup-continuous-mono] cont[THEN sup-continuousD]
   by (auto simp: mono-def)
qed

lemma sup-continuous-mult-right[order-continuous-intros]:
 0 ≤ c ⟹ c < ∞ ⟹ sup-continuous f ⟹ sup-continuous (λx. f x * c :: ereal)
 by (cases c) (auto simp: sup-continuous-def fun-eq-iff Sup-ereal-mult-right')

lemma sup-continuous-mult-left[order-continuous-intros]:
 0 ≤ c ⟹ c < ∞ ⟹ sup-continuous f ⟹ sup-continuous (λx. c * f x :: ereal)
 using sup-continuous-mult-right[of c f] by (simp add: mult-ac)

lemma sup-continuous-ereal-of-enat[order-continuous-intros]:
 assumes f: sup-continuous f shows sup-continuous (λx. ereal-of-enat (f x))
 by (rule sup-continuous-compose[OF - f])
    (auto simp: sup-continuous-def ereal-of-enat-SUP)
```

36.4.2 Sums

```
lemma sums-ereal-positive:
 fixes f :: nat ⇒ ereal
 assumes ⋀i. 0 ≤ f i
```

shows *f sums* (*SUP n.* $\sum i{<}n.\ f\ i$)
proof −
have *incseq* ($\lambda i.\ \sum j{=}0..{<}i.\ f\ j$)
using *ereal-add-mono*[*OF - assms*]
by (*auto intro*!: *incseq-SucI*)
from *LIMSEQ-SUP*[*OF this*]
show *?thesis* **unfolding** *sums-def*
by (*simp add*: *atLeast0LessThan*)
qed

lemma *summable-ereal-pos*:
fixes f :: *nat* ⇒ *ereal*
assumes $\bigwedge i.\ 0 \le f\ i$
shows *summable f*
using *sums-ereal-positive*[*of f*, *OF assms*]
unfolding *summable-def*
by *auto*

lemma *sums-ereal*: ($\lambda x.$ *ereal* (*f x*)) *sums ereal x* ⟷ *f sums x*
unfolding *sums-def* **by** *simp*

lemma *suminf-ereal-eq-SUP*:
fixes f :: *nat* ⇒ *ereal*
assumes $\bigwedge i.\ 0 \le f\ i$
shows ($\sum x.\ f\ x$) = (*SUP n.* $\sum i{<}n.\ f\ i$)
using *sums-ereal-positive*[*of f*, *OF assms*, *THEN sums-unique*]
by *simp*

lemma *suminf-bound*:
fixes f :: *nat* ⇒ *ereal*
assumes $\forall N.\ (\sum n{<}N.\ f\ n) \le x$
and *pos*: $\bigwedge n.\ 0 \le f\ n$
shows *suminf f* ≤ *x*
proof (*rule Lim-bounded-ereal*)
have *summable f* **using** *pos*[*THEN summable-ereal-pos*] **.**
then show ($\lambda N.\ \sum n{<}N.\ f\ n$) ⟶ *suminf f*
by (*auto dest*!: *summable-sums simp*: *sums-def atLeast0LessThan*)
show $\forall n{\ge}0.$ *setsum f* {..<*n*} ≤ *x*
using *assms* **by** *auto*
qed

lemma *suminf-bound-add*:
fixes f :: *nat* ⇒ *ereal*
assumes $\forall N.\ (\sum n{<}N.\ f\ n) + y \le x$
and *pos*: $\bigwedge n.\ 0 \le f\ n$
and $y \neq -\infty$
shows *suminf f* + *y* ≤ *x*
proof (*cases y*)
case (*real r*)

then have $\forall N.\ (\sum n{<}N.\ f\ n) \leq x - y$
using *assms* **by** (*simp add*: *ereal-le-minus*)
then have $(\sum\ n.\ f\ n) \leq x - y$
using *pos* **by** (*rule suminf-bound*)
then show $(\sum\ n.\ f\ n) + y \leq x$
using *assms real* **by** (*simp add*: *ereal-le-minus*)
qed (*insert assms*, *auto*)

lemma *suminf-upper*:
fixes f :: *nat* $\Rightarrow$ *ereal*
assumes $\bigwedge n.\ 0 \leq f\ n$
shows $(\sum n{<}N.\ f\ n) \leq (\sum n.\ f\ n)$
unfolding *suminf-ereal-eq-SUP* [*OF assms*]
by (*auto intro*: *complete-lattice-class.SUP-upper*)

lemma *suminf-0-le*:
fixes f :: *nat* $\Rightarrow$ *ereal*
assumes $\bigwedge n.\ 0 \leq f\ n$
shows $0 \leq (\sum n.\ f\ n)$
using *suminf-upper*[*of f 0*, *OF assms*]
by *simp*

lemma *suminf-le-pos*:
fixes $f\ g$ :: *nat* $\Rightarrow$ *ereal*
assumes $\bigwedge N.\ f\ N \leq g\ N$
and $\bigwedge N.\ 0 \leq f\ N$
shows *suminf f* $\leq$ *suminf g*
proof (*safe intro*!: *suminf-bound*)
fix *n*
{
fix *N*
have $0 \leq g\ N$
using *assms*(*2*,*1*)[*of N*] **by** *auto*
}
have *setsum f* $\{..{<}n\} \leq$ *setsum g* $\{..{<}n\}$
using *assms* **by** (*auto intro*: *setsum-mono*)
also have $\ldots \leq$ *suminf g*
using ‹$\bigwedge N.\ 0 \leq g\ N$›
by (*rule suminf-upper*)
finally show *setsum f* $\{..{<}n\} \leq$ *suminf g* **.**
qed (*rule assms*(*2*))

lemma *suminf-half-series-ereal*: $(\sum n.\ (1/2 :: ereal)\ \hat{}\ Suc\ n) = 1$
using *sums-ereal*[*THEN iffD2*, *OF power-half-series*, *THEN sums-unique*, *symmetric*]
by (*simp add*: *one-ereal-def*)

lemma *suminf-add-ereal*:
fixes $f\ g$ :: *nat* $\Rightarrow$ *ereal*

assumes $\bigwedge i.\ 0 \le f\ i$
and $\bigwedge i.\ 0 \le g\ i$
shows $(\sum i.\ f\ i + g\ i) = suminf\ f + suminf\ g$
apply (*subst* (*1 2 3*) *suminf-ereal-eq-SUP*)
unfolding *setsum.distrib*
apply (*intro assms ereal-add-nonneg-nonneg SUP-ereal-add-pos incseq-setsumI setsum-nonneg ballI*)+
done

lemma *suminf-cmult-ereal*:
fixes $f\ g :: nat \Rightarrow ereal$
assumes $\bigwedge i.\ 0 \le f\ i$
and $0 \le a$
shows $(\sum i.\ a * f\ i) = a * suminf\ f$
by (*auto simp*: *setsum-ereal-right-distrib*[*symmetric*] *assms ereal-zero-le-0-iff setsum-nonneg suminf-ereal-eq-SUP intro*!: *SUP-ereal-mult-left*)

lemma *suminf-PInfty*:
fixes $f :: nat \Rightarrow ereal$
assumes $\bigwedge i.\ 0 \le f\ i$
and $suminf\ f \neq \infty$
shows $f\ i \neq \infty$
proof −
from *suminf-upper*[*of f Suc i*, *OF assms*(*1*)] *assms*(*2*)
have $(\sum i{<}Suc\ i.\ f\ i) \neq \infty$
by *auto*
then show *?thesis*
unfolding *setsum-Pinfty* **by** *simp*
qed

lemma *suminf-PInfty-fun*:
assumes $\bigwedge i.\ 0 \le f\ i$
and $suminf\ f \neq \infty$
shows $\exists f'.\ f = (\lambda x.\ ereal\ (f'\ x))$
proof −
have $\forall i.\ \exists r.\ f\ i = ereal\ r$
proof
fix *i*
show $\exists r.\ f\ i = ereal\ r$
using *suminf-PInfty*[*OF assms*] *assms*(*1*)[*of i*]
by (*cases f i*) *auto*
qed
from *choice*[*OF this*] **show** *?thesis*
by *auto*
qed

lemma *summable-ereal*:
assumes $\bigwedge i.\ 0 \le f\ i$

```
    and (∑ i. ereal (f i)) ≠ ∞
  shows summable f
proof −
  have 0 ≤ (∑ i. ereal (f i))
    using assms by (intro suminf-0-le) auto
  with assms obtain r where r: (∑ i. ereal (f i)) = ereal r
    by (cases ∑ i. ereal (f i)) auto
  from summable-ereal-pos[of λx. ereal (f x)]
  have summable (λx. ereal (f x))
    using assms by auto
  from summable-sums[OF this]
  have (λx. ereal (f x)) sums (∑ x. ereal (f x))
    by auto
  then show summable f
    unfolding r sums-ereal summable-def ..
qed

lemma suminf-ereal:
  assumes ⋀i. 0 ≤ f i
    and (∑ i. ereal (f i)) ≠ ∞
  shows (∑ i. ereal (f i)) = ereal (suminf f)
proof (rule sums-unique[symmetric])
  from summable-ereal[OF assms]
  show (λx. ereal (f x)) sums (ereal (suminf f))
    unfolding sums-ereal
    using assms
    by (intro summable-sums summable-ereal)
qed

lemma suminf-ereal-minus:
  fixes f g :: nat ⇒ ereal
  assumes ord: ⋀i. g i ≤ f i ⋀i. 0 ≤ g i
    and fin: suminf f ≠ ∞ suminf g ≠ ∞
  shows (∑ i. f i − g i) = suminf f − suminf g
proof −
  {
    fix i
    have 0 ≤ f i
      using ord[of i] by auto
  }
  moreover
  from suminf-PInfty-fun[OF ‹⋀i. 0 ≤ f i› fin(1)] obtain f' where [simp]: f =
(λx. ereal (f' x)) ..
  from suminf-PInfty-fun[OF ‹⋀i. 0 ≤ g i› fin(2)] obtain g' where [simp]: g =
(λx. ereal (g' x)) ..
  {
    fix i
    have 0 ≤ f i − g i
      using ord[of i] by (auto simp: ereal-le-minus-iff)
```

```
  }
  moreover
  have suminf (λi. f i − g i) ≤ suminf f
    using assms by (auto intro!: suminf-le-pos simp: field-simps)
  then have suminf (λi. f i − g i) ≠ ∞
    using fin by auto
  ultimately show ?thesis
    using assms ‹⋀i. 0 ≤ f i›
    apply simp
    apply (subst (1 2 3) suminf-ereal)
    apply (auto intro!: suminf-diff[symmetric] summable-ereal)
    done
qed

lemma suminf-ereal-PInf [simp]: (∑ x. ∞::ereal) = ∞
proof −
  have (∑ i<Suc 0. ∞) ≤ (∑ x. ∞::ereal)
    by (rule suminf-upper) auto
  then show ?thesis
    by simp
qed

lemma summable-real-of-ereal:
  fixes f :: nat ⇒ ereal
  assumes f: ⋀i. 0 ≤ f i
    and fin: (∑ i. f i) ≠ ∞
  shows summable (λi. real-of-ereal (f i))
proof (rule summable-def[THEN iffD2])
  have 0 ≤ (∑ i. f i)
    using assms by (auto intro: suminf-0-le)
  with fin obtain r where r: ereal r = (∑ i. f i)
    by (cases (∑ i. f i)) auto
  {
    fix i
    have f i ≠ ∞
      using f by (intro suminf-PInfty[OF - fin]) auto
    then have |f i| ≠ ∞
      using f[of i] by auto
  }
  note fin = this
  have (λi. ereal (real-of-ereal (f i))) sums (∑ i. ereal (real-of-ereal (f i)))
    using f
    by (auto intro!: summable-ereal-pos simp: ereal-le-real-iff zero-ereal-def)
  also have ... = ereal r
    using fin r by (auto simp: ereal-real)
  finally show ∃ r. (λi. real-of-ereal (f i)) sums r
    by (auto simp: sums-ereal)
qed
```

lemma *suminf-SUP-eq*:
  fixes $f :: nat \Rightarrow nat \Rightarrow ereal$
  assumes $\bigwedge i.\ incseq\ (\lambda n.\ f\ n\ i)$
    and $\bigwedge n\ i.\ 0 \le f\ n\ i$
  shows $(\sum i.\ SUP\ n.\ f\ n\ i) = (SUP\ n.\ \sum i.\ f\ n\ i)$
proof −
  {
    fix $n :: nat$
    have $(\sum i{<}n.\ SUP\ k.\ f\ k\ i) = (SUP\ k.\ \sum i{<}n.\ f\ k\ i)$
      using *assms*
      by (*auto intro*!: *SUP-ereal-setsum* [*symmetric*])
  }
  note $* = this$
  show *?thesis*
    using *assms*
    apply (*subst* (*1 2*) *suminf-ereal-eq-SUP*)
    unfolding $*$
    apply (*auto intro*!: *SUP-upper2*)
    apply (*subst SUP-commute*)
    apply *rule*
    done
qed

lemma *suminf-setsum-ereal*:
  fixes $f :: - \Rightarrow - \Rightarrow ereal$
  assumes *nonneg*: $\bigwedge i\ a.\ a \in A \Longrightarrow 0 \le f\ i\ a$
  shows $(\sum i.\ \sum a{\in}A.\ f\ i\ a) = (\sum a{\in}A.\ \sum i.\ f\ i\ a)$
proof (*cases finite A*)
  case *True*
  then show *?thesis*
    using *nonneg*
    by *induct* (*simp-all add*: *suminf-add-ereal setsum-nonneg*)
next
  case *False*
  then show *?thesis* **by** *simp*
qed

lemma *suminf-ereal-eq-0*:
  fixes $f :: nat \Rightarrow ereal$
  assumes *nneg*: $\bigwedge i.\ 0 \le f\ i$
  shows $(\sum i.\ f\ i) = 0 \longleftrightarrow (\forall i.\ f\ i = 0)$
proof
  assume $(\sum i.\ f\ i) = 0$
  {
    fix i
    assume $f\ i \neq 0$
    with *nneg* **have** $0 < f\ i$
      by (*auto simp*: *less-le*)
    also have $f\ i = (\sum j.\ if\ j = i\ then\ f\ i\ else\ 0)$

```
    by (subst suminf-finite[where N={i}]) auto
  also have ... ≤ (∑ i. f i)
    using nneg
    by (auto intro!: suminf-le-pos)
  finally have False
    using ‹(∑ i. f i) = 0› by auto
 }
 then show ∀ i. f i = 0
  by auto
qed simp

lemma suminf-ereal-offset-le:
 fixes f :: nat ⇒ ereal
 assumes f: ⋀i. 0 ≤ f i
 shows (∑ i. f (i + k)) ≤ suminf f
proof −
 have (λn. ∑ i<n. f (i + k)) ⟶ (∑ i. f (i + k))
  using summable-sums[OF summable-ereal-pos] by (simp add: sums-def atLeast0LessThan
f)
 moreover have (λn. ∑ i<n. f i) ⟶ (∑ i. f i)
  using summable-sums[OF summable-ereal-pos] by (simp add: sums-def atLeast0LessThan
f)
 then have (λn. ∑ i<n + k. f i) ⟶ (∑ i. f i)
  by (rule LIMSEQ-ignore-initial-segment)
 ultimately show ?thesis
 proof (rule LIMSEQ-le, safe intro!: exI[of - k])
  fix n assume k ≤ n
  have (∑ i<n. f (i + k)) = (∑ i<n. (f ∘ (λi. i + k)) i)
   by simp
  also have ... = (∑ i∈(λi. i + k) ` {..<n}. f i)
   by (subst setsum.reindex) auto
  also have ... ≤ setsum f {..<n + k}
   by (intro setsum-mono3) (auto simp: f)
  finally show (∑ i<n. f (i + k)) ≤ setsum f {..<n + k} .
 qed
qed

lemma sums-suminf-ereal: f sums x ⟹ (∑ i. ereal (f i)) = ereal x
 by (metis sums-ereal sums-unique)

lemma suminf-ereal': summable f ⟹ (∑ i. ereal (f i)) = ereal (∑ i. f i)
 by (metis sums-ereal sums-unique summable-def)

lemma suminf-ereal-finite: summable f ⟹ (∑ i. ereal (f i)) ≠ ∞
 by (auto simp: sums-ereal[symmetric] summable-def sums-unique[symmetric])

lemma suminf-ereal-finite-neg:
 assumes summable f
 shows (∑ x. ereal (f x)) ≠ −∞
```

proof−
from *assms* **obtain** x **where** f *sums* x **by** *blast*
hence $(\lambda x.\ ereal\ (f\ x))$ *sums ereal* x **by** (*simp add*: *sums-ereal*)
from *sums-unique*[*OF this*] **have** $(\sum x.\ ereal\ (f\ x)) = ereal\ x$ **..**
thus $(\sum x.\ ereal\ (f\ x)) \neq -\infty$ **by** *simp-all*
qed

lemma *SUP-ereal-add-directed*:
fixes $f\ g :: 'a \Rightarrow ereal$
assumes *nonneg*: $\bigwedge i.\ i \in I \Longrightarrow 0 \leq f\ i\ \bigwedge i.\ i \in I \Longrightarrow 0 \leq g\ i$
assumes *directed*: $\bigwedge i\ j.\ i \in I \Longrightarrow j \in I \Longrightarrow \exists k{\in}I.\ f\ i + g\ j \leq f\ k + g\ k$
shows $(SUP\ i{:}I.\ f\ i + g\ i) = (SUP\ i{:}I.\ f\ i) + (SUP\ i{:}I.\ g\ i)$
proof *cases*
assume $I = \{\}$ **then show** *?thesis*
by (*simp add*: *bot-ereal-def*)
next
assume $I \neq \{\}$
show *?thesis*
proof (*rule antisym*)
show $(SUP\ i{:}I.\ f\ i + g\ i) \leq (SUP\ i{:}I.\ f\ i) + (SUP\ i{:}I.\ g\ i)$
by (*rule SUP-least*; *intro ereal-add-mono SUP-upper*)
next
have $bot < (SUP\ i{:}I.\ g\ i)$
using ‹$I \neq \{\}$› *nonneg*(*2*) **by** (*auto simp*: *bot-ereal-def less-SUP-iff*)
then have $(SUP\ i{:}I.\ f\ i) + (SUP\ i{:}I.\ g\ i) = (SUP\ i{:}I.\ f\ i + (SUP\ i{:}I.\ g\ i))$
by (*intro SUP-ereal-add-left*[*symmetric*] ‹$I \neq \{\}$›) *auto*
also have $\ldots = (SUP\ i{:}I.\ (SUP\ j{:}I.\ f\ i + g\ j))$
using *nonneg*(*1*) **by** (*intro SUP-cong refl SUP-ereal-add-right*[*symmetric*] ‹$I \neq \{\}$›) *auto*
also have $\ldots \leq (SUP\ i{:}I.\ f\ i + g\ i)$
using *directed* **by** (*intro SUP-least*) (*blast intro*: *SUP-upper2*)
finally show $(SUP\ i{:}I.\ f\ i) + (SUP\ i{:}I.\ g\ i) \leq (SUP\ i{:}I.\ f\ i + g\ i)$ **.**
qed
qed

lemma *SUP-ereal-setsum-directed*:
fixes $f\ g :: 'a \Rightarrow 'b \Rightarrow ereal$
assumes $I \neq \{\}$
assumes *directed*: $\bigwedge N\ i\ j.\ N \subseteq A \Longrightarrow i \in I \Longrightarrow j \in I \Longrightarrow \exists k{\in}I.\ \forall n{\in}N.\ f\ n\ i \leq f\ n\ k \wedge f\ n\ j \leq f\ n\ k$
assumes *nonneg*: $\bigwedge n\ i.\ i \in I \Longrightarrow n \in A \Longrightarrow 0 \leq f\ n\ i$
shows $(SUP\ i{:}I.\ \sum n{\in}A.\ f\ n\ i) = (\sum n{\in}A.\ SUP\ i{:}I.\ f\ n\ i)$
proof −
have $N \subseteq A \Longrightarrow (SUP\ i{:}I.\ \sum n{\in}N.\ f\ n\ i) = (\sum n{\in}N.\ SUP\ i{:}I.\ f\ n\ i)$ **for** N
proof (*induction N rule*: *infinite-finite-induct*)
case (*insert n N*)
moreover have $(SUP\ i{:}I.\ f\ n\ i + (\sum l{\in}N.\ f\ l\ i)) = (SUP\ i{:}I.\ f\ n\ i) + (SUP\ i{:}I.\ \sum l{\in}N.\ f\ l\ i)$
proof (*rule SUP-ereal-add-directed*)

fix i **assume** $i \in I$ **then show** $0 \leq f\ n\ i\ 0 \leq (\sum l{\in}N.\ f\ l\ i)$
using *insert* **by** (*auto intro*!: *setsum-nonneg nonneg*)
next
fix $i\ j$ **assume** $i \in I\ j \in I$
from *directed*[*OF* ‹*insert n N* ⊆ *A*› *this*] **guess** k **..**
then show $\exists k{\in}I.\ f\ n\ i + (\sum l{\in}N.\ f\ l\ j) \leq f\ n\ k + (\sum l{\in}N.\ f\ l\ k)$
by (*intro bexI*[*of* - *k*]) (*auto intro*!: *ereal-add-mono setsum-mono*)
qed
ultimately show *?case*
by *simp*
qed (*simp-all add*: *SUP-constant* ‹$I \neq \{\}$›)
from *this*[*of A*] **show** *?thesis* **by** *simp*
qed

lemma *suminf-SUP-eq-directed*:
fixes f :: - ⇒ *nat* ⇒ *ereal*
assumes $I \neq \{\}$
assumes *directed*: $\bigwedge N\ i\ j.\ i \in I \Longrightarrow j \in I \Longrightarrow$ *finite* $N \Longrightarrow \exists k{\in}I.\ \forall n{\in}N.\ f\ i\ n \leq f\ k\ n \wedge f\ j\ n \leq f\ k\ n$
assumes *nonneg*: $\bigwedge n\ i.\ 0 \leq f\ n\ i$
shows $(\sum i.\ SUP\ n{:}I.\ f\ n\ i) = (SUP\ n{:}I.\ \sum i.\ f\ n\ i)$
proof (*subst* (*1 2*) *suminf-ereal-eq-SUP*)
show $\bigwedge n\ i.\ 0 \leq f\ n\ i\ \bigwedge i.\ 0 \leq (SUP\ n{:}I.\ f\ n\ i)$
using ‹$I \neq \{\}$› *nonneg* **by** (*auto intro*: *SUP-upper2*)
show $(SUP\ n.\ \sum i{<}n.\ SUP\ n{:}I.\ f\ n\ i) = (SUP\ n{:}I.\ SUP\ j.\ \sum i{<}j.\ f\ n\ i)$
apply (*subst SUP-commute*)
apply (*subst SUP-ereal-setsum-directed*)
apply (*auto intro*!: *assms simp*: *finite-subset*)
done
qed

lemma *ereal-dense3*:
fixes $x\ y$ :: *ereal*
shows $x < y \Longrightarrow \exists r{::}rat.\ x < real\text{-}of\text{-}rat\ r \wedge real\text{-}of\text{-}rat\ r < y$
proof (*cases x y rule*: *ereal2-cases*, *simp-all*)
fix $r\ q$ :: *real*
assume $r < q$
from *Rats-dense-in-real*[*OF this*] **show** $\exists x.\ r < real\text{-}of\text{-}rat\ x \wedge real\text{-}of\text{-}rat\ x < q$
by (*fastforce simp*: *Rats-def*)
next
fix r :: *real*
show $\exists x.\ r < real\text{-}of\text{-}rat\ x\ \exists x.\ real\text{-}of\text{-}rat\ x < r$
using *gt-ex*[*of r*] *lt-ex*[*of r*] *Rats-dense-in-real*
by (*auto simp*: *Rats-def*)
qed

lemma *continuous-within-ereal*[*intro*, *simp*]: $x \in A \Longrightarrow$ *continuous* (*at x within A*) *ereal*
using *continuous-on-eq-continuous-within*[*of A ereal*]

by (*auto intro*: *continuous-on-ereal continuous-on-id*)

lemma *ereal-open-uminus*:
fixes S :: *ereal set*
assumes *open S*
shows *open* (*uminus ‘ S*)
using ‹*open S*›[*unfolded open-generated-order*]
proof *induct*
have *range uminus* = (*UNIV* :: *ereal set*)
by (*auto simp*: *image-iff ereal-uminus-eq-reorder*)
then show *open* (*range uminus* :: *ereal set*)
by *simp*
qed (*auto simp add*: *image-Union image-Int*)

lemma *ereal-uminus-complement*:
fixes S :: *ereal set*
shows *uminus ‘* (− *S*) = − *uminus ‘ S*
by (*auto intro*!: *bij-image-Compl-eq surjI*[*of - uminus*] *simp*: *bij-betw-def*)

lemma *ereal-closed-uminus*:
fixes S :: *ereal set*
assumes *closed S*
shows *closed* (*uminus ‘ S*)
using *assms*
unfolding *closed-def ereal-uminus-complement*[*symmetric*]
by (*rule ereal-open-uminus*)

lemma *ereal-open-affinity-pos*:
fixes S :: *ereal set*
assumes *open S*
and m: $m \neq \infty$ $0 < m$
and t: $|t| \neq \infty$
shows *open* ((λ*x*. *m* ∗ *x* + *t*) *‘ S*)
proof −
have *open* ((λ*x*. *inverse m* ∗ (*x* + −*t*)) −*‘ S*)
using *m t*
apply (*intro open-vimage* ‹*open S*›)
apply (*intro continuous-at-imp-continuous-on ballI tendsto-cmult-ereal continuous-at*[*THEN iffD2*]
tendsto-ident-at tendsto-add-left-ereal)
apply *auto*
done
also have (λ*x*. *inverse m* ∗ (*x* + −*t*)) −*‘ S* = (λ*x*. (*x* − *t*) / *m*) −*‘ S*
using *m t* **by** (*auto simp*: *divide-ereal-def mult.commute uminus-ereal.simps*[*symmetric*] *minus-ereal-def*
simp del: *uminus-ereal.simps*)
also have (λ*x*. (*x* − *t*) / *m*) −*‘ S* = (λ*x*. *m* ∗ *x* + *t*) *‘ S*
using *m t*
by (*simp add*: *set-eq-iff image-iff*)

(*metis abs-ereal-less0 abs-ereal-uminus ereal-divide-eq ereal-eq-minus ereal-minus*(7,8) *ereal-minus-less-minus ereal-mult-eq-PInfty ereal-uminus-uminus ereal-zero-mult*)
finally show *?thesis* **.**
qed

lemma *ereal-open-affinity*:
fixes S :: *ereal set*
assumes *open* S
and m: $|m| \neq \infty$ $m \neq 0$
and t: $|t| \neq \infty$
shows *open* $((\lambda x.\ m * x + t)\ `\ S)$
proof *cases*
assume $0 < m$
then show *?thesis*
using *ereal-open-affinity-pos*[*OF* ‹*open S*› - - *t*, *of m*] *m*
by *auto*
next
assume ¬ $0 < m$ **then**
have $0 < -m$
using ‹$m \neq 0$›
by (*cases m*) *auto*
then have m: $-m \neq \infty$ $0 < -m$
using ‹$|m| \neq \infty$›
by (*auto simp*: *ereal-uminus-eq-reorder*)
from *ereal-open-affinity-pos*[*OF ereal-open-uminus*[*OF* ‹*open S*›] *m t*] **show** *?thesis*
unfolding *image-image* **by** *simp*
qed

lemma *open-uminus-iff*:
fixes S :: *ereal set*
shows *open* (*uminus* ` S) ⟷ *open S*
using *ereal-open-uminus*[*of S*] *ereal-open-uminus*[*of uminus* ` *S*]
by *auto*

lemma *ereal-Liminf-uminus*:
fixes f :: $'a \Rightarrow$ *ereal*
shows *Liminf net* $(\lambda x.\ -\ (f\ x)) = -$ *Limsup net f*
using *ereal-Limsup-uminus*[*of* - $(\lambda x.\ -\ (f\ x))$] **by** *auto*

lemma *Liminf-PInfty*:
fixes f :: $'a \Rightarrow$ *ereal*
assumes ¬ *trivial-limit net*
shows $(f \longrightarrow \infty)$ *net* ⟷ *Liminf net f* $= \infty$
unfolding *tendsto-iff-Liminf-eq-Limsup*[*OF assms*]
using *Liminf-le-Limsup*[*OF assms*, *of f*]
by *auto*

lemma *Limsup-MInfty*:

fixes $f :: 'a \Rightarrow ereal$
assumes $\neg$ *trivial-limit net*
shows $(f \longrightarrow -\infty)$ *net* $\longleftrightarrow$ *Limsup net f* $= -\infty$
unfolding *tendsto-iff-Liminf-eq-Limsup*[*OF assms*]
using *Liminf-le-Limsup*[*OF assms*, *of f*]
by *auto*

lemma *convergent-ereal*: — RENAME
fixes $X :: nat \Rightarrow 'a :: \{$*complete-linorder*,*linorder-topology*$\}$
shows *convergent X* $\longleftrightarrow$ *limsup X* $=$ *liminf X*
using *tendsto-iff-Liminf-eq-Limsup*[*of sequentially*]
by (*auto simp*: *convergent-def*)

lemma *limsup-le-liminf-real*:
fixes $X :: nat \Rightarrow real$ **and** $L :: real$
assumes *1*: *limsup X* $\leq L$ **and** *2*: $L \leq$ *liminf X*
shows $X \longrightarrow L$
proof −
from *1 2* **have** *limsup X* $\leq$ *liminf X* **by** *auto*
hence *3*: *limsup X* $=$ *liminf X*
apply (*subst eq-iff*, *rule conjI*)
by (*rule Liminf-le-Limsup*, *auto*)
hence *4*: *convergent* $(\lambda n.\ ereal\ (X\ n))$
by (*subst convergent-ereal*)
hence *limsup X* $=$ *lim* $(\lambda n.\ ereal(X\ n))$
by (*rule convergent-limsup-cl*)
also from *1 2 3* **have** *limsup X* $= L$ **by** *auto*
finally have *lim* $(\lambda n.\ ereal(X\ n)) = L$ **..**
hence $(\lambda n.\ ereal\ (X\ n)) \longrightarrow L$
apply (*elim subst*)
by (*subst convergent-LIMSEQ-iff* [*symmetric*], *rule 4*)
thus *?thesis* **by** *simp*
qed

lemma *liminf-PInfty*:
fixes $X :: nat \Rightarrow ereal$
shows $X \longrightarrow \infty \longleftrightarrow$ *liminf X* $= \infty$
by (*metis Liminf-PInfty trivial-limit-sequentially*)

lemma *limsup-MInfty*:
fixes $X :: nat \Rightarrow ereal$
shows $X \longrightarrow -\infty \longleftrightarrow$ *limsup X* $= -\infty$
by (*metis Limsup-MInfty trivial-limit-sequentially*)

lemma *ereal-lim-mono*:
fixes $X\ Y :: nat \Rightarrow 'a$::*linorder-topology*
assumes $\bigwedge n.\ N \leq n \Longrightarrow X\ n \leq Y\ n$
and $X \longrightarrow x$
and $Y \longrightarrow y$

shows $x \leq y$
using *assms*(*1*) **by** (*intro LIMSEQ-le*[*OF assms*(*2*,*3*)]) *auto*

lemma *incseq-le-ereal*:
fixes $X :: nat \Rightarrow 'a$::*linorder-topology*
assumes *inc*: *incseq X*
and *lim*: $X \longrightarrow L$
shows $X\ N \leq L$
using *inc*
by (*intro ereal-lim-mono*[*of N*, *OF - tendsto-const lim*]) (*simp add*: *incseq-def*)

lemma *decseq-ge-ereal*:
assumes *dec*: *decseq X*
and *lim*: $X \longrightarrow (L::'a$::*linorder-topology*)
shows $X\ N \geq L$
using *dec* **by** (*intro ereal-lim-mono*[*of N*, *OF - lim tendsto-const*]) (*simp add*: *decseq-def*)

lemma *bounded-abs*:
fixes a :: *real*
assumes $a \leq x$
and $x \leq b$
shows $|x| \leq max\ |a|\ |b|$
by (*metis abs-less-iff assms leI le-max-iff-disj*
less-eq-real-def less-le-not-le less-minus-iff minus-minus)

lemma *ereal-Sup-lim*:
fixes $a :: 'a$::{*complete-linorder*,*linorder-topology*}
assumes $\bigwedge n.\ b\ n \in s$
and $b \longrightarrow a$
shows $a \leq Sup\ s$
by (*metis Lim-bounded-ereal assms complete-lattice-class.Sup-upper*)

lemma *ereal-Inf-lim*:
fixes $a :: 'a$::{*complete-linorder*,*linorder-topology*}
assumes $\bigwedge n.\ b\ n \in s$
and $b \longrightarrow a$
shows $Inf\ s \leq a$
by (*metis Lim-bounded2-ereal assms complete-lattice-class.Inf-lower*)

lemma *SUP-Lim-ereal*:
fixes $X :: nat \Rightarrow 'a$::{*complete-linorder*,*linorder-topology*}
assumes *inc*: *incseq X*
and l: $X \longrightarrow l$
shows $(SUP\ n.\ X\ n) = l$
using *LIMSEQ-SUP*[*OF inc*] *tendsto-unique*[*OF trivial-limit-sequentially l*]
by *simp*

lemma *INF-Lim-ereal*:

fixes $X :: nat \Rightarrow 'a::\{complete\text{-}linorder, linorder\text{-}topology\}$
assumes *dec*: *decseq X*
and *l*: $X \longrightarrow l$
shows $(INF\ n.\ X\ n) = l$
using *LIMSEQ-INF*[*OF dec*] *tendsto-unique*[*OF trivial-limit-sequentially l*]
by *simp*

lemma *SUP-eq-LIMSEQ*:
assumes *mono f*
shows $(SUP\ n.\ ereal\ (f\ n)) = ereal\ x \longleftrightarrow f \longrightarrow x$
proof
have *inc*: *incseq* $(\lambda i.\ ereal\ (f\ i))$
using ‹*mono f*› **unfolding** *mono-def incseq-def* **by** *auto*
{
assume $f \longrightarrow x$
then have $(\lambda i.\ ereal\ (f\ i)) \longrightarrow ereal\ x$
by *auto*
from *SUP-Lim-ereal*[*OF inc this*] **show** $(SUP\ n.\ ereal\ (f\ n)) = ereal\ x$ **.**
next
assume $(SUP\ n.\ ereal\ (f\ n)) = ereal\ x$
with *LIMSEQ-SUP*[*OF inc*] **show** $f \longrightarrow x$ **by** *auto*
}
qed

lemma *liminf-ereal-cminus*:
fixes $f :: nat \Rightarrow ereal$
assumes $c \neq -\infty$
shows $liminf\ (\lambda x.\ c - f\ x) = c - limsup\ f$
proof (*cases c*)
case *PInf*
then show *?thesis*
by (*simp add*: *Liminf-const*)
next
case (*real r*)
then show *?thesis*
unfolding *liminf-SUP-INF limsup-INF-SUP*
apply (*subst INF-ereal-minus-right*)
apply *auto*
apply (*subst SUP-ereal-minus-right*)
apply *auto*
done
qed (*insert* ‹$c \neq -\infty$›, *simp*)

36.4.3 Continuity

lemma *continuous-at-of-ereal*:
$|x0 :: ereal| \neq \infty \Longrightarrow continuous\ (at\ x0)\ real\text{-}of\text{-}ereal$
unfolding *continuous-at*
by (*rule lim-real-of-ereal*) (*simp add*: *ereal-real*)

lemma *nhds-ereal*: *nhds* (*ereal r*) = *filtermap ereal* (*nhds r*)
by (*simp add*: *filtermap-nhds-open-map open-ereal continuous-at-of-ereal*)

lemma *at-ereal*: *at* (*ereal r*) = *filtermap ereal* (*at r*)
by (*simp add*: *filter-eq-iff eventually-at-filter nhds-ereal eventually-filtermap*)

lemma *at-left-ereal*: *at-left* (*ereal r*) = *filtermap ereal* (*at-left r*)
by (*simp add*: *filter-eq-iff eventually-at-filter nhds-ereal eventually-filtermap*)

lemma *at-right-ereal*: *at-right* (*ereal r*) = *filtermap ereal* (*at-right r*)
by (*simp add*: *filter-eq-iff eventually-at-filter nhds-ereal eventually-filtermap*)

lemma
shows *at-left-PInf*: *at-left* ∞ = *filtermap ereal at-top*
and *at-right-MInf*: *at-right* ($-\infty$) = *filtermap ereal at-bot*
unfolding *filter-eq-iff eventually-filtermap eventually-at-top-dense eventually-at-bot-dense*
eventually-at-left[*OF ereal-less*(*5*)] *eventually-at-right*[*OF ereal-less*(*6*)]
by (*auto simp add*: *ereal-all-split ereal-ex-split*)

lemma *ereal-tendsto-simps1*:
((f ∘ *real-of-ereal*) ⟶ y) (*at-left* (*ereal x*)) ⟷ (f ⟶ y) (*at-left x*)
((f ∘ *real-of-ereal*) ⟶ y) (*at-right* (*ereal x*)) ⟷ (f ⟶ y) (*at-right x*)
((f ∘ *real-of-ereal*) ⟶ y) (*at-left* (∞::*ereal*)) ⟷ (f ⟶ y) *at-top*
((f ∘ *real-of-ereal*) ⟶ y) (*at-right* ($-\infty$::*ereal*)) ⟷ (f ⟶ y) *at-bot*
unfolding *tendsto-compose-filtermap at-left-ereal at-right-ereal at-left-PInf at-right-MInf*
by (*auto simp*: *filtermap-filtermap filtermap-ident*)

lemma *ereal-tendsto-simps2*:
(($ereal$ ∘ f) ⟶ *ereal a*) F ⟷ (f ⟶ a) F
(($ereal$ ∘ f) ⟶ ∞) F ⟷ (*LIM x F*. f x :> *at-top*)
(($ereal$ ∘ f) ⟶ $-\infty$) F ⟷ (*LIM x F*. f x :> *at-bot*)
unfolding *tendsto-PInfty filterlim-at-top-dense tendsto-MInfty filterlim-at-bot-dense*
using *lim-ereal* **by** (*simp-all add*: *comp-def*)

lemma *inverse-infty-ereal-tendsto-0*: *inverse* $-\infty\rightarrow$ (*0*::*ereal*)
proof −
have $**$: ((λx. *ereal* (*inverse x*)) ⟶ *ereal 0*) *at-infinity*
by (*intro tendsto-intros tendsto-inverse-0*)

show *?thesis*
by (*simp add*: *at-infty-ereal-eq-at-top tendsto-compose-filtermap*[*symmetric*]
comp-def)
(*auto simp*: *eventually-at-top-linorder exI*[*of - 1*] *zero-ereal-def at-top-le-at-infinity*
intro!: *filterlim-mono-eventually*[*OF* $**$])
qed

lemma *inverse-ereal-tendsto-pos*:
fixes x :: *ereal* **assumes** *0* < x

```
  shows inverse −x→ inverse x
proof (cases x)
  case (real r)
  with ‹0 < x› have ∗∗: (λx. ereal (inverse x)) −r→ ereal (inverse r)
    by (auto intro!: tendsto-inverse)
  from real ‹0 < x› show ?thesis
   by (auto simp: at-ereal tendsto-compose-filtermap[symmetric] eventually-at-filter
            intro!: Lim-transform-eventually[OF - ∗∗] t1-space-nhds)
qed (insert ‹0 < x›, auto intro!: inverse-infty-ereal-tendsto-0)

lemma inverse-ereal-tendsto-at-right-0: (inverse ⟶ ∞) (at-right (0::ereal))
  unfolding tendsto-compose-filtermap[symmetric] at-right-ereal zero-ereal-def
  by (subst filterlim-cong[OF refl refl, where g=λx. ereal (inverse x)])
    (auto simp: eventually-at-filter tendsto-PInfty-eq-at-top filterlim-inverse-at-top-right)

lemmas ereal-tendsto-simps = ereal-tendsto-simps1 ereal-tendsto-simps2

lemma continuous-at-iff-ereal:
  fixes f :: 'a::t2-space ⇒ real
  shows continuous (at x0 within s) f ⟷ continuous (at x0 within s) (ereal ∘ f)
  unfolding continuous-within comp-def lim-ereal ..

lemma continuous-on-iff-ereal:
  fixes f :: 'a::t2-space => real
  assumes open A
  shows continuous-on A f ⟷ continuous-on A (ereal ∘ f)
  unfolding continuous-on-def comp-def lim-ereal ..

lemma continuous-on-real: continuous-on (UNIV − {∞, −∞::ereal}) real-of-ereal
  using continuous-at-of-ereal continuous-on-eq-continuous-at open-image-ereal
  by auto

lemma continuous-on-iff-real:
  fixes f :: 'a::t2-space ⇒ ereal
  assumes ∗: ⋀x. x ∈ A ⟹ |f x| ≠ ∞
  shows continuous-on A f ⟷ continuous-on A (real-of-ereal ∘ f)
proof −
  have f ‘ A ⊆ UNIV − {∞, −∞}
    using assms by force
  then have ∗: continuous-on (f ‘ A) real-of-ereal
    using continuous-on-real by (simp add: continuous-on-subset)
  have ∗∗: continuous-on ((real-of-ereal ∘ f) ‘ A) ereal
    by (intro continuous-on-ereal continuous-on-id)
  {
    assume continuous-on A f
    then have continuous-on A (real-of-ereal ∘ f)
      apply (subst continuous-on-compose)
      using ∗
      apply auto
```

```
    done
  }
  moreover
  {
    assume continuous-on A (real-of-ereal ∘ f)
    then have continuous-on A (ereal ∘ (real-of-ereal ∘ f))
      apply (subst continuous-on-compose)
      using **
      apply auto
      done
    then have continuous-on A f
      apply (subst continuous-on-cong[of - A - ereal ∘ (real-of-ereal ∘ f)])
      using assms ereal-real
      apply auto
      done
  }
  ultimately show ?thesis
    by auto
qed

lemma continuous-uminus-ereal [continuous-intros]: continuous-on (A :: ereal set)
uminus
  unfolding continuous-on-def
  by (intro ballI tendsto-uminus-ereal[of λx. x::ereal]) simp

lemma ereal-uminus-atMost [simp]: uminus ‘ {..(a::ereal)} = {−a..}
proof (intro equalityI subsetI)
  fix x :: ereal assume x ∈ {−a..}
 hence −(−x) ∈ uminus ‘ {..a} by (intro imageI) (simp add: ereal-uminus-le-reorder)
  thus x ∈ uminus ‘ {..a} by simp
qed auto

lemma continuous-on-inverse-ereal [continuous-intros]:
  continuous-on {0::ereal ..} inverse
  unfolding continuous-on-def
proof clarsimp
  fix x :: ereal assume 0 ≤ x
  moreover have at 0 within {0 ..} = at-right (0::ereal)
    by (auto simp: filter-eq-iff eventually-at-filter le-less)
  moreover have at x within {0 ..} = at x if 0 < x
    using that by (intro at-within-nhd[of - {0<..}]) auto
  ultimately show (inverse ⟶ inverse x) (at x within {0..})
    by (auto simp: le-less inverse-ereal-tendsto-at-right-0 inverse-ereal-tendsto-pos)
qed

lemma continuous-inverse-ereal-nonpos: continuous-on ({..<0} :: ereal set) inverse
proof (subst continuous-on-cong[OF refl])
  have continuous-on {(0::ereal)<..} inverse
    by (rule continuous-on-subset[OF continuous-on-inverse-ereal]) auto
```

thus *continuous-on* {..<(*0*::*ereal*)} (*uminus* ∘ *inverse* ∘ *uminus*)
by (*intro continuous-intros*) *simp-all*
qed *simp*

lemma *tendsto-inverse-ereal*:
assumes (*f* ⟶ (*c* :: *ereal*)) *F*
assumes *eventually* (λ*x*. *f x* ≥ *0*) *F*
shows ((λ*x*. *inverse* (*f x*)) ⟶ *inverse c*) *F*
by (*cases F* = *bot*)
(*auto intro*!: *tendsto-le-const*[*of F*] *assms*
continuous-on-tendsto-compose[*OF continuous-on-inverse-ereal*])

36.4.4 liminf and limsup

lemma *Limsup-ereal-mult-right*:
assumes *F* ≠ *bot* (*c*::*real*) ≥ *0*
shows *Limsup F* (λ*n*. *f n* ∗ *ereal c*) = *Limsup F f* ∗ *ereal c*
proof (*rule Limsup-compose-continuous-mono*)
from *assms* **show** *continuous-on UNIV* (λ*a*. *a* ∗ *ereal c*)
using *tendsto-cmult-ereal*[*of ereal c* λ*x*. *x*]
by (*force simp*: *continuous-on-def mult-ac*)
qed (*insert assms*, *auto simp*: *mono-def ereal-mult-right-mono*)

lemma *Liminf-ereal-mult-right*:
assumes *F* ≠ *bot* (*c*::*real*) ≥ *0*
shows *Liminf F* (λ*n*. *f n* ∗ *ereal c*) = *Liminf F f* ∗ *ereal c*
proof (*rule Liminf-compose-continuous-mono*)
from *assms* **show** *continuous-on UNIV* (λ*a*. *a* ∗ *ereal c*)
using *tendsto-cmult-ereal*[*of ereal c* λ*x*. *x*]
by (*force simp*: *continuous-on-def mult-ac*)
qed (*insert assms*, *auto simp*: *mono-def ereal-mult-right-mono*)

lemma *Limsup-ereal-mult-left*:
assumes *F* ≠ *bot* (*c*::*real*) ≥ *0*
shows *Limsup F* (λ*n*. *ereal c* ∗ *f n*) = *ereal c* ∗ *Limsup F f*
using *Limsup-ereal-mult-right*[*OF assms*] **by** (*subst* (*1 2*) *mult.commute*)

lemma *limsup-ereal-mult-right*:
(*c*::*real*) ≥ *0* ⟹ *limsup* (λ*n*. *f n* ∗ *ereal c*) = *limsup f* ∗ *ereal c*
by (*rule Limsup-ereal-mult-right*) *simp-all*

lemma *limsup-ereal-mult-left*:
(*c*::*real*) ≥ *0* ⟹ *limsup* (λ*n*. *ereal c* ∗ *f n*) = *ereal c* ∗ *limsup f*
by (*subst* (*1 2*) *mult.commute*, *rule limsup-ereal-mult-right*) *simp-all*

lemma *Limsup-add-ereal-right*:
F ≠ *bot* ⟹ *abs c* ≠ ∞ ⟹ *Limsup F* (λ*n*. *g n* + (*c* :: *ereal*)) = *Limsup F g* + *c*
by (*rule Limsup-compose-continuous-mono*) (*auto simp*: *mono-def ereal-add-mono*

continuous-on-def)

lemma *Limsup-add-ereal-left*:
F ≠ *bot* ⟹ *abs c* ≠ ∞ ⟹ *Limsup F* (λ*n*. (*c* :: *ereal*) + *g n*) = *c* + *Limsup F g*
by (*subst* (*1 2*) *add.commute*) (*rule Limsup-add-ereal-right*)

lemma *Liminf-add-ereal-right*:
F ≠ *bot* ⟹ *abs c* ≠ ∞ ⟹ *Liminf F* (λ*n*. *g n* + (*c* :: *ereal*)) = *Liminf F g* + *c*
by (*rule Liminf-compose-continuous-mono*) (*auto simp*: *mono-def ereal-add-mono continuous-on-def*)

lemma *Liminf-add-ereal-left*:
F ≠ *bot* ⟹ *abs c* ≠ ∞ ⟹ *Liminf F* (λ*n*. (*c* :: *ereal*) + *g n*) = *c* + *Liminf F g*
by (*subst* (*1 2*) *add.commute*) (*rule Liminf-add-ereal-right*)

lemma
assumes *F* ≠ *bot*
assumes *nonneg*: *eventually* (λ*x*. *f x* ≥ (*0*::*ereal*)) *F*
shows *Liminf-inverse-ereal*: *Liminf F* (λ*x*. *inverse* (*f x*)) = *inverse* (*Limsup F f*)
and *Limsup-inverse-ereal*: *Limsup F* (λ*x*. *inverse* (*f x*)) = *inverse* (*Liminf F f*)
proof −
def *inv* ≡ λ*x*. *if x* ≤ *0 then* ∞ *else inverse x* :: *ereal*
have *continuous-on* ({..*0*} ∪ {*0*..}) *inv* **unfolding** *inv-def*
by (*intro continuous-on-If*) (*auto intro*!: *continuous-intros*)
also have {..*0*} ∪ {*0*..} = (*UNIV* :: *ereal set*) **by** *auto*
finally have *cont*: *continuous-on UNIV inv* **.**
have *antimono*: *antimono inv* **unfolding** *inv-def antimono-def*
by (*auto intro*!: *ereal-inverse-antimono*)

have *Liminf F* (λ*x*. *inverse* (*f x*)) = *Liminf F* (λ*x*. *inv* (*f x*)) **using** *nonneg*
by (*auto intro*!: *Liminf-eq elim*!: *eventually-mono simp*: *inv-def*)
also have ... = *inv* (*Limsup F f*)
by (*simp add*: *assms*(*1*) *Liminf-compose-continuous-antimono*[*OF cont antimono*])
also from *assms* **have** *Limsup F f* ≥ *0* **by** (*intro le-Limsup*) *simp-all*
hence *inv* (*Limsup F f*) = *inverse* (*Limsup F f*) **by** (*simp add*: *inv-def*)
finally show *Liminf F* (λ*x*. *inverse* (*f x*)) = *inverse* (*Limsup F f*) **.**

have *Limsup F* (λ*x*. *inverse* (*f x*)) = *Limsup F* (λ*x*. *inv* (*f x*)) **using** *nonneg*
by (*auto intro*!: *Limsup-eq elim*!: *eventually-mono simp*: *inv-def*)
also have ... = *inv* (*Liminf F f*)
by (*simp add*: *assms*(*1*) *Limsup-compose-continuous-antimono*[*OF cont antimono*])
also from *assms* **have** *Liminf F f* ≥ *0* **by** (*intro Liminf-bounded*) *simp-all*
hence *inv* (*Liminf F f*) = *inverse* (*Liminf F f*) **by** (*simp add*: *inv-def*)
finally show *Limsup F* (λ*x*. *inverse* (*f x*)) = *inverse* (*Liminf F f*) **.**

qed

36.4.5 Tests for code generator

value $-\infty$ :: *ereal*
value $|-\infty|$ :: *ereal*
value *4 + 5 / 4 − ereal 2* :: *ereal*
value *ereal 3 < ∞*
value *real-of-ereal* (∞::*ereal*) = *0*

end

37 Radius of Convergence and Summation Tests

theory *Summation*
imports
Complex-Main
~~/*src*/*HOL*/*Library*/*Extended-Real*
~~/*src*/*HOL*/*Library*/*Liminf-Limsup*
begin

The definition of the radius of convergence of a power series, various summability tests, lemmas to compute the radius of convergence etc.

37.1 Rounded dual logarithm

definition *natlog2 n* = (*if n* = *0 then 0 else nat* ⌊*log 2* (*real-of-nat n*)⌋)

lemma *natlog2-0* [*simp*]: *natlog2 0* = *0* **by** (*simp add*: *natlog2-def*)
lemma *natlog2-1* [*simp*]: *natlog2 1* = *0* **by** (*simp add*: *natlog2-def*)
lemma *natlog2-eq-0-iff*: *natlog2 n* = *0* ⟷ *n* < *2* **by** (*simp add*: *natlog2-def*)

lemma *natlog2-power-of-two* [*simp*]: *natlog2* (*2* ^ *n*) = *n*
by (*simp add*: *natlog2-def log-nat-power*)

lemma *natlog2-mono*: *m* ≤ *n* ⟹ *natlog2 m* ≤ *natlog2 n*
unfolding *natlog2-def* **by** (*simp-all add*: *nat-mono floor-mono*)

lemma *pow-natlog2-le*: *n* > *0* ⟹ *2* ^ *natlog2 n* ≤ *n*
proof −
assume *n*: *n* > *0*
from *n* **have** *of-nat* (*2* ^ *natlog2 n*) = *2 powr real-of-nat* (*nat* ⌊*log 2* (*real-of-nat n*)⌋)
by (*subst powr-realpow*) (*simp-all add*: *natlog2-def*)
also have ... = *2 powr of-int* ⌊*log 2* (*real-of-nat n*)⌋ **using** *n* **by** *simp*
also have ... ≤ *2 powr log 2* (*real-of-nat n*) **by** (*intro powr-mono*) (*linarith*, *simp-all*)
also have ... = *of-nat n* **using** *n* **by** *simp*
finally show *?thesis* **by** *simp*

qed

lemma *pow-natlog2-gt*: $n > 0 \Longrightarrow 2 * 2 \hat{} \mathit{natlog2}\ n > n$
and *pow-natlog2-ge*: $n > 0 \Longrightarrow 2 * 2 \hat{} \mathit{natlog2}\ n \geq n$
proof −
assume *n*: $n > 0$
from *n* **have** $\mathit{of\text{-}nat}\ n = 2\ \mathit{powr}\ \mathit{log}\ 2\ (\mathit{real\text{-}of\text{-}nat}\ n)$ **by** *simp*
also have $\ldots < 2\ \mathit{powr}\ (1 + \mathit{of\text{-}int}\ \lfloor \mathit{log}\ 2\ (\mathit{real\text{-}of\text{-}nat}\ n) \rfloor)$
by (*intro powr-less-mono*) (*linarith*, *simp-all*)
also from *n* **have** $\ldots = 2\ \mathit{powr}\ (1 + \mathit{real\text{-}of\text{-}nat}\ (\mathit{nat}\ \lfloor \mathit{log}\ 2\ (\mathit{real\text{-}of\text{-}nat}\ n) \rfloor))$
by *simp*
also from *n* **have** $\ldots = \mathit{of\text{-}nat}\ (2 * 2 \hat{} \mathit{natlog2}\ n)$
by (*simp-all add*: *natlog2-def powr-real-of-int powr-add*)
finally show $2 * 2 \hat{} \mathit{natlog2}\ n > n$ **by** (*rule of-nat-less-imp-less*)
thus $2 * 2 \hat{} \mathit{natlog2}\ n \geq n$ **by** *simp*
qed

lemma *natlog2-eqI*:
assumes $n > 0\ 2\hat{}k \leq n\ n < 2 * 2\hat{}k$
shows $\mathit{natlog2}\ n = k$
proof −
from *assms* **have** $\mathit{of\text{-}nat}\ (2 \hat{} k) \leq \mathit{real\text{-}of\text{-}nat}\ n$ **by** (*subst of-nat-le-iff*) *simp-all*
hence $\mathit{real\text{-}of\text{-}int}\ (\mathit{int}\ k) \leq \mathit{log}\ (\mathit{of\text{-}nat}\ 2)\ (\mathit{real\text{-}of\text{-}nat}\ n)$
by (*subst le-log-iff*) (*simp-all add*: *powr-realpow assms del*: *of-nat-le-iff*)
moreover from *assms* **have** $\mathit{real\text{-}of\text{-}nat}\ n < \mathit{of\text{-}nat}\ (2 \hat{} \mathit{Suc}\ k)$ **by** (*subst of-nat-less-iff*) *simp-all*
hence $\mathit{log}\ 2\ (\mathit{real\text{-}of\text{-}nat}\ n) < \mathit{of\text{-}nat}\ k + 1$
by (*subst log-less-iff*) (*simp-all add*: *assms powr-realpow powr-add*)
ultimately have $\lfloor \mathit{log}\ 2\ (\mathit{real\text{-}of\text{-}nat}\ n) \rfloor = \mathit{of\text{-}nat}\ k$ **by** (*intro floor-unique*) *simp-all*
with *assms* **show** *?thesis* **by** (*simp add*: *natlog2-def*)
qed

lemma *natlog2-rec*:
assumes $n \geq 2$
shows $\mathit{natlog2}\ n = 1 + \mathit{natlog2}\ (n\ \mathit{div}\ 2)$
proof (*rule natlog2-eqI*)
from *assms* **have** $2 \hat{} (1 + \mathit{natlog2}\ (n\ \mathit{div}\ 2)) \leq 2 * (n\ \mathit{div}\ 2)$
by (*simp add*: *pow-natlog2-le*)
also have $\ldots \leq n$ **by** *simp*
finally show $2 \hat{} (1 + \mathit{natlog2}\ (n\ \mathit{div}\ 2)) \leq n$ **.**
next
from *assms* **have** $n < 2 * (n\ \mathit{div}\ 2 + 1)$ **by** *simp*
also from *assms* **have** $(n\ \mathit{div}\ 2) < 2 \hat{} (1 + \mathit{natlog2}\ (n\ \mathit{div}\ 2))$
by (*simp add*: *pow-natlog2-gt*)
hence $2 * (n\ \mathit{div}\ 2 + 1) \leq 2 * (2 \hat{} (1 + \mathit{natlog2}\ (n\ \mathit{div}\ 2)))$
by (*intro mult-left-mono*) *simp-all*
finally show $n < 2 * 2 \hat{} (1 + \mathit{natlog2}\ (n\ \mathit{div}\ 2))$ **.**
qed (*insert assms*, *simp-all*)

fun *natlog2-aux* **where**
natlog2-aux n acc = (*if* (*n*::*nat*) < *2 then acc else natlog2-aux* (*n div 2*) (*acc* + *1*))

lemma *natlog2-aux-correct*:
natlog2-aux n acc = *acc* + *natlog2 n*
by (*induction n acc rule*: *natlog2-aux.induct*) (*auto simp*: *natlog2-rec natlog2-eq-0-iff*)

lemma *natlog2-code* [*code*]: *natlog2 n* = *natlog2-aux n 0*
by (*subst natlog2-aux-correct*) *simp*

37.2 Convergence tests for infinite sums

37.2.1 Root test

lemma *limsup-root-powser*:
fixes *f* :: *nat* ⇒ *'a* :: {*banach*, *real-normed-div-algebra*}
shows *limsup* (λ*n*. *ereal* (*root n* (*norm* (*f n* ∗ *z* ^ *n*)))) =
limsup (λ*n*. *ereal* (*root n* (*norm* (*f n*)))) ∗ *ereal* (*norm z*)
proof −
have *A*: (λ*n*. *ereal* (*root n* (*norm* (*f n* ∗ *z* ^ *n*)))) =
(λ*n*. *ereal* (*root n* (*norm* (*f n*))) ∗ *ereal* (*norm z*)) (**is** *?g* = *?h*)
proof
fix *n* **show** *?g n* = *?h n*
by (*cases n* = *0*) (*simp-all add*: *norm-mult real-root-mult real-root-pos2 norm-power*)
qed
show *?thesis* **by** (*subst A*, *subst limsup-ereal-mult-right*) *simp-all*
qed

lemma *limsup-root-limit*:
assumes (λ*n*. *ereal* (*root n* (*norm* (*f n*)))) ⟶ *l* (**is** *?g* ⟶ -)
shows *limsup* (λ*n*. *ereal* (*root n* (*norm* (*f n*)))) = *l*
proof −
from *assms* **have** *convergent ?g lim ?g* = *l*
unfolding *convergent-def* **by** (*blast intro*: *limI*)+
with *convergent-limsup-cl* **show** *?thesis* **by** *force*
qed

lemma *limsup-root-limit'*:
assumes (λ*n*. *root n* (*norm* (*f n*))) ⟶ *l*
shows *limsup* (λ*n*. *ereal* (*root n* (*norm* (*f n*)))) = *ereal l*
by (*intro limsup-root-limit tendsto-ereal assms*)

lemma *root-test-convergence'*:
fixes *f* :: *nat* ⇒ *'a* :: *banach*
defines *l* ≡ *limsup* (λ*n*. *ereal* (*root n* (*norm* (*f n*))))
assumes *l*: *l* < *1*
shows *summable f*
proof −

```
  have 0 = limsup (λn. 0) by (simp add: Limsup-const)
   also have ... ≤ l unfolding l-def by (intro Limsup-mono) (simp-all add:
real-root-ge-zero)
  finally have l ≥ 0 by simp
  with l obtain l' where l': l = ereal l' by (cases l) simp-all

  def c ≡ (1 - l') / 2
  from l and ‹l ≥ 0› have c: l + c > l l' + c ≥ 0 l' + c < 1 unfolding c-def
    by (simp-all add: field-simps l')
  have ∀ C>l. eventually (λn. ereal (root n (norm (f n))) < C) sequentially
    by (subst Limsup-le-iff[symmetric]) (simp add: l-def)
 with c have eventually (λn. ereal (root n (norm (f n))) < l + ereal c) sequentially
by simp
  with eventually-gt-at-top[of 0::nat]
    have eventually (λn. norm (f n) ≤ (l' + c) ^ n) sequentially
  proof eventually-elim
    fix n :: nat assume n: n > 0
    assume ereal (root n (norm (f n))) < l + ereal c
    hence root n (norm (f n)) ≤ l' + c by (simp add: l')
    with c n have root n (norm (f n)) ^ n ≤ (l' + c) ^ n
      by (intro power-mono) (simp-all add: real-root-ge-zero)
    also from n have root n (norm (f n)) ^ n = norm (f n) by simp
    finally show norm (f n) ≤ (l' + c) ^ n by simp
  qed
  thus ?thesis
    by (rule summable-comparison-test-ev[OF - summable-geometric]) (simp add:
c)
qed

lemma root-test-divergence:
  fixes f :: nat ⇒ 'a :: banach
  defines l ≡ limsup (λn. ereal (root n (norm (f n))))
  assumes l: l > 1
  shows   ¬summable f
proof
  assume summable f
  hence bounded: Bseq f by (simp add: summable-imp-Bseq)

  have 0 = limsup (λn. 0) by (simp add: Limsup-const)
   also have ... ≤ l unfolding l-def by (intro Limsup-mono) (simp-all add:
real-root-ge-zero)
  finally have l-nonneg: l ≥ 0 by simp

  def c ≡ if l = ∞ then 2 else 1 + (real-of-ereal l - 1) / 2
  from l l-nonneg consider l = ∞ | ∃ l'. l = ereal l' by (cases l) simp-all
  hence c: c > 1 ∧ ereal c < l by cases (insert l, auto simp: c-def field-simps)

  have unbounded: ¬bdd-above {n. root n (norm (f n)) > c}
  proof
```

assume *bdd-above* $\{n.\ root\ n\ (norm\ (f\ n)) > c\}$
then obtain N **where** $\forall n.\ root\ n\ (norm\ (f\ n)) > c \longrightarrow n \leq N$ **unfolding** *bdd-above-def* **by** *blast*
hence $\exists N.\ \forall n{\geq}N.\ root\ n\ (norm\ (f\ n)) \leq c$
by (*intro exI*[*of - N + 1*]) (*force simp*: *not-less-eq-eq*[*symmetric*])
hence *eventually* $(\lambda n.\ root\ n\ (norm\ (f\ n)) \leq c)$ *sequentially*
by (*auto simp*: *eventually-at-top-linorder*)
hence $l \leq c$ **unfolding** *l-def* **by** (*intro Limsup-bounded*) *simp-all*
with c **show** *False* **by** *auto*
qed

from *bounded* **obtain** K **where** K: $K > 0$ $\bigwedge n.\ norm\ (f\ n) \leq K$ **using** *BseqE* **by** *blast*
def $n \equiv nat\ \lceil log\ c\ K \rceil$
from *unbounded* **have** $\exists m{>}n.\ c < root\ m\ (norm\ (f\ m))$ **unfolding** *bdd-above-def*
by (*auto simp*: *not-le*)
then guess m **by** (*elim exE conjE*) **note** $m = this$
from $c\ K$ **have** $K = c\ powr\ log\ c\ K$ **by** (*simp add*: *powr-def log-def*)
also from c **have** $c\ powr\ log\ c\ K \leq c\ powr\ real\ n$ **unfolding** *n-def*
by (*intro powr-mono*, *linarith*, *simp*)
finally have $K \leq c\ \hat{}\ n$ **using** c **by** (*simp add*: *powr-realpow*)
also from $c\ m$ **have** $c\ \hat{}\ n < c\ \hat{}\ m$ **by** *simp*
also from $c\ m$ **have** $c\ \hat{}\ m < root\ m\ (norm\ (f\ m))\ \hat{}\ m$ **by** (*intro power-strict-mono*) *simp-all*
also from m **have** $... = norm\ (f\ m)$ **by** *simp*
finally show *False* **using** $K(2)$[*of m*] **by** *simp*
qed

37.2.2 Cauchy's condensation test

context
fixes $f :: nat \Rightarrow real$
begin

private lemma *condensation-inequality*:
assumes *mono*: $\bigwedge m\ n.\ 0 < m \Longrightarrow m \leq n \Longrightarrow f\ n \leq f\ m$
shows $(\sum k{=}1..{<}n.\ f\ k) \geq (\sum k{=}1..{<}n.\ f\ (2 * 2\ \hat{}\ natlog2\ k))$ (**is** *?thesis1*)
$(\sum k{=}1..{<}n.\ f\ k) \leq (\sum k{=}1..{<}n.\ f\ (2\ \hat{}\ natlog2\ k))$ (**is** *?thesis2*)
by (*intro setsum-mono mono pow-natlog2-ge pow-natlog2-le*, *simp*, *simp*)+

private lemma *condensation-condense1*: $(\sum k{=}1..{<}2\hat{}n.\ f\ (2\ \hat{}\ natlog2\ k)) = (\sum k{<}n.\ 2\hat{}k * f\ (2\ \hat{}\ k))$
proof (*induction n*)
case (*Suc n*)
have $\{1..{<}2\hat{}Suc\ n\} = \{1..{<}2\hat{}n\} \cup \{2\hat{}n..{<}(2\hat{}Suc\ n :: nat)\}$ **by** *auto*
also have $(\sum k{\in}....\ f\ (2\ \hat{}\ natlog2\ k)) =$
$(\sum k{<}n.\ 2\hat{}k * f\ (2\hat{}k)) + (\sum k = 2\hat{}n..{<}2\hat{}Suc\ n.\ f\ (2\hat{}natlog2\ k))$
by (*subst setsum.union-disjoint*) (*insert Suc*, *auto*)
also have *natlog2* $k = n$ **if** $k \in \{2\hat{}n..{<}2\hat{}Suc\ n\}$ **for** k **using** *that* **by** (*intro*

natlog2-eqI) *simp-all*
hence ($\sum k = 2\hat{}n..<2\hat{}Suc\ n.\ f\ (2\hat{}natlog2\ k)) = (\sum (-::nat) = 2\hat{}n..<2\hat{}Suc\ n.\ f\ (2\hat{}n))$
by (*intro setsum.cong*) *simp-all*
also have $\ldots = 2\hat{}n * f\ (2\hat{}n)$ **by** (*simp add*: *of-nat-power*)
finally show *?case* **by** *simp*
qed *simp*

private lemma *condensation-condense2*: $(\sum k{=}1..<2\hat{}n.\ f\ (2 * 2\ \hat{}\ natlog2\ k)) = (\sum k{<}n.\ 2\hat{}k * f\ (2\ \hat{}\ Suc\ k))$
proof (*induction n*)
case (*Suc n*)
have $\{1..<2\hat{}Suc\ n\} = \{1..<2\hat{}n\} \cup \{2\hat{}n..<(2\hat{}Suc\ n :: nat)\}$ **by** *auto*
also have $(\sum k{\in}\ldots.\ f\ (2 * 2\ \hat{}\ natlog2\ k)) = (\sum k{<}n.\ 2\hat{}k * f\ (2\hat{}Suc\ k)) + (\sum k = 2\hat{}n..<2\hat{}Suc\ n.\ f\ (2 * 2\hat{}natlog2\ k))$
by (*subst setsum.union-disjoint*) (*insert Suc*, *auto*)
also have *natlog2 k* = *n* **if** $k \in \{2\hat{}n..<2\hat{}Suc\ n\}$ **for** *k* **using** *that* **by** (*intro natlog2-eqI*) *simp-all*
hence $(\sum k = 2\hat{}n..<2\hat{}Suc\ n.\ f\ (2{*}2\hat{}natlog2\ k)) = (\sum (-::nat) = 2\hat{}n..<2\hat{}Suc\ n.\ f\ (2\hat{}Suc\ n))$
by (*intro setsum.cong*) *simp-all*
also have $\ldots = 2\hat{}n * f\ (2\hat{}Suc\ n)$ **by** (*simp add*: *of-nat-power*)
finally show *?case* **by** *simp*
qed *simp*

lemma *condensation-test*:
assumes *mono*: $\bigwedge m.\ 0 < m \Longrightarrow f\ (Suc\ m) \le f\ m$
assumes *nonneg*: $\bigwedge n.\ f\ n \ge 0$
shows *summable f* $\longleftrightarrow$ *summable* $(\lambda n.\ 2\hat{}n * f\ (2\hat{}n))$
proof −
def $f' \equiv \lambda n.\ if\ n = 0\ then\ 0\ else\ f\ n$
from *mono* **have** *mono'*: *decseq* $(\lambda n.\ f\ (Suc\ n))$ **by** (*intro decseq-SucI*) *simp*
hence *mono'*: $f\ n \le f\ m$ **if** $m \le n\ m > 0$ **for** *m n*
using *that decseqD*[*OF mono'*, *of m − 1 n − 1*] **by** *simp*

have $(\lambda n.\ f\ (Suc\ n)) = (\lambda n.\ f'\ (Suc\ n))$ **by** (*intro ext*) (*simp add*: *f'-def*)
hence *summable f* $\longleftrightarrow$ *summable f'*
by (*subst* (*1 2*) *summable-Suc-iff* [*symmetric*]) (*simp only*:)
also have $\ldots \longleftrightarrow$ *convergent* $(\lambda n.\ \sum k{<}n.\ f'\ k)$ **unfolding** *summable-iff-convergent*
..
also have *monoseq* $(\lambda n.\ \sum k{<}n.\ f'\ k)$ **unfolding** *f'-def*
by (*intro mono-SucI1*) (*auto intro*!: *mult-nonneg-nonneg nonneg*)
hence *convergent* $(\lambda n.\ \sum k{<}n.\ f'\ k) \longleftrightarrow$ *Bseq* $(\lambda n.\ \sum k{<}n.\ f'\ k)$
by (*rule monoseq-imp-convergent-iff-Bseq*)
also have $\ldots \longleftrightarrow$ *Bseq* $(\lambda n.\ \sum k{=}1..<n.\ f'\ k)$ **unfolding** *One-nat-def*
by (*subst setsum-shift-lb-Suc0-0-upt*) (*simp-all add*: *f'-def atLeast0LessThan*)
also have $\ldots \longleftrightarrow$ *Bseq* $(\lambda n.\ \sum k{=}1..<n.\ f\ k)$ **unfolding** *f'-def* **by** *simp*
also have $\ldots \longleftrightarrow$ *Bseq* $(\lambda n.\ \sum k{=}1..<2\hat{}n.\ f\ k)$

by (*rule nonneg-incseq-Bseq-subseq-iff*[*symmetric*])
(*auto intro*!: *setsum-nonneg incseq-SucI nonneg simp*: *subseq-def*)
also have ... ⟷ *Bseq* (λ*n*. ∑ *k*<*n*. *2*^*k* * *f* (*2*^*k*))
proof (*intro iffI*)
assume *A*: *Bseq* (λ*n*. ∑ *k*=*1*..<*2*^*n*. *f k*)
have *eventually* (λ*n*. *norm* (∑ *k*<*n*. *2*^*k* * *f* (*2*^*Suc k*)) ≤ *norm* (∑ *k*=*1*..<*2*^*n*. *f k*)) *sequentially*
proof (*intro always-eventually allI*)
fix *n* :: *nat*
have *norm* (∑ *k*<*n*. *2*^*k* * *f* (*2*^*Suc k*)) = (∑ *k*<*n*. *2*^*k* * *f* (*2*^*Suc k*)) **unfolding** *real-norm-def*
by (*intro abs-of-nonneg setsum-nonneg ballI mult-nonneg-nonneg nonneg*) *simp-all*
also have ... ≤ (∑ *k*=*1*..<*2*^*n*. *f k*)
by (*subst condensation-condense2* [*symmetric*]) (*intro condensation-inequality mono*′)
also have ... = *norm* ... **unfolding** *real-norm-def*
by (*intro abs-of-nonneg*[*symmetric*] *setsum-nonneg ballI mult-nonneg-nonneg nonneg*)
finally show *norm* (∑ *k*<*n*. *2* ^ *k* * *f* (*2* ^ *Suc k*)) ≤ *norm* (∑ *k*=*1*..<*2*^*n*. *f k*) **.**
qed
from *this* **and** *A* **have** *Bseq* (λ*n*. ∑ *k*<*n*. *2*^*k* * *f* (*2*^*Suc k*)) **by** (*rule Bseq-eventually-mono*)
from *Bseq-mult*[*OF Bfun-const*[*of 2*] *this*] **have** *Bseq* (λ*n*. ∑ *k*<*n*. *2*^*Suc k* * *f* (*2*^*Suc k*))
by (*simp add*: *setsum-right-distrib setsum-left-distrib mult-ac*)
hence *Bseq* (λ*n*. (∑ *k*=*Suc 0*..<*Suc n*. *2*^*k* * *f* (*2*^*k*)) + *f 1*)
by (*intro Bseq-add*, *subst setsum-shift-bounds-Suc-ivl*) (*simp add*: *atLeast0LessThan*)
hence *Bseq* (λ*n*. (∑ *k*=*0*..<*Suc n*. *2*^*k* * *f* (*2*^*k*)))
by (*subst setsum-head-upt-Suc*) (*simp-all add*: *add-ac*)
thus *Bseq* (λ*n*. (∑ *k*<*n*. *2*^*k* * *f* (*2*^*k*)))
by (*subst* (*asm*) *Bseq-Suc-iff*) (*simp add*: *atLeast0LessThan*)
next
assume *A*: *Bseq* (λ*n*. (∑ *k*<*n*. *2*^*k* * *f* (*2*^*k*)))
have *eventually* (λ*n*. *norm* (∑ *k*=*1*..<*2*^*n*. *f k*) ≤ *norm* (∑ *k*<*n*. *2*^*k* * *f* (*2*^*k*))) *sequentially*
proof (*intro always-eventually allI*)
fix *n* :: *nat*
have *norm* (∑ *k*=*1*..<*2*^*n*. *f k*) = (∑ *k*=*1*..<*2*^*n*. *f k*) **unfolding** *real-norm-def*
by (*intro abs-of-nonneg setsum-nonneg ballI mult-nonneg-nonneg nonneg*)
also have ... ≤ (∑ *k*<*n*. *2*^*k* * *f* (*2*^*k*))
by (*subst condensation-condense1* [*symmetric*]) (*intro condensation-inequality mono*′)
also have ... = *norm* ... **unfolding** *real-norm-def*
by (*intro abs-of-nonneg* [*symmetric*] *setsum-nonneg ballI mult-nonneg-nonneg nonneg*) *simp-all*
finally show *norm* (∑ *k*=*1*..<*2*^*n*. *f k*) ≤ *norm* (∑ *k*<*n*. *2*^*k* * *f* (*2*^*k*)) **.**
qed

from *this* **and** *A* **show** *Bseq* (λn. ∑ k=1..<2^n. f k) **by** (*rule Bseq-eventually-mono*)
qed
also have *monoseq* (λn. (∑ k<n. 2^k * f (2^k)))
by (*intro mono-SucI1*) (*auto intro*!: *mult-nonneg-nonneg nonneg*)
hence *Bseq* (λn. (∑ k<n. 2^k * f (2^k))) ⟷ *convergent* (λn. (∑ k<n. 2^k * f (2^k)))
by (*rule monoseq-imp-convergent-iff-Bseq* [*symmetric*])
also have ... ⟷ *summable* (λk. 2^k * f (2^k)) **by** (*simp only*: *summable-iff-convergent*)
finally show *?thesis* **.**
qed

end

37.2.3 Summability of powers

lemma *abs-summable-complex-powr-iff*:
summable (λn. *norm* (*exp* (*of-real* (*ln* (*of-nat n*)) * *s*))) ⟷ *Re s* < −*1*
proof (*cases Re s* ≤ *0*)
let *?l* = λn. *complex-of-real* (*ln* (*of-nat n*))
case *False*
with *eventually-gt-at-top*[*of 0*::*nat*]
have *eventually* (λn. *norm* (*1* :: *real*) ≤ *norm* (*exp* (*?l n* * *s*))) *sequentially*
by (*auto intro*!: *ge-one-powr-ge-zero elim*!: *eventually-mono*)
from *summable-comparison-test-ev*[*OF this*] *False* **show** *?thesis* **by** (*auto simp*: *summable-const-iff*)
next
let *?l* = λn. *complex-of-real* (*ln* (*of-nat n*))
case *True*
hence *summable* (λn. *norm* (*exp* (*?l n* * *s*))) ⟷ *summable* (λn. *2^n* * *norm* (*exp* (*?l* (*2^n*) * *s*)))
by (*intro condensation-test*) (*auto intro*!: *mult-right-mono-neg*)
also have (λn. *2^n* * *norm* (*exp* (*?l* (*2^n*) * *s*))) = (λn. (*2 powr* (*Re s* + *1*)) ^ *n*)
proof
fix *n* :: *nat*
have *2^n* * *norm* (*exp* (*?l* (*2^n*) * *s*)) = *exp* (*real n* * *ln 2*) * *exp* (*real n* * *ln 2* * *Re s*)
using *True* **by** (*subst exp-of-nat-mult*) (*simp add*: *ln-realpow algebra-simps*)
also have ... = *exp* (*real n* * (*ln 2* * (*Re s* + *1*)))
by (*simp add*: *algebra-simps exp-add*)
also have ... = *exp* (*ln 2* * (*Re s* + *1*)) ^ *n* **by** (*subst exp-of-nat-mult*) *simp*
also have *exp* (*ln 2* * (*Re s* + *1*)) = *2 powr* (*Re s* + *1*) **by** (*simp add*: *powr-def*)
finally show *2^n* * *norm* (*exp* (*?l* (*2^n*) * *s*)) = (*2 powr* (*Re s* + *1*)) ^ *n* **.**
qed
also have *summable* ... ⟷ *2 powr* (*Re s* + *1*) < *2 powr 0*
by (*subst summable-geometric-iff*) *simp*
also have ... ⟷ *Re s* < −*1* **by** (*subst powr-less-cancel-iff*) (*simp*, *linarith*)
finally show *?thesis* **.**

qed

lemma *summable-complex-powr-iff*:
assumes *Re s* < −*1*
shows *summable* (λ*n*. *exp* (*of-real* (*ln* (*of-nat n*)) ∗ *s*))
by (*rule summable-norm-cancel*, *subst abs-summable-complex-powr-iff*) *fact*

lemma *summable-real-powr-iff*: *summable* (λ*n*. *of-nat n powr s* :: *real*) ⟷ *s* < −*1*
proof −
from *eventually-gt-at-top*[*of 0*::*nat*]
have *summable* (λ*n*. *of-nat n powr s*) ⟷ *summable* (λ*n*. *exp* (*ln* (*of-nat n*) ∗ *s*))
by (*intro summable-cong*) (*auto elim*!: *eventually-mono simp*: *powr-def*)
also have ... ⟷ *s* < −*1* **using** *abs-summable-complex-powr-iff*[*of of-real s*]
by *simp*
finally show *?thesis* **.**
qed

lemma *inverse-power-summable*:
assumes *s*: *s* ≥ *2*
shows *summable* (λ*n*. *inverse* (*of-nat n* ^ *s* :: *'a* :: {*real-normed-div-algebra*,*banach*}))
proof (*rule summable-norm-cancel*, *subst summable-cong*)
from *eventually-gt-at-top*[*of 0*::*nat*]
show *eventually* (λ*n*. *norm* (*inverse* (*of-nat n* ^ *s*:: *'a*)) = *real-of-nat n powr* (−*real s*)) *at-top*
by *eventually-elim* (*simp add*: *norm-inverse norm-power powr-minus powr-realpow*)
qed (*insert s summable-real-powr-iff*[*of* −*s*], *simp-all*)

lemma *not-summable-harmonic*: ¬*summable* (λ*n*. *inverse* (*of-nat n*) :: *'a* :: *real-normed-field*)
proof
assume *summable* (λ*n*. *inverse* (*of-nat n*) :: *'a*)
hence *convergent* (λ*n*. *norm* (*of-real* (∑ *k*<*n*. *inverse* (*of-nat k*)) :: *'a*))
by (*simp add*: *summable-iff-convergent convergent-norm*)
hence *convergent* (λ*n*. *abs* (∑ *k*<*n*. *inverse* (*of-nat k*)) :: *real*) **by** (*simp only*: *norm-of-real*)
also have (λ*n*. *abs* (∑ *k*<*n*. *inverse* (*of-nat k*)) :: *real*) = (λ*n*. ∑ *k*<*n*. *inverse* (*of-nat k*))
by (*intro ext abs-of-nonneg setsum-nonneg*) *auto*
also have *convergent* ... ⟷ *summable* (λ*k*. *inverse* (*of-nat k*) :: *real*)
by (*simp add*: *summable-iff-convergent*)
finally show *False* **using** *summable-real-powr-iff*[*of* −*1*] **by** (*simp add*: *powr-minus*)
qed

37.2.4 Kummer's test

lemma *kummers-test-convergence*:
fixes *f p* :: *nat* ⇒ *real*
assumes *pos-f*: *eventually* (λ*n*. *f n* > *0*) *sequentially*

assumes *nonneg-p*: *eventually* $(\lambda n.\ p\ n \geq 0)$ *sequentially*
defines $l \equiv$ *liminf* $(\lambda n.$ *ereal* $(p\ n * f\ n\ /\ f\ (Suc\ n) - p\ (Suc\ n)))$
assumes *l*: $l > 0$
shows *summable f*
unfolding *summable-iff-convergent'*
proof −
def $r \equiv$ (*if* $l = \infty$ *then 1 else real-of-ereal l / 2*)
from *l* **have** $r > 0 \wedge$ *of-real* $r < l$ **by** (*cases l*) (*simp-all add*: *r-def*)
hence *r*: $r > 0$ *of-real* $r < l$ **by** *simp-all*
hence *eventually* $(\lambda n.\ p\ n * f\ n\ /\ f\ (Suc\ n) - p\ (Suc\ n) > r)$ *sequentially*
unfolding *l-def* **by** (*force dest*: *less-LiminfD*)
moreover from *pos-f* **have** *eventually* $(\lambda n.\ f\ (Suc\ n) > 0)$ *sequentially*
by (*subst eventually-sequentially-Suc*)
ultimately have *eventually* $(\lambda n.\ p\ n * f\ n - p\ (Suc\ n) * f\ (Suc\ n) > r * f\ (Suc\ n))$ *sequentially*
by *eventually-elim* (*simp add*: *field-simps*)
from *eventually-conj*[*OF pos-f eventually-conj*[*OF nonneg-p this*]]
obtain *m* **where** *m*: $\bigwedge n.\ n \geq m \Longrightarrow f\ n > 0\ \bigwedge n.\ n \geq m \Longrightarrow p\ n \geq 0$
$\bigwedge n.\ n \geq m \Longrightarrow p\ n * f\ n - p\ (Suc\ n) * f\ (Suc\ n) > r * f\ (Suc\ n)$
unfolding *eventually-at-top-linorder* **by** *blast*

let *?c* = $(norm\ (\sum k{\leq}m.\ r * f\ k) + p\ m * f\ m)\ /\ r$
have *Bseq* $(\lambda n.\ (\sum k{\leq}n + Suc\ m.\ f\ k))$
proof (*rule BseqI'*)
fix *k* :: *nat*
def $n \equiv k + Suc\ m$
have *n*: $n > m$ **by** (*simp add*: *n-def*)

from *r* **have** $r * norm\ (\sum k{\leq}n.\ f\ k) = norm\ (\sum k{\leq}n.\ r * f\ k)$
by (*simp add*: *setsum-right-distrib*[*symmetric*] *abs-mult*)
also from *n* **have** $\{..n\} = \{..m\} \cup \{Suc\ m..n\}$ **by** *auto*
hence $(\sum k{\leq}n.\ r * f\ k) = (\sum k{\in}\{..m\} \cup \{Suc\ m..n\}.\ r * f\ k)$ **by** (*simp only*:)
also have $\ldots = (\sum k{\leq}m.\ r * f\ k) + (\sum k{=}Suc\ m..n.\ r * f\ k)$
by (*subst setsum.union-disjoint*) *auto*
also have $norm\ \ldots \leq norm\ (\sum k{\leq}m.\ r * f\ k) + norm\ (\sum k{=}Suc\ m..n.\ r * f\ k)$
by (*rule norm-triangle-ineq*)
also from *r less-imp-le*[*OF m*(*1*)] **have** $(\sum k{=}Suc\ m..n.\ r * f\ k) \geq 0$
by (*intro setsum-nonneg*) *auto*
hence $norm\ (\sum k{=}Suc\ m..n.\ r * f\ k) = (\sum k{=}Suc\ m..n.\ r * f\ k)$ **by** *simp*
also have $(\sum k{=}Suc\ m..n.\ r * f\ k) = (\sum k{=}m..{<}n.\ r * f\ (Suc\ k))$
by (*subst setsum-shift-bounds-Suc-ivl* [*symmetric*])
(*simp only*: *atLeastLessThanSuc-atLeastAtMost*)
also from *m* **have** $\ldots \leq (\sum k{=}m..{<}n.\ p\ k * f\ k - p\ (Suc\ k) * f\ (Suc\ k))$
by (*intro setsum-mono*[*OF less-imp-le*]) *simp-all*
also have $\ldots = -(\sum k{=}m..{<}n.\ p\ (Suc\ k) * f\ (Suc\ k) - p\ k * f\ k)$
by (*simp add*: *setsum-negf* [*symmetric*] *algebra-simps*)
also from *n* **have** $\ldots = p\ m * f\ m - p\ n * f\ n$
by (*cases n*, *simp*, *simp only*: *atLeastLessThanSuc-atLeastAtMost*, *subst*

setsum-Suc-diff) *simp-all*
also from *less-imp-le*[*OF m*(*1*)] *m*(*2*) *n* **have** ... ≤ *p m* ∗ *f m* **by** *simp*
finally show *norm* (∑ *k*≤*n*. *f k*) ≤ (*norm* (∑ *k*≤*m*. *r* ∗ *f k*) + *p m* ∗ *f m*) / *r* **using** *r*
by (*subst pos-le-divide-eq*[*OF r*(*1*)]) (*simp only*: *mult-ac*)
qed
moreover have (∑ *k*≤*n*. *f k*) ≤ (∑ *k*≤*n*′. *f k*) **if** *Suc m* ≤ *n n* ≤ *n*′ **for** *n n*′
using *less-imp-le*[*OF m*(*1*)] *that* **by** (*intro setsum-mono2*) *auto*
ultimately show *convergent* (λ*n*. ∑ *k*≤*n*. *f k*) **by** (*rule Bseq-monoseq-convergent*′*-inc*)
qed

lemma *kummers-test-divergence*:
fixes *f p* :: *nat* ⇒ *real*
assumes *pos-f*: *eventually* (λ*n*. *f n* > *0*) *sequentially*
assumes *pos-p*: *eventually* (λ*n*. *p n* > *0*) *sequentially*
assumes *divergent-p*: ¬*summable* (λ*n*. *inverse* (*p n*))
defines *l* ≡ *limsup* (λ*n*. *ereal* (*p n* ∗ *f n* / *f* (*Suc n*) − *p* (*Suc n*)))
assumes *l*: *l* < *0*
shows ¬*summable f*
proof
assume *summable f*
from *eventually-conj*[*OF pos-f eventually-conj*[*OF pos-p Limsup-lessD*[*OF l*[*unfolded l-def*]]]]
obtain *N* **where** *N*: ⋀*n*. *n* ≥ *N* ⟹ *p n* > *0* ⋀*n*. *n* ≥ *N* ⟹ *f n* > *0*
⋀*n*. *n* ≥ *N* ⟹ *p n* ∗ *f n* / *f* (*Suc n*) − *p* (*Suc n*) < *0*
by (*auto simp*: *eventually-at-top-linorder*)
hence *A*: *p n* ∗ *f n* < *p* (*Suc n*) ∗ *f* (*Suc n*) **if** *n* ≥ *N* **for** *n* **using** *that N*[*of n*] *N*[*of Suc n*]
by (*simp add*: *field-simps*)
have *p n* ∗ *f n* ≥ *p N* ∗ *f N* **if** *n* ≥ *N* **for** *n* **using** *that* **and** *A*
by (*induction n rule*: *dec-induct*) (*auto intro*!: *less-imp-le elim*!: *order.trans*)
from *eventually-ge-at-top*[*of N*] *N this*
have *eventually* (λ*n*. *norm* (*p N* ∗ *f N* ∗ *inverse* (*p n*)) ≤ *f n*) *sequentially*
by (*auto elim*!: *eventually-mono simp*: *field-simps abs-of-pos*)
from *this* **and** ‹*summable f*› **have** *summable* (λ*n*. *p N* ∗ *f N* ∗ *inverse* (*p n*))
by (*rule summable-comparison-test-ev*)
from *summable-mult*[*OF this*, *of inverse* (*p N* ∗ *f N*)] *N*[*OF le-refl*]
have *summable* (λ*n*. *inverse* (*p n*)) **by** (*simp add*: *divide-simps*)
with *divergent-p* **show** *False* **by** *contradiction*
qed

37.2.5 Ratio test

lemma *ratio-test-convergence*:
fixes *f* :: *nat* ⇒ *real*
assumes *pos-f*: *eventually* (λ*n*. *f n* > *0*) *sequentially*
defines *l* ≡ *liminf* (λ*n*. *ereal* (*f n* / *f* (*Suc n*)))
assumes *l*: *l* > *1*

shows *summable f*
proof (*rule kummers-test-convergence*[*OF pos-f*])
note *l*
also have *l* = *liminf* (λ*n*. *ereal* (*f n* / *f* (*Suc n*) − *1*)) + *1*
by (*subst Liminf-add-ereal-right*[*symmetric*]) (*simp-all add*: *minus-ereal-def l-def one-ereal-def*)
finally show *liminf* (λ*n*. *ereal* (*1* ∗ *f n* / *f* (*Suc n*) − *1*)) > *0*
by (*cases liminf* (λ*n*. *ereal* (*1* ∗ *f n* / *f* (*Suc n*) − *1*))) *simp-all*
qed *simp*

lemma *ratio-test-divergence*:
fixes *f* :: *nat* ⇒ *real*
assumes *pos-f*: *eventually* (λ*n*. *f n* > *0*) *sequentially*
defines *l* ≡ *limsup* (λ*n*. *ereal* (*f n* / *f* (*Suc n*)))
assumes *l*: *l* < *1*
shows ¬*summable f*
proof (*rule kummers-test-divergence*[*OF pos-f*])
have *limsup* (λ*n*. *ereal* (*f n* / *f* (*Suc n*) − *1*)) + *1* = *l*
by (*subst Limsup-add-ereal-right*[*symmetric*]) (*simp-all add*: *minus-ereal-def l-def one-ereal-def*)
also note *l*
finally show *limsup* (λ*n*. *ereal* (*1* ∗ *f n* / *f* (*Suc n*) − *1*)) < *0*
by (*cases limsup* (λ*n*. *ereal* (*1* ∗ *f n* / *f* (*Suc n*) − *1*))) *simp-all*
qed (*simp-all add*: *summable-const-iff*)

37.2.6 Raabe's test

lemma *raabes-test-convergence*:
fixes *f* :: *nat* ⇒ *real*
assumes *pos*: *eventually* (λ*n*. *f n* > *0*) *sequentially*
defines *l* ≡ *liminf* (λ*n*. *ereal* (*of-nat n* ∗ (*f n* / *f* (*Suc n*) − *1*)))
assumes *l*: *l* > *1*
shows *summable f*
proof (*rule kummers-test-convergence*)
let *?l′* = *liminf* (λ*n*. *ereal* (*of-nat n* ∗ *f n* / *f* (*Suc n*) − *of-nat* (*Suc n*)))
have *1* < *l* **by** *fact*
also have *l* = *liminf* (λ*n*. *ereal* (*of-nat n* ∗ *f n* / *f* (*Suc n*) − *of-nat* (*Suc n*)) + *1*)
by (*simp add*: *l-def algebra-simps*)
also have ... = *?l′* + *1* **by** (*subst Liminf-add-ereal-right*) *simp-all*
finally show *?l′* > *0* **by** (*cases ?l′*) (*simp-all add*: *algebra-simps*)
qed (*simp-all add*: *pos*)

lemma *raabes-test-divergence*:
fixes *f* :: *nat* ⇒ *real*
assumes *pos*: *eventually* (λ*n*. *f n* > *0*) *sequentially*
defines *l* ≡ *limsup* (λ*n*. *ereal* (*of-nat n* ∗ (*f n* / *f* (*Suc n*) − *1*)))
assumes *l*: *l* < *1*
shows ¬*summable f*

proof (*rule kummers-test-divergence*)
let *?l'* = *limsup* (λ*n*. *ereal* (*of-nat n* ∗ *f n* / *f* (*Suc n*) − *of-nat* (*Suc n*)))
note *l*
also have *l* = *limsup* (λ*n*. *ereal* (*of-nat n* ∗ *f n* / *f* (*Suc n*) − *of-nat* (*Suc n*)) + *1*)
by (*simp add*: *l-def algebra-simps*)
also have ... = *?l'* + *1* **by** (*subst Limsup-add-ereal-right*) *simp-all*
finally show *?l'* < *0* **by** (*cases ?l'*) (*simp-all add*: *algebra-simps*)
qed (*insert pos eventually-gt-at-top*[*of 0*::*nat*] *not-summable-harmonic*, *simp-all*)

37.3 Radius of convergence

The radius of convergence of a power series. This value always exists, ranges from *0* to ∞, and the power series is guaranteed to converge for all inputs with a norm that is smaller than that radius and to diverge for all inputs with a norm that is greater.

definition *conv-radius* :: (*nat* ⇒ *'a* :: *banach*) ⇒ *ereal* **where**
conv-radius f = *inverse* (*limsup* (λ*n*. *ereal* (*root n* (*norm* (*f n*)))))

lemma *conv-radius-nonneg*: *conv-radius f* ≥ *0*
proof −
have *0* = *limsup* (λ*n*. *0*) **by** (*subst Limsup-const*) *simp-all*
also have ... ≤ *limsup* (λ*n*. *ereal* (*root n* (*norm* (*f n*))))
by (*intro Limsup-mono*) (*simp-all add*: *real-root-ge-zero*)
finally show *?thesis*
unfolding *conv-radius-def* **by** (*auto simp*: *ereal-inverse-nonneg-iff*)
qed

lemma *conv-radius-zero* [*simp*]: *conv-radius* (λ-. *0*) = ∞
by (*auto simp*: *conv-radius-def zero-ereal-def* [*symmetric*] *Limsup-const*)

lemma *conv-radius-cong*:
assumes *eventually* (λ*x*. *f x* = *g x*) *sequentially*
shows *conv-radius f* = *conv-radius g*
proof −
have *eventually* (λ*n*. *ereal* (*root n* (*norm* (*f n*))) = *ereal* (*root n* (*norm* (*g n*)))) *sequentially*
using *assms* **by** *eventually-elim simp*
from *Limsup-eq*[*OF this*] **show** *?thesis* **unfolding** *conv-radius-def* **by** *simp*
qed

lemma *conv-radius-altdef*:
conv-radius f = *liminf* (λ*n*. *inverse* (*ereal* (*root n* (*norm* (*f n*)))))
by (*subst Liminf-inverse-ereal*) (*simp-all add*: *real-root-ge-zero conv-radius-def*)

lemma *abs-summable-in-conv-radius*:
fixes *f* :: *nat* ⇒ *'a* :: {*banach*, *real-normed-div-algebra*}
assumes *ereal* (*norm z*) < *conv-radius f*

shows *summable* (λ*n*. *norm* (*f n* ∗ *z* ^ *n*))
proof (*rule root-test-convergence'*)
def *l* ≡ *limsup* (λ*n*. *ereal* (*root n* (*norm* (*f n*))))
have *0* = *limsup* (λ*n*. *0*) **by** (*simp add*: *Limsup-const*)
also have ... ≤ *l* **unfolding** *l-def* **by** (*intro Limsup-mono*) (*simp-all add*: *real-root-ge-zero*)
finally have *l-nonneg*: *l* ≥ *0* **.**

have *limsup* (λ*n*. *root n* (*norm* (*f n* ∗ *z*^*n*))) = *l* ∗ *ereal* (*norm z*) **unfolding** *l-def*
by (*rule limsup-root-powser*)
also from *l-nonneg* **consider** *l* = *0* | *l* = ∞ | ∃*l'*. *l* = *ereal l'* ∧ *l'* > *0*
by (*cases l*) (*auto simp*: *less-le*)
hence *l* ∗ *ereal* (*norm z*) < *1*
proof *cases*
assume *l* = ∞
hence *conv-radius f* = *0* **unfolding** *conv-radius-def l-def* **by** *simp*
with *assms* **show** *?thesis* **by** *simp*
next
assume ∃*l'*. *l* = *ereal l'* ∧ *l'* > *0*
then guess *l'* **by** (*elim exE conjE*) **note** *l'* = *this*
hence *l* ≠ ∞ **by** *auto*
have *l* ∗ *ereal* (*norm z*) < *l* ∗ *conv-radius f*
by (*intro ereal-mult-strict-left-mono*) (*simp-all add*: *l' assms*)
also have *conv-radius f* = *inverse l* **by** (*simp add*: *conv-radius-def l-def*)
also from *l'* **have** *l* ∗ *inverse l* = *1* **by** *simp*
finally show *?thesis* **.**
qed *simp-all*
finally show *limsup* (λ*n*. *ereal* (*root n* (*norm* (*norm* (*f n* ∗ *z* ^ *n*))))) < *1* **by** *simp*
qed

lemma *summable-in-conv-radius*:
fixes *f* :: *nat* ⇒ *'a* :: {*banach*, *real-normed-div-algebra*}
assumes *ereal* (*norm z*) < *conv-radius f*
shows *summable* (λ*n*. *f n* ∗ *z* ^ *n*)
by (*rule summable-norm-cancel*, *rule abs-summable-in-conv-radius*) *fact+*

lemma *not-summable-outside-conv-radius*:
fixes *f* :: *nat* ⇒ *'a* :: {*banach*, *real-normed-div-algebra*}
assumes *ereal* (*norm z*) > *conv-radius f*
shows ¬*summable* (λ*n*. *f n* ∗ *z* ^ *n*)
proof (*rule root-test-divergence*)
def *l* ≡ *limsup* (λ*n*. *ereal* (*root n* (*norm* (*f n*))))
have *0* = *limsup* (λ*n*. *0*) **by** (*simp add*: *Limsup-const*)
also have ... ≤ *l* **unfolding** *l-def* **by** (*intro Limsup-mono*) (*simp-all add*: *real-root-ge-zero*)
finally have *l-nonneg*: *l* ≥ *0* **.**
from *assms* **have** *l-nz*: *l* ≠ *0* **unfolding** *conv-radius-def l-def* **by** *auto*

have *limsup* (λ*n*. *ereal* (*root n* (*norm* (*f n* ∗ *z*^*n*)))) = *l* ∗ *ereal* (*norm z*)
unfolding *l-def* **by** (*rule limsup-root-powser*)
also have ... > *1*
proof (*cases l*)
assume *l* = ∞
with *assms conv-radius-nonneg*[*of f*] **show** *?thesis*
by (*auto simp*: *zero-ereal-def*[*symmetric*])
next
fix *l*′ **assume** *l*′: *l* = *ereal l*′
from *l-nonneg l-nz* **have** *1* = *l* ∗ *inverse l* **by** (*auto simp*: *l*′ *field-simps*)
also from *l-nz* **have** *inverse l* = *conv-radius f*
unfolding *l-def conv-radius-def* **by** *auto*
also from *l*′ *l-nz l-nonneg assms* **have** *l* ∗ ... < *l* ∗ *ereal* (*norm z*)
by (*intro ereal-mult-strict-left-mono*) (*auto simp*: *l*′)
finally show *?thesis* **.**
qed (*insert l-nonneg*, *simp-all*)
finally show *limsup* (λ*n*. *ereal* (*root n* (*norm* (*f n* ∗ *z*^*n*)))) > *1* **.**
qed

lemma *conv-radius-geI*:
assumes *summable* (λ*n*. *f n* ∗ *z* ^ *n* :: *'a* :: {*banach*, *real-normed-div-algebra*})
shows *conv-radius f* ≥ *norm z*
using *not-summable-outside-conv-radius*[*of f z*] *assms* **by** (*force simp*: *not-le*[*symmetric*])

lemma *conv-radius-leI*:
assumes ¬*summable* (λ*n*. *norm* (*f n* ∗ *z* ^ *n* :: *'a* :: {*banach*, *real-normed-div-algebra*}))
shows *conv-radius f* ≤ *norm z*
using *abs-summable-in-conv-radius*[*of z f*] *assms* **by** (*force simp*: *not-le*[*symmetric*])

lemma *conv-radius-leI'*:
assumes ¬*summable* (λ*n*. *f n* ∗ *z* ^ *n* :: *'a* :: {*banach*, *real-normed-div-algebra*})
shows *conv-radius f* ≤ *norm z*
using *summable-in-conv-radius*[*of z f*] *assms* **by** (*force simp*: *not-le*[*symmetric*])

lemma *conv-radius-geI-ex*:
fixes *f* :: *nat* ⇒ *'a* :: {*banach*, *real-normed-div-algebra*}
assumes ⋀*r*. *0* < *r* ⟹ *ereal r* < *R* ⟹ ∃ *z*. *norm z* = *r* ∧ *summable* (λ*n*. *f n* ∗ *z*^*n*)
shows *conv-radius f* ≥ *R*
proof (*rule linorder-cases*[*of conv-radius f R*])
assume *R*: *conv-radius f* < *R*
with *conv-radius-nonneg*[*of f*] **obtain** *conv-radius'*
where [*simp*]: *conv-radius f* = *ereal conv-radius'*
by (*cases conv-radius f*) *simp-all*
def *r* ≡ *if R* = ∞ *then conv-radius'* + *1 else* (*real-of-ereal R* + *conv-radius'*) / *2*
from *R conv-radius-nonneg*[*of f*] **have** *0* < *r* ∧ *ereal r* < *R* ∧ *ereal r* > *conv-radius f*

unfolding *r-def* **by** (*cases R*) (*auto simp*: *r-def field-simps*)
with *assms*(*1*)[*of r*] **obtain** *z* **where** *norm z* > *conv-radius f summable* (λ*n*. *f n* ∗ *z*^*n*) **by** *auto*
with *not-summable-outside-conv-radius*[*of f z*] **show** *?thesis* **by** *simp*
qed *simp-all*

lemma *conv-radius-geI-ex'*:
fixes *f* :: *nat* ⇒ *'a* :: {*banach*, *real-normed-div-algebra*}
assumes ⋀*r*. *0* < *r* ⟹ *ereal r* < *R* ⟹ *summable* (λ*n*. *f n* ∗ *of-real r*^*n*)
shows *conv-radius f* ≥ *R*
proof (*rule conv-radius-geI-ex*)
fix *r* **assume** *0* < *r ereal r* < *R*
with *assms*[*of r*] **show** ∃ *z*. *norm z* = *r* ∧ *summable* (λ*n*. *f n* ∗ *z* ^ *n*)
by (*intro exI*[*of - of-real r* :: *'a*]) *auto*
qed

lemma *conv-radius-leI-ex*:
fixes *f* :: *nat* ⇒ *'a* :: {*banach*, *real-normed-div-algebra*}
assumes *R* ≥ *0*
assumes ⋀*r*. *0* < *r* ⟹ *ereal r* > *R* ⟹ ∃ *z*. *norm z* = *r* ∧ ¬*summable* (λ*n*. *norm* (*f n* ∗ *z*^*n*))
shows *conv-radius f* ≤ *R*
proof (*rule linorder-cases*[*of conv-radius f R*])
assume *R*: *conv-radius f* > *R*
from *R assms*(*1*) **obtain** *R'* **where** *R'*: *R* = *ereal R'* **by** (*cases R*) *simp-all*
def *r* ≡ *if conv-radius f* = ∞ *then R'* + *1 else* (*R'* + *real-of-ereal* (*conv-radius f*)) / *2*
from *R conv-radius-nonneg*[*of f*] **have** *r* > *R* ∧ *r* < *conv-radius f* **unfolding** *r-def*
by (*cases conv-radius f*) (*auto simp*: *r-def field-simps R'*)
with *assms*(*1*) *assms*(*2*)[*of r*] *R'*
obtain *z* **where** *norm z* < *conv-radius f* ¬*summable* (λ*n*. *norm* (*f n* ∗ *z*^*n*)) **by** *auto*
with *abs-summable-in-conv-radius*[*of z f*] **show** *?thesis* **by** *auto*
qed *simp-all*

lemma *conv-radius-leI-ex'*:
fixes *f* :: *nat* ⇒ *'a* :: {*banach*, *real-normed-div-algebra*}
assumes *R* ≥ *0*
assumes ⋀*r*. *0* < *r* ⟹ *ereal r* > *R* ⟹ ¬*summable* (λ*n*. *f n* ∗ *of-real r*^*n*)
shows *conv-radius f* ≤ *R*
proof (*rule conv-radius-leI-ex*)
fix *r* **assume** *0* < *r ereal r* > *R*
with *assms*(*2*)[*of r*] **show** ∃ *z*. *norm z* = *r* ∧ ¬*summable* (λ*n*. *norm* (*f n* ∗ *z* ^ *n*))
by (*intro exI*[*of - of-real r* :: *'a*]) (*auto dest*: *summable-norm-cancel*)
qed *fact+*

lemma *conv-radius-eqI*:

fixes $f :: nat \Rightarrow 'a :: \{banach, real\text{-}normed\text{-}div\text{-}algebra\}$
assumes $R \geq 0$
assumes $\bigwedge r.\ 0 < r \Longrightarrow ereal\ r < R \Longrightarrow \exists z.\ norm\ z = r \wedge summable\ (\lambda n.\ f\ n * z\hat{}n)$
assumes $\bigwedge r.\ 0 < r \Longrightarrow ereal\ r > R \Longrightarrow \exists z.\ norm\ z = r \wedge \neg summable\ (\lambda n.\ norm\ (f\ n * z\hat{}n))$
shows *conv-radius* $f = R$
by (*intro antisym conv-radius-geI-ex conv-radius-leI-ex assms*)

lemma *conv-radius-eqI'*:
fixes $f :: nat \Rightarrow 'a :: \{banach, real\text{-}normed\text{-}div\text{-}algebra\}$
assumes $R \geq 0$
assumes $\bigwedge r.\ 0 < r \Longrightarrow ereal\ r < R \Longrightarrow summable\ (\lambda n.\ f\ n * (of\text{-}real\ r)\hat{}n)$
assumes $\bigwedge r.\ 0 < r \Longrightarrow ereal\ r > R \Longrightarrow \neg summable\ (\lambda n.\ norm\ (f\ n * (of\text{-}real\ r)\hat{}n))$
shows *conv-radius* $f = R$
proof (*intro conv-radius-eqI*[*OF assms*(*1*)])
fix r **assume** $0 < r\ ereal\ r < R$ **with** *assms*(*2*)[*OF this*]
show $\exists z.\ norm\ z = r \wedge summable\ (\lambda n.\ f\ n * z\ \hat{}\ n)$ **by** *force*
next
fix r **assume** $0 < r\ ereal\ r > R$ **with** *assms*(*3*)[*OF this*]
show $\exists z.\ norm\ z = r \wedge \neg summable\ (\lambda n.\ norm\ (f\ n * z\ \hat{}\ n))$ **by** *force*
qed

lemma *conv-radius-zeroI*:
fixes $f :: nat \Rightarrow 'a :: \{banach,real\text{-}normed\text{-}div\text{-}algebra\}$
assumes $\bigwedge z.\ z \neq 0 \Longrightarrow \neg summable\ (\lambda n.\ f\ n * z\hat{}n)$
shows *conv-radius* $f = 0$
proof (*rule ccontr*)
assume *conv-radius* $f \neq 0$
with *conv-radius-nonneg*[*of f*] **have** *pos*: *conv-radius* $f > 0$ **by** *simp*
def $r \equiv$ *if conv-radius* $f = \infty$ *then 1 else real-of-ereal* (*conv-radius f*) / *2*
from *pos* **have** *r*: $ereal\ r > 0 \wedge ereal\ r <$ *conv-radius f*
by (*cases conv-radius f*) (*simp-all add*: *r-def*)
hence *summable* $(\lambda n.\ f\ n * of\text{-}real\ r\ \hat{}\ n)$ **by** (*intro summable-in-conv-radius*) *simp*
moreover from *r* **and** *assms*[*of of-real r*] **have** $\neg summable\ (\lambda n.\ f\ n * of\text{-}real\ r\ \hat{}\ n)$ **by** *simp*
ultimately show *False* **by** *contradiction*
qed

lemma *conv-radius-inftyI'*:
fixes $f :: nat \Rightarrow 'a :: \{banach,real\text{-}normed\text{-}div\text{-}algebra\}$
assumes $\bigwedge r.\ r > c \Longrightarrow \exists z.\ norm\ z = r \wedge summable\ (\lambda n.\ f\ n * z\hat{}n)$
shows *conv-radius* $f = \infty$
proof −
{
fix $r :: real$
have $max\ r\ (c + 1) > c$ **by** (*auto simp*: *max-def*)

```
    from assms[OF this] obtain z where norm z = max r (c + 1) summable
(λn. f n * z^n) by blast
    from conv-radius-geI[OF this(2)] this(1) have conv-radius f ≥ r by simp
  }
  from this[of real-of-ereal (conv-radius f + 1)] show conv-radius f = ∞
    by (cases conv-radius f) simp-all
qed

lemma conv-radius-inftyI:
  fixes f :: nat ⇒ 'a :: {banach,real-normed-div-algebra}
  assumes ⋀r. ∃z. norm z = r ∧ summable (λn. f n * z^n)
  shows   conv-radius f = ∞
  using assms by (rule conv-radius-inftyI')

lemma conv-radius-inftyI'':
  fixes f :: nat ⇒ 'a :: {banach,real-normed-div-algebra}
  assumes ⋀z. summable (λn. f n * z^n)
  shows   conv-radius f = ∞
proof (rule conv-radius-inftyI')
  fix r :: real assume r > 0
  with assms show ∃z. norm z = r ∧ summable (λn. f n * z^n)
    by (intro exI[of - of-real r]) simp
qed

lemma conv-radius-ratio-limit-ereal:
  fixes f :: nat ⇒ 'a :: {banach,real-normed-div-algebra}
  assumes nz:  eventually (λn. f n ≠ 0) sequentially
  assumes lim: (λn. ereal (norm (f n) / norm (f (Suc n)))) ⟶ c
  shows   conv-radius f = c
proof (rule conv-radius-eqI')
  show c ≥ 0 by (intro Lim-bounded2-ereal[OF lim]) simp-all
next
  fix r assume r: 0 < r ereal r < c
  let ?l = liminf (λn. ereal (norm (f n * of-real r ^ n) / norm (f (Suc n) * of-real
r ^ Suc n)))
  have ?l = liminf (λn. ereal (norm (f n) / (norm (f (Suc n)))) * ereal (inverse
r))
    using r by (simp add: norm-mult norm-power divide-simps)
  also from r have ... = liminf (λn. ereal (norm (f n) / (norm (f (Suc n)))))
* ereal (inverse r)
    by (intro Liminf-ereal-mult-right) simp-all
  also have liminf (λn. ereal (norm (f n) / (norm (f (Suc n))))) = c
    by (intro lim-imp-Liminf lim) simp
  finally have l: ?l = c * ereal (inverse r) by simp
  from r have  l': c * ereal (inverse r) > 1 by (cases c) (simp-all add: field-simps)
  show summable (λn. f n * of-real r^n)
    by (rule summable-norm-cancel, rule ratio-test-convergence)
       (insert r nz l l', auto elim!: eventually-mono)
next
```

fix r **assume** r: $0 < r$ $ereal\ r > c$
let $?l = limsup\ (\lambda n.\ ereal\ (norm\ (f\ n) * of\text{-}real\ r\ \hat{}\ n)\ /\ norm\ (f\ (Suc\ n) * of\text{-}real\ r\ \hat{}\ Suc\ n)))$
have $?l = limsup\ (\lambda n.\ ereal\ (norm\ (f\ n)\ /\ (norm\ (f\ (Suc\ n)))) * ereal\ (inverse\ r))$
using r **by** (*simp add*: *norm-mult norm-power divide-simps*)
also from r **have** $\ldots = limsup\ (\lambda n.\ ereal\ (norm\ (f\ n)\ /\ (norm\ (f\ (Suc\ n))))) * ereal\ (inverse\ r)$
by (*intro Limsup-ereal-mult-right*) *simp-all*
also have $limsup\ (\lambda n.\ ereal\ (norm\ (f\ n)\ /\ (norm\ (f\ (Suc\ n))))) = c$
by (*intro lim-imp-Limsup lim*) *simp*
finally have l: $?l = c * ereal\ (inverse\ r)$ **by** *simp*
from r **have** l': $c * ereal\ (inverse\ r) < 1$ **by** (*cases c*) (*simp-all add*: *field-simps*)
show $\neg summable\ (\lambda n.\ norm\ (f\ n * of\text{-}real\ r\hat{}n))$
by (*rule ratio-test-divergence*) (*insert r nz l l'*, *auto elim*!: *eventually-mono*)
qed

lemma *conv-radius-ratio-limit-ereal-nonzero*:
fixes $f :: nat \Rightarrow 'a :: \{banach, real\text{-}normed\text{-}div\text{-}algebra\}$
assumes nz: $c \neq 0$
assumes lim: $(\lambda n.\ ereal\ (norm\ (f\ n)\ /\ norm\ (f\ (Suc\ n)))) \longrightarrow c$
shows *conv-radius f* $= c$
proof (*rule conv-radius-ratio-limit-ereal*[*OF - lim*], *rule ccontr*)
assume $\neg eventually\ (\lambda n.\ f\ n \neq 0)\ sequentially$
hence $frequently\ (\lambda n.\ f\ n = 0)\ sequentially$ **by** (*simp add*: *frequently-def*)
hence $frequently\ (\lambda n.\ ereal\ (norm\ (f\ n)\ /\ norm\ (f\ (Suc\ n))) = 0)\ sequentially$
by (*force elim*!: *frequently-elim1*)
hence $c = 0$ **by** (*intro limit-frequently-eq*[*OF - - lim*]) *auto*
with nz **show** *False* **by** *contradiction*
qed

lemma *conv-radius-ratio-limit*:
fixes $f :: nat \Rightarrow 'a :: \{banach, real\text{-}normed\text{-}div\text{-}algebra\}$
assumes $c' = ereal\ c$
assumes nz: $eventually\ (\lambda n.\ f\ n \neq 0)\ sequentially$
assumes lim: $(\lambda n.\ norm\ (f\ n)\ /\ norm\ (f\ (Suc\ n))) \longrightarrow c$
shows *conv-radius f* $= c'$
using *assms* **by** (*intro conv-radius-ratio-limit-ereal*) *simp-all*

lemma *conv-radius-ratio-limit-nonzero*:
fixes $f :: nat \Rightarrow 'a :: \{banach, real\text{-}normed\text{-}div\text{-}algebra\}$
assumes $c' = ereal\ c$
assumes nz: $c \neq 0$
assumes lim: $(\lambda n.\ norm\ (f\ n)\ /\ norm\ (f\ (Suc\ n))) \longrightarrow c$
shows *conv-radius f* $= c'$
using *assms* **by** (*intro conv-radius-ratio-limit-ereal-nonzero*) *simp-all*

lemma *conv-radius-mult-power*:
assumes $c \neq (0 :: 'a :: \{real\text{-}normed\text{-}div\text{-}algebra, banach\})$

shows *conv-radius* (λ*n*. *c* ^ *n* ∗ *f n*) = *conv-radius f* / *ereal* (*norm c*)
proof −
have *limsup* (λ*n*. *ereal* (*root n* (*norm* (*c* ^ *n* ∗ *f n*)))) =
limsup (λ*n*. *ereal* (*norm c*) ∗ *ereal* (*root n* (*norm* (*f n*))))
using *eventually-gt-at-top*[*of* 0::*nat*]
by (*intro Limsup-eq*)
(*auto elim*!: *eventually-mono simp*: *norm-mult norm-power real-root-mult real-root-power*)
also have ... = *ereal* (*norm c*) ∗ *limsup* (λ*n*. *ereal* (*root n* (*norm* (*f n*))))
using *assms* **by** (*subst Limsup-ereal-mult-left*[*symmetric*]) *simp-all*
finally have *A*: *limsup* (λ*n*. *ereal* (*root n* (*norm* (*c* ^ *n* ∗ *f n*)))) =
ereal (*norm c*) ∗ *limsup* (λ*n*. *ereal* (*root n* (*norm* (*f n*)))) .
show *?thesis* **using** *assms*
apply (*cases limsup* (λ*n*. *ereal* (*root n* (*norm* (*f n*)))) = 0)
apply (*simp add*: *A conv-radius-def*)
apply (*unfold conv-radius-def A divide-ereal-def*, *simp add*: *mult.commute ereal-inverse-mult*)
done
qed

lemma *conv-radius-mult-power-right*:
assumes *c* ≠ (0 :: 'a :: {*real-normed-div-algebra*,*banach*})
shows *conv-radius* (λ*n*. *f n* ∗ *c* ^ *n*) = *conv-radius f* / *ereal* (*norm c*)
using *conv-radius-mult-power*[*OF assms*, *of f*]
unfolding *conv-radius-def* **by** (*simp add*: *mult.commute norm-mult*)

lemma *conv-radius-divide-power*:
assumes *c* ≠ (0 :: 'a :: {*real-normed-div-algebra*,*banach*})
shows *conv-radius* (λ*n*. *f n* / *c*^*n*) = *conv-radius f* ∗ *ereal* (*norm c*)
proof −
from *assms* **have** *inverse c* ≠ 0 **by** *simp*
from *conv-radius-mult-power-right*[*OF this*, *of f*] **show** *?thesis*
by (*simp add*: *divide-inverse divide-ereal-def assms norm-inverse power-inverse*)
qed

lemma *conv-radius-add-ge*:
min (*conv-radius f*) (*conv-radius g*) ≤
conv-radius (λ*x*. *f x* + *g x* :: 'a :: {*banach*,*real-normed-div-algebra*})
by (*rule conv-radius-geI-ex'*)
(*auto simp*: *algebra-simps intro*!: *summable-add summable-in-conv-radius*)

lemma *conv-radius-mult-ge*:
fixes *f g* :: *nat* ⇒ ('a :: {*banach*,*real-normed-div-algebra*})
shows *conv-radius* (λ*x*. ∑ *i*≤*x*. *f i* ∗ *g* (*x* − *i*)) ≥ *min* (*conv-radius f*) (*conv-radius g*)
proof (*rule conv-radius-geI-ex'*)
fix *r* **assume** *r*: *r* > 0 *ereal r* < *min* (*conv-radius f*) (*conv-radius g*)
from *r* **have** *summable* (λ*n*. (∑ *i*≤*n*. (*f i* ∗ *of-real r*^*i*) ∗ (*g* (*n* − *i*) ∗ *of-real*

$r\hat{}(n - i))))$
by (*intro summable-Cauchy-product abs-summable-in-conv-radius*) *simp-all*
thus *summable* ($\lambda n.$ ($\sum i{\leq}n.$ *f i* $*$ *g* (*n* − *i*)) $*$ *of-real r* $\hat{}$ *n*)
by (*simp add*: *algebra-simps of-real-def power-add* [*symmetric*] *scaleR-setsum-right*)
qed

lemma *le-conv-radius-iff*:
fixes *a* :: *nat* ⇒ *'a*::{*real-normed-div-algebra*,*banach*}
shows *r* ≤ *conv-radius a* ⟷ (∀ *x*. *norm* (*x*−ξ) < *r* ⟶ *summable* (λ*i*. *a i* $*$ (*x* − ξ) $\hat{}$ *i*))
apply (*intro iffI allI impI summable-in-conv-radius conv-radius-geI-ex*)
apply (*meson less-ereal.simps*(*1*) *not-le order-trans*)
apply (*rule-tac x*=*of-real ra* **in** *exI*, *simp*)
apply (*metis abs-of-nonneg add-diff-cancel-left' less-eq-real-def norm-of-real*)
done

end

38 Uniform Limit and Uniform Convergence

theory *Uniform-Limit*
imports *Topology-Euclidean-Space Summation*
begin

definition *uniformly-on* :: *'a set* ⇒ (*'a* ⇒ *'b*::*metric-space*) ⇒ (*'a* ⇒ *'b*) *filter*
where *uniformly-on S l* = (*INF e*:{*0* <..}. *principal* {*f*. ∀ *x*∈*S*. *dist* (*f x*) (*l x*) < *e*})

abbreviation
uniform-limit S f l ≡ *filterlim f* (*uniformly-on S l*)

definition *uniformly-convergent-on* **where**
uniformly-convergent-on X f ⟷ (∃ *l*. *uniform-limit X f l sequentially*)

definition *uniformly-Cauchy-on* **where**
uniformly-Cauchy-on X f ⟷ (∀ *e*>*0*. ∃ *M*. ∀ *x*∈*X*. ∀ (*m*::*nat*)≥*M*. ∀ *n*≥*M*. *dist* (*f m x*) (*f n x*) < *e*)

lemma *uniform-limit-iff*:
uniform-limit S f l F ⟷ (∀ *e*>*0*. ∀$_F$ *n in F*. ∀ *x*∈*S*. *dist* (*f n x*) (*l x*) < *e*)
unfolding *filterlim-iff uniformly-on-def*
by (*subst eventually-INF-base*)
(*fastforce*
simp: *eventually-principal uniformly-on-def*
intro: *bexI*[**where** *x*=*min a b* **for** *a b*]
elim: *eventually-mono*)+

lemma *uniform-limitD*:
uniform-limit S f l F ⟹ *e* > *0* ⟹ ∀$_F$ *n in F*. ∀ *x*∈*S*. *dist* (*f n x*) (*l x*) < *e*

by (*simp add*: *uniform-limit-iff*)

lemma *uniform-limitI*:
(⋀*e*. *e* > *0* ⟹ ∀ *F* *n in F*. ∀ *x*∈*S*. *dist* (*f n x*) (*l x*) < *e*) ⟹ *uniform-limit S f l F*
by (*simp add*: *uniform-limit-iff*)

lemma *uniform-limit-sequentially-iff*:
uniform-limit S f l sequentially ⟷ (∀ *e*>*0*. ∃ *N*. ∀ *n*≥*N*. ∀ *x* ∈ *S*. *dist* (*f n x*) (*l x*) < *e*)
unfolding *uniform-limit-iff eventually-sequentially* **..**

lemma *uniform-limit-at-iff*:
uniform-limit S f l (*at x*) ⟷
(∀ *e*>*0*. ∃ *d*>*0*. ∀ *z*. *0* < *dist z x* ∧ *dist z x* < *d* ⟶ (∀ *x*∈*S*. *dist* (*f z x*) (*l x*) < *e*))
unfolding *uniform-limit-iff eventually-at* **by** *simp*

lemma *uniform-limit-at-le-iff*:
uniform-limit S f l (*at x*) ⟷
(∀ *e*>*0*. ∃ *d*>*0*. ∀ *z*. *0* < *dist z x* ∧ *dist z x* < *d* ⟶ (∀ *x*∈*S*. *dist* (*f z x*) (*l x*) ≤ *e*))
unfolding *uniform-limit-iff eventually-at*
by (*fastforce dest*: *spec*[**where** *x* = *e* / *2* **for** *e*])

lemma *metric-uniform-limit-imp-uniform-limit*:
assumes *f*: *uniform-limit S f a F*
assumes *le*: *eventually* (λ*x*. ∀ *y*∈*S*. *dist* (*g x y*) (*b y*) ≤ *dist* (*f x y*) (*a y*)) *F*
shows *uniform-limit S g b F*
proof (*rule uniform-limitI*)
fix *e* :: *real* **assume** *0* < *e*
from *uniform-limitD*[*OF f this*] *le*
show ∀ *F* *x in F*. ∀ *y*∈*S*. *dist* (*g x y*) (*b y*) < *e*
by *eventually-elim force*
qed

lemma *swap-uniform-limit*:
assumes *f*: ∀ *F* *n in F*. (*f n* ⟶ *g n*) (*at x within S*)
assumes *g*: (*g* ⟶ *l*) *F*
assumes *uc*: *uniform-limit S f h F*
assumes ¬*trivial-limit F*
shows (*h* ⟶ *l*) (*at x within S*)
proof (*rule tendstoI*)
fix *e* :: *real*
def *e'* ≡ *e*/*3*
assume *0* < *e*
then have *0* < *e'* **by** (*simp add*: *e'-def*)
from *uniform-limitD*[*OF uc* ‹*0* < *e'*›]
have ∀ *F* *n in F*. ∀ *x*∈*S*. *dist* (*h x*) (*f n x*) < *e'*

```
    by (simp add: dist-commute)
  moreover
  from f
  have ∀F n in F. ∀F x in at x within S. dist (g n) (f n x) < e'
    by eventually-elim (auto dest!: tendstoD[OF - ‹0 < e'›] simp: dist-commute)
  moreover
  from tendstoD[OF g ‹0 < e'›] have ∀F x in F. dist l (g x) < e'
    by (simp add: dist-commute)
  ultimately
  have ∀F - in F. ∀F x in at x within S. dist (h x) l < e
  proof eventually-elim
    case (elim n)
    note fh = elim(1)
    note gl = elim(3)
    have ∀F x in at x within S. x ∈ S
      by (auto simp: eventually-at-filter)
    with elim(2)
    show ?case
    proof eventually-elim
      case (elim x)
      from fh[rule-format, OF ‹x ∈ S›] elim(1)
      have dist (h x) (g n) < e' + e'
        by (rule dist-triangle-lt[OF add-strict-mono])
      from dist-triangle-lt[OF add-strict-mono, OF this gl]
      show ?case by (simp add: e'-def)
    qed
  qed
  thus ∀F x in at x within S. dist (h x) l < e
    using eventually-happens by (metis ‹¬trivial-limit F›)
qed

lemma
  tendsto-uniform-limitI:
  assumes uniform-limit S f l F
  assumes x ∈ S
  shows ((λy. f y x) ⟶ l x) F
  using assms
  by (auto intro!: tendstoI simp: eventually-mono dest!: uniform-limitD)

lemma uniform-limit-theorem:
  assumes c: ∀F n in F. continuous-on A (f n)
  assumes ul: uniform-limit A f l F
  assumes ¬ trivial-limit F
  shows continuous-on A l
  unfolding continuous-on-def
proof safe
  fix x assume x ∈ A
  then have ∀F n in F. (f n ⟶ f n x) (at x within A) ((λn. f n x) ⟶ l x)
F
```

using *c ul*
by (*auto simp*: *continuous-on-def eventually-mono tendsto-uniform-limitI*)
then show $(l \longrightarrow l\ x)\ (at\ x\ within\ A)$
by (*rule swap-uniform-limit*) *fact+*
qed

lemma *uniformly-Cauchy-onI*:
assumes $\bigwedge e.\ e > 0 \Longrightarrow \exists M.\ \forall x{\in}X.\ \forall m{\geq}M.\ \forall n{\geq}M.\ dist\ (f\ m\ x)\ (f\ n\ x) < e$
shows *uniformly-Cauchy-on X f*
using *assms* **unfolding** *uniformly-Cauchy-on-def* **by** *blast*

lemma *uniformly-Cauchy-onI'*:
assumes $\bigwedge e.\ e > 0 \Longrightarrow \exists M.\ \forall x{\in}X.\ \forall m{\geq}M.\ \forall n{>}m.\ dist\ (f\ m\ x)\ (f\ n\ x) < e$
shows *uniformly-Cauchy-on X f*
proof (*rule uniformly-Cauchy-onI*)
fix *e* :: *real* **assume** *e*: $e > 0$
from *assms*[*OF this*] **obtain** *M*
where *M*: $\bigwedge x\ m\ n.\ x \in X \Longrightarrow m \geq M \Longrightarrow n > m \Longrightarrow dist\ (f\ m\ x)\ (f\ n\ x) < e$ **by** *fast*
{
fix *x m n* **assume** *x*: $x \in X$ **and** *m*: $m \geq M$ **and** *n*: $n \geq M$
with *M*[*OF this*(*1*,*2*), *of n*] *M*[*OF this*(*1*,*3*), *of m*] *e* **have** $dist\ (f\ m\ x)\ (f\ n\ x) < e$
by (*cases m n rule*: *linorder-cases*) (*simp-all add*: *dist-commute*)
}
thus $\exists M.\ \forall x{\in}X.\ \forall m{\geq}M.\ \forall n{\geq}M.\ dist\ (f\ m\ x)\ (f\ n\ x) < e$ **by** *fast*
qed

lemma *uniformly-Cauchy-imp-Cauchy*:
uniformly-Cauchy-on X f $\Longrightarrow x \in X \Longrightarrow$ *Cauchy* $(\lambda n.\ f\ n\ x)$
unfolding *Cauchy-def uniformly-Cauchy-on-def* **by** *fast*

lemma *uniform-limit-cong*:
fixes *f g* :: $'a \Rightarrow 'b \Rightarrow ('c$ :: *metric-space*) **and** *h i* :: $'b \Rightarrow 'c$
assumes *eventually* $(\lambda y.\ \forall x{\in}X.\ f\ y\ x = g\ y\ x)\ F$
assumes $\bigwedge x.\ x \in X \Longrightarrow h\ x = i\ x$
shows *uniform-limit X f h F* $\longleftrightarrow$ *uniform-limit X g i F*
proof −
{
fix *f g* :: $'a \Rightarrow 'b \Rightarrow 'c$ **and** *h i* :: $'b \Rightarrow 'c$
assume *C*: *uniform-limit X f h F* **and** *A*: *eventually* $(\lambda y.\ \forall x{\in}X.\ f\ y\ x = g\ y\ x)\ F$
and *B*: $\bigwedge x.\ x \in X \Longrightarrow h\ x = i\ x$
{
fix *e* ::*real* **assume** $e > 0$
with *C* **have** *eventually* $(\lambda y.\ \forall x{\in}X.\ dist\ (f\ y\ x)\ (h\ x) < e)\ F$
unfolding *uniform-limit-iff* **by** *blast*
with *A* **have** *eventually* $(\lambda y.\ \forall x{\in}X.\ dist\ (g\ y\ x)\ (i\ x) < e)\ F$
by *eventually-elim* (*insert B*, *simp-all*)

```
    }
    hence uniform-limit X g i F unfolding uniform-limit-iff by blast
  } note A = this
  show ?thesis by (rule iffI) (erule A; insert assms; simp add: eq-commute)+
qed

lemma uniform-limit-cong':
  fixes f g :: 'a ⇒ 'b ⇒ ('c :: metric-space) and h i :: 'b ⇒ 'c
  assumes ⋀y x. x ∈ X ⟹ f y x = g y x
  assumes ⋀x. x ∈ X ⟹ h x = i x
  shows   uniform-limit X f h F ⟷ uniform-limit X g i F
  using assms by (intro uniform-limit-cong always-eventually) blast+

lemma uniformly-convergent-uniform-limit-iff:
  uniformly-convergent-on X f ⟷ uniform-limit X f (λx. lim (λn. f n x)) sequentially
proof
  assume uniformly-convergent-on X f
  then obtain l where l: uniform-limit X f l sequentially
    unfolding uniformly-convergent-on-def by blast
  from l have uniform-limit X f (λx. lim (λn. f n x)) sequentially ⟷
                uniform-limit X f l sequentially
    by (intro uniform-limit-cong' limI tendsto-uniform-limitI[of f X l]) simp-all
  also note l
  finally show uniform-limit X f (λx. lim (λn. f n x)) sequentially .
qed (auto simp: uniformly-convergent-on-def)

lemma uniformly-convergentI: uniform-limit X f l sequentially ⟹ uniformly-convergent-on
X f
  unfolding uniformly-convergent-on-def by blast

lemma uniformly-convergent-on-empty [iff]: uniformly-convergent-on {} f
  by (simp add: uniformly-convergent-on-def uniform-limit-sequentially-iff)

lemma Cauchy-uniformly-convergent:
  fixes f :: nat ⇒ 'a ⇒ 'b :: complete-space
  assumes uniformly-Cauchy-on X f
  shows   uniformly-convergent-on X f
unfolding uniformly-convergent-uniform-limit-iff uniform-limit-iff
proof safe
  let ?f = λx. lim (λn. f n x)
  fix e :: real assume e: e > 0
  hence e/2 > 0 by simp
  with assms obtain N where N: ⋀x m n. x ∈ X ⟹ m ≥ N ⟹ n ≥ N ⟹
dist (f m x) (f n x) < e/2
    unfolding uniformly-Cauchy-on-def by fast
  show eventually (λn. ∀x∈X. dist (f n x) (?f x) < e) sequentially
    using eventually-ge-at-top[of N]
  proof eventually-elim
    fix n assume n: n ≥ N
```

```
    show ∀x∈X. dist (f n x) (?f x) < e
    proof
      fix x assume x: x ∈ X
      with assms have (λn. f n x) ⟶ ?f x
        by (auto dest!: Cauchy-convergent uniformly-Cauchy-imp-Cauchy simp:
convergent-LIMSEQ-iff)
      with ‹e/2 > 0› have eventually (λm. m ≥ N ∧ dist (f m x) (?f x) < e/2)
sequentially
        by (intro tendstoD eventually-conj eventually-ge-at-top)
      then obtain m where m: m ≥ N dist (f m x) (?f x) < e/2
        unfolding eventually-at-top-linorder by blast
      have dist (f n x) (?f x) ≤ dist (f n x) (f m x) + dist (f m x) (?f x)
        by (rule dist-triangle)
      also from x n have ... < e/2 + e/2 by (intro add-strict-mono N m)
      finally show dist (f n x) (?f x) < e by simp
    qed
  qed
qed

lemma uniformly-convergent-imp-convergent:
  uniformly-convergent-on X f ⟹ x ∈ X ⟹ convergent (λn. f n x)
  unfolding uniformly-convergent-on-def convergent-def
  by (auto dest: tendsto-uniform-limitI)

lemma weierstrass-m-test-ev:
fixes f :: - ⇒ - ⇒ - :: banach
assumes eventually (λn. ∀x∈A. norm (f n x) ≤ M n) sequentially
assumes summable M
shows uniform-limit A (λn x. ∑i<n. f i x) (λx. suminf (λi. f i x)) sequentially
proof (rule uniform-limitI)
  fix e :: real
  assume 0 < e
  from suminf-exist-split[OF ‹0 < e› ‹summable M›]
  have ∀F k in sequentially. norm (∑i. M (i + k)) < e
    by (auto simp: eventually-sequentially)
  with eventually-all-ge-at-top[OF assms(1)]
    show ∀F n in sequentially. ∀x∈A. dist (∑i<n. f i x) (∑i. f i x) < e
  proof eventually-elim
    case (elim k)
    show ?case
    proof safe
      fix x assume x ∈ A
      have ∃N. ∀n≥N. norm (f n x) ≤ M n
        using assms(1) ‹x ∈ A› by (force simp: eventually-at-top-linorder)
      hence summable-norm-f: summable (λn. norm (f n x))
        by(rule summable-norm-comparison-test[OF - ‹summable M›])
      have summable-f: summable (λn. f n x)
        using summable-norm-cancel[OF summable-norm-f] .
      have summable-norm-f-plus-k: summable (λi. norm (f (i + k) x))
```

```
      using summable-ignore-initial-segment[OF summable-norm-f]
      by auto
    have summable-M-plus-k: summable (λi. M (i + k))
      using summable-ignore-initial-segment[OF ‹summable M›]
      by auto

    have dist (∑ i<k. f i x) (∑ i. f i x) = norm ((∑ i. f i x) − (∑ i<k. f i x))
      using dist-norm dist-commute by (subst dist-commute)
    also have ... = norm (∑ i. f (i + k) x)
      using suminf-minus-initial-segment[OF summable-f, where k=k] by simp
    also have ... ≤ (∑ i. norm (f (i + k) x))
      using summable-norm[OF summable-norm-f-plus-k] .
    also have ... ≤ (∑ i. M (i + k))
      by (rule suminf-le[OF - summable-norm-f-plus-k summable-M-plus-k])
        (insert elim(1) ‹x ∈ A›, simp)
    finally show dist (∑ i<k. f i x) (∑ i. f i x) < e
      using elim by auto
  qed
 qed
qed
```

Alternative version, formulated as in HOL Light

```
corollary series-comparison-uniform:
  fixes f :: - ⇒ nat ⇒ - :: banach
  assumes g: summable g and le: ⋀n x. N ≤ n ∧ x ∈ A ⟹ norm(f x n) ≤ g n
    shows ∃ l. ∀ e. 0 < e ⟶ (∃ N. ∀ n x. N ≤ n ∧ x ∈ A ⟶ dist(setsum (f x)
{..<n}) (l x) < e)
proof −
  have 1: ∀ F n in sequentially. ∀ x∈A. norm (f x n) ≤ g n
    using le eventually-sequentially by auto
  show ?thesis
    apply (rule-tac x=(λx. ∑ i. f x i) in exI)
     apply (metis (no-types, lifting) eventually-sequentially uniform-limitD [OF
weierstrass-m-test-ev [OF 1 g]])
    done
qed

corollary weierstrass-m-test:
  fixes f :: - ⇒ - ⇒ - :: banach
  assumes ⋀n x. x ∈ A ⟹ norm (f n x) ≤ M n
  assumes summable M
  shows uniform-limit A (λn x. ∑ i<n. f i x) (λx. suminf (λi. f i x)) sequentially
  using assms by (intro weierstrass-m-test-ev always-eventually) auto

corollary weierstrass-m-test'-ev:
  fixes f :: - ⇒ - ⇒ - :: banach
  assumes eventually (λn. ∀ x∈A. norm (f n x) ≤ M n) sequentially summable M

  shows   uniformly-convergent-on A (λn x. ∑ i<n. f i x)
```

unfolding *uniformly-convergent-on-def* **by** (*rule exI*, *rule weierstrass-m-test-ev*[*OF assms*])

corollary *weierstrass-m-test'*:
fixes f :: - ⇒ - ⇒ - :: *banach*
assumes $\bigwedge n\ x.\ x \in A \Longrightarrow$ *norm* $(f\ n\ x) \leq M\ n$ *summable* M
shows *uniformly-convergent-on* A $(\lambda n\ x.\ \sum i{<}n.\ f\ i\ x)$
unfolding *uniformly-convergent-on-def* **by** (*rule exI*, *rule weierstrass-m-test*[*OF assms*])

lemma *uniform-limit-eq-rhs*: *uniform-limit* $X\ f\ l\ F \Longrightarrow l = m \Longrightarrow$ *uniform-limit* $X\ f\ m\ F$
by *simp*

named-theorems *uniform-limit-intros introduction rules for uniform-limit*
setup ‹
Global-Theory.add-thms-dynamic (@{*binding uniform-limit-eq-intros*},
fn context =>
Named-Theorems.get (*Context.proof-of context*) @{*named-theorems uniform-limit-intros*}
|> *map-filter* (*try* (*fn thm =>* @{*thm uniform-limit-eq-rhs*} *OF* [*thm*])))
›

lemma (**in** *bounded-linear*) *uniform-limit*[*uniform-limit-intros*]:
assumes *uniform-limit* $X\ g\ l\ F$
shows *uniform-limit* X $(\lambda a\ b.\ f\ (g\ a\ b))$ $(\lambda a.\ f\ (l\ a))\ F$
proof (*rule uniform-limitI*)
fix e::*real*
from *pos-bounded* **obtain** K
where K: $\bigwedge x\ y.$ *dist* $(f\ x)\ (f\ y) \leq K * dist\ x\ y\ K > 0$
by (*auto simp*: *ac-simps dist-norm diff*[*symmetric*])
assume $0 < e$ **with** ‹$K > 0$› **have** $e\ /\ K > 0$ **by** *simp*
from *uniform-limitD*[*OF assms this*]
show $\forall_F\ n$ *in* $F.\ \forall x{\in}X.$ *dist* $(f\ (g\ n\ x))\ (f\ (l\ x)) < e$
by *eventually-elim* (*metis le-less-trans mult.commute pos-less-divide-eq* K)
qed

lemmas *bounded-linear-uniform-limit-intros*[*uniform-limit-intros*] =
bounded-linear.uniform-limit[*OF bounded-linear-Im*]
bounded-linear.uniform-limit[*OF bounded-linear-Re*]
bounded-linear.uniform-limit[*OF bounded-linear-cnj*]
bounded-linear.uniform-limit[*OF bounded-linear-fst*]
bounded-linear.uniform-limit[*OF bounded-linear-snd*]
bounded-linear.uniform-limit[*OF bounded-linear-zero*]
bounded-linear.uniform-limit[*OF bounded-linear-of-real*]
bounded-linear.uniform-limit[*OF bounded-linear-inner-left*]
bounded-linear.uniform-limit[*OF bounded-linear-inner-right*]
bounded-linear.uniform-limit[*OF bounded-linear-divide*]
bounded-linear.uniform-limit[*OF bounded-linear-scaleR-right*]
bounded-linear.uniform-limit[*OF bounded-linear-mult-left*]

bounded-linear.uniform-limit[*OF bounded-linear-mult-right*]
bounded-linear.uniform-limit[*OF bounded-linear-scaleR-left*]

lemmas *uniform-limit-uminus*[*uniform-limit-intros*] =
bounded-linear.uniform-limit[*OF bounded-linear-minus*[*OF bounded-linear-ident*]]

lemma *uniform-limit-const*[*uniform-limit-intros*]: *uniform-limit S* (λ*x y*. *c*) (λ*x*. *c*) *f*
by (*auto intro*!: *uniform-limitI*)

lemma *uniform-limit-add*[*uniform-limit-intros*]:
fixes *f g*::'*a* ⇒ '*b* ⇒ '*c*::*real-normed-vector*
assumes *uniform-limit X f l F*
assumes *uniform-limit X g m F*
shows *uniform-limit X* (λ*a b*. *f a b* + *g a b*) (λ*a*. *l a* + *m a*) *F*
proof (*rule uniform-limitI*)
fix *e*::*real*
assume *0* < *e*
hence *0* < *e* / *2* **by** *simp*
from
uniform-limitD[*OF assms*(*1*) *this*]
uniform-limitD[*OF assms*(*2*) *this*]
show ∀ *F n in F*. ∀*x*∈*X*. *dist* (*f n x* + *g n x*) (*l x* + *m x*) < *e*
by *eventually-elim* (*simp add*: *dist-triangle-add-half*)
qed

lemma *uniform-limit-minus*[*uniform-limit-intros*]:
fixes *f g*::'*a* ⇒ '*b* ⇒ '*c*::*real-normed-vector*
assumes *uniform-limit X f l F*
assumes *uniform-limit X g m F*
shows *uniform-limit X* (λ*a b*. *f a b* − *g a b*) (λ*a*. *l a* − *m a*) *F*
unfolding *diff-conv-add-uminus*
by (*rule uniform-limit-intros assms*)+

lemma *uniform-limit-norm*[*uniform-limit-intros*]:
assumes *uniform-limit S g l f*
shows *uniform-limit S* (λ*x y*. *norm* (*g x y*)) (λ*x*. *norm* (*l x*)) *f*
using *assms*
apply (*rule metric-uniform-limit-imp-uniform-limit*)
apply (*rule eventuallyI*)
by (*metis dist-norm norm-triangle-ineq3 real-norm-def*)

lemma (**in** *bounded-bilinear*) *bounded-uniform-limit*[*uniform-limit-intros*]:
assumes *uniform-limit X f l F*
assumes *uniform-limit X g m F*
assumes *bounded* (*m* ‘ *X*)
assumes *bounded* (*l* ‘ *X*)
shows *uniform-limit X* (λ*a b*. *prod* (*f a b*) (*g a b*)) (λ*a*. *prod* (*l a*) (*m a*)) *F*
proof (*rule uniform-limitI*)

fix *e*::*real*
from *pos-bounded* **obtain** *K* **where** *K*:
0 < K ⋀*a b*. *norm* (*prod a b*) ≤ *norm a* ∗ *norm b* ∗ *K*
by *auto*
hence *sqrt* (*K*∗*4*) > *0* **by** *simp*

from *assms* **obtain** *Km Kl*
where *Km*: *Km > 0* ⋀*x*. *x* ∈ *X* ⟹ *norm* (*m x*) ≤ *Km*
and *Kl*: *Kl > 0* ⋀*x*. *x* ∈ *X* ⟹ *norm* (*l x*) ≤ *Kl*
by (*auto simp*: *bounded-pos*)
hence *K* ∗ *Km* ∗ *4* > *0 K* ∗ *Kl* ∗ *4* > *0*
using ‹*K > 0*›
by *simp-all*
assume *0 < e*

hence *sqrt e > 0* **by** *simp*
from *uniform-limitD*[*OF assms*(*1*) *divide-pos-pos*[*OF this* ‹*sqrt* (*K*∗*4*) > *0*›]]
uniform-limitD[*OF assms*(*2*) *divide-pos-pos*[*OF this* ‹*sqrt* (*K*∗*4*) > *0*›]]
uniform-limitD[*OF assms*(*1*) *divide-pos-pos*[*OF* ‹*e > 0*› ‹*K* ∗ *Km* ∗ *4* > *0*›]]
uniform-limitD[*OF assms*(*2*) *divide-pos-pos*[*OF* ‹*e > 0*› ‹*K* ∗ *Kl* ∗ *4* > *0*›]]
show ∀ *F n in F*. ∀*x*∈*X*. *dist* (*prod* (*f n x*) (*g n x*)) (*prod* (*l x*) (*m x*)) < *e*
proof *eventually-elim*
case (*elim n*)
show *?case*
proof *safe*
fix *x* **assume** *x* ∈ *X*
have *dist* (*prod* (*f n x*) (*g n x*)) (*prod* (*l x*) (*m x*)) ≤
norm (*prod* (*f n x* − *l x*) (*g n x* − *m x*)) +
norm (*prod* (*f n x* − *l x*) (*m x*)) +
norm (*prod* (*l x*) (*g n x* − *m x*))
by (*auto simp*: *dist-norm prod-diff-prod intro*: *order-trans norm-triangle-ineq add-mono*)
also note *K*(*2*)[*of f n x* − *l x g n x* − *m x*]
also from *elim*(*1*)[*THEN bspec*, *OF* ‹*- ∈ X*›, *unfolded dist-norm*]
have *norm* (*f n x* − *l x*) ≤ *sqrt e* / *sqrt* (*K* ∗ *4*)
by *simp*
also from *elim*(*2*)[*THEN bspec*, *OF* ‹*- ∈ X*›, *unfolded dist-norm*]
have *norm* (*g n x* − *m x*) ≤ *sqrt e* / *sqrt* (*K* ∗ *4*)
by *simp*
also have *sqrt e* / *sqrt* (*K* ∗ *4*) ∗ (*sqrt e* / *sqrt* (*K* ∗ *4*)) ∗ *K* = *e* / *4*
using ‹*K > 0*› ‹*e > 0*› **by** *auto*
also note *K*(*2*)[*of f n x* − *l x m x*]
also note *K*(*2*)[*of l x g n x* − *m x*]
also from *elim*(*3*)[*THEN bspec*, *OF* ‹*- ∈ X*›, *unfolded dist-norm*]
have *norm* (*f n x* − *l x*) ≤ *e* / (*K* ∗ *Km* ∗ *4*)
by *simp*
also from *elim*(*4*)[*THEN bspec*, *OF* ‹*- ∈ X*›, *unfolded dist-norm*]
have *norm* (*g n x* − *m x*) ≤ *e* / (*K* ∗ *Kl* ∗ *4*)
by *simp*

```
    also note Kl(2)[OF ‹- ∈ X›]
    also note Km(2)[OF ‹- ∈ X›]
    also have e / (K * Km * 4) * Km * K = e / 4
      using ‹K > 0› ‹Km > 0› by simp
    also have  Kl * (e / (K * Kl * 4)) * K = e / 4
      using ‹K > 0› ‹Kl > 0› by simp
    also have e / 4 + e / 4 + e / 4 < e using ‹e > 0› by simp
    finally show dist (prod (f n x) (g n x)) (prod (l x) (m x)) < e
      using ‹K > 0› ‹Kl > 0› ‹Km > 0› ‹e > 0›
      by (simp add: algebra-simps mult-right-mono divide-right-mono)
  qed
 qed
qed

lemmas bounded-bilinear-bounded-uniform-limit-intros[uniform-limit-intros] =
 bounded-bilinear.bounded-uniform-limit[OF Inner-Product.bounded-bilinear-inner]
 bounded-bilinear.bounded-uniform-limit[OF Real-Vector-Spaces.bounded-bilinear-mult]
 bounded-bilinear.bounded-uniform-limit[OF Real-Vector-Spaces.bounded-bilinear-scaleR]

lemma uniform-limit-null-comparison:
 assumes ∀ F x in F. ∀ a∈S. norm (f x a) ≤ g x a
 assumes uniform-limit S g (λ-. 0) F
 shows uniform-limit S f (λ-. 0) F
 using assms(2)
proof (rule metric-uniform-limit-imp-uniform-limit)
 show ∀ F x in F. ∀ y∈S. dist (f x y) 0 ≤ dist (g x y) 0
   using assms(1) by (rule eventually-mono) (force simp add: dist-norm)
qed

lemma uniform-limit-on-union:
 uniform-limit I f g F ⟹ uniform-limit J f g F ⟹ uniform-limit (I ∪ J) f g F
 by (auto intro!: uniform-limitI dest!: uniform-limitD elim: eventually-elim2)

lemma uniform-limit-on-empty [iff]:
 uniform-limit {} f g F
 by (auto intro!: uniform-limitI)

lemma uniform-limit-on-UNION:
 assumes finite S
 assumes ⋀s. s ∈ S ⟹ uniform-limit (h s) f g F
 shows uniform-limit (UNION S h) f g F
 using assms
 by induct (auto intro: uniform-limit-on-empty uniform-limit-on-union)

lemma uniform-limit-on-Union:
 assumes finite I
 assumes ⋀J. J ∈ I ⟹ uniform-limit J f g F
 shows uniform-limit (Union I) f g F
 by (metis SUP-identity-eq assms uniform-limit-on-UNION)
```

lemma *uniform-limit-on-subset*:
uniform-limit J f g F $\Longrightarrow$ *I* $\subseteq$ *J* $\Longrightarrow$ *uniform-limit I f g F*
by (*auto intro*!: *uniform-limitI dest*!: *uniform-limitD intro*: *eventually-mono*)

lemma *uniformly-convergent-add*:
uniformly-convergent-on A f $\Longrightarrow$ *uniformly-convergent-on A g* $\Longrightarrow$
uniformly-convergent-on A ($\lambda k\ x.\ f\ k\ x + g\ k\ x :: 'a :: \{$*real-normed-algebra*$\}$)
unfolding *uniformly-convergent-on-def* **by** (*blast dest*: *uniform-limit-add*)

lemma *uniformly-convergent-minus*:
uniformly-convergent-on A f $\Longrightarrow$ *uniformly-convergent-on A g* $\Longrightarrow$
uniformly-convergent-on A ($\lambda k\ x.\ f\ k\ x - g\ k\ x :: 'a :: \{$*real-normed-algebra*$\}$)
unfolding *uniformly-convergent-on-def* **by** (*blast dest*: *uniform-limit-minus*)

lemma *uniformly-convergent-mult*:
uniformly-convergent-on A f $\Longrightarrow$
uniformly-convergent-on A ($\lambda k\ x.\ c * f\ k\ x :: 'a :: \{$*real-normed-algebra*$\}$)
unfolding *uniformly-convergent-on-def*
by (*blast dest*: *bounded-linear-uniform-limit-intros*(*13*))

38.1 Power series and uniform convergence

proposition *powser-uniformly-convergent*:
fixes *a* :: *nat* $\Rightarrow$ *'a*::{*real-normed-div-algebra*,*banach*}
assumes *r* < *conv-radius a*
shows *uniformly-convergent-on* (*cball* ξ *r*) ($\lambda n\ x.\ \sum i<n.\ a\ i * (x - \xi)\ \hat{}\ i$)
proof (*cases* 0 $\leq$ *r*)
case *True*
then have $*$: *summable* ($\lambda n.$ *norm* (*a n*) $*$ *r* $\hat{}$ *n*)
using *abs-summable-in-conv-radius* [*of of-real r a*] *assms*
by (*simp add*: *norm-mult norm-power*)
show *?thesis*
by (*simp add*: *weierstrass-m-test'-ev* [*OF* - $*$] *norm-mult norm-power*
mult-left-mono power-mono dist-norm norm-minus-commute)
next
case *False* **then show** *?thesis* **by** (*simp add*: *not-le*)
qed

lemma *powser-uniform-limit*:
fixes *a* :: *nat* $\Rightarrow$ *'a*::{*real-normed-div-algebra*,*banach*}
assumes *r* < *conv-radius a*
shows *uniform-limit* (*cball* ξ *r*) ($\lambda n\ x.\ \sum i<n.\ a\ i * (x - \xi)\ \hat{}\ i$) ($\lambda x.$ *suminf*
($\lambda i.\ a\ i * (x - \xi)\ \hat{}\ i$)) *sequentially*
using *powser-uniformly-convergent* [*OF assms*]
by (*simp add*: *Uniform-Limit.uniformly-convergent-uniform-limit-iff Series.suminf-eq-lim*)

lemma *powser-continuous-suminf*:
fixes *a* :: *nat* $\Rightarrow$ *'a*::{*real-normed-div-algebra*,*banach*}

assumes $r < \mathit{conv\text{-}radius}\ a$
shows $\mathit{continuous\text{-}on}\ (\mathit{cball}\ \xi\ r)\ (\lambda x.\ \mathit{suminf}\ (\lambda i.\ a\ i * (x - \xi)\ \hat{}\ i))$
apply (*rule uniform-limit-theorem* [*OF - powser-uniform-limit*])
apply (*rule eventuallyI continuous-intros assms*)+
apply (*simp add:*)
done

lemma *powser-continuous-sums*:
fixes $a :: \mathit{nat} \Rightarrow {'a}::\{\mathit{real\text{-}normed\text{-}div\text{-}algebra}, \mathit{banach}\}$
assumes *r*: $r < \mathit{conv\text{-}radius}\ a$
and *sm*: $\bigwedge x.\ x \in \mathit{cball}\ \xi\ r \Longrightarrow (\lambda n.\ a\ n * (x - \xi)\ \hat{}\ n)\ \mathit{sums}\ (f\ x)$
shows $\mathit{continuous\text{-}on}\ (\mathit{cball}\ \xi\ r)\ f$
apply (*rule continuous-on-cong* [*THEN iffD1*, *OF refl - powser-continuous-suminf* [*OF r*]])
using *sm sums-unique* **by** *fastforce*

end

39 Bounded Linear Function

theory *Bounded-Linear-Function*
imports
Topology-Euclidean-Space
Operator-Norm
begin

39.1 Intro rules for *bounded-linear*

named-theorems *bounded-linear-intros*

lemma *onorm-inner-left*:
assumes *bounded-linear r*
shows $\mathit{onorm}\ (\lambda x.\ r\ x \cdot f) \leq \mathit{onorm}\ r * \mathit{norm}\ f$
proof (*rule onorm-bound*)
fix *x*
have $\mathit{norm}\ (r\ x \cdot f) \leq \mathit{norm}\ (r\ x) * \mathit{norm}\ f$
by (*simp add*: *Cauchy-Schwarz-ineq2*)
also have $\ldots \leq \mathit{onorm}\ r * \mathit{norm}\ x * \mathit{norm}\ f$
by (*intro mult-right-mono onorm assms norm-ge-zero*)
finally show $\mathit{norm}\ (r\ x \cdot f) \leq \mathit{onorm}\ r * \mathit{norm}\ f * \mathit{norm}\ x$
by (*simp add*: *ac-simps*)
qed (*intro mult-nonneg-nonneg norm-ge-zero onorm-pos-le assms*)

lemma *onorm-inner-right*:
assumes *bounded-linear r*
shows $\mathit{onorm}\ (\lambda x.\ f \cdot r\ x) \leq \mathit{norm}\ f * \mathit{onorm}\ r$
apply (*subst inner-commute*)
apply (*rule onorm-inner-left*[*OF assms*, *THEN order-trans*])
apply *simp*

done

lemmas [*bounded-linear-intros*] =
bounded-linear-zero
bounded-linear-add
bounded-linear-const-mult
bounded-linear-mult-const
bounded-linear-scaleR-const
bounded-linear-const-scaleR
bounded-linear-ident
bounded-linear-setsum
bounded-linear-Pair
bounded-linear-sub
bounded-linear-fst-comp
bounded-linear-snd-comp
bounded-linear-inner-left-comp
bounded-linear-inner-right-comp

39.2 declaration of derivative/continuous/tendsto introduction rules for bounded linear functions

attribute-setup *bounded-linear* =
‹*Scan.succeed* (*Thm.declaration-attribute* (*fn thm* =>
fold (*fn* (*r*, *s*) => *Named-Theorems.add-thm s* (*thm RS r*))
[
(@{*thm bounded-linear.has-derivative*}, @{*named-theorems derivative-intros*}),
(@{*thm bounded-linear.tendsto*}, @{*named-theorems tendsto-intros*}),
(@{*thm bounded-linear.continuous*}, @{*named-theorems continuous-intros*}),
(@{*thm bounded-linear.continuous-on*}, @{*named-theorems continuous-intros*}),
(@{*thm bounded-linear.uniformly-continuous-on*}, @{*named-theorems continuous-intros*}),
(@{*thm bounded-linear-compose*}, @{*named-theorems bounded-linear-intros*})
]))›

attribute-setup *bounded-bilinear* =
‹*Scan.succeed* (*Thm.declaration-attribute* (*fn thm* =>
fold (*fn* (*r*, *s*) => *Named-Theorems.add-thm s* (*thm RS r*))
[
(@{*thm bounded-bilinear.FDERIV*}, @{*named-theorems derivative-intros*}),
(@{*thm bounded-bilinear.tendsto*}, @{*named-theorems tendsto-intros*}),
(@{*thm bounded-bilinear.continuous*}, @{*named-theorems continuous-intros*}),
(@{*thm bounded-bilinear.continuous-on*}, @{*named-theorems continuous-intros*}),
(@{*thm bounded-linear-compose*[*OF bounded-bilinear.bounded-linear-left*]},
@{*named-theorems bounded-linear-intros*}),
(@{*thm bounded-linear-compose*[*OF bounded-bilinear.bounded-linear-right*]},
@{*named-theorems bounded-linear-intros*}),
(@{*thm bounded-linear.uniformly-continuous-on*[*OF bounded-bilinear.bounded-linear-left*]},
@{*named-theorems continuous-intros*}),
(@{*thm bounded-linear.uniformly-continuous-on*[*OF bounded-bilinear.bounded-linear-right*]},
@{*named-theorems continuous-intros*})

]))⟩

39.3 type of bounded linear functions

typedef (**overloaded**) ('*a*, '*b*) *blinfun* ((- ⇒$_L$ /-) [*22*, *21*] *21*) =
{*f*::'*a*::*real-normed-vector*⇒'*b*::*real-normed-vector*. *bounded-linear f*}
morphisms *blinfun-apply Blinfun*
by (*blast intro*: *bounded-linear-intros*)

declare [[*coercion*
blinfun-apply :: ('*a*::*real-normed-vector* ⇒$_L$'*b*::*real-normed-vector*) ⇒ '*a* ⇒ '*b*]]

lemma *bounded-linear-blinfun-apply*[*bounded-linear-intros*]:
bounded-linear g ⟹ *bounded-linear* (λ*x*. *blinfun-apply f* (*g x*))
by (*metis blinfun-apply mem-Collect-eq bounded-linear-compose*)

setup-lifting *type-definition-blinfun*

lemma *blinfun-eqI*: (⋀*i*. *blinfun-apply x i* = *blinfun-apply y i*) ⟹ *x* = *y*
by *transfer auto*

lemma *bounded-linear-Blinfun-apply*: *bounded-linear f* ⟹ *blinfun-apply* (*Blinfun f*) = *f*
by (*auto simp*: *Blinfun-inverse*)

39.4 type class instantiations

instantiation *blinfun* :: (*real-normed-vector*, *real-normed-vector*) *real-normed-vector*
begin

lift-definition *norm-blinfun* :: '*a* ⇒$_L$ '*b* ⇒ *real* **is** *onorm* .

lift-definition *minus-blinfun* :: '*a* ⇒$_L$ '*b* ⇒ '*a* ⇒$_L$ '*b* ⇒ '*a* ⇒$_L$ '*b*
is λ*f g x*. *f x* − *g x*
by (*rule bounded-linear-sub*)

definition *dist-blinfun* :: '*a* ⇒$_L$ '*b* ⇒ '*a* ⇒$_L$ '*b* ⇒ *real*
where *dist-blinfun a b* = *norm* (*a* − *b*)

definition [*code del*]:
(*uniformity* :: (('*a* ⇒$_L$ '*b*) × ('*a* ⇒$_L$ '*b*)) *filter*) = (*INF e*:{*0* <..}. *principal* {(*x*, *y*). *dist x y* < *e*})

definition *open-blinfun* :: ('*a* ⇒$_L$ '*b*) *set* ⇒ *bool*
where [*code del*]: *open-blinfun S* = (∀ *x*∈*S*. ∀$_F$ (*x'*, *y*) *in uniformity*. *x'* = *x* ⟶ *y* ∈ *S*)

lift-definition *uminus-blinfun* :: '*a* ⇒$_L$ '*b* ⇒ '*a* ⇒$_L$ '*b* **is** λ*f x*. − *f x*
by (*rule bounded-linear-minus*)

lift-definition *zero-blinfun* :: *'a* $\Rightarrow_L$ *'b* **is** $\lambda x.\ 0$
by (*rule bounded-linear-zero*)

lift-definition *plus-blinfun* :: *'a* $\Rightarrow_L$ *'b* $\Rightarrow$ *'a* $\Rightarrow_L$ *'b* $\Rightarrow$ *'a* $\Rightarrow_L$ *'b*
is $\lambda f\ g\ x.\ f\ x + g\ x$
by (*metis bounded-linear-add*)

lift-definition *scaleR-blinfun*::*real* $\Rightarrow$ *'a* $\Rightarrow_L$ *'b* $\Rightarrow$ *'a* $\Rightarrow_L$ *'b* **is** $\lambda r\ f\ x.\ r *_R f\ x$
by (*metis bounded-linear-compose bounded-linear-scaleR-right*)

definition *sgn-blinfun* :: *'a* $\Rightarrow_L$ *'b* $\Rightarrow$ *'a* $\Rightarrow_L$ *'b*
where *sgn-blinfun x* = *scaleR* (*inverse* (*norm x*)) *x*

instance
apply *standard*
unfolding *dist-blinfun-def open-blinfun-def sgn-blinfun-def uniformity-blinfun-def*
apply (*rule refl* | (*transfer*, *force simp*: *onorm-triangle onorm-scaleR onorm-eq-0 algebra-simps*))+
done

end

declare *uniformity-Abort*[**where** *'a*=(*'a* :: *real-normed-vector*) $\Rightarrow_L$ (*'b* :: *real-normed-vector*), *code*]

lemma *norm-blinfun-eqI*:
assumes *n* $\leq$ *norm* (*blinfun-apply f x*) / *norm x*
assumes $\bigwedge x.$ *norm* (*blinfun-apply f x*) $\leq$ *n* $*$ *norm x*
assumes *0* $\leq$ *n*
shows *norm f* = *n*
by (*auto simp*: *norm-blinfun-def*
intro!: *antisym onorm-bound assms order-trans*[*OF - le-onorm*]
bounded-linear-intros)

lemma *norm-blinfun*: *norm* (*blinfun-apply f x*) $\leq$ *norm f* $*$ *norm x*
by *transfer* (*rule onorm*)

lemma *norm-blinfun-bound*: *0* $\leq$ *b* $\Longrightarrow$ ($\bigwedge x.$ *norm* (*blinfun-apply f x*) $\leq$ *b* $*$ *norm x*) $\Longrightarrow$ *norm f* $\leq$ *b*
by *transfer* (*rule onorm-bound*)

lemma *bounded-bilinear-blinfun-apply*[*bounded-bilinear*]: *bounded-bilinear blinfun-apply*
proof
fix *f g*::*'a* $\Rightarrow_L$ *'b* **and** *a b*::*'a* **and** *r*::*real*
show (*f* + *g*) *a* = *f a* + *g a* (*r* $*_R$ *f*) *a* = *r* $*_R$ *f a*
by (*transfer*, *simp*)+
interpret *bounded-linear f* **for** *f*::*'a* $\Rightarrow_L$ *'b*
by (*auto intro*!: *bounded-linear-intros*)
show *f* (*a* + *b*) = *f a* + *f b f* (*r* $*_R$ *a*) = *r* $*_R$ *f a*

by (*simp-all add*: *add scaleR*)
show $\exists K.\ \forall a\ b.\ norm\ (blinfun\text{-}apply\ a\ b) \leq norm\ a * norm\ b * K$
by (*auto intro*!: *exI*[**where** *x*=*1*] *norm-blinfun*)
qed

interpretation *blinfun*: *bounded-bilinear blinfun-apply*
by (*rule bounded-bilinear-blinfun-apply*)

lemmas *bounded-linear-apply-blinfun*[*intro*, *simp*] = *blinfun.bounded-linear-left*

context *bounded-bilinear*
begin

named-theorems *bilinear-simps*

lemmas [*bilinear-simps*] =
add-left
add-right
diff-left
diff-right
minus-left
minus-right
scaleR-left
scaleR-right
zero-left
zero-right
setsum-left
setsum-right

end

instance *blinfun* :: (*banach*, *banach*) *banach*
proof
fix *X*::*nat* ⇒ *'a* ⇒$_L$ *'b*
assume *Cauchy X*
{
fix *x*::*'a*
{
fix *x*::*'a*
assume *norm x* ≤ *1*
have *Cauchy* (λ*n*. *X n x*)
proof (*rule CauchyI*)
fix *e*::*real*
assume *0* < *e*
from *CauchyD*[*OF* ‹*Cauchy X*› ‹*0* < *e*›] **obtain** *M*
where *M*: ⋀*m n*. *m* ≥ *M* ⟹ *n* ≥ *M* ⟹ *norm* (*X m* − *X n*) < *e*
by *auto*

```
    show ∃ M. ∀ m≥M. ∀ n≥M. norm (X m x − X n x) < e
    proof (safe intro!: exI[where x=M])
      fix m n
      assume le: M ≤ m M ≤ n
      have norm (X m x − X n x) = norm ((X m − X n) x)
        by (simp add: blinfun.bilinear-simps)
      also have ... ≤ norm (X m − X n) * norm x
        by (rule norm-blinfun)
      also have ... ≤ norm (X m − X n) * 1
        using ‹norm x ≤ 1› norm-ge-zero by (rule mult-left-mono)
      also have ... = norm (X m − X n) by simp
      also have ... < e using le by fact
      finally show norm (X m x − X n x) < e .
    qed
  qed
  hence convergent (λn. X n x)
    by (metis Cauchy-convergent-iff)
 } note convergent-norm1 = this
 def y ≡ x /R norm x
 have y: norm y ≤ 1 and xy: x = norm x *R y
  by (simp-all add: y-def inverse-eq-divide)
 have convergent (λn. norm x *R X n y)
 by (intro bounded-bilinear.convergent[OF bounded-bilinear-scaleR] convergent-const
    convergent-norm1 y)
 also have (λn. norm x *R X n y) = (λn. X n x)
  by (subst xy) (simp add: blinfun.bilinear-simps)
 finally have convergent (λn. X n x) .
}
then obtain v where v: ⋀x. (λn. X n x) ⟶ v x
 unfolding convergent-def
 by metis

have Cauchy (λn. norm (X n))
proof (rule CauchyI)
 fix e::real
 assume e > 0
 from CauchyD[OF ‹Cauchy X› ‹0 < e›] obtain M
  where M: ⋀m n. m ≥ M ⟹ n ≥ M ⟹ norm (X m − X n) < e
  by auto
 show ∃ M. ∀ m≥M. ∀ n≥M. norm (norm (X m) − norm (X n)) < e
 proof (safe intro!: exI[where x=M])
  fix m n assume mn: m ≥ M n ≥ M
  have norm (norm (X m) − norm (X n)) ≤ norm (X m − X n)
   by (metis norm-triangle-ineq3 real-norm-def)
  also have ... < e using mn by fact
  finally show norm (norm (X m) − norm (X n)) < e .
 qed
qed
then obtain K where K: (λn. norm (X n)) ⟶ K
```

```
    unfolding Cauchy-convergent-iff convergent-def
    by metis

  have bounded-linear v
  proof
    fix x y and r::real
   from tendsto-add[OF v[of x] v [of y]] v[of x + y, unfolded blinfun.bilinear-simps]
        tendsto-scaleR[OF tendsto-const[of r] v[of x]] v[of r *R x, unfolded blin-
fun.bilinear-simps]
    show v (x + y) = v x + v y v (r *R x) = r *R v x
      by (metis (poly-guards-query) LIMSEQ-unique)+
    show ∃ K. ∀ x. norm (v x) ≤ norm x * K
    proof (safe intro!: exI[where x=K])
      fix x
      have norm (v x) ≤ K * norm x
       by (rule tendsto-le[OF - tendsto-mult[OF K tendsto-const] tendsto-norm[OF
v]])
          (auto simp: norm-blinfun)
      thus norm (v x) ≤ norm x * K
        by (simp add: ac-simps)
    qed
  qed
  hence Bv: ⋀x. (λn. X n x) ⟶ Blinfun v x
    by (auto simp: bounded-linear-Blinfun-apply v)

  have X ⟶ Blinfun v
  proof (rule LIMSEQ-I)
    fix r::real assume r > 0
    def r' ≡ r / 2
    have 0 < r' r' < r using ‹r > 0› by (simp-all add: r'-def)
    from CauchyD[OF ‹Cauchy X› ‹r' > 0›]
    obtain M where M: ⋀m n. m ≥ M ⟹ n ≥ M ⟹ norm (X m − X n) <
r'
      by metis
    show ∃ no. ∀ n≥no. norm (X n − Blinfun v) < r
    proof (safe intro!: exI[where x=M])
      fix n assume n: M ≤ n
      have norm (X n − Blinfun v) ≤ r'
      proof (rule norm-blinfun-bound)
        fix x
        have eventually (λm. m ≥ M) sequentially
          by (metis eventually-ge-at-top)
          hence ev-le: eventually (λm. norm (X n x − X m x) ≤ r' * norm x)
sequentially
        proof eventually-elim
          case (elim m)
          have norm (X n x − X m x) = norm ((X n − X m) x)
            by (simp add: blinfun.bilinear-simps)
          also have ... ≤ norm ((X n − X m)) * norm x
```

```
        by (rule norm-blinfun)
      also have ... ≤ r' * norm x
        using M[OF n elim] by (simp add: mult-right-mono)
      finally show ?case .
    qed
    have tendsto-v: (λm. norm (X n x − X m x)) ⟶ norm (X n x − Blinfun v x)
      by (auto intro!: tendsto-intros Bv)
    show norm ((X n − Blinfun v) x) ≤ r' * norm x
    by (auto intro!: tendsto-ge-const tendsto-v ev-le simp: blinfun.bilinear-simps)
  qed (simp add: ‹0 < r'› less-imp-le)
  thus norm (X n − Blinfun v) < r
    by (metis ‹r' < r› le-less-trans)
 qed
qed
thus convergent X
 by (rule convergentI)
qed
```

39.5 On Euclidean Space

```
lemma Zfun-setsum:
 assumes finite s
 assumes f: ⋀i. i ∈ s ⟹ Zfun (f i) F
 shows Zfun (λx. setsum (λi. f i x) s) F
 using assms by induct (auto intro!: Zfun-zero Zfun-add)

lemma norm-blinfun-euclidean-le:
 fixes a::'a::euclidean-space ⇒_L 'b::real-normed-vector
 shows norm a ≤ setsum (λx. norm (a x)) Basis
 apply (rule norm-blinfun-bound)
  apply (simp add: setsum-nonneg)
 apply (subst euclidean-representation[symmetric, where 'a='a])
 apply (simp only: blinfun.bilinear-simps setsum-left-distrib)
 apply (rule order.trans[OF norm-setsum setsum-mono])
 apply (simp add: abs-mult mult-right-mono ac-simps Basis-le-norm)
 done

lemma tendsto-componentwise1:
 fixes a::'a::euclidean-space ⇒_L 'b::real-normed-vector
   and b::'c ⇒ 'a ⇒_L 'b
 assumes (⋀j. j ∈ Basis ⟹ ((λn. b n j) ⟶ a j) F)
 shows (b ⟶ a) F
proof −
 have ⋀j. j ∈ Basis ⟹ Zfun (λx. norm (b x j − a j)) F
   using assms unfolding tendsto-Zfun-iff Zfun-norm-iff .
 hence Zfun (λx. ∑j∈Basis. norm (b x j − a j)) F
   by (auto intro!: Zfun-setsum)
 thus ?thesis
```

unfolding *tendsto-Zfun-iff*
by (*rule Zfun-le*)
(*auto intro*!: *order-trans*[*OF norm-blinfun-euclidean-le*] *simp*: *blinfun.bilinear-simps*)
qed

lift-definition
blinfun-of-matrix::('*b*::*euclidean-space* ⇒ '*a*::*euclidean-space* ⇒ *real*) ⇒ '*a* ⇒$_L$ '*b*
is λ*a x*. ∑*i*∈*Basis*. ∑*j*∈*Basis*. ((*x* · *j*) ∗ *a i j*) ∗$_R$ *i*
by (*intro bounded-linear-intros*)

lemma *blinfun-of-matrix-works*:
fixes *f*::'*a*::*euclidean-space* ⇒$_L$ '*b*::*euclidean-space*
shows *blinfun-of-matrix* (λ*i j*. (*f j*) · *i*) = *f*
proof (*transfer*, *rule*, *rule euclidean-eqI*)
fix *f*::'*a* ⇒ '*b* **and** *x*::'*a* **and** *b*::'*b* **assume** *bounded-linear f* **and** *b*: *b* ∈ *Basis*
then interpret *bounded-linear f* **by** *simp*
have (∑*j*∈*Basis*. ∑*i*∈*Basis*. (*x* · *i* ∗ (*f i* · *j*)) ∗$_R$ *j*) · *b*
= (∑*j*∈*Basis*. *if j* = *b then* (∑*i*∈*Basis*. (*x* · *i* ∗ (*f i* · *j*))) *else 0*)
using *b*
by (*auto simp add*: *algebra-simps inner-setsum-left inner-Basis split*: *if-split intro*!: *setsum.cong*)
also have ... = (∑*i*∈*Basis*. (*x* · *i* ∗ (*f i* · *b*)))
using *b* **by** (*simp add*: *setsum.delta*)
also have ... = *f x* · *b*
by (*subst linear-componentwise*[*symmetric*]) (*unfold-locales*, *rule*)
finally show (∑*j*∈*Basis*. ∑*i*∈*Basis*. (*x* · *i* ∗ (*f i* · *j*)) ∗$_R$ *j*) · *b* = *f x* · *b* .
qed

lemma *blinfun-of-matrix-apply*:
blinfun-of-matrix a x = (∑*i*∈*Basis*. ∑*j*∈*Basis*. ((*x* · *j*) ∗ *a i j*) ∗$_R$ *i*)
by *transfer simp*

lemma *blinfun-of-matrix-minus*: *blinfun-of-matrix x* − *blinfun-of-matrix y* = *blinfun-of-matrix* (*x* − *y*)
by *transfer* (*auto simp*: *algebra-simps setsum-subtractf*)

lemma *norm-blinfun-of-matrix*:
norm (*blinfun-of-matrix a*) ≤ (∑*i*∈*Basis*. ∑*j*∈*Basis*. |*a i j*|)
apply (*rule norm-blinfun-bound*)
apply (*simp add*: *setsum-nonneg*)
apply (*simp only*: *blinfun-of-matrix-apply setsum-left-distrib*)
apply (*rule order-trans*[*OF norm-setsum setsum-mono*])
apply (*rule order-trans*[*OF norm-setsum setsum-mono*])
apply (*simp add*: *abs-mult mult-right-mono ac-simps Basis-le-norm*)
done

lemma *tendsto-blinfun-of-matrix*:
assumes ⋀*i j*. *i* ∈ *Basis* ⟹ *j* ∈ *Basis* ⟹ (λ*n*. *b n i j*) ⟶ *a i j*) *F*
shows (λ*n*. *blinfun-of-matrix* (*b n*)) ⟶ *blinfun-of-matrix a*) *F*

proof −
have $\bigwedge i\ j.\ i \in Basis \Longrightarrow j \in Basis \Longrightarrow Zfun\ (\lambda x.\ norm\ (b\ x\ i\ j - a\ i\ j))\ F$
using *assms* **unfolding** *tendsto-Zfun-iff Zfun-norm-iff* **.**
hence $Zfun\ (\lambda x.\ (\sum i{\in}Basis.\ \sum j{\in}Basis.\ |b\ x\ i\ j - a\ i\ j|))\ F$
by (*auto intro*!: *Zfun-setsum*)
thus *?thesis*
unfolding *tendsto-Zfun-iff blinfun-of-matrix-minus*
by (*rule Zfun-le*) (*auto intro*!: *order-trans*[*OF norm-blinfun-of-matrix*])
qed

lemma *tendsto-componentwise*:
fixes $a{::}'a{::}euclidean\text{-}space \Rightarrow_L\ 'b{::}euclidean\text{-}space$
and $b{::}'c \Rightarrow\ 'a \Rightarrow_L\ 'b$
shows $(\bigwedge i\ j.\ i \in Basis \Longrightarrow j \in Basis \Longrightarrow ((\lambda n.\ b\ n\ j \cdot i) \longrightarrow a\ j \cdot i)\ F) \Longrightarrow (b \longrightarrow a)\ F$
apply (*subst blinfun-of-matrix-works*[*of a*, *symmetric*])
apply (*subst blinfun-of-matrix-works*[*of b x* **for** *x*, *symmetric*, *abs-def*])
by (*rule tendsto-blinfun-of-matrix*)

lemma
continuous-blinfun-componentwiseI:
fixes $f{::}\ 'b{::}t2\text{-}space \Rightarrow\ 'a{::}euclidean\text{-}space \Rightarrow_L\ 'c{::}euclidean\text{-}space$
assumes $\bigwedge i\ j.\ i \in Basis \Longrightarrow j \in Basis \Longrightarrow continuous\ F\ (\lambda x.\ (f\ x)\ j \cdot i)$
shows *continuous F f*
using *assms* **by** (*auto simp*: *continuous-def intro*!: *tendsto-componentwise*)

lemma
continuous-blinfun-componentwiseI1:
fixes $f{::}\ 'b{::}t2\text{-}space \Rightarrow\ 'a{::}euclidean\text{-}space \Rightarrow_L\ 'c{::}real\text{-}normed\text{-}vector$
assumes $\bigwedge i.\ i \in Basis \Longrightarrow continuous\ F\ (\lambda x.\ f\ x\ i)$
shows *continuous F f*
using *assms* **by** (*auto simp*: *continuous-def intro*!: *tendsto-componentwise1*)

lemma *bounded-linear-blinfun-matrix*: *bounded-linear* $(\lambda x.\ (x{::}\text{-}\Rightarrow_L\ \text{-})\ j \cdot i)$
by (*auto intro*!: *bounded-linearI'* *bounded-linear-intros*)

lemma *continuous-blinfun-matrix*:
fixes $f{::}\ 'b{::}t2\text{-}space \Rightarrow\ 'a{::}real\text{-}normed\text{-}vector \Rightarrow_L\ 'c{::}real\text{-}inner$
assumes *continuous F f*
shows *continuous F* $(\lambda x.\ (f\ x)\ j \cdot i)$
by (*rule bounded-linear.continuous*[*OF bounded-linear-blinfun-matrix assms*])

lemma *continuous-on-blinfun-matrix*:
fixes $f{::}'a{::}t2\text{-}space \Rightarrow\ 'b{::}real\text{-}normed\text{-}vector \Rightarrow_L\ 'c{::}real\text{-}inner$
assumes *continuous-on S f*
shows *continuous-on S* $(\lambda x.\ (f\ x)\ j \cdot i)$
using *assms*
by (*auto simp*: *continuous-on-eq-continuous-within continuous-blinfun-matrix*)

lemma *continuous-on-blinfun-of-matrix*[*continuous-intros*]:
assumes $\bigwedge i\ j.\ i \in Basis \Longrightarrow j \in Basis \Longrightarrow$ *continuous-on* $S\ (\lambda s.\ g\ s\ i\ j)$
shows *continuous-on* $S\ (\lambda s.$ *blinfun-of-matrix* $(g\ s))$
using *assms*
by (*auto simp*: *continuous-on intro*!: *tendsto-blinfun-of-matrix*)

lemma *mult-if-delta*:
(*if P then* $(1::'a::$*comm-semiring-1*$)$ *else* $0) * q =$ (*if P then q else* 0)
by *auto*

lemma *compact-blinfun-lemma*:
fixes f :: *nat* $\Rightarrow$ $'a::$*euclidean-space* $\Rightarrow_L$ $'b::$*euclidean-space*
assumes *bounded* (*range f*)
shows $\forall\, d \subseteq Basis.\ \exists\, l::'a \Rightarrow_L 'b.\ \exists\ r.$
subseq r $\wedge$ $(\forall\, e>0.$ *eventually* $(\lambda n.\ \forall\, i \in d.$ *dist* $(f\ (r\ n)\ i)\ (l\ i) < e)$ *sequentially*)
by (*rule compact-lemma-general*[**where** *unproj* $= \lambda e.$ *blinfun-of-matrix* $(\lambda i\ j.\ e\ j \cdot i)$])
(*auto intro*!: *euclidean-eqI*[**where** $'a='b$] *bounded-linear-image assms*
simp: *blinfun-of-matrix-works blinfun-of-matrix-apply inner-Basis mult-if-delta setsum.delta'*
scaleR-setsum-left[*symmetric*])

lemma *blinfun-euclidean-eqI*: $(\bigwedge i.\ i \in Basis \Longrightarrow$ *blinfun-apply* $x\ i =$ *blinfun-apply* $y\ i) \Longrightarrow x = y$
apply (*auto intro*!: *blinfun-eqI*)
apply (*subst* (*2*) *euclidean-representation*[*symmetric*, **where** $'a='a$])
apply (*subst* (*1*) *euclidean-representation*[*symmetric*, **where** $'a='a$])
apply (*simp add*: *blinfun.bilinear-simps*)
done

lemma *Blinfun-eq-matrix*: *bounded-linear f* $\Longrightarrow$ *Blinfun f* $=$ *blinfun-of-matrix* $(\lambda i\ j.\ f\ j \cdot i)$
by (*intro blinfun-euclidean-eqI*)
(*auto simp*: *blinfun-of-matrix-apply bounded-linear-Blinfun-apply inner-Basis if-distrib*
cond-application-beta setsum.delta' euclidean-representation
cong: *if-cong*)

TODO: generalize (via *compact* (*cball ?x ?e*))?

instance *blinfun* :: (*euclidean-space*, *euclidean-space*) *heine-borel*
proof
fix f :: *nat* $\Rightarrow$ $'a \Rightarrow_L 'b$
assume f: *bounded* (*range f*)
then obtain $l::'a \Rightarrow_L 'b$ **and** r **where** r: *subseq r*
and l: $\forall\, e>0.$ *eventually* $(\lambda n.\ \forall\, i \in Basis.$ *dist* $(f\ (r\ n)\ i)\ (l\ i) < e)$ *sequentially*
using *compact-blinfun-lemma* [*OF f*] **by** *blast*
{
fix $e::$*real*
let *?d* $=$ *real-of-nat DIM*$('a) *$ *real-of-nat DIM*$('b)$

```
    assume e > 0
    hence e / ?d > 0 by (simp add: DIM-positive)
     with l have eventually (λn. ∀i∈Basis. dist (f (r n) i) (l i) < e / ?d)
sequentially
     by simp
    moreover
    {
     fix n
     assume n: ∀i∈Basis. dist (f (r n) i) (l i) < e / ?d
     have norm (f (r n) − l) = norm (blinfun-of-matrix (λi j. (f (r n) − l) j ·
i))
       unfolding blinfun-of-matrix-works ..
     also note norm-blinfun-of-matrix
     also have (∑i∈Basis. ∑j∈Basis. |(f (r n) − l) j · i|) <
      (∑i∈(Basis::'b set). e / real-of-nat DIM('b))
     proof (rule setsum-strict-mono)
      fix i::'b assume i: i ∈ Basis
      have (∑j::'a∈Basis. |(f (r n) − l) j · i|) < (∑j::'a∈Basis. e / ?d)
      proof (rule setsum-strict-mono)
       fix j::'a assume j: j ∈ Basis
       have |(f (r n) − l) j · i| ≤ norm ((f (r n) − l) j)
        by (simp add: Basis-le-norm i)
       also have ... < e / ?d
        using n i j by (auto simp: dist-norm blinfun.bilinear-simps)
       finally show |(f (r n) − l) j · i| < e / ?d by simp
      qed simp-all
      also have ... ≤ e / real-of-nat DIM('b)
       by simp
      finally show (∑j∈Basis. |(f (r n) − l) j · i|) < e / real-of-nat DIM('b)
       by simp
     qed simp-all
     also have ... ≤ e by simp
     finally have dist (f (r n)) l < e
      by (auto simp: dist-norm)
    }
    ultimately have eventually (λn. dist (f (r n)) l < e) sequentially
     using eventually-elim2 by force
  }
  then have ∗: ((f ∘ r) ⟶ l) sequentially
   unfolding o-def tendsto-iff by simp
  with r show ∃l r. subseq r ∧ ((f ∘ r) ⟶ l) sequentially
   by auto
qed
```

39.6 concrete bounded linear functions

```
lemma transfer-bounded-bilinear-bounded-linearI:
  assumes g = (λi x. (blinfun-apply (f i) x))
  shows bounded-bilinear g = bounded-linear f
```

proof
assume *bounded-bilinear g*
then interpret *bounded-bilinear f* **by** (*simp add*: *assms*)
show *bounded-linear f*
proof (*unfold-locales*, *safe intro*!: *blinfun-eqI*)
fix *i*
show $f\ (x + y)\ i = (f\ x + f\ y)\ i\ f\ (r *_R x)\ i = (r *_R f\ x)\ i$ **for** *r x y*
by (*auto intro*!: *blinfun-eqI simp*: *blinfun.bilinear-simps*)
from - *nonneg-bounded* **show** $\exists K.\ \forall x.\ norm\ (f\ x) \leq norm\ x * K$
by (*rule ex-reg*) (*auto intro*!: *onorm-bound simp*: *norm-blinfun.rep-eq ac-simps*)
qed
qed (*auto simp*: *assms intro*!: *blinfun.comp*)

lemma *transfer-bounded-bilinear-bounded-linear*[*transfer-rule*]:
(*rel-fun* (*rel-fun op* = (*pcr-blinfun op* = *op* =)) *op* =) *bounded-bilinear bounded-linear*
by (*auto simp*: *pcr-blinfun-def cr-blinfun-def rel-fun-def OO-def*
intro!: *transfer-bounded-bilinear-bounded-linearI*)

context *bounded-bilinear*
begin

lift-definition *prod-left*::$'b \Rightarrow\ 'a \Rightarrow_L\ 'c$ **is** $(\lambda b\ a.\ prod\ a\ b)$
by (*rule bounded-linear-left*)
declare *prod-left.rep-eq*[*simp*]

lemma *bounded-linear-prod-left*[*bounded-linear*]: *bounded-linear prod-left*
by *transfer* (*rule flip*)

lift-definition *prod-right*::$'a \Rightarrow\ 'b \Rightarrow_L\ 'c$ **is** $(\lambda a\ b.\ prod\ a\ b)$
by (*rule bounded-linear-right*)
declare *prod-right.rep-eq*[*simp*]

lemma *bounded-linear-prod-right*[*bounded-linear*]: *bounded-linear prod-right*
by *transfer* (*rule bounded-bilinear-axioms*)

end

lift-definition *id-blinfun*::$'a$::*real-normed-vector* $\Rightarrow_L\ 'a$ **is** $\lambda x.\ x$
by (*rule bounded-linear-ident*)

lemmas *blinfun-apply-id-blinfun*[*simp*] = *id-blinfun.rep-eq*

lemma *norm-blinfun-id*[*simp*]:
norm (*id-blinfun*::$'a$::{*real-normed-vector*, *perfect-space*} $\Rightarrow_L\ 'a$) = *1*
by *transfer* (*auto simp*: *onorm-id*)

lemma *norm-blinfun-id-le*:
norm (*id-blinfun*::$'a$::*real-normed-vector* $\Rightarrow_L\ 'a$) $\leq$ *1*
by *transfer* (*auto simp*: *onorm-id-le*)

lift-definition *fst-blinfun*::(*'a*::*real-normed-vector* × *'b*::*real-normed-vector*) ⇒$_L$ *'a*
is *fst*
by (*rule bounded-linear-fst*)

lemma *blinfun-apply-fst-blinfun*[*simp*]: *blinfun-apply fst-blinfun* = *fst*
by *transfer* (*rule refl*)

lift-definition *snd-blinfun*::(*'a*::*real-normed-vector* × *'b*::*real-normed-vector*) ⇒$_L$
'b **is** *snd*
by (*rule bounded-linear-snd*)

lemma *blinfun-apply-snd-blinfun*[*simp*]: *blinfun-apply snd-blinfun* = *snd*
by *transfer* (*rule refl*)

lift-definition *blinfun-compose*::
'a::*real-normed-vector* ⇒$_L$ *'b*::*real-normed-vector* ⇒
'c::*real-normed-vector* ⇒$_L$ *'a* ⇒
'c ⇒$_L$ *'b* (**infixl** o_L 55) **is** *op o*
parametric *comp-transfer*
unfolding *o-def*
by (*rule bounded-linear-compose*)

lemma *blinfun-apply-blinfun-compose*[*simp*]: (*a* o_L *b*) *c* = *a* (*b c*)
by (*simp add*: *blinfun-compose.rep-eq*)

lemma *norm-blinfun-compose*:
norm (*f* o_L *g*) ≤ *norm f* ∗ *norm g*
by *transfer* (*rule onorm-compose*)

lemma *bounded-bilinear-blinfun-compose*[*bounded-bilinear*]: *bounded-bilinear op* o_L
by *unfold-locales*
(*auto intro*!: *blinfun-eqI exI*[**where** *x*=*1*] *simp*: *blinfun.bilinear-simps norm-blinfun-compose*)

lemma *blinfun-compose-zero*[*simp*]:
blinfun-compose 0 = (λ-. *0*)
blinfun-compose x 0 = *0*
by (*auto simp*: *blinfun.bilinear-simps intro*!: *blinfun-eqI*)

lift-definition *blinfun-inner-right*::*'a*::*real-inner* ⇒ *'a* ⇒$_L$ *real* **is** *op* ·
by (*rule bounded-linear-inner-right*)
declare *blinfun-inner-right.rep-eq*[*simp*]

lemma *bounded-linear-blinfun-inner-right*[*bounded-linear*]: *bounded-linear blinfun-inner-right*
by *transfer* (*rule bounded-bilinear-inner*)

lift-definition *blinfun-inner-left*::*'a*::*real-inner* ⇒ *'a* ⇒$_L$ *real* **is** λ*x y*. *y* · *x*
  by (*rule bounded-linear-inner-left*)
declare *blinfun-inner-left.rep-eq*[*simp*]

lemma *bounded-linear-blinfun-inner-left*[*bounded-linear*]: *bounded-linear blinfun-inner-left*
  by *transfer* (*rule bounded-bilinear.flip*[*OF bounded-bilinear-inner*])

lift-definition *blinfun-scaleR-right*::*real* ⇒ *'a* ⇒$_L$ *'a*::*real-normed-vector* **is** *op* *$_R$
  by (*rule bounded-linear-scaleR-right*)
declare *blinfun-scaleR-right.rep-eq*[*simp*]

lemma *bounded-linear-blinfun-scaleR-right*[*bounded-linear*]: *bounded-linear blinfun-scaleR-right*
  by *transfer* (*rule bounded-bilinear-scaleR*)

lift-definition *blinfun-scaleR-left*::*'a*::*real-normed-vector* ⇒ *real* ⇒$_L$ *'a* **is** λ*x y*. *y*
$_R$ x
  by (*rule bounded-linear-scaleR-left*)
lemmas [*simp*] = *blinfun-scaleR-left.rep-eq*

lemma *bounded-linear-blinfun-scaleR-left*[*bounded-linear*]: *bounded-linear blinfun-scaleR-left*
  by *transfer* (*rule bounded-bilinear.flip*[*OF bounded-bilinear-scaleR*])

lift-definition *blinfun-mult-right*::*'a* ⇒ *'a* ⇒$_L$ *'a*::*real-normed-algebra* **is** *op* *
  by (*rule bounded-linear-mult-right*)
declare *blinfun-mult-right.rep-eq*[*simp*]

lemma *bounded-linear-blinfun-mult-right*[*bounded-linear*]: *bounded-linear blinfun-mult-right*
  by *transfer* (*rule bounded-bilinear-mult*)

lift-definition *blinfun-mult-left*::*'a*::*real-normed-algebra* ⇒ *'a* ⇒$_L$ *'a* **is** λ*x y*. *y* *
x
  by (*rule bounded-linear-mult-left*)
lemmas [*simp*] = *blinfun-mult-left.rep-eq*

lemma *bounded-linear-blinfun-mult-left*[*bounded-linear*]: *bounded-linear blinfun-mult-left*
  by *transfer* (*rule bounded-bilinear.flip*[*OF bounded-bilinear-mult*])

end

40 Multivariate calculus in Euclidean space

theory *Derivative*
imports *Brouwer-Fixpoint Operator-Norm Uniform-Limit Bounded-Linear-Function*

begin

lemma *onorm-inner-left*:
assumes *bounded-linear r*
shows *onorm* $(\lambda x.\ r\ x \cdot f) \leq$ *onorm r* $*$ *norm f*
proof (*rule onorm-bound*)
fix *x*
have *norm* $(r\ x \cdot f) \leq$ *norm* $(r\ x) *$ *norm f*
by (*simp add*: *Cauchy-Schwarz-ineq2*)
also have $\ldots \leq$ *onorm r* $*$ *norm x* $*$ *norm f*
by (*intro mult-right-mono onorm assms norm-ge-zero*)
finally show *norm* $(r\ x \cdot f) \leq$ *onorm r* $*$ *norm f* $*$ *norm x*
by (*simp add*: *ac-simps*)
qed (*intro mult-nonneg-nonneg norm-ge-zero onorm-pos-le assms*)

lemma *onorm-inner-right*:
assumes *bounded-linear r*
shows *onorm* $(\lambda x.\ f \cdot r\ x) \leq$ *norm f* $*$ *onorm r*
apply (*subst inner-commute*)
apply (*rule onorm-inner-left*[*OF assms*, *THEN order-trans*])
apply *simp*
done

declare *has-derivative-bounded-linear*[*dest*]

40.1 Derivatives

40.1.1 Combining theorems.

lemmas *has-derivative-id* = *has-derivative-ident*
lemmas *has-derivative-neg* = *has-derivative-minus*
lemmas *has-derivative-sub* = *has-derivative-diff*
lemmas *scaleR-right-has-derivative* = *has-derivative-scaleR-right*
lemmas *scaleR-left-has-derivative* = *has-derivative-scaleR-left*
lemmas *inner-right-has-derivative* = *has-derivative-inner-right*
lemmas *inner-left-has-derivative* = *has-derivative-inner-left*
lemmas *mult-right-has-derivative* = *has-derivative-mult-right*
lemmas *mult-left-has-derivative* = *has-derivative-mult-left*

lemma *has-derivative-add-const*:
(*f has-derivative f'*) *net* $\Longrightarrow$ $((\lambda x.\ f\ x + c)$ *has-derivative f'*) *net*
by (*intro derivative-eq-intros*) *auto*

40.2 Derivative with composed bilinear function.

lemma *has-derivative-bilinear-within*:
assumes (*f has-derivative f'*) (*at x within s*)
and (*g has-derivative g'*) (*at x within s*)
and *bounded-bilinear h*

shows $((\lambda x.\ h\ (f\ x)\ (g\ x))$ *has-derivative* $(\lambda d.\ h\ (f\ x)\ (g'\ d) + h\ (f'\ d)\ (g\ x)))$ $(at\ x\ within\ s)$
using *bounded-bilinear.FDERIV*[*OF assms*(*3*,*1*,*2*)] .

lemma *has-derivative-bilinear-at*:
assumes $(f$ *has-derivative* $f')\ (at\ x)$
and $(g$ *has-derivative* $g')\ (at\ x)$
and *bounded-bilinear* h
shows $((\lambda x.\ h\ (f\ x)\ (g\ x))$ *has-derivative* $(\lambda d.\ h\ (f\ x)\ (g'\ d) + h\ (f'\ d)\ (g\ x)))$ $(at\ x)$
using *has-derivative-bilinear-within*[*of f f' x UNIV g g' h*] *assms* **by** *simp*

These are the only cases we'll care about, probably.

lemma *has-derivative-within*: $(f$ *has-derivative* $f')\ (at\ x\ within\ s) \longleftrightarrow$
bounded-linear $f' \wedge ((\lambda y.\ (1\ /\ norm(y - x)) *_R (f\ y - (f\ x + f'\ (y - x)))) \longrightarrow 0)\ (at\ x\ within\ s)$
unfolding *has-derivative-def Lim*
by (*auto simp add*: *netlimit-within field-simps*)

lemma *has-derivative-at*: $(f$ *has-derivative* $f')\ (at\ x) \longleftrightarrow$
bounded-linear $f' \wedge ((\lambda y.\ (1\ /\ (norm(y - x))) *_R (f\ y - (f\ x + f'\ (y - x)))) \longrightarrow 0)\ (at\ x)$
using *has-derivative-within* [*of f f' x UNIV*]
by *simp*

More explicit epsilon-delta forms.

lemma *has-derivative-within'*:
$(f$ *has-derivative* $f')(at\ x\ within\ s) \longleftrightarrow$
bounded-linear $f' \wedge$
$(\forall e{>}0.\ \exists d{>}0.\ \forall x'{\in}s.\ 0 < norm\ (x' - x) \wedge norm\ (x' - x) < d \longrightarrow$
$norm\ (f\ x' - f\ x - f'(x' - x))\ /\ norm\ (x' - x) < e)$
unfolding *has-derivative-within Lim-within dist-norm*
unfolding *diff-0-right*
by (*simp add*: *diff-diff-eq*)

lemma *has-derivative-at'*:
$(f$ *has-derivative* $f')\ (at\ x) \longleftrightarrow$ *bounded-linear* $f' \wedge$
$(\forall e{>}0.\ \exists d{>}0.\ \forall x'.\ 0 < norm\ (x' - x) \wedge norm\ (x' - x) < d \longrightarrow$
$norm\ (f\ x' - f\ x - f'(x' - x))\ /\ norm\ (x' - x) < e)$
using *has-derivative-within'* [*of f f' x UNIV*]
by *simp*

lemma *has-derivative-at-within*:
$(f$ *has-derivative* $f')\ (at\ x) \Longrightarrow (f$ *has-derivative* $f')\ (at\ x\ within\ s)$
unfolding *has-derivative-within' has-derivative-at'*
by *blast*

lemma *has-derivative-within-open*:
$a \in s \Longrightarrow$ *open* $s \Longrightarrow$

(*f has-derivative f'*) (*at a within s*) ⟷ (*f has-derivative f'*) (*at a*)
by (*simp only*: *at-within-interior interior-open*)

lemma *has-derivative-right*:
fixes *f* :: *real* ⇒ *real*
and *y* :: *real*
shows (*f has-derivative* (*op* ∗ *y*)) (*at x within* ({*x* <..} ∩ *I*)) ⟷
((λ*t*. (*f x* − *f t*) / (*x* − *t*)) ⟶ *y*) (*at x within* ({*x* <..} ∩ *I*))
proof −
have ((λ*t*. (*f t* − (*f x* + *y* ∗ (*t* − *x*))) / |*t* − *x*|) ⟶ *0*) (*at x within* ({*x*<..} ∩ *I*)) ⟷
((λ*t*. (*f t* − *f x*) / (*t* − *x*) − *y*) ⟶ *0*) (*at x within* ({*x*<..} ∩ *I*))
by (*intro Lim-cong-within*) (*auto simp add*: *diff-divide-distrib add-divide-distrib*)
also have ... ⟷ ((λ*t*. (*f t* − *f x*) / (*t* − *x*)) ⟶ *y*) (*at x within* ({*x*<..} ∩ *I*))
by (*simp add*: *Lim-null*[*symmetric*])
also have ... ⟷ ((λ*t*. (*f x* − *f t*) / (*x* − *t*)) ⟶ *y*) (*at x within* ({*x*<..} ∩ *I*))
by (*intro Lim-cong-within*) (*simp-all add*: *field-simps*)
finally show *?thesis*
by (*simp add*: *bounded-linear-mult-right has-derivative-within*)
qed

40.2.1 Caratheodory characterization

lemma *DERIV-within-iff*:
(*f has-field-derivative D*) (*at a within s*) ⟷ ((λ*z*. (*f z* − *f a*) / (*z* − *a*)) ⟶ *D*) (*at a within s*)
proof −
have *1*: ⋀*w y*. ~(*w* = *a*) ==> *y* / (*w* − *a*) − *D* = (*y* − (*w* − *a*)∗*D*)/(*w* − *a*)
by (*metis divide-diff-eq-iff eq-iff-diff-eq-0 mult.commute*)
show *?thesis*
apply (*simp add*: *has-field-derivative-def has-derivative-within bounded-linear-mult-right*)
apply (*simp add*: *LIM-zero-iff* [**where** *l* = *D*, *symmetric*])
apply (*simp add*: *Lim-within dist-norm*)
apply (*simp add*: *nonzero-norm-divide* [*symmetric*])
apply (*simp add*: *1 diff-diff-eq ac-simps*)
done
qed

lemma *DERIV-caratheodory-within*:
(*f has-field-derivative l*) (*at x within s*) ⟷
(∃ *g*. (∀ *z*. *f z* − *f x* = *g z* ∗ (*z* − *x*)) ∧ *continuous* (*at x within s*) *g* ∧ *g x* = *l*)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
show *?rhs*
proof (*intro exI conjI*)
let *?g* = (%*z*. *if z* = *x then l else* (*f z* − *f x*) / (*z*−*x*))

show $\forall z.\ f\ z - f\ x = ?g\ z * (z-x)$ **by** *simp*
show *continuous* (*at x within s*) *?g* **using** ‹*?lhs*›
by (*auto simp add*: *continuous-within DERIV-within-iff cong*: *Lim-cong-within*)
show *?g x* = *l* **by** *simp*
qed
next
assume *?rhs*
then obtain *g* **where**
$(\forall z.\ f\ z - f\ x = g\ z * (z-x))$ **and** *continuous* (*at x within s*) *g* **and** *g x* = *l*
by *blast*
thus *?lhs*
by (*auto simp add*: *continuous-within DERIV-within-iff cong*: *Lim-cong-within*)
qed

40.2.2 Limit transformation for derivatives

lemma *has-derivative-transform-within*:
assumes (*f has-derivative f'*) (*at x within s*)
and *0* < *d*
and *x* ∈ *s*
and $\bigwedge x'.\ [\![x' \in s;\ dist\ x'\ x < d]\!] \Longrightarrow f\ x' = g\ x'$
shows (*g has-derivative f'*) (*at x within s*)
using *assms*
unfolding *has-derivative-within*
by (*force simp add*: *intro*: *Lim-transform-within*)

lemma *has-derivative-transform-within-open*:
assumes (*f has-derivative f'*) (*at x*)
and *open s*
and *x* ∈ *s*
and $\bigwedge x.\ x \in s \Longrightarrow f\ x = g\ x$
shows (*g has-derivative f'*) (*at x*)
using *assms* **unfolding** *has-derivative-at*
by (*force simp add*: *intro*: *Lim-transform-within-open*)

40.3 Differentiability

definition
differentiable-on :: ('*a*::*real-normed-vector* ⇒ '*b*::*real-normed-vector*) ⇒ '*a set* ⇒ *bool*
(**infix** *differentiable'-on 50*)
where *f differentiable-on s* ⟷ $(\forall x \in s.\ f\ differentiable\ (at\ x\ within\ s))$

lemma *differentiableI*: (*f has-derivative f'*) *net* ⟹ *f differentiable net*
unfolding *differentiable-def*
by *auto*

lemma *differentiable-onD*: $[\![f\ differentiable\text{-}on\ S;\ x \in S]\!] \Longrightarrow f\ differentiable\ (at\ x\ within\ S)$
using *differentiable-on-def* **by** *blast*

lemma *differentiable-at-withinI*: *f differentiable* (*at x*) ⟹ *f differentiable* (*at x within s*)
unfolding *differentiable-def*
using *has-derivative-at-within*
by *blast*

lemma *differentiable-at-imp-differentiable-on*:
(⋀*x*. *x* ∈ *s* ⟹ *f differentiable at x*) ⟹ *f differentiable-on s*
by (*metis differentiable-at-withinI differentiable-on-def*)

corollary *differentiable-iff-scaleR*:
fixes *f* :: *real* ⇒ *'a*::*real-normed-vector*
shows *f differentiable F* ⟷ (∃ *d*. (*f has-derivative* (λ*x*. *x* ∗*R* *d*)) *F*)
by (*auto simp*: *differentiable-def dest*: *has-derivative-linear linear-imp-scaleR*)

lemma *differentiable-within-open*:
assumes *a* ∈ *s*
and *open s*
shows *f differentiable* (*at a within s*) ⟷ *f differentiable* (*at a*)
using *assms*
by (*simp only*: *at-within-interior interior-open*)

lemma *differentiable-on-eq-differentiable-at*:
open s ⟹ *f differentiable-on s* ⟷ (∀ *x*∈*s*. *f differentiable at x*)
unfolding *differentiable-on-def*
by (*metis at-within-interior interior-open*)

lemma *differentiable-transform-within*:
assumes *f differentiable* (*at x within s*)
and *0* < *d*
and *x* ∈ *s*
and ⋀*x'*. ⟦*x'*∈*s*; *dist x' x* < *d*⟧ ⟹ *f x'* = *g x'*
shows *g differentiable* (*at x within s*)
using *assms has-derivative-transform-within* **unfolding** *differentiable-def*
by *blast*

40.4 Frechet derivative and Jacobian matrix

definition *frechet-derivative f net* = (*SOME f'*. (*f has-derivative f'*) *net*)

lemma *frechet-derivative-works*:
f differentiable net ⟷ (*f has-derivative* (*frechet-derivative f net*)) *net*
unfolding *frechet-derivative-def differentiable-def*
unfolding *some-eq-ex*[*of* λ *f'* . (*f has-derivative f'*) *net*] **..**

lemma *linear-frechet-derivative*: *f differentiable net* ⟹ *linear* (*frechet-derivative f net*)
unfolding *frechet-derivative-works has-derivative-def*

by (*auto intro*: *bounded-linear.linear*)

40.5 Differentiability implies continuity

lemma *differentiable-imp-continuous-within*:
f differentiable (*at x within s*) $\Longrightarrow$ *continuous* (*at x within s*) *f*
by (*auto simp*: *differentiable-def intro*: *has-derivative-continuous*)

lemma *differentiable-imp-continuous-on*:
f differentiable-on s $\Longrightarrow$ *continuous-on s f*
unfolding *differentiable-on-def continuous-on-eq-continuous-within*
using *differentiable-imp-continuous-within* **by** *blast*

lemma *differentiable-on-subset*:
f differentiable-on t $\Longrightarrow$ $s \subseteq t$ $\Longrightarrow$ *f differentiable-on s*
unfolding *differentiable-on-def*
using *differentiable-within-subset*
by *blast*

lemma *differentiable-on-empty*: *f differentiable-on* {}
unfolding *differentiable-on-def*
by *auto*

Results about neighborhoods filter.

lemma *eventually-nhds-metric-le*:
eventually P (*nhds a*) = ($\exists$ *d*>*0*. $\forall$ *x*. *dist x a* $\leq$ *d* $\longrightarrow$ *P x*)
unfolding *eventually-nhds-metric* **by** (*safe*, *rule-tac x*=*d* / *2* **in** *exI*, *auto*)

lemma *le-nhds*: *F* $\leq$ *nhds a* $\longleftrightarrow$ ($\forall$ *S*. *open S* $\wedge$ *a* $\in$ *S* $\longrightarrow$ *eventually* (λ*x*. *x* $\in$ *S*) *F*)
unfolding *le-filter-def eventually-nhds* **by** (*fast elim*: *eventually-mono*)

lemma *le-nhds-metric*: *F* $\leq$ *nhds a* $\longleftrightarrow$ ($\forall$ *e*>*0*. *eventually* (λ*x*. *dist x a* < *e*) *F*)
unfolding *le-filter-def eventually-nhds-metric* **by** (*fast elim*: *eventually-mono*)

lemma *le-nhds-metric-le*: *F* $\leq$ *nhds a* $\longleftrightarrow$ ($\forall$ *e*>*0*. *eventually* (λ*x*. *dist x a* $\leq$ *e*) *F*)
unfolding *le-filter-def eventually-nhds-metric-le* **by** (*fast elim*: *eventually-mono*)

Several results are easier using a "multiplied-out" variant. (I got this idea from Dieudonne's proof of the chain rule).

lemma *has-derivative-within-alt*:
(*f has-derivative f'*) (*at x within s*) $\longleftrightarrow$ *bounded-linear f'* $\wedge$
($\forall$ *e*>*0*. $\exists$ *d*>*0*. $\forall$ *y*$\in$*s*. *norm*(*y* − *x*) < *d* $\longrightarrow$ *norm* (*f y* − *f x* − *f'* (*y* − *x*)) $\leq$ *e* ∗ *norm* (*y* − *x*))
unfolding *has-derivative-within filterlim-def le-nhds-metric-le eventually-filtermap eventually-at dist-norm diff-diff-eq*
by (*force simp add*: *linear-0 bounded-linear.linear pos-divide-le-eq*)

lemma *has-derivative-within-alt2*:
(*f has-derivative f'*) (*at x within s*) ⟷ *bounded-linear f'* ∧
(∀ *e*>*0*. *eventually* (λ*y*. *norm* (*f y* − *f x* − *f'* (*y* − *x*)) ≤ *e* ∗ *norm* (*y* − *x*))
(*at x within s*))
unfolding *has-derivative-within filterlim-def le-nhds-metric-le eventually-filtermap*
eventually-at dist-norm diff-diff-eq
by (*force simp add*: *linear-0 bounded-linear.linear pos-divide-le-eq*)

lemma *has-derivative-at-alt*:
(*f has-derivative f'*) (*at x*) ⟷
bounded-linear f' ∧
(∀ *e*>*0*. ∃ *d*>*0*. ∀ *y*. *norm*(*y* − *x*) < *d* ⟶ *norm* (*f y* − *f x* − *f'*(*y* − *x*)) ≤ *e*
∗ *norm* (*y* − *x*))
using *has-derivative-within-alt*[**where** *s*=*UNIV*]
by *simp*

40.6 The chain rule

lemma *diff-chain-within*[*derivative-intros*]:
assumes (*f has-derivative f'*) (*at x within s*)
and (*g has-derivative g'*) (*at* (*f x*) *within* (*f* ‘ *s*))
shows ((*g* ∘ *f*) *has-derivative* (*g'* ∘ *f'*))(*at x within s*)
using *has-derivative-in-compose*[*OF assms*]
by (*simp add*: *comp-def*)

lemma *diff-chain-at*[*derivative-intros*]:
(*f has-derivative f'*) (*at x*) ⟹
(*g has-derivative g'*) (*at* (*f x*)) ⟹ ((*g* ∘ *f*) *has-derivative* (*g'* ∘ *f'*)) (*at x*)
using *has-derivative-compose*[*of f f' x UNIV g g'*]
by (*simp add*: *comp-def*)

40.7 Composition rules stated just for differentiability

lemma *differentiable-chain-at*:
f differentiable (*at x*) ⟹
g differentiable (*at* (*f x*)) ⟹ (*g* ∘ *f*) *differentiable* (*at x*)
unfolding *differentiable-def*
by (*meson diff-chain-at*)

lemma *differentiable-chain-within*:
f differentiable (*at x within s*) ⟹
g differentiable (*at*(*f x*) *within* (*f* ‘ *s*)) ⟹ (*g* ∘ *f*) *differentiable* (*at x within s*)
unfolding *differentiable-def*
by (*meson diff-chain-within*)

40.8 Uniqueness of derivative

The general result is a bit messy because we need approachability of the limit point from any direction. But OK for nontrivial intervals etc.

```
lemma frechet-derivative-unique-within:
  fixes f :: 'a::euclidean-space ⇒ 'b::real-normed-vector
  assumes (f has-derivative f') (at x within s)
    and (f has-derivative f'') (at x within s)
    and ∀ i∈Basis. ∀ e>0. ∃ d. 0 < |d| ∧ |d| < e ∧ (x + d *R i) ∈ s
  shows f' = f''
proof −
  note as = assms(1,2)[unfolded has-derivative-def]
  then interpret f': bounded-linear f' by auto
  from as interpret f'': bounded-linear f'' by auto
  have x islimpt s unfolding islimpt-approachable
  proof (rule, rule)
    fix e :: real
    assume e > 0
    obtain d where 0 < |d| and |d| < e and x + d *R (SOME i. i ∈ Basis) ∈ s
      using assms(3) SOME-Basis ‹e>0› by blast
    then show ∃ x'∈s. x' ≠ x ∧ dist x' x < e
      apply (rule-tac x=x + d *R (SOME i. i ∈ Basis) in bexI)
      unfolding dist-norm
      apply (auto simp: SOME-Basis nonzero-Basis)
      done
  qed
  then have *: netlimit (at x within s) = x
    apply (auto intro!: netlimit-within)
    by (metis trivial-limit-within)
  show ?thesis
    apply (rule linear-eq-stdbasis)
    unfolding linear-conv-bounded-linear
    apply (rule as(1,2)[THEN conjunct1])+
  proof (rule, rule ccontr)
    fix i :: 'a
    assume i: i ∈ Basis
    def e ≡ norm (f' i − f'' i)
    assume f' i ≠ f'' i
    then have e > 0
      unfolding e-def by auto
    obtain d where d:
      0 < d
      (⋀xa. xa∈s ⟶ 0 < dist xa x ∧ dist xa x < d ⟶
          dist ((f xa − f x − f' (xa − x)) /R norm (xa − x) −
              (f xa − f x − f'' (xa − x)) /R norm (xa − x)) (0 − 0) < e)
      using tendsto-diff [OF as(1,2)[THEN conjunct2]]
      unfolding * Lim-within
      using ‹e>0› by blast
    obtain c where c: 0 < |c| |c| < d ∧ x + c *R i ∈ s
      using assms(3) i d(1) by blast
    have *: norm (− ((1 / |c|) *R f' (c *R i)) + (1 / |c|) *R f'' (c *R i)) =
        norm ((1 / |c|) *R (− (f' (c *R i)) + f'' (c *R i)))
      unfolding scaleR-right-distrib by auto
```

```
    also have ... = norm ((1 / |c|) *R (c *R (- (f' i) + f'' i)))
      unfolding f'.scaleR f''.scaleR
      unfolding scaleR-right-distrib scaleR-minus-right
      by auto
    also have ... = e
      unfolding e-def
      using c(1)
      using norm-minus-cancel[of f' i - f'' i]
      by auto
    finally show False
      using c
      using d(2)[of x + c *R i]
      unfolding dist-norm
      unfolding f'.scaleR f''.scaleR f'.add f''.add f'.diff f''.diff
        scaleR-scaleR scaleR-right-diff-distrib scaleR-right-distrib
      using i
      by (auto simp: inverse-eq-divide)
  qed
qed

lemma frechet-derivative-unique-at:
  (f has-derivative f') (at x) ⟹ (f has-derivative f'') (at x) ⟹ f' = f''
  by (rule has-derivative-unique)

lemma frechet-derivative-unique-within-closed-interval:
  fixes f::'a::euclidean-space ⇒ 'b::real-normed-vector
  assumes ∀ i∈Basis. a•i < b•i
    and x ∈ cbox a b
    and (f has-derivative f' ) (at x within cbox a b)
    and (f has-derivative f'') (at x within cbox a b)
  shows f' = f''
  apply(rule frechet-derivative-unique-within)
  apply(rule assms(3,4))+
proof (rule, rule, rule)
  fix e :: real
  fix i :: 'a
  assume e > 0 and i: i ∈ Basis
  then show ∃ d. 0 < |d| ∧ |d| < e ∧ x + d *R i ∈ cbox a b
  proof (cases x•i = a•i)
    case True
    then show ?thesis
      apply (rule-tac x=(min (b•i - a•i)  e) / 2 in exI)
      using assms(1)[THEN bspec[where x=i]] and ‹e>0› and assms(2)
      unfolding mem-box
      using i
      apply (auto simp add: field-simps inner-simps inner-Basis)
      done
  next
    note * = assms(2)[unfolded mem-box, THEN bspec, OF i]
```

```
    case False
    moreover have a · i < x · i
      using False ∗ by auto
    moreover {
      have a · i ∗ 2 + min (x · i − a · i) e ≤ a·i ∗2 + x·i − a·i
        by auto
      also have ... = a·i + x·i
        by auto
      also have ... ≤ 2 ∗ (x·i)
        using ∗ by auto
      finally have a · i ∗ 2 + min (x · i − a · i) e ≤ x · i ∗ 2
        by auto
    }
    moreover have min (x · i − a · i) e ≥ 0
      using ∗ and ‹e>0› by auto
    then have x · i ∗ 2 ≤ b · i ∗ 2 + min (x · i − a · i) e
      using ∗ by auto
    ultimately show ?thesis
      apply (rule-tac x=− (min (x·i − a·i) e) / 2 in exI)
      using assms(1)[THEN bspec, OF i] and ‹e>0› and assms(2)
      unfolding mem-box
      using i
      apply (auto simp add: field-simps inner-simps inner-Basis)
      done
  qed
qed

lemma frechet-derivative-unique-within-open-interval:
  fixes f::'a::euclidean-space ⇒ 'b::real-normed-vector
  assumes x ∈ box a b
    and (f has-derivative f' ) (at x within box a b)
    and (f has-derivative f'') (at x within box a b)
  shows f' = f''
proof −
  from assms(1) have ∗: at x within box a b = at x
    by (metis at-within-interior interior-open open-box)
  from assms(2,3) [unfolded ∗] show f' = f''
    by (rule frechet-derivative-unique-at)
qed

lemma frechet-derivative-at:
  (f has-derivative f') (at x) ⟹ f' = frechet-derivative f (at x)
  apply (rule frechet-derivative-unique-at[of f])
  apply assumption
  unfolding frechet-derivative-works[symmetric]
  using differentiable-def
  apply auto
  done
```

lemma *frechet-derivative-within-cbox*:
fixes $f :: 'a::euclidean\text{-}space \Rightarrow 'b::real\text{-}normed\text{-}vector$
assumes $\forall i{\in}Basis.\ a{\cdot}i < b{\cdot}i$
and $x \in cbox\ a\ b$
and $(f\ has\text{-}derivative\ f')\ (at\ x\ within\ cbox\ a\ b)$
shows $frechet\text{-}derivative\ f\ (at\ x\ within\ cbox\ a\ b) = f'$
using *assms*
by (*metis Derivative.differentiableI frechet-derivative-unique-within-closed-interval frechet-derivative-works*)

40.9 The traditional Rolle theorem in one dimension

Derivatives of local minima and maxima are zero.

lemma *has-derivative-local-min*:
fixes $f :: 'a::real\text{-}normed\text{-}vector \Rightarrow real$
assumes *deriv*: $(f\ has\text{-}derivative\ f')\ (at\ x)$
assumes *min*: $eventually\ (\lambda y.\ f\ x \le f\ y)\ (at\ x)$
shows $f' = (\lambda h.\ 0)$
proof
fix $h :: 'a$
interpret f': *bounded-linear* f'
using *deriv* **by** (*rule has-derivative-bounded-linear*)
show $f'\ h = 0$
proof (*cases* $h = 0$)
assume $h \neq 0$
from *min* **obtain** d **where** *d1*: $0 < d$ **and** *d2*: $\forall y{\in}ball\ x\ d.\ f\ x \le f\ y$
unfolding *eventually-at* **by** (*force simp*: *dist-commute*)
have $FDERIV\ (\lambda r.\ x + r *_R h)\ 0 :> (\lambda r.\ r *_R h)$
by (*intro derivative-eq-intros*) *auto*
then have $FDERIV\ (\lambda r.\ f\ (x + r *_R h))\ 0 :> (\lambda k.\ f'\ (k *_R h))$
by (*rule has-derivative-compose*, *simp add*: *deriv*)
then have $DERIV\ (\lambda r.\ f\ (x + r *_R h))\ 0 :> f'\ h$
unfolding *has-field-derivative-def* **by** (*simp add*: *f'.scaleR mult-commute-abs*)
moreover have $0 < d\ /\ norm\ h$ **using** *d1* **and** ‹$h \neq 0$› **by** *simp*
moreover have $\forall y.\ |0 - y| < d\ /\ norm\ h \longrightarrow f\ (x + 0 *_R h) \le f\ (x + y *_R h)$
using ‹$h \neq 0$› **by** (*auto simp add*: *d2 dist-norm pos-less-divide-eq*)
ultimately show $f'\ h = 0$
by (*rule DERIV-local-min*)
qed (*simp add*: *f'.zero*)
qed

lemma *has-derivative-local-max*:
fixes $f :: 'a::real\text{-}normed\text{-}vector \Rightarrow real$
assumes $(f\ has\text{-}derivative\ f')\ (at\ x)$
assumes $eventually\ (\lambda y.\ f\ y \le f\ x)\ (at\ x)$
shows $f' = (\lambda h.\ 0)$
using *has-derivative-local-min* [*of* $\lambda x.\ -f\ x$ $\lambda h.\ -f'\ h\ x$]
using *assms* **unfolding** *fun-eq-iff* **by** *simp*

lemma *differential-zero-maxmin*:
fixes f::$'a$::*real-normed-vector* $\Rightarrow$ *real*
assumes $x \in s$
and *open* s
and *deriv*: $(f$ *has-derivative* $f')$ $(at\ x)$
and *mono*: $(\forall\, y{\in}s.\ f\ y \leq f\ x) \vee (\forall\, y{\in}s.\ f\ x \leq f\ y)$
shows $f' = (\lambda v.\ 0)$
using *mono*
proof
assume $\forall\, y{\in}s.\ f\ y \leq f\ x$
with ‹$x \in s$› **and** ‹*open* s› **have** *eventually* $(\lambda y.\ f\ y \leq f\ x)$ $(at\ x)$
unfolding *eventually-at-topological* **by** *auto*
with *deriv* **show** *?thesis*
by (*rule has-derivative-local-max*)
next
assume $\forall\, y{\in}s.\ f\ x \leq f\ y$
with ‹$x \in s$› **and** ‹*open* s› **have** *eventually* $(\lambda y.\ f\ x \leq f\ y)$ $(at\ x)$
unfolding *eventually-at-topological* **by** *auto*
with *deriv* **show** *?thesis*
by (*rule has-derivative-local-min*)
qed

lemma *differential-zero-maxmin-component*:
fixes f :: $'a$::*euclidean-space* $\Rightarrow$ $'b$::*euclidean-space*
assumes k: $k \in Basis$
and *ball*: $0 < e$ $(\forall\, y \in ball\ x\ e.\ (f\ y){\cdot}k \leq (f\ x){\cdot}k) \vee (\forall\, y{\in}ball\ x\ e.\ (f\ x){\cdot}k \leq (f\ y){\cdot}k)$
and *diff*: f *differentiable* $(at\ x)$
shows $(\sum j{\in}Basis.\ (frechet\text{-}derivative\ f\ (at\ x)\ j \cdot k) *_R j) = (0::'a)$ (**is** *?D* $k = 0$)
proof −
let *?f'* = *frechet-derivative* f $(at\ x)$
have $x \in ball\ x\ e$ **using** ‹$0 < e$› **by** *simp*
moreover have *open* $(ball\ x\ e)$ **by** *simp*
moreover have $((\lambda x.\ f\ x \cdot k)$ *has-derivative* $(\lambda h.$ *?f'* $h \cdot k))$ $(at\ x)$
using *bounded-linear-inner-left diff*[*unfolded frechet-derivative-works*]
by (*rule bounded-linear.has-derivative*)
ultimately have $(\lambda h.$ *frechet-derivative* f $(at\ x)$ $h \cdot k) = (\lambda v.\ 0)$
using *ball*(*2*) **by** (*rule differential-zero-maxmin*)
then show *?thesis*
unfolding *fun-eq-iff* **by** *simp*
qed

lemma *rolle*:
fixes f :: *real* $\Rightarrow$ *real*
assumes $a < b$
and $f\ a = f\ b$
and *continuous-on* $\{a\ ..\ b\}$ f

```
    and ∀x∈{a <..< b}. (f has-derivative f' x) (at x)
  shows ∃x∈{a <..< b}. f' x = (λv. 0)
proof −
  have ∃x∈box a b. (∀y∈box a b. f x ≤ f y) ∨ (∀y∈box a b. f y ≤ f x)
  proof −
    have (a + b) / 2 ∈ {a .. b}
      using assms(1) by auto
    then have ∗: {a .. b} ≠ {}
      by auto
    obtain d where d:
        d ∈cbox a b
        ∀y∈cbox a b. f y ≤ f d
      using continuous-attains-sup[OF compact-Icc ∗ assms(3)] by auto
    obtain c where c:
        c ∈ cbox a b
        ∀y∈cbox a b. f c ≤ f y
      using continuous-attains-inf[OF compact-Icc ∗ assms(3)] by auto
    show ?thesis
    proof (cases d ∈ box a b ∨ c ∈ box a b)
      case True
      then show ?thesis
        by (metis c(2) d(2) box-subset-cbox subset-iff)
    next
      def e ≡ (a + b) /2
      case False
      then have f d = f c
        using d c assms(2) by auto
      then have ⋀x. x ∈ {a..b} ⟹ f x = f d
        using c d
        by force
      then show ?thesis
        apply (rule-tac x=e in bexI)
        unfolding e-def
        using assms(1)
        apply auto
        done
    qed
  qed
  then obtain x where x: x ∈ {a <..< b} (∀y∈{a <..< b}. f x ≤ f y) ∨ (∀y∈{a
<..< b}. f y ≤ f x)
    by auto
  then have f' x = (λv. 0)
    apply (rule-tac differential-zero-maxmin[of x box a b f f' x])
    using assms
    apply auto
    done
  then show ?thesis
    by (metis x(1))
qed
```

40.10 One-dimensional mean value theorem

lemma *mvt*:
fixes $f :: real \Rightarrow real$
assumes $a < b$
and *continuous-on* $\{a..b\}$ f
assumes $\forall x{\in}\{a<..<b\}.$ $(f$ *has-derivative* $(f'\ x))$ $(at\ x)$
shows $\exists x{\in}\{a<..<b\}.\ f\ b - f\ a = (f'\ x)\ (b - a)$
proof −
have $\exists x{\in}\{a <..< b\}.\ (\lambda xa.\ f'\ x\ xa - (f\ b - f\ a) / (b - a) * xa) = (\lambda v.\ 0)$
proof (*intro rolle*[*OF assms*(*1*), *of* $\lambda x.\ f\ x - (f\ b - f\ a) / (b - a) * x$] *ballI*)
fix x
assume x: $x \in \{a <..< b\}$
show $((\lambda x.\ f\ x - (f\ b - f\ a) / (b - a) * x)$ *has-derivative*
$(\lambda xa.\ f'\ x\ xa - (f\ b - f\ a) / (b - a) * xa))\ (at\ x)$
by (*intro derivative-intros assms*(*3*)[*rule-format*,*OF x*])
qed (*insert assms*(*1*,*2*), *auto intro*!: *continuous-intros simp*: *field-simps*)
then obtain x **where**
$x \in \{a <..< b\}$
$(\lambda xa.\ f'\ x\ xa - (f\ b - f\ a) / (b - a) * xa) = (\lambda v.\ 0)$ **..**
then show *?thesis*
by (*metis* (*hide-lams*) *assms*(*1*) *diff-gt-0-iff-gt eq-iff-diff-eq-0*
zero-less-mult-iff nonzero-mult-divide-cancel-right not-real-square-gt-zero
times-divide-eq-left)
qed

lemma *mvt-simple*:
fixes $f :: real \Rightarrow real$
assumes $a < b$
and $\forall x{\in}\{a..b\}.$ $(f$ *has-derivative* $f'\ x)$ $(at\ x\ within\ \{a..b\})$
shows $\exists x{\in}\{a<..<b\}.\ f\ b - f\ a = f'\ x\ (b - a)$
proof (*rule mvt*)
have f *differentiable-on* $\{a..b\}$
using *assms*(*2*) **unfolding** *differentiable-on-def differentiable-def* **by** *fast*
then show *continuous-on* $\{a..b\}$ f
by (*rule differentiable-imp-continuous-on*)
show $\forall x{\in}\{a<..<b\}.$ $(f$ *has-derivative* $f'\ x)$ $(at\ x)$
proof
fix x
assume x: $x \in \{a <..< b\}$
show $(f$ *has-derivative* $f'\ x)$ $(at\ x)$
unfolding *at-within-open*[*OF x open-greaterThanLessThan*,*symmetric*]
apply (*rule has-derivative-within-subset*)
apply (*rule assms*(*2*)[*rule-format*])
using x
apply *auto*
done
qed
qed (*rule assms*(*1*))

lemma *mvt-very-simple*:
fixes f :: *real* ⇒ *real*
assumes $a \leq b$
and $\forall x{\in}\{a\ ..\ b\}.\ (f\ has\text{-}derivative\ f'\ x)\ (at\ x\ within\ \{a\ ..\ b\})$
shows $\exists x{\in}\{a\ ..\ b\}.\ f\ b - f\ a = f'\ x\ (b - a)$
proof (*cases* $a = b$)
interpret *bounded-linear* f' b
using *assms*(*2*) *assms*(*1*) **by** *auto*
case *True*
then show *?thesis*
apply (*rule-tac* $x{=}a$ **in** *bexI*)
using *assms*(*2*)[*THEN bspec*[**where** $x{=}a$]]
unfolding *has-derivative-def*
unfolding *True*
using *zero*
apply *auto*
done
next
case *False*
then show *?thesis*
using *mvt-simple*[*OF* - *assms*(*2*)]
using *assms*(*1*)
by *auto*
qed

A nice generalization (see Havin's proof of 5.19 from Rudin's book).

lemma *mvt-general*:
fixes f :: *real* ⇒ 'a::*real-inner*
assumes $a < b$
and *continuous-on* $\{a\ ..\ b\}$ f
and $\forall x{\in}\{a{<}..{<}b\}.\ (f\ has\text{-}derivative\ f'(x))\ (at\ x)$
shows $\exists x{\in}\{a{<}..{<}b\}.\ norm\ (f\ b - f\ a) \leq norm\ (f'\ x\ (b - a))$
proof −
have $\exists x{\in}\{a{<}..{<}b\}.\ (f\ b - f\ a) \cdot f\ b - (f\ b - f\ a) \cdot f\ a = (f\ b - f\ a) \cdot f'\ x\ (b - a)$
apply (*rule mvt*)
apply (*rule assms*(*1*))
apply (*intro continuous-intros assms*(*2*))
using *assms*(*3*)
apply (*fast intro*: *has-derivative-inner-right*)
done
then obtain x **where** x:
$x \in \{a{<}..{<}b\}$
$(f\ b - f\ a) \cdot f\ b - (f\ b - f\ a) \cdot f\ a = (f\ b - f\ a) \cdot f'\ x\ (b - a)$ **..**
show *?thesis*
proof (*cases* $f\ a = f\ b$)
case *False*
have $norm\ (f\ b - f\ a) * norm\ (f\ b - f\ a) = (norm\ (f\ b - f\ a))^2$
by (*simp add*: *power2-eq-square*)

```
    also have ... = (f b − f a) · (f b − f a)
      unfolding power2-norm-eq-inner ..
    also have ... = (f b − f a) · f' x (b − a)
      using x(2) by (simp only: inner-diff-right)
    also have ... ≤ norm (f b − f a) * norm (f' x (b − a))
      by (rule norm-cauchy-schwarz)
    finally show ?thesis
      using False x(1)
      by (auto simp add: mult-left-cancel)
  next
    case True
    then show ?thesis
      using assms(1)
      apply (rule-tac x=(a + b) /2 in bexI)
      apply auto
      done
  qed
qed
```

40.11 More general bound theorems

```
lemma differentiable-bound-general:
  fixes f :: real ⇒ 'a::real-normed-vector
  assumes a < b
    and f-cont: continuous-on {a .. b} f
    and phi-cont: continuous-on {a .. b} φ
    and f': ⋀x. a < x ⟹ x < b ⟹ (f has-vector-derivative f' x) (at x)
    and phi': ⋀x. a < x ⟹ x < b ⟹ (φ has-vector-derivative φ' x) (at x)
    and bnd: ⋀x. a < x ⟹ x < b ⟹ norm (f' x) ≤ φ' x
  shows norm (f b − f a) ≤ φ b − φ a
proof −
  {
    fix x assume x: a < x x < b
    have 0 ≤ norm (f' x) by simp
    also have ... ≤ φ' x using x by (auto intro!: bnd)
    finally have 0 ≤ φ' x .
  } note phi'-nonneg = this
  note f-tendsto = assms(2)[simplified continuous-on-def, rule-format]
  note phi-tendsto = assms(3)[simplified continuous-on-def, rule-format]
  {
    fix e::real assume e > 0
    def e2 ≡ e / 2 with ‹e > 0› have e2 > 0 by simp
    let ?le = λx1. norm (f x1 − f a) ≤ φ x1 − φ a + e * (x1 − a) + e
    def A ≡ {x2. a ≤ x2 ∧ x2 ≤ b ∧ (∀x1∈{a ..< x2}. ?le x1)}
    have A-subset: A ⊆ {a .. b} by (auto simp: A-def)
    {
      fix x2
      assume a: a ≤ x2 x2 ≤ b and le: ∀x1∈{a..<x2}. ?le x1
      have ?le x2 using ‹e > 0›
```

```
    proof cases
      assume x2 ≠ a with a have a < x2 by simp
      have at x2 within {a <..<x2}≠ bot
        using ‹a < x2›
        by (auto simp: trivial-limit-within islimpt-in-closure)
      moreover
      have ((λx1. (φ x1 − φ a) + e ∗ (x1 − a) + e) ⟶ (φ x2 − φ a) + e ∗
(x2 − a) + e) (at x2 within {a <..<x2})
          ((λx1. norm (f x1 − f a)) ⟶ norm (f x2 − f a)) (at x2 within {a
<..<x2})
        using a
        by (auto intro!: tendsto-eq-intros f-tendsto phi-tendsto
          intro: tendsto-within-subset[where S={a .. b}])
      moreover
      have eventually (λx. x > a) (at x2 within {a <..<x2})
        by (auto simp: eventually-at-filter)
      hence eventually ?le (at x2 within {a <..<x2})
        unfolding eventually-at-filter
        by eventually-elim (insert le, auto)
      ultimately
      show ?thesis
        by (rule tendsto-le)
    qed simp
  } note le-cont = this
  have a ∈ A
    using assms by (auto simp: A-def)
  hence [simp]: A ≠ {} by auto
  have A-ivl: ⋀x1 x2. x2 ∈ A ⟹ x1 ∈ {a ..x2} ⟹ x1 ∈ A
    by (simp add: A-def)
  have [simp]: bdd-above A by (auto simp: A-def)
  def y ≡ Sup A
  have y ≤ b
    unfolding y-def
    by (simp add: cSup-le-iff) (simp add: A-def)
   have leI: ⋀x x1. a ≤ x1 ⟹ x ∈ A ⟹ x1 < x ⟹ ?le x1
     by (auto simp: A-def intro!: le-cont)
  have y-all-le: ∀ x1∈{a..<y}. ?le x1
    by (auto simp: y-def less-cSup-iff leI)
  have a ≤ y
    by (metis ‹a ∈ A› ‹bdd-above A› cSup-upper y-def)
  have y ∈ A
    using y-all-le ‹a ≤ y› ‹y ≤ b›
    by (auto simp: A-def)
  hence A = {a .. y}
    using A-subset
    by (auto simp: subset-iff y-def cSup-upper intro: A-ivl)
  from le-cont[OF ‹a ≤ y› ‹y ≤ b› y-all-le] have le-y: ?le y .
  {
    assume a ≠ y with ‹a ≤ y› have a < y by simp
```

have $y = b$
proof (*rule ccontr*)
assume $y \neq b$
hence $y < b$ **using** ‹$y \leq b$› **by** *simp*
let *?F* = *at y within* $\{y..<b\}$
from f' *phi'*
have (*f has-vector-derivative* f' *y*) *?F*
and (φ *has-vector-derivative* φ' *y*) *?F*
using ‹$a < y$› ‹$y < b$›
by (*auto simp add*: *at-within-open*[*of* - $\{a<..<b\}$] *has-vector-derivative-def*
intro!: *has-derivative-subset*[**where** s=$\{a<..<b\}$ **and** t=$\{y..<b\}$])
hence $\forall_F$ *x1 in ?F*. *norm* (*f x1* − *f y* − (*x1* − *y*) $*_R$ f' *y*) ≤ *e2* ∗ |*x1* − *y*|
$\forall_F$ *x1 in ?F*. *norm* (φ *x1* − φ *y* − (*x1* − *y*) $*_R$ φ' *y*) ≤ *e2* ∗ |*x1* − *y*|
using ‹*e2* > *0*›
by (*auto simp*: *has-derivative-within-alt2 has-vector-derivative-def*)
moreover
have $\forall_F$ *x1 in ?F*. *y* ≤ *x1* $\forall_F$ *x1 in ?F*. *x1* < *b*
by (*auto simp*: *eventually-at-filter*)
ultimately
have $\forall_F$ *x1 in ?F*. *norm* (*f x1* − *f y*) ≤ (φ *x1* − φ *y*) + *e* ∗ |*x1* − *y*|
(**is** $\forall_F$ *x1 in ?F*. *?le' x1*)
proof *eventually-elim*
case (*elim x1*)
from *norm-triangle-ineq2*[*THEN order-trans*, *OF elim*(*1*)]
have *norm* (*f x1* − *f y*) ≤ *norm* (*f'* *y*) ∗ |*x1* − *y*| + *e2* ∗ |*x1* − *y*|
by (*simp add*: *ac-simps*)
also have *norm* (*f'* *y*) ≤ φ' *y* **using** *bnd* ‹$a < y$› ‹$y < b$› **by** *simp*
also
from *elim* **have** φ' *y* ∗ |*x1* − *y*| ≤ φ *x1* − φ *y* + *e2* ∗ |*x1* − *y*|
by (*simp add*: *ac-simps*)
finally
have *norm* (*f x1* − *f y*) ≤ φ *x1* − φ *y* + *e2* ∗ |*x1* − *y*| + *e2* ∗ |*x1* − *y*|
by (*auto simp*: *mult-right-mono*)
thus *?case* **by** (*simp add*: *e2-def*)
qed
moreover have *?le' y* **by** *simp*
ultimately obtain *S*
where *S*: *open S y* ∈ *S* ⋀*x*. *x*∈*S* ⟹ *x* ∈ $\{y..<b\}$ ⟹ *?le' x*
unfolding *eventually-at-topological*
by *metis*
from ‹*open S*› **obtain** *d* **where** *d*: ⋀*x*. *dist x y* < *d* ⟹ *x* ∈ *S* *d* > *0*
by (*force simp*: *dist-commute open-dist ball-def dest*!: *bspec*[*OF* - ‹$y \in S$›])
def $d' \equiv$ *min* ((*y* + *b*)/*2*) (*y* + (*d*/*2*))
have $d' \in A$
unfolding *A-def*
proof *safe*
show *a* ≤ d' **using** ‹$a < y$› ‹*0* < *d*› ‹$y < b$› **by** (*simp add*: d'*-def*)
show d' ≤ *b* **using** ‹$y < b$› **by** (*simp add*: d'*-def min-def*)
fix *x1*

```
      assume x1: x1 ∈ {a..<d'}
      {
       assume x1 < y
       hence ?le x1
         using ‹x1 ∈ {a..<d'}› y-all-le by auto
      } moreover {
       assume x1 ≥ y
       hence x1': x1 ∈ S x1 ∈ {y..<b} using x1
         by (auto simp: d'-def dist-real-def intro!: d)
       have norm (f x1 − f a) ≤ norm (f x1 − f y) + norm (f y − f a)
         by (rule order-trans[OF - norm-triangle-ineq]) simp
       also note S(3)[OF x1']
       also note le-y
       finally have ?le x1
         using ‹x1 ≥ y› by (auto simp: algebra-simps)
      } ultimately show ?le x1 by arith
     qed
     hence d' ≤ y
      unfolding y-def
      by (rule cSup-upper) simp
     thus False using ‹d > 0› ‹y < b›
      by (simp add: d'-def min-def split: if-split-asm)
    qed
  } moreover {
    assume a = y
    with ‹a < b› have y < b by simp
    with ‹a = y› f-cont phi-cont ‹e2 > 0›
    have 1: ∀F x in at y within {y..b}. dist (f x) (f y) < e2
     and 2: ∀F x in at y within {y..b}. dist (φ x) (φ y) < e2
      by (auto simp: continuous-on-def tendsto-iff)
    have 3: eventually (λx. y < x) (at y within {y..b})
      by (auto simp: eventually-at-filter)
    have 4: eventually (λx::real. x < b) (at y within {y..b})
      using - ‹y < b›
      by (rule order-tendstoD) (auto intro!: tendsto-eq-intros)
    from 1 2 3 4
    have eventually-le: eventually (λx. ?le x) (at y within {y .. b})
    proof eventually-elim
      case (elim x1)
      have norm (f x1 − f a) = norm (f x1 − f y)
        by (simp add: ‹a = y›)
      also have norm (f x1 − f y) ≤ e2
        using elim ‹a = y› by (auto simp : dist-norm intro!: less-imp-le)
      also have ... ≤ e2 + (φ x1 − φ a + e2 + e * (x1 − a))
        using ‹0 < e› elim
        by (intro add-increasing2[OF add-nonneg-nonneg order.refl])
          (auto simp: ‹a = y› dist-norm intro!: mult-nonneg-nonneg)
      also have ... = φ x1 − φ a + e * (x1 − a) + e
        by (simp add: e2-def)
```

```
      finally show ?le x1 .
    qed
    from this[unfolded eventually-at-topological] ‹?le y›
    obtain S
    where S: open S y ∈ S ⋀x. x∈S ⟹ x ∈ {y..b} ⟹ ?le x
      by metis
    from ‹open S› obtain d where d: ⋀x. dist x y < d ⟹ x ∈ S d > 0
      by (force simp: dist-commute open-dist ball-def dest!: bspec[OF - ‹y ∈ S›])
    def d' ≡ min b (y + (d/2))
    have d' ∈ A
      unfolding A-def
    proof safe
      show a ≤ d' using ‹a = y› ‹0 < d› ‹y < b› by (simp add: d'-def)
      show d' ≤ b by (simp add: d'-def)
      fix x1
      assume x1 ∈ {a..<d'}
      hence x1 ∈ S x1 ∈ {y..b}
        by (auto simp: ‹a = y› d'-def dist-real-def intro!: d )
      thus ?le x1
        by (rule S)
    qed
    hence d' ≤ y
      unfolding y-def
      by (rule cSup-upper) simp
    hence y = b using ‹d > 0› ‹y < b›
      by (simp add: d'-def)
  } ultimately have y = b
    by auto
  with le-y have norm (f b − f a) ≤ φ b − φ a + e ∗ (b − a + 1)
    by (simp add: algebra-simps)
 } note ∗ = this
 {
  fix e::real assume e > 0
  hence norm (f b − f a) ≤ φ b − φ a + e
    using ∗[of e / (b − a + 1)] ‹a < b› by simp
 } thus ?thesis
  by (rule field-le-epsilon)
qed

lemma differentiable-bound:
 fixes f :: 'a::real-normed-vector ⇒ 'b::real-normed-vector
 assumes convex s
  and ∀x∈s. (f has-derivative f' x) (at x within s)
  and ∀x∈s. onorm (f' x) ≤ B
  and x: x ∈ s
  and y: y ∈ s
 shows norm (f x − f y) ≤ B ∗ norm (x − y)
proof −
 let ?p = λu. x + u ∗R (y − x)
```

let *?φ* = *λh. h ∗ B ∗ norm (x − y)*
have *∗*: *⋀u. u∈{0..1} ⟹ x + u ∗R (y − x) ∈ s*
using *assms(1)*[*unfolded convex-alt*,*rule-format*,*OF x y*]
unfolding *scaleR-left-diff-distrib scaleR-right-diff-distrib*
by (*auto simp add*: *algebra-simps*)
have *0*: *continuous-on* (*?p ‘ {0..1}*) *f*
using *∗*
unfolding *continuous-on-eq-continuous-within*
apply *−*
apply *rule*
apply (*rule differentiable-imp-continuous-within*)
unfolding *differentiable-def*
apply (*rule-tac x=f′ xa* **in** *exI*)
apply (*rule has-derivative-within-subset*)
apply (*rule assms(2)*[*rule-format*])
apply *auto*
done
from *∗* **have** *1*: *continuous-on {0 .. 1} (f ∘ ?p)*
by (*intro continuous-intros 0*)+
{
fix *u*::*real* **assume** *u*: *u ∈{0 <..< 1}*
let *?u = ?p u*
interpret *linear* (*f′ ?u*)
using *u* **by** (*auto intro*!: *has-derivative-linear assms(2)*[*rule-format*] *∗*)
have (*f ∘ ?p has-derivative* (*f′ ?u*) *∘* (*λu. 0 + u ∗R* (*y − x*))) (*at u within box 0 1*)
apply (*rule diff-chain-within*)
apply (*rule derivative-intros*)+
apply (*rule has-derivative-within-subset*)
apply (*rule assms(2)*[*rule-format*])
using *u ∗*
apply *auto*
done
hence ((*f ∘ ?p*) *has-vector-derivative f′ ?u* (*y − x*)) (*at u*)
by (*simp add*: *has-derivative-within-open*[*OF u open-greaterThanLessThan*]
scaleR has-vector-derivative-def o-def)
} note *2 = this*
{
have *continuous-on {0..1} ?φ*
by (*rule continuous-intros*)+
} note *3 = this*
{
fix *u*::*real* **assume** *u*: *u ∈{0 <..< 1}*
have (*?φ has-vector-derivative B ∗ norm* (*x − y*)) (*at u*)
by (*auto simp*: *has-vector-derivative-def intro*!: *derivative-eq-intros*)
} note *4 = this*
{
fix *u*::*real* **assume** *u*: *u ∈{0 <..< 1}*
let *?u = ?p u*

```
    interpret bounded-linear (f' ?u)
      using u by (auto intro!: has-derivative-bounded-linear assms(2)[rule-format]
*)
    have norm (f' ?u (y - x)) ≤ onorm (f' ?u) * norm (y - x)
      by (rule onorm) fact
    also have onorm (f' ?u) ≤ B
      using u by (auto intro!: assms(3)[rule-format] *)
    finally have norm ((f' ?u) (y - x)) ≤ B * norm (x - y)
      by (simp add: mult-right-mono norm-minus-commute)
  } note 5 = this
  have norm (f x - f y) = norm ((f ∘ (λu. x + u *R (y - x))) 1 - (f ∘ (λu. x
+ u *R (y - x))) 0)
    by (auto simp add: norm-minus-commute)
  also
  from differentiable-bound-general[OF zero-less-one 1, OF 3 2 4 5]
  have norm ((f ∘ ?p) 1 - (f ∘ ?p) 0) ≤ B * norm (x - y)
    by simp
  finally show ?thesis .
qed

lemma
  differentiable-bound-segment:
  fixes f::'a::real-normed-vector ⇒ 'b::real-normed-vector
  assumes ⋀t. t ∈ {0..1} ⟹ x0 + t *R a ∈ G
  assumes f': ⋀x. x ∈ G ⟹ (f has-derivative f' x) (at x within G)
  assumes B: ∀x∈{0..1}. onorm (f' (x0 + x *R a)) ≤ B
  shows norm (f (x0 + a) - f x0) ≤ norm a * B
proof -
  let ?G = (λx. x0 + x *R a) ` {0..1}
  have ?G = op + x0 ` (λx. x *R a) ` {0..1} by auto
  also have convex ...
    by (intro convex-translation convex-scaled convex-real-interval)
  finally have convex ?G .
  moreover have ?G ⊆ G x0 ∈ ?G x0 + a ∈ ?G using assms by (auto intro:
image-eqI[where x=1])
  ultimately show ?thesis
    using has-derivative-subset[OF f' ‹?G ⊆ G›] B
      differentiable-bound[of (λx. x0 + x *R a) ` {0..1} f f' B x0 + a x0]
    by (auto simp: ac-simps)
qed

lemma differentiable-bound-linearization:
  fixes f::'a::real-normed-vector ⇒ 'b::real-normed-vector
  assumes ⋀t. t ∈ {0..1} ⟹ a + t *R (b - a) ∈ S
  assumes f'[derivative-intros]: ⋀x. x ∈ S ⟹ (f has-derivative f' x) (at x within
S)
  assumes B: ∀x∈S. onorm (f' x - f' x0) ≤ B
  assumes x0 ∈ S
  shows norm (f b - f a - f' x0 (b - a)) ≤ norm (b - a) * B
```

proof −
def $g \equiv \lambda x.\ f\ x - f'\ x0\ x$
have g: $\bigwedge x.\ x \in S \Longrightarrow (g$ *has-derivative* $(\lambda i.\ f'\ x\ i - f'\ x0\ i))$ $(at\ x\ within\ S)$
unfolding *g-def* **using** *assms*
by (*auto intro*!: *derivative-eq-intros*
bounded-linear.has-derivative[*OF has-derivative-bounded-linear*, *OF f'*])
from B **have** B: $\forall x{\in}\{0..1\}.\ onorm\ (\lambda i.\ f'\ (a + x *_R (b - a))\ i - f'\ x0\ i) \leq B$
using *assms* **by** (*auto simp*: *fun-diff-def*)
from *differentiable-bound-segment*[*OF assms*(*1*) *g B*] ‹*x0* ∈ *S*›
show *?thesis*
by (*simp add*: *g-def field-simps linear-sub*[*OF has-derivative-linear*[*OF f'*]])
qed

In particular.

lemma *has-derivative-zero-constant*:
fixes f :: *'a*::*real-normed-vector* ⇒ *'b*::*real-normed-vector*
assumes *convex s*
and $\bigwedge x.\ x \in s \Longrightarrow (f$ *has-derivative* $(\lambda h.\ 0))$ $(at\ x\ within\ s)$
shows $\exists c.\ \forall x{\in}s.\ f\ x = c$
proof −
{ fix $x\ y$ **assume** $x \in s\ y \in s$
then have $norm\ (f\ x - f\ y) \leq 0 * norm\ (x - y)$
using *assms* **by** (*intro differentiable-bound*[*of s*]) (*auto simp*: *onorm-zero*)
then have $f\ x = f\ y$
by *simp* **}**
then show *?thesis*
by *metis*
qed

lemma *has-field-derivative-zero-constant*:
assumes *convex s* $\bigwedge x.\ x \in s \Longrightarrow (f$ *has-field-derivative 0*) $(at\ x\ within\ s)$
shows $\exists c.\ \forall x{\in}s.\ f\ (x) = (c :: 'a :: real\text{-}normed\text{-}field)$
proof (*rule has-derivative-zero-constant*)
have A: $op * 0 = (\lambda\text{-}.\ 0 :: 'a)$ **by** (*intro ext*) *simp*
fix x **assume** $x \in s$ **thus** $(f$ *has-derivative* $(\lambda h.\ 0))$ $(at\ x\ within\ s)$
using *assms*(*2*)[*of x*] **by** (*simp add*: *has-field-derivative-def A*)
qed *fact*

lemma *has-derivative-zero-unique*:
fixes f :: *'a*::*real-normed-vector* ⇒ *'b*::*real-normed-vector*
assumes *convex s*
and $\bigwedge x.\ x \in s \Longrightarrow (f$ *has-derivative* $(\lambda h.\ 0))$ $(at\ x\ within\ s)$
and $x \in s\ y \in s$
shows $f\ x = f\ y$
using *has-derivative-zero-constant*[*OF assms*(*1*,*2*)] *assms*(*3*−) **by** *force*

lemma *has-derivative-zero-unique-connected*:
fixes f :: *'a*::*real-normed-vector* ⇒ *'b*::*real-normed-vector*

assumes *open s connected s*
assumes *f*: $\bigwedge x.\ x \in s \Longrightarrow (f$ *has-derivative* $(\lambda x.\ 0))\ (at\ x)$
assumes $x \in s\ y \in s$
shows $f\ x = f\ y$
proof (*rule connected-local-const*[**where** *f*=*f*, *OF* ‹*connected s*› ‹*x*∈*s*› ‹*y*∈*s*›])
show $\forall\, a{\in}s.$ *eventually* $(\lambda b.\ f\ a = f\ b)\ (at\ a\ within\ s)$
proof
fix *a* **assume** $a \in s$
with ‹*open s*› **obtain** *e* **where** $0 < e$ *ball a e* $\subseteq s$
by (*rule openE*)
then have $\exists\, c.\ \forall\, x{\in}ball\ a\ e.\ f\ x = c$
by (*intro has-derivative-zero-constant*)
(*auto simp*: *at-within-open*[*OF - open-ball*] *f convex-ball*)
with ‹*0*<*e*› **have** $\forall\, x{\in}ball\ a\ e.\ f\ a = f\ x$
by *auto*
then show *eventually* $(\lambda b.\ f\ a = f\ b)\ (at\ a\ within\ s)$
using ‹*0*<*e*› **unfolding** *eventually-at-topological*
by (*intro exI*[*of - ball a e*]) *auto*
qed
qed

40.12 Differentiability of inverse function (most basic form)

lemma *has-derivative-inverse-basic*:
fixes *f* :: *'a*::*real-normed-vector* ⇒ *'b*::*real-normed-vector*
assumes (*f has-derivative f'*) $(at\ (g\ y))$
and *bounded-linear g'*
and $g' \circ f' = id$
and *continuous* $(at\ y)\ g$
and *open t*
and $y \in t$
and $\forall\, z{\in}t.\ f\ (g\ z) = z$
shows (*g has-derivative g'*) $(at\ y)$
proof −
interpret *f'*: *bounded-linear f'*
using *assms* **unfolding** *has-derivative-def* **by** *auto*
interpret *g'*: *bounded-linear g'*
using *assms* **by** *auto*
obtain *C* **where** *C*: $0 < C\ \bigwedge x.\ norm\ (g'\ x) \le norm\ x * C$
using *bounded-linear.pos-bounded*[*OF assms*(*2*)] **by** *blast*
have *lem1*: $\forall\, e{>}0.\ \exists\, d{>}0.\ \forall\, z.$
$norm\ (z - y) < d \longrightarrow norm\ (g\ z - g\ y - g'(z - y)) \le e * norm\ (g\ z - g\ y)$
proof (*rule*, *rule*)
fix *e* :: *real*
assume $e > 0$
with *C*(*1*) **have** ∗: $e\ /\ C > 0$ **by** *auto*
obtain *d0* **where** *d0*:
$0 < d0$
$\forall\, ya.\ norm\ (ya - g\ y) < d0 \longrightarrow norm\ (f\ ya - f\ (g\ y) - f'\ (ya - g\ y)) \le$

```
e / C * norm (ya - g y)
      using assms(1)
      unfolding has-derivative-at-alt
      using * by blast
    obtain d1 where d1:
        0 < d1
        ∀ x. 0 < dist x y ∧ dist x y < d1 ⟶ dist (g x) (g y) < d0
      using assms(4)
      unfolding continuous-at Lim-at
      using d0(1) by blast
    obtain d2 where d2:
        0 < d2
        ∀ ya. dist ya y < d2 ⟶ ya ∈ t
      using assms(5)
      unfolding open-dist
      using assms(6) by blast
    obtain d where d: 0 < d d < d1 d < d2
      using real-lbound-gt-zero[OF d1(1) d2(1)] by blast
    then show ∃ d>0. ∀ z. norm (z - y) < d ⟶ norm (g z - g y - g' (z - y))
≤ e * norm (g z - g y)
      apply (rule-tac x=d in exI)
      apply rule
      defer
      apply rule
      apply rule
    proof -
      fix z
      assume as: norm (z - y) < d
      then have z ∈ t
        using d2 d unfolding dist-norm by auto
      have norm (g z - g y - g' (z - y)) ≤ norm (g' (f (g z) - y - f' (g z - g
y)))
        unfolding g'.diff f'.diff
        unfolding assms(3)[unfolded o-def id-def, THEN fun-cong]
        unfolding assms(7)[rule-format,OF ‹z∈t›]
        apply (subst norm-minus-cancel[symmetric])
        apply auto
        done
      also have ... ≤ norm (f (g z) - y - f' (g z - g y)) * C
        by (rule C(2))
      also have ... ≤ (e / C) * norm (g z - g y) * C
        apply (rule mult-right-mono)
        apply (rule d0(2)[rule-format,unfolded assms(7)[rule-format,OF ‹y∈t›]])
        apply (cases z = y)
        defer
        apply (rule d1(2)[unfolded dist-norm,rule-format])
        using as d C d0
        apply auto
        done
```

```
    also have ... ≤ e * norm (g z - g y)
      using C by (auto simp add: field-simps)
    finally show norm (g z - g y - g' (z - y)) ≤ e * norm (g z - g y)
      by simp
  qed auto
 qed
 have *: (0::real) < 1 / 2
  by auto
 obtain d where d:
    0 < d
    ∀z. norm (z - y) < d ⟶ norm (g z - g y - g' (z - y)) ≤ 1 / 2 * norm
(g z - g y)
  using lem1 * by blast
 def B ≡ C * 2
 have B > 0
  unfolding B-def using C by auto
 have lem2: norm (g z - g y) ≤ B * norm (z - y) if z: norm(z - y) < d for z
 proof -
  have norm (g z - g y) ≤ norm(g' (z - y)) + norm ((g z - g y) - g'(z - y))
    by (rule norm-triangle-sub)
  also have ... ≤ norm (g' (z - y)) + 1 / 2 * norm (g z - g y)
    apply (rule add-left-mono)
    using d and z
    apply auto
    done
  also have ... ≤ norm (z - y) * C + 1 / 2 * norm (g z - g y)
    apply (rule add-right-mono)
    using C
    apply auto
    done
  finally show norm (g z - g y) ≤ B * norm (z - y)
    unfolding B-def
    by (auto simp add: field-simps)
 qed
 show ?thesis
  unfolding has-derivative-at-alt
  apply rule
  apply (rule assms)
  apply rule
  apply rule
 proof -
  fix e :: real
  assume e > 0
  then have *: e / B > 0 by (metis ‹B > 0› divide-pos-pos)
  obtain d' where d':
     0 < d'
    ∀z. norm (z - y) < d' ⟶ norm (g z - g y - g' (z - y)) ≤ e / B * norm
(g z - g y)
    using lem1 * by blast
```

```
    obtain k where k: 0 < k k < d k < d'
      using real-lbound-gt-zero[OF d(1) d'(1)] by blast
    show ∃ d>0. ∀ ya. norm (ya − y) < d ⟶ norm (g ya − g y − g' (ya − y))
≤ e * norm (ya − y)
      apply (rule-tac x=k in exI)
      apply auto
    proof −
      fix z
      assume as: norm (z − y) < k
      then have norm (g z − g y − g' (z − y)) ≤ e / B * norm(g z − g y)
        using d' k by auto
      also have ... ≤ e * norm (z − y)
        unfolding times-divide-eq-left pos-divide-le-eq[OF ‹B>0›]
        using lem2[of z]
        using k as using ‹e > 0›
        by (auto simp add: field-simps)
      finally show norm (g z − g y − g' (z − y)) ≤ e * norm (z − y)
        by simp
    qed(insert k, auto)
  qed
qed
```

Simply rewrite that based on the domain point x.

```
lemma has-derivative-inverse-basic-x:
  fixes f :: 'a::real-normed-vector ⇒ 'b::real-normed-vector
  assumes (f has-derivative f') (at x)
    and bounded-linear g'
    and g' ∘ f' = id
    and continuous (at (f x)) g
    and g (f x) = x
    and open t
    and f x ∈ t
    and ∀ y∈t. f (g y) = y
  shows (g has-derivative g') (at (f x))
  apply (rule has-derivative-inverse-basic)
  using assms
  apply auto
  done
```

This is the version in Dieudonne', assuming continuity of f and g.

```
lemma has-derivative-inverse-dieudonne:
  fixes f :: 'a::real-normed-vector ⇒ 'b::real-normed-vector
  assumes open s
    and open (f ‘ s)
    and continuous-on s f
    and continuous-on (f ‘ s) g
    and ∀ x∈s. g (f x) = x
    and x ∈ s
    and (f has-derivative f') (at x)
```

and *bounded-linear* g'
and $g' \circ f' = id$
shows (*g has-derivative g'*) (*at* (*f x*))
apply (*rule has-derivative-inverse-basic-x*[*OF assms*(7−9) - - *assms*(2)])
using *assms*(3−6)
unfolding *continuous-on-eq-continuous-at*[*OF assms*(1)] *continuous-on-eq-continuous-at*[*OF assms*(2)]
apply *auto*
done

Here's the simplest way of not assuming much about g.

lemma *has-derivative-inverse*:
fixes *f* :: 'a::*real-normed-vector* ⇒ 'b::*real-normed-vector*
assumes *compact s*
and $x \in s$
and $f\,x \in$ *interior* (*f* ' *s*)
and *continuous-on s f*
and $\forall\, y{\in}s.\ g\ (f\ y) = y$
and (*f has-derivative f'*) (*at x*)
and *bounded-linear* g'
and $g' \circ f' = id$
shows (*g has-derivative g'*) (*at* (*f x*))
proof −
{
fix *y*
assume $y \in$ *interior* (*f* ' *s*)
then obtain *x* **where** $x \in s$ **and** ∗: $y = f\,x$
unfolding *image-iff*
using *interior-subset*
by *auto*
have *f* (*g y*) = *y*
unfolding ∗ **and** *assms*(5)[*rule-format*,*OF* ‹*x*∈*s*›] **..**
} note ∗ = *this*
show *?thesis*
apply (*rule has-derivative-inverse-basic-x*[*OF assms*(6−8)])
apply (*rule continuous-on-interior*[*OF* - *assms*(3)])
apply (*rule continuous-on-inv*[*OF assms*(4,1)])
apply (*rule assms*(2,5) *assms*(5)[*rule-format*] *open-interior assms*(3))+
apply (*metis* ∗)
done
qed

40.13 Proving surjectivity via Brouwer fixpoint theorem

lemma *brouwer-surjective*:
fixes *f* :: 'n::*euclidean-space* ⇒ 'n
assumes *compact t*
and *convex t*
and $t \neq \{\}$

```
    and continuous-on t f
    and ∀x∈s. ∀y∈t. x + (y − f y) ∈ t
    and x ∈ s
  shows ∃y∈t. f y = x
proof −
  have ∗: ⋀x y. f y = x ⟷ x + (y − f y) = y
    by (auto simp add: algebra-simps)
  show ?thesis
    unfolding ∗
    apply (rule brouwer[OF assms(1−3), of λy. x + (y − f y)])
    apply (rule continuous-intros assms)+
    using assms(4−6)
    apply auto
    done
qed

lemma brouwer-surjective-cball:
  fixes f :: 'n::euclidean-space ⇒ 'n
  assumes e > 0
    and continuous-on (cball a e) f
    and ∀x∈s. ∀y∈cball a e. x + (y − f y) ∈ cball a e
    and x ∈ s
  shows ∃y∈cball a e. f y = x
  apply (rule brouwer-surjective)
  apply (rule compact-cball convex-cball)+
  unfolding cball-eq-empty
  using assms
  apply auto
  done
```

See Sussmann: "Multidifferential calculus", Theorem 2.1.1

```
lemma sussmann-open-mapping:
  fixes f :: 'a::real-normed-vector ⇒ 'b::euclidean-space
  assumes open s
    and continuous-on s f
    and x ∈ s
    and (f has-derivative f') (at x)
    and bounded-linear g' f' ∘ g' = id
    and t ⊆ s
    and x ∈ interior t
  shows f x ∈ interior (f ‘ t)
proof −
  interpret f': bounded-linear f'
    using assms
    unfolding has-derivative-def
    by auto
  interpret g': bounded-linear g'
    using assms
    by auto
```

```
  obtain B where B: 0 < B ∀ x. norm (g' x) ≤ norm x * B
    using bounded-linear.pos-bounded[OF assms(5)] by blast
  hence *: 1 / (2 * B) > 0 by auto
  obtain e0 where e0:
      0 < e0
     ∀ y. norm (y - x) < e0 ⟶ norm (f y - f x - f' (y - x)) ≤ 1 / (2 * B) *
norm (y - x)
    using assms(4)
    unfolding has-derivative-at-alt
    using * by blast
  obtain e1 where e1: 0 < e1 cball x e1 ⊆ t
    using assms(8)
    unfolding mem-interior-cball
    by blast
  have *: 0 < e0 / B 0 < e1 / B using e0 e1 B by auto
  obtain e where e: 0 < e e < e0 / B e < e1 / B
    using real-lbound-gt-zero[OF *] by blast
  have ∀ z∈cball (f x) (e / 2). ∃ y∈cball (f x) e. f (x + g' (y - f x)) = z
    apply rule
    apply (rule brouwer-surjective-cball[where s=cball (f x) (e/2)])
    prefer 3
    apply rule
    apply rule
  proof−
    show continuous-on (cball (f x) e) (λy. f (x + g' (y - f x)))
      unfolding g'.diff
      apply (rule continuous-on-compose[of - - f, unfolded o-def])
      apply (rule continuous-intros linear-continuous-on[OF assms(5)])+
      apply (rule continuous-on-subset[OF assms(2)])
      apply rule
      apply (unfold image-iff)
      apply (erule bexE)
    proof−
      fix y z
      assume as: y ∈cball (f x) e z = x + (g' y - g' (f x))
      have dist x z = norm (g' (f x) - g' y)
        unfolding as(2) and dist-norm by auto
      also have ... ≤ norm (f x - y) * B
        unfolding g'.diff[symmetric]
        using B
        by auto
      also have ... ≤ e * B
        using as(1)[unfolded mem-cball dist-norm]
        using B
        by auto
      also have ... ≤ e1
        using e
        unfolding less-divide-eq
        using B
```

```
      by auto
    finally have z ∈ cball x e1
      unfolding mem-cball
      by force
    then show z ∈ s
      using e1 assms(7) by auto
  qed
next
  fix y z
  assume as: y ∈ cball (f x) (e / 2) z ∈ cball (f x) e
  have norm (g' (z - f x)) ≤ norm (z - f x) * B
    using B by auto
  also have ... ≤ e * B
    apply (rule mult-right-mono)
    using as(2)[unfolded mem-cball dist-norm] and B
    unfolding norm-minus-commute
    apply auto
    done
  also have ... < e0
    using e and B
    unfolding less-divide-eq
    by auto
  finally have *: norm (x + g' (z - f x) - x) < e0
    by auto
  have **: f x + f' (x + g' (z - f x) - x) = z
    using assms(6)[unfolded o-def id-def,THEN cong]
    by auto
  have norm (f x - (y + (z - f (x + g' (z - f x))))) ≤
      norm (f (x + g' (z - f x)) - z) + norm (f x - y)
    using norm-triangle-ineq[of f (x + g'(z - f x)) - z f x - y]
    by (auto simp add: algebra-simps)
  also have ... ≤ 1 / (B * 2) * norm (g' (z - f x)) + norm (f x - y)
    using e0(2)[rule-format, OF *]
    unfolding algebra-simps **
    by auto
  also have ... ≤ 1 / (B * 2) * norm (g' (z - f x)) + e/2
    using as(1)[unfolded mem-cball dist-norm]
    by auto
  also have ... ≤ 1 / (B * 2) * B * norm (z - f x) + e/2
    using * and B
    by (auto simp add: field-simps)
  also have ... ≤ 1 / 2 * norm (z - f x) + e/2
    by auto
  also have ... ≤ e/2 + e/2
    apply (rule add-right-mono)
    using as(2)[unfolded mem-cball dist-norm]
    unfolding norm-minus-commute
    apply auto
    done
```

```
    finally show y + (z - f (x + g' (z - f x))) ∈ cball (f x) e
      unfolding mem-cball dist-norm
      by auto
  qed (insert e, auto) note lem = this
  show ?thesis
    unfolding mem-interior
    apply (rule-tac x=e/2 in exI)
    apply rule
    apply (rule divide-pos-pos)
    prefer 3
  proof
    fix y
    assume y ∈ ball (f x) (e / 2)
    then have *: y ∈ cball (f x) (e / 2)
      by auto
    obtain z where z: z ∈ cball (f x) e f (x + g' (z - f x)) = y
      using lem * by blast
    then have norm (g' (z - f x)) ≤ norm (z - f x) * B
      using B
      by (auto simp add: field-simps)
    also have ... ≤ e * B
      apply (rule mult-right-mono)
      using z(1)
      unfolding mem-cball dist-norm norm-minus-commute
      using B
      apply auto
      done
    also have ... ≤ e1
      using e B unfolding less-divide-eq by auto
    finally have x + g'(z - f x) ∈ t
      apply -
      apply (rule e1(2)[unfolded subset-eq,rule-format])
      unfolding mem-cball dist-norm
      apply auto
      done
    then show y ∈ f ' t
      using z by auto
  qed (insert e, auto)
qed
```

Hence the following eccentric variant of the inverse function theorem. This has no continuity assumptions, but we do need the inverse function. We could put $f' \circ g = I$ but this happens to fit with the minimal linear algebra theory I've set up so far.

```
lemma right-inverse-linear:
  fixes f :: 'a::euclidean-space ⇒ 'a
  assumes lf: linear f
    and gf: f ∘ g = id
  shows linear g
```

proof −
from *gf* **have** *fi*: *surj f*
by (*auto simp add*: *surj-def o-def id-def*) *metis*
from *linear-surjective-isomorphism*[*OF lf fi*]
obtain *h*:: *'a* ⇒ *'a* **where** *h*: *linear h* ∀ *x*. *h* (*f x*) = *x* ∀ *x*. *f* (*h x*) = *x*
by *blast*
have *h* = *g*
apply (*rule ext*)
using *gf h*(*2*,*3*)
apply (*simp add*: *o-def id-def fun-eq-iff*)
apply *metis*
done
with *h*(*1*) **show** *?thesis* **by** *blast*
qed

lemma *has-derivative-inverse-strong*:
fixes *f* :: *'n*::*euclidean-space* ⇒ *'n*
assumes *open s*
and *x* ∈ *s*
and *continuous-on s f*
and ∀ *x*∈*s*. *g* (*f x*) = *x*
and (*f has-derivative f'*) (*at x*)
and *f'* ∘ *g'* = *id*
shows (*g has-derivative g'*) (*at* (*f x*))
proof −
have *linf*: *bounded-linear f'*
using *assms*(*5*) **unfolding** *has-derivative-def* **by** *auto*
then have *ling*: *bounded-linear g'*
unfolding *linear-conv-bounded-linear*[*symmetric*]
apply −
apply (*rule right-inverse-linear*)
using *assms*(*6*)
apply *auto*
done
moreover have *g'* ∘ *f'* = *id*
using *assms*(*6*) *linf ling*
unfolding *linear-conv-bounded-linear*[*symmetric*]
using *linear-inverse-left*
by *auto*
moreover have ∗:∀ *t*⊆*s*. *x* ∈ *interior t* ⟶ *f x* ∈ *interior* (*f* ‘ *t*)
apply *clarify*
apply (*rule sussmann-open-mapping*)
apply (*rule assms ling*)+
apply *auto*
done
have *continuous* (*at* (*f x*)) *g*
unfolding *continuous-at Lim-at*
proof (*rule*, *rule*)
fix *e* :: *real*

```
    assume e > 0
    then have f x ∈ interior (f ‘ (ball x e ∩ s))
      using ∗[rule-format,of ball x e ∩ s] ‹x ∈ s›
      by (auto simp add: interior-open[OF open-ball] interior-open[OF assms(1)])
    then obtain d where d: 0 < d ball (f x) d ⊆ f ‘ (ball x e ∩ s)
      unfolding mem-interior by blast
    show ∃ d>0. ∀ y. 0 < dist y (f x) ∧ dist y (f x) < d ⟶ dist (g y) (g (f x))
< e
      apply (rule-tac x=d in exI)
      apply rule
      apply (rule d(1))
      apply rule
      apply rule
    proof −
      fix y
      assume 0 < dist y (f x) ∧ dist y (f x) < d
      then have g y ∈ g ‘ f ‘ (ball x e ∩ s)
        using d(2)[unfolded subset-eq,THEN bspec[where x=y]]
        by (auto simp add: dist-commute)
      then have g y ∈ ball x e ∩ s
        using assms(4) by auto
      then show dist (g y) (g (f x)) < e
        using assms(4)[rule-format,OF ‹x ∈ s›]
        by (auto simp add: dist-commute)
    qed
  qed
  moreover have f x ∈ interior (f ‘ s)
    apply (rule sussmann-open-mapping)
    apply (rule assms ling)+
    using interior-open[OF assms(1)] and ‹x ∈ s›
    apply auto
    done
  moreover have f (g y) = y if y ∈ interior (f ‘ s) for y
  proof −
    from that have y ∈ f ‘ s
      using interior-subset by auto
    then obtain z where z ∈ s y = f z unfolding image-iff ..
    then show ?thesis
      using assms(4) by auto
  qed
  ultimately show ?thesis using assms
    by (metis has-derivative-inverse-basic-x open-interior)
qed
```

A rewrite based on the other domain.

```
lemma has-derivative-inverse-strong-x:
  fixes f :: 'a::euclidean-space ⇒ 'a
  assumes open s
    and g y ∈ s
```

and *continuous-on s f*
and $\forall x{\in}s.\ g\ (f\ x) = x$
and (*f has-derivative f'*) (*at* (*g y*))
and $f' \circ g' = id$
and $f\ (g\ y) = y$
shows (*g has-derivative g'*) (*at y*)
using *has-derivative-inverse-strong*[*OF assms*(*1*−*6*)]
unfolding *assms*(*7*)
by *simp*

On a region.

lemma *has-derivative-inverse-on*:
fixes *f* :: *'n*::*euclidean-space* ⇒ *'n*
assumes *open s*
and $\forall x{\in}s.$ (*f has-derivative f'*(*x*)) (*at x*)
and $\forall x{\in}s.\ g\ (f\ x) = x$
and $f'\ x \circ g'\ x = id$
and $x \in s$
shows (*g has-derivative g'*(*x*)) (*at* (*f x*))
apply (*rule has-derivative-inverse-strong*[**where** *g'*=*g' x* **and** *f*=*f*])
apply (*rule assms*)+
unfolding *continuous-on-eq-continuous-at*[*OF assms*(*1*)]
apply *rule*
apply (*rule differentiable-imp-continuous-within*)
unfolding *differentiable-def*
using *assms*
apply *auto*
done

Invertible derivative continous at a point implies local injectivity. It's only for this we need continuity of the derivative, except of course if we want the fact that the inverse derivative is also continuous. So if we know for some other reason that the inverse function exists, it's OK.

proposition *has-derivative-locally-injective*:
fixes *f* :: *'n*::*euclidean-space* ⇒ *'m*::*euclidean-space*
assumes $a \in s$
and *open s*
and *bounded-linear g'*
and $g' \circ f'\ a = id$
and $\bigwedge x.\ x \in s \Longrightarrow$ (*f has-derivative f' x*) (*at x*)
and $\bigwedge e.\ e > 0 \Longrightarrow \exists\, d{>}0.\ \forall x.\ dist\ a\ x < d \longrightarrow onorm\ (\lambda v.\ f'\ x\ v - f'\ a\ v) < e$
obtains *r* **where** $r > 0$ *ball a r* ⊆ *s inj-on f* (*ball a r*)
proof −
interpret *bounded-linear g'*
using *assms* **by** *auto*
note $f'g' = assms(4)[unfolded\ id\text{-}def\ o\text{-}def, THEN\ cong]$
have $g'\ (f'\ a\ (\sum Basis)) = (\sum Basis)\ (\sum Basis) \neq (0::'n)$
defer

```
    apply (subst euclidean-eq-iff)
    using f'g'
    apply auto
    done
  then have ∗: 0 < onorm g'
    unfolding onorm-pos-lt[OF assms(3)]
    by fastforce
  def k ≡ 1 / onorm g' / 2
  have ∗: k > 0
    unfolding k-def using ∗ by auto
  obtain d1 where d1:
      0 < d1
      ⋀x. dist a x < d1 ⟹ onorm (λv. f' x v − f' a v) < k
    using assms(6) ∗ by blast
  from ‹open s› obtain d2 where d2 > 0 ball a d2 ⊆ s
    using ‹a∈s› ..
  obtain d2 where d2 > 0 ball a d2 ⊆ s
    using assms(2,1) ..
  obtain d2 where d2: 0 < d2 ball a d2 ⊆ s
    using assms(2)
    unfolding open-contains-ball
    using ‹a∈s› by blast
  obtain d where d: 0 < d d < d1 d < d2
    using real-lbound-gt-zero[OF d1(1) d2(1)] by blast
  show ?thesis
  proof
    show 0 < d by (fact d)
    show ball a d ⊆ s
      using ‹d < d2› ‹ball a d2 ⊆ s› by auto
    show inj-on f (ball a d)
    unfolding inj-on-def
    proof (intro strip)
      fix x y
      assume as: x ∈ ball a d y ∈ ball a d f x = f y
      def ph ≡ λw. w − g' (f w − f x)
      have ph':ph = g' ∘ (λw. f' a w − (f w − f x))
        unfolding ph-def o-def
        unfolding diff
        using f'g'
        by (auto simp: algebra-simps)
      have norm (ph x − ph y) ≤ (1 / 2) ∗ norm (x − y)
        apply (rule differentiable-bound[OF convex-ball - - as(1−2), where f'=λx
v. v − g'(f' x v)])
        apply (rule-tac[!] ballI)
      proof −
        fix u
        assume u: u ∈ ball a d
        then have u ∈ s
          using d d2 by auto
```

```
      have ∗: (λv. v − g′ (f′ u v)) = g′ ∘ (λw. f′ a w − f′ u w)
        unfolding o-def and diff
        using f′g′ by auto
      show (ph has-derivative (λv. v − g′ (f′ u v))) (at u within ball a d)
        unfolding ph′ ∗
        apply (simp add: comp-def)
        apply (rule bounded-linear.has-derivative[OF assms(3)])
        apply (rule derivative-intros)
        defer
        apply (rule has-derivative-sub[where g′=λx.0,unfolded diff-0-right])
        apply (rule has-derivative-at-within)
        using assms(5) and ‹u ∈ s› ‹a ∈ s›
       apply (auto intro!: derivative-intros bounded-linear.has-derivative[of - λx.
x] has-derivative-bounded-linear)
        done
      have ∗∗: bounded-linear (λx. f′ u x − f′ a x) bounded-linear (λx. f′ a x −
f′ u x)
        apply (rule-tac[!] bounded-linear-sub)
        apply (rule-tac[!] has-derivative-bounded-linear)
        using assms(5) ‹u ∈ s› ‹a ∈ s›
        apply auto
        done
      have onorm (λv. v − g′ (f′ u v)) ≤ onorm g′ ∗ onorm (λw. f′ a w − f′ u
w)
        unfolding ∗
        apply (rule onorm-compose)
        apply (rule assms(3) ∗∗)+
        done
      also have ... ≤ onorm g′ ∗ k
        apply (rule mult-left-mono)
        using d1(2)[of u]
        using onorm-neg[where f=λx. f′ u x − f′ a x]
        using d and u and onorm-pos-le[OF assms(3)]
        apply (auto simp: algebra-simps)
        done
      also have ... ≤ 1 / 2
        unfolding k-def by auto
      finally show onorm (λv. v − g′ (f′ u v)) ≤ 1 / 2 .
    qed
    moreover have norm (ph y − ph x) = norm (y − x)
      apply (rule arg-cong[where f=norm])
      unfolding ph-def
      using diff
      unfolding as
      apply auto
      done
    ultimately show x = y
      unfolding norm-minus-commute by auto
  qed
```

qed
qed

40.14 Uniformly convergent sequence of derivatives

lemma *has-derivative-sequence-lipschitz-lemma*:
fixes $f :: nat \Rightarrow 'a::real\text{-}normed\text{-}vector \Rightarrow 'b::real\text{-}normed\text{-}vector$
assumes *convex s*
and $\forall n.\ \forall x{\in}s.\ ((f\ n)\ has\text{-}derivative\ (f'\ n\ x))\ (at\ x\ within\ s)$
and $\forall n{\geq}N.\ \forall x{\in}s.\ \forall h.\ norm\ (f'\ n\ x\ h - g'\ x\ h) \leq e * norm\ h$
and $0 \leq e$
shows $\forall m{\geq}N.\ \forall n{\geq}N.\ \forall x{\in}s.\ \forall y{\in}s.\ norm\ ((f\ m\ x - f\ n\ x) - (f\ m\ y - f\ n\ y)) \leq 2 * e * norm\ (x - y)$
proof *rule*+
fix *m n x y*
assume *as*: $N \leq m\ N \leq n\ x \in s\ y \in s$
show $norm\ ((f\ m\ x - f\ n\ x) - (f\ m\ y - f\ n\ y)) \leq 2 * e * norm\ (x - y)$
apply (*rule differentiable-bound*[**where** $f'{=}\lambda x\ h.\ f'\ m\ x\ h - f'\ n\ x\ h$, *OF assms*(*1*) - - *as*(*3*−*4*)])
apply (*rule-tac*[!] *ballI*)
proof −
fix *x*
assume $x \in s$
show $((\lambda a.\ f\ m\ a - f\ n\ a)\ has\text{-}derivative\ (\lambda h.\ f'\ m\ x\ h - f'\ n\ x\ h))\ (at\ x\ within\ s)$
by (*rule derivative-intros assms*(*2*)[*rule-format*] ‹*x*∈*s*›)+
show $onorm\ (\lambda h.\ f'\ m\ x\ h - f'\ n\ x\ h) \leq 2 * e$
proof (*rule onorm-bound*)
fix *h*
have $norm\ (f'\ m\ x\ h - f'\ n\ x\ h) \leq norm\ (f'\ m\ x\ h - g'\ x\ h) + norm\ (f'\ n\ x\ h - g'\ x\ h)$
using *norm-triangle-ineq*[*of* $f'\ m\ x\ h - g'\ x\ h$ $-f'\ n\ x\ h + g'\ x\ h$]
unfolding *norm-minus-commute*
by (*auto simp add*: *algebra-simps*)
also have $\ldots \leq e * norm\ h + e * norm\ h$
using *assms*(*3*)[*rule-format*,*OF* ‹$N \leq m$› ‹$x \in s$›, *of h*]
using *assms*(*3*)[*rule-format*,*OF* ‹$N \leq n$› ‹$x \in s$›, *of h*]
by (*auto simp add*: *field-simps*)
finally show $norm\ (f'\ m\ x\ h - f'\ n\ x\ h) \leq 2 * e * norm\ h$
by *auto*
qed (*simp add*: ‹$0 \leq e$›)
qed
qed

lemma *has-derivative-sequence-lipschitz*:
fixes $f :: nat \Rightarrow 'a::real\text{-}normed\text{-}vector \Rightarrow 'b::real\text{-}normed\text{-}vector$
assumes *convex s*
and $\forall n.\ \forall x{\in}s.\ ((f\ n)\ has\text{-}derivative\ (f'\ n\ x))\ (at\ x\ within\ s)$
and $\forall e{>}0.\ \exists N.\ \forall n{\geq}N.\ \forall x{\in}s.\ \forall h.\ norm\ (f'\ n\ x\ h - g'\ x\ h) \leq e * norm\ h$

shows $\forall e{>}0.\ \exists N.\ \forall m{\geq}N.\ \forall n{\geq}N.\ \forall x{\in}s.\ \forall y{\in}s.$
$norm\ ((f\ m\ x - f\ n\ x) - (f\ m\ y - f\ n\ y)) \leq e * norm\ (x - y)$
proof (*rule*, *rule*)
fix e :: *real*
assume $e > 0$
then have $*$: $2 * (1/2 * e) = e\ 1/2 * e > 0$
by *auto*
obtain N **where** $\forall n{\geq}N.\ \forall x{\in}s.\ \forall h.\ norm\ (f'\ n\ x\ h - g'\ x\ h) \leq 1\ /\ 2 * e * norm\ h$
using *assms*(3) $*$(2) **by** *blast*
then show $\exists N.\ \forall m{\geq}N.\ \forall n{\geq}N.\ \forall x{\in}s.\ \forall y{\in}s.\ norm\ (f\ m\ x - f\ n\ x - (f\ m\ y - f\ n\ y)) \leq e * norm\ (x - y)$
apply (*rule-tac* $x{=}N$ **in** *exI*)
apply (*rule has-derivative-sequence-lipschitz-lemma*[**where** $e{=}1/2 * e$, *unfolded* $*$])
using *assms* ‹$e > 0$›
apply *auto*
done
qed

lemma *has-derivative-sequence*:
fixes f :: $nat \Rightarrow 'a{::}real\text{-}normed\text{-}vector \Rightarrow 'b{::}banach$
assumes *convex s*
and $\forall n.\ \forall x{\in}s.\ ((f\ n)\ has\text{-}derivative\ (f'\ n\ x))\ (at\ x\ within\ s)$
and $\forall e{>}0.\ \exists N.\ \forall n{\geq}N.\ \forall x{\in}s.\ \forall h.\ norm\ (f'\ n\ x\ h - g'\ x\ h) \leq e * norm\ h$
and $x0 \in s$
and $((\lambda n.\ f\ n\ x0) \longrightarrow l)\ sequentially$
shows $\exists g.\ \forall x{\in}s.\ ((\lambda n.\ f\ n\ x) \longrightarrow g\ x)\ sequentially \wedge (g\ has\text{-}derivative\ g'(x))\ (at\ x\ within\ s)$
proof −
have *lem1*: $\forall e{>}0.\ \exists N.\ \forall m{\geq}N.\ \forall n{\geq}N.\ \forall x{\in}s.\ \forall y{\in}s.$
$norm\ ((f\ m\ x - f\ n\ x) - (f\ m\ y - f\ n\ y)) \leq e * norm\ (x - y)$
using *assms*(1,2,3) **by** (*rule has-derivative-sequence-lipschitz*)
have $\exists g.\ \forall x{\in}s.\ ((\lambda n.\ f\ n\ x) \longrightarrow g\ x)\ sequentially$
apply (*rule bchoice*)
unfolding *convergent-eq-cauchy*
proof
fix x
assume $x \in s$
show *Cauchy* $(\lambda n.\ f\ n\ x)$
proof (*cases* $x = x0$)
case *True*
then show *?thesis*
using *LIMSEQ-imp-Cauchy*[*OF assms*(5)] **by** *auto*
next
case *False*
show *?thesis*
unfolding *Cauchy-def*
proof (*rule*, *rule*)

```
      fix e :: real
      assume e > 0
      hence *: e / 2 > 0 e / 2 / norm (x - x0) > 0 using False by auto
      obtain M where M: ∀ m≥M. ∀ n≥M. dist (f m x0) (f n x0) < e / 2
        using LIMSEQ-imp-Cauchy[OF assms(5)]
        unfolding Cauchy-def
        using *(1) by blast
      obtain N where N:
        ∀ m≥N. ∀ n≥N.
          ∀ xa∈s. ∀ y∈s. norm (f m xa - f n xa - (f m y - f n y)) ≤
            e / 2 / norm (x - x0) * norm (xa - y)
      using lem1 *(2) by blast
      show ∃ M. ∀ m≥M. ∀ n≥M. dist (f m x) (f n x) < e
        apply (rule-tac x=max M N in exI)
      proof rule+
        fix m n
        assume as: max M N ≤m max M N≤n
        have dist (f m x) (f n x) ≤
           norm (f m x0 - f n x0) + norm (f m x - f n x - (f m x0 - f n x0))
          unfolding dist-norm
          by (rule norm-triangle-sub)
        also have ... ≤ norm (f m x0 - f n x0) + e / 2
          using N[rule-format,OF - - ‹x∈s› ‹x0∈s›, of m n] and as and False
          by auto
        also have ... < e / 2 + e / 2
          apply (rule add-strict-right-mono)
          using as and M[rule-format]
          unfolding dist-norm
          apply auto
          done
        finally show dist (f m x) (f n x) < e
          by auto
      qed
    qed
  qed
  then obtain g where g: ∀ x∈s. (λn. f n x) ⟶ g x ..
  have lem2: ∀ e>0. ∃ N. ∀ n≥N. ∀ x∈s. ∀ y∈s. norm ((f n x - f n y) - (g x -
g y)) ≤ e * norm (x - y)
  proof (rule, rule)
    fix e :: real
    assume *: e > 0
    obtain N where
      N: ∀ m≥N. ∀ n≥N. ∀ x∈s. ∀ y∈s. norm (f m x - f n x - (f m y - f n y))
≤ e * norm (x - y)
      using lem1 * by blast
    show ∃ N. ∀ n≥N. ∀ x∈s. ∀ y∈s. norm (f n x - f n y - (g x - g y)) ≤ e *
norm (x - y)
      apply (rule-tac x=N in exI)
```

```
    proof rule+
      fix n x y
      assume as: N ≤ n x ∈ s y ∈ s
      have ((λm. norm (f n x − f n y − (f m x − f m y))) ⟶ norm (f n x − f
n y − (g x − g y))) sequentially
        by (intro tendsto-intros g[rule-format] as)
      moreover have eventually (λm. norm (f n x − f n y − (f m x − f m y)) ≤
e ∗ norm (x − y)) sequentially
        unfolding eventually-sequentially
        apply (rule-tac x=N in exI)
        apply rule
        apply rule
      proof −
        fix m
        assume N ≤ m
        then show norm (f n x − f n y − (f m x − f m y)) ≤ e ∗ norm (x − y)
          using N[rule-format, of n m x y] and as
          by (auto simp add: algebra-simps)
      qed
      ultimately show norm (f n x − f n y − (g x − g y)) ≤ e ∗ norm (x − y)
        by (rule tendsto-ge-const[OF trivial-limit-sequentially])
    qed
  qed
  have ∀ x∈s. ((λn. f n x) ⟶ g x) sequentially ∧ (g has-derivative g′ x) (at x
within s)
    unfolding has-derivative-within-alt2
  proof (intro ballI conjI)
    fix x
    assume x ∈ s
    then show ((λn. f n x) ⟶ g x) sequentially
      by (simp add: g)
    have lem3: ∀ u. ((λn. f′ n x u) ⟶ g′ x u) sequentially
      unfolding filterlim-def le-nhds-metric-le eventually-filtermap dist-norm
    proof (intro allI impI)
      fix u
      fix e :: real
      assume e > 0
      show eventually (λn. norm (f′ n x u − g′ x u) ≤ e) sequentially
      proof (cases u = 0)
        case True
        have eventually (λn. norm (f′ n x u − g′ x u) ≤ e ∗ norm u) sequentially
          using assms(3)[folded eventually-sequentially] and ‹0 < e› and ‹x ∈ s›
          by (fast elim: eventually-mono)
        then show ?thesis
          using ‹u = 0› and ‹0 < e› by (auto elim: eventually-mono)
      next
        case False
        with ‹0 < e› have 0 < e / norm u by simp
       then have eventually (λn. norm (f′ n x u − g′ x u) ≤ e / norm u ∗ norm
```

```
u) sequentially
        using assms(3)[folded eventually-sequentially] and ‹x ∈ s›
        by (fast elim: eventually-mono)
      then show ?thesis
        using ‹u ≠ 0› by simp
    qed
  qed
  show bounded-linear (g' x)
  proof
    fix x' y z :: 'a
    fix c :: real
  note lin = assms(2)[rule-format,OF ‹x∈s›,THEN has-derivative-bounded-linear]
    show g' x (c *R x') = c *R g' x x'
      apply (rule tendsto-unique[OF trivial-limit-sequentially])
      apply (rule lem3[rule-format])
      unfolding lin[THEN bounded-linear.linear, THEN linear-cmul]
      apply (intro tendsto-intros)
      apply (rule lem3[rule-format])
      done
    show g' x (y + z) = g' x y + g' x z
      apply (rule tendsto-unique[OF trivial-limit-sequentially])
      apply (rule lem3[rule-format])
      unfolding lin[THEN bounded-linear.linear, THEN linear-add]
      apply (rule tendsto-add)
      apply (rule lem3[rule-format])+
      done
    obtain N where N: ∀h. norm (f' N x h − g' x h) ≤ 1 * norm h
      using assms(3) ‹x ∈ s› by (fast intro: zero-less-one)
    have bounded-linear (f' N x)
      using assms(2) ‹x ∈ s› by fast
    from bounded-linear.bounded [OF this]
    obtain K where K: ∀h. norm (f' N x h) ≤ norm h * K ..
    {
      fix h
      have norm (g' x h) = norm (f' N x h − (f' N x h − g' x h))
        by simp
      also have ... ≤ norm (f' N x h) + norm (f' N x h − g' x h)
        by (rule norm-triangle-ineq4)
      also have ... ≤ norm h * K + 1 * norm h
        using N K by (fast intro: add-mono)
      finally have norm (g' x h) ≤ norm h * (K + 1)
        by (simp add: ring-distribs)
    }
    then show ∃K. ∀h. norm (g' x h) ≤ norm h * K by fast
  qed
  show ∀e>0. eventually (λy. norm (g y − g x − g' x (y − x)) ≤ e * norm (y
− x)) (at x within s)
  proof (rule, rule)
    fix e :: real
```

assume $e > 0$
then have $*$: $e / 3 > 0$
by *auto*
obtain *N1* **where** *N1*: $\forall n{\geq}N1.\ \forall x{\in}s.\ \forall h.\ norm\ (f'\ n\ x\ h - g'\ x\ h) \leq e / 3 * norm\ h$
using *assms*(3) $*$ **by** *blast*
obtain *N2* **where**
N2: $\forall n{\geq}N2.\ \forall x{\in}s.\ \forall y{\in}s.\ norm\ (f\ n\ x - f\ n\ y - (g\ x - g\ y)) \leq e / 3 * norm\ (x - y)$
using *lem2* $*$ **by** *blast*
let *?N* = *max N1 N2*
have *eventually* $(\lambda y.\ norm\ (f\ ?N\ y - f\ ?N\ x - f'\ ?N\ x\ (y - x)) \leq e / 3 * norm\ (y - x))\ (at\ x\ within\ s)$
using *assms*(2)[*unfolded has-derivative-within-alt2*] **and** ‹$x \in s$› **and** $*$ **by** *fast*
moreover have *eventually* $(\lambda y.\ y \in s)\ (at\ x\ within\ s)$
unfolding *eventually-at* **by** (*fast intro*: *zero-less-one*)
ultimately show $\forall_F\ y\ in\ at\ x\ within\ s.\ norm\ (g\ y - g\ x - g'\ x\ (y - x)) \leq e * norm\ (y - x)$
proof (*rule eventually-elim2*)
fix *y*
assume $y \in s$
assume *norm* $(f\ ?N\ y - f\ ?N\ x - f'\ ?N\ x\ (y - x)) \leq e / 3 * norm\ (y - x)$
moreover have *norm* $(g\ y - g\ x - (f\ ?N\ y - f\ ?N\ x)) \leq e / 3 * norm\ (y - x)$
using *N2*[*rule-format*, *OF* - ‹$y \in s$› ‹$x \in s$›]
by (*simp add*: *norm-minus-commute*)
ultimately have *norm* $(g\ y - g\ x - f'\ ?N\ x\ (y - x)) \leq 2 * e / 3 * norm\ (y - x)$
using *norm-triangle-le*[*of* $g\ y - g\ x - (f\ ?N\ y - f\ ?N\ x)\ f\ ?N\ y - f\ ?N\ x - f'\ ?N\ x\ (y - x)\ 2 * e / 3 * norm\ (y - x)$]
by (*auto simp add*: *algebra-simps*)
moreover
have *norm* $(f'\ ?N\ x\ (y - x) - g'\ x\ (y - x)) \leq e / 3 * norm\ (y - x)$
using *N1* ‹$x \in s$› **by** *auto*
ultimately show *norm* $(g\ y - g\ x - g'\ x\ (y - x)) \leq e * norm\ (y - x)$
using *norm-triangle-le*[*of* $g\ y - g\ x - f'\ (max\ N1\ N2)\ x\ (y - x)\ f'\ (max\ N1\ N2)\ x\ (y - x) - g'\ x\ (y - x)$]
by (*auto simp add*: *algebra-simps*)
qed
qed
qed
then show *?thesis* **by** *fast*
qed

Can choose to line up antiderivatives if we want.

lemma *has-antiderivative-sequence*:
fixes f :: *nat* $\Rightarrow$ *'a*::*real-normed-vector* $\Rightarrow$ *'b*::*banach*

assumes *convex s*
and $\forall n.\ \forall x{\in}s.\ ((f\ n)$ *has-derivative* $(f'\ n\ x))$ $(at\ x\ within\ s)$
and $\forall e{>}0.\ \exists N.\ \forall n{\geq}N.\ \forall x{\in}s.\ \forall h.\ norm\ (f'\ n\ x\ h - g'\ x\ h) \leq e * norm\ h$
shows $\exists g.\ \forall x{\in}s.\ (g$ *has-derivative* $g'\ x)$ $(at\ x\ within\ s)$
proof $(cases\ s = \{\})$
case *False*
then obtain a **where** $a \in s$
by *auto*
have $*$: $\bigwedge P\ Q.\ \exists g.\ \forall x{\in}s.\ P\ g\ x \wedge Q\ g\ x \Longrightarrow \exists g.\ \forall x{\in}s.\ Q\ g\ x$
by *auto*
show *?thesis*
apply $(rule\ *)$
apply $(rule$ *has-derivative-sequence*$[OF\ assms(1)$ - $assms(3),\ of\ \lambda n\ x.\ f\ n\ x + (f\ 0\ a - f\ n\ a)])$
apply $(metis\ assms(2)$ *has-derivative-add-const*$)$
apply $(rule\ \langle a \in s\rangle)$
apply *auto*
done
qed *auto*

lemma *has-antiderivative-limit*:
fixes g' :: *'a*::*real-normed-vector* $\Rightarrow$ *'a* $\Rightarrow$ *'b*::*banach*
assumes *convex s*
and $\forall e{>}0.\ \exists f\ f'.\ \forall x{\in}s.$
$(f$ *has-derivative* $(f'\ x))$ $(at\ x\ within\ s) \wedge (\forall h.\ norm\ (f'\ x\ h - g'\ x\ h) \leq e * norm\ h)$
shows $\exists g.\ \forall x{\in}s.\ (g$ *has-derivative* $g'\ x)$ $(at\ x\ within\ s)$
proof $-$
have $*$: $\forall n.\ \exists f\ f'.\ \forall x{\in}s.$
$(f$ *has-derivative* $(f'\ x))$ $(at\ x\ within\ s) \wedge$
$(\forall h.\ norm(f'\ x\ h - g'\ x\ h) \leq inverse\ (real\ (Suc\ n)) * norm\ h)$
by $(simp\ add{:}\ assms(2))$
obtain f **where**
$*$: $\forall x.\ \exists f'.\ \forall xa{\in}s.\ (f\ x$ *has-derivative* $f'\ xa)$ $(at\ xa\ within\ s) \wedge$
$(\forall h.\ norm\ (f'\ xa\ h - g'\ xa\ h) \leq inverse\ (real\ (Suc\ x)) * norm\ h)$
using $*[THEN\ choice]$ **..**
obtain f' **where**
f: $\forall x.\ \forall xa{\in}s.\ (f\ x$ *has-derivative* $f'\ x\ xa)$ $(at\ xa\ within\ s) \wedge$
$(\forall h.\ norm\ (f'\ x\ xa\ h - g'\ xa\ h) \leq inverse\ (real\ (Suc\ x)) * norm\ h)$
using $*[THEN\ choice]$ **..**
show *?thesis*
apply $(rule$ *has-antiderivative-sequence*$[OF\ assms(1),\ of\ f\ f'])$
defer
apply *rule*
apply *rule*
proof $-$
fix e :: *real*
assume $e > 0$
obtain N **where** N: $inverse\ (real\ (Suc\ N)) < e$

using *reals-Archimedean*[*OF* ‹*e*>*0*›] **..**
show ∃ *N*. ∀ *n*≥*N*. ∀ *x*∈*s*. ∀ *h*. *norm* (*f'* *n* *x* *h* − *g'* *x* *h*) ≤ *e* ∗ *norm* *h*
apply (*rule-tac* *x*=*N* **in** *exI*)
apply *rule*
apply *rule*
apply *rule*
apply *rule*
proof −
fix *n* *x* *h*
assume *n*: *N* ≤ *n* **and** *x*: *x* ∈ *s*
have ∗: *inverse* (*real* (*Suc* *n*)) ≤ *e*
apply (*rule* *order-trans*[*OF* - *N*[*THEN* *less-imp-le*]])
using *n*
apply (*auto* *simp* *add*: *field-simps*)
done
show *norm* (*f'* *n* *x* *h* − *g'* *x* *h*) ≤ *e* ∗ *norm* *h*
using *f*[*rule-format*,*THEN* *conjunct2*, *OF* *x*, *of* *n*, *THEN* *spec*[**where** *x*=*h*]]
apply (*rule* *order-trans*)
using *N* ∗
apply (*cases* *h* = *0*)
apply *auto*
done
qed
qed (*insert* *f*, *auto*)
qed

40.15 Differentiation of a series

lemma *has-derivative-series*:
fixes *f* :: *nat* ⇒ *'a*::*real-normed-vector* ⇒ *'b*::*banach*
assumes *convex* *s*
and ⋀*n* *x*. *x* ∈ *s* ⟹ ((*f* *n*) *has-derivative* (*f'* *n* *x*)) (*at* *x* *within* *s*)
and ∀ *e*>*0*. ∃ *N*. ∀ *n*≥*N*. ∀ *x*∈*s*. ∀ *h*. *norm* (*setsum* (λ*i*. *f'* *i* *x* *h*) {..<*n*} − *g'* *x* *h*) ≤ *e* ∗ *norm* *h*
and *x* ∈ *s*
and (λ*n*. *f* *n* *x*) *sums* *l*
shows ∃ *g*. ∀ *x*∈*s*. (λ*n*. *f* *n* *x*) *sums* (*g* *x*) ∧ (*g* *has-derivative* *g'* *x*) (*at* *x* *within* *s*)
unfolding *sums-def*
apply (*rule* *has-derivative-sequence*[*OF* *assms*(*1*) - *assms*(*3*)])
apply (*metis* *assms*(*2*) *has-derivative-setsum*)
using *assms*(*4*−*5*)
unfolding *sums-def*
apply *auto*
done

lemma *has-field-derivative-series*:
fixes *f* :: *nat* ⇒ (*'a* :: {*real-normed-field*,*banach*}) ⇒ *'a*
assumes *convex* *s*

assumes $\bigwedge n\ x.\ x \in s \Longrightarrow (f\ n\ \textit{has-field-derivative}\ f'\ n\ x)\ (\textit{at}\ x\ \textit{within}\ s)$
assumes *uniform-limit s* $(\lambda n\ x.\ \sum i{<}n.\ f'\ i\ x)\ g'$ *sequentially*
assumes $x0 \in s$ *summable* $(\lambda n.\ f\ n\ x0)$
shows $\exists g.\ \forall x{\in}s.\ (\lambda n.\ f\ n\ x)$ *sums* $g\ x \wedge (g\ \textit{has-field-derivative}\ g'\ x)\ (\textit{at}\ x\ \textit{within}\ s)$
unfolding *has-field-derivative-def*
proof (*rule has-derivative-series*)
show $\forall e{>}0.\ \exists N.\ \forall n{\geq}N.\ \forall x{\in}s.\ \forall h.\ norm\ ((\sum i{<}n.\ f'\ i\ x * h) - g'\ x * h) \leq e * norm\ h$
proof (*intro allI impI*)
fix e :: *real* **assume** $e > 0$
with *assms*(3) **obtain** N **where** N: $\bigwedge n\ x.\ n \geq N \Longrightarrow x \in s \Longrightarrow norm\ ((\sum i{<}n.\ f'\ i\ x) - g'\ x) < e$
unfolding *uniform-limit-iff eventually-at-top-linorder dist-norm* **by** *blast*
{
fix n :: *nat* **and** $x\ h$ :: $'a$ **assume** nx: $n \geq N\ x \in s$
have $norm\ ((\sum i{<}n.\ f'\ i\ x * h) - g'\ x * h) = norm\ ((\sum i{<}n.\ f'\ i\ x) - g'\ x) * norm\ h$
by (*simp add*: *norm-mult* [*symmetric*] *ring-distribs setsum-left-distrib*)
also from N[*OF nx*] **have** $norm\ ((\sum i{<}n.\ f'\ i\ x) - g'\ x) \leq e$ **by** *simp*
hence $norm\ ((\sum i{<}n.\ f'\ i\ x) - g'\ x) * norm\ h \leq e * norm\ h$
by (*intro mult-right-mono*) *simp-all*
finally have $norm\ ((\sum i{<}n.\ f'\ i\ x * h) - g'\ x * h) \leq e * norm\ h$ **.**
}
thus $\exists N.\ \forall n{\geq}N.\ \forall x{\in}s.\ \forall h.\ norm\ ((\sum i{<}n.\ f'\ i\ x * h) - g'\ x * h) \leq e * norm\ h$ **by** *blast*
qed
qed (*insert assms*, *auto simp*: *has-field-derivative-def*)

lemma *has-field-derivative-series'*:
fixes f :: $nat \Rightarrow ('a :: \{real\text{-}normed\text{-}field, banach\}) \Rightarrow 'a$
assumes *convex s*
assumes $\bigwedge n\ x.\ x \in s \Longrightarrow (f\ n\ \textit{has-field-derivative}\ f'\ n\ x)\ (\textit{at}\ x\ \textit{within}\ s)$
assumes *uniformly-convergent-on s* $(\lambda n\ x.\ \sum i{<}n.\ f'\ i\ x)$
assumes $x0 \in s$ *summable* $(\lambda n.\ f\ n\ x0)\ x \in interior\ s$
shows *summable* $(\lambda n.\ f\ n\ x)\ ((\lambda x.\ \sum n.\ f\ n\ x)\ \textit{has-field-derivative}\ (\sum n.\ f'\ n\ x))\ (\textit{at}\ x)$
proof −
from ‹$x \in interior\ s$› **have** $x \in s$ **using** *interior-subset* **by** *blast*
def $g' \equiv \lambda x.\ \sum i.\ f'\ i\ x$
from *assms*(3) **have** *uniform-limit s* $(\lambda n\ x.\ \sum i{<}n.\ f'\ i\ x)\ g'$ *sequentially*
by (*simp add*: *uniformly-convergent-uniform-limit-iff suminf-eq-lim g'-def*)
from *has-field-derivative-series*[*OF assms*(1,2) *this assms*(4,5)] **obtain** g **where** g:
$\bigwedge x.\ x \in s \Longrightarrow (\lambda n.\ f\ n\ x)$ *sums* $g\ x$
$\bigwedge x.\ x \in s \Longrightarrow (g\ \textit{has-field-derivative}\ g'\ x)\ (\textit{at}\ x\ \textit{within}\ s)$ **by** *blast*
from $g(1)$[*OF* ‹$x \in s$›] **show** *summable* $(\lambda n.\ f\ n\ x)$ **by** (*simp add*: *sums-iff*)
from $g(2)$[*OF* ‹$x \in s$›] ‹$x \in interior\ s$› **have** $(g\ \textit{has-field-derivative}\ g'\ x)\ (\textit{at}\ x)$
by (*simp add*: *at-within-interior*[*of x s*])

also have (*g has-field-derivative g′ x*) (*at x*) ⟷
((λ*x*. ∑ *n*. *f n x*) *has-field-derivative g′ x*) (*at x*)
using *eventually-nhds-in-nhd*[*OF* ‹*x* ∈ *interior s*›] *interior-subset*[*of s*] *g*(*1*)
by (*intro DERIV-cong-ev*) (*auto elim*!: *eventually-mono simp*: *sums-iff*)
finally show ((λ*x*. ∑ *n*. *f n x*) *has-field-derivative g′ x*) (*at x*) **.**
qed

lemma *differentiable-series*:
fixes *f* :: *nat* ⇒ ('*a* :: {*real-normed-field*,*banach*}) ⇒ '*a*
assumes *convex s open s*
assumes ⋀*n x*. *x* ∈ *s* ⟹ (*f n has-field-derivative f′ n x*) (*at x*)
assumes *uniformly-convergent-on s* (λ*n x*. ∑ *i*<*n*. *f′ i x*)
assumes *x0* ∈ *s summable* (λ*n*. *f n x0*) **and** *x*: *x* ∈ *s*
shows *summable* (λ*n*. *f n x*) **and** (λ*x*. ∑ *n*. *f n x*) *differentiable* (*at x*)
proof −
from *assms*(*4*) **obtain** *g′* **where** *A*: *uniform-limit s* (λ*n x*. ∑ *i*<*n*. *f′ i x*) *g′* *sequentially*
unfolding *uniformly-convergent-on-def* **by** *blast*
from *x* **and** ‹*open s*› **have** *s*: *at x within s* = *at x* **by** (*rule at-within-open*)
have ∃ *g*. ∀ *x*∈*s*. (λ*n*. *f n x*) *sums g x* ∧ (*g has-field-derivative g′ x*) (*at x within s*)
by (*intro has-field-derivative-series*[*of s f f′ g′ x0*] *assms A has-field-derivative-at-within*)
then obtain *g* **where** *g*: ⋀*x*. *x* ∈ *s* ⟹ (λ*n*. *f n x*) *sums g x*
⋀*x*. *x* ∈ *s* ⟹ (*g has-field-derivative g′ x*) (*at x within s*) **by** *blast*
from *g*[*OF x*] **show** *summable* (λ*n*. *f n x*) **by** (*auto simp*: *summable-def*)
from *g*(*2*)[*OF x*] **have** *g′*: (*g has-derivative op* ∗ (*g′ x*)) (*at x*)
by (*simp add*: *has-field-derivative-def s*)
have ((λ*x*. ∑ *n*. *f n x*) *has-derivative op* ∗ (*g′ x*)) (*at x*)
by (*rule has-derivative-transform-within-open*[*OF g′* ‹*open s*› *x*])
(*insert g*, *auto simp*: *sums-iff*)
thus (λ*x*. ∑ *n*. *f n x*) *differentiable* (*at x*) **unfolding** *differentiable-def*
by (*auto simp*: *summable-def differentiable-def has-field-derivative-def*)
qed

lemma *differentiable-series′*:
fixes *f* :: *nat* ⇒ ('*a* :: {*real-normed-field*,*banach*}) ⇒ '*a*
assumes *convex s open s*
assumes ⋀*n x*. *x* ∈ *s* ⟹ (*f n has-field-derivative f′ n x*) (*at x*)
assumes *uniformly-convergent-on s* (λ*n x*. ∑ *i*<*n*. *f′ i x*)
assumes *x0* ∈ *s summable* (λ*n*. *f n x0*)
shows (λ*x*. ∑ *n*. *f n x*) *differentiable* (*at x0*)
using *differentiable-series*[*OF assms*, *of x0*] ‹*x0* ∈ *s*› **by** *blast*+

Considering derivative *real* ⇒ '*b* as a vector.

definition *vector-derivative f net* = (*SOME f′*. (*f has-vector-derivative f′*) *net*)

lemma *vector-derivative-unique-within*:
assumes *not-bot*: *at x within s* ≠ *bot*
and *f′*: (*f has-vector-derivative f′*) (*at x within s*)

```
    and f'': (f has-vector-derivative f'') (at x within s)
  shows f' = f''
proof −
  have (λx. x *R f') = (λx. x *R f'')
  proof (rule frechet-derivative-unique-within)
    show ∀i∈Basis. ∀e>0. ∃d. 0 < |d| ∧ |d| < e ∧ x + d *R i ∈ s
    proof clarsimp
      fix e :: real assume 0 < e
      with islimpt-approachable-real[of x s] not-bot
      obtain x' where x' ∈ s x' ≠ x |x' − x| < e
        by (auto simp add: trivial-limit-within)
      then show ∃d. d ≠ 0 ∧ |d| < e ∧ x + d ∈ s
        by (intro exI[of - x' − x]) auto
    qed
  qed (insert f' f'', auto simp: has-vector-derivative-def)
  then show ?thesis
    unfolding fun-eq-iff by (metis scaleR-one)
qed

lemma vector-derivative-unique-at:
  (f has-vector-derivative f') (at x) ⟹ (f has-vector-derivative f'') (at x) ⟹ f'
= f''
  by (rule vector-derivative-unique-within) auto

lemma differentiableI-vector: (f has-vector-derivative y) F ⟹ f differentiable F
  by (auto simp: differentiable-def has-vector-derivative-def)

lemma vector-derivative-works:
  f differentiable net ⟷ (f has-vector-derivative (vector-derivative f net)) net
    (is ?l = ?r)
proof
  assume ?l
  obtain f' where f': (f has-derivative f') net
    using ‹?l› unfolding differentiable-def ..
  then interpret bounded-linear f'
    by auto
  show ?r
    unfolding vector-derivative-def has-vector-derivative-def
    by (rule someI[of - f' 1]) (simp add: scaleR[symmetric] f')
qed (auto simp: vector-derivative-def has-vector-derivative-def differentiable-def)

lemma vector-derivative-within:
  assumes not-bot: at x within s ≠ bot and y: (f has-vector-derivative y) (at x
within s)
  shows vector-derivative f (at x within s) = y
  using y
 by (intro vector-derivative-unique-within[OF not-bot vector-derivative-works[THEN
iffD1] y])
     (auto simp: differentiable-def has-vector-derivative-def)
```

lemma *frechet-derivative-eq-vector-derivative*:
assumes *f differentiable* (*at x*)
shows (*frechet-derivative f* (*at x*)) = (λ*r*. *r* ∗$_R$ *vector-derivative f* (*at x*))
using *assms*
by (*auto simp*: *differentiable-iff-scaleR vector-derivative-def has-vector-derivative-def*
intro: *someI frechet-derivative-at* [*symmetric*])

lemma *has-real-derivative*:
fixes *f* :: *real* ⇒ *real*
assumes (*f has-derivative f'*) *F*
obtains *c* **where** (*f has-real-derivative c*) *F*
proof −
obtain *c* **where** *f'* = (λ*x*. *x* ∗ *c*)
by (*metis assms has-derivative-bounded-linear real-bounded-linear*)
then show *?thesis*
by (*metis assms that has-field-derivative-def mult-commute-abs*)
qed

lemma *has-real-derivative-iff*:
fixes *f* :: *real* ⇒ *real*
shows (∃ *c*. (*f has-real-derivative c*) *F*) = (∃ *D*. (*f has-derivative D*) *F*)
by (*metis has-field-derivative-def has-real-derivative*)

definition *deriv* :: ('*a* ⇒ '*a*::*real-normed-field*) ⇒ '*a* ⇒ '*a* **where**
deriv f x ≡ *SOME D*. *DERIV f x* :> *D*

lemma *DERIV-imp-deriv*: *DERIV f x* :> *f'* ⟹ *deriv f x* = *f'*
unfolding *deriv-def* **by** (*metis some-equality DERIV-unique*)

lemma *DERIV-deriv-iff-has-field-derivative*:
DERIV f x :> *deriv f x* ⟷ (∃*f'*. (*f has-field-derivative f'*) (*at x*))
by (*auto simp*: *has-field-derivative-def DERIV-imp-deriv*)

lemma *DERIV-deriv-iff-real-differentiable*:
fixes *x* :: *real*
shows *DERIV f x* :> *deriv f x* ⟷ *f differentiable at x*
unfolding *differentiable-def* **by** (*metis DERIV-imp-deriv has-real-derivative-iff*)

lemma *real-derivative-chain*:
fixes *x* :: *real*
shows *f differentiable at x* ⟹ *g differentiable at* (*f x*)
⟹ *deriv* (*g o f*) *x* = *deriv g* (*f x*) ∗ *deriv f x*
by (*metis DERIV-deriv-iff-real-differentiable DERIV-chain DERIV-imp-deriv*)
lemma *field-derivative-eq-vector-derivative*:
(*deriv f x*) = *vector-derivative f* (*at x*)
by (*simp add*: *mult.commute deriv-def vector-derivative-def has-vector-derivative-def*
has-field-derivative-def)

lemma *islimpt-closure-open*:
fixes s :: *'a::perfect-space set*
assumes *open* s **and** t: t = *closure* s $x \in t$
shows x *islimpt* t
proof *cases*
assume $x \in s$
{ fix T **assume** $x \in T$ *open* T
then have *open* $(s \cap T)$
using ‹*open* s› **by** *auto*
then have $s \cap T \neq \{x\}$
using *not-open-singleton*[*of* x] **by** *auto*
with ‹$x \in T$› ‹$x \in s$› **have** $\exists y{\in}t.\ y \in T \wedge y \neq x$
using *closure-subset*[*of* s] **by** (*auto simp*: t) **}**
then show *?thesis*
by (*auto intro*!: *islimptI*)
next
assume $x \notin s$ **with** t **show** *?thesis*
unfolding t *closure-def* **by** (*auto intro*: *islimpt-subset*)
qed

lemma *vector-derivative-unique-within-closed-interval*:
assumes *ab*: $a < b$ $x \in$ *cbox* a b
assumes D: (f *has-vector-derivative* f') (*at* x *within cbox* a b) (f *has-vector-derivative* f'') (*at* x *within cbox* a b)
shows $f' = f''$
using *ab*
by (*intro vector-derivative-unique-within*[*OF - D*])
(*auto simp*: *trivial-limit-within intro*!: *islimpt-closure-open*[**where** s={a <..< b}])

lemma *vector-derivative-at*:
(f *has-vector-derivative* f') (*at* x) $\Longrightarrow$ *vector-derivative* f (*at* x) = f'
by (*intro vector-derivative-within at-neq-bot*)

lemma *has-vector-derivative-id-at* [*simp*]: *vector-derivative* ($\lambda x.\ x$) (*at* a) = *1*
by (*simp add*: *vector-derivative-at*)

lemma *vector-derivative-minus-at* [*simp*]:
f *differentiable at* a
$\Longrightarrow$ *vector-derivative* ($\lambda x.\ -f\ x$) (*at* a) = $-$ *vector-derivative* f (*at* a)
by (*simp add*: *vector-derivative-at has-vector-derivative-minus vector-derivative-works* [*symmetric*])

lemma *vector-derivative-add-at* [*simp*]:
⟦f *differentiable at* a; g *differentiable at* a⟧
$\Longrightarrow$ *vector-derivative* ($\lambda x.\ f\ x + g\ x$) (*at* a) = *vector-derivative* f (*at* a) + *vector-derivative* g (*at* a)
by (*simp add*: *vector-derivative-at has-vector-derivative-add vector-derivative-works* [*symmetric*])

lemma *vector-derivative-diff-at* [*simp*]:
⟦*f differentiable at a*; *g differentiable at a*⟧
⟹ *vector-derivative* (λ*x*. *f x* − *g x*) (*at a*) = *vector-derivative f* (*at a*) − *vector-derivative g* (*at a*)
by (*simp add*: *vector-derivative-at has-vector-derivative-diff vector-derivative-works* [*symmetric*])

lemma *vector-derivative-mult-at* [*simp*]:
fixes *f g* :: *real* ⇒ *'a* :: *real-normed-algebra*
shows ⟦*f differentiable at a*; *g differentiable at a*⟧
⟹ *vector-derivative* (λ*x*. *f x* ∗ *g x*) (*at a*) = *f a* ∗ *vector-derivative g* (*at a*) + *vector-derivative f* (*at a*) ∗ *g a*
by (*simp add*: *vector-derivative-at has-vector-derivative-mult vector-derivative-works* [*symmetric*])

lemma *vector-derivative-scaleR-at* [*simp*]:
⟦*f differentiable at a*; *g differentiable at a*⟧
⟹ *vector-derivative* (λ*x*. *f x* ∗$_R$ *g x*) (*at a*) = *f a* ∗$_R$ *vector-derivative g* (*at a*) + *vector-derivative f* (*at a*) ∗$_R$ *g a*
apply (*rule vector-derivative-at*)
apply (*rule has-vector-derivative-scaleR*)
apply (*auto simp*: *vector-derivative-works has-vector-derivative-def has-field-derivative-def mult-commute-abs*)
done

lemma *vector-derivative-within-closed-interval*:
assumes *ab*: *a* < *b x* ∈ *cbox a b*
assumes *f*: (*f has-vector-derivative f'*) (*at x within cbox a b*)
shows *vector-derivative f* (*at x within cbox a b*) = *f'*
by (*intro vector-derivative-unique-within-closed-interval*[*OF ab - f*]
vector-derivative-works[*THEN iffD1*] *differentiableI-vector*)
fact

lemma *has-vector-derivative-within-subset*:
(*f has-vector-derivative f'*) (*at x within s*) ⟹ *t* ⊆ *s* ⟹ (*f has-vector-derivative f'*) (*at x within t*)
by (*auto simp*: *has-vector-derivative-def intro*: *has-derivative-within-subset*)

lemma *has-vector-derivative-at-within*:
(*f has-vector-derivative f'*) (*at x*) ⟹ (*f has-vector-derivative f'*) (*at x within s*)
unfolding *has-vector-derivative-def*
by (*rule has-derivative-at-within*)

lemma *has-vector-derivative-weaken*:
fixes *x D* **and** *f g s t*
assumes *f*: (*f has-vector-derivative D*) (*at x within t*)
and *x* ∈ *s s* ⊆ *t*
and ⋀*x*. *x* ∈ *s* ⟹ *f x* = *g x*

shows (*g has-vector-derivative D*) (*at x within s*)
proof −
have (*f has-vector-derivative D*) (*at x within s*) ⟷ (*g has-vector-derivative D*) (*at x within s*)
unfolding *has-vector-derivative-def has-derivative-iff-norm*
using *assms* **by** (*intro conj-cong Lim-cong-within refl*) *auto*
then show *?thesis*
using *has-vector-derivative-within-subset*[*OF f* ‹*s* ⊆ *t*›] **by** *simp*
qed

lemma *has-vector-derivative-transform-within*:
assumes (*f has-vector-derivative f'*) (*at x within s*)
and *0* < *d*
and *x* ∈ *s*
and ⋀*x'*. ⟦*x'*∈*s*; *dist x' x* < *d*⟧ ⟹ *f x'* = *g x'*
shows (*g has-vector-derivative f'*) (*at x within s*)
using *assms*
unfolding *has-vector-derivative-def*
by (*rule has-derivative-transform-within*)

lemma *has-vector-derivative-transform-within-open*:
assumes (*f has-vector-derivative f'*) (*at x*)
and *open s*
and *x* ∈ *s*
and ⋀*y*. *y*∈*s* ⟹ *f y* = *g y*
shows (*g has-vector-derivative f'*) (*at x*)
using *assms*
unfolding *has-vector-derivative-def*
by (*rule has-derivative-transform-within-open*)

lemma *vector-diff-chain-at*:
assumes (*f has-vector-derivative f'*) (*at x*)
and (*g has-vector-derivative g'*) (*at* (*f x*))
shows ((*g* ∘ *f*) *has-vector-derivative* (*f'* *$_R$* *g'*)) (*at x*)
using *assms*(*2*)
unfolding *has-vector-derivative-def*
apply −
apply (*drule diff-chain-at*[*OF assms*(*1*)[*unfolded has-vector-derivative-def*]])
apply (*simp only*: *o-def real-scaleR-def scaleR-scaleR*)
done

lemma *vector-diff-chain-within*:
assumes (*f has-vector-derivative f'*) (*at x within s*)
and (*g has-vector-derivative g'*) (*at* (*f x*) *within f* ‘ *s*)
shows ((*g* ∘ *f*) *has-vector-derivative* (*f'* *$_R$* *g'*)) (*at x within s*)
using *assms*(*2*)
unfolding *has-vector-derivative-def*
apply −
apply (*drule diff-chain-within*[*OF assms*(*1*)[*unfolded has-vector-derivative-def*]])

apply (*simp only*: *o-def real-scaleR-def scaleR-scaleR*)
done

lemma *vector-derivative-const-at* [*simp*]: *vector-derivative* (λ*x*. *c*) (*at a*) = *0*
by (*simp add*: *vector-derivative-at*)

lemma *vector-derivative-at-within-ivl*:
(*f has-vector-derivative f'*) (*at x*) ⟹
a ≤ *x* ⟹ *x* ≤ *b* ⟹ *a*<*b* ⟹ *vector-derivative f* (*at x within* {*a*..*b*}) = *f'*
using *has-vector-derivative-at-within vector-derivative-within-closed-interval* **by** *fastforce*

lemma *vector-derivative-chain-at*:
assumes *f differentiable at x* (*g differentiable at* (*f x*))
shows *vector-derivative* (*g* ∘ *f*) (*at x*) =
vector-derivative f (*at x*) *∗R vector-derivative g* (*at* (*f x*))
by (*metis vector-diff-chain-at vector-derivative-at vector-derivative-works assms*)

lemma *field-vector-diff-chain-at*:
assumes *Df*: (*f has-vector-derivative f'*) (*at x*)
and *Dg*: (*g has-field-derivative g'*) (*at* (*f x*))
shows ((*g* ∘ *f*) *has-vector-derivative* (*f' ∗ g'*)) (*at x*)
using *diff-chain-at*[*OF Df*[*unfolded has-vector-derivative-def*]
Dg [*unfolded has-field-derivative-def*]]
by (*auto simp*: *o-def mult.commute has-vector-derivative-def*)

lemma *vector-derivative-chain-at-general*:
assumes *f differentiable at x* (∃ *g'*. (*g has-field-derivative g'*) (*at* (*f x*)))
shows *vector-derivative* (*g* ∘ *f*) (*at x*) =
vector-derivative f (*at x*) ∗ *deriv g* (*f x*)
apply (*rule vector-derivative-at* [*OF field-vector-diff-chain-at*])
using *assms*
by (*auto simp*: *vector-derivative-works DERIV-deriv-iff-has-field-derivative*)

lemma *exp-scaleR-has-vector-derivative-right*:
((λ*t*. *exp* (*t ∗R A*)) *has-vector-derivative exp* (*t ∗R A*) ∗ *A*) (*at t within T*)
unfolding *has-vector-derivative-def*
proof (*rule has-derivativeI*)
let *?F* = *at t within* (*T* ∩ {*t* − *1* <..< *t* + *1*})
have ∗: *at t within T* = *?F*
by (*rule at-within-nhd*[**where** *S*={*t* − *1* <..< *t* + *1*}]) *auto*
let *?e* = λ*i x*. (*inverse* (*1* + *real i*) ∗ *inverse* (*fact i*) ∗ (*x* − *t*) ^ *i*) *∗R* (*A* ∗ *A* ^ *i*)
have ∀ *F n in sequentially*.
∀ *x*∈*T* ∩ {*t* − *1*<..<*t* + *1*}. *norm* (*?e n x*) ≤ *norm* (*A* ^ (*n* + *1*) */R fact* (*n* + *1*))
by (*auto simp*: *divide-simps power-abs intro*!: *mult-left-le-one-le power-le-one eventuallyI*)
then have *uniform-limit* (*T* ∩ {*t* − *1*<..<*t* + *1*}) (λ*n x*. ∑ *i*<*n*. *?e i x*) (λ*x*. ∑ *i*. *?e i x*) *sequentially*

by (*rule weierstrass-m-test-ev*) (*intro summable-ignore-initial-segment summable-norm-exp*)
moreover
have $\forall_F\ x$ *in sequentially*. $x > 0$
by (*metis eventually-gt-at-top*)
then have
$\forall_F\ n$ *in sequentially*. $((\lambda x.\ \sum i{<}n.\ ?e\ i\ x) \longrightarrow A)\ ?F$
by *eventually-elim*
(*auto intro*!: *tendsto-eq-intros*
simp: *power-0-left if-distrib cond-application-beta setsum.delta*
cong: *if-cong*)
ultimately
have [*tendsto-intros*]: $((\lambda x.\ \sum i.\ ?e\ i\ x) \longrightarrow A)\ ?F$
by (*auto intro*!: *swap-uniform-limit*[**where** $f{=}\lambda n\ x.\ \sum i < n.\ ?e\ i\ x$ **and** $F = sequentially$])
have [*tendsto-intros*]: $((\lambda x.\ if\ x = t\ then\ 0\ else\ 1) \longrightarrow 1)\ ?F$
by (*rule Lim-eventually*) (*simp add*: *eventually-at-filter*)
have $((\lambda y.\ ((y - t)\ /\ abs\ (y - t)) *_R ((\sum n.\ ?e\ n\ y) - A)) \longrightarrow 0)\ (at\ t\ within\ T)$
unfolding $*$
by (*rule tendsto-norm-zero-cancel*) (*auto intro*!: *tendsto-eq-intros*)

moreover

have $\forall_F\ x$ *in at t within T*. $x \neq t$
by (*simp add*: *eventually-at-filter*)
then have $\forall_F\ x$ *in at t within T*. $((x - t)\ /\ |x - t|) *_R ((\sum n.\ ?e\ n\ x) - A) =$
$(exp\ ((x - t) *_R A) - 1 - (x - t) *_R A) /_R\ norm\ (x - t)$
proof *eventually-elim*
case (*elim x*)
have $(exp\ ((x - t) *_R A) - 1 - (x - t) *_R A) /_R\ norm\ (x - t) =$
$((\sum n.\ (x - t) *_R\ ?e\ n\ x) - (x - t) *_R A) /_R\ norm\ (x - t)$
unfolding *exp-first-term*
by (*simp add*: *ac-simps*)
also
have *summable* $(\lambda n.\ ?e\ n\ x)$
proof −
from *elim* **have** $?e\ n\ x = (((x - t) *_R A)\ \hat{}\ (n + 1)) /_R\ fact\ (n + 1) /_R (x - t)$ **for** *n*
by *simp*
then show *?thesis*
by (*auto simp only*:
intro!: *summable-scaleR-right summable-ignore-initial-segment summable-exp-generic*)
qed
then have $(\sum n.\ (x - t) *_R\ ?e\ n\ x) = (x - t) *_R (\sum n.\ ?e\ n\ x)$
by (*rule suminf-scaleR-right*[*symmetric*])
also have $(\ldots - (x - t) *_R A) /_R\ norm\ (x - t) = (x - t) *_R ((\sum n.\ ?e\ n\ x) - A) /_R\ norm\ (x - t)$
by (*simp add*: *algebra-simps*)
finally show *?case*

by (*simp add*: *divide-simps*)
qed

ultimately

have ((λ*y*. (*exp* ((*y* − *t*) ∗$_R$ *A*) − *1* − (*y* − *t*) ∗$_R$ *A*) /$_R$ *norm* (*y* − *t*)) ⟶ *0*) (*at t within T*)
by (*rule Lim-transform-eventually*[*rotated*])
from *tendsto-mult-right-zero*[*OF this*, **where** *c*=*exp* (*t* ∗$_R$ *A*)]
show ((λ*y*. (*exp* (*y* ∗$_R$ *A*) − *exp* (*t* ∗$_R$ *A*) − (*y* − *t*) ∗$_R$ (*exp* (*t* ∗$_R$ *A*) ∗ *A*)) /$_R$ *norm* (*y* − *t*)) ⟶ *0*)
(*at t within T*)
by (*rule Lim-transform-eventually*[*rotated*])
(*auto simp*: *algebra-simps divide-simps exp-add-commuting*[*symmetric*])
qed (*rule bounded-linear-scaleR-left*)

lemma *exp-times-scaleR-commute*: *exp* (*t* ∗$_R$ *A*) ∗ *A* = *A* ∗ *exp* (*t* ∗$_R$ *A*)
using *exp-times-arg-commute*[*symmetric*, *of t* ∗$_R$ *A*]
by (*auto simp*: *algebra-simps*)

lemma *exp-scaleR-has-vector-derivative-left*: ((λ*t*. *exp* (*t* ∗$_R$ *A*)) *has-vector-derivative* *A* ∗ *exp* (*t* ∗$_R$ *A*)) (*at t*)
using *exp-scaleR-has-vector-derivative-right*[*of A t*]
by (*simp add*: *exp-times-scaleR-commute*)

40.16 Relation between convexity and derivative

lemma *convex-on-imp-above-tangent*:
assumes *convex*: *convex-on A f* **and** *connected*: *connected A*
assumes *c*: *c* ∈ *interior A* **and** *x* : *x* ∈ *A*
assumes *deriv*: (*f has-field-derivative f'*) (*at c within A*)
shows *f x* − *f c* ≥ *f'* ∗ (*x* − *c*)
proof (*cases x c rule*: *linorder-cases*)
assume *xc*: *x* > *c*
let *?A'* = *interior A* ∩ {*c*<..}
from *c* **have** *c* ∈ *interior A* ∩ *closure* {*c*<..} **by** *auto*
also have ... ⊆ *closure* (*interior A* ∩ {*c*<..}) **by** (*intro open-inter-closure-subset*) *auto*
finally have *at c within ?A'* ≠ *bot* **by** (*subst at-within-eq-bot-iff*) *auto*
moreover from *deriv* **have** ((λ*y*. (*f y* − *f c*) / (*y* − *c*)) ⟶ *f'*) (*at c within ?A'*)
unfolding *DERIV-within-iff* **using** *interior-subset*[*of A*] **by** (*blast intro*: *tendsto-mono at-le*)
moreover from *eventually-at-right-real*[*OF xc*]
have *eventually* (λ*y*. (*f y* − *f c*) / (*y* − *c*) ≤ (*f x* − *f c*) / (*x* − *c*)) (*at-right c*)
proof *eventually-elim*
fix *y* **assume** *y*: *y* ∈ {*c*<..<*x*}
with *convex connected x c* **have** *f y* ≤ (*f x* − *f c*) / (*x* − *c*) ∗ (*y* − *c*) + *f c*
using *interior-subset*[*of A*]

by (*intro convex-onD-Icc′ convex-on-subset*[*OF convex*] *connected-contains-Icc*) *auto*
hence $f\ y - f\ c \leq (f\ x - f\ c)\ /\ (x - c) * (y - c)$ **by** *simp*
thus $(f\ y - f\ c)\ /\ (y - c) \leq (f\ x - f\ c)\ /\ (x - c)$ **using** *y xc* **by** (*simp add*: *divide-simps*)
qed
hence *eventually* $(\lambda y.\ (f\ y - f\ c)\ /\ (y - c) \leq (f\ x - f\ c)\ /\ (x - c))$ (*at c within ?A′*)
by (*blast intro*: *filter-leD at-le*)
ultimately have $f' \leq (f\ x - f\ c)\ /\ (x - c)$ **by** (*rule tendsto-ge-const*)
thus *?thesis* **using** *xc* **by** (*simp add*: *field-simps*)
next
assume *xc*: $x < c$
let *?A′* = *interior A* ∩ {..<*c*}
from *c* **have** $c \in$ *interior A* ∩ *closure* {..<*c*} **by** *auto*
also have ... ⊆ *closure* (*interior A* ∩ {..<*c*}) **by** (*intro open-inter-closure-subset*) *auto*
finally have *at c within ?A′* ≠ *bot* **by** (*subst at-within-eq-bot-iff*) *auto*
moreover from *deriv* **have** $((\lambda y.\ (f\ y - f\ c)\ /\ (y - c)) \longrightarrow f')$ (*at c within ?A′*)
unfolding *DERIV-within-iff* **using** *interior-subset*[*of A*] **by** (*blast intro*: *tendsto-mono at-le*)
moreover from *eventually-at-left-real*[*OF xc*]
have *eventually* $(\lambda y.\ (f\ y - f\ c)\ /\ (y - c) \geq (f\ x - f\ c)\ /\ (x - c))$ (*at-left c*)
proof *eventually-elim*
fix *y* **assume** *y*: $y \in \{x<..<c\}$
with *convex connected x c* **have** $f\ y \leq (f\ x - f\ c)\ /\ (c - x) * (c - y) + f\ c$
using *interior-subset*[*of A*]
by (*intro convex-onD-Icc′′ convex-on-subset*[*OF convex*] *connected-contains-Icc*) *auto*
hence $f\ y - f\ c \leq (f\ x - f\ c) * ((c - y)\ /\ (c - x))$ **by** *simp*
also have $(c - y)\ /\ (c - x) = (y - c)\ /\ (x - c)$ **using** *y xc* **by** (*simp add*: *field-simps*)
finally show $(f\ y - f\ c)\ /\ (y - c) \geq (f\ x - f\ c)\ /\ (x - c)$ **using** *y xc*
by (*simp add*: *divide-simps*)
qed
hence *eventually* $(\lambda y.\ (f\ y - f\ c)\ /\ (y - c) \geq (f\ x - f\ c)\ /\ (x - c))$ (*at c within ?A′*)
by (*blast intro*: *filter-leD at-le*)
ultimately have $f' \geq (f\ x - f\ c)\ /\ (x - c)$ **by** (*rule tendsto-le-const*)
thus *?thesis* **using** *xc* **by** (*simp add*: *field-simps*)
qed *simp-all*

40.17 Partial derivatives

lemma *eventually-at-Pair-within-TimesI1*:
fixes *x*::′*a*::*metric-space*
assumes $\forall_F\ x'$ *in at x within X*. *P x′*
assumes *P x*

shows $\forall_F\ (x', y')$ *in at* (x, y) *within* $X \times Y.\ P\ x'$
proof −
from *assms*[*unfolded eventually-at-topological*]
obtain S **where** S: *open* $S\ x \in S\ \bigwedge x'.\ x' \in X \Longrightarrow x' \in S \Longrightarrow P\ x'$
by *metis*
show $\forall_F\ (x', y')$ *in at* (x, y) *within* $X \times Y.\ P\ x'$
unfolding *eventually-at-topological*
by (*auto intro*!: *exI*[**where** $x{=}S \times UNIV$] S *open-Times*)
qed

lemma *eventually-at-Pair-within-TimesI2*:
fixes $x{::}'a{::}$*metric-space*
assumes $\forall_F\ y'$ *in at* y *within* $Y.\ P\ y'$
assumes $P\ y$
shows $\forall_F\ (x', y')$ *in at* (x, y) *within* $X \times Y.\ P\ y'$
proof −
from *assms*[*unfolded eventually-at-topological*]
obtain S **where** S: *open* $S\ y \in S\ \bigwedge y'.\ y' \in Y \Longrightarrow y' \in S \Longrightarrow P\ y'$
by *metis*
show $\forall_F\ (x', y')$ *in at* (x, y) *within* $X \times Y.\ P\ y'$
unfolding *eventually-at-topological*
by (*auto intro*!: *exI*[**where** $x{=}UNIV \times S$] S *open-Times*)
qed

lemma *has-derivative-partialsI*:
assumes *fx*: $\bigwedge x\ y.\ x \in X \Longrightarrow y \in Y \Longrightarrow ((\lambda x.\ f\ x\ y)$ *has-derivative blinfun-apply* $(fx\ x\ y))$ $(at\ x\ within\ X)$
assumes *fy*: $\bigwedge x\ y.\ x \in X \Longrightarrow y \in Y \Longrightarrow ((\lambda y.\ f\ x\ y)$ *has-derivative blinfun-apply* $(fy\ x\ y))$ $(at\ y\ within\ Y)$
assumes *fx-cont*: *continuous-on* $(X \times Y)\ (\lambda(x, y).\ fx\ x\ y)$
assumes *fy-cont*: *continuous-on* $(X \times Y)\ (\lambda(x, y).\ fy\ x\ y)$
assumes $x \in X\ y \in Y$
assumes *convex X convex Y*
shows $((\lambda(x, y).\ f\ x\ y)$ *has-derivative* $(\lambda(tx, ty).\ fx\ x\ y\ tx + fy\ x\ y\ ty))$ $(at\ (x, y)\ within\ X \times Y)$
proof (*safe intro*!: *has-derivativeI tendstoI*, *goal-cases*)
case $(2\ e')$
def $e{\equiv}e'$ / *9*
have $e > 0$ **using** $\langle e' > 0\rangle$ **by** (*simp add*: *e-def*)

have $(x, y) \in X \times Y$ **using** *assms* **by** *auto*
from *fy-cont*[*unfolded continuous-on-eq-continuous-within*, *rule-format*, *OF this*, *unfolded continuous-within*, *THEN tendstoD*, *OF* $\langle e > 0\rangle$]
have $\forall_F\ (x', y')$ *in at* (x, y) *within* $X \times Y.\ dist\ (fy\ x'\ y')\ (fy\ x\ y) < e$
by (*auto simp*: *split-beta'*)
from *this*[*unfolded eventually-at*] **obtain** d' **where**
$d' > 0$
$\bigwedge x'\ y'.\ x' \in X \Longrightarrow y' \in Y \Longrightarrow (x', y') \neq (x, y) \Longrightarrow dist\ (x', y')\ (x, y) < d'$
$\Longrightarrow$

```
    dist (fy x' y') (fy x y) < e
   by auto
 then
 have d': x' ∈ X ⟹ y' ∈ Y ⟹ dist (x', y') (x, y) < d' ⟹ dist (fy x' y') (fy
x y) < e
   for x' y'
   using ‹0 < e›
   by (cases (x', y') = (x, y)) auto
 def d ≡ d' / sqrt 2
 have d > 0 using ‹0 < d'› by (simp add: d-def)
 have d: x' ∈ X ⟹ y' ∈ Y ⟹ dist x' x < d ⟹ dist y' y < d ⟹ dist (fy x'
y') (fy x y) < e
   for x' y'
   by (auto simp: dist-prod-def d-def intro!: d' real-sqrt-sum-squares-less)

 let ?S = ball y d ∩ Y
 have convex ?S
   by (auto intro!: convex-Int ‹convex Y›)
 {
   fix x'::'a and y'::'b
   assume x': x' ∈ X and y': y' ∈ Y
   assume dx': dist x' x < d and dy': dist y' y < d
   have norm (fy x' y' − fy x' y) ≤ dist (fy x' y') (fy x y) + dist (fy x' y) (fy x
y)
     by norm
   also have dist (fy x' y') (fy x y) < e
     by (rule d; fact)
   also have dist (fy x' y) (fy x y) < e
     by (auto intro!: d simp: dist-prod-def x' ‹d > 0› ‹y ∈ Y› dx')
   finally
   have norm (fy x' y' − fy x' y) < e + e
     by arith
   then have onorm (blinfun-apply (fy x' y') − blinfun-apply (fy x' y)) < e + e
     by (auto simp: norm-blinfun.rep-eq blinfun.diff-left[abs-def] fun-diff-def)
 } note onorm = this

 have ev-mem: ∀F (x', y') in at (x, y) within X × Y. (x', y') ∈ X × Y
   using ‹x ∈ X› ‹y ∈ Y›
   by (auto simp: eventually-at intro!: zero-less-one)
 moreover
 have ev-dist: ∀F xy in at (x, y) within X × Y. dist xy (x, y) < d if d > 0 for
d
   using eventually-at-ball[OF that]
   by (rule eventually-elim2) (auto simp: dist-commute intro!: eventually-True)
 note ev-dist[OF ‹0 < d›]
 ultimately
 have ∀F (x', y') in at (x, y) within X × Y.
   norm (f x' y' − f x' y − (fy x' y) (y' − y)) ≤ norm (y' − y) * (e + e)
 proof (eventually-elim, safe)
```

```
    fix x' y'
    assume x' ∈ X and y': y' ∈ Y
    assume dist: dist (x', y') (x, y) < d
    then have dx: dist x' x < d and dy: dist y' y < d
      unfolding dist-prod-def fst-conv snd-conv atomize-conj
      by (metis le-less-trans real-sqrt-sum-squares-ge1 real-sqrt-sum-squares-ge2)
    {
      fix t::real
      assume t ∈ {0 .. 1}
      then have y + t *R (y' − y) ∈ closed-segment y y'
        by (auto simp: closed-segment-def algebra-simps intro!: exI[where x=t])
      also
      have ... ⊆ ball y d ∩ Y
        using ‹y ∈ Y› ‹0 < d› dy y'
        by (intro ‹convex ?S›[unfolded convex-contains-segment, rule-format, of y
y'])
           (auto simp: dist-commute)
      finally have y + t *R (y' − y) ∈ ?S .
    } note seg = this

    have ∀x∈ball y d ∩ Y. onorm (blinfun-apply (fy x' x) − blinfun-apply (fy x'
y)) ≤ e + e
      by (safe intro!: onorm less-imp-le ‹x' ∈ X› dx) (auto simp: dist-commute ‹0
< d› ‹y ∈ Y›)
    with seg has-derivative-within-subset[OF assms(2)[OF ‹x' ∈ X›]]
    show norm (f x' y' − f x' y − (fy x' y) (y' − y)) ≤ norm (y' − y) * (e + e)
      by (rule differentiable-bound-linearization[where S=?S])
        (auto intro!: ‹0 < d› ‹y ∈ Y›)
  qed
  moreover
  let ?le = λx'. norm (f x' y − f x y − (fx x y) (x' − x)) ≤ norm (x' − x) * e
  from fx[OF ‹x ∈ X› ‹y ∈ Y›, unfolded has-derivative-within, THEN conjunct2,
THEN tendstoD, OF ‹0 < e›]
  have ∀F x' in at x within X. ?le x'
    by eventually-elim
       (auto simp: dist-norm divide-simps blinfun.bilinear-simps field-simps split:
if-split-asm)
  then have ∀F (x', y') in at (x, y) within X × Y. ?le x'
    by (rule eventually-at-Pair-within-TimesI1)
       (simp add: blinfun.bilinear-simps)
  moreover have ∀F (x', y') in at (x, y) within X × Y. norm ((x', y') − (x,
y)) ≠ 0
    unfolding norm-eq-zero right-minus-eq
    by (auto simp: eventually-at intro!: zero-less-one)
  moreover
  from fy-cont[unfolded continuous-on-eq-continuous-within, rule-format, OF Sig-
maI[OF ‹x ∈ X› ‹y ∈ Y›],
      unfolded continuous-within, THEN tendstoD, OF ‹0 < e›]
  have ∀F x' in at x within X. norm (fy x' y − fy x y) < e
```

unfolding *eventually-at*
using ‹$y \in Y$›
by (*auto simp*: *dist-prod-def dist-norm*)
then have $\forall_F\ (x', y')$ *in at* (x, y) *within* $X \times Y$. *norm* $(fy\ x'\ y - fy\ x\ y) < e$
by (*rule eventually-at-Pair-within-TimesI1*)
(*simp add*: *blinfun.bilinear-simps* ‹$0 < e$›)
ultimately
have $\forall_F\ (x', y')$ *in at* (x, y) *within* $X \times Y$.
norm $((f\ x'\ y' - f\ x\ y - (fx\ x\ y\ (x' - x) + fy\ x\ y\ (y' - y))) /_R$
norm $((x', y') - (x, y)))$
$< e'$
apply *eventually-elim*
proof *safe*
fix $x'\ y'$
have *norm* $(f\ x'\ y' - f\ x\ y - (fx\ x\ y\ (x' - x) + fy\ x\ y\ (y' - y))) \leq$
norm $(f\ x'\ y' - f\ x'\ y - fy\ x'\ y\ (y' - y)) +$
norm $(fy\ x\ y\ (y' - y) - fy\ x'\ y\ (y' - y)) +$
norm $(f\ x'\ y - f\ x\ y - fx\ x\ y\ (x' - x))$
by *norm*
also
assume *nz*: *norm* $((x', y') - (x, y)) \neq 0$
and *nfy*: *norm* $(fy\ x'\ y - fy\ x\ y) < e$
assume *norm* $(f\ x'\ y' - f\ x'\ y - blinfun\text{-}apply\ (fy\ x'\ y)\ (y' - y)) \leq norm\ (y' - y) * (e + e)$
also assume *norm* $(f\ x'\ y - f\ x\ y - blinfun\text{-}apply\ (fx\ x\ y)\ (x' - x)) \leq norm\ (x' - x) * e$
also
have *norm* $((fy\ x\ y)\ (y' - y) - (fy\ x'\ y)\ (y' - y)) \leq norm\ ((fy\ x\ y) - (fy\ x'\ y)) * norm\ (y' - y)$
by (*auto simp*: *blinfun.bilinear-simps*[*symmetric*] *intro*!: *norm-blinfun*)
also have $\ldots \leq (e + e) * norm\ (y' - y)$
using ‹$e > 0$› *nfy*
by (*auto simp*: *norm-minus-commute intro*!: *mult-right-mono*)
also have *norm* $(x' - x) * e \leq norm\ (x' - x) * (e + e)$
using ‹$0 < e$› **by** *simp*
also have *norm* $(y' - y) * (e + e) + (e + e) * norm\ (y' - y) + norm\ (x' - x) * (e + e) \leq$
$(norm\ (y' - y) + norm\ (x' - x)) * (4 * e)$
using ‹$e > 0$›
by (*simp add*: *algebra-simps*)
also have $\ldots \leq 2 * norm\ ((x', y') - (x, y)) * (4 * e)$
using ‹$0 < e$› *real-sqrt-sum-squares-ge1*[*of norm* $(x' - x)$ *norm* $(y' - y)$]
real-sqrt-sum-squares-ge2[*of norm* $(y' - y)$ *norm* $(x' - x)$]
by (*auto intro*!: *mult-right-mono simp*: *norm-prod-def*
simp del: *real-sqrt-sum-squares-ge1 real-sqrt-sum-squares-ge2*)
also have $\ldots \leq norm\ ((x', y') - (x, y)) * (8 * e)$
by *simp*
also have $\ldots < norm\ ((x', y') - (x, y)) * e'$
using ‹$0 < e'$› *nz*

by (*auto simp*: *e-def*)
finally show *norm* (($f\ x'\ y' - f\ x\ y - (fx\ x\ y\ (x' - x) + fy\ x\ y\ (y' - y))) /_R$
norm ((x', y') − (x, y))) < e'
by (*auto simp*: *divide-simps dist-norm mult.commute*)
qed
then show *?case*
by *eventually-elim* (*auto simp*: *dist-norm field-simps*)
qed (*auto intro*!: *bounded-linear-intros simp*: *split-beta'*)

end

41 Kurzweil-Henstock Gauge Integration in many dimensions.

theory *Integration*
imports
Derivative
Uniform-Limit
~~/*src*/*HOL*/*Library*/*Indicator-Function*
begin

lemmas *scaleR-simps* = *scaleR-zero-left scaleR-minus-left scaleR-left-diff-distrib*
scaleR-zero-right scaleR-minus-right scaleR-right-diff-distrib scaleR-eq-0-iff
scaleR-cancel-left scaleR-cancel-right scaleR-add-right scaleR-add-left real-vector-class.scaleR-one

41.1 Sundries

lemma *conjunctD2*: **assumes** $a \wedge b$ **shows** a b **using** *assms* **by** *auto*
lemma *conjunctD3*: **assumes** $a \wedge b \wedge c$ **shows** a b c **using** *assms* **by** *auto*
lemma *conjunctD4*: **assumes** $a \wedge b \wedge c \wedge d$ **shows** a b c d **using** *assms* **by** *auto*

declare *norm-triangle-ineq4*[*intro*]

lemma *simple-image*: {$f\ x$ |x . $x \in s$} = f ‘ s
by *blast*

lemma *linear-simps*:
assumes *bounded-linear f*
shows
$f\ (a + b) = f\ a + f\ b$
$f\ (a - b) = f\ a - f\ b$
$f\ 0 = 0$
$f\ (-\ a) = -\ f\ a$
$f\ (s *_R v) = s *_R (f\ v)$
proof −
interpret f: *bounded-linear f* **by** *fact*
show $f\ (a + b) = f\ a + f\ b$ **by** (*rule f.add*)

```
  show f (a - b) = f a - f b by (rule f.diff)
  show f 0 = 0 by (rule f.zero)
  show f (- a) = - f a by (rule f.minus)
  show f (s *R v) = s *R (f v) by (rule f.scaleR)
qed

lemma bounded-linearI:
  assumes ⋀x y. f (x + y) = f x + f y
    and ⋀r x. f (r *R x) = r *R f x
    and ⋀x. norm (f x) ≤ norm x * K
  shows bounded-linear f
  using assms by (rule bounded-linear-intro)

lemma bounded-linear-component [intro]: bounded-linear (λx::'a::euclidean-space.
x • k)
  by (rule bounded-linear-inner-left)

lemma transitive-stepwise-lt-eq:
  assumes (⋀x y z::nat. R x y ⟹ R y z ⟹ R x z)
  shows ((∀ m. ∀ n>m. R m n) ⟷ (∀ n. R n (Suc n)))
  (is ?l = ?r)
proof safe
  assume ?r
  fix n m :: nat
  assume m < n
  then show R m n
  proof (induct n arbitrary: m)
    case 0
    then show ?case by auto
  next
    case (Suc n)
    show ?case
    proof (cases m < n)
      case True
      show ?thesis
        apply (rule assms[OF Suc(1)[OF True]])
        using ‹?r›
        apply auto
        done
    next
      case False
      then have m = n
        using Suc(2) by auto
      then show ?thesis
        using ‹?r› by auto
    qed
  qed
qed auto
```

lemma *transitive-stepwise-gt*:
assumes $\bigwedge x\ y\ z.\ R\ x\ y \Longrightarrow R\ y\ z \Longrightarrow R\ x\ z\ \bigwedge n.\ R\ n\ (Suc\ n)$
shows $\forall\, n{>}m.\ R\ m\ n$
proof −
have $\forall\, m.\ \forall\, n{>}m.\ R\ m\ n$
apply (*subst transitive-stepwise-lt-eq*)
apply (*blast intro*: *assms*)+
done
then show *?thesis* **by** *auto*
qed

lemma *transitive-stepwise-le-eq*:
assumes $\bigwedge x.\ R\ x\ x\ \bigwedge x\ y\ z.\ R\ x\ y \Longrightarrow R\ y\ z \Longrightarrow R\ x\ z$
shows $(\forall\, m.\ \forall\, n{\geq}m.\ R\ m\ n) \longleftrightarrow (\forall\, n.\ R\ n\ (Suc\ n))$
(**is** *?l* = *?r*)
proof *safe*
assume *?r*
fix *m n* :: *nat*
assume $m \leq n$
then show *R m n*
proof (*induct n arbitrary*: *m*)
case *0*
with *assms* **show** *?case* **by** *auto*
next
case (*Suc n*)
show *?case*
proof (*cases* $m \leq n$)
case *True*
with *Suc.hyps* ‹$\forall\, n.\ R\ n\ (Suc\ n)$› *assms* **show** *?thesis*
by *blast*
next
case *False*
then have $m = Suc\ n$
using *Suc*(*2*) **by** *auto*
then show *?thesis*
using *assms*(*1*) **by** *auto*
qed
qed
qed *auto*

lemma *transitive-stepwise-le*:
assumes $\bigwedge x.\ R\ x\ x\ \bigwedge x\ y\ z.\ R\ x\ y \Longrightarrow R\ y\ z \Longrightarrow R\ x\ z$
and $\bigwedge n.\ R\ n\ (Suc\ n)$
shows $\forall\, n{\geq}m.\ R\ m\ n$
proof −
have $\forall\, m.\ \forall\, n{\geq}m.\ R\ m\ n$
apply (*subst transitive-stepwise-le-eq*)
apply (*blast intro*: *assms*)+
done

then show *?thesis* **by** *auto*
qed

41.2 Some useful lemmas about intervals.

lemma *empty-as-interval*: $\{\} = cbox\ One\ (0::'a::euclidean\text{-}space)$
using *nonempty-Basis*
by (*fastforce simp add*: *set-eq-iff mem-box*)

lemma *interior-subset-union-intervals*:
assumes $i = cbox\ a\ b$
and $j = cbox\ c\ d$
and $interior\ j \neq \{\}$
and $i \subseteq j \cup s$
and $interior\ i \cap interior\ j = \{\}$
shows $interior\ i \subseteq interior\ s$
proof −
have $box\ a\ b \cap cbox\ c\ d = \{\}$
using *inter-interval-mixed-eq-empty*[*of c d a b*] **and** *assms*(*3*,*5*)
unfolding *assms*(*1*,*2*) *interior-cbox* **by** *auto*
moreover
have $box\ a\ b \subseteq cbox\ c\ d \cup s$
apply (*rule order-trans*,*rule box-subset-cbox*)
using *assms*(*4*) **unfolding** *assms*(*1*,*2*)
apply *auto*
done
ultimately
show *?thesis*
unfolding *assms interior-cbox*
by *auto* (*metis IntI UnE empty-iff interior-maximal open-box subsetCE subsetI*)
qed

lemma *inter-interior-unions-intervals*:
fixes $f::('a::euclidean\text{-}space)\ set\ set$
assumes *finite f*
and *open s*
and $\forall t{\in}f.\ \exists a\ b.\ t = cbox\ a\ b$
and $\forall t{\in}f.\ s \cap (interior\ t) = \{\}$
shows $s \cap interior\ (\bigcup f) = \{\}$
proof (*clarsimp simp only*: *all-not-in-conv* [*symmetric*])
fix *x*
assume *x*: $x \in s\ x \in interior\ (\bigcup f)$
have *lem1*: $\bigwedge x\ e\ s\ U.\ ball\ x\ e \subseteq s \cap interior\ U \longleftrightarrow ball\ x\ e \subseteq s \cap U$
using *interior-subset*
by *auto* (*meson Topology-Euclidean-Space.open-ball contra-subsetD interior-maximal mem-ball*)
have $\exists t{\in}f.\ \exists x.\ \exists e{>}0.\ ball\ x\ e \subseteq s \cap t$
if *finite f* **and** $\forall t{\in}f.\ \exists a\ b.\ t = cbox\ a\ b$ **and** $\exists x.\ x \in s \cap interior\ (\bigcup f)$ **for** *f*
using *that*

```
  proof (induct rule: finite-induct)
    case empty
    obtain x where x ∈ s ∩ interior (⋃{})
      using empty(2) ..
    then have False
      unfolding Union-empty interior-empty by auto
    then show ?case by auto
  next
    case (insert i f)
    obtain x where x: x ∈ s ∩ interior (⋃insert i f)
      using insert(5) ..
    then obtain e where e: 0 < e ∧ ball x e ⊆ s ∩ interior (⋃insert i f)
        unfolding open-contains-ball-eq[OF open-Int[OF assms(2) open-interior],
rule-format] ..
    obtain a where ∃b. i = cbox a b
      using insert(4)[rule-format,OF insertI1] ..
    then obtain b where ab: i = cbox a b ..
    show ?case
    proof (cases x ∈ i)
      case False
      then have x ∈ UNIV − cbox a b
        unfolding ab by auto
      then obtain d where 0 < d ∧ ball x d ⊆ UNIV − cbox a b
      unfolding open-contains-ball-eq[OF open-Diff[OF open-UNIV closed-cbox],rule-format]
..
      then have 0 < d ball x (min d e) ⊆ UNIV − i
        unfolding ab ball-min-Int by auto
      then have ball x (min d e) ⊆ s ∩ interior (⋃f)
        using e unfolding lem1 unfolding  ball-min-Int by auto
      then have x ∈ s ∩ interior (⋃f) using ‹d>0› e by auto
      then have ∃t∈f. ∃x e. 0 < e ∧ ball x e ⊆ s ∩ t
        using insert.hyps(3) insert.prems(1) by blast
      then show ?thesis by auto
    next
      case True show ?thesis
      proof (cases x∈box a b)
        case True
        then obtain d where 0 < d ∧ ball x d ⊆ box a b
          unfolding open-contains-ball-eq[OF open-box,rule-format] ..
        then show ?thesis
          apply (rule-tac x=i in bexI, rule-tac x=x in exI, rule-tac x=min d e in
exI)
          unfolding ab
          using box-subset-cbox[of a b] and e
          apply fastforce+
          done
      next
        case False
        then obtain k where x·k ≤ a·k ∨ x·k ≥ b·k and k: k ∈ Basis
```

```
  unfolding mem-box by (auto simp add: not-less)
then have x·k = a·k ∨ x·k = b·k
  using True unfolding ab and mem-box
    apply (erule-tac x = k in ballE)
    apply auto
    done
then have ∃x. ball x (e/2) ⊆ s ∩ (⋃f)
proof (rule disjE)
  let ?z = x − (e/2) *R k
  assume as: x·k = a·k
  have ball ?z (e / 2) ∩ i = {}
  proof (clarsimp simp only: all-not-in-conv [symmetric])
    fix y
    assume y ∈ ball ?z (e / 2) and yi: y ∈ i
    then have dist ?z y < e/2 by auto
    then have |(?z − y) · k| < e/2
      using Basis-le-norm[OF k, of ?z − y] unfolding dist-norm by auto
    then have y·k < a·k
      using e k
      by (auto simp add: field-simps abs-less-iff as inner-simps)
    then have y ∉ i
      unfolding ab mem-box by (auto intro!: bexI[OF - k])
    then show False using yi by auto
  qed
  moreover
  have ball ?z (e/2) ⊆ s ∩ (⋃ insert i f)
    apply (rule order-trans[OF - e[THEN conjunct2, unfolded lem1]])
  proof
    fix y
    assume as: y ∈ ball ?z (e/2)
    have norm (x − y) ≤ |e| / 2 + norm (x − y − (e / 2) *R k)
      apply (rule order-trans,rule norm-triangle-sub[of x − y (e/2) *R k])
      unfolding norm-scaleR norm-Basis[OF k]
      apply auto
      done
    also have ... < |e| / 2 + |e| / 2
      apply (rule add-strict-left-mono)
      using as e
      apply (auto simp add: field-simps dist-norm)
      done
    finally show y ∈ ball x e
     unfolding mem-ball dist-norm using e by (auto simp add:field-simps)
  qed
  ultimately show ?thesis
    apply (rule-tac x=?z in exI)
    unfolding Union-insert
    apply auto
    done
next
```

```
  let ?z = x + (e/2) *R k
  assume as: x•k = b•k
  have ball ?z (e / 2) ∩ i = {}
  proof (clarsimp simp only: all-not-in-conv [symmetric])
    fix y
    assume y ∈ ball ?z (e / 2) and yi: y ∈ i
    then have dist ?z y < e/2
      by auto
    then have |(?z − y) • k| < e/2
      using Basis-le-norm[OF k, of ?z − y]
      unfolding dist-norm by auto
    then have y•k > b•k
      using e k
      by (auto simp add:field-simps inner-simps inner-Basis as)
    then have y ∉ i
      unfolding ab mem-box by (auto intro!: bexI[OF - k])
    then show False using yi by auto
  qed
  moreover
  have ball ?z (e/2) ⊆ s ∩ (⋃ insert i f)
    apply (rule order-trans[OF - e[THEN conjunct2, unfolded lem1]])
  proof
    fix y
    assume as: y∈ ball ?z (e/2)
    have norm (x − y) ≤ |e| / 2 + norm (x − y + (e / 2) *R k)
     apply (rule order-trans,rule norm-triangle-sub[of x − y − (e/2) *R k])
      unfolding norm-scaleR
      apply (auto simp: k)
      done
    also have ... < |e| / 2 + |e| / 2
      apply (rule add-strict-left-mono)
      using as unfolding mem-ball dist-norm
      using e apply (auto simp add: field-simps)
      done
    finally show y ∈ ball x e
     unfolding mem-ball dist-norm using e by (auto simp add:field-simps)
  qed
  ultimately show ?thesis
    apply (rule-tac x=?z in exI)
    unfolding Union-insert
    apply auto
    done
qed
then obtain x where ball x (e / 2) ⊆ s ∩ ⋃f ..
then have x ∈ s ∩ interior (⋃f)
  unfolding lem1[where U=⋃f, symmetric]
  using centre-in-ball e by auto
then show ?thesis
  using insert.hyps(3) insert.prems(1) by blast
```

qed
qed
qed
from *this*[*OF assms*(*1*,*3*)] *x*
obtain *t x e* **where** $t \in f\ 0 < e$ *ball x e* $\subseteq s \cap t$
by *blast*
then have $x \in s\ x \in$ *interior t*
using *open-subset-interior*[*OF open-ball*, *of x e t*]
by *auto*
then show *False*
using ‹$t \in f$› *assms*(*4*) **by** *auto*
qed

41.3 Bounds on intervals where they exist.

definition *interval-upperbound* :: ('*a*::*euclidean-space*) *set* ⇒ '*a*
where *interval-upperbound s* = ($\sum$ *i*∈*Basis*. (*SUP x*:*s*. *x*•*i*) *∗*$_R$ *i*)

definition *interval-lowerbound* :: ('*a*::*euclidean-space*) *set* ⇒ '*a*
where *interval-lowerbound s* = ($\sum$ *i*∈*Basis*. (*INF x*:*s*. *x*•*i*) *∗*$_R$ *i*)

lemma *interval-upperbound*[*simp*]:
∀ *i*∈*Basis*. *a*•*i* ≤ *b*•*i* ⟹
interval-upperbound (*cbox a b*) = (*b*::'*a*::*euclidean-space*)
unfolding *interval-upperbound-def euclidean-representation-setsum cbox-def*
by (*safe intro*!: *cSup-eq*) *auto*

lemma *interval-lowerbound*[*simp*]:
∀ *i*∈*Basis*. *a*•*i* ≤ *b*•*i* ⟹
interval-lowerbound (*cbox a b*) = (*a*::'*a*::*euclidean-space*)
unfolding *interval-lowerbound-def euclidean-representation-setsum cbox-def*
by (*safe intro*!: *cInf-eq*) *auto*

lemmas *interval-bounds* = *interval-upperbound interval-lowerbound*

lemma
fixes *X*::*real set*
shows *interval-upperbound-real*[*simp*]: *interval-upperbound X* = *Sup X*
and *interval-lowerbound-real*[*simp*]: *interval-lowerbound X* = *Inf X*
by (*auto simp*: *interval-upperbound-def interval-lowerbound-def*)

lemma *interval-bounds'*[*simp*]:
assumes *cbox a b* ≠ {}
shows *interval-upperbound* (*cbox a b*) = *b*
and *interval-lowerbound* (*cbox a b*) = *a*
using *assms* **unfolding** *box-ne-empty* **by** *auto*

lemma *interval-upperbound-Times*:

assumes $A \neq \{\}$ **and** $B \neq \{\}$
shows *interval-upperbound* $(A \times B) = (\textit{interval-upperbound}\ A, \textit{interval-upperbound}\ B)$
proof−
from *assms* **have** *fst-image-times'*: $A = fst\ `\ (A \times B)$ **by** *simp*
have $(\sum i{\in}Basis.\ (SUP\ x{:}A \times B.\ x \cdot (i,\ 0)) *_R i) = (\sum i{\in}Basis.\ (SUP\ x{:}A.\ x \cdot i) *_R i)$
by (*subst* (*2*) *fst-image-times'*) (*simp del*: *fst-image-times add*: *o-def inner-Pair-0*)
moreover from *assms* **have** *snd-image-times'*: $B = snd\ `\ (A \times B)$ **by** *simp*
have $(\sum i{\in}Basis.\ (SUP\ x{:}A \times B.\ x \cdot (0,\ i)) *_R i) = (\sum i{\in}Basis.\ (SUP\ x{:}B.\ x \cdot i) *_R i)$
by (*subst* (*2*) *snd-image-times'*) (*simp del*: *snd-image-times add*: *o-def inner-Pair-0*)
ultimately show *?thesis* **unfolding** *interval-upperbound-def*
by (*subst setsum-Basis-prod-eq*) (*auto simp add*: *setsum-prod*)
qed

lemma *interval-lowerbound-Times*:
assumes $A \neq \{\}$ **and** $B \neq \{\}$
shows *interval-lowerbound* $(A \times B) = (\textit{interval-lowerbound}\ A, \textit{interval-lowerbound}\ B)$
proof−
from *assms* **have** *fst-image-times'*: $A = fst\ `\ (A \times B)$ **by** *simp*
have $(\sum i{\in}Basis.\ (INF\ x{:}A \times B.\ x \cdot (i,\ 0)) *_R i) = (\sum i{\in}Basis.\ (INF\ x{:}A.\ x \cdot i) *_R i)$
by (*subst* (*2*) *fst-image-times'*) (*simp del*: *fst-image-times add*: *o-def inner-Pair-0*)
moreover from *assms* **have** *snd-image-times'*: $B = snd\ `\ (A \times B)$ **by** *simp*
have $(\sum i{\in}Basis.\ (INF\ x{:}A \times B.\ x \cdot (0,\ i)) *_R i) = (\sum i{\in}Basis.\ (INF\ x{:}B.\ x \cdot i) *_R i)$
by (*subst* (*2*) *snd-image-times'*) (*simp del*: *snd-image-times add*: *o-def inner-Pair-0*)
ultimately show *?thesis* **unfolding** *interval-lowerbound-def*
by (*subst setsum-Basis-prod-eq*) (*auto simp add*: *setsum-prod*)
qed

41.4 Content (length, area, volume...) of an interval.

definition *content* $(s{::}('a{::}euclidean\text{-}space)\ set) =$
$(if\ s = \{\}\ then\ 0\ else\ (\prod i{\in}Basis.\ (\textit{interval-upperbound}\ s){\cdot}i - (\textit{interval-lowerbound}\ s){\cdot}i))$

lemma *interval-not-empty*: $\forall\, i{\in}Basis.\ a{\cdot}i \leq b{\cdot}i \Longrightarrow cbox\ a\ b \neq \{\}$
unfolding *box-eq-empty* **unfolding** *not-ex not-less* **by** *auto*

lemma *content-cbox*:
fixes $a :: 'a{::}euclidean\text{-}space$
assumes $\forall\, i{\in}Basis.\ a{\cdot}i \leq b{\cdot}i$
shows *content* $(cbox\ a\ b) = (\prod i{\in}Basis.\ b{\cdot}i - a{\cdot}i)$
using *interval-not-empty*[*OF assms*]

unfolding *content-def*
by *auto*

lemma *content-cbox'*:
fixes *a* :: *'a*::*euclidean-space*
assumes *cbox a b* ≠ {}
shows *content* (*cbox a b*) = (∏ *i*∈*Basis*. *b*•*i* − *a*•*i*)
using *assms box-ne-empty*(*1*) *content-cbox* **by** *blast*

lemma *content-real*: *a* ≤ *b* ⟹ *content* {*a*..*b*} = *b* − *a*
by (*auto simp*: *interval-upperbound-def interval-lowerbound-def content-def*)

lemma *abs-eq-content*: |*y* − *x*| = (*if x*≤*y then content* {*x* .. *y*} *else content* {*y*..*x*})
by (*auto simp*: *content-real*)

lemma *content-singleton*[*simp*]: *content* {*a*} = *0*
proof −
have *content* (*cbox a a*) = *0*
by (*subst content-cbox*) (*auto simp*: *ex-in-conv*)
then show *?thesis* **by** (*simp add*: *cbox-sing*)
qed

lemma *content-unit*[*iff*]: *content*(*cbox 0* (*One*::*'a*::*euclidean-space*)) = *1*
proof −
have ∗: ∀ *i*∈*Basis*. (*0*::*'a*)•*i* ≤ (*One*::*'a*)•*i*
by *auto*
have *0* ∈ *cbox 0* (*One*::*'a*)
unfolding *mem-box* **by** *auto*
then show *?thesis*
unfolding *content-def interval-bounds*[*OF* ∗] **using** *setprod.neutral-const* **by** *auto*
qed

lemma *content-pos-le*[*intro*]:
fixes *a*::*'a*::*euclidean-space*
shows *0* ≤ *content* (*cbox a b*)
proof (*cases cbox a b* = {})
case *False*
then have ∗: ∀ *i*∈*Basis*. *a* · *i* ≤ *b* · *i*
unfolding *box-ne-empty* **.**
have *0* ≤ (∏ *i*∈*Basis*. *interval-upperbound* (*cbox a b*) · *i* − *interval-lowerbound* (*cbox a b*) · *i*)
apply (*rule setprod-nonneg*)
unfolding *interval-bounds*[*OF* ∗]
using ∗
apply *auto*
done
also have ... = *content* (*cbox a b*) **using** *False* **by** (*simp add*: *content-def*)
finally show *?thesis* **.**

qed (*simp add*: *content-def*)

corollary *content-nonneg* [*simp*]:
fixes *a*::*'a*::*euclidean-space*
shows ~ *content* (*cbox a b*) < *0*
using *not-le* **by** *blast*

lemma *content-pos-lt*:
fixes *a* :: *'a*::*euclidean-space*
assumes ∀ *i*∈*Basis*. *a*•*i* < *b*•*i*
shows *0* < *content* (*cbox a b*)
using *assms*
by (*auto simp*: *content-def box-eq-empty intro*!: *setprod-pos*)

lemma *content-eq-0*:
content (*cbox a b*) = *0* ⟷ (∃ *i*∈*Basis*. *b*•*i* ≤ *a*•*i*)
by (*auto simp*: *content-def box-eq-empty intro*!: *setprod-pos bexI*)

lemma *cond-cases*: (*P* ⟹ *Q x*) ⟹ (¬ *P* ⟹ *Q y*) ⟹ *Q* (*if P then x else y*)
by *auto*

lemma *content-cbox-cases*:
content (*cbox a* (*b*::*'a*::*euclidean-space*)) =
(*if* ∀ *i*∈*Basis*. *a*•*i* ≤ *b*•*i then setprod* (λ*i*. *b*•*i* − *a*•*i*) *Basis else 0*)
by (*auto simp*: *not-le content-eq-0 intro*: *less-imp-le content-cbox*)

lemma *content-eq-0-interior*: *content* (*cbox a b*) = *0* ⟷ *interior*(*cbox a b*) = {}
unfolding *content-eq-0 interior-cbox box-eq-empty*
by *auto*

lemma *content-pos-lt-eq*:
0 < *content* (*cbox a* (*b*::*'a*::*euclidean-space*)) ⟷ (∀ *i*∈*Basis*. *a*•*i* < *b*•*i*)
proof (*rule iffI*)
assume *0* < *content* (*cbox a b*)
then have *content* (*cbox a b*) ≠ *0* **by** *auto*
then show ∀ *i*∈*Basis*. *a*•*i* < *b*•*i*
unfolding *content-eq-0 not-ex not-le* **by** *fastforce*
next
assume ∀ *i*∈*Basis*. *a* • *i* < *b* • *i*
then show *0* < *content* (*cbox a b*)
by (*metis content-pos-lt*)
qed

lemma *content-empty* [*simp*]: *content* {} = *0*
unfolding *content-def* **by** *auto*

lemma *content-real-if* [*simp*]: *content* {*a*..*b*} = (*if a* ≤ *b then b* − *a else 0*)
by (*simp add*: *content-real*)

lemma *content-subset*:
assumes *cbox a b* ⊆ *cbox c d*
shows *content* (*cbox a b*) ≤ *content* (*cbox c d*)
proof (*cases cbox a b* = {})
case *True*
then show *?thesis*
using *content-pos-le*[*of c d*] **by** *auto*
next
case *False*
then have *ab-ne*: ∀ *i*∈*Basis*. *a* · *i* ≤ *b* · *i*
unfolding *box-ne-empty* **by** *auto*
then have *ab-ab*: *a*∈*cbox a b b*∈*cbox a b*
unfolding *mem-box* **by** *auto*
have *cbox c d* ≠ {} **using** *assms False* **by** *auto*
then have *cd-ne*: ∀ *i*∈*Basis*. *c* · *i* ≤ *d* · *i*
using *assms* **unfolding** *box-ne-empty* **by** *auto*
have ⋀*i*. *i* ∈ *Basis* ⟹ *0* ≤ *b* · *i* − *a* · *i*
using *ab-ne* **by** *auto*
moreover
have ⋀*i*. *i* ∈ *Basis* ⟹ *b* · *i* − *a* · *i* ≤ *d* · *i* − *c* · *i*
using *assms*[*unfolded subset-eq mem-box*,*rule-format*,*OF ab-ab*(*2*)]
assms[*unfolded subset-eq mem-box*,*rule-format*,*OF ab-ab*(*1*)]
by (*metis diff-mono*)
ultimately show *?thesis*
unfolding *content-def interval-bounds*[*OF ab-ne*] *interval-bounds*[*OF cd-ne*]
by (*simp add*: *setprod-mono if-not-P*[*OF False*] *if-not-P*[*OF* ‹*cbox c d* ≠ {}›])
qed

lemma *content-lt-nz*: *0* < *content* (*cbox a b*) ⟷ *content* (*cbox a b*) ≠ *0*
unfolding *content-pos-lt-eq content-eq-0* **unfolding** *not-ex not-le* **by** *fastforce*

lemma *content-times*[*simp*]: *content* (*A* × *B*) = *content A* ∗ *content B*
proof (*cases A* × *B* = {})
let *?ub1* = *interval-upperbound* **and** *?lb1* = *interval-lowerbound*
let *?ub2* = *interval-upperbound* **and** *?lb2* = *interval-lowerbound*
assume *nonempty*: *A* × *B* ≠ {}
hence *content* (*A* × *B*) = (∏ *i*∈*Basis*. (*?ub1 A*, *?ub2 B*) · *i* − (*?lb1 A*, *?lb2 B*)
· *i*)
unfolding *content-def* **by** (*simp add*: *interval-upperbound-Times interval-lowerbound-Times*)
also have ... = *content A* ∗ *content B* **unfolding** *content-def* **using** *nonempty*
apply (*subst Basis-prod-def*, *subst setprod.union-disjoint*, *force*, *force*, *force*,
simp)
apply (*subst* (*1 2*) *setprod.reindex*, *auto intro*: *inj-onI*)
done
finally show *?thesis* **.**
qed (*auto simp*: *content-def*)

lemma *content-Pair*: *content* (*cbox* (*a*,*c*) (*b*,*d*)) = *content* (*cbox a b*) ∗ *content*
(*cbox c d*)

by (*simp add*: *cbox-Pair-eq*)

lemma *content-cbox-pair-eq0-D*:
content (*cbox* (*a*,*c*) (*b*,*d*)) = *0* ⟹ *content* (*cbox a b*) = *0* ∨ *content* (*cbox c d*) = *0*
by (*simp add*: *content-Pair*)

lemma *content-eq-0-gen*:
fixes *s* :: *'a*::*euclidean-space set*
assumes *bounded s*
shows *content s* = *0* ⟷ (∃ *i*∈*Basis*. ∃ *v*. ∀ *x* ∈ *s*. *x* ∙ *i* = *v*) (**is** *- = ?rhs*)
proof *safe*
assume *content s* = *0* **then show** *?rhs*
apply (*clarsimp simp*: *ex-in-conv content-def split*: *if-split-asm*)
apply (*rule-tac x*=*a* **in** *bexI*)
apply (*rule-tac x*=*interval-lowerbound s* ∙ *a* **in** *exI*)
apply (*clarsimp simp*: *interval-upperbound-def interval-lowerbound-def*)
apply (*drule cSUP-eq-cINF-D*)
apply (*auto simp*: *bounded-inner-imp-bdd-above* [*OF assms*] *bounded-inner-imp-bdd-below* [*OF assms*])
done
next
fix *i a*
assume *i* ∈ *Basis* ∀ *x*∈*s*. *x* ∙ *i* = *a*
then show *content s* = *0*
apply (*clarsimp simp*: *content-def*)
apply (*rule-tac x*=*i* **in** *bexI*)
apply (*auto simp*: *interval-upperbound-def interval-lowerbound-def*)
done
qed

lemma *content-0-subset-gen*:
fixes *a* :: *'a*::*euclidean-space*
assumes *content t* = *0 s* ⊆ *t bounded t* **shows** *content s* = *0*
proof −
have *bounded s*
using *assms* **by** (*metis bounded-subset*)
then show *?thesis*
using *assms*
by (*auto simp*: *content-eq-0-gen*)
qed

lemma *content-0-subset*: ⟦*content*(*cbox a b*) = *0*; *s* ⊆ *cbox a b*⟧ ⟹ *content s* = *0*
by (*simp add*: *content-0-subset-gen bounded-cbox*)

41.5 The notion of a gauge — simply an open set containing the point.

definition *gauge d* ⟷ (∀ *x*. *x* ∈ *d x* ∧ *open* (*d x*))

lemma *gaugeI*:
assumes $\bigwedge x.\ x \in g\ x$
and $\bigwedge x.\ open\ (g\ x)$
shows *gauge g*
using *assms* **unfolding** *gauge-def* **by** *auto*

lemma *gaugeD*[*dest*]:
assumes *gauge d*
shows $x \in d\ x$
and *open* $(d\ x)$
using *assms* **unfolding** *gauge-def* **by** *auto*

lemma *gauge-ball-dependent*: $\forall x.\ 0 < e\ x \Longrightarrow gauge\ (\lambda x.\ ball\ x\ (e\ x))$
unfolding *gauge-def* **by** *auto*

lemma *gauge-ball*[*intro*]: $0 < e \Longrightarrow gauge\ (\lambda x.\ ball\ x\ e)$
unfolding *gauge-def* **by** *auto*

lemma *gauge-trivial*[*intro*!]: *gauge* $(\lambda x.\ ball\ x\ 1)$
by (*rule gauge-ball*) *auto*

lemma *gauge-inter*[*intro*]: *gauge d1* $\Longrightarrow$ *gauge d2* $\Longrightarrow$ *gauge* $(\lambda x.\ d1\ x \cap d2\ x)$
unfolding *gauge-def* **by** *auto*

lemma *gauge-inters*:
assumes *finite s*
and $\forall d{\in}s.\ gauge\ (f\ d)$
shows *gauge* $(\lambda x.\ \bigcap \{f\ d\ x \mid d.\ d \in s\})$
proof −
have $*$: $\bigwedge x.\ \{f\ d\ x \mid d.\ d \in s\} = (\lambda d.\ f\ d\ x)\ `\ s$
by *auto*
show *?thesis*
unfolding *gauge-def* **unfolding** $*$
using *assms* **unfolding** *Ball-def Inter-iff mem-Collect-eq gauge-def* **by** *auto*
qed

lemma *gauge-existence-lemma*:
$(\forall x.\ \exists d :: real.\ p\ x \longrightarrow 0 < d \wedge q\ d\ x) \longleftrightarrow (\forall x.\ \exists d{>}0.\ p\ x \longrightarrow q\ d\ x)$
by (*metis zero-less-one*)

41.6 Divisions.

definition *division-of* (**infixl** *division'-of 40*)
where
s division-of i $\longleftrightarrow$
finite s $\wedge$
$(\forall k{\in}s.\ k \subseteq i \wedge k \neq \{\} \wedge (\exists a\ b.\ k = cbox\ a\ b)) \wedge$
$(\forall k1{\in}s.\ \forall k2{\in}s.\ k1 \neq k2 \longrightarrow interior(k1) \cap interior(k2) = \{\}) \wedge$

$(\bigcup s = i)$

lemma *division-ofD*[*dest*]:
assumes *s division-of i*
shows *finite s*
and $\bigwedge k.\ k \in s \Longrightarrow k \subseteq i$
and $\bigwedge k.\ k \in s \Longrightarrow k \neq \{\}$
and $\bigwedge k.\ k \in s \Longrightarrow \exists\, a\ b.\ k = cbox\ a\ b$
and $\bigwedge k1\ k2.\ k1 \in s \Longrightarrow k2 \in s \Longrightarrow k1 \neq k2 \Longrightarrow interior(k1) \cap interior(k2) = \{\}$
and $\bigcup s = i$
using *assms* **unfolding** *division-of-def* **by** *auto*

lemma *division-ofI*:
assumes *finite s*
and $\bigwedge k.\ k \in s \Longrightarrow k \subseteq i$
and $\bigwedge k.\ k \in s \Longrightarrow k \neq \{\}$
and $\bigwedge k.\ k \in s \Longrightarrow \exists\, a\ b.\ k = cbox\ a\ b$
and $\bigwedge k1\ k2.\ k1 \in s \Longrightarrow k2 \in s \Longrightarrow k1 \neq k2 \Longrightarrow interior\ k1 \cap interior\ k2 = \{\}$
and $\bigcup s = i$
shows *s division-of i*
using *assms* **unfolding** *division-of-def* **by** *auto*

lemma *division-of-finite*: *s division-of i* $\Longrightarrow$ *finite s*
unfolding *division-of-def* **by** *auto*

lemma *division-of-self*[*intro*]: *cbox a b* $\neq \{\} \Longrightarrow \{cbox\ a\ b\}$ *division-of* $(cbox\ a\ b)$
unfolding *division-of-def* **by** *auto*

lemma *division-of-trivial*[*simp*]: *s division-of* $\{\} \longleftrightarrow s = \{\}$
unfolding *division-of-def* **by** *auto*

lemma *division-of-sing*[*simp*]:
s division-of cbox a $(a::'a::euclidean\text{-}space) \longleftrightarrow s = \{cbox\ a\ a\}$
(**is** *?l* = *?r*)
proof
assume *?r*
moreover
{ fix *k*
assume $s = \{\{a\}\}$ $k \in s$
then have $\exists\, x\ y.\ k = cbox\ x\ y$
apply (*rule-tac x*=*a* **in** *exI*)+
apply (*force simp*: *cbox-sing*)
done
}
ultimately show *?l*
unfolding *division-of-def cbox-sing* **by** *auto*
next

assume *?l*
note $*$ = *conjunctD4*[*OF this*[*unfolded division-of-def cbox-sing*]]
{
fix *x*
assume *x*: $x \in s$ **have** $x = \{a\}$
using $*(2)$[*rule-format*,*OF x*] **by** *auto*
}
moreover have $s \neq \{\}$
using $*(4)$ **by** *auto*
ultimately show *?r*
unfolding *cbox-sing* **by** *auto*
qed

lemma *elementary-empty*: **obtains** *p* **where** *p division-of* $\{\}$
unfolding *division-of-trivial* **by** *auto*

lemma *elementary-interval*: **obtains** *p* **where** *p division-of* (*cbox a b*)
by (*metis division-of-trivial division-of-self*)

lemma *division-contains*: *s division-of i* $\Longrightarrow \forall\, x{\in}i.\ \exists\, k{\in}s.\ x \in k$
unfolding *division-of-def* **by** *auto*

lemma *forall-in-division*:
d division-of i $\Longrightarrow (\forall\, x{\in}d.\ P\ x) \longleftrightarrow (\forall\, a\ b.\ cbox\ a\ b \in d \longrightarrow P\ (cbox\ a\ b))$
unfolding *division-of-def* **by** *fastforce*

lemma *division-of-subset*:
assumes *p division-of* $(\bigcup p)$
and $q \subseteq p$
shows *q division-of* $(\bigcup q)$
proof (*rule division-ofI*)
note $*$ = *division-ofD*[*OF assms*(*1*)]
show *finite q*
using $*(1)$ *assms*(*2*) *infinite-super* **by** *auto*
{
fix *k*
assume $k \in q$
then have *kp*: $k \in p$
using *assms*(*2*) **by** *auto*
show $k \subseteq \bigcup q$
using ‹$k \in q$› **by** *auto*
show $\exists\, a\ b.\ k = cbox\ a\ b$
using $*(4)$[*OF kp*] **by** *auto*
show $k \neq \{\}$
using $*(3)$[*OF kp*] **by** *auto*
}
fix *k1 k2*
assume $k1 \in q\ k2 \in q\ k1 \neq k2$
then have $**$: $k1 \in p\ k2 \in p\ k1 \neq k2$

```
    using assms(2) by auto
  show interior k1 ∩ interior k2 = {}
    using ∗(5)[OF ∗∗] by auto
qed auto

lemma division-of-union-self[intro]: p division-of s ⟹ p division-of (⋃p)
  unfolding division-of-def by auto

lemma division-of-content-0:
  assumes content (cbox a b) = 0 d division-of (cbox a b)
  shows ∀k∈d. content k = 0
  unfolding forall-in-division[OF assms(2)]
  by (metis antisym-conv assms content-pos-le content-subset division-ofD(2))

lemma division-inter:
  fixes s1 s2 :: 'a::euclidean-space set
  assumes p1 division-of s1
    and p2 division-of s2
  shows {k1 ∩ k2 | k1 k2 .k1 ∈ p1 ∧ k2 ∈ p2 ∧ k1 ∩ k2 ≠ {}} division-of (s1
∩ s2)
  (is ?A' division-of -)
proof −
  let ?A = {s. s ∈ (λ(k1,k2). k1 ∩ k2) ‘ (p1 × p2) ∧ s ≠ {}}
  have ∗: ?A' = ?A by auto
  show ?thesis
    unfolding ∗
  proof (rule division-ofI)
    have ?A ⊆ (λ(x, y). x ∩ y) ‘ (p1 × p2)
      by auto
    moreover have finite (p1 × p2)
      using assms unfolding division-of-def by auto
    ultimately show finite ?A by auto
    have ∗: ⋀s. ⋃{x∈s. x ≠ {}} = ⋃s
      by auto
    show ⋃?A = s1 ∩ s2
      apply (rule set-eqI)
      unfolding ∗ and UN-iff
      using division-ofD(6)[OF assms(1)] and division-ofD(6)[OF assms(2)]
      apply auto
      done
    {
      fix k
      assume k ∈ ?A
      then obtain k1 k2 where k: k = k1 ∩ k2 k1 ∈ p1 k2 ∈ p2 k ≠ {}
        by auto
      then show k ≠ {}
        by auto
      show k ⊆ s1 ∩ s2
       using division-ofD(2)[OF assms(1) k(2)] and division-ofD(2)[OF assms(2)
```

```
k(3)]
      unfolding k by auto
    obtain a1 b1 where k1: k1 = cbox a1 b1
      using division-ofD(4)[OF assms(1) k(2)] by blast
    obtain a2 b2 where k2: k2 = cbox a2 b2
      using division-ofD(4)[OF assms(2) k(3)] by blast
    show ∃ a b. k = cbox a b
      unfolding k k1 k2 unfolding inter-interval by auto
  }
  fix k1 k2
  assume k1 ∈ ?A
  then obtain x1 y1 where k1: k1 = x1 ∩ y1 x1 ∈ p1 y1 ∈ p2 k1 ≠ {}
    by auto
  assume k2 ∈ ?A
  then obtain x2 y2 where k2: k2 = x2 ∩ y2 x2 ∈ p1 y2 ∈ p2 k2 ≠ {}
    by auto
  assume k1 ≠ k2
  then have th: x1 ≠ x2 ∨ y1 ≠ y2
    unfolding k1 k2 by auto
  have ∗: interior x1 ∩ interior x2 = {} ∨ interior y1 ∩ interior y2 = {} ⟹
    interior (x1 ∩ y1) ⊆ interior x1 ⟹ interior (x1 ∩ y1) ⊆ interior y1 ⟹
    interior (x2 ∩ y2) ⊆ interior x2 ⟹ interior (x2 ∩ y2) ⊆ interior y2 ⟹
    interior (x1 ∩ y1) ∩ interior (x2 ∩ y2) = {} by auto
  show interior k1 ∩ interior k2 = {}
    unfolding k1 k2
    apply (rule ∗)
    using assms division-ofD(5) k1 k2(2) k2(3) th apply auto
    done
 qed
qed

lemma division-inter-1:
 assumes d division-of i
   and cbox a (b::'a::euclidean-space) ⊆ i
 shows {cbox a b ∩ k | k. k ∈ d ∧ cbox a b ∩ k ≠ {}} division-of (cbox a b)
proof (cases cbox a b = {})
 case True
 show ?thesis
   unfolding True and division-of-trivial by auto
next
 case False
 have ∗: cbox a b ∩ i = cbox a b using assms(2) by auto
 show ?thesis
   using division-inter[OF division-of-self[OF False] assms(1)]
   unfolding ∗ by auto
qed

lemma elementary-inter:
 fixes s t :: 'a::euclidean-space set
```

```
  assumes p1 division-of s
    and p2 division-of t
  shows ∃p. p division-of (s ∩ t)
using assms division-inter by blast

lemma elementary-inters:
  assumes finite f
    and f ≠ {}
    and ∀s∈f. ∃p. p division-of (s::('a::euclidean-space) set)
  shows ∃p. p division-of (⋂f)
  using assms
proof (induct f rule: finite-induct)
  case (insert x f)
  show ?case
  proof (cases f = {})
    case True
    then show ?thesis
      unfolding True using insert by auto
  next
    case False
    obtain p where p division-of ⋂f
      using insert(3)[OF False insert(5)[unfolded ball-simps,THEN conjunct2]] ..
    moreover obtain px where px division-of x
      using insert(5)[rule-format,OF insertI1] ..
    ultimately show ?thesis
      by (simp add: elementary-inter Inter-insert)
  qed
qed auto

lemma division-disjoint-union:
  assumes p1 division-of s1
    and p2 division-of s2
    and interior s1 ∩ interior s2 = {}
  shows (p1 ∪ p2) division-of (s1 ∪ s2)
proof (rule division-ofI)
  note d1 = division-ofD[OF assms(1)]
  note d2 = division-ofD[OF assms(2)]
  show finite (p1 ∪ p2)
    using d1(1) d2(1) by auto
  show ⋃(p1 ∪ p2) = s1 ∪ s2
    using d1(6) d2(6) by auto
  {
    fix k1 k2
    assume as: k1 ∈ p1 ∪ p2 k2 ∈ p1 ∪ p2 k1 ≠ k2
    moreover
    let ?g=interior k1 ∩ interior k2 = {}
    {
      assume as: k1∈p1 k2∈p2
      have ?g
```

```
        using interior-mono[OF d1(2)[OF as(1)]] interior-mono[OF d2(2)[OF
as(2)]]
      using assms(3) by blast
   }
   moreover
   {
    assume as: k1∈p2 k2∈p1
    have ?g
        using interior-mono[OF d1(2)[OF as(2)]] interior-mono[OF d2(2)[OF
as(1)]]
      using assms(3) by blast
   }
   ultimately show ?g
    using d1(5)[OF - - as(3)] and d2(5)[OF - - as(3)] by auto
 }
 fix k
 assume k: k ∈ p1 ∪ p2
 show k ⊆ s1 ∪ s2
   using k d1(2) d2(2) by auto
 show k ≠ {}
   using k d1(3) d2(3) by auto
 show ∃ a b. k = cbox a b
   using k d1(4) d2(4) by auto
qed

lemma partial-division-extend-1:
 fixes a b c d :: 'a::euclidean-space
 assumes incl: cbox c d ⊆ cbox a b
   and nonempty: cbox c d ≠ {}
 obtains p where p division-of (cbox a b) cbox c d ∈ p
proof
 let ?B = λf::'a⇒'a × 'a.
   cbox (∑ i∈Basis. (fst (f i) · i) *R i) (∑ i∈Basis. (snd (f i) · i) *R i)
 def p ≡ ?B ‘ (Basis →E {(a, c), (c, d), (d, b)})

 show cbox c d ∈ p
   unfolding p-def
  by (auto simp add: box-eq-empty cbox-def intro!: image-eqI[where x=λ(i::'a)∈Basis.
(c, d)])
 {
  fix i :: 'a
  assume i ∈ Basis
  with incl nonempty have a · i ≤ c · i c · i ≤ d · i d · i ≤ b · i
    unfolding box-eq-empty subset-box by (auto simp: not-le)
 }
 note ord = this

 show p division-of (cbox a b)
 proof (rule division-ofI)
```

```
  show finite p
    unfolding p-def by (auto intro!: finite-PiE)
  {
    fix k
    assume k ∈ p
    then obtain f where f: f ∈ Basis →E {(a, c), (c, d), (d, b)} and k: k =
?B f
      by (auto simp: p-def)
    then show ∃ a b. k = cbox a b
      by auto
    have k ⊆ cbox a b ∧ k ≠ {}
    proof (simp add: k box-eq-empty subset-box not-less, safe)
      fix i :: 'a
      assume i: i ∈ Basis
      with f have f i = (a, c) ∨ f i = (c, d) ∨ f i = (d, b)
        by (auto simp: PiE-iff)
      with i ord[of i]
      show a · i ≤ fst (f i) · i snd (f i) · i ≤ b · i fst (f i) · i ≤ snd (f i) · i
        by auto
    qed
    then show k ≠ {} k ⊆ cbox a b
      by auto
    {
      fix l
      assume l ∈ p
      then obtain g where g: g ∈ Basis →E {(a, c), (c, d), (d, b)} and l: l =
?B g
        by (auto simp: p-def)
      assume l ≠ k
      have ∃ i∈Basis. f i ≠ g i
      proof (rule ccontr)
        assume ¬ ?thesis
        with f g have f = g
          by (auto simp: PiE-iff extensional-def intro!: ext)
        with ‹l ≠ k› show False
          by (simp add: l k)
      qed
      then obtain i where ∗: i ∈ Basis f i ≠ g i ..
      then have f i = (a, c) ∨ f i = (c, d) ∨ f i = (d, b)
                g i = (a, c) ∨ g i = (c, d) ∨ g i = (d, b)
        using f g by (auto simp: PiE-iff)
      with ∗ ord[of i] show interior l ∩ interior k = {}
        by (auto simp add: l k interior-cbox disjoint-interval intro!: bexI[of - i])
    }
    note ‹k ⊆ cbox a b›
  }
  moreover
  {
    fix x assume x: x ∈ cbox a b
```

```
    have ∀ i∈Basis. ∃ l. x · i ∈ {fst l · i .. snd l · i} ∧ l ∈ {(a, c), (c, d), (d, b)}
     proof
       fix i :: 'a
       assume i ∈ Basis
       with x ord[of i]
       have (a · i ≤ x · i ∧ x · i ≤ c · i) ∨ (c · i ≤ x · i ∧ x · i ≤ d · i) ∨
           (d · i ≤ x · i ∧ x · i ≤ b · i)
         by (auto simp: cbox-def)
       then show ∃ l. x · i ∈ {fst l · i .. snd l · i} ∧ l ∈ {(a, c), (c, d), (d, b)}
         by auto
     qed
     then obtain f where
       f: ∀ i∈Basis. x · i ∈ {fst (f i) · i..snd (f i) · i} ∧ f i ∈ {(a, c), (c, d), (d,
b)}
       unfolding bchoice-iff ..
     moreover from f have restrict f Basis ∈ Basis →_E {(a, c), (c, d), (d, b)}
       by auto
     moreover from f have x ∈ ?B (restrict f Basis)
       by (auto simp: mem-box)
     ultimately have ∃ k∈p. x ∈ k
       unfolding p-def by blast
   }
   ultimately show ⋃p = cbox a b
     by auto
 qed
qed

lemma partial-division-extend-interval:
 assumes p division-of (⋃p) (⋃p) ⊆ cbox a b
 obtains q where p ⊆ q q division-of cbox a (b::'a::euclidean-space)
proof (cases p = {})
 case True
 obtain q where q division-of (cbox a b)
   by (rule elementary-interval)
 then show ?thesis
   using True that by blast
next
 case False
 note p = division-ofD[OF assms(1)]
 have div-cbox: ∀ k∈p. ∃ q. q division-of cbox a b ∧ k ∈ q
 proof
   fix k
   assume kp: k ∈ p
   obtain c d where k: k = cbox c d
     using p(4)[OF kp] by blast
   have ∗: cbox c d ⊆ cbox a b cbox c d ≠ {}
     using p(2,3)[OF kp, unfolded k] using assms(2)
     by (blast intro: order.trans)+
   obtain q where q division-of cbox a b cbox c d ∈ q
```

```
    by (rule partial-division-extend-1[OF *])
  then show ∃ q. q division-of cbox a b ∧ k ∈ q
    unfolding k by auto
 qed
 obtain q where q: ⋀x. x ∈ p ⟹ q x division-of cbox a b ⋀x. x ∈ p ⟹ x ∈
q x
  using bchoice[OF div-cbox] by blast
 { fix x
  assume x: x ∈ p
  have q x division-of ⋃ q x
   apply (rule division-ofI)
   using division-ofD[OF q(1)[OF x]]
   apply auto
   done }
 then have ⋀x. x ∈ p ⟹ ∃ d. d division-of ⋃(q x - {x})
  by (meson Diff-subset division-of-subset)
 then have ∃ d. d division-of ⋂((λi. ⋃(q i - {i})) ` p)
  apply -
  apply (rule elementary-inters [OF finite-imageI[OF p(1)]])
  apply (auto simp: False elementary-inters [OF finite-imageI[OF p(1)]])
  done
 then obtain d where d: d division-of ⋂((λi. ⋃(q i - {i})) ` p) ..
 have d ∪ p division-of cbox a b
 proof -
  have te: ⋀s f t. s ≠ {} ⟹ ∀ i∈s. f i ∪ i = t ⟹ t = ⋂(f ` s) ∪ ⋃s by auto
  have cbox-eq: cbox a b = ⋂((λi. ⋃(q i - {i})) ` p) ∪ ⋃p
  proof (rule te[OF False], clarify)
   fix i
   assume i: i ∈ p
   show ⋃(q i - {i}) ∪ i = cbox a b
    using division-ofD(6)[OF q(1)[OF i]] using q(2)[OF i] by auto
  qed
  { fix k
   assume k: k ∈ p
   have *: ⋀u t s. t ∩ s = {} ⟹ u ⊆ s ⟹ u ∩ t = {}
    by auto
   have interior (⋂ i∈p. ⋃(q i - {i})) ∩ interior k = {}
   proof (rule *[OF inter-interior-unions-intervals])
    note qk=division-ofD[OF q(1)[OF k]]
    show finite (q k - {k}) open (interior k) ∀ t∈q k - {k}. ∃ a b. t = cbox a b
     using qk by auto
    show ∀ t∈q k - {k}. interior k ∩ interior t = {}
     using qk(5) using q(2)[OF k] by auto
    show interior (⋂ i∈p. ⋃(q i - {i})) ⊆ interior (⋃(q k - {k}))
     apply (rule interior-mono)+
     using k
     apply auto
     done
   qed } note [simp] = this
```

```
    show d ∪ p division-of (cbox a b)
      unfolding cbox-eq
      apply (rule division-disjoint-union[OF d assms(1)])
      apply (rule inter-interior-unions-intervals)
      apply (rule p open-interior ballI)+
      apply simp-all
      done
  qed
  then show ?thesis
    by (meson Un-upper2 that)
qed

lemma elementary-bounded[dest]:
  fixes s :: 'a::euclidean-space set
  shows p division-of s ⟹ bounded s
  unfolding division-of-def by (metis bounded-Union bounded-cbox)

lemma elementary-subset-cbox:
  p division-of s ⟹ ∃ a b. s ⊆ cbox a (b::'a::euclidean-space)
  by (meson elementary-bounded bounded-subset-cbox)

lemma division-union-intervals-exists:
  fixes a b :: 'a::euclidean-space
  assumes cbox a b ≠ {}
  obtains p where (insert (cbox a b) p) division-of (cbox a b ∪ cbox c d)
proof (cases cbox c d = {})
  case True
  show ?thesis
    apply (rule that[of {}])
    unfolding True
    using assms
    apply auto
    done
next
  case False
  show ?thesis
  proof (cases cbox a b ∩ cbox c d = {})
    case True
    then show ?thesis
      by (metis that False assms division-disjoint-union division-of-self insert-is-Un
interior-Int interior-empty)
  next
    case False
    obtain u v where uv: cbox a b ∩ cbox c d = cbox u v
      unfolding inter-interval by auto
    have uv-sub: cbox u v ⊆ cbox c d using uv by auto
    obtain p where p division-of cbox c d cbox u v ∈ p
      by (rule partial-division-extend-1[OF uv-sub False[unfolded uv]])
    note p = this division-ofD[OF this(1)]
```

```
    have interior (cbox a b ∩ ⋃(p − {cbox u v})) = interior(cbox u v ∩ ⋃(p −
{cbox u v}))
      apply (rule arg-cong[of - - interior])
      using p(8) uv by auto
    also have ... = {}
      unfolding interior-Int
      apply (rule inter-interior-unions-intervals)
      using p(6) p(7)[OF p(2)] p(3)
      apply auto
      done
    finally have [simp]: interior (cbox a b) ∩ interior (⋃(p − {cbox u v})) = {}
by simp
    have cbe: cbox a b ∪ cbox c d = cbox a b ∪ ⋃(p − {cbox u v})
      using p(8) unfolding uv[symmetric] by auto
    have insert (cbox a b) (p − {cbox u v}) division-of cbox a b ∪ ⋃(p − {cbox u
v})
    proof −
      have {cbox a b} division-of cbox a b
        by (simp add: assms division-of-self)
      then show insert (cbox a b) (p − {cbox u v}) division-of cbox a b ∪ ⋃(p −
{cbox u v})
        by (metis (no-types) Diff-subset ‹interior (cbox a b) ∩ interior (⋃(p − {cbox
u v})) = {}› division-disjoint-union division-of-subset insert-is-Un p(1) p(8))
    qed
    with that[of p − {cbox u v}] show ?thesis by (simp add: cbe)
  qed
qed

lemma division-of-unions:
  assumes finite f
    and ⋀p. p ∈ f ⟹ p division-of (⋃p)
    and ⋀k1 k2. k1 ∈ ⋃f ⟹ k2 ∈ ⋃f ⟹ k1 ≠ k2 ⟹ interior k1 ∩ interior
k2 = {}
  shows ⋃f division-of ⋃⋃f
  using assms
  by (auto intro!: division-ofI)

lemma elementary-union-interval:
  fixes a b :: 'a::euclidean-space
  assumes p division-of ⋃p
  obtains q where q division-of (cbox a b ∪ ⋃p)
proof −
  note assm = division-ofD[OF assms]
  have lem1: ⋀f s. ⋃⋃(f ‘ s) = ⋃((λx. ⋃(f x)) ‘ s)
    by auto
  have lem2: ⋀f s. f ≠ {} ⟹ ⋃{s ∪ t |t. t ∈ f} = s ∪ ⋃f
    by auto
  {
    presume p = {} ⟹ thesis
```

```
    cbox a b = {} ⟹ thesis
    cbox a b ≠ {} ⟹ interior (cbox a b) = {} ⟹ thesis
    p ≠ {} ⟹ interior (cbox a b)≠{} ⟹ cbox a b ≠ {} ⟹ thesis
  then show thesis by auto
next
  assume as: p = {}
  obtain p where p division-of (cbox a b)
    by (rule elementary-interval)
  then show thesis
    using as that by auto
next
  assume as: cbox a b = {}
  show thesis
    using as assms that by auto
next
  assume as: interior (cbox a b) = {} cbox a b ≠ {}
  show thesis
    apply (rule that[of insert (cbox a b) p],rule division-ofI)
    unfolding finite-insert
    apply (rule assm(1)) unfolding Union-insert
    using assm(2-4) as
    apply -
    apply (fast dest: assm(5))+
    done
next
  assume as: p ≠ {} interior (cbox a b) ≠ {} cbox a b ≠ {}
  have ∀ k∈p. ∃ q. (insert (cbox a b) q) division-of (cbox a b ∪ k)
  proof
    fix k
    assume kp: k ∈ p
    from assm(4)[OF kp] obtain c d where k = cbox c d by blast
    then show ∃ q. (insert (cbox a b) q) division-of (cbox a b ∪ k)
      by (meson as(3) division-union-intervals-exists)
  qed
 from bchoice[OF this] obtain q where ∀ x∈p. insert (cbox a b) (q x) division-of
(cbox a b) ∪ x ..
  note q = division-ofD[OF this[rule-format]]
  let ?D = ⋃{insert (cbox a b) (q k) | k. k ∈ p}
  show thesis
  proof (rule that[OF division-ofI])
    have *: {insert (cbox a b) (q k) |k. k ∈ p} = (λk. insert (cbox a b) (q k)) ' p
      by auto
    show finite ?D
      using * assm(1) q(1) by auto
    show ⋃ ?D = cbox a b ∪ ⋃ p
      unfolding * lem1
      unfolding lem2[OF as(1), of cbox a b, symmetric]
      using q(6)
      by auto
```

fix k
assume k: $k \in ?D$
then show $k \subseteq cbox\ a\ b \cup \bigcup p$
  using $q(2)$ **by** *auto*
show $k \neq \{\}$
  using $q(3)\ k$ **by** *auto*
show $\exists\, a\ b.\ k = cbox\ a\ b$
  using $q(4)\ k$ **by** *auto*
fix k'
assume k': $k' \in ?D\ k \neq k'$
obtain x **where** x: $k \in insert\ (cbox\ a\ b)\ (q\ x)\ x{\in}p$
  using k **by** *auto*
obtain x' **where** x': $k'{\in}insert\ (cbox\ a\ b)\ (q\ x')\ x'{\in}p$
  using k' **by** *auto*
show $interior\ k \cap interior\ k' = \{\}$
proof $(cases\ x = x')$
  case *True*
  show *?thesis*
    using *True* $k'\ q(5)\ x'\ x$ **by** *auto*
next
  case *False*
  {
    presume $k = cbox\ a\ b \Longrightarrow ?thesis$
      and $k' = cbox\ a\ b \Longrightarrow ?thesis$
      and $k \neq cbox\ a\ b \Longrightarrow k' \neq cbox\ a\ b \Longrightarrow ?thesis$
    then show *?thesis* **by** *linarith*
  next
    assume as': $k\ = cbox\ a\ b$
    show *?thesis*
      using $as'\ k'\ q(5)\ x'$ **by** *blast*
  next
    assume as': $k' = cbox\ a\ b$
    show *?thesis*
      using $as'\ k'(2)\ q(5)\ x$ **by** *blast*
  }
  assume as': $k \neq cbox\ a\ b\ k' \neq cbox\ a\ b$
  obtain $c\ d$ **where** k: $k = cbox\ c\ d$
    using $q(4)[OF\ x(2,1)]$ **by** *blast*
  have $interior\ k \cap interior\ (cbox\ a\ b) = \{\}$
    using $as'\ k'(2)\ q(5)\ x$ **by** *blast*
  then have $interior\ k \subseteq interior\ x$
  using *interior-subset-union-intervals*
    by $(metis\ as(2)\ k\ q(2)\ x\ interior\text{-}subset\text{-}union\text{-}intervals)$
  moreover
  obtain $c\ d$ **where** c-d: $k' = cbox\ c\ d$
    using $q(4)[OF\ x'(2,1)]$ **by** *blast*
  have $interior\ k' \cap interior\ (cbox\ a\ b) = \{\}$
    using $as'(2)\ q(5)\ x'$ **by** *blast*
  then have $interior\ k' \subseteq interior\ x'$

```
        by (metis as(2) c-d interior-subset-union-intervals q(2) x'(1) x'(2))
      ultimately show ?thesis
        using assm(5)[OF x(2) x'(2) False] by auto
    qed
  qed
 }
qed

lemma elementary-unions-intervals:
  assumes fin: finite f
    and ⋀s. s ∈ f ⟹ ∃a b. s = cbox a (b::'a::euclidean-space)
  obtains p where p division-of (⋃f)
proof −
  have ∃p. p division-of (⋃f)
  proof (induct-tac f rule:finite-subset-induct)
    show ∃p. p division-of ⋃{} using elementary-empty by auto
  next
    fix x F
    assume as: finite F x ∉ F ∃p. p division-of ⋃F x∈f
    from this(3) obtain p where p: p division-of ⋃F ..
    from assms(2)[OF as(4)] obtain a b where x: x = cbox a b by blast
    have ∗: ⋃F = ⋃p
      using division-ofD[OF p] by auto
    show ∃p. p division-of ⋃insert x F
      using elementary-union-interval[OF p[unfolded ∗], of a b]
      unfolding Union-insert x ∗ by metis
  qed (insert assms, auto)
  then show ?thesis
    using that by auto
qed

lemma elementary-union:
  fixes s t :: 'a::euclidean-space set
  assumes ps division-of s pt division-of t
  obtains p where p division-of (s ∪ t)
proof −
  have ∗: s ∪ t = ⋃ps ∪ ⋃pt
    using assms unfolding division-of-def by auto
  show ?thesis
    apply (rule elementary-unions-intervals[of ps ∪ pt])
    using assms apply auto
    by (simp add: ∗ that)
qed

lemma partial-division-extend:
  fixes t :: 'a::euclidean-space set
  assumes p division-of s
    and q division-of t
    and s ⊆ t
```

```
  obtains r where p ⊆ r and r division-of t
proof −
  note divp = division-ofD[OF assms(1)] and divq = division-ofD[OF assms(2)]
  obtain a b where ab: t ⊆ cbox a b
    using elementary-subset-cbox[OF assms(2)] by auto
  obtain r1 where p ⊆ r1 r1 division-of (cbox a b)
    using assms
    by (metis ab dual-order.trans partial-division-extend-interval divp(6))
  note r1 = this division-ofD[OF this(2)]
  obtain p' where p' division-of ⋃(r1 − p)
    apply (rule elementary-unions-intervals[of r1 − p])
    using r1(3,6)
    apply auto
    done
  then obtain r2 where r2: r2 division-of (⋃(r1 − p)) ∩ (⋃q)
    by (metis assms(2) divq(6) elementary-inter)
  {
    fix x
    assume x: x ∈ t x ∉ s
    then have x∈⋃r1
      unfolding r1 using ab by auto
    then obtain r where r: r ∈ r1 x ∈ r
      unfolding Union-iff ..
    moreover
    have r ∉ p
    proof
      assume r ∈ p
      then have x ∈ s using divp(2) r by auto
      then show False using x by auto
    qed
    ultimately have x∈⋃(r1 − p) by auto
  }
  then have ∗: t = ⋃p ∪ (⋃(r1 − p) ∩ ⋃q)
    unfolding divp divq using assms(3) by auto
  show ?thesis
    apply (rule that[of p ∪ r2])
    unfolding ∗
    defer
    apply (rule division-disjoint-union)
    unfolding divp(6)
    apply(rule assms r2)+
  proof −
    have interior s ∩ interior (⋃(r1−p)) = {}
    proof (rule inter-interior-unions-intervals)
      show finite (r1 − p) and open (interior s) and ∀ t∈r1−p. ∃ a b. t = cbox a
b
        using r1 by auto
      have ∗: ⋀s. (⋀x. x ∈ s ⟹ False) ⟹ s = {}
        by auto
```

show $\forall\, t{\in}r1-p.\ \mathit{interior}\ s \cap \mathit{interior}\ t = \{\}$
proof
fix *m x*
assume *as*: $m \in r1 - p$
have $\mathit{interior}\ m \cap \mathit{interior}\ (\bigcup p) = \{\}$
proof (*rule inter-interior-unions-intervals*)
show *finite p* **and** *open* (*interior m*) **and** $\forall\, t{\in}p.\ \exists\, a\ b.\ t = \mathit{cbox}\ a\ b$
using *divp* **by** *auto*
show $\forall\, t{\in}p.\ \mathit{interior}\ m \cap \mathit{interior}\ t = \{\}$
by (*metis DiffD1 DiffD2 as r1*(*1*) *r1*(*7*) *set-rev-mp*)
qed
then show $\mathit{interior}\ s \cap \mathit{interior}\ m = \{\}$
unfolding *divp* **by** *auto*
qed
qed
then show $\mathit{interior}\ s \cap \mathit{interior}\ (\bigcup(r1-p) \cap (\bigcup q)) = \{\}$
using *interior-subset* **by** *auto*
qed *auto*
qed

41.7 Tagged (partial) divisions.

definition *tagged-partial-division-of* (**infixr** *tagged'-partial'-division'-of 40*)
where *s tagged-partial-division-of i* $\longleftrightarrow$
finite s $\wedge$
$(\forall\, x\ k.\ (x,\ k) \in s \longrightarrow x \in k \wedge k \subseteq i \wedge (\exists\, a\ b.\ k = \mathit{cbox}\ a\ b)) \wedge$
$(\forall\, x1\ k1\ x2\ k2.\ (x1,\ k1) \in s \wedge (x2,\ k2) \in s \wedge (x1,\ k1) \neq (x2,\ k2) \longrightarrow$
$\mathit{interior}\ k1 \cap \mathit{interior}\ k2 = \{\})$

lemma *tagged-partial-division-ofD*[*dest*]:
assumes *s tagged-partial-division-of i*
shows *finite s*
and $\bigwedge x\ k.\ (x,k) \in s \Longrightarrow x \in k$
and $\bigwedge x\ k.\ (x,k) \in s \Longrightarrow k \subseteq i$
and $\bigwedge x\ k.\ (x,k) \in s \Longrightarrow \exists\, a\ b.\ k = \mathit{cbox}\ a\ b$
and $\bigwedge x1\ k1\ x2\ k2.\ (x1,k1) \in s \Longrightarrow$
$(x2,\ k2) \in s \Longrightarrow (x1,\ k1) \neq (x2,\ k2) \Longrightarrow \mathit{interior}\ k1 \cap \mathit{interior}\ k2 = \{\}$
using *assms* **unfolding** *tagged-partial-division-of-def* **by** *blast*+

definition *tagged-division-of* (**infixr** *tagged'-division'-of 40*)
where *s tagged-division-of i* $\longleftrightarrow$ *s tagged-partial-division-of i* $\wedge$ $(\bigcup\{k.\ \exists x.\ (x,k) \in s\} = i)$

lemma *tagged-division-of-finite*: *s tagged-division-of i* $\Longrightarrow$ *finite s*
unfolding *tagged-division-of-def tagged-partial-division-of-def* **by** *auto*

lemma *tagged-division-of*:
s tagged-division-of i $\longleftrightarrow$
finite s $\wedge$

$(\forall\, x\ k.\ (x,\ k) \in s \longrightarrow x \in k \wedge k \subseteq i \wedge (\exists\, a\ b.\ k = cbox\ a\ b)) \wedge$
$(\forall\, x1\ k1\ x2\ k2.\ (x1,\ k1) \in s \wedge (x2,\ k2) \in s \wedge (x1,\ k1) \neq (x2,\ k2) \longrightarrow$
$\quad interior\ k1 \cap interior\ k2 = \{\}) \wedge$
$(\bigcup \{k.\ \exists\, x.\ (x,k) \in s\} = i)$
unfolding *tagged-division-of-def tagged-partial-division-of-def* **by** *auto*

lemma *tagged-division-ofI*:
assumes *finite s*
and $\bigwedge x\ k.\ (x,k) \in s \Longrightarrow x \in k$
and $\bigwedge x\ k.\ (x,k) \in s \Longrightarrow k \subseteq i$
and $\bigwedge x\ k.\ (x,k) \in s \Longrightarrow \exists\, a\ b.\ k = cbox\ a\ b$
and $\bigwedge x1\ k1\ x2\ k2.\ (x1,k1) \in s \Longrightarrow (x2,\ k2) \in s \Longrightarrow (x1,\ k1) \neq (x2,\ k2) \Longrightarrow$
$\quad interior\ k1 \cap interior\ k2 = \{\}$
and $(\bigcup \{k.\ \exists\, x.\ (x,k) \in s\} = i)$
shows *s tagged-division-of i*
unfolding *tagged-division-of*
using *assms*
apply *auto*
apply *fastforce+*
done

lemma *tagged-division-ofD*[*dest*]:
assumes *s tagged-division-of i*
shows *finite s*
and $\bigwedge x\ k.\ (x,k) \in s \Longrightarrow x \in k$
and $\bigwedge x\ k.\ (x,k) \in s \Longrightarrow k \subseteq i$
and $\bigwedge x\ k.\ (x,k) \in s \Longrightarrow \exists\, a\ b.\ k = cbox\ a\ b$
and $\bigwedge x1\ k1\ x2\ k2.\ (x1,\ k1) \in s \Longrightarrow (x2,\ k2) \in s \Longrightarrow (x1,\ k1) \neq (x2,\ k2) \Longrightarrow$
$\quad interior\ k1 \cap interior\ k2 = \{\}$
and $(\bigcup \{k.\ \exists\, x.\ (x,k) \in s\} = i)$
using *assms* **unfolding** *tagged-division-of* **by** *blast+*

lemma *division-of-tagged-division*:
assumes *s tagged-division-of i*
shows (*snd* ' *s*) *division-of i*
proof (*rule division-ofI*)
note *assm* = *tagged-division-ofD*[*OF assms*]
show $\bigcup$(*snd* ' *s*) = *i finite* (*snd* ' *s*)
using *assm* **by** *auto*
fix *k*
assume *k*: *k* ∈ *snd* ' *s*
then obtain *xk* **where** *xk*: (*xk*, *k*) ∈ *s*
by *auto*
then show *k* ⊆ *i k* ≠ {} ∃ *a b*. *k* = *cbox a b*
using *assm* **by** *fastforce+*
fix *k'*
assume *k'*: *k'* ∈ *snd* ' *s k* ≠ *k'*
from *this*(*1*) **obtain** *xk'* **where** *xk'*: (*xk'*, *k'*) ∈ *s*
by *auto*

```
  then show interior k ∩ interior k' = {}
    using assm(5) k'(2) xk by blast
qed

lemma partial-division-of-tagged-division:
  assumes s tagged-partial-division-of i
  shows (snd ` s) division-of ⋃(snd ` s)
proof (rule division-ofI)
  note assm = tagged-partial-division-ofD[OF assms]
  show finite (snd ` s) ⋃(snd ` s) = ⋃(snd ` s)
    using assm by auto
  fix k
  assume k: k ∈ snd ` s
  then obtain xk where xk: (xk, k) ∈ s
    by auto
  then show k ≠ {} ∃ a b. k = cbox a b k ⊆ ⋃(snd ` s)
    using assm by auto
  fix k'
  assume k': k' ∈ snd ` s k ≠ k'
  from this(1) obtain xk' where xk': (xk', k') ∈ s
    by auto
  then show interior k ∩ interior k' = {}
    using assm(5) k'(2) xk by auto
qed

lemma tagged-partial-division-subset:
  assumes s tagged-partial-division-of i
    and t ⊆ s
  shows t tagged-partial-division-of i
  using assms
  unfolding tagged-partial-division-of-def
  using finite-subset[OF assms(2)]
  by blast

lemma setsum-over-tagged-division-lemma:
  assumes p tagged-division-of i
    and ⋀u v. cbox u v ≠ {} ⟹ content (cbox u v) = 0 ⟹ d (cbox u v) = 0
  shows setsum (λ(x,k). d k) p = setsum d (snd ` p)
proof −
  have ∗: (λ(x,k). d k) = d ∘ snd
    unfolding o-def by (rule ext) auto
  note assm = tagged-division-ofD[OF assms(1)]
  show ?thesis
    unfolding ∗
  proof (rule setsum.reindex-nontrivial[symmetric])
    show finite p
      using assm by auto
    fix x y
    assume x∈p y∈p x≠y snd x = snd y
```

```
    obtain a b where ab: snd x = cbox a b
      using assm(4)[of fst x snd x] ‹x∈p› by auto
    have (fst x, snd y) ∈ p (fst x, snd y) ≠ y
      by (metis prod.collapse ‹x∈p› ‹snd x = snd y› ‹x ≠ y›)+
    with ‹x∈p› ‹y∈p› have interior (snd x) ∩ interior (snd y) = {}
      by (intro assm(5)[of fst x - fst y]) auto
    then have content (cbox a b) = 0
      unfolding ‹snd x = snd y›[symmetric] ab content-eq-0-interior by auto
    then have d (cbox a b) = 0
      using assm(2)[of fst x snd x] ‹x∈p› ab[symmetric] by (intro assms(2)) auto
    then show d (snd x) = 0
      unfolding ab by auto
  qed
qed

lemma tag-in-interval: p tagged-division-of i ⟹ (x, k) ∈ p ⟹ x ∈ i
  by auto

lemma tagged-division-of-empty: {} tagged-division-of {}
  unfolding tagged-division-of by auto

lemma tagged-partial-division-of-trivial[simp]: p tagged-partial-division-of {} ⟷
p = {}
  unfolding tagged-partial-division-of-def by auto

lemma tagged-division-of-trivial[simp]: p tagged-division-of {} ⟷ p = {}
  unfolding tagged-division-of by auto

lemma tagged-division-of-self: x ∈ cbox a b ⟹ {(x,cbox a b)} tagged-division-of
(cbox a b)
  by (rule tagged-division-ofI) auto

lemma tagged-division-of-self-real: x ∈ {a .. b::real} ⟹ {(x,{a .. b})} tagged-division-of
{a .. b}
  unfolding box-real[symmetric]
  by (rule tagged-division-of-self)

lemma tagged-division-union:
  assumes p1 tagged-division-of s1
    and p2 tagged-division-of s2
    and interior s1 ∩ interior s2 = {}
  shows (p1 ∪ p2) tagged-division-of (s1 ∪ s2)
proof (rule tagged-division-ofI)
  note p1 = tagged-division-ofD[OF assms(1)]
  note p2 = tagged-division-ofD[OF assms(2)]
  show finite (p1 ∪ p2)
    using p1(1) p2(1) by auto
  show ⋃{k. ∃x. (x, k) ∈ p1 ∪ p2} = s1 ∪ s2
    using p1(6) p2(6) by blast
```

fix $x\ k$
assume xk: $(x,\ k) \in p1 \cup p2$
show $x \in k\ \exists\, a\ b.\ k = cbox\ a\ b$
using $xk\ p1(2,4)\ p2(2,4)$ **by** *auto*
show $k \subseteq s1 \cup s2$
using $xk\ p1(3)\ p2(3)$ **by** *blast*
fix $x'\ k'$
assume xk': $(x',\ k') \in p1 \cup p2\ (x,\ k) \neq (x',\ k')$
have $*$: $\bigwedge a\ b.\ a \subseteq s1 \Longrightarrow b \subseteq s2 \Longrightarrow interior\ a \cap interior\ b = \{\}$
using $assms(3)$ *interior-mono* **by** *blast*
show $interior\ k \cap interior\ k' = \{\}$
apply $(cases\ (x,\ k) \in p1)$
apply $(meson * UnE\ assms(1)\ assms(2)\ p1(5)\ tagged\text{-}division\text{-}ofD(3)\ xk'(1)\ xk'(2))$
by $(metis * UnE\ assms(1)\ assms(2)\ inf\text{-}sup\text{-}aci(1)\ p2(5)\ tagged\text{-}division\text{-}ofD(3)\ xk\ xk'(1)\ xk'(2))$
qed

lemma *tagged-division-unions*:
assumes *finite iset*
and $\forall\, i{\in}iset.\ pfn\ i\ tagged\text{-}division\text{-}of\ i$
and $\forall\, i1{\in}iset.\ \forall\, i2{\in}iset.\ i1 \neq i2 \longrightarrow interior(i1) \cap interior(i2) = \{\}$
shows $\bigcup(pfn\ `\ iset)\ tagged\text{-}division\text{-}of\ (\bigcup iset)$
proof $(rule\ tagged\text{-}division\text{-}ofI)$
note $assm = tagged\text{-}division\text{-}ofD[OF\ assms(2)[rule\text{-}format]]$
show $finite\ (\bigcup(pfn\ `\ iset))$
apply $(rule\ finite\text{-}Union)$
using *assms*
apply *auto*
done
have $\bigcup\{k.\ \exists x.\ (x,\ k) \in \bigcup(pfn\ `\ iset)\} = \bigcup((\lambda i.\ \bigcup\{k.\ \exists x.\ (x,\ k) \in pfn\ i\})\ `\ iset)$
by *blast*
also have $\ldots = \bigcup iset$
using $assm(6)$ **by** *auto*
finally show $\bigcup\{k.\ \exists x.\ (x,\ k) \in \bigcup(pfn\ `\ iset)\} = \bigcup iset$ **.**
fix $x\ k$
assume xk: $(x,\ k) \in \bigcup(pfn\ `\ iset)$
then obtain i **where** i: $i \in iset\ (x,\ k) \in pfn\ i$
by *auto*
show $x \in k\ \exists\, a\ b.\ k = cbox\ a\ b\ k \subseteq \bigcup iset$
using $assm(2{-}4)[OF\ i]$ **using** $i(1)$ **by** *auto*
fix $x'\ k'$
assume xk': $(x',\ k') \in \bigcup(pfn\ `\ iset)\ (x,\ k) \neq (x',\ k')$
then obtain i' **where** i': $i' \in iset\ (x',\ k') \in pfn\ i'$
by *auto*
have $*$: $\bigwedge a\ b.\ i \neq i' \Longrightarrow a \subseteq i \Longrightarrow b \subseteq i' \Longrightarrow interior\ a \cap interior\ b = \{\}$
using $i(1)\ i'(1)$
using $assms(3)[rule\text{-}format]\ interior\text{-}mono$

by *blast*
show *interior k ∩ interior k' = {}*
apply (*cases i = i'*)
using *assm(5) i' i(2) xk'(2)* **apply** *blast*
using ∗ *assm(3) i' i* **by** *auto*
qed

lemma *tagged-partial-division-of-union-self*:
assumes *p tagged-partial-division-of s*
shows *p tagged-division-of* (⋃(*snd* ' *p*))
apply (*rule tagged-division-ofI*)
using *tagged-partial-division-ofD*[*OF assms*]
apply *auto*
done

lemma *tagged-division-of-union-self*:
assumes *p tagged-division-of s*
shows *p tagged-division-of* (⋃(*snd* ' *p*))
apply (*rule tagged-division-ofI*)
using *tagged-division-ofD*[*OF assms*]
apply *auto*
done

41.8 Fine-ness of a partition w.r.t. a gauge.

definition *fine* (**infixr** *fine 46*)
where *d fine s* ⟷ (∀(*x*,*k*) ∈ *s*. *k* ⊆ *d x*)

lemma *fineI*:
assumes ⋀*x k*. (*x*, *k*) ∈ *s* ⟹ *k* ⊆ *d x*
shows *d fine s*
using *assms* **unfolding** *fine-def* **by** *auto*

lemma *fineD*[*dest*]:
assumes *d fine s*
shows ⋀*x k*. (*x*,*k*) ∈ *s* ⟹ *k* ⊆ *d x*
using *assms* **unfolding** *fine-def* **by** *auto*

lemma *fine-inter*: (λ*x*. *d1 x* ∩ *d2 x*) *fine p* ⟷ *d1 fine p* ∧ *d2 fine p*
unfolding *fine-def* **by** *auto*

lemma *fine-inters*:
(λ*x*. ⋂{*f d x* | *d*. *d* ∈ *s*}) *fine p* ⟷ (∀ *d*∈*s*. (*f d*) *fine p*)
unfolding *fine-def* **by** *blast*

lemma *fine-union*: *d fine p1* ⟹ *d fine p2* ⟹ *d fine* (*p1* ∪ *p2*)
unfolding *fine-def* **by** *blast*

lemma *fine-unions*: (⋀*p*. *p* ∈ *ps* ⟹ *d fine p*) ⟹ *d fine* (⋃*ps*)

unfolding *fine-def* **by** *auto*

lemma *fine-subset*: $p \subseteq q \Longrightarrow d$ *fine* $q \Longrightarrow d$ *fine* p
unfolding *fine-def* **by** *blast*

41.9 Gauge integral. Define on compact intervals first, then use a limit.

definition *has-integral-compact-interval* (**infixr** *has'-integral'-compact'-interval* 46)
where (f *has-integral-compact-interval* y) $i \longleftrightarrow$
$(\forall e{>}0.\ \exists d.\ gauge\ d \wedge$
$(\forall p.\ p$ *tagged-division-of* $i \wedge d$ *fine* $p \longrightarrow$
norm (*setsum* ($\lambda(x,k).$ *content* $k *_R f\ x$) $p - y) < e))$

definition *has-integral* ::
($'n$::*euclidean-space* $\Rightarrow$ $'b$::*real-normed-vector*) $\Rightarrow$ $'b \Rightarrow$ $'n$ *set* $\Rightarrow$ *bool*
(**infixr** *has'-integral* 46)
where (f *has-integral* y) $i \longleftrightarrow$
(*if* $\exists a\ b.\ i =$ *cbox* $a\ b$
then (f *has-integral-compact-interval* y) i
else $(\forall e{>}0.\ \exists B{>}0.\ \forall a\ b.$ *ball* $0\ B \subseteq$ *cbox* $a\ b \longrightarrow$
$(\exists z.$ $((\lambda x.$ *if* $x \in i$ *then* $f\ x$ *else* $0)$ *has-integral-compact-interval* $z)$ (*cbox* $a\ b$)
$\wedge$
norm $(z - y) < e)))$

lemma *has-integral*:
(f *has-integral* y) (*cbox* $a\ b$) $\longleftrightarrow$
$(\forall e{>}0.\ \exists d.\ gauge\ d \wedge$
$(\forall p.\ p$ *tagged-division-of* (*cbox* $a\ b$) $\wedge\ d$ *fine* $p \longrightarrow$
norm (*setsum* ($\lambda(x,k).$ *content*(k) $*_R f\ x$) $p - y) < e))$
unfolding *has-integral-def has-integral-compact-interval-def*
by *auto*

lemma *has-integral-real*:
(f *has-integral* y) $\{a$ .. b::*real*$\} \longleftrightarrow$
$(\forall e{>}0.\ \exists d.\ gauge\ d \wedge$
$(\forall p.\ p$ *tagged-division-of* $\{a$ .. $b\} \wedge d$ *fine* $p \longrightarrow$
norm (*setsum* ($\lambda(x,k).$ *content*(k) $*_R f\ x$) $p - y) < e))$
unfolding *box-real*[*symmetric*]
by (*rule has-integral*)

lemma *has-integralD*[*dest*]:
assumes (f *has-integral* y) (*cbox* $a\ b$)
and $e > 0$
obtains d **where** *gauge* d
and $\bigwedge p.\ p$ *tagged-division-of* (*cbox* $a\ b$) $\Longrightarrow d$ *fine* $p \Longrightarrow$
norm (*setsum* ($\lambda(x,k).$ *content*(k) $*_R f(x)$) $p - y) < e$
using *assms* **unfolding** *has-integral* **by** *auto*

lemma *has-integral-alt*:
(*f has-integral y*) *i* ⟷
(*if* ∃ *a b*. *i* = *cbox a b*
then (*f has-integral y*) *i*
else (∀ *e*>0. ∃ *B*>0. ∀ *a b*. *ball 0 B* ⊆ *cbox a b* ⟶
(∃ *z*. ((λ*x*. *if x* ∈ *i then f*(*x*) *else 0*) *has-integral z*) (*cbox a b*) ∧ *norm* (*z* − *y*) < *e*)))
unfolding *has-integral*
unfolding *has-integral-compact-interval-def has-integral-def*
by *auto*

lemma *has-integral-altD*:
assumes (*f has-integral y*) *i*
and ¬ (∃ *a b*. *i* = *cbox a b*)
and *e*>0
obtains *B* **where** *B* > 0
and ∀ *a b*. *ball 0 B* ⊆ *cbox a b* ⟶
(∃ *z*. ((λ*x*. *if x* ∈ *i then f*(*x*) *else 0*) *has-integral z*) (*cbox a b*) ∧ *norm*(*z* − *y*) < *e*)
using *assms*
unfolding *has-integral*
unfolding *has-integral-compact-interval-def has-integral-def*
by *auto*

definition *integrable-on* (**infixr** *integrable′-on* 46)
where *f integrable-on i* ⟷ (∃ *y*. (*f has-integral y*) *i*)

definition *integral i f* = (*SOME y*. (*f has-integral y*) *i* ∨ ~ *f integrable-on i* ∧ *y*=0)

lemma *integrable-integral*[*dest*]: *f integrable-on i* ⟹ (*f has-integral* (*integral i f*)) *i*
unfolding *integrable-on-def integral-def* **by** (*metis* (*mono-tags*, *lifting*) *someI-ex*)

lemma *not-integrable-integral*: ~ *f integrable-on i* ⟹ *integral i f* = 0
unfolding *integrable-on-def integral-def* **by** *blast*

lemma *has-integral-integrable*[*intro*]: (*f has-integral i*) *s* ⟹ *f integrable-on s*
unfolding *integrable-on-def* **by** *auto*

lemma *has-integral-integral*: *f integrable-on s* ⟷ (*f has-integral* (*integral s f*)) *s*
by *auto*

lemma *setsum-content-null*:
assumes *content* (*cbox a b*) = 0
and *p tagged-division-of* (*cbox a b*)
shows *setsum* (λ(*x*,*k*). *content k* ∗$_R$ *f x*) *p* = (0::′*a*::*real-normed-vector*)
proof (*rule setsum.neutral*, *rule*)
fix *y*

assume *y*: $y \in p$
obtain *x k* **where** *xk*: $y = (x, k)$
using *surj-pair*[*of y*] **by** *blast*
note *assm* = *tagged-division-ofD*(*3*−*4*)[*OF assms*(*2*) *y*[*unfolded xk*]]
from *this*(*2*) **obtain** *c d* **where** *k*: $k = cbox\ c\ d$ **by** *blast*
have $(\lambda(x, k).\ content\ k *_R f\ x)\ y = content\ k *_R f\ x$
unfolding *xk* **by** *auto*
also have $\ldots = 0$
using *content-subset*[*OF assm*(*1*)[*unfolded k*]] *content-pos-le*[*of c d*]
unfolding *assms*(*1*) *k*
by *auto*
finally show $(\lambda(x, k).\ content\ k *_R f\ x)\ y = 0$ **.**
qed

41.10 Some basic combining lemmas.

lemma *tagged-division-unions-exists*:
assumes *finite iset*
and $\forall i{\in}iset.\ \exists p.\ p$ *tagged-division-of* $i \wedge d$ *fine* p
and $\forall i1{\in}iset.\ \forall i2{\in}iset.\ i1 \neq i2 \longrightarrow interior\ i1 \cap interior\ i2 = \{\}$
and $\bigcup iset = i$
obtains *p* **where** *p tagged-division-of i* **and** *d fine p*
proof −
obtain *pfn* **where** *pfn*:
$\bigwedge x.\ x \in iset \Longrightarrow pfn\ x$ *tagged-division-of* x
$\bigwedge x.\ x \in iset \Longrightarrow d$ *fine* $pfn\ x$
using *bchoice*[*OF assms*(*2*)] **by** *auto*
show *thesis*
apply (*rule-tac* $p{=}\bigcup(pfn\ `\ iset)$ **in** *that*)
using *assms*(*1*) *assms*(*3*) *assms*(*4*) *pfn*(*1*) *tagged-division-unions* **apply** *force*
by (*metis* (*mono-tags*, *lifting*) *fine-unions imageE pfn*(*2*))
qed

41.11 The set we're concerned with must be closed.

lemma *division-of-closed*:
fixes $i :: 'n{::}euclidean\text{-}space\ set$
shows s *division-of* $i \Longrightarrow$ *closed* i
unfolding *division-of-def* **by** *fastforce*

41.12 General bisection principle for intervals; might be useful elsewhere.

lemma *interval-bisection-step*:
fixes $type :: 'a{::}euclidean\text{-}space$
assumes $P\ \{\}$
and $\forall s\ t.\ P\ s \wedge P\ t \wedge interior(s) \cap interior(t) = \{\} \longrightarrow P\ (s \cup t)$
and $\neg\ P\ (cbox\ a\ (b{::}'a))$
obtains $c\ d$ **where** $\neg\ P\ (cbox\ c\ d)$

```
    and ∀ i∈Basis. a·i ≤ c·i ∧ c·i ≤ d·i ∧ d·i ≤ b·i ∧ 2 * (d·i − c·i) ≤ b·i −
a·i
proof −
  have cbox a b ≠ {}
    using assms(1,3) by metis
  then have ab: ⋀i. i∈Basis ⟹ a · i ≤ b · i
    by (force simp: mem-box)
  { fix f
    have ⟦finite f;
          ⋀s. s∈f ⟹ P s;
          ⋀s. s∈f ⟹ ∃ a b. s = cbox a b;
           ⋀s t. s∈f ⟹ t∈f ⟹ s ≠ t ⟹ interior s ∩ interior t = {}⟧ ⟹ P
(⋃f)
    proof (induct f rule: finite-induct)
      case empty
      show ?case
        using assms(1) by auto
    next
      case (insert x f)
      show ?case
        unfolding Union-insert
        apply (rule assms(2)[rule-format])
        using inter-interior-unions-intervals [of f interior x]
        apply (auto simp: insert)
        by (metis IntI empty-iff insert.hyps(2) insert.prems(3) insert-iff)
    qed
  } note UN-cases = this
  let ?A = {cbox c d | c d::'a. ∀ i∈Basis. (c·i = a·i) ∧ (d·i = (a·i + b·i) / 2) ∨
    (c·i = (a·i + b·i) / 2) ∧ (d·i = b·i)}
  let ?PP = λc d. ∀ i∈Basis. a·i ≤ c·i ∧ c·i ≤ d·i ∧ d·i ≤ b·i ∧ 2 * (d·i − c·i)
≤ b·i − a·i
  {
    presume ∀ c d. ?PP c d ⟶ P (cbox c d) ⟹ False
    then show thesis
      unfolding atomize-not not-all
      by (blast intro: that)
  }
  assume as: ∀ c d. ?PP c d ⟶ P (cbox c d)
  have P (⋃ ?A)
  proof (rule UN-cases)
    let ?B = (λs. cbox (∑ i∈Basis. (if i ∈ s then a·i else (a·i + b·i) / 2) *R i::'a)
      (∑ i∈Basis. (if i ∈ s then (a·i + b·i) / 2 else b·i) *R i)) ` {s. s ⊆ Basis}
    have ?A ⊆ ?B
    proof
      fix x
      assume x ∈ ?A
      then obtain c d
        where x: x = cbox c d
                 ⋀i. i ∈ Basis ⟹
```

```
                c · i = a · i ∧ d · i = (a · i + b · i) / 2 ∨
                c · i = (a · i + b · i) / 2 ∧ d · i = b · i by blast
    show x ∈ ?B
      unfolding image-iff x
      apply (rule-tac x={i. i∈Basis ∧ c·i = a·i} in bexI)
      apply (rule arg-cong2 [where f = cbox])
      using x(2) ab
      apply (auto simp add: euclidean-eq-iff[where 'a='a])
      by fastforce
  qed
  then show finite ?A
    by (rule finite-subset) auto
next
  fix s
  assume s ∈ ?A
  then obtain c d
    where s: s = cbox c d
          ⋀i. i ∈ Basis ⟹
              c · i = a · i ∧ d · i = (a · i + b · i) / 2 ∨
              c · i = (a · i + b · i) / 2 ∧ d · i = b · i
    by blast
  show P s
    unfolding s
    apply (rule as[rule-format])
    using ab s(2) by force
  show ∃ a b. s = cbox a b
    unfolding s by auto
  fix t
  assume t ∈ ?A
  then obtain e f where t:
    t = cbox e f
    ⋀i. i ∈ Basis ⟹
      e · i = a · i ∧ f · i = (a · i + b · i) / 2 ∨
      e · i = (a · i + b · i) / 2 ∧ f · i = b · i
    by blast
  assume s ≠ t
  then have ¬ (c = e ∧ d = f)
    unfolding s t by auto
  then obtain i where c·i ≠ e·i ∨ d·i ≠ f·i and i': i ∈ Basis
    unfolding euclidean-eq-iff[where 'a='a] by auto
  then have i: c·i ≠ e·i d·i ≠ f·i
    using s(2) t(2) apply fastforce
    using t(2)[OF i'] ‹c · i ≠ e · i ∨ d · i ≠ f · i› i' s(2) t(2) by fastforce
  have *: ⋀s t. (⋀a. a ∈ s ⟹ a ∈ t ⟹ False) ⟹ s ∩ t = {}
    by auto
  show interior s ∩ interior t = {}
    unfolding s t interior-cbox
  proof (rule *)
    fix x
```

assume $x \in box\ c\ d\ x \in box\ e\ f$
then have x: $c{\cdot}i < d{\cdot}i\ e{\cdot}i < f{\cdot}i\ c{\cdot}i < f{\cdot}i\ e{\cdot}i < d{\cdot}i$
unfolding *mem-box* **using** i'
by *force+*
show *False* **using** $s(2)[OF\ i']$
proof *safe*
assume *as*: $c \cdot i = a \cdot i\ d \cdot i = (a \cdot i + b \cdot i)\ /\ 2$
show *False*
using $t(2)[OF\ i']$ **and** $i\ x$ **unfolding** *as* **by** (*fastforce simp add*:*field-simps*)
next
assume *as*: $c \cdot i = (a \cdot i + b \cdot i)\ /\ 2\ d \cdot i = b \cdot i$
show *False*
using $t(2)[OF\ i']$ **and** $i\ x$ **unfolding** *as* **by**(*fastforce simp add*:*field-simps*)
qed
qed
qed
also have $\bigcup ?A = cbox\ a\ b$
proof (*rule set-eqI*,*rule*)
fix x
assume $x \in \bigcup ?A$
then obtain $c\ d$ **where** x:
$x \in cbox\ c\ d$
$\bigwedge i.\ i \in Basis \Longrightarrow$
$c \cdot i = a \cdot i \wedge d \cdot i = (a \cdot i + b \cdot i)\ /\ 2 \vee$
$c \cdot i = (a \cdot i + b \cdot i)\ /\ 2 \wedge d \cdot i = b \cdot i$
by *blast*
show $x{\in}cbox\ a\ b$
unfolding *mem-box*
proof *safe*
fix $i :: 'a$
assume i: $i \in Basis$
then show $a \cdot i \leq x \cdot i\ x \cdot i \leq b \cdot i$
using $x(2)[OF\ i]$ $x(1)$[*unfolded mem-box*,*THEN bspec*, *OF i*] **by** *auto*
qed
next
fix x
assume x: $x \in cbox\ a\ b$
have $\forall i{\in}Basis.$
$\exists c\ d.\ (c = a{\cdot}i \wedge d = (a{\cdot}i + b{\cdot}i)\ /\ 2 \vee c = (a{\cdot}i + b{\cdot}i)\ /\ 2 \wedge d = b{\cdot}i) \wedge$
$c{\leq}x{\cdot}i \wedge x{\cdot}i \leq d$
(**is** $\forall i{\in}Basis.\ \exists c\ d.\ ?P\ i\ c\ d$)
unfolding *mem-box*
proof
fix $i :: 'a$
assume i: $i \in Basis$
have $?P\ i\ (a{\cdot}i)\ ((a \cdot i + b \cdot i)\ /\ 2) \vee ?P\ i\ ((a \cdot i + b \cdot i)\ /\ 2)\ (b{\cdot}i)$
using x[*unfolded mem-box*,*THEN bspec*, *OF i*] **by** *auto*
then show $\exists c\ d.\ ?P\ i\ c\ d$
by *blast*

```
    qed
    then show x∈⋃ ?A
      unfolding Union-iff Bex-def mem-Collect-eq choice-Basis-iff
      apply auto
      apply (rule-tac x=cbox xa xaa in exI)
      unfolding mem-box
      apply auto
      done
  qed
  finally show False
    using assms by auto
qed

lemma interval-bisection:
  fixes type :: 'a::euclidean-space
  assumes P {}
    and (∀ s t. P s ∧ P t ∧ interior(s) ∩ interior(t) = {} ⟶ P(s ∪ t))
    and ¬ P (cbox a (b::'a))
  obtains x where x ∈ cbox a b
    and ∀ e>0. ∃ c d. x ∈ cbox c d ∧ cbox c d ⊆ ball x e ∧ cbox c d ⊆ cbox a b ∧
¬ P (cbox c d)
proof −
  have ∀ x. ∃ y. ¬ P (cbox (fst x) (snd x)) ⟶ (¬ P (cbox (fst y) (snd y)) ∧
    (∀ i∈Basis. fst x•i ≤ fst y•i ∧ fst y•i ≤ snd y•i ∧ snd y•i ≤ snd x•i ∧
      2 ∗ (snd y•i − fst y•i) ≤ snd x•i − fst x•i)) (is ∀ x. ?P x)
  proof
    show ?P x for x
    proof (cases P (cbox (fst x) (snd x)))
      case True
      then show ?thesis by auto
    next
      case as: False
      obtain c d where ¬ P (cbox c d)
        ∀ i∈Basis.
          fst x • i ≤ c • i ∧
          c • i ≤ d • i ∧
          d • i ≤ snd x • i ∧
          2 ∗ (d • i − c • i) ≤ snd x • i − fst x • i
        by (rule interval-bisection-step[of P, OF assms(1−2) as])
      then show ?thesis
        apply −
        apply (rule-tac x=(c,d) in exI)
        apply auto
        done
    qed
  qed
  then obtain f where f:
    ∀ x.
      ¬ P (cbox (fst x) (snd x)) ⟶
```

```
    ¬ P (cbox (fst (f x)) (snd (f x))) ∧
     (∀ i∈Basis.
        fst x · i ≤ fst (f x) · i ∧
        fst (f x) · i ≤ snd (f x) · i ∧
        snd (f x) · i ≤ snd x · i ∧
        2 * (snd (f x) · i − fst (f x) · i) ≤ snd x · i − fst x · i)
   apply −
   apply (drule choice)
   apply blast
   done
 def AB ≡ λn. (f ^^ n) (a,b)
 def A ≡ λn. fst(AB n)
 def B ≡ λn. snd(AB n)
 note ab-def = A-def B-def AB-def
 have A 0 = a B 0 = b ⋀n. ¬ P (cbox (A(Suc n)) (B(Suc n))) ∧
   (∀ i∈Basis. A(n)·i ≤ A(Suc n)·i ∧ A(Suc n)·i ≤ B(Suc n)·i ∧ B(Suc n)·i ≤
B(n)·i ∧
   2 * (B(Suc n)·i − A(Suc n)·i) ≤ B(n)·i − A(n)·i) (is ⋀n. ?P n)
 proof −
   show A 0 = a B 0 = b
     unfolding ab-def by auto
   note S = ab-def funpow.simps o-def id-apply
   show ?P n for n
   proof (induct n)
     case 0
     then show ?case
       unfolding S
       apply (rule f[rule-format]) using assms(3)
       apply auto
       done
   next
     case (Suc n)
     show ?case
       unfolding S
       apply (rule f[rule-format])
       using Suc
       unfolding S
       apply auto
       done
   qed
 qed
 note AB = this(1−2) conjunctD2[OF this(3),rule-format]

 have interv: ∃ n. ∀ x∈cbox (A n) (B n). ∀ y∈cbox (A n) (B n). dist x y < e
   if e: 0 < e for e
 proof −
   obtain n where n: (∑ i∈Basis. b · i − a · i) / e < 2 ^ n
     using real-arch-pow[of 2 (setsum (λi. b·i − a·i) Basis) / e] by auto
   show ?thesis
```

```
  proof (rule exI [where x=n], clarify)
    fix x y
    assume xy: x∈cbox (A n) (B n) y∈cbox (A n) (B n)
    have dist x y ≤ setsum (λi. |(x − y)·i|) Basis
      unfolding dist-norm by(rule norm-le-l1)
    also have ... ≤ setsum (λi. B n·i − A n·i) Basis
    proof (rule setsum-mono)
      fix i :: 'a
      assume i: i ∈ Basis
      show |(x − y) · i| ≤ B n · i − A n · i
        using xy[unfolded mem-box,THEN bspec, OF i]
        by (auto simp: inner-diff-left)
    qed
    also have ... ≤ setsum (λi. b·i − a·i) Basis / 2^n
      unfolding setsum-divide-distrib
    proof (rule setsum-mono)
      show B n · i − A n · i ≤ (b · i − a · i) / 2 ^ n if i: i ∈ Basis for i
      proof (induct n)
        case 0
        then show ?case
          unfolding AB by auto
      next
        case (Suc n)
        have B (Suc n) · i − A (Suc n) · i ≤ (B n · i − A n · i) / 2
          using AB(4)[of i n] using i by auto
        also have ... ≤ (b · i − a · i) / 2 ^ Suc n
          using Suc by (auto simp add: field-simps)
        finally show ?case .
      qed
    qed
    also have ... < e
      using n using e by (auto simp add: field-simps)
    finally show dist x y < e .
  qed
qed
{
  fix n m :: nat
  assume m ≤ n then have cbox (A n) (B n) ⊆ cbox (A m) (B m)
  proof (induction rule: inc-induct)
    case (step i)
    show ?case
      using AB(4) by (intro order-trans[OF step.IH] subset-box-imp) auto
  qed simp
} note ABsubset = this
have ∃a. ∀n. a∈ cbox (A n) (B n)
  by (rule decreasing-closed-nest[rule-format,OF closed-cbox - ABsubset interv])
    (metis nat.exhaust AB(1−3) assms(1,3))
then obtain x0 where x0: ⋀n. x0 ∈ cbox (A n) (B n)
  by blast
```

show *thesis*
proof (*rule that*[*rule-format*, *of x0*])
show *x0*∈*cbox a b*
using *x0*[*of 0*] **unfolding** *AB* **.**
fix *e* :: *real*
assume *e* > *0*
from *interv*[*OF this*] **obtain** *n*
where *n*: ∀ *x*∈*cbox* (*A n*) (*B n*). ∀ *y*∈*cbox* (*A n*) (*B n*). *dist x y* < *e* **..**
have ¬ *P* (*cbox* (*A n*) (*B n*))
apply (*cases 0* < *n*)
using *AB*(*3*)[*of n* − *1*] *assms*(*3*) *AB*(*1*−*2*)
apply *auto*
done
moreover have *cbox* (*A n*) (*B n*) ⊆ *ball x0 e*
using *n* **using** *x0*[*of n*] **by** *auto*
moreover have *cbox* (*A n*) (*B n*) ⊆ *cbox a b*
unfolding *AB*(*1*−*2*)[*symmetric*] **by** (*rule ABsubset*) *auto*
ultimately show ∃ *c d*. *x0* ∈ *cbox c d* ∧ *cbox c d* ⊆ *ball x0 e* ∧ *cbox c d* ⊆ *cbox a b* ∧ ¬ *P* (*cbox c d*)
apply (*rule-tac x*=*A n* **in** *exI*)
apply (*rule-tac x*=*B n* **in** *exI*)
apply (*auto simp*: *x0*)
done
qed
qed

41.13 Cousin's lemma.

lemma *fine-division-exists*:
fixes *a b* :: *'a*::*euclidean-space*
assumes *gauge g*
obtains *p* **where** *p tagged-division-of* (*cbox a b*) *g fine p*
proof −
presume ¬ (∃ *p*. *p tagged-division-of* (*cbox a b*) ∧ *g fine p*) ⟹ *False*
then obtain *p* **where** *p tagged-division-of* (*cbox a b*) *g fine p*
by *blast*
then show *thesis* **..**
next
assume *as*: ¬ (∃ *p*. *p tagged-division-of* (*cbox a b*) ∧ *g fine p*)
obtain *x* **where** *x*:
x ∈ (*cbox a b*)
⋀*e*. *0* < *e* ⟹
∃ *c d*.
x ∈ *cbox c d* ∧
cbox c d ⊆ *ball x e* ∧
cbox c d ⊆ (*cbox a b*) ∧
¬ (∃ *p*. *p tagged-division-of cbox c d* ∧ *g fine p*)
apply (*rule interval-bisection*[*of* λ*s*. ∃ *p*. *p tagged-division-of s* ∧ *g fine p*, *OF* - - *as*])

apply (*simp add*: *fine-def*)
apply (*metis tagged-division-union fine-union*)
apply (*auto simp*:)
done
obtain *e* **where** *e*: *e* > *0* *ball x e* ⊆ *g x*
using *gaugeD*[*OF assms*, *of x*] **unfolding** *open-contains-ball* **by** *auto*
from *x*(*2*)[*OF e*(*1*)]
obtain *c d* **where** *c-d*: *x* ∈ *cbox c d*
cbox c d ⊆ *ball x e*
cbox c d ⊆ *cbox a b*
¬ (∃*p*. *p tagged-division-of cbox c d* ∧ *g fine p*)
by *blast*
have *g fine* {(*x*, *cbox c d*)}
unfolding *fine-def* **using** *e* **using** *c-d*(*2*) **by** *auto*
then show *False*
using *tagged-division-of-self*[*OF c-d*(*1*)] **using** *c-d* **by** *auto*
qed

lemma *fine-division-exists-real*:
fixes *a b* :: *real*
assumes *gauge g*
obtains *p* **where** *p tagged-division-of* {*a* .. *b*} *g fine p*
by (*metis assms box-real*(*2*) *fine-division-exists*)

41.14 Basic theorems about integrals.

lemma *has-integral-unique*:
fixes *f* :: *'n*::*euclidean-space* ⇒ *'a*::*real-normed-vector*
assumes (*f has-integral k1*) *i*
and (*f has-integral k2*) *i*
shows *k1* = *k2*
proof (*rule ccontr*)
let *?e* = *norm* (*k1* − *k2*) / *2*
assume *as*: *k1* ≠ *k2*
then have *e*: *?e* > *0*
by *auto*
have *lem*: *False*
if *f-k1*: (*f has-integral k1*) (*cbox a b*)
and *f-k2*: (*f has-integral k2*) (*cbox a b*)
and *k1* ≠ *k2*
for *f* :: *'n* ⇒ *'a* **and** *a b k1 k2*
proof −
let *?e* = *norm* (*k1* − *k2*) / *2*
from ‹*k1* ≠ *k2*› **have** *e*: *?e* > *0* **by** *auto*
obtain *d1* **where** *d1*:
gauge d1
⋀*p*. *p tagged-division-of cbox a b* ⟹
d1 fine p ⟹ *norm* ((∑ (*x*, *k*)∈*p*. *content k* ∗*R* *f x*) − *k1*) < *norm* (*k1* − *k2*) / *2*

```
    by (rule has-integralD[OF f-k1 e]) blast
  obtain d2 where d2:
    gauge d2
    ⋀p. p tagged-division-of cbox a b ⟹
     d2 fine p ⟹ norm ((∑(x, k)∈p. content k *R f x) − k2) < norm (k1 −
k2) / 2
    by (rule has-integralD[OF f-k2 e]) blast
  obtain p where p:
    p tagged-division-of cbox a b
    (λx. d1 x ∩ d2 x) fine p
    by (rule fine-division-exists[OF gauge-inter[OF d1(1) d2(1)]])
  let ?c = (∑(x, k)∈p. content k *R f x)
  have norm (k1 − k2) ≤ norm (?c − k2) + norm (?c − k1)
    using norm-triangle-ineq4[of k1 − ?c k2 − ?c]
    by (auto simp add:algebra-simps norm-minus-commute)
  also have ... < norm (k1 − k2) / 2 + norm (k1 − k2) / 2
    apply (rule add-strict-mono)
    apply (rule-tac[!] d2(2) d1(2))
    using p unfolding fine-def
    apply auto
    done
  finally show False by auto
 qed
 {
  presume ¬ (∃ a b. i = cbox a b) ⟹ False
  then show False
    using as assms lem by blast
 }
 assume as: ¬ (∃ a b. i = cbox a b)
 obtain B1 where B1:
    0 < B1
    ⋀a b. ball 0 B1 ⊆ cbox a b ⟹
     ∃ z. ((λx. if x ∈ i then f x else 0) has-integral z) (cbox a b) ∧
       norm (z − k1) < norm (k1 − k2) / 2
  by (rule has-integral-altD[OF assms(1) as,OF e]) blast
 obtain B2 where B2:
    0 < B2
    ⋀a b. ball 0 B2 ⊆ cbox a b ⟹
     ∃ z. ((λx. if x ∈ i then f x else 0) has-integral z) (cbox a b) ∧
       norm (z − k2) < norm (k1 − k2) / 2
  by (rule has-integral-altD[OF assms(2) as,OF e]) blast
 have ∃ a b::'n. ball 0 B1 ∪ ball 0 B2 ⊆ cbox a b
  apply (rule bounded-subset-cbox)
  using bounded-Un bounded-ball
  apply auto
  done
 then obtain a b :: 'n where ab: ball 0 B1 ⊆ cbox a b ball 0 B2 ⊆ cbox a b
  by blast
 obtain w where w:
```

```
    ((λx. if x ∈ i then f x else 0) has-integral w) (cbox a b)
    norm (w − k1) < norm (k1 − k2) / 2
    using B1(2)[OF ab(1)] by blast
  obtain z where z:
    ((λx. if x ∈ i then f x else 0) has-integral z) (cbox a b)
    norm (z − k2) < norm (k1 − k2) / 2
    using B2(2)[OF ab(2)] by blast
  have z = w
    using lem[OF w(1) z(1)] by auto
  then have norm (k1 − k2) ≤ norm (z − k2) + norm (w − k1)
    using norm-triangle-ineq4 [of k1 − w k2 − z]
    by (auto simp add: norm-minus-commute)
  also have ... < norm (k1 − k2) / 2 + norm (k1 − k2) / 2
    apply (rule add-strict-mono)
    apply (rule-tac[!] z(2) w(2))
    done
  finally show False by auto
qed

lemma integral-unique [intro]: (f has-integral y) k ⟹ integral k f = y
  unfolding integral-def
  by (rule some-equality) (auto intro: has-integral-unique)

lemma eq-integralD: integral k f = y ⟹ (f has-integral y) k ∨ ~ f integrable-on
k ∧ y=0
  unfolding integral-def integrable-on-def
  apply (erule subst)
  apply (rule someI-ex)
  by blast

lemma has-integral-is-0:
  fixes f :: 'n::euclidean-space ⇒ 'a::real-normed-vector
  assumes ∀x∈s. f x = 0
  shows (f has-integral 0) s
proof −
  have lem: ⋀a b. ⋀f::'n ⇒ 'a.
    (∀x∈cbox a b. f(x) = 0) ⟹ (f has-integral 0) (cbox a b)
    unfolding has-integral
  proof clarify
    fix a b e
    fix f :: 'n ⇒ 'a
    assume as: ∀x∈cbox a b. f x = 0 0 < (e::real)
    have norm ((∑(x, k)∈p. content k *R f x) − 0) < e
      if p: p tagged-division-of cbox a b for p
    proof −
      have (∑(x, k)∈p. content k *R f x) = 0
      proof (rule setsum.neutral, rule)
        fix x
        assume x: x ∈ p
```

```
      have f (fst x) = 0
        using tagged-division-ofD(2−3)[OF p, of fst x snd x] using as x by auto
      then show (λ(x, k). content k *R f x) x = 0
        apply (subst surjective-pairing[of x])
        unfolding split-conv
        apply auto
        done
    qed
    then show ?thesis
      using as by auto
  qed
  then show ∃ d. gauge d ∧
      (∀ p. p tagged-division-of (cbox a b) ∧ d fine p ⟶ norm ((∑(x, k)∈p.
content k *R f x) − 0) < e)
    by auto
 qed
 {
  presume ¬ (∃ a b. s = cbox a b) ⟹ ?thesis
  with assms lem show ?thesis
   by blast
 }
 have *: (λx. if x ∈ s then f x else 0) = (λx. 0)
  apply (rule ext)
  using assms
  apply auto
  done
 assume ¬ (∃ a b. s = cbox a b)
 then show ?thesis
  using lem
  by (subst has-integral-alt) (force simp add: *)
qed

lemma has-integral-0[simp]: ((λx::'n::euclidean-space. 0) has-integral 0) s
 by (rule has-integral-is-0) auto

lemma has-integral-0-eq[simp]: ((λx. 0) has-integral i) s ⟷ i = 0
 using has-integral-unique[OF has-integral-0] by auto

lemma has-integral-linear:
 fixes f :: 'n::euclidean-space ⇒ 'a::real-normed-vector
 assumes (f has-integral y) s
  and bounded-linear h
 shows ((h ∘ f) has-integral ((h y))) s
proof −
 interpret bounded-linear h
  using assms(2) .
 from pos-bounded obtain B where B: 0 < B ⋀x. norm (h x) ≤ norm x * B
  by blast
 have lem: ⋀(f :: 'n ⇒ 'a) y a b.
```

```
    (f has-integral y) (cbox a b) ⟹ ((h ∘ f) has-integral h y) (cbox a b)
    unfolding has-integral
  proof (clarify, goal-cases)
    case prems: (1 f y a b e)
    from pos-bounded
    obtain B where B: 0 < B ⋀x. norm (h x) ≤ norm x * B
      by blast
    have e / B > 0 using prems(2) B by simp
    then obtain g
      where g: gauge g
            ⋀p. p tagged-division-of (cbox a b) ⟹ g fine p ⟹
                norm ((∑(x, k)∈p. content k *R f x) − y) < e / B
      using prems(1) by auto
    {
      fix p
      assume as: p tagged-division-of (cbox a b) g fine p
      have hc: ⋀x k. h ((λ(x, k). content k *R f x) x) = (λ(x, k). h (content k *R
f x)) x
        by auto
      then have (∑(x, k)∈p. content k *R (h ∘ f) x) = setsum (h ∘ (λ(x, k).
content k *R f x)) p
        unfolding o-def unfolding scaleR[symmetric] hc by simp
      also have ... = h (∑(x, k)∈p. content k *R f x)
        using setsum[of λ(x,k). content k *R f x p] using as by auto
      finally have (∑(x, k)∈p. content k *R (h ∘ f) x) = h (∑(x, k)∈p. content
k *R f x) .
      then have norm ((∑(x, k)∈p. content k *R (h ∘ f) x) − h y) < e
        apply (simp add: diff[symmetric])
        apply (rule le-less-trans[OF B(2)])
        using g(2)[OF as] B(1)
        apply (auto simp add: field-simps)
        done
    }
    with g show ?case
      by (rule-tac x=g in exI) auto
  qed
  {
    presume ¬ (∃ a b. s = cbox a b) ⟹ ?thesis
    then show ?thesis
      using assms(1) lem by blast
  }
  assume as: ¬ (∃ a b. s = cbox a b)
  then show ?thesis
  proof (subst has-integral-alt, clarsimp)
    fix e :: real
    assume e: e > 0
    have *: 0 < e/B using e B(1) by simp
    obtain M where M:
      M > 0
```

```
    ⋀a b. ball 0 M ⊆ cbox a b ⟹
     ∃z. ((λx. if x ∈ s then f x else 0) has-integral z) (cbox a b) ∧ norm (z − y)
< e / B
    using has-integral-altD[OF assms(1) as *] by blast
  show ∃B>0. ∀a b. ball 0 B ⊆ cbox a b ⟶
   (∃z. ((λx. if x ∈ s then (h ∘ f) x else 0) has-integral z) (cbox a b) ∧ norm (z
− h y) < e)
  proof (rule-tac x=M in exI, clarsimp simp add: M, goal-cases)
    case prems: (1 a b)
    obtain z where z:
      ((λx. if x ∈ s then f x else 0) has-integral z) (cbox a b)
      norm (z − y) < e / B
      using M(2)[OF prems(1)] by blast
    have *: (λx. if x ∈ s then (h ∘ f) x else 0) = h ∘ (λx. if x ∈ s then f x else
0)
      using zero by auto
    show ?case
      apply (rule-tac x=h z in exI)
      apply (simp add: * lem z(1))
      apply (metis B diff le-less-trans pos-less-divide-eq z(2))
      done
  qed
 qed
qed

lemma has-integral-scaleR-left:
 (f has-integral y) s ⟹ ((λx. f x *R c) has-integral (y *R c)) s
 using has-integral-linear[OF - bounded-linear-scaleR-left] by (simp add: comp-def)

lemma has-integral-mult-left:
 fixes c :: - :: real-normed-algebra
 shows (f has-integral y) s ⟹ ((λx. f x * c) has-integral (y * c)) s
 using has-integral-linear[OF - bounded-linear-mult-left] by (simp add: comp-def)
```

The case analysis eliminates the condition *f integrable-on s* at the cost of the type class constraint *division-ring*

```
corollary integral-mult-left [simp]:
 fixes c:: 'a::{real-normed-algebra,division-ring}
 shows integral s (λx. f x * c) = integral s f * c
proof (cases f integrable-on s ∨ c = 0)
 case True then show ?thesis
   by (force intro: has-integral-mult-left)
next
 case False then have ~ (λx. f x * c) integrable-on s
   using has-integral-mult-left [of (λx. f x * c) - s inverse c]
   by (force simp add: mult.assoc)
 with False show ?thesis by (simp add: not-integrable-integral)
qed
```

corollary *integral-mult-right* [*simp*]:
fixes *c*:: *'a*::{*real-normed-field*}
shows *integral s* (λ*x*. *c* ∗ *f x*) = *c* ∗ *integral s f*
by (*simp add*: *mult.commute* [*of c*])

corollary *integral-divide* [*simp*]:
fixes *z* :: *'a*::*real-normed-field*
shows *integral S* (λ*x*. *f x* / *z*) = *integral S* (λ*x*. *f x*) / *z*
using *integral-mult-left* [*of S f inverse z*]
by (*simp add*: *divide-inverse-commute*)

lemma *has-integral-mult-right*:
fixes *c* :: *'a* :: *real-normed-algebra*
shows (*f has-integral y*) *i* ⟹ ((λ*x*. *c* ∗ *f x*) *has-integral* (*c* ∗ *y*)) *i*
using *has-integral-linear*[*OF - bounded-linear-mult-right*] **by** (*simp add*: *comp-def*)

lemma *has-integral-cmul*: (*f has-integral k*) *s* ⟹ ((λ*x*. *c* ∗$_R$ *f x*) *has-integral* (*c* ∗$_R$ *k*)) *s*
unfolding *o-def*[*symmetric*]
by (*metis has-integral-linear bounded-linear-scaleR-right*)

lemma *has-integral-cmult-real*:
fixes *c* :: *real*
assumes *c* ≠ *0* ⟹ (*f has-integral x*) *A*
shows ((λ*x*. *c* ∗ *f x*) *has-integral c* ∗ *x*) *A*
proof (*cases c* = *0*)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
from *has-integral-cmul*[*OF assms*[*OF this*], *of c*] **show** *?thesis*
unfolding *real-scaleR-def* **.**
qed

lemma *has-integral-neg*: (*f has-integral k*) *s* ⟹ ((λ*x*. −(*f x*)) *has-integral* −*k*) *s*
by (*drule-tac c*=−*1* **in** *has-integral-cmul*) *auto*

lemma *has-integral-add*:
fixes *f* :: *'n*::*euclidean-space* ⇒ *'a*::*real-normed-vector*
assumes (*f has-integral k*) *s*
and (*g has-integral l*) *s*
shows ((λ*x*. *f x* + *g x*) *has-integral* (*k* + *l*)) *s*
proof −
have *lem*: ((λ*x*. *f x* + *g x*) *has-integral* (*k* + *l*)) (*cbox a b*)
if *f-k*: (*f has-integral k*) (*cbox a b*)
and *g-l*: (*g has-integral l*) (*cbox a b*)
for *f* :: *'n* ⇒ *'a* **and** *g a b k l*
unfolding *has-integral*
proof *clarify*

```
    fix e :: real
    assume e: e > 0
    then have *: e / 2 > 0
      by auto
    obtain d1 where d1:
      gauge d1
      ⋀p. p tagged-division-of (cbox a b) ⟹ d1 fine p ⟹
        norm ((∑(x, k)∈p. content k *R f x) − k) < e / 2
      using has-integralD[OF f-k *] by blast
    obtain d2 where d2:
      gauge d2
      ⋀p. p tagged-division-of (cbox a b) ⟹ d2 fine p ⟹
        norm ((∑(x, k)∈p. content k *R g x) − l) < e / 2
      using has-integralD[OF g-l *] by blast
    show ∃d. gauge d ∧ (∀p. p tagged-division-of (cbox a b) ∧ d fine p ⟶
            norm ((∑(x, k)∈p. content k *R (f x + g x)) − (k + l)) < e)
  proof (rule exI [where x=λx. (d1 x) ∩ (d2 x)], clarsimp simp add: gauge-inter[OF
d1(1) d2(1)])
      fix p
      assume as: p tagged-division-of (cbox a b) (λx. d1 x ∩ d2 x) fine p
      have *: (∑(x, k)∈p. content k *R (f x + g x)) =
        (∑(x, k)∈p. content k *R f x) + (∑(x, k)∈p. content k *R g x)
          unfolding scaleR-right-distrib setsum.distrib[of λ(x,k). content k *R f x
λ(x,k). content k *R g x p,symmetric]
        by (rule setsum.cong) auto
      from as have fine: d1 fine p d2 fine p
        unfolding fine-inter by auto
      have norm ((∑(x, k)∈p. content k *R (f x + g x)) − (k + l)) =
          norm (((∑(x, k)∈p. content k *R f x) − k) + ((∑(x, k)∈p. content k
*R g x) − l))
        unfolding * by (auto simp add: algebra-simps)
      also have ... < e/2 + e/2
        apply (rule le-less-trans[OF norm-triangle-ineq])
        using as d1 d2 fine
        apply (blast intro: add-strict-mono)
        done
      finally show norm ((∑(x, k)∈p. content k *R (f x + g x)) − (k + l)) < e
        by auto
    qed
  qed
  {
    presume ¬ (∃a b. s = cbox a b) ⟹ ?thesis
    then show ?thesis
      using assms lem by force
  }
  assume as: ¬ (∃a b. s = cbox a b)
  then show ?thesis
  proof (subst has-integral-alt, clarsimp, goal-cases)
    case (1 e)
```

then have $*$: $e / 2 > 0$
by *auto*
from *has-integral-altD*[*OF assms*(1) *as* $*$]
obtain *B1* **where** *B1*:
$0 < B1$
$\bigwedge a\ b.\ ball\ 0\ B1 \subseteq cbox\ a\ b \Longrightarrow$
$\exists z.\ ((\lambda x.\ if\ x \in s\ then\ f\ x\ else\ 0)\ has\text{-}integral\ z)\ (cbox\ a\ b) \wedge norm\ (z - k) < e / 2$
by *blast*
from *has-integral-altD*[*OF assms*(2) *as* $*$]
obtain *B2* **where** *B2*:
$0 < B2$
$\bigwedge a\ b.\ ball\ 0\ B2 \subseteq (cbox\ a\ b) \Longrightarrow$
$\exists z.\ ((\lambda x.\ if\ x \in s\ then\ g\ x\ else\ 0)\ has\text{-}integral\ z)\ (cbox\ a\ b) \wedge norm\ (z - l) < e / 2$
by *blast*
show *?case*
proof (*rule-tac x*=*max B1 B2* **in** *exI*, *clarsimp simp add*: *max.strict-coboundedI1 B1*)
fix *a b*
assume *ball 0* (*max B1 B2*) $\subseteq$ *cbox a* (*b*::′*n*)
then have $*$: *ball 0 B1* $\subseteq$ *cbox a* (*b*::′*n*) *ball 0 B2* $\subseteq$ *cbox a* (*b*::′*n*)
by *auto*
obtain *w* **where** *w*:
((λ*x*. *if x* ∈ *s then f x else 0*) *has-integral w*) (*cbox a b*)
norm (*w* − *k*) < *e* / *2*
using *B1*(*2*)[*OF* $*$(*1*)] **by** *blast*
obtain *z* **where** *z*:
((λ*x*. *if x* ∈ *s then g x else 0*) *has-integral z*) (*cbox a b*)
norm (*z* − *l*) < *e* / *2*
using *B2*(*2*)[*OF* $*$(*2*)] **by** *blast*
have $*$: $\bigwedge x.$ (*if x* ∈ *s then f x* + *g x else 0*) =
(*if x* ∈ *s then f x else 0*) + (*if x* ∈ *s then g x else 0*)
by *auto*
show ∃*z*. ((λ*x*. *if x* ∈ *s then f x* + *g x else 0*) *has-integral z*) (*cbox a b*) ∧ *norm* (*z* − (*k* + *l*)) < *e*
apply (*rule-tac x*=*w* + *z* **in** *exI*)
apply (*simp add*: *lem*[*OF w*(*1*) *z*(*1*), *unfolded* $*$[*symmetric*]])
using *norm-triangle-ineq*[*of w* − *k z* − *l*] *w*(*2*) *z*(*2*)
apply (*auto simp add*: *field-simps*)
done
qed
qed
qed

lemma *has-integral-sub*:
(*f has-integral k*) *s* ⟹ (*g has-integral l*) *s* ⟹
((λ*x*. *f x* − *g x*) *has-integral* (*k* − *l*)) *s*
using *has-integral-add*[*OF* - *has-integral-neg*, *of f k s g l*]

unfolding *algebra-simps*
by *auto*

lemma *integral-0* [*simp*]:
integral s ($\lambda x::'n::euclidean\text{-}space.\ 0::'m::real\text{-}normed\text{-}vector$) = *0*
by (*rule integral-unique has-integral-0*)+

lemma *integral-add*: *f integrable-on s* $\Longrightarrow$ *g integrable-on s* $\Longrightarrow$
integral s ($\lambda x.\ f\ x + g\ x$) = *integral s f* + *integral s g*
by (*rule integral-unique*) (*metis integrable-integral has-integral-add*)

lemma *integral-cmul* [*simp*]: *integral s* ($\lambda x.\ c *_R f\ x$) = $c *_R$ *integral s f*
proof (*cases f integrable-on s* $\vee$ *c* = *0*)
case *True* **with** *has-integral-cmul* **show** *?thesis* **by** *force*
next
case *False* **then have** $\sim$ ($\lambda x.\ c *_R f\ x$) *integrable-on s*
using *has-integral-cmul* [*of* ($\lambda x.\ c *_R f\ x$) - *s inverse c*]
by *force*
with *False* **show** *?thesis* **by** (*simp add*: *not-integrable-integral*)
qed

lemma *integral-neg* [*simp*]: *integral s* ($\lambda x.\ -\ f\ x$) = − *integral s f*
proof (*cases f integrable-on s*)
case *True* **then show** *?thesis*
by (*simp add*: *has-integral-neg integrable-integral integral-unique*)
next
case *False* **then have** $\sim$ ($\lambda x.\ -\ f\ x$) *integrable-on s*
using *has-integral-neg* [*of* ($\lambda x.\ -\ f\ x$) - *s*]
by *force*
with *False* **show** *?thesis* **by** (*simp add*: *not-integrable-integral*)
qed

lemma *integral-diff*: *f integrable-on s* $\Longrightarrow$ *g integrable-on s* $\Longrightarrow$
integral s ($\lambda x.\ f\ x - g\ x$) = *integral s f* − *integral s g*
by (*rule integral-unique*) (*metis integrable-integral has-integral-sub*)

lemma *integrable-0*: ($\lambda x.\ 0$) *integrable-on s*
unfolding *integrable-on-def* **using** *has-integral-0* **by** *auto*

lemma *integrable-add*: *f integrable-on s* $\Longrightarrow$ *g integrable-on s* $\Longrightarrow$ ($\lambda x.\ f\ x + g\ x$) *integrable-on s*
unfolding *integrable-on-def* **by**(*auto intro*: *has-integral-add*)

lemma *integrable-cmul*: *f integrable-on s* $\Longrightarrow$ ($\lambda x.\ c *_R f(x)$) *integrable-on s*
unfolding *integrable-on-def* **by**(*auto intro*: *has-integral-cmul*)

lemma *integrable-on-cmult-iff*:
fixes *c* :: *real*
assumes $c \neq 0$

shows $(\lambda x.\ c * f\ x)$ *integrable-on* $s \longleftrightarrow f$ *integrable-on* s
using *integrable-cmul*[*of* $\lambda x.\ c * f\ x\ s\ 1\ /\ c$] *integrable-cmul*[*of* $f\ s\ c$] ‹$c \neq 0$›
by *auto*

lemma *integrable-on-cmult-left*:
assumes f *integrable-on* s
shows $(\lambda x.$ *of-real* $c * f\ x)$ *integrable-on* s
using *integrable-cmul*[*of* $f\ s$ *of-real* c] *assms*
by (*simp add*: *scaleR-conv-of-real*)

lemma *integrable-neg*: f *integrable-on* $s \Longrightarrow (\lambda x.\ -f(x))$ *integrable-on* s
unfolding *integrable-on-def* **by**(*auto intro*: *has-integral-neg*)

lemma *integrable-diff*:
f *integrable-on* $s \Longrightarrow g$ *integrable-on* $s \Longrightarrow (\lambda x.\ f\ x - g\ x)$ *integrable-on* s
unfolding *integrable-on-def* **by**(*auto intro*: *has-integral-sub*)

lemma *integrable-linear*:
f *integrable-on* $s \Longrightarrow$ *bounded-linear* $h \Longrightarrow (h \circ f)$ *integrable-on* s
unfolding *integrable-on-def* **by**(*auto intro*: *has-integral-linear*)

lemma *integral-linear*:
f *integrable-on* $s \Longrightarrow$ *bounded-linear* $h \Longrightarrow$ *integral* $s\ (h \circ f) = h\ (\textit{integral}\ s\ f)$
apply (*rule has-integral-unique* [**where** $i{=}s$ **and** $f = h \circ f$])
apply (*simp-all add*: *integrable-integral integrable-linear has-integral-linear*)
done

lemma *integral-component-eq*[*simp*]:
fixes $f :: 'n{::}$*euclidean-space* $\Rightarrow 'm{::}$*euclidean-space*
assumes f *integrable-on* s
shows *integral* $s\ (\lambda x.\ f\ x \cdot k) =$ *integral* $s\ f \cdot k$
unfolding *integral-linear*[*OF assms*(*1*) *bounded-linear-component*,*unfolded o-def*]
..

lemma *has-integral-setsum*:
assumes *finite* t
and $\forall\ a{\in}t.\ ((f\ a)$ *has-integral* $(i\ a))\ s$
shows $((\lambda x.$ *setsum* $(\lambda a.\ f\ a\ x)\ t)$ *has-integral* $(\textit{setsum}\ i\ t))\ s$
using *assms*(*1*) *subset-refl*[*of* t]
proof (*induct rule*: *finite-subset-induct*)
case *empty*
then show *?case* **by** *auto*
next
case (*insert* $x\ F$)
with *assms* **show** *?case*
by (*simp add*: *has-integral-add*)
qed

lemma *integral-setsum*:

⟦*finite t*; ∀ *a*∈*t*. (*f a*) *integrable-on s*⟧ ⟹
integral s (λ*x*. *setsum* (λ*a*. *f a x*) *t*) = *setsum* (λ*a*. *integral s* (*f a*)) *t*
by (*auto intro*: *has-integral-setsum integrable-integral*)

lemma *integrable-setsum*:
⟦*finite t*; ∀ *a*∈*t*. (*f a*) *integrable-on s*⟧ ⟹ (λ*x*. *setsum* (λ*a*. *f a x*) *t*) *integrable-on s*
unfolding *integrable-on-def*
apply (*drule bchoice*)
using *has-integral-setsum*[*of t*]
apply *auto*
done

lemma *has-integral-eq*:
assumes ⋀*x*. *x* ∈ *s* ⟹ *f x* = *g x*
and (*f has-integral k*) *s*
shows (*g has-integral k*) *s*
using *has-integral-sub*[*OF assms*(*2*), *of* λ*x*. *f x* − *g x 0*]
using *has-integral-is-0*[*of s* λ*x*. *f x* − *g x*]
using *assms*(*1*)
by *auto*

lemma *integrable-eq*: (⋀*x*. *x* ∈ *s* ⟹ *f x* = *g x*) ⟹ *f integrable-on s* ⟹ *g integrable-on s*
unfolding *integrable-on-def*
using *has-integral-eq*[*of s f g*] *has-integral-eq* **by** *blast*

lemma *has-integral-cong*:
assumes ⋀*x*. *x* ∈ *s* ⟹ *f x* = *g x*
shows (*f has-integral i*) *s* = (*g has-integral i*) *s*
using *has-integral-eq*[*of s f g*] *has-integral-eq*[*of s g f*] *assms*
by *auto*

lemma *integral-cong*:
assumes ⋀*x*. *x* ∈ *s* ⟹ *f x* = *g x*
shows *integral s f* = *integral s g*
unfolding *integral-def*
by (*metis* (*full-types*, *hide-lams*) *assms has-integral-cong integrable-eq*)

lemma *integrable-on-cmult-left-iff* [*simp*]:
assumes *c* ≠ *0*
shows (λ*x*. *of-real c* ∗ *f x*) *integrable-on s* ⟷ *f integrable-on s*
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then have (λ*x*. *of-real* (*1* / *c*) ∗ (*of-real c* ∗ *f x*)) *integrable-on s*
using *integrable-cmul*[*of* λ*x*. *of-real c* ∗ *f x s 1* / *of-real c*]
by (*simp add*: *scaleR-conv-of-real*)
then have (λ*x*. (*of-real* (*1* / *c*) ∗ *of-real c* ∗ *f x*)) *integrable-on s*

by (*simp add*: *algebra-simps*)
with ‹$c \neq 0$› **show** *?rhs*
by (*metis* (*no-types*, *lifting*) *integrable-eq mult.left-neutral nonzero-divide-eq-eq of-real-1 of-real-mult*)
qed (*blast intro*: *integrable-on-cmult-left*)

lemma *integrable-on-cmult-right*:
fixes *f* :: - ⇒ *'b* :: {*comm-ring*,*real-algebra-1*,*real-normed-vector*}
assumes *f integrable-on s*
shows (λ*x*. *f x* ∗ *of-real c*) *integrable-on s*
using *integrable-on-cmult-left* [*OF assms*] **by** (*simp add*: *mult.commute*)

lemma *integrable-on-cmult-right-iff* [*simp*]:
fixes *f* :: - ⇒ *'b* :: {*comm-ring*,*real-algebra-1*,*real-normed-vector*}
assumes *c* ≠ *0*
shows (λ*x*. *f x* ∗ *of-real c*) *integrable-on s* ⟷ *f integrable-on s*
using *integrable-on-cmult-left-iff* [*OF assms*] **by** (*simp add*: *mult.commute*)

lemma *integrable-on-cdivide*:
fixes *f* :: - ⇒ *'b* :: *real-normed-field*
assumes *f integrable-on s*
shows (λ*x*. *f x* / *of-real c*) *integrable-on s*
by (*simp add*: *integrable-on-cmult-right divide-inverse assms of-real-inverse* [*symmetric*] *del*: *of-real-inverse*)

lemma *integrable-on-cdivide-iff* [*simp*]:
fixes *f* :: - ⇒ *'b* :: *real-normed-field*
assumes *c* ≠ *0*
shows (λ*x*. *f x* / *of-real c*) *integrable-on s* ⟷ *f integrable-on s*
by (*simp add*: *divide-inverse assms of-real-inverse* [*symmetric*] *del*: *of-real-inverse*)

lemma *has-integral-null* [*intro*]:
assumes *content*(*cbox a b*) = *0*
shows (*f has-integral 0*) (*cbox a b*)
proof −
have *gauge* (λ*x*. *ball x 1*)
by *auto*
moreover
{
fix *e* :: *real*
fix *p*
assume *e*: *e* > *0*
assume *p*: *p tagged-division-of* (*cbox a b*)
have *norm* ((∑ (*x*, *k*)∈*p*. *content k* ∗$_R$ *f x*) − *0*) = *0*
unfolding *norm-eq-zero diff-0-right*
using *setsum-content-null*[*OF assms*(*1*) *p*, *of f*] **.**
then have *norm* ((∑ (*x*, *k*)∈*p*. *content k* ∗$_R$ *f x*) − *0*) < *e*
using *e* **by** *auto*
}

ultimately show *?thesis*
by (*auto simp*: *has-integral*)
qed

lemma *has-integral-null-real* [*intro*]:
assumes *content* {*a* .. *b*::*real*} = *0*
shows (*f has-integral 0*) {*a* .. *b*}
by (*metis assms box-real*(*2*) *has-integral-null*)

lemma *has-integral-null-eq*[*simp*]: *content* (*cbox a b*) = *0* ⟹ (*f has-integral i*) (*cbox a b*) ⟷ *i* = *0*
by (*auto simp add*: *has-integral-null dest*!: *integral-unique*)

lemma *integral-null* [*simp*]: *content* (*cbox a b*) = *0* ⟹ *integral* (*cbox a b*) *f* = *0*
by (*metis has-integral-null integral-unique*)

lemma *integrable-on-null* [*intro*]: *content* (*cbox a b*) = *0* ⟹ *f integrable-on* (*cbox a b*)
by (*simp add*: *has-integral-integrable*)

lemma *has-integral-empty*[*intro*]: (*f has-integral 0*) {}
by (*simp add*: *has-integral-is-0*)

lemma *has-integral-empty-eq*[*simp*]: (*f has-integral i*) {} ⟷ *i* = *0*
by (*auto simp add*: *has-integral-empty has-integral-unique*)

lemma *integrable-on-empty*[*intro*]: *f integrable-on* {}
unfolding *integrable-on-def* **by** *auto*

lemma *integral-empty*[*simp*]: *integral* {} *f* = *0*
by (*rule integral-unique*) (*rule has-integral-empty*)

lemma *has-integral-refl*[*intro*]:
fixes *a* :: *'a*::*euclidean-space*
shows (*f has-integral 0*) (*cbox a a*)
and (*f has-integral 0*) {*a*}
proof −
have ∗: {*a*} = *cbox a a*
apply (*rule set-eqI*)
unfolding *mem-box singleton-iff euclidean-eq-iff*[**where** *'a*=*'a*]
apply *safe*
prefer *3*
apply (*erule-tac x*=*b* **in** *ballE*)
apply (*auto simp add*: *field-simps*)
done
show (*f has-integral 0*) (*cbox a a*) (*f has-integral 0*) {*a*}
unfolding ∗
apply (*rule-tac*[!] *has-integral-null*)
unfolding *content-eq-0-interior*

unfolding *interior-cbox*
using *box-sing*
apply *auto*
done
qed

lemma *integrable-on-refl*[*intro*]: *f integrable-on cbox a a*
unfolding *integrable-on-def* **by** *auto*

lemma *integral-refl* [*simp*]: *integral* (*cbox a a*) *f* = *0*
by (*rule integral-unique*) *auto*

lemma *integral-singleton* [*simp*]: *integral* {*a*} *f* = *0*
by *auto*

lemma *integral-blinfun-apply*:
assumes *f integrable-on s*
shows *integral s* (λ*x*. *blinfun-apply h* (*f x*)) = *blinfun-apply h* (*integral s f*)
by (*subst integral-linear*[*symmetric*, *OF assms blinfun.bounded-linear-right*]) (*simp add*: *o-def*)

lemma *blinfun-apply-integral*:
assumes *f integrable-on s*
shows *blinfun-apply* (*integral s f*) *x* = *integral s* (λ*y*. *blinfun-apply* (*f y*) *x*)
by (*metis* (*no-types*, *lifting*) *assms blinfun.prod-left.rep-eq integral-blinfun-apply integral-cong*)

41.15 Cauchy-type criterion for integrability.

lemma *integrable-cauchy*:
fixes *f* :: *'n*::*euclidean-space* ⇒ *'a*::{*real-normed-vector*,*complete-space*}
shows *f integrable-on cbox a b* ⟷
(∀ *e*>*0*.∃ *d*. *gauge d* ∧
(∀ *p1 p2*. *p1 tagged-division-of* (*cbox a b*) ∧ *d fine p1* ∧
p2 tagged-division-of (*cbox a b*) ∧ *d fine p2* ⟶
norm (*setsum* (λ(*x*,*k*). *content k* ∗*R f x*) *p1* −
setsum (λ(*x*,*k*). *content k* ∗*R f x*) *p2*) < *e*))
(**is** *?l* = (∀ *e*>*0*. ∃ *d*. *?P e d*))
proof
assume *?l*
then guess *y* **unfolding** *integrable-on-def has-integral* **..** **note** *y*=*this*
show ∀ *e*>*0*. ∃ *d*. *?P e d*
proof (*clarify*, *goal-cases*)
case (*1 e*)
then have *e*/*2* > *0* **by** *auto*
then guess *d*
apply −
apply (*drule y*[*rule-format*])
apply (*elim exE conjE*)

```
    done
  note d=this[rule-format]
  show ?case
  proof (rule-tac x=d in exI, clarsimp simp: d)
    fix p1 p2
    assume as: p1 tagged-division-of (cbox a b) d fine p1
           p2 tagged-division-of (cbox a b) d fine p2
    show norm ((∑(x, k)∈p1. content k *R f x) − (∑(x, k)∈p2. content k *R
f x)) < e
      apply (rule dist-triangle-half-l[where y=y,unfolded dist-norm])
      using d(2)[OF conjI[OF as(1−2)]] d(2)[OF conjI[OF as(3−4)]] .
  qed
 qed
next
 assume ∀ e>0. ∃ d. ?P e d
 then have ∀ n::nat. ∃ d. ?P (inverse(of-nat (n + 1))) d
  by auto
 from choice[OF this] guess d .. note d=conjunctD2[OF this[rule-format],rule-format]
 have ⋀n. gauge (λx. ⋂{d i x |i. i ∈ {0..n}})
  apply (rule gauge-inters)
  using d(1)
  apply auto
  done
 then have ∀ n. ∃ p. p tagged-division-of (cbox a b) ∧ (λx. ⋂{d i x |i. i ∈ {0..n}})
fine p
  by (meson fine-division-exists)
 from choice[OF this] guess p .. note p = conjunctD2[OF this[rule-format]]
 have dp: ⋀i n. i≤n ⟹ d i fine p n
  using p(2) unfolding fine-inters by auto
 have Cauchy (λn. setsum (λ(x,k). content k *R (f x)) (p n))
 proof (rule CauchyI, goal-cases)
  case (1 e)
  then guess N unfolding real-arch-inverse[of e] .. note N=this
  show ?case
   apply (rule-tac x=N in exI)
  proof clarify
   fix m n
   assume mn: N ≤ m N ≤ n
   have ∗: N = (N − 1) + 1 using N by auto
   show norm ((∑(x, k)∈p m. content k *R f x) − (∑(x, k)∈p n. content k
*R f x)) < e
    apply (rule less-trans[OF - N[THEN conjunct2,THEN conjunct2]])
    apply(subst ∗)
    using dp p(1) mn d(2) by auto
  qed
 qed
 then guess y unfolding convergent-eq-cauchy[symmetric] .. note y=this[THEN
LIMSEQ-D]
 show ?l
```

unfolding *integrable-on-def has-integral*
proof (*rule-tac x=y* **in** *exI*, *clarify*)
fix *e* :: *real*
assume *e>0*
then have ∗:*e/2 > 0* **by** *auto*
then guess *N1* **unfolding** *real-arch-inverse*[*of e/2*] **.. note** *N1=this*
then have *N1'*: *N1 = N1 − 1 + 1*
by *auto*
guess *N2* **using** *y*[*OF* ∗] **.. note** *N2=this*
have *gauge* (*d* (*N1 + N2*))
using *d* **by** *auto*
moreover
{
fix *q*
assume *as*: *q tagged-division-of* (*cbox a b*) *d* (*N1 + N2*) *fine q*
have ∗: *inverse* (*of-nat* (*N1 + N2 + 1*)) < *e / 2*
apply (*rule less-trans*)
using *N1*
apply *auto*
done
have *norm* ((∑(*x*, *k*)∈*q*. *content k* ∗$_R$ *f x*) − *y*) < *e*
apply (*rule norm-triangle-half-r*)
apply (*rule less-trans*[*OF* - ∗])
apply (*subst N1'*, *rule d*(*2*)[*of p* (*N1+N2*)])
using *N1' as*(*1*) *as*(*2*) *dp*
apply (*simp add*: ‹∀ *x*. *p x tagged-division-of cbox a b* ∧ (λ*xa*. ⋂{*d i xa* |*i*. *i* ∈ {*0*..*x*}}) *fine p x*›)
using *N2 le-add2* **by** *blast*
}
ultimately show ∃ *d*. *gauge d* ∧
(∀ *p*. *p tagged-division-of* (*cbox a b*) ∧ *d fine p* ⟶
norm ((∑(*x*, *k*)∈*p*. *content k* ∗$_R$ *f x*) − *y*) < *e*)
by (*rule-tac x=d* (*N1 + N2*) **in** *exI*) *auto*
qed
qed

41.16 Additivity of integral on abutting intervals.

lemma *interval-split*:
fixes *a* :: *'a*::*euclidean-space*
assumes *k* ∈ *Basis*
shows
cbox a b ∩ {*x*. *x*•*k* ≤ *c*} = *cbox a* (∑ *i*∈*Basis*. (*if i = k then min* (*b*•*k*) *c else b*•*i*) ∗$_R$ *i*)
cbox a b ∩ {*x*. *x*•*k* ≥ *c*} = *cbox* (∑ *i*∈*Basis*. (*if i = k then max* (*a*•*k*) *c else a*•*i*) ∗$_R$ *i*) *b*
apply (*rule-tac*[!] *set-eqI*)
unfolding *Int-iff mem-box mem-Collect-eq*
using *assms*

apply *auto*
done

lemma *content-split*:
fixes $a :: 'a::euclidean\text{-}space$
assumes $k \in Basis$
shows $content\ (cbox\ a\ b) = content(cbox\ a\ b \cap \{x.\ x \cdot k \leq c\}) + content(cbox\ a\ b \cap \{x.\ x \cdot k \geq c\})$
proof *cases*
note *simps* = *interval-split*[*OF assms*] *content-cbox-cases*
have $*$: $Basis = insert\ k\ (Basis - \{k\})\ \bigwedge x.\ finite\ (Basis - \{x\})\ \bigwedge x.\ x \notin Basis - \{x\}$
using *assms* **by** *auto*
have $*$: $\bigwedge X\ Y\ Z.\ (\prod i \in Basis.\ Z\ i\ (if\ i = k\ then\ X\ else\ Y\ i)) = Z\ k\ X * (\prod i \in Basis - \{k\}.\ Z\ i\ (Y\ i))$
$(\prod i \in Basis.\ b \cdot i - a \cdot i) = (\prod i \in Basis - \{k\}.\ b \cdot i - a \cdot i) * (b \cdot k - a \cdot k)$
apply (*subst* $*(1)$)
defer
apply (*subst* $*(1)$)
unfolding *setprod.insert*[*OF* $*(2-)$]
apply *auto*
done
assume *as*: $\forall i \in Basis.\ a \cdot i \leq b \cdot i$
moreover
have $\bigwedge x.\ min\ (b \cdot k)\ c = max\ (a \cdot k)\ c \Longrightarrow$
$x * (b \cdot k - a \cdot k) = x * (max\ (a \cdot k)\ c - a \cdot k) + x * (b \cdot k - max\ (a \cdot k)\ c)$
by (*auto simp add*: *field-simps*)
moreover
have $**$: $(\prod i \in Basis.\ ((\sum i \in Basis.\ (if\ i = k\ then\ min\ (b \cdot k)\ c\ else\ b \cdot i) *_R i) \cdot i - a \cdot i)) =$
$(\prod i \in Basis.\ (if\ i = k\ then\ min\ (b \cdot k)\ c\ else\ b \cdot i) - a \cdot i)$
$(\prod i \in Basis.\ b \cdot i - ((\sum i \in Basis.\ (if\ i = k\ then\ max\ (a \cdot k)\ c\ else\ a \cdot i) *_R i) \cdot i)) =$
$(\prod i \in Basis.\ b \cdot i - (if\ i = k\ then\ max\ (a \cdot k)\ c\ else\ a \cdot i))$
by (*auto intro*!: *setprod.cong*)
have $\neg\ a \cdot k \leq c \Longrightarrow \neg\ c \leq b \cdot k \Longrightarrow False$
unfolding *not-le*
using *as*[*unfolded* ,*rule-format*,*of k*] *assms*
by *auto*
ultimately show *?thesis*
using *assms*
unfolding *simps* $**$
unfolding $*(1)$[*of* $\lambda i\ x.\ b \cdot i - x$] $*(1)$[*of* $\lambda i\ x.\ x - a \cdot i$]
unfolding $*(2)$
by *auto*
next
assume $\neg\ (\forall i \in Basis.\ a \cdot i \leq b \cdot i)$
then have $cbox\ a\ b = \{\}$
unfolding *box-eq-empty* **by** (*auto simp*: *not-le*)
then show *?thesis*

```
    by (auto simp: not-le)
qed

lemma division-split-left-inj:
  fixes type :: 'a::euclidean-space
  assumes d division-of i
    and k1 ∈ d
    and k2 ∈ d
    and k1 ≠ k2
    and k1 ∩ {x::'a. x•k ≤ c} = k2 ∩ {x. x•k ≤ c}
    and k: k∈Basis
  shows content(k1 ∩ {x. x•k ≤ c}) = 0
proof −
  note d=division-ofD[OF assms(1)]
  have ∗: ⋀(a::'a) b c. content (cbox a b ∩ {x. x•k ≤ c}) = 0 ⟷
    interior(cbox a b ∩ {x. x•k ≤ c}) = {}
    unfolding interval-split[OF k] content-eq-0-interior by auto
  guess u1 v1 using d(4)[OF assms(2)] by (elim exE) note uv1=this
  guess u2 v2 using d(4)[OF assms(3)] by (elim exE) note uv2=this
  have ∗∗: ⋀s t u. s ∩ t = {} ⟹ u ⊆ s ⟹ u ⊆ t ⟹ u = {}
    by auto
  show ?thesis
    unfolding uv1 uv2 ∗
    apply (rule ∗∗[OF d(5)[OF assms(2−4)]])
    apply (simp add: uv1)
    using assms(5) uv1 by auto
qed

lemma division-split-right-inj:
  fixes type :: 'a::euclidean-space
  assumes d division-of i
    and k1 ∈ d
    and k2 ∈ d
    and k1 ≠ k2
    and k1 ∩ {x::'a. x•k ≥ c} = k2 ∩ {x. x•k ≥ c}
    and k: k ∈ Basis
  shows content (k1 ∩ {x. x•k ≥ c}) = 0
proof −
  note d=division-ofD[OF assms(1)]
  have ∗: ⋀a b::'a. ⋀c. content(cbox a b ∩ {x. x•k ≥ c}) = 0 ⟷
    interior(cbox a b ∩ {x. x•k ≥ c}) = {}
    unfolding interval-split[OF k] content-eq-0-interior by auto
  guess u1 v1 using d(4)[OF assms(2)] by (elim exE) note uv1=this
  guess u2 v2 using d(4)[OF assms(3)] by (elim exE) note uv2=this
  have ∗∗: ⋀s t u. s ∩ t = {} ⟹ u ⊆ s ⟹ u ⊆ t ⟹ u = {}
    by auto
  show ?thesis
    unfolding uv1 uv2 ∗
    apply (rule ∗∗[OF d(5)[OF assms(2−4)]])
```

apply (*simp add*: *uv1*)
using *assms*(*5*) *uv1* **by** *auto*
qed

lemma *tagged-division-split-left-inj*:
fixes *x1* :: *'a*::*euclidean-space*
assumes *d*: *d tagged-division-of i*
and *k12*: (*x1*, *k1*) ∈ *d*
(*x2*, *k2*) ∈ *d*
k1 ≠ *k2*
k1 ∩ {*x*. *x*•*k* ≤ *c*} = *k2* ∩ {*x*. *x*•*k* ≤ *c*}
k ∈ *Basis*
shows *content* (*k1* ∩ {*x*. *x*•*k* ≤ *c*}) = *0*
proof −
have ∗: ⋀*a b c*. (*a*,*b*) ∈ *c* ⟹ *b* ∈ *snd* ‘ *c*
by *force*
show *?thesis*
using *k12*
by (*fastforce intro*!: *division-split-left-inj*[*OF division-of-tagged-division*[*OF d*]]
∗)
qed

lemma *tagged-division-split-right-inj*:
fixes *x1* :: *'a*::*euclidean-space*
assumes *d*: *d tagged-division-of i*
and *k12*: (*x1*, *k1*) ∈ *d*
(*x2*, *k2*) ∈ *d*
k1 ≠ *k2*
k1 ∩ {*x*. *x*•*k* ≥ *c*} = *k2* ∩ {*x*. *x*•*k* ≥ *c*}
k ∈ *Basis*
shows *content* (*k1* ∩ {*x*. *x*•*k* ≥ *c*}) = *0*
proof −
have ∗: ⋀*a b c*. (*a*,*b*) ∈ *c* ⟹ *b* ∈ *snd* ‘ *c*
by *force*
show *?thesis*
using *k12*
by (*fastforce intro*!: *division-split-right-inj*[*OF division-of-tagged-division*[*OF*
d]] ∗)
qed

lemma *division-split*:
fixes *a* :: *'a*::*euclidean-space*
assumes *p division-of* (*cbox a b*)
and *k*: *k*∈*Basis*
shows {*l* ∩ {*x*. *x*•*k* ≤ *c*} | *l*. *l* ∈ *p* ∧ *l* ∩ {*x*. *x*•*k* ≤ *c*} ≠ {}} *division-of*(*cbox a*
b ∩ {*x*. *x*•*k* ≤ *c*})
(**is** *?p1 division-of ?I1*)
and {*l* ∩ {*x*. *x*•*k* ≥ *c*} | *l*. *l* ∈ *p* ∧ *l* ∩ {*x*. *x*•*k* ≥ *c*} ≠ {}} *division-of* (*cbox a*
b ∩ {*x*. *x*•*k* ≥ *c*})

```
    (is ?p2 division-of ?I2)
proof (rule-tac[!] division-ofI)
  note p = division-ofD[OF assms(1)]
  show finite ?p1 finite ?p2
    using p(1) by auto
  show ⋃ ?p1 = ?I1 ⋃ ?p2 = ?I2
    unfolding p(6)[symmetric] by auto
  {
    fix k
    assume k ∈ ?p1
    then guess l unfolding mem-Collect-eq by (elim exE conjE) note l=this
    guess u v using p(4)[OF l(2)] by (elim exE) note uv=this
    show k ⊆ ?I1
      using l p(2) uv by force
    show  k ≠ {}
      by (simp add: l)
    show  ∃ a b. k = cbox a b
      apply (simp add: l uv p(2-3)[OF l(2)])
      apply (subst interval-split[OF k])
      apply (auto intro: order.trans)
      done
    fix k'
    assume k' ∈ ?p1
    then guess l' unfolding mem-Collect-eq by (elim exE conjE) note l'=this
    assume k ≠ k'
    then show interior k ∩ interior k' = {}
      unfolding l l' using p(5)[OF l(2) l'(2)] by auto
  }
  {
    fix k
    assume k ∈ ?p2
    then guess l unfolding mem-Collect-eq by (elim exE conjE) note l=this
    guess u v using p(4)[OF l(2)] by (elim exE) note uv=this
    show k ⊆ ?I2
      using l p(2) uv by force
    show  k ≠ {}
      by (simp add: l)
    show  ∃ a b. k = cbox a b
      apply (simp add: l uv p(2-3)[OF l(2)])
      apply (subst interval-split[OF k])
      apply (auto intro: order.trans)
      done
    fix k'
    assume k' ∈ ?p2
    then guess l' unfolding mem-Collect-eq by (elim exE conjE) note l'=this
    assume k ≠ k'
    then show interior k ∩ interior k' = {}
      unfolding l l' using p(5)[OF l(2) l'(2)] by auto
  }
```

qed

lemma *has-integral-split*:
fixes *f* :: *'a*::*euclidean-space* ⇒ *'b*::*real-normed-vector*
assumes *fi*: (*f has-integral i*) (*cbox a b* ∩ {*x*. *x*·*k* ≤ *c*})
and *fj*: (*f has-integral j*) (*cbox a b* ∩ {*x*. *x*·*k* ≥ *c*})
and *k*: *k* ∈ *Basis*
shows (*f has-integral* (*i* + *j*)) (*cbox a b*)
proof (*unfold has-integral*, *rule*, *rule*, *goal-cases*)
case (*1 e*)
then have *e*: *e*/*2* > *0*
by *auto*
obtain *d1*
where *d1*: *gauge d1*
and *d1norm*:
⋀*p*. ⟦*p tagged-division-of cbox a b* ∩ {*x*. *x* · *k* ≤ *c*};
d1 fine p⟧ ⟹ *norm* ((∑ (*x*, *k*) ∈ *p*. *content k* ∗*R f x*) − *i*) < *e* / *2*
apply (*rule has-integralD*[*OF fi*[*unfolded interval-split*[*OF k*]] *e*])
apply (*simp add*: *interval-split*[*symmetric*] *k*)
done
obtain *d2*
where *d2*: *gauge d2*
and *d2norm*:
⋀*p*. ⟦*p tagged-division-of cbox a b* ∩ {*x*. *c* ≤ *x* · *k*};
d2 fine p⟧ ⟹ *norm* ((∑ (*x*, *k*) ∈ *p*. *content k* ∗*R f x*) − *j*) < *e* / *2*
apply (*rule has-integralD*[*OF fj*[*unfolded interval-split*[*OF k*]] *e*])
apply (*simp add*: *interval-split*[*symmetric*] *k*)
done
let *?d* = λ*x*. *if x*·*k* = *c then* (*d1 x* ∩ *d2 x*) *else ball x* |*x*·*k* − *c*| ∩ *d1 x* ∩ *d2 x*
have *gauge ?d*
using *d1 d2* **unfolding** *gauge-def* **by** *auto*
then show *?case*
proof (*rule-tac x*=*?d* **in** *exI*, *safe*)
fix *p*
assume *p tagged-division-of* (*cbox a b*) *?d fine p*
note *p* = *this tagged-division-ofD*[*OF this*(*1*)]
have *xk-le-c*: ⋀*x kk*. (*x*, *kk*) ∈ *p* ⟹ *kk* ∩ {*x*. *x*·*k* ≤ *c*} ≠ {} ⟹ *x*·*k* ≤ *c*
proof −
fix *x kk*
assume *as*: (*x*, *kk*) ∈ *p* **and** *kk*: *kk* ∩ {*x*. *x*·*k* ≤ *c*} ≠ {}
show *x*·*k* ≤ *c*
proof (*rule ccontr*)
assume ∗∗: ¬ *?thesis*
from *this*[*unfolded not-le*]
have *kk* ⊆ *ball x* |*x* · *k* − *c*|
using *p*(*2*)[*unfolded fine-def*, *rule-format*,*OF as*] **by** *auto*
with *kk* **obtain** *y* **where** *y*: *y* ∈ *ball x* |*x* · *k* − *c*| *y*·*k* ≤ *c*
by *blast*
then have |*x* · *k* − *y* · *k*| < |*x* · *k* − *c*|

```
        using Basis-le-norm[OF k, of x − y]
        by (auto simp add: dist-norm inner-diff-left intro: le-less-trans)
      with y show False
        using ∗∗ by (auto simp add: field-simps)
    qed
  qed
  have xk-ge-c: ⋀x kk. (x, kk) ∈ p ⟹ kk ∩ {x. x•k ≥ c} ≠ {} ⟹ x•k ≥ c
  proof −
    fix x kk
    assume as: (x, kk) ∈ p and kk: kk ∩ {x. x•k ≥ c} ≠ {}
    show x•k ≥ c
    proof (rule ccontr)
      assume ∗∗: ¬ ?thesis
      from this[unfolded not-le] have kk ⊆ ball x |x • k − c|
        using p(2)[unfolded fine-def,rule-format,OF as,unfolded split-conv] by
auto
      with kk obtain y where y: y ∈ ball x |x • k − c| y•k ≥ c
        by blast
      then have |x • k − y • k| < |x • k − c|
        using Basis-le-norm[OF k, of x − y]
        by (auto simp add: dist-norm inner-diff-left intro: le-less-trans)
      with y show False
        using ∗∗ by (auto simp add: field-simps)
    qed
  qed

  have lem1: ⋀f P Q. (∀x k. (x, k) ∈ {(x, f k) | x k. P x k} ⟶ Q x k) ⟷
                  (∀x k. P x k ⟶ Q x (f k))
    by auto
  have fin-finite: finite {(x,f k) | x k. (x,k) ∈ s ∧ P x k} if finite s for f s P
  proof −
    from that have finite ((λ(x, k). (x, f k)) ‘ s)
      by auto
    then show ?thesis
      by (rule rev-finite-subset) auto
  qed
  { fix g :: 'a set ⇒ 'a set
    fix i :: 'a × 'a set
    assume i ∈ (λ(x, k). (x, g k)) ‘ p − {(x, g k) |x k. (x, k) ∈ p ∧ g k ≠ {}}
    then obtain x k where xk:
          i = (x, g k)  (x, k) ∈ p
          (x, g k) ∉ {(x, g k) |x k. (x, k) ∈ p ∧ g k ≠ {}}
      by auto
    have content (g k) = 0
      using xk using content-empty by auto
    then have (λ(x, k). content k ∗R f x) i = 0
      unfolding xk split-conv by auto
  } note [simp] = this
  have lem3: ⋀g :: 'a set ⇒ 'a set. finite p ⟹
```

setsum (λ(*x*, *k*). *content k* ∗$_R$ *f x*) {(*x*,*g k*) |*x k*. (*x*,*k*) ∈ *p* ∧ *g k* ≠ {}} =
setsum (λ(*x*, *k*). *content k* ∗$_R$ *f x*) ((λ(*x*, *k*). (*x*, *g k*)) ‘ *p*)
by (*rule setsum.mono-neutral-left*) *auto*
let *?M1* = {(*x*, *kk* ∩ {*x*. *x*⋅*k* ≤ *c*}) |*x kk*. (*x*, *kk*) ∈ *p* ∧ *kk* ∩ {*x*. *x*⋅*k* ≤ *c*} ≠ {}}
have *d1-fine*: *d1 fine ?M1*
by (*force intro*: *fineI dest*: *fineD*[*OF p*(*2*)] *simp add*: *split*: *if-split-asm*)
have *norm* ((∑ (*x*, *k*)∈*?M1*. *content k* ∗$_R$ *f x*) − *i*) < *e*/*2*
proof (*rule d1norm* [*OF tagged-division-ofI d1-fine*])
show *finite ?M1*
by (*rule fin-finite p*(*3*))+
show ⋃{*k*. ∃ *x*. (*x*, *k*) ∈ *?M1*} = *cbox a b* ∩ {*x*. *x*⋅*k* ≤ *c*}
unfolding *p*(*8*)[*symmetric*] **by** *auto*
fix *x l*
assume *xl*: (*x*, *l*) ∈ *?M1*
then guess *x′ l′* **unfolding** *mem-Collect-eq* **unfolding** *prod.inject* **by** (*elim exE conjE*) **note** *xl′*=*this*
show *x* ∈ *l l* ⊆ *cbox a b* ∩ {*x*. *x* ⋅ *k* ≤ *c*}
unfolding *xl′*
using *p*(*4*−*6*)[*OF xl′*(*3*)] **using** *xl′*(*4*)
using *xk-le-c*[*OF xl′*(*3*−*4*)] **by** *auto*
show ∃ *a b*. *l* = *cbox a b*
unfolding *xl′*
using *p*(*6*)[*OF xl′*(*3*)]
by (*fastforce simp add*: *interval-split*[*OF k*,**where** *c*=*c*])
fix *y r*
let *?goal* = *interior l* ∩ *interior r* = {}
assume *yr*: (*y*, *r*) ∈ *?M1*
then guess *y′ r′* **unfolding** *mem-Collect-eq* **unfolding** *prod.inject* **by** (*elim exE conjE*) **note** *yr′*=*this*
assume *as*: (*x*, *l*) ≠ (*y*, *r*)
show *interior l* ∩ *interior r* = {}
proof (*cases l′* = *r′* ⟶ *x′* = *y′*)
case *False*
then show *?thesis*
using *p*(*7*)[*OF xl′*(*3*) *yr′*(*3*)] **using** *as* **unfolding** *xl′ yr′* **by** *auto*
next
case *True*
then have *l′* ≠ *r′*
using *as* **unfolding** *xl′ yr′* **by** *auto*
then show *?thesis*
using *p*(*7*)[*OF xl′*(*3*) *yr′*(*3*)] **using** *as* **unfolding** *xl′ yr′* **by** *auto*
qed
qed
moreover
let *?M2* = {(*x*,*kk* ∩ {*x*. *x*⋅*k* ≥ *c*}) |*x kk*. (*x*,*kk*) ∈ *p* ∧ *kk* ∩ {*x*. *x*⋅*k* ≥ *c*} ≠ {}}
have *d2-fine*: *d2 fine ?M2*

by (*force intro*: *fineI dest*: *fineD*[*OF p*(*2*)] *simp add*: *split*: *if-split-asm*)
have *norm* ((∑ (*x*, *k*)∈*?M2*. *content k* ∗$_R$ *f x*) − *j*) < *e*/*2*
proof (*rule d2norm* [*OF tagged-division-ofI d2-fine*])
show *finite ?M2*
by (*rule fin-finite p*(*3*))+
show ⋃{*k*. ∃ *x*. (*x*, *k*) ∈ *?M2*} = *cbox a b* ∩ {*x*. *x*·*k* ≥ *c*}
unfolding *p*(*8*)[*symmetric*] **by** *auto*
fix *x l*
assume *xl*: (*x*, *l*) ∈ *?M2*
then guess *x′ l′* **unfolding** *mem-Collect-eq* **unfolding** *prod.inject* **by** (*elim exE conjE*) **note** *xl′*=*this*
show *x* ∈ *l l* ⊆ *cbox a b* ∩ {*x*. *x* · *k* ≥ *c*}
unfolding *xl′*
using *p*(*4*−*6*)[*OF xl′*(*3*)] *xl′*(*4*) *xk-ge-c*[*OF xl′*(*3*−*4*)]
by *auto*
show ∃ *a b*. *l* = *cbox a b*
unfolding *xl′*
using *p*(*6*)[*OF xl′*(*3*)]
by (*fastforce simp add*: *interval-split*[*OF k*, **where** *c*=*c*])
fix *y r*
let *?goal* = *interior l* ∩ *interior r* = {}
assume *yr*: (*y*, *r*) ∈ *?M2*
then guess *y′ r′* **unfolding** *mem-Collect-eq* **unfolding** *prod.inject* **by** (*elim exE conjE*) **note** *yr′*=*this*
assume *as*: (*x*, *l*) ≠ (*y*, *r*)
show *interior l* ∩ *interior r* = {}
proof (*cases l′* = *r′* ⟶ *x′* = *y′*)
case *False*
then show *?thesis*
using *p*(*7*)[*OF xl′*(*3*) *yr′*(*3*)] **using** *as* **unfolding** *xl′ yr′* **by** *auto*
next
case *True*
then have *l′* ≠ *r′*
using *as* **unfolding** *xl′ yr′* **by** *auto*
then show *?thesis*
using *p*(*7*)[*OF xl′*(*3*) *yr′*(*3*)] **using** *as* **unfolding** *xl′ yr′* **by** *auto*
qed
qed
ultimately
have *norm* (((∑ (*x*, *k*)∈*?M1*. *content k* ∗$_R$ *f x*) − *i*) + ((∑ (*x*, *k*)∈*?M2*. *content k* ∗$_R$ *f x*) − *j*)) < *e*/*2* + *e*/*2*
using *norm-add-less* **by** *blast*
also **{**
have *eq0*: ⋀*x y*. *x* = (*0*::*real*) ⟹ *x* ∗$_R$ (*y*::′*b*) = *0*
using *scaleR-zero-left* **by** *auto*
have *cont-eq*: ⋀*g*. (λ(*x*,*l*). *content l* ∗$_R$ *f x*) ∘ (λ(*x*,*l*). (*x*,*g l*)) = (λ(*x*,*l*). *content* (*g l*) ∗$_R$ *f x*)
by *auto*
have ((∑ (*x*, *k*)∈*?M1*. *content k* ∗$_R$ *f x*) − *i*) + ((∑ (*x*, *k*)∈*?M2*. *content k*

$*_R\ f\ x) - j) =$
$(\sum(x,\ k)\in ?M1.\ content\ k\ *_R\ f\ x) + (\sum(x,\ k)\in ?M2.\ content\ k\ *_R\ f\ x) -$
$(i + j)$
by *auto*
also have $\ldots = (\sum(x,\ ka)\in p.\ content\ (ka \cap \{x.\ x \cdot k \le c\})\ *_R\ f\ x) +$
$(\sum(x,\ ka)\in p.\ content\ (ka \cap \{x.\ c \le x \cdot k\})\ *_R\ f\ x) - (i + j)$
unfolding *lem3*[*OF p*(*3*)]
by (*subst setsum.reindex-nontrivial*[*OF p*(*3*)], *auto intro*!: *k eq0 tagged-division-split-left-inj*[*OF p*(*1*)] *tagged-division-split-right-inj*[*OF p*(*1*)]
simp: *cont-eq*)+
also note *setsum.distrib*[*symmetric*]
also have $\bigwedge x.\ x \in p \Longrightarrow$
$(\lambda(x,ka).\ content\ (ka \cap \{x.\ x \cdot k \le c\})\ *_R\ f\ x)\ x +$
$(\lambda(x,ka).\ content\ (ka \cap \{x.\ c \le x \cdot k\})\ *_R\ f\ x)\ x =$
$(\lambda(x,ka).\ content\ ka\ *_R\ f\ x)\ x$
proof *clarify*
fix *a b*
assume $(a,\ b) \in p$
from *p*(*6*)[*OF this*] **guess** *u v* **by** (*elim exE*) **note** *uv*=*this*
then show *content* $(b \cap \{x.\ x \cdot k \le c\})\ *_R\ f\ a +$ *content* $(b \cap \{x.\ c \le x \cdot k\})\ *_R\ f\ a =$
content b $*_R$ *f a*
unfolding *scaleR-left-distrib*[*symmetric*]
unfolding *uv content-split*[*OF k*,*of u v c*]
by *auto*
qed
note *setsum.cong* [*OF - this*]
finally have $(\sum(x,\ k)\in\{(x,\ kk \cap \{x.\ x \cdot k \le c\})\ |x\ kk.\ (x,\ kk) \in p \wedge kk \cap \{x.\ x \cdot k \le c\} \ne \{\}\}.\ content\ k\ *_R\ f\ x) - i +$
$((\sum(x,\ k)\in\{(x,\ kk \cap \{x.\ c \le x \cdot k\})\ |x\ kk.\ (x,\ kk) \in p \wedge kk \cap \{x.\ c \le x \cdot k\} \ne \{\}\}.\ content\ k\ *_R\ f\ x) - j) =$
$(\sum(x,\ ka)\in p.\ content\ ka\ *_R\ f\ x) - (i + j)$
by *auto*
}
finally show *norm* $((\sum(x,\ k)\in p.\ content\ k\ *_R\ f\ x) - (i + j)) < e$
by *auto*
qed
qed

41.17 A sort of converse, integrability on subintervals.

lemma *tagged-division-union-interval*:
fixes *a* :: *'a*::*euclidean-space*
assumes *p1 tagged-division-of* $(cbox\ a\ b \cap \{x.\ x \cdot k \le (c::real)\})$
and *p2 tagged-division-of* $(cbox\ a\ b \cap \{x.\ x \cdot k \ge c\})$
and *k*: $k \in Basis$
shows (*p1* $\cup$ *p2*) *tagged-division-of* (*cbox a b*)
proof −
have $*$: $cbox\ a\ b = (cbox\ a\ b \cap \{x.\ x \cdot k \le c\}) \cup (cbox\ a\ b \cap \{x.\ x \cdot k \ge c\})$

```
    by auto
  show ?thesis
    apply (subst *)
    apply (rule tagged-division-union[OF assms(1−2)])
    unfolding interval-split[OF k] interior-cbox
    using k
    apply (auto simp add: box-def elim!: ballE[where x=k])
    done
qed

lemma tagged-division-union-interval-real:
  fixes a :: real
  assumes p1 tagged-division-of ({a .. b} ∩ {x. x•k ≤ (c::real)})
    and p2 tagged-division-of ({a .. b} ∩ {x. x•k ≥ c})
    and k: k ∈ Basis
  shows (p1 ∪ p2) tagged-division-of {a .. b}
  using assms
  unfolding box-real[symmetric]
  by (rule tagged-division-union-interval)

lemma has-integral-separate-sides:
  fixes f :: 'a::euclidean-space ⇒ 'b::real-normed-vector
  assumes (f has-integral i) (cbox a b)
    and e > 0
    and k: k ∈ Basis
  obtains d where gauge d
    ∀ p1 p2. p1 tagged-division-of (cbox a b ∩ {x. x•k ≤ c}) ∧ d fine p1 ∧
        p2 tagged-division-of (cbox a b ∩ {x. x•k ≥ c}) ∧ d fine p2 ⟶
        norm ((setsum (λ(x,k). content k *R f x) p1 + setsum (λ(x,k). content k
*R f x) p2) − i) < e
proof −
  guess d using has-integralD[OF assms(1−2)] . note d=this
  { fix p1 p2
    assume p1 tagged-division-of (cbox a b) ∩ {x. x • k ≤ c} d fine p1
    note p1=tagged-division-ofD[OF this(1)] this
    assume p2 tagged-division-of (cbox a b) ∩ {x. c ≤ x • k} d fine p2
    note p2=tagged-division-ofD[OF this(1)] this
   note tagged-division-union-interval[OF p1(7) p2(7)] note p12 = tagged-division-ofD[OF
this] this
    { fix a b
      assume ab: (a, b) ∈ p1 ∩ p2
      have (a, b) ∈ p1
        using ab by auto
      with p1 obtain u v where uv: b = cbox u v by auto
      have b ⊆ {x. x•k = c}
        using ab p1(3)[of a b] p2(3)[of a b] by fastforce
      moreover
      have interior {x::'a. x • k = c} = {}
      proof (rule ccontr)
```

assume ¬ *?thesis*
then obtain *x* **where** *x*: $x \in$ *interior* $\{x::'a.\ x \cdot k = c\}$
by *auto*
then guess *e* **unfolding** *mem-interior* **.. note** *e=this*
have *x*: $x \cdot k = c$
using *x interior-subset* **by** *fastforce*
have ∗: $\bigwedge i.\ i \in Basis \Longrightarrow |(x - (x + (e\ /\ 2) *_R k)) \cdot i| = (if\ i = k\ then\ e/2\ else\ 0)$
using *e k* **by** (*auto simp*: *inner-simps inner-not-same-Basis*)
have $(\sum i \in Basis.\ |(x - (x + (e\ /\ 2\ ) *_R k)) \cdot i|) =$
$(\sum i \in Basis.\ (if\ i = k\ then\ e\ /\ 2\ else\ 0))$
using ∗ **by** (*blast intro*: *setsum.cong*)
also have ... < *e*
apply (*subst setsum.delta*)
using *e*
apply *auto*
done
finally have $x + (e/2) *_R k \in ball\ x\ e$
unfolding *mem-ball dist-norm* **by**(*rule le-less-trans*[*OF norm-le-l1*])
then have $x + (e/2) *_R k \in \{x.\ x \cdot k = c\}$
using *e* **by** *auto*
then show *False*
unfolding *mem-Collect-eq* **using** *e x k* **by** (*auto simp*: *inner-simps*)
qed
ultimately have *content b* = *0*
unfolding *uv content-eq-0-interior*
using *interior-mono* **by** *blast*
then have *content b* $*_R$ *f a* = *0*
by *auto*
}
then have *norm* $((\sum (x,\ k) \in p1.\ content\ k *_R f\ x) + (\sum (x,\ k) \in p2.\ content\ k *_R f\ x) - i) =$
norm $((\sum (x,\ k) \in p1 \cup p2.\ content\ k *_R f\ x) - i)$
by (*subst setsum.union-inter-neutral*) (*auto simp*: *p1 p2*)
also have ... < *e*
by (*rule k d*(*2*) *p12 fine-union p1 p2*)+
finally have *norm* $((\sum (x,\ k) \in p1.\ content\ k *_R f\ x) + (\sum (x,\ k) \in p2.\ content\ k *_R f\ x) - i) < e$ **.**
}
then show *?thesis*
by (*auto intro*: *that*[*of d*] *d elim*:)
qed

lemma *integrable-split*[*intro*]:
fixes *f* :: $'a$::*euclidean-space* ⇒ $'b$::{*real-normed-vector*,*complete-space*}
assumes *f integrable-on cbox a b*
and *k*: $k \in Basis$
shows *f integrable-on* (*cbox a b* ∩ $\{x.\ x \cdot k \leq c\}$) (**is** *?t1*)
and *f integrable-on* (*cbox a b* ∩ $\{x.\ x \cdot k \geq c\}$) (**is** *?t2*)

```
proof −
  guess y using assms(1) unfolding integrable-on-def .. note y=this
  def b' ≡ ∑ i∈Basis. (if i = k then min (b•k) c else b•i)*R i::'a
  def a' ≡ ∑ i∈Basis. (if i = k then max (a•k) c else a•i)*R i::'a
  show ?t1 ?t2
    unfolding interval-split[OF k] integrable-cauchy
    unfolding interval-split[symmetric,OF k]
  proof (rule-tac[!] allI impI)+
    fix e :: real
    assume e > 0
    then have e/2>0
      by auto
   from has-integral-separate-sides[OF y this k,of c] guess d . note d=this[rule-format]
    let ?P = λA. ∃ d. gauge d ∧ (∀ p1 p2. p1 tagged-division-of (cbox a b) ∩ A ∧
d fine p1 ∧
      p2 tagged-division-of (cbox a b) ∩ A ∧ d fine p2 ⟶
      norm ((∑ (x, k)∈p1. content k *R f x) − (∑ (x, k)∈p2. content k *R f x))
< e)
    show ?P {x. x • k ≤ c}
    proof (rule-tac x=d in exI, clarsimp simp add: d)
      fix p1 p2
      assume as: p1 tagged-division-of (cbox a b) ∩ {x. x • k ≤ c} d fine p1
                p2 tagged-division-of (cbox a b) ∩ {x. x • k ≤ c} d fine p2
      show norm ((∑ (x, k)∈p1. content k *R f x) − (∑ (x, k)∈p2. content k *R
f x)) < e
      proof (rule fine-division-exists[OF d(1), of a' b] )
        fix p
        assume p tagged-division-of cbox a' b d fine p
        then show ?thesis
          using as norm-triangle-half-l[OF d(2)[of p1 p] d(2)[of p2 p]]
          unfolding interval-split[OF k] b'-def[symmetric] a'-def[symmetric]
          by (auto simp add: algebra-simps)
      qed
    qed
    show ?P {x. x • k ≥ c}
    proof (rule-tac x=d in exI, clarsimp simp add: d)
      fix p1 p2
      assume as: p1 tagged-division-of (cbox a b) ∩ {x. x • k ≥ c} d fine p1
                p2 tagged-division-of (cbox a b) ∩ {x. x • k ≥ c} d fine p2
      show norm ((∑ (x, k)∈p1. content k *R f x) − (∑ (x, k)∈p2. content k *R
f x)) < e
      proof (rule fine-division-exists[OF d(1), of a b'] )
        fix p
        assume p tagged-division-of cbox a b' d fine p
        then show ?thesis
          using as norm-triangle-half-l[OF d(2)[of p p1] d(2)[of p p2]]
          unfolding interval-split[OF k] b'-def[symmetric] a'-def[symmetric]
          by (auto simp add: algebra-simps)
      qed
```

qed
qed
qed

41.18 Generalized notion of additivity.

definition *neutral opp* = (*SOME x*. ∀ *y*. *opp x y* = *y* ∧ *opp y x* = *y*)

definition *operative* :: ('*a* ⇒ '*a* ⇒ '*a*) ⇒ (('*b*::*euclidean-space*) *set* ⇒ '*a*) ⇒ *bool*
where *operative opp f* ⟷
(∀ *a b*. *content* (*cbox a b*) = *0* ⟶ *f* (*cbox a b*) = *neutral opp*) ∧
(∀ *a b c*. ∀ *k*∈*Basis*. *f* (*cbox a b*) = *opp* (*f*(*cbox a b* ∩ {*x*. *x*•*k* ≤ *c*})) (*f* (*cbox a b* ∩ {*x*. *x*•*k* ≥ *c*})))

lemma *operativeD*[*dest*]:
fixes *type* :: '*a*::*euclidean-space*
assumes *operative opp f*
shows ⋀*a b*::'*a*. *content* (*cbox a b*) = *0* ⟹ *f* (*cbox a b*) = *neutral opp*
and ⋀*a b c k*. *k* ∈ *Basis* ⟹ *f* (*cbox a b*) =
opp (*f* (*cbox a b* ∩ {*x*. *x*•*k* ≤ *c*})) (*f* (*cbox a b* ∩ {*x*. *x*•*k* ≥ *c*}))
using *assms* **unfolding** *operative-def* **by** *auto*

lemma *property-empty-interval*: ∀ *a b*. *content* (*cbox a b*) = *0* ⟶ *P* (*cbox a b*) ⟹ *P* {}
using *content-empty* **unfolding** *empty-as-interval* **by** *auto*

lemma *operative-empty*: *operative opp f* ⟹ *f* {} = *neutral opp*
unfolding *operative-def* **by** (*rule property-empty-interval*) *auto*

41.19 Using additivity of lifted function to encode definedness.

fun *lifted* **where**
lifted (*opp* :: '*a* ⇒ '*a* ⇒ '*b*) (*Some x*) (*Some y*) = *Some* (*opp x y*)
| *lifted opp None* - = (*None*::'*b option*)
| *lifted opp* - *None* = *None*

lemma *lifted-simp-1*[*simp*]: *lifted opp v None* = *None*
by (*induct v*) *auto*

definition *monoidal opp* ⟷
(∀ *x y*. *opp x y* = *opp y x*) ∧
(∀ *x y z*. *opp x* (*opp y z*) = *opp* (*opp x y*) *z*) ∧
(∀ *x*. *opp* (*neutral opp*) *x* = *x*)

lemma *monoidalI*:
assumes ⋀*x y*. *opp x y* = *opp y x*
and ⋀*x y z*. *opp x* (*opp y z*) = *opp* (*opp x y*) *z*
and ⋀*x*. *opp* (*neutral opp*) *x* = *x*

shows *monoidal opp*
unfolding *monoidal-def* **using** *assms* **by** *fastforce*

lemma *monoidal-ac*:
assumes *monoidal opp*
shows [*simp*]: *opp* (*neutral opp*) *a* = *a*
and [*simp*]: *opp a* (*neutral opp*) = *a*
and *opp a b* = *opp b a*
and *opp* (*opp a b*) *c* = *opp a* (*opp b c*)
and *opp a* (*opp b c*) = *opp b* (*opp a c*)
using *assms* **unfolding** *monoidal-def* **by** *metis*+

lemma *neutral-lifted* [*cong*]:
assumes *monoidal opp*
shows *neutral* (*lifted opp*) = *Some* (*neutral opp*)
proof −
{ fix *x*
assume ∀ *y*. *lifted opp x y* = *y* ∧ *lifted opp y x* = *y*
then have *Some* (*neutral opp*) = *x*
apply (*induct x*)
apply *force*
by (*metis assms lifted.simps*(*1*) *monoidal-ac*(*2*) *option.inject*) **}**
note [*simp*] = *this*
show *?thesis*
apply (*subst neutral-def*)
apply (*intro some-equality allI*)
apply (*induct-tac y*)
apply (*auto simp add*:*monoidal-ac*[*OF assms*])
done
qed

lemma *monoidal-lifted*[*intro*]:
assumes *monoidal opp*
shows *monoidal* (*lifted opp*)
unfolding *monoidal-def split-option-all neutral-lifted*[*OF assms*]
using *monoidal-ac*[*OF assms*]
by *auto*

definition *support opp f s* = {*x*. *x*∈*s* ∧ *f x* ≠ *neutral opp*}
definition *fold' opp e s* = (*if finite s then Finite-Set.fold opp e s else e*)
definition *iterate opp s f* = *fold'* (λ*x a*. *opp* (*f x*) *a*) (*neutral opp*) (*support opp f s*)

lemma *support-subset*[*intro*]: *support opp f s* ⊆ *s*
unfolding *support-def* **by** *auto*

lemma *support-empty*[*simp*]: *support opp f* {} = {}
using *support-subset*[*of opp f* {}] **by** *auto*

lemma *comp-fun-commute-monoidal*[*intro*]:
assumes *monoidal opp*
shows *comp-fun-commute opp*
unfolding *comp-fun-commute-def*
using *monoidal-ac*[*OF assms*]
by *auto*

lemma *support-clauses*:
$\bigwedge f\ g\ s.\ support\ opp\ f\ \{\} = \{\}$
$\bigwedge f\ g\ s.\ support\ opp\ f\ (insert\ x\ s) =$
$(if\ f(x) = neutral\ opp\ then\ support\ opp\ f\ s\ else\ insert\ x\ (support\ opp\ f\ s))$
$\bigwedge f\ g\ s.\ support\ opp\ f\ (s - \{x\}) = (support\ opp\ f\ s) - \{x\}$
$\bigwedge f\ g\ s.\ support\ opp\ f\ (s \cup t) = (support\ opp\ f\ s) \cup (support\ opp\ f\ t)$
$\bigwedge f\ g\ s.\ support\ opp\ f\ (s \cap t) = (support\ opp\ f\ s) \cap (support\ opp\ f\ t)$
$\bigwedge f\ g\ s.\ support\ opp\ f\ (s - t) = (support\ opp\ f\ s) - (support\ opp\ f\ t)$
$\bigwedge f\ g\ s.\ support\ opp\ g\ (f\ `\ s) = f\ `\ (support\ opp\ (g \circ f)\ s)$
unfolding *support-def* **by** *auto*

lemma *finite-support*[*intro*]: *finite s* $\Longrightarrow$ *finite* (*support opp f s*)
unfolding *support-def* **by** *auto*

lemma *iterate-empty*[*simp*]: *iterate opp* $\{\}$ *f* = *neutral opp*
unfolding *iterate-def fold'-def* **by** *auto*

lemma *iterate-insert*[*simp*]:
assumes *monoidal opp*
and *finite s*
shows *iterate opp* (*insert x s*) *f* =
(*if x* $\in$ *s then iterate opp s f else opp* (*f x*) (*iterate opp s f*))
proof (*cases x* $\in$ *s*)
case *True*
then show *?thesis* **by** (*auto simp*: *insert-absorb iterate-def*)
next
case *False*
note $*$ = *comp-fun-commute.comp-comp-fun-commute* [*OF comp-fun-commute-monoidal*[*OF assms*(*1*)]]
show *?thesis*
proof (*cases f x* = *neutral opp*)
case *True*
then show *?thesis*
using *assms* ‹*x* $\notin$ *s*›
by (*auto simp*: *iterate-def support-clauses*)
next
case *False*
with ‹*x* $\notin$ *s*› ‹*finite s*› *support-subset* **show** *?thesis*
apply (*simp add*: *iterate-def fold'-def support-clauses*)
apply (*subst comp-fun-commute.fold-insert*[*OF* $*$ *finite-support*, *simplified comp-def*])
apply (*force simp add*:)+

done
qed
qed

lemma *iterate-some*:
⟦*monoidal opp*; *finite s*⟧ ⟹ *iterate* (*lifted opp*) *s* (λ*x*. *Some*(*f x*)) = *Some* (*iterate opp s f*)
by (*erule finite-induct*) (*auto simp*: *monoidal-lifted*)

41.20 Two key instances of additivity.

lemma *neutral-add*[*simp*]: *neutral op* + = (*0*::'*a*::*comm-monoid-add*)
unfolding *neutral-def*
by (*force elim*: *allE* [**where** *x*=*0*])

lemma *operative-content*[*intro*]: *operative* (*op* +) *content*
by (*force simp add*: *operative-def content-split*[*symmetric*])

lemma *monoidal-monoid*[*intro*]: *monoidal* ((*op* +)::('*a*::*comm-monoid-add*) ⇒ '*a* ⇒ '*a*)
unfolding *monoidal-def neutral-add*
by (*auto simp add*: *algebra-simps*)

lemma *operative-integral*:
fixes *f* :: '*a*::*euclidean-space* ⇒ '*b*::*banach*
shows *operative* (*lifted*(*op* +)) (λ*i*. *if f integrable-on i then Some*(*integral i f*) *else None*)
unfolding *operative-def neutral-lifted*[*OF monoidal-monoid*] *neutral-add*
proof *safe*
fix *a b c*
fix *k* :: '*a*
assume *k*: *k* ∈ *Basis*
show (*if f integrable-on cbox a b then Some* (*integral* (*cbox a b*) *f*) *else None*) =
lifted op + (*if f integrable-on cbox a b* ∩ {*x*. *x* • *k* ≤ *c*} *then Some* (*integral* (*cbox a b* ∩ {*x*. *x* • *k* ≤ *c*}) *f*) *else None*)
(*if f integrable-on cbox a b* ∩ {*x*. *c* ≤ *x* • *k*} *then Some* (*integral* (*cbox a b* ∩ {*x*. *c* ≤ *x* • *k*}) *f*) *else None*)
proof (*cases f integrable-on cbox a b*)
case *True*
with *k* **show** *?thesis*
apply (*simp add*: *integrable-split*)
apply (*rule integral-unique* [*OF has-integral-split*[*OF - - k*]])
apply (*auto intro*: *integrable-integral*)
done
next
case *False*
have ¬ (*f integrable-on cbox a b* ∩ {*x*. *x* • *k* ≤ *c*}) ∨ ¬ (*f integrable-on cbox a b* ∩ {*x*. *c* ≤ *x* • *k*})
proof (*rule ccontr*)

assume ¬ *?thesis*
then have *f integrable-on cbox a b*
unfolding *integrable-on-def*
apply (*rule-tac x=integral* (*cbox a b* ∩ {*x*. *x* ∙ *k* ≤ *c*}) *f* + *integral* (*cbox a b* ∩ {*x*. *x* ∙ *k* ≥ *c*}) *f* **in** *exI*)
apply (*rule has-integral-split*[*OF - - k*])
apply (*auto intro*: *integrable-integral*)
done
then show *False*
using *False* **by** *auto*
qed
then show *?thesis*
using *False* **by** *auto*
qed
next
fix *a b* :: *'a*
assume *content* (*cbox a b*) = *0*
then show (*if f integrable-on cbox a b then Some* (*integral* (*cbox a b*) *f*) *else None*) = *Some 0*
using *has-integral-null-eq*
by (*auto simp*: *integrable-on-null*)
qed

41.21 Points of division of a partition.

definition *division-points* (*k*::(*'a*::*euclidean-space*) *set*) *d* =
{(*j*,*x*). *j* ∈ *Basis* ∧ (*interval-lowerbound k*)∙*j* < *x* ∧ *x* < (*interval-upperbound k*)∙*j* ∧
(∃ *i*∈*d*. (*interval-lowerbound i*)∙*j* = *x* ∨ (*interval-upperbound i*)∙*j* = *x*)}

lemma *division-points-finite*:
fixes *i* :: *'a*::*euclidean-space set*
assumes *d division-of i*
shows *finite* (*division-points i d*)
proof −
note *assm* = *division-ofD*[*OF assms*]
let *?M* = λ*j*. {(*j*,*x*)|*x*. (*interval-lowerbound i*)∙*j* < *x* ∧ *x* < (*interval-upperbound i*)∙*j* ∧
(∃ *i*∈*d*. (*interval-lowerbound i*)∙*j* = *x* ∨ (*interval-upperbound i*)∙*j* = *x*)}
have ∗: *division-points i d* = ⋃ (*?M* ' *Basis*)
unfolding *division-points-def* **by** *auto*
show *?thesis*
unfolding ∗ **using** *assm* **by** *auto*
qed

lemma *division-points-subset*:
fixes *a* :: *'a*::*euclidean-space*
assumes *d division-of* (*cbox a b*)
and ∀ *i*∈*Basis*. *a*∙*i* < *b*∙*i* *a*∙*k* < *c* *c* < *b*∙*k*

```
    and k: k ∈ Basis
  shows division-points (cbox a b ∩ {x. x•k ≤ c}) {l ∩ {x. x•k ≤ c} | l . l ∈ d ∧
l ∩ {x. x•k ≤ c} ≠ {}} ⊆
      division-points (cbox a b) d (is ?t1)
    and division-points (cbox a b ∩ {x. x•k ≥ c}) {l ∩ {x. x•k ≥ c} | l . l ∈ d ∧
~(l ∩ {x. x•k ≥ c} = {})} ⊆
      division-points (cbox a b) d (is ?t2)
proof −
  note assm = division-ofD[OF assms(1)]
  have ∗: ∀ i∈Basis. a•i ≤ b•i
    ∀ i∈Basis. a•i ≤ (∑ i∈Basis. (if i = k then min (b • k) c else  b • i) ∗R i) • i
    ∀ i∈Basis. (∑ i∈Basis. (if i = k then max (a • k) c else a • i) ∗R i) • i ≤ b•i
    min (b • k) c = c max (a • k) c = c
    using assms using less-imp-le by auto
  show ?t1
    unfolding division-points-def interval-split[OF k, of a b]
   unfolding interval-bounds[OF ∗(1)] interval-bounds[OF ∗(2)] interval-bounds[OF
∗(3)]
    unfolding ∗
    apply (rule subsetI)
    unfolding mem-Collect-eq split-beta
    apply (erule bexE conjE)+
    apply (simp add: )
    apply (erule exE conjE)+
  proof
    fix i l x
    assume as:
      a • fst x < snd x snd x < (if fst x = k then c else b • fst x)
      interval-lowerbound i • fst x = snd x ∨ interval-upperbound i • fst x = snd x
      i = l ∩ {x. x • k ≤ c} l ∈ d l ∩ {x. x • k ≤ c} ≠ {}
      and fstx: fst x ∈ Basis
    from assm(4)[OF this(5)] guess u v apply−by(erule exE)+ note l=this
    have ∗: ∀ i∈Basis. u • i ≤ (∑ i∈Basis. (if i = k then min (v • k) c else v • i)
∗R i) • i
      using as(6) unfolding l interval-split[OF k] box-ne-empty as .
    have ∗∗: ∀ i∈Basis. u•i ≤ v•i
      using l using as(6) unfolding box-ne-empty[symmetric] by auto
   show ∃ i∈d. interval-lowerbound i • fst x = snd x ∨ interval-upperbound i • fst
x = snd x
      apply (rule bexI[OF - ‹l ∈ d›])
      using as(1−3,5) fstx
       unfolding l interval-bounds[OF ∗∗] interval-bounds[OF ∗] interval-split[OF
k] as
      apply (auto split: if-split-asm)
      done
    show snd x < b • fst x
      using as(2) ‹c < b•k› by (auto split: if-split-asm)
  qed
  show ?t2
```

unfolding *division-points-def interval-split*[*OF k, of a b*]
unfolding *interval-bounds*[*OF ∗*(*1*)] *interval-bounds*[*OF ∗*(*2*)] *interval-bounds*[*OF ∗*(*3*)]
unfolding ∗
unfolding *subset-eq*
apply *rule*
unfolding *mem-Collect-eq split-beta*
apply (*erule bexE conjE*)+
apply (*simp only*: *mem-Collect-eq inner-setsum-left-Basis simp-thms*)
apply (*erule exE conjE*)+
proof
fix *i l x*
assume *as*:
(*if fst x* = *k then c else a* · *fst x*) < *snd x snd x* < *b* · *fst x*
interval-lowerbound i · *fst x* = *snd x* ∨ *interval-upperbound i* · *fst x* = *snd x*
i = *l* ∩ {*x*. *c* ≤ *x* · *k*} *l* ∈ *d l* ∩ {*x*. *c* ≤ *x* · *k*} ≠ {}
and *fstx*: *fst x* ∈ *Basis*
from *assm*(*4*)[*OF this*(*5*)] **guess** *u v* **by** (*elim exE*) **note** *l*=*this*
have ∗: ∀ *i*∈*Basis*. (∑ *i*∈*Basis*. (*if i* = *k then max* (*u* · *k*) *c else u* · *i*) ∗$_R$ *i*) · *i* ≤ *v* · *i*
using *as*(*6*) **unfolding** *l interval-split*[*OF k*] *box-ne-empty as* **.**
have ∗∗: ∀ *i*∈*Basis*. *u*·*i* ≤ *v*·*i*
using *l* **using** *as*(*6*) **unfolding** *box-ne-empty*[*symmetric*] **by** *auto*
show ∃ *i*∈*d*. *interval-lowerbound i* · *fst x* = *snd x* ∨ *interval-upperbound i* · *fst x* = *snd x*
apply (*rule bexI*[*OF* - ‹*l* ∈ *d*›])
using *as*(*1*−*3*,*5*) *fstx*
unfolding *l interval-bounds*[*OF ∗∗*] *interval-bounds*[*OF ∗*] *interval-split*[*OF k*] *as*
apply (*auto split*: *if-split-asm*)
done
show *a* · *fst x* < *snd x*
using *as*(*1*) ‹*a*·*k* < *c*› **by** (*auto split*: *if-split-asm*)
qed
qed

lemma *division-points-psubset*:
fixes *a* :: *'a*::*euclidean-space*
assumes *d division-of* (*cbox a b*)
and ∀ *i*∈*Basis*. *a*·*i* < *b*·*i* *a*·*k* < *c c* < *b*·*k*
and *l* ∈ *d*
and *interval-lowerbound l*·*k* = *c* ∨ *interval-upperbound l*·*k* = *c*
and *k*: *k* ∈ *Basis*
shows *division-points* (*cbox a b* ∩ {*x*. *x*·*k* ≤ *c*}) {*l* ∩ {*x*. *x*·*k* ≤ *c*} | *l*. *l*∈*d* ∧ *l* ∩ {*x*. *x*·*k* ≤ *c*} ≠ {}} ⊂
division-points (*cbox a b*) *d* (**is** *?D1* ⊂ *?D*)
and *division-points* (*cbox a b* ∩ {*x*. *x*·*k* ≥ *c*}) {*l* ∩ {*x*. *x*·*k* ≥ *c*} | *l*. *l*∈*d* ∧ *l* ∩ {*x*. *x*·*k* ≥ *c*} ≠ {}} ⊂
division-points (*cbox a b*) *d* (**is** *?D2* ⊂ *?D*)

proof −
have *ab*: ∀ *i*∈*Basis*. *a*•*i* ≤ *b*•*i*
using *assms*(*2*) **by** (*auto intro*!:*less-imp-le*)
guess *u v* **using** *division-ofD*(*4*)[*OF assms*(*1*,*5*)] **by** (*elim exE*) **note** *l*=*this*
have *uv*: ∀ *i*∈*Basis*. *u*•*i* ≤ *v*•*i* ∀ *i*∈*Basis*. *a*•*i* ≤ *u*•*i* ∧ *v*•*i* ≤ *b*•*i*
using *division-ofD*(*2*,*2*,*3*)[*OF assms*(*1*,*5*)] **unfolding** *l box-ne-empty*
using *subset-box*(*1*)
apply *auto*
apply *blast*+
done
have ∗: *interval-upperbound* (*cbox a b* ∩ {*x*. *x* • *k* ≤ *interval-upperbound l* • *k*}) • *k* = *interval-upperbound l* • *k*
interval-upperbound (*cbox a b* ∩ {*x*. *x* • *k* ≤ *interval-lowerbound l* • *k*}) • *k* = *interval-lowerbound l* • *k*
unfolding *l interval-split*[*OF k*] *interval-bounds*[*OF uv*(*1*)]
using *uv*[*rule-format*, *of k*] *ab k*
by *auto*
have ∃ *x*. *x* ∈ *?D* − *?D1*
using *assms*(*3*−)
unfolding *division-points-def interval-bounds*[*OF ab*]
apply −
apply (*erule disjE*)
apply (*rule-tac x*=(*k*,(*interval-lowerbound l*)•*k*) **in** *exI*, *force simp add*: ∗)
apply (*rule-tac x*=(*k*,(*interval-upperbound l*)•*k*) **in** *exI*, *force simp add*: ∗)
done
moreover have *?D1* ⊆ *?D*
by (*auto simp add*: *assms division-points-subset*)
ultimately show *?D1* ⊂ *?D*
by *blast*
have ∗: *interval-lowerbound* (*cbox a b* ∩ {*x*. *x* • *k* ≥ *interval-lowerbound l* • *k*}) • *k* = *interval-lowerbound l* • *k*
interval-lowerbound (*cbox a b* ∩ {*x*. *x* • *k* ≥ *interval-upperbound l* • *k*}) • *k* = *interval-upperbound l* • *k*
unfolding *l interval-split*[*OF k*] *interval-bounds*[*OF uv*(*1*)]
using *uv*[*rule-format*, *of k*] *ab k*
by *auto*
have ∃ *x*. *x* ∈ *?D* − *?D2*
using *assms*(*3*−)
unfolding *division-points-def interval-bounds*[*OF ab*]
apply −
apply (*erule disjE*)
apply (*rule-tac x*=(*k*,(*interval-lowerbound l*)•*k*) **in** *exI*, *force simp add*: ∗)
apply (*rule-tac x*=(*k*,(*interval-upperbound l*)•*k*) **in** *exI*, *force simp add*: ∗)
done
moreover have *?D2* ⊆ *?D*
by (*auto simp add*: *assms division-points-subset*)
ultimately show *?D2* ⊂ *?D*
by *blast*
qed

41.22 Preservation by divisions and tagged divisions.

lemma *support-support*[*simp*]:*support opp f* (*support opp f s*) = *support opp f s*
unfolding *support-def* **by** *auto*

lemma *iterate-support*[*simp*]: *iterate opp* (*support opp f s*) *f* = *iterate opp s f*
unfolding *iterate-def support-support* **by** *auto*

lemma *iterate-expand-cases*:
iterate opp s f = (*if finite*(*support opp f s*) *then iterate opp* (*support opp f s*) *f else neutral opp*)
by (*simp add*: *iterate-def fold'-def*)

lemma *iterate-image*:
assumes *monoidal opp*
and *inj-on f s*
shows *iterate opp* (*f* ‘ *s*) *g* = *iterate opp s* (*g* ∘ *f*)
proof −
have ∗: *iterate opp* (*f* ‘ *s*) *g* = *iterate opp s* (*g* ∘ *f*)
if *finite s* ∀ *x*∈*s*. ∀ *y*∈*s*. *f x* = *f y* ⟶ *x* = *y* **for** *s*
using *that*
proof (*induct s*)
case *empty*
then show *?case* **by** *simp*
next
case *insert*
with *assms*(*1*) **show** *?case* **by** *auto*
qed
show *?thesis*
apply (*cases finite* (*support opp g* (*f* ‘ *s*)))
prefer *2*
apply (*metis finite-imageI iterate-expand-cases support-clauses*(*7*))
apply (*subst* (*1*) *iterate-support*[*symmetric*], *subst* (*2*) *iterate-support*[*symmetric*])
unfolding *support-clauses*
apply (*rule* ∗)
apply (*meson assms*(*2*) *finite-imageD subset-inj-on support-subset*)
apply (*meson assms*(*2*) *inj-on-contraD rev-subsetD support-subset*)
done
qed

lemma *iterate-nonzero-image-lemma*:
assumes *monoidal opp*
and *finite s g*(*a*) = *neutral opp*
and ∀ *x*∈*s*. ∀ *y*∈*s*. *f x* = *f y* ∧ *x* ≠ *y* ⟶ *g*(*f x*) = *neutral opp*
shows *iterate opp* {*f x* | *x*. *x* ∈ *s* ∧ *f x* ≠ *a*} *g* = *iterate opp s* (*g* ∘ *f*)
proof −
have ∗: {*f x* |*x*. *x* ∈ *s* ∧ *f x* ≠ *a*} = *f* ‘ {*x*. *x* ∈ *s* ∧ *f x* ≠ *a*}
by *auto*

```
  have **: support opp (g ∘ f) {x ∈ s. f x ≠ a} = support opp (g ∘ f) s
    unfolding support-def using assms(3) by auto
  have inj: inj-on f (support opp (g ∘ f) {x ∈ s. f x ≠ a})
    apply (simp add: inj-on-def)
   apply (metis (mono-tags, lifting) assms(4) comp-def mem-Collect-eq support-def)
    done
  show ?thesis
    apply (subst iterate-support[symmetric])
    apply (simp add: * support-clauses iterate-image[OF assms(1) inj])
    apply (simp add: iterate-def **)
    done
qed

lemma iterate-eq-neutral:
  assumes monoidal opp
      and ⋀x. x ∈ s ⟹ f x = neutral opp
    shows iterate opp s f = neutral opp
proof −
  have [simp]: support opp f s = {}
    unfolding support-def using assms(2) by auto
  show ?thesis
    by (subst iterate-support[symmetric]) simp
qed

lemma iterate-op:
   ⟦monoidal opp; finite s⟧
    ⟹ iterate opp s (λx. opp (f x) (g x)) = opp (iterate opp s f) (iterate opp s g)
by (erule finite-induct) (auto simp: monoidal-ac(4) monoidal-ac(5))

lemma iterate-eq:
  assumes monoidal opp
    and ⋀x. x ∈ s ⟹ f x = g x
  shows iterate opp s f = iterate opp s g
proof −
  have *: support opp g s = support opp f s
    unfolding support-def using assms(2) by auto
  show ?thesis
  proof (cases finite (support opp f s))
    case False
    then show ?thesis
      by (simp add: * iterate-expand-cases)
  next
    case True
    def su ≡ support opp f s
    have fsu: finite su
      using True by (simp add: su-def)
    moreover
    { assume finite su su ⊆ s
      then have iterate opp su f = iterate opp su g
```

```
      by (induct su) (auto simp: assms)
    }
    ultimately have iterate opp (support opp f s) f = iterate opp (support opp g
s) g
      by (simp add: ∗ su-def support-subset)
    then show ?thesis
      by simp
  qed
qed

lemma nonempty-witness:
  assumes s ≠ {}
  obtains x where x ∈ s
  using assms by auto

lemma operative-division:
  fixes f :: 'a::euclidean-space set ⇒ 'b
  assumes monoidal opp
      and operative opp f
      and d division-of (cbox a b)
    shows iterate opp d f = f (cbox a b)
proof −
  def C ≡ card (division-points (cbox a b) d)
  then show ?thesis
    using assms
  proof (induct C arbitrary: a b d rule: full-nat-induct)
    case (1 a b d)
    show ?case
    proof (cases content (cbox a b) = 0)
      case True
      show iterate opp d f = f (cbox a b)
        unfolding operativeD(1)[OF assms(2) True]
      proof (rule iterate-eq-neutral[OF ‹monoidal opp›])
        fix x
        assume x: x ∈ d
        then show f x = neutral opp
            by (metis division-ofD(4) 1(4) division-of-content-0[OF True] opera-
tiveD(1)[OF assms(2)] x)
      qed
    next
      case False
      note ab = this[unfolded content-lt-nz[symmetric] content-pos-lt-eq]
      then have ab': ∀ i∈Basis. a•i ≤ b•i
        by (auto intro!: less-imp-le)
        show iterate opp d f = f (cbox a b)
      proof (cases division-points (cbox a b) d = {})
        case True
        { fix u v and j :: 'a
          assume j: j ∈ Basis and as: cbox u v ∈ d
```

```
        then have cbox u v ≠ {}
          using 1.prems(3) by blast
        then have uv: ∀ i∈Basis. u•i ≤ v•i u•j ≤ v•j
          using j unfolding box-ne-empty by auto
       have *: ⋀p r Q. ¬ j∈Basis ∨ p ∨ r ∨ (∀ x∈d. Q x) ⟹ p ∨ r ∨ Q (cbox
u v)
          using as j by auto
        have (j, u•j) ∉ division-points (cbox a b) d
             (j, v•j) ∉ division-points (cbox a b) d using True by auto
           note this[unfolded de-Morgan-conj division-points-def mem-Collect-eq
split-conv interval-bounds[OF ab'] bex-simps]
         note *[OF this(1)] *[OF this(2)] note this[unfolded interval-bounds[OF
uv(1)]]
        moreover
        have a•j ≤ u•j v•j ≤ b•j
          using division-ofD(2,2,3)[OF ‹d division-of cbox a b› as]
          apply (metis j subset-box(1) uv(1))
          by (metis ‹cbox u v ⊆ cbox a b› j subset-box(1) uv(1))
       ultimately have u•j = a•j ∧ v•j = a•j ∨ u•j = b•j ∧ v•j = b•j ∨ u•j =
a•j ∧ v•j = b•j
          unfolding not-less de-Morgan-disj using ab[rule-format,of j] uv(2) j
by force }
      then have d': ∀ i∈d. ∃ u v. i = cbox u v ∧
       (∀ j∈Basis. u•j = a•j ∧ v•j = a•j ∨ u•j = b•j ∧ v•j = b•j ∨ u•j = a•j ∧
v•j = b•j)
        unfolding forall-in-division[OF 1(4)]
        by blast
      have (1/2) *R (a+b) ∈ cbox a b
      unfolding mem-box using ab by(auto intro!: less-imp-le simp: inner-simps)
       note this[unfolded division-ofD(6)[OF ‹d division-of cbox a b›,symmetric]
Union-iff]
      then guess i .. note i=this
     guess u v using d'[rule-format,OF i(1)] by (elim exE conjE) note uv=this
      have cbox a b ∈ d
      proof −
        have u = a v = b
          unfolding euclidean-eq-iff[where 'a='a]
        proof safe
          fix j :: 'a
          assume j: j ∈ Basis
          note i(2)[unfolded uv mem-box,rule-format,of j]
          then show u • j = a • j and v • j = b • j
            using uv(2)[rule-format,of j] j by (auto simp: inner-simps)
        qed
        then have i = cbox a b using uv by auto
        then show ?thesis using i by auto
      qed
      then have deq: d = insert (cbox a b) (d − {cbox a b})
        by auto
```

```
      have iterate opp (d − {cbox a b}) f = neutral opp
      proof (rule iterate-eq-neutral[OF 1(2)])
        fix x
        assume x: x ∈ d − {cbox a b}
        then have x∈d
          by auto note d'[rule-format,OF this]
        then guess u v by (elim exE conjE) note uv=this
        have u ≠ a ∨ v ≠ b
          using x[unfolded uv] by auto
        then obtain j where u•j ≠ a•j ∨ v•j ≠ b•j and j: j ∈ Basis
          unfolding euclidean-eq-iff[where 'a='a] by auto
        then have u•j = v•j
          using uv(2)[rule-format,OF j] by auto
        then have content (cbox u v) = 0
          unfolding content-eq-0 using j
          by force
        then show f x = neutral opp
          unfolding uv(1) by (rule operativeD(1)[OF 1(3)])
      qed
      then show iterate opp d f = f (cbox a b)
        apply (subst deq)
        apply (subst iterate-insert[OF 1(2)])
        using 1
        apply auto
        done
    next
      case False
      then have ∃x. x ∈ division-points (cbox a b) d
        by auto
      then guess k c
        unfolding split-paired-Ex division-points-def mem-Collect-eq split-conv
        apply (elim exE conjE)
        done
      note this(2−4,1) note kc=this[unfolded interval-bounds[OF ab']]
      from this(3) guess j .. note j=this
      def d1 ≡ {l ∩ {x. x•k ≤ c} | l. l ∈ d ∧ l ∩ {x. x•k ≤ c} ≠ {}}
      def d2 ≡ {l ∩ {x. x•k ≥ c} | l. l ∈ d ∧ l ∩ {x. x•k ≥ c} ≠ {}}
      def cb ≡ (∑ i∈Basis. (if i = k then c else b•i) *R i)::'a
      def ca ≡ (∑ i∈Basis. (if i = k then c else a•i) *R i)::'a
      note division-points-psubset[OF ‹d division-of cbox a b› ab kc(1−2) j]
      note psubset-card-mono[OF - this(1)] psubset-card-mono[OF - this(2)]
      then have *: (iterate opp d1 f) = f (cbox a b ∩ {x. x•k ≤ c})
        (iterate opp d2 f) = f (cbox a b ∩ {x. x•k ≥ c})
        unfolding interval-split[OF kc(4)]
        apply (rule-tac[!] 1.hyps[rule-format])
        using division-split[OF ‹d division-of cbox a b›, where k=k and c=c]
    apply (simp-all add: interval-split 1 kc d1-def d2-def division-points-finite[OF
‹d division-of cbox a b›])
      done
```

```
      { fix l y
        assume as: l ∈ d y ∈ d l ∩ {x. x • k ≤ c} = y ∩ {x. x • k ≤ c} l ≠ y
         from division-ofD(4)[OF ‹d division-of cbox a b› this(1)] guess u v by
(elim exE) note leq=this
        have f (l ∩ {x. x • k ≤ c}) = neutral opp
          unfolding leq interval-split[OF kc(4)]
          apply (rule operativeD(1) 1)+
          unfolding interval-split[symmetric,OF kc(4)]
          using division-split-left-inj 1 as kc leq by blast
      } note fxk-le = this
      { fix l y
        assume as: l ∈ d y ∈ d l ∩ {x. c ≤ x • k} = y ∩ {x. c ≤ x • k} l ≠ y
         from division-ofD(4)[OF ‹d division-of cbox a b› this(1)] guess u v by
(elim exE) note leq=this
        have f (l ∩ {x. x • k ≥ c}) = neutral opp
          unfolding leq interval-split[OF kc(4)]
          apply (rule operativeD(1) 1)+
          unfolding interval-split[symmetric,OF kc(4)]
          using division-split-right-inj 1 leq as kc by blast
      } note fxk-ge = this
      have f (cbox a b) = opp (iterate opp d1 f) (iterate opp d2 f) (is - = ?prev)
        unfolding ∗
        using assms(2) kc(4) by blast
      also have iterate opp d1 f = iterate opp d (λl. f(l ∩ {x. x•k ≤ c}))
        unfolding d1-def empty-as-interval
        apply (rule iterate-nonzero-image-lemma[unfolded o-def])
        apply (rule 1 division-of-finite operativeD[OF 1(3)])+
        apply (force simp add: empty-as-interval[symmetric] fxk-le)+
        done
      also have iterate opp d2 f = iterate opp d (λl. f(l ∩ {x. x•k ≥ c}))
        unfolding d2-def empty-as-interval
        apply (rule iterate-nonzero-image-lemma[unfolded o-def])
        apply (rule 1 division-of-finite operativeD[OF 1(3)])+
        apply (force simp add: empty-as-interval[symmetric] fxk-ge)+
        done
      also have ∗: ∀x∈d. f x = opp (f (x ∩ {x. x • k ≤ c})) (f (x ∩ {x. c ≤ x
• k}))
        unfolding forall-in-division[OF ‹d division-of cbox a b›]
        using assms(2) kc(4) by blast
       have opp (iterate opp d (λl. f (l ∩ {x. x • k ≤ c}))) (iterate opp d (λl. f
(l ∩ {x. c ≤ x • k}))) =
        iterate opp d f
        apply (subst(3) iterate-eq[OF - ∗[rule-format]])
        using 1
        apply (auto simp: iterate-op[symmetric])
        done
      finally show ?thesis by auto
    qed
  qed
```

qed
qed

lemma *iterate-image-nonzero*:
assumes *monoidal opp*
and *finite s*
and $\bigwedge x\ y.\ \forall x{\in}s.\ \forall y{\in}s.\ x \neq y \wedge f\ x = f\ y \longrightarrow g\ (f\ x) = neutral\ opp$
shows *iterate opp* $(f\ `\ s)\ g = iterate\ opp\ s\ (g \circ f)$
using *assms*
by (*induct rule*: *finite-subset-induct*[*OF assms*(*2*) *subset-refl*]) *auto*

lemma *operative-tagged-division*:
assumes *monoidal opp*
and *operative opp f*
and *d tagged-division-of* (*cbox a b*)
shows *iterate opp d* $(\lambda(x,l).\ f\ l) = f\ (cbox\ a\ b)$
proof −
have $*$: $(\lambda(x,l).\ f\ l) = f \circ snd$
unfolding *o-def* **by** *rule auto* **note** *tagged* = *tagged-division-ofD*[*OF assms*(*3*)]
{ fix *a b a′*
assume *as*: $(a,\ b) \in d\ (a',\ b) \in d\ (a,\ b) \neq (a',\ b)$
have *f b* = *neutral opp*
using *tagged*(*4*)[*OF as*(*1*)]
apply *clarify*
apply (*rule operativeD*(*1*)[*OF assms*(*2*)])
by (*metis content-eq-0-interior inf.idem tagged-division-ofD*(*5*)[*OF assms*(*3*) *as*(*1*−*3*)])
}
then have *iterate opp d* $(\lambda(x,l).\ f\ l) = iterate\ opp\ (snd\ `\ d)\ f$
unfolding $*$
by (*force intro*!: *assms iterate-image-nonzero*[*symmetric*, *OF - tagged-division-of-finite*])
also have $\ldots = f\ (cbox\ a\ b)$
using *operative-division*[*OF assms*(*1*−*2*) *division-of-tagged-division*[*OF assms*(*3*)]]
.
finally show *?thesis* **.**
qed

41.23 Additivity of content.

lemma *setsum-iterate*:
assumes *finite s*
shows *setsum f s* = *iterate op* + *s f*
proof −
have *setsum f s* = *setsum f* (*support op* + *f s*)
using *assms*
by (*auto simp*: *support-def intro*: *setsum.mono-neutral-right*)
then show *?thesis* **unfolding** *iterate-def fold′-def setsum.eq-fold*
by (*simp add*: *comp-def*)
qed

lemma *additive-content-division*:
d division-of (*cbox a b*) ⟹ *setsum content d* = *content* (*cbox a b*)
by (*metis division-ofD*(1) *monoidal-monoid operative-content operative-division setsum-iterate*)

lemma *additive-content-tagged-division*:
d tagged-division-of (*cbox a b*) ⟹ *setsum* (λ(*x*,*l*). *content l*) *d* = *content* (*cbox a b*)
unfolding *operative-tagged-division*[*OF monoidal-monoid operative-content assms*,*symmetric*]
using *setsum-iterate* **by** *blast*

41.24 Finally, the integral of a constant

lemma *has-integral-const* [*intro*]:
fixes *a b* :: *'a*::*euclidean-space*
shows ((λ*x*. *c*) *has-integral* (*content* (*cbox a b*) ∗$_R$ *c*)) (*cbox a b*)
apply (*auto intro*!: *exI* [**where** *x*=λ*x*. *ball x 1*] *simp*: *split-def has-integral*)
apply (*subst scaleR-left.setsum*[*symmetric*, *unfolded o-def*])
apply (*subst additive-content-tagged-division*[*unfolded split-def*])
apply *auto*
done

lemma *has-integral-const-real* [*intro*]:
fixes *a b* :: *real*
shows ((λ*x*. *c*) *has-integral* (*content* {*a* .. *b*} ∗$_R$ *c*)) {*a* .. *b*}
by (*metis box-real*(2) *has-integral-const*)

lemma *integral-const* [*simp*]:
fixes *a b* :: *'a*::*euclidean-space*
shows *integral* (*cbox a b*) (λ*x*. *c*) = *content* (*cbox a b*) ∗$_R$ *c*
by (*rule integral-unique*) (*rule has-integral-const*)

lemma *integral-const-real* [*simp*]:
fixes *a b* :: *real*
shows *integral* {*a* .. *b*} (λ*x*. *c*) = *content* {*a* .. *b*} ∗$_R$ *c*
by (*metis box-real*(2) *integral-const*)

41.25 Bounds on the norm of Riemann sums and the integral itself.

lemma *dsum-bound*:
assumes *p division-of* (*cbox a b*)
and *norm c* ≤ *e*
shows *norm* (*setsum* (λ*l*. *content l* ∗$_R$ *c*) *p*) ≤ *e* ∗ *content*(*cbox a b*)
proof −
have *sumeq*: (∑ *i*∈*p*. |*content i*|) = *setsum content p*
apply (*rule setsum.cong*)
using *assms*

```
    apply simp
    apply (metis abs-of-nonneg assms(1) content-pos-le division-ofD(4))
    done
  have e: 0 ≤ e
    using assms(2) norm-ge-zero order-trans by blast
  have norm (setsum (λl. content l *R c) p) ≤ (∑ i∈p. norm (content i *R c))
    using norm-setsum by blast
  also have ... ≤ e * (∑ i∈p. |content i|)
    apply (simp add: setsum-right-distrib[symmetric] mult.commute)
    using assms(2) mult-right-mono by blast
  also have ... ≤ e * content (cbox a b)
    apply (rule mult-left-mono [OF - e])
    apply (simp add: sumeq)
    using additive-content-division assms(1) eq-iff apply blast
    done
  finally show ?thesis .
qed

lemma rsum-bound:
  assumes p: p tagged-division-of (cbox a b)
      and ∀ x∈cbox a b. norm (f x) ≤ e
    shows norm (setsum (λ(x,k). content k *R f x) p) ≤ e * content (cbox a b)
proof (cases cbox a b = {})
  case True show ?thesis
    using p unfolding True tagged-division-of-trivial by auto
next
  case False
  then have e: e ≥ 0
    by (metis assms(2) norm-ge-zero order-trans nonempty-witness)
  have setsum-le: setsum (content ∘ snd) p ≤ content (cbox a b)
    unfolding additive-content-tagged-division[OF p, symmetric] split-def
    by (auto intro: eq-refl)
  have con: ⋀xk. xk ∈ p ⟹ 0 ≤ content (snd xk)
    using tagged-division-ofD(4) [OF p] content-pos-le
    by force
  have norm: ⋀xk. xk ∈ p ⟹ norm (f (fst xk)) ≤ e
    unfolding fst-conv using tagged-division-ofD(2,3)[OF p] assms
    by (metis prod.collapse subset-eq)
  have norm (setsum (λ(x,k). content k *R f x) p) ≤ (∑ i∈p. norm (case i of (x,
k) ⇒ content k *R f x))
    by (rule norm-setsum)
  also have ... ≤ e * content (cbox a b)
    unfolding split-def norm-scaleR
    apply (rule order-trans[OF setsum-mono])
    apply (rule mult-left-mono[OF - abs-ge-zero, of - e])
    apply (metis norm)
    unfolding setsum-left-distrib[symmetric]
    using con setsum-le
    apply (auto simp: mult.commute intro: mult-left-mono [OF - e])
```

```
    done
  finally show ?thesis .
qed

lemma rsum-diff-bound:
  assumes p tagged-division-of (cbox a b)
    and ∀x∈cbox a b. norm (f x − g x) ≤ e
  shows norm (setsum (λ(x,k). content k *R f x) p − setsum (λ(x,k). content k
*R g x) p) ≤
        e * content (cbox a b)
  apply (rule order-trans[OF - rsum-bound[OF assms]])
  apply (simp add: split-def scaleR-diff-right setsum-subtractf eq-refl)
  done

lemma has-integral-bound:
  fixes f :: 'a::euclidean-space ⇒ 'b::real-normed-vector
  assumes 0 ≤ B
      and (f has-integral i) (cbox a b)
      and ∀x∈cbox a b. norm (f x) ≤ B
    shows norm i ≤ B * content (cbox a b)
proof (rule ccontr)
  assume ¬ ?thesis
  then have *: norm i − B * content (cbox a b) > 0
    by auto
  from assms(2)[unfolded has-integral,rule-format,OF *]
  guess d by (elim exE conjE) note d=this[rule-format]
  from fine-division-exists[OF this(1), of a b] guess p . note p=this
  have *: ⋀s B. norm s ≤ B ⟹ ¬ norm (s − i) < norm i − B
    unfolding not-less
   by (metis norm-triangle-sub[of i] add.commute le-less-trans less-diff-eq linorder-not-le
norm-minus-commute)
  show False
    using d(2)[OF conjI[OF p]] *[OF rsum-bound[OF p(1) assms(3)]] by auto
qed

corollary has-integral-bound-real:
  fixes f :: real ⇒ 'b::real-normed-vector
  assumes 0 ≤ B
      and (f has-integral i) {a .. b}
      and ∀x∈{a .. b}. norm (f x) ≤ B
    shows norm i ≤ B * content {a .. b}
  by (metis assms box-real(2) has-integral-bound)

corollary integrable-bound:
  fixes f :: 'a::euclidean-space ⇒ 'b::real-normed-vector
  assumes 0 ≤ B
      and f integrable-on (cbox a b)
      and ⋀x. x∈cbox a b ⟹ norm (f x) ≤ B
    shows norm (integral (cbox a b) f) ≤ B * content (cbox a b)
```

by (*metis integrable-integral has-integral-bound assms*)

41.26 Similar theorems about relationship among components.

lemma *rsum-component-le*:
fixes $f :: 'a::euclidean\text{-}space \Rightarrow 'b::euclidean\text{-}space$
assumes *p tagged-division-of* (*cbox a b*)
and $\forall x{\in}cbox\ a\ b.\ (f\ x){\cdot}i \le (g\ x){\cdot}i$
shows (*setsum* ($\lambda(x,k).$ *content k* $*_R$ *f x*) *p*)$\cdot i \le$ (*setsum* ($\lambda(x,k).$ *content k* $*_R$ *g x*) *p*)$\cdot i$
unfolding *inner-setsum-left*
proof (*rule setsum-mono*, *clarify*)
fix *a b*
assume *ab*: $(a, b) \in p$
note *tagged* = *tagged-division-ofD*(*2*−*4*)[*OF assms*(*1*) *ab*]
from *this*(*3*) **guess** *u v* **by** (*elim exE*) **note** *b*=*this*
show (*content b* $*_R$ *f a*) $\cdot$ *i* $\le$ (*content b* $*_R$ *g a*) $\cdot$ *i*
unfolding *b inner-simps real-scaleR-def*
apply (*rule mult-left-mono*)
using *assms*(*2*) *tagged*
by (*auto simp add*: *content-pos-le*)
qed

lemma *has-integral-component-le*:
fixes *f g* :: $'a::euclidean\text{-}space \Rightarrow 'b::euclidean\text{-}space$
assumes *k*: $k \in Basis$
assumes (*f has-integral i*) *s* (*g has-integral j*) *s*
and $\forall x{\in}s.\ (f\ x){\cdot}k \le (g\ x){\cdot}k$
shows $i{\cdot}k \le j{\cdot}k$
proof −
have *lem*: $i{\cdot}k \le j{\cdot}k$
if *f-i*: (*f has-integral i*) (*cbox a b*)
and *g-j*: (*g has-integral j*) (*cbox a b*)
and *le*: $\forall x{\in}cbox\ a\ b.\ (f\ x){\cdot}k \le (g\ x){\cdot}k$
for *a b i* **and** *j* :: $'b$ **and** *f g* :: $'a \Rightarrow 'b$
proof (*rule ccontr*)
assume ¬ *?thesis*
then have ∗: $0 < (i{\cdot}k - j{\cdot}k)\ /\ 3$
by *auto*
guess *d1* **using** *f-i*[*unfolded has-integral*,*rule-format*,*OF* ∗] **by** (*elim exE conjE*)
note *d1*=*this*[*rule-format*]
guess *d2* **using** *g-j*[*unfolded has-integral*,*rule-format*,*OF* ∗] **by** (*elim exE conjE*)
note *d2*=*this*[*rule-format*]
obtain *p* **where** *p*: *p tagged-division-of cbox a b d1 fine p d2 fine p*
using *fine-division-exists*[*OF gauge-inter*[*OF d1*(*1*) *d2*(*1*)], *of a b*] **unfolding** *fine-inter*
by *metis*
note *le-less-trans*[*OF Basis-le-norm*[*OF k*]]

```
    then have |((∑(x, k)∈p. content k *R f x) − i) · k| < (i · k − j · k) / 3
            |((∑(x, k)∈p. content k *R g x) − j) · k| < (i · k − j · k) / 3
      using k norm-bound-Basis-lt d1 d2 p
      by blast+
    then show False
      unfolding inner-simps
      using rsum-component-le[OF p(1) le]
      by (simp add: abs-real-def split: if-split-asm)
  qed
  show ?thesis
  proof (cases ∃ a b. s = cbox a b)
    case True
    with lem assms show ?thesis
      by auto
  next
    case False
    show ?thesis
    proof (rule ccontr)
      assume ¬ i·k ≤ j·k
      then have ij: (i·k − j·k) / 3 > 0
        by auto
      note has-integral-altD[OF - False this]
    from this[OF assms(2)] this[OF assms(3)] guess B1 B2 . note B=this[rule-format]
      have bounded (ball 0 B1 ∪ ball (0::'a) B2)
        unfolding bounded-Un by(rule conjI bounded-ball)+
      from bounded-subset-cbox[OF this] guess a b by (elim exE)
      note ab = conjunctD2[OF this[unfolded Un-subset-iff]]
      guess w1 using B(2)[OF ab(1)] .. note w1=conjunctD2[OF this]
      guess w2 using B(4)[OF ab(2)] .. note w2=conjunctD2[OF this]
     have *: ⋀w1 w2 j i::real .|w1 − i| < (i − j) / 3 ⟹ |w2 − j| < (i − j) / 3
⟹ w1 ≤ w2 ⟹ False
        by (simp add: abs-real-def split: if-split-asm)
      note le-less-trans[OF Basis-le-norm[OF k]]
      note this[OF w1(2)] this[OF w2(2)]
      moreover
      have w1·k ≤ w2·k
        by (rule lem[OF w1(1) w2(1)]) (simp add: assms(4))
      ultimately show False
        unfolding inner-simps by(rule *)
    qed
  qed
qed

lemma integral-component-le:
  fixes g f :: 'a::euclidean-space ⇒ 'b::euclidean-space
  assumes k ∈ Basis
    and f integrable-on s g integrable-on s
    and ∀ x∈s. (f x)·k ≤ (g x)·k
  shows (integral s f)·k ≤ (integral s g)·k
```

apply (*rule has-integral-component-le*)
using *integrable-integral assms*
apply *auto*
done

lemma *has-integral-component-nonneg*:
fixes $f :: 'a::euclidean\text{-}space \Rightarrow 'b::euclidean\text{-}space$
assumes $k \in Basis$
and $(f\ has\text{-}integral\ i)\ s$
and $\forall x{\in}s.\ 0 \le (f\ x){\cdot}k$
shows $0 \le i{\cdot}k$
using *has-integral-component-le*[*OF assms*(*1*) *has-integral-0 assms*(*2*)]
using *assms*(*3*−)
by *auto*

lemma *integral-component-nonneg*:
fixes $f :: 'a::euclidean\text{-}space \Rightarrow 'b::euclidean\text{-}space$
assumes $k \in Basis$
and $\forall x{\in}s.\ 0 \le (f\ x){\cdot}k$
shows $0 \le (integral\ s\ f){\cdot}k$
proof (*cases f integrable-on s*)
case *True* **show** *?thesis*
apply (*rule has-integral-component-nonneg*)
using *assms True*
apply *auto*
done
next
case *False* **then show** *?thesis* **by** (*simp add*: *not-integrable-integral*)
qed

lemma *has-integral-component-neg*:
fixes $f :: 'a::euclidean\text{-}space \Rightarrow 'b::euclidean\text{-}space$
assumes $k \in Basis$
and $(f\ has\text{-}integral\ i)\ s$
and $\forall x{\in}s.\ (f\ x){\cdot}k \le 0$
shows $i{\cdot}k \le 0$
using *has-integral-component-le*[*OF assms*(*1*,*2*) *has-integral-0*] *assms*(*2*−)
by *auto*

lemma *has-integral-component-lbound*:
fixes $f :: 'a::euclidean\text{-}space \Rightarrow 'b::euclidean\text{-}space$
assumes $(f\ has\text{-}integral\ i)\ (cbox\ a\ b)$
and $\forall x{\in}cbox\ a\ b.\ B \le f(x){\cdot}k$
and $k \in Basis$
shows $B * content\ (cbox\ a\ b) \le i{\cdot}k$
using *has-integral-component-le*[*OF assms*(*3*) *has-integral-const assms*(*1*),*of* $(\sum i{\in}Basis.\ B *_R i)::'b$] *assms*(*2*−)
by (*auto simp add*: *field-simps*)

lemma *has-integral-component-ubound*:
fixes f::*'a*::*euclidean-space* => *'b*::*euclidean-space*
assumes (*f has-integral i*) (*cbox a b*)
and $\forall x \in cbox\ a\ b.\ f\ x \cdot k \leq B$
and $k \in Basis$
shows $i \cdot k \leq B * content\ (cbox\ a\ b)$
using *has-integral-component-le*[*OF assms*(*3*,*1*) *has-integral-const*, *of* $\sum i \in Basis.\ B *_R i$] *assms*(*2*−)
by (*auto simp add*: *field-simps*)

lemma *integral-component-lbound*:
fixes f :: *'a*::*euclidean-space* ⇒ *'b*::*euclidean-space*
assumes *f integrable-on cbox a b*
and $\forall x \in cbox\ a\ b.\ B \leq f(x) \cdot k$
and $k \in Basis$
shows $B * content\ (cbox\ a\ b) \leq (integral(cbox\ a\ b)\ f) \cdot k$
apply (*rule has-integral-component-lbound*)
using *assms*
unfolding *has-integral-integral*
apply *auto*
done

lemma *integral-component-lbound-real*:
assumes *f integrable-on* {*a* ::*real* .. *b*}
and $\forall x \in \{a\ ..\ b\}.\ B \leq f(x) \cdot k$
and $k \in Basis$
shows $B * content\ \{a\ ..\ b\} \leq (integral\ \{a\ ..\ b\}\ f) \cdot k$
using *assms*
by (*metis box-real*(*2*) *integral-component-lbound*)

lemma *integral-component-ubound*:
fixes f :: *'a*::*euclidean-space* ⇒ *'b*::*euclidean-space*
assumes *f integrable-on cbox a b*
and $\forall x \in cbox\ a\ b.\ f\ x \cdot k \leq B$
and $k \in Basis$
shows $(integral\ (cbox\ a\ b)\ f) \cdot k \leq B * content\ (cbox\ a\ b)$
apply (*rule has-integral-component-ubound*)
using *assms*
unfolding *has-integral-integral*
apply *auto*
done

lemma *integral-component-ubound-real*:
fixes f :: *real* ⇒ *'a*::*euclidean-space*
assumes *f integrable-on* {*a* .. *b*}
and $\forall x \in \{a\ ..\ b\}.\ f\ x \cdot k \leq B$
and $k \in Basis$
shows $(integral\ \{a\ ..\ b\}\ f) \cdot k \leq B * content\ \{a\ ..\ b\}$
using *assms*

by (*metis box-real*(*2*) *integral-component-ubound*)

41.27 Uniform limit of integrable functions is integrable.

lemma *real-arch-invD*:
0 < (*e*::*real*) ⟹ (∃ *n*::*nat*. *n* ≠ *0* ∧ *0* < *inverse* (*real n*) ∧ *inverse* (*real n*) < *e*)
by (*subst*(*asm*) *real-arch-inverse*)

lemma *integrable-uniform-limit*:
fixes *f* :: '*a*::*euclidean-space* ⇒ '*b*::*banach*
assumes ∀ *e*>*0*. ∃ *g*. (∀ *x*∈*cbox a b*. *norm* (*f x* − *g x*) ≤ *e*) ∧ *g integrable-on cbox a b*
shows *f integrable-on cbox a b*
proof (*cases content* (*cbox a b*) > *0*)
case *False* **then show** *?thesis*
using *has-integral-null*
by (*simp add*: *content-lt-nz integrable-on-def*)
next
case *True*
have ∗: ⋀*P*. ∀ *e*>(*0*::*real*). *P e* ⟹ ∀ *n*::*nat*. *P* (*inverse* (*real n* + *1*))
by *auto*
from *choice*[*OF* ∗[*OF assms*]] **guess** *g* **.. note** *g*=*conjunctD2*[*OF this*[*rule-format*],*rule-format*]
from *choice*[*OF allI*[*OF g*(*2*)[*unfolded integrable-on-def*], *of* λ*x*. *x*]]
obtain *i* **where** *i*: ⋀*x*. (*g x has-integral i x*) (*cbox a b*)
by *auto*
have *Cauchy i*
unfolding *Cauchy-def*
proof *clarify*
fix *e* :: *real*
assume *e*>*0*
then have *e* / *4* / *content* (*cbox a b*) > *0*
using *True* **by** (*auto simp add*: *field-simps*)
then obtain *M* :: *nat*
where *M*: *M* ≠ *0* *0* < *inverse* (*real-of-nat M*) *inverse* (*of-nat M*) < *e* / *4* / *content* (*cbox a b*)
by (*subst* (*asm*) *real-arch-inverse*) *auto*
show ∃ *M*. ∀ *m*≥*M*. ∀ *n*≥*M*. *dist* (*i m*) (*i n*) < *e*
proof (*rule exI* [**where** *x*=*M*], *clarify*)
fix *m n*
assume *m*: *M* ≤ *m* **and** *n*: *M* ≤ *n*
have *e*/*4*>*0* **using** ‹*e*>*0*› **by** *auto*
note ∗ = *i*[*unfolded has-integral*,*rule-format*,*OF this*]
from ∗[*of m*] **guess** *gm* **by** (*elim conjE exE*) **note** *gm*=*this*[*rule-format*]
from ∗[*of n*] **guess** *gn* **by** (*elim conjE exE*) **note** *gn*=*this*[*rule-format*]
from *fine-division-exists*[*OF gauge-inter*[*OF gm*(*1*) *gn*(*1*)], *of a b*]
obtain *p* **where** *p*: *p tagged-division-of cbox a b* (λ*x*. *gm x* ∩ *gn x*) *fine p*
by *auto*
{ fix *s1 s2 i1* **and** *i2*::'*b*

```
      assume no: norm(s2 − s1) ≤ e/2 norm (s1 − i1) < e/4 norm (s2 − i2)
< e/4
      have norm (i1 − i2) ≤ norm (i1 − s1) + norm (s1 − s2) + norm (s2 −
i2)
        using norm-triangle-ineq[of i1 − s1 s1 − i2]
        using norm-triangle-ineq[of s1 − s2 s2 − i2]
        by (auto simp add: algebra-simps)
      also have ... < e
        using no
        unfolding norm-minus-commute
        by (auto simp add: algebra-simps)
      finally have norm (i1 − i2) < e .
    } note triangle3 = this
    have finep: gm fine p gn fine p
      using fine-inter p  by auto
    { fix x
      assume x: x ∈ cbox a b
      have norm (f x − g n x) + norm (f x − g m x) ≤ inverse (real n + 1) +
inverse (real m + 1)
        using g(1)[OF x, of n] g(1)[OF x, of m] by auto
      also have ... ≤ inverse (real M) + inverse (real M)
        apply (rule add-mono)
        using M(2) m n by auto
      also have ... = 2 / real M
        unfolding divide-inverse by auto
      finally have norm (g n x − g m x) ≤ 2 / real M
        using norm-triangle-le[of g n x − f x f x − g m x 2 / real M]
        by (auto simp add: algebra-simps simp add: norm-minus-commute)
    } note norm-le = this
    have le-e2: norm ((∑(x, k)∈p. content k *R g n x) − (∑(x, k)∈p. content
k *R g m x)) ≤ e / 2
      apply (rule order-trans [OF rsum-diff-bound[OF p(1), where e=2 / real
M]])
      apply (blast intro: norm-le)
      using M True
      by (auto simp add: field-simps)
    then show dist (i m) (i n) < e
      unfolding dist-norm
      using gm gn p finep
      by (auto intro!: triangle3)
  qed
 qed
 then obtain s where s: i ⟶ s
  using convergent-eq-cauchy[symmetric] by blast
 show ?thesis
  unfolding integrable-on-def has-integral
 proof (rule-tac x=s in exI, clarify)
  fix e::real
  assume e: 0 < e
```

then have $*$: $e/3 > 0$ **by** *auto*
then obtain *N1* **where** *N1*: $\forall\, n{\geq}N1.\ norm\ (i\ n - s) < e\ /\ 3$
using *LIMSEQ-D* [*OF s*] **by** *metis*
from *e True* **have** $e\ /\ 3\ /\ content\ (cbox\ a\ b) > 0$
by (*auto simp add*: *field-simps*)
from *real-arch-invD*[*OF this*] **guess** *N2* **by** (*elim exE conjE*) **note** *N2*=*this*
from *i*[*of N1 + N2*,*unfolded has-integral*,*rule-format*,*OF* $*$] **guess** g' **.. note** g'=*conjunctD2*[*OF this*,*rule-format*]
{ fix *sf sg i*
assume *no*: $norm\ (sf - sg) \leq e\ /\ 3$
$norm(i - s) < e\ /\ 3$
$norm\ (sg - i) < e\ /\ 3$
have $norm\ (sf - s) \leq norm\ (sf - sg) + norm\ (sg - i) + norm\ (i - s)$
using *norm-triangle-ineq*[*of sf − sg sg − s*]
using *norm-triangle-ineq*[*of sg − i i − s*]
by (*auto simp add*: *algebra-simps*)
also have $\ldots < e$
using *no*
unfolding *norm-minus-commute*
by (*auto simp add*: *algebra-simps*)
finally have $norm\ (sf - s) < e$ **.**
} note *lem* = *this*
{ fix *p*
assume *p*: *p tagged-division-of* $(cbox\ a\ b) \wedge g'$ *fine p*
then have *norm-less*: $norm\ ((\sum (x,\ k){\in}p.\ content\ k *_R g\ (N1 + N2)\ x) - i\ (N1 + N2)) < e\ /\ 3$
using g' **by** *blast*
have $content\ (cbox\ a\ b) < e\ /\ 3 * (of\text{-}nat\ N2)$
using *N2* **unfolding** *inverse-eq-divide* **using** *True* **by** (*auto simp add*: *field-simps*)
moreover have $e\ /\ 3 * of\text{-}nat\ N2 \leq e\ /\ 3 * (of\text{-}nat\ (N1 + N2) + 1)$
using ‹*e*>*0*› **by** *auto*
ultimately have $content\ (cbox\ a\ b) < e\ /\ 3 * (of\text{-}nat\ (N1 + N2) + 1)$
by *linarith*
then have *le-e3*: $inverse\ (real\ (N1 + N2) + 1) * content\ (cbox\ a\ b) \leq e\ /\ 3$
unfolding *inverse-eq-divide*
by (*auto simp add*: *field-simps*)
have *ne3*: $norm\ (i\ (N1 + N2) - s) < e\ /\ 3$
using *N1* **by** *auto*
have $norm\ ((\sum (x,\ k){\in}p.\ content\ k *_R f\ x) - s) < e$
apply (*rule lem*[*OF order-trans* [*OF - le-e3*] *ne3 norm-less*])
apply (*rule rsum-diff-bound*[*OF p*[*THEN conjunct1*]])
apply (*blast intro*: *g*)
done }
then show $\exists\, d.\ gauge\ d \wedge$
$(\forall\, p.\ p\ tagged\text{-}division\text{-}of\ cbox\ a\ b \wedge d\ fine\ p \longrightarrow norm\ ((\sum (x,\ k){\in}p.\ content\ k *_R f\ x) - s) < e)$
by (*blast intro*: g')
qed

qed

lemmas *integrable-uniform-limit-real* = *integrable-uniform-limit* [**where** *'a=real*, *simplified*]

41.28 Negligible sets.

definition *negligible* (*s*:: *'a*::*euclidean-space set*) ⟷
(∀ *a b*. ((*indicator s* :: *'a*⇒*real*) *has-integral 0*) (*cbox a b*))

41.29 Negligibility of hyperplane.

lemma *setsum-nonzero-image-lemma*:
assumes *finite s*
and *g a* = *0*
and ∀ *x*∈*s*. ∀ *y*∈*s*. *f x* = *f y* ∧ *x* ≠ *y* ⟶ *g* (*f x*) = *0*
shows *setsum g* {*f x* |*x*. *x* ∈ *s* ∧ *f x* ≠ *a*} = *setsum* (*g* ∘ *f*) *s*
apply (*subst setsum-iterate*)
using *assms monoidal-monoid*
unfolding *setsum-iterate*[*OF assms*(*1*)]
apply (*auto intro*!: *iterate-nonzero-image-lemma*)
done

lemma *interval-doublesplit*:
fixes *a* :: *'a*::*euclidean-space*
assumes *k* ∈ *Basis*
shows *cbox a b* ∩ {*x* . |*x*•*k* − *c*| ≤ (*e*::*real*)} =
cbox (∑ *i*∈*Basis*. (*if i* = *k then max* (*a*•*k*) (*c* − *e*) *else a*•*i*) ∗*R i*)
(∑ *i*∈*Basis*. (*if i* = *k then min* (*b*•*k*) (*c* + *e*) *else b*•*i*) ∗*R i*)
proof −
have ∗: ⋀*x c e*::*real*. |*x* − *c*| ≤ *e* ⟷ *x* ≥ *c* − *e* ∧ *x* ≤ *c* + *e*
by *auto*
have ∗∗: ⋀*s P Q*. *s* ∩ {*x*. *P x* ∧ *Q x*} = (*s* ∩ {*x*. *Q x*}) ∩ {*x*. *P x*}
by *blast*
show *?thesis*
unfolding ∗ ∗∗ *interval-split*[*OF assms*] **by** (*rule refl*)
qed

lemma *division-doublesplit*:
fixes *a* :: *'a*::*euclidean-space*
assumes *p division-of* (*cbox a b*)
and *k*: *k* ∈ *Basis*
shows {*l* ∩ {*x*. |*x*•*k* − *c*| ≤ *e*} |*l*. *l* ∈ *p* ∧ *l* ∩ {*x*. |*x*•*k* − *c*| ≤ *e*} ≠ {}}
division-of (*cbox a b* ∩ {*x*. |*x*•*k* − *c*| ≤ *e*})
proof −
have ∗: ⋀*x c*. |*x* − *c*| ≤ *e* ⟷ *x* ≥ *c* − *e* ∧ *x* ≤ *c* + *e*
by *auto*
have ∗∗: ⋀*p q p*′ *q*′. *p division-of q* ⟹ *p* = *p*′ ⟹ *q* = *q*′ ⟹ *p*′ *division-of q*′
by *auto*
note *division-split*(*1*)[*OF assms*, **where** *c*=*c*+*e*,*unfolded interval-split*[*OF k*]]

```
  note division-split(2)[OF this, where c=c−e and k=k,OF k]
  then show ?thesis
    apply (rule **)
    unfolding interval-doublesplit [OF k]
    using k
    apply (simp-all add: * interval-split)
    apply (rule equalityI, blast)
    apply clarsimp
    apply (rule-tac x=l ∩ {x. c + e ≥ x · k} in exI)
    apply auto
    done
qed

lemma content-doublesplit:
  fixes a :: 'a::euclidean-space
  assumes 0 < e
    and k: k ∈ Basis
  obtains d where 0 < d and content (cbox a b ∩ {x. |x·k − c| ≤ d}) < e
proof (cases content (cbox a b) = 0)
  case True
  then have ce: content (cbox a b) < e
    by (metis ‹0 < e›)
  show ?thesis
    apply (rule that[of 1])
    apply simp
    unfolding interval-doublesplit[OF k]
    apply (rule le-less-trans[OF content-subset ce])
    apply (auto simp: interval-doublesplit[symmetric] k)
    done
next
  case False
  def d ≡ e / 3 / setprod (λi. b·i − a·i) (Basis − {k})
  note False[unfolded content-eq-0 not-ex not-le, rule-format]
  then have ⋀x. x ∈ Basis ⟹ b·x > a·x
    by (auto simp add:not-le)
  then have prod0: 0 < setprod (λi. b·i − a·i) (Basis − {k})
    by (force simp add: setprod-pos field-simps)
  then have d > 0
    using assms
    by (auto simp add: d-def field-simps)
  then show ?thesis
  proof (rule that[of d])
    have *: Basis = insert k (Basis − {k})
      using k by auto
    have less-e: (min (b · k) (c + d) − max (a · k) (c − d)) * (∏ i∈Basis − {k}.
b · i − a · i) < e
    proof −
      have (min (b · k) (c + d) − max (a · k) (c − d)) ≤ 2 * d
        by auto
```

```
    also have ... < e / (∏ i∈Basis - {k}. b · i - a · i)
      unfolding d-def
      using assms prod0
      by (auto simp add: field-simps)
    finally show (min (b · k) (c + d) - max (a · k) (c - d)) * (∏ i∈Basis -
{k}. b · i - a · i) < e
      unfolding pos-less-divide-eq[OF prod0] .
  qed
  show content (cbox a b ∩ {x. |x · k - c| ≤ d}) < e
  proof (cases cbox a b ∩ {x. |x · k - c| ≤ d} = {})
    case True
    then show ?thesis
      using assms by simp
  next
    case False
    then have
        (∏ i∈Basis - {k}. interval-upperbound (cbox a b ∩ {x. |x · k - c| ≤ d})
· i -
              interval-lowerbound (cbox a b ∩ {x. |x · k - c| ≤ d}) · i)
        = (∏ i∈Basis - {k}. b·i - a·i)
      by (simp add: box-eq-empty interval-doublesplit[OF k])
    then show content (cbox a b ∩ {x. |x · k - c| ≤ d}) < e
      unfolding content-def
      using assms False
      apply (subst *)
      apply (subst setprod.insert)
     apply (simp-all add: interval-doublesplit[OF k] box-eq-empty not-less less-e)
      done
  qed
 qed
qed

lemma negligible-standard-hyperplane[intro]:
 fixes k :: 'a::euclidean-space
 assumes k: k ∈ Basis
 shows negligible {x. x·k = c}
 unfolding negligible-def has-integral
proof (clarify, goal-cases)
 case (1 a b e)
 from this and k obtain d where d: 0 < d content (cbox a b ∩ {x. |x · k - c|
≤ d}) < e
   by (rule content-doublesplit)
 let ?i = indicator {x::'a. x·k = c} :: 'a⇒real
 show ?case
   apply (rule-tac x=λx. ball x d in exI)
   apply rule
   apply (rule gauge-ball)
   apply (rule d)
 proof (rule, rule)
```

fix p
assume p: p *tagged-division-of* $(cbox\ a\ b) \wedge (\lambda x.\ ball\ x\ d)$ *fine* p
have $*$: $(\sum (x,\ ka) \in p.\ content\ ka *_R\ ?i\ x) =$
$(\sum (x,\ ka) \in p.\ content\ (ka \cap \{x.\ |x \cdot k - c| \leq d\}) *_R\ ?i\ x)$
apply (*rule setsum.cong*)
apply (*rule refl*)
unfolding *split-paired-all real-scaleR-def mult-cancel-right split-conv*
apply *cases*
apply (*rule disjI1*)
apply *assumption*
apply (*rule disjI2*)
proof −
fix $x\ l$
assume *as*: $(x,\ l) \in p\ ?i\ x \neq 0$
then have *xk*: $x \cdot k = c$
unfolding *indicator-def*
apply −
apply (*rule ccontr*)
apply *auto*
done
show $content\ l = content\ (l \cap \{x.\ |x \cdot k - c| \leq d\})$
apply (*rule arg-cong*[**where** f=*content*])
apply (*rule set-eqI*)
apply *rule*
apply *rule*
unfolding *mem-Collect-eq*
proof −
fix y
assume y: $y \in l$
note *p*[*THEN conjunct2*,*unfolded fine-def*,*rule-format*,*OF as*(*1*),*unfolded split-conv*]
note *this*[*unfolded subset-eq mem-ball dist-norm*,*rule-format*,*OF y*]
note *le-less-trans*[*OF Basis-le-norm*[*OF k*] *this*]
then show $|y \cdot k - c| \leq d$
unfolding *inner-simps xk* **by** *auto*
qed *auto*
qed
note p'= *tagged-division-ofD*[*OF p*[*THEN conjunct1*]] **and** p''=*division-of-tagged-division*[*OF p*[*THEN conjunct1*]]
show $norm\ ((\sum (x,\ ka) \in p.\ content\ ka *_R\ ?i\ x) - 0) < e$
unfolding *diff-0-right* $*$
unfolding *real-scaleR-def real-norm-def*
apply (*subst abs-of-nonneg*)
apply (*rule setsum-nonneg*)
apply *rule*
unfolding *split-paired-all split-conv*
apply (*rule mult-nonneg-nonneg*)
apply (*drule* $p'(4)$)
apply (*erule exE*)+

apply(*rule-tac b=b* **in** *back-subst*)
prefer *2*
apply (*subst*(*asm*) *eq-commute*)
apply *assumption*
apply (*subst interval-doublesplit*[*OF k*])
apply (*rule content-pos-le*)
apply (*rule indicator-pos-le*)
proof −
have ($\sum$(*x*, *ka*)∈*p*. *content* (*ka* ∩ {*x*. |*x* · *k* − *c*| ≤ *d*}) ∗ *?i x*) ≤
($\sum$(*x*, *ka*)∈*p*. *content* (*ka* ∩ {*x*. |*x* · *k* − *c*| ≤ *d*}))
apply (*rule setsum-mono*)
unfolding *split-paired-all split-conv*
apply (*rule mult-right-le-one-le*)
apply (*drule p'*(*4*))
apply (*auto simp add*:*interval-doublesplit*[*OF k*])
done
also have ... < *e*
proof (*subst setsum-over-tagged-division-lemma*[*OF p*[*THEN conjunct1*]],
goal-cases)
case *prems*: (*1 u v*)
have *content* (*cbox u v* ∩ {*x*. |*x* · *k* − *c*| ≤ *d*}) ≤ *content* (*cbox u v*)
unfolding *interval-doublesplit*[*OF k*]
apply (*rule content-subset*)
unfolding *interval-doublesplit*[*symmetric*,*OF k*]
apply *auto*
done
then show *?case*
unfolding *prems interval-doublesplit*[*OF k*]
by (*blast intro*: *antisym*)
next
have ∗: *setsum content* {*l* ∩ {*x*. |*x* · *k* − *c*| ≤ *d*} |*l*. *l* ∈ *snd* ' *p* ∧ *l* ∩ {*x*.
|*x* · *k* − *c*| ≤ *d*} ≠ {}} ≥ *0*
apply (*rule setsum-nonneg*)
apply *rule*
unfolding *mem-Collect-eq image-iff*
apply (*erule exE bexE conjE*)+
unfolding *split-paired-all*
proof −
fix *x l a b*
assume *as*: *x* = *l* ∩ {*x*. |*x* · *k* − *c*| ≤ *d*} (*a*, *b*) ∈ *p l* = *snd* (*a*, *b*)
guess *u v* **using** *p'*(*4*)[*OF as*(*2*)] **by** (*elim exE*) **note** ∗ = *this*
show *content x* ≥ *0*
unfolding *as snd-conv* ∗ *interval-doublesplit*[*OF k*]
by (*rule content-pos-le*)
qed
have ∗∗: *norm* (*1*::*real*) ≤ *1*
by *auto*
note *division-doublesplit*[*OF p'' k*,*unfolded interval-doublesplit*[*OF k*]]
note *dsum-bound*[*OF this* ∗∗,*unfolded interval-doublesplit*[*symmetric*,*OF k*]]

note *this*[*unfolded real-scaleR-def real-norm-def mult-1-right mult-1*, *of c d*]
note *le-less-trans*[*OF this d*(*2*)]
from *this*[*unfolded abs-of-nonneg*[*OF* ∗]]
show $(\sum ka{\in}snd\ `\ p.\ content\ (ka \cap \{x.\ |x \cdot k - c| \leq d\})) < e$
apply (*subst setsum-nonzero-image-lemma*[*of snd ` p content* {}, *unfolded o-def*,*symmetric*])
apply (*rule finite-imageI p′ content-empty*)+
unfolding *forall-in-division*[*OF p′′*]
proof (*rule*,*rule*,*rule*,*rule*,*rule*,*rule*,*rule*,*erule conjE*)
fix *m n u v*
assume *as*:
cbox m n ∈ *snd ` p cbox u v* ∈ *snd ` p*
cbox m n ≠ *cbox u v*
$cbox\ m\ n \cap \{x.\ |x \cdot k - c| \leq d\} = cbox\ u\ v \cap \{x.\ |x \cdot k - c| \leq d\}$
have $(cbox\ m\ n \cap \{x.\ |x \cdot k - c| \leq d\}) \cap (cbox\ u\ v \cap \{x.\ |x \cdot k - c| \leq d\}) \subseteq cbox\ m\ n \cap cbox\ u\ v$
by *blast*
note *interior-mono*[*OF this*, *unfolded division-ofD*(*5*)[*OF p′′ as*(*1*−*3*)] *interior-Int*[*of cbox m n*]]
then have $interior\ (cbox\ m\ n \cap \{x.\ |x \cdot k - c| \leq d\}) = \{\}$
unfolding *as Int-absorb* **by** *auto*
then show $content\ (cbox\ m\ n \cap \{x.\ |x \cdot k - c| \leq d\}) = 0$
unfolding *interval-doublesplit*[*OF k*] *content-eq-0-interior*[*symmetric*] .
qed
qed
finally show $(\sum (x,\ ka){\in}p.\ content\ (ka \cap \{x.\ |x \cdot k - c| \leq d\}) * ?i\ x) < e$
.
qed
qed
qed

41.30 A technical lemma about "refinement" of division.

lemma *tagged-division-finer*:
fixes *p* :: ('*a*::*euclidean-space* × ('*a*::*euclidean-space set*)) *set*
assumes *p tagged-division-of* (*cbox a b*)
and *gauge d*
obtains *q* **where** *q tagged-division-of* (*cbox a b*)
and *d fine q*
and ∀(*x*,*k*) ∈ *p*. *k* ⊆ *d*(*x*) ⟶ (*x*,*k*) ∈ *q*
proof −
let *?P* = λ*p*. *p tagged-partial-division-of* (*cbox a b*) ⟶ *gauge d* ⟶
(∃*q*. *q tagged-division-of* $(\bigcup \{k.\ \exists x.\ (x,k) \in p\})$ ∧ *d fine q* ∧
(∀(*x*,*k*) ∈ *p*. *k* ⊆ *d*(*x*) ⟶ (*x*,*k*) ∈ *q*))
{
have ∗: *finite p p tagged-partial-division-of* (*cbox a b*)
using *assms*(*1*)
unfolding *tagged-division-of-def*
by *auto*

```
  presume ⋀p. finite p ⟹ ?P p
  from this[rule-format,OF * assms(2)] guess q .. note q=this
  then show ?thesis
    apply −
    apply (rule that[of q])
    unfolding tagged-division-ofD[OF assms(1)]
    apply auto
    done
}
fix p :: ('a::euclidean-space × ('a::euclidean-space set)) set
assume as: finite p
show ?P p
  apply rule
  apply rule
  using as
proof (induct p)
  case empty
  show ?case
    apply (rule-tac x={} in exI)
    unfolding fine-def
    apply auto
    done
next
  case (insert xk p)
  guess x k using surj-pair[of xk] by (elim exE) note xk=this
  note tagged-partial-division-subset[OF insert(4) subset-insertI]
  from insert(3)[OF this insert(5)] guess q1 .. note q1 = conjunctD3[OF this]
  have *: ⋃{l. ∃y. (y,l) ∈ insert xk p} = k ∪ ⋃{l. ∃y. (y,l) ∈ p}
    unfolding xk by auto
  note p = tagged-partial-division-ofD[OF insert(4)]
  from p(4)[unfolded xk, OF insertI1] guess u v by (elim exE) note uv=this

  have finite {k. ∃x. (x, k) ∈ p}
    apply (rule finite-subset[of - snd ` p])
    using p
    apply safe
    apply (metis image-iff snd-conv)
    apply auto
    done
  then have int: interior (cbox u v) ∩ interior (⋃{k. ∃x. (x, k) ∈ p}) = {}
    apply (rule inter-interior-unions-intervals)
    apply (rule open-interior)
    apply (rule-tac[!] ballI)
    unfolding mem-Collect-eq
    apply (erule-tac[!] exE)
    apply (drule p(4)[OF insertI2])
    apply assumption
    apply (rule p(5))
    unfolding uv xk
```

```
    apply (rule insertI1)
    apply (rule insertI2)
    apply assumption
    using insert(2)
    unfolding uv xk
    apply auto
    done
  show ?case
  proof (cases cbox u v ⊆ d x)
    case True
    then show ?thesis
      apply (rule-tac x={(x,cbox u v)} ∪ q1 in exI)
      apply rule
      unfolding ∗ uv
      apply (rule tagged-division-union)
      apply (rule tagged-division-of-self)
      apply (rule p[unfolded xk uv] insertI1)+
      apply (rule q1)
      apply (rule int)
      apply rule
      apply (rule fine-union)
      apply (subst fine-def)
      defer
      apply (rule q1)
      unfolding Ball-def split-paired-All split-conv
      apply rule
      apply rule
      apply rule
      apply rule
      apply (erule insertE)
      apply (simp add: uv xk)
      apply (rule UnI2)
      apply (drule q1(3)[rule-format])
      unfolding xk uv
      apply auto
      done
  next
    case False
    from fine-division-exists[OF assms(2), of u v] guess q2 . note q2=this
    show ?thesis
      apply (rule-tac x=q2 ∪ q1 in exI)
      apply rule
      unfolding ∗ uv
      apply (rule tagged-division-union q2 q1 int fine-union)+
      unfolding Ball-def split-paired-All split-conv
      apply rule
      apply (rule fine-union)
      apply (rule q1 q2)+
      apply rule
```

```
      apply rule
      apply rule
      apply rule
      apply (erule insertE)
      apply (rule UnI2)
      apply (simp add: False uv xk)
      apply (drule q1(3)[rule-format])
      using False
      unfolding xk uv
      apply auto
      done
    qed
  qed
qed
```

41.31 Hence the main theorem about negligible sets.

```
lemma finite-product-dependent:
  assumes finite s
    and ⋀x. x ∈ s ⟹ finite (t x)
  shows finite {(i, j) |i j. i ∈ s ∧ j ∈ t i}
  using assms
proof induct
  case (insert x s)
  have ∗: {(i, j) |i j. i ∈ insert x s ∧ j ∈ t i} =
    (λy. (x,y)) ‘ (t x) ∪ {(i, j) |i j. i ∈ s ∧ j ∈ t i} by auto
  show ?case
    unfolding ∗
    apply (rule finite-UnI)
    using insert
    apply auto
    done
qed auto

lemma sum-sum-product:
  assumes finite s
    and ∀ i∈s. finite (t i)
  shows setsum (λi. setsum (x i) (t i)::real) s =
    setsum (λ(i,j). x i j) {(i,j) | i j. i ∈ s ∧ j ∈ t i}
  using assms
proof induct
  case (insert a s)
  have ∗: {(i, j) |i j. i ∈ insert a s ∧ j ∈ t i} =
    (λy. (a,y)) ‘ (t a) ∪ {(i, j) |i j. i ∈ s ∧ j ∈ t i} by auto
  show ?case
    unfolding ∗
    apply (subst setsum.union-disjoint)
    unfolding setsum.insert[OF insert(1−2)]
    prefer 4
```

```
    apply (subst insert(3))
    unfolding add-right-cancel
  proof −
    show setsum (x a) (t a) = (∑(xa, y)∈ Pair a ' t a. x xa y)
      apply (subst setsum.reindex)
      unfolding inj-on-def
      apply auto
      done
    show finite {(i, j) |i j. i ∈ s ∧ j ∈ t i}
      apply (rule finite-product-dependent)
      using insert
      apply auto
      done
  qed (insert insert, auto)
qed auto

lemma has-integral-negligible:
  fixes f :: 'b::euclidean-space ⇒ 'a::real-normed-vector
  assumes negligible s
    and ∀x∈(t − s). f x = 0
  shows (f has-integral 0) t
proof −
  presume P: ⋀f::'b::euclidean-space ⇒ 'a.
    ⋀a b. ∀x. x ∉ s ⟶ f x = 0 ⟹ (f has-integral 0) (cbox a b)
  let ?f = (λx. if x ∈ t then f x else 0)
  show ?thesis
    apply (rule-tac f=?f in has-integral-eq)
    unfolding if-P
    apply (rule refl)
    apply (subst has-integral-alt)
    apply cases
    apply (subst if-P, assumption)
    unfolding if-not-P
  proof −
    assume ∃a b. t = cbox a b
    then guess a b apply − by (erule exE)+ note t = this
    show (?f has-integral 0) t
      unfolding t
      apply (rule P)
      using assms(2)
      unfolding t
      apply auto
      done
  next
    show ∀e>0. ∃B>0. ∀a b. ball 0 B ⊆ cbox a b ⟶
      (∃z. ((λx. if x ∈ t then ?f x else 0) has-integral z) (cbox a b) ∧ norm (z −
0) < e)
      apply safe
      apply (rule-tac x=1 in exI)
```

```
      apply rule
      apply (rule zero-less-one)
      apply safe
      apply (rule-tac x=0 in exI)
      apply rule
      apply (rule P)
      using assms(2)
      apply auto
      done
  qed
next
  fix f :: 'b ⇒ 'a
  fix a b :: 'b
  assume assm: ∀ x. x ∉ s ⟶ f x = 0
  show (f has-integral 0) (cbox a b)
    unfolding has-integral
  proof (safe, goal-cases)
    case prems: (1 e)
    then have ⋀n. e / 2 / ((real n+1) * (2 ^ n)) > 0
      apply −
      apply (rule divide-pos-pos)
      defer
      apply (rule mult-pos-pos)
      apply (auto simp add:field-simps)
      done
    note assms(1)[unfolded negligible-def has-integral,rule-format,OF this,of a b]
    note allI[OF this,of λx. x]
    from choice[OF this] guess d .. note d=conjunctD2[OF this[rule-format]]
    show ?case
      apply (rule-tac x=λx. d (nat ⌊norm (f x)⌋) x in exI)
    proof safe
      show gauge (λx. d (nat ⌊norm (f x)⌋) x)
        using d(1) unfolding gauge-def by auto
      fix p
      assume as: p tagged-division-of (cbox a b) (λx. d (nat ⌊norm (f x)⌋) x) fine
p
      let ?goal = norm ((∑(x, k)∈p. content k *R f x) − 0) < e
      {
        presume p ≠ {} ⟹ ?goal
        then show ?goal
          apply (cases p = {})
          using prems
          apply auto
          done
      }
      assume as': p ≠ {}
      from real-arch-simple[of Max((λ(x,k). norm(f x)) ` p)] guess N ..
      then have N: ∀ x∈(λ(x, k). norm (f x)) ` p. x ≤ real N
      by (meson Max-ge as(1) dual-order.trans finite-imageI tagged-division-of-finite)
```

```
    have ∀ i. ∃ q. q tagged-division-of (cbox a b) ∧ (d i) fine q ∧ (∀(x, k)∈p. k
⊆ (d i) x ⟶ (x, k) ∈ q)
      by (auto intro: tagged-division-finer[OF as(1) d(1)])
    from choice[OF this] guess q .. note q=conjunctD3[OF this[rule-format]]
    have *: ⋀i. (∑(x, k)∈q i. content k *R indicator s x) ≥ (0::real)
      apply (rule setsum-nonneg)
      apply safe
      unfolding real-scaleR-def
      apply (drule tagged-division-ofD(4)[OF q(1)])
      apply (auto intro: mult-nonneg-nonneg)
      done
    have **: finite s ⟹ finite t ⟹ (∀(x,y) ∈ t. (0::real) ≤ g(x,y)) ⟹
      (∀y∈s. ∃x. (x,y) ∈ t ∧ f(y) ≤ g(x,y)) ⟹ setsum f s ≤ setsum g t for f g
s t
      apply (rule setsum-le-included[of s t g snd f])
      prefer 4
      apply safe
      apply (erule-tac x=x in ballE)
      apply (erule exE)
      apply (rule-tac x=(xa,x) in bexI)
      apply auto
      done
    have norm ((∑(x, k)∈p. content k *R f x) - 0) ≤ setsum (λi. (real i + 1)
*
      norm (setsum (λ(x,k). content k *R indicator s x :: real) (q i))) {..N+1}
    unfolding real-norm-def setsum-right-distrib abs-of-nonneg[OF *] diff-0-right
      apply (rule order-trans)
      apply (rule norm-setsum)
      apply (subst sum-sum-product)
      prefer 3
    proof (rule **, safe)
      show finite {(i, j) |i j. i ∈ {..N + 1} ∧ j ∈ q i}
        apply (rule finite-product-dependent)
        using q
        apply auto
        done
      fix i a b
      assume as'': (a, b) ∈ q i
      show 0 ≤ (real i + 1) * (content b *R indicator s a)
        unfolding real-scaleR-def
        using tagged-division-ofD(4)[OF q(1) as'']
        by (auto intro!: mult-nonneg-nonneg)
    next
      fix i :: nat
      show finite (q i)
        using q by auto
    next
      fix x k
      assume xk: (x, k) ∈ p
```

```
      def n ≡ nat ⌊norm (f x)⌋
      have ∗: norm (f x) ∈ (λ(x, k). norm (f x)) ‘ p
        using xk by auto
      have nfx: real n ≤ norm (f x) norm (f x) ≤ real n + 1
        unfolding n-def by auto
      then have n ∈ {0..N + 1}
        using N[rule-format,OF ∗] by auto
      moreover
      note as(2)[unfolded fine-def,rule-format,OF xk,unfolded split-conv]
      note q(3)[rule-format,OF xk,unfolded split-conv,rule-format,OF this]
      note this[unfolded n-def[symmetric]]
      moreover
      have norm (content k ∗R f x) ≤ (real n + 1) ∗ (content k ∗ indicator s x)
      proof (cases x ∈ s)
        case False
        then show ?thesis
          using assm by auto
      next
        case True
        have ∗: content k ≥ 0
          using tagged-division-ofD(4)[OF as(1) xk] by auto
        moreover
        have content k ∗ norm (f x) ≤ content k ∗ (real n + 1)
          apply (rule mult-mono)
          using nfx ∗
          apply auto
          done
        ultimately
        show ?thesis
          unfolding abs-mult
          using nfx True
          by (auto simp add: field-simps)
      qed
      ultimately show ∃ y. (y, x, k) ∈ {(i, j) |i j. i ∈ {..N + 1} ∧ j ∈ q i} ∧
norm (content k ∗R f x) ≤
        (real y + 1) ∗ (content k ∗R indicator s x)
        apply (rule-tac x=n in exI)
        apply safe
        apply (rule-tac x=n in exI)
        apply (rule-tac x=(x,k) in exI)
        apply safe
        apply auto
        done
    qed (insert as, auto)
    also have ... ≤ setsum (λi. e / 2 / 2 ^ i) {..N+1}
    proof (rule setsum-mono, goal-cases)
      case (1 i)
      then show ?case
        apply (subst mult.commute, subst pos-le-divide-eq[symmetric])
```

```
        using d(2)[rule-format, of q i i]
        using q[rule-format]
        apply (auto simp add: field-simps)
        done
    qed
    also have ... < e * inverse 2 * 2
      unfolding divide-inverse setsum-right-distrib[symmetric]
      apply (rule mult-strict-left-mono)
      unfolding power-inverse [symmetric] lessThan-Suc-atMost[symmetric]
      apply (subst geometric-sum)
      using prems
      apply auto
      done
    finally show ?goal by auto
  qed
 qed
qed

lemma has-integral-spike:
 fixes f :: 'b::euclidean-space ⇒ 'a::real-normed-vector
 assumes negligible s
   and (∀ x∈(t - s). g x = f x)
   and (f has-integral y) t
 shows (g has-integral y) t
proof -
 {
   fix a b :: 'b
   fix f g :: 'b ⇒ 'a
   fix y :: 'a
   assume as: ∀ x ∈ cbox a b - s. g x = f x (f has-integral y) (cbox a b)
   have ((λx. f x + (g x - f x)) has-integral (y + 0)) (cbox a b)
     apply (rule has-integral-add[OF as(2)])
     apply (rule has-integral-negligible[OF assms(1)])
     using as
     apply auto
     done
   then have (g has-integral y) (cbox a b)
     by auto
 } note * = this
 show ?thesis
   apply (subst has-integral-alt)
   using assms(2-)
   apply -
   apply (rule cond-cases)
   apply safe
   apply (rule *)
   apply assumption+
   apply (subst(asm) has-integral-alt)
   unfolding if-not-P
```

```
    apply (erule-tac x=e in allE)
    apply safe
    apply (rule-tac x=B in exI)
    apply safe
    apply (erule-tac x=a in allE)
    apply (erule-tac x=b in allE)
    apply safe
    apply (rule-tac x=z in exI)
    apply safe
    apply (rule *[where fa2=λx. if x∈t then f x else 0])
    apply auto
    done
qed

lemma has-integral-spike-eq:
  assumes negligible s
    and ∀ x∈(t - s). g x = f x
  shows ((f has-integral y) t ⟷ (g has-integral y) t)
  apply rule
  apply (rule-tac[!] has-integral-spike[OF assms(1)])
  using assms(2)
  apply auto
  done

lemma integrable-spike:
  assumes negligible s
    and ∀ x∈(t - s). g x = f x
    and f integrable-on t
  shows g integrable-on  t
  using assms
  unfolding integrable-on-def
  apply -
  apply (erule exE)
  apply rule
  apply (rule has-integral-spike)
  apply fastforce+
  done

lemma integral-spike:
  assumes negligible s
    and ∀ x∈(t - s). g x = f x
  shows integral t f = integral t g
 using has-integral-spike-eq[OF assms] by (simp add: integral-def integrable-on-def)
```

41.32 Some other trivialities about negligible sets.

```
lemma negligible-subset[intro]:
  assumes negligible s
    and t ⊆ s
```

shows *negligible t*
unfolding *negligible-def*
proof (*safe*, *goal-cases*)
case (*1 a b*)
show *?case*
using *assms*(*1*)[*unfolded negligible-def*,*rule-format*,*of a b*]
apply −
apply (*rule has-integral-spike*[*OF assms*(*1*)])
defer
apply *assumption*
using *assms*(*2*)
unfolding *indicator-def*
apply *auto*
done
qed

lemma *negligible-diff*[*intro?*]:
assumes *negligible s*
shows *negligible* (*s* − *t*)
using *assms* **by** *auto*

lemma *negligible-inter*:
assumes *negligible s* ∨ *negligible t*
shows *negligible* (*s* ∩ *t*)
using *assms* **by** *auto*

lemma *negligible-union*:
assumes *negligible s*
and *negligible t*
shows *negligible* (*s* ∪ *t*)
unfolding *negligible-def*
proof (*safe*, *goal-cases*)
case (*1 a b*)
note *assm* = *assms*[*unfolded negligible-def*,*rule-format*,*of a b*]
then show *?case*
apply (*subst has-integral-spike-eq*[*OF assms*(*2*)])
defer
apply *assumption*
unfolding *indicator-def*
apply *auto*
done
qed

lemma *negligible-union-eq*[*simp*]: *negligible* (*s* ∪ *t*) ⟷ *negligible s* ∧ *negligible t*
using *negligible-union* **by** *auto*

lemma *negligible-sing*[*intro*]: *negligible* {*a*::′*a*::*euclidean-space*}
using *negligible-standard-hyperplane*[*OF SOME-Basis*, *of a* · (*SOME i*. *i* ∈ *Basis*)] **by** *auto*

lemma *negligible-insert*[*simp*]: *negligible* (*insert a s*) ⟷ *negligible s*
apply (*subst insert-is-Un*)
unfolding *negligible-union-eq*
apply *auto*
done

lemma *negligible-empty*[*iff*]: *negligible* {}
by *auto*

lemma *negligible-finite*[*intro*]:
assumes *finite s*
shows *negligible s*
using *assms* **by** (*induct s*) *auto*

lemma *negligible-unions*[*intro*]:
assumes *finite s*
and ∀ *t*∈*s*. *negligible t*
shows *negligible*(⋃ *s*)
using *assms* **by** *induct auto*

lemma *negligible*:
negligible s ⟷ (∀ *t*::('*a*::*euclidean-space*) *set*. ((*indicator s*::'*a*⇒*real*) *has-integral 0*) *t*)
apply *safe*
defer
apply (*subst negligible-def*)
proof −
fix *t* :: '*a set*
assume *as*: *negligible s*
have ∗: (λ*x*. *if x* ∈ *s* ∩ *t then 1 else 0*) = (λ*x*. *if x*∈*t then if x*∈*s then 1 else 0 else 0*)
by *auto*
show ((*indicator s*::'*a*⇒*real*) *has-integral 0*) *t*
apply (*subst has-integral-alt*)
apply *cases*
apply (*subst if-P*,*assumption*)
unfolding *if-not-P*
apply *safe*
apply (*rule as*[*unfolded negligible-def*,*rule-format*])
apply (*rule-tac x=1* **in** *exI*)
apply *safe*
apply (*rule zero-less-one*)
apply (*rule-tac x=0* **in** *exI*)
using *negligible-subset*[*OF as*,*of s* ∩ *t*]
unfolding *negligible-def indicator-def* [*abs-def*]
unfolding ∗
apply *auto*
done

qed *auto*

41.33 Finite case of the spike theorem is quite commonly needed.

lemma *has-integral-spike-finite*:
assumes *finite s*
and $\forall x{\in}t{-}s.\ g\ x = f\ x$
and (*f has-integral y*) *t*
shows (*g has-integral y*) *t*
apply (*rule has-integral-spike*)
using *assms*
apply *auto*
done

lemma *has-integral-spike-finite-eq*:
assumes *finite s*
and $\forall x{\in}t{-}s.\ g\ x = f\ x$
shows ((*f has-integral y*) *t* $\longleftrightarrow$ (*g has-integral y*) *t*)
apply *rule*
apply (*rule-tac*[!] *has-integral-spike-finite*)
using *assms*
apply *auto*
done

lemma *integrable-spike-finite*:
assumes *finite s*
and $\forall x{\in}t{-}s.\ g\ x = f\ x$
and *f integrable-on t*
shows *g integrable-on t*
using *assms*
unfolding *integrable-on-def*
apply *safe*
apply (*rule-tac x=y* **in** *exI*)
apply (*rule has-integral-spike-finite*)
apply *auto*
done

41.34 In particular, the boundary of an interval is negligible.

lemma *negligible-frontier-interval*: *negligible*(*cbox* (*a*::'*a*::*euclidean-space*) *b* − *box a b*)
proof −
let *?A* = $\bigcup((\lambda k.\ \{x.\ x{\cdot}k = a{\cdot}k\} \cup \{x{::}'a.\ x{\cdot}k = b{\cdot}k\})$ ' *Basis*)
have *cbox a b* − *box a b* ⊆ *?A*
apply *rule* **unfolding** *Diff-iff mem-box*
apply *simp*
apply(*erule conjE bexE*)+
apply(*rule-tac x=i* **in** *bexI*)

apply *auto*
done
then show *?thesis*
apply −
apply (*rule negligible-subset*[*of ?A*])
apply (*rule negligible-unions*[*OF finite-imageI*])
apply *auto*
done
qed

lemma *has-integral-spike-interior*:
assumes ∀ *x*∈*box a b*. *g x* = *f x*
and (*f has-integral y*) (*cbox a b*)
shows (*g has-integral y*) (*cbox a b*)
apply (*rule has-integral-spike*[*OF negligible-frontier-interval - assms*(*2*)])
using *assms*(*1*)
apply *auto*
done

lemma *has-integral-spike-interior-eq*:
assumes ∀ *x*∈*box a b*. *g x* = *f x*
shows (*f has-integral y*) (*cbox a b*) ⟷ (*g has-integral y*) (*cbox a b*)
apply *rule*
apply (*rule-tac*[!] *has-integral-spike-interior*)
using *assms*
apply *auto*
done

lemma *integrable-spike-interior*:
assumes ∀ *x*∈*box a b*. *g x* = *f x*
and *f integrable-on cbox a b*
shows *g integrable-on cbox a b*
using *assms*
unfolding *integrable-on-def*
using *has-integral-spike-interior*[*OF assms*(*1*)]
by *auto*

41.35 Integrability of continuous functions.

lemma *neutral-and*[*simp*]: *neutral op* ∧ = *True*
unfolding *neutral-def* **by** (*rule some-equality*) *auto*

lemma *monoidal-and*[*intro*]: *monoidal op* ∧
unfolding *monoidal-def* **by** *auto*

lemma *iterate-and*[*simp*]:
assumes *finite s*
shows (*iterate op* ∧) *s p* ⟷ (∀ *x*∈*s*. *p x*)
using *assms*

apply *induct*
unfolding *iterate-insert*[*OF monoidal-and*]
apply *auto*
done

lemma *operative-division-and*:
assumes *operative op* ∧ *P*
and *d division-of* (*cbox a b*)
shows (∀ *i*∈*d*. *P i*) ⟷ *P* (*cbox a b*)
using *operative-division*[*OF monoidal-and assms*] *division-of-finite*[*OF assms*(*2*)]
by *auto*

lemma *operative-approximable*:
fixes *f* :: *'b*::*euclidean-space* ⇒ *'a*::*banach*
assumes *0* ≤ *e*
shows *operative op* ∧ (λ*i*. ∃ *g*. (∀ *x*∈*i*. *norm* (*f x* − *g* (*x*::*'b*)) ≤ *e*) ∧ *g integrable-on* *i*)
unfolding *operative-def neutral-and*
proof *safe*
fix *a b* :: *'b*
show ∃ *g*. (∀ *x*∈*cbox a b*. *norm* (*f x* − *g x*) ≤ *e*) ∧ *g integrable-on cbox a b*
if *content* (*cbox a b*) = *0*
apply (*rule-tac x*=*f* **in** *exI*)
using *assms that*
apply (*auto intro*!: *integrable-on-null*)
done
{
fix *c g*
fix *k* :: *'b*
assume *as*: ∀ *x*∈*cbox a b*. *norm* (*f x* − *g x*) ≤ *e g integrable-on cbox a b*
assume *k*: *k* ∈ *Basis*
show ∃ *g*. (∀ *x*∈*cbox a b* ∩ {*x*. *x* • *k* ≤ *c*}. *norm* (*f x* − *g x*) ≤ *e*) ∧ *g integrable-on* *cbox a b* ∩ {*x*. *x* • *k* ≤ *c*}
∃ *g*. (∀ *x*∈*cbox a b* ∩ {*x*. *c* ≤ *x* • *k*}. *norm* (*f x* − *g x*) ≤ *e*) ∧ *g integrable-on* *cbox a b* ∩ {*x*. *c* ≤ *x* • *k*}
apply (*rule-tac*[!] *x*=*g* **in** *exI*)
using *as*(*1*) *integrable-split*[*OF as*(*2*) *k*]
apply *auto*
done
}
fix *c k g1 g2*
assume *as*: ∀ *x*∈*cbox a b* ∩ {*x*. *x* • *k* ≤ *c*}. *norm* (*f x* − *g1 x*) ≤ *e g1 integrable-on* *cbox a b* ∩ {*x*. *x* • *k* ≤ *c*}
∀ *x*∈*cbox a b* ∩ {*x*. *c* ≤ *x* • *k*}. *norm* (*f x* − *g2 x*) ≤ *e g2 integrable-on cbox a* *b* ∩ {*x*. *c* ≤ *x* • *k*}
assume *k*: *k* ∈ *Basis*
let *?g* = λ*x*. *if x*•*k* = *c then f x else if x*•*k* ≤ *c then g1 x else g2 x*
show ∃ *g*. (∀ *x*∈*cbox a b*. *norm* (*f x* − *g x*) ≤ *e*) ∧ *g integrable-on cbox a b*
apply (*rule-tac x*=*?g* **in** *exI*)

```
    apply safe
  proof goal-cases
    case (1 x)
    then show ?case
      apply −
      apply (cases x·k=c)
      apply (case-tac x·k < c)
      using as assms
      apply auto
      done
  next
    case 2
    presume ?g integrable-on cbox a b ∩ {x. x · k ≤ c}
      and ?g integrable-on cbox a b ∩ {x. x · k ≥ c}
    then guess h1 h2 unfolding integrable-on-def by auto
    from has-integral-split[OF this k] show ?case
      unfolding integrable-on-def by auto
  next
    show ?g integrable-on cbox a b ∩ {x. x · k ≤ c} ?g integrable-on cbox a b ∩
{x. x · k ≥ c}
      apply(rule-tac[!] integrable-spike[OF negligible-standard-hyperplane[of k c]])
      using k as(2,4)
      apply auto
      done
  qed
qed

lemma approximable-on-division:
  fixes f :: 'b::euclidean-space ⇒ 'a::banach
  assumes 0 ≤ e
    and d division-of (cbox a b)
    and ∀ i∈d. ∃ g. (∀ x∈i. norm (f x − g x) ≤ e) ∧ g integrable-on i
  obtains g where ∀ x∈cbox a b. norm (f x − g x) ≤ e g integrable-on cbox a b
proof −
  note ∗ = operative-division[OF monoidal-and operative-approximable[OF assms(1)]
assms(2)]
  note this[unfolded iterate-and[OF division-of-finite[OF assms(2)]]]
  from assms(3)[unfolded this[of f]] guess g ..
  then show thesis
    apply −
    apply (rule that[of g])
    apply auto
    done
qed

lemma integrable-continuous:
  fixes f :: 'b::euclidean-space ⇒ 'a::banach
  assumes continuous-on (cbox a b) f
  shows f integrable-on cbox a b
```

proof (*rule integrable-uniform-limit*, *safe*)
fix *e* :: *real*
assume *e*: *e* > *0*
from *compact-uniformly-continuous*[*OF assms compact-cbox*,*unfolded uniformly-continuous-on-def*,*rule-form*
e] **guess** *d* **..**
note *d*=*conjunctD2*[*OF this*,*rule-format*]
from *fine-division-exists*[*OF gauge-ball*[*OF d*(*1*)], *of a b*] **guess** *p* **. note** *p*=*this*
note *p'* = *tagged-division-ofD*[*OF p*(*1*)]
have ∗: ∀ *i*∈*snd* ' *p*. ∃ *g*. (∀ *x*∈*i*. *norm* (*f x* − *g x*) ≤ *e*) ∧ *g integrable-on i*
proof (*safe*, *unfold snd-conv*)
fix *x l*
assume *as*: (*x*, *l*) ∈ *p*
from *p'*(*4*)[*OF this*] **guess** *a b* **by** (*elim exE*) **note** *l*=*this*
show ∃ *g*. (∀ *x*∈*l*. *norm* (*f x* − *g x*) ≤ *e*) ∧ *g integrable-on l*
apply (*rule-tac x*=λ*y*. *f x* **in** *exI*)
proof *safe*
show (λ*y*. *f x*) *integrable-on l*
unfolding *integrable-on-def l*
apply *rule*
apply (*rule has-integral-const*)
done
fix *y*
assume *y*: *y* ∈ *l*
note *fineD*[*OF p*(*2*) *as*,*unfolded subset-eq*,*rule-format*,*OF this*]
note *d*(*2*)[*OF - - this*[*unfolded mem-ball*]]
then show *norm* (*f y* − *f x*) ≤ *e*
using *y p'*(*2*−*3*)[*OF as*] **unfolding** *dist-norm l norm-minus-commute* **by**
fastforce
qed
qed
from *e* **have** *e* ≥ *0*
by *auto*
from *approximable-on-division*[*OF this division-of-tagged-division*[*OF p*(*1*)] ∗]
guess *g* **.**
then show ∃ *g*. (∀ *x*∈*cbox a b*. *norm* (*f x* − *g x*) ≤ *e*) ∧ *g integrable-on cbox a b*
by *auto*
qed

lemma *integrable-continuous-real*:
fixes *f* :: *real* ⇒ *'a*::*banach*
assumes *continuous-on* {*a* .. *b*} *f*
shows *f integrable-on* {*a* .. *b*}
by (*metis assms box-real*(*2*) *integrable-continuous*)

41.36 Specialization of additivity to one dimension.

lemma
shows *real-inner-1-left*: *inner 1 x* = *x*
and *real-inner-1-right*: *inner x 1* = *x*

by *simp-all*

lemma *content-real-eq-0*: *content* $\{a$ .. b::*real*$\} = 0 \longleftrightarrow a \geq b$
by (*metis atLeastatMost-empty-iff2 content-empty content-real diff-self eq-iff le-cases le-iff-diff-le-0*)

lemma *interval-real-split*:
$\{a$ .. b::*real*$\} \cap \{x.\ x \leq c\} = \{a$ .. *min* $b\ c\}$
$\{a$ .. $b\} \cap \{x.\ c \leq x\} = \{max\ a\ c$ .. $b\}$
apply (*metis Int-atLeastAtMostL1 atMost-def*)
apply (*metis Int-atLeastAtMostL2 atLeast-def*)
done

lemma *operative-1-lt*:
assumes *monoidal opp*
shows *operative opp f* $\longleftrightarrow$ $((\forall\ a\ b.\ b \leq a \longrightarrow f\ \{a$ .. b::*real*$\} = neutral\ opp) \wedge$
$(\forall\ a\ b\ c.\ a < c \wedge c < b \longrightarrow opp\ (f\ \{a$ .. $c\})\ (f\ \{c$ .. $b\}) = f\ \{a$ .. $b\}))$
apply (*simp add*: *operative-def content-real-eq-0 del*: *content-real-if*)
proof *safe*
fix *a b c* :: *real*
assume *as*:
$\forall\ a\ b\ c.\ f\ \{a..b\} = opp\ (f\ (\{a..b\} \cap \{x.\ x \leq c\}))\ (f\ (\{a..b\} \cap Collect\ (op \leq c)))$
$a < c$
$c < b$
from *this*(*2*−) **have** *cbox a b* $\cap \{x.\ x \leq c\} = cbox\ a\ c\ cbox\ a\ b \cap \{x.\ x \geq c\} = cbox\ c\ b$
by (*auto simp*: *mem-box*)
then show *opp* $(f\ \{a..c\})\ (f\ \{c..b\}) = f\ \{a..b\}$
unfolding *as*(*1*)[*rule-format*,*of a b c*] **by** *auto*
next
fix *a b c* :: *real*
assume *as*: $\forall\ a\ b.\ b \leq a \longrightarrow f\ \{a..b\} = neutral\ opp$
$\forall\ a\ b\ c.\ a < c \wedge c < b \longrightarrow opp\ (f\ \{a..c\})\ (f\ \{c..b\}) = f\ \{a..b\}$
show $f\ \{a..b\} = opp\ (f\ (\{a..b\} \cap \{x.\ x \leq c\}))\ (f\ (\{a..b\} \cap Collect\ (op \leq c)))$
proof (*cases* $c \in \{a..b\}$)
case *False*
then have $c < a \vee c > b$ **by** *auto*
then show *?thesis*
proof
assume $c < a$
then have $*$: $\{a..b\} \cap \{x.\ x \leq c\} = \{1..0\}\ \{a..b\} \cap \{x.\ c \leq x\} = \{a..b\}$
by *auto*
show *?thesis*
unfolding $*$
apply (*subst as*(*1*)[*rule-format*,*of 0 1*])
using *assms*
apply *auto*
done

```
  next
    assume b < c
    then have ∗: {a..b} ∩ {x. x ≤ c} = {a..b} {a..b} ∩ {x. c ≤ x} = {1 .. 0}
      by auto
    show ?thesis
      unfolding ∗
      apply (subst as(1)[rule-format,of 0 1])
      using assms
      apply auto
      done
  qed
next
  case True
  then have ∗: min (b) c = c max a c = c
    by auto
  have ∗∗: (1::real) ∈ Basis
    by simp
  have ∗∗∗: ⋀P Q. (∑ i∈Basis. (if i = 1 then P i else Q i) ∗R i) = (P 1::real)
    by simp
  show ?thesis
    unfolding interval-real-split unfolding ∗
  proof (cases c = a ∨ c = b)
    case False
    then show f {a..b} = opp (f {a..c}) (f {c..b})
      apply −
      apply (subst as(2)[rule-format])
      using True
      apply auto
      done
  next
    case True
    then show f {a..b} = opp (f {a..c}) (f {c..b})
    proof
      assume ∗: c = a
      then have f {a .. c} = neutral opp
        apply −
        apply (rule as(1)[rule-format])
        apply auto
        done
      then show ?thesis
        using assms unfolding ∗ by auto
    next
      assume ∗: c = b
      then have f {c .. b} = neutral opp
        apply −
        apply (rule as(1)[rule-format])
        apply auto
        done
      then show ?thesis
```

```
        using assms unfolding ∗ by auto
      qed
    qed
  qed
qed

lemma operative-1-le:
  assumes monoidal opp
  shows operative opp f ⟷ ((∀ a b. b ≤ a ⟶ f {a .. b::real} = neutral opp) ∧
    (∀ a b c. a ≤ c ∧ c ≤ b ⟶ opp (f {a .. c}) (f {c .. b}) = f {a .. b}))
  unfolding operative-1-lt[OF assms]
proof safe
  fix a b c :: real
  assume as:
    ∀ a b c. a ≤ c ∧ c ≤ b ⟶ opp (f {a..c}) (f {c..b}) = f {a..b}
    a < c
    c < b
  show opp (f {a..c}) (f {c..b}) = f {a..b}
    apply (rule as(1)[rule-format])
    using as(2−)
    apply auto
    done
next
  fix a b c :: real
  assume ∀ a b. b ≤ a ⟶ f {a .. b} = neutral opp
    and ∀ a b c. a < c ∧ c < b ⟶ opp (f {a..c}) (f {c..b}) = f {a..b}
    and a ≤ c
    and c ≤ b
  note as = this[rule-format]
  show opp (f {a..c}) (f {c..b}) = f {a..b}
  proof (cases c = a ∨ c = b)
    case False
    then show ?thesis
      apply −
      apply (subst as(2))
      using as(3−)
      apply auto
      done
  next
    case True
    then show ?thesis
    proof
      assume ∗: c = a
      then have f {a .. c} = neutral opp
        apply −
        apply (rule as(1)[rule-format])
        apply auto
        done
      then show ?thesis
```

```
      using assms unfolding ∗ by auto
    next
     assume ∗: c = b
     then have f {c .. b} = neutral opp
      apply −
      apply (rule as(1)[rule-format])
      apply auto
      done
     then show ?thesis
      using assms unfolding ∗ by auto
    qed
  qed
qed
```

41.37 Special case of additivity we need for the FCT.

```
lemma additive-tagged-division-1:
  fixes f :: real ⇒ 'a::real-normed-vector
  assumes a ≤ b
    and p tagged-division-of {a..b}
  shows setsum (λ(x,k). f(Sup k) − f(Inf k)) p = f b − f a
proof −
   let ?f = (λk::(real) set. if k = {} then 0 else f(interval-upperbound k) −
f(interval-lowerbound k))
  have ∗∗∗: ∀ i∈Basis. a · i ≤ b · i
    using assms by auto
  have ∗: operative op + ?f
    unfolding operative-1-lt[OF monoidal-monoid] box-eq-empty
    by auto
  have ∗∗: cbox a b ≠ {}
    using assms(1) by auto
 note operative-tagged-division[OF monoidal-monoid ∗ assms(2)[simplified box-real[symmetric]]]
  note ∗ = this[unfolded if-not-P[OF ∗∗] interval-bounds[OF ∗∗∗],symmetric]
  show ?thesis
    unfolding ∗
    apply (subst setsum-iterate[symmetric])
    defer
    apply (rule setsum.cong)
    unfolding split-paired-all split-conv
    using assms(2)
    apply auto
    done
qed
```

41.38 A useful lemma allowing us to factor out the content size.

```
lemma has-integral-factor-content:
  (f has-integral i) (cbox a b) ⟷
```

```
    (∀ e>0. ∃ d. gauge d ∧ (∀ p. p tagged-division-of (cbox a b) ∧ d fine p ⟶
      norm (setsum (λ(x,k). content k *R f x) p - i) ≤ e * content (cbox a b)))
proof (cases content (cbox a b) = 0)
  case True
  show ?thesis
    unfolding has-integral-null-eq[OF True]
    apply safe
    apply (rule, rule, rule gauge-trivial, safe)
    unfolding setsum-content-null[OF True] True
    defer
    apply (erule-tac x=1 in allE)
    apply safe
    defer
    apply (rule fine-division-exists[of - a b])
    apply assumption
    apply (erule-tac x=p in allE)
    unfolding setsum-content-null[OF True]
    apply auto
    done
next
  case False
  note F = this[unfolded content-lt-nz[symmetric]]
  let ?P = λe opp. ∃ d. gauge d ∧
    (∀ p. p tagged-division-of (cbox a b) ∧ d fine p ⟶ opp (norm ((∑(x, k)∈p.
content k *R f x) - i)) e)
  show ?thesis
    apply (subst has-integral)
  proof safe
    fix e :: real
    assume e: e > 0
    {
      assume ∀ e>0. ?P e op <
      then show ?P (e * content (cbox a b)) op ≤
        apply (erule-tac x=e * content (cbox a b) in allE)
        apply (erule impE)
        defer
        apply (erule exE,rule-tac x=d in exI)
        using F e
        apply (auto simp add:field-simps)
        done
    }
    {
      assume ∀ e>0. ?P (e * content (cbox a b)) op ≤
      then show ?P e op <
        apply (erule-tac x=e / 2 / content (cbox a b) in allE)
        apply (erule impE)
        defer
        apply (erule exE,rule-tac x=d in exI)
        using F e
```

```
      apply (auto simp add: field-simps)
      done
  }
 qed
qed

lemma has-integral-factor-content-real:
 (f has-integral i) {a .. b::real} ⟷
  (∀ e>0. ∃ d. gauge d ∧ (∀ p. p tagged-division-of {a .. b} ∧ d fine p ⟶
    norm (setsum (λ(x,k). content k *R f x) p − i) ≤ e * content {a .. b} ))
 unfolding box-real[symmetric]
 by (rule has-integral-factor-content)
```

41.39 Fundamental theorem of calculus.

```
lemma interval-bounds-real:
 fixes q b :: real
 assumes a ≤ b
 shows Sup {a..b} = b
  and Inf {a..b} = a
 using assms by auto

lemma fundamental-theorem-of-calculus:
 fixes f :: real ⇒ 'a::banach
 assumes a ≤ b
  and ∀ x∈{a .. b}. (f has-vector-derivative f' x) (at x within {a .. b})
 shows (f' has-integral (f b − f a)) {a .. b}
 unfolding has-integral-factor-content box-real[symmetric]
proof safe
 fix e :: real
 assume e: e > 0
 note assm = assms(2)[unfolded has-vector-derivative-def has-derivative-within-alt]
 have *: ⋀P Q. ∀ x∈{a .. b}. P x ∧ (∀ e>0. ∃ d>0. Q x e d) ⟹ ∀ x. ∃ (d::real)>0.
x∈{a .. b} ⟶ Q x e d
  using e by blast
 note this[OF assm,unfolded gauge-existence-lemma]
 from choice[OF this,unfolded Ball-def[symmetric]] guess d ..
 note d=conjunctD2[OF this[rule-format],rule-format]
 show ∃ d. gauge d ∧ (∀ p. p tagged-division-of (cbox a b) ∧ d fine p ⟶
  norm ((∑(x, k)∈p. content k *R f' x) − (f b − f a)) ≤ e * content (cbox a
b))
  apply (rule-tac x=λx. ball x (d x) in exI)
  apply safe
  apply (rule gauge-ball-dependent)
  apply rule
  apply (rule d(1))
 proof −
  fix p
  assume as: p tagged-division-of cbox a b (λx. ball x (d x)) fine p
```

show *norm* $((\sum (x,\ k){\in}p.\ content\ k *_R f'\ x) - (f\ b - f\ a)) \leq e * content\ (cbox\ a\ b)$
unfolding *content-real*[*OF assms*(*1*), *simplified box-real*[*symmetric*]] *additive-tagged-division-1*[*OF assms*(*1*) *as*(*1*)[*simplified box-real*],*of f*,*symmetric*]
unfolding *additive-tagged-division-1*[*OF assms*(*1*) *as*(*1*)[*simplified box-real*],*of* $\lambda x.\ x$,*symmetric*]
unfolding *setsum-right-distrib*
defer
unfolding *setsum-subtractf*[*symmetric*]
proof (*rule setsum-norm-le*,*safe*)
fix *x k*
assume $(x,\ k) \in p$
note *xk* = *tagged-division-ofD*(*2*−*4*)[*OF as*(*1*) *this*]
from *this*(*3*) **guess** *u v* **by** (*elim exE*) **note** *k*=*this*
have ∗: $u \leq v$
using *xk* **unfolding** *k* **by** *auto*
have *ball*: $\forall xa{\in}k.\ xa \in ball\ x\ (d\ x)$
using *as*(*2*)[*unfolded fine-def*,*rule-format*,*OF* ‹(*x*,*k*)∈*p*›,*unfolded split-conv subset-eq*] **.**
have *norm* $((v - u) *_R f'\ x - (f\ v - f\ u)) \leq$
norm $(f\ u - f\ x - (u - x) *_R f'\ x) + norm\ (f\ v - f\ x - (v - x) *_R f'\ x)$
apply (*rule order-trans*[*OF - norm-triangle-ineq4*])
apply (*rule eq-refl*)
apply (*rule arg-cong*[**where** *f*=*norm*])
unfolding *scaleR-diff-left*
apply (*auto simp add*:*algebra-simps*)
done
also have $\ldots \leq e * norm\ (u - x) + e * norm\ (v - x)$
apply (*rule add-mono*)
apply (*rule d*(*2*)[*of x u*,*unfolded o-def*])
prefer *4*
apply (*rule d*(*2*)[*of x v*,*unfolded o-def*])
using *ball*[*rule-format*,*of u*] *ball*[*rule-format*,*of v*]
using *xk*(*1*−*2*)
unfolding *k subset-eq*
apply (*auto simp add*:*dist-real-def*)
done
also have $\ldots \leq e * (Sup\ k - Inf\ k)$
unfolding *k interval-bounds-real*[*OF* ∗]
using *xk*(*1*)
unfolding *k*
by (*auto simp add*: *dist-real-def field-simps*)
finally show *norm* $(content\ k *_R f'\ x - (f\ (Sup\ k) - f\ (Inf\ k))) \leq$
$e * (Sup\ k - Inf\ k)$
unfolding *box-real k interval-bounds-real*[*OF* ∗] *content-real*[*OF* ∗]
interval-upperbound-real interval-lowerbound-real
.
qed
qed

qed

lemma *ident-has-integral*:
fixes *a*::*real*
assumes $a \leq b$
shows $((\lambda x.\ x)$ *has-integral* $(b^2 - a^2)$ */* $2)$ $\{a..b\}$
proof −
have $((\lambda x.\ x)$ *has-integral inverse* $2 * b^2 -$ *inverse* $2 * a^2)$ $\{a..b\}$
apply (*rule fundamental-theorem-of-calculus* [*OF assms*], *clarify*)
unfolding *power2-eq-square*
by (*rule derivative-eq-intros* | *simp*)+
then show *?thesis*
by (*simp add*: *field-simps*)
qed

lemma *integral-ident* [*simp*]:
fixes *a*::*real*
assumes $a \leq b$
shows *integral* $\{a..b\}$ $(\lambda x.\ x) =$ (*if* $a \leq b$ *then* $(b^2 - a^2)$ */* 2 *else* 0)
using *ident-has-integral integral-unique* **by** *fastforce*

lemma *ident-integrable-on*:
fixes *a*::*real*
shows $(\lambda x.\ x)$ *integrable-on* $\{a..b\}$
by (*metis atLeastatMost-empty-iff integrable-on-def has-integral-empty ident-has-integral*)

41.40 Taylor series expansion

lemma (**in** *bounded-bilinear*) *setsum-prod-derivatives-has-vector-derivative*:
assumes $p>0$
and *f0*: *Df* $0 = f$
and *Df*: $\bigwedge m\ t.\ m < p \Longrightarrow a \leq t \Longrightarrow t \leq b \Longrightarrow$
(*Df m has-vector-derivative Df* (*Suc m*) *t*) (*at t within* $\{a\ ..\ b\}$)
and *g0*: *Dg* $0 = g$
and *Dg*: $\bigwedge m\ t.\ m < p \Longrightarrow a \leq t \Longrightarrow t \leq b \Longrightarrow$
(*Dg m has-vector-derivative Dg* (*Suc m*) *t*) (*at t within* $\{a\ ..\ b\}$)
and *ivl*: $a \leq t\ t \leq b$
shows $((\lambda t.\ \sum i{<}p.\ (-1)\ \hat{}\ i *_R$ *prod* (*Df i t*) (*Dg* $(p -$ *Suc i*) *t*))
has-vector-derivative
prod (*f t*) (*Dg p t*) $-$ $(-1)\ \hat{}\ p *_R$ *prod* (*Df p t*) (*g t*))
(*at t within* $\{a\ ..\ b\}$)
using *assms*
proof *cases*
assume *p*: $p \neq 1$
def $p' \equiv p - 2$
from *assms p* **have** p': $\{..{<}p\} = \{..Suc\ p'\}$ $p =$ *Suc* (*Suc* p')
by (*auto simp*: p'-*def*)
have $*$: $\bigwedge i.\ i \leq p' \Longrightarrow$ *Suc* (*Suc* $p' - i$) = (*Suc* (*Suc* p') $- i$)
by *auto*

```
  let ?f = λi. (−1) ^ i *R (prod (Df i t) (Dg ((p − i)) t))
  have (∑ i<p. (−1) ^ i *R (prod (Df i t) (Dg (Suc (p − Suc i)) t) +
    prod (Df (Suc i) t) (Dg (p − Suc i) t))) =
    (∑ i≤(Suc p'). ?f i − ?f (Suc i))
    by (auto simp: algebra-simps p'(2) numeral-2-eq-2 * lessThan-Suc-atMost)
  also note setsum-telescope
  finally
  have (∑ i<p. (−1) ^ i *R (prod (Df i t) (Dg (Suc (p − Suc i)) t) +
    prod (Df (Suc i) t) (Dg (p − Suc i) t)))
    = prod (f t) (Dg p t) − (− 1) ^ p *R prod (Df p t) (g t)
    unfolding p'[symmetric]
    by (simp add: assms)
  thus ?thesis
    using assms
    by (auto intro!: derivative-eq-intros has-vector-derivative)
qed (auto intro!: derivative-eq-intros has-vector-derivative)

lemma
  fixes f::real⇒'a::banach
  assumes p>0
  and f0: Df 0 = f
  and Df: ⋀m t. m < p ⟹ a ≤ t ⟹ t ≤ b ⟹
    (Df m has-vector-derivative Df (Suc m) t) (at t within {a .. b})
  and ivl: a ≤ b
  defines i ≡ λx. ((b − x) ^ (p − 1) / fact (p − 1)) *R Df p x
  shows taylor-has-integral:
    (i has-integral f b − (∑ i<p. ((b − a) ^ i / fact i) *R Df i a)) {a..b}
  and taylor-integral:
    f b = (∑ i<p. ((b − a) ^ i / fact i) *R Df i a) + integral {a..b} i
  and taylor-integrable:
    i integrable-on {a .. b}
proof goal-cases
  case 1
  interpret bounded-bilinear scaleR::real⇒'a⇒'a
    by (rule bounded-bilinear-scaleR)
  def g ≡ λs. (b − s)^(p − 1)/fact (p − 1)
  def Dg ≡ λn s. if n < p then (−1)^n * (b − s)^(p − 1 − n) / fact (p − 1 −
n) else 0
  have g0: Dg 0 = g
    using ‹p > 0›
    by (auto simp add: Dg-def divide-simps g-def split: if-split-asm)
  {
    fix m
    assume p > Suc m
    hence p − Suc m = Suc (p − Suc (Suc m))
      by auto
    hence real (p − Suc m) * fact (p − Suc (Suc m)) = fact (p − Suc m)
      by auto
  } note fact-eq = this
```

have *Dg*: $\bigwedge m\ t.\ m < p \Longrightarrow a \leq t \Longrightarrow t \leq b \Longrightarrow$
(*Dg m has-vector-derivative Dg* (*Suc m*) *t*) (*at t within* {*a* .. *b*})
unfolding *Dg-def*
by (*auto intro*!: *derivative-eq-intros simp*: *has-vector-derivative-def fact-eq divide-simps*)
let *?sum* = $\lambda t.\ \sum i{<}p.\ (-1)\ \hat{}\ i *_R Dg\ i\ t *_R Df\ (p - Suc\ i)\ t$
from *setsum-prod-derivatives-has-vector-derivative*[*of - Dg - - - Df*,
OF ‹*p* > *0*› *g0 Dg f0 Df*]
have *deriv*: $\bigwedge t.\ a \leq t \Longrightarrow t \leq b \Longrightarrow$
(*?sum has-vector-derivative*
g t $*_R$ *Df p t* − (− *1*) ^ *p* $*_R$ *Dg p t* $*_R$ *f t*) (*at t within* {*a*..*b*})
by *auto*
from *fundamental-theorem-of-calculus*[*rule-format*, *OF* ‹*a* ≤ *b*› *deriv*]
have (*i has-integral ?sum b* − *?sum a*) {*a* .. *b*}
by (*simp add*: *i-def g-def Dg-def*)
also
have *one*: (− *1*) ^ *p′* ∗ (− *1*) ^ *p′* = (*1*::*real*)
and {..<*p*} ∩ {*i*. *p* = *Suc i*} = {*p* − *1*}
for *p′*
using ‹*p* > *0*›
by (*auto simp*: *power-mult-distrib*[*symmetric*])
then have *?sum b* = *f b*
using *Suc-pred′*[*OF* ‹*p* > *0*›]
by (*simp add*: *diff-eq-eq Dg-def power-0-left le-Suc-eq if-distrib*
cond-application-beta setsum.If-cases f0)
also
have {..<*p*} = (λ*x*. *p* − *x* − *1*) ‘ {..<*p*}
proof *safe*
fix *x*
assume *x* < *p*
thus *x* ∈ (λ*x*. *p* − *x* − *1*) ‘ {..<*p*}
by (*auto intro*!: *image-eqI*[**where** *x* = *p* − *x* − *1*])
qed *simp*
from - *this*
have *?sum a* = (∑ *i*<*p*. ((*b* − *a*) ^ *i* / *fact i*) $*_R$ *Df i a*)
by (*rule setsum.reindex-cong*) (*auto simp add*: *inj-on-def Dg-def one*)
finally show *c*: *?case* **.**
case *2* **show** *?case* **using** *c integral-unique* **by** *force*
case *3* **show** *?case* **using** *c* **by** *force*
qed

41.41 Attempt a systematic general set of "offset" results for components.

lemma *gauge-modify*:
assumes (∀ *s*. *open s* ⟶ *open* {*x*. *f*(*x*) ∈ *s*}) *gauge d*
shows *gauge* (λ*x*. {*y*. *f y* ∈ *d* (*f x*)})
using *assms*
unfolding *gauge-def*
apply *safe*

defer
apply (*erule-tac x=f x* **in** *allE*)
apply (*erule-tac x=d (f x)* **in** *allE*)
apply *auto*
done

41.42 Only need trivial subintervals if the interval itself is trivial.

lemma *division-of-nontrivial*:
fixes *s* :: *'a::euclidean-space set set*
assumes *s division-of (cbox a b)*
and *content (cbox a b) ≠ 0*
shows {*k*. *k ∈ s ∧ content k ≠ 0*} *division-of (cbox a b)*
using *assms(1)*
apply −
proof (*induct card s arbitrary*: *s rule*: *nat-less-induct*)
fix *s::'a set set*
assume *assm*: *s division-of (cbox a b)*
∀ m<card s. ∀ x. m = card x ⟶
x division-of (cbox a b) ⟶ {k ∈ x. content k ≠ 0} division-of (cbox a b)
note *s = division-ofD[OF assm(1)]*
let *?thesis = {k ∈ s. content k ≠ 0} division-of (cbox a b)*
{
presume ∗: *{k ∈ s. content k ≠ 0} ≠ s ⟹ ?thesis*
show *?thesis*
apply *cases*
defer
apply (*rule* ∗)
apply *assumption*
using *assm(1)*
apply *auto*
done
}
assume *noteq*: *{k ∈ s. content k ≠ 0} ≠ s*
then obtain *k* **where** *k*: *k ∈ s content k = 0*
by *auto*
from *s(4)[OF k(1)]* **guess** *c d* **by** (*elim exE*) **note** *k=k this*
from *k* **have** *card s > 0*
unfolding *card-gt-0-iff* **using** *assm(1)* **by** *auto*
then have *card*: *card (s − {k}) < card s*
using *assm(1) k(1)*
apply (*subst card-Diff-singleton-if*)
apply *auto*
done
have ∗: *closed (⋃(s − {k}))*
apply (*rule closed-Union*)
defer
apply *rule*

```
  apply (drule Diff_D1,drule s(4))
  using assm(1)
  apply auto
  done
have k ⊆ ⋃(s - {k})
  apply safe
  apply (rule *[unfolded closed_limpt,rule_format])
  unfolding islimpt_approachable
proof safe
  fix x
  fix e :: real
  assume as: x ∈ k e > 0
  from k(2)[unfolded k content_eq_0] guess i ..
  then have i:c•i = d•i i∈Basis
    using s(3)[OF k(1),unfolded k] unfolding box_ne_empty by auto
  then have xi: x•i = d•i
    using as unfolding k mem_box by (metis antisym)
  def y ≡ ∑j∈Basis. (if j = i then if c•i ≤ (a•i + b•i) / 2 then c•i +
    min e (b•i - c•i) / 2 else c•i - min e (c•i - a•i) / 2 else x•j) *R j
  show ∃x'∈⋃(s - {k}). x' ≠ x ∧ dist x' x < e
    apply (rule_tac x=y in bexI)
  proof
    have d ∈ cbox c d
      using s(3)[OF k(1)]
      unfolding k box_eq_empty mem_box
      by (fastforce simp add: not_less)
    then have d ∈ cbox a b
      using s(2)[OF k(1)]
      unfolding k
      by auto
    note di = this[unfolded mem_box,THEN bspec[where x=i]]
    then have xyi: y•i ≠ x•i
      unfolding y_def i xi
      using as(2) assms(2)[unfolded content_eq_0] i(2)
      by (auto elim!: ballE[of - - i])
    then show y ≠ x
      unfolding euclidean_eq_iff[where 'a='a] using i by auto
    have *: Basis = insert i (Basis - {i})
      using i by auto
    have norm (y - x) < e + setsum (λi. 0) Basis
      apply (rule le_less_trans[OF norm_le_l1])
      apply (subst *)
      apply (subst setsum.insert)
      prefer 3
      apply (rule add_less_le_mono)
    proof -
      show |(y - x) • i| < e
        using di as(2) y_def i xi by (auto simp: inner_simps)
      show (∑i∈Basis - {i}. |(y - x) • i|) ≤ (∑i∈Basis. 0)
```

```
      unfolding y-def by (auto simp: inner-simps)
    qed auto
    then show dist y x < e
      unfolding dist-norm by auto
    have y ∉ k
      unfolding k mem-box
      apply rule
      apply (erule-tac x=i in ballE)
      using xyi k i xi
      apply auto
      done
    moreover
    have y ∈ ⋃ s
      using set-rev-mp[OF as(1) s(2)[OF k(1)]] as(2) di i
      unfolding s mem-box y-def
      by (auto simp: field-simps elim!: ballE[of - - i])
    ultimately
    show y ∈ ⋃ (s - {k}) by auto
  qed
 qed
 then have ⋃ (s - {k}) = cbox a b
  unfolding s(6)[symmetric] by auto
 then have {ka ∈ s - {k}. content ka ≠ 0} division-of (cbox a b)
  apply -
  apply (rule assm(2)[rule-format,OF card refl])
  apply (rule division-ofI)
  defer
  apply (rule-tac[1-4] s)
  using assm(1)
  apply auto
  done
 moreover
 have {ka ∈ s - {k}. content ka ≠ 0} = {k ∈ s. content k ≠ 0}
  using k by auto
 ultimately show ?thesis by auto
qed
```

41.43 Integrability on subintervals.

```
lemma operative-integrable:
 fixes f :: 'b::euclidean-space ⇒ 'a::banach
 shows operative op ∧ (λi. f integrable-on i)
 unfolding operative-def neutral-and
 apply safe
 apply (subst integrable-on-def)
 unfolding has-integral-null-eq
 apply (rule, rule refl)
 apply (rule, assumption, assumption)+
 unfolding integrable-on-def
```

by (*auto intro*!: *has-integral-split*)

lemma *integrable-subinterval*:
fixes *f* :: *'b*::*euclidean-space* ⇒ *'a*::*banach*
assumes *f integrable-on cbox a b*
and *cbox c d* ⊆ *cbox a b*
shows *f integrable-on cbox c d*
apply (*cases cbox c d* = {})
defer
apply (*rule partial-division-extend-1*[*OF assms*(*2*)],*assumption*)
using *operative-division-and*[*OF operative-integrable*,*symmetric*,*of - - - f*] *assms*(*1*)
apply *auto*
done

lemma *integrable-subinterval-real*:
fixes *f* :: *real* ⇒ *'a*::*banach*
assumes *f integrable-on* {*a* .. *b*}
and {*c* .. *d*} ⊆ {*a* .. *b*}
shows *f integrable-on* {*c* .. *d*}
by (*metis assms*(*1*) *assms*(*2*) *box-real*(*2*) *integrable-subinterval*)

41.44 Combining adjacent intervals in 1 dimension.

lemma *has-integral-combine*:
fixes *a b c* :: *real*
assumes *a* ≤ *c*
and *c* ≤ *b*
and (*f has-integral i*) {*a* .. *c*}
and (*f has-integral* (*j*::*'a*::*banach*)) {*c* .. *b*}
shows (*f has-integral* (*i* + *j*)) {*a* .. *b*}
proof −
note *operative-integral*[*of f*, *unfolded operative-1-le*[*OF monoidal-lifted*[*OF monoidal-monoid*]]]
note *conjunctD2*[*OF this*,*rule-format*]
note ∗ = *this*(*2*)[*OF conjI*[*OF assms*(*1*−*2*)],*unfolded if-P*[*OF assms*(*3*)]]
then have *f integrable-on cbox a b*
apply −
apply (*rule ccontr*)
apply (*subst*(*asm*) *if-P*)
defer
apply (*subst*(*asm*) *if-P*)
using *assms*(*3*−)
apply *auto*
done
with ∗
show *?thesis*
apply −
apply (*subst*(*asm*) *if-P*)
defer
apply (*subst*(*asm*) *if-P*)

defer
apply (*subst*(*asm*) *if-P*)
unfolding *lifted.simps*
using *assms*(3−)
apply (*auto simp add*: *integrable-on-def integral-unique*)
done
qed

lemma *integral-combine*:
fixes *f* :: *real* ⇒ 'a::*banach*
assumes *a* ≤ *c*
and *c* ≤ *b*
and *f integrable-on* {*a* .. *b*}
shows *integral* {*a* .. *c*} *f* + *integral* {*c* .. *b*} *f* = *integral* {*a* .. *b*} *f*
apply (*rule integral-unique*[*symmetric*])
apply (*rule has-integral-combine*[*OF assms*(1−2)])
apply (*metis assms*(2) *assms*(3) *atLeastatMost-subset-iff box-real*(2) *content-pos-le*
content-real-eq-0 integrable-integral integrable-subinterval le-add-same-cancel2 monoid-add-class.add.left-neutra
by (*metis assms*(1) *assms*(3) *atLeastatMost-subset-iff box-real*(2) *content-pos-le*
content-real-eq-0 integrable-integral integrable-subinterval le-add-same-cancel1 monoid-add-class.add.right-neut

lemma *integrable-combine*:
fixes *f* :: *real* ⇒ 'a::*banach*
assumes *a* ≤ *c*
and *c* ≤ *b*
and *f integrable-on* {*a* .. *c*}
and *f integrable-on* {*c* .. *b*}
shows *f integrable-on* {*a* .. *b*}
using *assms*
unfolding *integrable-on-def*
by (*fastforce intro*!:*has-integral-combine*)

41.45 Reduce integrability to "local" integrability.

lemma *integrable-on-little-subintervals*:
fixes *f* :: 'b::*euclidean-space* ⇒ 'a::*banach*
assumes ∀ *x*∈*cbox a b*. ∃ *d*>0. ∀ *u v*. *x* ∈ *cbox u v* ∧ *cbox u v* ⊆ *ball x d* ∧ *cbox*
u v ⊆ *cbox a b* ⟶
f integrable-on cbox u v
shows *f integrable-on cbox a b*
proof −
have ∀ *x*. ∃ *d*. *x*∈*cbox a b* ⟶ *d*>0 ∧ (∀ *u v*. *x* ∈ *cbox u v* ∧ *cbox u v* ⊆ *ball x d*
∧ *cbox u v* ⊆ *cbox a b* ⟶
f integrable-on cbox u v)
using *assms* **by** *auto*
note *this*[*unfolded gauge-existence-lemma*]
from *choice*[*OF this*] **guess** *d* **.. note** *d*=*this*[*rule-format*]
guess *p*
apply (*rule fine-division-exists*[*OF gauge-ball-dependent*,*of d a b*])

```
    using d
    by auto
  note p=this(1-2)
  note division-of-tagged-division[OF this(1)]
  note * = operative-division-and[OF operative-integrable,OF this,symmetric,of f]
  show ?thesis
    unfolding *
    apply safe
    unfolding snd-conv
  proof -
    fix x k
    assume (x, k) ∈ p
    note tagged-division-ofD(2-4)[OF p(1) this] fineD[OF p(2) this]
    then show f integrable-on k
      apply safe
      apply (rule d[THEN conjunct2,rule-format,of x])
      apply (auto intro: order.trans)
      done
  qed
qed
```

41.46 Second FCT or existence of antiderivative.

```
lemma integrable-const[intro]: (λx. c) integrable-on cbox a b
  unfolding integrable-on-def
  apply rule
  apply (rule has-integral-const)
  done

lemma integral-has-vector-derivative-continuous-at:
  fixes f :: real ⇒ 'a::banach
  assumes f: f integrable-on {a..b}
      and x: x ∈ {a..b}
      and fx: continuous (at x within {a..b}) f
  shows ((λu. integral {a..u} f) has-vector-derivative f x) (at x within {a..b})
proof -
  let ?I = λa b. integral {a..b} f
  { fix e::real
    assume e > 0
    obtain d where d>0 and d: ⋀x'. ⟦x' ∈ {a..b}; |x' - x| < d⟧ ⟹ norm(f x'
- f x) ≤ e
      using ‹e>0› fx by (auto simp: continuous-within-eps-delta dist-norm less-imp-le)
      have norm (integral {a..y} f - integral {a..x} f - (y - x) *R f x) ≤ e * |y
- x|
          if y: y ∈ {a..b} and yx: |y - x| < d for y
      proof (cases y < x)
        case False
        have f integrable-on {a..y}
          using f y by (simp add: integrable-subinterval-real)
```

then have *Idiff*: *?I a y − ?I a x = ?I x y*
using *False x* **by** (*simp add*: *algebra-simps integral-combine*)
have *fux-int*: ((λ*u*. *f u − f x*) *has-integral integral* {*x..y*} *f − (y − x) ∗R f x*) {*x..y*}
apply (*rule has-integral-sub*)
using *x y* **apply** (*force intro*: *integrable-integral* [*OF integrable-subinterval-real* [*OF f*]])
using *has-integral-const-real* [*of f x x y*] *False*
apply (*simp add*:)
done
show *?thesis*
using *False*
apply (*simp add*: *abs-eq-content del*: *content-real-if*)
apply (*rule has-integral-bound-real*[**where** *f*=(λ*u*. *f u − f x*)])
using *yx False d x y* ‹*e*>*0*› **apply** (*auto simp add*: *Idiff fux-int*)
done
next
case *True*
have *f integrable-on* {*a..x*}
using *f x* **by** (*simp add*: *integrable-subinterval-real*)
then have *Idiff*: *?I a x − ?I a y = ?I y x*
using *True x y* **by** (*simp add*: *algebra-simps integral-combine*)
have *fux-int*: ((λ*u*. *f u − f x*) *has-integral integral* {*y..x*} *f − (x − y) ∗R f x*) {*y..x*}
apply (*rule has-integral-sub*)
using *x y* **apply** (*force intro*: *integrable-integral* [*OF integrable-subinterval-real* [*OF f*]])
using *has-integral-const-real* [*of f x y x*] *True*
apply (*simp add*:)
done
have *norm* (*integral* {*a..x*} *f − integral* {*a..y*} *f − (x − y) ∗R f x*) ≤ *e ∗* |*y − x*|
using *True*
apply (*simp add*: *abs-eq-content del*: *content-real-if*)
apply (*rule has-integral-bound-real*[**where** *f*=(λ*u*. *f u − f x*)])
using *yx True d x y* ‹*e*>*0*› **apply** (*auto simp add*: *Idiff fux-int*)
done
then show *?thesis*
by (*simp add*: *algebra-simps norm-minus-commute*)
qed
then have ∃ *d*>*0*. ∀ *y*∈{*a..b*}. |*y − x*| < *d* ⟶ *norm* (*integral* {*a..y*} *f − integral* {*a..x*} *f − (y − x) ∗R f x*) ≤ *e ∗* |*y − x*|
using ‹*d*>*0*› **by** *blast*
}
then show *?thesis*
by (*simp add*: *has-vector-derivative-def has-derivative-within-alt bounded-linear-scaleR-left*)
qed

lemma *integral-has-vector-derivative*:

fixes $f :: real \Rightarrow 'a::banach$
assumes *continuous-on* $\{a .. b\}$ f
and $x \in \{a .. b\}$
shows $((\lambda u.\ integral\ \{a .. u\}\ f)\ has\text{-}vector\text{-}derivative\ f(x))\ (at\ x\ within\ \{a .. b\})$
apply (*rule integral-has-vector-derivative-continuous-at* [*OF integrable-continuous-real*])
using *assms*
apply (*auto simp*: *continuous-on-eq-continuous-within*)
done

lemma *antiderivative-continuous*:
fixes $q\ b :: real$
assumes *continuous-on* $\{a .. b\}$ f
obtains g **where** $\forall x{\in}\{a .. b\}.\ (g\ has\text{-}vector\text{-}derivative\ (f\ x::-::banach))\ (at\ x\ within\ \{a .. b\})$
apply (*rule that*)
apply *rule*
using *integral-has-vector-derivative*[*OF assms*]
apply *auto*
done

41.47 Combined fundamental theorem of calculus.

lemma *antiderivative-integral-continuous*:
fixes $f :: real \Rightarrow 'a::banach$
assumes *continuous-on* $\{a .. b\}$ f
obtains g **where** $\forall u{\in}\{a .. b\}.\ \forall v \in \{a .. b\}.\ u \le v \longrightarrow (f\ has\text{-}integral\ (g\ v - g\ u))\ \{u .. v\}$
proof −
from *antiderivative-continuous*[*OF assms*] **guess** g **. note** $g{=}this$
show *?thesis*
apply (*rule that*[*of g*])
apply *safe*
proof *goal-cases*
case *prems*: (*1 u v*)
have $\forall x{\in}cbox\ u\ v.\ (g\ has\text{-}vector\text{-}derivative\ f\ x)\ (at\ x\ within\ cbox\ u\ v)$
apply *rule*
apply (*rule has-vector-derivative-within-subset*)
apply (*rule g*[*rule-format*])
using *prems*(*1*,*2*)
apply *auto*
done
then show *?case*
using *fundamental-theorem-of-calculus*[*OF prems*(*3*), *of g f*] **by** *auto*
qed
qed

41.48 General "twiddling" for interval-to-interval function image.

lemma *has-integral-twiddle*:
assumes $0 < r$
and $\forall x.\ h(g\ x) = x$
and $\forall x.\ g(h\ x) = x$
and $\forall x.\ \textit{continuous}\ (\textit{at}\ x)\ g$
and $\forall u\ v.\ \exists w\ z.\ g\ `\ \textit{cbox}\ u\ v = \textit{cbox}\ w\ z$
and $\forall u\ v.\ \exists w\ z.\ h\ `\ \textit{cbox}\ u\ v = \textit{cbox}\ w\ z$
and $\forall u\ v.\ \textit{content}(g\ `\ \textit{cbox}\ u\ v) = r * \textit{content}\ (\textit{cbox}\ u\ v)$
and $(f\ \textit{has-integral}\ i)\ (\textit{cbox}\ a\ b)$
shows $((\lambda x.\ f(g\ x))\ \textit{has-integral}\ (1\ /\ r) *_R i)\ (h\ `\ \textit{cbox}\ a\ b)$
proof −
show *?thesis* **when** $*$: $\textit{cbox}\ a\ b \neq \{\} \Longrightarrow$ *?thesis*
apply *cases*
defer
apply (*rule* $*$)
apply *assumption*
proof *goal-cases*
case *prems*: *1*
then show *?thesis*
unfolding *prems assms*(*8*)[*unfolded prems has-integral-empty-eq*] **by** *auto*
qed
assume $\textit{cbox}\ a\ b \neq \{\}$
from *assms*(*6*)[*rule-format*,*of a b*] **guess** *w z* **by** (*elim exE*) **note** *wz*=*this*
have *inj*: *inj g inj h*
unfolding *inj-on-def*
apply *safe*
apply(*rule-tac*[!] *ccontr*)
using *assms*(*2*)
apply(*erule-tac x*=*x* **in** *allE*)
using *assms*(*2*)
apply(*erule-tac x*=*y* **in** *allE*)
defer
using *assms*(*3*)
apply (*erule-tac x*=*x* **in** *allE*)
using *assms*(*3*)
apply(*erule-tac x*=*y* **in** *allE*)
apply *auto*
done
show *?thesis*
unfolding *has-integral-def has-integral-compact-interval-def*
apply (*subst if-P*)
apply *rule*
apply *rule*
apply (*rule wz*)
proof *safe*
fix *e* :: *real*
assume *e*: $e > 0$

```
  with assms(1) have e * r > 0 by simp
   from assms(8)[unfolded has-integral,rule-format,OF this] guess d by (elim
exE conjE) note d=this[rule-format]
  def d' ≡ λx. {y. g y ∈ d (g x)}
  have d': ⋀x. d' x = {y. g y ∈ (d (g x))}
    unfolding d'-def ..
  show ∃d. gauge d ∧ (∀p. p tagged-division-of h ‘ cbox a b ∧ d fine p ⟶ norm
((∑(x, k)∈p. content k *R f (g x)) − (1 / r) *R i) < e)
  proof (rule-tac x=d' in exI, safe)
    show gauge d'
      using d(1)
      unfolding gauge-def d'
      using continuous-open-preimage-univ[OF assms(4)]
      by auto
    fix p
    assume as: p tagged-division-of h ‘ cbox a b d' fine p
    note p = tagged-division-ofD[OF as(1)]
   have (λ(x, k). (g x, g ‘ k)) ‘ p tagged-division-of (cbox a b) ∧ d fine (λ(x, k).
(g x, g ‘ k)) ‘ p
      unfolding tagged-division-of
    proof safe
      show finite ((λ(x, k). (g x, g ‘ k)) ‘ p)
        using as by auto
      show d fine (λ(x, k). (g x, g ‘ k)) ‘ p
        using as(2) unfolding fine-def d' by auto
      fix x k
      assume xk[intro]: (x, k) ∈ p
      show g x ∈ g ‘ k
        using p(2)[OF xk] by auto
      show ∃u v. g ‘ k = cbox u v
        using p(4)[OF xk] using assms(5−6) by auto
      {
        fix y
        assume y ∈ k
        then show g y ∈ cbox a b g y ∈ cbox a b
          using p(3)[OF xk,unfolded subset-eq,rule-format,of h (g y)]
          using assms(2)[rule-format,of y]
          unfolding inj-image-mem-iff[OF inj(2)]
          by auto
      }
      fix x' k'
      assume xk': (x', k') ∈ p
      fix z
      assume z ∈ interior (g ‘ k) and z ∈ interior (g ‘ k')
      then have *: interior (g ‘ k) ∩ interior (g ‘ k') ≠ {}
        by auto
      have same: (x, k) = (x', k')
        apply −
        apply (rule ccontr)
```

```
      apply (drule p(5)[OF xk xk'])
     proof −
      assume as: interior k ∩ interior k' = {}
      from nonempty-witness[OF *] guess z .
      then have z ∈ g ‘ (interior k ∩ interior k')
       using interior-image-subset[OF assms(4) inj(1)]
       unfolding image-Int[OF inj(1)]
       by auto
      then show False
       using as by blast
     qed
     then show g x = g x'
      by auto
     {
      fix z
      assume z ∈ k
      then show g z ∈ g ‘ k'
       using same by auto
     }
     {
      fix z
      assume z ∈ k'
      then show g z ∈ g ‘ k
       using same by auto
     }
    next
     fix x
     assume x ∈ cbox a b
     then have h x ∈ ⋃{k. ∃x. (x, k) ∈ p}
      using p(6) by auto
     then guess X unfolding Union-iff .. note X=this
     from this(1) guess y unfolding mem-Collect-eq ..
     then show x ∈ ⋃{k. ∃x. (x, k) ∈ (λ(x, k). (g x, g ‘ k)) ‘ p}
      apply −
      apply (rule-tac X=g ‘ X in UnionI)
      defer
      apply (rule-tac x=h x in image-eqI)
      using X(2) assms(3)[rule-format,of x]
      apply auto
      done
    qed
     note ** = d(2)[OF this]
     have *: inj-on (λ(x, k). (g x, g ‘ k)) p
      using inj(1) unfolding inj-on-def by fastforce
     have (∑(x, k)∈(λ(x, k). (g x, g ‘ k)) ‘ p. content k *R f x) − i = r *R
(∑(x, k)∈p. content k *R f (g x)) − i (is ?l = -)
      using assms(7)
      unfolding algebra-simps add-left-cancel scaleR-right.setsum
      by (subst setsum.reindex-bij-betw[symmetric, where h=λ(x, k). (g x, g ‘
```

```
k) and S=p])
          (auto intro!: * setsum.cong simp: bij-betw-def dest!: p(4))
     also have ... = r *R ((∑(x, k)∈p. content k *R f (g x)) - (1 / r) *R i)
(is - = ?r)
       unfolding scaleR-diff-right scaleR-scaleR
       using assms(1)
       by auto
     finally have *: ?l = ?r .
     show norm ((∑(x, k)∈p. content k *R f (g x)) - (1 / r) *R i) < e
       using **
       unfolding *
       unfolding norm-scaleR
       using assms(1)
       by (auto simp add:field-simps)
   qed
  qed
qed
```

41.49 Special case of a basic affine transformation.

```
lemma interval-image-affinity-interval:
  ∃u v. (λx. m *R (x::'a::euclidean-space) + c) ' cbox a b = cbox u v
  unfolding image-affinity-cbox
  by auto

lemma content-image-affinity-cbox:
  content((λx::'a::euclidean-space. m *R x + c) ' cbox a b) =
    |m| ^ DIM('a) * content (cbox a b) (is ?l = ?r)
proof (cases cbox a b = {})
  case True then show ?thesis by simp
next
  case False
  show ?thesis
  proof (cases m ≥ 0)
    case True
    with ‹cbox a b ≠ {}› have cbox (m *R a + c) (m *R b + c) ≠ {}
      unfolding box-ne-empty
      apply (intro ballI)
      apply (erule-tac x=i in ballE)
      apply (auto simp: inner-simps mult-left-mono)
      done
    moreover from True have *: ⋀i. (m *R b + c) · i - (m *R a + c) · i = m
*R (b - a) · i
      by (simp add: inner-simps field-simps)
    ultimately show ?thesis
      by (simp add: image-affinity-cbox True content-cbox'
        setprod.distrib setprod-constant inner-diff-left)
  next
    case False
```

with ‹*cbox a b* ≠ {}› **have** *cbox* (*m* *$_R$ *b* + *c*) (*m* *$_R$ *a* + *c*) ≠ {}
unfolding *box-ne-empty*
apply (*intro ballI*)
apply (*erule-tac x=i* **in** *ballE*)
apply (*auto simp*: *inner-simps mult-left-mono*)
done
moreover from *False* **have** *: ⋀*i*. (*m* *$_R$ *a* + *c*) ∙ *i* − (*m* *$_R$ *b* + *c*) ∙ *i* = (−*m*) *$_R$ (*b* − *a*) ∙ *i*
by (*simp add*: *inner-simps field-simps*)
ultimately show *?thesis* **using** *False*
by (*simp add*: *image-affinity-cbox content-cbox′*
setprod.distrib[*symmetric*] *setprod-constant*[*symmetric*] *inner-diff-left*)
qed
qed

lemma *has-integral-affinity*:
fixes *a* :: *′a*::*euclidean-space*
assumes (*f has-integral i*) (*cbox a b*)
and *m* ≠ *0*
shows ((*λx*. *f*(*m* *$_R$ *x* + *c*)) *has-integral* ((*1* / (|*m*| ^ *DIM*(*′a*))) *$_R$ *i*)) ((*λx*. (*1* / *m*) *$_R$ *x* + −((*1* / *m*) *$_R$ *c*)) ‘ *cbox a b*)
apply (*rule has-integral-twiddle*)
using *assms*
apply (*safe intro*!: *interval-image-affinity-interval content-image-affinity-cbox*)
apply (*rule zero-less-power*)
unfolding *scaleR-right-distrib*
apply *auto*
done

lemma *integrable-affinity*:
assumes *f integrable-on cbox a b*
and *m* ≠ *0*
shows (*λx*. *f*(*m* *$_R$ *x* + *c*)) *integrable-on* ((*λx*. (*1* / *m*) *$_R$ *x* + −((*1*/*m*) *$_R$ *c*)) ‘ *cbox a b*)
using *assms*
unfolding *integrable-on-def*
apply *safe*
apply (*drule has-integral-affinity*)
apply *auto*
done

lemmas *has-integral-affinity01* = *has-integral-affinity* [*of* - - *0 1*::*real*, *simplified*]

41.50 Special case of stretching coordinate axes separately.

lemma *image-stretch-interval*:
(*λx*. ∑*k*∈*Basis*. (*m k* ∗ (*x*∙*k*)) *$_R$ *k*) ‘ *cbox a* (*b*::*′a*::*euclidean-space*) =
(*if* (*cbox a b*) = {} *then* {} *else*
cbox (∑*k*∈*Basis*. (*min* (*m k* ∗ (*a*∙*k*)) (*m k* ∗ (*b*∙*k*))) *$_R$ *k*::*′a*)

```
    (∑ k∈Basis. (max (m k * (a•k)) (m k * (b•k))) *R k))
proof cases
  assume *: cbox a b ≠ {}
  show ?thesis
    unfolding box-ne-empty if-not-P[OF *]
   apply (simp add: cbox-def image-Collect set-eq-iff euclidean-eq-iff [where 'a='a]
ball-conj-distrib[symmetric])
    apply (subst choice-Basis-iff[symmetric])
  proof (intro allI ball-cong refl)
    fix x i :: 'a assume i ∈ Basis
    with * have a-le-b: a • i ≤ b • i
      unfolding box-ne-empty by auto
    show (∃ xa. x • i = m i * xa ∧ a • i ≤ xa ∧ xa ≤ b • i) ⟷
        min (m i * (a • i)) (m i * (b • i)) ≤ x • i ∧ x • i ≤ max (m i * (a • i))
(m i * (b • i))
    proof (cases m i = 0)
      case True
      with a-le-b show ?thesis by auto
    next
      case False
      then have *: ⋀a b. a = m i * b ⟷ b = a / m i
        by (auto simp add: field-simps)
      from False have
          min (m i * (a • i)) (m i * (b • i)) = (if 0 < m i then m i * (a • i) else m
i * (b • i))
          max (m i * (a • i)) (m i * (b • i)) = (if 0 < m i then m i * (b • i) else m
i * (a • i))
        using a-le-b by (auto simp: min-def max-def mult-le-cancel-left)
      with False show ?thesis using a-le-b
        unfolding * by (auto simp add: le-divide-eq divide-le-eq ac-simps)
    qed
  qed
qed simp

lemma interval-image-stretch-interval:
  ∃ u v. (λx. ∑ k∈Basis. (m k * (x•k))*R k) ' cbox a (b::'a::euclidean-space) =
cbox u (v::'a::euclidean-space)
  unfolding image-stretch-interval by auto

lemma content-image-stretch-interval:
  content ((λx::'a::euclidean-space. (∑ k∈Basis. (m k * (x•k))*R k)::'a) ' cbox a
b) =
    |setprod m Basis| * content (cbox a b)
proof (cases cbox a b = {})
  case True
  then show ?thesis
    unfolding content-def image-is-empty image-stretch-interval if-P[OF True] by
auto
next
```

```
  case False
  then have (λx. (∑ k∈Basis. (m k * (x•k))*R k)) ` cbox a b ≠ {}
    by auto
  then show ?thesis
    using False
    unfolding content-def image-stretch-interval
    apply −
    unfolding interval-bounds' if-not-P
    unfolding abs-setprod setprod.distrib[symmetric]
    apply (rule setprod.cong)
    apply (rule refl)
    unfolding lessThan-iff
    apply (simp only: inner-setsum-left-Basis)
  proof −
    fix i :: 'a
    assume i: i ∈ Basis
    have (m i < 0 ∨ m i > 0) ∨ m i = 0
      by auto
    then show max (m i * (a • i)) (m i * (b • i)) − min (m i * (a • i)) (m i *
(b • i)) =
      |m i| * (b • i − a • i)
      apply −
      apply (erule disjE)+
      unfolding min-def max-def
      using False[unfolded box-ne-empty,rule-format,of i] i
    apply (auto simp add:field-simps not-le mult-le-cancel-left-neg mult-le-cancel-left-pos)
      done
  qed
qed

lemma has-integral-stretch:
  fixes f :: 'a::euclidean-space ⇒ 'b::real-normed-vector
  assumes (f has-integral i) (cbox a b)
    and ∀ k∈Basis. m k ≠ 0
  shows ((λx. f (∑ k∈Basis. (m k * (x•k))*R k)) has-integral
    ((1/ |setprod m Basis|) *R i)) ((λx. (∑ k∈Basis. (1 / m k * (x•k))*R k)) `
cbox a b)
  apply (rule has-integral-twiddle[where f=f])
  unfolding zero-less-abs-iff content-image-stretch-interval
 unfolding image-stretch-interval empty-as-interval euclidean-eq-iff[where 'a='a]
  using assms
proof −
  show ∀ y::'a. continuous (at y) (λx. (∑ k∈Basis. (m k * (x•k))*R k))
    apply rule
    apply (rule linear-continuous-at)
    unfolding linear-linear
    unfolding linear-iff inner-simps euclidean-eq-iff[where 'a='a]
    apply (auto simp add: field-simps)
    done
```

qed *auto*

lemma *integrable-stretch*:
fixes $f :: 'a{::}euclidean\text{-}space \Rightarrow 'b{::}real\text{-}normed\text{-}vector$
assumes f *integrable-on cbox a b*
and $\forall k{\in}Basis.\ m\ k \neq 0$
shows $(\lambda x{::}'a.\ f\ (\sum k{\in}Basis.\ (m\ k * (x{\cdot}k))*_R\ k))$ *integrable-on*
$((\lambda x.\ \sum k{\in}Basis.\ (1\ /\ m\ k * (x{\cdot}k))*_R\ k)$ ‘ *cbox a b*)
using *assms*
unfolding *integrable-on-def*
apply −
apply (*erule exE*)
apply (*drule has-integral-stretch*)
apply *assumption*
apply *auto*
done

41.51 even more special cases.

lemma *uminus-interval-vector*[*simp*]:
fixes $a\ b :: 'a{::}euclidean\text{-}space$
shows *uminus* ‘ *cbox a b* = *cbox* $(-b)$ $(-a)$
apply (*rule set-eqI*)
apply *rule*
defer
unfolding *image-iff*
apply (*rule-tac* $x{=}-x$ **in** *bexI*)
apply (*auto simp add*:*minus-le-iff le-minus-iff mem-box*)
done

lemma *has-integral-reflect-lemma*[*intro*]:
assumes (*f has-integral i*) (*cbox a b*)
shows $((\lambda x.\ f(-x))$ *has-integral i*) (*cbox* $(-b)$ $(-a)$)
using *has-integral-affinity*[*OF assms*, *of* $-1\ 0$]
by *auto*

lemma *has-integral-reflect-lemma-real*[*intro*]:
assumes (*f has-integral i*) $\{a\ ..\ b{::}real\}$
shows $((\lambda x.\ f(-x))$ *has-integral i*) $\{-b\ ..\ -a\}$
using *assms*
unfolding *box-real*[*symmetric*]
by (*rule has-integral-reflect-lemma*)

lemma *has-integral-reflect*[*simp*]:
$((\lambda x.\ f\ (-x))$ *has-integral i*) (*cbox* $(-b)$ $(-a)$) $\longleftrightarrow$ (*f has-integral i*) (*cbox a b*)
apply *rule*
apply (*drule-tac*[!] *has-integral-reflect-lemma*)
apply *auto*
done

lemma *integrable-reflect*[*simp*]: $(\lambda x.\ f(-x))$ *integrable-on cbox* $(-b)$ $(-a) \longleftrightarrow f$ *integrable-on cbox a b*
unfolding *integrable-on-def* **by** *auto*

lemma *integrable-reflect-real*[*simp*]: $(\lambda x.\ f(-x))$ *integrable-on* $\{-b\ ..\ -a\} \longleftrightarrow f$ *integrable-on* $\{a\ ..\ b{::}real\}$
unfolding *box-real*[*symmetric*]
by (*rule integrable-reflect*)

lemma *integral-reflect*[*simp*]: *integral* $(cbox\ (-b)\ (-a))\ (\lambda x.\ f\ (-x)) =$ *integral* $(cbox\ a\ b)\ f$
unfolding *integral-def* **by** *auto*

lemma *integral-reflect-real*[*simp*]: *integral* $\{-b\ ..\ -a\}\ (\lambda x.\ f\ (-x)) =$ *integral* $\{a\ ..\ b{::}real\}\ f$
unfolding *box-real*[*symmetric*]
by (*rule integral-reflect*)

41.52 Stronger form of FCT; quite a tedious proof.

lemma *bgauge-existence-lemma*: $(\forall x{\in}s.\ \exists d{::}real.\ 0 < d \wedge q\ d\ x) \longleftrightarrow (\forall x.\ \exists d{>}0.\ x{\in}s \longrightarrow q\ d\ x)$
by (*meson zero-less-one*)

lemma *additive-tagged-division-1'*:
fixes f :: *real* $\Rightarrow$ *'a*::*real-normed-vector*
assumes $a \le b$
and p *tagged-division-of* $\{a..b\}$
shows *setsum* $(\lambda(x,k).\ f\ (Sup\ k) - f(Inf\ k))\ p = f\ b - f\ a$
using *additive-tagged-division-1*[*OF - assms*(*2*), *of f*]
using *assms*(*1*)
by *auto*

lemma *split-minus*[*simp*]: $(\lambda(x,\ k).\ f\ x\ k)\ x - (\lambda(x,\ k).\ g\ x\ k)\ x = (\lambda(x,\ k).\ f\ x\ k - g\ x\ k)\ x$
by (*simp add*: *split-def*)

lemma *norm-triangle-le-sub*: *norm* x + *norm* $y \le e \Longrightarrow$ *norm* $(x - y) \le e$
apply (*subst*(*asm*)(*2*) *norm-minus-cancel*[*symmetric*])
apply (*drule norm-triangle-le*)
apply (*auto simp add*: *algebra-simps*)
done

lemma *fundamental-theorem-of-calculus-interior*:
fixes f :: *real* $\Rightarrow$ *'a*::*real-normed-vector*
assumes $a \le b$
and *continuous-on* $\{a\ ..\ b\}\ f$
and $\forall x{\in}\{a <..< b\}.$ $(f$ *has-vector-derivative* $f'(x))$ $(at\ x)$

```
  shows (f' has-integral (f b − f a)) {a .. b}
proof −
  {
    presume ∗: a < b ⟹ ?thesis
    show ?thesis
    proof (cases a < b)
      case True
      then show ?thesis by (rule ∗)
    next
      case False
      then have a = b
        using assms(1) by auto
      then have ∗: cbox a b = {b} f b − f a = 0
        by (auto simp add:  order-antisym)
      show ?thesis
        unfolding ∗(2)
        unfolding content-eq-0
        using ∗ ‹a = b›
        by (auto simp: ex-in-conv)
    qed
  }
  assume ab: a < b
  let ?P = λe. ∃ d. gauge d ∧ (∀ p. p tagged-division-of {a .. b} ∧ d fine p ⟶
    norm ((∑(x, k)∈p. content k ∗R f' x) − (f b − f a)) ≤ e ∗ content {a .. b})
  { presume ⋀e. e > 0 ⟹ ?P e then show ?thesis unfolding has-integral-factor-content-real
by auto }
  fix e :: real
  assume e: e > 0
  note assms(3)[unfolded has-vector-derivative-def has-derivative-at-alt ball-conj-distrib]
  note conjunctD2[OF this]
  note bounded=this(1) and this(2)
  from this(2) have ∀ x∈box a b. ∃ d>0. ∀ y. norm (y − x) < d ⟶
    norm (f y − f x − (y − x) ∗R f' x) ≤ e/2 ∗ norm (y − x)
    apply −
    apply safe
    apply (erule-tac x=x in ballE)
    apply (erule-tac x=e/2 in allE)
    using e
    apply auto
    done
  note this[unfolded bgauge-existence-lemma]
  from choice[OF this] guess d ..
  note conjunctD2[OF this[rule-format]]
  note d = this[rule-format]
  have bounded (f ‘ cbox a b)
    apply (rule compact-imp-bounded compact-continuous-image)+
    using compact-cbox assms
    apply auto
    done
```

from *this*[*unfolded bounded-pos*] **guess** B **.. note** B = *this*[*rule-format*]

have $\exists da.\ 0 < da \wedge (\forall c.\ a \leq c \wedge \{a\ ..\ c\} \subseteq \{a\ ..\ b\} \wedge \{a\ ..\ c\} \subseteq \mathit{ball}\ a\ da \longrightarrow$
norm (*content* $\{a\ ..\ c\}$ $*_R$ $f'\ a - (f\ c - f\ a)) \leq (e * (b - a))\ /\ 4)$
proof −
have $a \in \{a\ ..\ b\}$
using *ab* **by** *auto*
note *assms*(2)[*unfolded continuous-on-eq-continuous-within*,*rule-format*,*OF this*]
note $*$ = *this*[*unfolded continuous-within Lim-within*,*rule-format*]
have $(e * (b - a))\ /\ 8 > 0$
using *e ab* **by** (*auto simp add*: *field-simps*)
from $*$[*OF this*] **guess** k **.. note** k = *conjunctD2*[*OF this*,*rule-format*]
have $\exists l.\ 0 < l \wedge \mathit{norm}(l *_R f'\ a) \leq (e * (b - a))\ /\ 8$
proof (*cases* $f'\ a = 0$)
case *True*
thus *?thesis* **using** *ab e* **by** *auto*
next
case *False*
then show *?thesis*
apply (*rule-tac x*=$(e * (b - a))\ /\ 8\ /\ \mathit{norm}\ (f'\ a)$ **in** *exI*)
using *ab e*
apply (*auto simp add*: *field-simps*)
done
qed
then guess l **.. note** l = *conjunctD2*[*OF this*]
show *?thesis*
apply (*rule-tac x*=*min k l* **in** *exI*)
apply *safe*
unfolding *min-less-iff-conj*
apply *rule*
apply (*rule l k*)+
proof −
fix *c*
assume *as*: $a \leq c$ $\{a\ ..\ c\} \subseteq \{a\ ..\ b\}$ $\{a\ ..\ c\} \subseteq \mathit{ball}\ a\ (\mathit{min}\ k\ l)$
note *as'* = *this*[*unfolded subset-eq Ball-def mem-ball dist-real-def mem-box*]
have *norm* $((c - a) *_R f'\ a - (f\ c - f\ a)) \leq \mathit{norm}\ ((c - a) *_R f'\ a) +$
norm $(f\ c - f\ a)$
by (*rule norm-triangle-ineq4*)
also have $\ldots \leq e * (b - a)\ /\ 8 + e * (b - a)\ /\ 8$
proof (*rule add-mono*)
have $|c - a| \leq |l|$
using *as'* **by** *auto*
then show *norm* $((c - a) *_R f'\ a) \leq e * (b - a)\ /\ 8$
apply −
apply (*rule order-trans*[*OF - l*(2)])
unfolding *norm-scaleR*
apply (*rule mult-right-mono*)
apply *auto*
done

```
    next
      show norm (f c − f a) ≤ e * (b − a) / 8
        apply (rule less-imp-le)
        apply (cases a = c)
        defer
        apply (rule k(2)[unfolded dist-norm])
        using as' e ab
        apply (auto simp add: field-simps)
        done
    qed
    finally show norm (content {a .. c} *R f' a − (f c − f a)) ≤ e * (b − a) /
4
      unfolding content-real[OF as(1)] by auto
  qed
 qed
 then guess da .. note da=conjunctD2[OF this,rule-format]

 have ∃ db>0. ∀ c≤b. {c .. b} ⊆ {a .. b} ∧ {c .. b} ⊆ ball b db ⟶
   norm (content {c .. b} *R f' b − (f b − f c)) ≤ (e * (b − a)) / 4
 proof −
   have b ∈ {a .. b}
     using ab by auto
  note assms(2)[unfolded continuous-on-eq-continuous-within,rule-format,OF this]
   note * = this[unfolded continuous-within Lim-within,rule-format] have (e * (b
− a)) / 8 > 0
     using e ab by (auto simp add: field-simps)
   from *[OF this] guess k .. note k = conjunctD2[OF this,rule-format]
   have ∃ l. 0 < l ∧ norm (l *R f' b) ≤ (e * (b − a)) / 8
   proof (cases f' b = 0)
     case True
     thus ?thesis using ab e by auto
   next
     case False
     then show ?thesis
       apply (rule-tac x=(e * (b − a)) / 8 / norm (f' b) in exI)
       using ab e
       apply (auto simp add: field-simps)
       done
   qed
   then guess l .. note l = conjunctD2[OF this]
   show ?thesis
     apply (rule-tac x=min k l in exI)
     apply safe
     unfolding min-less-iff-conj
     apply rule
     apply (rule l k)+
   proof −
     fix c
     assume as: c ≤ b {c..b} ⊆ {a..b} {c..b} ⊆ ball b (min k l)
```

```
      note as' = this[unfolded subset-eq Ball-def mem-ball dist-real-def mem-box]
     have norm ((b − c) *R f' b − (f b − f c)) ≤ norm ((b − c) *R f' b) + norm
(f b − f c)
       by (rule norm-triangle-ineq4)
     also have ... ≤ e * (b − a) / 8 + e * (b − a) / 8
     proof (rule add-mono)
       have |c − b| ≤ |l|
         using as' by auto
       then show norm ((b − c) *R f' b) ≤ e * (b − a) / 8
         apply −
         apply (rule order-trans[OF - l(2)])
         unfolding norm-scaleR
         apply (rule mult-right-mono)
         apply auto
         done
     next
       show norm (f b − f c) ≤ e * (b − a) / 8
         apply (rule less-imp-le)
         apply (cases b = c)
         defer
         apply (subst norm-minus-commute)
         apply (rule k(2)[unfolded dist-norm])
         using as' e ab
         apply (auto simp add: field-simps)
         done
     qed
     finally show norm (content {c .. b} *R f' b − (f b − f c)) ≤ e * (b − a) / 4
       unfolding content-real[OF as(1)] by auto
   qed
 qed
 then guess db .. note db=conjunctD2[OF this,rule-format]

 let ?d = (λx. ball x (if x=a then da else if x=b then db else d x))
 show ?P e
   apply (rule-tac x=?d in exI)
 proof (safe, goal-cases)
   case 1
   show ?case
     apply (rule gauge-ball-dependent)
     using ab db(1) da(1) d(1)
     apply auto
     done
 next
   case as: (2 p)
   let ?A = {t. fst t ∈ {a, b}}
   note p = tagged-division-ofD[OF as(1)]
   have pA: p = (p ∩ ?A) ∪ (p − ?A) finite (p ∩ ?A) finite (p − ?A) (p ∩ ?A)
∩ (p − ?A) = {}
     using as by auto
```

```
    note ∗ = additive-tagged-division-1'[OF assms(1) as(1), symmetric]
    have ∗∗: ⋀n1 s1 n2 s2::real. n2 ≤ s2 / 2 ⟹ n1 − s1 ≤ s2 / 2 ⟹ n1 +
n2 ≤ s1 + s2
      by arith
    show ?case
    unfolding content-real[OF assms(1)] and ∗[of λx. x] ∗[of f] setsum-subtractf[symmetric]
split-minus
      unfolding setsum-right-distrib
      apply (subst(2) pA)
      apply (subst pA)
      unfolding setsum.union-disjoint[OF pA(2−)]
    proof (rule norm-triangle-le, rule ∗∗, goal-cases)
      case 1
      show ?case
        apply (rule order-trans)
        apply (rule setsum-norm-le)
        defer
        apply (subst setsum-divide-distrib)
        apply (rule order-refl)
        apply safe
        apply (unfold not-le o-def split-conv fst-conv)
      proof (rule ccontr)
        fix x k
        assume xk: (x, k) ∈ p
          e ∗ (Sup k − Inf k) / 2 <
            norm (content k ∗R f' x − (f (Sup k) − f (Inf k)))
        from p(4)[OF this(1)] guess u v by (elim exE) note k=this
        then have u ≤ v and uv: {u, v} ⊆ cbox u v
          using p(2)[OF xk(1)] by auto
          note result = xk(2)[unfolded k box-real interval-bounds-real[OF this(1)]
content-real[OF this(1)]]

        assume as': x ≠ a x ≠ b
        then have x ∈ box a b
          using p(2−3)[OF xk(1)] by (auto simp: mem-box)
        note ∗ = d(2)[OF this]
        have norm ((v − u) ∗R f' (x) − (f (v) − f (u))) =
          norm ((f (u) − f (x) − (u − x) ∗R f' (x)) − (f (v) − f (x) − (v − x)
∗R f' (x)))
          apply (rule arg-cong[of - - norm])
          unfolding scaleR-left.diff
          apply auto
          done
        also have ... ≤ e / 2 ∗ norm (u − x) + e / 2 ∗ norm (v − x)
          apply (rule norm-triangle-le-sub)
          apply (rule add-mono)
          apply (rule-tac[!] ∗)
          using fineD[OF as(2) xk(1)] as'
          unfolding k subset-eq
```

```
        apply −
        apply (erule-tac x=u in ballE)
        apply (erule-tac[3] x=v in ballE)
        using uv
        apply (auto simp:dist-real-def)
        done
      also have ... ≤ e / 2 * norm (v − u)
        using p(2)[OF xk(1)]
        unfolding k
        by (auto simp add: field-simps)
      finally have e * (v − u) / 2 < e * (v − u) / 2
        apply −
        apply (rule less-le-trans[OF result])
        using uv
        apply auto
        done
      then show False by auto
    qed
  next
    have *: ⋀x s1 s2::real. 0 ≤ s1 ⟹ x ≤ (s1 + s2) / 2 ⟹ x − s1 ≤ s2 / 2
      by auto
    case 2
    show ?case
      apply (rule *)
      apply (rule setsum-nonneg)
      apply rule
      apply (unfold split-paired-all split-conv)
      defer
      unfolding setsum.union-disjoint[OF pA(2−),symmetric] pA(1)[symmetric]
      unfolding setsum-right-distrib[symmetric]
      apply (subst additive-tagged-division-1[OF - as(1)])
      apply (rule assms)
    proof −
      fix x k
      assume (x, k) ∈ p ∩ {t. fst t ∈ {a, b}}
      note xk=IntD1[OF this]
      from p(4)[OF this] guess u v by (elim exE) note uv=this
      with p(2)[OF xk] have cbox u v ≠ {}
        by auto
      then show 0 ≤ e * ((Sup k) − (Inf k))
        unfolding uv using e by (auto simp add: field-simps)
    next
      have *: ⋀s f t e. setsum f s = setsum f t ⟹ norm (setsum f t) ≤ e ⟹
norm (setsum f s) ≤ e
        by auto
      show norm (∑(x, k)∈p ∩ ?A. content k *R f' x −
        (f ((Sup k)) − f ((Inf k)))) ≤ e * (b − a) / 2
        apply (rule *[where t1=p ∩ {t. fst t ∈ {a, b} ∧ content(snd t) ≠ 0}])
        apply (rule setsum.mono-neutral-right[OF pA(2)])
```

defer
apply *rule*
unfolding *split-paired-all split-conv o-def*
proof *goal-cases*
fix x k
assume $(x, k) \in p \cap \{t.\ fst\ t \in \{a, b\}\} - p \cap \{t.\ fst\ t \in \{a, b\} \wedge content\ (snd\ t) \neq 0\}$
then have *xk*: $(x, k) \in p$ $content\ k = 0$
by *auto*
from *p(4)[OF xk(1)]* **guess** u v **by** (*elim exE*) **note** *uv=this*
have $k \neq \{\}$
using *p(2)[OF xk(1)]* **by** *auto*
then have $*$: $u = v$
using *xk*
unfolding *uv content-eq-0 box-eq-empty*
by *auto*
then show $content\ k *_R (f'(x)) - (f\ ((Sup\ k)) - f\ ((Inf\ k))) = 0$
using *xk* **unfolding** *uv* **by** *auto*
next
have $*$: $p \cap \{t.\ fst\ t \in \{a, b\} \wedge content(snd\ t) \neq 0\} =$
$\{t.\ t{\in}p \wedge fst\ t = a \wedge content(snd\ t) \neq 0\} \cup \{t.\ t{\in}p \wedge fst\ t = b \wedge content(snd\ t) \neq 0\}$
by *blast*
have $**$: $norm\ (setsum\ f\ s) \leq e$
if $\forall x\ y.\ x \in s \wedge y \in s \longrightarrow x = y$
and $\forall x.\ x \in s \longrightarrow norm\ (f\ x) \leq e$
and $e > 0$
for s f **and** e :: *real*
proof (*cases* $s = \{\}$)
case *True*
with *that* **show** *?thesis* **by** *auto*
next
case *False*
then obtain x **where** $x \in s$
by *auto*
then have $*$: $s = \{x\}$
using *that(1)* **by** *auto*
then show *?thesis*
using ‹$x \in s$› *that(2)* **by** *auto*
qed
case *2*
show *?case*
apply (*subst* $*$)
apply (*subst setsum.union-disjoint*)
prefer *4*
apply (*rule order-trans*[*of* - $e * (b - a)/4 + e * (b - a)/4$])
apply (*rule norm-triangle-le*,*rule add-mono*)
apply (*rule-tac*[*1*−*2*] $**$)
proof −

```
let ?B = λx. {t ∈ p. fst t = x ∧ content (snd t) ≠ 0}
have pa: ∃ v. k = cbox a v ∧ a ≤ v if (a, k) ∈ p for k
proof −
  guess u v using p(4)[OF that] by (elim exE) note uv=this
  have ∗: u ≤ v
    using p(2)[OF that] unfolding uv by auto
  have u: u = a
  proof (rule ccontr)
    have u ∈ cbox u v
      using p(2−3)[OF that(1)] unfolding uv by auto
    have u ≥ a
      using p(2−3)[OF that(1)] unfolding uv subset-eq by auto
    moreover assume ¬ ?thesis
    ultimately have u > a by auto
    then show False
      using p(2)[OF that(1)] unfolding uv by (auto simp add:)
  qed
  then show ?thesis
    apply (rule-tac x=v in exI)
    unfolding uv
    using ∗
    apply auto
    done
qed
have pb: ∃ v. k = cbox v b ∧ b ≥ v if (b, k) ∈ p for k
proof −
  guess u v using p(4)[OF that] by (elim exE) note uv=this
  have ∗: u ≤ v
    using p(2)[OF that] unfolding uv by auto
  have u: v = b
  proof (rule ccontr)
    have u ∈ cbox u v
      using p(2−3)[OF that(1)] unfolding uv by auto
    have v ≤ b
      using p(2−3)[OF that(1)] unfolding uv subset-eq by auto
    moreover assume ¬ ?thesis
    ultimately have v < b by auto
    then show False
      using p(2)[OF that(1)] unfolding uv by (auto simp add:)
  qed
  then show ?thesis
    apply (rule-tac x=u in exI)
    unfolding uv
    using ∗
    apply auto
    done
qed
show ∀ x y. x ∈ ?B a ∧ y ∈ ?B a ⟶ x = y
  apply (rule,rule,rule,unfold split-paired-all)
```

```
            unfolding mem-Collect-eq fst-conv snd-conv
            apply safe
          proof −
            fix x k k'
            assume k: (a, k) ∈ p (a, k') ∈ p content k ≠ 0 content k' ≠ 0
            guess v using pa[OF k(1)] .. note v = conjunctD2[OF this]
            guess v' using pa[OF k(2)] .. note v' = conjunctD2[OF this] let ?v
= min v v'
            have box a ?v ⊆ k ∩ k'
              unfolding v v' by (auto simp add: mem-box)
            note interior-mono[OF this,unfolded interior-Int]
            moreover have (a + ?v)/2 ∈ box a ?v
              using k(3−)
              unfolding v v' content-eq-0 not-le
              by (auto simp add: mem-box)
            ultimately have (a + ?v)/2 ∈ interior k ∩ interior k'
              unfolding interior-open[OF open-box] by auto
            then have ∗: k = k'
              apply −
              apply (rule ccontr)
              using p(5)[OF k(1−2)]
              apply auto
              done
            { assume x ∈ k then show x ∈ k' unfolding ∗ . }
            { assume x ∈ k' then show x ∈ k unfolding ∗ . }
          qed
          show ∀ x y. x ∈ ?B b ∧ y ∈ ?B b ⟶ x = y
            apply rule
            apply rule
            apply rule
            apply (unfold split-paired-all)
            unfolding mem-Collect-eq fst-conv snd-conv
            apply safe
          proof −
            fix x k k'
            assume k: (b, k) ∈ p (b, k') ∈ p content k ≠ 0 content k' ≠ 0
            guess v using pb[OF k(1)] .. note v = conjunctD2[OF this]
            guess v' using pb[OF k(2)] .. note v' = conjunctD2[OF this]
            let ?v = max v v'
            have box ?v b ⊆ k ∩ k'
              unfolding v v' by (auto simp: mem-box)
              note interior-mono[OF this,unfolded interior-Int]
            moreover have ((b + ?v)/2) ∈ box ?v b
                using k(3−) unfolding v v' content-eq-0 not-le by (auto simp:
mem-box)
            ultimately have ((b + ?v)/2) ∈ interior k ∩ interior k'
              unfolding interior-open[OF open-box] by auto
            then have ∗: k = k'
              apply −
```

```
      apply (rule ccontr)
      using p(5)[OF k(1–2)]
      apply auto
      done
    { assume x ∈ k then show x ∈ k' unfolding * . }
    { assume x ∈ k' then show x∈k unfolding * . }
  qed

  let ?a = a and ?b = b
   show ∀x. x ∈ ?B a ⟶ norm ((λ(x, k). content k *R f' x − (f (Sup
k) −
    f (Inf k))) x) ≤ e * (b − a) / 4
    apply rule
    apply rule
    unfolding mem-Collect-eq
    unfolding split-paired-all fst-conv snd-conv
  proof (safe, goal-cases)
    case prems: 1
    guess v using pa[OF prems(1)] .. note v = conjunctD2[OF this]
    have ?a ∈ {?a..v}
      using v(2) by auto
    then have v ≤ ?b
      using p(3)[OF prems(1)] unfolding subset-eq v by auto
    moreover have {?a..v} ⊆ ball ?a da
      using fineD[OF as(2) prems(1)]
      apply –
      apply (subst(asm) if-P)
      apply (rule refl)
      unfolding subset-eq
      apply safe
      apply (erule-tac x= x in ballE)
      apply (auto simp add:subset-eq dist-real-def v)
      done
    ultimately show ?case
      unfolding v interval-bounds-real[OF v(2)] box-real
      apply –
      apply(rule da(2)[of v])
      using prems fineD[OF as(2) prems(1)]
      unfolding v content-eq-0
      apply auto
      done
  qed
  show ∀x. x ∈ ?B b ⟶ norm ((λ(x, k). content k *R f' x −
    (f (Sup k) − f (Inf k))) x) ≤ e * (b − a) / 4
    apply rule
    apply rule
    unfolding mem-Collect-eq
    unfolding split-paired-all fst-conv snd-conv
  proof (safe, goal-cases)
```

```
          case prems: 1
          guess v using pb[OF prems(1)] .. note v = conjunctD2[OF this]
          have ?b ∈ {v.. ?b}
            using v(2) by auto
          then have v ≥ ?a using p(3)[OF prems(1)]
            unfolding subset-eq v by auto
          moreover have {v..?b} ⊆ ball ?b db
            using fineD[OF as(2) prems(1)]
            apply −
            apply (subst(asm) if-P, rule refl)
            unfolding subset-eq
            apply safe
            apply (erule-tac x= x in ballE)
            using ab
            apply (auto simp add:subset-eq v dist-real-def)
            done
          ultimately show ?case
            unfolding v
            unfolding interval-bounds-real[OF v(2)] box-real
            apply −
            apply(rule db(2)[of v])
            using prems fineD[OF as(2) prems(1)]
            unfolding v content-eq-0
            apply auto
            done
        qed
      qed (insert p(1) ab e, auto simp add: field-simps)
    qed auto
  qed
 qed
qed
```

41.53 Stronger form with finite number of exceptional points.

```
lemma fundamental-theorem-of-calculus-interior-strong:
 fixes f :: real ⇒ 'a::banach
 assumes finite s
   and a ≤ b
   and continuous-on {a .. b} f
   and ∀x∈{a <..< b} − s. (f has-vector-derivative f'(x)) (at x)
 shows (f' has-integral (f b − f a)) {a .. b}
 using assms
proof (induct card s arbitrary: s a b)
 case 0
 show ?case
   apply (rule fundamental-theorem-of-calculus-interior)
   using 0
   apply auto
```

```
    done
next
  case (Suc n)
  from this(2) guess c s'
    apply -
    apply (subst(asm) eq-commute)
    unfolding card-Suc-eq
    apply (subst(asm)(2) eq-commute)
    apply (elim exE conjE)
    done
  note cs = this[rule-format]
  show ?case
  proof (cases c ∈ box a b)
    case False
    then show ?thesis
      apply -
      apply (rule Suc(1)[OF cs(3) - Suc(4,5)])
      apply safe
      defer
      apply (rule Suc(6)[rule-format])
      using Suc(3)
      unfolding cs
      apply auto
      done
  next
    have *: f b - f a = (f c - f a) + (f b - f c)
      by auto
    case True
    then have a ≤ c c ≤ b
      by (auto simp: mem-box)
    then show ?thesis
      apply (subst *)
      apply (rule has-integral-combine)
      apply assumption+
      apply (rule-tac[!] Suc(1)[OF cs(3)])
      using Suc(3)
      unfolding cs
    proof -
      show continuous-on {a .. c} f continuous-on {c .. b} f
        apply (rule-tac[!] continuous-on-subset[OF Suc(5)])
        using True
        apply (auto simp: mem-box)
        done
      let ?P = λi j. ∀x∈{i <..< j} - s'. (f has-vector-derivative f' x) (at x)
      show ?P a c ?P c b
        apply safe
        apply (rule-tac[!] Suc(6)[rule-format])
        using True
        unfolding cs
```

```
      apply (auto simp: mem-box)
      done
   qed auto
  qed
qed

lemma fundamental-theorem-of-calculus-strong:
  fixes f :: real ⇒ 'a::banach
  assumes finite s
    and a ≤ b
    and continuous-on {a .. b} f
    and ∀x∈{a .. b} − s. (f has-vector-derivative f'(x)) (at x)
  shows (f' has-integral (f b − f a)) {a .. b}
  apply (rule fundamental-theorem-of-calculus-interior-strong[OF assms(1−3), of
f ])
  using assms(4)
  apply (auto simp: mem-box)
  done

lemma indefinite-integral-continuous-left:
  fixes f:: real ⇒ 'a::banach
  assumes f integrable-on {a .. b}
    and a < c
    and c ≤ b
    and e > 0
  obtains d where d > 0
    and ∀t. c − d < t ∧ t ≤ c ⟶ norm (integral {a .. c} f − integral {a .. t}
f) < e
proof −
  have ∃w>0. ∀t. c − w < t ∧ t < c ⟶ norm (f c) ∗ norm(c − t) < e / 3
  proof (cases f c = 0)
    case False
    hence 0 < e / 3 / norm (f c) using ‹e>0› by simp
    then show ?thesis
      apply −
      apply rule
      apply rule
      apply assumption
      apply safe
    proof −
      fix t
      assume as: t < c and c − e / 3 / norm (f c) < t
      then have c − t < e / 3 / norm (f c)
        by auto
      then have norm (c − t) < e / 3 / norm (f c)
        using as by auto
      then show norm (f c) ∗ norm (c − t) < e / 3
        using False
        apply −
```

```
      apply (subst mult.commute)
      apply (subst pos-less-divide-eq[symmetric])
      apply auto
      done
  qed
 next
  case True
  show ?thesis
   apply (rule-tac x=1 in exI)
   unfolding True
   using ‹e > 0›
   apply auto
   done
 qed
 then guess w .. note w = conjunctD2[OF this,rule-format]

 have *: e / 3 > 0
  using assms by auto
 have f integrable-on {a .. c}
  apply (rule integrable-subinterval-real[OF assms(1)])
  using assms(2-3)
  apply auto
  done
 from integrable-integral[OF this,unfolded has-integral-real,rule-format,OF *] guess
d1 ..
 note d1 = conjunctD2[OF this,rule-format]
 def d ≡ λx. ball x w ∩ d1 x
 have gauge d
  unfolding d-def using w(1) d1 by auto
 note this[unfolded gauge-def,rule-format,of c]
 note conjunctD2[OF this]
 from this(2)[unfolded open-contains-ball,rule-format,OF this(1)] guess k ..
 note k=conjunctD2[OF this]

 let ?d = min k (c - a) / 2
 show ?thesis
  apply (rule that[of ?d])
  apply safe
 proof -
  show ?d > 0
   using k(1) using assms(2) by auto
  fix t
  assume as: c - ?d < t t ≤ c
  let ?thesis = norm (integral ({a .. c}) f - integral ({a .. t}) f) < e
  {
   presume *: t < c ⟹ ?thesis
   show ?thesis
    apply (cases t = c)
    defer
```

```
      apply (rule *)
      apply (subst less-le)
      using ‹e > 0› as(2)
      apply auto
      done
    }
    assume t < c

    have f integrable-on {a .. t}
      apply (rule integrable-subinterval-real[OF assms(1)])
      using assms(2−3) as(2)
      apply auto
      done
     from integrable-integral[OF this,unfolded has-integral-real,rule-format,OF *]
guess d2 ..
    note d2 = conjunctD2[OF this,rule-format]
    def d3 ≡ λx. if x ≤ t then d1 x ∩ d2 x else d1 x
    have gauge d3
      using d2(1) d1(1) unfolding d3-def gauge-def by auto
    from fine-division-exists-real[OF this, of a t] guess p . note p=this
    note p'=tagged-division-ofD[OF this(1)]
    have pt: ∀(x,k)∈p. x ≤ t
    proof (safe, goal-cases)
      case prems: 1
      from p'(2,3)[OF prems] show ?case
        by auto
    qed
    with p(2) have d2 fine p
      unfolding fine-def d3-def
      apply safe
      apply (erule-tac x=(a,b) in ballE)+
      apply auto
      done
    note d2-fin = d2(2)[OF conjI[OF p(1) this]]

    have *: {a .. c} ∩ {x. x • 1 ≤ t} = {a .. t} {a .. c} ∩ {x. x • 1 ≥ t} = {t ..
c}
      using assms(2−3) as by (auto simp add: field-simps)
   have p ∪ {(c, {t .. c})} tagged-division-of {a .. c} ∧ d1 fine p ∪ {(c, {t .. c})}
      apply rule
      apply (rule tagged-division-union-interval-real[of - - - 1 t])
      unfolding *
      apply (rule p)
      apply (rule tagged-division-of-self-real)
      unfolding fine-def
      apply safe
    proof −
      fix x k y
      assume (x,k) ∈ p and y ∈ k
```

```
    then show y ∈ d1 x
      using p(2) pt
      unfolding fine-def d3-def
      apply −
      apply (erule-tac x=(x,k) in ballE)+
      apply auto
      done
  next
    fix x assume x ∈ {t..c}
    then have dist c x < k
      unfolding dist-real-def
      using as(1)
      by (auto simp add: field-simps)
    then show x ∈ d1 c
      using k(2)
      unfolding d-def
      by auto
  qed (insert as(2), auto) note d1-fin = d1(2)[OF this]

  have ∗: integral {a .. c} f − integral {a .. t} f = −(((c − t) ∗R f c + (∑(x,
k)∈p. content k ∗R f x)) −
    integral {a .. c} f) + ((∑(x, k)∈p. content k ∗R f x) − integral {a .. t} f)
+ (c − t) ∗R f c
    e = (e/3 + e/3) + e/3
    by auto
  have ∗∗: (∑(x, k)∈p ∪ {(c, {t .. c})}. content k ∗R f x) =
    (c − t) ∗R f c + (∑(x, k)∈p. content k ∗R f x)
  proof −
    have ∗∗: ⋀x F. F ∪ {x} = insert x F
      by auto
    have (c, cbox t c) ∉ p
    proof (safe, goal-cases)
      case prems: 1
      from p'(2−3)[OF prems] have c ∈ cbox a t
        by auto
      then show False using ‹t < c›
        by auto
    qed
    then show ?thesis
      unfolding ∗∗ box-real
      apply −
      apply (subst setsum.insert)
      apply (rule p')
      unfolding split-conv
      defer
      apply (subst content-real)
      using as(2)
      apply auto
      done
```

```
    qed
    have ***: c − w < t ∧ t < c
    proof −
      have c − k < t
        using ‹k>0› as(1) by (auto simp add: field-simps)
      moreover have k ≤ w
        apply (rule ccontr)
        using k(2)
        unfolding subset-eq
        apply (erule-tac x=c + ((k + w)/2) in ballE)
        unfolding d-def
        using ‹k > 0› ‹w > 0›
        apply (auto simp add: field-simps not-le not-less dist-real-def)
        done
      ultimately show ?thesis using ‹t < c›
        by (auto simp add: field-simps)
    qed
    show ?thesis
      unfolding *(1)
      apply (subst *(2))
      apply (rule norm-triangle-lt add-strict-mono)+
      unfolding norm-minus-cancel
      apply (rule d1-fin[unfolded **])
      apply (rule d2-fin)
      using w(2)[OF ***]
      unfolding norm-scaleR
      apply (auto simp add: field-simps)
      done
  qed
qed

lemma indefinite-integral-continuous-right:
  fixes f :: real ⇒ 'a::banach
  assumes f integrable-on {a .. b}
    and a ≤ c
    and c < b
    and e > 0
  obtains d where 0 < d
    and ∀ t. c ≤ t ∧ t < c + d ⟶ norm (integral {a .. c} f − integral {a .. t}
f) < e
proof −
  have *: (λx. f (− x)) integrable-on {−b .. −a} − b < − c − c ≤ − a
    using assms by auto
  from indefinite-integral-continuous-left[OF * ‹e>0›] guess d . note d = this
  let ?d = min d (b − c)
  show ?thesis
    apply (rule that[of ?d])
    apply safe
  proof −
```

```
    show 0 < ?d
      using d(1) assms(3) by auto
    fix t :: real
    assume as: c ≤ t t < c + ?d
    have ∗: integral {a .. c} f = integral {a .. b} f − integral {c .. b} f
      integral {a .. t} f = integral {a .. b} f − integral {t .. b} f
      unfolding algebra-simps
      apply (rule-tac[!] integral-combine)
      using assms as
      apply auto
      done
    have (− c) − d < (− t) ∧ − t ≤ − c
      using as by auto note d(2)[rule-format,OF this]
    then show norm (integral {a .. c} f − integral {a .. t} f) < e
      unfolding ∗
      unfolding integral-reflect
      apply (subst norm-minus-commute)
      apply (auto simp add: algebra-simps)
      done
  qed
qed

lemma indefinite-integral-continuous:
  fixes f :: real ⇒ 'a::banach
  assumes f integrable-on {a .. b}
  shows continuous-on {a .. b} (λx. integral {a .. x} f)
proof (unfold continuous-on-iff, safe)
  fix x e :: real
  assume as: x ∈ {a .. b} e > 0
  let ?thesis = ∃ d>0. ∀ x'∈{a .. b}. dist x' x < d ⟶ dist (integral {a .. x'} f)
(integral {a .. x} f) < e
  {
    presume ∗: a < b ⟹ ?thesis
    show ?thesis
      apply cases
      apply (rule ∗)
      apply assumption
    proof goal-cases
      case 1
      then have cbox a b = {x}
        using as(1)
        apply −
        apply (rule set-eqI)
        apply auto
        done
      then show ?case using ‹e > 0› by auto
    qed
  }
  assume a < b
```

```
  have (x = a ∨ x = b) ∨ (a < x ∧ x < b)
    using as(1) by auto
  then show ?thesis
    apply (elim disjE)
  proof −
    assume x = a
    have a ≤ a ..
   from indefinite-integral-continuous-right[OF assms(1) this ‹a<b› ‹e>0›] guess
d . note d=this
    show ?thesis
      apply rule
      apply rule
      apply (rule d)
      apply safe
      apply (subst dist-commute)
      unfolding ‹x = a› dist-norm
      apply (rule d(2)[rule-format])
      apply auto
      done
  next
    assume x = b
    have b ≤ b ..
    from indefinite-integral-continuous-left[OF assms(1) ‹a<b› this ‹e>0›] guess
d . note d=this
    show ?thesis
      apply rule
      apply rule
      apply (rule d)
      apply safe
      apply (subst dist-commute)
      unfolding ‹x = b› dist-norm
      apply (rule d(2)[rule-format])
      apply auto
      done
  next
    assume a < x ∧ x < b
    then have xl: a < x x ≤ b and xr: a ≤ x x < b
      by auto
     from indefinite-integral-continuous-left [OF assms(1) xl ‹e>0›] guess d1 .
note d1=this
     from indefinite-integral-continuous-right[OF assms(1) xr ‹e>0›] guess d2 .
note d2=this
    show ?thesis
      apply (rule-tac x=min d1 d2 in exI)
    proof safe
      show 0 < min d1 d2
        using d1 d2 by auto
      fix y
      assume y ∈ {a .. b} and dist y x < min d1 d2
```

```
    then show dist (integral {a .. y} f) (integral {a .. x} f) < e
      apply (subst dist-commute)
      apply (cases y < x)
      unfolding dist-norm
      apply (rule d1(2)[rule-format])
      defer
      apply (rule d2(2)[rule-format])
      unfolding not-less
      apply (auto simp add: field-simps)
      done
  qed
 qed
qed
```

41.54 This doesn't directly involve integration, but that gives an easy proof.

```
lemma has-derivative-zero-unique-strong-interval:
 fixes f :: real ⇒ 'a::banach
 assumes finite k
   and continuous-on {a .. b} f
   and f a = y
   and ∀x∈({a .. b} − k). (f has-derivative (λh. 0)) (at x within {a .. b}) x ∈
{a .. b}
 shows f x = y
proof −
 have ab: a ≤ b
   using assms by auto
 have ∗: a ≤ x
   using assms(5) by auto
 have ((λx. 0::'a) has-integral f x − f a) {a .. x}
   apply (rule fundamental-theorem-of-calculus-interior-strong[OF assms(1) ∗])
   apply (rule continuous-on-subset[OF assms(2)])
   defer
   apply safe
   unfolding has-vector-derivative-def
   apply (subst has-derivative-within-open[symmetric])
   apply assumption
   apply (rule open-greaterThanLessThan)
   apply (rule has-derivative-within-subset[where s={a .. b}])
   using assms(4) assms(5)
   apply (auto simp: mem-box)
   done
 note this[unfolded ∗]
 note has-integral-unique[OF has-integral-0 this]
 then show ?thesis
   unfolding assms by auto
qed
```

41.55 Generalize a bit to any convex set.

lemma *has-derivative-zero-unique-strong-convex*:
fixes $f :: 'a::euclidean\text{-}space \Rightarrow 'b::banach$
assumes *convex s*
and *finite k*
and *continuous-on s f*
and $c \in s$
and $f\ c = y$
and $\forall x{\in}(s - k).\ (f\ has\text{-}derivative\ (\lambda h.\ 0))\ (at\ x\ within\ s)$
and $x \in s$
shows $f\ x = y$
proof −
{
presume $*$: $x \neq c \Longrightarrow ?thesis$
show *?thesis*
apply *cases*
apply (*rule* $*$)
apply *assumption*
unfolding *assms*(*5*)[*symmetric*]
apply *auto*
done
}
assume $x \neq c$
note *conv* = *assms*(*1*)[*unfolded convex-alt*,*rule-format*]
have *as1*: *continuous-on* $\{0\ ..1\}\ (f \circ (\lambda t.\ (1 - t) *_R c + t *_R x))$
apply (*rule continuous-intros*)+
apply (*rule continuous-on-subset*[*OF assms*(*3*)])
apply *safe*
apply (*rule conv*)
using *assms*(*4*,*7*)
apply *auto*
done
have $*$: $t = xa$ **if** $(1 - t) *_R c + t *_R x = (1 - xa) *_R c + xa *_R x$ **for** $t\ xa$
proof −
from *that* **have** $(t - xa) *_R x = (t - xa) *_R c$
unfolding *scaleR-simps* **by** (*auto simp add*: *algebra-simps*)
then show *?thesis*
using ‹$x \neq c$› **by** *auto*
qed
have *as2*: *finite* $\{t.\ ((1 - t) *_R c + t *_R x) \in k\}$
using *assms*(*2*)
apply (*rule finite-surj*[**where** $f{=}\lambda z.\ SOME\ t.\ (1{-}t) *_R c + t *_R x = z$])
apply *safe*
unfolding *image-iff*
apply *rule*
defer
apply *assumption*
apply (*rule sym*)
apply (*rule some-equality*)

```
    defer
    apply (drule *)
    apply auto
    done
  have (f ∘ (λt. (1 - t) *R c + t *R x)) 1 = y
    apply (rule has-derivative-zero-unique-strong-interval[OF as2 as1, of ])
    unfolding o-def
    using assms(5)
    defer
    apply -
    apply rule
  proof -
    fix t
    assume as: t ∈ {0 .. 1} - {t. (1 - t) *R c + t *R x ∈ k}
    have *: c - t *R c + t *R x ∈ s - k
      apply safe
      apply (rule conv[unfolded scaleR-simps])
      using ‹x ∈ s› ‹c ∈ s› as
      by (auto simp add: algebra-simps)
    have (f ∘ (λt. (1 - t) *R c + t *R x) has-derivative (λx. 0) ∘ (λz. (0 - z *R
c) + z *R x))
      (at t within {0 .. 1})
      apply (intro derivative-eq-intros)
      apply simp-all
      apply (simp add: field-simps)
      unfolding scaleR-simps
      apply (rule has-derivative-within-subset,rule assms(6)[rule-format])
      apply (rule *)
      apply safe
      apply (rule conv[unfolded scaleR-simps])
      using ‹x ∈ s› ‹c ∈ s›
      apply auto
      done
    then show ((λxa. f ((1 - xa) *R c + xa *R x)) has-derivative (λh. 0)) (at t
within {0 .. 1})
      unfolding o-def .
  qed auto
  then show ?thesis
    by auto
qed
```

Also to any open connected set with finite set of exceptions. Could generalize to locally convex set with limpt-free set of exceptions.

```
lemma has-derivative-zero-unique-strong-connected:
  fixes f :: 'a::euclidean-space ⇒ 'b::banach
  assumes connected s
    and open s
    and finite k
    and continuous-on s f
```

```
    and c ∈ s
    and f c = y
    and ∀x∈(s − k). (f has-derivative (λh. 0)) (at x within s)
    and x∈s
  shows f x = y
proof −
  have {x ∈ s. f x ∈ {y}} = {} ∨ {x ∈ s. f x ∈ {y}} = s
    apply (rule assms(1)[unfolded connected-clopen,rule-format])
    apply rule
    defer
    apply (rule continuous-closedin-preimage[OF assms(4) closed-singleton])
    apply (rule open-openin-trans[OF assms(2)])
    unfolding open-contains-ball
  proof safe
    fix x
    assume x ∈ s
   from assms(2)[unfolded open-contains-ball,rule-format,OF this] guess e .. note
e=conjunctD2[OF this]
    show ∃e>0. ball x e ⊆ {xa ∈ s. f xa ∈ {f x}}
      apply rule
      apply rule
      apply (rule e)
    proof safe
      fix y
      assume y: y ∈ ball x e
      then show y ∈ s
        using e by auto
      show f y = f x
        apply (rule has-derivative-zero-unique-strong-convex[OF convex-ball])
        apply (rule assms)
        apply (rule continuous-on-subset)
        apply (rule assms)
        apply (rule e)+
        apply (subst centre-in-ball)
        apply (rule e)
        apply rule
        apply safe
        apply (rule has-derivative-within-subset)
        apply (rule assms(7)[rule-format])
        using y e
        apply auto
        done
    qed
  qed
  then show ?thesis
    using ‹x ∈ s› ‹f c = y› ‹c ∈ s› by auto
qed

lemma has-derivative-zero-connected-constant:
```

```
  fixes f :: 'a::euclidean-space ⇒ 'b::banach
  assumes connected s
     and open s
     and finite k
     and continuous-on s f
     and ∀x∈(s − k). (f has-derivative (λh. 0)) (at x within s)
    obtains c where ⋀x. x ∈ s ⟹ f(x) = c
proof (cases s = {})
  case True
  then show ?thesis
by (metis empty-iff that)
next
  case False
  then obtain c where c ∈ s
    by (metis equals0I)
  then show ?thesis
    by (metis has-derivative-zero-unique-strong-connected assms that)
qed
```

41.56 Integrating characteristic function of an interval

```
lemma has-integral-restrict-open-subinterval:
  fixes f :: 'a::euclidean-space ⇒ 'b::banach
  assumes (f has-integral i) (cbox c d)
    and cbox c d ⊆ cbox a b
  shows ((λx. if x ∈ box c d then f x else 0) has-integral i) (cbox a b)
proof −
  def g ≡ λx. if x ∈box c d then f x else 0
  {
    presume ∗: cbox c d ≠ {} ⟹ ?thesis
    show ?thesis
      apply cases
      apply (rule ∗)
      apply assumption
    proof goal-cases
      case prems: 1
      then have ∗: box c d = {}
        by (metis bot.extremum-uniqueI box-subset-cbox)
      show ?thesis
        using assms(1)
        unfolding ∗
        using prems
        by auto
    qed
  }
  assume cbox c d ≠ {}
  from partial-division-extend-1[OF assms(2) this] guess p . note p=this
  note mon = monoidal-lifted[OF monoidal-monoid]
  note operat = operative-division[OF this operative-integral p(1), symmetric]
```

let *?P* = (*if g integrable-on cbox a b then Some* (*integral* (*cbox a b*) *g*) *else None*) = *Some i*
{
presume *?P*
then have *g integrable-on cbox a b* ∧ *integral* (*cbox a b*) *g* = *i*
apply −
apply *cases*
apply (*subst*(*asm*) *if-P*)
apply *assumption*
apply *auto*
done
then show *?thesis*
using *integrable-integral*
unfolding *g-def*
by *auto*
}

note *iterate-eq-neutral*[*OF mon*,*unfolded neutral-lifted*[*OF monoidal-monoid*]]
note ∗ = *this*[*unfolded neutral-add*]
have *iterate*:*iterate* (*lifted op* +) (*p* − {*cbox c d*})
(λ*i*. *if g integrable-on i then Some* (*integral i g*) *else None*) = *Some 0*
proof (*rule* ∗)
fix *x*
assume *x*: *x* ∈ *p* − {*cbox c d*}
then have *x* ∈ *p*
by *auto*
note *div* = *division-ofD*(*2*−*5*)[*OF p*(*1*) *this*]
from *div*(*3*) **guess** *u v* **by** (*elim exE*) **note** *uv*=*this*
have *interior x* ∩ *interior* (*cbox c d*) = {}
using *div*(*4*)[*OF p*(*2*)] *x* **by** *auto*
then have (*g has-integral 0*) *x*
unfolding *uv*
apply −
apply (*rule has-integral-spike-interior*[**where** *f*=λ*x*. *0*])
unfolding *g-def interior-cbox*
apply *auto*
done
then show (*if g integrable-on x then Some* (*integral x g*) *else None*) = *Some 0*
by *auto*
qed

have ∗: *p* = *insert* (*cbox c d*) (*p* − {*cbox c d*})
using *p* **by** *auto*
have ∗∗: *g integrable-on cbox c d*
apply (*rule integrable-spike-interior*[**where** *f*=*f*])
unfolding *g-def* **using** *assms*(*1*)
apply *auto*
done
moreover

```
  have integral (cbox c d) g = i
    apply (rule has-integral-unique[OF - assms(1)])
    apply (rule has-integral-spike-interior[where f=g])
    defer
    apply (rule integrable-integral[OF **])
    unfolding g-def
    apply auto
    done
  ultimately show ?P
    unfolding operat
    apply (subst *)
    apply (subst iterate-insert)
    apply rule+
    unfolding iterate
    defer
    apply (subst if-not-P)
    defer
    using p
    apply auto
    done
qed

lemma has-integral-restrict-closed-subinterval:
  fixes f :: 'a::euclidean-space ⇒ 'b::banach
  assumes (f has-integral i) (cbox c d)
    and cbox c d ⊆ cbox a b
  shows ((λx. if x ∈ cbox c d then f x else 0) has-integral i) (cbox a b)
proof −
  note has-integral-restrict-open-subinterval[OF assms]
  note * = has-integral-spike[OF negligible-frontier-interval - this]
  show ?thesis
    apply (rule *[of c d])
    using box-subset-cbox[of c d]
    apply auto
    done
qed

lemma has-integral-restrict-closed-subintervals-eq:
  fixes f :: 'a::euclidean-space ⇒ 'b::banach
  assumes cbox c d ⊆ cbox a b
  shows ((λx. if x ∈ cbox c d then f x else 0) has-integral i) (cbox a b) ⟷ (f
has-integral i) (cbox c d)
  (is ?l = ?r)
proof (cases cbox c d = {})
  case False
  let ?g = λx. if x ∈ cbox c d then f x else 0
  show ?thesis
    apply rule
    defer
```

apply (*rule has-integral-restrict-closed-subinterval*[*OF* - *assms*])
apply *assumption*
proof −
assume *?l*
then have *?g integrable-on cbox c d*
using *assms has-integral-integrable integrable-subinterval* **by** *blast*
then have ∗: *f integrable-on cbox c d*
apply −
apply (*rule integrable-eq*)
apply *auto*
done
then have *i* = *integral* (*cbox c d*) *f*
apply −
apply (*rule has-integral-unique*)
apply (*rule* ‹*?l*›)
apply (*rule has-integral-restrict-closed-subinterval*[*OF* - *assms*])
apply *auto*
done
then show *?r*
using ∗ **by** *auto*
qed
qed *auto*

Hence we can apply the limit process uniformly to all integrals.

lemma *has-integral′*:
fixes *f* :: *′n*::*euclidean-space* ⇒ *′a*::*banach*
shows (*f has-integral i*) *s* ⟷
(∀ *e*>*0*. ∃ *B*>*0*. ∀ *a b*. *ball 0 B* ⊆ *cbox a b* ⟶
(∃ *z*. ((λ*x*. *if x* ∈ *s then f*(*x*) *else 0*) *has-integral z*) (*cbox a b*) ∧ *norm*(*z* − *i*) < *e*))
(**is** *?l* ⟷ (∀ *e*>*0*. *?r e*))
proof −
{
presume ∗: ∃ *a b*. *s* = *cbox a b* ⟹ *?thesis*
show *?thesis*
apply *cases*
apply (*rule* ∗)
apply *assumption*
apply (*subst has-integral-alt*)
apply *auto*
done
}
assume ∃ *a b*. *s* = *cbox a b*
then guess *a b* **by** (*elim exE*) **note** *s*=*this*
from *bounded-cbox*[*of a b*, *unfolded bounded-pos*] **guess** *B* **..**
note *B* = *conjunctD2*[*OF this*,*rule-format*] **show** *?thesis*
apply *safe*
proof −
fix *e* :: *real*

assume *?l* **and** $e > 0$
show *?r e*
apply (*rule-tac x=B+1* **in** *exI*)
apply *safe*
defer
apply (*rule-tac x=i* **in** *exI*)
proof
fix *c d* :: $'n$
assume *as*: *ball 0 (B+1)* $\subseteq$ *cbox c d*
then show ((λ*x*. *if x* $\in$ *s then f x else 0*) *has-integral i*) (*cbox c d*)
unfolding *s*
apply −
apply (*rule has-integral-restrict-closed-subinterval*)
apply (*rule* ‹*?l*›[*unfolded s*])
apply *safe*
apply (*drule B(2)*[*rule-format*])
unfolding *subset-eq*
apply (*erule-tac x=x* **in** *ballE*)
apply (*auto simp add*: *dist-norm*)
done
qed (*insert B* ‹*e>0*›, *auto*)
next
assume *as*: ∀ *e>0*. *?r e*
from *this*[*rule-format*,*OF zero-less-one*] **guess** *C* **..** **note** *C=conjunctD2*[*OF this*,*rule-format*]
def *c* ≡ (∑ *i*∈*Basis*. (− *max B C*) ∗$_R$ *i*)::$'n$
def *d* ≡ (∑ *i*∈*Basis*. *max B C* ∗$_R$ *i*)::$'n$
have *c-d*: *cbox a b* $\subseteq$ *cbox c d*
apply *safe*
apply (*drule B(2)*)
unfolding *mem-box*
proof
fix *x i*
show *c* · *i* ≤ *x* · *i* ∧ *x* · *i* ≤ *d* · *i* **if** *norm x* ≤ *B* **and** *i* ∈ *Basis*
using *that* **and** *Basis-le-norm*[*OF* ‹*i*∈*Basis*›, *of x*]
unfolding *c-def d-def*
by (*auto simp add*: *field-simps setsum-negf*)
qed
have *ball 0 C* $\subseteq$ *cbox c d*
apply (*rule subsetI*)
unfolding *mem-box mem-ball dist-norm*
proof
fix *x i* :: $'n$
assume *x*: *norm* (*0* − *x*) < *C* **and** *i*: *i* ∈ *Basis*
show *c* · *i* ≤ *x* · *i* ∧ *x* · *i* ≤ *d* · *i*
using *Basis-le-norm*[*OF i*, *of x*] **and** *x i*
unfolding *c-def d-def*
by (*auto simp*: *setsum-negf*)
qed

from *C(2)*[*OF this*] **have** ∃ *y*. (*f has-integral y*) (*cbox a b*)
unfolding *has-integral-restrict-closed-subintervals-eq*[*OF c-d*,*symmetric*]
unfolding *s*
by *auto*
then guess *y* **.. note** *y*=*this*

have *y* = *i*
proof (*rule ccontr*)
assume ¬ *?thesis*
then have *0* < *norm* (*y* − *i*)
by *auto*
from *as*[*rule-format*,*OF this*] **guess** *C* **.. note** *C*=*conjunctD2*[*OF this*,*rule-format*]
def *c* ≡ (∑ *i*∈*Basis*. (− *max B C*) ∗$_R$ *i*)::′*n*
def *d* ≡ (∑ *i*∈*Basis*. *max B C* ∗$_R$ *i*)::′*n*
have *c-d*: *cbox a b* ⊆ *cbox c d*
apply *safe*
apply (*drule B(2)*)
unfolding *mem-box*
proof
fix *x i* :: ′*n*
assume *norm x* ≤ *B* **and** *i* ∈ *Basis*
then show *c* · *i* ≤ *x* · *i* ∧ *x* · *i* ≤ *d* · *i*
using *Basis-le-norm*[*of i x*]
unfolding *c-def d-def*
by (*auto simp add*: *field-simps setsum-negf*)
qed
have *ball 0 C* ⊆ *cbox c d*
apply (*rule subsetI*)
unfolding *mem-box mem-ball dist-norm*
proof
fix *x i* :: ′*n*
assume *norm* (*0* − *x*) < *C* **and** *i* ∈ *Basis*
then show *c* · *i* ≤ *x* · *i* ∧ *x* · *i* ≤ *d* · *i*
using *Basis-le-norm*[*of i x*]
unfolding *c-def d-def*
by (*auto simp*: *setsum-negf*)
qed
note *C(2)*[*OF this*] **then guess** *z* **.. note** *z* = *conjunctD2*[*OF this*, *unfolded s*]
note *this*[*unfolded has-integral-restrict-closed-subintervals-eq*[*OF c-d*]]
then have *z* = *y* **and** *norm* (*z* − *i*) < *norm* (*y* − *i*)
apply −
apply (*rule has-integral-unique*[*OF - y(1)*])
apply *assumption*
apply *assumption*
done
then show *False*
by *auto*
qed

```
    then show ?l
      using y
      unfolding s
      by auto
  qed
qed

lemma has-integral-le:
  fixes f :: 'n::euclidean-space ⇒ real
  assumes (f has-integral i) s
    and (g has-integral j) s
    and ∀x∈s. f x ≤ g x
  shows i ≤ j
  using has-integral-component-le[OF - assms(1−2), of 1]
  using assms(3)
  by auto

lemma integral-le:
  fixes f :: 'n::euclidean-space ⇒ real
  assumes f integrable-on s
    and g integrable-on s
    and ∀x∈s. f x ≤ g x
  shows integral s f ≤ integral s g
  by (rule has-integral-le[OF assms(1,2)[unfolded has-integral-integral] assms(3)])

lemma has-integral-nonneg:
  fixes f :: 'n::euclidean-space ⇒ real
  assumes (f has-integral i) s
    and ∀x∈s. 0 ≤ f x
  shows 0 ≤ i
  using has-integral-component-nonneg[of 1 f i s]
  unfolding o-def
  using assms
  by auto

lemma integral-nonneg:
  fixes f :: 'n::euclidean-space ⇒ real
  assumes f integrable-on s
    and ∀x∈s. 0 ≤ f x
  shows 0 ≤ integral s f
  by (rule has-integral-nonneg[OF assms(1)[unfolded has-integral-integral] assms(2)])
```

Hence a general restriction property.

```
lemma has-integral-restrict[simp]:
  assumes s ⊆ t
   shows ((λx. if x ∈ s then f x else (0::'a::banach)) has-integral i) t ⟷ (f
has-integral i) s
proof −
  have ∗: ⋀x. (if x ∈ t then if x ∈ s then f x else 0 else 0) = (if x∈s then f x else
```

0)
using *assms* **by** *auto*
show *?thesis*
apply (*subst*(*2*) *has-integral'*)
apply (*subst has-integral'*)
unfolding ∗
apply *rule*
done
qed

lemma *has-integral-restrict-univ*:
fixes *f* :: *'n*::*euclidean-space* ⇒ *'a*::*banach*
shows ((λ*x*. *if x* ∈ *s then f x else 0*) *has-integral i*) *UNIV* ⟷ (*f has-integral i*) *s*
by *auto*

lemma *has-integral-on-superset*:
fixes *f* :: *'n*::*euclidean-space* ⇒ *'a*::*banach*
assumes ∀ *x*. *x* ∉ *s* ⟶ *f x* = *0*
and *s* ⊆ *t*
and (*f has-integral i*) *s*
shows (*f has-integral i*) *t*
proof −
have (λ*x*. *if x* ∈ *s then f x else 0*) = (λ*x*. *if x* ∈ *t then f x else 0*)
apply *rule*
using *assms*(*1*−*2*)
apply *auto*
done
then show *?thesis*
using *assms*(*3*)
apply (*subst has-integral-restrict-univ*[*symmetric*])
apply (*subst*(*asm*) *has-integral-restrict-univ*[*symmetric*])
apply *auto*
done
qed

lemma *integrable-on-superset*:
fixes *f* :: *'n*::*euclidean-space* ⇒ *'a*::*banach*
assumes ∀ *x*. *x* ∉ *s* ⟶ *f x* = *0*
and *s* ⊆ *t*
and *f integrable-on s*
shows *f integrable-on t*
using *assms*
unfolding *integrable-on-def*
by (*auto intro*:*has-integral-on-superset*)

lemma *integral-restrict-univ*[*intro*]:
fixes *f* :: *'n*::*euclidean-space* ⇒ *'a*::*banach*
shows *f integrable-on s* ⟹ *integral UNIV* (λ*x*. *if x* ∈ *s then f x else 0*) = *integral*

s f
apply (*rule integral-unique*)
unfolding *has-integral-restrict-univ*
apply *auto*
done

lemma *integrable-restrict-univ*:
fixes *f* :: *'n*::*euclidean-space* ⇒ *'a*::*banach*
shows (λ*x*. *if x* ∈ *s then f x else 0*) *integrable-on UNIV* ⟷ *f integrable-on s*
unfolding *integrable-on-def*
by *auto*

lemma *negligible-on-intervals*: *negligible s* ⟷ (∀ *a b*. *negligible*(*s* ∩ *cbox a b*)) (**is** *?l* ⟷ *?r*)
proof
assume *?r*
show *?l*
unfolding *negligible-def*
proof *safe*
fix *a b*
show (*indicator s has-integral 0*) (*cbox a b*)
apply (*rule has-integral-negligible*[*OF* ‹*?r*›[*rule-format*,*of a b*]])
unfolding *indicator-def*
apply *auto*
done
qed
qed *auto*

lemma *has-integral-spike-set-eq*:
fixes *f* :: *'n*::*euclidean-space* ⇒ *'a*::*banach*
assumes *negligible* ((*s* − *t*) ∪ (*t* − *s*))
shows (*f has-integral y*) *s* ⟷ (*f has-integral y*) *t*
unfolding *has-integral-restrict-univ*[*symmetric*,*of f*]
apply (*rule has-integral-spike-eq*[*OF assms*])
by (*auto split*: *if-split-asm*)

lemma *has-integral-spike-set*[*dest*]:
fixes *f* :: *'n*::*euclidean-space* ⇒ *'a*::*banach*
assumes *negligible* ((*s* − *t*) ∪ (*t* − *s*))
and (*f has-integral y*) *s*
shows (*f has-integral y*) *t*
using *assms has-integral-spike-set-eq*
by *auto*

lemma *integrable-spike-set*[*dest*]:
fixes *f* :: *'n*::*euclidean-space* ⇒ *'a*::*banach*
assumes *negligible* ((*s* − *t*) ∪ (*t* − *s*))
and *f integrable-on s*
shows *f integrable-on t*

using *assms(2)*
unfolding *integrable-on-def*
unfolding *has-integral-spike-set-eq*[*OF assms(1)*] **.**

lemma *integrable-spike-set-eq*:
fixes $f :: 'n::euclidean\text{-}space \Rightarrow 'a::banach$
assumes *negligible* $((s - t) \cup (t - s))$
shows *f integrable-on s* $\longleftrightarrow$ *f integrable-on t*
apply *rule*
apply (*rule-tac*[!] *integrable-spike-set*)
using *assms*
apply *auto*
done

41.57 More lemmas that are useful later

lemma *has-integral-subset-component-le*:
fixes $f :: 'n::euclidean\text{-}space \Rightarrow 'm::euclidean\text{-}space$
assumes k: $k \in Basis$
and *as*: $s \subseteq t$ (*f has-integral i*) s (*f has-integral j*) t $\forall x{\in}t.\ 0 \le f(x){\cdot}k$
shows $i{\cdot}k \le j{\cdot}k$
proof −
note *has-integral-restrict-univ*[*symmetric*, *of f*]
note *as(2−3)*[*unfolded this*] **note** $*$ = *has-integral-component-le*[*OF k this*]
show *?thesis*
apply (*rule* $*$)
using *as(1,4)*
apply *auto*
done
qed

lemma *has-integral-subset-le*:
fixes $f :: 'n::euclidean\text{-}space \Rightarrow real$
assumes $s \subseteq t$
and (*f has-integral i*) s
and (*f has-integral j*) t
and $\forall x{\in}t.\ 0 \le f\ x$
shows $i \le j$
using *has-integral-subset-component-le*[*OF - assms(1), of 1 f i j*]
using *assms*
by *auto*

lemma *integral-subset-component-le*:
fixes $f :: 'n::euclidean\text{-}space \Rightarrow 'm::euclidean\text{-}space$
assumes $k \in Basis$
and $s \subseteq t$
and *f integrable-on s*
and *f integrable-on t*
and $\forall x \in t.\ 0 \le f\ x \cdot k$

shows (*integral s f*)·*k* ≤ (*integral t f*)·*k*
apply (*rule has-integral-subset-component-le*)
using *assms*
apply *auto*
done

lemma *integral-subset-le*:
fixes *f* :: *'n::euclidean-space* ⇒ *real*
assumes *s* ⊆ *t*
and *f integrable-on s*
and *f integrable-on t*
and ∀ *x* ∈ *t*. *0* ≤ *f x*
shows *integral s f* ≤ *integral t f*
apply (*rule has-integral-subset-le*)
using *assms*
apply *auto*
done

lemma *has-integral-alt'*:
fixes *f* :: *'n::euclidean-space* ⇒ *'a::banach*
shows (*f has-integral i*) *s* ⟷ (∀ *a b*. (λ*x*. *if x* ∈ *s then f x else 0*) *integrable-on cbox a b*) ∧
(∀ *e*>*0*. ∃ *B*>*0*. ∀ *a b*. *ball 0 B* ⊆ *cbox a b* ⟶
norm (*integral* (*cbox a b*) (λ*x*. *if x* ∈ *s then f x else 0*) − *i*) < *e*)
(**is** *?l* = *?r*)
proof
assume *?r*
show *?l*
apply (*subst has-integral'*)
apply *safe*
proof *goal-cases*
case (*1 e*)
from ‹*?r*›[*THEN conjunct2*,*rule-format*,*OF this*] **guess** *B* **.. note** *B*=*conjunctD2*[*OF this*]
show *?case*
apply *rule*
apply *rule*
apply (*rule B*)
apply *safe*
apply (*rule-tac x*=*integral* (*cbox a b*) (λ*x*. *if x* ∈ *s then f x else 0*) **in** *exI*)
apply (*drule B*(*2*)[*rule-format*])
using *integrable-integral*[*OF* ‹*?r*›[*THEN conjunct1*,*rule-format*]]
apply *auto*
done
qed
next
assume *?l* **note** *as* = *this*[*unfolded has-integral'*[*of f*],*rule-format*]
let *?f* = λ*x*. *if x* ∈ *s then f x else 0*
show *?r*

```
 proof safe
  fix a b :: 'n
 from as[OF zero-less-one] guess B .. note B=conjunctD2[OF this,rule-format]
  let ?a = ∑ i∈Basis. min (a•i) (−B) *R i::'n
  let ?b = ∑ i∈Basis. max (b•i) B *R i::'n
  show ?f integrable-on cbox a b
  proof (rule integrable-subinterval[of - ?a ?b])
   have ball 0 B ⊆ cbox ?a ?b
    apply (rule subsetI)
    unfolding mem-ball mem-box dist-norm
   proof (rule, goal-cases)
    case (1 x i)
    then show ?case using Basis-le-norm[of i x]
     by (auto simp add:field-simps)
   qed
   from B(2)[OF this] guess z .. note conjunct1[OF this]
   then show ?f integrable-on cbox ?a ?b
    unfolding integrable-on-def by auto
   show cbox a b ⊆ cbox ?a ?b
    apply safe
    unfolding mem-box
    apply rule
    apply (erule-tac x=i in ballE)
    apply auto
    done
  qed

  fix e :: real
  assume e > 0
  from as[OF this] guess B .. note B=conjunctD2[OF this,rule-format]
  show ∃ B>0. ∀ a b. ball 0 B ⊆ cbox a b ⟶
   norm (integral (cbox a b) (λx. if x ∈ s then f x else 0) − i) < e
   apply rule
   apply rule
   apply (rule B)
   apply safe
  proof goal-cases
   case 1
   from B(2)[OF this] guess z .. note z=conjunctD2[OF this]
   from integral-unique[OF this(1)] show ?case
    using z(2) by auto
  qed
 qed
qed
```

41.58 Continuity of the integral (for a 1-dimensional interval).

lemma *integrable-alt*:

```
  fixes f :: 'n::euclidean-space ⇒ 'a::banach
  shows f integrable-on s ⟷
    (∀ a b. (λx. if x ∈ s then f x else 0) integrable-on cbox a b) ∧
    (∀ e>0. ∃ B>0. ∀ a b c d. ball 0 B ⊆ cbox a b ∧ ball 0 B ⊆ cbox c d ⟶
    norm (integral (cbox a b) (λx. if x ∈ s then f x else 0) −
      integral (cbox c d) (λx. if x ∈ s then f x else 0)) < e)
  (is ?l = ?r)
proof
  assume ?l
 then guess y unfolding integrable-on-def .. note this[unfolded has-integral-alt'[of
f]]
  note y=conjunctD2[OF this,rule-format]
  show ?r
    apply safe
    apply (rule y)
  proof goal-cases
    case (1 e)
    then have e/2 > 0
      by auto
    from y(2)[OF this] guess B .. note B=conjunctD2[OF this,rule-format]
    show ?case
      apply rule
      apply rule
      apply (rule B)
      apply safe
    proof goal-cases
      case prems: (1 a b c d)
      show ?case
        apply (rule norm-triangle-half-l)
        using B(2)[OF prems(1)] B(2)[OF prems(2)]
        apply auto
        done
    qed
  qed
next
  assume ?r
  note as = conjunctD2[OF this,rule-format]
  let ?cube = λn. cbox (∑ i∈Basis. − real n *R i::'n) (∑ i∈Basis. real n *R i)
  have Cauchy (λn. integral (?cube n) (λx. if x ∈ s then f x else 0))
  proof (unfold Cauchy-def, safe, goal-cases)
    case (1 e)
    from as(2)[OF this] guess B .. note B = conjunctD2[OF this,rule-format]
    from real-arch-simple[of B] guess N .. note N = this
    {
      fix n
      assume n: n ≥ N
      have ball 0 B ⊆ ?cube n
        apply (rule subsetI)
        unfolding mem-ball mem-box dist-norm
```

```
    proof (rule, goal-cases)
      case (1 x i)
      then show ?case
        using Basis-le-norm[of i x] ‹i∈Basis›
        using n N
        by (auto simp add: field-simps setsum-negf)
    qed
  }
  then show ?case
    apply −
    apply (rule-tac x=N in exI)
    apply safe
    unfolding dist-norm
    apply (rule B(2))
    apply auto
    done
qed
from this[unfolded convergent-eq-cauchy[symmetric]] guess i ..
note i = this[THEN LIMSEQ-D]

show ?l unfolding integrable-on-def has-integral-alt'[of f]
  apply (rule-tac x=i in exI)
  apply safe
  apply (rule as(1)[unfolded integrable-on-def])
proof goal-cases
  case (1 e)
  then have ∗: e/2 > 0 by auto
  from i[OF this] guess N .. note N =this[rule-format]
  from as(2)[OF ∗] guess B .. note B=conjunctD2[OF this,rule-format]
  let ?B = max (real N) B
  show ?case
    apply (rule-tac x=?B in exI)
  proof safe
    show 0 < ?B
      using B(1) by auto
    fix a b :: 'n
    assume ab: ball 0 ?B ⊆ cbox a b
    from real-arch-simple[of ?B] guess n .. note n=this
    show norm (integral (cbox a b) (λx. if x ∈ s then f x else 0) − i) < e
      apply (rule norm-triangle-half-l)
      apply (rule B(2))
      defer
      apply (subst norm-minus-commute)
      apply (rule N[of n])
    proof safe
      show N ≤ n
        using n by auto
      fix x :: 'n
      assume x: x ∈ ball 0 B
```

```
      then have x ∈ ball 0 ?B
        by auto
      then show x ∈ cbox a b
        using ab by blast
      show x ∈ ?cube n
        using x
        unfolding mem-box mem-ball dist-norm
        apply −
      proof (rule, goal-cases)
        case (1 i)
        then show ?case
          using Basis-le-norm[of i x] ‹i ∈ Basis›
          using n
          by (auto simp add: field-simps setsum-negf)
      qed
    qed
  qed
qed

lemma integrable-altD:
  fixes f :: 'n::euclidean-space ⇒ 'a::banach
  assumes f integrable-on s
  shows ⋀a b. (λx. if x ∈ s then f x else 0) integrable-on cbox a b
    and ⋀e. e > 0 ⟹ ∃B>0. ∀a b c d. ball 0 B ⊆ cbox a b ∧ ball 0 B ⊆ cbox c
d ⟶
      norm (integral (cbox a b) (λx. if x ∈ s then f x else 0) − integral (cbox c d)
(λx. if x ∈ s then f x else 0)) < e
  using assms[unfolded integrable-alt[of f]] by auto

lemma integrable-on-subcbox:
  fixes f :: 'n::euclidean-space ⇒ 'a::banach
  assumes f integrable-on s
    and cbox a b ⊆ s
  shows f integrable-on cbox a b
  apply (rule integrable-eq)
  defer
  apply (rule integrable-altD(1)[OF assms(1)])
  using assms(2)
  apply auto
  done
```

41.59 A straddling criterion for integrability

```
lemma integrable-straddle-interval:
  fixes f :: 'n::euclidean-space ⇒ real
  assumes ∀e>0. ∃g  h i j. (g has-integral i) (cbox a b) ∧ (h has-integral j) (cbox
a b) ∧
    norm (i − j) < e ∧ (∀x∈cbox a b. (g x) ≤ f x ∧ f x ≤ h x)
```

shows *f integrable-on cbox a b*
proof (*subst integrable-cauchy*, *safe*, *goal-cases*)
case (*1 e*)
then have *e*: *e/3 > 0*
by *auto*
note *assms*[*rule-format*,*OF this*]
then guess *g h i j* **by** (*elim exE conjE*) **note** *obt = this*
from *obt*(*1*)[*unfolded has-integral*[*of g*], *rule-format*, *OF e*] **guess** *d1* **..** **note** *d1=conjunctD2*[*OF this*,*rule-format*]
from *obt*(*2*)[*unfolded has-integral*[*of h*], *rule-format*, *OF e*] **guess** *d2* **..** **note** *d2=conjunctD2*[*OF this*,*rule-format*]
show *?case*
apply (*rule-tac x=λx. d1 x ∩ d2 x* **in** *exI*)
apply (*rule conjI gauge-inter d1 d2*)+
unfolding *fine-inter*
proof (*safe*, *goal-cases*)
have ∗∗: ⋀*i j g1 g2 h1 h2 f1 f2. g1 − h2 ≤ f1 − f2 ⟹ f1 − f2 ≤ h1 − g2 ⟹*
|i − j| < e / 3 ⟹ |g2 − i| < e / 3 ⟹ |g1 − i| < e / 3 ⟹
|h2 − j| < e / 3 ⟹ |h1 − j| < e / 3 ⟹ |f1 − f2| < e
using ‹*e > 0*› **by** *arith*
case *prems*: (*1 p1 p2*)
note *tagged-division-ofD*(*2−4*) **note** ∗ = *this*[*OF prems*(*1*)] *this*[*OF prems*(*4*)]

have (∑(*x*, *k*)∈*p1*. *content k* ∗*R f x*) − (∑(*x*, *k*)∈*p1*. *content k* ∗*R g x*) ≥ *0*
and *0* ≤ (∑(*x*, *k*)∈*p2*. *content k* ∗*R h x*) − (∑(*x*, *k*)∈*p2*. *content k* ∗*R f x*)
and (∑(*x*, *k*)∈*p2*. *content k* ∗*R f x*) − (∑(*x*, *k*)∈*p2*. *content k* ∗*R g x*) ≥ *0*
and *0* ≤ (∑(*x*, *k*)∈*p1*. *content k* ∗*R h x*) − (∑(*x*, *k*)∈*p1*. *content k* ∗*R f x*)
unfolding *setsum-subtractf*[*symmetric*]
apply −
apply (*rule-tac*[!] *setsum-nonneg*)
apply *safe*
unfolding *real-scaleR-def right-diff-distrib*[*symmetric*]
apply (*rule-tac*[!] *mult-nonneg-nonneg*)
proof −
fix *a b*
assume *ab*: (*a*, *b*) ∈ *p1*
show *0* ≤ *content b*
using ∗(*3*)[*OF ab*]
apply *safe*
apply (*rule content-pos-le*)
done
then show *0* ≤ *content b* **.**
show *0* ≤ *f a − g a 0* ≤ *h a − f a*
using ∗(*1−2*)[*OF ab*]
using *obt*(*4*)[*rule-format*,*of a*]
by *auto*
next
fix *a b*

```
      assume ab: (a, b) ∈ p2
      show 0 ≤ content b
        using ∗(6)[OF ab]
        apply safe
        apply (rule content-pos-le)
        done
      then show 0 ≤ content b .
      show 0 ≤ f a − g a and 0 ≤ h a − f a
        using ∗(4−5)[OF ab] using obt(4)[rule-format,of a] by auto
    qed
    then show ?case
      apply −
      unfolding real-norm-def
      apply (rule ∗∗)
      defer
      defer
      unfolding real-norm-def[symmetric]
      apply (rule obt(3))
      apply (rule d1(2)[OF conjI[OF prems(4,5)]])
      apply (rule d1(2)[OF conjI[OF prems(1,2)]])
      apply (rule d2(2)[OF conjI[OF prems(4,6)]])
      apply (rule d2(2)[OF conjI[OF prems(1,3)]])
      apply auto
      done
  qed
qed

lemma integrable-straddle:
  fixes f :: 'n::euclidean-space ⇒ real
  assumes ∀ e>0. ∃ g h i j. (g has-integral i) s ∧ (h has-integral j) s ∧
    norm (i − j) < e ∧ (∀ x∈s. g x ≤ f x ∧ f x ≤ h x)
  shows f integrable-on s
proof −
  have ⋀a b. (λx. if x ∈ s then f x else 0) integrable-on cbox a b
  proof (rule integrable-straddle-interval, safe, goal-cases)
    case (1 a b e)
    then have ∗: e/4 > 0
      by auto
     from assms[rule-format,OF this] guess g h i j by (elim exE conjE) note
obt=this
    note obt(1)[unfolded has-integral-alt'[of g]]
    note conjunctD2[OF this, rule-format]
    note g = this(1) and this(2)[OF ∗]
    from this(2) guess B1 .. note B1 = conjunctD2[OF this,rule-format]
    note obt(2)[unfolded has-integral-alt'[of h]]
    note conjunctD2[OF this, rule-format]
    note h = this(1) and this(2)[OF ∗]
    from this(2) guess B2 .. note B2 = conjunctD2[OF this,rule-format]
    def c ≡ ∑ i∈Basis. min (a•i) (− (max B1 B2)) ∗R i::'n
```

def $d \equiv \sum i{\in}Basis.\ max\ (b{\cdot}i)\ (max\ B1\ B2) *_R i::'n$
have $*$: *ball 0 B1* $\subseteq$ *cbox c d ball 0 B2* $\subseteq$ *cbox c d*
apply *safe*
unfolding *mem-ball mem-box dist-norm*
apply (*rule-tac*[!] *ballI*)
proof *goal-cases*
case (*1 x i*)
then show *?case* **using** *Basis-le-norm*[*of i x*]
unfolding *c-def d-def* **by** *auto*
next
case (*2 x i*)
then show *?case* **using** *Basis-le-norm*[*of i x*]
unfolding *c-def d-def* **by** *auto*
qed
have $**$: $\bigwedge ch\ cg\ ag\ ah::real.\ norm\ (ah - ag) \leq norm\ (ch - cg) \Longrightarrow norm\ (cg - i) < e\ /\ 4 \Longrightarrow$
norm $(ch - j) < e\ /\ 4 \Longrightarrow norm\ (ag - ah) < e$
using *obt*(*3*)
unfolding *real-norm-def*
by *arith*
show *?case*
apply (*rule-tac x*=$\lambda x.$ *if x* $\in$ *s then g x else 0* **in** *exI*)
apply (*rule-tac x*=$\lambda x.$ *if x* $\in$ *s then h x else 0* **in** *exI*)
apply (*rule-tac x*=*integral* (*cbox a b*) ($\lambda x.$ *if x* $\in$ *s then g x else 0*) **in** *exI*)
apply (*rule-tac x*=*integral* (*cbox a b*) ($\lambda x.$ *if x* $\in$ *s then h x else 0*) **in** *exI*)
apply *safe*
apply (*rule-tac*[*1*−*2*] *integrable-integral*,*rule g*)
apply (*rule h*)
apply (*rule* $**$[*OF* - *B1*(*2*)[*OF* $*$(*1*)] *B2*(*2*)[*OF* $*$(*2*)]])
proof −
have $*$: $\bigwedge x\ f\ g.$ (*if x* $\in$ *s then f x else 0*) − (*if x* $\in$ *s then g x else 0*) =
(*if x* $\in$ *s then f x* − *g x else* (*0*::*real*))
by *auto*
note $**$ = *abs-of-nonneg*[*OF integral-nonneg*[*OF integrable-diff*, *OF h g*]]
show *norm* (*integral* (*cbox a b*) ($\lambda x.$ *if x* $\in$ *s then h x else 0*) −
integral (*cbox a b*) ($\lambda x.$ *if x* $\in$ *s then g x else 0*)) $\leq$
norm (*integral* (*cbox c d*) ($\lambda x.$ *if x* $\in$ *s then h x else 0*) −
integral (*cbox c d*) ($\lambda x.$ *if x* $\in$ *s then g x else 0*))
unfolding *integral-diff*[*OF h g*,*symmetric*] *real-norm-def*
apply (*subst* $**$)
defer
apply (*subst* $**$)
defer
apply (*rule has-integral-subset-le*)
defer
apply (*rule integrable-integral integrable-diff h g*)+
proof *safe*
fix *x*
assume *x* $\in$ *cbox a b*

```
      then show x ∈ cbox c d
        unfolding mem-box c-def d-def
        apply −
        apply rule
        apply (erule-tac x=i in ballE)
        apply auto
        done
    qed (insert obt(4), auto)
  qed (insert obt(4), auto)
 qed
 note interv = this

 show ?thesis
  unfolding integrable-alt[of f]
  apply safe
  apply (rule interv)
 proof goal-cases
  case (1 e)
  then have ∗: e/3 > 0
   by auto
  from assms[rule-format,OF this] guess g h i j by (elim exE conjE) note
obt=this
  note obt(1)[unfolded has-integral-alt'[of g]]
  note conjunctD2[OF this, rule-format]
  note g = this(1) and this(2)[OF ∗]
  from this(2) guess B1 .. note B1 = conjunctD2[OF this,rule-format]
  note obt(2)[unfolded has-integral-alt'[of h]]
  note conjunctD2[OF this, rule-format]
  note h = this(1) and this(2)[OF ∗]
  from this(2) guess B2 .. note B2 = conjunctD2[OF this,rule-format]
  show ?case
   apply (rule-tac x=max B1 B2 in exI)
   apply safe
   apply (rule max.strict-coboundedI1)
   apply (rule B1)
  proof −
   fix a b c d :: 'n
   assume as: ball 0 (max B1 B2) ⊆ cbox a b ball 0 (max B1 B2) ⊆ cbox c d
   have ∗∗: ball 0 B1 ⊆ ball (0::'n) (max B1 B2) ball 0 B2 ⊆ ball (0::'n) (max
B1 B2)
    by auto
   have ∗: ⋀ga gc ha hc fa fc::real.
    |ga − i| < e / 3 ∧ |gc − i| < e / 3 ∧ |ha − j| < e / 3 ∧
    |hc − j| < e / 3 ∧ |i − j| < e / 3 ∧ ga ≤ fa ∧ fa ≤ ha ∧ gc ≤ fc ∧ fc ≤
hc ⟹
    |fa − fc| < e
    by (simp add: abs-real-def split: if-split-asm)
   show norm (integral (cbox a b) (λx. if x ∈ s then f x else 0) − integral (cbox
c d)
```

```
      (λx. if x ∈ s then f x else 0)) < e
      unfolding real-norm-def
      apply (rule *)
      apply safe
      unfolding real-norm-def[symmetric]
      apply (rule B1(2))
      apply (rule order-trans)
      apply (rule **)
      apply (rule as(1))
      apply (rule B1(2))
      apply (rule order-trans)
      apply (rule **)
      apply (rule as(2))
      apply (rule B2(2))
      apply (rule order-trans)
      apply (rule **)
      apply (rule as(1))
      apply (rule B2(2))
      apply (rule order-trans)
      apply (rule **)
      apply (rule as(2))
      apply (rule obt)
      apply (rule-tac[!] integral-le)
      using obt
      apply (auto intro!: h g interv)
      done
    qed
  qed
qed
```

41.60 Adding integrals over several sets

```
lemma has-integral-union:
  fixes f :: 'n::euclidean-space ⇒ 'a::banach
  assumes (f has-integral i) s
    and (f has-integral j) t
    and negligible (s ∩ t)
  shows (f has-integral (i + j)) (s ∪ t)
proof −
  note * = has-integral-restrict-univ[symmetric, of f]
  show ?thesis
    unfolding *
    apply (rule has-integral-spike[OF assms(3)])
    defer
    apply (rule has-integral-add[OF assms(1−2)[unfolded *]])
    apply auto
    done
qed
```

lemma *has-integral-unions*:
fixes $f :: 'n::euclidean\text{-}space \Rightarrow 'a::banach$
assumes *finite t*
and $\forall s{\in}t.\ (f\ has\text{-}integral\ (i\ s))\ s$
and $\forall s{\in}t.\ \forall s'{\in}t.\ s \neq s' \longrightarrow negligible\ (s \cap s')$
shows $(f\ has\text{-}integral\ (setsum\ i\ t))\ (\bigcup t)$
proof −
note $* = has\text{-}integral\text{-}restrict\text{-}univ[symmetric,\ of\ f]$
have $**$: $negligible\ (\bigcup((\lambda(a,b).\ a \cap b)\ `\ \{(a,b).\ a \in t \wedge b \in \{y.\ y \in t \wedge a \neq y\}\}))$
apply (*rule negligible-unions*)
apply (*rule finite-imageI*)
apply (*rule finite-subset*[*of* - $t \times t$])
defer
apply (*rule finite-cartesian-product*[*OF assms*(*1*,*1*)])
using *assms*(*3*)
apply *auto*
done
note *assms*(*2*)[*unfolded* $*$]
note *has-integral-setsum*[*OF assms*(*1*) *this*]
then show *?thesis*
unfolding $*$
apply −
apply (*rule has-integral-spike*[*OF* $**$])
defer
apply *assumption*
apply *safe*
proof *goal-cases*
case *prems*: (*1 x*)
then show *?case*
proof (*cases* $x \in \bigcup t$)
case *True*
then guess *s* **unfolding** *Union-iff* **..** **note** *s*=*this*
then have $*$: $\forall b{\in}t.\ x \in b \longleftrightarrow b = s$
using *prems*(*3*) **by** *blast*
show *?thesis*
unfolding *if-P*[*OF True*]
apply (*rule trans*)
defer
apply (*rule setsum.cong*)
apply (*rule refl*)
apply (*subst* $*$)
apply *assumption*
apply (*rule refl*)
unfolding *setsum.delta*[*OF assms*(*1*)]
using *s*
apply *auto*
done
qed *auto*

qed
qed

In particular adding integrals over a division, maybe not of an interval.

lemma *has-integral-combine-division*:
fixes *f* :: *'n*::*euclidean-space* ⇒ *'a*::*banach*
assumes *d division-of s*
and ∀ *k*∈*d*. (*f has-integral* (*i k*)) *k*
shows (*f has-integral* (*setsum i d*)) *s*
proof −
note *d* = *division-ofD*[*OF assms*(*1*)]
show *?thesis*
unfolding *d*(*6*)[*symmetric*]
apply (*rule has-integral-unions*)
apply (*rule d assms*)+
apply *rule*
apply *rule*
apply *rule*
proof *goal-cases*
case *prems*: (*1 s s'*)
from *d*(*4*)[*OF this*(*1*)] *d*(*4*)[*OF this*(*2*)] **guess** *a c b d* **by** (*elim exE*) **note** *obt*=*this*
from *d*(*5*)[*OF prems*] **show** *?case*
unfolding *obt interior-cbox*
apply −
apply (*rule negligible-subset*[*of* (*cbox a b*−*box a b*) ∪ (*cbox c d*−*box c d*)])
apply (*rule negligible-union negligible-frontier-interval*)+
apply *auto*
done
qed
qed

lemma *integral-combine-division-bottomup*:
fixes *f* :: *'n*::*euclidean-space* ⇒ *'a*::*banach*
assumes *d division-of s*
and ∀ *k*∈*d*. *f integrable-on k*
shows *integral s f* = *setsum* (λ*i*. *integral i f*) *d*
apply (*rule integral-unique*)
apply (*rule has-integral-combine-division*[*OF assms*(*1*)])
using *assms*(*2*)
unfolding *has-integral-integral*
apply *assumption*
done

lemma *has-integral-combine-division-topdown*:
fixes *f* :: *'n*::*euclidean-space* ⇒ *'a*::*banach*
assumes *f integrable-on s*
and *d division-of k*
and *k* ⊆ *s*

shows (*f has-integral* (*setsum* (λ*i*. *integral i f*) *d*)) *k*
apply (*rule has-integral-combine-division*[*OF assms*(*2*)])
apply *safe*
unfolding *has-integral-integral*[*symmetric*]
proof *goal-cases*
case (*1 k*)
from *division-ofD*(*2*,*4*)[*OF assms*(*2*) *this*]
show *?case*
apply *safe*
apply (*rule integrable-on-subcbox*)
apply (*rule assms*)
using *assms*(*3*)
apply *auto*
done
qed

lemma *integral-combine-division-topdown*:
fixes *f* :: *'n*::*euclidean-space* ⇒ *'a*::*banach*
assumes *f integrable-on s*
and *d division-of s*
shows *integral s f* = *setsum* (λ*i*. *integral i f*) *d*
apply (*rule integral-unique*)
apply (*rule has-integral-combine-division-topdown*)
using *assms*
apply *auto*
done

lemma *integrable-combine-division*:
fixes *f* :: *'n*::*euclidean-space* ⇒ *'a*::*banach*
assumes *d division-of s*
and ∀ *i*∈*d*. *f integrable-on i*
shows *f integrable-on s*
using *assms*(*2*)
unfolding *integrable-on-def*
by (*metis has-integral-combine-division*[*OF assms*(*1*)])

lemma *integrable-on-subdivision*:
fixes *f* :: *'n*::*euclidean-space* ⇒ *'a*::*banach*
assumes *d division-of i*
and *f integrable-on s*
and *i* ⊆ *s*
shows *f integrable-on i*
apply (*rule integrable-combine-division assms*)+
apply *safe*
proof *goal-cases*
case *1*
note *division-ofD*(*2*,*4*)[*OF assms*(*1*) *this*]
then show *?case*
apply *safe*

apply (*rule integrable-on-subcbox*[*OF assms*(*2*)])
using *assms*(*3*)
apply *auto*
done
qed

41.61 Also tagged divisions

lemma *has-integral-combine-tagged-division*:
fixes *f* :: *'n*::*euclidean-space* ⇒ *'a*::*banach*
assumes *p tagged-division-of s*
and ∀(*x*,*k*) ∈ *p*. (*f has-integral* (*i k*)) *k*
shows (*f has-integral* (*setsum* (λ(*x*,*k*). *i k*) *p*)) *s*
proof −
have ∗: (*f has-integral* (*setsum* (λ*k*. *integral k f*) (*snd* ' *p*))) *s*
apply (*rule has-integral-combine-division*)
apply (*rule division-of-tagged-division*[*OF assms*(*1*)])
using *assms*(*2*)
unfolding *has-integral-integral*[*symmetric*]
apply *safe*
apply *auto*
done
then show *?thesis*
apply −
apply (*rule subst*[**where** *P*=λ*i*. (*f has-integral i*) *s*])
defer
apply *assumption*
apply (*rule trans*[*of* - *setsum* (λ(*x*,*k*). *integral k f*) *p*])
apply (*subst eq-commute*)
apply (*rule setsum-over-tagged-division-lemma*[*OF assms*(*1*)])
apply (*rule integral-null*)
apply *assumption*
apply (*rule setsum.cong*)
using *assms*(*2*)
apply *auto*
done
qed

lemma *integral-combine-tagged-division-bottomup*:
fixes *f* :: *'n*::*euclidean-space* ⇒ *'a*::*banach*
assumes *p tagged-division-of* (*cbox a b*)
and ∀(*x*,*k*)∈*p*. *f integrable-on k*
shows *integral* (*cbox a b*) *f* = *setsum* (λ(*x*,*k*). *integral k f*) *p*
apply (*rule integral-unique*)
apply (*rule has-integral-combine-tagged-division*[*OF assms*(*1*)])
using *assms*(*2*)
apply *auto*
done

lemma *has-integral-combine-tagged-division-topdown*:
fixes f :: *'n::euclidean-space* ⇒ *'a::banach*
assumes *f integrable-on cbox a b*
and *p tagged-division-of* (*cbox a b*)
shows (*f has-integral* (*setsum* (λ(*x*,*k*). *integral k f*) *p*)) (*cbox a b*)
apply (*rule has-integral-combine-tagged-division*[*OF assms*(*2*)])
apply *safe*
proof *goal-cases*
case *1*
note *tagged-division-ofD*(*3*−*4*)[*OF assms*(*2*) *this*]
then show *?case*
using *integrable-subinterval*[*OF assms*(*1*)] **by** *blast*
qed

lemma *integral-combine-tagged-division-topdown*:
fixes f :: *'n::euclidean-space* ⇒ *'a::banach*
assumes *f integrable-on cbox a b*
and *p tagged-division-of* (*cbox a b*)
shows *integral* (*cbox a b*) *f* = *setsum* (λ(*x*,*k*). *integral k f*) *p*
apply (*rule integral-unique*)
apply (*rule has-integral-combine-tagged-division-topdown*)
using *assms*
apply *auto*
done

41.62 Henstock's lemma

lemma *henstock-lemma-part1*:
fixes f :: *'n::euclidean-space* ⇒ *'a::banach*
assumes *f integrable-on cbox a b*
and *e* > *0*
and *gauge d*
and (∀ *p*. *p tagged-division-of* (*cbox a b*) ∧ *d fine p* ⟶
norm (*setsum* (λ(*x*,*k*). *content k* ∗$_R$ *f x*) *p* − *integral*(*cbox a b*) *f*) < *e*)
and *p*: *p tagged-partial-division-of* (*cbox a b*) *d fine p*
shows *norm* (*setsum* (λ(*x*,*k*). *content k* ∗$_R$ *f x* − *integral k f*) *p*) ≤ *e*
(**is** *?x* ≤ *e*)
proof −
{ **presume** ⋀*k*. *0*<*k* ⟹ *?x* ≤ *e* + *k* **then show** *?thesis* **by** (*blast intro*: *field-le-epsilon*) }
fix *k* :: *real*
assume *k*: *k* > *0*
note *p'* = *tagged-partial-division-ofD*[*OF p*(*1*)]
have ⋃(*snd* ' *p*) ⊆ *cbox a b*
using *p'*(*3*) **by** *fastforce*
note *partial-division-of-tagged-division*[*OF p*(*1*)] *this*
from *partial-division-extend-interval*[*OF this*] **guess** *q* **. note** *q*=*this* **and** *q'* = *division-ofD*[*OF this*(*2*)]
def *r* ≡ *q* − *snd* ' *p*

```
  have snd ` p ∩ r = {}
    unfolding r-def by auto
  have r: finite r
    using q' unfolding r-def by auto

  have ∀i∈r. ∃p. p tagged-division-of i ∧ d fine p ∧
    norm (setsum (λ(x,j). content j *R f x) p − integral i f) < k / (real (card r)
+ 1)
    apply safe
  proof goal-cases
    case (1 i)
    then have i: i ∈ q
      unfolding r-def by auto
    from q'(4)[OF this] guess u v by (elim exE) note uv=this
    have *: k / (real (card r) + 1) > 0 using k by simp
    have f integrable-on cbox u v
      apply (rule integrable-subinterval[OF assms(1)])
      using q'(2)[OF i]
      unfolding uv
      apply auto
      done
    note integrable-integral[OF this, unfolded has-integral[of f]]
   from this[rule-format,OF *] guess dd .. note dd=conjunctD2[OF this,rule-format]
    note gauge-inter[OF ‹gauge d› dd(1)]
    from fine-division-exists[OF this,of u v] guess qq .
    then show ?case
      apply (rule-tac x=qq in exI)
      using dd(2)[of qq]
      unfolding fine-inter uv
      apply auto
      done
  qed
  from bchoice[OF this] guess qq .. note qq=this[rule-format]

  let ?p = p ∪ ⋃(qq ` r)
  have norm ((∑(x, k)∈?p. content k *R f x) − integral (cbox a b) f) < e
    apply (rule assms(4)[rule-format])
  proof
    show d fine ?p
      apply (rule fine-union)
      apply (rule p)
      apply (rule fine-unions)
      using qq
      apply auto
      done
    note * = tagged-partial-division-of-union-self[OF p(1)]
    have p ∪ ⋃(qq ` r) tagged-division-of ⋃(snd ` p) ∪ ⋃r
      using r
    proof (rule tagged-division-union[OF * tagged-division-unions], goal-cases)
```

```
    case 1
    then show ?case
      using qq by auto
  next
    case 2
    then show ?case
      apply rule
      apply rule
      apply rule
      apply(rule q'(5))
      unfolding r-def
      apply auto
      done
  next
    case 3
    then show ?case
      apply (rule inter-interior-unions-intervals)
      apply fact
      apply rule
      apply rule
      apply (rule q')
      defer
      apply rule
      apply (subst Int-commute)
      apply (rule inter-interior-unions-intervals)
      apply (rule finite-imageI)
      apply (rule p')
      apply rule
      defer
      apply rule
      apply (rule q')
      using q(1) p'
      unfolding r-def
      apply auto
      done
  qed
  moreover have ⋃(snd ` p) ∪ ⋃r = cbox a b and {qq i |i. i ∈ r} = qq ` r
    unfolding Union-Un-distrib[symmetric] r-def
    using q
    by auto
  ultimately show ?p tagged-division-of (cbox a b)
    by fastforce
qed

then have norm ((∑(x, k)∈p. content k *R f x) + (∑(x, k)∈⋃(qq ` r). content
k *R f x) −
  integral (cbox a b) f) < e
  apply (subst setsum.union-inter-neutral[symmetric])
  apply (rule p')
```

```
    prefer 3
    apply assumption
    apply rule
    apply (rule r)
    apply safe
    apply (drule qq)
  proof −
    fix x l k
    assume as: (x, l) ∈ p (x, l) ∈ qq k k ∈ r
    note qq[OF this(3)]
    note tagged-division-ofD(3,4)[OF conjunct1[OF this] as(2)]
    from this(2) guess u v by (elim exE) note uv=this
   have l∈snd ‘ p unfolding image-iff apply(rule-tac x=(x,l) in bexI) using as
by auto
    then have l ∈ q k ∈ q l ≠ k
      using as(1,3) q(1) unfolding r-def by auto
    note q′(5)[OF this]
    then have interior l = {}
      using interior-mono[OF ‹l ⊆ k›] by blast
    then show content l ∗R f x = 0
      unfolding uv content-eq-0-interior[symmetric] by auto
  qed auto

  then have norm ((∑(x, k)∈p. content k ∗R f x) + setsum (setsum (λ(x, k).
content k ∗R f x))
    (qq ‘ r) − integral (cbox a b) f) < e
    apply (subst (asm) setsum.Union-comp)
    prefer 2
    unfolding split-paired-all split-conv image-iff
    apply (erule bexE)+
  proof −
    fix x m k l T1 T2
    assume (x, m) ∈ T1 (x, m) ∈ T2 T1 ≠ T2 k ∈ r l ∈ r T1 = qq k T2 = qq l
    note as = this(1−5)[unfolded this(6−)]
    note kl = tagged-division-ofD(3,4)[OF qq[THEN conjunct1]]
    from this(2)[OF as(4,1)] guess u v by (elim exE) note uv=this
    have ∗: interior (k ∩ l) = {}
      by (metis DiffE ‹T1 = qq k› ‹T1 ≠ T2› ‹T2 = qq l› as(4) as(5) interior-Int
q′(5) r-def)
    have interior m = {}
      unfolding subset-empty[symmetric]
      unfolding ∗[symmetric]
      apply (rule interior-mono)
      using kl(1)[OF as(4,1)] kl(1)[OF as(5,2)]
      apply auto
      done
    then show content m ∗R f x = 0
      unfolding uv content-eq-0-interior[symmetric]
      by auto
```

```
  qed (insert qq, auto)

  then have **: norm ((∑(x, k)∈p. content k *R f x) + setsum (setsum (λ(x,
k). content k *R f x) ∘ qq) r −
    integral (cbox a b) f) < e
    apply (subst (asm) setsum.reindex-nontrivial)
    apply fact
    apply (rule setsum.neutral)
    apply rule
    unfolding split-paired-all split-conv
    defer
    apply assumption
  proof −
    fix k l x m
    assume as: k ∈ r l ∈ r k ≠ l qq k = qq l (x, m) ∈ qq k
    note tagged-division-ofD(6)[OF qq[THEN conjunct1]]
    from this[OF as(1)] this[OF as(2)] show content m *R f x = 0
      using as(3) unfolding as by auto
  qed

  have *: norm (cp − ip) ≤ e + k
    if norm ((cp + cr) − i) < e
    and norm (cr − ir) < k
    and ip + ir = i
    for ir ip i cr cp
  proof −
    from that show ?thesis
      using norm-triangle-le[of cp + cr − i − (cr − ir)]
      unfolding that(3)[symmetric] norm-minus-cancel
      by (auto simp add: algebra-simps)
  qed

  have ?x =  norm ((∑(x, k)∈p. content k *R f x) − (∑(x, k)∈p. integral k f))
    unfolding split-def setsum-subtractf ..
  also have ... ≤ e + k
    apply (rule *[OF **, where ir1=setsum (λk. integral k f) r])
  proof goal-cases
    case 1
    have *: k * real (card r) / (1 + real (card r)) < k
      using k by (auto simp add: field-simps)
    show ?case
      apply (rule le-less-trans[of - setsum (λx. k / (real (card r) + 1)) r])
      unfolding setsum-subtractf[symmetric]
      apply (rule setsum-norm-le)
      apply rule
      apply (drule qq)
      defer
      unfolding divide-inverse setsum-left-distrib[symmetric]
      unfolding divide-inverse[symmetric]
```

```
    using ∗ apply (auto simp add: field-simps)
    done
  next
    case 2
    have ∗: (∑(x, k)∈p. integral k f) = (∑k∈snd ‘ p. integral k f)
      apply (subst setsum.reindex-nontrivial)
      apply fact
      unfolding split-paired-all snd-conv split-def o-def
    proof −
      fix x l y m
      assume as: (x, l) ∈ p (y, m) ∈ p (x, l) ≠ (y, m) l = m
      from p′(4)[OF as(1)] guess u v by (elim exE) note uv=this
      show integral l f = 0
        unfolding uv
        apply (rule integral-unique)
        apply (rule has-integral-null)
        unfolding content-eq-0-interior
        using p′(5)[OF as(1−3)]
        unfolding uv as(4)[symmetric]
        apply auto
        done
    qed auto
    from q(1) have ∗∗: snd ‘ p ∪ q = q by auto
    show ?case
      unfolding integral-combine-division-topdown[OF assms(1) q(2)] ∗ r-def
     using ∗∗ q′(1) p′(1) setsum.union-disjoint [of snd ‘ p q − snd ‘ p λk. integral
k f, symmetric]
      by simp
  qed
  finally show ?x ≤ e + k .
qed

lemma henstock-lemma-part2:
  fixes f :: 'm::euclidean-space ⇒ 'n::euclidean-space
  assumes f integrable-on cbox a b
    and e > 0
    and gauge d
    and ∀p. p tagged-division-of (cbox a b) ∧ d fine p ⟶
      norm (setsum (λ(x,k). content k ∗R f x) p − integral (cbox a b) f) < e
    and p tagged-partial-division-of (cbox a b)
    and d fine p
  shows setsum (λ(x,k). norm (content k ∗R f x − integral k f)) p ≤ 2 ∗ real
(DIM('n)) ∗ e
  unfolding split-def
  apply (rule setsum-norm-allsubsets-bound)
  defer
  apply (rule henstock-lemma-part1[unfolded split-def,OF assms(1−3)])
  apply safe
  apply (rule assms[rule-format,unfolded split-def])
```

defer
apply (*rule tagged-partial-division-subset*)
apply (*rule assms*)
apply *assumption*
apply (*rule fine-subset*)
apply *assumption*
apply (*rule assms*)
using *assms*(*5*)
apply *auto*
done

lemma *henstock-lemma*:
fixes *f* :: *'m*::*euclidean-space* ⇒ *'n*::*euclidean-space*
assumes *f integrable-on cbox a b*
and *e* > *0*
obtains *d* **where** *gauge d*
and ∀ *p*. *p tagged-partial-division-of* (*cbox a b*) ∧ *d fine p* ⟶
setsum (λ(*x*,*k*). *norm*(*content k* ∗$_R$ *f x* − *integral k f*)) *p* < *e*
proof −
have ∗: *e* / (*2* ∗ (*real DIM*(*'n*) + *1*)) > *0* **using** *assms*(*2*) **by** *simp*
from *integrable-integral*[*OF assms*(*1*),*unfolded has-integral*[*of f*],*rule-format*,*OF this*]
guess *d* **..** **note** *d* = *conjunctD2*[*OF this*]
show *thesis*
apply (*rule that*)
apply (*rule d*)
proof (*safe*, *goal-cases*)
case (*1 p*)
note ∗ = *henstock-lemma-part2*[*OF assms*(*1*) ∗ *d this*]
show *?case*
apply (*rule le-less-trans*[*OF* ∗])
using ‹*e* > *0*›
apply (*auto simp add*: *field-simps*)
done
qed
qed

41.63 Geometric progression

FIXME: Should one or more of these theorems be moved to `~~/src/HOL/Set_Interval.thy`, alongside *geometric-sum*?

lemma *sum-gp-basic*:
fixes *x* :: *'a*::*ring-1*
shows (*1* − *x*) ∗ *setsum* (λ*i*. *x*^*i*) {*0* .. *n*} = (*1* − *x*^(*Suc n*))
proof −
def *y* ≡ *1* − *x*
have *y* ∗ (∑ *i*=*0*..*n*. (*1* − *y*) ^ *i*) = *1* − (*1* − *y*) ^ *Suc n*
by (*induct n*) (*simp-all add*: *algebra-simps*)
then show *?thesis*

```
    unfolding y-def by simp
qed

lemma sum-gp-multiplied:
  assumes mn: m ≤ n
  shows ((1::'a::{field}) − x) ∗ setsum (op ^ x) {m..n} = x^m − x^ Suc n
  (is ?lhs = ?rhs)
proof −
  let ?S = {0..(n − m)}
  from mn have mn': n − m ≥ 0
    by arith
  let ?f = op + m
  have i: inj-on ?f ?S
    unfolding inj-on-def by auto
  have f: ?f ' ?S = {m..n}
    using mn
    apply (auto simp add: image-iff Bex-def)
    apply presburger
    done
  have th: op ^ x ∘ op + m = (λi. x^m ∗ x^i)
    by (rule ext) (simp add: power-add power-mult)
 from setsum.reindex[OF i, of op ^ x, unfolded f th setsum-right-distrib[symmetric]]
  have ?lhs = x^m ∗ ((1 − x) ∗ setsum (op ^ x) {0..n − m})
    by simp
  then show ?thesis
    unfolding sum-gp-basic
    using mn
    by (simp add: field-simps power-add[symmetric])
qed

lemma sum-gp:
  setsum (op ^ (x::'a::{field})) {m .. n} =
    (if n < m then 0
     else if x = 1 then of-nat ((n + 1) − m)
     else (x^ m − x^ (Suc n)) / (1 − x))
proof −
  {
    assume nm: n < m
    then have ?thesis by simp
  }
  moreover
  {
    assume ¬ n < m
    then have nm: m ≤ n
      by arith
    {
      assume x: x = 1
      then have ?thesis
        by simp
```

```
    }
    moreover
    {
      assume x: x ≠ 1
      then have nz: 1 − x ≠ 0
        by simp
      from sum-gp-multiplied[OF nm, of x] nz have ?thesis
        by (simp add: field-simps)
    }
    ultimately have ?thesis by blast
  }
  ultimately show ?thesis by blast
qed

lemma sum-gp-offset:
  setsum (op ^ (x::'a::{field})) {m .. m+n} =
    (if x = 1 then of-nat n + 1 else x^m * (1 − x^Suc n) / (1 − x))
  unfolding sum-gp[of x m m + n] power-Suc
  by (simp add: field-simps power-add)
```

41.64 Monotone convergence (bounded interval first)

```
lemma monotone-convergence-interval:
  fixes f :: nat ⇒ 'n::euclidean-space ⇒ real
  assumes ∀k. (f k) integrable-on cbox a b
    and ∀k. ∀x∈cbox a b.(f k x) ≤ f (Suc k) x
    and ∀x∈cbox a b. ((λk. f k x) ⟶ g x) sequentially
    and bounded {integral (cbox a b) (f k) | k . k ∈ UNIV}
  shows g integrable-on cbox a b ∧ ((λk. integral (cbox a b) (f k)) ⟶ integral
(cbox a b) g) sequentially
proof (cases content (cbox a b) = 0)
  case True
  show ?thesis
    using integrable-on-null[OF True]
    unfolding integral-null[OF True]
    using tendsto-const
    by auto
next
  case False
  have fg: ∀x∈cbox a b. ∀k. (f k x) · 1 ≤ (g x) · 1
  proof safe
    fix x k
    assume x: x ∈ cbox a b
   note * = Lim-component-ge[OF assms(3)[rule-format, OF x] trivial-limit-sequentially]
    show f k x · 1 ≤ g x · 1
      apply (rule *)
      unfolding eventually-sequentially
      apply (rule-tac x=k in exI)
      apply −
```

```
    apply (rule transitive-stepwise-le)
    using assms(2)[rule-format, OF x]
    apply auto
    done
 qed
 have ∃i. ((λk. integral (cbox a b) (f k)) ⟶ i) sequentially
  apply (rule bounded-increasing-convergent)
  defer
  apply rule
  apply (rule integral-le)
  apply safe
  apply (rule assms(1−2)[rule-format])+
  using assms(4)
  apply auto
  done
 then guess i .. note i=this
 have i': ⋀k. (integral(cbox a b) (f k)) ≤ i•1
  apply (rule Lim-component-ge)
  apply (rule i)
  apply (rule trivial-limit-sequentially)
  unfolding eventually-sequentially
  apply (rule-tac x=k in exI)
  apply (rule transitive-stepwise-le)
  prefer 3
  unfolding inner-simps real-inner-1-right
  apply (rule integral-le)
  apply (rule assms(1−2)[rule-format])+
  using assms(2)
  apply auto
  done

 have (g has-integral i) (cbox a b)
  unfolding has-integral
 proof (safe, goal-cases)
  case e: (1 e)
  then have ∀k. (∃d. gauge d ∧ (∀p. p tagged-division-of (cbox a b) ∧ d fine p
⟶
    norm ((∑(x, ka)∈p. content ka *R f k x) − integral (cbox a b) (f k)) < e /
2 ^ (k + 2)))
    apply −
    apply rule
    apply (rule assms(1)[unfolded has-integral-integral has-integral,rule-format])
    apply auto
    done
  from choice[OF this] guess c .. note c=conjunctD2[OF this[rule-format],rule-format]

  have ∃r. ∀k≥r. 0 ≤ i•1 − (integral (cbox a b) (f k)) ∧ i•1 − (integral (cbox
a b) (f k)) < e / 4
  proof −
```

```
    have e/4 > 0
      using e by auto
    from LIMSEQ-D [OF i this] guess r ..
    then show ?thesis
      apply (rule-tac x=r in exI)
      apply rule
      apply (erule-tac x=k in allE)
      subgoal for k using i'[of k] by auto
      done
  qed
  then guess r .. note r=conjunctD2[OF this[rule-format]]

  have ∀x∈cbox a b. ∃n≥r. ∀k≥n. 0 ≤ (g x)·1 - (f k x)·1 ∧
    (g x)·1 - (f k x)·1 < e / (4 * content(cbox a b))
  proof (rule, goal-cases)
    case prems: (1 x)
    have e / (4 * content (cbox a b)) > 0
      using ‹e>0› False content-pos-le[of a b] by auto
    from assms(3)[rule-format, OF prems, THEN LIMSEQ-D, OF this]
    guess n .. note n=this
    then show ?case
      apply (rule-tac x=n + r in exI)
      apply safe
      apply (erule-tac[2-3] x=k in allE)
      unfolding dist-real-def
      using fg[rule-format, OF prems]
      apply (auto simp add: field-simps)
      done
  qed
 from bchoice[OF this] guess m .. note m=conjunctD2[OF this[rule-format],rule-format]
  def d ≡ λx. c (m x) x

  show ?case
    apply (rule-tac x=d in exI)
  proof safe
    show gauge d
      using c(1) unfolding gauge-def d-def by auto
  next
    fix p
    assume p: p tagged-division-of (cbox a b) d fine p
    note p'=tagged-division-ofD[OF p(1)]
    have ∃a. ∀x∈p. m (fst x) ≤ a
      by (metis finite-imageI finite-nat-set-iff-bounded-le p'(1) rev-image-eqI)
    then guess s .. note s=this
    have *: ∀a b c d. norm(a - b) ≤ e / 4 ∧ norm(b - c) < e / 2 ∧
      norm (c - d) < e / 4 ⟶ norm (a - d) < e
    proof (safe, goal-cases)
      case (1 a b c d)
      then show ?case
```

```
      using norm-triangle-lt[of a − b b − c 3* e/4]
        norm-triangle-lt[of a − b + (b − c) c − d e]
      unfolding norm-minus-cancel
      by (auto simp add: algebra-simps)
    qed
    show norm ((∑(x, k)∈p. content k *R g x) − i) < e
     apply (rule *[rule-format,where
       b=∑(x, k)∈p. content k *R f (m x) x and c=∑(x, k)∈p. integral k (f
(m x))])
    proof (safe, goal-cases)
     case 1
     show ?case
       apply (rule order-trans[of - ∑(x, k)∈p. content k * (e / (4 * content
(cbox a b)))])
      unfolding setsum-subtractf[symmetric]
      apply (rule order-trans)
      apply (rule norm-setsum)
      apply (rule setsum-mono)
      unfolding split-paired-all split-conv
    unfolding split-def setsum-left-distrib[symmetric] scaleR-diff-right[symmetric]
      unfolding additive-content-tagged-division[OF p(1), unfolded split-def]
     proof −
      fix x k
      assume xk: (x, k) ∈ p
      then have x: x ∈ cbox a b
       using p'(2−3)[OF xk] by auto
      from p'(4)[OF xk] guess u v by (elim exE) note uv=this
       show norm (content k *R (g x − f (m x) x)) ≤ content k * (e / (4 *
content (cbox a b)))
        unfolding norm-scaleR uv
        unfolding abs-of-nonneg[OF content-pos-le]
        apply (rule mult-left-mono)
        using m(2)[OF x,of m x]
        apply auto
        done
     qed (insert False, auto)

    next
     case 2
     show ?case
      apply (rule le-less-trans[of - norm (∑j = 0..s.
        ∑(x, k)∈{xk∈p. m (fst xk) = j}. content k *R f (m x) x − integral k
(f (m x)))])
      apply (subst setsum-group)
      apply fact
      apply (rule finite-atLeastAtMost)
      defer
      apply (subst split-def)+
      unfolding setsum-subtractf
```

```
      apply rule
     proof −
      show norm (∑j = 0..s. ∑(x, k)∈{xk ∈ p.
       m (fst xk) = j}. content k *R f (m x) x − integral k (f (m x))) < e / 2
       apply (rule le-less-trans[of - setsum (λi. e / 2^(i+2)) {0..s}])
       apply (rule setsum-norm-le)
      proof
       show (∑ i = 0..s. e / 2 ^ (i + 2)) < e / 2
        unfolding power-add divide-inverse inverse-mult-distrib
       unfolding setsum-right-distrib[symmetric] setsum-left-distrib[symmetric]
        unfolding power-inverse [symmetric] sum-gp
        apply(rule mult-strict-left-mono[OF - e])
        unfolding power2-eq-square
        apply auto
        done
       fix t
       assume t ∈ {0..s}
       show norm (∑(x, k)∈{xk ∈ p. m (fst xk) = t}. content k *R f (m x)
x −
        integral k (f (m x))) ≤ e / 2 ^ (t + 2)
        apply (rule order-trans
          [of - norm (setsum (λ(x,k). content k *R f t x − integral k (f t)) {xk
∈ p. m (fst xk) = t})])
        apply (rule eq-refl)
        apply (rule arg-cong[where f=norm])
        apply (rule setsum.cong)
        apply (rule refl)
        defer
        apply (rule henstock-lemma-part1)
        apply (rule assms(1)[rule-format])
        apply (simp add: e)
        apply safe
        apply (rule c)+
        apply rule
        apply assumption+
        apply (rule tagged-partial-division-subset[of p])
        apply (rule p(1)[unfolded tagged-division-of-def,THEN conjunct1])
        defer
        unfolding fine-def
        apply safe
        apply (drule p(2)[unfolded fine-def,rule-format])
        unfolding d-def
        apply auto
        done
      qed
     qed (insert s, auto)
    next
     case 3
    note comb = integral-combine-tagged-division-topdown[OF assms(1)[rule-format]
```

```
p(1)]
      have ∗: ⋀sr sx ss ks kr::real. kr = sr ⟶ ks = ss ⟶
        ks ≤ i ∧ sr ≤ sx ∧ sx ≤ ss ∧ 0 ≤ i•1 − kr•1 ∧ i•1 − kr•1 < e/4 ⟶
|sx − i| < e/4
        by auto
      show ?case
        unfolding real-norm-def
        apply (rule ∗[rule-format])
        apply safe
        apply (rule comb[of r])
        apply (rule comb[of s])
        apply (rule i'[unfolded real-inner-1-right])
        apply (rule-tac[1−2] setsum-mono)
        unfolding split-paired-all split-conv
        apply (rule-tac[1−2] integral-le[OF ])
      proof safe
        show 0 ≤ i•1 − (integral (cbox a b) (f r))•1
          using r(1) by auto
        show i•1 − (integral (cbox a b) (f r))•1 < e / 4
          using r(2) by auto
        fix x k
        assume xk: (x, k) ∈ p
        from p'(4)[OF this] guess u v by (elim exE) note uv=this
        show f r integrable-on k
          and f s integrable-on k
          and f (m x) integrable-on k
          and f (m x) integrable-on k
          unfolding uv
          apply (rule-tac[!] integrable-on-subcbox[OF assms(1)[rule-format]])
          using p'(3)[OF xk]
          unfolding uv
          apply auto
          done
        fix y
        assume y ∈ k
        then have y ∈ cbox a b
          using p'(3)[OF xk] by auto
        then have ∗: ⋀m. ∀ n≥m. f m y ≤ f n y
          apply −
          apply (rule transitive-stepwise-le)
          using assms(2)
          apply auto
          done
        show f r y ≤ f (m x) y and f (m x) y ≤ f s y
          apply (rule-tac[!] ∗[rule-format])
          using s[rule-format,OF xk] m(1)[of x] p'(2−3)[OF xk]
          apply auto
          done
      qed
```

```
      qed
    qed
  qed note ∗ = this

  have integral (cbox a b) g = i
    by (rule integral-unique) (rule ∗)
  then show ?thesis
    using i ∗ by auto
qed

lemma monotone-convergence-increasing:
  fixes f :: nat ⇒ 'n::euclidean-space ⇒ real
  assumes ∀ k. (f k) integrable-on s
    and ∀ k. ∀ x∈s. (f k x) ≤ (f (Suc k) x)
    and ∀ x∈s. ((λk. f k x) ⟶ g x) sequentially
    and bounded {integral s (f k)| k. True}
  shows g integrable-on s ∧ ((λk. integral s (f k)) ⟶ integral s g) sequentially
proof −
  have lem: g integrable-on s ∧ ((λk. integral s (f k)) ⟶ integral s g) sequentially
    if ∀ k. ∀ x∈s. 0 ≤ f k x
    and ∀ k. (f k) integrable-on s
    and ∀ k. ∀ x∈s. f k x ≤ f (Suc k) x
    and ∀ x∈s. ((λk. f k x) ⟶ g x) sequentially
    and bounded {integral s (f k)| k. True}
    for f :: nat ⇒ 'n::euclidean-space ⇒ real and g s
  proof −
    note assms=that[rule-format]
    have ∀ x∈s. ∀ k. (f k x)•1 ≤ (g x)•1
      apply safe
      apply (rule Lim-component-ge)
      apply (rule that(4)[rule-format])
      apply assumption
      apply (rule trivial-limit-sequentially)
      unfolding eventually-sequentially
      apply (rule-tac x=k in exI)
      apply (rule transitive-stepwise-le)
      using that(3)
      apply auto
      done
    note fg=this[rule-format]

    have ∃ i. ((λk. integral s (f k)) ⟶ i) sequentially
      apply (rule bounded-increasing-convergent)
      apply (rule that(5))
      apply rule
      apply (rule integral-le)
      apply (rule that(2)[rule-format])+
      using that(3)
      apply auto
```

done
then guess i **..** **note** i=*this*
have $\bigwedge k.\ \forall x{\in}s.\ \forall n{\geq}k.\ f\ k\ x \leq f\ n\ x$
apply *rule*
apply (*rule transitive-stepwise-le*)
using *that*(*3*)
apply *auto*
done
then have i': $\forall k.\ (\mathit{integral}\ s\ (f\ k)){\cdot}1 \leq i{\cdot}1$
apply −
apply *rule*
apply (*rule Lim-component-ge*)
apply (*rule i*)
apply (*rule trivial-limit-sequentially*)
unfolding *eventually-sequentially*
apply (*rule-tac x*=*k* **in** *exI*)
apply *safe*
apply (*rule integral-component-le*)
apply *simp*
apply (*rule that*(*2*)[*rule-format*])+
apply *auto*
done

note *int* = *assms*(*2*)[*unfolded integrable-alt*[*of - s*],*THEN conjunct1*,*rule-format*]
have *ifif*: $\bigwedge k\ t.\ (\lambda x.\ \mathit{if}\ x \in t\ \mathit{then}\ \mathit{if}\ x \in s\ \mathit{then}\ f\ k\ x\ \mathit{else}\ 0\ \mathit{else}\ 0) =$
$(\lambda x.\ \mathit{if}\ x \in t \cap s\ \mathit{then}\ f\ k\ x\ \mathit{else}\ 0)$
by (*rule ext*) *auto*
have *int'*: $\bigwedge k\ a\ b.\ f\ k\ \mathit{integrable\text{-}on}\ \mathit{cbox}\ a\ b \cap s$
apply (*subst integrable-restrict-univ*[*symmetric*])
apply (*subst ifif*[*symmetric*])
apply (*subst integrable-restrict-univ*)
apply (*rule int*)
done
have $\bigwedge a\ b.\ (\lambda x.\ \mathit{if}\ x \in s\ \mathit{then}\ g\ x\ \mathit{else}\ 0)\ \mathit{integrable\text{-}on}\ \mathit{cbox}\ a\ b \wedge$
$((\lambda k.\ \mathit{integral}\ (\mathit{cbox}\ a\ b)\ (\lambda x.\ \mathit{if}\ x \in s\ \mathit{then}\ f\ k\ x\ \mathit{else}\ 0)) \longrightarrow$
$\mathit{integral}\ (\mathit{cbox}\ a\ b)\ (\lambda x.\ \mathit{if}\ x \in s\ \mathit{then}\ g\ x\ \mathit{else}\ 0))\ \mathit{sequentially}$
proof (*rule monotone-convergence-interval*, *safe*, *goal-cases*)
case *1*
show *?case* **by** (*rule int*)
next
case (*2 - - - x*)
then show *?case*
apply (*cases* $x \in s$)
using *assms*(*3*)
apply *auto*
done
next
case (*3 - - x*)
then show *?case*

```
      apply (cases x ∈ s)
      using assms(4)
      apply auto
      done
  next
    case (4 a b)
    note ∗ = integral-nonneg
    have ⋀k. norm (integral (cbox a b) (λx. if x ∈ s then f k x else 0)) ≤ norm
(integral s (f k))
      unfolding real-norm-def
      apply (subst abs-of-nonneg)
      apply (rule ∗[OF int])
      apply safe
      apply (case-tac x ∈ s)
      apply (drule assms(1))
      prefer 3
      apply (subst abs-of-nonneg)
      apply (rule ∗[OF assms(2) that(1)[THEN spec]])
      apply (subst integral-restrict-univ[symmetric,OF int])
      unfolding ifif
      unfolding integral-restrict-univ[OF int']
      apply (rule integral-subset-le[OF - int' assms(2)])
      using assms(1)
      apply auto
      done
    then show ?case
      using assms(5)
      unfolding bounded-iff
      apply safe
      apply (rule-tac x=aa in exI)
      apply safe
      apply (erule-tac x=integral s (f k) in ballE)
      apply (rule order-trans)
      apply assumption
      apply auto
      done
  qed
  note g = conjunctD2[OF this]

  have (g has-integral i) s
    unfolding has-integral-alt'
    apply safe
    apply (rule g(1))
  proof goal-cases
    case (1 e)
    then have e/4>0
      by auto
    from LIMSEQ-D [OF i this] guess N .. note N=this
    note assms(2)[of N,unfolded has-integral-integral has-integral-alt'[of f N]]
```

```
      from this[THEN conjunct2,rule-format,OF ‹e/4>0›] guess B .. note
B=conjunctD2[OF this]
    show ?case
     apply rule
     apply rule
     apply (rule B)
     apply safe
    proof −
     fix a b :: 'n
     assume ab: ball 0 B ⊆ cbox a b
     from ‹e > 0› have e/2 > 0
      by auto
     from LIMSEQ-D [OF g(2)[of a b] this] guess M .. note M=this
     have **: norm (integral (cbox a b) (λx. if x ∈ s then f N x else 0) − i) <
e/2
      apply (rule norm-triangle-half-l)
      using B(2)[rule-format,OF ab] N[rule-format,of N]
      apply −
      defer
      apply (subst norm-minus-commute)
      apply auto
      done
     have *: ⋀f1 f2 g. |f1 − i| < e / 2 ⟶ |f2 − g| < e / 2 ⟶
      f1 ≤ f2 ⟶ f2 ≤ i ⟶ |g − i| < e
      unfolding real-inner-1-right by arith
     show norm (integral (cbox a b) (λx. if x ∈ s then g x else 0) − i) < e
      unfolding real-norm-def
      apply (rule *[rule-format])
      apply (rule **[unfolded real-norm-def])
      apply (rule M[rule-format,of M + N,unfolded real-norm-def])
      apply (rule le-add1)
      apply (rule integral-le[OF int int])
      defer
    apply (rule order-trans[OF - i'[rule-format,of M + N,unfolded real-inner-1-right]])
     proof (safe, goal-cases)
      case (2 x)
      have ⋀m. x ∈ s ⟹ ∀n≥m. (f m x)•1 ≤ (f n x)•1
       apply (rule transitive-stepwise-le)
       using assms(3)
       apply auto
       done
      then show ?case
       by auto
     next
      case 1
      show ?case
       apply (subst integral-restrict-univ[symmetric,OF int])
       unfolding ifif integral-restrict-univ[OF int']
       apply (rule integral-subset-le[OF - int'])
```

```
          using assms
          apply auto
          done
      qed
    qed
  qed
  then show ?thesis
    apply safe
    defer
    apply (drule integral-unique)
    using i
    apply auto
    done
qed

  have sub: ⋀k. integral s (λx. f k x − f 0 x) = integral s (f k) − integral s (f 0)
    apply (subst integral-diff)
    apply (rule assms(1)[rule-format])+
    apply rule
    done
  have ⋀x m. x ∈ s ⟹ ∀ n≥m. f m x ≤ f n x
    apply (rule transitive-stepwise-le)
    using assms(2)
    apply auto
    done
  note ∗ = this[rule-format]
  have (λx. g x − f 0 x) integrable-on s ∧ ((λk. integral s (λx. f (Suc k) x − f 0
x)) ⟶
    integral s (λx. g x − f 0 x)) sequentially
    apply (rule lem)
    apply safe
  proof goal-cases
    case (1 k x)
    then show ?case
      using ∗[of x 0 Suc k] by auto
  next
    case (2 k)
    then show ?case
      apply (rule integrable-diff)
      using assms(1)
      apply auto
      done
  next
    case (3 k x)
    then show ?case
      using ∗[of x Suc k Suc (Suc k)] by auto
  next
    case (4 x)
    then show ?case
```

```
    apply −
    apply (rule tendsto-diff)
    using LIMSEQ-ignore-initial-segment[OF assms(3)[rule-format],of x 1]
    apply auto
    done
  next
    case 5
    then show ?case
    using assms(4)
    unfolding bounded-iff
    apply safe
    apply (rule-tac x=a + norm (integral s (λx. f 0 x)) in exI)
    apply safe
    apply (erule-tac x=integral s (λx. f (Suc k) x) in ballE)
    unfolding sub
    apply (rule order-trans[OF norm-triangle-ineq4])
    apply auto
    done
  qed
  note conjunctD2[OF this]
  note tendsto-add[OF this(2) tendsto-const[of integral s (f 0)]]
    integrable-add[OF this(1) assms(1)[rule-format,of 0]]
  then show ?thesis
    unfolding sub
    apply −
    apply rule
    defer
    apply (subst(asm) integral-diff)
    using assms(1)
    apply auto
    apply (rule LIMSEQ-imp-Suc)
    apply assumption
    done
qed

lemma has-integral-monotone-convergence-increasing:
  fixes f :: nat ⇒ 'a::euclidean-space ⇒ real
  assumes f: ⋀k. (f k has-integral x k) s
  assumes ⋀k x. x ∈ s ⟹ f k x ≤ f (Suc k) x
  assumes ⋀x. x ∈ s ⟹ (λk. f k x) ⟶ g x
  assumes x ⟶ x'
  shows (g has-integral x') s
proof −
  have x-eq: x = (λi. integral s (f i))
    by (simp add: integral-unique[OF f])
  then have x: {integral s (f k) |k. True} = range x
    by auto

  have g integrable-on s ∧ (λk. integral s (f k)) ⟶ integral s g
```

```
  proof (intro monotone-convergence-increasing allI ballI assms)
    show bounded {integral s (f k) |k. True}
      unfolding x by (rule convergent-imp-bounded) fact
  qed (auto intro: f)
  moreover then have integral s g = x'
    by (intro LIMSEQ-unique[OF - ‹x ⟶ x'›]) (simp add: x-eq)
  ultimately show ?thesis
    by (simp add: has-integral-integral)
qed

lemma monotone-convergence-decreasing:
  fixes f :: nat ⇒ 'n::euclidean-space ⇒ real
  assumes ∀ k. (f k) integrable-on s
    and ∀ k. ∀ x∈s. f (Suc k) x ≤ f k x
    and ∀ x∈s. ((λk. f k x) ⟶ g x) sequentially
    and bounded {integral s (f k)| k. True}
  shows g integrable-on s ∧ ((λk. integral s (f k)) ⟶ integral s g) sequentially
proof −
  note assm = assms[rule-format]
  have ∗: {integral s (λx. − f k x) |k. True} = op ∗R (− 1) ‘ {integral s (f k)| k.
True}
    apply safe
    unfolding image-iff
    apply (rule-tac x=integral s (f k) in bexI)
    prefer 3
    apply (rule-tac x=k in exI)
    apply auto
    done
  have (λx. − g x) integrable-on s ∧
    ((λk. integral s (λx. − f k x)) ⟶ integral s (λx. − g x)) sequentially
    apply (rule monotone-convergence-increasing)
    apply safe
    apply (rule integrable-neg)
    apply (rule assm)
    defer
    apply (rule tendsto-minus)
    apply (rule assm)
    apply assumption
    unfolding ∗
    apply (rule bounded-scaling)
    using assm
    apply auto
    done
  note ∗ = conjunctD2[OF this]
  show ?thesis
    using integrable-neg[OF ∗(1)] tendsto-minus[OF ∗(2)]
    by auto
qed
```

41.65 Absolute integrability (this is the same as Lebesgue integrability)

definition *absolutely-integrable-on* (**infixr** *absolutely'-integrable'-on 46*)
where *f absolutely-integrable-on s* $\longleftrightarrow$ *f integrable-on s* $\wedge$ $(\lambda x.\ (norm(f\ x)))$ *integrable-on s*

lemma *absolutely-integrable-onI*[*intro?*]:
f integrable-on s $\Longrightarrow$
$(\lambda x.\ (norm(f\ x)))$ *integrable-on s* $\Longrightarrow$ *f absolutely-integrable-on s*
unfolding *absolutely-integrable-on-def*
by *auto*

lemma *absolutely-integrable-onD*[*dest*]:
assumes *f absolutely-integrable-on s*
shows *f integrable-on s*
and $(\lambda x.\ norm\ (f\ x))$ *integrable-on s*
using *assms*
unfolding *absolutely-integrable-on-def*
by *auto*

lemma *integral-norm-bound-integral*:
fixes *f* :: *'n*::*euclidean-space* $\Rightarrow$ *'a*::*banach*
assumes *f integrable-on s*
and *g integrable-on s*
and $\forall\, x{\in}s.\ norm\ (f\ x) \leq g\ x$
shows *norm* (*integral s f*) $\leq$ *integral s g*
proof −
have $*$: $\bigwedge x\ y.\ (\forall\, e{::}real.\ 0 < e \longrightarrow x < y + e) \Longrightarrow x \leq y$
apply (*rule ccontr*)
apply (*erule-tac x*=*x* − *y* **in** *allE*)
apply *auto*
done
have *norm*: *norm ig* < *dia* + *e*
if *norm sg* $\leq$ *dsa*
and |*dsa* − *dia*| < *e* / *2*
and *norm* (*sg* − *ig*) < *e* / *2*
for *e dsa dia* **and** *sg ig* :: *'a*
apply (*rule le-less-trans*[*OF norm-triangle-sub*[*of ig sg*]])
apply (*subst real-sum-of-halves*[*of e*,*symmetric*])
unfolding *add.assoc*[*symmetric*]
apply (*rule add-le-less-mono*)
defer
apply (*subst norm-minus-commute*)
apply (*rule that*(*3*))
apply (*rule order-trans*[*OF that*(*1*)])
using *that*(*2*)
apply *arith*

```
    done
  have lem: norm (integral(cbox a b) f) ≤ integral (cbox a b) g
    if f integrable-on cbox a b
    and g integrable-on cbox a b
    and ∀x∈cbox a b. norm (f x) ≤ g x
    for f :: 'n ⇒ 'a and g a b
  proof (rule *[rule-format])
    fix e :: real
    assume e > 0
    then have *: e/2 > 0
      by auto
    from integrable-integral[OF that(1),unfolded has-integral[of f],rule-format,OF
*]
    guess d1 .. note d1 = conjunctD2[OF this,rule-format]
    from integrable-integral[OF that(2),unfolded has-integral[of g],rule-format,OF
*]
    guess d2 .. note d2 = conjunctD2[OF this,rule-format]
    note gauge-inter[OF d1(1) d2(1)]
    from fine-division-exists[OF this, of a b] guess p . note p=this
    show norm (integral (cbox a b) f) < integral (cbox a b) g + e
      apply (rule norm)
      defer
      apply (rule d2(2)[OF conjI[OF p(1)],unfolded real-norm-def])
      defer
      apply (rule d1(2)[OF conjI[OF p(1)]])
      defer
      apply (rule setsum-norm-le)
    proof safe
      fix x k
      assume (x, k) ∈ p
      note as = tagged-division-ofD(2–4)[OF p(1) this]
      from this(3) guess u v by (elim exE) note uv=this
      show norm (content k *R f x) ≤ content k *R g x
        unfolding uv norm-scaleR
        unfolding abs-of-nonneg[OF content-pos-le] real-scaleR-def
        apply (rule mult-left-mono)
        using that(3) as
        apply auto
        done
    qed (insert p[unfolded fine-inter], auto)
  qed

  { presume ⋀e. 0 < e ⟹ norm (integral s f) < integral s g + e
    then show ?thesis by (rule *[rule-format]) auto }
  fix e :: real
  assume e > 0
  then have e: e/2 > 0
    by auto
  note assms(1)[unfolded integrable-alt[of f]] note f=this[THEN conjunct1,rule-format]
```

note *assms(2)*[*unfolded integrable-alt*[*of g*]] **note** *g=this*[*THEN conjunct1*,*rule-format*]
from *integrable-integral*[*OF assms(1)*,*unfolded has-integral'*[*of f*],*rule-format*,*OF e*]
guess *B1* **..** **note** *B1=conjunctD2*[*OF this*[*rule-format*],*rule-format*]
from *integrable-integral*[*OF assms(2)*,*unfolded has-integral'*[*of g*],*rule-format*,*OF e*]
guess *B2* **..** **note** *B2=conjunctD2*[*OF this*[*rule-format*],*rule-format*]
from *bounded-subset-cbox*[*OF bounded-ball*, *of 0::'n max B1 B2*]
guess *a b* **by** (*elim exE*) **note** *ab=this*[*unfolded ball-max-Un*]

have *ball 0 B1* ⊆ *cbox a b*
using *ab* **by** *auto*
from *B1(2)*[*OF this*] **guess** *z* **..** **note** *z=conjunctD2*[*OF this*]
have *ball 0 B2* ⊆ *cbox a b*
using *ab* **by** *auto*
from *B2(2)*[*OF this*] **guess** *w* **..** **note** *w=conjunctD2*[*OF this*]

show *norm* (*integral s f*) < *integral s g* + *e*
apply (*rule norm*)
apply (*rule lem*[*OF f g*, *of a b*])
unfolding *integral-unique*[*OF z(1)*] *integral-unique*[*OF w(1)*]
defer
apply (*rule w(2)*[*unfolded real-norm-def*])
apply (*rule z(2)*)
apply *safe*
apply (*case-tac x* ∈ *s*)
unfolding *if-P*
apply (*rule assms(3)*[*rule-format*])
apply *auto*
done
qed

lemma *integral-norm-bound-integral-component*:
fixes *f* :: *'n*::*euclidean-space* ⇒ *'a*::*banach*
fixes *g* :: *'n* ⇒ *'b*::*euclidean-space*
assumes *f integrable-on s*
and *g integrable-on s*
and ∀ *x*∈*s*. *norm*(*f x*) ≤ (*g x*)·*k*
shows *norm* (*integral s f*) ≤ (*integral s g*)·*k*
proof −
have *norm* (*integral s f*) ≤ *integral s* ((λ*x*. *x* · *k*) ∘ *g*)
apply (*rule integral-norm-bound-integral*[*OF assms(1)*])
apply (*rule integrable-linear*[*OF assms(2)*])
apply *rule*
unfolding *o-def*
apply (*rule assms*)
done
then show *?thesis*
unfolding *o-def integral-component-eq*[*OF assms(2)*] **.**

qed

lemma *has-integral-norm-bound-integral-component*:
fixes $f :: 'n::euclidean\text{-}space \Rightarrow 'a::banach$
fixes $g :: 'n \Rightarrow 'b::euclidean\text{-}space$
assumes $(f\ has\text{-}integral\ i)\ s$
and $(g\ has\text{-}integral\ j)\ s$
and $\forall x{\in}s.\ norm\ (f\ x) \leq (g\ x){\cdot}k$
shows $norm\ i \leq j{\cdot}k$
using *integral-norm-bound-integral-component*[*of f s g k*]
unfolding *integral-unique*[*OF assms*(*1*)] *integral-unique*[*OF assms*(*2*)]
using *assms*
by *auto*

lemma *absolutely-integrable-le*:
fixes $f :: 'n::euclidean\text{-}space \Rightarrow 'a::banach$
assumes *f absolutely-integrable-on s*
shows $norm\ (integral\ s\ f) \leq integral\ s\ (\lambda x.\ norm\ (f\ x))$
apply (*rule integral-norm-bound-integral*)
using *assms*
apply *auto*
done

lemma *absolutely-integrable-0*[*intro*]:
$(\lambda x.\ 0)$ *absolutely-integrable-on s*
unfolding *absolutely-integrable-on-def*
by *auto*

lemma *absolutely-integrable-cmul*[*intro*]:
f absolutely-integrable-on s $\Longrightarrow$
$(\lambda x.\ c *_R f\ x)$ *absolutely-integrable-on s*
unfolding *absolutely-integrable-on-def*
using *integrable-cmul*[*of f s c*]
using *integrable-cmul*[*of* $\lambda x.\ norm\ (f\ x)$ *s* $|c|$]
by *auto*

lemma *absolutely-integrable-neg*[*intro*]:
f absolutely-integrable-on s $\Longrightarrow$
$(\lambda x.\ -f(x))$ *absolutely-integrable-on s*
apply (*drule absolutely-integrable-cmul*[**where** *c*=-1])
apply *auto*
done

lemma *absolutely-integrable-norm*[*intro*]:
f absolutely-integrable-on s $\Longrightarrow$
$(\lambda x.\ norm\ (f\ x))$ *absolutely-integrable-on s*
unfolding *absolutely-integrable-on-def*
by *auto*

lemma *absolutely-integrable-abs*[*intro*]:
  f absolutely-integrable-on s $\Longrightarrow$
    $(\lambda x.\ |f\ x::real|)$ *absolutely-integrable-on s*
  apply (*drule absolutely-integrable-norm*)
  unfolding *real-norm-def*
  apply *assumption*
  done

lemma *absolutely-integrable-on-subinterval*:
  fixes *f* :: *'n::euclidean-space* $\Rightarrow$ *'a::banach*
  shows *f absolutely-integrable-on s* $\Longrightarrow$
    cbox a b $\subseteq$ *s* $\Longrightarrow$ *f absolutely-integrable-on cbox a b*
  unfolding *absolutely-integrable-on-def*
  by (*metis integrable-on-subcbox*)

lemma *absolutely-integrable-bounded-variation*:
  fixes *f* :: *'n::euclidean-space* $\Rightarrow$ *'a::banach*
  assumes *f absolutely-integrable-on UNIV*
  obtains *B* **where** $\forall d.\ d\ division\text{-}of\ (\bigcup d) \longrightarrow setsum\ (\lambda k.\ norm(integral\ k\ f))\ d \leq B$
  apply (*rule that*[*of integral UNIV* $(\lambda x.\ norm\ (f\ x))$])
  apply *safe*
proof *goal-cases*
  case *prems*: (*1 d*)
  note *d* = *division-ofD*[*OF prems*(*2*)]
  have $(\sum k{\in}d.\ norm\ (integral\ k\ f)) \leq (\sum i{\in}d.\ integral\ i\ (\lambda x.\ norm\ (f\ x)))$
    apply (*rule setsum-mono*,*rule absolutely-integrable-le*)
    apply (*drule d*(*4*))
    apply *safe*
    apply (*rule absolutely-integrable-on-subinterval*[*OF assms*])
    apply *auto*
    done
  also have ... $\leq$ *integral* $(\bigcup d)$ $(\lambda x.\ norm\ (f\ x))$
    apply (*subst integral-combine-division-topdown*[*OF - prems*(*2*)])
    using *integrable-on-subdivision*[*OF prems*(*2*)]
    using *assms*
    apply *auto*
    done
  also have ... $\leq$ *integral UNIV* $(\lambda x.\ norm\ (f\ x))$
    apply (*rule integral-subset-le*)
    using *integrable-on-subdivision*[*OF prems*(*2*)]
    using *assms*
    apply *auto*
    done
  finally show *?case* **.**
qed

lemma *helplemma*:
  assumes *setsum* $(\lambda x.\ norm\ (f\ x - g\ x))$ *s* $<$ *e*

```
    and finite s
  shows |setsum (λx. norm(f x)) s − setsum (λx. norm(g x)) s| < e
  unfolding setsum-subtractf[symmetric]
  apply (rule le-less-trans[OF setsum-abs])
  apply (rule le-less-trans[OF - assms(1)])
  apply (rule setsum-mono)
  apply (rule norm-triangle-ineq3)
  done

lemma bounded-variation-absolutely-integrable-interval:
  fixes f :: 'n::euclidean-space ⇒ 'm::euclidean-space
  assumes f: f integrable-on cbox a b
    and ∗: ∀ d. d division-of (cbox a b) ⟶ setsum (λk. norm(integral k f)) d ≤ B
  shows f absolutely-integrable-on cbox a b
proof −
  let ?f = λd. ∑ k∈d. norm (integral k f) and ?D = {d. d division-of (cbox a
b)}
  have D-1: ?D ≠ {}
    by (rule elementary-interval[of a b]) auto
  have D-2: bdd-above (?f`?D)
    by (metis ∗ mem-Collect-eq bdd-aboveI2)
  note D = D-1 D-2
  let ?S = SUP x:?D. ?f x
  show ?thesis
    apply (rule absolutely-integrable-onI [OF f has-integral-integrable])
    apply (subst has-integral[of - ?S])
    apply safe
  proof goal-cases
    case e: (1 e)
    then have ?S − e / 2 < ?S by simp
   then obtain d where d: d division-of (cbox a b) ?S − e / 2 < (∑ k∈d. norm
(integral k f))
      unfolding less-cSUP-iff[OF D] by auto
    note d′ = division-ofD[OF this(1)]

    have ∀ x. ∃ e>0. ∀ i∈d. x ∉ i ⟶ ball x e ∩ i = {}
    proof
      fix x
      have ∃ da>0. ∀ xa∈⋃{i ∈ d. x ∉ i}. da ≤ dist x xa
        apply (rule separate-point-closed)
        apply (rule closed-Union)
        apply (rule finite-subset[OF - d′(1)])
        using d′(4)
        apply auto
        done
      then show ∃ e>0. ∀ i∈d. x ∉ i ⟶ ball x e ∩ i = {}
        by force
    qed
   from choice[OF this] guess k .. note k=conjunctD2[OF this[rule-format],rule-format]
```

```
have e/2 > 0
  using e by auto
from henstock-lemma[OF assms(1) this] guess g . note g=this[rule-format]
let ?g = λx. g x ∩ ball x (k x)
show ?case
  apply (rule-tac x=?g in exI)
  apply safe
proof −
  show gauge ?g
    using g(1) k(1)
    unfolding gauge-def
    by auto
  fix p
  assume p tagged-division-of (cbox a b) and ?g fine p
  note p = this(1) conjunctD2[OF this(2)[unfolded fine-inter]]
  note p' = tagged-division-ofD[OF p(1)]
  def p' ≡ {(x,k) | x k. ∃i l. x ∈ i ∧ i ∈ d ∧ (x,l) ∈ p ∧ k = i ∩ l}
  have gp': g fine p'
    using p(2)
    unfolding p'-def fine-def
    by auto
  have p'': p' tagged-division-of (cbox a b)
    apply (rule tagged-division-ofI)
  proof −
    show finite p'
      apply (rule finite-subset[of - (λ(k,(x,l)). (x,k ∩ l)) `
        {(k,xl) | k xl. k ∈ d ∧ xl ∈ p}])
      unfolding p'-def
      defer
      apply (rule finite-imageI,rule finite-product-dependent[OF d'(1) p'(1)])
      apply safe
      unfolding image-iff
      apply (rule-tac x=(i,x,l) in bexI)
      apply auto
      done
    fix x k
    assume (x, k) ∈ p'
    then have ∃i l. x ∈ i ∧ i ∈ d ∧ (x, l) ∈ p ∧ k = i ∩ l
      unfolding p'-def by auto
    then guess i l by (elim exE) note il=conjunctD4[OF this]
    show x ∈ k and k ⊆ cbox a b
      using p'(2−3)[OF il(3)] il by auto
    show ∃a b. k = cbox a b
      unfolding il using p'(4)[OF il(3)] d'(4)[OF il(2)]
      apply safe
      unfolding inter-interval
      apply auto
      done
```

next
fix *x1 k1*
assume $(x1,\ k1) \in p'$
then have $\exists\, i\ l.\ x1 \in i \wedge i \in d \wedge (x1,\ l) \in p \wedge k1 = i \cap l$
unfolding *p'-def* **by** *auto*
then guess *i1 l1* **by** (*elim exE*) **note** *il1=conjunctD4*[*OF this*]
fix *x2 k2*
assume $(x2,k2)\in p'$
then have $\exists\, i\ l.\ x2 \in i \wedge i \in d \wedge (x2,\ l) \in p \wedge k2 = i \cap l$
unfolding *p'-def* **by** *auto*
then guess *i2 l2* **by** (*elim exE*) **note** *il2=conjunctD4*[*OF this*]
assume $(x1,\ k1) \neq (x2,\ k2)$
then have *interior i1* $\cap$ *interior i2* $= \{\} \vee$ *interior l1* $\cap$ *interior l2* $= \{\}$
using *d'*(*5*)[*OF il1*(*2*) *il2*(*2*)] *p'*(*5*)[*OF il1*(*3*) *il2*(*3*)]
unfolding *il1 il2*
by *auto*
then show *interior k1* $\cap$ *interior k2* $= \{\}$
unfolding *il1 il2* **by** *auto*
next
have $*$: $\forall\,(x,\ X) \in p'.\ X \subseteq$ *cbox a b*
unfolding *p'-def* **using** *d'* **by** *auto*
show $\bigcup\{k.\ \exists\, x.\ (x,\ k) \in p'\} =$ *cbox a b*
apply *rule*
apply (*rule Union-least*)
unfolding *mem-Collect-eq*
apply (*erule exE*)
apply (*drule* $*$[*rule-format*])
apply *safe*
proof −
fix *y*
assume *y*: $y \in$ *cbox a b*
then have $\exists\, x\ l.\ (x,\ l) \in p \wedge y \in l$
unfolding *p'*(*6*)[*symmetric*] **by** *auto*
then guess *x l* **by** (*elim exE*) **note** *xl=conjunctD2*[*OF this*]
then have $\exists\, k.\ k \in d \wedge y \in k$
using *y* **unfolding** *d'*(*6*)[*symmetric*] **by** *auto*
then guess *i* **..** **note** $i =$ *conjunctD2*[*OF this*]
have $x \in i$
using *fineD*[*OF p*(*3*) *xl*(*1*)]
using *k*(*2*)[*OF i*(*1*), *of x*]
using *i*(*2*) *xl*(*2*)
by *auto*
then show $y \in \bigcup\{k.\ \exists\, x.\ (x,\ k) \in p'\}$
unfolding *p'-def Union-iff*
apply (*rule-tac x=i* $\cap$ *l* **in** *bexI*)
using *i xl*
apply *auto*
done
qed

qed

then have $(\sum (x,\ k){\in}p'.\ norm\ (content\ k\ *_R\ f\ x\ -\ integral\ k\ f)) < e\ /\ 2$
apply −
apply (*rule g(2)*[*rule-format*])
unfolding *tagged-division-of-def*
apply *safe*
apply (*rule gp'*)
done
then have $**$: $|(\sum (x,k){\in}p'.\ norm\ (content\ k\ *_R\ f\ x)) - (\sum (x,k){\in}p'.\ norm\ (integral\ k\ f))| < e\ /\ 2$
unfolding *split-def*
using *p''*
by (*force intro*!: *helplemma*)

have *p'alt*: $p' = \{(x,(i \cap l)) \mid x\ i\ l.\ (x,l) \in p \wedge i \in d \wedge i \cap l \neq \{\}\}$
proof (*safe*, *goal-cases*)
case *prems*: (*2* - - *x i l*)
have $x \in i$
using *fineD*[*OF p(3) prems(1)*] *k(2)*[*OF prems(2), of x*] *prems(4−)*
by *auto*
then have $(x,\ i \cap l) \in p'$
unfolding *p'-def*
using *prems*
apply *safe*
apply (*rule-tac x=x* **in** *exI*)
apply (*rule-tac x=i ∩ l* **in** *exI*)
apply *safe*
using *prems*
apply *auto*
done
then show *?case*
using *prems(3)* **by** *auto*
next
fix *x k*
assume $(x,\ k) \in p'$
then have $\exists\, i\ l.\ x \in i \wedge i \in d \wedge (x,\ l) \in p \wedge k = i \cap l$
unfolding *p'-def* **by** *auto*
then guess *i l* **by** (*elim exE*) **note** *il=conjunctD4*[*OF this*]
then show $\exists\, y\ i\ l.\ (x,\ k) = (y,\ i \cap l) \wedge (y,\ l) \in p \wedge i \in d \wedge i \cap l \neq \{\}$
apply (*rule-tac x=x* **in** *exI*)
apply (*rule-tac x=i* **in** *exI*)
apply (*rule-tac x=l* **in** *exI*)
using *p'(2)*[*OF il(3)*]
apply *auto*
done
qed
have *sum-p'*: $(\sum (x,\ k){\in}p'.\ norm\ (integral\ k\ f)) = (\sum k{\in}snd\ `\ p'.\ norm\ (integral\ k\ f))$

apply (*subst setsum-over-tagged-division-lemma*[*OF p''*,*of* λ*k*. *norm* (*integral k f*)])
unfolding *norm-eq-zero*
apply (*rule integral-null*)
apply *assumption*
apply *rule*
done
note *snd-p* = *division-ofD*[*OF division-of-tagged-division*[*OF p*(*1*)]]

have ∗: ⋀*sni sni' sf sf'*. |*sf'* − *sni'*| < *e* / *2* ⟶ *?S* − *e* / *2* < *sni* ∧ *sni'* ≤ *?S* ∧
sni ≤ *sni'* ∧ *sf'* = *sf* ⟶ |*sf* − *?S*| < *e*
by *arith*
show *norm* ((∑ (*x*, *k*)∈*p*. *content k* ∗$_R$ *norm* (*f x*)) − *?S*) < *e*
unfolding *real-norm-def*
apply (*rule* ∗[*rule-format*,*OF* ∗∗])
apply *safe*
apply(*rule d*(*2*))
proof *goal-cases*
case *1*
show *?case*
by (*auto simp*: *sum-p' division-of-tagged-division*[*OF p'*] *D intro*!: *cSUP-upper*)
next
case *2*
have ∗: {*k* ∩ *l* | *k l*. *k* ∈ *d* ∧ *l* ∈ *snd* ‘ *p*} =
(λ(*k*,*l*). *k* ∩ *l*) ‘ {(*k*,*l*)|*k l*. *k* ∈ *d* ∧ *l* ∈ *snd* ‘ *p*}
by *auto*
have (∑ *k*∈*d*. *norm* (*integral k f*)) ≤ (∑ *i*∈*d*. ∑ *l*∈*snd* ‘ *p*. *norm* (*integral* (*i* ∩ *l*) *f*))
proof (*rule setsum-mono*, *goal-cases*)
case *k*: (*1 k*)
from *d'*(*4*)[*OF this*] **guess** *u v* **by** (*elim exE*) **note** *uv*=*this*
def *d'* ≡ {*cbox u v* ∩ *l* |*l*. *l* ∈ *snd* ‘ *p* ∧ *cbox u v* ∩ *l* ≠ {}}
note *uvab* = *d'*(*2*)[*OF k*[*unfolded uv*]]
have *d' division-of cbox u v*
apply (*subst d'-def*)
apply (*rule division-inter-1*)
apply (*rule division-of-tagged-division*[*OF p*(*1*)])
apply (*rule uvab*)
done
then have *norm* (*integral k f*) ≤ *setsum* (λ*k*. *norm* (*integral k f*)) *d'*
unfolding *uv*
apply (*subst integral-combine-division-topdown*[*of - - d'*])
apply (*rule integrable-on-subcbox*[*OF assms*(*1*) *uvab*])
apply *assumption*
apply (*rule setsum-norm-le*)
apply *auto*
done

```
      also have ... = (∑ k∈{k ∩ l |l. l ∈ snd ` p}. norm (integral k f))
        apply (rule setsum.mono-neutral-left)
        apply (subst simple-image)
        apply (rule finite-imageI)+
        apply fact
        unfolding d'-def uv
        apply blast
      proof (rule, goal-cases)
        case prems: (1 i)
        then have i ∈ {cbox u v ∩ l |l. l ∈ snd ` p}
          by auto
        from this[unfolded mem-Collect-eq] guess l .. note l=this
        then have cbox u v ∩ l = {}
          using prems by auto
        then show ?case
          using l by auto
      qed
      also have ... = (∑ l∈snd ` p. norm (integral (k ∩ l) f))
        unfolding simple-image
        apply (rule setsum.reindex-nontrivial [unfolded o-def])
        apply (rule finite-imageI)
        apply (rule p')
      proof goal-cases
        case prems: (1 l y)
        have interior (k ∩ l) ⊆ interior (l ∩ y)
          apply (subst(2) interior-Int)
          apply (rule Int-greatest)
          defer
          apply (subst prems(4))
          apply auto
          done
        then have *: interior (k ∩ l) = {}
          using snd-p(5)[OF prems(1-3)] by auto
        from d'(4)[OF k] snd-p(4)[OF prems(1)] guess u1 v1 u2 v2 by (elim
exE) note uv=this
        show ?case
          using *
          unfolding uv inter-interval content-eq-0-interior[symmetric]
          by auto
      qed
      finally show ?case .
    qed
    also have ... = (∑ (i,l)∈{(i, l) |i l. i ∈ d ∧ l ∈ snd ` p}. norm (integral
(i∩l) f))
      apply (subst sum-sum-product[symmetric])
      apply fact
      using p'(1)
      apply auto
      done
```

also have ... = $(\sum x{\in}\{(i,\ l)\ |i\ l.\ i \in d \wedge l \in snd\ `\ p\}.\ norm\ (integral\ (case\text{-}prod\ op \cap x)\ f))$
unfolding *split-def* **..**
also have ... = $(\sum k{\in}\{i \cap l\ |i\ l.\ i \in d \wedge l \in snd\ `\ p\}.\ norm\ (integral\ k\ f))$
unfolding $*$
apply (*rule setsum.reindex-nontrivial* [*symmetric*, *unfolded o-def*])
apply (*rule finite-product-dependent*)
apply *fact*
apply (*rule finite-imageI*)
apply (*rule* p')
unfolding *split-paired-all mem-Collect-eq split-conv o-def*
proof −
note $*$ = *division-ofD*(*4*,*5*)[*OF division-of-tagged-division*,*OF* $p(1)$]
fix *l1 l2 k1 k2*
assume *as*:
$(l1,\ k1) \neq (l2,\ k2)$
$l1 \cap k1 = l2 \cap k2$
$\exists i\ l.\ (l1,\ k1) = (i,\ l) \wedge i \in d \wedge l \in snd\ `\ p$
$\exists i\ l.\ (l2,\ k2) = (i,\ l) \wedge i \in d \wedge l \in snd\ `\ p$
then have $l1 \in d$ **and** $k1 \in snd\ `\ p$
by *auto* **from** $d'(4)$[*OF this*(*1*)] $*(1)$[*OF this*(*2*)]
guess *u1 v1 u2 v2* **by** (*elim exE*) **note** *uv*=*this*
have $l1 \neq l2 \vee k1 \neq k2$
using *as* **by** *auto*
then have *interior k1* $\cap$ *interior k2* = $\{\} \vee$ *interior l1* $\cap$ *interior l2* = $\{\}$
apply −
apply (*erule disjE*)
apply (*rule disjI2*)
apply (*rule* $d'(5)$)
prefer *4*
apply (*rule disjI1*)
apply (*rule* $*$)
using *as*
apply *auto*
done
moreover have *interior* $(l1 \cap k1)$ = *interior* $(l2 \cap k2)$
using *as*(*2*) **by** *auto*
ultimately have *interior*$(l1 \cap k1) = \{\}$
by *auto*
then show *norm* (*integral* $(l1 \cap k1)\ f) = 0$
unfolding *uv inter-interval*
unfolding *content-eq-0-interior*[*symmetric*]
by *auto*
qed
also have ... = $(\sum (x,\ k){\in}p'.\ norm\ (integral\ k\ f))$
unfolding *sum-p'*
apply (*rule setsum.mono-neutral-right*)
apply (*subst* $*$)
apply (*rule finite-imageI*[*OF finite-product-dependent*])

```
      apply (rule finite-imageI[OF p'(1)])
      apply safe
    proof goal-cases
      case (2 i ia l a b)
      then have ia ∩ b = {}
        unfolding p'alt image-iff Bex-def not-ex
        apply (erule-tac x=(a, ia ∩ b) in allE)
        apply auto
        done
      then show ?case
        by auto
    next
      case (1 x a b)
      then show ?case
        unfolding p'-def
        apply safe
        apply (rule-tac x=i in exI)
        apply (rule-tac x=l in exI)
        unfolding snd-conv image-iff
        apply safe
        apply (rule-tac x=(a,l) in bexI)
        apply auto
        done
    qed
    finally show ?case .
  next
    case 3
    let ?S = {(x, i ∩ l) |x i l. (x, l) ∈ p ∧ i ∈ d}
    have Sigma-alt: ⋀s t. s × t = {(i, j) |i j. i ∈ s ∧ j ∈ t}
      by auto
    have *: ?S = (λ(xl,i). (fst xl, snd xl ∩ i)) ' (p × d)
      apply safe
      unfolding image-iff
      apply (rule-tac x=((x,l),i) in bexI)
      apply auto
      done
    note pdfin = finite-cartesian-product[OF p'(1) d'(1)]
    have (∑(x, k)∈p'. norm (content k *R f x)) = (∑(x, k)∈?S. |content k|
* norm (f x))
      unfolding norm-scaleR
      apply (rule setsum.mono-neutral-left)
      apply (subst *)
      apply (rule finite-imageI)
      apply fact
      unfolding p'alt
      apply blast
      apply safe
      apply (rule-tac x=x in exI)
```

```
      apply (rule-tac x=i in exI)
      apply (rule-tac x=l in exI)
      apply auto
      done
    also have ... = (∑((x,l),i)∈p × d. |content (l ∩ i)| * norm (f x))
      unfolding *
      apply (subst setsum.reindex-nontrivial)
      apply fact
      unfolding split-paired-all
      unfolding o-def split-def snd-conv fst-conv mem-Sigma-iff prod.inject
      apply (elim conjE)
    proof −
      fix x1 l1 k1 x2 l2 k2
      assume as: (x1, l1) ∈ p (x2, l2) ∈ p k1 ∈ d k2 ∈ d
        x1 = x2 l1 ∩ k1 = l2 ∩ k2 ¬ ((x1 = x2 ∧ l1 = l2) ∧ k1 = k2)
      from d'(4)[OF as(3)] p'(4)[OF as(1)] guess u1 v1 u2 v2 by (elim exE)
note uv=this
      from as have l1 ≠ l2 ∨ k1 ≠ k2
        by auto
     then have interior k1 ∩ interior k2 = {} ∨ interior l1 ∩ interior l2 = {}
        apply −
        apply (erule disjE)
        apply (rule disjI2)
        defer
        apply (rule disjI1)
        apply (rule d'(5)[OF as(3−4)])
        apply assumption
        apply (rule p'(5)[OF as(1−2)])
        apply auto
        done
      moreover have interior (l1 ∩ k1) = interior (l2 ∩ k2)
        unfolding as ..
      ultimately have interior (l1 ∩ k1) = {}
        by auto
      then show |content (l1 ∩ k1)| * norm (f x1) = 0
        unfolding uv inter-interval
        unfolding content-eq-0-interior[symmetric]
        by auto
    qed safe
    also have ... = (∑(x, k)∈p. content k *R norm (f x))
      unfolding Sigma-alt
      apply (subst sum-sum-product[symmetric])
      apply (rule p')
      apply rule
      apply (rule d')
      apply (rule setsum.cong)
      apply (rule refl)
      unfolding split-paired-all split-conv
    proof −
```

fix *x l*
assume *as*: $(x, l) \in p$
note $xl = p'(2-4)[OF\ this]$
from *this*(*3*) **guess** *u v* **by** (*elim exE*) **note** *uv=this*
have $(\sum i{\in}d.\ |content\ (l \cap i)|) = (\sum k{\in}d.\ content\ (k \cap cbox\ u\ v))$
apply (*rule setsum.cong*)
apply (*rule refl*)
apply (*drule d'*(*4*))
apply *safe*
apply (*subst Int-commute*)
unfolding *inter-interval uv*
apply (*subst abs-of-nonneg*)
apply *auto*
done
also have $\ldots = setsum\ content\ \{k \cap cbox\ u\ v|\ k.\ k \in d\}$
unfolding *simple-image*
apply (*rule setsum.reindex-nontrivial* [*unfolded o-def*, *symmetric*])
apply (*rule d'*)
proof *goal-cases*
case *prems*: (*1 k y*)
from $d'(4)[OF\ this(1)]$ $d'(4)[OF\ this(2)]$
guess *u1 v1 u2 v2* **by** (*elim exE*) **note** *uv=this*
have $\{\} = interior\ ((k \cap y) \cap cbox\ u\ v)$
apply (*subst interior-Int*)
using $d'(5)[OF\ prems(1-3)]$
apply *auto*
done
also have $\ldots = interior\ (y \cap (k \cap cbox\ u\ v))$
by *auto*
also have $\ldots = interior\ (k \cap cbox\ u\ v)$
unfolding *prems*(*4*) **by** *auto*
finally show *?case*
unfolding *uv inter-interval content-eq-0-interior* **..**
qed
also have $\ldots = setsum\ content\ \{cbox\ u\ v \cap k\ |k.\ k \in d \wedge cbox\ u\ v \cap k \neq \{\}\}$
apply (*rule setsum.mono-neutral-right*)
unfolding *simple-image*
apply (*rule finite-imageI*)
apply (*rule d'*)
apply *blast*
apply *safe*
apply (*rule-tac x=k* **in** *exI*)
proof *goal-cases*
case *prems*: (*1 i k*)
from $d'(4)[OF\ this(1)]$ **guess** *a b* **by** (*elim exE*) **note** *ab=this*
have $interior\ (k \cap cbox\ u\ v) \neq \{\}$
using *prems*(*2*)
unfolding *ab inter-interval content-eq-0-interior*

```
            by auto
          then show ?case
            using prems(1)
            using interior-subset[of k ∩ cbox u v]
            by auto
        qed
       finally show (∑ i∈d. |content (l ∩ i)| * norm (f x)) = content l *R norm
(f x)
          unfolding setsum-left-distrib[symmetric] real-scaleR-def
            apply (subst(asm) additive-content-division[OF division-inter-1[OF
d(1)]])
          using xl(2)[unfolded uv]
          unfolding uv
          apply auto
          done
      qed
      finally show ?case .
    qed
  qed
 qed
qed

lemma bounded-variation-absolutely-integrable:
  fixes f :: 'n::euclidean-space ⇒ 'm::euclidean-space
  assumes f integrable-on UNIV
    and ∀ d. d division-of (⋃ d) ⟶ setsum (λk. norm (integral k f)) d ≤ B
  shows f absolutely-integrable-on UNIV
proof (rule absolutely-integrable-onI, fact, rule)
  let ?f = λd. ∑ k∈d. norm (integral k f) and ?D = {d. d division-of (⋃ d)}
  have D-1: ?D ≠ {}
    by (rule elementary-interval) auto
  have D-2: bdd-above (?f`?D)
    by (intro bdd-aboveI2[where M=B] assms(2)[rule-format]) simp
  note D = D-1 D-2
  let ?S = SUP d:?D. ?f d
  have f-int: ⋀a b. f absolutely-integrable-on cbox a b
    apply (rule bounded-variation-absolutely-integrable-interval[where B=B])
    apply (rule integrable-on-subcbox[OF assms(1)])
    defer
    apply safe
    apply (rule assms(2)[rule-format])
    apply auto
    done
  show ((λx. norm (f x)) has-integral ?S) UNIV
    apply (subst has-integral-alt')
    apply safe
  proof goal-cases
    case (1 a b)
    show ?case
```

```
    using f-int[of a b] by auto
  next
    case prems: (2 e)
    have ∃y∈setsum (λk. norm (integral k f)) ` {d. d division-of ⋃d}. ¬ y ≤ ?S
− e
    proof (rule ccontr)
      assume ¬ ?thesis
      then have ?S ≤ ?S − e
        by (intro cSUP-least[OF D(1)]) auto
      then show False
        using prems by auto
    qed
     then obtain K where ∗: ∃x∈{d. d division-of ⋃d}. K = (∑k∈x. norm
(integral k f))
      SUPREMUM {d. d division-of ⋃d} (setsum (λk. norm (integral k f))) − e
< K
      by (auto simp add: image-iff not-le)
    from this(1) obtain d where d division-of ⋃d
      and K = (∑k∈d. norm (integral k f))
      by auto
    note d = this(1) ∗(2)[unfolded this(2)]
    note d'=division-ofD[OF this(1)]
    have bounded (⋃d)
      by (rule elementary-bounded,fact)
    from this[unfolded bounded-pos] obtain K where
      K: 0 < K ∀x∈⋃d. norm x ≤ K by auto
    show ?case
      apply (rule-tac x=K + 1 in exI)
      apply safe
    proof −
      fix a b :: 'n
      assume ab: ball 0 (K + 1) ⊆ cbox a b
      have ∗: ∀s s1. ?S − e < s1 ∧ s1 ≤ s ∧ s < ?S + e ⟶ |s − ?S| < e
        by arith
      show norm (integral (cbox a b) (λx. if x ∈ UNIV then norm (f x) else 0) −
?S) < e
        unfolding real-norm-def
        apply (rule ∗[rule-format])
        apply safe
        apply (rule d(2))
      proof goal-cases
        case 1
         have (∑k∈d. norm (integral k f)) ≤ setsum (λk. integral k (λx. norm (f
x))) d
          apply (rule setsum-mono)
          apply (rule absolutely-integrable-le)
          apply (drule d'(4))
          apply safe
          apply (rule f-int)
```

```
      done
    also have ... = integral (⋃ d) (λx. norm (f x))
      apply (rule integral-combine-division-bottomup[symmetric])
      apply (rule d)
      unfolding forall-in-division[OF d(1)]
      using f-int
      apply auto
      done
    also have ... ≤ integral (cbox a b) (λx. if x ∈ UNIV then norm (f x) else
0)
    proof −
      have ⋃ d ⊆ cbox a b
        apply rule
        apply (drule K(2)[rule-format])
        apply (rule ab[unfolded subset-eq,rule-format])
        apply (auto simp add: dist-norm)
        done
      then show ?thesis
        apply −
        apply (subst if-P)
        apply rule
        apply (rule integrable-on-subdivision[of - - - cbox a b])
        apply (rule d)
        using f-int[of a b]
        apply auto
        done
    qed
    finally show ?case .
  next
    note f = absolutely-integrable-onD[OF f-int[of a b]]
    note ∗ = this(2)[unfolded has-integral-integral has-integral[of λx. norm (f
x)],rule-format]
    have e/2>0
      using ‹e > 0› by auto
    from ∗ [OF this] obtain d1 where
      d1: gauge d1 ∀ p. p tagged-division-of (cbox a b) ∧ d1 fine p ⟶
        norm ((∑ (x, k)∈p. content k ∗R norm (f x)) − integral (cbox a b) (λx.
norm (f x))) < e / 2
      by auto
    from henstock-lemma [OF f(1) ‹e/2>0›] obtain d2 where
      d2: gauge d2 ∀ p. p tagged-partial-division-of (cbox a b) ∧ d2 fine p ⟶
        (∑ (x, k)∈p. norm (content k ∗R f x − integral k f)) < e / 2 .
    obtain p where
      p: p tagged-division-of (cbox a b) d1 fine p d2 fine p
      by (rule fine-division-exists [OF gauge-inter [OF d1(1) d2(1)], of a b])
        (auto simp add: fine-inter)
    have ∗: ⋀sf sf' si di. sf' = sf ⟶ si ≤ ?S ⟶ |sf − si| < e / 2 ⟶
```

```
        |sf' − di| < e / 2 ⟶ di < ?S + e
        by arith
      show integral (cbox a b) (λx. if x ∈ UNIV then norm (f x) else 0) < ?S +
e
        apply (subst if-P)
        apply rule
       proof (rule *[rule-format])
       show |(∑(x,k)∈p. norm (content k *R f x)) − (∑(x,k)∈p. norm (integral
k f))| < e / 2
          unfolding split-def
          apply (rule helplemma)
          using d2(2)[rule-format,of p]
          using p(1,3)
          unfolding tagged-division-of-def split-def
          apply auto
          done
         show |(∑(x, k)∈p. content k *R norm (f x)) − integral (cbox a b) (λx.
norm(f x))| < e / 2
          using d1(2)[rule-format,OF conjI[OF p(1,2)]]
          by (simp only: real-norm-def)
       show (∑(x, k)∈p. content k *R norm (f x)) = (∑(x, k)∈p. norm (content
k *R f x))
          apply (rule setsum.cong)
          apply (rule refl)
          unfolding split-paired-all split-conv
          apply (drule tagged-division-ofD(4)[OF p(1)])
          unfolding norm-scaleR
          apply (subst abs-of-nonneg)
          apply auto
          done
        show (∑(x, k)∈p. norm (integral k f)) ≤ ?S
          using partial-division-of-tagged-division[of p cbox a b] p(1)
          apply (subst setsum-over-tagged-division-lemma[OF p(1)])
          apply (simp add: integral-null)
          apply (intro cSUP-upper2[OF D(2), of snd ` p])
          apply (auto simp: tagged-partial-division-of-def)
          done
       qed
     qed
   qed (insert K, auto)
  qed
qed

lemma absolutely-integrable-restrict-univ:
  (λx. if x ∈ s then f x else (0::'a::banach)) absolutely-integrable-on UNIV ⟷
    f absolutely-integrable-on s
 unfolding absolutely-integrable-on-def if-distrib norm-zero integrable-restrict-univ
..
```

lemma *absolutely-integrable-add*[*intro*]:
fixes *f g* :: *'n*::*euclidean-space* ⇒ *'m*::*euclidean-space*
assumes *f absolutely-integrable-on s*
and *g absolutely-integrable-on s*
shows (λ*x*. *f x* + *g x*) *absolutely-integrable-on s*
proof −
let *?P* = ⋀*f g*::*'n* ⇒ *'m*. *f absolutely-integrable-on UNIV* ⟹ *g absolutely-integrable-on UNIV* ⟹ (λ*x*. *f x* + *g x*) *absolutely-integrable-on UNIV*
{
presume *as*: *PROP ?P*
note *a* = *absolutely-integrable-restrict-univ*[*symmetric*]
have ∗: ⋀*x*. (*if x* ∈ *s then f x else 0*) + (*if x* ∈ *s then g x else 0*) = (*if x* ∈ *s then f x* + *g x else 0*) **by** *auto*
show *?thesis*
apply (*subst a*)
using *as*[*OF assms*[*unfolded a*[*of f*] *a*[*of g*]]]
apply (*simp only*: ∗)
done
}
fix *f g* :: *'n* ⇒ *'m*
assume *assms*: *f absolutely-integrable-on UNIV g absolutely-integrable-on UNIV*
note *absolutely-integrable-bounded-variation*
from *this*[*OF assms*(*1*)] *this*[*OF assms*(*2*)] **guess** *B1 B2* **. note** *B*=*this*[*rule-format*]
show (λ*x*. *f x* + *g x*) *absolutely-integrable-on UNIV*
apply (*rule bounded-variation-absolutely-integrable*[*of - B1*+*B2*])
apply (*rule integrable-add*)
prefer *3*
apply *safe*
proof *goal-cases*
case *prems*: (*1 d*)
have ⋀*k*. *k* ∈ *d* ⟹ *f integrable-on k* ∧ *g integrable-on k*
apply (*drule division-ofD*(*4*)[*OF prems*])
apply *safe*
apply (*rule-tac*[!] *integrable-on-subcbox*[*of - UNIV*])
using *assms*
apply *auto*
done
then have (∑*k*∈*d*. *norm* (*integral k* (λ*x*. *f x* + *g x*))) ≤ (∑*k*∈*d*. *norm* (*integral k f*)) + (∑*k*∈*d*. *norm* (*integral k g*))
apply −
unfolding *setsum.distrib* [*symmetric*]
apply (*rule setsum-mono*)
apply (*subst integral-add*)
prefer *3*
apply (*rule norm-triangle-ineq*)
apply *auto*
done
also have ... ≤ *B1* + *B2*

```
    using B(1)[OF prems] B(2)[OF prems] by auto
   finally show ?case .
 qed (insert assms, auto)
qed

lemma absolutely-integrable-sub[intro]:
 fixes f g :: 'n::euclidean-space ⇒ 'm::euclidean-space
 assumes f absolutely-integrable-on s
   and g absolutely-integrable-on s
 shows (λx. f x − g x) absolutely-integrable-on s
 using absolutely-integrable-add[OF assms(1) absolutely-integrable-neg[OF assms(2)]]
 by (simp add: algebra-simps)

lemma absolutely-integrable-linear:
 fixes f :: 'm::euclidean-space ⇒ 'n::euclidean-space
   and h :: 'n::euclidean-space ⇒ 'p::euclidean-space
 assumes f absolutely-integrable-on s
   and bounded-linear h
 shows (h ∘ f) absolutely-integrable-on s
proof −
 {
   presume as: ⋀f::'m ⇒ 'n. ⋀h::'n ⇒ 'p. f absolutely-integrable-on UNIV ⟹
     bounded-linear h ⟹ (h ∘ f) absolutely-integrable-on UNIV
   note a = absolutely-integrable-restrict-univ[symmetric]
   show ?thesis
     apply (subst a)
     using as[OF assms[unfolded a[of f] a[of g]]]
     apply (simp only: o-def if-distrib linear-simps[OF assms(2)])
     done
 }
 fix f :: 'm ⇒ 'n
 fix h :: 'n ⇒ 'p
 assume assms: f absolutely-integrable-on UNIV bounded-linear h
 from absolutely-integrable-bounded-variation[OF assms(1)] guess B . note B=this
 from bounded-linear.pos-bounded[OF assms(2)] guess b .. note b=conjunctD2[OF
this]
 show (h ∘ f) absolutely-integrable-on UNIV
   apply (rule bounded-variation-absolutely-integrable[of - B * b])
   apply (rule integrable-linear[OF - assms(2)])
   apply safe
 proof goal-cases
   case prems: (2 d)
   have (∑ k∈d. norm (integral k (h ∘ f))) ≤ setsum (λk. norm(integral k f)) d
* b
     unfolding setsum-left-distrib
     apply (rule setsum-mono)
   proof goal-cases
     case (1 k)
     from division-ofD(4)[OF prems this]
```

```
    guess u v by (elim exE) note uv=this
    have ∗: f integrable-on k
      unfolding uv
      apply (rule integrable-on-subcbox[of - UNIV])
      using assms
      apply auto
      done
    note this[unfolded has-integral-integral]
    note has-integral-linear[OF this assms(2)] integrable-linear[OF ∗ assms(2)]
   note ∗ = has-integral-unique[OF this(2)[unfolded has-integral-integral] this(1)]
    show ?case
      unfolding ∗ using b by auto
  qed
  also have ... ≤ B ∗ b
    apply (rule mult-right-mono)
    using B prems b
    apply auto
    done
  finally show ?case .
 qed (insert assms, auto)
qed

lemma absolutely-integrable-setsum:
 fixes f :: 'a ⇒ 'n::euclidean-space ⇒ 'm::euclidean-space
 assumes finite t
   and ⋀a. a ∈ t ⟹ (f a) absolutely-integrable-on s
 shows (λx. setsum (λa. f a x) t) absolutely-integrable-on s
 using assms(1,2)
 by induct auto

lemma absolutely-integrable-vector-abs:
 fixes f :: 'a::euclidean-space => 'b::euclidean-space
   and T :: 'c::euclidean-space ⇒ 'b
 assumes f: f absolutely-integrable-on s
 shows (λx. (∑ i∈Basis. |f x•T i| ∗R i)) absolutely-integrable-on s
 (is ?Tf absolutely-integrable-on s)
proof −
 have if-distrib: ⋀P A B x. (if P then A else B) ∗R x = (if P then A ∗R x else
B ∗R x)
   by simp
 have ∗: ⋀x. ?Tf x = (∑ i∈Basis.
   ((λy. (∑ j∈Basis. (if j = i then y else 0) ∗R j)) o
    (λx. (norm (∑ j∈Basis. (if j = i then f x•T i else 0) ∗R j)))) x)
   by (simp add: comp-def if-distrib setsum.If-cases)
 show ?thesis
   unfolding ∗
   apply (rule absolutely-integrable-setsum[OF finite-Basis])
   apply (rule absolutely-integrable-linear)
   apply (rule absolutely-integrable-norm)
```

apply (*rule absolutely-integrable-linear*[*OF f*, *unfolded o-def*])
apply (*auto simp*: *linear-linear euclidean-eq-iff*[**where** *'a='c*] *inner-simps intro*!: *linearI*)
done
qed

lemma *absolutely-integrable-max*:
fixes *f g* :: *'m*::*euclidean-space* ⇒ *'n*::*euclidean-space*
assumes *f absolutely-integrable-on s*
and *g absolutely-integrable-on s*
shows (λ*x*. (∑ *i*∈*Basis*. *max* (*f*(*x*)•*i*) (*g*(*x*)•*i*) *∗*$_R$ *i*)::*'n*) *absolutely-integrable-on s*
proof −
have *∗*:⋀*x*. (*1* / *2*) *∗*$_R$ (((∑ *i*∈*Basis*. |(*f x* − *g x*) • *i*| *∗*$_R$ *i*)::*'n*) + (*f x* + *g x*)) =
(∑ *i*∈*Basis*. *max* (*f*(*x*)•*i*) (*g*(*x*)•*i*) *∗*$_R$ *i*)
unfolding *euclidean-eq-iff*[**where** *'a='n*] **by** (*auto simp*: *inner-simps*)
note *absolutely-integrable-sub*[*OF assms*] *absolutely-integrable-add*[*OF assms*]
note *absolutely-integrable-vector-abs*[*OF this*(*1*), **where** *T*=λ*x*. *x*] *this*(*2*)
note *absolutely-integrable-add*[*OF this*]
note *absolutely-integrable-cmul*[*OF this*, *of 1/2*]
then show *?thesis* **unfolding** *∗* **.**
qed

lemma *absolutely-integrable-min*:
fixes *f g*::*'m*::*euclidean-space* ⇒ *'n*::*euclidean-space*
assumes *f absolutely-integrable-on s*
and *g absolutely-integrable-on s*
shows (λ*x*. (∑ *i*∈*Basis*. *min* (*f*(*x*)•*i*) (*g*(*x*)•*i*) *∗*$_R$ *i*)::*'n*) *absolutely-integrable-on s*
proof −
have *∗*:⋀*x*. (*1* / *2*) *∗*$_R$ ((*f x* + *g x*) − (∑ *i*∈*Basis*. |(*f x* − *g x*) • *i*| *∗*$_R$ *i*::*'n*)) =
(∑ *i*∈*Basis*. *min* (*f*(*x*)•*i*) (*g*(*x*)•*i*) *∗*$_R$ *i*)
unfolding *euclidean-eq-iff*[**where** *'a='n*] **by** (*auto simp*: *inner-simps*)

note *absolutely-integrable-add*[*OF assms*] *absolutely-integrable-sub*[*OF assms*]
note *this*(*1*) *absolutely-integrable-vector-abs*[*OF this*(*2*), **where** *T*=λ*x*. *x*]
note *absolutely-integrable-sub*[*OF this*]
note *absolutely-integrable-cmul*[*OF this*,*of 1/2*]
then show *?thesis* **unfolding** *∗* **.**
qed

lemma *absolutely-integrable-abs-eq*:
fixes *f*::*'n*::*euclidean-space* ⇒ *'m*::*euclidean-space*
shows *f absolutely-integrable-on s* ⟷ *f integrable-on s* ∧
(λ*x*. (∑ *i*∈*Basis*. |*f x*•*i*| *∗*$_R$ *i*)::*'m*) *integrable-on s*
(**is** *?l* = *?r*)
proof
assume *?l*

```
  then show ?r
    apply −
    apply rule
    defer
    apply (drule absolutely-integrable-vector-abs)
    apply auto
    done
next
  assume ?r
  {
    presume lem: ⋀f::'n ⇒ 'm. f integrable-on UNIV ⟹
      (λx. (∑ i∈Basis. |f x•i| *R i)::'m) integrable-on UNIV ⟹
        f absolutely-integrable-on UNIV
    have *: ⋀x. (∑ i∈Basis. |(if x ∈ s then f x else 0) • i| *R i) =
      (if x ∈ s then (∑ i∈Basis. |f x • i| *R i) else (0::'m))
      unfolding euclidean-eq-iff[where 'a='m]
      by auto
    show ?l
      apply (subst absolutely-integrable-restrict-univ[symmetric])
      apply (rule lem)
      unfolding integrable-restrict-univ *
      using ‹?r›
      apply auto
      done
  }
  fix f :: 'n ⇒ 'm
 assume assms: f integrable-on UNIV (λx. (∑ i∈Basis. |f x•i| *R i)::'m) integrable-on
UNIV
  let ?B = ∑ i∈Basis. integral UNIV (λx. (∑ i∈Basis. |f x•i| *R i)::'m) • i
  show f absolutely-integrable-on UNIV
   apply (rule bounded-variation-absolutely-integrable[OF assms(1), where B=?B])
    apply safe
  proof goal-cases
    case d: (1 d)
    note d'=division-ofD[OF d]
    have (∑ k∈d. norm (integral k f)) ≤
      (∑ k∈d. setsum (op • (integral k (λx. (∑ i∈Basis. |f x•i| *R i)::'m))) Basis)
      apply (rule setsum-mono)
      apply (rule order-trans[OF norm-le-l1])
      apply (rule setsum-mono)
      unfolding lessThan-iff
    proof −
      fix k
      fix i :: 'm
      assume k ∈ d and i: i ∈ Basis
      from d'(4)[OF this(1)] guess a b by (elim exE) note ab=this
      show |integral k f • i| ≤ integral k (λx. (∑ i∈Basis. |f x•i| *R i)::'m) • i
        apply (rule abs-leI)
        unfolding inner-minus-left[symmetric]
```

```
      defer
      apply (subst integral-neg[symmetric])
      apply (rule-tac[1-2] integral-component-le[OF i])
      using integrable-on-subcbox[OF assms(1),of a b]
        integrable-on-subcbox[OF assms(2),of a b] i  integrable-neg
      unfolding ab
      apply auto
      done
    qed
    also have ... ≤ setsum (op · (integral UNIV (λx. (∑i∈Basis. |f x•i| *R
i)::'m))) Basis
      apply (subst setsum.commute)
      apply (rule setsum-mono)
    proof goal-cases
      case (1 j)
      have *: (λx. ∑i∈Basis. |f x•i| *R i::'m) integrable-on ⋃ d
        using integrable-on-subdivision[OF d assms(2)] by auto
      have (∑i∈d. integral i (λx. ∑i∈Basis. |f x•i| *R i::'m) · j) =
        integral (⋃ d) (λx. ∑i∈Basis. |f x•i| *R i::'m) · j
      unfolding inner-setsum-left[symmetric] integral-combine-division-topdown[OF
* d] ..
      also have ... ≤ integral UNIV (λx. ∑i∈Basis. |f x•i| *R i::'m) · j
        apply (rule integral-subset-component-le)
        using assms * ‹j ∈ Basis›
        apply auto
        done
      finally show ?case .
    qed
    finally show ?case .
  qed
qed

lemma nonnegative-absolutely-integrable:
  fixes f :: 'n::euclidean-space ⇒ 'm::euclidean-space
  assumes ∀x∈s. ∀i∈Basis. 0 ≤ f x · i
    and f integrable-on s
  shows f absolutely-integrable-on s
  unfolding absolutely-integrable-abs-eq
  apply rule
  apply (rule assms)thm integrable-eq
  apply (rule integrable-eq[of - f])
  using assms
  apply (auto simp: euclidean-eq-iff[where 'a='m])
  done

lemma absolutely-integrable-integrable-bound:
  fixes f :: 'n::euclidean-space ⇒ 'm::euclidean-space
  assumes ∀x∈s. norm (f x) ≤ g x
    and f integrable-on s
```

```
    and g integrable-on s
  shows f absolutely-integrable-on s
proof −
  {
    presume ∗: ⋀f::'n ⇒ 'm. ⋀g. ∀x. norm (f x) ≤ g x ⟹ f integrable-on UNIV
⟹
      g integrable-on UNIV ⟹ f absolutely-integrable-on UNIV
    show ?thesis
      apply (subst absolutely-integrable-restrict-univ[symmetric])
      apply (rule ∗[of - λx. if x∈s then g x else 0])
      using assms
      unfolding integrable-restrict-univ
      apply auto
      done
  }
  fix g
  fix f :: 'n ⇒ 'm
  assume assms: ∀x. norm (f x) ≤ g x f integrable-on UNIV g integrable-on UNIV
  show f absolutely-integrable-on UNIV
   apply (rule bounded-variation-absolutely-integrable[OF assms(2),where B=integral
UNIV g])
    apply safe
  proof goal-cases
    case d: (1 d)
    note d'=division-ofD[OF d]
    have (∑k∈d. norm (integral k f)) ≤ (∑k∈d. integral k g)
      apply (rule setsum-mono)
      apply (rule integral-norm-bound-integral)
      apply (drule-tac[!] d'(4))
      apply safe
      apply (rule-tac[1−2] integrable-on-subcbox)
      using assms
      apply auto
      done
    also have ... = integral (⋃d) g
      apply (rule integral-combine-division-bottomup[symmetric])
      apply (rule d)
      apply safe
      apply (drule d'(4))
      apply safe
      apply (rule integrable-on-subcbox[OF assms(3)])
      apply auto
      done
    also have ... ≤ integral UNIV g
      apply (rule integral-subset-le)
      defer
      apply (rule integrable-on-subdivision[OF d,of - UNIV])
      prefer 4
      apply rule
```

```
    apply (rule-tac y=norm (f x) in order-trans)
    using assms
    apply auto
    done
  finally show ?case .
 qed
qed

lemma absolutely-integrable-absolutely-integrable-bound:
 fixes f :: 'n::euclidean-space ⇒ 'm::euclidean-space
   and g :: 'n::euclidean-space ⇒ 'p::euclidean-space
 assumes ∀x∈s. norm (f x) ≤ norm (g x)
   and f integrable-on s
   and g absolutely-integrable-on s
 shows f absolutely-integrable-on s
 apply (rule absolutely-integrable-integrable-bound[of s f λx. norm (g x)])
 using assms
 apply auto
 done

lemma absolutely-integrable-inf-real:
 assumes finite k
   and k ≠ {}
   and ∀i∈k. (λx. (fs x i)::real) absolutely-integrable-on s
 shows (λx. (Inf ((fs x) ` k))) absolutely-integrable-on s
 using assms
proof induct
 case (insert a k)
 let ?P = (λx.
   if fs x ` k = {} then fs x a
   else min (fs x a) (Inf (fs x ` k))) absolutely-integrable-on s
 show ?case
   unfolding image-insert
   apply (subst Inf-insert-finite)
   apply (rule finite-imageI[OF insert(1)])
 proof (cases k = {})
   case True
   then show ?P
     apply (subst if-P)
     defer
     apply (rule insert(5)[rule-format])
     apply auto
     done
 next
   case False
   then show ?P
     apply (subst if-not-P)
     defer
    apply (rule absolutely-integrable-min[where 'n=real, unfolded Basis-real-def,
```

simplified])
defer
apply(*rule insert*(*3*)[*OF False*])
using *insert*(*5*)
apply *auto*
done
qed
next
case *empty*
then show *?case* **by** *auto*
qed

lemma *absolutely-integrable-sup-real*:
assumes *finite k*
and $k \neq \{\}$
and $\forall\, i{\in}k.$ ($\lambda x.$ (*fs x i*)::*real*) *absolutely-integrable-on s*
shows ($\lambda x.$ (*Sup* ((*fs x*) ' *k*))) *absolutely-integrable-on s*
using *assms*
proof *induct*
case (*insert a k*)
let *?P* = ($\lambda x.$
if fs x ' k = {} *then fs x a*
else max (*fs x a*) (*Sup* (*fs x ' k*))) *absolutely-integrable-on s*
show *?case*
unfolding *image-insert*
apply (*subst Sup-insert-finite*)
apply (*rule finite-imageI*[*OF insert*(*1*)])
proof (*cases k* = {})
case *True*
then show *?P*
apply (*subst if-P*)
defer
apply (*rule insert*(*5*)[*rule-format*])
apply *auto*
done
next
case *False*
then show *?P*
apply (*subst if-not-P*)
defer
apply (*rule absolutely-integrable-max*[**where** *'n*=*real*, *unfolded Basis-real-def*, *simplified*])
defer
apply (*rule insert*(*3*)[*OF False*])
using *insert*(*5*)
apply *auto*
done
qed
qed *auto*

41.66 differentiation under the integral sign

lemma *tube-lemma*:
assumes *compact K*
assumes *open W*
assumes $\{x0\} \times K \subseteq W$
shows $\exists X0.\ x0 \in X0 \wedge$ *open X0* $\wedge\ X0 \times K \subseteq W$
proof −
{
fix *y* **assume** $y \in K$
then have $(x0,\ y) \in W$ **using** *assms* **by** *auto*
with ‹*open W*› **have** $\exists X0\ Y.$ *open X0* $\wedge$ *open Y* $\wedge\ x0 \in X0 \wedge y \in Y \wedge X0 \times Y \subseteq W$
by (*rule open-prod-elim*) *blast*
} then obtain *X0 Y* **where**
$\forall y \in K.$ *open* $(X0\ y) \wedge$ *open* $(Y\ y) \wedge x0 \in X0\ y \wedge y \in Y\ y \wedge X0\ y \times Y\ y \subseteq W$
by *metis*
moreover
then have $\forall t{\in}Y\ `\ K.$ *open* $t\ K \subseteq \bigcup (Y\ `\ K)$ **by** *auto*
with ‹*compact K*› **obtain** *CC* **where** $CC \subseteq Y\ `\ K$ *finite CC* $K \subseteq \bigcup CC$
by (*rule compactE*)
moreover
then obtain *c* **where** *c*:
$\bigwedge C.\ C \in CC \Longrightarrow c\ C \in K \wedge C = Y\ (c\ C)$
by (*force intro*!: *choice*)
ultimately show *?thesis*
by (*force intro*!: *exI*[**where** $x{=}\bigcap C{\in}CC.$ *X0* $(c\ C)$])
qed

lemma *continuous-on-prod-compactE*:
fixes *fx*::*'a*::*topological-space* × *'b*::*topological-space* ⇒ *'c*::*metric-space*
and *e*::*real*
assumes *cont-fx*: *continuous-on* $(U \times C)$ *fx*
assumes *compact C*
assumes [*intro*]: $x0 \in U$
notes [*continuous-intros*] = *continuous-on-compose2*[*OF cont-fx*]
assumes $e > 0$
obtains *X0* **where** $x0 \in X0$ *open X0*
$\forall x{\in}X0 \cap U.\ \forall t \in C.$ *dist* $(fx\ (x,\ t))\ (fx\ (x0,\ t)) \leq e$
proof −
def *psi* $\equiv \lambda(x,\ t).$ *dist* $(fx\ (x,\ t))\ (fx\ (x0,\ t))$
def *W0* $\equiv \{(x,\ t) \in U \times C.\ psi\ (x,\ t) < e\}$
have *W0-eq*: *W0* = *psi* $-`\ \{..{<}e\} \cap U \times C$
by (*auto simp*: *vimage-def W0-def*)
have *open* $\{..{<}e\}$ **by** *simp*
have *continuous-on* $(U \times C)$ *psi*
by (*auto intro*!: *continuous-intros simp*: *psi-def split-beta'*)
from *this*[*unfolded continuous-on-open-invariant*, *rule-format*, *OF* ‹*open* $\{..{<}e\}$›]
obtain *W* **where** *W*: *open W* $W \cap U \times C = W0 \cap U \times C$

```
    unfolding W0-eq by blast
  have {x0} × C ⊆ W ∩ U × C
    unfolding W
    by (auto simp: W0-def psi-def ‹0 < e›)
  then have {x0} × C ⊆ W by blast
  from tube-lemma[OF ‹compact C› ‹open W› this]
  obtain X0 where X0: x0 ∈ X0 open X0 X0 × C ⊆ W
    by blast

  have ∀x∈X0 ∩ U. ∀t ∈ C. dist (fx (x, t)) (fx (x0, t)) ≤ e
  proof safe
    fix x assume x: x ∈ X0 x ∈ U
    fix t assume t: t ∈ C
    have dist (fx (x, t)) (fx (x0, t)) = psi (x, t)
      by (auto simp: psi-def)
    also
    {
      have (x, t) ∈ X0 × C
        using t x
        by auto
      also note ‹... ⊆ W›
      finally have (x, t) ∈ W .
      with t x have (x, t) ∈ W ∩ U × C
        by blast
      also note ‹W ∩ U × C = W0 ∩ U × C›
      finally  have psi (x, t) < e
        by (auto simp: W0-def)
    }
    finally show dist (fx (x, t)) (fx (x0, t)) ≤ e by simp
  qed
  from X0(1,2) this show ?thesis ..
qed

lemma integral-continuous-on-param:
  fixes f::'a::topological-space ⇒ 'b::euclidean-space ⇒ 'c::banach
  assumes cont-fx: continuous-on (U × cbox a b) (λ(x, t). f x t)
  shows continuous-on U (λx. integral (cbox a b) (f x))
proof cases
  assume content (cbox a b) ≠ 0
  then have ne: cbox a b ≠ {} by auto

  note [continuous-intros] =
    continuous-on-compose2[OF cont-fx, where f=λy. Pair x y for x,
      unfolded split-beta fst-conv snd-conv]

  show ?thesis
    unfolding continuous-on-def
  proof (safe intro!: tendstoI)
    fix e'::real and x
```

```
    assume e' > 0
    def e ≡ e' / (content (cbox a b) + 1)
   have e > 0 using ‹e' > 0› by (auto simp: e-def intro!: divide-pos-pos add-nonneg-pos)
    assume x ∈ U
    from continuous-on-prod-compactE[OF cont-fx compact-cbox ‹x ∈ U› ‹0 < e›]
    obtain X0 where X0: x ∈ X0 open X0
      and fx-bound: ⋀y t. y ∈ X0 ∩ U ⟹ t ∈ cbox a b ⟹ norm (f y t − f x t)
≤ e
      unfolding split-beta fst-conv snd-conv dist-norm
      by metis
    have ∀F y in at x within U. y ∈ X0 ∩ U
      using X0(1) X0(2) eventually-at-topological by auto
     then show ∀F y in at x within U. dist (integral (cbox a b) (f y)) (integral
(cbox a b) (f x)) < e'
    proof eventually-elim
      case (elim y)
      have dist (integral (cbox a b) (f y)) (integral (cbox a b) (f x)) =
        norm (integral (cbox a b) (λt. f y t − f x t))
        using elim ‹x ∈ U›
        unfolding dist-norm
        by (subst integral-diff)
           (auto intro!: integrable-continuous continuous-intros)
      also have ... ≤ e * content (cbox a b)
        using elim ‹x ∈ U›
        by (intro integrable-bound)
           (auto intro!: fx-bound ‹x ∈ U › less-imp-le[OF ‹0 < e›]
              integrable-continuous continuous-intros)
      also have ... < e'
        using ‹0 < e'› ‹e > 0›
        by (auto simp: e-def divide-simps)
      finally show dist (integral (cbox a b) (f y)) (integral (cbox a b) (f x)) < e' .
    qed
  qed
qed (auto intro!: continuous-on-const)

lemma eventually-closed-segment:
  fixes x0::'a::real-normed-vector
  assumes open X0 x0 ∈ X0
  shows ∀F x in at x0 within U. closed-segment x0 x ⊆ X0
proof −
  from openE[OF assms]
  obtain e where e: 0 < e ball x0 e ⊆ X0 .
  then have ∀F x in at x0 within U. x ∈ ball x0 e
    by (auto simp: dist-commute eventually-at)
  then show ?thesis
  proof eventually-elim
    case (elim x)
    have x0 ∈ ball x0 e using ‹e > 0› by simp
    from convex-ball[unfolded convex-contains-segment, rule-format, OF this elim]
```

```
    have closed-segment x0 x ⊆ ball x0 e .
    also note ‹... ⊆ X0›
    finally show ?case .
  qed
qed

lemma leibniz-rule:
  fixes f::'a::banach ⇒ 'b::euclidean-space ⇒ 'c::banach
  assumes fx: ⋀x t. x ∈ U ⟹ t ∈ cbox a b ⟹
    ((λx. f x t) has-derivative blinfun-apply (fx x t)) (at x within U)
  assumes integrable-f2: ⋀x. x ∈ U ⟹ f x integrable-on cbox a b
  assumes cont-fx: continuous-on (U × (cbox a b)) (λ(x, t). fx x t)
  assumes [intro]: x0 ∈ U
  assumes convex U
  shows
    ((λx. integral (cbox a b) (f x)) has-derivative integral (cbox a b) (fx x0)) (at x0
within U)
    (is (?F has-derivative ?dF) -)
proof cases
  assume content (cbox a b) ≠ 0
  then have ne: cbox a b ≠ {} by auto
  note [continuous-intros] =
    continuous-on-compose2[OF cont-fx, where f=λy. Pair x y for x,
      unfolded split-beta fst-conv snd-conv]
  show ?thesis
  proof (intro has-derivativeI bounded-linear-scaleR-left tendstoI, fold norm-conv-dist)
    have cont-f1: ⋀t. t ∈ cbox a b ⟹ continuous-on U (λx. f x t)
    by (auto simp: continuous-on-eq-continuous-within intro!: has-derivative-continuous
fx)
    note [continuous-intros] = continuous-on-compose2[OF cont-f1]
    fix e'::real
    assume e' > 0
    def e ≡ e' / (content (cbox a b) + 1)
   have e > 0 using ‹e' > 0› by (auto simp: e-def intro!: divide-pos-pos add-nonneg-pos)
    from continuous-on-prod-compactE[OF cont-fx compact-cbox ‹x0 ∈ U› ‹e > 0›]
    obtain X0 where X0: x0 ∈ X0 open X0
      and fx-bound: ⋀x t. x ∈ X0 ∩ U ⟹ t ∈ cbox a b ⟹ norm (fx x t − fx x0
t) ≤ e
      unfolding split-beta fst-conv snd-conv
      by (metis dist-norm)

    note eventually-closed-segment[OF ‹open X0› ‹x0 ∈ X0›, of U]
    moreover
    have ∀F x in at x0 within U. x ∈ X0
      using ‹open X0› ‹x0 ∈ X0› eventually-at-topological by blast
    moreover have ∀F x in at x0 within U. x ≠ x0
      by (auto simp: eventually-at-filter)
    moreover have ∀F x in at x0 within U. x ∈ U
      by (auto simp: eventually-at-filter)
```

```
  ultimately
   show ∀F x in at x0 within U. norm ((?F x − ?F x0 − ?dF (x − x0)) /R
norm (x − x0)) < e'
  proof eventually-elim
    case (elim x)
    from elim have 0 < norm (x − x0) by simp
    have closed-segment x0 x ⊆ U
      by (rule ‹convex U›[unfolded convex-contains-segment, rule-format, OF ‹x0
∈ U› ‹x ∈ U›])
    from elim have [intro]: x ∈ U by auto

    have ?F x − ?F x0 − ?dF (x − x0) =
      integral (cbox a b) (λy. f x y − f x0 y − fx x0 y (x − x0))
      (is - = ?id)
      using ‹x ≠ x0›
      by (subst blinfun-apply-integral integral-diff,
          auto intro!: integrable-diff integrable-f2 continuous-intros
            intro: integrable-continuous)+
    also
    {
      fix t assume t: t ∈ (cbox a b)
      have seg: ⋀t. t ∈ {0..1} ⟹ x0 + t *R (x − x0) ∈ X0 ∩ U
        using ‹closed-segment x0 x ⊆ U›
          ‹closed-segment x0 x ⊆ X0›
        by (force simp: closed-segment-def algebra-simps)
      from t have deriv:
        ((λx. f x t) has-derivative (fx y t)) (at y within X0 ∩ U)
        if y ∈ X0 ∩ U for y
        unfolding has-vector-derivative-def[symmetric]
        using that ‹x ∈ X0›
        by (intro has-derivative-within-subset[OF fx]) auto
      have ∀x ∈ X0 ∩ U. onorm (blinfun-apply (fx x t) − (fx x0 t)) ≤ e
        using fx-bound t
     by (auto simp add: norm-blinfun-def fun-diff-def blinfun.bilinear-simps[symmetric])
      from differentiable-bound-linearization[OF seg deriv this] X0
      have norm (f x t − f x0 t − fx x0 t (x − x0)) ≤ e * norm (x − x0)
        by (auto simp add: ac-simps)
    }
    then have norm ?id ≤ integral (cbox a b) (λ-. e * norm (x − x0))
      by (intro integral-norm-bound-integral)
        (auto intro!: continuous-intros integrable-diff integrable-f2
          intro: integrable-continuous)
    also have ... = content (cbox a b) * e * norm (x − x0)
      by simp
    also have ... < e' * norm (x − x0)
      using ‹e' > 0› content-pos-le[of a b]
      by (intro mult-strict-right-mono[OF - ‹0 < norm (x − x0)›])
        (auto simp: divide-simps e-def)
    finally have norm (?F x − ?F x0 − ?dF (x − x0)) < e' * norm (x − x0) .
```

```
      then show ?case
        by (auto simp: divide-simps)
    qed
  qed (rule blinfun.bounded-linear-right)
qed (auto intro!: derivative-eq-intros simp: blinfun.bilinear-simps)

lemma
  has-vector-derivative-eq-has-derivative-blinfun:
  (f has-vector-derivative f') (at x within U) ⟷
    (f has-derivative blinfun-scaleR-left f') (at x within U)
  by (simp add: has-vector-derivative-def)

lemma leibniz-rule-vector-derivative:
  fixes f::real ⇒ 'b::euclidean-space ⇒ 'c::banach
  assumes fx: ⋀x t. x ∈ U ⟹ t ∈ cbox a b ⟹
      ((λx. f x t) has-vector-derivative (fx x t)) (at x within U)
  assumes integrable-f2: ⋀x. x ∈ U ⟹ (f x) integrable-on cbox a b
  assumes cont-fx: continuous-on (U × cbox a b) (λ(x, t). fx x t)
  assumes U: x0 ∈ U convex U
  shows ((λx. integral (cbox a b) (f x)) has-vector-derivative integral (cbox a b) (fx
x0))
      (at x0 within U)
proof −
  note [continuous-intros] =
    continuous-on-compose2[OF cont-fx, where f=λy. Pair x y for x,
      unfolded split-beta fst-conv snd-conv]
  have ∗: blinfun-scaleR-left (integral (cbox a b) (fx x0)) =
    integral (cbox a b) (λt. blinfun-scaleR-left (fx x0 t))
    by (subst integral-linear[symmetric])
      (auto simp: has-vector-derivative-def o-def
        intro!: integrable-continuous U continuous-intros bounded-linear-intros)
  show ?thesis
    unfolding has-vector-derivative-eq-has-derivative-blinfun
    apply (rule has-derivative-eq-rhs)
   apply (rule leibniz-rule[OF - integrable-f2 - U, where fx=λx t. blinfun-scaleR-left
(fx x t)])
    using fx cont-fx
   apply (auto simp: has-vector-derivative-def ∗ split-beta intro!: continuous-intros)
    done
qed

lemma
  has-field-derivative-eq-has-derivative-blinfun:
  (f has-field-derivative f') (at x within U) ⟷ (f has-derivative blinfun-mult-right
f') (at x within U)
  by (simp add: has-field-derivative-def)

lemma leibniz-rule-field-derivative:
  fixes f::'a::{real-normed-field, banach} ⇒ 'b::euclidean-space ⇒ 'a
```

assumes *fx*: $\bigwedge x\ t.\ x \in U \Longrightarrow t \in cbox\ a\ b \Longrightarrow ((\lambda x.\ f\ x\ t)\ has\text{-}field\text{-}derivative\ fx\ x\ t)\ (at\ x\ within\ U)$
assumes *integrable-f2*: $\bigwedge x.\ x \in U \Longrightarrow (f\ x)\ integrable\text{-}on\ cbox\ a\ b$
assumes *cont-fx*: *continuous-on* $(U \times (cbox\ a\ b))\ (\lambda(x,\ t).\ fx\ x\ t)$
assumes U: $x0 \in U\ convex\ U$
shows $((\lambda x.\ integral\ (cbox\ a\ b)\ (f\ x))\ has\text{-}field\text{-}derivative\ integral\ (cbox\ a\ b)\ (fx\ x0))\ (at\ x0\ within\ U)$
proof −
note [*continuous-intros*] =
continuous-on-compose2[*OF cont-fx*, **where** $f{=}\lambda y.\ Pair\ x\ y$ **for** x,
unfolded split-beta fst-conv snd-conv]
have $*$: *blinfun-mult-right* $(integral\ (cbox\ a\ b)\ (fx\ x0)) =$
$integral\ (cbox\ a\ b)\ (\lambda t.\ blinfun\text{-}mult\text{-}right\ (fx\ x0\ t))$
by (*subst integral-linear*[*symmetric*])
(*auto simp*: *has-vector-derivative-def o-def*
intro!: *integrable-continuous U continuous-intros bounded-linear-intros*)
show *?thesis*
unfolding *has-field-derivative-eq-has-derivative-blinfun*
apply (*rule has-derivative-eq-rhs*)
apply (*rule leibniz-rule*[*OF - integrable-f2 - U*, **where** $fx{=}\lambda x\ t.\ blinfun\text{-}mult\text{-}right\ (fx\ x\ t)$])
using *fx cont-fx*
apply (*auto simp*: *has-field-derivative-def* $*$ *split-beta intro*!: *continuous-intros*)
done
qed

41.67 Exchange uniform limit and integral

lemma
uniform-limit-integral:
fixes $f{::}'a \Rightarrow 'b{::}euclidean\text{-}space \Rightarrow 'c{::}banach$
assumes u: *uniform-limit* $(cbox\ a\ b)\ f\ g\ F$
assumes c: $\bigwedge n.\ continuous\text{-}on\ (cbox\ a\ b)\ (f\ n)$
assumes [*simp*]: $F \neq bot$
obtains $I\ J$ **where**
$\bigwedge n.\ (f\ n\ has\text{-}integral\ I\ n)\ (cbox\ a\ b)$
$(g\ has\text{-}integral\ J)\ (cbox\ a\ b)$
$(I \longrightarrow J)\ F$
proof −
have *fi*[*simp*]: $f\ n\ integrable\text{-}on\ (cbox\ a\ b)$ **for** n
by (*auto intro*!: *integrable-continuous assms*)
then obtain I **where** I: $\bigwedge n.\ (f\ n\ has\text{-}integral\ I\ n)\ (cbox\ a\ b)$
by *atomize-elim* (*auto simp*: *integrable-on-def intro*!: *choice*)

moreover

have *gi*[*simp*]: $g\ integrable\text{-}on\ (cbox\ a\ b)$
by (*auto intro*!: *integrable-continuous uniform-limit-theorem*[*OF - u*] *eventuallyI c*)

```
  then obtain J where J: (g has-integral J) (cbox a b)
    by blast

  moreover

  have (I ⟶ J) F
  proof cases
    assume content (cbox a b) = 0
    hence I = (λ-. 0) J = 0
      by (auto intro!: has-integral-unique I J)
    thus ?thesis by simp
  next
    assume content-nonzero: content (cbox a b) ≠ 0
    show ?thesis
    proof (rule tendstoI)
      fix e::real
      assume e > 0
      def e' ≡ e / 2
      with ‹e > 0› have e' > 0 by simp
     then have ∀⇩F n in F. ∀x∈cbox a b. norm (f n x − g x) < e' / content (cbox
a b)
        using u content-nonzero content-pos-le[of a b]
        by (auto simp: uniform-limit-iff dist-norm)
      then show ∀⇩F n in F. dist (I n) J < e
      proof eventually-elim
        case (elim n)
        have I n = integral (cbox a b) (f n)
            J = integral (cbox a b) g
          using I[of n] J by (simp-all add: integral-unique)
        then have dist (I n) J = norm (integral (cbox a b) (λx. f n x − g x))
          by (simp add: integral-diff dist-norm)
        also have ... ≤ integral (cbox a b) (λx. (e' / content (cbox a b)))
          using elim
          by (intro integral-norm-bound-integral)
            (auto intro!: integrable-diff absolutely-integrable-onI)
        also have ... < e
          using ‹0 < e›
          by (simp add: e'-def)
        finally show ?case .
      qed
    qed
  qed
  ultimately show ?thesis ..
qed
```

41.68 Dominated convergence

```
lemma dominated-convergence:
  fixes f :: nat ⇒ 'n::euclidean-space ⇒ real
```

```
  assumes ⋀k. (f k) integrable-on s h integrable-on s
    and ⋀k. ∀ x ∈ s. norm (f k x) ≤ h x
    and ∀ x ∈ s. ((λk. f k x) ⟶ g x) sequentially
  shows g integrable-on s
    and ((λk. integral s (f k)) ⟶ integral s g) sequentially
proof −
  have bdd-below[simp]: ⋀x P. x ∈ s ⟹ bdd-below {f n x |n. P n}
  proof (safe intro!: bdd-belowI)
    fix n x show x ∈ s ⟹ − h x ≤ f n x
      using assms(3)[rule-format, of x n] by simp
  qed
  have bdd-above[simp]: ⋀x P. x ∈ s ⟹ bdd-above {f n x |n. P n}
  proof (safe intro!: bdd-aboveI)
    fix n x show x ∈ s ⟹ f n x ≤ h x
      using assms(3)[rule-format, of x n] by simp
  qed

  have ⋀m. (λx. Inf {f j x |j. m ≤ j}) integrable-on s ∧
    ((λk. integral s (λx. Inf {f j x |j. j ∈ {m..m + k}})) ⟶
    integral s (λx. Inf {f j x |j. m ≤ j}))sequentially
  proof (rule monotone-convergence-decreasing, safe)
    fix m :: nat
    show bounded {integral s (λx. Inf {f j x |j. j ∈ {m..m + k}}) |k. True}
      unfolding bounded-iff
      apply (rule-tac x=integral s h in exI)
    proof safe
      fix k :: nat
      show norm (integral s (λx. Inf {f j x |j. j ∈ {m..m + k}})) ≤ integral s h
        apply (rule integral-norm-bound-integral)
        unfolding simple-image
        apply (rule absolutely-integrable-onD)
        apply (rule absolutely-integrable-inf-real)
        prefer 5
        unfolding real-norm-def
        apply rule
        apply (rule cInf-abs-ge)
        prefer 5
        apply rule
        apply (rule-tac g = h in absolutely-integrable-integrable-bound)
        using assms
        unfolding real-norm-def
        apply auto
        done
    qed
    fix k :: nat
    show (λx. Inf {f j x |j. j ∈ {m..m + k}}) integrable-on s
      unfolding simple-image
      apply (rule absolutely-integrable-onD)
      apply (rule absolutely-integrable-inf-real)
```

```
    prefer 3
    using absolutely-integrable-integrable-bound[OF assms(3,1,2)]
    apply auto
    done
  fix x
  assume x: x ∈ s
  show Inf {f j x |j. j ∈ {m..m + Suc k}} ≤ Inf {f j x |j. j ∈ {m..m + k}}
    by (rule cInf-superset-mono) auto
  let ?S = {f j x| j. m ≤ j}
  show ((λk. Inf {f j x |j. j ∈ {m..m + k}}) ⟶ Inf ?S) sequentially
  proof (rule LIMSEQ-I, goal-cases)
    case r: (1 r)

    have ∃ y∈?S. y < Inf ?S + r
      by (subst cInf-less-iff[symmetric]) (auto simp: ‹x∈s› r)
    then obtain N where N: f N x < Inf ?S + r m ≤ N
      by blast

    show ?case
      apply (rule-tac x=N in exI)
      apply safe
    proof goal-cases
      case prems: (1 n)
      have ∗: ⋀y ix. y < Inf ?S + r ⟶ Inf ?S ≤ ix ⟶ ix ≤ y ⟶ |ix − Inf
?S| < r
        by arith
      show ?case
        unfolding real-norm-def
        apply (rule ∗[rule-format, OF N(1)])
        apply (rule cInf-superset-mono, auto simp: ‹x∈s›) []
        apply (rule cInf-lower)
        using N prems
        apply auto []
        apply simp
        done
    qed
  qed
qed
note dec1 = conjunctD2[OF this]

have ⋀m. (λx. Sup {f j x |j. m ≤ j}) integrable-on s ∧
  ((λk. integral s (λx. Sup {f j x |j. j ∈ {m..m + k}})) ⟶
  integral s (λx. Sup {f j x |j. m ≤ j})) sequentially
proof (rule monotone-convergence-increasing,safe)
  fix m :: nat
  show bounded {integral s (λx. Sup {f j x |j. j ∈ {m..m + k}}) |k. True}
    unfolding bounded-iff
    apply (rule-tac x=integral s h in exI)
  proof safe
```

fix k :: *nat*
show *norm* (*integral s* (λx. *Sup* {*f j x* |*j*. *j* ∈ {*m..m* + *k*}})) ≤ *integral s h*
apply (*rule integral-norm-bound-integral*) **unfolding** *simple-image*
apply (*rule absolutely-integrable-onD*)
apply(*rule absolutely-integrable-sup-real*)
prefer *5* **unfolding** *real-norm-def*
apply *rule*
apply (*rule cSup-abs-le*)
using *assms*
apply (*force simp add*:)
prefer *4*
apply *rule*
apply (*rule-tac g=h* **in** *absolutely-integrable-integrable-bound*)
using *assms*
unfolding *real-norm-def*
apply *auto*
done
qed
fix k :: *nat*
show (λx. *Sup* {*f j x* |*j*. *j* ∈ {*m..m* + *k*}}) *integrable-on s*
unfolding *simple-image*
apply (*rule absolutely-integrable-onD*)
apply (*rule absolutely-integrable-sup-real*)
prefer *3*
using *absolutely-integrable-integrable-bound*[*OF assms*(*3*,*1*,*2*)]
apply *auto*
done
fix x
assume x: $x \in s$
show *Sup* {*f j x* |*j*. *j* ∈ {*m..m* + *Suc k*}} ≥ *Sup* {*f j x* |*j*. *j* ∈ {*m..m* + *k*}}
by (*rule cSup-subset-mono*) *auto*
let *?S* = {*f j x*| *j*. *m* ≤ *j*}
show ((λk. *Sup* {*f j x* |*j*. *j* ∈ {*m..m* + *k*}}) ⟶ *Sup ?S*) *sequentially*
proof (*rule LIMSEQ-I*, *goal-cases*)
case *r*: (*1 r*)
have ∃ *y*∈*?S*. *Sup ?S* − *r* < *y*
by (*subst less-cSup-iff*[*symmetric*]) (*auto simp*: *r* ‹*x*∈*s*›)
then obtain *N* **where** *N*: *Sup ?S* − *r* < *f N x m* ≤ *N*
by *blast*

show *?case*
apply (*rule-tac x=N* **in** *exI*)
apply *safe*
proof *goal-cases*
case *prems*: (*1 n*)
have ∗: ⋀*y ix*. *Sup ?S* − *r* < *y* ⟶ *ix* ≤ *Sup ?S* ⟶ *y* ≤ *ix* ⟶ |*ix* − *Sup ?S*| < *r*
by *arith*
show *?case*

```
        apply simp
        apply (rule ∗[rule-format, OF N(1)])
        apply (rule cSup-subset-mono, auto simp: ‹x∈s›) []
        apply (subst cSup-upper)
        using N prems
        apply auto
        done
    qed
  qed
 qed
 note inc1 = conjunctD2[OF this]

 have g integrable-on s ∧
  ((λk. integral s (λx. Inf {f j x |j. k ≤ j})) ⟶ integral s g) sequentially
  apply (rule monotone-convergence-increasing,safe)
  apply fact
 proof −
  show bounded {integral s (λx. Inf {f j x |j. k ≤ j}) |k. True}
   unfolding bounded-iff apply(rule-tac x=integral s h in exI)
  proof safe
   fix k :: nat
   show norm (integral s (λx. Inf {f j x |j. k ≤ j})) ≤ integral s h
    apply (rule integral-norm-bound-integral)
    apply fact+
    unfolding real-norm-def
    apply rule
    apply (rule cInf-abs-ge)
    using assms(3)
    apply auto
    done
  qed
  fix k :: nat and x
  assume x: x ∈ s

  have ∗: ⋀x y::real. x ≥ − y ⟹ − x ≤ y by auto
  show Inf {f j x |j. k ≤ j} ≤ Inf {f j x |j. Suc k ≤ j}
   by (intro cInf-superset-mono) (auto simp: ‹x∈s›)

  show (λk::nat. Inf {f j x |j. k ≤ j}) ⟶ g x
  proof (rule LIMSEQ-I, goal-cases)
   case r: (1 r)
   then have 0<r/2
    by auto
  from assms(4)[THEN bspec, THEN LIMSEQ-D, OF x this] guess N .. note
N = this
   show ?case
    apply (rule-tac x=N in exI)
    apply safe
    unfolding real-norm-def
```

```
      apply (rule le-less-trans[of - r/2])
      apply (rule cInf-asclose)
      apply safe
      defer
      apply (rule less-imp-le)
      using N r
      apply auto
      done
  qed
 qed
 note inc2 = conjunctD2[OF this]

 have g integrable-on s ∧
  ((λk. integral s (λx. Sup {f j x |j. k ≤ j})) ⟶ integral s g) sequentially
  apply (rule monotone-convergence-decreasing,safe)
  apply fact
 proof −
  show bounded {integral s (λx. Sup {f j x |j. k ≤ j}) |k. True}
   unfolding bounded-iff
   apply (rule-tac x=integral s h in exI)
  proof safe
   fix k :: nat
   show norm (integral s (λx. Sup {f j x |j. k ≤ j})) ≤ integral s h
    apply (rule integral-norm-bound-integral)
    apply fact+
    unfolding real-norm-def
    apply rule
    apply (rule cSup-abs-le)
    using assms(3)
    apply auto
    done
  qed
  fix k :: nat
  fix x
  assume x: x ∈ s

  show Sup {f j x |j. k ≤ j} ≥ Sup {f j x |j. Suc k ≤ j}
   by (rule cSup-subset-mono) (auto simp: ‹x∈s›)
  show ((λk. Sup {f j x |j. k ≤ j}) ⟶ g x) sequentially
  proof (rule LIMSEQ-I, goal-cases)
   case r: (1 r)
   then have 0<r/2
    by auto
  from assms(4)[THEN bspec, THEN LIMSEQ-D, OF x this] guess N .. note
N=this
   show ?case
    apply (rule-tac x=N in exI,safe)
    unfolding real-norm-def
    apply (rule le-less-trans[of - r/2])
```

```
      apply (rule cSup-asclose)
      apply safe
      defer
      apply (rule less-imp-le)
      using N r
      apply auto
      done
    qed
  qed
  note dec2 = conjunctD2[OF this]

  show g integrable-on s by fact
  show ((λk. integral s (f k)) ⟶ integral s g) sequentially
  proof (rule LIMSEQ-I, goal-cases)
    case r: (1 r)
   from LIMSEQ-D [OF inc2(2) r] guess N1 .. note N1=this[unfolded real-norm-def]
   from LIMSEQ-D [OF dec2(2) r] guess N2 .. note N2=this[unfolded real-norm-def]
    show ?case
    proof (rule-tac x=N1+N2 in exI, safe)
      fix n
      assume n: n ≥ N1 + N2
     have ∗: ⋀i0 i i1 g. |i0 − g| < r ⟶ |i1 − g| < r ⟶ i0 ≤ i ⟶ i ≤ i1 ⟶
|i − g| < r
       by arith
      show norm (integral s (f n) − integral s g) < r
       unfolding real-norm-def
      proof (rule ∗[rule-format,OF N1[rule-format] N2[rule-format], of n n])
        show integral s (λx. Inf {f j x |j. n ≤ j}) ≤ integral s (f n)
          by (rule integral-le[OF dec1(1) assms(1)]) (auto intro!: cInf-lower)
        show integral s (f n) ≤ integral s (λx. Sup {f j x |j. n ≤ j})
          by (rule integral-le[OF assms(1) inc1(1)]) (auto intro!: cSup-upper)
      qed (insert n, auto)
    qed
  qed
qed

lemma has-integral-dominated-convergence:
  fixes f :: nat ⇒ 'n::euclidean-space ⇒ real
  assumes ⋀k. (f k has-integral y k) s h integrable-on s
    ⋀k. ∀x∈s. norm (f k x) ≤ h x ∀x∈s. (λk. f k x) ⟶ g x
    and x: y ⟶ x
  shows (g has-integral x) s
proof −
  have int-f: ⋀k. (f k) integrable-on s
    using assms by (auto simp: integrable-on-def)
  have (g has-integral (integral s g)) s
    by (intro integrable-integral dominated-convergence[OF int-f assms(2)]) fact+
  moreover have integral s g = x
  proof (rule LIMSEQ-unique)
```

show (λ*i*. *integral s* (*f i*)) ⟶ *x*
using *integral-unique*[*OF assms*(*1*)] *x* **by** *simp*
show (λ*i*. *integral s* (*f i*)) ⟶ *integral s g*
by (*intro dominated-convergence*[*OF int-f assms*(*2*)]) *fact+*
qed
ultimately show *?thesis*
by *simp*
qed

41.69 Compute a double integral using iterated integrals and switching the order of integration

lemma *setcomp-dot1*: {*z*. *P* (*z* · (*i*,*0*))} = {(*x*,*y*). *P*(*x* · *i*)}
by *auto*

lemma *setcomp-dot2*: {*z*. *P* (*z* · (*0*,*i*))} = {(*x*,*y*). *P*(*y* · *i*)}
by *auto*

lemma *Sigma-Int-Paircomp1*: (*Sigma A B*) ∩ {(*x*, *y*). *P x*} = *Sigma* (*A* ∩ {*x*. *P x*}) *B*
by *blast*

lemma *Sigma-Int-Paircomp2*: (*Sigma A B*) ∩ {(*x*, *y*). *P y*} = *Sigma A* (λ*z*. *B z* ∩ {*y*. *P y*})
by *blast*

lemma *continuous-on-imp-integrable-on-Pair1*:
fixes *f* :: - ⇒ *'b*::*banach*
assumes *con*: *continuous-on* (*cbox* (*a*,*c*) (*b*,*d*)) *f* **and** *x*: *x* ∈ *cbox a b*
shows (λ*y*. *f* (*x*, *y*)) *integrable-on* (*cbox c d*)
proof −
have *f* ∘ (λ*y*. (*x*, *y*)) *integrable-on* (*cbox c d*)
apply (*rule integrable-continuous*)
apply (*rule continuous-on-compose* [*OF* - *continuous-on-subset* [*OF con*]])
using *x*
apply (*auto intro*: *continuous-on-Pair continuous-on-const continuous-on-id continuous-on-subset con*)
done
then show *?thesis*
by (*simp add*: *o-def*)
qed

lemma *integral-integrable-2dim*:
fixes *f* :: (*'a*::*euclidean-space* ∗ *'b*::*euclidean-space*) ⇒ *'c*::*banach*
assumes *continuous-on* (*cbox* (*a*,*c*) (*b*,*d*)) *f*
shows (λ*x*. *integral* (*cbox c d*) (λ*y*. *f* (*x*,*y*))) *integrable-on cbox a b*
proof (*cases content*(*cbox c d*) = *0*)
case *True*
then show *?thesis*

```
    by (simp add: True integrable-const)
next
  case False
  have uc: uniformly-continuous-on (cbox (a,c) (b,d)) f
    by (simp add: assms compact-cbox compact-uniformly-continuous)
  { fix x::'a and e::real
    assume x: x ∈ cbox a b and e: 0 < e
    then have e2-gt: 0 < e / 2 / content (cbox c d) and e2-less: e / 2 / content
(cbox c d) * content (cbox c d) < e
      by (auto simp: False content-lt-nz e)
    then obtain dd
    where dd: ⋀x x'. ⟦x∈cbox (a, c) (b, d); x'∈cbox (a, c) (b, d); norm (x' − x)
< dd⟧
                    ⟹ norm (f x' − f x) ≤ e / (2 * content (cbox c d))  dd>0
      using uc [unfolded uniformly-continuous-on-def, THEN spec, of e / (2 *
content (cbox c d))]
      by (auto simp: dist-norm intro: less-imp-le)
    have ∃ delta>0. ∀ x'∈cbox a b. norm (x' − x) < delta ⟶ norm (integral (cbox
c d) (λu. f (x', u) − f (x, u))) < e
      apply (rule-tac x=dd in exI)
      using dd e2-gt assms x
      apply clarify
      apply (rule le-less-trans [OF - e2-less])
      apply (rule integrable-bound)
      apply (auto intro: integrable-diff continuous-on-imp-integrable-on-Pair1)
      done
  } note * = this
  show ?thesis
    apply (rule integrable-continuous)
   apply (simp add: * continuous-on-iff dist-norm integral-diff [symmetric] continuous-on-imp-integrable-on-Pa
[OF assms])
    done
qed

lemma norm-diff2: ⟦y = y1 + y2; x = x1 + x2; e = e1 + e2; norm(y1 − x1)
≤ e1; norm(y2 − x2) ≤ e2⟧
          ⟹ norm(y − x) ≤ e
using norm-triangle-mono [of y1 − x1 e1 y2 − x2 e2]
  by (simp add: add-diff-add)

lemma integral-split:
  fixes f :: 'a::euclidean-space ⇒ 'b::{real-normed-vector,complete-space}
  assumes f: f integrable-on (cbox a b)
      and k: k ∈ Basis
  shows integral (cbox a b) f =
          integral (cbox a b ∩ {x. x•k ≤ c}) f +
          integral (cbox a b ∩ {x. x•k ≥ c}) f
  apply (rule integral-unique [OF has-integral-split [where c=c]])
  using k f
```

apply (*auto simp*: *has-integral-integral* [*symmetric*])
done

lemma *integral-swap-operative*:
fixes $f :: ('a::euclidean\text{-}space * 'b::euclidean\text{-}space) \Rightarrow 'c::banach$
assumes f: *continuous-on s f* **and** e: $0 < e$
shows *operative*$(op \wedge)$
$(\lambda k.\ \forall\ a\ b\ c\ d.$
cbox (a,c) $(b,d) \subseteq k \wedge$ *cbox* (a,c) $(b,d) \subseteq s$
$\longrightarrow$ *norm*(*integral* (*cbox* (a,c) (b,d)) f $-$
integral (*cbox a b*) $(\lambda x.$ *integral* (*cbox c d*) $(\lambda y.\ f((x,y)))))$
$\leq e *$ *content* (*cbox* (a,c) (b,d))))
proof (*auto simp*: *operative-def*)
fix $a::'a$ **and** $c::'b$ **and** $b::'a$ **and** $d::'b$ **and** $u::'a$ **and** $v::'a$ **and** $w::'b$ **and** $z::'b$
assume *c0*: *content* (*cbox* $(a,\ c)$ $(b,\ d)$) $= 0$
and *cb1*: *cbox* $(u,\ w)$ $(v,\ z) \subseteq$ *cbox* $(a,\ c)$ $(b,\ d)$
and *cb2*: *cbox* $(u,\ w)$ $(v,\ z) \subseteq s$
have *c0'*: *content* (*cbox* $(u,\ w)$ $(v,\ z)$) $= 0$
by (*fact content-0-subset* [*OF c0 cb1*])
show *norm* (*integral* (*cbox* (u,w) (v,z)) f $-$ *integral* (*cbox u v*) $(\lambda x.$ *integral* (*cbox w z*) $(\lambda y.\ f\ (x,\ y))))$
$\leq e *$ *content* (*cbox* (u,w) (v,z))
using *content-cbox-pair-eq0-D* [*OF c0'*]
by (*force simp add*: *c0'*)
next
fix $a::'a$ **and** $c::'b$ **and** $b::'a$ **and** $d::'b$
and $M::real$ **and** $i::'a$ **and** $j::'b$
and $u::'a$ **and** $v::'a$ **and** $w::'b$ **and** $z::'b$
assume *ij*: $(i,j) \in Basis$
and *n1*: $\forall\ a'\ b'\ c'\ d'.$
cbox (a',c') $(b',d') \subseteq$ *cbox* (a,c) $(b,d) \wedge$
cbox (a',c') $(b',d') \subseteq \{x.\ x \cdot (i,j) \leq M\} \wedge$ *cbox* (a',c') $(b',d') \subseteq s$
$\longrightarrow$
norm (*integral* (*cbox* (a',c') (b',d')) f $-$ *integral* (*cbox a' b'*) $(\lambda x.$ *integral* (*cbox c' d'*) $(\lambda y.\ f\ (x,y))))$
$\leq e *$ *content* (*cbox* (a',c') (b',d'))
and *n2*: $\forall\ a'\ b'\ c'\ d'.$
cbox (a',c') $(b',d') \subseteq$ *cbox* (a,c) $(b,d) \wedge$
cbox (a',c') $(b',d') \subseteq \{x.\ M \leq x \cdot (i,j)\} \wedge$ *cbox* (a',c') $(b',d') \subseteq s$
$\longrightarrow$
norm (*integral* (*cbox* (a',c') (b',d')) f $-$ *integral* (*cbox a' b'*) $(\lambda x.$ *integral* (*cbox c' d'*) $(\lambda y.\ f\ (x,y))))$
$\leq e *$ *content* (*cbox* (a',c') (b',d'))
and *subs*: *cbox* (u,w) $(v,z) \subseteq$ *cbox* (a,c) (b,d) *cbox* (u,w) $(v,z) \subseteq s$
have *fcont*: *continuous-on* (*cbox* $(u,\ w)$ $(v,\ z)$) f
using *assms*(*1*) *continuous-on-subset subs*(*2*) **by** *blast*
then have *fint*: f *integrable-on cbox* $(u,\ w)$ $(v,\ z)$
by (*metis integrable-continuous*)
consider $i \in Basis\ j{=}0 \mid j \in Basis\ i{=}0$ **using** *ij*

by (*auto simp*: *Euclidean-Space.Basis-prod-def*)
then show *norm* (*integral* (*cbox* (*u*,*w*) (*v*,*z*)) *f* − *integral* (*cbox u v*) (λ*x*. *integral* (*cbox w z*) (λ*y*. *f* (*x*,*y*))))
≤ *e* ∗ *content* (*cbox* (*u*,*w*) (*v*,*z*)) (**is** *?normle*)
proof *cases*
case *1*
then have *e*: *e* ∗ *content* (*cbox* (*u*, *w*) (*v*, *z*)) =
e ∗ (*content* (*cbox u v* ∩ {*x*. *x* · *i* ≤ *M*}) ∗ *content* (*cbox w z*)) +
e ∗ (*content* (*cbox u v* ∩ {*x*. *M* ≤ *x* · *i*}) ∗ *content* (*cbox w z*))
by (*simp add*: *content-split* [**where** *c*=*M*] *content-Pair algebra-simps*)
have ∗: *integral* (*cbox u v*) (λ*x*. *integral* (*cbox w z*) (λ*y*. *f* (*x*, *y*))) =
integral (*cbox u v* ∩ {*x*. *x* · *i* ≤ *M*}) (λ*x*. *integral* (*cbox w z*) (λ*y*. *f* (*x*, *y*))) +
integral (*cbox u v* ∩ {*x*. *M* ≤ *x* · *i*}) (λ*x*. *integral* (*cbox w z*) (λ*y*. *f* (*x*, *y*)))
using *1 f subs integral-integrable-2dim continuous-on-subset*
by (*blast intro*: *integral-split*)
show *?normle*
apply (*rule norm-diff2* [*OF integral-split* [**where** *c*=*M*, *OF fint ij*] ∗ *e*])
using *1 subs*
apply (*simp-all add*: *cbox-Pair-eq setcomp-dot1* [*of* λ*u*. *M*≤*u*] *setcomp-dot1* [*of* λ*u*. *u*≤*M*] *Sigma-Int-Paircomp1*)
apply (*simp-all add*: *interval-split ij*)
apply (*simp-all add*: *cbox-Pair-eq* [*symmetric*] *content-Pair* [*symmetric*])
apply (*force simp add*: *interval-split* [*symmetric*] *intro*!: *n1* [*rule-format*])
apply (*force simp add*: *interval-split* [*symmetric*] *intro*!: *n2* [*rule-format*])
done
next
case *2*
then have *e*: *e* ∗ *content* (*cbox* (*u*, *w*) (*v*, *z*)) =
e ∗ (*content* (*cbox u v*) ∗ *content* (*cbox w z* ∩ {*x*. *x* · *j* ≤ *M*})) +
e ∗ (*content* (*cbox u v*) ∗ *content* (*cbox w z* ∩ {*x*. *M* ≤ *x* · *j*}))
by (*simp add*: *content-split* [**where** *c*=*M*] *content-Pair algebra-simps*)
have (λ*x*. *integral* (*cbox w z* ∩ {*x*. *x* · *j* ≤ *M*}) (λ*y*. *f* (*x*, *y*))) *integrable-on cbox u v*
(λ*x*. *integral* (*cbox w z* ∩ {*x*. *M* ≤ *x* · *j*}) (λ*y*. *f* (*x*, *y*))) *integrable-on cbox u v*
using *2 subs*
apply (*simp-all add*: *interval-split*)
apply (*rule-tac* [!] *integral-integrable-2dim* [*OF continuous-on-subset* [*OF f*]])
apply (*auto simp*: *cbox-Pair-eq interval-split* [*symmetric*])
done
with *2* **have** ∗: *integral* (*cbox u v*) (λ*x*. *integral* (*cbox w z*) (λ*y*. *f* (*x*, *y*))) =
integral (*cbox u v*) (λ*x*. *integral* (*cbox w z* ∩ {*x*. *x* · *j* ≤ *M*}) (λ*y*. *f* (*x*, *y*))) +
integral (*cbox u v*) (λ*x*. *integral* (*cbox w z* ∩ {*x*. *M* ≤ *x* · *j*}) (λ*y*. *f* (*x*, *y*)))
by (*simp add*: *integral-add* [*symmetric*] *integral-split* [*symmetric*]
continuous-on-imp-integrable-on-Pair1 [*OF fcont*] *cong*: *integral-cong*)

```
    show ?normle
      apply (rule norm-diff2 [OF integral-split [where c=M, OF fint ij] * e])
      using 2 subs
      apply (simp-all add: cbox-Pair-eq setcomp-dot2 [of λu. M≤u] setcomp-dot2
[of λu. u≤M] Sigma-Int-Paircomp2)
      apply (simp-all add: interval-split ij)
      apply (simp-all add: cbox-Pair-eq [symmetric] content-Pair [symmetric])
      apply (force simp add: interval-split [symmetric] intro!: n1 [rule-format])
      apply (force simp add: interval-split [symmetric] intro!: n2 [rule-format])
      done
  qed
qed

lemma integral-Pair-const:
    integral (cbox (a,c) (b,d)) (λx. k) =
     integral (cbox a b) (λx. integral (cbox c d) (λy. k))
  by (simp add: content-Pair)

lemma norm-minus2: norm (x1−x2, y1−y2) = norm (x2−x1, y2−y1)
  by (simp add: norm-minus-eqI)

lemma integral-prod-continuous:
  fixes f :: ('a::euclidean-space * 'b::euclidean-space) ⇒ 'c::banach
  assumes continuous-on (cbox (a,c) (b,d)) f (is continuous-on ?CBOX f)
    shows integral (cbox (a,c) (b,d)) f = integral (cbox a b) (λx. integral (cbox c
d) (λy. f(x,y)))
proof (cases content ?CBOX = 0)
  case True
  then show ?thesis
    by (auto simp: content-Pair)
next
  case False
  then have cbp: content ?CBOX > 0
    using content-lt-nz by blast
  have norm (integral ?CBOX f − integral (cbox a b) (λx. integral (cbox c d) (λy.
f (x,y)))) = 0
  proof (rule dense-eq0-I, simp)
    fix e::real  assume 0 < e
    with cbp have e': 0 < e / content ?CBOX
      by simp
    have f-int-acbd: f integrable-on cbox (a,c) (b,d)
      by (rule integrable-continuous [OF assms])
    { fix p
      assume p: p division-of cbox (a,c) (b,d)
       note opd1 = operative-division-and [OF integral-swap-operative [OF assms
e'], THEN iffD1,
                        THEN spec, THEN spec, THEN spec, THEN spec, of p (a,c)
(b,d) a c b d]
      have (⋀t u v w z.
```

```
          ⟦t ∈ p; cbox (u,w) (v,z) ⊆ t;  cbox (u,w) (v,z) ⊆ cbox (a,c) (b,d)⟧ ⟹
           norm (integral (cbox (u,w) (v,z)) f − integral (cbox u v) (λx. integral
(cbox w z) (λy. f (x,y))))
           ≤ e * content (cbox (u,w) (v,z)) / content?CBOX)
          ⟹
          norm (integral ?CBOX f − integral (cbox a b) (λx. integral (cbox c d)
(λy. f (x,y)))) ≤ e
       using opd1 [OF p] False  by simp
   } note op-acbd = this
   { fix k::real and p and u::'a and v w and z::'b and t1 t2 l
     assume k: 0 < k
       and nf: ⋀x y u v.
             ⟦x ∈ cbox a b; y ∈ cbox c d; u ∈ cbox a b; v∈cbox c d; norm (u−x,
v−y) < k⟧
             ⟹ norm (f(u,v) − f(x,y)) < e / (2 * (content ?CBOX))
       and p-acbd: p tagged-division-of cbox (a,c) (b,d)
       and fine: (λx. ball x k) fine p  ((t1,t2), l) ∈ p
       and uwvz-sub: cbox (u,w) (v,z) ⊆ l cbox (u,w) (v,z) ⊆ cbox (a,c) (b,d)
     have t: t1 ∈ cbox a b t2 ∈ cbox c d
       by (meson fine p-acbd cbox-Pair-iff tag-in-interval)+
     have ls: l ⊆ ball (t1,t2) k
       using fine by (simp add: fine-def Ball-def)
     { fix x1 x2
       assume xuvwz: x1 ∈ cbox u v x2 ∈ cbox w z
       then have x: x1 ∈ cbox a b x2 ∈ cbox c d
         using uwvz-sub by auto
       have norm (x1 − t1, x2 − t2) < k
         using xuvwz ls uwvz-sub unfolding ball-def
         by (force simp add: cbox-Pair-eq dist-norm norm-minus2)
       then have norm (f (x1,x2) − f (t1,t2)) ≤ e / content ?CBOX / 2
         using nf [OF t x]  by simp
     } note nf' = this
     have f-int-uwvz: f integrable-on cbox (u,w) (v,z)
       using f-int-acbd uwvz-sub integrable-on-subcbox by blast
     have f-int-uv: ⋀x. x ∈ cbox u v ⟹ (λy. f (x,y)) integrable-on cbox w z
       using assms continuous-on-subset uwvz-sub
       by (blast intro: continuous-on-imp-integrable-on-Pair1)
      have 1: norm (integral (cbox (u,w) (v,z)) f − integral (cbox (u,w) (v,z))
(λx. f (t1,t2)))
       ≤ e * content (cbox (u,w) (v,z)) / content ?CBOX / 2
       apply (simp only: integral-diff [symmetric] f-int-uwvz integrable-const)
      apply (rule order-trans [OF integrable-bound [of e / content ?CBOX / 2]])
       using cbp e' nf'
       apply (auto simp: integrable-diff f-int-uwvz integrable-const)
       done
     have int-integrable: (λx. integral (cbox w z) (λy. f (x, y))) integrable-on cbox
u v
       using assms integral-integrable-2dim continuous-on-subset uwvz-sub(2) by
blast
```

have *normint-wz*:
⋀*x*. *x* ∈ *cbox u v* ⟹
norm (*integral* (*cbox w z*) (λ*y*. *f* (*x*, *y*)) − *integral* (*cbox w z*) (λ*y*. *f* (*t1*, *t2*)))
≤ *e* ∗ *content* (*cbox w z*) / *content* (*cbox* (*a*, *c*) (*b*, *d*)) / *2*
apply (*simp only*: *integral-diff* [*symmetric*] *f-int-uv integrable-const*)
apply (*rule order-trans* [*OF integrable-bound* [*of e* / *content ?CBOX* / *2*]])
using *cbp e′ nf′*
apply (*auto simp*: *integrable-diff f-int-uv integrable-const*)
done
have *norm* (*integral* (*cbox u v*)
(λ*x*. *integral* (*cbox w z*) (λ*y*. *f* (*x*,*y*)) − *integral* (*cbox w z*) (λ*y*. *f* (*t1*,*t2*))))
≤ *e* ∗ *content* (*cbox w z*) / *content ?CBOX* / *2* ∗ *content* (*cbox u v*)
apply (*rule integrable-bound* [*OF* - - *normint-wz*])
using *cbp e′*
apply (*auto simp*: *divide-simps content-pos-le integrable-diff int-integrable integrable-const*)
done
also have ... ≤ *e* ∗ *content* (*cbox* (*u*,*w*) (*v*,*z*)) / *content ?CBOX* / *2*
by (*simp add*: *content-Pair divide-simps*)
finally have *2*: *norm* (*integral* (*cbox u v*) (λ*x*. *integral* (*cbox w z*) (λ*y*. *f* (*x*,*y*))) −
integral (*cbox u v*) (λ*x*. *integral* (*cbox w z*) (λ*y*. *f* (*t1*,*t2*))))
≤ *e* ∗ *content* (*cbox* (*u*,*w*) (*v*,*z*)) / *content ?CBOX* / *2*
by (*simp only*: *integral-diff* [*symmetric*] *int-integrable integrable-const*)
have *norm-xx*: ⟦*x′* = *y′*; *norm*(*x* − *x′*) ≤ *e*/*2*; *norm*(*y* − *y′*) ≤ *e*/*2*⟧ ⟹ *norm*(*x* − *y*) ≤ *e* **for** *x*::*′c* **and** *y x′ y′ e*
using *norm-triangle-mono* [*of x*−*y′ e*/*2 y′*−*y e*/*2*] *real-sum-of-halves*
by (*simp add*: *norm-minus-commute*)
have *norm* (*integral* (*cbox* (*u*,*w*) (*v*,*z*)) *f* − *integral* (*cbox u v*) (λ*x*. *integral* (*cbox w z*) (λ*y*. *f* (*x*,*y*))))
≤ *e* ∗ *content* (*cbox* (*u*,*w*) (*v*,*z*)) / *content ?CBOX*
by (*rule norm-xx* [*OF integral-Pair-const 1 2*])
} **note** ∗ = *this*
show *norm* (*integral ?CBOX f* − *integral* (*cbox a b*) (λ*x*. *integral* (*cbox c d*) (λ*y*. *f* (*x*,*y*)))) ≤ *e*
using *compact-uniformly-continuous* [*OF assms compact-cbox*]
apply (*simp add*: *uniformly-continuous-on-def dist-norm*)
apply (*drule-tac x*=*e* / *2* / *content?CBOX* **in** *spec*)
using *cbp* ‹*0* < *e*›
apply (*auto simp*: *zero-less-mult-iff*)
apply (*rename-tac k*)
apply (*rule-tac e1*=*k* **in** *fine-division-exists* [*OF gauge-ball*, **where** *a* = (*a*,*c*) **and** *b* = (*b*,*d*)])
apply *assumption*
apply (*rule op-acbd*)
apply (*erule division-of-tagged-division*)
using ∗

```
    apply auto
    done
 qed
 then show ?thesis
   by simp
qed

lemma swap-continuous:
 assumes continuous-on (cbox (a,c) (b,d)) (λ(x,y). f x y)
   shows continuous-on (cbox (c,a) (d,b)) (λ(x, y). f y x)
proof −
 have (λ(x, y). f y x) = (λ(x, y). f x y) ∘ prod.swap
   by auto
 then show ?thesis
   apply (rule ssubst)
   apply (rule continuous-on-compose)
   apply (simp add: split-def)
   apply (rule continuous-intros | simp add: assms)+
   done
qed

lemma integral-swap-2dim:
 fixes f :: ['a::euclidean-space, 'b::euclidean-space] ⇒ 'c::banach
 assumes continuous-on (cbox (a,c) (b,d)) (λ(x,y). f x y)
   shows integral (cbox (a, c) (b, d)) (λ(x, y). f x y) = integral (cbox (c, a) (d,
b)) (λ(x, y). f y x)
proof −
 have ((λ(x, y). f x y) has-integral integral (cbox (c, a) (d, b)) (λ(x, y). f y x))
(prod.swap ‘ (cbox (c, a) (d, b)))
  apply (rule has-integral-twiddle [of 1 prod.swap prod.swap λ(x,y). f y x integral
(cbox (c, a) (d, b)) (λ(x, y). f y x), simplified])
   apply (auto simp: isCont-swap content-Pair has-integral-integral [symmetric]
integrable-continuous swap-continuous assms)
   done
 then show ?thesis
  by force
qed

theorem integral-swap-continuous:
 fixes f :: ['a::euclidean-space, 'b::euclidean-space] ⇒ 'c::banach
 assumes continuous-on (cbox (a,c) (b,d)) (λ(x,y). f x y)
   shows integral (cbox a b) (λx. integral (cbox c d) (f x)) =
         integral (cbox c d) (λy. integral (cbox a b) (λx. f x y))
proof −
 have integral (cbox a b) (λx. integral (cbox c d) (f x)) = integral (cbox (a, c)
(b, d)) (λ(x, y). f x y)
   using integral-prod-continuous [OF assms] by auto
 also have ... = integral (cbox (c, a) (d, b)) (λ(x, y). f y x)
   by (rule integral-swap-2dim [OF assms])
```

also have ... = *integral* (*cbox c d*) (λ*y*. *integral* (*cbox a b*) (λ*x*. *f x y*))
using *integral-prod-continuous* [*OF swap-continuous*] *assms*
by *auto*
finally show *?thesis* **.**
qed

end

42 Instantiates the finite Cartesian product of Euclidean spaces as a Euclidean space.

theory *Cartesian-Euclidean-Space*
imports *Finite-Cartesian-Product Integration*
begin

lemma *delta-mult-idempotent*:
(*if k=a then 1 else* (*0::'a::semiring-1*)) ∗ (*if k=a then 1 else 0*) = (*if k=a then 1 else 0*)
by (*cases k=a*) *auto*

lemma *setsum-UNIV-sum*:
fixes *g* :: *'a::finite* + *'b::finite* ⇒ -
shows (∑ *x*∈*UNIV*. *g x*) = (∑ *x*∈*UNIV*. *g* (*Inl x*)) + (∑ *x*∈*UNIV*. *g* (*Inr x*))
apply (*subst UNIV-Plus-UNIV* [*symmetric*])
apply (*subst setsum.Plus*)
apply *simp-all*
done

lemma *setsum-mult-product*:
setsum h {..<*A* ∗ *B* :: *nat*} = (∑ *i*∈{..<*A*}. ∑ *j*∈{..<*B*}. *h* (*j* + *i* ∗ *B*))
unfolding *setsum-nat-group*[*of h B A*, *unfolded atLeast0LessThan*, *symmetric*]
proof (*rule setsum.cong*, *simp*, *rule setsum.reindex-cong*)
fix *i*
show *inj-on* (λ*j*. *j* + *i* ∗ *B*) {..<*B*} **by** (*auto intro*!: *inj-onI*)
show {*i* ∗ *B*..<*i* ∗ *B* + *B*} = (λ*j*. *j* + *i* ∗ *B*) ‘ {..<*B*}
proof *safe*
fix *j* **assume** *j* ∈ {*i* ∗ *B*..<*i* ∗ *B* + *B*}
then show *j* ∈ (λ*j*. *j* + *i* ∗ *B*) ‘ {..<*B*}
by (*auto intro*!: *image-eqI*[*of - - j* − *i* ∗ *B*])
qed *simp*
qed *simp*

42.1 Basic componentwise operations on vectors.

instantiation *vec* :: (*times*, *finite*) *times*
begin

definition *op* ∗ ≡ (λ *x y*. (χ *i*. (*x*$*i*) ∗ (*y*$*i*)))

instance ..

end

instantiation *vec* :: (*one*, *finite*) *one*
begin

definition *1* ≡ (χ *i*. *1*)
instance ..

end

instantiation *vec* :: (*ord*, *finite*) *ord*
begin

definition $x \leq y \longleftrightarrow (\forall\, i.\ x\$i \leq y\$i)$
definition $x < (y::'a\hat{}'b) \longleftrightarrow x \leq y \wedge \neg\ y \leq x$
instance ..

end

The ordering on one-dimensional vectors is linear.

class *cart-one* =
  assumes *UNIV-one*: *card* (*UNIV* :: *'a set*) = *Suc 0*
begin

subclass *finite*
proof
  from *UNIV-one* **show** *finite* (*UNIV* :: *'a set*)
    by (*auto intro*!: *card-ge-0-finite*)
qed

end

instance *vec*:: (*order*, *finite*) *order*
  by *standard* (*auto simp*: *less-eq-vec-def less-vec-def vec-eq-iff*
      intro: *order.trans order.antisym order.strict-implies-order*)

instance *vec* :: (*linorder*, *cart-one*) *linorder*
proof
  obtain *a* :: *'b* **where** *all*: $\bigwedge P.\ (\forall\, i.\ P\ i) \longleftrightarrow P\ a$
  proof −
    have *card* (*UNIV* :: *'b set*) = *Suc 0* **by** (*rule UNIV-one*)
    then obtain *b* :: *'b* **where** *UNIV* = {*b*} **by** (*auto iff*: *card-Suc-eq*)
    then have $\bigwedge P.\ (\forall\, i{\in}UNIV.\ P\ i) \longleftrightarrow P\ b$ **by** *auto*
    then show *thesis* **by** (*auto intro*: *that*)
  qed
  fix *x y* :: *'a*^*'b*::*cart-one*
  note [*simp*] = *less-eq-vec-def less-vec-def all vec-eq-iff field-simps*

show $x \leq y \vee y \leq x$ **by** *auto*
qed

Constant Vectors

definition *vec x* = $(\chi\ i.\ x)$

lemma *interval-cbox-cart*: $\{a::real\hat{}'n..b\} = cbox\ a\ b$
by (*auto simp add*: *less-eq-vec-def mem-box Basis-vec-def inner-axis*)

Also the scalar-vector multiplication.

definition *vector-scalar-mult*:: *'a*::*times* ⇒ *'a* ^ *'n* ⇒ *'a* ^ *'n* (**infixl** ∗*s* *70*)
where *c* ∗*s x* = $(\chi\ i.\ c * (x\$i))$

42.2 A naive proof procedure to lift really trivial arithmetic stuff from the basis of the vector space.

lemma *setsum-cong-aux*:
$(\bigwedge x.\ x \in A \Longrightarrow f\ x = g\ x) \Longrightarrow setsum\ f\ A = setsum\ g\ A$
by (*auto intro*: *setsum.cong*)

hide-fact (**open**) *setsum-cong-aux*

method-setup *vector* = ‹
let
val ss1 =
simpset-of (*put-simpset HOL-basic-ss* @{*context*}
addsimps [@{*thm setsum.distrib*} *RS sym*,
@{*thm setsum-subtractf*} *RS sym*, @{*thm setsum-right-distrib*},
@{*thm setsum-left-distrib*}, @{*thm setsum-negf*} *RS sym*])
val ss2 =
simpset-of (@{*context*} *addsimps*
[@{*thm plus-vec-def*}, @{*thm times-vec-def*},
@{*thm minus-vec-def*}, @{*thm uminus-vec-def*},
@{*thm one-vec-def*}, @{*thm zero-vec-def*}, @{*thm vec-def*},
@{*thm scaleR-vec-def*},
@{*thm vec-lambda-beta*}, @{*thm vector-scalar-mult-def*}])
fun vector-arith-tac ctxt ths =
simp-tac (*put-simpset ss1 ctxt*)
THEN' (*fn i => resolve-tac ctxt* @{*thms Cartesian-Euclidean-Space.setsum-cong-aux*}
i
ORELSE resolve-tac ctxt @{*thms setsum.neutral*} *i*
ORELSE simp-tac (*put-simpset HOL-basic-ss ctxt addsimps* [@{*thm vec-eq-iff*}]) *i*)
(∗ *THEN' TRY o clarify-tac HOL-cs THEN'* (*TRY o rtac* @{*thm iffI*}) ∗)
THEN' asm-full-simp-tac (*put-simpset ss2 ctxt addsimps ths*)
in
Attrib.thms >> (*fn ths => fn ctxt => SIMPLE-METHOD'* (*vector-arith-tac ctxt ths*))
end

› *lift trivial vector statements to real arith statements*

lemma *vec-0*[*simp*]: *vec 0 = 0* **by** *vector*
lemma *vec-1*[*simp*]: *vec 1 = 1* **by** *vector*

lemma *vec-inj*[*simp*]: *vec x = vec y ⟷ x = y* **by** *vector*

lemma *vec-in-image-vec*: *vec x ∈ (vec ‘ S) ⟷ x ∈ S* **by** *auto*

lemma *vec-add*: *vec*(*x + y*) = *vec x + vec y* **by** *vector*
lemma *vec-sub*: *vec*(*x − y*) = *vec x − vec y* **by** *vector*
lemma *vec-cmul*: *vec*(*c ∗ x*) = *c ∗s vec x* **by** *vector*
lemma *vec-neg*: *vec*(*− x*) = *− vec x* **by** *vector*

lemma *vec-setsum*:
assumes *finite S*
shows *vec*(*setsum f S*) = *setsum* (*vec ∘ f*) *S*
using *assms*
proof *induct*
case *empty*
then show *?case* **by** *simp*
next
case *insert*
then show *?case* **by** (*auto simp add*: *vec-add*)
qed

Obvious "component-pushing".

lemma *vec-component* [*simp*]: *vec x $ i = x*
by *vector*

lemma *vector-mult-component* [*simp*]: (*x ∗ y*)$*i* = *x*$*i ∗ y*$*i*
by *vector*

lemma *vector-smult-component* [*simp*]: (*c ∗s y*)$*i* = *c ∗* (*y*$*i*)
by *vector*

lemma *cond-component*: (*if b then x else y*)$*i* = (*if b then x*$*i else y*$*i*) **by** *vector*

lemmas *vector-component* =
vec-component vector-add-component vector-mult-component
vector-smult-component vector-minus-component vector-uminus-component
vector-scaleR-component cond-component

42.3 Some frequently useful arithmetic lemmas over vectors.

instance *vec* :: (*semigroup-mult*, *finite*) *semigroup-mult*
by *standard* (*vector mult.assoc*)

instance *vec* :: (*monoid-mult*, *finite*) *monoid-mult*

by *standard vector+*

instance *vec* :: (*ab-semigroup-mult*, *finite*) *ab-semigroup-mult*
by *standard* (*vector mult.commute*)

instance *vec* :: (*comm-monoid-mult*, *finite*) *comm-monoid-mult*
by *standard vector*

instance *vec* :: (*semiring*, *finite*) *semiring*
by *standard* (*vector field-simps*)+

instance *vec* :: (*semiring-0*, *finite*) *semiring-0*
by *standard* (*vector field-simps*)+
instance *vec* :: (*semiring-1*, *finite*) *semiring-1*
by *standard vector*
instance *vec* :: (*comm-semiring*, *finite*) *comm-semiring*
by *standard* (*vector field-simps*)+

instance *vec* :: (*comm-semiring-0*, *finite*) *comm-semiring-0* **..**
instance *vec* :: (*cancel-comm-monoid-add*, *finite*) *cancel-comm-monoid-add* **..**
instance *vec* :: (*semiring-0-cancel*, *finite*) *semiring-0-cancel* **..**
instance *vec* :: (*comm-semiring-0-cancel*, *finite*) *comm-semiring-0-cancel* **..**
instance *vec* :: (*ring*, *finite*) *ring* **..**
instance *vec* :: (*semiring-1-cancel*, *finite*) *semiring-1-cancel* **..**
instance *vec* :: (*comm-semiring-1*, *finite*) *comm-semiring-1* **..**

instance *vec* :: (*ring-1*, *finite*) *ring-1* **..**

instance *vec* :: (*real-algebra*, *finite*) *real-algebra*
by *standard* (*simp-all add*: *vec-eq-iff*)

instance *vec* :: (*real-algebra-1*, *finite*) *real-algebra-1* **..**

lemma *of-nat-index*: (*of-nat n* :: *'a*::*semiring-1* ^ *'n*)$*i* = *of-nat n*
proof (*induct n*)
case *0*
then show *?case* **by** *vector*
next
case *Suc*
then show *?case* **by** *vector*
qed

lemma *one-index* [*simp*]: (*1* :: *'a* :: *one* ^ *'n*) $ *i* = *1*
by *vector*

lemma *neg-one-index* [*simp*]: (*− 1* :: *'a* :: {*one*, *uminus*} ^ *'n*) $ *i* = *− 1*
by *vector*

instance *vec* :: (*semiring-char-0*, *finite*) *semiring-char-0*

proof
fix *m n* :: *nat*
show *inj* (*of-nat* :: *nat* ⇒ *'a ^ 'b*)
by (*auto intro*!: *injI simp add*: *vec-eq-iff of-nat-index*)
qed

instance *vec* :: (*numeral*, *finite*) *numeral* **..**
instance *vec* :: (*semiring-numeral*, *finite*) *semiring-numeral* **..**

lemma *numeral-index* [*simp*]: *numeral w $ i = numeral w*
by (*induct w*) (*simp-all only*: *numeral.simps vector-add-component one-index*)

lemma *neg-numeral-index* [*simp*]: *− numeral w $ i = − numeral w*
by (*simp only*: *vector-uminus-component numeral-index*)

instance *vec* :: (*comm-ring-1*, *finite*) *comm-ring-1* **..**
instance *vec* :: (*ring-char-0*, *finite*) *ring-char-0* **..**

lemma *vector-smult-assoc*: *a *s (b *s x) = ((a::'a::semigroup-mult) * b) *s x*
by (*vector mult.assoc*)
lemma *vector-sadd-rdistrib*: *((a::'a::semiring) + b) *s x = a *s x + b *s x*
by (*vector field-simps*)
lemma *vector-add-ldistrib*: *(c::'a::semiring) *s (x + y) = c *s x + c *s y*
by (*vector field-simps*)
lemma *vector-smult-lzero*[*simp*]: *(0::'a::mult-zero) *s x = 0* **by** *vector*
lemma *vector-smult-lid*[*simp*]: *(1::'a::monoid-mult) *s x = x* **by** *vector*
lemma *vector-ssub-ldistrib*: *(c::'a::ring) *s (x − y) = c *s x − c *s y*
by (*vector field-simps*)
lemma *vector-smult-rneg*: *(c::'a::ring) *s −x = −(c *s x)* **by** *vector*
lemma *vector-smult-lneg*: *− (c::'a::ring) *s x = −(c *s x)* **by** *vector*
lemma *vector-sneg-minus1*: *−x = (−1::'a::ring-1) *s x* **by** *vector*
lemma *vector-smult-rzero*[*simp*]: *c *s 0 = (0::'a::mult-zero ^ 'n)* **by** *vector*
lemma *vector-sub-rdistrib*: *((a::'a::ring) − b) *s x = a *s x − b *s x*
by (*vector field-simps*)

lemma *vec-eq*[*simp*]: *(vec m = vec n) ⟷ (m = n)*
by (*simp add*: *vec-eq-iff*)

lemma *norm-eq-0-imp*: *norm x = 0 ==> x = (0::real ^'n)* **by** (*metis norm-eq-zero*)
lemma *vector-mul-eq-0*[*simp*]: *(a *s x = 0) ⟷ a = (0::'a::idom) ∨ x = 0*
by *vector*
lemma *vector-mul-lcancel*[*simp*]: *a *s x = a *s y ⟷ a = (0::real) ∨ x = y*
by (*metis eq-iff-diff-eq-0 vector-mul-eq-0 vector-ssub-ldistrib*)
lemma *vector-mul-rcancel*[*simp*]: *a *s x = b *s x ⟷ (a::real) = b ∨ x = 0*
by (*metis eq-iff-diff-eq-0 vector-mul-eq-0 vector-sub-rdistrib*)
lemma *vector-mul-lcancel-imp*: *a ≠ (0::real) ==> a *s x = a *s y ==> (x = y)*
by (*metis vector-mul-lcancel*)
lemma *vector-mul-rcancel-imp*: *x ≠ 0 ⟹ (a::real) *s x = b *s x ==> a = b*

by (*metis vector-mul-rcancel*)

lemma *component-le-norm-cart*: $|x\$i| <= norm\ x$
apply (*simp add*: *norm-vec-def*)
apply (*rule member-le-setL2*, *simp-all*)
done

lemma *norm-bound-component-le-cart*: $norm\ x <= e ==> |x\$i| <= e$
by (*metis component-le-norm-cart order-trans*)

lemma *norm-bound-component-lt-cart*: $norm\ x < e ==> |x\$i| < e$
by (*metis component-le-norm-cart le-less-trans*)

lemma *norm-le-l1-cart*: $norm\ x <= setsum(\lambda i.\ |x\$i|)\ UNIV$
by (*simp add*: *norm-vec-def setL2-le-setsum*)

lemma *scalar-mult-eq-scaleR*: $c *s x = c *_R x$
unfolding *scaleR-vec-def vector-scalar-mult-def* **by** *simp*

lemma *dist-mul*[*simp*]: $dist\ (c *s x)\ (c *s y) = |c| * dist\ x\ y$
unfolding *dist-norm scalar-mult-eq-scaleR*
unfolding *scaleR-right-diff-distrib*[*symmetric*] **by** *simp*

lemma *setsum-component* [*simp*]:
fixes f:: $'a \Rightarrow ('b$::*comm-monoid-add*$)\ \hat{}\ 'n$
shows $(setsum\ f\ S)\$i = setsum\ (\lambda x.\ (f\ x)\$i)\ S$
proof (*cases finite S*)
case *True*
then show *?thesis* **by** *induct simp-all*
next
case *False*
then show *?thesis* **by** *simp*
qed

lemma *setsum-eq*: $setsum\ f\ S = (\chi\ i.\ setsum\ (\lambda x.\ (f\ x)\$i\ )\ S)$
by (*simp add*: *vec-eq-iff*)

lemma *setsum-cmul*:
fixes f:: $'c \Rightarrow ('a$::*semiring-1*$)\ \hat{}\ 'n$
shows $setsum\ (\lambda x.\ c *s f\ x)\ S = c *s setsum\ f\ S$
by (*simp add*: *vec-eq-iff setsum-right-distrib*)

lemma *setsum-norm-allsubsets-bound-cart*:
fixes f:: $'a \Rightarrow real\ \hat{}\ 'n$
assumes *fP*: *finite P* **and** *fPs*: $\bigwedge Q.\ Q \subseteq P \Longrightarrow norm\ (setsum\ f\ Q) \leq e$
shows $setsum\ (\lambda x.\ norm\ (f\ x))\ P \leq 2 * real\ CARD('n) *\ e$
using *setsum-norm-allsubsets-bound*[*OF assms*]
by *simp*

42.4 Closures and interiors of halfspaces

lemma *interior-halfspace-le* [*simp*]:
assumes $a \neq 0$
shows *interior* $\{x.\ a \cdot x \leq b\} = \{x.\ a \cdot x < b\}$
proof −
have $*$: $a \cdot x < b$ **if** x: $x \in S$ **and** S: $S \subseteq \{x.\ a \cdot x \leq b\}$ **and** *open* S **for** $S\ x$
proof −
obtain e **where** $e{>}0$ **and** e: *cball* $x\ e \subseteq S$
using ‹*open S*› *open-contains-cball* x **by** *blast*
then have $x + (e\ /\ norm\ a) *_R\ a \in cball\ x\ e$
by (*simp add*: *dist-norm*)
then have $x + (e\ /\ norm\ a) *_R\ a \in S$
using e **by** *blast*
then have $x + (e\ /\ norm\ a) *_R\ a \in \{x.\ a \cdot x \leq b\}$
using S **by** *blast*
moreover have $e * (a \cdot a)\ /\ norm\ a > 0$
by (*simp add*: ‹$0 < e$› *assms*)
ultimately show *?thesis*
by (*simp add*: *algebra-simps*)
qed
show *?thesis*
by (*rule interior-unique*) (*auto simp*: *open-halfspace-lt* $*$)
qed

lemma *interior-halfspace-ge* [*simp*]:
$a \neq 0 \Longrightarrow$ *interior* $\{x.\ a \cdot x \geq b\} = \{x.\ a \cdot x > b\}$
using *interior-halfspace-le* [*of* $-a$ $-b$] **by** *simp*

lemma *interior-halfspace-component-le* [*simp*]:
interior $\{x.\ x\$k \leq a\} = \{x :: (real,'n::finite)\ vec.\ x\$k < a\}$ (**is** *?LE*)
and *interior-halfspace-component-ge* [*simp*]:
interior $\{x.\ x\$k \geq a\} = \{x :: (real,'n::finite)\ vec.\ x\$k > a\}$ (**is** *?GE*)
proof −
have *axis* k $(1::real) \neq 0$
by (*simp add*: *axis-def vec-eq-iff*)
moreover have *axis* k $(1::real) \cdot x = x\$k$ **for** x
by (*simp add*: *cart-eq-inner-axis inner-commute*)
ultimately show *?LE ?GE*
using *interior-halfspace-le* [*of axis* k $(1::real)$ a]
interior-halfspace-ge [*of axis* k $(1::real)$ a] **by** *auto*
qed

lemma *closure-halfspace-lt* [*simp*]:
assumes $a \neq 0$
shows *closure* $\{x.\ a \cdot x < b\} = \{x.\ a \cdot x \leq b\}$
proof −
have [*simp*]: $-\{x.\ a \cdot x < b\} = \{x.\ a \cdot x \geq b\}$
by (*force simp*:)
then show *?thesis*

using *interior-halfspace-ge* [*of a b*] *assms*
by (*force simp*: *closure-interior*)
qed

lemma *closure-halfspace-gt* [*simp*]:
$a \neq 0 \Longrightarrow$ *closure* $\{x.\ a \cdot x > b\} = \{x.\ a \cdot x \geq b\}$
using *closure-halfspace-lt* [*of* $-a$ $-b$] **by** *simp*

lemma *closure-halfspace-component-lt* [*simp*]:
closure $\{x.\ x\$k < a\} = \{x :: (real,'n::finite)\ vec.\ x\$k \leq a\}$ (**is** *?LE*)
and *closure-halfspace-component-gt* [*simp*]:
closure $\{x.\ x\$k > a\} = \{x :: (real,'n::finite)\ vec.\ x\$k \geq a\}$ (**is** *?GE*)
proof −
have *axis k* (*1*::*real*) $\neq$ *0*
by (*simp add*: *axis-def vec-eq-iff*)
moreover have *axis k* (*1*::*real*) $\cdot$ x = $x\$k$ **for** x
by (*simp add*: *cart-eq-inner-axis inner-commute*)
ultimately show *?LE ?GE*
using *closure-halfspace-lt* [*of axis k* (*1*::*real*) *a*]
closure-halfspace-gt [*of axis k* (*1*::*real*) *a*] **by** *auto*
qed

lemma *interior-hyperplane* [*simp*]:
assumes $a \neq 0$
shows *interior* $\{x.\ a \cdot x = b\} = \{\}$
proof −
have [*simp*]: $\{x.\ a \cdot x = b\} = \{x.\ a \cdot x \leq b\} \cap \{x.\ a \cdot x \geq b\}$
by (*force simp*:)
then show *?thesis*
by (*auto simp*: *assms*)
qed

lemma *frontier-halfspace-le*:
assumes $a \neq 0 \vee b \neq 0$
shows *frontier* $\{x.\ a \cdot x \leq b\} = \{x.\ a \cdot x = b\}$
proof (*cases* $a = 0$)
case *True* **with** *assms* **show** *?thesis* **by** *simp*
next
case *False* **then show** *?thesis*
by (*force simp*: *frontier-def closed-halfspace-le*)
qed

lemma *frontier-halfspace-ge*:
assumes $a \neq 0 \vee b \neq 0$
shows *frontier* $\{x.\ a \cdot x \geq b\} = \{x.\ a \cdot x = b\}$
proof (*cases* $a = 0$)
case *True* **with** *assms* **show** *?thesis* **by** *simp*
next
case *False* **then show** *?thesis*

by (*force simp*: *frontier-def closed-halfspace-ge*)
qed

lemma *frontier-halfspace-lt*:
assumes $a \neq 0 \vee b \neq 0$
shows *frontier* $\{x.\ a \cdot x < b\} = \{x.\ a \cdot x = b\}$
proof (*cases* $a = 0$)
case *True* **with** *assms* **show** *?thesis* **by** *simp*
next
case *False* **then show** *?thesis*
by (*force simp*: *frontier-def interior-open open-halfspace-lt*)
qed

lemma *frontier-halfspace-gt*:
assumes $a \neq 0 \vee b \neq 0$
shows *frontier* $\{x.\ a \cdot x > b\} = \{x.\ a \cdot x = b\}$
proof (*cases* $a = 0$)
case *True* **with** *assms* **show** *?thesis* **by** *simp*
next
case *False* **then show** *?thesis*
by (*force simp*: *frontier-def interior-open open-halfspace-gt*)
qed

lemma *interior-standard-hyperplane*:
interior $\{x :: (real,'n::finite)\ vec.\ x\$k = a\} = \{\}$
proof −
have *axis k* (*1*::*real*) $\neq 0$
by (*simp add*: *axis-def vec-eq-iff*)
moreover have *axis k* (*1*::*real*) $\cdot\ x = x\$k$ **for** *x*
by (*simp add*: *cart-eq-inner-axis inner-commute*)
ultimately show *?thesis*
using *interior-hyperplane* [*of axis k* (*1*::*real*) *a*]
by *force*
qed

42.5 Matrix operations

Matrix notation. NB: an MxN matrix is of type $(('a, 'n)\ vec, 'm)\ vec$, not $(('a, 'm)\ vec, 'n)\ vec$

definition *matrix-matrix-mult* :: $('a::semiring\text{-}1)\ \hat{}'n\hat{}'m \Rightarrow 'a\ \hat{}'p\hat{}'n \Rightarrow 'a\ \hat{}\ 'p\ \hat{}'m$
(**infixl** $**$ *70*)
where $m ** m' == (\chi\ i\ j.\ setsum\ (\lambda k.\ ((m\$i)\$k) * ((m'\$k)\$j))\ (UNIV :: 'n\ set)) ::'a\ \hat{}\ 'p\ \hat{}'m$

definition *matrix-vector-mult* :: $('a::semiring\text{-}1)\ \hat{}'n\hat{}'m \Rightarrow 'a\ \hat{}'n \Rightarrow 'a\ \hat{}\ 'm$
(**infixl** $*v$ *70*)
where $m *v x \equiv (\chi\ i.\ setsum\ (\lambda j.\ ((m\$i)\$j) * (x\$j))\ (UNIV ::'n\ set)) :: 'a\hat{}'m$

definition *vector-matrix-mult* :: *'a ^ 'm ⇒ ('a::semiring-1) ^'n^'m ⇒ 'a ^'n*
(**infixl** *v∗ 70*)
where *v v∗ m == (χ j. setsum (λi. ((m$i)$j) ∗ (v$i)) (UNIV :: 'm set)) :: 'a^'n*

definition (*mat::'a::zero => 'a ^'n^'n*) *k = (χ i j. if i = j then k else 0)*
definition *transpose* **where**
(*transpose::'a^'n^'m ⇒ 'a^'m^'n*) *A = (χ i j. ((A$j)$i))*
definition (*row::'m => 'a ^'n^'m ⇒ 'a ^'n*) *i A = (χ j. ((A$i)$j))*
definition (*column::'n =>'a^'n^'m =>'a^'m*) *j A = (χ i. ((A$i)$j))*
definition *rows(A::'a^'n^'m) = { row i A | i. i ∈ (UNIV :: 'm set)}*
definition *columns(A::'a^'n^'m) = { column i A | i. i ∈ (UNIV :: 'n set)}*

lemma *mat-0[simp]*: *mat 0 = 0* **by** (*vector mat-def*)
lemma *matrix-add-ldistrib*: *(A ∗∗ (B + C)) = (A ∗∗ B) + (A ∗∗ C)*
by (*vector matrix-matrix-mult-def setsum.distrib[symmetric] field-simps*)

lemma *matrix-mul-lid*:
fixes *A :: 'a::semiring-1 ^ 'm ^ 'n*
shows *mat 1 ∗∗ A = A*
apply (*simp add*: *matrix-matrix-mult-def mat-def*)
apply *vector*
apply (*auto simp only*: *if-distrib cond-application-beta setsum.delta'[OF finite]*
mult-1-left mult-zero-left if-True UNIV-I)
done

lemma *matrix-mul-rid*:
fixes *A :: 'a::semiring-1 ^ 'm ^ 'n*
shows *A ∗∗ mat 1 = A*
apply (*simp add*: *matrix-matrix-mult-def mat-def*)
apply *vector*
apply (*auto simp only*: *if-distrib cond-application-beta setsum.delta[OF finite]*
mult-1-right mult-zero-right if-True UNIV-I cong: *if-cong*)
done

lemma *matrix-mul-assoc*: *A ∗∗ (B ∗∗ C) = (A ∗∗ B) ∗∗ C*
apply (*vector matrix-matrix-mult-def setsum-right-distrib setsum-left-distrib mult.assoc*)
apply (*subst setsum.commute*)
apply *simp*
done

lemma *matrix-vector-mul-assoc*: *A ∗v (B ∗v x) = (A ∗∗ B) ∗v x*
apply (*vector matrix-matrix-mult-def matrix-vector-mult-def*
setsum-right-distrib setsum-left-distrib mult.assoc)
apply (*subst setsum.commute*)
apply *simp*
done

lemma *matrix-vector-mul-lid*: *mat 1 ∗v x = (x::'a::semiring-1 ^ 'n)*
apply (*vector matrix-vector-mult-def mat-def*)
apply (*simp add*: *if-distrib cond-application-beta setsum.delta' cong del*: *if-weak-cong*)
done

lemma *matrix-transpose-mul*:
transpose(A ∗∗ B) = transpose B ∗∗ transpose (*A::'a::comm-semiring-1^-^-*)
by (*simp add*: *matrix-matrix-mult-def transpose-def vec-eq-iff mult.commute*)

lemma *matrix-eq*:
fixes *A B* :: *'a::semiring-1 ^ 'n ^ 'm*
shows *A = B ⟷* (∀ *x*. *A ∗v x = B ∗v x*) (**is** *?lhs ⟷ ?rhs*)
apply *auto*
apply (*subst vec-eq-iff*)
apply *clarify*
apply (*clarsimp simp add*: *matrix-vector-mult-def if-distrib cond-application-beta vec-eq-iff cong del*: *if-weak-cong*)
apply (*erule-tac x=axis ia 1* **in** *allE*)
apply (*erule-tac x=i* **in** *allE*)
apply (*auto simp add*: *if-distrib cond-application-beta axis-def setsum.delta*[*OF finite*] *cong del*: *if-weak-cong*)
done

lemma *matrix-vector-mul-component*: ((*A::real^-^-*) *∗v x*)$*k* = (*A*$*k*) ⋅ *x*
by (*simp add*: *matrix-vector-mult-def inner-vec-def*)

lemma *dot-lmul-matrix*: ((*x::real ^-*) *v∗ A*) ⋅ *y* = *x* ⋅ (*A ∗v y*)
apply (*simp add*: *inner-vec-def matrix-vector-mult-def vector-matrix-mult-def setsum-left-distrib setsum-right-distrib ac-simps*)
apply (*subst setsum.commute*)
apply *simp*
done

lemma *transpose-mat*: *transpose* (*mat n*) = *mat n*
by (*vector transpose-def mat-def*)

lemma *transpose-transpose*: *transpose*(*transpose A*) = *A*
by (*vector transpose-def*)

lemma *row-transpose*:
fixes *A*:: *'a::semiring-1^-^-*
shows *row i* (*transpose A*) = *column i A*
by (*simp add*: *row-def column-def transpose-def vec-eq-iff*)

lemma *column-transpose*:
fixes *A*:: *'a::semiring-1^-^-*
shows *column i* (*transpose A*) = *row i A*
by (*simp add*: *row-def column-def transpose-def vec-eq-iff*)

lemma *rows-transpose*: *rows*(*transpose* (*A*::*'a*::*semiring-1^-^-*)) = *columns A*
by (*auto simp add*: *rows-def columns-def row-transpose intro*: *set-eqI*)

lemma *columns-transpose*: *columns*(*transpose* (*A*::*'a*::*semiring-1^-^-*)) = *rows A*
by (*metis transpose-transpose rows-transpose*)

Two sometimes fruitful ways of looking at matrix-vector multiplication.

lemma *matrix-mult-dot*: *A* ∗*v x* = (*χ i*. *A*$*i* · *x*)
by (*simp add*: *matrix-vector-mult-def inner-vec-def*)

lemma *matrix-mult-vsum*:
(*A*::*'a*::*comm-semiring-1^'n^'m*) ∗*v x* = *setsum* (*λi*. (*x*$*i*) ∗*s column i A*) (*UNIV*::
'n set)
by (*simp add*: *matrix-vector-mult-def vec-eq-iff column-def mult.commute*)

lemma *vector-componentwise*:
(*x*::*'a*::*ring-1^'n*) = (*χ j*. ∑ *i*∈*UNIV*. (*x*$*i*) ∗ (*axis i 1* :: *'a^'n*) $ *j*)
by (*simp add*: *axis-def if-distrib setsum.If-cases vec-eq-iff*)

lemma *basis-expansion*: *setsum* (*λi*. (*x*$*i*) ∗*s axis i 1*) *UNIV* = (*x*::(*'a*::*ring-1*)
^'n)
by (*auto simp add*: *axis-def vec-eq-iff if-distrib setsum.If-cases cong del*: *if-weak-cong*)

lemma *linear-componentwise*:
fixes *f*:: *real ^'m* ⇒ *real ^ -*
assumes *lf*: *linear f*
shows (*f x*)$*j* = *setsum* (*λi*. (*x*$*i*) ∗ (*f* (*axis i 1*)$*j*)) (*UNIV* :: *'m set*) (**is** *?lhs*
= *?rhs*)
proof −
let *?M* = (*UNIV* :: *'m set*)
let *?N* = (*UNIV* :: *'n set*)
have *?rhs* = (*setsum* (*λi*.(*x*$*i*) ∗*R f* (*axis i 1*)) *?M*)$*j*
unfolding *setsum-component* **by** *simp*
then show *?thesis*
unfolding *linear-setsum-mul*[*OF lf*, *symmetric*]
unfolding *scalar-mult-eq-scaleR*[*symmetric*]
unfolding *basis-expansion*
by *simp*
qed

Inverse matrices (not necessarily square)

definition
invertible(*A*::*'a*::*semiring-1^'n^'m*) ⟷ (∃ *A'*::*'a^'m^'n*. *A* ∗∗ *A'* = *mat 1* ∧ *A'*
∗∗ *A* = *mat 1*)

definition
matrix-inv(*A*:: *'a*::*semiring-1^'n^'m*) =
(*SOME A'*::*'a^'m^'n*. *A* ∗∗ *A'* = *mat 1* ∧ *A'* ∗∗ *A* = *mat 1*)

Correspondence between matrices and linear operators.

definition *matrix* :: ('a::{*plus*,*times*, *one*, *zero*}^'m ⇒ 'a ^ 'n) ⇒ 'a^'m^'n
where *matrix f* = (χ *i j*. (*f*(*axis j 1*))$*i*)

lemma *matrix-vector-mul-linear*: *linear*(λ*x*. *A* ∗*v* (*x*::*real* ^ -))
by (*simp add*: *linear-iff matrix-vector-mult-def vec-eq-iff*
field-simps setsum-right-distrib setsum.distrib)

lemma *matrix-works*:
assumes *lf*: *linear f*
shows *matrix f* ∗*v x* = *f* (*x*::*real* ^ 'n)
apply (*simp add*: *matrix-def matrix-vector-mult-def vec-eq-iff mult.commute*)
apply *clarify*
apply (*rule linear-componentwise*[*OF lf*, *symmetric*])
done

lemma *matrix-vector-mul*: *linear f* ==> *f* = (λ*x*. *matrix f* ∗*v* (*x*::*real* ^ 'n))
by (*simp add*: *ext matrix-works*)

lemma *matrix-of-matrix-vector-mul*: *matrix*(λ*x*. *A* ∗*v* (*x* :: *real* ^ 'n)) = *A*
by (*simp add*: *matrix-eq matrix-vector-mul-linear matrix-works*)

lemma *matrix-compose*:
assumes *lf*: *linear* (*f*::*real*^'n ⇒ *real*^'m)
and *lg*: *linear* (*g*::*real*^'m ⇒ *real*^-)
shows *matrix* (*g* ∘ *f*) = *matrix g* ∗∗ *matrix f*
using *lf lg linear-compose*[*OF lf lg*] *matrix-works*[*OF linear-compose*[*OF lf lg*]]
by (*simp add*: *matrix-eq matrix-works matrix-vector-mul-assoc*[*symmetric*] *o-def*)

lemma *matrix-vector-column*:
(*A*::'a::*comm-semiring-1*^'n^-) ∗*v x* = *setsum* (λ*i*. (*x*$*i*) ∗*s* ((*transpose A*)$*i*))
(*UNIV*:: 'n *set*)
by (*simp add*: *matrix-vector-mult-def transpose-def vec-eq-iff mult.commute*)

lemma *adjoint-matrix*: *adjoint*(λ*x*. (*A*::*real*^'n^'m) ∗*v x*) = (λ*x*. *transpose A* ∗*v x*)
apply (*rule adjoint-unique*)
apply (*simp add*: *transpose-def inner-vec-def matrix-vector-mult-def*
setsum-left-distrib setsum-right-distrib)
apply (*subst setsum.commute*)
apply (*auto simp add*: *ac-simps*)
done

lemma *matrix-adjoint*: **assumes** *lf*: *linear* (*f* :: *real*^'n ⇒ *real* ^'m)
shows *matrix*(*adjoint f*) = *transpose*(*matrix f*)
apply (*subst matrix-vector-mul*[*OF lf*])
unfolding *adjoint-matrix matrix-of-matrix-vector-mul*
apply *rule*
done

42.6 lambda skolemization on cartesian products

lemma *lambda-skolem*: $(\forall\, i.\ \exists\, x.\ P\ i\ x) \longleftrightarrow$
$(\exists\, x{::}'a\ \hat{}\ 'n.\ \forall\, i.\ P\ i\ (x\ \$\ i))$ (**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof −
let *?S* = (*UNIV* :: *'n set*)
{ assume *H*: *?rhs*
then have *?lhs* **by** *auto* **}**
moreover
{ assume *H*: *?lhs*
then obtain *f* **where** *f*:$\forall\, i.\ P\ i\ (f\ i)$ **unfolding** *choice-iff* **by** *metis*
let *?x* = $(\chi\ i.\ (f\ i))$ :: *'a* $\hat{}$ *'n*
{ fix *i*
from *f* **have** *P i* (*f i*) **by** *metis*
then have *P i* (*?x* $ *i*) **by** *auto*
}
hence $\forall\, i.\ P\ i$ (*?x*$*i*) **by** *metis*
hence *?rhs* **by** *metis* **}**
ultimately show *?thesis* **by** *metis*
qed

lemma *vector-sub-project-orthogonal-cart*: $(b{::}real\hat{}'n) \cdot (x - ((b \cdot x) / (b \cdot b)) *s\ b) = 0$
unfolding *inner-simps scalar-mult-eq-scaleR* **by** *auto*

lemma *left-invertible-transpose*:
$(\exists\,(B).\ B ** transpose\ (A) = mat\ (1{::}'a{::}comm\text{-}semiring\text{-}1)) \longleftrightarrow (\exists\,(B).\ A ** B = mat\ 1)$
by (*metis matrix-transpose-mul transpose-mat transpose-transpose*)

lemma *right-invertible-transpose*:
$(\exists\,(B).\ transpose\ (A) ** B = mat\ (1{::}'a{::}comm\text{-}semiring\text{-}1)) \longleftrightarrow (\exists\,(B).\ B ** A = mat\ 1)$
by (*metis matrix-transpose-mul transpose-mat transpose-transpose*)

lemma *matrix-left-invertible-injective*:
$(\exists\, B.\ (B{::}real\hat{}'m\hat{}'n) ** (A{::}real\hat{}'n\hat{}'m) = mat\ 1) \longleftrightarrow (\forall\, x\ y.\ A *v\ x = A *v\ y \longrightarrow x = y)$
proof −
{ fix *B*:: *real*$\hat{}$*'m*$\hat{}$*'n* **and** *x y* **assume** *B*: *B ** A = mat 1* **and** *xy*: *A *v x = A*v y*
from *xy* **have** *B*v* (*A *v x*) = *B *v* (*A*v y*) **by** *simp*
hence *x = y*
unfolding *matrix-vector-mul-assoc B matrix-vector-mul-lid* **. }**
moreover
{ assume *A*: $\forall\, x\ y.\ A *v\ x = A *v\ y \longrightarrow x = y$
hence *i*: *inj* (*op *v A*) **unfolding** *inj-on-def* **by** *auto*
from *linear-injective-left-inverse*[*OF matrix-vector-mul-linear i*]
obtain *g* **where** *g*: *linear g g* ∘ *op *v A = id* **by** *blast*
have *matrix g ** A = mat 1*

```
    unfolding matrix-eq matrix-vector-mul-lid matrix-vector-mul-assoc[symmetric]
matrix-works[OF g(1)]
      using g(2) by (simp add: fun-eq-iff)
    then have ∃ B. (B::real ^'m^'n) ** A = mat 1 by blast }
  ultimately show ?thesis by blast
qed

lemma matrix-left-invertible-ker:
  (∃ B. (B::real ^'m^'n) ** (A::real^'n^'m) = mat 1) ⟷ (∀ x. A *v x = 0 ⟶ x
= 0)
  unfolding matrix-left-invertible-injective
  using linear-injective-0[OF matrix-vector-mul-linear, of A]
  by (simp add: inj-on-def)

lemma matrix-right-invertible-surjective:
  (∃ B. (A::real^'n^'m) ** (B::real^'m^'n) = mat 1) ⟷ surj (λx. A *v x)
proof −
  { fix B :: real ^'m^'n
    assume AB: A ** B = mat 1
    { fix x :: real ^ 'm
      have A *v (B *v x) = x
        by (simp add: matrix-vector-mul-lid matrix-vector-mul-assoc AB) }
    hence surj (op *v A) unfolding surj-def by metis }
  moreover
  { assume sf: surj (op *v A)
    from linear-surjective-right-inverse[OF matrix-vector-mul-linear sf]
    obtain g:: real ^'m ⇒ real ^'n where g: linear g op *v A ∘ g = id
      by blast

    have A ** (matrix g) = mat 1
      unfolding matrix-eq  matrix-vector-mul-lid
        matrix-vector-mul-assoc[symmetric] matrix-works[OF g(1)]
      using g(2) unfolding o-def fun-eq-iff id-def
      .
    hence ∃ B. A ** (B::real^'m^'n) = mat 1 by blast
  }
  ultimately show ?thesis unfolding surj-def by blast
qed

lemma matrix-left-invertible-independent-columns:
  fixes A :: real^'n^'m
  shows (∃(B::real ^'m^'n). B ** A = mat 1) ⟷
      (∀ c. setsum (λi. c i *s column i A) (UNIV :: 'n set) = 0 ⟶ (∀ i. c i = 0))
    (is ?lhs ⟷ ?rhs)
proof −
  let ?U = UNIV :: 'n set
  { assume k: ∀ x. A *v x = 0 ⟶ x = 0
    { fix c i
      assume c: setsum (λi. c i *s column i A) ?U = 0 and i: i ∈ ?U
```

```
    let ?x = χ i. c i
    have th0:A *v ?x = 0
      using c
      unfolding matrix-mult-vsum vec-eq-iff
      by auto
    from k[rule-format, OF th0] i
    have c i = 0 by (vector vec-eq-iff)}
   hence ?rhs by blast }
  moreover
  { assume H: ?rhs
   { fix x assume x: A *v x = 0
    let ?c = λi. ((x$i ):: real)
    from H[rule-format, of ?c, unfolded matrix-mult-vsum[symmetric], OF x]
    have x = 0 by vector }
  }
  ultimately show ?thesis unfolding matrix-left-invertible-ker by blast
qed

lemma matrix-right-invertible-independent-rows:
  fixes A :: real^'n^'m
  shows (∃(B::real^'m^'n). A ** B = mat 1) ⟷
    (∀ c. setsum (λi. c i *s row i A) (UNIV :: 'm set) = 0 ⟶ (∀ i. c i = 0))
  unfolding left-invertible-transpose[symmetric]
    matrix-left-invertible-independent-columns
  by (simp add: column-transpose)

lemma matrix-right-invertible-span-columns:
  (∃(B::real ^'n^'m). (A::real ^'m^'n) ** B = mat 1) ⟷
    span (columns A) = UNIV (is ?lhs = ?rhs)
proof −
  let ?U = UNIV :: 'm set
  have fU: finite ?U by simp
  have lhseq: ?lhs ⟷ (∀ y. ∃(x::real^'m). setsum (λi. (x$i) *s column i A) ?U
= y)
    unfolding matrix-right-invertible-surjective matrix-mult-vsum surj-def
    apply (subst eq-commute)
    apply rule
    done
  have rhseq: ?rhs ⟷ (∀ x. x ∈ span (columns A)) by blast
  { assume h: ?lhs
   { fix x:: real ^'n
    from h[unfolded lhseq, rule-format, of x] obtain y :: real ^'m
      where y: setsum (λi. (y$i) *s column i A) ?U = x by blast
    have x ∈ span (columns A)
      unfolding y[symmetric]
      apply (rule span-setsum)
      apply clarify
      unfolding scalar-mult-eq-scaleR
      apply (rule span-mul)
```

```
      apply (rule span-superset)
      unfolding columns-def
      apply blast
      done
   }
   then have ?rhs unfolding rhseq by blast }
  moreover
  { assume h:?rhs
   let ?P = λ(y::real ^'n). ∃(x::real^'m). setsum (λi. (x$i) *s column i A) ?U =
y
   { fix y
     have ?P y
     proof (rule span-induct-alt[of ?P columns A, folded scalar-mult-eq-scaleR])
       show ∃x::real ^ 'm. setsum (λi. (x$i) *s column i A) ?U = 0
         by (rule exI[where x=0], simp)
     next
       fix c y1 y2
       assume y1: y1 ∈ columns A and y2: ?P y2
       from y1 obtain i where i: i ∈ ?U y1 = column i A
         unfolding columns-def by blast
       from y2 obtain x:: real ^'m where
         x: setsum (λi. (x$i) *s column i A) ?U = y2 by blast
       let ?x = (χ j. if j = i then c + (x$i) else (x$j))::real^'m
       show ?P (c*s y1 + y2)
         proof (rule exI[where x= ?x], vector, auto simp add: i x[symmetric]
if-distrib distrib-left cond-application-beta cong del: if-weak-cong)
         fix j
         have th: ∀ xa ∈ ?U. (if xa = i then (c + (x$i)) * ((column xa A)$j)
           else (x$xa) * ((column xa A$j))) = (if xa = i then c * ((column i A)$j)
else 0) + ((x$xa) * ((column xa A)$j))
           using i(1) by (simp add: field-simps)
         have setsum (λxa. if xa = i then (c + (x$i)) * ((column xa A)$j)
             else (x$xa) * ((column xa A$j))) ?U = setsum (λxa. (if xa = i then c
* ((column i A)$j) else 0) + ((x$xa) * ((column xa A)$j))) ?U
           apply (rule setsum.cong[OF refl])
           using th apply blast
           done
         also have ... = setsum (λxa. if xa = i then c * ((column i A)$j) else 0)
?U + setsum (λxa. ((x$xa) * ((column xa A)$j))) ?U
           by (simp add: setsum.distrib)
        also have ... = c * ((column i A)$j) + setsum (λxa. ((x$xa) * ((column
xa A)$j))) ?U
           unfolding setsum.delta[OF fU]
           using i(1) by simp
        finally show setsum (λxa. if xa = i then (c + (x$i)) * ((column xa A)$j)
           else (x$xa) * ((column xa A$j))) ?U = c * ((column i A)$j) + setsum
(λxa. ((x$xa) * ((column xa A)$j))) ?U .
       qed
     next
```

```
      show y ∈ span (columns A)
        unfolding h by blast
    qed
   }
   then have ?lhs unfolding lhseq ..
  }
  ultimately show ?thesis by blast
qed

lemma matrix-left-invertible-span-rows:
  (∃(B::real^'m^'n). B ** (A::real^'n^'m) = mat 1) ⟷ span (rows A) = UNIV
  unfolding right-invertible-transpose[symmetric]
  unfolding columns-transpose[symmetric]
  unfolding matrix-right-invertible-span-columns
  ..
```

The same result in terms of square matrices.

```
lemma matrix-left-right-inverse:
  fixes A A' :: real ^'n^'n
  shows A ** A' = mat 1 ⟷ A' ** A = mat 1
proof −
  { fix A A' :: real ^'n^'n
    assume AA': A ** A' = mat 1
    have sA: surj (op *v A)
      unfolding surj-def
      apply clarify
      apply (rule-tac x=(A' *v y) in exI)
      apply (simp add: matrix-vector-mul-assoc AA' matrix-vector-mul-lid)
      done
    from linear-surjective-isomorphism[OF matrix-vector-mul-linear sA]
    obtain f' :: real ^'n ⇒ real ^'n
      where f': linear f' ∀ x. f' (A *v x) = x ∀ x. A *v f' x = x by blast
    have th: matrix f' ** A = mat 1
      by (simp add: matrix-eq matrix-works[OF f'(1)]
       matrix-vector-mul-assoc[symmetric] matrix-vector-mul-lid f'(2)[rule-format])
    hence (matrix f' ** A) ** A' = mat 1 ** A' by simp
    hence matrix f' = A'
    by (simp add: matrix-mul-assoc[symmetric] AA' matrix-mul-rid matrix-mul-lid)
    hence matrix f' ** A = A' ** A by simp
    hence A' ** A = mat 1 by (simp add: th)
  }
  then show ?thesis by blast
qed
```

Considering an n-element vector as an n-by-1 or 1-by-n matrix.

```
definition rowvector v = (χ i j. (v$j))

definition columnvector v = (χ i j. (v$i))
```

lemma *transpose-columnvector*: *transpose*(*columnvector v*) = *rowvector v*
by (*simp add*: *transpose-def rowvector-def columnvector-def vec-eq-iff*)

lemma *transpose-rowvector*: *transpose*(*rowvector v*) = *columnvector v*
by (*simp add*: *transpose-def columnvector-def rowvector-def vec-eq-iff*)

lemma *dot-rowvector-columnvector*: *columnvector* (*A* ∗*v v*) = *A* ∗∗ *columnvector v*
by (*vector columnvector-def matrix-matrix-mult-def matrix-vector-mult-def*)

lemma *dot-matrix-product*:
(*x*::*real*^′*n*) · *y* = (((*rowvector x* ::*real*^′*n*^*1*) ∗∗ (*columnvector y* :: *real*^*1*^′*n*))$*1*)$*1*
by (*vector matrix-matrix-mult-def rowvector-def columnvector-def inner-vec-def*)

lemma *dot-matrix-vector-mul*:
fixes *A B* :: *real* ^′*n* ^′*n* **and** *x y* :: *real* ^′*n*
shows (*A* ∗*v x*) · (*B* ∗*v y*) =
(((*rowvector x* :: *real*^′*n*^*1*) ∗∗ ((*transpose A* ∗∗ *B*) ∗∗ (*columnvector y* :: *real*^*1*^′*n*)))$*1*)$*1*
unfolding *dot-matrix-product transpose-columnvector*[*symmetric*]
dot-rowvector-columnvector matrix-transpose-mul matrix-mul-assoc **..**

lemma *infnorm-cart*:*infnorm* (*x*::*real*^′*n*) = *Sup* {|*x*$*i*| |*i*. *i*∈*UNIV*}
by (*simp add*: *infnorm-def inner-axis Basis-vec-def*) (*metis* (*lifting*) *inner-axis real-inner-1-right*)

lemma *component-le-infnorm-cart*: |*x*$*i*| ≤ *infnorm* (*x*::*real*^′*n*)
using *Basis-le-infnorm*[*of axis i 1 x*]
by (*simp add*: *Basis-vec-def axis-eq-axis inner-axis*)

lemma *continuous-component*: *continuous F f* ⟹ *continuous F* (λ*x*. *f x* $ *i*)
unfolding *continuous-def* **by** (*rule tendsto-vec-nth*)

lemma *continuous-on-component*: *continuous-on s f* ⟹ *continuous-on s* (λ*x*. *f x* $ *i*)
unfolding *continuous-on-def* **by** (*fast intro*: *tendsto-vec-nth*)

lemma *closed-positive-orthant*: *closed* {*x*::*real*^′*n*. ∀ *i*. *0* ≤*x*$*i*}
by (*simp add*: *Collect-all-eq closed-INT closed-Collect-le*)

lemma *bounded-component-cart*: *bounded s* ⟹ *bounded* ((λ*x*. *x* $ *i*) ‘ *s*)
unfolding *bounded-def*
apply *clarify*
apply (*rule-tac x*=*x* $ *i* **in** *exI*)
apply (*rule-tac x*=*e* **in** *exI*)
apply *clarify*
apply (*rule order-trans* [*OF dist-vec-nth-le*], *simp*)
done

lemma *compact-lemma-cart*:
fixes $f :: nat \Rightarrow 'a::heine\text{-}borel\ \hat{}\ 'n$
assumes f: *bounded* (*range* f)
shows $\exists\, l\ r.$ *subseq* $r \wedge$
$(\forall\, e{>}0.$ *eventually* $(\lambda n.\ \forall\, i{\in}d.$ *dist* $(f\ (r\ n)\ \$\ i)\ (l\ \$\ i) < e)$ *sequentially*)
(**is** *?th d*)
proof −
have $\forall\, d' \subseteq d.$ *?th* d'
by (*rule compact-lemma-general*[**where** *unproj*=*vec-lambda*])
(*auto intro*!: *f bounded-component-cart simp*: *vec-lambda-eta*)
then show *?th d* **by** *simp*
qed

instance *vec* :: (*heine-borel*, *finite*) *heine-borel*
proof
fix $f :: nat \Rightarrow 'a\ \hat{}\ 'b$
assume f: *bounded* (*range* f)
then obtain $l\ r$ **where** r: *subseq* r
and l: $\forall\, e{>}0.$ *eventually* $(\lambda n.\ \forall\, i{\in}UNIV.$ *dist* $(f\ (r\ n)\ \$\ i)\ (l\ \$\ i) < e)$ *sequentially*
using *compact-lemma-cart* [*OF f*] **by** *blast*
let *?d* = *UNIV*::'*b set*
{ fix *e*::*real* **assume** *e*>*0*
hence *0* < *e* / (*real-of-nat* (*card ?d*))
using *zero-less-card-finite divide-pos-pos*[*of e*, *of real-of-nat* (*card ?d*)] **by** *auto*
with *l* **have** *eventually* $(\lambda n.\ \forall\, i.$ *dist* $(f\ (r\ n)\ \$\ i)\ (l\ \$\ i) < e$ / (*real-of-nat* (*card ?d*))) *sequentially*
by *simp*
moreover
{ fix *n*
assume *n*: $\forall\, i.$ *dist* $(f\ (r\ n)\ \$\ i)\ (l\ \$\ i) <$ *e* / (*real-of-nat* (*card ?d*))
have *dist* $(f\ (r\ n))\ l \leq (\sum i{\in}?d.$ *dist* $(f\ (r\ n)\ \$\ i)\ (l\ \$\ i))$
unfolding *dist-vec-def* **using** *zero-le-dist* **by** (*rule setL2-le-setsum*)
also have ... < $(\sum i{\in}?d.$ *e* / (*real-of-nat* (*card ?d*)))
by (*rule setsum-strict-mono*) (*simp-all add*: *n*)
finally have *dist* $(f\ (r\ n))\ l <$ *e* **by** *simp*
}
ultimately have *eventually* $(\lambda n.$ *dist* $(f\ (r\ n))\ l <$ *e*) *sequentially*
by (*rule eventually-mono*)
}
hence $((f \circ r) \longrightarrow l)$ *sequentially* **unfolding** *o-def tendsto-iff* **by** *simp*
with *r* **show** $\exists\, l\ r.$ *subseq* $r \wedge ((f \circ r) \longrightarrow l)$ *sequentially* **by** *auto*
qed

lemma *interval-cart*:
fixes *a* :: *real*^'*n*
shows *box a b* = $\{x::real\hat{}'n.\ \forall\, i.\ a\$i < x\$i \wedge x\$i < b\$i\}$
and *cbox a b* = $\{x::real\hat{}'n.\ \forall\, i.\ a\$i \leq x\$i \wedge x\$i \leq b\$i\}$

by (*auto simp add*: *set-eq-iff less-vec-def less-eq-vec-def mem-box Basis-vec-def inner-axis*)

lemma *mem-interval-cart*:
fixes $a :: real\hat{}'n$
shows $x \in box\ a\ b \longleftrightarrow (\forall i.\ a\$i < x\$i \wedge x\$i < b\$i)$
and $x \in cbox\ a\ b \longleftrightarrow (\forall i.\ a\$i \leq x\$i \wedge x\$i \leq b\$i)$
using *interval-cart*[*of a b*] **by** (*auto simp add*: *set-eq-iff less-vec-def less-eq-vec-def*)

lemma *interval-eq-empty-cart*:
fixes $a :: real\hat{}'n$
shows $(box\ a\ b = \{\} \longleftrightarrow (\exists i.\ b\$i \leq a\$i))$ (**is** *?th1*)
and $(cbox\ a\ b = \{\} \longleftrightarrow (\exists i.\ b\$i < a\$i))$ (**is** *?th2*)
proof −
{ fix *i x* **assume** *as*:$b\$i \leq a\$i$ **and** *x*:$x \in box\ a\ b$
hence $a\ \$\ i < x\ \$\ i \wedge x\ \$\ i < b\ \$\ i$ **unfolding** *mem-interval-cart* **by** *auto*
hence $a\$i < b\$i$ **by** *auto*
hence *False* **using** *as* **by** *auto* **}**
moreover
{ assume *as*:$\forall i.\ \neg\ (b\$i \leq a\$i)$
let *?x* = $(1/2) *_R (a + b)$
{ fix *i*
have $a\$i < b\$i$ **using** *as*[*THEN spec*[**where** *x*=*i*]] **by** *auto*
hence $a\$i < ((1/2) *_R (a+b))\ \$\ i\ ((1/2) *_R (a+b))\ \$\ i < b\$i$
unfolding *vector-smult-component* **and** *vector-add-component*
by *auto* **}**
hence $box\ a\ b \neq \{\}$ **using** *mem-interval-cart*(*1*)[*of ?x a b*] **by** *auto* **}**
ultimately show *?th1* **by** *blast*

{ fix *i x* **assume** *as*:$b\$i < a\$i$ **and** *x*:$x \in cbox\ a\ b$
hence $a\ \$\ i \leq x\ \$\ i \wedge x\ \$\ i \leq b\ \$\ i$ **unfolding** *mem-interval-cart* **by** *auto*
hence $a\$i \leq b\$i$ **by** *auto*
hence *False* **using** *as* **by** *auto* **}**
moreover
{ assume *as*:$\forall i.\ \neg\ (b\$i < a\$i)$
let *?x* = $(1/2) *_R (a + b)$
{ fix *i*
have $a\$i \leq b\$i$ **using** *as*[*THEN spec*[**where** *x*=*i*]] **by** *auto*
hence $a\$i \leq ((1/2) *_R (a+b))\ \$\ i\ ((1/2) *_R (a+b))\ \$\ i \leq b\$i$
unfolding *vector-smult-component* **and** *vector-add-component*
by *auto* **}**
hence $cbox\ a\ b \neq \{\}$ **using** *mem-interval-cart*(*2*)[*of ?x a b*] **by** *auto* **}**
ultimately show *?th2* **by** *blast*
qed

lemma *interval-ne-empty-cart*:
fixes $a :: real\hat{}'n$
shows $cbox\ a\ b \neq \{\} \longleftrightarrow (\forall i.\ a\$i \leq b\$i)$
and $box\ a\ b \neq \{\} \longleftrightarrow (\forall i.\ a\$i < b\$i)$

unfolding *interval-eq-empty-cart*[*of a b*] **by** (*auto simp add*: *not-less not-le*)

lemma *subset-interval-imp-cart*:
fixes *a* :: *real^'n*
shows $(\forall i.\ a\$i \leq c\$i \wedge d\$i \leq b\$i) \Longrightarrow cbox\ c\ d \subseteq cbox\ a\ b$
and $(\forall i.\ a\$i < c\$i \wedge d\$i < b\$i) \Longrightarrow cbox\ c\ d \subseteq box\ a\ b$
and $(\forall i.\ a\$i \leq c\$i \wedge d\$i \leq b\$i) \Longrightarrow box\ c\ d \subseteq cbox\ a\ b$
and $(\forall i.\ a\$i \leq c\$i \wedge d\$i \leq b\$i) \Longrightarrow box\ c\ d \subseteq box\ a\ b$
unfolding *subset-eq*[*unfolded Ball-def*] **unfolding** *mem-interval-cart*
by (*auto intro*: *order-trans less-le-trans le-less-trans less-imp-le*)

lemma *interval-sing*:
fixes *a* :: *'a::linorder^'n*
shows $\{a\ ..\ a\} = \{a\} \wedge \{a<..<a\} = \{\}$
apply (*auto simp add*: *set-eq-iff less-vec-def less-eq-vec-def vec-eq-iff*)
done

lemma *subset-interval-cart*:
fixes *a* :: *real^'n*
shows *cbox c d* $\subseteq$ *cbox a b* $\longleftrightarrow$ $(\forall i.\ c\$i \leq d\$i)$ *--->* $(\forall i.\ a\$i \leq c\$i \wedge d\$i \leq b\$i)$ (**is** *?th1*)
and *cbox c d* $\subseteq$ *box a b* $\longleftrightarrow$ $(\forall i.\ c\$i \leq d\$i)$ *--->* $(\forall i.\ a\$i < c\$i \wedge d\$i < b\$i)$ (**is** *?th2*)
and *box c d* $\subseteq$ *cbox a b* $\longleftrightarrow$ $(\forall i.\ c\$i < d\$i)$ *--->* $(\forall i.\ a\$i \leq c\$i \wedge d\$i \leq b\$i)$ (**is** *?th3*)
and *box c d* $\subseteq$ *box a b* $\longleftrightarrow$ $(\forall i.\ c\$i < d\$i)$ *--->* $(\forall i.\ a\$i \leq c\$i \wedge d\$i \leq b\$i)$ (**is** *?th4*)
using *subset-box*[*of c d a b*] **by** (*simp-all add*: *Basis-vec-def inner-axis*)

lemma *disjoint-interval-cart*:
fixes *a*::*real^'n*
shows *cbox a b* $\cap$ *cbox c d* $= \{\} \longleftrightarrow (\exists i.\ (b\$i < a\$i \vee d\$i < c\$i \vee b\$i < c\$i \vee d\$i < a\$i))$ (**is** *?th1*)
and *cbox a b* $\cap$ *box c d* $= \{\} \longleftrightarrow (\exists i.\ (b\$i < a\$i \vee d\$i \leq c\$i \vee b\$i \leq c\$i \vee d\$i \leq a\$i))$ (**is** *?th2*)
and *box a b* $\cap$ *cbox c d* $= \{\} \longleftrightarrow (\exists i.\ (b\$i \leq a\$i \vee d\$i < c\$i \vee b\$i \leq c\$i \vee d\$i \leq a\$i))$ (**is** *?th3*)
and *box a b* $\cap$ *box c d* $= \{\} \longleftrightarrow (\exists i.\ (b\$i \leq a\$i \vee d\$i \leq c\$i \vee b\$i \leq c\$i \vee d\$i \leq a\$i))$ (**is** *?th4*)
using *disjoint-interval*[*of a b c d*] **by** (*simp-all add*: *Basis-vec-def inner-axis*)

lemma *inter-interval-cart*:
fixes *a* :: *real^'n*
shows *cbox a b* $\cap$ *cbox c d* $= \{(\chi\ i.\ max\ (a\$i)\ (c\$i))\ ..\ (\chi\ i.\ min\ (b\$i)\ (d\$i))\}$
unfolding *inter-interval*
by (*auto simp*: *mem-box less-eq-vec-def*)
(*auto simp*: *Basis-vec-def inner-axis*)

lemma *closed-interval-left-cart*:
fixes $b :: real\hat{}'n$
shows *closed* $\{x::real\hat{}'n.\ \forall\, i.\ x\$i \leq b\$i\}$
by (*simp add*: *Collect-all-eq closed-INT closed-Collect-le*)

lemma *closed-interval-right-cart*:
fixes $a::real\hat{}'n$
shows *closed* $\{x::real\hat{}'n.\ \forall\, i.\ a\$i \leq x\$i\}$
by (*simp add*: *Collect-all-eq closed-INT closed-Collect-le*)

lemma *is-interval-cart*:
is-interval $(s::(real\hat{}'n)\ set) \longleftrightarrow$
$(\forall\, a{\in}s.\ \forall\, b{\in}s.\ \forall\, x.\ (\forall\, i.\ ((a\$i \leq x\$i \wedge x\$i \leq b\$i) \vee (b\$i \leq x\$i \wedge x\$i \leq a\$i))) \longrightarrow x \in s)$
by (*simp add*: *is-interval-def Ball-def Basis-vec-def inner-axis imp-ex*)

lemma *closed-halfspace-component-le-cart*: *closed* $\{x::real\hat{}'n.\ x\$i \leq a\}$
by (*simp add*: *closed-Collect-le*)

lemma *closed-halfspace-component-ge-cart*: *closed* $\{x::real\hat{}'n.\ x\$i \geq a\}$
by (*simp add*: *closed-Collect-le*)

lemma *open-halfspace-component-lt-cart*: *open* $\{x::real\hat{}'n.\ x\$i < a\}$
by (*simp add*: *open-Collect-less*)

lemma *open-halfspace-component-gt-cart*: *open* $\{x::real\hat{}'n.\ x\$i > a\}$
by (*simp add*: *open-Collect-less*)

lemma *Lim-component-le-cart*:
fixes $f :: 'a \Rightarrow real\hat{}'n$
assumes $(f \longrightarrow l)\ net\ \neg\ (trivial\text{-}limit\ net)$ *eventually* $(\lambda x.\ f\ x\ \$i \leq b)\ net$
shows $l\$i \leq b$
by (*rule tendsto-le*[*OF assms*(*2*) *tendsto-const tendsto-vec-nth*, *OF assms*(*1*, *3*)])

lemma *Lim-component-ge-cart*:
fixes $f :: 'a \Rightarrow real\hat{}'n$
assumes $(f \longrightarrow l)\ net\ \neg\ (trivial\text{-}limit\ net)$ *eventually* $(\lambda x.\ b \leq (f\ x)\$i)\ net$
shows $b \leq l\$i$
by (*rule tendsto-le*[*OF assms*(*2*) *tendsto-vec-nth tendsto-const*, *OF assms*(*1*, *3*)])

lemma *Lim-component-eq-cart*:
fixes $f :: 'a \Rightarrow real\hat{}'n$
assumes *net*: $(f \longrightarrow l)\ net\ \sim(trivial\text{-}limit\ net)$ **and** *ev*:*eventually* $(\lambda x.\ f(x)\$i = b)\ net$
shows $l\$i = b$
using *ev*[*unfolded order-eq-iff eventually-conj-iff*] **and**
Lim-component-ge-cart[*OF net*, *of b i*] **and**
Lim-component-le-cart[*OF net*, *of i b*] **by** *auto*

lemma *connected-ivt-component-cart*:
fixes $x :: real\hat{}'n$
shows $connected\ s \Longrightarrow x \in s \Longrightarrow y \in s \Longrightarrow x\$k \leq a \Longrightarrow a \leq y\$k \Longrightarrow (\exists z{\in}s.\ z\$k = a)$
using *connected-ivt-hyperplane*[*of s x y axis k 1 a*]
by (*auto simp add*: *inner-axis inner-commute*)

lemma *subspace-substandard-cart*: $subspace\ \{x::real\hat{}\text{-}.\ (\forall i.\ P\ i \longrightarrow x\$i = 0)\}$
unfolding *subspace-def* **by** *auto*

lemma *closed-substandard-cart*:
$closed\ \{x::'a::real\text{-}normed\text{-}vector\ \hat{}\ 'n.\ \forall i.\ P\ i \longrightarrow x\$i = 0\}$
proof −
{ fix $i::'n$
have $closed\ \{x::'a\ \hat{}\ 'n.\ P\ i \longrightarrow x\$i = 0\}$
by (*cases P i*) (*simp-all add*: *closed-Collect-eq*) **}**
thus *?thesis*
unfolding *Collect-all-eq* **by** (*simp add*: *closed-INT*)
qed

lemma *dim-substandard-cart*: $dim\ \{x::real\hat{}'n.\ \forall i.\ i \notin d \longrightarrow x\$i = 0\} = card\ d$
(**is** *dim ?A* = -)
proof −
let *?a* = $\lambda x.\ axis\ x\ 1 :: real\hat{}'n$
have ∗: $\{x.\ \forall i{\in}Basis.\ i \notin ?a\ `\ d \longrightarrow x \cdot i = 0\} = ?A$
by (*auto simp*: *image-iff Basis-vec-def axis-eq-axis inner-axis*)
have *?a ` d* ⊆ *Basis*
by (*auto simp*: *Basis-vec-def*)
thus *?thesis*
using *dim-substandard*[*of ?a ` d*] *card-image*[*of ?a d*]
by (*auto simp*: *axis-eq-axis inj-on-def* ∗)
qed

lemma *affinity-inverses*:
assumes *m0*: $m \neq (0::'a::field)$
shows $(\lambda x.\ m\ *s\ x + c) \circ (\lambda x.\ inverse(m)\ *s\ x + (-(inverse(m)\ *s\ c))) = id$
$(\lambda x.\ inverse(m)\ *s\ x + (-(inverse(m)\ *s\ c))) \circ (\lambda x.\ m\ *s\ x + c) = id$
using *m0*
apply (*auto simp add*: *fun-eq-iff vector-add-ldistrib diff-conv-add-uminus simp del*: *add-uminus-conv-diff*)
apply (*simp-all add*: *vector-smult-lneg*[*symmetric*] *vector-smult-assoc vector-sneg-minus1* [*symmetric*])
done

lemma *vector-affinity-eq*:
assumes *m0*: $(m::'a::field) \neq 0$
shows $m\ *s\ x + c = y \longleftrightarrow x = inverse\ m\ *s\ y + -(inverse\ m\ *s\ c)$
proof
assume *h*: $m\ *s\ x + c = y$

hence $m *s x = y - c$ **by** (*simp add*: *field-simps*)
hence *inverse* $m *s (m *s x) =$ *inverse* $m *s (y - c)$ **by** *simp*
then show $x =$ *inverse* $m *s y + - ($*inverse* $m *s c)$
using *m0* **by** (*simp add*: *vector-smult-assoc vector-ssub-ldistrib*)
next
assume *h*: $x =$ *inverse* $m *s y + - ($*inverse* $m *s c)$
show $m *s x + c = y$ **unfolding** *h*
using *m0* **by** (*simp add*: *vector-smult-assoc vector-ssub-ldistrib*)
qed

lemma *vector-eq-affinity*:
$(m::'a::field) \neq 0 ==> (y = m *s x + c \longleftrightarrow$ *inverse*$(m) *s y + -($*inverse*$(m) *s c) = x)$
using *vector-affinity-eq*[**where** $m=m$ **and** $x=x$ **and** $y=y$ **and** $c=c$]
by *metis*

lemma *vector-cart*:
fixes $f :: real\hat{}'n \Rightarrow real$
shows $(\chi\ i.\ f\ (axis\ i\ 1)) = (\sum i{\in}Basis.\ f\ i *_R i)$
unfolding *euclidean-eq-iff*[**where** $'a=real\hat{}'n$]
by *simp* (*simp add*: *Basis-vec-def inner-axis*)

lemma *const-vector-cart*:$((\chi\ i.\ d)::real\hat{}'n) = (\sum i{\in}Basis.\ d *_R i)$
by (*rule vector-cart*)

42.7 Convex Euclidean Space

lemma *Cart-1*:$(1::real\hat{}'n) = \sum Basis$
using *const-vector-cart*[*of 1*] **by** (*simp add*: *one-vec-def*)

declare *vector-add-ldistrib*[*simp*] *vector-ssub-ldistrib*[*simp*] *vector-smult-assoc*[*simp*] *vector-smult-rneg*[*simp*]
declare *vector-sadd-rdistrib*[*simp*] *vector-sub-rdistrib*[*simp*]

lemmas *vector-component-simps* = *vector-minus-component vector-smult-component vector-add-component less-eq-vec-def vec-lambda-beta vector-uminus-component*

lemma *convex-box-cart*:
assumes $\bigwedge i.$ *convex* $\{x.\ P\ i\ x\}$
shows *convex* $\{x.\ \forall i.\ P\ i\ (x\$i)\}$
using *assms* **unfolding** *convex-def* **by** *auto*

lemma *convex-positive-orthant-cart*: *convex* $\{x::real\hat{}'n.\ (\forall i.\ 0 \leq x\$i)\}$
by (*rule convex-box-cart*) (*simp add*: *atLeast-def*[*symmetric*] *convex-real-interval*)

lemma *unit-interval-convex-hull-cart*:
cbox $(0::real\hat{}'n)\ 1 =$ *convex hull* $\{x.\ \forall i.\ (x\$i = 0) \vee (x\$i = 1)\}$
unfolding *Cart-1 unit-interval-convex-hull*[**where** $'a=real\hat{}'n$] *box-real*[*symmetric*]
by (*rule arg-cong*[**where** $f=\lambda x.$ *convex hull* x]) (*simp add*: *Basis-vec-def inner-axis*)

lemma *cube-convex-hull-cart*:
assumes *0 < d*
obtains *s*::(*real^'n*) *set*
where *finite s cbox* (*x* − (*χ i. d*)) (*x* + (*χ i. d*)) = *convex hull s*
proof −
from *assms* **obtain** *s* **where** *finite s*
and *cbox* (*x* − *setsum* (*op* ∗*R d*) *Basis*) (*x* + *setsum* (*op* ∗*R d*) *Basis*) = *convex hull s*
by (*rule cube-convex-hull*)
with *that*[*of s*] **show** *thesis*
by (*simp add*: *const-vector-cart*)
qed

42.8 Derivative

definition *jacobian f net* = *matrix*(*frechet-derivative f net*)

lemma *jacobian-works*:
(*f*::(*real^'a*) ⇒ (*real^'b*)) *differentiable net* ⟷
(*f has-derivative* (*λh.* (*jacobian f net*) ∗*v h*)) *net*
apply *rule*
unfolding *jacobian-def*
apply (*simp only*: *matrix-works*[*OF linear-frechet-derivative*]) **defer**
apply (*rule differentiableI*)
apply *assumption*
unfolding *frechet-derivative-works*
apply *assumption*
done

42.9 Component of the differential must be zero if it exists at a local maximum or minimum for that corresponding component.

lemma *differential-zero-maxmin-cart*:
fixes *f*::*real^'a* ⇒ *real^'b*
assumes *0 < e* ((∀ *y* ∈ *ball x e.* (*f y*)$*k* ≤ (*f x*)$*k*) ∨ (∀ *y*∈*ball x e.* (*f x*)$*k* ≤ (*f y*)$*k*))
f differentiable (*at x*)
shows *jacobian f* (*at x*) $ *k* = *0*
using *differential-zero-maxmin-component*[*of axis k 1 e x f*] *assms*
vector-cart[*of λj. frechet-derivative f* (*at x*) *j* $ *k*]
by (*simp add*: *Basis-vec-def axis-eq-axis inner-axis jacobian-def matrix-def*)

42.10 Lemmas for working on (*real*, *1*) *vec*

lemma *forall-1*[*simp*]: (∀ *i*::*1. P i*) ⟷ *P 1*
by (*metis* (*full-types*) *num1-eq-iff*)

lemma *ex-1*[*simp*]: $(\exists x{::}1.\ P\ x) \longleftrightarrow P\ 1$
by *auto* (*metis* (*full-types*) *num1-eq-iff*)

lemma *exhaust-2*:
fixes $x :: 2$
shows $x = 1 \vee x = 2$
proof (*induct x*)
case (*of-int z*)
then have $0 <= z$ **and** $z < 2$ **by** *simp-all*
then have $z = 0 \mid z = 1$ **by** *arith*
then show *?case* **by** *auto*
qed

lemma *forall-2*: $(\forall i{::}2.\ P\ i) \longleftrightarrow P\ 1 \wedge P\ 2$
by (*metis exhaust-2*)

lemma *exhaust-3*:
fixes $x :: 3$
shows $x = 1 \vee x = 2 \vee x = 3$
proof (*induct x*)
case (*of-int z*)
then have $0 <= z$ **and** $z < 3$ **by** *simp-all*
then have $z = 0 \vee z = 1 \vee z = 2$ **by** *arith*
then show *?case* **by** *auto*
qed

lemma *forall-3*: $(\forall i{::}3.\ P\ i) \longleftrightarrow P\ 1 \wedge P\ 2 \wedge P\ 3$
by (*metis exhaust-3*)

lemma *UNIV-1* [*simp*]: $UNIV = \{1{::}1\}$
by (*auto simp add*: *num1-eq-iff*)

lemma *UNIV-2*: $UNIV = \{1{::}2, 2{::}2\}$
using *exhaust-2* **by** *auto*

lemma *UNIV-3*: $UNIV = \{1{::}3, 2{::}3, 3{::}3\}$
using *exhaust-3* **by** *auto*

lemma *setsum-1*: $setsum\ f\ (UNIV{::}1\ set) = f\ 1$
unfolding *UNIV-1* **by** *simp*

lemma *setsum-2*: $setsum\ f\ (UNIV{::}2\ set) = f\ 1 + f\ 2$
unfolding *UNIV-2* **by** *simp*

lemma *setsum-3*: $setsum\ f\ (UNIV{::}3\ set) = f\ 1 + f\ 2 + f\ 3$
unfolding *UNIV-3* **by** (*simp add*: *ac-simps*)

instantiation *num1* :: *cart-one*
begin

instance
proof
show *CARD(1) = Suc 0* **by** *auto*
qed

end

42.11 The collapse of the general concepts to dimension one.

lemma *vector-one*: $(x::'a\ \hat{}\ 1) = (\chi\ i.\ (x\$1))$
by (*simp add*: *vec-eq-iff*)

lemma *forall-one*: $(\forall (x::'a\ \hat{}\ 1).\ P\ x) \longleftrightarrow (\forall x.\ P(\chi\ i.\ x))$
apply *auto*
apply (*erule-tac x*= *x*$*1* **in** *allE*)
apply (*simp only*: *vector-one*[*symmetric*])
done

lemma *norm-vector-1*: *norm* $(x :: -\hat{}1) =$ *norm* $(x\$1)$
by (*simp add*: *norm-vec-def*)

lemma *norm-real*: $norm(x::real\ \hat{}\ 1) = |x\$1|$
by (*simp add*: *norm-vector-1*)

lemma *dist-real*: $dist(x::real\ \hat{}\ 1)\ y = |(x\$1) - (y\$1)|$
by (*auto simp add*: *norm-real dist-norm*)

42.12 Explicit vector construction from lists.

definition $vector\ l = (\chi\ i.\ foldr\ (\lambda x\ f\ n.\ fun\text{-}upd\ (f\ (n+1))\ n\ x)\ l\ (\lambda n\ x.\ 0)\ 1\ i)$

lemma *vector-1*: $(vector[x])\ \$1 = x$
unfolding *vector-def* **by** *simp*

lemma *vector-2*:
$(vector[x,y])\ \$1 = x$
$(vector[x,y] :: 'a\hat{}2)\$2 = (y::'a::zero)$
unfolding *vector-def* **by** *simp-all*

lemma *vector-3*:
$(vector\ [x,y,z] ::('a::zero)\hat{}3)\$1 = x$
$(vector\ [x,y,z] ::('a::zero)\hat{}3)\$2 = y$
$(vector\ [x,y,z] ::('a::zero)\hat{}3)\$3 = z$
unfolding *vector-def* **by** *simp-all*

lemma *forall-vector-1*: $(\forall v::'a::zero\hat{}1.\ P\ v) \longleftrightarrow (\forall x.\ P(vector[x]))$
apply *auto*
apply (*erule-tac x*=*v*$*1* **in** *allE*)
apply (*subgoal-tac vector* $[v\$1] = v$)

apply *simp*
apply (*vector vector-def*)
apply *simp*
done

lemma *forall-vector-2*: $(\forall v::'a::zero\hat{}2.\ P\ v) \longleftrightarrow (\forall x\ y.\ P(vector[x,\ y]))$
apply *auto*
apply (*erule-tac x=v$1* **in** *allE*)
apply (*erule-tac x=v$2* **in** *allE*)
apply (*subgoal-tac vector* $[v\$1,\ v\$2] = v$)
apply *simp*
apply (*vector vector-def*)
apply (*simp add*: *forall-2*)
done

lemma *forall-vector-3*: $(\forall v::'a::zero\hat{}3.\ P\ v) \longleftrightarrow (\forall x\ y\ z.\ P(vector[x,\ y,\ z]))$
apply *auto*
apply (*erule-tac x=v$1* **in** *allE*)
apply (*erule-tac x=v$2* **in** *allE*)
apply (*erule-tac x=v$3* **in** *allE*)
apply (*subgoal-tac vector* $[v\$1,\ v\$2,\ v\$3] = v$)
apply *simp*
apply (*vector vector-def*)
apply (*simp add*: *forall-3*)
done

lemma *bounded-linear-component-cart*[*intro*]: *bounded-linear* $(\lambda x::real\hat{}'n.\ x\ \$\ k)$
apply (*rule bounded-linearI*[**where** *K=1*])
using *component-le-norm-cart*[*of - k*] **unfolding** *real-norm-def* **by** *auto*

lemma *integral-component-eq-cart*[*simp*]:
fixes *f* :: *'n::euclidean-space* ⇒ *real^'m*
assumes *f integrable-on s*
shows *integral s* $(\lambda x.\ f\ x\ \$\ k) =$ *integral s f* $\$\ k$
using *integral-linear*[*OF assms*(*1*) *bounded-linear-component-cart*,*unfolded o-def*]
.

lemma *interval-split-cart*:
$\{a..b::real\hat{}'n\} \cap \{x.\ x\$k \le c\} = \{a\ ..\ (\chi\ i.\ if\ i = k\ then\ min\ (b\$k)\ c\ else\ b\$i)\}$
$cbox\ a\ b \cap \{x.\ x\$k \ge c\} = \{(\chi\ i.\ if\ i = k\ then\ max\ (a\$k)\ c\ else\ a\$i)\ ..\ b\}$
apply (*rule-tac*[!] *set-eqI*)
unfolding *Int-iff mem-interval-cart mem-Collect-eq interval-cbox-cart*
unfolding *vec-lambda-beta*
by *auto*

end

43 Fashoda meet theorem

theory *Fashoda*
imports *Brouwer-Fixpoint Path-Connected Cartesian-Euclidean-Space*
begin

43.1 Bijections between intervals.

definition *interval-bij* :: *'a* × *'a* ⇒ *'a* × *'a* ⇒ *'a* ⇒ *'a*::*euclidean-space*
where *interval-bij* =
(λ(*a*, *b*) (*u*, *v*) *x*. (∑ *i*∈*Basis*. (*u*•*i* + (*x*•*i* − *a*•*i*) / (*b*•*i* − *a*•*i*) ∗ (*v*•*i* − *u*•*i*)) ∗*R* *i*))

lemma *interval-bij-affine*:
interval-bij (*a*,*b*) (*u*,*v*) = (λ*x*. (∑ *i*∈*Basis*. ((*v*•*i* − *u*•*i*) / (*b*•*i* − *a*•*i*) ∗ (*x*•*i*)) ∗*R* *i*) +
(∑ *i*∈*Basis*. (*u*•*i* − (*v*•*i* − *u*•*i*) / (*b*•*i* − *a*•*i*) ∗ (*a*•*i*)) ∗*R* *i*))
by (*auto simp*: *setsum.distrib*[*symmetric*] *scaleR-add-left*[*symmetric*] *interval-bij-def fun-eq-iff*
field-simps inner-simps add-divide-distrib[*symmetric*] *intro*!: *setsum.cong*)

lemma *continuous-interval-bij*:
fixes *a b* :: *'a*::*euclidean-space*
shows *continuous* (*at x*) (*interval-bij* (*a*, *b*) (*u*, *v*))
by (*auto simp add*: *divide-inverse interval-bij-def intro*!: *continuous-setsum continuous-intros*)

lemma *continuous-on-interval-bij*: *continuous-on s* (*interval-bij* (*a*, *b*) (*u*, *v*))
apply(*rule continuous-at-imp-continuous-on*)
apply (*rule*, *rule continuous-interval-bij*)
done

lemma *in-interval-interval-bij*:
fixes *a b u v x* :: *'a*::*euclidean-space*
assumes *x* ∈ *cbox a b*
and *cbox u v* ≠ {}
shows *interval-bij* (*a*, *b*) (*u*, *v*) *x* ∈ *cbox u v*
apply (*simp only*: *interval-bij-def split-conv mem-box inner-setsum-left-Basis cong*: *ball-cong*)
apply *safe*
proof −
fix *i* :: *'a*
assume *i*: *i* ∈ *Basis*
have *cbox a b* ≠ {}
using *assms* **by** *auto*
with *i* **have** ∗: *a*•*i* ≤ *b*•*i u*•*i* ≤ *v*•*i*
using *assms*(*2*) **by** (*auto simp add*: *box-eq-empty*)
have *x*: *a*•*i*≤*x*•*i x*•*i*≤*b*•*i*
using *assms*(*1*)[*unfolded mem-box*] **using** *i* **by** *auto*
have *0* ≤ (*x* • *i* − *a* • *i*) / (*b* • *i* − *a* • *i*) ∗ (*v* • *i* − *u* • *i*)
using ∗ *x* **by** *auto*

then show $u \cdot i \leq u \cdot i + (x \cdot i - a \cdot i) \,/\, (b \cdot i - a \cdot i) * (v \cdot i - u \cdot i)$
using $*$ **by** *auto*
have $((x \cdot i - a \cdot i) \,/\, (b \cdot i - a \cdot i)) * (v \cdot i - u \cdot i) \leq 1 * (v \cdot i - u \cdot i)$
apply (*rule mult-right-mono*)
unfolding *divide-le-eq-1*
using $*$ *x*
apply *auto*
done
then show $u \cdot i + (x \cdot i - a \cdot i) \,/\, (b \cdot i - a \cdot i) * (v \cdot i - u \cdot i) \leq v \cdot i$
using $*$ **by** *auto*
qed

lemma *interval-bij-bij*:
$\forall (i::'a::euclidean\text{-}space) \in Basis.\ a \cdot i < b \cdot i \wedge u \cdot i < v \cdot i \Longrightarrow$
interval-bij (*a*, *b*) (*u*, *v*) (*interval-bij* (*u*, *v*) (*a*, *b*) *x*) = *x*
by (*auto simp*: *interval-bij-def euclidean-eq-iff* [**where** *'a*='*a*])

lemma *interval-bij-bij-cart*: **fixes** *x*::*real^'n* **assumes** $\forall\, i.\ a\$i < b\$i \wedge u\$i < v\$i$
shows *interval-bij* (*a*,*b*) (*u*,*v*) (*interval-bij* (*u*,*v*) (*a*,*b*) *x*) = *x*
using *assms* **by** (*intro interval-bij-bij*) (*auto simp*: *Basis-vec-def inner-axis*)

43.2 Fashoda meet theorem

lemma *infnorm-2*:
fixes *x* :: *real^2*
shows *infnorm x* = *max* |*x*$*1*| |*x*$*2*|
unfolding *infnorm-cart UNIV-2* **by** (*rule cSup-eq*) *auto*

lemma *infnorm-eq-1-2*:
fixes *x* :: *real^2*
shows *infnorm x* = *1* $\longleftrightarrow$
|*x*$*1*| $\leq$ *1* $\wedge$ |*x*$*2*| $\leq$ *1* $\wedge$ (*x*$*1* = -1 $\vee$ *x*$*1* = *1* $\vee$ *x*$*2* = -1 $\vee$ *x*$*2* = *1*)
unfolding *infnorm-2* **by** *auto*

lemma *infnorm-eq-1-imp*:
fixes *x* :: *real^2*
assumes *infnorm x* = *1*
shows |*x*$*1*| $\leq$ *1* **and** |*x*$*2*| $\leq$ *1*
using *assms* **unfolding** *infnorm-eq-1-2* **by** *auto*

lemma *fashoda-unit*:
fixes *f g* :: *real* $\Rightarrow$ *real^2*
assumes *f* ' {-1 .. *1*} $\subseteq$ *cbox* (-1) *1*
and *g* ' {-1 .. *1*} $\subseteq$ *cbox* (-1) *1*
and *continuous-on* {-1 .. *1*} *f*
and *continuous-on* {-1 .. *1*} *g*
and *f* ($-$ *1*)$*1* = $-$ *1*
and *f 1*$*1* = *1 g* ($-$ *1*) $*2* = -1
and *g 1* $*2* = *1*

```
  shows ∃ s∈{−1 .. 1}. ∃ t∈{−1 .. 1}. f s = g t
proof (rule ccontr)
  assume ¬ ?thesis
  note as = this[unfolded bex-simps,rule-format]
  def sqprojection ≡ λz::real^2. (inverse (infnorm z)) *R z
  def negatex ≡ λx::real^2. (vector [−(x$1), x$2])::real^2
  have lem1: ∀ z::real^2. infnorm (negatex z) = infnorm z
    unfolding negatex-def infnorm-2 vector-2 by auto
  have lem2: ∀ z. z ≠ 0 ⟶ infnorm (sqprojection z) = 1
    unfolding sqprojection-def
    unfolding infnorm-mul[unfolded scalar-mult-eq-scaleR]
    unfolding abs-inverse real-abs-infnorm
    apply (subst infnorm-eq-0[symmetric])
    apply auto
    done
  let ?F = λw::real^2. (f ∘ (λx. x$1)) w − (g ∘ (λx. x$2)) w
  have *: ⋀i. (λx::real^2. x $ i) ` cbox (− 1) 1 = {−1 .. 1}
    apply (rule set-eqI)
    unfolding image-iff Bex-def mem-interval-cart interval-cbox-cart
    apply rule
    defer
    apply (rule-tac x=vec x in exI)
    apply auto
    done
  {
    fix x
    assume x ∈ (λw. (f ∘ (λx. x $ 1)) w − (g ∘ (λx. x $ 2)) w) ` (cbox (− 1)
(1::real^2))
    then obtain w :: real^2 where w:
        w ∈ cbox (− 1) 1
        x = (f ∘ (λx. x $ 1)) w − (g ∘ (λx. x $ 2)) w
      unfolding image-iff ..
    then have x ≠ 0
      using as[of w$1 w$2]
      unfolding mem-interval-cart atLeastAtMost-iff
      by auto
  } note x0 = this
  have 21: ⋀i::2. i ≠ 1 ⟹ i = 2
    using UNIV-2 by auto
  have 1: box (− 1) (1::real^2) ≠ {}
    unfolding interval-eq-empty-cart by auto
  have 2: continuous-on (cbox (− 1) 1) (negatex ∘ sqprojection ∘ ?F)
    apply (intro continuous-intros continuous-on-component)
    unfolding *
    apply (rule assms)+
    apply (subst sqprojection-def)
    apply (intro continuous-intros)
    apply (simp add: infnorm-eq-0 x0)
    apply (rule linear-continuous-on)
```

```
  proof −
    show bounded-linear negatex
      apply (rule bounded-linearI')
      unfolding vec-eq-iff
    proof (rule-tac[!] allI)
      fix i :: 2
      fix x y :: real^2
      fix c :: real
      show negatex (x + y) $ i =
        (negatex x + negatex y) $ i negatex (c *R x) $ i = (c *R negatex x) $ i
        apply −
        apply (case-tac[!] i≠1)
        prefer 3
        apply (drule-tac[1−2] 21)
        unfolding negatex-def
        apply (auto simp add:vector-2)
        done
    qed
  qed
  have 3: (negatex ∘ sqprojection ∘ ?F) ' cbox (−1) 1 ⊆ cbox (−1) 1
    unfolding subset-eq
  proof (rule, goal-cases)
    case (1 x)
    then obtain y :: real^2 where y:
        y ∈ cbox (− 1) 1
        x = (negatex ∘ sqprojection ∘ (λw. (f ∘ (λx. x $ 1)) w − (g ∘ (λx. x $ 2))
w)) y
      unfolding image-iff ..
    have ?F y ≠ 0
      apply (rule x0)
      using y(1)
      apply auto
      done
    then have ∗: infnorm (sqprojection (?F y)) = 1
      unfolding y o-def
      by − (rule lem2[rule-format])
    have infnorm x = 1
      unfolding ∗[symmetric] y o-def
      by (rule lem1[rule-format])
    then show x ∈ cbox (−1) 1
      unfolding mem-interval-cart interval-cbox-cart infnorm-2
      apply −
      apply rule
    proof −
      fix i
      assume max |x $ 1| |x $ 2| = 1
      then show (− 1) $ i ≤ x $ i ∧ x $ i ≤ 1 $ i
        apply (cases i = 1)
        defer
```

```
      apply (drule 21)
      apply auto
      done
    qed
  qed
  obtain x :: real^2 where x:
     x ∈ cbox (− 1) 1
     (negatex ∘ sqprojection ∘ (λw. (f ∘ (λx. x $ 1)) w − (g ∘ (λx. x $ 2)) w)) x
= x
    apply (rule brouwer-weak[of cbox (− 1) (1::real^2) negatex ∘ sqprojection ∘
?F])
    apply (rule compact-cbox convex-box)+
    unfolding interior-cbox
    apply (rule 1 2 3)+
    apply blast
    done
  have ?F x ≠ 0
    apply (rule x0)
    using x(1)
    apply auto
    done
  then have ∗: infnorm (sqprojection (?F x)) = 1
    unfolding o-def
    by (rule lem2[rule-format])
  have nx: infnorm x = 1
    apply (subst x(2)[symmetric])
    unfolding ∗[symmetric] o-def
    apply (rule lem1[rule-format])
    done
  have ∀x i. x ≠ 0 ⟶ (0 < (sqprojection x)$i ⟷ 0 < x$i)
    and ∀x i. x ≠ 0 ⟶ ((sqprojection x)$i < 0 ⟷ x$i < 0)
    apply −
    apply (rule-tac[!] allI impI)+
  proof −
    fix x :: real^2
    fix i :: 2
    assume x: x ≠ 0
    have inverse (infnorm x) > 0
      using x[unfolded infnorm-pos-lt[symmetric]] by auto
    then show (0 < sqprojection x $ i) = (0 < x $ i)
      and (sqprojection x $ i < 0) = (x $ i < 0)
      unfolding sqprojection-def vector-component-simps vector-scaleR-component
real-scaleR-def
      unfolding zero-less-mult-iff mult-less-0-iff
      by (auto simp add: field-simps)
  qed
  note lem3 = this[rule-format]
  have x1: x $ 1 ∈ {− 1..1::real} x $ 2 ∈ {− 1..1::real}
    using x(1) unfolding mem-interval-cart by auto
```

```
then have nz: f (x $ 1) - g (x $ 2) ≠ 0
  unfolding right-minus-eq
  apply -
  apply (rule as)
  apply auto
  done
have x $ 1 = -1 ∨ x $ 1 = 1 ∨ x $ 2 = -1 ∨ x $ 2 = 1
  using nx unfolding infnorm-eq-1-2 by auto
then show False
proof -
  fix P Q R S
  presume P ∨ Q ∨ R ∨ S
    and P ⟹ False
    and Q ⟹ False
    and R ⟹ False
    and S ⟹ False
  then show False by auto
next
  assume as: x$1 = 1
  then have *: f (x $ 1) $ 1 = 1
    using assms(6) by auto
  have sqprojection (f (x$1) - g (x$2)) $ 1 < 0
    using x(2)[unfolded o-def vec-eq-iff,THEN spec[where x=1]]
    unfolding as negatex-def vector-2
    by auto
  moreover
  from x1 have g (x $ 2) ∈ cbox (-1) 1
    apply -
    apply (rule assms(2)[unfolded subset-eq,rule-format])
    apply auto
    done
  ultimately show False
    unfolding lem3[OF nz] vector-component-simps * mem-interval-cart
    apply (erule-tac x=1 in allE)
    apply auto
    done
next
  assume as: x$1 = -1
  then have *: f (x $ 1) $ 1 = - 1
    using assms(5) by auto
  have sqprojection (f (x$1) - g (x$2)) $ 1 > 0
    using x(2)[unfolded o-def vec-eq-iff,THEN spec[where x=1]]
    unfolding as negatex-def vector-2
    by auto
  moreover
  from x1 have g (x $ 2) ∈ cbox (-1) 1
    apply -
    apply (rule assms(2)[unfolded subset-eq,rule-format])
    apply auto
```

```
    done
  ultimately show False
    unfolding lem3[OF nz] vector-component-simps * mem-interval-cart
    apply (erule-tac x=1 in allE)
    apply auto
    done
next
  assume as: x$2 = 1
  then have *: g (x $ 2) $ 2 = 1
    using assms(8) by auto
  have sqprojection (f (x$1) - g (x$2)) $ 2 > 0
    using x(2)[unfolded o-def vec-eq-iff,THEN spec[where x=2]]
    unfolding as negatex-def vector-2
    by auto
  moreover
  from x1 have f (x $ 1) ∈ cbox (-1) 1
    apply -
    apply (rule assms(1)[unfolded subset-eq,rule-format])
    apply auto
    done
  ultimately show False
    unfolding lem3[OF nz] vector-component-simps * mem-interval-cart
    apply (erule-tac x=2 in allE)
    apply auto
    done
next
  assume as: x$2 = -1
  then have *: g (x $ 2) $ 2 = - 1
    using assms(7) by auto
  have sqprojection (f (x$1) - g (x$2)) $ 2 < 0
    using x(2)[unfolded o-def vec-eq-iff,THEN spec[where x=2]]
    unfolding as negatex-def vector-2
    by auto
  moreover
  from x1 have f (x $ 1) ∈ cbox (-1) 1
    apply -
    apply (rule assms(1)[unfolded subset-eq,rule-format])
    apply auto
    done
  ultimately show False
    unfolding lem3[OF nz] vector-component-simps * mem-interval-cart
    apply (erule-tac x=2 in allE)
    apply auto
    done
qed auto
qed

lemma fashoda-unit-path:
  fixes f g :: real ⇒ real^2
```

```
  assumes path f
    and path g
    and path-image f ⊆ cbox (−1) 1
    and path-image g ⊆ cbox (−1) 1
    and (pathstart f)$1 = −1
    and (pathfinish f)$1 = 1
    and (pathstart g)$2 = −1
    and (pathfinish g)$2 = 1
  obtains z where z ∈ path-image f and z ∈ path-image g
proof −
  note assms=assms[unfolded path-def pathstart-def pathfinish-def path-image-def]
  def iscale ≡ λz::real. inverse 2 *R (z + 1)
  have isc: iscale ‘ {− 1..1} ⊆ {0..1}
    unfolding iscale-def by auto
  have ∃s∈{− 1..1}. ∃t∈{− 1..1}. (f ∘ iscale) s = (g ∘ iscale) t
  proof (rule fashoda-unit)
    show (f ∘ iscale) ‘ {− 1..1} ⊆ cbox (− 1) 1 (g ∘ iscale) ‘ {− 1..1} ⊆ cbox
(− 1) 1
      using isc and assms(3−4) by (auto simp add: image-comp [symmetric])
    have *: continuous-on {− 1..1} iscale
      unfolding iscale-def by (rule continuous-intros)+
    show continuous-on {− 1..1} (f ∘ iscale) continuous-on {− 1..1} (g ∘ iscale)
      apply −
      apply (rule-tac[!] continuous-on-compose[OF *])
      apply (rule-tac[!] continuous-on-subset[OF - isc])
      apply (rule assms)+
      done
    have *: (1 / 2) *R (1 + (1::real^1)) = 1
      unfolding vec-eq-iff by auto
    show (f ∘ iscale) (− 1) $ 1 = − 1
      and (f ∘ iscale) 1 $ 1 = 1
      and (g ∘ iscale) (− 1) $ 2 = −1
      and (g ∘ iscale) 1 $ 2 = 1
      unfolding o-def iscale-def
      using assms
      by (auto simp add: *)
  qed
  then obtain s t where st:
      s ∈ {− 1..1}
      t ∈ {− 1..1}
      (f ∘ iscale) s = (g ∘ iscale) t
    by auto
  show thesis
    apply (rule-tac z = f (iscale s) in that)
    using st
    unfolding o-def path-image-def image-iff
    apply −
    apply (rule-tac x=iscale s in bexI)
    prefer 3
```

```
    apply (rule-tac x=iscale t in bexI)
    using isc[unfolded subset-eq, rule-format]
    apply auto
    done
qed

lemma fashoda:
  fixes b :: real^2
  assumes path f
    and path g
    and path-image f ⊆ cbox a b
    and path-image g ⊆ cbox a b
    and (pathstart f)$1 = a$1
    and (pathfinish f)$1 = b$1
    and (pathstart g)$2 = a$2
    and (pathfinish g)$2 = b$2
  obtains z where z ∈ path-image f and z ∈ path-image g
proof −
  fix P Q S
  presume P ∨ Q ∨ S P ⟹ thesis and Q ⟹ thesis and S ⟹ thesis
  then show thesis
    by auto
next
  have cbox a b ≠ {}
    using assms(3) using path-image-nonempty[of f] by auto
  then have a ≤ b
    unfolding interval-eq-empty-cart less-eq-vec-def by (auto simp add: not-less)
  then show a$1 = b$1 ∨ a$2 = b$2 ∨ (a$1 < b$1 ∧ a$2 < b$2)
    unfolding less-eq-vec-def forall-2 by auto
next
  assume as: a$1 = b$1
  have ∃z∈path-image g. z$2 = (pathstart f)$2
    apply (rule connected-ivt-component-cart)
    apply (rule connected-path-image assms)+
    apply (rule pathstart-in-path-image)
    apply (rule pathfinish-in-path-image)
   unfolding assms using assms(3)[unfolded path-image-def subset-eq,rule-format,of
f 0]
    unfolding pathstart-def
    apply (auto simp add: less-eq-vec-def mem-interval-cart)
    done
  then obtain z :: real^2 where z: z ∈ path-image g z $ 2 = pathstart f $ 2 ..
  have z ∈ cbox a b
    using z(1) assms(4)
    unfolding path-image-def
    by blast
  then have z = f 0
    unfolding vec-eq-iff forall-2
    unfolding z(2) pathstart-def
```

using *assms(3)*[*unfolded path-image-def subset-eq mem-interval-cart*,*rule-format*,*of f 0 1*]
unfolding *mem-interval-cart*
apply (*erule-tac x=1* **in** *allE*)
using *as*
apply *auto*
done
then show *thesis*
apply −
apply (*rule that*[*OF - z(1)*])
unfolding *path-image-def*
apply *auto*
done
next
assume *as*: *a$2 = b$2*
have ∃*z*∈*path-image f*. *z$1 = (pathstart g)$1*
apply (*rule connected-ivt-component-cart*)
apply (*rule connected-path-image assms*)+
apply (*rule pathstart-in-path-image*)
apply (*rule pathfinish-in-path-image*)
unfolding *assms*
using *assms(4)*[*unfolded path-image-def subset-eq*,*rule-format*,*of g 0*]
unfolding *pathstart-def*
apply (*auto simp add*: *less-eq-vec-def mem-interval-cart*)
done
then obtain *z* **where** *z*: *z ∈ path-image f z $ 1 = pathstart g $ 1* **..**
have *z ∈ cbox a b*
using *z(1) assms(3)*
unfolding *path-image-def*
by *blast*
then have *z = g 0*
unfolding *vec-eq-iff forall-2*
unfolding *z(2) pathstart-def*
using *assms(4)*[*unfolded path-image-def subset-eq mem-interval-cart*,*rule-format*,*of g 0 2*]
unfolding *mem-interval-cart*
apply (*erule-tac x=2* **in** *allE*)
using *as*
apply *auto*
done
then show *thesis*
apply −
apply (*rule that*[*OF z(1)*])
unfolding *path-image-def*
apply *auto*
done
next
assume *as*: *a $ 1 < b $ 1 ∧ a $ 2 < b $ 2*
have *int-nem*: *cbox (−1) (1*::*real^2) ≠ {}*

```
    unfolding interval-eq-empty-cart by auto
  obtain z :: real^2 where z:
      z ∈ (interval-bij (a, b) (− 1, 1) ∘ f) ‘ {0..1}
      z ∈ (interval-bij (a, b) (− 1, 1) ∘ g) ‘ {0..1}
    apply (rule fashoda-unit-path[of interval-bij (a,b) (− 1,1) ∘ f interval-bij (a,b)
(− 1,1) ∘ g])
    unfolding path-def path-image-def pathstart-def pathfinish-def
    apply (rule-tac[1−2] continuous-on-compose)
    apply (rule assms[unfolded path-def] continuous-on-interval-bij)+
    unfolding subset-eq
    apply(rule-tac[1−2] ballI)
  proof −
    fix x
    assume x ∈ (interval-bij (a, b) (− 1, 1) ∘ f) ‘ {0..1}
    then obtain y where y:
        y ∈ {0..1}
        x = (interval-bij (a, b) (− 1, 1) ∘ f) y
      unfolding image-iff ..
    show x ∈ cbox (− 1) 1
      unfolding y o-def
      apply (rule in-interval-interval-bij)
      using y(1)
      using assms(3)[unfolded path-image-def subset-eq] int-nem
      apply auto
      done
  next
    fix x
    assume x ∈ (interval-bij (a, b) (− 1, 1) ∘ g) ‘ {0..1}
    then obtain y where y:
        y ∈ {0..1}
        x = (interval-bij (a, b) (− 1, 1) ∘ g) y
      unfolding image-iff ..
    show x ∈ cbox (− 1) 1
      unfolding y o-def
      apply (rule in-interval-interval-bij)
      using y(1)
      using assms(4)[unfolded path-image-def subset-eq] int-nem
      apply auto
      done
  next
    show (interval-bij (a, b) (− 1, 1) ∘ f) 0 $ 1 = −1
      and (interval-bij (a, b) (− 1, 1) ∘ f) 1 $ 1 = 1
      and (interval-bij (a, b) (− 1, 1) ∘ g) 0 $ 2 = −1
      and (interval-bij (a, b) (− 1, 1) ∘ g) 1 $ 2 = 1
      using assms as
      by (simp-all add: axis-in-Basis cart-eq-inner-axis pathstart-def pathfinish-def
interval-bij-def)
         (simp-all add: inner-axis)
  qed
```

```
  from z(1) obtain zf where zf:
      zf ∈ {0..1}
      z = (interval-bij (a, b) (− 1, 1) ∘ f) zf
    unfolding image-iff ..
  from z(2) obtain zg where zg:
      zg ∈ {0..1}
      z = (interval-bij (a, b) (− 1, 1) ∘ g) zg
    unfolding image-iff ..
  have ∗: ∀ i. (− 1) $ i < (1::real^2) $ i ∧ a $ i < b $ i
    unfolding forall-2
    using as
    by auto
  show thesis
    apply (rule-tac z=interval-bij (− 1,1) (a,b) z in that)
    apply (subst zf)
    defer
    apply (subst zg)
    unfolding o-def interval-bij-bij-cart[OF ∗] path-image-def
    using zf(1) zg(1)
    apply auto
    done
qed
```

43.3 Some slightly ad hoc lemmas I use below

```
lemma segment-vertical:
  fixes a :: real^2
  assumes a$1 = b$1
  shows x ∈ closed-segment a b ⟷
    x$1 = a$1 ∧ x$1 = b$1 ∧ (a$2 ≤ x$2 ∧ x$2 ≤ b$2 ∨ b$2 ≤ x$2 ∧ x$2 ≤
a$2)
  (is - = ?R)
proof −
  let ?L = ∃ u. (x $ 1 = (1 − u) ∗ a $ 1 + u ∗ b $ 1 ∧ x $ 2 = (1 − u) ∗ a $ 2
+ u ∗ b $ 2) ∧ 0 ≤ u ∧ u ≤ 1
  {
    presume ?L ⟹ ?R and ?R ⟹ ?L
    then show ?thesis
      unfolding closed-segment-def mem-Collect-eq
    unfolding vec-eq-iff forall-2 scalar-mult-eq-scaleR[symmetric] vector-component-simps
      by blast
  }
  {
    assume ?L
    then obtain u where u:
        x $ 1 = (1 − u) ∗ a $ 1 + u ∗ b $ 1
        x $ 2 = (1 − u) ∗ a $ 2 + u ∗ b $ 2
        0 ≤ u
        u ≤ 1
```

```
    by blast
  { fix b a
    assume b + u * a > a + u * b
    then have (1 − u) * b > (1 − u) * a
      by (auto simp add:field-simps)
    then have b ≥ a
      apply (drule-tac mult-left-less-imp-less)
      using u
      apply auto
      done
    then have u * a ≤ u * b
      apply −
      apply (rule mult-left-mono[OF - u(3)])
      using u(3−4)
      apply (auto simp add: field-simps)
      done
  } note * = this
  {
    fix a b
    assume u * b > u * a
    then have (1 − u) * a ≤ (1 − u) * b
      apply −
      apply (rule mult-left-mono)
      apply (drule mult-left-less-imp-less)
      using u
      apply auto
      done
    then have a + u * b ≤ b + u * a
      by (auto simp add: field-simps)
  } note ** = this
  then show ?R
    unfolding u assms
    using u
    by (auto simp add:field-simps not-le intro: * **)
}
{
  assume ?R
  then show ?L
  proof (cases x$2 = b$2)
    case True
    then show ?L
      apply (rule-tac x=(x$2 − a$2) / (b$2 − a$2) in exI)
      unfolding assms True
      using ‹?R›
      apply (auto simp add: field-simps)
      done
  next
    case False
    then show ?L
```

```
      apply (rule-tac x=1 - (x$2 - b$2) / (a$2 - b$2) in exI)
      unfolding assms
      using ‹?R›
      apply (auto simp add: field-simps)
      done
   qed
  }
qed

lemma segment-horizontal:
  fixes a :: real^2
  assumes a$2 = b$2
  shows x ∈ closed-segment a b ⟷
    x$2 = a$2 ∧ x$2 = b$2 ∧ (a$1 ≤ x$1 ∧ x$1 ≤ b$1 ∨ b$1 ≤ x$1 ∧ x$1 ≤
a$1)
  (is - = ?R)
proof -
  let ?L = ∃ u. (x $ 1 = (1 - u) * a $ 1 + u * b $ 1 ∧ x $ 2 = (1 - u) * a $ 2
+ u * b $ 2) ∧ 0 ≤ u ∧ u ≤ 1
  {
    presume ?L ⟹ ?R and ?R ⟹ ?L
    then show ?thesis
      unfolding closed-segment-def mem-Collect-eq
    unfolding vec-eq-iff forall-2 scalar-mult-eq-scaleR[symmetric] vector-component-simps
      by blast
  }
  {
    assume ?L
    then obtain u where u:
        x $ 1 = (1 - u) * a $ 1 + u * b $ 1
        x $ 2 = (1 - u) * a $ 2 + u * b $ 2
        0 ≤ u
        u ≤ 1
      by blast
    {
      fix b a
      assume b + u * a > a + u * b
      then have (1 - u) * b > (1 - u) * a
        by (auto simp add: field-simps)
      then have b ≥ a
        apply (drule-tac mult-left-less-imp-less)
        using u
        apply auto
        done
      then have u * a ≤ u * b
        apply -
        apply (rule mult-left-mono[OF - u(3)])
        using u(3-4)
        apply (auto simp add: field-simps)
```

```
      done
  } note ∗ = this
  {
   fix a b
   assume u ∗ b > u ∗ a
   then have (1 − u) ∗ a ≤ (1 − u) ∗ b
    apply −
    apply (rule mult-left-mono)
    apply (drule mult-left-less-imp-less)
    using u
    apply auto
    done
   then have a + u ∗ b ≤ b + u ∗ a
    by (auto simp add: field-simps)
  } note ∗∗ = this
  then show ?R
   unfolding u assms
   using u
   by (auto simp add: field-simps not-le intro: ∗ ∗∗)
 }
 {
  assume ?R
  then show ?L
  proof (cases x$1 = b$1)
   case True
   then show ?L
    apply (rule-tac x=(x$1 − a$1) / (b$1 − a$1) in exI)
    unfolding assms True
    using ‹?R›
    apply (auto simp add: field-simps)
    done
  next
   case False
   then show ?L
    apply (rule-tac x=1 − (x$1 − b$1) / (a$1 − b$1) in exI)
    unfolding assms
    using ‹?R›
    apply (auto simp add: field-simps)
    done
  qed
 }
qed
```

43.4 Useful Fashoda corollary pointed out to me by Tom Hales

```
lemma fashoda-interlace:
 fixes a :: real^2
 assumes path f
```

and *path g*
and *path-image f* ⊆ *cbox a b*
and *path-image g* ⊆ *cbox a b*
and (*pathstart f*)$*2* = *a*$*2*
and (*pathfinish f*)$*2* = *a*$*2*
and (*pathstart g*)$*2* = *a*$*2*
and (*pathfinish g*)$*2* = *a*$*2*
and (*pathstart f*)$*1* < (*pathstart g*)$*1*
and (*pathstart g*)$*1* < (*pathfinish f*)$*1*
and (*pathfinish f*)$*1* < (*pathfinish g*)$*1*
obtains *z* **where** *z* ∈ *path-image f* **and** *z* ∈ *path-image g*
proof −
have *cbox a b* ≠ {}
using *path-image-nonempty*[*of f*] **using** *assms*(*3*) **by** *auto*
note *ab*=*this*[*unfolded interval-eq-empty-cart not-ex forall-2 not-less*]
have *pathstart f* ∈ *cbox a b*
and *pathfinish f* ∈ *cbox a b*
and *pathstart g* ∈ *cbox a b*
and *pathfinish g* ∈ *cbox a b*
using *pathstart-in-path-image pathfinish-in-path-image*
using *assms*(*3*−*4*)
by *auto*
note *startfin* = *this*[*unfolded mem-interval-cart forall-2*]
let *?P1* = *linepath* (*vector*[*a*$*1* − *2*, *a*$*2* − *2*]) (*vector*[(*pathstart f*)$*1*,*a*$*2* − *2*]) +++
linepath(*vector*[(*pathstart f*)$*1*,*a*$*2* − *2*])(*pathstart f*) +++ *f* +++
linepath(*pathfinish f*)(*vector*[(*pathfinish f*)$*1*,*a*$*2* − *2*]) +++
linepath(*vector*[(*pathfinish f*)$*1*,*a*$*2* − *2*])(*vector*[*b*$*1* + *2*,*a*$*2* − *2*])
let *?P2* = *linepath*(*vector*[(*pathstart g*)$*1*, (*pathstart g*)$*2* − *3*])(*pathstart g*) +++ *g* +++
linepath(*pathfinish g*)(*vector*[(*pathfinish g*)$*1*,*a*$*2* − *1*]) +++
linepath(*vector*[(*pathfinish g*)$*1*,*a*$*2* − *1*])(*vector*[*b*$*1* + *1*,*a*$*2* − *1*]) +++
linepath(*vector*[*b*$*1* + *1*,*a*$*2* − *1*])(*vector*[*b*$*1* + *1*,*b*$*2* + *3*])
let *?a* = *vector*[*a*$*1* − *2*, *a*$*2* − *3*]
let *?b* = *vector*[*b*$*1* + *2*, *b*$*2* + *3*]
have *P1P2*: *path-image ?P1* = *path-image* (*linepath* (*vector*[*a*$*1* − *2*, *a*$*2* − *2*]) (*vector*[(*pathstart f*)$*1*,*a*$*2* − *2*])) ∪
path-image (*linepath*(*vector*[(*pathstart f*)$*1*,*a*$*2* − *2*])(*pathstart f*)) ∪ *path-image f* ∪
path-image (*linepath*(*pathfinish f*)(*vector*[(*pathfinish f*)$*1*,*a*$*2* − *2*])) ∪
path-image (*linepath*(*vector*[(*pathfinish f*)$*1*,*a*$*2* − *2*])(*vector*[*b*$*1* + *2*,*a*$*2* − *2*]))
path-image ?P2 = *path-image*(*linepath*(*vector*[(*pathstart g*)$*1*, (*pathstart g*)$*2* − *3*])(*pathstart g*)) ∪ *path-image g* ∪
path-image(*linepath*(*pathfinish g*)(*vector*[(*pathfinish g*)$*1*,*a*$*2* − *1*])) ∪
path-image(*linepath*(*vector*[(*pathfinish g*)$*1*,*a*$*2* − *1*])(*vector*[*b*$*1* + *1*,*a*$*2* − *1*])) ∪
path-image(*linepath*(*vector*[*b*$*1* + *1*,*a*$*2* − *1*])(*vector*[*b*$*1* + *1*,*b*$*2* + *3*]))
using *assms*(*1*−*2*)

```
    by(auto simp add: path-image-join path-linepath)
  have abab: cbox a b ⊆ cbox ?a ?b
    unfolding interval-cbox-cart[symmetric]
    by (auto simp add:less-eq-vec-def forall-2 vector-2)
  obtain z where
    z ∈ path-image
          (linepath (vector [a $ 1 − 2, a $ 2 − 2]) (vector [pathstart f $ 1, a $ 2
− 2]) +++
           linepath (vector [pathstart f $ 1, a $ 2 − 2]) (pathstart f) +++
           f +++
           linepath (pathfinish f) (vector [pathfinish f $ 1, a $ 2 − 2]) +++
           linepath (vector [pathfinish f $ 1, a $ 2 − 2]) (vector [b $ 1 + 2, a $ 2
− 2]))
    z ∈ path-image
         (linepath (vector [pathstart g $ 1, pathstart g $ 2 − 3]) (pathstart g) +++
           g +++
           linepath (pathfinish g) (vector [pathfinish g $ 1, a $ 2 − 1]) +++
           linepath (vector [pathfinish g $ 1, a $ 2 − 1]) (vector [b $ 1 + 1, a $ 2
− 1]) +++
          linepath (vector [b $ 1 + 1, a $ 2 − 1]) (vector [b $ 1 + 1, b $ 2 + 3]))
    apply (rule fashoda[of ?P1 ?P2 ?a ?b])
     unfolding pathstart-join pathfinish-join pathstart-linepath pathfinish-linepath
vector-2
  proof −
    show path ?P1 and path ?P2
      using assms by auto
    have path-image ?P1 ⊆ cbox ?a ?b
      unfolding P1P2 path-image-linepath
      apply (rule Un-least)+
      defer 3
    apply (rule-tac[1−4] convex-box(1)[unfolded convex-contains-segment,rule-format])
      unfolding mem-interval-cart forall-2 vector-2
      using ab startfin abab assms(3)
      using assms(9−)
      unfolding assms
      apply (auto simp add: field-simps box-def)
      done
    then show path-image ?P1 ⊆ cbox ?a ?b .
    have path-image ?P2 ⊆ cbox ?a ?b
      unfolding P1P2 path-image-linepath
      apply (rule Un-least)+
      defer 2
    apply (rule-tac[1−4] convex-box(1)[unfolded convex-contains-segment,rule-format])
      unfolding mem-interval-cart forall-2 vector-2
      using ab startfin abab assms(4)
      using assms(9−)
      unfolding assms
      apply (auto simp add: field-simps box-def)
      done
```

```
    then show path-image ?P2 ⊆ cbox ?a ?b .
    show a $ 1 − 2 = a $ 1 − 2
      and b $ 1 + 2 = b $ 1 + 2
      and pathstart g $ 2 − 3 = a $ 2 − 3
      and b $ 2 + 3 = b $ 2 + 3
      by (auto simp add: assms)
  qed
  note z=this[unfolded P1P2 path-image-linepath]
  show thesis
    apply (rule that[of z])
  proof −
    have (z ∈ closed-segment (vector [a $ 1 − 2, a $ 2 − 2]) (vector [pathstart f
$ 1, a $ 2 − 2]) ∨
      z ∈ closed-segment (vector [pathstart f $ 1, a $ 2 − 2]) (pathstart f)) ∨
      z ∈ closed-segment (pathfinish f) (vector [pathfinish f $ 1, a $ 2 − 2]) ∨
      z ∈ closed-segment (vector [pathfinish f $ 1, a $ 2 − 2]) (vector [b $ 1 + 2,
a $ 2 − 2]) ⟹
    (((z ∈ closed-segment (vector [pathstart g $ 1, pathstart g $ 2 − 3]) (pathstart
g)) ∨
      z ∈ closed-segment (pathfinish g) (vector [pathfinish g $ 1, a $ 2 − 1])) ∨
      z ∈ closed-segment (vector [pathfinish g $ 1, a $ 2 − 1]) (vector [b $ 1 + 1,
a $ 2 − 1])) ∨
      z ∈ closed-segment (vector [b $ 1 + 1, a $ 2 − 1]) (vector [b $ 1 + 1, b $
2 + 3]) ⟹ False
    proof (simp only: segment-vertical segment-horizontal vector-2, goal-cases)
      case prems: 1
      have pathfinish f ∈ cbox a b
        using assms(3) pathfinish-in-path-image[of f] by auto
      then have 1 + b $ 1 ≤ pathfinish f $ 1 ⟹ False
        unfolding mem-interval-cart forall-2 by auto
      then have z$1 ≠ pathfinish f$1
        using prems(2)
        using assms ab
        by (auto simp add: field-simps)
      moreover have pathstart f ∈ cbox a b
        using assms(3) pathstart-in-path-image[of f]
        by auto
      then have 1 + b $ 1 ≤ pathstart f $ 1 ⟹ False
        unfolding mem-interval-cart forall-2
        by auto
      then have z$1 ≠ pathstart f$1
        using prems(2) using assms ab
        by (auto simp add: field-simps)
      ultimately have ∗: z$2 = a$2 − 2
        using prems(1)
        by auto
      have z$1 ≠ pathfinish g$1
        using prems(2)
        using assms ab
```

```
      by (auto simp add: field-simps *)
    moreover have pathstart g ∈ cbox a b
      using assms(4) pathstart-in-path-image[of g]
      by auto
    note this[unfolded mem-interval-cart forall-2]
    then have z$1 ≠ pathstart g$1
      using prems(1)
      using assms ab
      by (auto simp add: field-simps *)
    ultimately have a $ 2 - 1 ≤ z $ 2 ∧ z $ 2 ≤ b $ 2 + 3 ∨ b $ 2 + 3 ≤ z
$ 2 ∧ z $ 2 ≤ a $ 2 - 1
      using prems(2)
      unfolding * assms
      by (auto simp add: field-simps)
    then show False
      unfolding * using ab by auto
  qed
  then have z ∈ path-image f ∨ z ∈ path-image g
    using z unfolding Un-iff by blast
  then have z': z ∈ cbox a b
    using assms(3-4)
    by auto
  have a $ 2 = z $ 2 ⟹ (z $ 1 = pathstart f $ 1 ∨ z $ 1 = pathfinish f $ 1)
⟹
    z = pathstart f ∨ z = pathfinish f
    unfolding vec-eq-iff forall-2 assms
    by auto
  with z' show z ∈ path-image f
    using z(1)
    unfolding Un-iff mem-interval-cart forall-2
    apply -
    apply (simp only: segment-vertical segment-horizontal vector-2)
    unfolding assms
    apply auto
    done
  have a $ 2 = z $ 2 ⟹ (z $ 1 = pathstart g $ 1 ∨ z $ 1 = pathfinish g $ 1)
⟹
    z = pathstart g ∨ z = pathfinish g
    unfolding vec-eq-iff forall-2 assms
    by auto
  with z' show z ∈ path-image g
    using z(2)
    unfolding Un-iff mem-interval-cart forall-2
    apply (simp only: segment-vertical segment-horizontal vector-2)
    unfolding assms
    apply auto
    done
 qed
qed
```

end

43.5 The type of non-negative extended real numbers

theory *Extended-Nonnegative-Real*
imports *Extended-Real Indicator-Function*
begin

lemma *ereal-ineq-diff-add*:
assumes $b \neq (-\infty::ereal)$ $a \geq b$
shows $a = b + (a-b)$
by (*metis add.commute assms*(*1*) *assms*(*2*) *ereal-eq-minus-iff ereal-minus-le-iff ereal-plus-eq-PInfty*)

lemma *Limsup-const-add*:
fixes $c :: 'a::\{complete\text{-}linorder, linorder\text{-}topology, topological\text{-}monoid\text{-}add, ordered\text{-}ab\text{-}semigroup\text{-}add\}$
shows $F \neq bot \Longrightarrow Limsup\ F\ (\lambda x.\ c + f\ x) = c + Limsup\ F\ f$
by (*rule Limsup-compose-continuous-mono*)
(*auto intro*!: *monoI add-mono continuous-on-add continuous-on-id continuous-on-const*)

lemma *Liminf-const-add*:
fixes $c :: 'a::\{complete\text{-}linorder, linorder\text{-}topology, topological\text{-}monoid\text{-}add, ordered\text{-}ab\text{-}semigroup\text{-}add\}$
shows $F \neq bot \Longrightarrow Liminf\ F\ (\lambda x.\ c + f\ x) = c + Liminf\ F\ f$
by (*rule Liminf-compose-continuous-mono*)
(*auto intro*!: *monoI add-mono continuous-on-add continuous-on-id continuous-on-const*)

lemma *Liminf-add-const*:
fixes $c :: 'a::\{complete\text{-}linorder, linorder\text{-}topology, topological\text{-}monoid\text{-}add, ordered\text{-}ab\text{-}semigroup\text{-}add\}$
shows $F \neq bot \Longrightarrow Liminf\ F\ (\lambda x.\ f\ x + c) = Liminf\ F\ f + c$
by (*rule Liminf-compose-continuous-mono*)
(*auto intro*!: *monoI add-mono continuous-on-add continuous-on-id continuous-on-const*)

lemma *sums-offset*:
fixes $f\ g :: nat \Rightarrow 'a :: \{t2\text{-}space, topological\text{-}comm\text{-}monoid\text{-}add\}$
assumes $(\lambda n.\ f\ (n + i))\ sums\ l$ **shows** $f\ sums\ (l + (\sum j<i.\ f\ j))$
proof −
have $(\lambda k.\ (\sum n<k.\ f\ (n + i)) + (\sum j<i.\ f\ j)) \longrightarrow l + (\sum j<i.\ f\ j)$
using *assms* **by** (*auto intro*!: *tendsto-add simp*: *sums-def*)
moreover
{ fix $k :: nat$
have $(\sum j<k + i.\ f\ j) = (\sum j=i..<k + i.\ f\ j) + (\sum j=0..<i.\ f\ j)$
by (*subst setsum.union-disjoint*[*symmetric*]) (*auto intro*!: *setsum.cong*)
also have $(\sum j=i..<k + i.\ f\ j) = (\sum j\in(\lambda n.\ n + i)`\{0..<k\}.\ f\ j)$
unfolding *image-add-atLeastLessThan* **by** *simp*
finally have $(\sum j<k + i.\ f\ j) = (\sum n<k.\ f\ (n + i)) + (\sum j<i.\ f\ j)$
by (*auto simp*: *inj-on-def atLeast0LessThan setsum.reindex*) **}**

ultimately have $(\lambda k.\ (\sum n{<}k + i.\ f\ n)) \longrightarrow l + (\sum j{<}i.\ f\ j)$
by *simp*
then show *?thesis*
unfolding *sums-def* **by** (*rule LIMSEQ-offset*)
qed

lemma *suminf-offset*:
fixes $f\ g :: nat \Rightarrow 'a :: \{t2\text{-}space, topological\text{-}comm\text{-}monoid\text{-}add\}$
shows *summable* $(\lambda j.\ f\ (j + i)) \Longrightarrow suminf\ f = (\sum j.\ f\ (j + i)) + (\sum j{<}i.\ f\ j)$
by (*intro sums-unique*[*symmetric*] *sums-offset summable-sums*)

lemma *eventually-at-left-1*: $(\bigwedge z{::}real.\ 0 < z \Longrightarrow z < 1 \Longrightarrow P\ z) \Longrightarrow eventually\ P\ (at\text{-}left\ 1)$
by (*subst eventually-at-left*[*of 0*]) (*auto intro*: *exI*[*of - 0*])

lemma *mult-eq-1*:
fixes $a\ b :: 'a :: \{ordered\text{-}semiring, comm\text{-}monoid\text{-}mult\}$
shows $0 \le a \Longrightarrow a \le 1 \Longrightarrow b \le 1 \Longrightarrow a * b = 1 \longleftrightarrow (a = 1 \wedge b = 1)$
by (*metis mult.left-neutral eq-iff mult.commute mult-right-mono*)

lemma *ereal-add-diff-cancel*:
fixes $a\ b :: ereal$
shows $|b| \neq \infty \Longrightarrow (a + b) - b = a$
by (*cases a b rule*: *ereal2-cases*) *auto*

lemma *add-top*:
fixes $x :: 'a{::}\{order\text{-}top, ordered\text{-}comm\text{-}monoid\text{-}add\}$
shows $0 \le x \Longrightarrow x + top = top$
by (*intro top-le add-increasing order-refl*)

lemma *top-add*:
fixes $x :: 'a{::}\{order\text{-}top, ordered\text{-}comm\text{-}monoid\text{-}add\}$
shows $0 \le x \Longrightarrow top + x = top$
by (*intro top-le add-increasing2 order-refl*)

lemma *le-lfp*: $mono\ f \Longrightarrow x \le lfp\ f \Longrightarrow f\ x \le lfp\ f$
by (*subst lfp-unfold*) (*auto dest*: *monoD*)

lemma *lfp-transfer*:
assumes α: *sup-continuous* α **and** f: *sup-continuous* f **and** *mg*: *mono g*
assumes *bot*: $\alpha\ bot \le lfp\ g$ **and** *eq*: $\bigwedge x.\ x \le lfp\ f \Longrightarrow \alpha\ (f\ x) = g\ (\alpha\ x)$
shows $\alpha\ (lfp\ f) = lfp\ g$
proof (*rule antisym*)
note *mf* = *sup-continuous-mono*[*OF f*]
have *f-le-lfp*: $(f\ \hat{}\hat{}\ i)\ bot \le lfp\ f$ **for** i
by (*induction i*) (*auto intro*: *le-lfp mf*)

have $\alpha\ ((f\ \hat{}\hat{}\ i)\ bot) \le lfp\ g$ **for** i
by (*induction i*) (*auto simp*: *bot eq f-le-lfp intro*!: *le-lfp mg*)

```
  then show α (lfp f) ≤ lfp g
    unfolding sup-continuous-lfp[OF f]
    by (subst α[THEN sup-continuousD])
       (auto intro!: mono-funpow sup-continuous-mono[OF f] SUP-least)

  show lfp g ≤ α (lfp f)
    by (rule lfp-lowerbound) (simp add: eq[symmetric] lfp-unfold[OF mf, symmetric])
qed

lemma sup-continuous-applyD: sup-continuous f ⟹ sup-continuous (λx. f x h)
  using sup-continuous-apply[THEN sup-continuous-compose] .

lemma sup-continuous-SUP[order-continuous-intros]:
  fixes M :: - ⇒ - ⇒ 'a::complete-lattice
  assumes M: ⋀i. i ∈ I ⟹ sup-continuous (M i)
  shows  sup-continuous (SUP i:I. M i)
 unfolding sup-continuous-def by (auto simp add: sup-continuousD[OF M] intro: SUP-commute)

lemma sup-continuous-apply-SUP[order-continuous-intros]:
  fixes M :: - ⇒ - ⇒ 'a::complete-lattice
  shows (⋀i. i ∈ I ⟹ sup-continuous (M i)) ⟹ sup-continuous (λx. SUP i:I. M i x)
  unfolding SUP-apply[symmetric] by (rule sup-continuous-SUP)

lemma sup-continuous-lfp'[order-continuous-intros]:
  assumes 1: sup-continuous f
  assumes 2: ⋀g. sup-continuous g ⟹ sup-continuous (f g)
  shows sup-continuous (lfp f)
proof −
  have sup-continuous ((f ^^ i) bot) for i
  proof (induction i)
    case (Suc i) then show ?case
      by (auto intro!: 2)
  qed (simp add: bot-fun-def sup-continuous-const)
  then show ?thesis
    unfolding sup-continuous-lfp[OF 1] by (intro order-continuous-intros)
qed

lemma sup-continuous-lfp''[order-continuous-intros]:
  assumes 1: ⋀s. sup-continuous (f s)
  assumes 2: ⋀g. sup-continuous g ⟹ sup-continuous (λs. f s (g s))
  shows sup-continuous (λx. lfp (f x))
proof −
  have sup-continuous (λx. (f x ^^ i) bot) for i
  proof (induction i)
    case (Suc i) then show ?case
      by (auto intro!: 2)
```

qed (*simp add*: *bot-fun-def sup-continuous-const*)
then show *?thesis*
unfolding *sup-continuous-lfp*[*OF 1*] **by** (*intro order-continuous-intros*)
qed

lemma *mono-INF-fun*:
(⋀*x y*. *mono* (*F x y*)) ⟹ *mono* (λ*z x*. *INF y* : *X x*. *F x y z* :: *'a* :: *complete-lattice*)
by (*auto intro*!: *INF-mono*[*OF bexI*] *simp*: *le-fun-def mono-def*)

lemma *continuous-on-max*:
fixes *f g* :: *'a*::*topological-space* ⇒ *'b*::*linorder-topology*
shows *continuous-on A f* ⟹ *continuous-on A g* ⟹ *continuous-on A* (λ*x*. *max* (*f x*) (*g x*))
by (*auto simp*: *continuous-on-def intro*!: *tendsto-max*)

lemma *continuous-on-cmult-ereal*:
|*c*::*ereal*| ≠ ∞ ⟹ *continuous-on A f* ⟹ *continuous-on A* (λ*x*. *c* ∗ *f x*)
using *tendsto-cmult-ereal*[*of c f f x at x within A* **for** *x*]
by (*auto simp*: *continuous-on-def simp del*: *tendsto-cmult-ereal*)

context *linordered-nonzero-semiring*
begin

lemma *of-nat-nonneg* [*simp*]: *0* ≤ *of-nat n*
by (*induct n*) *simp-all*

lemma *of-nat-mono*[*simp*]: *i* ≤ *j* ⟹ *of-nat i* ≤ *of-nat j*
by (*auto simp add*: *le-iff-add intro*!: *add-increasing2*)

end

lemma *real-of-nat-Sup*:
assumes *A* ≠ {} *bdd-above A*
shows *of-nat* (*Sup A*) = (*SUP a*:*A*. *of-nat a* :: *real*)
proof (*intro antisym*)
show (*SUP a*:*A*. *of-nat a*::*real*) ≤ *of-nat* (*Sup A*)
using *assms* **by** (*intro cSUP-least of-nat-mono*) (*auto intro*: *cSup-upper*)
have *Sup A* ∈ *A*
unfolding *Sup-nat-def* **using** *assms* **by** (*intro Max-in*) (*auto simp*: *bdd-above-nat*)
then show *of-nat* (*Sup A*) ≤ (*SUP a*:*A*. *of-nat a*::*real*)
by (*intro cSUP-upper bdd-above-image-mono assms*) (*auto simp*: *mono-def*)
qed

lemma *of-nat-less*[*simp*]:
i < *j* ⟹ *of-nat i* < (*of-nat j*::*'a*::{*linordered-nonzero-semiring*, *semiring-char-0*})
by (*auto simp*: *less-le*)

lemma *of-nat-le-iff* [*simp*]:

of-nat i ≤ (*of-nat j*::'*a*::{*linordered-nonzero-semiring*, *semiring-char-0*}) ⟷ *i* ≤ *j*
proof (*safe intro*!: *of-nat-mono*)
assume *of-nat i* ≤ (*of-nat j*::'*a*) **then show** *i* ≤ *j*
proof (*intro leI notI*)
assume *j* < *i* **from** *less-le-trans*[*OF of-nat-less*[*OF this*] ‹*of-nat i* ≤ *of-nat j*›]
show *False*
by *blast*
qed
qed

lemma (**in** *complete-lattice*) *SUP-sup-const1*:
I ≠ {} ⟹ (*SUP i*:*I*. *sup c* (*f i*)) = *sup c* (*SUP i*:*I*. *f i*)
using *SUP-sup-distrib*[*of* λ-. *c I f*] **by** *simp*

lemma (**in** *complete-lattice*) *SUP-sup-const2*:
I ≠ {} ⟹ (*SUP i*:*I*. *sup* (*f i*) *c*) = *sup* (*SUP i*:*I*. *f i*) *c*
using *SUP-sup-distrib*[*of f I* λ-. *c*] **by** *simp*

lemma *one-less-of-natD*:
(*1*::'*a*::*linordered-semidom*) < *of-nat n* ⟹ *1* < *n*
using *zero-le-one*[**where** *'a*='*a*]
apply (*cases n*)
apply *simp*
subgoal for *n'*
apply (*cases n'*)
apply *simp*
apply *simp*
done
done

lemma *setsum-le-suminf*:
fixes *f* :: *nat* ⇒ '*a*::{*ordered-comm-monoid-add*, *linorder-topology*}
shows *summable f* ⟹ *finite I* ⟹ ∀ *m*∈− *I*. *0* ≤ *f m* ⟹ *setsum f I* ≤ *suminf f*
by (*rule sums-le*[*OF - sums-If-finite-set summable-sums*]) *auto*

43.6 Defining the extended non-negative reals

Basic definitions and type class setup

typedef *ennreal* = {*x* :: *ereal*. *0* ≤ *x*}
morphisms *enn2ereal e2ennreal'*
by *auto*

definition *e2ennreal x* = *e2ennreal'* (*max 0 x*)

lemma *enn2ereal-range*: *e2ennreal* ‘ {*0*..} = *UNIV*
proof −
have ∃ *y*≥*0*. *x* = *e2ennreal y* **for** *x*

by (*cases x*) (*auto simp*: *e2ennreal-def max-absorb2*)
then show *?thesis*
by (*auto simp*: *image-iff Bex-def*)
qed

lemma *type-definition-ennreal'*: *type-definition enn2ereal e2ennreal* {*x*. *0* ≤ *x*}
using *type-definition-ennreal*
by (*auto simp*: *type-definition-def e2ennreal-def max-absorb2*)

setup-lifting *type-definition-ennreal'*

declare [[*coercion e2ennreal*]]

instantiation *ennreal* :: *complete-linorder*
begin

lift-definition *top-ennreal* :: *ennreal* **is** *top* **by** (*rule top-greatest*)
lift-definition *bot-ennreal* :: *ennreal* **is** *0* **by** (*rule order-refl*)
lift-definition *sup-ennreal* :: *ennreal* ⇒ *ennreal* ⇒ *ennreal* **is** *sup* **by** (*rule le-supI1*)
lift-definition *inf-ennreal* :: *ennreal* ⇒ *ennreal* ⇒ *ennreal* **is** *inf* **by** (*rule le-infI*)

lift-definition *Inf-ennreal* :: *ennreal set* ⇒ *ennreal* **is** *Inf*
by (*rule Inf-greatest*)

lift-definition *Sup-ennreal* :: *ennreal set* ⇒ *ennreal* **is** *sup 0* ∘ *Sup*
by *auto*

lift-definition *less-eq-ennreal* :: *ennreal* ⇒ *ennreal* ⇒ *bool* **is** *op* ≤ **.**
lift-definition *less-ennreal* :: *ennreal* ⇒ *ennreal* ⇒ *bool* **is** *op* < **.**

instance
by *standard*
(*transfer* ; *auto simp*: *Inf-lower Inf-greatest Sup-upper Sup-least le-max-iff-disj max.absorb1*)+

end

lemma *pcr-ennreal-enn2ereal*[*simp*]: *pcr-ennreal* (*enn2ereal x*) *x*
by (*simp add*: *ennreal.pcr-cr-eq cr-ennreal-def*)

lemma *rel-fun-eq-pcr-ennreal*: *rel-fun op* = *pcr-ennreal f g* ⟷ *f* = *enn2ereal* ∘ *g*
by (*auto simp*: *rel-fun-def ennreal.pcr-cr-eq cr-ennreal-def*)

instantiation *ennreal* :: *infinity*
begin

definition *infinity-ennreal* :: *ennreal*
where
[*simp*]: ∞ = (*top*::*ennreal*)

instance ..

end

instantiation *ennreal* :: {*semiring-1-no-zero-divisors*, *comm-semiring-1*}
begin

lift-definition *one-ennreal* :: *ennreal* **is** *1* **by** *simp*
lift-definition *zero-ennreal* :: *ennreal* **is** *0* **by** *simp*
lift-definition *plus-ennreal* :: *ennreal* ⇒ *ennreal* ⇒ *ennreal* **is** *op +* **by** *simp*
lift-definition *times-ennreal* :: *ennreal* ⇒ *ennreal* ⇒ *ennreal* **is** *op ∗* **by** *simp*

instance
by *standard* (*transfer*; *auto simp*: *field-simps ereal-right-distrib*)+

end

instantiation *ennreal* :: *minus*
begin

lift-definition *minus-ennreal* :: *ennreal* ⇒ *ennreal* ⇒ *ennreal* **is** λ*a b*. *max 0* (*a* − *b*)
by *simp*

instance ..

end

instance *ennreal* :: *numeral* **..**

instantiation *ennreal* :: *inverse*
begin

lift-definition *inverse-ennreal* :: *ennreal* ⇒ *ennreal* **is** *inverse*
by (*rule inverse-ereal-ge0I*)

definition *divide-ennreal* :: *ennreal* ⇒ *ennreal* ⇒ *ennreal*
where *x div y* = *x ∗ inverse* (*y* :: *ennreal*)

instance ..

end

lemma *ennreal-zero-less-one*: *0* < (*1*::*ennreal*) — TODO: remove
by *transfer auto*

instance *ennreal* :: *dioid*
proof (*standard*; *transfer*)

fix *a b* :: *ereal* **assume** *0* ≤ *a 0* ≤ *b* **then show** (*a* ≤ *b*) = (∃ *c*∈*Collect* (*op* ≤ *0*). *b* = *a* + *c*)
unfolding *ereal-ex-split Bex-def*
by (*cases a b rule*: *ereal2-cases*) (*auto intro*!: *exI*[*of* - *real-of-ereal* (*b* − *a*)])
qed

instance *ennreal* :: *ordered-comm-semiring*
by *standard*
(*transfer* ; *auto intro*: *add-mono mult-mono mult-ac ereal-left-distrib ereal-mult-left-mono*)+

instance *ennreal* :: *linordered-nonzero-semiring*
proof qed (*transfer*; *simp*)

instance *ennreal* :: *strict-ordered-ab-semigroup-add*
proof
fix *a b c d* :: *ennreal* **show** *a* < *b* ⟹ *c* < *d* ⟹ *a* + *c* < *b* + *d*
by *transfer* (*auto intro*!: *ereal-add-strict-mono*)
qed

declare [[*coercion of-nat* :: *nat* ⇒ *ennreal*]]

lemma *e2ennreal-neg*: *x* ≤ *0* ⟹ *e2ennreal x* = *0*
unfolding *zero-ennreal-def e2ennreal-def* **by** (*simp add*: *max-absorb1*)

lemma *e2ennreal-mono*: *x* ≤ *y* ⟹ *e2ennreal x* ≤ *e2ennreal y*
by (*cases 0* ≤ *x 0* ≤ *y rule*: *bool.exhaust*[*case-product bool.exhaust*])
(*auto simp*: *e2ennreal-neg less-eq-ennreal.abs-eq eq-onp-def*)

lemma *enn2ereal-nonneg*[*simp*]: *0* ≤ *enn2ereal x*
using *ennreal.enn2ereal*[*of x*] **by** *simp*

lemma *ereal-ennreal-cases*:
obtains *b* **where** *0* ≤ *a a* = *enn2ereal b* | *a* < *0*
using *e2ennreal'-inverse*[*of a*, *symmetric*] **by** (*cases 0* ≤ *a*) (*auto intro*: *enn2ereal-nonneg*)

lemma *rel-fun-liminf*[*transfer-rule*]: *rel-fun* (*rel-fun op* = *pcr-ennreal*) *pcr-ennreal liminf liminf*
proof −
have *rel-fun* (*rel-fun op* = *pcr-ennreal*) *pcr-ennreal* (λ*x*. *sup 0* (*liminf x*)) *liminf*
unfolding *liminf-SUP-INF*[*abs-def*] **by** (*transfer-prover-start*, *transfer-step*+; *simp*)
then show *?thesis*
apply (*subst* (*asm*) (*2*) *rel-fun-def*)
apply (*subst* (*2*) *rel-fun-def*)
apply (*auto simp*: *comp-def max.absorb2 Liminf-bounded rel-fun-eq-pcr-ennreal*)
done
qed

lemma *rel-fun-limsup*[*transfer-rule*]: *rel-fun* (*rel-fun op* = *pcr-ennreal*) *pcr-ennreal*

limsup limsup
proof −
have *rel-fun* (*rel-fun op* = *pcr-ennreal*) *pcr-ennreal* (λ*x*. *INF n*. *sup 0* (*SUP i*:{*n*..}. *x i*)) *limsup*
unfolding *limsup-INF-SUP*[*abs-def*] **by** (*transfer-prover-start*, *transfer-step*+; *simp*)
then show *?thesis*
unfolding *limsup-INF-SUP*[*abs-def*]
apply (*subst* (*asm*) (*2*) *rel-fun-def*)
apply (*subst* (*2*) *rel-fun-def*)
apply (*auto simp*: *comp-def max.absorb2 Sup-upper2 rel-fun-eq-pcr-ennreal*)
apply (*subst* (*asm*) *max.absorb2*)
apply (*rule SUP-upper2*)
apply *auto*
done
qed

lemma *setsum-enn2ereal*[*simp*]: (⋀*i*. *i* ∈ *I* ⟹ *0* ≤ *f i*) ⟹ (∑ *i*∈*I*. *enn2ereal* (*f i*)) = *enn2ereal* (*setsum f I*)
by (*induction I rule*: *infinite-finite-induct*) (*auto simp*: *setsum-nonneg zero-ennreal.rep-eq plus-ennreal.rep-eq*)

lemma *transfer-e2ennreal-setsum* [*transfer-rule*]:
rel-fun (*rel-fun op* = *pcr-ennreal*) (*rel-fun op* = *pcr-ennreal*) *setsum setsum*
by (*auto intro*!: *rel-funI simp*: *rel-fun-eq-pcr-ennreal comp-def*)

lemma *enn2ereal-of-nat*[*simp*]: *enn2ereal* (*of-nat n*) = *ereal n*
by (*induction n*) (*auto simp*: *zero-ennreal.rep-eq one-ennreal.rep-eq plus-ennreal.rep-eq*)

lemma *enn2ereal-numeral*[*simp*]: *enn2ereal* (*numeral a*) = *numeral a*
apply (*subst of-nat-numeral*[*of a*, *symmetric*])
apply (*subst enn2ereal-of-nat*)
apply *simp*
done

lemma *transfer-numeral*[*transfer-rule*]: *pcr-ennreal* (*numeral a*) (*numeral a*)
unfolding *cr-ennreal-def pcr-ennreal-def* **by** *auto*

43.7 Cancellation simprocs

lemma *ennreal-add-left-cancel*: *a* + *b* = *a* + *c* ⟷ *a* = (∞::*ennreal*) ∨ *b* = *c*
unfolding *infinity-ennreal-def* **by** *transfer* (*simp add*: *top-ereal-def ereal-add-cancel-left*)

lemma *ennreal-add-left-cancel-le*: *a* + *b* ≤ *a* + *c* ⟷ *a* = (∞::*ennreal*) ∨ *b* ≤ *c*
unfolding *infinity-ennreal-def* **by** *transfer* (*simp add*: *ereal-add-le-add-iff top-ereal-def disj-commute*)

lemma *ereal-add-left-cancel-less*:
fixes *a b c* :: *ereal*

shows $0 \leq a \Longrightarrow 0 \leq b \Longrightarrow a + b < a + c \longleftrightarrow a \neq \infty \wedge b < c$
by (*cases a b c rule*: *ereal3-cases*) *auto*

lemma *ennreal-add-left-cancel-less*: $a + b < a + c \longleftrightarrow a \neq (\infty::ennreal) \wedge b < c$
unfolding *infinity-ennreal-def*
by *transfer* (*simp add*: *top-ereal-def ereal-add-left-cancel-less*)

ML ‹
```
structure Cancel-Ennreal-Common =
struct
  (* copied from src/HOL/Tools/nat-numeral-simprocs.ML *)
  fun find-first-t - - [] = raise TERM(find-first-t, [])
    | find-first-t past u (t::terms) =
        if u aconv t then (rev past @ terms)
        else find-first-t (t::past) u terms

  fun dest-summing (Const (@{const-name Groups.plus}, -) $ t $ u, ts) =
        dest-summing (t, dest-summing (u, ts))
    | dest-summing (t, ts) = t :: ts

  val mk-sum = Arith-Data.long-mk-sum
  fun dest-sum t = dest-summing (t, [])
  val find-first = find-first-t []
  val trans-tac = Numeral-Simprocs.trans-tac
  val norm-ss =
    simpset-of (put-simpset HOL-basic-ss @{context}
      addsimps @{thms ac-simps add-0-left add-0-right})
  fun norm-tac ctxt = ALLGOALS (simp-tac (put-simpset norm-ss ctxt))
  fun simplify-meta-eq ctxt cancel-th th =
    Arith-Data.simplify-meta-eq [] ctxt
      ([th, cancel-th] MRS trans)
  fun mk-eq (a, b) = HOLogic.mk-Trueprop (HOLogic.mk-eq (a, b))
end

structure Eq-Ennreal-Cancel = ExtractCommonTermFun
(open Cancel-Ennreal-Common
  val mk-bal = HOLogic.mk-eq
  val dest-bal = HOLogic.dest-bin @{const-name HOL.eq} @{typ ennreal}
  fun simp-conv - - = SOME @{thm ennreal-add-left-cancel}
)

structure Le-Ennreal-Cancel = ExtractCommonTermFun
(open Cancel-Ennreal-Common
  val mk-bal = HOLogic.mk-binrel @{const-name Orderings.less-eq}
  val dest-bal = HOLogic.dest-bin @{const-name Orderings.less-eq} @{typ ennreal}
  fun simp-conv - - = SOME @{thm ennreal-add-left-cancel-le}
)
```

structure Less-Ennreal-Cancel = ExtractCommonTermFun
(*open Cancel-Ennreal-Common*
val mk-bal = HOLogic.mk-binrel @{const-name Orderings.less}
val dest-bal = HOLogic.dest-bin @{const-name Orderings.less} @{typ ennreal}
fun simp-conv - - = SOME @{thm ennreal-add-left-cancel-less}
)
›

simproc-setup *ennreal-eq-cancel*
((*l*::*ennreal*) + *m* = *n* | (*l*::*ennreal*) = *m* + *n*) =
‹*fn phi => fn ctxt => fn ct => Eq-Ennreal-Cancel.proc ctxt* (*Thm.term-of ct*)›

simproc-setup *ennreal-le-cancel*
((*l*::*ennreal*) + *m* ≤ *n* | (*l*::*ennreal*) ≤ *m* + *n*) =
‹*fn phi => fn ctxt => fn ct => Le-Ennreal-Cancel.proc ctxt* (*Thm.term-of ct*)›

simproc-setup *ennreal-less-cancel*
((*l*::*ennreal*) + *m* < *n* | (*l*::*ennreal*) < *m* + *n*) =
‹*fn phi => fn ctxt => fn ct => Less-Ennreal-Cancel.proc ctxt* (*Thm.term-of ct*)›

43.8 Order with top

lemma *ennreal-zero-less-top*[*simp*]: *0* < (*top*::*ennreal*)
by *transfer* (*simp add*: *top-ereal-def*)

lemma *ennreal-one-less-top*[*simp*]: *1* < (*top*::*ennreal*)
by *transfer* (*simp add*: *top-ereal-def*)

lemma *ennreal-zero-neq-top*[*simp*]: *0* ≠ (*top*::*ennreal*)
by *transfer* (*simp add*: *top-ereal-def*)

lemma *ennreal-top-neq-zero*[*simp*]: (*top*::*ennreal*) ≠ *0*
by *transfer* (*simp add*: *top-ereal-def*)

lemma *ennreal-top-neq-one*[*simp*]: *top* ≠ (*1*::*ennreal*)
by *transfer* (*simp add*: *top-ereal-def one-ereal-def ereal-max*[*symmetric*] *del*: *ereal-max*)

lemma *ennreal-one-neq-top*[*simp*]: *1* ≠ (*top*::*ennreal*)
by *transfer* (*simp add*: *top-ereal-def one-ereal-def ereal-max*[*symmetric*] *del*: *ereal-max*)

lemma *ennreal-add-less-top*[*simp*]:
fixes *a b* :: *ennreal*
shows *a* + *b* < *top* ⟷ *a* < *top* ∧ *b* < *top*
by *transfer* (*auto simp*: *top-ereal-def*)

lemma *ennreal-add-eq-top*[*simp*]:
fixes *a b* :: *ennreal*
shows *a* + *b* = *top* ⟷ *a* = *top* ∨ *b* = *top*

by *transfer* (*auto simp*: *top-ereal-def*)

lemma *ennreal-setsum-less-top*[*simp*]:
fixes $f :: 'a \Rightarrow ennreal$
shows $finite\ I \Longrightarrow (\sum i{\in}I.\ f\ i) < top \longleftrightarrow (\forall\, i{\in}I.\ f\ i < top)$
by (*induction I rule*: *finite-induct*) *auto*

lemma *ennreal-setsum-eq-top*[*simp*]:
fixes $f :: 'a \Rightarrow ennreal$
shows $finite\ I \Longrightarrow (\sum i{\in}I.\ f\ i) = top \longleftrightarrow (\exists\, i{\in}I.\ f\ i = top)$
by (*induction I rule*: *finite-induct*) *auto*

lemma *ennreal-mult-eq-top-iff*:
fixes $a\ b :: ennreal$
shows $a * b = top \longleftrightarrow (a = top \wedge b \neq 0) \vee (b = top \wedge a \neq 0)$
by *transfer* (*auto simp*: *top-ereal-def*)

lemma *ennreal-top-eq-mult-iff*:
fixes $a\ b :: ennreal$
shows $top = a * b \longleftrightarrow (a = top \wedge b \neq 0) \vee (b = top \wedge a \neq 0)$
using *ennreal-mult-eq-top-iff*[*of a b*] **by** *auto*

lemma *ennreal-mult-less-top*:
fixes $a\ b :: ennreal$
shows $a * b < top \longleftrightarrow (a = 0 \vee b = 0 \vee (a < top \wedge b < top))$
by *transfer* (*auto simp add*: *top-ereal-def*)

lemma *top-power-ennreal*: $top\ \hat{}\ n = (if\ n = 0\ then\ 1\ else\ top :: ennreal)$
by (*induction n*) (*simp-all add*: *ennreal-mult-eq-top-iff*)

lemma *ennreal-setprod-eq-0*[*simp*]:
fixes $f :: 'a \Rightarrow ennreal$
shows $(setprod\ f\ A = 0) = (finite\ A \wedge (\exists\, i{\in}A.\ f\ i = 0))$
by (*induction A rule*: *infinite-finite-induct*) *auto*

lemma *ennreal-setprod-eq-top*:
fixes $f :: 'a \Rightarrow ennreal$
shows $(\prod i{\in}I.\ f\ i) = top \longleftrightarrow (finite\ I \wedge ((\forall\, i{\in}I.\ f\ i \neq 0) \wedge (\exists\, i{\in}I.\ f\ i = top)))$
by (*induction I rule*: *infinite-finite-induct*) (*auto simp*: *ennreal-mult-eq-top-iff*)

lemma *ennreal-top-mult*: $top * a = (if\ a = 0\ then\ 0\ else\ top :: ennreal)$
by (*simp add*: *ennreal-mult-eq-top-iff*)

lemma *ennreal-mult-top*: $a * top = (if\ a = 0\ then\ 0\ else\ top :: ennreal)$
by (*simp add*: *ennreal-mult-eq-top-iff*)

lemma *enn2ereal-eq-top-iff*[*simp*]: $enn2ereal\ x = \infty \longleftrightarrow x = top$
by *transfer* (*simp add*: *top-ereal-def*)

lemma *enn2ereal-top*: *enn2ereal top* = ∞
by *transfer* (*simp add*: *top-ereal-def*)

lemma *e2ennreal-infty*: *e2ennreal* ∞ = *top*
by (*simp add*: *top-ennreal.abs-eq top-ereal-def*)

lemma *ennreal-top-minus*[*simp*]: *top* − *x* = (*top*::*ennreal*)
by *transfer* (*auto simp*: *top-ereal-def max-def*)

lemma *minus-top-ennreal*: *x* − *top* = (*if x* = *top then top else 0*:: *ennreal*)
apply *transfer*
subgoal for *x*
by (*cases x*) (*auto simp*: *top-ereal-def max-def*)
done

lemma *bot-ennreal*: *bot* = (*0*::*ennreal*)
by *transfer rule*

lemma *ennreal-of-nat-neq-top*[*simp*]: *of-nat i* ≠ (*top*::*ennreal*)
by (*induction i*) *auto*

lemma *numeral-eq-of-nat*: (*numeral a*::*ennreal*) = *of-nat* (*numeral a*)
by *simp*

lemma *of-nat-less-top*: *of-nat i* < (*top*::*ennreal*)
using *less-le-trans*[*of of-nat i of-nat* (*Suc i*) *top*::*ennreal*]
by *simp*

lemma *top-neq-numeral*[*simp*]: *top* ≠ (*numeral i*::*ennreal*)
using *of-nat-less-top*[*of numeral i*] **by** *simp*

lemma *ennreal-numeral-less-top*[*simp*]: *numeral i* < (*top*::*ennreal*)
using *of-nat-less-top*[*of numeral i*] **by** *simp*

lemma *ennreal-add-bot*[*simp*]: *bot* + *x* = (*x*::*ennreal*)
by *transfer simp*

instance *ennreal* :: *semiring-char-0*
proof (*standard*, *safe intro*!: *linorder-injI*)
have ∗: *1* + *of-nat k* ≠ (*0*::*ennreal*) **for** *k*
using *add-pos-nonneg*[*OF zero-less-one*, *of of-nat k* :: *ennreal*] **by** *auto*
fix *x y* :: *nat* **assume** *x* < *y of-nat x* = (*of-nat y*::*ennreal*) **then show** *False*
by (*auto simp add*: *less-iff-Suc-add* ∗)
qed

43.9 Arithmetic

lemma *ennreal-minus-zero*[*simp*]: *a* − (*0*::*ennreal*) = *a*
by *transfer* (*auto simp*: *max-def*)

lemma *ennreal-add-diff-cancel-right*[*simp*]:
fixes $x\ y\ z$ :: *ennreal* **shows** $y \neq top \Longrightarrow (x + y) - y = x$
apply *transfer*
subgoal for $x\ y$
apply (*cases x y rule*: *ereal2-cases*)
apply (*auto split*: *split-max simp*: *top-ereal-def*)
done
done

lemma *ennreal-add-diff-cancel-left*[*simp*]:
fixes $x\ y\ z$ :: *ennreal* **shows** $y \neq top \Longrightarrow (y + x) - y = x$
by (*simp add*: *add.commute*)

lemma
fixes $a\ b$ :: *ennreal*
shows $a - b = 0 \Longrightarrow a \leq b$
apply *transfer*
subgoal for $a\ b$
apply (*cases a b rule*: *ereal2-cases*)
apply (*auto simp*: *not-le max-def split*: *if-splits*)
done
done

lemma *ennreal-minus-cancel*:
fixes $a\ b\ c$ :: *ennreal*
shows $c \neq top \Longrightarrow a \leq c \Longrightarrow b \leq c \Longrightarrow c - a = c - b \Longrightarrow a = b$
apply *transfer*
subgoal for $a\ b\ c$
by (*cases a b c rule*: *ereal3-cases*)
(*auto simp*: *top-ereal-def max-def split*: *if-splits*)
done

lemma *sup-const-add-ennreal*:
fixes $a\ b\ c$ :: *ennreal*
shows *sup* $(c + a)\ (c + b) = c + sup\ a\ b$
apply *transfer*
subgoal for $a\ b\ c$
apply (*cases a b c rule*: *ereal3-cases*)
apply (*auto simp*: *ereal-max*[*symmetric*] *simp del*: *ereal-max*)
apply (*auto simp*: *top-ereal-def*[*symmetric*] *sup-ereal-def*[*symmetric*]
simp del: *sup-ereal-def*)
apply (*auto simp add*: *top-ereal-def*)
done
done

lemma *ennreal-diff-add-assoc*:
fixes $a\ b\ c$ :: *ennreal*
shows $a \leq b \Longrightarrow c + b - a = c + (b - a)$

apply *transfer*
subgoal for *a b c*
by (*cases a b c rule*: *ereal3-cases*) (*auto simp*: *field-simps max-absorb2*)
done

lemma *mult-divide-eq-ennreal*:
fixes *a b* :: *ennreal*
shows $b \neq 0 \Longrightarrow b \neq top \Longrightarrow (a * b) / b = a$
unfolding *divide-ennreal-def*
apply *transfer*
apply (*subst mult.assoc*)
apply (*simp add*: *top-ereal-def divide-ereal-def*[*symmetric*])
done

lemma *divide-mult-eq*: $a \neq 0 \Longrightarrow a \neq \infty \Longrightarrow x * a / (b * a) = x / (b::ennreal)$
unfolding *divide-ennreal-def infinity-ennreal-def*
apply *transfer*
subgoal for *a b c*
apply (*cases a b c rule*: *ereal3-cases*)
apply (*auto simp*: *top-ereal-def*)
done
done

lemma *ennreal-mult-divide-eq*:
fixes *a b* :: *ennreal*
shows $b \neq 0 \Longrightarrow b \neq top \Longrightarrow (a * b) / b = a$
unfolding *divide-ennreal-def*
apply *transfer*
apply (*subst mult.assoc*)
apply (*simp add*: *top-ereal-def divide-ereal-def*[*symmetric*])
done

lemma *ennreal-add-diff-cancel*:
fixes *a b* :: *ennreal*
shows $b \neq \infty \Longrightarrow (a + b) - b = a$
unfolding *infinity-ennreal-def*
by *transfer* (*simp add*: *max-absorb2 top-ereal-def ereal-add-diff-cancel*)

lemma *ennreal-minus-eq-0*:
$a - b = 0 \Longrightarrow a \leq (b::ennreal)$
apply *transfer*
subgoal for *a b*
apply (*cases a b rule*: *ereal2-cases*)
apply (*auto simp*: *zero-ereal-def ereal-max*[*symmetric*] *max.absorb2 simp del*: *ereal-max*)
done
done

lemma *ennreal-mono-minus-cancel*:

fixes *a b c* :: *ennreal*
shows $a - b \leq a - c \Longrightarrow a < top \Longrightarrow b \leq a \Longrightarrow c \leq a \Longrightarrow c \leq b$
by *transfer*
(*auto simp add*: *max.absorb2 ereal-diff-positive top-ereal-def dest*: *ereal-mono-minus-cancel*)

lemma *ennreal-mono-minus*:
fixes *a b c* :: *ennreal*
shows $c \leq b \Longrightarrow a - b \leq a - c$
apply *transfer*
apply (*rule max.mono*)
apply *simp*
subgoal for *a b c*
by (*cases a b c rule*: *ereal3-cases*) *auto*
done

lemma *ennreal-minus-pos-iff*:
fixes *a b* :: *ennreal*
shows $a < top \vee b < top \Longrightarrow 0 < a - b \Longrightarrow b < a$
apply *transfer*
subgoal for *a b*
by (*cases a b rule*: *ereal2-cases*) (*auto simp*: *less-max-iff-disj*)
done

lemma *ennreal-inverse-top*[*simp*]: *inverse top* = (*0*::*ennreal*)
by *transfer* (*simp add*: *top-ereal-def ereal-inverse-eq-0*)

lemma *ennreal-inverse-zero*[*simp*]: *inverse 0* = (*top*::*ennreal*)
by *transfer* (*simp add*: *top-ereal-def ereal-inverse-eq-0*)

lemma *ennreal-top-divide*: *top* / (*x*::*ennreal*) = (*if x* = *top then 0 else top*)
unfolding *divide-ennreal-def*
by *transfer* (*simp add*: *top-ereal-def ereal-inverse-eq-0 ereal-0-gt-inverse*)

lemma *ennreal-zero-divide*[*simp*]: *0* / (*x*::*ennreal*) = *0*
by (*simp add*: *divide-ennreal-def*)

lemma *ennreal-divide-zero*[*simp*]: *x* / (*0*::*ennreal*) = (*if x* = *0 then 0 else top*)
by (*simp add*: *divide-ennreal-def ennreal-mult-top*)

lemma *ennreal-divide-top*[*simp*]: *x* / (*top*::*ennreal*) = *0*
by (*simp add*: *divide-ennreal-def ennreal-top-mult*)

lemma *ennreal-times-divide*: *a* * (*b* / *c*) = *a* * *b* / (*c*::*ennreal*)
unfolding *divide-ennreal-def*
by *transfer* (*simp add*: *divide-ereal-def*[*symmetric*] *ereal-times-divide-eq*)

lemma *ennreal-zero-less-divide*: *0* < *a* / *b* ⟷ (*0* < *a* ∧ *b* < (*top*::*ennreal*))
unfolding *divide-ennreal-def*
by *transfer* (*auto simp*: *ereal-zero-less-0-iff top-ereal-def ereal-0-gt-inverse*)

lemma *divide-right-mono-ennreal*:
fixes *a b c* :: *ennreal*
shows $a \leq b \Longrightarrow a \mathbin{/} c \leq b \mathbin{/} c$
unfolding *divide-ennreal-def* **by** (*intro mult-mono*) *auto*

lemma *ennreal-mult-strict-right-mono*: (*a*::*ennreal*) < *c* ⟹ *0* < *b* ⟹ *b* < *top* ⟹ *a* * *b* < *c* * *b*
by *transfer* (*auto intro*!: *ereal-mult-strict-right-mono*)

lemma *ennreal-indicator-less*[*simp*]:
indicator A x ≤ (*indicator B x*::*ennreal*) ⟷ (*x* ∈ *A* ⟶ *x* ∈ *B*)
by (*simp add*: *indicator-def not-le*)

lemma *ennreal-inverse-positive*: *0* < *inverse x* ⟷ (*x*::*ennreal*) ≠ *top*
by *transfer* (*simp add*: *ereal-0-gt-inverse top-ereal-def*)

lemma *ennreal-inverse-mult'*: ((*0* < *b* ∨ *a* < *top*) ∧ (*0* < *a* ∨ *b* < *top*)) ⟹ *inverse* (*a* * *b*::*ennreal*) = *inverse a* * *inverse b*
apply *transfer*
subgoal for *a b*
by (*cases a b rule*: *ereal2-cases*) (*auto simp*: *top-ereal-def*)
done

lemma *ennreal-inverse-mult*: *a* < *top* ⟹ *b* < *top* ⟹ *inverse* (*a* * *b*::*ennreal*) = *inverse a* * *inverse b*
apply *transfer*
subgoal for *a b*
by (*cases a b rule*: *ereal2-cases*) (*auto simp*: *top-ereal-def*)
done

lemma *ennreal-inverse-1*[*simp*]: *inverse* (*1*::*ennreal*) = *1*
by *transfer simp*

lemma *ennreal-inverse-eq-0-iff*[*simp*]: *inverse* (*a*::*ennreal*) = *0* ⟷ *a* = *top*
by *transfer* (*simp add*: *ereal-inverse-eq-0 top-ereal-def*)

lemma *ennreal-inverse-eq-top-iff*[*simp*]: *inverse* (*a*::*ennreal*) = *top* ⟷ *a* = *0*
by *transfer* (*simp add*: *top-ereal-def*)

lemma *ennreal-divide-eq-0-iff*[*simp*]: (*a*::*ennreal*) / *b* = *0* ⟷ (*a* = *0* ∨ *b* = *top*)
by (*simp add*: *divide-ennreal-def*)

lemma *ennreal-divide-eq-top-iff*: (*a*::*ennreal*) / *b* = *top* ⟷ ((*a* ≠ *0* ∧ *b* = *0*) ∨ (*a* = *top* ∧ *b* ≠ *top*))
by (*auto simp add*: *divide-ennreal-def ennreal-mult-eq-top-iff*)

lemma *one-divide-one-divide-ennreal*[*simp*]: *1* / (*1* / *c*) = (*c*::*ennreal*)
including *ennreal.lifting*

unfolding *divide-ennreal-def*
by *transfer auto*

lemma *ennreal-mult-left-cong*:
$((a::ennreal) \neq 0 \Longrightarrow b = c) \Longrightarrow a * b = a * c$
by (*cases* $a = 0$) *simp-all*

lemma *ennreal-mult-right-cong*:
$((a::ennreal) \neq 0 \Longrightarrow b = c) \Longrightarrow b * a = c * a$
by (*cases* $a = 0$) *simp-all*

lemma *ennreal-zero-less-mult-iff*: $0 < a * b \longleftrightarrow 0 < a \wedge 0 < (b::ennreal)$
by *transfer* (*auto simp add*: *ereal-zero-less-0-iff le-less*)

lemma *less-diff-eq-ennreal*:
fixes *a b c* :: *ennreal*
shows $b < top \vee c < top \Longrightarrow a < b - c \longleftrightarrow a + c < b$
apply *transfer*
subgoal for *a b c*
by (*cases a b c rule*: *ereal3-cases*) (*auto split*: *split-max*)
done

lemma *diff-add-cancel-ennreal*:
fixes *a b* :: *ennreal* **shows** $a \leq b \Longrightarrow b - a + a = b$
unfolding *infinity-ennreal-def*
apply *transfer*
subgoal for *a b*
by (*cases a b rule*: *ereal2-cases*) (*auto simp*: *max-absorb2*)
done

lemma *ennreal-diff-self*[*simp*]: $a \neq top \Longrightarrow a - a = (0::ennreal)$
by *transfer* (*simp add*: *top-ereal-def*)

lemma *ennreal-minus-mono*:
fixes *a b c* :: *ennreal*
shows $a \leq c \Longrightarrow d \leq b \Longrightarrow a - b \leq c - d$
apply *transfer*
apply (*rule max.mono*)
apply *simp*
subgoal for *a b c d*
by (*cases a b c d rule*: *ereal3-cases*[*case-product ereal-cases*]) *auto*
done

lemma *ennreal-minus-eq-top*[*simp*]: $a - (b::ennreal) = top \longleftrightarrow a = top$
by *transfer* (*auto simp*: *top-ereal-def max.absorb2 ereal-minus-eq-PInfty-iff split*: *split-max*)

lemma *ennreal-divide-self*[*simp*]: $a \neq 0 \Longrightarrow a < top \Longrightarrow a \mathbin{/} a = (1::ennreal)$
unfolding *divide-ennreal-def*

apply *transfer*
subgoal for *a*
by (*cases a*) (*auto simp*: *top-ereal-def*)
done

43.10 Coercion from *real* to *ennreal*

lift-definition *ennreal* :: *real* ⇒ *ennreal* **is** *sup 0* ∘ *ereal*
by *simp*

declare [[*coercion ennreal*]]

lemma *ennreal-cases*[*cases type*: *ennreal*]:
fixes *x* :: *ennreal*
obtains (*real*) *r* :: *real* **where** *0* ≤ *r x* = *ennreal r* | (*top*) *x* = *top*
apply *transfer*
subgoal for *x thesis*
by (*cases x*) (*auto simp*: *max.absorb2 top-ereal-def*)
done

lemmas *ennreal2-cases* = *ennreal-cases*[*case-product ennreal-cases*]
lemmas *ennreal3-cases* = *ennreal-cases*[*case-product ennreal2-cases*]

lemma *ennreal-neq-top*[*simp*]: *ennreal r* ≠ *top*
by *transfer* (*simp add*: *top-ereal-def zero-ereal-def ereal-max*[*symmetric*] *del*: *ereal-max*)

lemma *top-neq-ennreal*[*simp*]: *top* ≠ *ennreal r*
using *ennreal-neq-top*[*of r*] **by** (*auto simp del*: *ennreal-neq-top*)

lemma *ennreal-less-top*[*simp*]: *ennreal x* < *top*
by *transfer* (*simp add*: *top-ereal-def max-def*)

lemma *ennreal-neg*: *x* ≤ *0* ⟹ *ennreal x* = *0*
by *transfer* (*simp add*: *max.absorb1*)

lemma *ennreal-inj*[*simp*]:
0 ≤ *a* ⟹ *0* ≤ *b* ⟹ *ennreal a* = *ennreal b* ⟷ *a* = *b*
by (*transfer fixing*: *a b*) (*auto simp*: *max-absorb2*)

lemma *ennreal-le-iff*[*simp*]: *0* ≤ *y* ⟹ *ennreal x* ≤ *ennreal y* ⟷ *x* ≤ *y*
by (*auto simp*: *ennreal-def zero-ereal-def less-eq-ennreal.abs-eq eq-onp-def split*: *split-max*)

lemma *le-ennreal-iff*: *0* ≤ *r* ⟹ *x* ≤ *ennreal r* ⟷ (∃ *q*≥*0*. *x* = *ennreal q* ∧ *q* ≤ *r*)
by (*cases x*) (*auto simp*: *top-unique*)

lemma *ennreal-less-iff*: *0* ≤ *r* ⟹ *ennreal r* < *ennreal q* ⟷ *r* < *q*

unfolding *not-le*[*symmetric*] **by** *auto*

lemma *ennreal-eq-zero-iff*[*simp*]: $0 \leq x \Longrightarrow ennreal\ x = 0 \longleftrightarrow x = 0$
by *transfer* (*auto simp*: *max-absorb2*)

lemma *ennreal-less-zero-iff*[*simp*]: $0 < ennreal\ x \longleftrightarrow 0 < x$
by *transfer* (*auto simp*: *max-def*)

lemma *ennreal-lessI*: $0 < q \Longrightarrow r < q \Longrightarrow ennreal\ r < ennreal\ q$
by (*cases* $0 \leq r$) (*auto simp*: *ennreal-less-iff ennreal-neg*)

lemma *ennreal-leI*: $x \leq y \Longrightarrow ennreal\ x \leq ennreal\ y$
by (*cases* $0 \leq y$) (*auto simp*: *ennreal-neg*)

lemma *enn2ereal-ennreal*[*simp*]: $0 \leq x \Longrightarrow enn2ereal\ (ennreal\ x) = x$
by *transfer* (*simp add*: *max-absorb2*)

lemma *e2ennreal-enn2ereal*[*simp*]: $e2ennreal\ (enn2ereal\ x) = x$
by (*simp add*: *e2ennreal-def max-absorb2 ennreal.enn2ereal-inverse*)

lemma *ennreal-0*[*simp*]: $ennreal\ 0 = 0$
by (*simp add*: *ennreal-def max.absorb1 zero-ennreal.abs-eq*)

lemma *ennreal-1*[*simp*]: $ennreal\ 1 = 1$
by *transfer* (*simp add*: *max-absorb2*)

lemma *ennreal-eq-0-iff*: $ennreal\ x = 0 \longleftrightarrow x \leq 0$
by (*cases* $0 \leq x$) (*auto simp*: *ennreal-neg*)

lemma *ennreal-le-iff2*: $ennreal\ x \leq ennreal\ y \longleftrightarrow ((0 \leq y \wedge x \leq y) \vee (x \leq 0 \wedge y \leq 0))$
by (*cases* $0 \leq y$) (*auto simp*: *ennreal-eq-0-iff ennreal-neg*)

lemma *ennreal-eq-1*[*simp*]: $ennreal\ x = 1 \longleftrightarrow x = 1$
by (*cases* $0 \leq x$)
(*auto simp*: *ennreal-neg ennreal-1*[*symmetric*] *simp del*: *ennreal-1*)

lemma *ennreal-le-1*[*simp*]: $ennreal\ x \leq 1 \longleftrightarrow x \leq 1$
by (*cases* $0 \leq x$)
(*auto simp*: *ennreal-neg ennreal-1*[*symmetric*] *simp del*: *ennreal-1*)

lemma *ennreal-ge-1*[*simp*]: $ennreal\ x \geq 1 \longleftrightarrow x \geq 1$
by (*cases* $0 \leq x$)
(*auto simp*: *ennreal-neg ennreal-1*[*symmetric*] *simp del*: *ennreal-1*)

lemma *ennreal-plus*[*simp*]:
$0 \leq a \Longrightarrow 0 \leq b \Longrightarrow ennreal\ (a + b) = ennreal\ a + ennreal\ b$
by (*transfer fixing*: *a b*) (*auto simp*: *max-absorb2*)

lemma *setsum-ennreal*[*simp*]: $(\bigwedge i.\ i \in I \Longrightarrow 0 \leq f\ i) \Longrightarrow (\sum i{\in}I.\ ennreal\ (f\ i)) = ennreal\ (setsum\ f\ I)$
by (*induction I rule*: *infinite-finite-induct*) (*auto simp*: *setsum-nonneg*)

lemma *ennreal-of-nat-eq-real-of-nat*: *of-nat i* = *ennreal* (*of-nat i*)
by (*induction i*) *simp-all*

lemma *of-nat-le-ennreal-iff*[*simp*]: $0 \leq r \Longrightarrow$ *of-nat i* $\leq$ *ennreal r* $\longleftrightarrow$ *of-nat i* $\leq r$
by (*simp add*: *ennreal-of-nat-eq-real-of-nat*)

lemma *ennreal-le-of-nat-iff*[*simp*]: *ennreal r* $\leq$ *of-nat i* $\longleftrightarrow$ $r \leq$ *of-nat i*
by (*simp add*: *ennreal-of-nat-eq-real-of-nat*)

lemma *ennreal-indicator*: *ennreal* (*indicator A x*) = *indicator A x*
by (*auto split*: *split-indicator*)

lemma *ennreal-numeral*[*simp*]: *ennreal* (*numeral n*) = *numeral n*
using *ennreal-of-nat-eq-real-of-nat*[*of numeral n*] **by** *simp*

lemma *min-ennreal*: $0 \leq x \Longrightarrow 0 \leq y \Longrightarrow$ *min* (*ennreal x*) (*ennreal y*) = *ennreal* (*min x y*)
by (*auto split*: *split-min*)

lemma *ennreal-half*[*simp*]: *ennreal* (*1 / 2*) = *inverse 2*
by *transfer* (*simp add*: *max.absorb2*)

lemma *ennreal-minus*: $0 \leq q \Longrightarrow$ *ennreal r* $-$ *ennreal q* = *ennreal* $(r - q)$
by *transfer*
(*simp add*: *max.absorb2 zero-ereal-def ereal-max*[*symmetric*] *del*: *ereal-max*)

lemma *ennreal-minus-top*[*simp*]: *ennreal a* $-$ *top* = *0*
by (*simp add*: *minus-top-ennreal*)

lemma *ennreal-mult*: $0 \leq a \Longrightarrow 0 \leq b \Longrightarrow$ *ennreal* $(a * b)$ = *ennreal a* $*$ *ennreal b*
by *transfer* (*simp add*: *max-absorb2*)

lemma *ennreal-mult'*: $0 \leq a \Longrightarrow$ *ennreal* $(a * b)$ = *ennreal a* $*$ *ennreal b*
by (*cases* $0 \leq b$) (*auto simp*: *ennreal-mult ennreal-neg mult-nonneg-nonpos*)

lemma *indicator-mult-ennreal*: *indicator A x* $*$ *ennreal r* = *ennreal* (*indicator A x* $*$ *r*)
by (*simp split*: *split-indicator*)

lemma *ennreal-mult''*: $0 \leq b \Longrightarrow$ *ennreal* $(a * b)$ = *ennreal a* $*$ *ennreal b*
by (*cases* $0 \leq a$) (*auto simp*: *ennreal-mult ennreal-neg mult-nonpos-nonneg*)

lemma *numeral-mult-ennreal*: $0 \leq x \Longrightarrow$ *numeral b* $*$ *ennreal x* = *ennreal* (*numeral*

$b * x$)
by (*simp add*: *ennreal-mult*)

lemma *ennreal-power*: $0 \leq r \Longrightarrow \mathit{ennreal}\ r\ \hat{}\ n = \mathit{ennreal}\ (r\ \hat{}\ n)$
by (*induction n*) (*auto simp*: *ennreal-mult*)

lemma *power-eq-top-ennreal*: $x\ \hat{}\ n = \mathit{top} \longleftrightarrow (n \neq 0 \wedge (x{::}\mathit{ennreal}) = \mathit{top})$
by (*cases x rule*: *ennreal-cases*)
(*auto simp*: *ennreal-power top-power-ennreal*)

lemma *inverse-ennreal*: $0 < r \Longrightarrow \mathit{inverse}\ (\mathit{ennreal}\ r) = \mathit{ennreal}\ (\mathit{inverse}\ r)$
by *transfer* (*simp add*: *max.absorb2*)

lemma *divide-ennreal*: $0 \leq r \Longrightarrow 0 < q \Longrightarrow \mathit{ennreal}\ r\ /\ \mathit{ennreal}\ q = \mathit{ennreal}\ (r\ /\ q)$
by (*simp add*: *divide-ennreal-def inverse-ennreal ennreal-mult*[*symmetric*] *inverse-eq-divide*)

lemma *ennreal-inverse-power*: $\mathit{inverse}\ (x\ \hat{}\ n :: \mathit{ennreal}) = \mathit{inverse}\ x\ \hat{}\ n$
proof (*cases x rule*: *ennreal-cases*)
case *top* **with** *power-eq-top-ennreal*[*of x n*] **show** *?thesis*
by (*cases n* = *0*) *auto*
next
case (*real r*) **then show** *?thesis*
proof *cases*
assume $x = 0$ **then show** *?thesis*
using *power-eq-top-ennreal*[*of top n* − *1*]
by (*cases n*) (*auto simp*: *ennreal-top-mult*)
next
assume $x \neq 0$
with *real* **have** $0 < r$ **by** *auto*
with *real* **show** *?thesis*
by (*induction n*)
(*auto simp add*: *ennreal-power ennreal-mult*[*symmetric*] *inverse-ennreal*)
qed
qed

lemma *ennreal-divide-numeral*: $0 \leq x \Longrightarrow \mathit{ennreal}\ x\ /\ \mathit{numeral}\ b = \mathit{ennreal}\ (x\ /\ \mathit{numeral}\ b)$
by (*subst divide-ennreal*[*symmetric*]) *auto*

lemma *setprod-ennreal*: $(\bigwedge i.\ i \in A \Longrightarrow 0 \leq f\ i) \Longrightarrow (\prod i{\in}A.\ \mathit{ennreal}\ (f\ i)) = \mathit{ennreal}\ (\mathit{setprod}\ f\ A)$
by (*induction A rule*: *infinite-finite-induct*)
(*auto simp*: *ennreal-mult setprod-nonneg*)

lemma *mult-right-ennreal-cancel*: $a * \mathit{ennreal}\ c = b * \mathit{ennreal}\ c \longleftrightarrow (a = b \vee c \leq 0)$
apply (*cases* $0 \leq c$)
apply (*cases a b rule*: *ennreal2-cases*)

apply (*auto simp*: *ennreal-mult*[*symmetric*] *ennreal-neg ennreal-top-mult*)
done

lemma *ennreal-le-epsilon*:
(⋀*e*::*real*. *y* < *top* ⟹ 0 < *e* ⟹ *x* ≤ *y* + *ennreal e*) ⟹ *x* ≤ *y*
apply (*cases y rule*: *ennreal-cases*)
apply (*cases x rule*: *ennreal-cases*)
apply (*auto simp del*: *ennreal-plus simp add*: *top-unique ennreal-plus*[*symmetric*]
intro: *zero-less-one field-le-epsilon*)
done

lemma *ennreal-rat-dense*:
fixes *x y* :: *ennreal*
shows *x* < *y* ⟹ ∃ *r*::*rat*. *x* < *real-of-rat r* ∧ *real-of-rat r* < *y*
proof *transfer*
fix *x y* :: *ereal* **assume** *xy*: 0 ≤ *x* 0 ≤ *y x* < *y*
moreover
from *ereal-dense3*[*OF* ‹*x* < *y*›]
obtain *r* **where** *x* < *ereal* (*real-of-rat r*) *ereal* (*real-of-rat r*) < *y*
by *auto*
moreover then have 0 ≤ *r*
using *le-less-trans*[*OF* ‹0 ≤ *x*› ‹*x* < *ereal* (*real-of-rat r*)›] **by** *auto*
ultimately show ∃ *r*. *x* < (*sup* 0 ∘ *ereal*) (*real-of-rat r*) ∧ (*sup* 0 ∘ *ereal*)
(*real-of-rat r*) < *y*
by (*intro exI*[*of* - *r*]) (*auto simp*: *max-absorb2*)
qed

lemma *ennreal-Ex-less-of-nat*: (*x*::*ennreal*) < *top* ⟹ ∃ *n*. *x* < *of-nat n*
by (*cases x rule*: *ennreal-cases*)
(*auto simp*: *ennreal-of-nat-eq-real-of-nat ennreal-less-iff reals-Archimedean2*)

43.11 Coercion from *ennreal* to *real*

definition *enn2real x* = *real-of-ereal* (*enn2ereal x*)

lemma *enn2real-nonneg*[*simp*]: 0 ≤ *enn2real x*
by (*auto simp*: *enn2real-def intro*!: *real-of-ereal-pos enn2ereal-nonneg*)

lemma *enn2real-mono*: *a* ≤ *b* ⟹ *b* < *top* ⟹ *enn2real a* ≤ *enn2real b*
by (*auto simp add*: *enn2real-def less-eq-ennreal.rep-eq intro*!: *real-of-ereal-positive-mono*
enn2ereal-nonneg)

lemma *enn2real-of-nat*[*simp*]: *enn2real* (*of-nat n*) = *n*
by (*auto simp*: *enn2real-def*)

lemma *enn2real-ennreal*[*simp*]: 0 ≤ *r* ⟹ *enn2real* (*ennreal r*) = *r*
by (*simp add*: *enn2real-def*)

lemma *ennreal-enn2real*[*simp*]: *r* < *top* ⟹ *ennreal* (*enn2real r*) = *r*

by (*cases r rule*: *ennreal-cases*) *auto*

lemma *real-of-ereal-enn2ereal*[*simp*]: *real-of-ereal* (*enn2ereal x*) = *enn2real x*
by (*simp add*: *enn2real-def*)

lemma *enn2real-top*[*simp*]: *enn2real top* = *0*
unfolding *enn2real-def top-ennreal.rep-eq top-ereal-def* **by** *simp*

lemma *enn2real-0*[*simp*]: *enn2real 0* = *0*
unfolding *enn2real-def zero-ennreal.rep-eq* **by** *simp*

lemma *enn2real-1*[*simp*]: *enn2real 1* = *1*
unfolding *enn2real-def one-ennreal.rep-eq* **by** *simp*

lemma *enn2real-numeral*[*simp*]: *enn2real* (*numeral n*) = (*numeral n*)
unfolding *enn2real-def* **by** *simp*

lemma *enn2real-mult*: *enn2real* (*a* ∗ *b*) = *enn2real a* ∗ *enn2real b*
unfolding *enn2real-def*
by (*simp del*: *real-of-ereal-enn2ereal add*: *times-ennreal.rep-eq*)

lemma *enn2real-leI*: *0* ≤ *B* ⟹ *x* ≤ *ennreal B* ⟹ *enn2real x* ≤ *B*
by (*cases x rule*: *ennreal-cases*) (*auto simp*: *top-unique*)

lemma *enn2real-positive-iff*: *0* < *enn2real x* ⟷ (*0* < *x* ∧ *x* < *top*)
by (*cases x rule*: *ennreal-cases*) *auto*

43.12 Coercion from *enat* to *ennreal*

definition *ennreal-of-enat* :: *enat* ⇒ *ennreal*
where
ennreal-of-enat n = (*case n of* ∞ ⇒ *top* | *enat n* ⇒ *of-nat n*)

declare [[*coercion ennreal-of-enat*]]
declare [[*coercion of-nat* :: *nat* ⇒ *ennreal*]]

lemma *ennreal-of-enat-infty*[*simp*]: *ennreal-of-enat* ∞ = ∞
by (*simp add*: *ennreal-of-enat-def*)

lemma *ennreal-of-enat-enat*[*simp*]: *ennreal-of-enat* (*enat n*) = *of-nat n*
by (*simp add*: *ennreal-of-enat-def*)

lemma *ennreal-of-enat-0*[*simp*]: *ennreal-of-enat 0* = *0*
using *ennreal-of-enat-enat*[*of 0*] **unfolding** *enat-0* **by** *simp*

lemma *ennreal-of-enat-1*[*simp*]: *ennreal-of-enat 1* = *1*
using *ennreal-of-enat-enat*[*of 1*] **unfolding** *enat-1* **by** *simp*

lemma *ennreal-top-neq-of-nat*[*simp*]: (*top*::*ennreal*) ≠ *of-nat i*

using *ennreal-of-nat-neq-top*[*of i*] **by** *metis*

lemma *ennreal-of-enat-inj*[*simp*]: *ennreal-of-enat i* = *ennreal-of-enat j* ⟷ *i* = *j*
by (*cases i j rule*: *enat.exhaust*[*case-product enat.exhaust*]) *auto*

lemma *ennreal-of-enat-le-iff*[*simp*]: *ennreal-of-enat m* ≤ *ennreal-of-enat n* ⟷ *m* ≤ *n*
by (*auto simp*: *ennreal-of-enat-def top-unique split*: *enat.split*)

lemma *of-nat-less-ennreal-of-nat*[*simp*]: *of-nat n* ≤ *ennreal-of-enat x* ⟷ *of-nat n* ≤ *x*
by (*cases x*) (*auto simp*: *of-nat-eq-enat*)

lemma *ennreal-of-enat-Sup*: *ennreal-of-enat* (*Sup X*) = (*SUP x*:*X*. *ennreal-of-enat x*)
proof −
have *ennreal-of-enat* (*Sup X*) ≤ (*SUP x* : *X*. *ennreal-of-enat x*)
unfolding *Sup-enat-def*
proof (*clarsimp*, *intro conjI impI*)
fix *x* **assume** *finite X X* ≠ {}
then show *ennreal-of-enat* (*Max X*) ≤ (*SUP x* : *X*. *ennreal-of-enat x*)
by (*intro SUP-upper Max-in*)
next
assume *infinite X X* ≠ {}
have ∃ *y*∈*X*. *r* < *ennreal-of-enat y* **if** *r*: *r* < *top* **for** *r*
proof −
from *ennreal-Ex-less-of-nat*[*OF r*] **guess** *n* **.. note** *n* = *this*
have ¬ (*X* ⊆ *enat* ' {.. *n*})
using ‹*infinite X*› **by** (*auto dest*: *finite-subset*)
then obtain *x* **where** *x* ∈ *X x* ∉ *enat* ' {..*n*}
by *blast*
moreover then have *of-nat n* ≤ *x*
by (*cases x*) (*auto simp*: *of-nat-eq-enat*)
ultimately show *?thesis*
by (*auto intro*!: *bexI*[*of* - *x*] *less-le-trans*[*OF n*])
qed
then have (*SUP x* : *X*. *ennreal-of-enat x*) = *top*
by *simp*
then show *top* ≤ (*SUP x* : *X*. *ennreal-of-enat x*)
unfolding *top-unique* **by** *simp*
qed
then show *?thesis*
by (*auto intro*!: *antisym Sup-least intro*: *Sup-upper*)
qed

lemma *ennreal-of-enat-eSuc*[*simp*]: *ennreal-of-enat* (*eSuc x*) = *1* + *ennreal-of-enat x*
by (*cases x*) (*auto simp*: *eSuc-enat*)

43.13 Topology on *ennreal*

lemma *enn2ereal-Iio*: *enn2ereal −‘ {..<a} = (if 0 ≤ a then {..< e2ennreal a} else {})*
using *enn2ereal-nonneg*
by (*cases a rule*: *ereal-ennreal-cases*)
(*auto simp add*: *vimage-def set-eq-iff ennreal.enn2ereal-inverse less-ennreal.rep-eq e2ennreal-def max-absorb2*
simp del: *enn2ereal-nonneg*
intro: *le-less-trans less-imp-le*)

lemma *enn2ereal-Ioi*: *enn2ereal −‘ {a <..} = (if 0 ≤ a then {e2ennreal a <..} else UNIV)*
by (*cases a rule*: *ereal-ennreal-cases*)
(*auto simp add*: *vimage-def set-eq-iff ennreal.enn2ereal-inverse less-ennreal.rep-eq e2ennreal-def max-absorb2*
intro: *less-le-trans*)

instantiation *ennreal* :: *linear-continuum-topology*
begin

definition *open-ennreal* :: *ennreal set ⇒ bool*
where (*open* :: *ennreal set ⇒ bool*) = *generate-topology* (*range lessThan ∪ range greaterThan*)

instance
proof
show ∃ *a b*::*ennreal*. *a ≠ b*
using *zero-neq-one* **by** (*intro exI*)
show ⋀*x y*::*ennreal*. *x < y ⟹* ∃ *z*>*x*. *z < y*
proof *transfer*
fix *x y* :: *ereal* **assume** *0 ≤ x x < y*
moreover from *dense*[*OF this*(*2*)] **guess** *z* **..**
ultimately show ∃ *z*∈*Collect* (*op ≤ 0*). *x < z ∧ z < y*
by (*intro bexI*[*of - z*]) *auto*
qed
qed (*rule open-ennreal-def*)

end

lemma *continuous-on-e2ennreal*: *continuous-on A e2ennreal*
proof (*rule continuous-on-subset*)
show *continuous-on* ({*0..*} ∪ {*..0*}) *e2ennreal*
proof (*rule continuous-on-closed-Un*)
show *continuous-on* {*0 ..*} *e2ennreal*
by (*rule continuous-onI-mono*)
(*auto simp add*: *less-eq-ennreal.abs-eq eq-onp-def enn2ereal-range*)
show *continuous-on* {*.. 0*} *e2ennreal*
by (*subst continuous-on-cong*[*OF refl*, *of - - λ-. 0*])
(*auto simp add*: *e2ennreal-neg continuous-on-const*)

qed *auto*
show $A \subseteq \{0..\} \cup \{..0::ereal\}$
by *auto*
qed

lemma *continuous-at-e2ennreal*: *continuous* (*at x within A*) *e2ennreal*
by (*rule continuous-on-imp-continuous-within*[*OF continuous-on-e2ennreal*, *of - UNIV*]) *auto*

lemma *continuous-on-enn2ereal*: *continuous-on UNIV enn2ereal*
by (*rule continuous-on-generate-topology*[*OF open-generated-order*])
(*auto simp add*: *enn2ereal-Iio enn2ereal-Ioi*)

lemma *continuous-at-enn2ereal*: *continuous* (*at x within A*) *enn2ereal*
by (*rule continuous-on-imp-continuous-within*[*OF continuous-on-enn2ereal*]) *auto*

lemma *sup-continuous-e2ennreal*[*order-continuous-intros*]:
assumes *f*: *sup-continuous f* **shows** *sup-continuous* (λ*x*. *e2ennreal* (*f x*))
apply (*rule sup-continuous-compose*[*OF - f*])
apply (*rule continuous-at-left-imp-sup-continuous*)
apply (*auto simp*: *mono-def e2ennreal-mono continuous-at-e2ennreal*)
done

lemma *sup-continuous-enn2ereal*[*order-continuous-intros*]:
assumes *f*: *sup-continuous f* **shows** *sup-continuous* (λ*x*. *enn2ereal* (*f x*))
apply (*rule sup-continuous-compose*[*OF - f*])
apply (*rule continuous-at-left-imp-sup-continuous*)
apply (*simp-all add*: *mono-def less-eq-ennreal.rep-eq continuous-at-enn2ereal*)
done

lemma *sup-continuous-mult-left-ennreal'*:
fixes *c* :: *ennreal*
shows *sup-continuous* (λ*x*. *c* ∗ *x*)
unfolding *sup-continuous-def*
by *transfer* (*auto simp*: *SUP-ereal-mult-left max.absorb2 SUP-upper2*)

lemma *sup-continuous-mult-left-ennreal*[*order-continuous-intros*]:
sup-continuous f ⟹ *sup-continuous* (λ*x*. *c* ∗ *f x* :: *ennreal*)
by (*rule sup-continuous-compose*[*OF sup-continuous-mult-left-ennreal'*])

lemma *sup-continuous-mult-right-ennreal*[*order-continuous-intros*]:
sup-continuous f ⟹ *sup-continuous* (λ*x*. *f x* ∗ *c* :: *ennreal*)
using *sup-continuous-mult-left-ennreal*[*of f c*] **by** (*simp add*: *mult.commute*)

lemma *sup-continuous-divide-ennreal*[*order-continuous-intros*]:
fixes *f g* :: *'a*::*complete-lattice* ⇒ *ennreal*
shows *sup-continuous f* ⟹ *sup-continuous* (λ*x*. *f x* / *c*)
unfolding *divide-ennreal-def* **by** (*rule sup-continuous-mult-right-ennreal*)

lemma *transfer-enn2ereal-continuous-on* [*transfer-rule*]:
rel-fun (*op* =) (*rel-fun* (*rel-fun op* = *pcr-ennreal*) *op* =) *continuous-on continuous-on*
proof −
have *continuous-on A f* **if** *continuous-on A* (λ*x*. *enn2ereal* (*f x*)) **for** *A* **and** *f* :: *'a* ⇒ *ennreal*
using *continuous-on-compose2*[*OF continuous-on-e2ennreal*[*of* {*0*..}] *that*]
by (*auto simp*: *ennreal.enn2ereal-inverse subset-eq e2ennreal-def max-absorb2*)
moreover
have *continuous-on A* (λ*x*. *enn2ereal* (*f x*)) **if** *continuous-on A f* **for** *A* **and** *f* :: *'a* ⇒ *ennreal*
using *continuous-on-compose2*[*OF continuous-on-enn2ereal that*] **by** *auto*
ultimately
show *?thesis*
by (*auto simp add*: *rel-fun-def ennreal.pcr-cr-eq cr-ennreal-def*)
qed

lemma *transfer-sup-continuous*[*transfer-rule*]:
(*rel-fun* (*rel-fun* (*op* =) *pcr-ennreal*) *op* =) *sup-continuous sup-continuous*
proof (*safe intro*!: *rel-funI dest*!: *rel-fun-eq-pcr-ennreal*[*THEN iffD1*])
show *sup-continuous* (*enn2ereal* ∘ *f*) ⟹ *sup-continuous f* **for** *f* :: *'a* ⇒ -
using *sup-continuous-e2ennreal*[*of enn2ereal* ∘ *f*] **by** *simp*
show *sup-continuous f* ⟹ *sup-continuous* (*enn2ereal* ∘ *f*) **for** *f* :: *'a* ⇒ -
using *sup-continuous-enn2ereal*[*of f*] **by** (*simp add*: *comp-def*)
qed

lemma *continuous-on-ennreal*[*tendsto-intros*]:
continuous-on A f ⟹ *continuous-on A* (λ*x*. *ennreal* (*f x*))
by *transfer* (*auto intro*!: *continuous-on-max continuous-on-const continuous-on-ereal*)

lemma *tendsto-ennrealD*:
assumes *lim*: ((λ*x*. *ennreal* (*f x*)) ⟶ *ennreal x*) *F*
assumes ∗: ∀ *F x in F*. *0* ≤ *f x* **and** *x*: *0* ≤ *x*
shows (*f* ⟶ *x*) *F*
using *continuous-on-tendsto-compose*[*OF continuous-on-enn2ereal lim*]
apply *simp*
apply (*subst* (*asm*) *tendsto-cong*)
using ∗
apply *eventually-elim*
apply (*auto simp*: *max-absorb2* ‹*0* ≤ *x*›)
done

lemma *tendsto-ennreal-iff*[*simp*]:
∀ *F x in F*. *0* ≤ *f x* ⟹ *0* ≤ *x* ⟹ ((λ*x*. *ennreal* (*f x*)) ⟶ *ennreal x*) *F* ⟷ (*f* ⟶ *x*) *F*
by (*auto dest*: *tendsto-ennrealD*)
(*auto simp*: *ennreal-def*
intro!: *continuous-on-tendsto-compose*[*OF continuous-on-e2ennreal*[*of UNIV*]] *tendsto-max*)

lemma *tendsto-enn2ereal-iff*[*simp*]: $((\lambda i.\ enn2ereal\ (f\ i)) \longrightarrow enn2ereal\ x)\ F \longleftrightarrow (f \longrightarrow x)\ F$
using *continuous-on-enn2ereal*[*THEN continuous-on-tendsto-compose*, *of f x F*]
continuous-on-e2ennreal[*THEN continuous-on-tendsto-compose*, *of* $\lambda x.\ enn2ereal\ (f\ x)\ enn2ereal\ x\ F\ UNIV$]
by *auto*

lemma *continuous-on-add-ennreal*:
fixes *f g* :: *'a*::*topological-space* ⇒ *ennreal*
shows *continuous-on A f* ⟹ *continuous-on A g* ⟹ *continuous-on A* $(\lambda x.\ f\ x + g\ x)$
by (*transfer fixing*: *A*) (*auto intro*!: *tendsto-add-ereal-nonneg simp*: *continuous-on-def*)

lemma *continuous-on-inverse-ennreal*[*continuous-intros*]:
fixes *f* :: *'a*::*topological-space* ⇒ *ennreal*
shows *continuous-on A f* ⟹ *continuous-on A* $(\lambda x.\ inverse\ (f\ x))$
proof (*transfer fixing*: *A*)
show *pred-fun* (λ-. *True*) (*op* ≤ *0*) *f* ⟹ *continuous-on A* $(\lambda x.\ inverse\ (f\ x))$
if *continuous-on A f*
for *f* :: *'a* ⇒ *ereal*
using *continuous-on-compose2*[*OF continuous-on-inverse-ereal that*] **by** (*auto simp*: *subset-eq*)
qed

instance *ennreal* :: *topological-comm-monoid-add*
proof
show $((\lambda x.\ fst\ x + snd\ x) \longrightarrow a + b)\ (nhds\ a \times_F nhds\ b)$ **for** *a b* :: *ennreal*
using *continuous-on-add-ennreal*[*of UNIV fst snd*]
using *tendsto-at-iff-tendsto-nhds*[*symmetric*, *of* λx::(*ennreal* × *ennreal*). *fst x + snd x*]
by (*auto simp*: *continuous-on-eq-continuous-at*)
(*simp add*: *isCont-def nhds-prod*[*symmetric*])
qed

lemma *sup-continuous-add-ennreal*[*order-continuous-intros*]:
fixes *f g* :: *'a*::*complete-lattice* ⇒ *ennreal*
shows *sup-continuous f* ⟹ *sup-continuous g* ⟹ *sup-continuous* $(\lambda x.\ f\ x + g\ x)$
by *transfer* (*auto intro*!: *sup-continuous-add*)

lemma *ennreal-suminf-lessD*: $(\sum i.\ f\ i :: ennreal) < x \Longrightarrow f\ i < x$
using *le-less-trans*[*OF setsum-le-suminf*[*OF summableI*, *of* {*i*} *f*]] **by** *simp*

lemma *sums-ennreal*[*simp*]: $(\bigwedge i.\ 0 \le f\ i) \Longrightarrow 0 \le x \Longrightarrow (\lambda i.\ ennreal\ (f\ i))\ sums\ ennreal\ x \longleftrightarrow f\ sums\ x$
unfolding *sums-def* **by** (*simp add*: *always-eventually setsum-nonneg*)

lemma *summable-suminf-not-top*: $(\bigwedge i.\ 0 \le f\ i) \Longrightarrow (\sum i.\ ennreal\ (f\ i)) \neq top \Longrightarrow summable\ f$

using *summable-sums*[*OF summableI*, *of* $\lambda i.$ *ennreal* (*f i*)]
by (*cases* $\sum i.$ *ennreal* (*f i*) *rule*: *ennreal-cases*)
(*auto simp*: *summable-def*)

lemma *suminf-ennreal*[*simp*]:
$(\bigwedge i.\ 0 \le f\ i) \Longrightarrow (\sum i.\ ennreal\ (f\ i)) \neq top \Longrightarrow (\sum i.\ ennreal\ (f\ i)) = ennreal\ (\sum i.\ f\ i)$
by (*rule sums-unique*[*symmetric*]) (*simp add*: *summable-suminf-not-top suminf-nonneg summable-sums*)

lemma *sums-enn2ereal*[*simp*]: $(\lambda i.\ enn2ereal\ (f\ i))\ sums\ enn2ereal\ x \longleftrightarrow f\ sums\ x$
unfolding *sums-def* **by** (*simp add*: *always-eventually setsum-nonneg*)

lemma *suminf-enn2ereal*[*simp*]: $(\sum i.\ enn2ereal\ (f\ i)) = enn2ereal\ (suminf\ f)$
by (*rule sums-unique*[*symmetric*]) (*simp add*: *summable-sums*)

lemma *transfer-e2ennreal-suminf* [*transfer-rule*]: *rel-fun* (*rel-fun op* = *pcr-ennreal*) *pcr-ennreal suminf suminf*
by (*auto simp*: *rel-funI rel-fun-eq-pcr-ennreal comp-def*)

lemma *ennreal-suminf-cmult*[*simp*]: $(\sum i.\ r * f\ i) = r * (\sum i.\ f\ i::ennreal)$
by *transfer* (*auto intro*!: *suminf-cmult-ereal*)

lemma *ennreal-suminf-multc*[*simp*]: $(\sum i.\ f\ i * r) = (\sum i.\ f\ i::ennreal) * r$
using *ennreal-suminf-cmult*[*of r f*] **by** (*simp add*: *ac-simps*)

lemma *ennreal-suminf-divide*[*simp*]: $(\sum i.\ f\ i\ /\ r) = (\sum i.\ f\ i::ennreal)\ /\ r$
by (*simp add*: *divide-ennreal-def*)

lemma *ennreal-suminf-neq-top*: *summable f* $\Longrightarrow (\bigwedge i.\ 0 \le f\ i) \Longrightarrow (\sum i.\ ennreal\ (f\ i)) \neq top$
using *sums-ennreal*[*of f suminf f*]
by (*simp add*: *suminf-nonneg sums-unique*[*symmetric*] *summable-sums-iff*[*symmetric*] *del*: *sums-ennreal*)

lemma *suminf-ennreal-eq*:
$(\bigwedge i.\ 0 \le f\ i) \Longrightarrow f\ sums\ x \Longrightarrow (\sum i.\ ennreal\ (f\ i)) = ennreal\ x$
using *suminf-nonneg*[*of f*] *sums-unique*[*of f x*]
by (*intro sums-unique*[*symmetric*]) (*auto simp*: *summable-sums-iff*)

lemma *ennreal-suminf-bound-add*:
fixes *f* :: *nat* ⇒ *ennreal*
shows $(\bigwedge N.\ (\sum n{<}N.\ f\ n) + y \le x) \Longrightarrow suminf\ f + y \le x$
by *transfer* (*auto intro*!: *suminf-bound-add*)

lemma *ennreal-suminf-SUP-eq-directed*:
fixes *f* :: *'a* ⇒ *nat* ⇒ *ennreal*
assumes ∗: $\bigwedge N\ i\ j.\ i \in I \Longrightarrow j \in I \Longrightarrow finite\ N \Longrightarrow \exists k{\in}I.\ \forall n{\in}N.\ f\ i\ n \le f\ k$

$n \wedge f\ j\ n \leq f\ k\ n$
shows $(\sum n.\ SUP\ i{:}I.\ f\ i\ n) = (SUP\ i{:}I.\ \sum n.\ f\ i\ n)$
proof *cases*
assume $I \neq \{\}$
then obtain i **where** $i \in I$ **by** *auto*
from $*$ **show** *?thesis*
by (*transfer fixing*: I)
(*auto simp*: *max-absorb2 SUP-upper2*[*OF* ‹$i \in I$›] *suminf-nonneg summable-ereal-pos* ‹$I \neq \{\}$›
intro!: *suminf-SUP-eq-directed*)
qed (*simp add*: *bot-ennreal*)

lemma *INF-ennreal-add-const*:
fixes $f\ g :: nat \Rightarrow ennreal$
shows $(INF\ i.\ f\ i + c) = (INF\ i.\ f\ i) + c$
using *continuous-at-Inf-mono*[*of* $\lambda x.\ x + c$ *f'UNIV*]
using *continuous-add*[*of at-right* (*Inf* (*range f*)), *of* $\lambda x.\ x$ $\lambda x.\ c$]
by (*auto simp*: *mono-def*)

lemma *INF-ennreal-const-add*:
fixes $f\ g :: nat \Rightarrow ennreal$
shows $(INF\ i.\ c + f\ i) = c + (INF\ i.\ f\ i)$
using *INF-ennreal-add-const*[*of f c*] **by** (*simp add*: *ac-simps*)

lemma *SUP-mult-left-ennreal*: $c * (SUP\ i{:}I.\ f\ i) = (SUP\ i{:}I.\ c * f\ i :: ennreal)$
proof *cases*
assume $I \neq \{\}$ **then show** *?thesis*
by *transfer* (*auto simp add*: *SUP-ereal-mult-left max-absorb2 SUP-upper2*)
qed (*simp add*: *bot-ennreal*)

lemma *SUP-mult-right-ennreal*: $(SUP\ i{:}I.\ f\ i) * c = (SUP\ i{:}I.\ f\ i * c :: ennreal)$
using *SUP-mult-left-ennreal* **by** (*simp add*: *mult.commute*)

lemma *SUP-divide-ennreal*: $(SUP\ i{:}I.\ f\ i) / c = (SUP\ i{:}I.\ f\ i / c :: ennreal)$
using *SUP-mult-right-ennreal* **by** (*simp add*: *divide-ennreal-def*)

lemma *ennreal-SUP-of-nat-eq-top*: $(SUP\ x.\ of\text{-}nat\ x :: ennreal) = top$
proof (*intro antisym top-greatest le-SUP-iff*[*THEN iffD2*] *allI impI*)
fix $y :: ennreal$ **assume** $y < top$
then obtain r **where** $y = ennreal\ r$
by (*cases y rule*: *ennreal-cases*) *auto*
then show $\exists i{\in}UNIV.\ y < of\text{-}nat\ i$
using *reals-Archimedean2*[*of max 1 r*] *zero-less-one*
by (*auto simp*: *ennreal-of-nat-eq-real-of-nat ennreal-def less-ennreal.abs-eq eq-onp-def max.absorb2*
dest!: *one-less-of-natD intro*: *less-trans*)
qed

lemma *ennreal-SUP-eq-top*:

fixes $f :: 'a \Rightarrow ennreal$
assumes $\bigwedge n.\ \exists\, i{\in}I.\ of\text{-}nat\ n \le f\ i$
shows $(SUP\ i : I.\ f\ i) = top$
proof −
have $(SUP\ x.\ of\text{-}nat\ x :: ennreal) \le (SUP\ i : I.\ f\ i)$
using *assms* **by** (*auto intro*!: *SUP-least intro*: *SUP-upper2*)
then show *?thesis*
by (*auto simp*: *ennreal-SUP-of-nat-eq-top top-unique*)
qed

lemma *ennreal-INF-const-minus*:
fixes $f :: 'a \Rightarrow ennreal$
shows $I \neq \{\} \Longrightarrow (SUP\ x{:}I.\ c - f\ x) = c - (INF\ x{:}I.\ f\ x)$
by (*transfer fixing*: *I*)
(*simp add*: *sup-max*[*symmetric*] *SUP-sup-const1 SUP-ereal-minus-right del*: *sup-ereal-def*)

lemma *of-nat-Sup-ennreal*:
assumes $A \neq \{\}$ *bdd-above A*
shows *of-nat* (*Sup A*) = (*SUP a*:*A*. *of-nat a* :: *ennreal*)
proof (*intro antisym*)
show (*SUP a*:*A*. *of-nat a*::*ennreal*) ≤ *of-nat* (*Sup A*)
by (*intro SUP-least of-nat-mono*) (*auto intro*: *cSup-upper assms*)
have *Sup A* ∈ *A*
unfolding *Sup-nat-def* **using** *assms* **by** (*intro Max-in*) (*auto simp*: *bdd-above-nat*)
then show *of-nat* (*Sup A*) ≤ (*SUP a*:*A*. *of-nat a*::*ennreal*)
by (*intro SUP-upper*)
qed

lemma *ennreal-tendsto-const-minus*:
fixes $g :: 'a \Rightarrow ennreal$
assumes *ae*: $\forall_F\ x\ in\ F.\ g\ x \le c$
assumes *g*: $((\lambda x.\ c - g\ x) \longrightarrow 0)\ F$
shows $(g \longrightarrow c)\ F$
proof (*cases c rule*: *ennreal-cases*)
case *top* **with** *tendsto-unique*[*OF - g*, *of top*] **show** *?thesis*
by (*cases F = bot*) *auto*
next
case (*real r*)
then have $\forall x.\ \exists\, q{\ge}0.\ g\ x \le c \longrightarrow (g\ x = ennreal\ q \wedge q \le r)$
by (*auto simp*: *le-ennreal-iff*)
then obtain *f* **where** ∗: $\bigwedge x.\ g\ x \le c \Longrightarrow 0 \le f\ x\ \bigwedge x.\ g\ x \le c \Longrightarrow g\ x = ennreal\ (f\ x)\ \bigwedge x.\ g\ x \le c \Longrightarrow f\ x \le r$
by *metis*
from *ae* **have** *ae2*: $\forall_F\ x\ in\ F.\ c - g\ x = ennreal\ (r - f\ x) \wedge f\ x \le r \wedge g\ x = ennreal\ (f\ x) \wedge 0 \le f\ x$
proof *eventually-elim*
fix *x* **assume** $g\ x \le c$ **with** ∗[*of x*] ‹$0 \le r$› **show** $c - g\ x = ennreal\ (r - f\ x) \wedge f\ x \le r \wedge g\ x = ennreal\ (f\ x) \wedge 0 \le f\ x$

by (*auto simp*: *real ennreal-minus*)
qed
with *g* **have** ((λ*x*. *ennreal* (*r* − *f x*)) ⟶ *ennreal 0*) *F*
by (*auto simp add*: *tendsto-cong eventually-conj-iff*)
with *ae2* **have** ((λ*x*. *r* − *f x*) ⟶ *0*) *F*
by (*subst* (*asm*) *tendsto-ennreal-iff*) (*auto elim*: *eventually-mono*)
then have (*f* ⟶ *r*) *F*
by (*rule Lim-transform2*[*OF tendsto-const*])
with *ae2* **have** ((λ*x*. *ennreal* (*f x*)) ⟶ *ennreal r*) *F*
by (*subst tendsto-ennreal-iff*) (*auto elim*: *eventually-mono simp*: *real*)
with *ae2* **show** *?thesis*
by (*auto simp*: *real tendsto-cong eventually-conj-iff*)
qed

lemma *ennreal-SUP-add*:
fixes *f g* :: *nat* ⇒ *ennreal*
shows *incseq f* ⟹ *incseq g* ⟹ (*SUP i*. *f i* + *g i*) = *SUPREMUM UNIV f* + *SUPREMUM UNIV g*
unfolding *incseq-def le-fun-def*
by *transfer*
(*simp add*: *SUP-ereal-add incseq-def le-fun-def max-absorb2 SUP-upper2*)

lemma *ennreal-SUP-setsum*:
fixes *f* :: *'a* ⇒ *nat* ⇒ *ennreal*
shows (⋀*i*. *i* ∈ *I* ⟹ *incseq* (*f i*)) ⟹ (*SUP n*. ∑ *i*∈*I*. *f i n*) = (∑ *i*∈*I*. *SUP n*. *f i n*)
unfolding *incseq-def*
by *transfer*
(*simp add*: *SUP-ereal-setsum incseq-def SUP-upper2 max-absorb2 setsum-nonneg*)

lemma *ennreal-liminf-minus*:
fixes *f* :: *nat* ⇒ *ennreal*
shows (⋀*n*. *f n* ≤ *c*) ⟹ *liminf* (λ*n*. *c* − *f n*) = *c* − *limsup f*
apply *transfer*
apply (*simp add*: *ereal-diff-positive max.absorb2 liminf-ereal-cminus*)
apply (*subst max.absorb2*)
apply (*rule ereal-diff-positive*)
apply (*rule Limsup-bounded*)
apply *auto*
done

lemma *ennreal-continuous-on-cmult*:
(*c*::*ennreal*) < *top* ⟹ *continuous-on A f* ⟹ *continuous-on A* (λ*x*. *c* ∗ *f x*)
by (*transfer fixing*: *A*) (*auto intro*: *continuous-on-cmult-ereal*)

lemma *ennreal-tendsto-cmult*:
(*c*::*ennreal*) < *top* ⟹ (*f* ⟶ *x*) *F* ⟹ ((λ*x*. *c* ∗ *f x*) ⟶ *c* ∗ *x*) *F*
by (*rule continuous-on-tendsto-compose*[**where** *g*=*f*, *OF ennreal-continuous-on-cmult*, **where** *s*=*UNIV*])

(*auto simp*: *continuous-on-id*)

lemma *tendsto-ennrealI*[*intro*, *simp*]:
(*f* ⟶ *x*) *F* ⟹ ((λ*x*. *ennreal* (*f x*)) ⟶ *ennreal x*) *F*
by (*auto simp*: *ennreal-def*
intro!: *continuous-on-tendsto-compose*[*OF continuous-on-e2ennreal*[*of UNIV*]] *tendsto-max*)

lemma *ennreal-suminf-minus*:
fixes *f g* :: *nat* ⇒ *ennreal*
shows (⋀*i*. *g i* ≤ *f i*) ⟹ *suminf f* ≠ *top* ⟹ *suminf g* ≠ *top* ⟹ (∑ *i*. *f i* − *g i*) = *suminf f* − *suminf g*
by *transfer*
(*auto simp add*: *max.absorb2 ereal-diff-positive suminf-le-pos top-ereal-def intro*!: *suminf-ereal-minus*)

lemma *ennreal-Sup-countable-SUP*:
A ≠ {} ⟹ ∃*f*::*nat* ⇒ *ennreal*. *incseq f* ∧ *range f* ⊆ *A* ∧ *Sup A* = (*SUP i*. *f i*)
unfolding *incseq-def*
apply *transfer*
subgoal for *A*
using *Sup-countable-SUP*[*of A*]
apply (*clarsimp simp add*: *incseq-def*[*symmetric*] *SUP-upper2 max.absorb2 image-subset-iff Sup-upper2 cong*: *conj-cong*)
subgoal for *f*
by (*intro exI*[*of* - *f*]) *auto*
done
done

lemma *ennreal-SUP-countable-SUP*:
A ≠ {} ⟹ ∃*f*::*nat* ⇒ *ennreal*. *range f* ⊆ *g*‘*A* ∧ *SUPREMUM A g* = *SUPREMUM UNIV f*
using *ennreal-Sup-countable-SUP* [*of g*‘*A*] **by** *auto*

lemma *of-nat-tendsto-top-ennreal*: (λ*n*::*nat*. *of-nat n* :: *ennreal*) ⟶ *top*
using *LIMSEQ-SUP*[*of of-nat* :: *nat* ⇒ *ennreal*]
by (*simp add*: *ennreal-SUP-of-nat-eq-top incseq-def*)

lemma *SUP-sup-continuous-ennreal*:
fixes *f* :: *ennreal* ⇒ *'a*::*complete-lattice*
assumes *f*: *sup-continuous f* **and** *I* ≠ {}
shows (*SUP i*:*I*. *f* (*g i*)) = *f* (*SUP i*:*I*. *g i*)
proof (*rule antisym*)
show (*SUP i*:*I*. *f* (*g i*)) ≤ *f* (*SUP i*:*I*. *g i*)
by (*rule mono-SUP*[*OF sup-continuous-mono*[*OF f*]])
from *ennreal-Sup-countable-SUP*[*of g*‘*I*] ‹*I* ≠ {}›
obtain *M* :: *nat* ⇒ *ennreal* **where** *incseq M* **and** *M*: *range M* ⊆ *g* ‘ *I* **and** *eq*: (*SUP i* : *I*. *g i*) = (*SUP i*. *M i*)
by *auto*

```
  have f (SUP i : I. g i) = (SUP i : range M. f i)
    unfolding eq sup-continuousD[OF f ‹mono M›] by simp
  also have ... ≤ (SUP i : I. f (g i))
    by (insert M, drule SUP-subset-mono) auto
  finally show f (SUP i : I. g i) ≤ (SUP i : I. f (g i)) .
qed

lemma ennreal-suminf-SUP-eq:
  fixes f :: nat ⇒ nat ⇒ ennreal
  shows (⋀i. incseq (λn. f n i)) ⟹ (∑ i. SUP n. f n i) = (SUP n. ∑ i. f n i)
  apply (rule ennreal-suminf-SUP-eq-directed)
  subgoal for N n j
    by (auto simp: incseq-def intro!:exI[of - max n j])
  done

lemma ennreal-SUP-add-left:
  fixes c :: ennreal
  shows I ≠ {} ⟹ (SUP i:I. f i + c) = (SUP i:I. f i) + c
  apply transfer
  apply (simp add: SUP-ereal-add-left)
  apply (subst (1 2) max.absorb2)
  apply (auto intro: SUP-upper2 ereal-add-nonneg-nonneg)
  done

lemma ennreal-SUP-const-minus:
  fixes f :: 'a ⇒ ennreal
  shows I ≠ {} ⟹ c < top ⟹ (INF x:I. c - f x) = c - (SUP x:I. f x)
  apply (transfer fixing: I)
  unfolding ex-in-conv[symmetric]
  apply (auto simp add: sup-max[symmetric] SUP-upper2 sup-absorb2
              simp del: sup-ereal-def)
  apply (subst INF-ereal-minus-right[symmetric])
  apply (auto simp del: sup-ereal-def simp add: sup-INF)
  done
```

43.14 Approximation lemmas

```
lemma INF-approx-ennreal:
  fixes x::ennreal and e::real
  assumes e > 0
  assumes INF: x = (INF i : A. f i)
  assumes x ≠ ∞
  shows ∃ i ∈ A. f i < x + e
proof -
  have (INF i : A. f i) < x + e
    unfolding INF[symmetric] using ‹0<e› ‹x ≠ ∞› by (cases x) auto
  then show ?thesis
    unfolding INF-less-iff .
qed
```

lemma *SUP-approx-ennreal*:
fixes *x*::*ennreal* **and** *e*::*real*
assumes $e > 0$ $A \neq \{\}$
assumes *SUP*: $x = (SUP\ i : A.\ f\ i)$
assumes $x \neq \infty$
shows $\exists\, i \in A.\ x < f\ i + e$
proof −
have $x < x + e$
using ‹*0*<*e*› ‹$x \neq \infty$› **by** (*cases x*) *auto*
also have $x + e = (SUP\ i : A.\ f\ i + e)$
unfolding *SUP ennreal-SUP-add-left*[*OF* ‹$A \neq \{\}$›] **..**
finally show *?thesis*
unfolding *less-SUP-iff* **.**
qed

lemma *ennreal-approx-SUP*:
fixes *x*::*ennreal*
assumes *f-bound*: $\bigwedge i.\ i \in A \Longrightarrow f\ i \leq x$
assumes *approx*: $\bigwedge e.\ (e{::}real) > 0 \Longrightarrow \exists\, i \in A.\ x \leq f\ i + e$
shows $x = (SUP\ i : A.\ f\ i)$
proof (*rule antisym*)
show $x \leq (SUP\ i{:}A.\ f\ i)$
proof (*rule ennreal-le-epsilon*)
fix *e* :: *real* **assume** $0 < e$
from *approx*[*OF this*] **guess** *i* **..**
then have $x \leq f\ i + e$
by *simp*
also have $\ldots \leq (SUP\ i{:}A.\ f\ i) + e$
by (*intro add-mono* ‹$i \in A$› *SUP-upper order-refl*)
finally show $x \leq (SUP\ i{:}A.\ f\ i) + e$ **.**
qed
qed (*intro SUP-least f-bound*)

lemma *ennreal-approx-INF*:
fixes *x*::*ennreal*
assumes *f-bound*: $\bigwedge i.\ i \in A \Longrightarrow x \leq f\ i$
assumes *approx*: $\bigwedge e.\ (e{::}real) > 0 \Longrightarrow \exists\, i \in A.\ f\ i \leq x + e$
shows $x = (INF\ i : A.\ f\ i)$
proof (*rule antisym*)
show $(INF\ i{:}A.\ f\ i) \leq x$
proof (*rule ennreal-le-epsilon*)
fix *e* :: *real* **assume** $0 < e$
from *approx*[*OF this*] **guess** *i* **.. note** $i = this$
then have $(INF\ i{:}A.\ f\ i) \leq f\ i$
by (*intro INF-lower*)
also have $\ldots \leq x + e$
by *fact*
finally show $(INF\ i{:}A.\ f\ i) \leq x + e$ **.**

qed
qed (*intro INF-greatest f-bound*)

lemma *ennreal-approx-unit*:
(⋀*a*::*ennreal*. *0* < *a* ⟹ *a* < *1* ⟹ *a* * *z* ≤ *y*) ⟹ *z* ≤ *y*
apply (*subst SUP-mult-right-ennreal*[*of* λ*x*. *x* {*0* <..< *1*} *z*, *simplified*])
apply (*rule SUP-least*)
apply *auto*
done

lemma *suminf-ennreal2*:
(⋀*i*. *0* ≤ *f i*) ⟹ *summable f* ⟹ (∑ *i*. *ennreal* (*f i*)) = *ennreal* (∑ *i*. *f i*)
using *suminf-ennreal-eq* **by** *blast*

lemma *less-top-ennreal*: *x* < *top* ⟷ (∃ *r*≥*0*. *x* = *ennreal r*)
by (*cases x*) *auto*

lemma *tendsto-top-iff-ennreal*:
fixes *f* :: *'a* ⇒ *ennreal*
shows (*f* ⟶ *top*) *F* ⟷ (∀ *l*≥*0*. *eventually* (λ*x*. *ennreal l* < *f x*) *F*)
by (*auto simp*: *less-top-ennreal order-tendsto-iff*)

lemma *ennreal-tendsto-top-eq-at-top*:
((λ*z*. *ennreal* (*f z*)) ⟶ *top*) *F* ⟷ (*LIM z F*. *f z* :> *at-top*)
unfolding *filterlim-at-top-dense tendsto-top-iff-ennreal*
apply (*auto simp*: *ennreal-less-iff*)
subgoal for *y*
by (*auto elim*!: *eventually-mono allE*[*of* - *max 0 y*])
done

lemma *tendsto-0-if-Limsup-eq-0-ennreal*:
fixes *f* :: - ⇒ *ennreal*
shows *Limsup F f* = *0* ⟹ (*f* ⟶ *0*) *F*
using *Liminf-le-Limsup*[*of F f*] *tendsto-iff-Liminf-eq-Limsup*[*of F f 0*]
by (*cases F* = *bot*) *auto*

lemma *diff-le-self-ennreal*[*simp*]: *a* − *b* ≤ (*a*::*ennreal*)
by (*cases a b rule*: *ennreal2-cases*) (*auto simp*: *ennreal-minus*)

lemma *ennreal-ineq-diff-add*: *b* ≤ *a* ⟹ *a* = *b* + (*a* − *b*::*ennreal*)
by *transfer* (*auto simp*: *ereal-diff-positive max.absorb2 ereal-ineq-diff-add*)

lemma *ennreal-mult-strict-left-mono*: (*a*::*ennreal*) < *c* ⟹ *0* < *b* ⟹ *b* < *top* ⟹ *b* * *a* < *b* * *c*
by *transfer* (*auto intro*!: *ereal-mult-strict-left-mono*)

lemma *ennreal-between*: *0* < *e* ⟹ *0* < *x* ⟹ *x* < *top* ⟹ *x* − *e* < (*x*::*ennreal*)
by *transfer* (*auto intro*!: *ereal-between*)

lemma *minus-less-iff-ennreal*: $b < top \Longrightarrow b \leq a \Longrightarrow a - b < c \longleftrightarrow a < c + (b::ennreal)$
by *transfer*
(*auto simp*: *top-ereal-def ereal-minus-less le-less*)

lemma *tendsto-zero-ennreal*:
assumes *ev*: $\bigwedge r.\ 0 < r \Longrightarrow \forall_F\ x\ in\ F.\ f\ x < ennreal\ r$
shows $(f \longrightarrow 0)\ F$
proof (*rule order-tendstoI*)
fix *e*::*ennreal* **assume** $e > 0$
obtain *e'*::*real* **where** $e' > 0\ ennreal\ e' < e$
using ‹$0 < e$› *dense*[*of 0 if e = top then 1 else* (*enn2real e*)]
by (*cases e*) (*auto simp*: *ennreal-less-iff*)
from *ev*[*OF* ‹$e' > 0$›] **show** $\forall_F\ x\ in\ F.\ f\ x < e$
by *eventually-elim* (*insert* ‹*ennreal e'* $< e$›, *auto*)
qed *simp*

lifting-update *ennreal.lifting*
lifting-forget *ennreal.lifting*

end

44 Limits on the Extended real number line

theory *Extended-Real-Limits*
imports
Topology-Euclidean-Space
~~/*src*/*HOL*/*Library*/*Extended-Real*
~~/*src*/*HOL*/*Library*/*Extended-Nonnegative-Real*
~~/*src*/*HOL*/*Library*/*Indicator-Function*
begin

lemma *compact-UNIV*:
compact (*UNIV* :: *'a*::{*complete-linorder*,*linorder-topology*,*second-countable-topology*} *set*)
using *compact-complete-linorder*
by (*auto simp*: *seq-compact-eq-compact*[*symmetric*] *seq-compact-def*)

lemma *compact-eq-closed*:
fixes *S* :: *'a*::{*complete-linorder*,*linorder-topology*,*second-countable-topology*} *set*
shows *compact S* $\longleftrightarrow$ *closed S*
using *closed-Int-compact*[*of S*, *OF - compact-UNIV*] *compact-imp-closed*
by *auto*

lemma *closed-contains-Sup-cl*:
fixes *S* :: *'a*::{*complete-linorder*,*linorder-topology*,*second-countable-topology*} *set*
assumes *closed S*
and $S \neq \{\}$
shows *Sup S* $\in S$

proof −
from *compact-eq-closed*[*of S*] *compact-attains-sup*[*of S*] *assms*
obtain *s* **where** *S*: $s \in S \; \forall\, t{\in}S.\; t \leq s$
by *auto*
then have $Sup\ S = s$
by (*auto intro*!: *Sup-eqI*)
with *S* **show** *?thesis*
by *simp*
qed

lemma *closed-contains-Inf-cl*:
fixes *S* :: *'a*::{*complete-linorder*,*linorder-topology*,*second-countable-topology*} *set*
assumes *closed S*
and $S \neq \{\}$
shows $Inf\ S \in S$
proof −
from *compact-eq-closed*[*of S*] *compact-attains-inf*[*of S*] *assms*
obtain *s* **where** *S*: $s \in S \; \forall\, t{\in}S.\; s \leq t$
by *auto*
then have $Inf\ S = s$
by (*auto intro*!: *Inf-eqI*)
with *S* **show** *?thesis*
by *simp*
qed

instance *ereal* :: *second-countable-topology*
proof (*standard*, *intro exI conjI*)
let *?B* = $(\bigcup r{\in}\mathbb{Q}.\ \{\{..< r\}, \{r <..\}\} :: ereal\ set\ set)$
show *countable ?B*
by (*auto intro*: *countable-rat*)
show *open* = *generate-topology ?B*
proof (*intro ext iffI*)
fix *S* :: *ereal set*
assume *open S*
then show *generate-topology ?B S*
unfolding *open-generated-order*
proof *induct*
case (*Basis b*)
then obtain *e* **where** $b = \{..{<}e\} \vee b = \{e{<}..\}$
by *auto*
moreover have $\{..{<}e\} = \bigcup\{\{..{<}x\}|x.\ x \in \mathbb{Q} \wedge x < e\}\ \{e{<}..\} = \bigcup\{\{x{<}..\}|x.\ x \in \mathbb{Q} \wedge e < x\}$
by (*auto dest*: *ereal-dense3*
simp del: *ex-simps*
simp add: *ex-simps*[*symmetric*] *conj-commute Rats-def image-iff*)
ultimately show *?case*
by (*auto intro*: *generate-topology.intros*)
qed (*auto intro*: *generate-topology.intros*)
next

```
    fix S
    assume generate-topology ?B S
    then show open S
      by induct auto
  qed
qed
```

This is a copy from *ereal* :: *second-countable-topology*. Maybe find a common super class of topological spaces where the rational numbers are densely embedded ?

```
instance ennreal :: second-countable-topology
proof (standard, intro exI conjI)
  let ?B = (⋃r∈ℚ. {{..< r}, {r <..}} :: ennreal set set)
  show countable ?B
    by (auto intro: countable-rat)
  show open = generate-topology ?B
  proof (intro ext iffI)
    fix S :: ennreal set
    assume open S
    then show generate-topology ?B S
      unfolding open-generated-order
    proof induct
      case (Basis b)
      then obtain e where b = {..<e} ∨ b = {e<..}
        by auto
    moreover have {..<e} = ⋃{{..<x}|x. x ∈ ℚ ∧ x < e} {e<..} = ⋃{{x<..}|x.
x ∈ ℚ ∧ e < x}
        by (auto dest: ennreal-rat-dense
                 simp del: ex-simps
                 simp add: ex-simps[symmetric] conj-commute Rats-def image-iff)
      ultimately show ?case
        by (auto intro: generate-topology.intros)
    qed (auto intro: generate-topology.intros)
  next
    fix S
    assume generate-topology ?B S
    then show open S
      by induct auto
  qed
qed

lemma ereal-open-closed-aux:
  fixes S :: ereal set
  assumes open S
    and closed S
    and S: (−∞) ∉ S
  shows S = {}
proof (rule ccontr)
  assume ¬ ?thesis
```

```
  then have ∗: Inf S ∈ S

    by (metis assms(2) closed-contains-Inf-cl)
  {
    assume Inf S = −∞
    then have False
      using ∗ assms(3) by auto
  }
  moreover
  {
    assume Inf S = ∞
    then have S = {∞}
      by (metis Inf-eq-PInfty ‹S ≠ {}›)
    then have False
      by (metis assms(1) not-open-singleton)
  }
  moreover
  {
    assume fin: |Inf S| ≠ ∞
    from ereal-open-cont-interval[OF assms(1) ∗ fin]
    obtain e where e: e > 0 {Inf S − e<..<Inf S + e} ⊆ S .
    then obtain b where b: Inf S − e < b b < Inf S
      using fin ereal-between[of Inf S e] dense[of Inf S − e]
      by auto
    then have b: {Inf S − e <..< Inf S + e}
      using e fin ereal-between[of Inf S e]
      by auto
    then have b ∈ S
      using e by auto
    then have False
      using b by (metis complete-lattice-class.Inf-lower leD)
  }
  ultimately show False
    by auto
qed

lemma ereal-open-closed:
  fixes S :: ereal set
  shows open S ∧ closed S ⟷ S = {} ∨ S = UNIV
proof −
  {
    assume lhs: open S ∧ closed S
    {
      assume −∞ ∉ S
      then have S = {}
        using lhs ereal-open-closed-aux by auto
    }
    moreover
    {
```

```
    assume −∞ ∈ S
    then have − S = {}
      using lhs ereal-open-closed-aux[of −S] by auto
   }
   ultimately have S = {} ∨ S = UNIV
    by auto
 }
 then show ?thesis
   by auto
qed

lemma ereal-open-atLeast:
 fixes x :: ereal
 shows open {x..} ⟷ x = −∞
proof
 assume x = −∞
 then have {x..} = UNIV
   by auto
 then show open {x..}
   by auto
next
 assume open {x..}
 then have open {x..} ∧ closed {x..}
   by auto
 then have {x..} = UNIV
   unfolding ereal-open-closed by auto
 then show x = −∞
   by (simp add: bot-ereal-def atLeast-eq-UNIV-iff)
qed

lemma mono-closed-real:
 fixes S :: real set
 assumes mono: ∀ y z. y ∈ S ∧ y ≤ z ⟶ z ∈ S
   and closed S
 shows S = {} ∨ S = UNIV ∨ (∃ a. S = {a..})
proof −
 {
   assume S ≠ {}
   { assume ex: ∃ B. ∀ x∈S. B ≤ x
     then have ∗: ∀ x∈S. Inf S ≤ x
       using cInf-lower[of - S] ex by (metis bdd-below-def)
     then have Inf S ∈ S
       apply (subst closed-contains-Inf)
       using ex ‹S ≠ {}› ‹closed S›
       apply auto
       done
     then have ∀ x. Inf S ≤ x ⟷ x ∈ S
       using mono[rule-format, of Inf S] ∗
       by auto
```

```
      then have S = {Inf S ..}
        by auto
      then have ∃ a. S = {a ..}
        by auto
    }
    moreover
    {
      assume ¬ (∃ B. ∀ x∈S. B ≤ x)
      then have nex: ∀ B. ∃ x∈S. x < B
        by (simp add: not-le)
      {
        fix y
        obtain x where x∈S and x < y
          using nex by auto
        then have y ∈ S
          using mono[rule-format, of x y] by auto
      }
      then have S = UNIV
        by auto
    }
    ultimately have S = UNIV ∨ (∃ a. S = {a ..})
      by blast
  }
  then show ?thesis
    by blast
qed

lemma mono-closed-ereal:
  fixes S :: real set
  assumes mono: ∀ y z. y ∈ S ∧ y ≤ z ⟶ z ∈ S
    and closed S
  shows ∃ a. S = {x. a ≤ ereal x}
proof −
  {
    assume S = {}
    then have ?thesis
      apply (rule-tac x=PInfty in exI)
      apply auto
      done
  }
  moreover
  {
    assume S = UNIV
    then have ?thesis
      apply (rule-tac x=−∞ in exI)
      apply auto
      done
  }
  moreover
```

```
  {
    assume ∃ a. S = {a ..}
    then obtain a where S = {a ..}
      by auto
    then have ?thesis
      apply (rule-tac x=ereal a in exI)
      apply auto
      done
  }
  ultimately show ?thesis
    using mono-closed-real[of S] assms by auto
qed

lemma Liminf-within:
  fixes f :: 'a::metric-space ⇒ 'b::complete-lattice
  shows Liminf (at x within S) f = (SUP e:{0<..}. INF y:(S ∩ ball x e − {x}).
f y)
  unfolding Liminf-def eventually-at
proof (rule SUP-eq, simp-all add: Ball-def Bex-def, safe)
  fix P d
  assume 0 < d and ∀ y. y ∈ S ⟶ y ≠ x ∧ dist y x < d ⟶ P y
  then have S ∩ ball x d − {x} ⊆ {x. P x}
    by (auto simp: zero-less-dist-iff dist-commute)
  then show ∃ r>0. INFIMUM (Collect P) f ≤ INFIMUM (S ∩ ball x r − {x})
f
    by (intro exI[of - d] INF-mono conjI ‹0 < d›) auto
next
  fix d :: real
  assume 0 < d
  then show ∃ P. (∃ d>0. ∀ xa. xa ∈ S ⟶ xa ≠ x ∧ dist xa x < d ⟶ P xa) ∧
    INFIMUM (S ∩ ball x d − {x}) f ≤ INFIMUM (Collect P) f
    by (intro exI[of - λy. y ∈ S ∩ ball x d − {x}])
       (auto intro!: INF-mono exI[of - d] simp: dist-commute)
qed

lemma Limsup-within:
  fixes f :: 'a::metric-space ⇒ 'b::complete-lattice
  shows Limsup (at x within S) f = (INF e:{0<..}. SUP y:(S ∩ ball x e − {x}).
f y)
  unfolding Limsup-def eventually-at
proof (rule INF-eq, simp-all add: Ball-def Bex-def, safe)
  fix P d
  assume 0 < d and ∀ y. y ∈ S ⟶ y ≠ x ∧ dist y x < d ⟶ P y
  then have S ∩ ball x d − {x} ⊆ {x. P x}
    by (auto simp: zero-less-dist-iff dist-commute)
 then show ∃ r>0. SUPREMUM (S ∩ ball x r − {x}) f ≤ SUPREMUM (Collect
P) f
    by (intro exI[of - d] SUP-mono conjI ‹0 < d›) auto
next
```

fix d :: *real*
assume $0 < d$
then show $\exists P.\ (\exists\, d{>}0.\ \forall xa.\ xa \in S \longrightarrow xa \neq x \wedge dist\ xa\ x < d \longrightarrow P\ xa) \wedge$
$SUPREMUM\ (Collect\ P)\ f \leq SUPREMUM\ (S \cap ball\ x\ d - \{x\})\ f$
by (*intro exI*[*of* - $\lambda y.\ y \in S \cap ball\ x\ d - \{x\}$])
(*auto intro*!: *SUP-mono exI*[*of* - *d*] *simp*: *dist-commute*)
qed

lemma *Liminf-at*:
fixes f :: $'a$::*metric-space* $\Rightarrow$ $'b$::*complete-lattice*
shows $Liminf\ (at\ x)\ f = (SUP\ e{:}\{0{<}..\}.\ INF\ y{:}(ball\ x\ e - \{x\}).\ f\ y)$
using *Liminf-within*[*of x UNIV f*] **by** *simp*

lemma *Limsup-at*:
fixes f :: $'a$::*metric-space* $\Rightarrow$ $'b$::*complete-lattice*
shows $Limsup\ (at\ x)\ f = (INF\ e{:}\{0{<}..\}.\ SUP\ y{:}(ball\ x\ e - \{x\}).\ f\ y)$
using *Limsup-within*[*of x UNIV f*] **by** *simp*

lemma *min-Liminf-at*:
fixes f :: $'a$::*metric-space* $\Rightarrow$ $'b$::*complete-linorder*
shows $min\ (f\ x)\ (Liminf\ (at\ x)\ f) = (SUP\ e{:}\{0{<}..\}.\ INF\ y{:}ball\ x\ e.\ f\ y)$
unfolding *inf-min*[*symmetric*] *Liminf-at*
apply (*subst inf-commute*)
apply (*subst SUP-inf*)
apply (*intro SUP-cong*[*OF refl*])
apply (*cut-tac A*=*ball x xa* $-$ $\{x\}$ **and** *B*=$\{x\}$ **and** *M*=*f* **in** *INF-union*)
apply (*drule sym*)
apply *auto*
apply (*metis INF-absorb centre-in-ball*)
done

44.1 monoset

definition (**in** *order*) *mono-set*:
$mono\text{-}set\ S \longleftrightarrow (\forall x\ y.\ x \leq y \longrightarrow x \in S \longrightarrow y \in S)$

lemma (**in** *order*) *mono-greaterThan* [*intro*, *simp*]: *mono-set* $\{B{<}..\}$ **unfolding** *mono-set* **by** *auto*
lemma (**in** *order*) *mono-atLeast* [*intro*, *simp*]: *mono-set* $\{B..\}$ **unfolding** *mono-set* **by** *auto*
lemma (**in** *order*) *mono-UNIV* [*intro*, *simp*]: *mono-set UNIV* **unfolding** *mono-set* **by** *auto*
lemma (**in** *order*) *mono-empty* [*intro*, *simp*]: *mono-set* $\{\}$ **unfolding** *mono-set* **by** *auto*

lemma (**in** *complete-linorder*) *mono-set-iff*:
fixes S :: $'a$ *set*
defines $a \equiv Inf\ S$
shows $mono\text{-}set\ S \longleftrightarrow S = \{a\ {<}..\} \vee S = \{a..\}$ (**is** - = *?c*)

proof
assume *mono-set S*
then have *mono*: $\bigwedge x\ y.\ x \leq y \Longrightarrow x \in S \Longrightarrow y \in S$
by (*auto simp*: *mono-set*)
show *?c*
proof *cases*
assume $a \in S$
show *?c*
using *mono*[*OF* - ‹$a \in S$›]
by (*auto intro*: *Inf-lower simp*: *a-def*)
next
assume $a \notin S$
have $S = \{a <..\}$
proof *safe*
fix x **assume** $x \in S$
then have $a \leq x$
unfolding *a-def* **by** (*rule Inf-lower*)
then show $a < x$
using ‹$x \in S$› ‹$a \notin S$› **by** (*cases* $a = x$) *auto*
next
fix x **assume** $a < x$
then obtain y **where** $y < x\ y \in S$
unfolding *a-def Inf-less-iff* **..**
with *mono*[*of y x*] **show** $x \in S$
by *auto*
qed
then show *?c* **..**
qed
qed *auto*

lemma *ereal-open-mono-set*:
fixes S :: *ereal set*
shows *open* $S \wedge$ *mono-set* $S \longleftrightarrow S = UNIV \vee S = \{Inf\ S <..\}$
by (*metis Inf-UNIV atLeast-eq-UNIV-iff ereal-open-atLeast*
ereal-open-closed mono-set-iff open-ereal-greaterThan)

lemma *ereal-closed-mono-set*:
fixes S :: *ereal set*
shows *closed* $S \wedge$ *mono-set* $S \longleftrightarrow S = \{\} \vee S = \{Inf\ S\ ..\}$
by (*metis Inf-UNIV atLeast-eq-UNIV-iff closed-ereal-atLeast*
ereal-open-closed mono-empty mono-set-iff open-ereal-greaterThan)

lemma *ereal-Liminf-Sup-monoset*:
fixes f :: $'a \Rightarrow$ *ereal*
shows *Liminf net f* =
Sup $\{l.\ \forall S.$ *open* $S \longrightarrow$ *mono-set* $S \longrightarrow l \in S \longrightarrow$ *eventually* $(\lambda x.\ f\ x \in S)$ *net*$\}$
(**is** - = *Sup ?A*)
proof (*safe intro*!: *Liminf-eqI complete-lattice-class.Sup-upper complete-lattice-class.Sup-least*)

```
  fix P
  assume P: eventually P net
  fix S
  assume S: mono-set S INFIMUM (Collect P) f ∈ S
  {
    fix x
    assume P x
    then have INFIMUM (Collect P) f ≤ f x
      by (intro complete-lattice-class.INF-lower) simp
    with S have f x ∈ S
      by (simp add: mono-set)
  }
  with P show eventually (λx. f x ∈ S) net
    by (auto elim: eventually-mono)
next
  fix y l
  assume S: ∀S. open S ⟶ mono-set S ⟶ l ∈ S ⟶ eventually (λx. f x ∈ S)
net
  assume P: ∀P. eventually P net ⟶ INFIMUM (Collect P) f ≤ y
  show l ≤ y
  proof (rule dense-le)
    fix B
    assume B < l
    then have eventually (λx. f x ∈ {B <..}) net
      by (intro S[rule-format]) auto
    then have INFIMUM {x. B < f x} f ≤ y
      using P by auto
    moreover have B ≤ INFIMUM {x. B < f x} f
      by (intro INF-greatest) auto
    ultimately show B ≤ y
      by simp
  qed
qed

lemma ereal-Limsup-Inf-monoset:
  fixes f :: 'a ⇒ ereal
  shows Limsup net f =
    Inf {l. ∀S. open S ⟶ mono-set (uminus ' S) ⟶ l ∈ S ⟶ eventually (λx.
f x ∈ S) net}
    (is - = Inf ?A)
proof (safe intro!: Limsup-eqI complete-lattice-class.Inf-lower complete-lattice-class.Inf-greatest)
  fix P
  assume P: eventually P net
  fix S
  assume S: mono-set (uminus'S) SUPREMUM (Collect P) f ∈ S
  {
    fix x
    assume P x
    then have f x ≤ SUPREMUM (Collect P) f
```

```
    by (intro complete-lattice-class.SUP-upper) simp
   with S(1)[unfolded mono-set, rule-format, of − SUPREMUM (Collect P) f
− f x] S(2)
   have f x ∈ S
    by (simp add: inj-image-mem-iff) }
 with P show eventually (λx. f x ∈ S) net
   by (auto elim: eventually-mono)
next
 fix y l
 assume S: ∀ S. open S ⟶ mono-set (uminus ‘ S) ⟶ l ∈ S ⟶ eventually
(λx. f x ∈ S) net
 assume P: ∀ P. eventually P net ⟶ y ≤ SUPREMUM (Collect P) f
 show y ≤ l
 proof (rule dense-ge)
   fix B
   assume l < B
   then have eventually (λx. f x ∈ {..< B}) net
    by (intro S[rule-format]) auto
   then have y ≤ SUPREMUM {x. f x < B} f
    using P by auto
   moreover have SUPREMUM {x. f x < B} f ≤ B
    by (intro SUP-least) auto
   ultimately show y ≤ B
    by simp
 qed
qed

lemma liminf-bounded-open:
 fixes x :: nat ⇒ ereal
 shows x0 ≤ liminf x ⟷ (∀ S. open S ⟶ mono-set S ⟶ x0 ∈ S ⟶ (∃ N.
∀ n≥N. x n ∈ S))
 (is - ⟷ ?P x0)
proof
 assume ?P x0
 then show x0 ≤ liminf x
   unfolding ereal-Liminf-Sup-monoset eventually-sequentially
   by (intro complete-lattice-class.Sup-upper) auto
next
 assume x0 ≤ liminf x
 {
   fix S :: ereal set
   assume om: open S mono-set S x0 ∈ S
   {
     assume S = UNIV
     then have ∃ N. ∀ n≥N. x n ∈ S
       by auto
   }
   moreover
   {
```

assume $S \neq UNIV$
then obtain B **where** B: $S = \{B<..\}$
using *om ereal-open-mono-set* **by** *auto*
then have $B < x0$
using *om* **by** *auto*
then have $\exists N.\ \forall n{\geq}N.\ x\ n \in S$
unfolding B
using ‹$x0 \leq liminf\ x$› *liminf-bounded-iff*
by *auto*
}
ultimately have $\exists N.\ \forall n{\geq}N.\ x\ n \in S$
by *auto*
}
then show *?P x0*
by *auto*
qed

44.2 Relate extended reals and the indicator function

lemma *ereal-indicator-le-0*: $(indicator\ S\ x{::}ereal) \leq 0 \longleftrightarrow x \notin S$
by (*auto split*: *split-indicator simp*: *one-ereal-def*)

lemma *ereal-indicator*: $ereal\ (indicator\ A\ x) = indicator\ A\ x$
by (*auto simp*: *indicator-def one-ereal-def*)

lemma *ereal-mult-indicator*: $ereal\ (x * indicator\ A\ y) = ereal\ x * indicator\ A\ y$
by (*simp split*: *split-indicator*)

lemma *ereal-indicator-mult*: $ereal\ (indicator\ A\ y * x) = indicator\ A\ y * ereal\ x$
by (*simp split*: *split-indicator*)

lemma *ereal-indicator-nonneg*[*simp*, *intro*]: $0 \leq (indicator\ A\ x\ ::ereal)$
unfolding *indicator-def* **by** *auto*

lemma *indicator-inter-arith-ereal*: $indicator\ A\ x * indicator\ B\ x = (indicator\ (A \cap B)\ x :: ereal)$
by (*simp split*: *split-indicator*)

end

45 Permutations, both general and specifically on finite sets.

theory *Permutations*
imports *Binomial*
begin

45.1 Transpositions

lemma *swap-id-idempotent* [*simp*]:
Fun.swap a b id ∘ *Fun.swap a b id* = *id*
by (*rule ext*, *auto simp add*: *Fun.swap-def*)

lemma *inv-swap-id*:
inv (*Fun.swap a b id*) = *Fun.swap a b id*
by (*rule inv-unique-comp*) *simp-all*

lemma *swap-id-eq*:
Fun.swap a b id x = (*if x* = *a then b else if x* = *b then a else x*)
by (*simp add*: *Fun.swap-def*)

45.2 Basic consequences of the definition

definition *permutes* (**infixr** *permutes 41*)
where (*p permutes S*) ⟷ (∀ *x*. *x* ∉ *S* ⟶ *p x* = *x*) ∧ (∀ *y*. ∃!*x*. *p x* = *y*)

lemma *permutes-in-image*: *p permutes S* ⟹ *p x* ∈ *S* ⟷ *x* ∈ *S*
unfolding *permutes-def* **by** *metis*

lemma *permutes-image*: *p permutes S* ⟹ *p* ‘ *S* = *S*
unfolding *permutes-def*
apply (*rule set-eqI*)
apply (*simp add*: *image-iff*)
apply *metis*
done

lemma *permutes-inj*: *p permutes S* ⟹ *inj p*
unfolding *permutes-def inj-on-def* **by** *blast*

lemma *permutes-surj*: *p permutes s* ⟹ *surj p*
unfolding *permutes-def surj-def* **by** *metis*

lemma *permutes-bij*: *p permutes s* ⟹ *bij p*
unfolding *bij-def* **by** (*metis permutes-inj permutes-surj*)

lemma *permutes-imp-bij*: *p permutes S* ⟹ *bij-betw p S S*
by (*metis UNIV-I bij-betw-subset permutes-bij permutes-image subsetI*)

lemma *bij-imp-permutes*: *bij-betw p S S* ⟹ (⋀*x*. *x* ∉ *S* ⟹ *p x* = *x*) ⟹ *p permutes S*
unfolding *permutes-def bij-betw-def inj-on-def*
by *auto* (*metis image-iff*)+

lemma *permutes-inv-o*:
assumes *pS*: *p permutes S*
shows *p* ∘ *inv p* = *id*
and *inv p* ∘ *p* = *id*

using *permutes-inj*[*OF pS*] *permutes-surj*[*OF pS*]
unfolding *inj-iff*[*symmetric*] *surj-iff*[*symmetric*] **by** *blast+*

lemma *permutes-inverses*:
fixes $p :: 'a \Rightarrow 'a$
assumes *pS*: *p permutes S*
shows $p\ (inv\ p\ x) = x$
and $inv\ p\ (p\ x) = x$
using *permutes-inv-o*[*OF pS*, *unfolded fun-eq-iff o-def*] **by** *auto*

lemma *permutes-subset*: *p permutes S* $\Longrightarrow$ $S \subseteq T$ $\Longrightarrow$ *p permutes T*
unfolding *permutes-def* **by** *blast*

lemma *permutes-empty*[*simp*]: *p permutes* $\{\}$ $\longleftrightarrow$ $p = id$
unfolding *fun-eq-iff permutes-def* **by** *simp metis*

lemma *permutes-sing*[*simp*]: *p permutes* $\{a\}$ $\longleftrightarrow$ $p = id$
unfolding *fun-eq-iff permutes-def* **by** *simp metis*

lemma *permutes-univ*: *p permutes UNIV* $\longleftrightarrow$ $(\forall y.\ \exists!x.\ p\ x = y)$
unfolding *permutes-def* **by** *simp*

lemma *permutes-inv-eq*: *p permutes S* $\Longrightarrow$ $inv\ p\ y = x \longleftrightarrow p\ x = y$
unfolding *permutes-def inv-def*
apply *auto*
apply (*erule allE*[**where** *x*=*y*])
apply (*erule allE*[**where** *x*=*y*])
apply (*rule someI-ex*)
apply *blast*
apply (*rule some1-equality*)
apply *blast*
apply *blast*
done

lemma *permutes-swap-id*: $a \in S \Longrightarrow b \in S \Longrightarrow$ *Fun.swap a b id permutes S*
unfolding *permutes-def Fun.swap-def fun-upd-def* **by** *auto metis*

lemma *permutes-superset*: *p permutes S* $\Longrightarrow$ $(\forall x \in S - T.\ p\ x = x) \Longrightarrow$ *p permutes T*
by (*simp add*: *Ball-def permutes-def*) *metis*

45.3 Group properties

lemma *permutes-id*: *id permutes S*
unfolding *permutes-def* **by** *simp*

lemma *permutes-compose*: *p permutes S* $\Longrightarrow$ *q permutes S* $\Longrightarrow$ $q \circ p$ *permutes S*
unfolding *permutes-def o-def* **by** *metis*

lemma *permutes-inv*:
assumes *pS*: *p permutes S*
shows *inv p permutes S*
using *pS* **unfolding** *permutes-def permutes-inv-eq*[*OF pS*] **by** *metis*

lemma *permutes-inv-inv*:
assumes *pS*: *p permutes S*
shows *inv* (*inv p*) = *p*
unfolding *fun-eq-iff permutes-inv-eq*[*OF pS*] *permutes-inv-eq*[*OF permutes-inv*[*OF pS*]]
by *blast*

45.4 The number of permutations on a finite set

lemma *permutes-insert-lemma*:
assumes *pS*: *p permutes* (*insert a S*)
shows *Fun.swap a* (*p a*) *id* ∘ *p permutes S*
apply (*rule permutes-superset*[**where** *S* = *insert a S*])
apply (*rule permutes-compose*[*OF pS*])
apply (*rule permutes-swap-id*, *simp*)
using *permutes-in-image*[*OF pS*, *of a*]
apply *simp*
apply (*auto simp add*: *Ball-def Fun.swap-def*)
done

lemma *permutes-insert*: {*p*. *p permutes* (*insert a S*)} =
(λ(*b*,*p*). *Fun.swap a b id* ∘ *p*) ' {(*b*,*p*). *b* ∈ *insert a S* ∧ *p* ∈ {*p*. *p permutes S*}}
proof −
{
fix *p*
{
assume *pS*: *p permutes insert a S*
let *?b* = *p a*
let *?q* = *Fun.swap a* (*p a*) *id* ∘ *p*
have *th0*: *p* = *Fun.swap a ?b id* ∘ *?q*
unfolding *fun-eq-iff o-assoc* **by** *simp*
have *th1*: *?b* ∈ *insert a S*
unfolding *permutes-in-image*[*OF pS*] **by** *simp*
from *permutes-insert-lemma*[*OF pS*] *th0 th1*
have ∃ *b q*. *p* = *Fun.swap a b id* ∘ *q* ∧ *b* ∈ *insert a S* ∧ *q permutes S* **by** *blast*
}
moreover
{
fix *b q*
assume *bq*: *p* = *Fun.swap a b id* ∘ *q b* ∈ *insert a S q permutes S*
from *permutes-subset*[*OF bq*(*3*), *of insert a S*]
have *qS*: *q permutes insert a S*
by *auto*

```
    have aS: a ∈ insert a S
      by simp
    from bq(1) permutes-compose[OF qS permutes-swap-id[OF aS bq(2)]]
    have p permutes insert a S
      by simp
  }
  ultimately have p permutes insert a S ⟷
      (∃ b q. p = Fun.swap a b id ∘ q ∧ b ∈ insert a S ∧ q permutes S)
    by blast
 }
 then show ?thesis
  by auto
qed

lemma card-permutations:
 assumes Sn: card S = n
  and fS: finite S
 shows card {p. p permutes S} = fact n
 using fS Sn
proof (induct arbitrary: n)
 case empty
 then show ?case by simp
next
 case (insert x F)
 {
  fix n
  assume H0: card (insert x F) = n
  let ?xF = {p. p permutes insert x F}
  let ?pF = {p. p permutes F}
  let ?pF' = {(b, p). b ∈ insert x F ∧ p ∈ ?pF}
  let ?g = (λ(b, p). Fun.swap x b id ∘ p)
  from permutes-insert[of x F]
  have xfgpF': ?xF = ?g ` ?pF' .
  have Fs: card F = n - 1
    using ‹x ∉ F› H0 ‹finite F› by auto
  from insert.hyps Fs have pFs: card ?pF = fact (n - 1)
    using ‹finite F› by auto
  then have finite ?pF
    by (auto intro: card-ge-0-finite)
  then have pF'f: finite ?pF'
    using H0 ‹finite F›
    apply (simp only: Collect-case-prod Collect-mem-eq)
    apply (rule finite-cartesian-product)
    apply simp-all
    done

  have ginj: inj-on ?g ?pF'
  proof -
   {
```

```
    fix b p c q
    assume bp: (b,p) ∈ ?pF'
    assume cq: (c,q) ∈ ?pF'
    assume eq: ?g (b,p) = ?g (c,q)
    from bp cq have ths: b ∈ insert x F c ∈ insert x F x ∈ insert x F
      p permutes F q permutes F
      by auto
    from ths(4) ‹x ∉ F› eq have b = ?g (b,p) x
      unfolding permutes-def
      by (auto simp add: Fun.swap-def fun-upd-def fun-eq-iff)
    also have ... = ?g (c,q) x
      using ths(5) ‹x ∉ F› eq
      by (auto simp add: swap-def fun-upd-def fun-eq-iff)
    also have ... = c
      using ths(5) ‹x ∉ F›
      unfolding permutes-def
      by (auto simp add: Fun.swap-def fun-upd-def fun-eq-iff)
    finally have bc: b = c .
    then have Fun.swap x b id = Fun.swap x c id
      by simp
    with eq have Fun.swap x b id ∘ p = Fun.swap x b id ∘ q
      by simp
    then have Fun.swap x b id ∘ (Fun.swap x b id ∘ p) =
      Fun.swap x b id ∘ (Fun.swap x b id ∘ q)
      by simp
    then have p = q
      by (simp add: o-assoc)
    with bc have (b, p) = (c, q)
      by simp
  }
  then show ?thesis
    unfolding inj-on-def by blast
qed
from ‹x ∉ F› H0 have n0: n ≠ 0
  using ‹finite F› by auto
then have ∃ m. n = Suc m
  by presburger
then obtain m where n[simp]: n = Suc m
  by blast
from pFs H0 have xFc: card ?xF = fact n
  unfolding xfgpF' card-image[OF ginj]
  using ‹finite F› ‹finite ?pF›
  apply (simp only: Collect-case-prod Collect-mem-eq card-cartesian-product)
  apply simp
  done
from finite-imageI[OF pF'f, of ?g] have xFf: finite ?xF
  unfolding xfgpF' by simp
have card ?xF = fact n
  using xFf xFc unfolding xFf by blast
```

}
then show *?case*
using *insert* **by** *simp*
qed

lemma *finite-permutations*:
assumes *fS*: *finite S*
shows *finite* {*p*. *p permutes S*}
using *card-permutations*[*OF refl fS*]
by (*auto intro*: *card-ge-0-finite*)

45.5 Permutations of index set for iterated operations

lemma (**in** *comm-monoid-set*) *permute*:
assumes *p permutes S*
shows *F g S* = *F* (*g* ∘ *p*) *S*
proof −
from ‹*p permutes S*› **have** *inj p*
by (*rule permutes-inj*)
then have *inj-on p S*
by (*auto intro*: *subset-inj-on*)
then have *F g* (*p* ‘ *S*) = *F* (*g* ∘ *p*) *S*
by (*rule reindex*)
moreover from ‹*p permutes S*› **have** *p* ‘ *S* = *S*
by (*rule permutes-image*)
ultimately show *?thesis*
by *simp*
qed

45.6 Various combinations of transpositions with 2, 1 and 0 common elements

lemma *swap-id-common*: *a* ≠ *c* ⟹ *b* ≠ *c* ⟹
Fun.swap a b id ∘ *Fun.swap a c id* = *Fun.swap b c id* ∘ *Fun.swap a b id*
by (*simp add*: *fun-eq-iff Fun.swap-def*)

lemma *swap-id-common'*: *a* ≠ *b* ⟹ *a* ≠ *c* ⟹
Fun.swap a c id ∘ *Fun.swap b c id* = *Fun.swap b c id* ∘ *Fun.swap a b id*
by (*simp add*: *fun-eq-iff Fun.swap-def*)

lemma *swap-id-independent*: *a* ≠ *c* ⟹ *a* ≠ *d* ⟹ *b* ≠ *c* ⟹ *b* ≠ *d* ⟹
Fun.swap a b id ∘ *Fun.swap c d id* = *Fun.swap c d id* ∘ *Fun.swap a b id*
by (*simp add*: *fun-eq-iff Fun.swap-def*)

45.7 Permutations as transposition sequences

inductive *swapidseq* :: *nat* ⇒ (′*a* ⇒ ′*a*) ⇒ *bool*
where
id[*simp*]: *swapidseq 0 id*

| *comp-Suc*: *swapidseq n p* $\Longrightarrow$ *a* $\neq$ *b* $\Longrightarrow$ *swapidseq* (*Suc n*) (*Fun.swap a b id* $\circ$ *p*)

declare *id*[*unfolded id-def*, *simp*]

definition *permutation p* $\longleftrightarrow$ ($\exists$ *n*. *swapidseq n p*)

45.8 Some closure properties of the set of permutations, with lengths

lemma *permutation-id*[*simp*]: *permutation id*
unfolding *permutation-def* **by** (*rule exI*[**where** *x=0*]) *simp*

declare *permutation-id*[*unfolded id-def*, *simp*]

lemma *swapidseq-swap*: *swapidseq* (*if a = b then 0 else 1*) (*Fun.swap a b id*)
apply *clarsimp*
using *comp-Suc*[*of 0 id a b*]
apply *simp*
done

lemma *permutation-swap-id*: *permutation* (*Fun.swap a b id*)
apply (*cases a = b*)
apply *simp-all*
unfolding *permutation-def*
using *swapidseq-swap*[*of a b*]
apply *blast*
done

lemma *swapidseq-comp-add*: *swapidseq n p* $\Longrightarrow$ *swapidseq m q* $\Longrightarrow$ *swapidseq* (*n + m*) (*p* $\circ$ *q*)
proof (*induct n p arbitrary*: *m q rule*: *swapidseq.induct*)
case (*id m q*)
then show *?case* **by** *simp*
next
case (*comp-Suc n p a b m q*)
have *th*: *Suc n + m = Suc* (*n + m*)
by *arith*
show *?case*
unfolding *th comp-assoc*
apply (*rule swapidseq.comp-Suc*)
using *comp-Suc.hyps*(*2*)[*OF comp-Suc.prems*] *comp-Suc.hyps*(*3*)
apply *blast+*
done
qed

lemma *permutation-compose*: *permutation p* $\Longrightarrow$ *permutation q* $\Longrightarrow$ *permutation* (*p* $\circ$ *q*)
unfolding *permutation-def* **using** *swapidseq-comp-add*[*of - p - q*] **by** *metis*

lemma *swapidseq-endswap*: *swapidseq n p* $\Longrightarrow$ *a* $\neq$ *b* $\Longrightarrow$ *swapidseq* (*Suc n*) (*p* $\circ$ *Fun.swap a b id*)
apply (*induct n p rule*: *swapidseq.induct*)
using *swapidseq-swap*[*of a b*]
apply (*auto simp add*: *comp-assoc intro*: *swapidseq.comp-Suc*)
done

lemma *swapidseq-inverse-exists*: *swapidseq n p* $\Longrightarrow$ $\exists$ *q*. *swapidseq n q* $\wedge$ *p* $\circ$ *q* = *id* $\wedge$ *q* $\circ$ *p* = *id*
proof (*induct n p rule*: *swapidseq.induct*)
case *id*
then show *?case*
by (*rule exI*[**where** *x*=*id*]) *simp*
next
case (*comp-Suc n p a b*)
from *comp-Suc.hyps* **obtain** *q* **where** *q*: *swapidseq n q p* $\circ$ *q* = *id q* $\circ$ *p* = *id*
by *blast*
let *?q* = *q* $\circ$ *Fun.swap a b id*
note *H* = *comp-Suc.hyps*
from *swapidseq-swap*[*of a b*] *H*(*3*) **have** *th0*: *swapidseq 1* (*Fun.swap a b id*)
by *simp*
from *swapidseq-comp-add*[*OF q*(*1*) *th0*] **have** *th1*: *swapidseq* (*Suc n*) *?q*
by *simp*
have *Fun.swap a b id* $\circ$ *p* $\circ$ *?q* = *Fun.swap a b id* $\circ$ (*p* $\circ$ *q*) $\circ$ *Fun.swap a b id*
by (*simp add*: *o-assoc*)
also have ... = *id*
by (*simp add*: *q*(*2*))
finally have *th2*: *Fun.swap a b id* $\circ$ *p* $\circ$ *?q* = *id* **.**
have *?q* $\circ$ (*Fun.swap a b id* $\circ$ *p*) = *q* $\circ$ (*Fun.swap a b id* $\circ$ *Fun.swap a b id*) $\circ$ *p*
by (*simp only*: *o-assoc*)
then have *?q* $\circ$ (*Fun.swap a b id* $\circ$ *p*) = *id*
by (*simp add*: *q*(*3*))
with *th1 th2* **show** *?case*
by *blast*
qed

lemma *swapidseq-inverse*:
assumes *H*: *swapidseq n p*
shows *swapidseq n* (*inv p*)
using *swapidseq-inverse-exists*[*OF H*] *inv-unique-comp*[*of p*] **by** *auto*

lemma *permutation-inverse*: *permutation p* $\Longrightarrow$ *permutation* (*inv p*)
using *permutation-def swapidseq-inverse* **by** *blast*

45.9 The identity map only has even transposition sequences

lemma *symmetry-lemma*:
assumes $\bigwedge$*a b c d*. *P a b c d* $\Longrightarrow$ *P a b d c*

and $\bigwedge a\ b\ c\ d.\ a \neq b \Longrightarrow c \neq d \Longrightarrow$
$a = c \wedge b = d \vee a = c \wedge b \neq d \vee a \neq c \wedge b = d \vee a \neq c \wedge a \neq d \wedge b \neq c \wedge b \neq d \Longrightarrow$
$P\ a\ b\ c\ d$
shows $\bigwedge a\ b\ c\ d.\ a \neq b \longrightarrow c \neq d \longrightarrow P\ a\ b\ c\ d$
using *assms* **by** *metis*

lemma *swap-general*: $a \neq b \Longrightarrow c \neq d \Longrightarrow$
Fun.swap a b id $\circ$ *Fun.swap c d id* $=$ *id* $\vee$
$(\exists x\ y\ z.\ x \neq a \wedge y \neq a \wedge z \neq a \wedge x \neq y \wedge$
Fun.swap a b id $\circ$ *Fun.swap c d id* $=$ *Fun.swap x y id* $\circ$ *Fun.swap a z id*)
proof −
assume *H*: $a \neq b\ c \neq d$
have $a \neq b \longrightarrow c \neq d \longrightarrow$
(*Fun.swap a b id* $\circ$ *Fun.swap c d id* $=$ *id* $\vee$
$(\exists x\ y\ z.\ x \neq a \wedge y \neq a \wedge z \neq a \wedge x \neq y \wedge$
Fun.swap a b id $\circ$ *Fun.swap c d id* $=$ *Fun.swap x y id* $\circ$ *Fun.swap a z id*))
apply (*rule symmetry-lemma*[**where** a=a **and** b=b **and** c=c **and** d=d])
apply (*simp-all only*: *swap-commute*)
apply (*case-tac* $a = c \wedge b = d$)
apply (*clarsimp simp only*: *swap-commute swap-id-idempotent*)
apply (*case-tac* $a = c \wedge b \neq d$)
apply (*rule disjI2*)
apply (*rule-tac* x=b **in** *exI*)
apply (*rule-tac* x=d **in** *exI*)
apply (*rule-tac* x=b **in** *exI*)
apply (*clarsimp simp add*: *fun-eq-iff Fun.swap-def*)
apply (*case-tac* $a \neq c \wedge b = d$)
apply (*rule disjI2*)
apply (*rule-tac* x=c **in** *exI*)
apply (*rule-tac* x=d **in** *exI*)
apply (*rule-tac* x=c **in** *exI*)
apply (*clarsimp simp add*: *fun-eq-iff Fun.swap-def*)
apply (*rule disjI2*)
apply (*rule-tac* x=c **in** *exI*)
apply (*rule-tac* x=d **in** *exI*)
apply (*rule-tac* x=b **in** *exI*)
apply (*clarsimp simp add*: *fun-eq-iff Fun.swap-def*)
done
with *H* **show** *?thesis* **by** *metis*
qed

lemma *swapidseq-id-iff*[*simp*]: *swapidseq 0 p* $\longleftrightarrow$ $p = id$
using *swapidseq.cases*[*of 0 p p* $= id$]
by *auto*

lemma *swapidseq-cases*: *swapidseq n p* $\longleftrightarrow$
$n = 0 \wedge p = id \vee (\exists\ a\ b\ q\ m.\ n = Suc\ m \wedge p =$ *Fun.swap a b id* $\circ\ q \wedge$ *swapidseq m q* $\wedge\ a \neq b$)

```
  apply (rule iffI)
  apply (erule swapidseq.cases[of n p])
  apply simp
  apply (rule disjI2)
  apply (rule-tac x= a in exI)
  apply (rule-tac x= b in exI)
  apply (rule-tac x= pa in exI)
  apply (rule-tac x= na in exI)
  apply simp
  apply auto
  apply (rule comp-Suc, simp-all)
  done

lemma fixing-swapidseq-decrease:
  assumes spn: swapidseq n p
    and ab: a ≠ b
    and pa: (Fun.swap a b id ∘ p) a = a
  shows n ≠ 0 ∧ swapidseq (n − 1) (Fun.swap a b id ∘ p)
  using spn ab pa
proof (induct n arbitrary: p a b)
  case 0
  then show ?case
    by (auto simp add: Fun.swap-def fun-upd-def)
next
  case (Suc n p a b)
  from Suc.prems(1) swapidseq-cases[of Suc n p]
  obtain c d q m where
    cdqm: Suc n = Suc m p = Fun.swap c d id ∘ q swapidseq m q c ≠ d n = m
    by auto
  {
    assume H: Fun.swap a b id ∘ Fun.swap c d id = id
    have ?case by (simp only: cdqm o-assoc H) (simp add: cdqm)
  }
  moreover
  {
    fix x y z
    assume H: x ≠ a y ≠ a z ≠ a x ≠ y
      Fun.swap a b id ∘ Fun.swap c d id = Fun.swap x y id ∘ Fun.swap a z id
    from H have az: a ≠ z
      by simp

    {
      fix h
      have (Fun.swap x y id ∘ h) a = a ⟷ h a = a
        using H by (simp add: Fun.swap-def)
    }
    note th3 = this
    from cdqm(2) have Fun.swap a b id ∘ p = Fun.swap a b id ∘ (Fun.swap c d
id ∘ q)
```

```
    by simp
  then have Fun.swap a b id ∘ p = Fun.swap x y id ∘ (Fun.swap a z id ∘ q)
    by (simp add: o-assoc H)
  then have (Fun.swap a b id ∘ p) a = (Fun.swap x y id ∘ (Fun.swap a z id ∘
q)) a
    by simp
  then have (Fun.swap x y id ∘ (Fun.swap a z id ∘ q)) a = a
    unfolding Suc by metis
  then have th1: (Fun.swap a z id ∘ q) a = a
    unfolding th3 .
  from Suc.hyps[OF cdqm(3)[ unfolded cdqm(5)[symmetric]] az th1]
  have th2: swapidseq (n − 1) (Fun.swap a z id ∘ q) n ≠ 0
    by blast+
  have th: Suc n − 1 = Suc (n − 1)
    using th2(2) by auto
  have ?case
    unfolding cdqm(2) H o-assoc th
    apply (simp only: Suc-not-Zero simp-thms comp-assoc)
    apply (rule comp-Suc)
    using th2 H
    apply blast+
    done
 }
 ultimately show ?case
  using swap-general[OF Suc.prems(2) cdqm(4)] by metis
qed

lemma swapidseq-identity-even:
 assumes swapidseq n (id :: 'a ⇒ 'a)
 shows even n
 using ‹swapidseq n id›
proof (induct n rule: nat-less-induct)
 fix n
 assume H: ∀ m<n. swapidseq m (id::'a ⇒ 'a) ⟶ even m swapidseq n (id :: 'a
⇒ 'a)
 {
  assume n = 0
  then have even n by presburger
 }
 moreover
 {
  fix a b :: 'a and q m
  assume h: n = Suc m (id :: 'a ⇒ 'a) = Fun.swap a b id ∘ q swapidseq m q a
≠ b
  from fixing-swapidseq-decrease[OF h(3,4), unfolded h(2)[symmetric]]
  have m: m ≠ 0 swapidseq (m − 1) (id :: 'a ⇒ 'a)
    by auto
  from h m have mn: m − 1 < n
    by arith
```

```
    from H(1)[rule-format, OF mn m(2)] h(1) m(1) have even n
      by presburger
  }
  ultimately show even n
    using H(2)[unfolded swapidseq-cases[of n id]] by auto
qed
```

45.10 Therefore we have a welldefined notion of parity

```
definition evenperm p = even (SOME n. swapidseq n p)

lemma swapidseq-even-even:
  assumes m: swapidseq m p
    and n: swapidseq n p
  shows even m ⟷ even n
proof −
  from swapidseq-inverse-exists[OF n]
  obtain q where q: swapidseq n q p ∘ q = id q ∘ p = id
    by blast
  from swapidseq-identity-even[OF swapidseq-comp-add[OF m q(1), unfolded q]]
  show ?thesis
    by arith
qed

lemma evenperm-unique:
  assumes p: swapidseq n p
    and n:even n = b
  shows evenperm p = b
  unfolding n[symmetric] evenperm-def
  apply (rule swapidseq-even-even[where p = p])
  apply (rule someI[where x = n])
  using p
  apply blast+
  done
```

45.11 And it has the expected composition properties

```
lemma evenperm-id[simp]: evenperm id = True
  by (rule evenperm-unique[where n = 0]) simp-all

lemma evenperm-swap: evenperm (Fun.swap a b id) = (a = b)
   by (rule evenperm-unique[where n=if a = b then 0 else 1]) (simp-all add:
swapidseq-swap)

lemma evenperm-comp:
  assumes p: permutation p
    and q:permutation q
  shows evenperm (p ∘ q) = (evenperm p = evenperm q)
proof −
  from p q obtain n m where n: swapidseq n p and m: swapidseq m q
```

unfolding *permutation-def* **by** *blast*
note *nm* = *swapidseq-comp-add*[*OF n m*]
have *th*: *even* (*n* + *m*) = (*even n* ⟷ *even m*)
by *arith*
from *evenperm-unique*[*OF n refl*] *evenperm-unique*[*OF m refl*]
evenperm-unique[*OF nm th*]
show *?thesis*
by *blast*
qed

lemma *evenperm-inv*:
assumes *p*: *permutation p*
shows *evenperm* (*inv p*) = *evenperm p*
proof −
from *p* **obtain** *n* **where** *n*: *swapidseq n p*
unfolding *permutation-def* **by** *blast*
from *evenperm-unique*[*OF swapidseq-inverse*[*OF n*] *evenperm-unique*[*OF n refl*, *symmetric*]]
show *?thesis* **.**
qed

45.12 A more abstract characterization of permutations

lemma *bij-iff*: *bij f* ⟷ (∀ *x*. ∃!*y*. *f y* = *x*)
unfolding *bij-def inj-on-def surj-def*
apply *auto*
apply *metis*
apply *metis*
done

lemma *permutation-bijective*:
assumes *p*: *permutation p*
shows *bij p*
proof −
from *p* **obtain** *n* **where** *n*: *swapidseq n p*
unfolding *permutation-def* **by** *blast*
from *swapidseq-inverse-exists*[*OF n*]
obtain *q* **where** *q*: *swapidseq n q p* ∘ *q* = *id q* ∘ *p* = *id*
by *blast*
then show *?thesis* **unfolding** *bij-iff*
apply (*auto simp add*: *fun-eq-iff*)
apply *metis*
done
qed

lemma *permutation-finite-support*:
assumes *p*: *permutation p*
shows *finite* {*x*. *p x* ≠ *x*}
proof −

```
  from p obtain n where n: swapidseq n p
    unfolding permutation-def by blast
  from n show ?thesis
  proof (induct n p rule: swapidseq.induct)
    case id
    then show ?case by simp
  next
    case (comp-Suc n p a b)
    let ?S = insert a (insert b {x. p x ≠ x})
    from comp-Suc.hyps(2) have fS: finite ?S
      by simp
    from ‹a ≠ b› have th: {x. (Fun.swap a b id ∘ p) x ≠ x} ⊆ ?S
      by (auto simp add: Fun.swap-def)
    from finite-subset[OF th fS] show ?case .
  qed
qed

lemma bij-inv-eq-iff: bij p ⟹ x = inv p y ⟷ p x = y
  using surj-f-inv-f[of p] by (auto simp add: bij-def)

lemma bij-swap-comp:
  assumes bp: bij p
  shows Fun.swap a b id ∘ p = Fun.swap (inv p a) (inv p b) p
  using surj-f-inv-f[OF bij-is-surj[OF bp]]
  by (simp add: fun-eq-iff Fun.swap-def bij-inv-eq-iff[OF bp])

lemma bij-swap-ompose-bij: bij p ⟹ bij (Fun.swap a b id ∘ p)
proof −
  assume H: bij p
  show ?thesis
    unfolding bij-swap-comp[OF H] bij-swap-iff
    using H .
qed

lemma permutation-lemma:
  assumes fS: finite S
    and p: bij p
    and pS: ∀ x. x∉ S ⟶ p x = x
  shows permutation p
  using fS p pS
proof (induct S arbitrary: p rule: finite-induct)
  case (empty p)
  then show ?case by simp
next
  case (insert a F p)
  let ?r = Fun.swap a (p a) id ∘ p
  let ?q = Fun.swap a (p a) id ∘ ?r
  have raa: ?r a = a
    by (simp add: Fun.swap-def)
```

```
  from bij-swap-ompose-bij[OF insert(4)]
  have br: bij ?r .

  from insert raa have th: ∀ x. x ∉ F ⟶ ?r x = x
    apply (clarsimp simp add: Fun.swap-def)
    apply (erule-tac x=x in allE)
    apply auto
    unfolding bij-iff
    apply metis
    done
  from insert(3)[OF br th]
  have rp: permutation ?r .
  have permutation ?q
    by (simp add: permutation-compose permutation-swap-id rp)
  then show ?case
    by (simp add: o-assoc)
qed

lemma permutation: permutation p ⟷ bij p ∧ finite {x. p x ≠ x}
  (is ?lhs ⟷ ?b ∧ ?f)
proof
  assume p: ?lhs
  from p permutation-bijective permutation-finite-support show ?b ∧ ?f
    by auto
next
  assume ?b ∧ ?f
  then have ?f ?b by blast+
  from permutation-lemma[OF this] show ?lhs
    by blast
qed

lemma permutation-inverse-works:
  assumes p: permutation p
  shows inv p ∘ p = id
    and p ∘ inv p = id
  using permutation-bijective [OF p]
  unfolding bij-def inj-iff surj-iff by auto

lemma permutation-inverse-compose:
  assumes p: permutation p
    and q: permutation q
  shows inv (p ∘ q) = inv q ∘ inv p
proof −
  note ps = permutation-inverse-works[OF p]
  note qs = permutation-inverse-works[OF q]
  have p ∘ q ∘ (inv q ∘ inv p) = p ∘ (q ∘ inv q) ∘ inv p
    by (simp add: o-assoc)
  also have ... = id
    by (simp add: ps qs)
```

finally have *th0*: $p \circ q \circ (inv\ q \circ inv\ p) = id$ **.**
have $inv\ q \circ inv\ p \circ (p \circ q) = inv\ q \circ (inv\ p \circ p) \circ q$
by (*simp add*: *o-assoc*)
also have $\ldots = id$
by (*simp add*: *ps qs*)
finally have *th1*: $inv\ q \circ inv\ p \circ (p \circ q) = id$ **.**
from *inv-unique-comp*[*OF th0 th1*] **show** *?thesis* **.**
qed

45.13 Relation to "permutes"

lemma *permutation-permutes*: *permutation* $p \longleftrightarrow (\exists S.\ finite\ S \wedge p\ permutes\ S)$
unfolding *permutation permutes-def bij-iff*[*symmetric*]
apply (*rule iffI*, *clarify*)
apply (*rule exI*[**where** $x=\{x.\ p\ x \neq x\}$])
apply *simp*
apply *clarsimp*
apply (*rule-tac* $B=S$ **in** *finite-subset*)
apply *auto*
done

45.14 Hence a sort of induction principle composing by swaps

lemma *permutes-induct*: *finite* $S \Longrightarrow P\ id \Longrightarrow$
$(\bigwedge a\ b\ p.\ a \in S \Longrightarrow b \in S \Longrightarrow P\ p \Longrightarrow P\ p \Longrightarrow permutation\ p \Longrightarrow P\ (Fun.swap\ a\ b\ id \circ p)) \Longrightarrow$
$(\bigwedge p.\ p\ permutes\ S \Longrightarrow P\ p)$
proof (*induct S rule*: *finite-induct*)
case *empty*
then show *?case* **by** *auto*
next
case (*insert x F p*)
let *?r* = *Fun.swap x* (*p x*) *id* $\circ$ *p*
let *?q* = *Fun.swap x* (*p x*) *id* $\circ$ *?r*
have *qp*: *?q* = *p*
by (*simp add*: *o-assoc*)
from *permutes-insert-lemma*[*OF insert.prems*(*3*)] *insert* **have** *Pr*: *P ?r*
by *blast*
from *permutes-in-image*[*OF insert.prems*(*3*), *of x*]
have *pxF*: *p x* $\in$ *insert x F*
by *simp*
have *xF*: *x* $\in$ *insert x F*
by *simp*
have *rp*: *permutation ?r*
unfolding *permutation-permutes* **using** *insert.hyps*(*1*)
permutes-insert-lemma[*OF insert.prems*(*3*)]
by *blast*
from *insert.prems*(*2*)[*OF xF pxF Pr Pr rp*]
show *?case*
unfolding *qp* **.**

qed

45.15 Sign of a permutation as a real number

definition *sign p* = (*if evenperm p then* (*1*::*int*) *else* −*1*)

lemma *sign-nz*: *sign p* ≠ *0*
by (*simp add*: *sign-def*)

lemma *sign-id*: *sign id* = *1*
by (*simp add*: *sign-def*)

lemma *sign-inverse*: *permutation p* ⟹ *sign* (*inv p*) = *sign p*
by (*simp add*: *sign-def evenperm-inv*)

lemma *sign-compose*: *permutation p* ⟹ *permutation q* ⟹ *sign* (*p* ∘ *q*) = *sign p* ∗ *sign q*
by (*simp add*: *sign-def evenperm-comp*)

lemma *sign-swap-id*: *sign* (*Fun.swap a b id*) = (*if a* = *b then 1 else* −*1*)
by (*simp add*: *sign-def evenperm-swap*)

lemma *sign-idempotent*: *sign p* ∗ *sign p* = *1*
by (*simp add*: *sign-def*)

45.16 More lemmas about permutations

lemma *permutes-natset-le*:
fixes *S* :: *'a*::*wellorder set*
assumes *p*: *p permutes S*
and *le*: ∀ *i* ∈ *S*. *p i* ≤ *i*
shows *p* = *id*
proof −
{
fix *n*
have *p n* = *n*
using *p le*
proof (*induct n arbitrary*: *S rule*: *less-induct*)
fix *n S*
assume *H*:
⋀*m S*. *m* < *n* ⟹ *p permutes S* ⟹ ∀ *i*∈*S*. *p i* ≤ *i* ⟹ *p m* = *m*
p permutes S ∀ *i* ∈*S*. *p i* ≤ *i*
{
assume *n* ∉ *S*
with *H*(*2*) **have** *p n* = *n*
unfolding *permutes-def* **by** *metis*
}
moreover
{
assume *ns*: *n* ∈ *S*

```
      from H(3) ns have p n < n ∨ p n = n
        by auto
      moreover {
        assume h: p n < n
        from H h have p (p n) = p n
          by metis
        with permutes-inj[OF H(2)] have p n = n
          unfolding inj-on-def by blast
        with h have False
          by simp
      }
      ultimately have p n = n
        by blast
    }
    ultimately show p n = n
      by blast
  qed
 }
 then show ?thesis
  by (auto simp add: fun-eq-iff)
qed

lemma permutes-natset-ge:
 fixes S :: 'a::wellorder set
 assumes p: p permutes S
  and le: ∀ i ∈ S. p i ≥ i
 shows p = id
proof −
 {
  fix i
  assume i: i ∈ S
  from i permutes-in-image[OF permutes-inv[OF p]] have inv p i ∈ S
    by simp
  with le have p (inv p i) ≥ inv p i
    by blast
  with permutes-inverses[OF p] have i ≥ inv p i
    by simp
 }
 then have th: ∀ i∈S. inv p i ≤ i
  by blast
 from permutes-natset-le[OF permutes-inv[OF p] th]
 have inv p = inv id
  by simp
 then show ?thesis
  apply (subst permutes-inv-inv[OF p, symmetric])
  apply (rule inv-unique-comp)
  apply simp-all
  done
qed
```

lemma *image-inverse-permutations*: $\{inv\ p\ |p.\ p\ permutes\ S\} = \{p.\ p\ permutes\ S\}$
apply (*rule set-eqI*)
apply *auto*
using *permutes-inv-inv permutes-inv*
apply *auto*
apply (*rule-tac x=inv x* **in** *exI*)
apply *auto*
done

lemma *image-compose-permutations-left*:
assumes *q*: *q permutes S*
shows $\{q \circ p \mid p.\ p\ permutes\ S\} = \{p\ .\ p\ permutes\ S\}$
apply (*rule set-eqI*)
apply *auto*
apply (*rule permutes-compose*)
using *q*
apply *auto*
apply (*rule-tac x = inv q ∘ x* **in** *exI*)
apply (*simp add*: *o-assoc permutes-inv permutes-compose permutes-inv-o*)
done

lemma *image-compose-permutations-right*:
assumes *q*: *q permutes S*
shows $\{p \circ q \mid p.\ p\ permutes\ S\} = \{p\ .\ p\ permutes\ S\}$
apply (*rule set-eqI*)
apply *auto*
apply (*rule permutes-compose*)
using *q*
apply *auto*
apply (*rule-tac x = x ∘ inv q* **in** *exI*)
apply (*simp add*: *o-assoc permutes-inv permutes-compose permutes-inv-o comp-assoc*)
done

lemma *permutes-in-seg*: $p\ permutes\ \{1\ ..n\} \Longrightarrow i \in \{1..n\} \Longrightarrow 1 \leq p\ i \wedge p\ i \leq n$
by (*simp add*: *permutes-def*) *metis*

lemma *setsum-permutations-inverse*:
$setsum\ f\ \{p.\ p\ permutes\ S\} = setsum\ (\lambda p.\ f(inv\ p))\ \{p.\ p\ permutes\ S\}$
(**is** *?lhs = ?rhs*)
proof −
let *?S* = $\{p\ .\ p\ permutes\ S\}$
have *th0*: *inj-on inv ?S*
proof (*auto simp add*: *inj-on-def*)
fix *q r*
assume *q*: *q permutes S*
and *r*: *r permutes S*

```
    and qr: inv q = inv r
  then have inv (inv q) = inv (inv r)
    by simp
  with permutes-inv-inv[OF q] permutes-inv-inv[OF r] show q = r
    by metis
 qed
 have th1: inv ' ?S = ?S
  using image-inverse-permutations by blast
 have th2: ?rhs = setsum (f ∘ inv) ?S
  by (simp add: o-def)
 from setsum.reindex[OF th0, of f] show ?thesis unfolding th1 th2 .
qed

lemma setum-permutations-compose-left:
 assumes q: q permutes S
 shows setsum f {p. p permutes S} = setsum (λp. f(q ∘ p)) {p. p permutes S}
 (is ?lhs = ?rhs)
proof −
 let ?S = {p. p permutes S}
 have th0: ?rhs = setsum (f ∘ (op ∘ q)) ?S
  by (simp add: o-def)
 have th1: inj-on (op ∘ q) ?S
 proof (auto simp add: inj-on-def)
  fix p r
  assume p permutes S
    and r: r permutes S
    and rp: q ∘ p = q ∘ r
  then have inv q ∘ q ∘ p = inv q ∘ q ∘ r
    by (simp add: comp-assoc)
  with permutes-inj[OF q, unfolded inj-iff] show p = r
    by simp
 qed
 have th3: (op ∘ q) ' ?S = ?S
  using image-compose-permutations-left[OF q] by auto
 from setsum.reindex[OF th1, of f] show ?thesis unfolding th0 th1 th3 .
qed

lemma sum-permutations-compose-right:
 assumes q: q permutes S
 shows setsum f {p. p permutes S} = setsum (λp. f(p ∘ q)) {p. p permutes S}
 (is ?lhs = ?rhs)
proof −
 let ?S = {p. p permutes S}
 have th0: ?rhs = setsum (f ∘ (λp. p ∘ q)) ?S
  by (simp add: o-def)
 have th1: inj-on (λp. p ∘ q) ?S
 proof (auto simp add: inj-on-def)
  fix p r
  assume p permutes S
```

```
    and r: r permutes S
    and rp: p ∘ q = r ∘ q
  then have p ∘ (q ∘ inv q) = r ∘ (q ∘ inv q)
    by (simp add: o-assoc)
  with permutes-surj[OF q, unfolded surj-iff] show p = r
    by simp
 qed
 have th3: (λp. p ∘ q) ' ?S = ?S
  using image-compose-permutations-right[OF q] by auto
 from setsum.reindex[OF th1, of f]
 show ?thesis unfolding th0 th1 th3 .
qed
```

45.17 Sum over a set of permutations (could generalize to iteration)

```
lemma setsum-over-permutations-insert:
 assumes fS: finite S
  and aS: a ∉ S
 shows setsum f {p. p permutes (insert a S)} =
  setsum (λb. setsum (λq. f (Fun.swap a b id ∘ q)) {p. p permutes S}) (insert a
S)
proof −
 have th0: ⋀f a b. (λ(b,p). f (Fun.swap a b id ∘ p)) = f ∘ (λ(b,p). Fun.swap a
b id ∘ p)
  by (simp add: fun-eq-iff)
 have th1: ⋀P Q. P × Q = {(a,b). a ∈ P ∧ b ∈ Q}
  by blast
 have th2: ⋀P Q. P ⟹ (P ⟹ Q) ⟹ P ∧ Q
  by blast
 show ?thesis
  unfolding permutes-insert
  unfolding setsum.cartesian-product
  unfolding th1[symmetric]
  unfolding th0
 proof (rule setsum.reindex)
  let ?f = (λ(b, y). Fun.swap a b id ∘ y)
  let ?P = {p. p permutes S}
  {
   fix b c p q
   assume b: b ∈ insert a S
   assume c: c ∈ insert a S
   assume p: p permutes S
   assume q: q permutes S
   assume eq: Fun.swap a b id ∘ p = Fun.swap a c id ∘ q
   from p q aS have pa: p a = a and qa: q a = a
    unfolding permutes-def by metis+
   from eq have (Fun.swap a b id ∘ p) a = (Fun.swap a c id ∘ q) a
    by simp
```

```
    then have bc: b = c
      by (simp add: permutes-def pa qa o-def fun-upd-def Fun.swap-def id-def
          cong del: if-weak-cong split: if-split-asm)
    from eq[unfolded bc] have (λp. Fun.swap a c id ∘ p) (Fun.swap a c id ∘ p)
=
      (λp. Fun.swap a c id ∘ p) (Fun.swap a c id ∘ q) by simp
    then have p = q
      unfolding o-assoc swap-id-idempotent
      by (simp add: o-def)
    with bc have b = c ∧ p = q
      by blast
  }
  then show inj-on ?f (insert a S × ?P)
    unfolding inj-on-def by clarify metis
 qed
qed

end
```

46 Traces, Determinant of square matrices and some properties

```
theory Determinants
imports
  Cartesian-Euclidean-Space
  ~~/src/HOL/Library/Permutations
begin
```

46.1 Trace

```
definition trace :: 'a::semiring-1^'n^'n ⇒ 'a
  where trace A = setsum (λi. ((A$i)$i)) (UNIV::'n set)

lemma trace-0: trace (mat 0) = 0
  by (simp add: trace-def mat-def)

lemma trace-I: trace (mat 1 :: 'a::semiring-1^'n^'n) = of-nat(CARD('n))
  by (simp add: trace-def mat-def)

lemma trace-add: trace ((A::'a::comm-semiring-1^'n^'n) + B) = trace A + trace
B
  by (simp add: trace-def setsum.distrib)

lemma trace-sub: trace ((A::'a::comm-ring-1^'n^'n) - B) = trace A - trace B
  by (simp add: trace-def setsum-subtractf)

lemma trace-mul-sym: trace ((A::'a::comm-semiring-1^'n^'m) ** B) = trace (B**A)
  apply (simp add: trace-def matrix-matrix-mult-def)
```

apply (*subst setsum.commute*)
apply (*simp add*: *mult.commute*)
done

Definition of determinant.

definition *det*:: *'a*::*comm-ring-1^'n^'n* ⇒ *'a* **where**
det A =
setsum (λ*p*. *of-int* (*sign p*) ∗ *setprod* (λ*i*. *A*$*i*$*p i*) (*UNIV* :: *'n set*))
{*p*. *p permutes* (*UNIV* :: *'n set*)}

A few general lemmas we need below.

lemma *setprod-permute*:
assumes *p*: *p permutes S*
shows *setprod f S* = *setprod* (*f* ∘ *p*) *S*
using *assms* **by** (*fact setprod.permute*)

lemma *setproduct-permute-nat-interval*:
fixes *m n* :: *nat*
shows *p permutes* {*m*..*n*} ⟹ *setprod f* {*m*..*n*} = *setprod* (*f* ∘ *p*) {*m*..*n*}
by (*blast intro*!: *setprod-permute*)

Basic determinant properties.

lemma *det-transpose*: *det* (*transpose A*) = *det* (*A*::*'a*::*comm-ring-1* ^*'n^'n*)
proof −
let *?di* = λ*A i j*. *A*$*i*$*j*
let *?U* = (*UNIV* :: *'n set*)
have *fU*: *finite ?U* **by** *simp*
{
fix *p*
assume *p*: *p* ∈ {*p*. *p permutes ?U*}
from *p* **have** *pU*: *p permutes ?U*
by *blast*
have *sth*: *sign* (*inv p*) = *sign p*
by (*metis sign-inverse fU p mem-Collect-eq permutation-permutes*)
from *permutes-inj*[*OF pU*]
have *pi*: *inj-on p ?U*
by (*blast intro*: *subset-inj-on*)
from *permutes-image*[*OF pU*]
have *setprod* (λ*i*. *?di* (*transpose A*) *i* (*inv p i*)) *?U* =
setprod (λ*i*. *?di* (*transpose A*) *i* (*inv p i*)) (*p* ' *?U*)
by *simp*
also have ... = *setprod* ((λ*i*. *?di* (*transpose A*) *i* (*inv p i*)) ∘ *p*) *?U*
unfolding *setprod.reindex*[*OF pi*] **..**
also have ... = *setprod* (λ*i*. *?di A i* (*p i*)) *?U*
proof −
{
fix *i*
assume *i*: *i* ∈ *?U*
from *i permutes-inv-o*[*OF pU*] *permutes-in-image*[*OF pU*]

```
      have ((λi. ?di (transpose A) i (inv p i)) ∘ p) i = ?di A i (p i)
        unfolding transpose-def by (simp add: fun-eq-iff)
    }
    then show setprod ((λi. ?di (transpose A) i (inv p i)) ∘ p) ?U =
      setprod (λi. ?di A i (p i)) ?U
      by (auto intro: setprod.cong)
  qed
  finally have of-int (sign (inv p)) * (setprod (λi. ?di (transpose A) i (inv p i))
?U) =
    of-int (sign p) * (setprod (λi. ?di A i (p i)) ?U)
    using sth by simp
 }
 then show ?thesis
  unfolding det-def
  apply (subst setsum-permutations-inverse)
  apply (rule setsum.cong)
  apply (rule refl)
  apply blast
  done
qed

lemma det-lowerdiagonal:
 fixes A :: 'a::comm-ring-1^('n::{finite,wellorder})^('n::{finite,wellorder})
 assumes ld: ⋀i j. i < j ⟹ A$i$j = 0
 shows det A = setprod (λi. A$i$i) (UNIV:: 'n set)
proof −
 let ?U = UNIV:: 'n set
 let ?PU = {p. p permutes ?U}
 let ?pp = λp. of-int (sign p) * setprod (λi. A$i$p i) (UNIV :: 'n set)
 have fU: finite ?U
  by simp
 from finite-permutations[OF fU] have fPU: finite ?PU .
 have id0: {id} ⊆ ?PU
  by (auto simp add: permutes-id)
 {
  fix p
  assume p: p ∈ ?PU − {id}
  from p have pU: p permutes ?U and pid: p ≠ id
   by blast+
  from permutes-natset-le[OF pU] pid obtain i where i: p i > i
   by (metis not-le)
  from ld[OF i] have ex:∃ i ∈ ?U. A$i$p i = 0
   by blast
  from setprod-zero[OF fU ex] have ?pp p = 0
   by simp
 }
 then have p0: ∀ p ∈ ?PU − {id}. ?pp p = 0
  by blast
 from setsum.mono-neutral-cong-left[OF fPU id0 p0] show ?thesis
```

```
  unfolding det-def by (simp add: sign-id)
qed

lemma det-upperdiagonal:
  fixes A :: 'a::comm-ring-1^'n::{finite,wellorder}^'n::{finite,wellorder}
  assumes ld: ⋀i j. i > j ⟹ A$i$j = 0
  shows det A = setprod (λi. A$i$i) (UNIV:: 'n set)
proof −
  let ?U = UNIV:: 'n set
  let ?PU = {p. p permutes ?U}
  let ?pp = (λp. of-int (sign p) * setprod (λi. A$i$p i) (UNIV :: 'n set))
  have fU: finite ?U
    by simp
  from finite-permutations[OF fU] have fPU: finite ?PU .
  have id0: {id} ⊆ ?PU
    by (auto simp add: permutes-id)
  {
    fix p
    assume p: p ∈ ?PU − {id}
    from p have pU: p permutes ?U and pid: p ≠ id
      by blast+
    from permutes-natset-ge[OF pU] pid obtain i where i: p i < i
      by (metis not-le)
    from ld[OF i] have ex:∃ i ∈ ?U. A$i$p i = 0
      by blast
    from setprod-zero[OF fU ex] have ?pp p = 0
      by simp
  }
  then have p0: ∀ p ∈ ?PU −{id}. ?pp p = 0
    by blast
  from setsum.mono-neutral-cong-left[OF fPU id0 p0] show ?thesis
    unfolding det-def by (simp add: sign-id)
qed

lemma det-diagonal:
  fixes A :: 'a::comm-ring-1^'n^'n
  assumes ld: ⋀i j. i ≠ j ⟹ A$i$j = 0
  shows det A = setprod (λi. A$i$i) (UNIV::'n set)
proof −
  let ?U = UNIV:: 'n set
  let ?PU = {p. p permutes ?U}
  let ?pp = λp. of-int (sign p) * setprod (λi. A$i$p i) (UNIV :: 'n set)
  have fU: finite ?U by simp
  from finite-permutations[OF fU] have fPU: finite ?PU .
  have id0: {id} ⊆ ?PU
    by (auto simp add: permutes-id)
  {
    fix p
    assume p: p ∈ ?PU − {id}
```

```
    then have p ≠ id
      by simp
    then obtain i where i: p i ≠ i
      unfolding fun-eq-iff by auto
    from ld [OF i [symmetric]] have ex:∃ i ∈ ?U. A$i$p i = 0
      by blast
    from setprod-zero [OF fU ex] have ?pp p = 0
      by simp
  }
  then have p0: ∀ p ∈ ?PU - {id}. ?pp p = 0
    by blast
  from setsum.mono-neutral-cong-left[OF fPU id0 p0] show ?thesis
    unfolding det-def by (simp add: sign-id)
qed

lemma det-I: det (mat 1 :: 'a::comm-ring-1^'n^'n) = 1
proof -
  let ?A = mat 1 :: 'a::comm-ring-1^'n^'n
  let ?U = UNIV :: 'n set
  let ?f = λi j. ?A$i$j
  {
    fix i
    assume i: i ∈ ?U
    have ?f i i = 1
      using i by (vector mat-def)
  }
  then have th: setprod (λi. ?f i i) ?U = setprod (λx. 1) ?U
    by (auto intro: setprod.cong)
  {
    fix i j
    assume i: i ∈ ?U and j: j ∈ ?U and ij: i ≠ j
    have ?f i j = 0 using i j ij
      by (vector mat-def)
  }
  then have det ?A = setprod (λi. ?f i i) ?U
    using det-diagonal by blast
  also have ... = 1
    unfolding th setprod.neutral-const ..
  finally show ?thesis .
qed

lemma det-0: det (mat 0 :: 'a::comm-ring-1^'n^'n) = 0
  by (simp add: det-def setprod-zero)

lemma det-permute-rows:
  fixes A :: 'a::comm-ring-1^'n^'n
  assumes p: p permutes (UNIV :: 'n::finite set)
  shows det (χ i. A$p i :: 'a^'n^'n) = of-int (sign p) * det A
  apply (simp add: det-def setsum-right-distrib mult.assoc[symmetric])
```

```
  apply (subst sum-permutations-compose-right[OF p])
proof (rule setsum.cong)
  let ?U = UNIV :: 'n set
  let ?PU = {p. p permutes ?U}
  fix q
  assume qPU: q ∈ ?PU
  have fU: finite ?U
    by simp
  from qPU have q: q permutes ?U
    by blast
  from p q have pp: permutation p and qp: permutation q
    by (metis fU permutation-permutes)+
  from permutes-inv[OF p] have ip: inv p permutes ?U .
  have setprod (λi. A$p i$ (q ∘ p) i) ?U = setprod ((λi. A$p i$(q ∘ p) i) ∘ inv
p) ?U
    by (simp only: setprod-permute[OF ip, symmetric])
  also have ... = setprod (λi. A $ (p ∘ inv p) i $ (q ∘ (p ∘ inv p)) i) ?U
    by (simp only: o-def)
  also have ... = setprod (λi. A$i$q i) ?U
    by (simp only: o-def permutes-inverses[OF p])
  finally have thp: setprod (λi. A$p i$ (q ∘ p) i) ?U = setprod (λi. A$i$q i) ?U
    by blast
  show of-int (sign (q ∘ p)) * setprod (λi. A$ p i$ (q ∘ p) i) ?U =
    of-int (sign p) * of-int (sign q) * setprod (λi. A$i$q i) ?U
    by (simp only: thp sign-compose[OF qp pp] mult.commute of-int-mult)
qed rule

lemma det-permute-columns:
  fixes A :: 'a::comm-ring-1^'n^'n
  assumes p: p permutes (UNIV :: 'n set)
  shows det(χ i j. A$i$ p j :: 'a^'n^'n) = of-int (sign p) * det A
proof −
  let ?Ap = χ i j. A$i$ p j :: 'a^'n^'n
  let ?At = transpose A
  have of-int (sign p) * det A = det (transpose (χ i. transpose A $ p i))
    unfolding det-permute-rows[OF p, of ?At] det-transpose ..
  moreover
  have ?Ap = transpose (χ i. transpose A $ p i)
    by (simp add: transpose-def vec-eq-iff)
  ultimately show ?thesis
    by simp
qed

lemma det-identical-rows:
  fixes A :: 'a::linordered-idom^'n^'n
  assumes ij: i ≠ j
    and r: row i A = row j A
  shows det A = 0
proof−
```

have *tha*: $\bigwedge(a::'a)\ b.\ a = b \Longrightarrow b = -\ a \Longrightarrow a = 0$
by *simp*
have *th1*: *of-int* $(-1) = -\ 1$ **by** *simp*
let *?p* = *Fun.swap i j id*
let *?A* = χ *i. A $?p i*
from *r* **have** *A* = *?A* **by** (*simp add*: *vec-eq-iff row-def Fun.swap-def*)
then have *det A* = *det ?A* **by** *simp*
moreover have *det A* = $-$ *det ?A*
by (*simp add*: *det-permute-rows*[*OF permutes-swap-id*] *sign-swap-id ij th1*)
ultimately show *det A* = *0* **by** (*metis tha*)
qed

lemma *det-identical-columns*:
fixes *A* :: *'a*::*linordered-idom^'n^'n*
assumes *ij*: *i* ≠ *j*
and *r*: *column i A* = *column j A*
shows *det A* = *0*
apply (*subst det-transpose*[*symmetric*])
apply (*rule det-identical-rows*[*OF ij*])
apply (*metis row-transpose r*)
done

lemma *det-zero-row*:
fixes *A* :: *'a*::{*idom*, *ring-char-0*}*^'n^'n*
assumes *r*: *row i A* = *0*
shows *det A* = *0*
using *r*
apply (*simp add*: *row-def det-def vec-eq-iff*)
apply (*rule setsum.neutral*)
apply (*auto simp*: *sign-nz*)
done

lemma *det-zero-column*:
fixes *A* :: *'a*::{*idom*,*ring-char-0*}*^'n^'n*
assumes *r*: *column i A* = *0*
shows *det A* = *0*
apply (*subst det-transpose*[*symmetric*])
apply (*rule det-zero-row* [*of i*])
apply (*metis row-transpose r*)
done

lemma *det-row-add*:
fixes *a b c* :: *'n*::*finite* ⇒ - *^ 'n*
shows *det*((χ *i. if i* = *k then a i* + *b i else c i*)::*'a*::*comm-ring-1^'n^'n*) =
det((χ *i. if i* = *k then a i else c i*)::*'a*::*comm-ring-1^'n^'n*) +
det((χ *i. if i* = *k then b i else c i*)::*'a*::*comm-ring-1^'n^'n*)
unfolding *det-def vec-lambda-beta setsum.distrib*[*symmetric*]
proof (*rule setsum.cong*)
let *?U* = *UNIV* :: *'n set*

let *?pU* = {*p*. *p permutes ?U*}
let *?f* = (λ*i*. *if i* = *k then a i* + *b i else c i*)::*'n* ⇒ *'a*::*comm-ring-1*^*'n*
let *?g* = (λ *i*. *if i* = *k then a i else c i*)::*'n* ⇒ *'a*::*comm-ring-1*^*'n*
let *?h* = (λ *i*. *if i* = *k then b i else c i*)::*'n* ⇒ *'a*::*comm-ring-1*^*'n*
fix *p*
assume *p*: *p* ∈ *?pU*
let *?Uk* = *?U* − {*k*}
from *p* **have** *pU*: *p permutes ?U*
  by *blast*
have *kU*: *?U* = *insert k ?Uk*
  by *blast*
{
  fix *j*
  assume *j*: *j* ∈ *?Uk*
  from *j* **have** *?f j* $ *p j* = *?g j* $ *p j* **and** *?f j* $ *p j*= *?h j* $ *p j*
    by *simp-all*
}
then have *th1*: *setprod* (λ*i*. *?f i* $ *p i*) *?Uk* = *setprod* (λ*i*. *?g i* $ *p i*) *?Uk*
  and *th2*: *setprod* (λ*i*. *?f i* $ *p i*) *?Uk* = *setprod* (λ*i*. *?h i* $ *p i*) *?Uk*
  apply −
  apply (*rule setprod.cong*, *simp-all*)+
  done
have *th3*: *finite ?Uk k* ∉ *?Uk*
  by *auto*
have *setprod* (λ*i*. *?f i* $ *p i*) *?U* = *setprod* (λ*i*. *?f i* $ *p i*) (*insert k ?Uk*)
  unfolding *kU*[*symmetric*] **..**
also have ... = *?f k* $ *p k* ∗ *setprod* (λ*i*. *?f i* $ *p i*) *?Uk*
  apply (*rule setprod.insert*)
  apply *simp*
  apply *blast*
  done
also have ... = (*a k* $ *p k* ∗ *setprod* (λ*i*. *?f i* $ *p i*) *?Uk*) + (*b k*$ *p k* ∗ *setprod* (λ*i*. *?f i* $ *p i*) *?Uk*)
  by (*simp add*: *field-simps*)
also have ... = (*a k* $ *p k* ∗ *setprod* (λ*i*. *?g i* $ *p i*) *?Uk*) + (*b k*$ *p k* ∗ *setprod* (λ*i*. *?h i* $ *p i*) *?Uk*)
  by (*metis th1 th2*)
also have ... = *setprod* (λ*i*. *?g i* $ *p i*) (*insert k ?Uk*) + *setprod* (λ*i*. *?h i* $ *p i*) (*insert k ?Uk*)
  unfolding *setprod.insert*[*OF th3*] **by** *simp*
finally have *setprod* (λ*i*. *?f i* $ *p i*) *?U* = *setprod* (λ*i*. *?g i* $ *p i*) *?U* + *setprod* (λ*i*. *?h i* $ *p i*) *?U*
  unfolding *kU*[*symmetric*] **.**
then show *of-int* (*sign p*) ∗ *setprod* (λ*i*. *?f i* $ *p i*) *?U* =
  of-int (*sign p*) ∗ *setprod* (λ*i*. *?g i* $ *p i*) *?U* + *of-int* (*sign p*) ∗ *setprod* (λ*i*. *?h i* $ *p i*) *?U*
  by (*simp add*: *field-simps*)
qed *rule*

lemma *det-row-mul*:
  fixes *a b* :: *'n::finite* ⇒ - ^ *'n*
  shows *det*((χ *i*. *if i* = *k then c* ∗*s a i else b i*)::*'a::comm-ring-1^'n^'n*) =
    c ∗ *det*((χ *i*. *if i* = *k then a i else b i*)::*'a::comm-ring-1^'n^'n*)
  unfolding *det-def vec-lambda-beta setsum-right-distrib*
proof (*rule setsum.cong*)
  let *?U* = *UNIV* :: *'n set*
  let *?pU* = {*p*. *p permutes ?U*}
  let *?f* = (λ*i*. *if i* = *k then c*∗*s a i else b i*)::*'n* ⇒ *'a::comm-ring-1^'n*
  let *?g* = (λ *i*. *if i* = *k then a i else b i*)::*'n* ⇒ *'a::comm-ring-1^'n*
  fix *p*
  assume *p*: *p* ∈ *?pU*
  let *?Uk* = *?U* − {*k*}
  from *p* **have** *pU*: *p permutes ?U*
    by *blast*
  have *kU*: *?U* = *insert k ?Uk*
    by *blast*
  {
    fix *j*
    assume *j*: *j* ∈ *?Uk*
    from *j* **have** *?f j* $ *p j* = *?g j* $ *p j*
      by *simp*
  }
  then have *th1*: *setprod* (λ*i*. *?f i* $ *p i*) *?Uk* = *setprod* (λ*i*. *?g i* $ *p i*) *?Uk*
    apply −
    apply (*rule setprod.cong*)
    apply *simp-all*
    done
  have *th3*: *finite ?Uk k* ∉ *?Uk*
    by *auto*
  have *setprod* (λ*i*. *?f i* $ *p i*) *?U* = *setprod* (λ*i*. *?f i* $ *p i*) (*insert k ?Uk*)
    unfolding *kU*[*symmetric*] **..**
  also have ... = *?f k* $ *p k* ∗ *setprod* (λ*i*. *?f i* $ *p i*) *?Uk*
    apply (*rule setprod.insert*)
    apply *simp*
    apply *blast*
    done
  also have ... = (*c*∗*s a k*) $ *p k* ∗ *setprod* (λ*i*. *?f i* $ *p i*) *?Uk*
    by (*simp add*: *field-simps*)
  also have ... = *c*∗ (*a k* $ *p k* ∗ *setprod* (λ*i*. *?g i* $ *p i*) *?Uk*)
    unfolding *th1* **by** (*simp add*: *ac-simps*)
  also have ... = *c*∗ (*setprod* (λ*i*. *?g i* $ *p i*) (*insert k ?Uk*))
    unfolding *setprod.insert*[*OF th3*] **by** *simp*
  finally have *setprod* (λ*i*. *?f i* $ *p i*) *?U* = *c*∗ (*setprod* (λ*i*. *?g i* $ *p i*) *?U*)
    unfolding *kU*[*symmetric*] **.**
  then show *of-int* (*sign p*) ∗ *setprod* (λ*i*. *?f i* $ *p i*) *?U* =
    c ∗ (*of-int* (*sign p*) ∗ *setprod* (λ*i*. *?g i* $ *p i*) *?U*)
    by (*simp add*: *field-simps*)
qed *rule*

```
lemma det-row-0:
  fixes b :: 'n::finite ⇒ - ^ 'n
  shows det((χ i. if i = k then 0 else b i)::'a::comm-ring-1^'n^'n) = 0
  using det-row-mul[of k 0 λi. 1 b]
  apply simp
  apply (simp only: vector-smult-lzero)
  done

lemma det-row-operation:
  fixes A :: 'a::linordered-idom^'n^'n
  assumes ij: i ≠ j
  shows det (χ k. if k = i then row i A + c *s row j A else row k A) = det A
proof −
  let ?Z = (χ k. if k = i then row j A else row k A) :: 'a ^'n^'n
  have th: row i ?Z = row j ?Z by (vector row-def)
  have th2: ((χ k. if k = i then row i A else row k A) :: 'a^'n^'n) = A
    by (vector row-def)
  show ?thesis
    unfolding det-row-add [of i] det-row-mul[of i] det-identical-rows[OF ij th] th2
    by simp
qed

lemma det-row-span:
  fixes A :: real^'n^'n
  assumes x: x ∈ span {row j A |j. j ≠ i}
  shows det (χ k. if k = i then row i A + x else row k A) = det A
proof −
  let ?U = UNIV :: 'n set
  let ?S = {row j A |j. j ≠ i}
  let ?d = λx. det (χ k. if k = i then x else row k A)
  let ?P = λx. ?d (row i A + x) = det A
  {
    fix k
    have (if k = i then row i A + 0 else row k A) = row k A
      by simp
  }
  then have P0: ?P 0
    apply −
    apply (rule cong[of det, OF refl])
    apply (vector row-def)
    done
  moreover
  {
    fix c z y
    assume zS: z ∈ ?S and Py: ?P y
    from zS obtain j where j: z = row j A i ≠ j
      by blast
    let ?w = row i A + y
```

```
    have th0: row i A + (c*s z + y) = ?w + c*s z
      by vector
    have thz: ?d z = 0
      apply (rule det-identical-rows[OF j(2)])
      using j
      apply (vector row-def)
      done
    have ?d (row i A + (c*s z + y)) = ?d (?w + c*s z)
      unfolding th0 ..
    then have ?P (c*s z + y)
      unfolding thz Py det-row-mul[of i] det-row-add[of i]
      by simp
  }
  ultimately show ?thesis
    apply -
    apply (rule span-induct-alt[of ?P ?S, OF P0, folded scalar-mult-eq-scaleR])
    apply blast
    apply (rule x)
    done
qed
```

May as well do this, though it's a bit unsatisfactory since it ignores exact duplicates by considering the rows/columns as a set.

```
lemma det-dependent-rows:
  fixes A:: real^'n^'n
  assumes d: dependent (rows A)
  shows det A = 0
proof -
  let ?U = UNIV :: 'n set
  from d obtain i where i: row i A ∈ span (rows A - {row i A})
    unfolding dependent-def rows-def by blast
  {
    fix j k
    assume jk: j ≠ k and c: row j A = row k A
    from det-identical-rows[OF jk c] have ?thesis .
  }
  moreover
  {
    assume H: ⋀ i j. i ≠ j ⟹ row i A ≠ row j A
    have th0: - row i A ∈ span {row j A|j. j ≠ i}
      apply (rule span-neg)
      apply (rule set-rev-mp)
      apply (rule i)
      apply (rule span-mono)
      using H i
      apply (auto simp add: rows-def)
      done
    from det-row-span[OF th0]
    have det A = det (χ k. if k = i then 0 *s 1 else row k A)
```

```
    unfolding right-minus vector-smult-lzero ..
   with det-row-mul[of i 0::real λi. 1]
   have det A = 0 by simp
 }
 ultimately show ?thesis by blast
qed

lemma det-dependent-columns:
 assumes d: dependent (columns (A::real^'n^'n))
 shows det A = 0
 by (metis d det-dependent-rows rows-transpose det-transpose)
```

Multilinearity and the multiplication formula.

```
lemma Cart-lambda-cong: (⋀x. f x = g x) ⟹ (vec-lambda f::'a^'n) = (vec-lambda
g :: 'a^'n)
 by (rule iffD1[OF vec-lambda-unique]) vector

lemma det-linear-row-setsum:
 assumes fS: finite S
 shows det ((χ i. if i = k then setsum (a i) S else c i)::'a::comm-ring-1^'n^'n)
=
   setsum (λj. det ((χ i. if i = k then a  i j else c i)::'a^'n^'n)) S
proof (induct rule: finite-induct[OF fS])
 case 1
 then show ?case
   apply simp
   unfolding setsum.empty det-row-0[of k]
   apply rule
   done
next
 case (2 x F)
 then show ?case
   by (simp add: det-row-add cong del: if-weak-cong)
qed

lemma finite-bounded-functions:
 assumes fS: finite S
 shows finite {f. (∀ i ∈ {1.. (k::nat)}. f i ∈ S) ∧ (∀ i. i ∉ {1 .. k} ⟶ f i = i)}
proof (induct k)
 case 0
 have th: {f. ∀ i. f i = i} = {id}
   by auto
 show ?case
   by (auto simp add: th)
next
 case (Suc k)
 let ?f = λ(y::nat,g) i. if i = Suc k then y else g i
 let ?S = ?f ` (S × {f. (∀ i∈{1..k}. f i ∈ S) ∧ (∀ i. i ∉ {1..k} ⟶ f i = i)})
 have ?S = {f. (∀ i∈{1.. Suc k}. f i ∈ S) ∧ (∀ i. i ∉ {1.. Suc k} ⟶ f i = i)}
```

```
    apply (auto simp add: image-iff)
    apply (rule-tac x=x (Suc k) in bexI)
    apply (rule-tac x = λi. if i = Suc k then i else x i in exI)
    apply auto
    done
  with finite-imageI[OF finite-cartesian-product[OF fS Suc.hyps(1)], of ?f]
  show ?case
    by metis
qed


lemma det-linear-rows-setsum-lemma:
  assumes fS: finite S
    and fT: finite T
  shows det ((χ i. if i ∈ T then setsum (a i) S else c i):: 'a::comm-ring-1^'n^'n)
=
    setsum (λf. det((χ i. if i ∈ T then a i (f i) else c i)::'a^'n^'n))
      {f. (∀ i ∈ T. f i ∈ S) ∧ (∀ i. i ∉ T ⟶ f i = i)}
  using fT
proof (induct T arbitrary: a c set: finite)
  case empty
  have th0: ⋀x y. (χ i. if i ∈ {} then x i else y i) = (χ i. y i)
    by vector
  from empty.prems show ?case
    unfolding th0 by (simp add: eq-id-iff)
next
  case (insert z T a c)
  let ?F = λT. {f. (∀ i ∈ T. f i ∈ S) ∧ (∀ i. i ∉ T ⟶ f i = i)}
  let ?h = λ(y,g) i. if i = z then y else g i
  let ?k = λh. (h(z),(λi. if i = z then i else h i))
  let ?s = λ k a c f. det((χ i. if i ∈ T then a i (f i) else c i)::'a^'n^'n)
  let ?c = λj i. if i = z then a i j else c i
  have thif: ⋀a b c d. (if a ∨ b then c else d) = (if a then c else if b then c else d)
    by simp
  have thif2: ⋀a b c d e. (if a then b else if c then d else e) =
    (if c then (if a then b else d) else (if a then b else e))
    by simp
  from ‹z ∉ T› have nz: ⋀i. i ∈ T ⟹ i = z ⟷ False
    by auto
  have det (χ i. if i ∈ insert z T then setsum (a i) S else c i) =
    det (χ i. if i = z then setsum (a i) S else if i ∈ T then setsum (a i) S else c i)
    unfolding insert-iff thif ..
  also have ... = (∑j∈S. det (χ i. if i ∈ T then setsum (a i) S else if i = z then
a i j else c i))
    unfolding det-linear-row-setsum[OF fS]
    apply (subst thif2)
    using nz
    apply (simp cong del: if-weak-cong cong add: if-cong)
    done
```

```
  finally have tha:
    det (χ i. if i ∈ insert z T then setsum (a i) S else c i) =
     (∑(j, f)∈S × ?F T. det (χ i. if i ∈ T then a i (f i)
                                  else if i = z then a i j
                                  else c i))
    unfolding insert.hyps unfolding setsum.cartesian-product by blast
  show ?case unfolding tha
    using ‹z ∉ T›
    by (intro setsum.reindex-bij-witness[where i=?k and j=?h])
       (auto intro!: cong[OF refl[of det]] simp: vec-eq-iff)
qed

lemma det-linear-rows-setsum:
  fixes S :: 'n::finite set
  assumes fS: finite S
  shows det (χ i. setsum (a i) S) =
    setsum (λf. det (χ i. a i (f i) :: 'a::comm-ring-1 ^ 'n^'n)) {f. ∀i. f i ∈ S}
proof −
  have th0: ⋀x y. ((χ i. if i ∈ (UNIV:: 'n set) then x i else y i) :: 'a^'n^'n) = (χ
i. x i)
    by vector
  from det-linear-rows-setsum-lemma[OF fS, of UNIV :: 'n set a, unfolded th0,
OF finite]
  show ?thesis by simp
qed

lemma matrix-mul-setsum-alt:
  fixes A B :: 'a::comm-ring-1^'n^'n
  shows A ** B = (χ i. setsum (λk. A$i$k *s B $ k) (UNIV :: 'n set))
  by (vector matrix-matrix-mult-def setsum-component)

lemma det-rows-mul:
  det((χ i. c i *s a i)::'a::comm-ring-1^'n^'n) =
    setprod (λi. c i) (UNIV:: 'n set) * det((χ i. a i)::'a^'n^'n)
proof (simp add: det-def setsum-right-distrib cong add: setprod.cong, rule set-
sum.cong)
  let ?U = UNIV :: 'n set
  let ?PU = {p. p permutes ?U}
  fix p
  assume pU: p ∈ ?PU
  let ?s = of-int (sign p)
  from pU have p: p permutes ?U
    by blast
  have setprod (λi. c i * a i $ p i) ?U = setprod c ?U * setprod (λi. a i $ p i) ?U
    unfolding setprod.distrib ..
  then show ?s * (∏xa∈?U. c xa * a xa $ p xa) =
    setprod c ?U * (?s* (∏xa∈?U. a xa $ p xa))
    by (simp add: field-simps)
qed rule
```

lemma *det-mul*:
fixes *A B* :: *'a::linordered-idom^'n^'n*
shows *det* (*A* ** *B*) = *det A* * *det B*
proof −
let *?U* = *UNIV* :: *'n set*
let *?F* = {*f*. (∀ *i*∈ *?U*. *f i* ∈ *?U*) ∧ (∀ *i*. *i* ∉ *?U* ⟶ *f i* = *i*)}
let *?PU* = {*p*. *p permutes ?U*}
have *fU*: *finite ?U*
by *simp*
have *fF*: *finite ?F*
by (*rule finite*)
{
fix *p*
assume *p*: *p permutes ?U*
have *p* ∈ *?F* **unfolding** *mem-Collect-eq permutes-in-image*[*OF p*]
using *p*[*unfolded permutes-def*] **by** *simp*
}
then have *PUF*: *?PU* ⊆ *?F* **by** *blast*
{
fix *f*
assume *fPU*: *f* ∈ *?F* − *?PU*
have *fUU*: *f* ' *?U* ⊆ *?U*
using *fPU* **by** *auto*
from *fPU* **have** *f*: ∀ *i* ∈ *?U*. *f i* ∈ *?U* ∀ *i*. *i* ∉ *?U* ⟶ *f i* = *i* ¬(∀ *y*. ∃!*x*. *f x* = *y*)
unfolding *permutes-def* **by** *auto*

let *?A* = (χ *i*. *A*$*i*$*f i* ** *s B*$*f i*) :: *'a^'n^'n*
let *?B* = (χ *i*. *B*$*f i*) :: *'a^'n^'n*
{
assume *fni*: ¬ *inj-on f ?U*
then obtain *i j* **where** *ij*: *f i* = *f j i* ≠ *j*
unfolding *inj-on-def* **by** *blast*
from *ij*
have *rth*: *row i ?B* = *row j ?B*
by (*vector row-def*)
from *det-identical-rows*[*OF ij*(*2*) *rth*]
have *det* (χ *i*. *A*$*i*$*f i* ** *s B*$*f i*) = *0*
unfolding *det-rows-mul* **by** *simp*
}
moreover
{
assume *fi*: *inj-on f ?U*
from *f fi* **have** *fith*: ⋀*i j*. *f i* = *f j* ⟹ *i* = *j*
unfolding *inj-on-def* **by** *metis*
note *fs* = *fi*[*unfolded surjective-iff-injective-gen*[*OF fU fU refl fUU*, *symmetric*]]
{
fix *y*

```
      from fs f have ∃x. f x = y
        by blast
      then obtain x where x: f x = y
        by blast
      {
        fix z
        assume z: f z = y
        from fith x z have z = x
          by metis
      }
      with x have ∃!x. f x = y
        by blast
    }
    with f(3) have det (χ i. A$i$f i *s B$f i) = 0
      by blast
  }
  ultimately have det (χ i. A$i$f i *s B$f i) = 0
    by blast
 }
 then have zth: ∀ f∈ ?F - ?PU. det (χ i. A$i$f i *s B$f i) = 0
  by simp
 {
  fix p
  assume pU: p ∈ ?PU
  from pU have p: p permutes ?U
    by blast
  let ?s = λp. of-int (sign p)
  let ?f = λq. ?s p * (∏ i∈ ?U. A $ i $ p i) * (?s q * (∏ i∈ ?U. B $ i $ q i))
  have (setsum (λq. ?s q *
      (∏ i∈ ?U. (χ i. A $ i $ p i *s B $ p i :: 'a^'n^'n) $ i $ q i)) ?PU) =
    (setsum (λq. ?s p * (∏ i∈ ?U. A $ i $ p i) * (?s q * (∏ i∈ ?U. B $ i $ q
i))) ?PU)
    unfolding sum-permutations-compose-right[OF permutes-inv[OF p], of ?f]
  proof (rule setsum.cong)
    fix q
    assume qU: q ∈ ?PU
    then have q: q permutes ?U
      by blast
    from p q have pp: permutation p and pq: permutation q
      unfolding permutation-permutes by auto
    have th00: of-int (sign p) * of-int (sign p) = (1::'a)
     ⋀a. of-int (sign p) * (of-int (sign p) * a) = a
      unfolding mult.assoc[symmetric]
      unfolding of-int-mult[symmetric]
      by (simp-all add: sign-idempotent)
    have ths: ?s q = ?s p * ?s (q ∘ inv p)
      using pp pq permutation-inverse[OF pp] sign-inverse[OF pp]
      by (simp add:  th00 ac-simps sign-idempotent sign-compose)
    have th001: setprod (λi. B$i$ q (inv p i)) ?U = setprod ((λi. B$i$ q (inv p
```

```
i)) ∘ p) ?U
      by (rule setprod-permute[OF p])
    have thp: setprod (λi. (χ i. A$i$p i *s B$p i :: 'a^'n^'n) $i $ q i) ?U =
      setprod (λi. A$i$p i) ?U * setprod (λi. B$i$ q (inv p i)) ?U
      unfolding th001 setprod.distrib[symmetric] o-def permutes-inverses[OF p]
      apply (rule setprod.cong[OF refl])
      using permutes-in-image[OF q]
      apply vector
      done
    show ?s q * setprod (λi. (((χ i. A$i$p i *s B$p i) :: 'a^'n^'n)$i$q i)) ?U =
      ?s p * (setprod (λi. A$i$p i) ?U) * (?s (q ∘ inv p) * setprod (λi. B$i$(q ∘
inv p) i) ?U)
      using ths thp pp pq permutation-inverse[OF pp] sign-inverse[OF pp]
      by (simp add: sign-nz th00 field-simps sign-idempotent sign-compose)
  qed rule
 }
 then have th2: setsum (λf. det (χ i. A$i$f i *s B$f i)) ?PU = det A * det B
  unfolding det-def setsum-product
  by (rule setsum.cong [OF refl])
 have det (A**B) = setsum (λf. det (χ i. A $ i $ f i *s B $ f i)) ?F
  unfolding matrix-mul-setsum-alt det-linear-rows-setsum[OF fU]
  by simp
 also have ... = setsum (λf. det (χ i. A$i$f i *s B$f i)) ?PU
  using setsum.mono-neutral-cong-left[OF fF PUF zth, symmetric]
  unfolding det-rows-mul by auto
 finally show ?thesis unfolding th2 .
qed
```

Relation to invertibility.

```
lemma invertible-left-inverse:
 fixes A :: real^'n^'n
 shows invertible A ⟷ (∃(B::real^'n^'n). B** A = mat 1)
 by (metis invertible-def matrix-left-right-inverse)

lemma invertible-righ-inverse:
 fixes A :: real^'n^'n
 shows invertible A ⟷ (∃(B::real^'n^'n). A** B = mat 1)
 by (metis invertible-def matrix-left-right-inverse)

lemma invertible-det-nz:
 fixes A::real ^'n^'n
 shows invertible A ⟷ det A ≠ 0
proof −
 {
  assume invertible A
  then obtain B :: real ^'n^'n where B: A ** B = mat 1
   unfolding invertible-righ-inverse by blast
  then have det (A ** B) = det (mat 1 :: real ^'n^'n)
   by simp
```

```
  then have det A ≠ 0
    by (simp add: det-mul det-I) algebra
 }
 moreover
 {
  assume H: ¬ invertible A
  let ?U = UNIV :: 'n set
  have fU: finite ?U
    by simp
  from H obtain c i where c: setsum (λi. c i *s row i A) ?U = 0
    and iU: i ∈ ?U
    and ci: c i ≠ 0
    unfolding invertible-righ-inverse
    unfolding matrix-right-invertible-independent-rows
    by blast
  have *: ⋀(a::real^'n) b. a + b = 0 ⟹ −a = b
    apply (drule-tac f=op + (− a) in cong[OF refl])
    apply (simp only: ab-left-minus add.assoc[symmetric])
    apply simp
    done
  from c ci
  have thr0: − row i A = setsum (λj. (1/ c i) *s (c j *s row j A)) (?U − {i})
    unfolding setsum.remove[OF fU iU] setsum-cmul
    apply −
    apply (rule vector-mul-lcancel-imp[OF ci])
    apply (auto simp add: field-simps)
    unfolding *
    apply rule
    done
  have thr: − row i A ∈ span {row j A| j. j ≠ i}
    unfolding thr0
    apply (rule span-setsum)
    apply simp
    apply (rule ballI)
    apply (rule span-mul [where 'a=real^'n, folded scalar-mult-eq-scaleR])+
    apply (rule span-superset)
    apply auto
    done
  let ?B = (χ k. if k = i then 0 else row k A) :: real ^'n^'n
  have thrb: row i ?B = 0 using iU by (vector row-def)
  have det A = 0
    unfolding det-row-span[OF thr, symmetric] right-minus
    unfolding det-zero-row[OF thrb] ..
 }
 ultimately show ?thesis
   by blast
qed
```

Cramer's rule.

lemma *cramer-lemma-transpose*:
fixes *A*:: *real^'n^'n*
and *x* :: *real^'n*
shows *det* ((*χ i. if i = k then setsum* (*λi. x$i ∗s row i A*) (*UNIV*::*'n set*)
else row i A)::*real^'n^'n*) = *x$k ∗ det A*
(**is** *?lhs* = *?rhs*)
proof −
let *?U* = *UNIV* :: *'n set*
let *?Uk* = *?U* − {*k*}
have *U*: *?U* = *insert k ?Uk*
by *blast*
have *fUk*: *finite ?Uk*
by *simp*
have *kUk*: *k* ∉ *?Uk*
by *simp*
have *th00*: ⋀*k s. x$k ∗s row k A* + *s* = (*x$k* − *1*) *∗s row k A* + *row k A* + *s*
by (*vector field-simps*)
have *th001*: ⋀*f k* . (*λx. if x* = *k then f k else f x*) = *f*
by *auto*
have (*χ i. row i A*) = *A* **by** (*vector row-def*)
then have *thd1*: *det* (*χ i. row i A*) = *det A*
by *simp*
have *thd0*: *det* (*χ i. if i* = *k then row k A* + (∑ *i* ∈ *?Uk. x $ i ∗s row i A*) *else row i A*) = *det A*
apply (*rule det-row-span*)
apply (*rule span-setsum*)
apply (*rule ballI*)
apply (*rule span-mul* [**where** *'a*=*real^'n*, *folded scalar-mult-eq-scaleR*])+
apply (*rule span-superset*)
apply *auto*
done
show *?lhs* = *x$k ∗ det A*
apply (*subst U*)
unfolding *setsum.insert*[*OF fUk kUk*]
apply (*subst th00*)
unfolding *add.assoc*
apply (*subst det-row-add*)
unfolding *thd0*
unfolding *det-row-mul*
unfolding *th001*[*of k λi. row i A*]
unfolding *thd1*
apply (*simp add*: *field-simps*)
done
qed

lemma *cramer-lemma*:
fixes *A* :: *real^'n^'n*
shows *det*((*χ i j. if j* = *k then* (*A ∗v x*)$*i else A$i$j*):: *real^'n^'n*) = *x$k ∗ det A*

proof −
let *?U* = *UNIV* :: *'n set*
have ∗: ⋀*c*. *setsum* (λ*i*. *c i* ∗*s row i* (*transpose A*)) *?U* = *setsum* (λ*i*. *c i* ∗*s column i A*) *?U*
by (*auto simp add*: *row-transpose intro*: *setsum.cong*)
show *?thesis*
unfolding *matrix-mult-vsum*
unfolding *cramer-lemma-transpose*[*of k x transpose A*, *unfolded det-transpose*, *symmetric*]
unfolding ∗[*of* λ*i*. *x*$*i*]
apply (*subst det-transpose*[*symmetric*])
apply (*rule cong*[*OF refl*[*of det*]])
apply (*vector transpose-def column-def row-def*)
done
qed

lemma *cramer*:
fixes *A* ::*real^'n^'n*
assumes *d0*: *det A* ≠ *0*
shows *A* ∗*v x* = *b* ⟷ *x* = (χ *k*. *det*(χ *i j*. *if j*=*k then b*$*i else A*$*i*$*j*) / *det A*)
proof −
from *d0* **obtain** *B* **where** *B*: *A* ∗∗ *B* = *mat 1 B* ∗∗ *A* = *mat 1*
unfolding *invertible-det-nz*[*symmetric*] *invertible-def*
by *blast*
have (*A* ∗∗ *B*) ∗*v b* = *b*
by (*simp add*: *B matrix-vector-mul-lid*)
then have *A* ∗*v* (*B* ∗*v b*) = *b*
by (*simp add*: *matrix-vector-mul-assoc*)
then have *xe*: ∃ *x*. *A* ∗*v x* = *b*
by *blast*
{
fix *x*
assume *x*: *A* ∗*v x* = *b*
have *x* = (χ *k*. *det*(χ *i j*. *if j*=*k then b*$*i else A*$*i*$*j*) / *det A*)
unfolding *x*[*symmetric*]
using *d0* **by** (*simp add*: *vec-eq-iff cramer-lemma field-simps*)
}
with *xe* **show** *?thesis*
by *auto*
qed

Orthogonality of a transformation and matrix.

definition *orthogonal-transformation f* ⟷ *linear f* ∧ (∀ *v w*. *f v* ∙ *f w* = *v* ∙ *w*)

lemma *orthogonal-transformation*:
orthogonal-transformation f ⟷ *linear f* ∧ (∀ (*v*::*real* ^-). *norm* (*f v*) = *norm v*)
unfolding *orthogonal-transformation-def*
apply *auto*
apply (*erule-tac x*=*v* **in** *allE*)+

apply (*simp add*: *norm-eq-sqrt-inner*)
apply (*simp add*: *dot-norm linear-add*[*symmetric*])
done

definition *orthogonal-matrix* (*Q*::'*a*::*semiring-1*^'*n*^'*n*) ⟷
transpose Q ** *Q* = *mat 1* ∧ *Q* ** *transpose Q* = *mat 1*

lemma *orthogonal-matrix*: *orthogonal-matrix* (*Q*:: *real* ^'*n*^'*n*) ⟷ *transpose Q* ** *Q* = *mat 1*
by (*metis matrix-left-right-inverse orthogonal-matrix-def*)

lemma *orthogonal-matrix-id*: *orthogonal-matrix* (*mat 1* :: -^'*n*^'*n*)
by (*simp add*: *orthogonal-matrix-def transpose-mat matrix-mul-lid*)

lemma *orthogonal-matrix-mul*:
fixes *A* :: *real* ^'*n*^'*n*
assumes *oA* : *orthogonal-matrix A*
and *oB*: *orthogonal-matrix B*
shows *orthogonal-matrix*(*A* ** *B*)
using *oA oB*
unfolding *orthogonal-matrix matrix-transpose-mul*
apply (*subst matrix-mul-assoc*)
apply (*subst matrix-mul-assoc*[*symmetric*])
apply (*simp add*: *matrix-mul-rid*)
done

lemma *orthogonal-transformation-matrix*:
fixes *f*:: *real*^'*n* ⇒ *real*^'*n*
shows *orthogonal-transformation f* ⟷ *linear f* ∧ *orthogonal-matrix*(*matrix f*)
(**is** *?lhs* ⟷ *?rhs*)
proof −
let *?mf* = *matrix f*
let *?ot* = *orthogonal-transformation f*
let *?U* = *UNIV* :: '*n set*
have *fU*: *finite ?U* **by** *simp*
let *?m1* = *mat 1* :: *real* ^'*n*^'*n*
{
assume *ot*: *?ot*
from *ot* **have** *lf*: *linear f* **and** *fd*: ∀ *v w*. *f v* ⋅ *f w* = *v* ⋅ *w*
unfolding *orthogonal-transformation-def orthogonal-matrix* **by** *blast*+
{
fix *i j*
let *?A* = *transpose ?mf* ** *?mf*
have *th0*: ⋀*b* (*x*::'*a*::*comm-ring-1*). (*if b then 1 else 0*)∗*x* = (*if b then x else 0*)
⋀*b* (*x*::'*a*::*comm-ring-1*). *x*∗(*if b then 1 else 0*) = (*if b then x else 0*)
by *simp-all*
from *fd*[*rule-format*, *of axis i 1 axis j 1*, *unfolded matrix-works*[*OF lf*, *symmetric*] *dot-matrix-vector-mul*]

```
    have ?A$i$j = ?m1 $ i $ j
    by (simp add: inner-vec-def matrix-matrix-mult-def columnvector-def rowvector-def
        th0 setsum.delta[OF fU] mat-def axis-def)
  }
  then have orthogonal-matrix ?mf
    unfolding orthogonal-matrix
    by vector
  with lf have ?rhs
    by blast
 }
 moreover
 {
  assume lf: linear f and om: orthogonal-matrix ?mf
  from lf om have ?lhs
    unfolding orthogonal-matrix-def norm-eq orthogonal-transformation
    unfolding matrix-works[OF lf, symmetric]
    apply (subst dot-matrix-vector-mul)
    apply (simp add: dot-matrix-product matrix-mul-lid)
    done
 }
 ultimately show ?thesis
  by blast
qed

lemma det-orthogonal-matrix:
 fixes Q:: 'a::linordered-idom^'n^'n
 assumes oQ: orthogonal-matrix Q
 shows det Q = 1 ∨ det Q = - 1
proof -
 have th: ⋀x::'a. x = 1 ∨ x = - 1 ⟷ x*x = 1 (is ⋀x::'a. ?ths x)
 proof -
  fix x:: 'a
  have th0: x * x - 1 = (x - 1) * (x + 1)
   by (simp add: field-simps)
  have th1: ⋀(x::'a) y. x = - y ⟷ x + y = 0
   apply (subst eq-iff-diff-eq-0)
   apply simp
   done
  have x * x = 1 ⟷ x * x - 1 = 0
   by simp
  also have ... ⟷ x = 1 ∨ x = - 1
   unfolding th0 th1 by simp
  finally show ?ths x ..
 qed
 from oQ have Q ** transpose Q = mat 1
  by (metis orthogonal-matrix-def)
 then have det (Q ** transpose Q) = det (mat 1:: 'a^'n^'n)
  by simp
 then have det Q * det Q = 1
```

```
    by (simp add: det-mul det-I det-transpose)
  then show ?thesis unfolding th .
qed
```

Linearity of scaling, and hence isometry, that preserves origin.

```
lemma scaling-linear:
  fixes f :: real ^'n ⇒ real ^'n
  assumes f0: f 0 = 0
    and fd: ∀ x y. dist (f x) (f y) = c * dist x y
  shows linear f
proof −
  {
    fix v w
    {
      fix x
      note fd[rule-format, of x 0, unfolded dist-norm f0 diff-0-right]
    }
    note th0 = this
    have f v · f w = c^2 * (v · w)
      unfolding dot-norm-neg dist-norm[symmetric]
      unfolding th0 fd[rule-format] by (simp add: power2-eq-square field-simps)}
  note fc = this
  show ?thesis
    unfolding linear-iff vector-eq[where 'a=real^'n] scalar-mult-eq-scaleR
    by (simp add: inner-add fc field-simps)
qed

lemma isometry-linear:
  f (0:: real^'n) = (0:: real^'n) ⟹ ∀ x y. dist(f x) (f y) = dist x y ⟹ linear f
  by (rule scaling-linear[where c=1]) simp-all
```

Hence another formulation of orthogonal transformation.

```
lemma orthogonal-transformation-isometry:
  orthogonal-transformation f ⟷ f(0::real^'n) = (0::real^'n) ∧ (∀ x y. dist(f x)
(f y) = dist x y)
  unfolding orthogonal-transformation
  apply (rule iffI)
  apply clarify
  apply (clarsimp simp add: linear-0 linear-sub[symmetric] dist-norm)
  apply (rule conjI)
  apply (rule isometry-linear)
  apply simp
  apply simp
  apply clarify
  apply (erule-tac x=v in allE)
  apply (erule-tac x=0 in allE)
  apply (simp add: dist-norm)
  done
```

Can extend an isometry from unit sphere.

lemma *isometry-sphere-extend*:
fixes f:: *real ^'n* $\Rightarrow$ *real ^'n*
assumes *f1*: $\forall x.$ *norm* $x = 1 \longrightarrow$ *norm* $(f\ x) = 1$
and *fd1*: $\forall\ x\ y.$ *norm* $x = 1 \longrightarrow$ *norm* $y = 1 \longrightarrow$ *dist* $(f\ x)\ (f\ y) =$ *dist* $x\ y$
shows $\exists g.$ *orthogonal-transformation* $g \wedge (\forall x.$ *norm* $x = 1 \longrightarrow g\ x = f\ x)$
proof −
{
fix $x\ y\ x'\ y'\ x0\ y0\ x0'\ y0'$ :: *real ^'n*
assume H:
$x =$ *norm* $x *_R x0$
$y =$ *norm* $y *_R y0$
$x' =$ *norm* $x *_R x0'\ y' =$ *norm* $y *_R y0'$
norm $x0 = 1$ *norm* $x0' = 1$ *norm* $y0 = 1$ *norm* $y0' = 1$
$norm(x0' - y0') = norm(x0 - y0)$
then have $*$: $x0 \cdot y0 = x0' \cdot y0' + y0' \cdot x0' - y0 \cdot x0$
by (*simp add*: *norm-eq norm-eq-1 inner-add inner-diff*)
have $norm(x' - y') = norm(x - y)$
apply (*subst H*(*1*))
apply (*subst H*(*2*))
apply (*subst H*(*3*))
apply (*subst H*(*4*))
using $H(5-9)$
apply (*simp add*: *norm-eq norm-eq-1*)
apply (*simp add*: *inner-diff scalar-mult-eq-scaleR*)
unfolding $*$
apply (*simp add*: *field-simps*)
done
}
note *th0* = *this*
let *?g* = $\lambda x.$ *if* $x = 0$ *then 0 else norm* $x *_R f$ (*inverse* (*norm* $x) *_R x$)
{
fix x:: *real ^'n*
assume *nx*: *norm* $x = 1$
have *?g* $x = f\ x$
using *nx* **by** *auto*
}
then have *thfg*: $\forall x.$ *norm* $x = 1 \longrightarrow$ *?g* $x = f\ x$
by *blast*
have *g0*: *?g* $0 = 0$
by *simp*
{
fix $x\ y$ :: *real ^'n*
{
assume $x = 0\ y = 0$
then have *dist* (*?g* x) (*?g* y) = *dist* $x\ y$
by *simp*
}
moreover
{

```
    assume x = 0 y ≠ 0
    then have dist (?g x) (?g y) = dist x y
      apply (simp add: dist-norm)
      apply (rule f1[rule-format])
      apply (simp add: field-simps)
      done
  }
  moreover
  {
    assume x ≠ 0 y = 0
    then have dist (?g x) (?g y) = dist x y
      apply (simp add: dist-norm)
      apply (rule f1[rule-format])
      apply (simp add: field-simps)
      done
  }
  moreover
  {
    assume z: x ≠ 0 y ≠ 0
    have th00:
      x = norm x *R (inverse (norm x) *R x)
      y = norm y *R (inverse (norm y) *R y)
      norm x *R f ((inverse (norm x) *R x)) = norm x *R f (inverse (norm x)
*R x)
      norm y *R f (inverse (norm y) *R y) = norm y *R f (inverse (norm y) *R
y)
      norm (inverse (norm x) *R x) = 1
      norm (f (inverse (norm x) *R x)) = 1
      norm (inverse (norm y) *R y) = 1
      norm (f (inverse (norm y) *R y)) = 1
      norm (f (inverse (norm x) *R x) − f (inverse (norm y) *R y)) =
        norm (inverse (norm x) *R x − inverse (norm y) *R y)
      using z
    by (auto simp add: field-simps intro: f1[rule-format] fd1[rule-format, unfolded
dist-norm])
    from z th0[OF th00] have dist (?g x) (?g y) = dist x y
      by (simp add: dist-norm)
  }
  ultimately have dist (?g x) (?g y) = dist x y
    by blast
 }
 note thd = this
  show ?thesis
  apply (rule exI[where x= ?g])
  unfolding orthogonal-transformation-isometry
  using g0 thfg thd
  apply metis
  done
qed
```

Rotation, reflection, rotoinversion.

definition *rotation-matrix Q* $\longleftrightarrow$ *orthogonal-matrix Q* $\wedge$ *det Q = 1*
definition *rotoinversion-matrix Q* $\longleftrightarrow$ *orthogonal-matrix Q* $\wedge$ *det Q = − 1*

lemma *orthogonal-rotation-or-rotoinversion*:
fixes *Q* :: *'a::linordered-idom^'n^'n*
shows *orthogonal-matrix Q* $\longleftrightarrow$ *rotation-matrix Q* $\vee$ *rotoinversion-matrix Q*
by (*metis rotoinversion-matrix-def rotation-matrix-def det-orthogonal-matrix*)

Explicit formulas for low dimensions.

lemma *setprod-neutral-const*: *setprod f {(1::nat)..1} = f 1*
by *simp*

lemma *setprod-2*: *setprod f {(1::nat)..2} = f 1 * f 2*
by (*simp add*: *eval-nat-numeral atLeastAtMostSuc-conv mult.commute*)

lemma *setprod-3*: *setprod f {(1::nat)..3} = f 1 * f 2 * f 3*
by (*simp add*: *eval-nat-numeral atLeastAtMostSuc-conv mult.commute*)

lemma *det-1*: *det (A::'a::comm-ring-1^1^1) = A$1$1*
by (*simp add*: *det-def of-nat-Suc sign-id*)

lemma *det-2*: *det (A::'a::comm-ring-1^2^2) = A$1$1 * A$2$2 − A$1$2 * A$2$1*
proof −
have *f12*: *finite {2::2} 1 ∉ {2::2}* **by** *auto*
show *?thesis*
unfolding *det-def UNIV-2*
unfolding *setsum-over-permutations-insert[OF f12]*
unfolding *permutes-sing*
by (*simp add*: *sign-swap-id sign-id swap-id-eq*)
qed

lemma *det-3*:
det (A::'a::comm-ring-1^3^3) =
*A$1$1 * A$2$2 * A$3$3 +*
*A$1$2 * A$2$3 * A$3$1 +*
*A$1$3 * A$2$1 * A$3$2 −*
*A$1$1 * A$2$3 * A$3$2 −*
*A$1$2 * A$2$1 * A$3$3 −*
*A$1$3 * A$2$2 * A$3$1*
proof −
have *f123*: *finite {2::3, 3} 1 ∉ {2::3, 3}*
by *auto*
have *f23*: *finite {3::3} 2 ∉ {3::3}*
by *auto*

show *?thesis*
unfolding *det-def UNIV-3*
unfolding *setsum-over-permutations-insert[OF f123]*

unfolding *setsum-over-permutations-insert*[*OF f23*]
unfolding *permutes-sing*
by (*simp add*: *sign-swap-id permutation-swap-id sign-compose sign-id swap-id-eq*)
qed

end

47 Pointwise order on product types

theory *Product-Order*
imports *Product-plus Conditionally-Complete-Lattices*
begin

47.1 Pointwise ordering

instantiation *prod* :: (*ord*, *ord*) *ord*
begin

definition
$x \leq y \longleftrightarrow fst\ x \leq fst\ y \wedge snd\ x \leq snd\ y$

definition
$(x::'a \times 'b) < y \longleftrightarrow x \leq y \wedge \neg\ y \leq x$

instance ..

end

lemma *fst-mono*: $x \leq y \Longrightarrow fst\ x \leq fst\ y$
unfolding *less-eq-prod-def* **by** *simp*

lemma *snd-mono*: $x \leq y \Longrightarrow snd\ x \leq snd\ y$
unfolding *less-eq-prod-def* **by** *simp*

lemma *Pair-mono*: $x \leq x' \Longrightarrow y \leq y' \Longrightarrow (x, y) \leq (x', y')$
unfolding *less-eq-prod-def* **by** *simp*

lemma *Pair-le* [*simp*]: $(a, b) \leq (c, d) \longleftrightarrow a \leq c \wedge b \leq d$
unfolding *less-eq-prod-def* **by** *simp*

instance *prod* :: (*preorder*, *preorder*) *preorder*
proof
fix $x\ y\ z :: 'a \times 'b$
show $x < y \longleftrightarrow x \leq y \wedge \neg\ y \leq x$
by (*rule less-prod-def*)
show $x \leq x$
unfolding *less-eq-prod-def*
by *fast*
assume $x \leq y$ **and** $y \leq z$ **thus** $x \leq z$

unfolding *less-eq-prod-def*
by (*fast elim*: *order-trans*)
qed

instance *prod* :: (*order*, *order*) *order*
by *standard auto*

47.2 Binary infimum and supremum

instantiation *prod* :: (*inf*, *inf*) *inf*
begin

definition *inf x y* = (*inf* (*fst x*) (*fst y*), *inf* (*snd x*) (*snd y*))

lemma *inf-Pair-Pair* [*simp*]: *inf* (*a*, *b*) (*c*, *d*) = (*inf a c*, *inf b d*)
unfolding *inf-prod-def* **by** *simp*

lemma *fst-inf* [*simp*]: *fst* (*inf x y*) = *inf* (*fst x*) (*fst y*)
unfolding *inf-prod-def* **by** *simp*

lemma *snd-inf* [*simp*]: *snd* (*inf x y*) = *inf* (*snd x*) (*snd y*)
unfolding *inf-prod-def* **by** *simp*

instance ..

end

instance *prod* :: (*semilattice-inf*, *semilattice-inf*) *semilattice-inf*
by *standard auto*

instantiation *prod* :: (*sup*, *sup*) *sup*
begin

definition
sup x y = (*sup* (*fst x*) (*fst y*), *sup* (*snd x*) (*snd y*))

lemma *sup-Pair-Pair* [*simp*]: *sup* (*a*, *b*) (*c*, *d*) = (*sup a c*, *sup b d*)
unfolding *sup-prod-def* **by** *simp*

lemma *fst-sup* [*simp*]: *fst* (*sup x y*) = *sup* (*fst x*) (*fst y*)
unfolding *sup-prod-def* **by** *simp*

lemma *snd-sup* [*simp*]: *snd* (*sup x y*) = *sup* (*snd x*) (*snd y*)
unfolding *sup-prod-def* **by** *simp*

instance ..

end

instance *prod* :: (*semilattice-sup*, *semilattice-sup*) *semilattice-sup*
by *standard auto*

instance *prod* :: (*lattice*, *lattice*) *lattice* **..**

instance *prod* :: (*distrib-lattice*, *distrib-lattice*) *distrib-lattice*
by *standard* (*auto simp add*: *sup-inf-distrib1*)

47.3 Top and bottom elements

instantiation *prod* :: (*top*, *top*) *top*
begin

definition
top = (*top*, *top*)

instance ..

end

lemma *fst-top* [*simp*]: *fst top* = *top*
unfolding *top-prod-def* **by** *simp*

lemma *snd-top* [*simp*]: *snd top* = *top*
unfolding *top-prod-def* **by** *simp*

lemma *Pair-top-top*: (*top*, *top*) = *top*
unfolding *top-prod-def* **by** *simp*

instance *prod* :: (*order-top*, *order-top*) *order-top*
by *standard* (*auto simp add*: *top-prod-def*)

instantiation *prod* :: (*bot*, *bot*) *bot*
begin

definition
bot = (*bot*, *bot*)

instance ..

end

lemma *fst-bot* [*simp*]: *fst bot* = *bot*
unfolding *bot-prod-def* **by** *simp*

lemma *snd-bot* [*simp*]: *snd bot* = *bot*
unfolding *bot-prod-def* **by** *simp*

lemma *Pair-bot-bot*: (*bot*, *bot*) = *bot*
unfolding *bot-prod-def* **by** *simp*

instance *prod* :: (*order-bot*, *order-bot*) *order-bot*
by *standard* (*auto simp add*: *bot-prod-def*)

instance *prod* :: (*bounded-lattice*, *bounded-lattice*) *bounded-lattice* **..**

instance *prod* :: (*boolean-algebra*, *boolean-algebra*) *boolean-algebra*
by *standard* (*auto simp add*: *prod-eqI diff-eq*)

47.4 Complete lattice operations

instantiation *prod* :: (*Inf*, *Inf*) *Inf*
begin

definition *Inf A* = (*INF x*:*A*. *fst x*, *INF x*:*A*. *snd x*)

instance ..

end

instantiation *prod* :: (*Sup*, *Sup*) *Sup*
begin

definition *Sup A* = (*SUP x*:*A*. *fst x*, *SUP x*:*A*. *snd x*)

instance ..

end

instance *prod* :: (*conditionally-complete-lattice*, *conditionally-complete-lattice*)
conditionally-complete-lattice
by *standard* (*force simp*: *less-eq-prod-def Inf-prod-def Sup-prod-def bdd-below-def bdd-above-def*
intro!: *cInf-lower cSup-upper cInf-greatest cSup-least*)+

instance *prod* :: (*complete-lattice*, *complete-lattice*) *complete-lattice*
by *standard* (*simp-all add*: *less-eq-prod-def Inf-prod-def Sup-prod-def*
INF-lower SUP-upper le-INF-iff SUP-le-iff bot-prod-def top-prod-def)

lemma *fst-Sup*: *fst* (*Sup A*) = (*SUP x*:*A*. *fst x*)
unfolding *Sup-prod-def* **by** *simp*

lemma *snd-Sup*: *snd* (*Sup A*) = (*SUP x*:*A*. *snd x*)
unfolding *Sup-prod-def* **by** *simp*

lemma *fst-Inf*: *fst* (*Inf A*) = (*INF x*:*A*. *fst x*)
unfolding *Inf-prod-def* **by** *simp*

lemma *snd-Inf*: *snd* (*Inf A*) = (*INF x*:*A*. *snd x*)
unfolding *Inf-prod-def* **by** *simp*

lemma *fst-SUP*: *fst* (*SUP x*:*A*. *f x*) = (*SUP x*:*A*. *fst* (*f x*))
using *fst-Sup* [*of f ' A*, *symmetric*] **by** (*simp add*: *comp-def*)

lemma *snd-SUP*: *snd* (*SUP x*:*A*. *f x*) = (*SUP x*:*A*. *snd* (*f x*))
using *snd-Sup* [*of f ' A*, *symmetric*] **by** (*simp add*: *comp-def*)

lemma *fst-INF*: *fst* (*INF x*:*A*. *f x*) = (*INF x*:*A*. *fst* (*f x*))
using *fst-Inf* [*of f ' A*, *symmetric*] **by** (*simp add*: *comp-def*)

lemma *snd-INF*: *snd* (*INF x*:*A*. *f x*) = (*INF x*:*A*. *snd* (*f x*))
using *snd-Inf* [*of f ' A*, *symmetric*] **by** (*simp add*: *comp-def*)

lemma *SUP-Pair*: (*SUP x*:*A*. (*f x*, *g x*)) = (*SUP x*:*A*. *f x*, *SUP x*:*A*. *g x*)
unfolding *Sup-prod-def* **by** (*simp add*: *comp-def*)

lemma *INF-Pair*: (*INF x*:*A*. (*f x*, *g x*)) = (*INF x*:*A*. *f x*, *INF x*:*A*. *g x*)
unfolding *Inf-prod-def* **by** (*simp add*: *comp-def*)

Alternative formulations for set infima and suprema over the product of two complete lattices:

lemma *INF-prod-alt-def*:
INFIMUM A f = (*INFIMUM A* (*fst o f*), *INFIMUM A* (*snd o f*))
unfolding *Inf-prod-def* **by** *simp*

lemma *SUP-prod-alt-def*:
SUPREMUM A f = (*SUPREMUM A* (*fst o f*), *SUPREMUM A* (*snd o f*))
unfolding *Sup-prod-def* **by** *simp*

47.5 Complete distributive lattices

instance *prod* :: (*complete-distrib-lattice*, *complete-distrib-lattice*) *complete-distrib-lattice*
proof (*standard*, *goal-cases*)
case *1*
then show *?case*
by (*auto simp*: *sup-prod-def Inf-prod-def INF-prod-alt-def sup-Inf sup-INF comp-def*)
next
case *2*
then show *?case*
by (*auto simp*: *inf-prod-def Sup-prod-def SUP-prod-alt-def inf-Sup inf-SUP comp-def*)
qed

end
theory *Ordered-Euclidean-Space*

imports
Cartesian-Euclidean-Space
~~/src/HOL/Library/Product-Order
begin

47.6 An ordering on euclidean spaces that will allow us to talk about intervals

class *ordered-euclidean-space* = *ord* + *inf* + *sup* + *abs* + *Inf* + *Sup* + *euclidean-space* +
assumes *eucl-le*: $x \leq y \longleftrightarrow (\forall i{\in}Basis.\ x \cdot i \leq y \cdot i)$
assumes *eucl-less-le-not-le*: $x < y \longleftrightarrow x \leq y \wedge \neg\ y \leq x$
assumes *eucl-inf*: $inf\ x\ y = (\sum i{\in}Basis.\ inf\ (x \cdot i)\ (y \cdot i) *_R i)$
assumes *eucl-sup*: $sup\ x\ y = (\sum i{\in}Basis.\ sup\ (x \cdot i)\ (y \cdot i) *_R i)$
assumes *eucl-Inf*: $Inf\ X = (\sum i{\in}Basis.\ (INF\ x{:}X.\ x \cdot i) *_R i)$
assumes *eucl-Sup*: $Sup\ X = (\sum i{\in}Basis.\ (SUP\ x{:}X.\ x \cdot i) *_R i)$
assumes *eucl-abs*: $|x| = (\sum i{\in}Basis.\ |x \cdot i| *_R i)$
begin

subclass *order*
by *standard*
(*auto simp*: *eucl-le eucl-less-le-not-le intro*!: *euclidean-eqI antisym intro*: *order.trans*)

subclass *ordered-ab-group-add-abs*
by *standard* (*auto simp*: *eucl-le inner-add-left eucl-abs abs-leI*)

subclass *ordered-real-vector*
by *standard* (*auto simp*: *eucl-le intro*!: *mult-left-mono mult-right-mono*)

subclass *lattice*
by *standard* (*auto simp*: *eucl-inf eucl-sup eucl-le*)

subclass *distrib-lattice*
by *standard* (*auto simp*: *eucl-inf eucl-sup sup-inf-distrib1 intro*!: *euclidean-eqI*)

subclass *conditionally-complete-lattice*
proof
fix $z{::}'a$ **and** $X{::}'a\ set$
assume $X \neq \{\}$
hence $\bigwedge i.\ (\lambda x.\ x \cdot i)\ `\ X \neq \{\}$ **by** *simp*
thus $(\bigwedge x.\ x \in X \Longrightarrow z \leq x) \Longrightarrow z \leq Inf\ X\ (\bigwedge x.\ x \in X \Longrightarrow x \leq z) \Longrightarrow Sup\ X \leq z$
by (*auto simp*: *eucl-Inf eucl-Sup eucl-le*
intro!: *cInf-greatest cSup-least*)
qed (*force intro*!: *cInf-lower cSup-upper*
simp: *bdd-below-def bdd-above-def preorder-class.bdd-below-def preorder-class.bdd-above-def*
eucl-Inf eucl-Sup eucl-le)+

lemma *inner-Basis-inf-left*: $i \in Basis \Longrightarrow inf\ x\ y \cdot i = inf\ (x \cdot i)\ (y \cdot i)$
and *inner-Basis-sup-left*: $i \in Basis \Longrightarrow sup\ x\ y \cdot i = sup\ (x \cdot i)\ (y \cdot i)$
by (*simp-all add*: *eucl-inf eucl-sup inner-setsum-left inner-Basis if-distrib comm-monoid-add-class.setsum.del*
cong: *if-cong*)

lemma *inner-Basis-INF-left*: $i \in Basis \Longrightarrow (INF\ x{:}X.\ f\ x) \cdot i = (INF\ x{:}X.\ f\ x \cdot i)$
and *inner-Basis-SUP-left*: $i \in Basis \Longrightarrow (SUP\ x{:}X.\ f\ x) \cdot i = (SUP\ x{:}X.\ f\ x \cdot i)$
using *eucl-Sup* [*of f ' X*] *eucl-Inf* [*of f ' X*] **by** (*simp-all add*: *comp-def*)

lemma *abs-inner*: $i \in Basis \Longrightarrow |x| \cdot i = |x \cdot i|$
by (*auto simp*: *eucl-abs*)

lemma
abs-scaleR: $|a *_R b| = |a| *_R |b|$
by (*auto simp*: *eucl-abs abs-mult intro*!: *euclidean-eqI*)

lemma *interval-inner-leI*:
assumes $x \in \{a\ ..\ b\}\ 0 \le i$
shows $a \cdot i \le x \cdot i\ x \cdot i \le b \cdot i$
using *assms*
unfolding *euclidean-inner*[*of a i*] *euclidean-inner*[*of x i*] *euclidean-inner*[*of b i*]
by (*auto intro*!: *ordered-comm-monoid-add-class.setsum-mono mult-right-mono simp*: *eucl-le*)

lemma *inner-nonneg-nonneg*:
shows $0 \le a \Longrightarrow 0 \le b \Longrightarrow 0 \le a \cdot b$
using *interval-inner-leI*[*of a 0 a b*]
by *auto*

lemma *inner-Basis-mono*:
shows $a \le b \Longrightarrow c \in Basis \Longrightarrow a \cdot c \le b \cdot c$
by (*simp add*: *eucl-le*)

lemma *Basis-nonneg*[*intro*, *simp*]: $i \in Basis \Longrightarrow 0 \le i$
by (*auto simp*: *eucl-le inner-Basis*)

lemma *Sup-eq-maximum-componentwise*:
fixes *s*::'*a set*
assumes *i*: $\bigwedge b.\ b \in Basis \Longrightarrow X \cdot b = i\ b \cdot b$
assumes *sup*: $\bigwedge b\ x.\ b \in Basis \Longrightarrow x \in s \Longrightarrow x \cdot b \le X \cdot b$
assumes *i-s*: $\bigwedge b.\ b \in Basis \Longrightarrow (i\ b \cdot b) \in (\lambda x.\ x \cdot b)\ `\ s$
shows $Sup\ s = X$
using *assms*
unfolding *eucl-Sup euclidean-representation-setsum*
by (*auto intro*!: *conditionally-complete-lattice-class.cSup-eq-maximum*)

lemma *Inf-eq-minimum-componentwise*:

assumes *i*: $\bigwedge b.\ b \in Basis \Longrightarrow X \cdot b = i\ b \cdot b$
assumes *sup*: $\bigwedge b\ x.\ b \in Basis \Longrightarrow x \in s \Longrightarrow X \cdot b \leq x \cdot b$
assumes *i-s*: $\bigwedge b.\ b \in Basis \Longrightarrow (i\ b \cdot b) \in (\lambda x.\ x \cdot b)\ `\ s$
shows $Inf\ s = X$
using *assms*
unfolding *eucl-Inf euclidean-representation-setsum*
by (*auto intro*!: *conditionally-complete-lattice-class.cInf-eq-minimum*)

end

lemma
compact-attains-Inf-componentwise:
fixes *b*::*'a*::*ordered-euclidean-space*
assumes $b \in Basis$ **assumes** $X \neq \{\}$ *compact X*
obtains *x* **where** $x \in X\ x \cdot b = Inf\ X \cdot b\ \bigwedge y.\ y \in X \Longrightarrow x \cdot b \leq y \cdot b$
proof *atomize-elim*
let *?proj* $= (\lambda x.\ x \cdot b)\ `\ X$
from *assms* **have** *compact ?proj* $?proj \neq \{\}$
by (*auto intro*!: *compact-continuous-image continuous-intros*)
from *compact-attains-inf*[*OF this*]
obtain *s x*
where *s*: $s{\in}(\lambda x.\ x \cdot b)\ `\ X\ \bigwedge t.\ t{\in}(\lambda x.\ x \cdot b)\ `\ X \Longrightarrow s \leq t$
and *x*: $x \in X\ s = x \cdot b\ \bigwedge y.\ y \in X \Longrightarrow x \cdot b \leq y \cdot b$
by *auto*
hence $Inf\ ?proj = x \cdot b$
by (*auto intro*!: *conditionally-complete-lattice-class.cInf-eq-minimum*)
hence $x \cdot b = Inf\ X \cdot b$
by (*auto simp*: *eucl-Inf inner-setsum-left inner-Basis if-distrib* ‹$b \in Basis$› *setsum.delta*
cong: *if-cong*)
with *x* **show** $\exists x.\ x \in X \wedge x \cdot b = Inf\ X \cdot b \wedge (\forall y.\ y \in X \longrightarrow x \cdot b \leq y \cdot b)$
by *blast*
qed

lemma
compact-attains-Sup-componentwise:
fixes *b*::*'a*::*ordered-euclidean-space*
assumes $b \in Basis$ **assumes** $X \neq \{\}$ *compact X*
obtains *x* **where** $x \in X\ x \cdot b = Sup\ X \cdot b\ \bigwedge y.\ y \in X \Longrightarrow y \cdot b \leq x \cdot b$
proof *atomize-elim*
let *?proj* $= (\lambda x.\ x \cdot b)\ `\ X$
from *assms* **have** *compact ?proj* $?proj \neq \{\}$
by (*auto intro*!: *compact-continuous-image continuous-intros*)
from *compact-attains-sup*[*OF this*]
obtain *s x*
where *s*: $s{\in}(\lambda x.\ x \cdot b)\ `\ X\ \bigwedge t.\ t{\in}(\lambda x.\ x \cdot b)\ `\ X \Longrightarrow t \leq s$
and *x*: $x \in X\ s = x \cdot b\ \bigwedge y.\ y \in X \Longrightarrow y \cdot b \leq x \cdot b$
by *auto*
hence $Sup\ ?proj = x \cdot b$

by (*auto intro*!: *cSup-eq-maximum*)
hence $x \cdot b = Sup\ X \cdot b$
by (*auto simp*: *eucl-Sup*[**where** *'a*=*'a*] *inner-setsum-left inner-Basis if-distrib* ‹$b \in Basis$› *setsum.delta*
cong: *if-cong*)
with x **show** $\exists x.\ x \in X \wedge x \cdot b = Sup\ X \cdot b \wedge (\forall y.\ y \in X \longrightarrow y \cdot b \leq x \cdot b)$
by *blast*
qed

lemma (**in** *order*) *atLeastatMost-empty'*[*simp*]:
$(\sim\ a <= b) \Longrightarrow \{a..b\} = \{\}$
by (*auto*)

instance *real* :: *ordered-euclidean-space*
by *standard auto*

lemma *in-Basis-prod-iff*:
fixes *i*::*'a*::*euclidean-space***'b*::*euclidean-space*
shows $i \in Basis \longleftrightarrow fst\ i = 0 \wedge snd\ i \in Basis \vee snd\ i = 0 \wedge fst\ i \in Basis$
by (*cases i*) (*auto simp*: *Basis-prod-def*)

instantiation *prod* :: (*abs*, *abs*) *abs*
begin

definition $|x| = (|fst\ x|, |snd\ x|)$

instance ..

end

instance *prod* :: (*ordered-euclidean-space*, *ordered-euclidean-space*) *ordered-euclidean-space*
by *standard*
(*auto intro*!: *add-mono simp add*: *euclidean-representation-setsum'* *Ball-def inner-prod-def*
in-Basis-prod-iff inner-Basis-inf-left inner-Basis-sup-left inner-Basis-INF-left Inf-prod-def
inner-Basis-SUP-left Sup-prod-def less-prod-def less-eq-prod-def eucl-le[**where** *'a*=*'a*]
eucl-le[**where** *'a*=*'b*] *abs-prod-def abs-inner*)

Instantiation for intervals on *ordered-euclidean-space*

lemma
fixes *a* :: *'a*::*ordered-euclidean-space*
shows *cbox-interval*: *cbox a b* = $\{a..b\}$
and *interval-cbox*: $\{a..b\}$ = *cbox a b*
and *eucl-le-atMost*: $\{x.\ \forall i{\in}Basis.\ x \cdot i <= a \cdot i\} = \{..a\}$
and *eucl-le-atLeast*: $\{x.\ \forall i{\in}Basis.\ a \cdot i <= x \cdot i\} = \{a..\}$
by (*auto simp*: *eucl-le*[**where** *'a*=*'a*] *eucl-less-def box-def cbox-def*)

lemma *closed-eucl-atLeastAtMost*[*simp*, *intro*]:
fixes *a* :: *'a*::*ordered-euclidean-space*
shows *closed* {*a*..*b*}
by (*simp add*: *cbox-interval*[*symmetric*] *closed-cbox*)

lemma *closed-eucl-atMost*[*simp*, *intro*]:
fixes *a* :: *'a*::*ordered-euclidean-space*
shows *closed* {..*a*}
by (*simp add*: *eucl-le-atMost*[*symmetric*])

lemma *closed-eucl-atLeast*[*simp*, *intro*]:
fixes *a* :: *'a*::*ordered-euclidean-space*
shows *closed* {*a*..}
by (*simp add*: *eucl-le-atLeast*[*symmetric*])

lemma *bounded-closed-interval*:
fixes *a* :: *'a*::*ordered-euclidean-space*
shows *bounded* {*a* .. *b*}
using *bounded-cbox*[*of a b*]
by (*metis interval-cbox*)

lemma *convex-closed-interval*:
fixes *a* :: *'a*::*ordered-euclidean-space*
shows *convex* {*a* .. *b*}
using *convex-box*[*of a b*]
by (*metis interval-cbox*)

lemma *image-smult-interval*:(λ*x*. *m* ∗$_R$ (*x*::-::*ordered-euclidean-space*)) ' {*a* .. *b*}
=
(*if* {*a* .. *b*} = {} *then* {} *else if* 0 ≤ *m then* {*m* ∗$_R$ *a* .. *m* ∗$_R$ *b*} *else* {*m* ∗$_R$ *b*
.. *m* ∗$_R$ *a*})
using *image-smult-cbox*[*of m a b*]
by (*simp add*: *cbox-interval*)

lemma *is-interval-closed-interval*:
is-interval {*a* .. (*b*::*'a*::*ordered-euclidean-space*)}
by (*metis cbox-interval is-interval-cbox*)

lemma *compact-interval*:
fixes *a b*::*'a*::*ordered-euclidean-space*
shows *compact* {*a* .. *b*}
by (*metis compact-cbox interval-cbox*)

lemma *homeomorphic-closed-intervals*:
fixes *a* :: *'a*::*euclidean-space* **and** *b* **and** *c* :: *'a*::*euclidean-space* **and** *d*
assumes *box a b* ≠ {} **and** *box c d* ≠ {}
shows (*cbox a b*) *homeomorphic* (*cbox c d*)
apply (*rule homeomorphic-convex-compact*)
using *assms* **apply** *auto*

done

lemma *homeomorphic-closed-intervals-real*:
fixes *a*::*real* **and** *b* **and** *c*::*real* **and** *d*
assumes *a*<*b* **and** *c*<*d*
shows {*a*..*b*} *homeomorphic* {*c*..*d*}
by (*metis assms compact-interval continuous-on-id convex-real-interval*(*5*) *emptyE homeomorphic-convex-compact interior-atLeastAtMost-real mvt*)

no-notation
eucl-less (**infix** <*e* *50*)

lemma *One-nonneg*: *0* ≤ (∑ *Basis*::*'a*::*ordered-euclidean-space*)
by (*auto intro*: *setsum-nonneg*)

lemma *content-closed-interval*:
fixes *a* :: *'a*::*ordered-euclidean-space*
assumes *a* ≤ *b*
shows *content* {*a* .. *b*} = (∏ *i*∈*Basis*. *b*•*i* − *a*•*i*)
using *content-cbox*[*of a b*] *assms*
by (*simp add*: *cbox-interval eucl-le*[**where** *'a*=*'a*])

lemma *integrable-const-ivl*[*intro*]:
fixes *a*::*'a*::*ordered-euclidean-space*
shows (λ*x*. *c*) *integrable-on* {*a* .. *b*}
unfolding *cbox-interval*[*symmetric*]
by (*rule integrable-const*)

lemma *integrable-on-subinterval*:
fixes *f* :: *'n*::*ordered-euclidean-space* ⇒ *'a*::*banach*
assumes *f integrable-on s*
and {*a* .. *b*} ⊆ *s*
shows *f integrable-on* {*a* .. *b*}
using *integrable-on-subcbox*[*of f s a b*] *assms*
by (*simp add*: *cbox-interval*)

lemma
fixes *a b*::*'a*::*ordered-euclidean-space*
shows *bdd-above-cbox*[*intro*, *simp*]: *bdd-above* (*cbox a b*)
and *bdd-below-cbox*[*intro*, *simp*]: *bdd-below* (*cbox a b*)
and *bdd-above-box*[*intro*, *simp*]: *bdd-above* (*box a b*)
and *bdd-below-box*[*intro*, *simp*]: *bdd-below* (*box a b*)
unfolding *atomize-conj*
by (*metis bdd-above-Icc bdd-above-mono bdd-below-Icc bdd-below-mono bounded-box bounded-subset-cbox interval-cbox*)

instantiation *vec* :: (*ordered-euclidean-space*, *finite*) *ordered-euclidean-space*
begin

definition *inf x y* = (χ *i*. *inf* (*x* $ *i*) (*y* $ *i*))
definition *sup x y* = (χ *i*. *sup* (*x* $ *i*) (*y* $ *i*))
definition *Inf X* = (χ *i*. (*INF x*:*X*. *x* $ *i*))
definition *Sup X* = (χ *i*. (*SUP x*:*X*. *x* $ *i*))
definition |*x*| = (χ *i*. |*x* $ *i*|)

instance
apply *standard*
unfolding *euclidean-representation-setsum'*
apply (*auto simp*: *less-eq-vec-def inf-vec-def sup-vec-def Inf-vec-def Sup-vec-def inner-axis*
Basis-vec-def inner-Basis-inf-left inner-Basis-sup-left inner-Basis-INF-left
inner-Basis-SUP-left eucl-le[**where** *'a='a*] *less-le-not-le abs-vec-def abs-inner*)
done

end

end

48 Bounded Continuous Functions

theory *Bounded-Continuous-Function*
imports *Integration*
begin

48.1 Definition

definition *bcontfun* :: (*'a*::*topological-space* ⇒ *'b*::*metric-space*) *set*
where *bcontfun* = {*f*. *continuous-on UNIV f* ∧ *bounded* (*range f*)}

typedef (**overloaded**) (*'a*, *'b*) *bcontfun* =
bcontfun :: (*'a*::*topological-space* ⇒ *'b*::*metric-space*) *set*
by (*auto simp*: *bcontfun-def intro*: *continuous-intros simp*: *bounded-def*)

lemma *bcontfunE*:
assumes *f* ∈ *bcontfun*
obtains *y* **where** *continuous-on UNIV f* ⋀*x*. *dist* (*f x*) *u* ≤ *y*
using *assms* **unfolding** *bcontfun-def*
by (*metis* (*lifting*) *bounded-any-center dist-commute mem-Collect-eq rangeI*)

lemma *bcontfunE'*:
assumes *f* ∈ *bcontfun*
obtains *y* **where** *continuous-on UNIV f* ⋀*x*. *dist* (*f x*) *undefined* ≤ *y*
using *assms bcontfunE*
by *metis*

lemma *bcontfunI*: *continuous-on UNIV f* ⟹ (⋀*x*. *dist* (*f x*) *u* ≤ *b*) ⟹ *f* ∈ *bcontfun*
unfolding *bcontfun-def*

by (*metis* (*lifting*, *no-types*) *bounded-def dist-commute mem-Collect-eq rangeE*)

lemma *bcontfunI'*: *continuous-on UNIV f* $\Longrightarrow$ ($\bigwedge x$. *dist* (*f x*) *undefined* $\leq$ *b*) $\Longrightarrow$ *f* $\in$ *bcontfun*
using *bcontfunI* **by** *metis*

lemma *continuous-on-Rep-bcontfun*[*intro*, *simp*]: *continuous-on T* (*Rep-bcontfun x*)
using *Rep-bcontfun*[*of x*]
by (*auto simp*: *bcontfun-def intro*: *continuous-on-subset*)

instantiation *bcontfun* :: (*topological-space*, *metric-space*) *metric-space*
begin

definition *dist-bcontfun* :: ('*a*, '*b*) *bcontfun* $\Rightarrow$ ('*a*, '*b*) *bcontfun* $\Rightarrow$ *real*
where *dist-bcontfun f g* = (*SUP x*. *dist* (*Rep-bcontfun f x*) (*Rep-bcontfun g x*))

definition *uniformity-bcontfun* :: (('*a*, '*b*) *bcontfun* $\times$ ('*a*, '*b*) *bcontfun*) *filter*
where *uniformity-bcontfun* = (*INF e*:{*0* <..}. *principal* {(*x*, *y*). *dist x y* < *e*})

definition *open-bcontfun* :: ('*a*, '*b*) *bcontfun set* $\Rightarrow$ *bool*
where *open-bcontfun S* = ($\forall$ *x*$\in$*S*. $\forall_F$ (*x'*, *y*) *in uniformity*. *x'* = *x* $\longrightarrow$ *y* $\in$ *S*)

lemma *dist-bounded*:
fixes *f* :: ('*a*, '*b*) *bcontfun*
shows *dist* (*Rep-bcontfun f x*) (*Rep-bcontfun g x*) $\leq$ *dist f g*
proof −
have *Rep-bcontfun f* $\in$ *bcontfun* **by** (*rule Rep-bcontfun*)
from *bcontfunE'*[*OF this*] **obtain** *y* **where** *y*:
continuous-on UNIV (*Rep-bcontfun f*)
$\bigwedge x$. *dist* (*Rep-bcontfun f x*) *undefined* $\leq$ *y*
by *auto*
have *Rep-bcontfun g* $\in$ *bcontfun* **by** (*rule Rep-bcontfun*)
from *bcontfunE'*[*OF this*] **obtain** *z* **where** *z*:
continuous-on UNIV (*Rep-bcontfun g*)
$\bigwedge x$. *dist* (*Rep-bcontfun g x*) *undefined* $\leq$ *z*
by *auto*
show *?thesis*
unfolding *dist-bcontfun-def*
proof (*intro cSUP-upper bdd-aboveI2*)
fix *x*
have *dist* (*Rep-bcontfun f x*) (*Rep-bcontfun g x*) $\leq$
dist (*Rep-bcontfun f x*) *undefined* + *dist* (*Rep-bcontfun g x*) *undefined*
by (*rule dist-triangle2*)
also have ... $\leq$ *y* + *z*
using *y*(*2*)[*of x*] *z*(*2*)[*of x*] **by** (*rule add-mono*)
finally show *dist* (*Rep-bcontfun f x*) (*Rep-bcontfun g x*) $\leq$ *y* + *z* .
qed *simp*

qed

lemma *dist-bound*:
fixes f :: $('a, 'b)$ *bcontfun*
assumes $\bigwedge x$. *dist* (*Rep-bcontfun f x*) (*Rep-bcontfun g x*) $\leq b$
shows *dist f g* $\leq b$
using *assms* **by** (*auto simp*: *dist-bcontfun-def intro*: *cSUP-least*)

lemma *dist-bounded-Abs*:
fixes $f\ g$ :: $'a \Rightarrow 'b$
assumes $f \in$ *bcontfun* $g \in$ *bcontfun*
shows *dist* (*f x*) (*g x*) $\leq$ *dist* (*Abs-bcontfun f*) (*Abs-bcontfun g*)
by (*metis Abs-bcontfun-inverse assms dist-bounded*)

lemma *const-bcontfun*: $(\lambda x{::}'a.\ b{::}'b) \in$ *bcontfun*
by (*auto intro*: *bcontfunI continuous-on-const*)

lemma *dist-fun-lt-imp-dist-val-lt*:
assumes *dist f g* $< e$
shows *dist* (*Rep-bcontfun f x*) (*Rep-bcontfun g x*) $< e$
using *dist-bounded assms* **by** (*rule le-less-trans*)

lemma *dist-val-lt-imp-dist-fun-le*:
assumes $\forall x$. *dist* (*Rep-bcontfun f x*) (*Rep-bcontfun g x*) $< e$
shows *dist f g* $\leq e$
unfolding *dist-bcontfun-def*
proof (*intro cSUP-least*)
fix x
show *dist* (*Rep-bcontfun f x*) (*Rep-bcontfun g x*) $\leq e$
using *assms*[*THEN spec*[**where** $x{=}x$]] **by** (*simp add*: *dist-norm*)
qed *simp*

instance
proof
fix $f\ g\ h$ :: $('a, 'b)$ *bcontfun*
show *dist f g* $= 0 \longleftrightarrow f = g$
proof
have $\bigwedge x$. *dist* (*Rep-bcontfun f x*) (*Rep-bcontfun g x*) $\leq$ *dist f g*
by (*rule dist-bounded*)
also assume *dist f g* $= 0$
finally show $f = g$
by (*auto simp*: *Rep-bcontfun-inject*[*symmetric*] *Abs-bcontfun-inverse*)
qed (*auto simp*: *dist-bcontfun-def intro*!: *cSup-eq*)
show *dist f g* $\leq$ *dist f h* $+$ *dist g h*
proof (*subst dist-bcontfun-def*, *safe intro*!: *cSUP-least*)
fix x
have *dist* (*Rep-bcontfun f x*) (*Rep-bcontfun g x*) $\leq$
dist (*Rep-bcontfun f x*) (*Rep-bcontfun h x*) $+$ *dist* (*Rep-bcontfun g x*) (*Rep-bcontfun h x*)

```
    by (rule dist-triangle2)
  also have dist (Rep-bcontfun f x) (Rep-bcontfun h x) ≤ dist f h
    by (rule dist-bounded)
  also have dist (Rep-bcontfun g x) (Rep-bcontfun h x) ≤ dist g h
    by (rule dist-bounded)
  finally show dist (Rep-bcontfun f x) (Rep-bcontfun g x) ≤ dist f h + dist g h
    by simp
 qed
qed (rule open-bcontfun-def uniformity-bcontfun-def)+

end

lemma closed-Pi-bcontfun:
 fixes I :: 'a::metric-space set
  and X :: 'a ⇒ 'b::complete-space set
 assumes ⋀i. i ∈ I ⟹ closed (X i)
 shows closed (Abs-bcontfun ' (Pi I X ∩ bcontfun))
 unfolding closed-sequential-limits
proof safe
 fix f l
 assume seq: ∀ n. f n ∈ Abs-bcontfun ' (Pi I X ∩ bcontfun) and lim: f ⟶ l
 have lim-fun: ∀ e>0. ∃ N. ∀ n≥N. ∀ x. dist (Rep-bcontfun (f n) x) (Rep-bcontfun
l x) < e
  using LIMSEQ-imp-Cauchy[OF lim, simplified Cauchy-def] metric-LIMSEQ-D[OF
lim]
  by (intro uniformly-cauchy-imp-uniformly-convergent[where P=λ-. True, sim-
plified])
     (metis dist-fun-lt-imp-dist-val-lt)+
 show l ∈ Abs-bcontfun ' (Pi I X ∩ bcontfun)
 proof (rule, safe)
  fix x assume x ∈ I
  then have closed (X x)
   using assms by simp
  moreover have eventually (λxa. Rep-bcontfun (f xa) x ∈ X x) sequentially
  proof (rule always-eventually, safe)
   fix i
   from seq[THEN spec, of i] ‹x ∈ I›
   show Rep-bcontfun (f i) x ∈ X x
    by (auto simp: Abs-bcontfun-inverse)
  qed
  moreover note sequentially-bot
  moreover have (λn. Rep-bcontfun (f n) x) ⟶ Rep-bcontfun l x
   using lim-fun by (blast intro!: metric-LIMSEQ-I)
  ultimately show Rep-bcontfun l x ∈ X x
   by (rule Lim-in-closed-set)
 qed (auto simp: Rep-bcontfun Rep-bcontfun-inverse)
qed
```

48.2 Complete Space

instance *bcontfun* :: (*metric-space*, *complete-space*) *complete-space*
proof
fix *f* :: *nat* ⇒ ('*a*, '*b*) *bcontfun*
assume *Cauchy f* — Cauchy equals uniform convergence
then obtain *g* **where** *limit-function*:
∀ *e*>*0*. ∃ *N*. ∀ *n*≥*N*. ∀ *x*. *dist* (*Rep-bcontfun* (*f n*) *x*) (*g x*) < *e*
using *uniformly-convergent-eq-cauchy*[*of* λ-. *True*
λ*n*. *Rep-bcontfun* (*f n*)]
unfolding *Cauchy-def*
by (*metis dist-fun-lt-imp-dist-val-lt*)

then obtain *N* **where** *fg-dist*: — for an upper bound on *g*
∀ *n*≥*N*. ∀ *x*. *dist* (*g x*) (*Rep-bcontfun* (*f n*) *x*) < *1*
by (*force simp add*: *dist-commute*)
from *bcontfunE'*[*OF Rep-bcontfun*, *of f N*] **obtain** *b* **where**
f-bound: ∀ *x*. *dist* (*Rep-bcontfun* (*f N*) *x*) *undefined* ≤ *b*
by *force*
have *g* ∈ *bcontfun* — The limit function is bounded and continuous
proof (*intro bcontfunI*)
show *continuous-on UNIV g*
using *bcontfunE*[*OF Rep-bcontfun*] *limit-function*
by (*intro continuous-uniform-limit*[**where** *f*=λ*n*. *Rep-bcontfun* (*f n*) **and** *F*=*sequentially*])
(*auto simp add*: *eventually-sequentially trivial-limit-def dist-norm*)
next
fix *x*
from *fg-dist* **have** *dist* (*g x*) (*Rep-bcontfun* (*f N*) *x*) < *1*
by (*simp add*: *dist-norm norm-minus-commute*)
with *dist-triangle*[*of g x undefined Rep-bcontfun* (*f N*) *x*]
show *dist* (*g x*) *undefined* ≤ *1* + *b* **using** *f-bound*[*THEN spec*, *of x*]
by *simp*
qed
show *convergent f*
proof (*rule convergentI*, *subst lim-sequentially*, *safe*)
— The limit function converges according to its norm
fix *e* :: *real*
assume *e* > *0*
then have *e*/*2* > *0* **by** *simp*
with *limit-function*[*THEN spec*, *ofe*/*2*]
have ∃ *N*. ∀ *n*≥*N*. ∀ *x*. *dist* (*Rep-bcontfun* (*f n*) *x*) (*g x*) < *e*/*2*
by *simp*
then obtain *N* **where** *N*: ∀ *n*≥*N*. ∀ *x*. *dist* (*Rep-bcontfun* (*f n*) *x*) (*g x*) < *e* / *2* **by** *auto*
show ∃ *N*. ∀ *n*≥*N*. *dist* (*f n*) (*Abs-bcontfun g*) < *e*
proof (*rule*, *safe*)
fix *n*
assume *N* ≤ *n*
with *N* **show** *dist* (*f n*) (*Abs-bcontfun g*) < *e*

using *dist-val-lt-imp-dist-fun-le*[*of*
f n Abs-bcontfun g e/2]
Abs-bcontfun-inverse[*OF* ‹*g* ∈ *bcontfun*›] ‹*e* > *0*› **by** *simp*
qed
qed
qed

48.3 Supremum norm for a normed vector space

instantiation *bcontfun* :: (*topological-space*, *real-normed-vector*) *real-vector*
begin

definition $-f$ = *Abs-bcontfun* ($\lambda x.$ $-$(*Rep-bcontfun f x*))

definition $f + g$ = *Abs-bcontfun* ($\lambda x.$ *Rep-bcontfun f x* + *Rep-bcontfun g x*)

definition $f - g$ = *Abs-bcontfun* ($\lambda x.$ *Rep-bcontfun f x* − *Rep-bcontfun g x*)

definition *0* = *Abs-bcontfun* ($\lambda x.$ *0*)

definition *scaleR r f* = *Abs-bcontfun* ($\lambda x.$ *r* $*_R$ *Rep-bcontfun f x*)

lemma *plus-cont*:
fixes *f g* :: *'a* ⇒ *'b*
assumes *f*: *f* ∈ *bcontfun*
and *g*: *g* ∈ *bcontfun*
shows ($\lambda x.$ *f x* + *g x*) ∈ *bcontfun*
proof −
from *bcontfunE'*[*OF f*] **obtain** *y* **where** *continuous-on UNIV f* ⋀*x*. *dist* (*f x*) *undefined* ≤ *y*
by *auto*
moreover
from *bcontfunE'*[*OF g*] **obtain** *z* **where** *continuous-on UNIV g* ⋀*x*. *dist* (*g x*) *undefined* ≤ *z*
by *auto*
ultimately show *?thesis*
proof (*intro bcontfunI*)
fix *x*
have *dist* (*f x* + *g x*) *0* = *dist* (*f x* + *g x*) (*0* + *0*)
by *simp*
also have ... ≤ *dist* (*f x*) *0* + *dist* (*g x*) *0*
by (*rule dist-triangle-add*)
also have ... ≤ *dist* (*Abs-bcontfun f*) *0* + *dist* (*Abs-bcontfun g*) *0*
unfolding *zero-bcontfun-def* **using** *assms*
by (*metis add-mono const-bcontfun dist-bounded-Abs*)
finally show *dist* (*f x* + *g x*) *0* ≤ *dist* (*Abs-bcontfun f*) *0* + *dist* (*Abs-bcontfun g*) *0* **.**
qed (*simp add*: *continuous-on-add*)
qed

lemma *Rep-bcontfun-plus*[*simp*]: *Rep-bcontfun* (*f* + *g*) *x* = *Rep-bcontfun f x* + *Rep-bcontfun g x*
by (*simp add*: *plus-bcontfun-def Abs-bcontfun-inverse plus-cont Rep-bcontfun*)

lemma *uminus-cont*:
fixes *f* :: *'a* ⇒ *'b*
assumes *f* ∈ *bcontfun*
shows (λ*x*. − *f x*) ∈ *bcontfun*
proof −
from *bcontfunE*[*OF assms*, *of 0*] **obtain** *y*
where *continuous-on UNIV f* ⋀*x*. *dist* (*f x*) *0* ≤ *y*
by *auto*
then show *?thesis*
proof (*intro bcontfunI*)
fix *x*
assume ⋀*x*. *dist* (*f x*) *0* ≤ *y*
then show *dist* (− *f x*) *0* ≤ *y*
by (*subst dist-minus*[*symmetric*]) *simp*
qed (*simp add*: *continuous-on-minus*)
qed

lemma *Rep-bcontfun-uminus*[*simp*]: *Rep-bcontfun* (− *f*) *x* = − *Rep-bcontfun f x*
by (*simp add*: *uminus-bcontfun-def Abs-bcontfun-inverse uminus-cont Rep-bcontfun*)

lemma *minus-cont*:
fixes *f g* :: *'a* ⇒ *'b*
assumes *f*: *f* ∈ *bcontfun*
and *g*: *g* ∈ *bcontfun*
shows (λ*x*. *f x* − *g x*) ∈ *bcontfun*
using *plus-cont* [*of f* − *g*] *assms*
by (*simp add*: *uminus-cont fun-Compl-def*)

lemma *Rep-bcontfun-minus*[*simp*]: *Rep-bcontfun* (*f* − *g*) *x* = *Rep-bcontfun f x* − *Rep-bcontfun g x*
by (*simp add*: *minus-bcontfun-def Abs-bcontfun-inverse minus-cont Rep-bcontfun*)

lemma *scaleR-cont*:
fixes *a* :: *real*
and *f* :: *'a* ⇒ *'b*
assumes *f* ∈ *bcontfun*
shows (λ*x*. *a* ∗*R* *f x*) ∈ *bcontfun*
proof −
from *bcontfunE*[*OF assms*, *of 0*] **obtain** *y*
where *continuous-on UNIV f* ⋀*x*. *dist* (*f x*) *0* ≤ *y*
by *auto*
then show *?thesis*
proof (*intro bcontfunI*)
fix *x*

```
    assume ⋀x. dist (f x) 0 ≤ y
    then show dist (a *R f x) 0 ≤ |a| * y
      by (metis norm-cmul-rule-thm norm-conv-dist)
  qed (simp add: continuous-intros)
qed

lemma Rep-bcontfun-scaleR[simp]: Rep-bcontfun (a *R g) x = a *R Rep-bcontfun
g x
  by (simp add: scaleR-bcontfun-def Abs-bcontfun-inverse scaleR-cont Rep-bcontfun)

instance
  by standard
   (simp-all add: plus-bcontfun-def zero-bcontfun-def minus-bcontfun-def scaleR-bcontfun-def
      Abs-bcontfun-inverse Rep-bcontfun-inverse Rep-bcontfun algebra-simps
      plus-cont const-bcontfun minus-cont scaleR-cont)

end

instantiation bcontfun :: (topological-space, real-normed-vector) real-normed-vector
begin

definition norm-bcontfun :: ('a, 'b) bcontfun ⇒ real
  where norm-bcontfun f = dist f 0

definition sgn (f::('a,'b) bcontfun) = f /R norm f

instance
proof
  fix a :: real
  fix f g :: ('a, 'b) bcontfun
  show dist f g = norm (f - g)
    by (simp add: norm-bcontfun-def dist-bcontfun-def zero-bcontfun-def
      Abs-bcontfun-inverse const-bcontfun dist-norm)
  show norm (f + g) ≤ norm f + norm g
    unfolding norm-bcontfun-def
  proof (subst dist-bcontfun-def, safe intro!: cSUP-least)
    fix x
    have dist (Rep-bcontfun (f + g) x) (Rep-bcontfun 0 x) ≤
      dist (Rep-bcontfun f x) 0 + dist (Rep-bcontfun g x) 0
    by (metis (hide-lams, no-types) Rep-bcontfun-minus Rep-bcontfun-plus diff-0-right
dist-norm
        le-less-linear less-irrefl norm-triangle-lt)
    also have dist (Rep-bcontfun f x) 0 ≤ dist f 0
      using dist-bounded[of f x 0]
      by (simp add: Abs-bcontfun-inverse const-bcontfun zero-bcontfun-def)
    also have dist (Rep-bcontfun g x) 0 ≤ dist g 0 using dist-bounded[of g x 0]
      by (simp add: Abs-bcontfun-inverse const-bcontfun zero-bcontfun-def)
    finally show dist (Rep-bcontfun (f + g) x) (Rep-bcontfun 0 x) ≤ dist f 0 +
dist g 0 by simp
```

qed
show *norm* $(a *_R f) = |a| * norm\ f$
proof −
have $|a| * Sup\ (range\ (\lambda x.\ dist\ (Rep\text{-}bcontfun\ f\ x)\ 0)) =$
$(SUP\ i{:}range\ (\lambda x.\ dist\ (Rep\text{-}bcontfun\ f\ x)\ 0).\ |a| * i)$
proof (*intro continuous-at-Sup-mono bdd-aboveI2*)
fix *x*
show *dist* (*Rep-bcontfun f x*) *0* ≤ *norm f* **using** *dist-bounded*[*of f x 0*]
by (*simp add*: *norm-bcontfun-def Abs-bcontfun-inverse zero-bcontfun-def const-bcontfun*)
qed (*auto intro*!: *monoI mult-left-mono continuous-intros*)
moreover
have *range* $(\lambda x.\ dist\ (Rep\text{-}bcontfun\ (a *_R f)\ x)\ 0) =$
$(\lambda x.\ |a| * x)$ ‘ (*range* $(\lambda x.\ dist\ (Rep\text{-}bcontfun\ f\ x)\ 0)$)
by *auto*
ultimately
show *norm* $(a *_R f) = |a| * norm\ f$
by (*simp add*: *norm-bcontfun-def dist-bcontfun-def Abs-bcontfun-inverse zero-bcontfun-def const-bcontfun image-image*)
qed
qed (*auto simp*: *norm-bcontfun-def sgn-bcontfun-def*)

end

lemma *bcontfun-normI*: *continuous-on UNIV f* ⟹ $(\bigwedge x.\ norm\ (f\ x) \le b)$ ⟹ *f* ∈ *bcontfun*
by (*metis bcontfunI dist-0-norm dist-commute*)

lemma *norm-bounded*:
fixes *f* :: ('*a*::*topological-space*, '*b*::*real-normed-vector*) *bcontfun*
shows *norm* (*Rep-bcontfun f x*) ≤ *norm f*
using *dist-bounded*[*of f x 0*]
by (*simp add*: *norm-bcontfun-def Abs-bcontfun-inverse zero-bcontfun-def const-bcontfun*)

lemma *norm-bound*:
fixes *f* :: ('*a*::*topological-space*, '*b*::*real-normed-vector*) *bcontfun*
assumes $\bigwedge x.\ norm\ (Rep\text{-}bcontfun\ f\ x) \le b$
shows *norm f* ≤ *b*
using *dist-bound*[*of f 0 b*] *assms*
by (*simp add*: *norm-bcontfun-def Abs-bcontfun-inverse zero-bcontfun-def const-bcontfun*)

48.4 Continuously Extended Functions

definition *clamp* :: '*a*::*euclidean-space* ⇒ '*a* ⇒ '*a* ⇒ '*a* **where**
clamp a b x = $(\sum i{\in}Basis.\ (if\ x{\cdot}i < a{\cdot}i\ then\ a{\cdot}i\ else\ if\ x{\cdot}i \le b{\cdot}i\ then\ x{\cdot}i\ else\ b{\cdot}i) *_R i)$

definition *ext-cont* :: ('*a*::*euclidean-space* ⇒ '*b*::*real-normed-vector*) ⇒ '*a* ⇒ '*a* ⇒

```
('a, 'b) bcontfun
  where ext-cont f a b = Abs-bcontfun ((λx. f (clamp a b x)))

lemma ext-cont-def':
  ext-cont f a b = Abs-bcontfun (λx.
    f (∑ i∈Basis. (if x•i < a•i then a•i else if x•i ≤ b•i then x•i else b•i) *R i))
  unfolding ext-cont-def clamp-def ..

lemma clamp-in-interval:
  assumes ⋀i. i ∈ Basis ⟹ a • i ≤ b • i
  shows clamp a b x ∈ cbox a b
  unfolding clamp-def
  using box-ne-empty(1)[of a b] assms by (auto simp: cbox-def)

lemma dist-clamps-le-dist-args:
  fixes x :: 'a::euclidean-space
  assumes ⋀i. i ∈ Basis ⟹ a • i ≤ b • i
  shows dist (clamp a b y) (clamp a b x) ≤ dist y x
proof −
  from box-ne-empty(1)[of a b] assms have (∀ i∈Basis. a • i ≤ b • i)
    by (simp add: cbox-def)
  then have (∑ i∈Basis. (dist (clamp a b y • i) (clamp a b x • i))^2) ≤
    (∑ i∈Basis. (dist (y • i) (x • i))^2)
   by (auto intro!: setsum-mono simp: clamp-def dist-real-def abs-le-square-iff[symmetric])
  then show ?thesis
    by (auto intro: real-sqrt-le-mono
      simp: euclidean-dist-l2[where y=x] euclidean-dist-l2[where y=clamp a b x]
setL2-def)
qed

lemma clamp-continuous-at:
  fixes f :: 'a::euclidean-space ⇒ 'b::metric-space
    and x :: 'a
  assumes ⋀i. i ∈ Basis ⟹ a • i ≤ b • i
    and f-cont: continuous-on (cbox a b) f
  shows continuous (at x) (λx. f (clamp a b x))
  unfolding continuous-at-eps-delta
proof safe
  fix x :: 'a
  fix e :: real
  assume e > 0
  moreover have clamp a b x ∈ cbox a b
    by (simp add: clamp-in-interval assms)
  moreover note f-cont[simplified continuous-on-iff]
  ultimately
  obtain d where d: 0 < d
    ⋀x'. x' ∈ cbox a b ⟹ dist x' (clamp a b x) < d ⟹ dist (f x') (f (clamp a b
x)) < e
    by force
```

show $\exists\, d{>}0.\ \forall\, x'.\ dist\ x'\ x < d \longrightarrow$
dist (*f* (*clamp a b x'*)) (*f* (*clamp a b x*)) < *e*
by (*auto intro*!: *d clamp-in-interval assms dist-clamps-le-dist-args*[*THEN le-less-trans*])
qed

lemma *clamp-continuous-on*:
fixes *f* :: *'a*::*euclidean-space* ⇒ *'b*::*metric-space*
assumes ⋀*i*. *i* ∈ *Basis* ⟹ *a* · *i* ≤ *b* · *i*
and *f-cont*: *continuous-on* (*cbox a b*) *f*
shows *continuous-on UNIV* (λ*x*. *f* (*clamp a b x*))
using *assms*
by (*auto intro*: *continuous-at-imp-continuous-on clamp-continuous-at*)

lemma *clamp-bcontfun*:
fixes *f* :: *'a*::*euclidean-space* ⇒ *'b*::*real-normed-vector*
assumes ⋀*i*. *i* ∈ *Basis* ⟹ *a* · *i* ≤ *b* · *i*
and *continuous*: *continuous-on* (*cbox a b*) *f*
shows (λ*x*. *f* (*clamp a b x*)) ∈ *bcontfun*
proof −
have *bounded* (*f* ‘ (*cbox a b*))
by (*rule compact-continuous-image*[*OF continuous compact-cbox*[*of a b*], *THEN compact-imp-bounded*])
then obtain *c* **where** *f-bound*: ∀ *x*∈*f* ‘ *cbox a b*. *norm x* ≤ *c*
by (*auto simp add*: *bounded-pos*)
show (λ*x*. *f* (*clamp a b x*)) ∈ *bcontfun*
proof (*intro bcontfun-normI*)
fix *x*
show *norm* (*f* (*clamp a b x*)) ≤ *c*
using *clamp-in-interval*[*OF assms*(*1*), *of x*] *f-bound*
by (*simp add*: *ext-cont-def*)
qed (*simp add*: *clamp-continuous-on assms*)
qed

lemma *integral-clamp*:
integral {*t0*::*real*..*clamp t0 t1 x*} *f* =
(*if x* < *t0 then 0 else if x* ≤ *t1 then integral* {*t0*..*x*} *f else integral* {*t0*..*t1*} *f*)
by (*auto simp*: *clamp-def*)

declare [[*coercion Rep-bcontfun*]]

lemma *ext-cont-cancel*[*simp*]:
fixes *x a b* :: *'a*::*euclidean-space*
assumes *x*: *x* ∈ *cbox a b*
and *continuous-on* (*cbox a b*) *f*
shows *ext-cont f a b x* = *f x*
using *assms*
unfolding *ext-cont-def*
proof (*subst Abs-bcontfun-inverse*[*OF clamp-bcontfun*])

show *f* (*clamp a b x*) = *f x*
using *x* **unfolding** *clamp-def mem-box*
by (*intro arg-cong*[**where** *f*=*f*] *euclidean-eqI*[**where** *'a*=*'a*]) (*simp add*: *not-less*)
qed (*auto simp*: *cbox-def*)

lemma *ext-cont-cong*:
assumes *t0* = *s0*
and *t1* = *s1*
and ⋀*t*. *t* ∈ (*cbox t0 t1*) ⟹ *f t* = *g t*
and *continuous-on* (*cbox t0 t1*) *f*
and *continuous-on* (*cbox s0 s1*) *g*
and *ord*: ⋀*i*. *i* ∈ *Basis* ⟹ *t0* · *i* ≤ *t1* · *i*
shows *ext-cont f t0 t1* = *ext-cont g s0 s1*
unfolding *assms ext-cont-def*
using *assms clamp-in-interval*[*OF ord*]
by (*subst Rep-bcontfun-inject*[*symmetric*]) *simp*

end

49 The Bernstein-Weierstrass and Stone-Weierstrass Theorems

By L C Paulson (2015)

theory *Weierstrass*
imports *Uniform-Limit Path-Connected*
begin

49.1 Bernstein polynomials

definition *Bernstein* :: [*nat*,*nat*,*real*] ⇒ *real* **where**
Bernstein n k x ≡ *of-nat* (*n choose k*) ∗ *x* ^ *k* ∗ (*1* − *x*) ^ (*n* − *k*)

lemma *Bernstein-nonneg*: ⟦*0* ≤ *x*; *x* ≤ *1*⟧ ⟹ *0* ≤ *Bernstein n k x*
by (*simp add*: *Bernstein-def*)

lemma *Bernstein-pos*: ⟦*0* < *x*; *x* < *1*; *k* ≤ *n*⟧ ⟹ *0* < *Bernstein n k x*
by (*simp add*: *Bernstein-def*)

lemma *sum-Bernstein* [*simp*]: (∑ *k* = *0*..*n*. *Bernstein n k x*) = *1*
using *binomial-ring* [*of x 1*−*x n*]
by (*simp add*: *Bernstein-def*)

lemma *binomial-deriv1*:
(∑*k*=*0*..*n*. (*of-nat k* ∗ *of-nat* (*n choose k*)) ∗ *a*^(*k*−*1*) ∗ *b*^(*n*−*k*)) = *real-of-nat n* ∗ (*a*+*b*) ^ (*n*−*1*)
apply (*rule DERIV-unique* [**where** *f* = λ*a*. (*a*+*b*)^*n* **and** *x*=*a*])
apply (*subst binomial-ring*)
apply (*rule derivative-eq-intros setsum.cong* | *simp*)+

done

lemma *binomial-deriv2*:
(∑ k=0..n. (*of-nat* k * *of-nat* (k−1) * *of-nat* (n *choose* k)) * a^(k−2) * b^(n−k)) =
of-nat n * *of-nat* (n−1) * (a+b::*real*) ^ (n−2)
apply (*rule DERIV-unique* [**where** f = λa. *of-nat* n * (a+b::*real*) ^ (n−1) **and** x=a])
apply (*subst binomial-deriv1* [*symmetric*])
apply (*rule derivative-eq-intros setsum.cong* | *simp add*: *Num.numeral-2-eq-2*)+
done

lemma *sum-k-Bernstein* [*simp*]: (∑ k = 0..n. *real* k * *Bernstein* n k x) = *of-nat* n * x
apply (*subst binomial-deriv1* [*of* n x 1−x, *simplified*, *symmetric*])
apply (*simp add*: *setsum-left-distrib*)
apply (*auto simp*: *Bernstein-def algebra-simps realpow-num-eq-if intro*!: *setsum.cong*)
done

lemma *sum-kk-Bernstein* [*simp*]: (∑ k = 0..n. *real* k * (*real* k − 1) * *Bernstein* n k x) = *real* n * (*real* n − 1) * x^2
proof −
have (∑ k = 0..n. *real* k * (*real* k − 1) * *Bernstein* n k x) = *real-of-nat* n * *real-of-nat* (n − *Suc* 0) * x^2
apply (*subst binomial-deriv2* [*of* n x 1−x, *simplified*, *symmetric*])
apply (*simp add*: *setsum-left-distrib*)
apply (*rule setsum.cong* [*OF refl*])
apply (*simp add*: *Bernstein-def power2-eq-square algebra-simps*)
apply (*rename-tac* k)
apply (*subgoal-tac* k = 0 ∨ k = 1 ∨ (∃ k′. k = *Suc* (*Suc* k′)))
apply (*force simp add*: *field-simps of-nat-Suc power2-eq-square*)
by *presburger*
also have ... = n * (n − 1) * x^2
by *auto*
finally show *?thesis*
by *auto*
qed

49.2 Explicit Bernstein version of the 1D Weierstrass approximation theorem

lemma *Bernstein-Weierstrass*:
fixes f :: *real* ⇒ *real*
assumes *contf*: *continuous-on* {0..1} f **and** e: 0 < e
shows ∃ N. ∀ n x. N ≤ n ∧ x ∈ {0..1}
⟶ |f x − (∑ k = 0..n. f(k/n) * *Bernstein* n k x)| < e
proof −
have *bounded* (f ‘ {0..1})

using *compact-continuous-image compact-imp-bounded contf* **by** *blast*
then obtain M **where** M: $\bigwedge x.\ 0 \le x \Longrightarrow x \le 1 \Longrightarrow |f\ x| \le M$
by (*force simp add*: *bounded-iff*)
then have *Mge0*: $0 \le M$ **by** *force*
have *ucontf*: *uniformly-continuous-on* $\{0..1\}$ f
using *compact-uniformly-continuous contf* **by** *blast*
then obtain d **where** d: $d{>}0$ $\bigwedge x\ x'.\ [\![x \in \{0..1\};\ x' \in \{0..1\};\ |x' - x| < d]\!] \Longrightarrow |f\ x' - f\ x| < e/2$
apply (*rule uniformly-continuous-onE* [**where** $e = e/2$])
using e **by** (*auto simp*: *dist-norm*)
{ fix n::*nat* **and** x::*real*
assume n: $Suc\ (nat\lceil 4{*}M/(e{*}d^2)\rceil) \le n$ **and** x: $0 \le x\ x \le 1$
have $0 < n$ **using** n **by** *simp*
have *ed0*: $-\ (e * d^2) < 0$
using e ‹$0{<}d$› **by** *simp*
also have ... $\le M * 4$
using ‹$0{\le}M$› **by** *simp*
finally have [*simp*]: *real-of-int* $(nat\ \lceil 4 * M\ /\ (e * d^2)\rceil) =$ *real-of-int* $\lceil 4 * M\ /\ (e * d^2)\rceil$
using ‹$0{\le}M$› e ‹$0{<}d$›
by (*simp add*: *of-nat-Suc field-simps*)
have $4{*}M/(e{*}d^2) + 1 \le$ *real* $(Suc\ (nat\lceil 4{*}M/(e{*}d^2)\rceil))$
by (*simp add*: *of-nat-Suc real-nat-ceiling-ge*)
also have ... $\le$ *real* n
using n **by** (*simp add*: *of-nat-Suc field-simps*)
finally have *nbig*: $4{*}M/(e{*}d^2) + 1 \le$ *real* n **.**
have *sum-bern*: $(\sum k = 0..n.\ (x - k/n)^2 * Bernstein\ n\ k\ x) = x * (1 - x)\ /\ n$
proof −
have $*$: $\bigwedge a\ b\ x$::*real*. $(a - b)^2 * x = a * (a - 1) * x + (1 - 2 * b) * a * x + b * b * x$
by (*simp add*: *algebra-simps power2-eq-square*)
have $(\sum\ k = 0..n.\ (k - n * x)^2 * Bernstein\ n\ k\ x) = n * x * (1 - x)$
apply (*simp add*: $*$ *setsum.distrib*)
apply (*simp add*: *setsum-right-distrib* [*symmetric*] *mult.assoc*)
apply (*simp add*: *algebra-simps power2-eq-square*)
done
then have $(\sum\ k = 0..n.\ (k - n * x)^2 * Bernstein\ n\ k\ x)/n\hat{}2 = x * (1 - x)\ /\ n$
by (*simp add*: *power2-eq-square*)
then show *?thesis*
using n **by** (*simp add*: *setsum-divide-distrib divide-simps mult.commute power2-commute*)
qed
{ fix k
assume k: $k \le n$
then have *kn*: $0 \le k\ /\ n\ k\ /\ n \le 1$
by (*auto simp*: *divide-simps*)
consider (*lessd*) $|x - k\ /\ n| < d$ | (*ged*) $d \le |x - k\ /\ n|$

```
      by linarith
    then have |(f x − f (k/n))| ≤ e/2 + 2 * M / d² * (x − k/n)²
    proof cases
      case lessd
      then have |(f x − f (k/n))| < e/2
        using d x kn by (simp add: abs-minus-commute)
      also have ... ≤ (e/2 + 2 * M / d² * (x − k/n)²)
        using Mge0 d by simp
      finally show ?thesis by simp
    next
      case ged
      then have dle: d² ≤ (x − k/n)²
        by (metis d(1) less-eq-real-def power2-abs power-mono)
      have |(f x − f (k/n))| ≤ |f x| + |f (k/n)|
        by (rule abs-triangle-ineq4)
      also have ... ≤ M+M
        by (meson M add-mono-thms-linordered-semiring(1) kn x)
      also have ... ≤ 2 * M * ((x − k/n)² / d²)
        apply simp
       apply (rule Rings.ordered-semiring-class.mult-left-mono [of 1 ((x − k/n)²
/ d²), simplified])
        using dle ‹d>0› ‹M≥0› by auto
      also have ... ≤ e/2 + 2 * M / d² * (x − k/n)²
        using e  by simp
      finally show ?thesis .
      qed
  } note * = this
  have |f x − (∑ k = 0..n. f(k / n) * Bernstein n k x)| ≤ |∑ k = 0..n. (f x
− f(k / n)) * Bernstein n k x|
   by (simp add: setsum-subtractf setsum-right-distrib [symmetric] algebra-simps)
  also have ... ≤ (∑ k = 0..n. (e/2 + (2 * M / d²) * (x − k / n)²) * Bernstein
n k x)
    apply (rule order-trans [OF setsum-abs setsum-mono])
    using *
    apply (simp add: abs-mult Bernstein-nonneg x mult-right-mono)
    done
  also have ... ≤ e/2 + (2 * M) / (d² * n)
   apply (simp only: setsum.distrib Rings.semiring-class.distrib-right setsum-right-distrib
[symmetric] mult.assoc sum-bern)
    using ‹d>0› x
    apply (simp add: divide-simps Mge0 mult-le-one mult-left-le)
    done
  also have ... < e
    apply (simp add: field-simps)
    using ‹d>0› nbig e ‹n>0›
    apply (simp add: divide-simps algebra-simps)
    using ed0 by linarith
  finally have |f x − (∑k = 0..n. f (real k / real n) * Bernstein n k x)| < e .
 }
```

then show *?thesis*
by *auto*
qed

49.3 General Stone-Weierstrass theorem

Source: Bruno Brosowski and Frank Deutsch. An Elementary Proof of the Stone-Weierstrass Theorem. Proceedings of the American Mathematical Society Volume 81, Number 1, January 1981. DOI: 10.2307/2043993 http://www.jstor.org/stable/2043993

locale *function-ring-on* =
fixes $R :: ('a::t2\text{-}space \Rightarrow real)\ set$ **and** $s :: 'a\ set$
assumes *compact*: *compact s*
assumes *continuous*: $f \in R \Longrightarrow$ *continuous-on s f*
assumes *add*: $f \in R \Longrightarrow g \in R \Longrightarrow (\lambda x.\ f\ x + g\ x) \in R$
assumes *mult*: $f \in R \Longrightarrow g \in R \Longrightarrow (\lambda x.\ f\ x * g\ x) \in R$
assumes *const*: $(\lambda\text{-}.\ c) \in R$
assumes *separable*: $x \in s \Longrightarrow y \in s \Longrightarrow x \neq y \Longrightarrow \exists f{\in}R.\ f\ x \neq f\ y$

begin
lemma *minus*: $f \in R \Longrightarrow (\lambda x.\ -\ f\ x) \in R$
by (*frule mult* [*OF const* [*of* -1]]) *simp*

lemma *diff*: $f \in R \Longrightarrow g \in R \Longrightarrow (\lambda x.\ f\ x - g\ x) \in R$
unfolding *diff-conv-add-uminus* **by** (*metis add minus*)

lemma *power*: $f \in R \Longrightarrow (\lambda x.\ f\ x\ \hat{}\ n) \in R$
by (*induct n*) (*auto simp*: *const mult*)

lemma *setsum*: $[\![finite\ I;\ \bigwedge i.\ i \in I \Longrightarrow f\ i \in R]\!] \Longrightarrow (\lambda x.\ \sum i \in I.\ f\ i\ x) \in R$
by (*induct I rule*: *finite-induct*; *simp add*: *const add*)

lemma *setprod*: $[\![finite\ I;\ \bigwedge i.\ i \in I \Longrightarrow f\ i \in R]\!] \Longrightarrow (\lambda x.\ \prod i \in I.\ f\ i\ x) \in R$
by (*induct I rule*: *finite-induct*; *simp add*: *const mult*)

definition *normf* $:: ('a::t2\text{-}space \Rightarrow real) \Rightarrow real$
where *normf f* $\equiv$ *SUP x*:*s*. $|f\ x|$

lemma *normf-upper*: $[\![continuous\text{-}on\ s\ f;\ x \in s]\!] \Longrightarrow |f\ x| \leq normf\ f$
apply (*simp add*: *normf-def*)
apply (*rule cSUP-upper*, *assumption*)
by (*simp add*: *bounded-imp-bdd-above compact compact-continuous-image compact-imp-bounded continuous-on-rabs*)

lemma *normf-least*: $s \neq \{\} \Longrightarrow (\bigwedge x.\ x \in s \Longrightarrow |f\ x| \leq M) \Longrightarrow normf\ f \leq M$
by (*simp add*: *normf-def cSUP-least*)

end

lemma (**in** *function-ring-on*) *one*:
assumes *U*: *open U* **and** *t0*: *t0* ∈ *s t0* ∈ *U* **and** *t1*: *t1* ∈ *s*−*U*
shows ∃ *V*. *open V* ∧ *t0* ∈ *V* ∧ *s* ∩ *V* ⊆ *U* ∧
(∀ *e*>*0*. ∃ *f* ∈ *R*. *f* ‘ *s* ⊆ {*0*..*1*} ∧ (∀ *t* ∈ *s* ∩ *V*. *f t* < *e*) ∧ (∀ *t* ∈ *s* − *U*. *f t* > *1* − *e*))
proof −
have ∃ *pt* ∈ *R*. *pt t0* = *0* ∧ *pt t* > *0* ∧ *pt* ‘ *s* ⊆ {*0*..*1*} **if** *t*: *t* ∈ *s* − *U* **for** *t*
proof −
have *t* ≠ *t0* **using** *t t0* **by** *auto*
then obtain *g* **where** *g*: *g* ∈ *R g t* ≠ *g t0*
using *separable t0* **by** (*metis Diff-subset subset-eq t*)
def *h* ≡ λ*x*. *g x* − *g t0*
have *h* ∈ *R*
unfolding *h-def* **by** (*fast intro*: *g const diff*)
then have *hsq*: (λ*w*. (*h w*)^2) ∈ *R*
by (*simp add*: *power2-eq-square mult*)
have *h t* ≠ *h t0*
by (*simp add*: *h-def g*)
then have *h t* ≠ *0*
by (*simp add*: *h-def*)
then have *ht2*: *0* < (*h t*) ^ *2*
by *simp*
also have ... ≤ *normf* (λ*w*. (*h w*)^2)
using *t normf-upper* [**where** *x*=*t*] *continuous* [*OF hsq*] **by** *force*
finally have *nfp*: *0* < *normf* (λ*w*. (*h w*)^2) **.**
def *p* ≡ λ*x*. (*1* / *normf* (λ*w*. (*h w*)^2)) ∗ (*h x*) ^ *2*
have *p* ∈ *R*
unfolding *p-def* **by** (*fast intro*: *hsq const mult*)
moreover have *p t0* = *0*
by (*simp add*: *p-def h-def*)
moreover have *p t* > *0*
using *nfp ht2* **by** (*simp add*: *p-def*)
moreover have ⋀*x*. *x* ∈ *s* ⟹ *p x* ∈ {*0*..*1*}
using *nfp normf-upper* [*OF continuous* [*OF hsq*]] **by** (*auto simp*: *p-def*)
ultimately show ∃ *pt* ∈ *R*. *pt t0* = *0* ∧ *pt t* > *0* ∧ *pt* ‘ *s* ⊆ {*0*..*1*}
by *auto*
qed
then obtain *pf* **where** *pf*: ⋀*t*. *t* ∈ *s*−*U* ⟹ *pf t* ∈ *R* ∧ *pf t t0* = *0* ∧ *pf t t* > *0*
and *pf01*: ⋀*t*. *t* ∈ *s*−*U* ⟹ *pf t* ‘ *s* ⊆ {*0*..*1*}
by *metis*
have *com-sU*: *compact* (*s*−*U*)
using *compact closed-Int-compact U* **by** (*simp add*: *Diff-eq compact-Int-closed open-closed*)
have ⋀*t*. *t* ∈ *s*−*U* ⟹ ∃ *A*. *open A* ∧ *A* ∩ *s* = {*x*∈*s*. *0* < *pf t x*}
apply (*rule open-Collect-positive*)
by (*metis pf continuous*)
then obtain *Uf* **where** *Uf*: ⋀*t*. *t* ∈ *s*−*U* ⟹ *open* (*Uf t*) ∧ (*Uf t*) ∩ *s* = {*x*∈*s*.

0 < pf t x}
by *metis*
then have *open-Uf*: $\bigwedge t.\ t \in s - U \Longrightarrow open\ (Uf\ t)$
by *blast*
have *tUft*: $\bigwedge t.\ t \in s - U \Longrightarrow t \in Uf\ t$
using *pf Uf* **by** *blast*
then have $*$: $s - U \subseteq (\bigcup x \in s - U.\ Uf\ x)$
by *blast*
obtain *subU* **where** *subU*: *subU* $\subseteq$ *s − U finite subU s − U* $\subseteq$ $(\bigcup x \in subU.$ *Uf x*)
by (*blast intro*: *that open-Uf compactE-image* [*OF com-sU - ∗*])
then have [*simp*]: *subU* $\neq$ {}
using *t1* **by** *auto*
then have *cardp*: *card subU > 0* **using** *subU*
by (*simp add*: *card-gt-0-iff*)
def $p \equiv \lambda x.\ (1\ /\ card\ subU) * (\sum t \in subU.\ pf\ t\ x)$
have *pR*: $p \in R$
unfolding *p-def* **using** *subU pf* **by** (*fast intro*: *pf const mult setsum*)
have *pt0* [*simp*]: *p t0 = 0*
using *subU pf* **by** (*auto simp*: *p-def intro*: *setsum.neutral*)
have *pt-pos*: *p t > 0* **if** *t*: $t \in s - U$ **for** *t*
proof −
obtain *i* **where** *i*: $i \in subU\ t \in Uf\ i$ **using** *subU t* **by** *blast*
show *?thesis*
using *subU i t*
apply (*clarsimp simp*: *p-def divide-simps*)
apply (*rule setsum-pos2* [*OF* ‹*finite subU*›])
using *Uf t pf01* **apply** *auto*
apply (*force elim*!: *subsetCE*)
done
qed
have *p01*: *p x* $\in \{0..1\}$ **if** *t*: $x \in s$ **for** *x*
proof −
have *0* $\leq$ *p x*
using *subU cardp t*
apply (*simp add*: *p-def divide-simps setsum-nonneg*)
apply (*rule setsum-nonneg*)
using *pf01* **by** *force*
moreover have *p x* $\leq$ *1*
using *subU cardp t*
apply (*simp add*: *p-def divide-simps setsum-nonneg*)
apply (*rule setsum-bounded-above* [**where** *'a=real* **and** *K=1*, *simplified*])
using *pf01* **by** *force*
ultimately show *?thesis*
by *auto*
qed
have *compact* (*p* ‘ (*s − U*))
by (*meson Diff-subset com-sU compact-continuous-image continuous continuous-on-subset pR*)

then have *open* (− (*p* ‘ (*s*−*U*)))
by (*simp add*: *compact-imp-closed open-Compl*)
moreover have *0* ∈ − (*p* ‘ (*s*−*U*))
by (*metis* (*no-types*) *ComplI image-iff not-less-iff-gr-or-eq pt-pos*)
ultimately obtain *delta0* **where** *delta0*: *delta0* > *0 ball 0 delta0* ⊆ − (*p* ‘ (*s*−*U*))
by (*auto simp*: *elim*!: *openE*)
then have *pt-delta*: ⋀*x*. *x* ∈ *s*−*U* ⟹ *p x* ≥ *delta0*
by (*force simp*: *ball-def dist-norm dest*: *p01*)
def *δ* ≡ *delta0*/*2*
have *delta0* ≤ *1* **using** *delta0 p01* [*of t1*] *t1*
by (*force simp*: *ball-def dist-norm dest*: *p01*)
with *delta0* **have** *δ01*: *0* < *δ δ* < *1*
by (*auto simp*: *δ-def*)
have *pt-δ*: ⋀*x*. *x* ∈ *s*−*U* ⟹ *p x* ≥ *δ*
using *pt-delta delta0* **by** (*force simp*: *δ-def*)
have ∃ *A*. *open A* ∧ *A* ∩ *s* = {*x*∈*s*. *p x* < *δ*/*2*}
by (*rule open-Collect-less-Int* [*OF continuous* [*OF pR*] *continuous-on-const*])
then obtain *V* **where** *V*: *open V V* ∩ *s* = {*x*∈*s*. *p x* < *δ*/*2*}
by *blast*
def *k* ≡ *nat*⌊*1*/*δ*⌋ + *1*
have *k*>*0* **by** (*simp add*: *k-def*)
have *k*−*1* ≤ *1*/*δ*
using *δ01* **by** (*simp add*: *k-def*)
with *δ01* **have** *k* ≤ (*1*+*δ*)/*δ*
by (*auto simp*: *algebra-simps add-divide-distrib*)
also have ... < *2*/*δ*
using *δ01* **by** (*auto simp*: *divide-simps*)
finally have *k2δ*: *k* < *2*/*δ* **.**
have *1*/*δ* < *k*
using *δ01* **unfolding** *k-def* **by** *linarith*
with *δ01 k2δ* **have** *kδ*: *1* < *k*∗*δ k*∗*δ* < *2*
by (*auto simp*: *divide-simps*)
def *q* ≡ λ*n t*. (*1* − *p t* ^ *n*) ^ (*k*^*n*)
have *qR*: *q n* ∈ *R* **for** *n*
by (*simp add*: *q-def const diff power pR*)
have *q01*: ⋀*n t*. *t* ∈ *s* ⟹ *q n t* ∈ {*0*..*1*}
using *p01* **by** (*simp add*: *q-def power-le-one algebra-simps*)
have *qt0* [*simp*]: ⋀*n*. *n*>*0* ⟹ *q n t0* = *1*
using *t0 pf* **by** (*simp add*: *q-def power-0-left*)
{ fix *t* **and** *n*::*nat*
assume *t*: *t* ∈ *s* ∩ *V*
with ‹*k*>*0*› *V* **have** *k* ∗ *p t* < *k* ∗ *δ* / *2*
by *force*
then have *1* − (*k* ∗ *δ* / *2*) ^*n* ≤ *1* − (*k* ∗ *p t*) ^*n*
using ‹*k*>*0*› *p01 t* **by** (*simp add*: *power-mono*)
also have ... ≤ *q n t*
using *Bernoulli-inequality* [*of* − ((*p t*) ^*n*) *k*^*n*]
apply (*simp add*: *q-def*)

```
    by (metis IntE atLeastAtMost-iff p01 power-le-one power-mult-distrib t)
   finally have 1 − (k ∗ δ / 2) ^ n ≤ q n t .
  } note limitV = this
  { fix t and n::nat
   assume t: t ∈ s − U
   with ‹k>0› U have k ∗ δ ≤ k ∗ p t
    by (simp add: pt-δ)
   with kδ have kpt: 1 < k ∗ p t
    by (blast intro: less-le-trans)
   have ptn-pos: 0 < p t ^ n
    using pt-pos [OF t] by simp
   have ptn-le: p t ^ n ≤ 1
    by (meson DiffE atLeastAtMost-iff p01 power-le-one t)
   have q n t = (1/(k^n ∗ (p t)^n)) ∗ (1 − p t ^ n) ^ (k^n) ∗ k^n ∗ (p t)^n
    using pt-pos [OF t] ‹k>0› by (simp add: q-def)
   also have ... ≤ (1/(k ∗ (p t))^n) ∗ (1 − p t ^ n) ^ (k^n) ∗ (1 + k^n ∗ (p
t)^n)
    using pt-pos [OF t] ‹k>0›
    apply simp
    apply (simp only: times-divide-eq-right [symmetric])
    apply (rule mult-left-mono [of 1::real, simplified])
    apply (simp-all add: power-mult-distrib)
    apply (rule zero-le-power)
    using ptn-le by linarith
   also have ... ≤ (1/(k ∗ (p t))^n) ∗ (1 − p t ^ n) ^ (k^n) ∗ (1 + (p t)^n) ^
(k^n)
    apply (rule mult-left-mono [OF Bernoulli-inequality [of p t ^ n k^n]])
    using ‹k>0› ptn-pos ptn-le
    apply (auto simp: power-mult-distrib)
    done
   also have ... = (1/(k ∗ (p t))^n) ∗ (1 − p t ^ (2∗n)) ^ (k^n)
    using pt-pos [OF t] ‹k>0›
    by (simp add: algebra-simps power-mult power2-eq-square power-mult-distrib
[symmetric])
   also have ... ≤ (1/(k ∗ (p t))^n) ∗ 1
    apply (rule mult-left-mono [OF power-le-one])
    using pt-pos ‹k>0› p01 power-le-one t apply auto
    done
   also have ... ≤ (1 / (k∗δ))^n
    using ‹k>0› δ01  power-mono pt-δ t
    by (fastforce simp: field-simps)
   finally have q n t ≤ (1 / (real k ∗ δ)) ^ n  .
  } note limitNonU = this
  def NN ≡ λe. 1 + nat ⌈max (ln e / ln (real k ∗ δ / 2)) (− ln e / ln (real k ∗
δ))⌉
  have NN: of-nat (NN e) > ln e / ln (real k ∗ δ / 2)  of-nat (NN e) > − ln e /
ln (real k ∗ δ)
            if 0<e for e
    unfolding NN-def  by linarith+
```

```
  have NN1: ⋀e. e>0 ⟹ (k * δ / 2) ^NN e < e
    apply (subst Transcendental.ln-less-cancel-iff [symmetric])
      prefer 3 apply (subst ln-realpow)
    using ‹k>0› ‹δ>0› NN  kδ
    apply (force simp add: field-simps)+
    done
  have NN0: ⋀e. e>0 ⟹ (1/(k*δ)) ^NN e < e
    apply (subst Transcendental.ln-less-cancel-iff [symmetric])
      prefer 3 apply (subst ln-realpow)
    using ‹k>0› ‹δ>0› NN kδ
    apply (force simp add: field-simps ln-div)+
    done
  { fix t and e::real
    assume e>0
    have t ∈ s ∩ V ⟹ 1 − q (NN e) t < e t ∈ s − U ⟹ q (NN e) t < e
    proof −
      assume t: t ∈ s ∩ V
      show 1 − q (NN e) t < e
       by (metis add.commute diff-le-eq not-le limitV [OF t] less-le-trans [OF NN1
[OF ‹e>0›]])
    next
      assume t: t ∈ s − U
      show q (NN e) t < e
      using  limitNonU [OF t] less-le-trans [OF NN0 [OF ‹e>0›]] not-le by blast
    qed
  } then have ⋀e. e > 0 ⟹ ∃f∈R. f ‘ s ⊆ {0..1} ∧ (∀t ∈ s ∩ V. f t < e) ∧
(∀t ∈ s − U. 1 − e < f t)
    using q01
    by (rule-tac x=λx. 1 − q (NN e) x in bexI) (auto simp: algebra-simps intro:
diff const qR)
  moreover have t0V: t0 ∈ V  s ∩ V ⊆ U
    using pt-δ t0 U V δ01  by fastforce+
  ultimately show ?thesis using V t0V
    by blast
qed
```

Non-trivial case, with A and B both non-empty

```
lemma (in function-ring-on) two-special:
  assumes A: closed A A ⊆ s a ∈ A
      and B: closed B B ⊆ s b ∈ B
      and disj: A ∩ B = {}
      and e: 0 < e e < 1
    shows ∃f ∈ R. f ‘ s ⊆ {0..1} ∧ (∀x ∈ A. f x < e) ∧ (∀x ∈ B. f x > 1 − e)
proof −
  { fix w
    assume w ∈ A
    then have open ( − B) b ∈ s w ∉ B w ∈ s
      using assms by auto
    then have ∃ V. open V ∧ w ∈ V ∧ s ∩ V ⊆ −B ∧
```

```
            (∀ e>0. ∃f ∈ R. f ‘ s ⊆ {0..1} ∧ (∀ x ∈ s ∩ V. f x < e) ∧ (∀ x ∈ s
∩ B. f x > 1 − e))
      using one [of −B w b] assms ‹w ∈ A› by simp
  }
  then obtain Vf where Vf:
        ⋀w. w ∈ A ⟹ open (Vf w) ∧ w ∈ Vf w ∧ s ∩ Vf w ⊆ −B ∧
                    (∀ e>0. ∃f ∈ R. f ‘ s ⊆ {0..1} ∧ (∀ x ∈ s ∩ Vf w. f x < e) ∧
(∀ x ∈ s ∩ B. f x > 1 − e))
    by metis
  then have open-Vf: ⋀w. w ∈ A ⟹ open (Vf w)
    by blast
  have tVft: ⋀w. w ∈ A ⟹ w ∈ Vf w
    using Vf by blast
  then have setsum-max-0: A ⊆ (⋃ x ∈ A. Vf x)
    by blast
  have com-A: compact A using A
    by (metis compact compact-Int-closed inf.absorb-iff2)
  obtain subA where subA: subA ⊆ A finite subA A ⊆ (⋃ x ∈ subA. Vf x)
    by (blast intro: that open-Vf compactE-image [OF com-A - setsum-max-0])
  then have [simp]: subA ≠ {}
    using ‹a ∈ A› by auto
  then have cardp: card subA > 0 using subA
    by (simp add: card-gt-0-iff)
  have ⋀w. w ∈ A ⟹ ∃f ∈ R. f ‘ s ⊆ {0..1} ∧ (∀ x ∈ s ∩ Vf w. f x < e / card
subA) ∧ (∀ x ∈ s ∩ B. f x > 1 − e / card subA)
    using Vf e cardp by simp
  then obtain ff where ff:
        ⋀w. w ∈ A ⟹ ff w ∈ R ∧ ff w ‘ s ⊆ {0..1} ∧
                    (∀ x ∈ s ∩ Vf w. ff w x < e / card subA) ∧ (∀ x ∈ s ∩ B. ff w
x > 1 − e / card subA)
    by metis
  def pff ≡ λx. (∏ w ∈ subA. ff w x)
  have pffR: pff ∈ R
    unfolding pff-def using subA ff by (auto simp: intro: setprod)
  moreover
  have pff01: pff x ∈ {0..1} if t: x ∈ s for x
  proof −
    have 0 ≤ pff x
      using subA cardp t
      apply (simp add: pff-def divide-simps setsum-nonneg)
      apply (rule Groups-Big.linordered-semidom-class.setprod-nonneg)
      using ff by fastforce
    moreover have pff x ≤ 1
      using subA cardp t
      apply (simp add: pff-def divide-simps setsum-nonneg)
      apply (rule setprod-mono [where g = λx. 1, simplified])
      using ff by fastforce
    ultimately show ?thesis
      by auto
```

```
  qed
  moreover
  { fix v x
    assume v: v ∈ subA and x: x ∈ Vf v x ∈ s
    from subA v have pff x = ff v x * (∏ w ∈ subA − {v}. ff w x)
      unfolding pff-def by (metis setprod.remove)
    also have ... ≤ ff v x * 1
      apply (rule Rings.ordered-semiring-class.mult-left-mono)
      apply (rule setprod-mono [where g = λx. 1, simplified])
      using ff [THEN conjunct2, THEN conjunct1] v subA x
      apply auto
      apply (meson atLeastAtMost-iff contra-subsetD imageI)
      apply (meson atLeastAtMost-iff contra-subsetD image-eqI)
      using atLeastAtMost-iff by blast
    also have ... < e / card subA
      using ff [THEN conjunct2, THEN conjunct2, THEN conjunct1] v subA x
      by auto
    also have ... ≤ e
      using cardp e by (simp add: divide-simps)
    finally have pff x < e .
  }
  then have ⋀x. x ∈ A ⟹ pff x < e
    using A Vf subA by (metis UN-E contra-subsetD)
  moreover
  { fix x
    assume x: x ∈ B
    then have x ∈ s
      using B by auto
    have 1 − e ≤ (1 − e / card subA) ^ card subA
      using Bernoulli-inequality [of −e / card subA card subA] e cardp
      by (auto simp: field-simps)
    also have ... = (∏ w ∈ subA. 1 − e / card subA)
      by (simp add: setprod-constant subA(2))
    also have ... < pff x
      apply (simp add: pff-def)
     apply (rule setprod-mono-strict [where f = λx. 1 − e / card subA, simplified])
      apply (simp-all add: subA(2))
      apply (intro ballI conjI)
      using e apply (force simp: divide-simps)
      using ff [THEN conjunct2, THEN conjunct2, THEN conjunct2] subA B x
      apply blast
      done
    finally have 1 − e < pff x .
  }
  ultimately
  show ?thesis by blast
qed

lemma (in function-ring-on) two:
```

```
  assumes A: closed A A ⊆ s
      and B: closed B B ⊆ s
      and disj: A ∩ B = {}
      and e: 0 < e e < 1
    shows ∃f ∈ R. f ' s ⊆ {0..1} ∧ (∀x ∈ A. f x < e) ∧ (∀x ∈ B. f x > 1 − e)
proof (cases A ≠ {} ∧ B ≠ {})
  case True then show ?thesis
    apply (simp add: ex-in-conv [symmetric])
    using assms
    apply safe
    apply (force simp add: intro!: two-special)
    done
next
  case False with e show ?thesis
    apply simp
    apply (erule disjE)
    apply (rule-tac [2] x=λx. 0 in bexI)
    apply (rule-tac x=λx. 1 in bexI)
    apply (auto simp: const)
    done
qed
```

The special case where f is non-negative and $e < (1::'a) \,/\, (3::'a)$

```
lemma (in function-ring-on) Stone-Weierstrass-special:
  assumes f: continuous-on s f and fpos: ⋀x. x ∈ s ⟹ f x ≥ 0
      and e: 0 < e e < 1/3
  shows ∃g ∈ R. ∀x∈s. |f x − g x| < 2∗e
proof −
  def n ≡ 1 + nat ⌈normf f / e⌉
  def A ≡ λj::nat. {x ∈ s. f x ≤ (j − 1/3)∗e}
  def B ≡ λj::nat. {x ∈ s. f x ≥ (j + 1/3)∗e}
  have ngt: (n−1) ∗ e ≥ normf f n≥1
    using e
    apply (simp-all add: n-def field-simps of-nat-Suc)
    by (metis real-nat-ceiling-ge mult.commute not-less pos-less-divide-eq)
  then have ge-fx: (n−1) ∗ e ≥ f x if x ∈ s for x
    using f normf-upper that by fastforce
  { fix j
    have A: closed (A j) A j ⊆ s
      apply (simp-all add: A-def Collect-restrict)
      apply (rule continuous-on-closed-Collect-le [OF f continuous-on-const])
      apply (simp add: compact compact-imp-closed)
      done
    have B: closed (B j) B j ⊆ s
      apply (simp-all add: B-def Collect-restrict)
      apply (rule continuous-on-closed-Collect-le [OF continuous-on-const f])
      apply (simp add: compact compact-imp-closed)
      done
    have disj: (A j) ∩ (B j) = {}
```

using *e* **by** (*auto simp*: *A-def B-def field-simps*)
have $\exists f \in R.\ f\ `\ s \subseteq \{0..1\} \wedge (\forall x \in A\ j.\ f\ x < e/n) \wedge (\forall x \in B\ j.\ f\ x > 1 - e/n)$
apply (*rule two*)
using *e A B disj ngt*
apply *simp-all*
done
}
then obtain *xf* **where** *xfR*: $\bigwedge j.\ xf\ j \in R$ **and** *xf01*: $\bigwedge j.\ xf\ j\ `\ s \subseteq \{0..1\}$
and *xfA*: $\bigwedge x\ j.\ x \in A\ j \Longrightarrow xf\ j\ x < e/n$
and *xfB*: $\bigwedge x\ j.\ x \in B\ j \Longrightarrow xf\ j\ x > 1 - e/n$
by *metis*
def $g \equiv \lambda x.\ e * (\sum i{\le}n.\ xf\ i\ x)$
have *gR*: $g \in R$
unfolding *g-def* **by** (*fast intro*: *mult const setsum xfR*)
have *gge0*: $\bigwedge x.\ x \in s \Longrightarrow g\ x \ge 0$
using *e xf01* **by** (*simp add*: *g-def zero-le-mult-iff image-subset-iff setsum-nonneg*)
have *A0*: $A\ 0 = \{\}$
using *fpos e* **by** (*fastforce simp*: *A-def*)
have *An*: $A\ n = s$
using *e ngt f normf-upper* **by** (*fastforce simp*: *A-def field-simps of-nat-diff*)
have *Asub*: $A\ j \subseteq A\ i$ **if** $i{\ge}j$ **for** *i j*
using *e that* **apply** (*clarsimp simp*: *A-def*)
apply (*erule order-trans*, *simp*)
done
{ fix *t*
assume *t*: $t \in s$
def $j \equiv LEAST\ j.\ t \in A\ j$
have *jn*: $j \le n$
using *t An* **by** (*simp add*: *Least-le j-def*)
have *Aj*: $t \in A\ j$
using *t An* **by** (*fastforce simp add*: *j-def intro*: *LeastI*)
then have *Ai*: $t \in A\ i$ **if** $i{\ge}j$ **for** *i*
using *Asub* [*OF that*] **by** *blast*
then have *fj1*: $f\ t \le (j - 1/3)*e$
by (*simp add*: *A-def*)
then have *Anj*: $t \notin A\ i$ **if** $i{<}j$ **for** *i*
using *Aj* ‹*i*<*j*›
apply (*simp add*: *j-def*)
using *not-less-Least* **by** *blast*
have *j1*: $1 \le j$
using *A0 Aj j-def not-less-eq-eq* **by** (*fastforce simp add*: *j-def*)
then have *Anj*: $t \notin A\ (j{-}1)$
using *Least-le* **by** (*fastforce simp add*: *j-def*)
then have *fj2*: $(j - 4/3)*e < f\ t$
using *j1 t* **by** (*simp add*: *A-def of-nat-diff*)
have ***: $xf\ i\ t \le e/n$ **if** $i{\ge}j$ **for** *i*
using *xfA* [*OF Ai*] *that* **by** (*simp add*: *less-eq-real-def*)
{ fix *i*

```
    assume i+2 ≤ j
    then obtain d where i+2+d = j
      using le-Suc-ex that by blast
    then have t ∈ B i
      using Anj e ge-fx [OF t] ‹1 ≤ n› fpos [OF t] t
      apply (simp add: A-def B-def)
      apply (clarsimp simp add: field-simps of-nat-diff not-le of-nat-Suc)
      apply (rule order-trans [of - e * 2 + (e * (real d * 3) + e * (real i * 3))])
      apply auto
      done
    then have xf i t > 1 - e/n
      by (rule xfB)
  } note **** = this
  have xf-le1: ⋀i. xf i t ≤ 1
    using xf01 t by force
  have g t = e * (∑ i<j. xf i t) + e * (∑ i=j..n. xf i t)
    using j1 jn e
    apply (simp add: g-def distrib-left [symmetric])
    apply (subst setsum.union-disjoint [symmetric])
    apply (auto simp: ivl-disj-un)
    done
  also have ... ≤ e*j + e * ((Suc n - j)*e/n)
    apply (rule add-mono)
    apply (simp-all only: mult-le-cancel-left-pos e)
     apply (rule setsum-bounded-above [OF xf-le1, where A = lessThan j, simplified])
    using setsum-bounded-above [of {j..n} λi. xf i t, OF ***]
    apply simp
    done
  also have ... ≤ j*e + e*(n - j + 1)*e/n
    using ‹1 ≤ n› e by (simp add: field-simps del: of-nat-Suc)
  also have ... ≤ j*e + e*e
    using ‹1 ≤ n› e j1 by (simp add: field-simps del: of-nat-Suc)
  also have ... < (j + 1/3)*e
    using e by (auto simp: field-simps)
  finally have gj1: g t < (j + 1 / 3) * e .
  have gj2: (j - 4/3)*e < g t
  proof (cases 2 ≤ j)
    case False
    then have j=1 using j1 by simp
    with t gge0 e show ?thesis by force
  next
    case True
    then have (j - 4/3)*e < (j-1)*e - e^2
      using e by (auto simp: of-nat-diff algebra-simps power2-eq-square)
    also have ... < (j-1)*e - ((j - 1)/n) * e^2
      using e True jn by (simp add: power2-eq-square field-simps)
    also have ... = e * (j-1) * (1 - e/n)
      by (simp add: power2-eq-square field-simps)
```

```
    also have ... ≤ e * (∑ i≤j−2. xf i t)
      using e
      apply simp
       apply (rule order-trans [OF - setsum-bounded-below [OF less-imp-le [OF
****]]])
      using True
      apply (simp-all add: of-nat-Suc of-nat-diff)
      done
    also have ... ≤ g t
      using jn e
      using e xf01 t
      apply (simp add: g-def zero-le-mult-iff image-subset-iff setsum-nonneg)
      apply (rule Groups-Big.setsum-mono2, auto)
      done
    finally show ?thesis .
  qed
  have |f t − g t| < 2 * e
    using fj1 fj2 gj1 gj2 by (simp add: abs-less-iff field-simps)
 }
 then show ?thesis
  by (rule-tac x=g in bexI) (auto intro: gR)
qed
```

The "unpretentious" formulation

```
lemma (in function-ring-on) Stone-Weierstrass-basic:
 assumes f: continuous-on s f and e: e > 0
 shows ∃ g ∈ R. ∀ x∈s. |f x − g x| < e
proof −
 have ∃ g ∈ R. ∀ x∈s. |(f x + normf f) − g x| < 2 * min (e/2) (1/4)
  apply (rule Stone-Weierstrass-special)
  apply (rule Limits.continuous-on-add [OF f Topological-Spaces.continuous-on-const])
  using normf-upper [OF f] apply force
  apply (simp add: e, linarith)
  done
 then obtain g where g ∈ R ∀ x∈s. |g x − (f x + normf f)| < e
  by force
 then show ?thesis
  apply (rule-tac x=λx. g x − normf f in bexI)
  apply (auto simp: algebra-simps intro: diff const)
  done
qed


theorem (in function-ring-on) Stone-Weierstrass:
 assumes f: continuous-on s f
 shows ∃ F∈UNIV → R. LIM n sequentially. F n :> uniformly-on s f
proof −
 { fix e::real
  assume e: 0 < e
```

```
    then obtain N::nat where N: 0 < N 0 < inverse N inverse N < e
      by (auto simp: real-arch-inverse [of e])
    { fix n :: nat and x :: 'a and g :: 'a ⇒ real
      assume n: N ≤ n  ∀x∈s. |f x − g x| < 1 / (1 + real n)
      assume x: x ∈ s
      have ¬ real (Suc n) < inverse e
        using ‹N ≤ n› N using less-imp-inverse-less by force
      then have 1 / (1 + real n) ≤ e
        using e by (simp add: field-simps of-nat-Suc)
      then have |f x − g x| < e
        using n(2) x by auto
    } note ∗ = this
    have ∀F n in sequentially. ∀x∈s. |f x − (SOME g. g ∈ R ∧ (∀x∈s. |f x − g
x| < 1 / (1 + real n))) x| < e
      apply (rule eventually-sequentiallyI [of N])
      apply (auto intro: someI2-bex [OF Stone-Weierstrass-basic [OF f]] ∗)
      done
  } then
  show ?thesis
    apply (rule-tac x=λn::nat. SOME g. g ∈ R ∧ (∀x∈s. |f x − g x| < 1 / (1 +
n)) in bexI)
    prefer 2 apply (force intro: someI2-bex [OF Stone-Weierstrass-basic [OF f]])
    unfolding uniform-limit-iff
    apply (auto simp: dist-norm abs-minus-commute)
    done
qed
```

A HOL Light formulation

```
corollary Stone-Weierstrass-HOL:
  fixes R :: ('a::t2-space ⇒ real) set and s :: 'a set
  assumes compact s  ⋀c. P(λx. c::real)
          ⋀f. P f ⟹ continuous-on s f
          ⋀f g. P(f) ∧ P(g) ⟹ P(λx. f x + g x)  ⋀f g. P(f) ∧ P(g) ⟹ P(λx. f
x ∗ g x)
          ⋀x y. x ∈ s ∧ y ∈ s ∧ ~(x = y) ⟹ ∃f. P(f) ∧ ~(f x = f y)
          continuous-on s f
        0 < e
    shows ∃g. P(g) ∧ (∀x ∈ s. |f x − g x| < e)
proof −
  interpret PR: function-ring-on Collect P
    apply unfold-locales
    using assms
    by auto
  show ?thesis
    using PR.Stone-Weierstrass-basic [OF ‹continuous-on s f› ‹0 < e›]
    by blast
qed
```

49.4 Polynomial functions

inductive *real-polynomial-function* :: (*'a*::*real-normed-vector* ⇒ *real*) ⇒ *bool* **where**
  linear: *bounded-linear f* ⟹ *real-polynomial-function f*
 | *const*: *real-polynomial-function* (λ*x*. *c*)
 | *add*: ⟦*real-polynomial-function f*; *real-polynomial-function g*⟧ ⟹ *real-polynomial-function* (λ*x*. *f x* + *g x*)
 | *mult*: ⟦*real-polynomial-function f*; *real-polynomial-function g*⟧ ⟹ *real-polynomial-function* (λ*x*. *f x* ∗ *g x*)

declare *real-polynomial-function.intros* [*intro*]

definition *polynomial-function* :: (*'a*::*real-normed-vector* ⇒ *'b*::*real-normed-vector*) ⇒ *bool*
 where
  polynomial-function p ≡ (∀*f*. *bounded-linear f* ⟶ *real-polynomial-function* (*f o p*))

lemma *real-polynomial-function-eq*: *real-polynomial-function p* = *polynomial-function p*
unfolding *polynomial-function-def*
proof
 assume *real-polynomial-function p*
 then show ∀*f*. *bounded-linear f* ⟶ *real-polynomial-function* (*f* ∘ *p*)
 proof (*induction p rule*: *real-polynomial-function.induct*)
  case (*linear h*) **then show** *?case*
   by (*auto simp*: *bounded-linear-compose real-polynomial-function.linear*)
 next
  case (*const h*) **then show** *?case*
   by (*simp add*: *real-polynomial-function.const*)
 next
  case (*add h*) **then show** *?case*
   by (*force simp add*: *bounded-linear-def linear-add real-polynomial-function.add*)
 next
  case (*mult h*) **then show** *?case*
   by (*force simp add*: *real-bounded-linear const real-polynomial-function.mult*)
 qed
next
 assume [*rule-format*, *OF bounded-linear-ident*]: ∀*f*. *bounded-linear f* ⟶ *real-polynomial-function* (*f* ∘ *p*)
 then show *real-polynomial-function p*
  by (*simp add*: *o-def*)
qed

lemma *polynomial-function-const* [*iff*]: *polynomial-function* (λ*x*. *c*)
 by (*simp add*: *polynomial-function-def o-def const*)

lemma *polynomial-function-bounded-linear*:
 bounded-linear f ⟹ *polynomial-function f*
by (*simp add*: *polynomial-function-def o-def bounded-linear-compose real-polynomial-function.linear*)

lemma *polynomial-function-id* [*iff*]: *polynomial-function*(λ*x*. *x*)
by (*simp add*: *polynomial-function-bounded-linear*)

lemma *polynomial-function-add* [*intro*]:
⟦*polynomial-function f*; *polynomial-function g*⟧ ⟹ *polynomial-function* (λ*x*. *f x* + *g x*)
by (*auto simp*: *polynomial-function-def bounded-linear-def linear-add real-polynomial-function.add o-def*)

lemma *polynomial-function-mult* [*intro*]:
assumes *f*: *polynomial-function f* **and** *g*: *polynomial-function g*
shows *polynomial-function* (λ*x*. *f x* ∗*R* *g x*)
using *g*
apply (*auto simp*: *polynomial-function-def bounded-linear-def Real-Vector-Spaces.linear.scaleR const real-polynomial-function.mult o-def*)
apply (*rule mult*)
using *f*
apply (*auto simp*: *real-polynomial-function-eq*)
done

lemma *polynomial-function-cmul* [*intro*]:
assumes *f*: *polynomial-function f*
shows *polynomial-function* (λ*x*. *c* ∗*R* *f x*)
by (*rule polynomial-function-mult* [*OF polynomial-function-const f*])

lemma *polynomial-function-minus* [*intro*]:
assumes *f*: *polynomial-function f*
shows *polynomial-function* (λ*x*. − *f x*)
using *polynomial-function-cmul* [*OF f*, *of* −*1*] **by** *simp*

lemma *polynomial-function-diff* [*intro*]:
⟦*polynomial-function f*; *polynomial-function g*⟧ ⟹ *polynomial-function* (λ*x*. *f x* − *g x*)
unfolding *add-uminus-conv-diff* [*symmetric*]
by (*metis polynomial-function-add polynomial-function-minus*)

lemma *polynomial-function-setsum* [*intro*]:
⟦*finite I*; ⋀*i*. *i* ∈ *I* ⟹ *polynomial-function* (λ*x*. *f x i*)⟧ ⟹ *polynomial-function* (λ*x*. *setsum* (*f x*) *I*)
by (*induct I rule*: *finite-induct*) *auto*

lemma *real-polynomial-function-minus* [*intro*]:
real-polynomial-function f ⟹ *real-polynomial-function* (λ*x*. − *f x*)
using *polynomial-function-minus* [*of f*]
by (*simp add*: *real-polynomial-function-eq*)

lemma *real-polynomial-function-diff* [*intro*]:
⟦*real-polynomial-function f*; *real-polynomial-function g*⟧ ⟹ *real-polynomial-function*

$(\lambda x.\ f\ x\ -\ g\ x)$
using *polynomial-function-diff* [*of f*]
by (*simp add*: *real-polynomial-function-eq*)

lemma *real-polynomial-function-setsum* [*intro*]:
$[\![$*finite I*; $\bigwedge i.\ i \in I \Longrightarrow$ *real-polynomial-function* $(\lambda x.\ f\ x\ i)]\!] \Longrightarrow$ *real-polynomial-function* $(\lambda x.\ setsum\ (f\ x)\ I)$
using *polynomial-function-setsum* [*of I f*]
by (*simp add*: *real-polynomial-function-eq*)

lemma *real-polynomial-function-power* [*intro*]:
real-polynomial-function f $\Longrightarrow$ *real-polynomial-function* $(\lambda x.\ f\ x\ \hat{}\ n)$
by (*induct n*) (*simp-all add*: *const mult*)

lemma *real-polynomial-function-compose* [*intro*]:
assumes *f*: *polynomial-function f* **and** *g*: *real-polynomial-function g*
shows *real-polynomial-function* (*g o f*)
using *g*
apply (*induction g rule*: *real-polynomial-function.induct*)
using *f*
apply (*simp-all add*: *polynomial-function-def o-def const add mult*)
done

lemma *polynomial-function-compose* [*intro*]:
assumes *f*: *polynomial-function f* **and** *g*: *polynomial-function g*
shows *polynomial-function* (*g o f*)
using *g real-polynomial-function-compose* [*OF f*]
by (*auto simp*: *polynomial-function-def o-def*)

lemma *setsum-max-0*:
fixes *x*::*real*
shows $(\sum i = 0..max\ m\ n.\ x\hat{}i * (if\ i \le m\ then\ a\ i\ else\ 0)) = (\sum i = 0..m.\ x\hat{}i * a\ i)$
proof −
have $(\sum i = 0..max\ m\ n.\ x\hat{}i * (if\ i \le m\ then\ a\ i\ else\ 0)) = (\sum i = 0..max\ m\ n.\ (if\ i \le m\ then\ x\hat{}i * a\ i\ else\ 0))$
by (*auto simp*: *algebra-simps intro*: *setsum.cong*)
also have ... $= (\sum i = 0..m.\ (if\ i \le m\ then\ x\hat{}i * a\ i\ else\ 0))$
by (*rule setsum.mono-neutral-right*) *auto*
also have ... $= (\sum i = 0..m.\ x\hat{}i * a\ i)$
by (*auto simp*: *algebra-simps intro*: *setsum.cong*)
finally show *?thesis* **.**
qed

lemma *real-polynomial-function-imp-setsum*:
assumes *real-polynomial-function f*
shows $\exists\, a\ n{::}nat.\ f = (\lambda x.\ \sum i{=}0..n.\ a\ i * x\ \hat{}\ i)$
using *assms*
proof (*induct f*)

```
  case (linear f)
  then show ?case
    apply (clarsimp simp add: real-bounded-linear)
    apply (rule-tac x=λi. if i=0 then 0 else c in exI)
    apply (rule-tac x=1 in exI)
    apply (simp add: mult-ac)
    done
next
  case (const c)
  show ?case
    apply (rule-tac x=λi. c in exI)
    apply (rule-tac x=0 in exI)
    apply (auto simp: mult-ac of-nat-Suc)
    done
  case (add f1 f2)
  then obtain a1 n1 a2 n2 where
    f1 = (λx. ∑ i = 0..n1. a1 i * x ^ i) f2 = (λx. ∑ i = 0..n2. a2 i * x ^ i)
    by auto
  then show ?case
    apply (rule-tac x=λi. (if i ≤ n1 then a1 i else 0) + (if i ≤ n2 then a2 i else
0) in exI)
    apply (rule-tac x=max n1 n2 in exI)
    using setsum-max-0 [where m=n1 and n=n2] setsum-max-0 [where m=n2
and n=n1]
    apply (simp add: setsum.distrib algebra-simps max.commute)
    done
  case (mult f1 f2)
  then obtain a1 n1 a2 n2 where
    f1 = (λx. ∑ i = 0..n1. a1 i * x ^ i) f2 = (λx. ∑ i = 0..n2. a2 i * x ^ i)
    by auto
  then obtain b1 b2 where
    f1 = (λx. ∑ i = 0..n1. b1 i * x ^ i) f2 = (λx. ∑ i = 0..n2. b2 i * x ^ i)
    b1 = (λi. if i≤n1 then a1 i else 0) b2 = (λi. if i≤n2 then a2 i else 0)
    by auto
  then show ?case
    apply (rule-tac x=λi. ∑ k≤i. b1 k * b2 (i − k) in exI)
    apply (rule-tac x=n1+n2 in exI)
    using polynomial-product [of n1 b1 n2 b2]
    apply (simp add: Set-Interval.atLeast0AtMost)
    done
qed

lemma real-polynomial-function-iff-setsum:
     real-polynomial-function f ⟷ (∃ a n::nat. f = (λx. ∑ i=0..n. a i * x ^ i))
  apply (rule iffI)
  apply (erule real-polynomial-function-imp-setsum)
 apply (auto simp: linear mult const real-polynomial-function-power real-polynomial-function-setsum)
  done
```

lemma *polynomial-function-iff-Basis-inner*:
fixes f :: *'a*::*real-normed-vector* ⇒ *'b*::*euclidean-space*
shows *polynomial-function f* ⟷ (∀ *b*∈*Basis*. *real-polynomial-function* (λ*x*. *inner* (*f x*) *b*))
(**is** *?lhs* = *?rhs*)
unfolding *polynomial-function-def*
proof (*intro iffI allI impI*)
assume ∀ *h*. *bounded-linear h* ⟶ *real-polynomial-function* (*h* ∘ *f*)
then show *?rhs*
by (*force simp add*: *bounded-linear-inner-left o-def*)
next
fix *h* :: *'b* ⇒ *real*
assume *rp*: ∀ *b*∈*Basis*. *real-polynomial-function* (λ*x*. *f x* • *b*) **and** *h*: *bounded-linear h*
have *real-polynomial-function* (*h* ∘ (λ*x*. ∑ *b*∈*Basis*. (*f x* • *b*) *∗*$_R$ *b*))
apply (*rule real-polynomial-function-compose* [*OF* - *linear* [*OF h*]])
using *rp*
apply (*auto simp*: *real-polynomial-function-eq polynomial-function-mult*)
done
then show *real-polynomial-function* (*h* ∘ *f*)
by (*simp add*: *euclidean-representation-setsum-fun*)
qed

49.5 Stone-Weierstrass theorem for polynomial functions

First, we need to show that they are continous, differentiable and separable.

lemma *continuous-real-polymonial-function*:
assumes *real-polynomial-function f*
shows *continuous* (*at x*) *f*
using *assms*
by (*induct f*) (*auto simp*: *linear-continuous-at*)

lemma *continuous-polymonial-function*:
fixes *f* :: *'a*::*real-normed-vector* ⇒ *'b*::*euclidean-space*
assumes *polynomial-function f*
shows *continuous* (*at x*) *f*
apply (*rule euclidean-isCont*)
using *assms* **apply** (*simp add*: *polynomial-function-iff-Basis-inner*)
apply (*force dest*: *continuous-real-polymonial-function intro*: *isCont-scaleR*)
done

lemma *continuous-on-polymonial-function*:
fixes *f* :: *'a*::*real-normed-vector* ⇒ *'b*::*euclidean-space*
assumes *polynomial-function f*
shows *continuous-on s f*
using *continuous-polymonial-function* [*OF assms*] *continuous-at-imp-continuous-on*
by *blast*

lemma *has-real-derivative-polynomial-function*:

```
  assumes real-polynomial-function p
    shows ∃p'. real-polynomial-function p' ∧
                (∀x. (p has-real-derivative (p' x)) (at x))
using assms
proof (induct p)
  case (linear p)
  then show ?case
    by (force simp: real-bounded-linear const intro!: derivative-eq-intros)
next
  case (const c)
  show ?case
    by (rule-tac x=λx. 0 in exI) auto
  case (add f1 f2)
  then obtain p1 p2 where
    real-polynomial-function p1 ⋀x. (f1 has-real-derivative p1 x) (at x)
    real-polynomial-function p2 ⋀x. (f2 has-real-derivative p2 x) (at x)
    by auto
  then show ?case
    apply (rule-tac x=λx. p1 x + p2 x in exI)
    apply (auto intro!: derivative-eq-intros)
    done
  case (mult f1 f2)
  then obtain p1 p2 where
    real-polynomial-function p1 ⋀x. (f1 has-real-derivative p1 x) (at x)
    real-polynomial-function p2 ⋀x. (f2 has-real-derivative p2 x) (at x)
    by auto
  then show ?case
    using mult
    apply (rule-tac x=λx. f1 x * p2 x + f2 x * p1 x in exI)
    apply (auto intro!: derivative-eq-intros)
    done
qed

lemma has-vector-derivative-polynomial-function:
  fixes p :: real ⇒ 'a::euclidean-space
  assumes polynomial-function p
    shows ∃p'. polynomial-function p' ∧
                (∀x. (p has-vector-derivative (p' x)) (at x))
proof −
  { fix b :: 'a
    assume b ∈ Basis
    then
     obtain p' where p': real-polynomial-function p' and pd: ⋀x. ((λx. p x • b)
has-real-derivative p' x) (at x)
      using assms [unfolded polynomial-function-iff-Basis-inner, rule-format] ‹b ∈
Basis›
       has-real-derivative-polynomial-function
       by blast
   have ∃q. polynomial-function q ∧ (∀x. ((λu. (p u • b) *R b) has-vector-derivative
```

```
q x) (at x))
      apply (rule-tac x=λx. p' x *R b in exI)
      using ‹b ∈ Basis› p'
      apply (simp add: polynomial-function-iff-Basis-inner inner-Basis)
      apply (auto intro: derivative-eq-intros pd)
      done
  }
  then obtain qf where qf:
      ⋀b. b ∈ Basis ⟹ polynomial-function (qf b)
      ⋀b x. b ∈ Basis ⟹ ((λu. (p u • b) *R b) has-vector-derivative qf b x) (at x)
    by metis
  show ?thesis
    apply (subst euclidean-representation-setsum-fun [of p, symmetric])
    apply (rule-tac x=λx. ∑ b∈Basis. qf b x in exI)
    apply (auto intro: has-vector-derivative-setsum qf)
    done
qed

lemma real-polynomial-function-separable:
  fixes x :: 'a::euclidean-space
  assumes x ≠ y shows ∃f. real-polynomial-function f ∧ f x ≠ f y
proof −
  have real-polynomial-function (λu. ∑ b∈Basis. (inner (x−u) b)^2)
    apply (rule real-polynomial-function-setsum)
   apply (auto simp: algebra-simps real-polynomial-function-power real-polynomial-function-diff
                const linear bounded-linear-inner-left)
    done
  then show ?thesis
    apply (intro exI conjI, assumption)
    using assms
   apply (force simp add: euclidean-eq-iff [of x y] setsum-nonneg-eq-0-iff algebra-simps)
    done
qed

lemma Stone-Weierstrass-real-polynomial-function:
  fixes f :: 'a::euclidean-space ⇒ real
  assumes compact s continuous-on s f 0 < e
    shows ∃g. real-polynomial-function g ∧ (∀x ∈ s. |f x − g x| < e)
proof −
  interpret PR: function-ring-on Collect real-polynomial-function
    apply unfold-locales
    using assms continuous-on-polymonial-function real-polynomial-function-eq
    apply (auto intro: real-polynomial-function-separable)
    done
  show ?thesis
    using PR.Stone-Weierstrass-basic [OF ‹continuous-on s f› ‹0 < e›]
    by blast
qed
```

lemma *Stone-Weierstrass-polynomial-function*:
fixes $f :: 'a::euclidean\text{-}space \Rightarrow 'b::euclidean\text{-}space$
assumes s: *compact* s
and f: *continuous-on* s f
and e: $0 < e$
shows $\exists g.$ *polynomial-function* $g \wedge (\forall x \in s.$ $norm(f\ x - g\ x) < e)$
proof −
{ fix $b :: 'b$
assume $b \in Basis$
have $\exists p.$ *real-polynomial-function* $p \wedge (\forall x \in s.$ $|f\ x \cdot b - p\ x| < e\ /\ DIM('b))$
apply (*rule exE* [*OF Stone-Weierstrass-real-polynomial-function* [*OF s -, of* $\lambda x.$ $f\ x \cdot b\ e\ /$ *card Basis*]])
using *e f*
apply (*auto simp*: *Euclidean-Space.DIM-positive intro*: *continuous-intros*)
done
}
then obtain *pf* **where** *pf*:
$\bigwedge b.$ $b \in Basis \Longrightarrow$ *real-polynomial-function* $(pf\ b) \wedge (\forall x \in s.$ $|f\ x \cdot b - pf\ b\ x| < e\ /\ DIM('b))$
apply (*rule bchoice* [*rule-format*, *THEN exE*])
apply *assumption*
apply (*force simp add*: *intro*: *that*)
done
have *polynomial-function* $(\lambda x.$ $\sum b \in Basis.$ $pf\ b\ x *_R b)$
using *pf*
by (*simp add*: *polynomial-function-setsum polynomial-function-mult real-polynomial-function-eq*)
moreover
{ fix x
assume $x \in s$
have $norm\ (\sum b \in Basis.\ (f\ x \cdot b) *_R b - pf\ b\ x *_R b) \leq (\sum b \in Basis.\ norm\ ((f\ x \cdot b) *_R b - pf\ b\ x *_R b))$
by (*rule norm-setsum*)
also have ... $<$ *of-nat* $DIM('b) * (e\ /\ DIM('b))$
apply (*rule setsum-bounded-above-strict*)
apply (*simp add*: *Real-Vector-Spaces.scaleR-diff-left* [*symmetric*] *pf* ‹$x \in s$›)
apply (*rule DIM-positive*)
done
also have ... $= e$
using *DIM-positive* **by** (*simp add*: *field-simps*)
finally have $norm\ (\sum b \in Basis.\ (f\ x \cdot b) *_R b - pf\ b\ x *_R b) < e$ **.**
}
ultimately
show *?thesis*
apply (*subst euclidean-representation-setsum-fun* [*of f*, *symmetric*])
apply (*rule-tac* $x = \lambda x.$ $\sum b \in Basis.$ $pf\ b\ x *_R b$ **in** *exI*)
apply (*auto simp*: *setsum-subtractf* [*symmetric*])
done
qed

49.6 Polynomial functions as paths

One application is to pick a smooth approximation to a path, or just pick a smooth path anyway in an open connected set

lemma *path-polynomial-function*:
  fixes *g* :: *real* ⇒ *'b*::*euclidean-space*
  shows *polynomial-function g* ⟹ *path g*
 by (*simp add*: *path-def continuous-on-polymonial-function*)

lemma *path-approx-polynomial-function*:
  fixes *g* :: *real* ⇒ *'b*::*euclidean-space*
  assumes *path g 0* < *e*
  shows ∃*p*. *polynomial-function p* ∧
        pathstart p = *pathstart g* ∧
        pathfinish p = *pathfinish g* ∧
        (∀ *t* ∈ {*0..1*}. *norm*(*p t* − *g t*) < *e*)
proof −
 obtain *q* **where** *poq*: *polynomial-function q* **and** *noq*: ⋀*x*. *x* ∈ {*0..1*} ⟹ *norm* (*g x* − *q x*) < *e*/*4*
  using *Stone-Weierstrass-polynomial-function* [*of* {*0..1*} *g e*/*4*] *assms*
  by (*auto simp*: *path-def*)
 have *pf*: *polynomial-function* (λ*t*. *q t* + (*g 0* − *q 0*) + *t* ∗*R* (*g 1* − *q 1* − (*g 0* − *q 0*)))
  by (*force simp add*: *poq*)
 **have* : ⋀*t*. *t* ∈ {*0..1*} ⟹ *norm* (((*q t* − *g t*) + (*g 0* − *q 0*)) + (*t* ∗*R* (*g 1* − *q 1*) + *t* ∗*R* (*q 0* − *g 0*))) < (*e*/*4* + *e*/*4*) + (*e*/*4*+*e*/*4*)
  apply (*intro Real-Vector-Spaces.norm-add-less*)
  using *noq*
  apply (*auto simp*: *norm-minus-commute intro*: *le-less-trans* [*OF mult-left-le-one-le noq*] *simp del*: *less-divide-eq-numeral1*)
  done
 show *?thesis*
  apply (*intro exI conjI*)
  apply (*rule pf*)
  using ∗
  apply (*auto simp add*: *pathstart-def pathfinish-def algebra-simps*)
  done
qed

lemma *connected-open-polynomial-connected*:
 fixes *s* :: *'a*::*euclidean-space set*
 assumes *s*: *open s connected s*
   and *x* ∈ *s y* ∈ *s*
  shows ∃*g*. *polynomial-function g* ∧ *path-image g* ⊆ *s* ∧
        pathstart g = *x* ∧ *pathfinish g* = *y*
proof −
 have *path-connected s* **using** *assms*
  by (*simp add*: *connected-open-path-connected*)
 with ‹*x* ∈ *s*› ‹*y* ∈ *s*› **obtain** *p* **where** *p*: *path p path-image p* ⊆ *s pathstart p* =

x pathfinish p = y
by (*force simp*: *path-connected-def*)
have ∃ *e*. *0 < e* ∧ (∀ *x* ∈ *path-image p*. *ball x e* ⊆ *s*)
proof (*cases s = UNIV*)
case *True* **then show** *?thesis*
by (*simp add*: *gt-ex*)
next
case *False*
then have − *s* ≠ {} **by** *blast*
then show *?thesis*
apply (*rule-tac x*=*setdist* (*path-image p*) (−*s*) **in** *exI*)
using *s p*
apply (*simp add*: *setdist-gt-0-compact-closed compact-path-image open-closed*)
using *setdist-le-dist* [*of* - *path-image p* - −*s*]
by *fastforce*
qed
then obtain *e* **where** *0 < e***and** *eb*: ⋀*x*. *x* ∈ *path-image p* ⟹ *ball x e* ⊆ *s*
by *auto*
show *?thesis*
using *path-approx-polynomial-function* [*OF* ‹*path p*› ‹*0 < e*›]
apply *clarify*
apply (*intro exI conjI*, *assumption*)
using *p*
apply (*fastforce simp add*: *dist-norm path-image-def norm-minus-commute intro*: *eb* [*THEN subsetD*])+
done
qed

hide-fact *linear add mult const*

end

50 Non-negative, non-positive integers and reals

theory *Nonpos-Ints*
imports *Complex-Main*
begin

50.1 Non-positive integers

The set of non-positive integers on a ring. (in analogy to the set of non-negative integers ℕ) This is useful e.g. for the Gamma function.

definition *nonpos-Ints* ($\mathbb{Z}_{\leq 0}$) **where** $\mathbb{Z}_{\leq 0} = \{of\text{-}int\ n\ |n.\ n \leq 0\}$

lemma *zero-in-nonpos-Ints* [*simp*,*intro*]: $0 \in \mathbb{Z}_{\leq 0}$
unfolding *nonpos-Ints-def* **by** (*auto intro*!: *exI*[*of* - *0*::*int*])

lemma *neg-one-in-nonpos-Ints* [*simp*,*intro*]: $-1 \in \mathbb{Z}_{\leq 0}$

unfolding *nonpos-Ints-def* **by** (*auto intro*!: *exI*[*of* - -1::*int*])

lemma *neg-numeral-in-nonpos-Ints* [*simp*,*intro*]: $-$*numeral n* $\in \mathbb{Z}_{\leq 0}$
unfolding *nonpos-Ints-def* **by** (*auto intro*!: *exI*[*of* - $-$*numeral n*::*int*])

lemma *one-notin-nonpos-Ints* [*simp*]: (*1* :: *'a* :: *ring-char-0*) $\notin \mathbb{Z}_{\leq 0}$
by (*auto simp*: *nonpos-Ints-def*)

lemma *numeral-notin-nonpos-Ints* [*simp*]: (*numeral n* :: *'a* :: *ring-char-0*) $\notin \mathbb{Z}_{\leq 0}$
by (*auto simp*: *nonpos-Ints-def*)

lemma *minus-of-nat-in-nonpos-Ints* [*simp*, *intro*]: $-$ *of-nat n* $\in \mathbb{Z}_{\leq 0}$
proof $-$
have $-$ *of-nat n* = *of-int* ($-$*int n*) **by** *simp*
also have $-$*int n* $\leq$ *0* **by** *simp*
hence *of-int* ($-$*int n*) $\in \mathbb{Z}_{\leq 0}$ **unfolding** *nonpos-Ints-def* **by** *blast*
finally show *?thesis* **.**
qed

lemma *of-nat-in-nonpos-Ints-iff*: (*of-nat n* :: *'a* :: {*ring-1*,*ring-char-0*}) $\in \mathbb{Z}_{\leq 0}$ $\longleftrightarrow$ *n* = *0*
proof
assume (*of-nat n* :: *'a*) $\in \mathbb{Z}_{\leq 0}$
then obtain *m* **where** *of-nat n* = (*of-int m* :: *'a*) *m* $\leq$ *0* **by** (*auto simp*: *nonpos-Ints-def*)
hence (*of-int m* :: *'a*) = *of-nat n* **by** *simp*
also have ... = *of-int* (*int n*) **by** *simp*
finally have *m* = *int n* **by** (*subst* (*asm*) *of-int-eq-iff*)
with ‹*m* $\leq$ *0*› **show** *n* = *0* **by** *auto*
qed *simp*

lemma *nonpos-Ints-of-int*: *n* $\leq$ *0* $\Longrightarrow$ *of-int n* $\in \mathbb{Z}_{\leq 0}$
unfolding *nonpos-Ints-def* **by** *blast*

lemma *nonpos-IntsI*:
x $\in \mathbb{Z} \Longrightarrow$ *x* $\leq$ *0* $\Longrightarrow$ (*x* :: *'a* :: *linordered-idom*) $\in \mathbb{Z}_{\leq 0}$
using *assms* **unfolding** *nonpos-Ints-def Ints-def* **by** *auto*

lemma *nonpos-Ints-subset-Ints*: $\mathbb{Z}_{\leq 0} \subseteq \mathbb{Z}$
unfolding *nonpos-Ints-def Ints-def* **by** *blast*

lemma *nonpos-Ints-nonpos* [*dest*]: *x* $\in \mathbb{Z}_{\leq 0} \Longrightarrow$ *x* $\leq$ (*0* :: *'a* :: *linordered-idom*)
unfolding *nonpos-Ints-def* **by** *auto*

lemma *nonpos-Ints-Int* [*dest*]: *x* $\in \mathbb{Z}_{\leq 0} \Longrightarrow$ *x* $\in \mathbb{Z}$
unfolding *nonpos-Ints-def Ints-def* **by** *blast*

lemma *nonpos-Ints-cases*:
assumes *x* $\in \mathbb{Z}_{\leq 0}$

obtains n **where** $x = \textit{of-int}\ n\ n \leq 0$
using *assms* **unfolding** *nonpos-Ints-def* **by** (*auto elim*!: *Ints-cases*)

lemma *nonpos-Ints-cases'*:
assumes $x \in \mathbb{Z}_{\leq 0}$
obtains n **where** $x = -\textit{of-nat}\ n$
proof −
from *assms* **obtain** m **where** $x = \textit{of-int}\ m$ **and** m: $m \leq 0$ **by** (*auto elim*!: *nonpos-Ints-cases*)
hence $x = -\ \textit{of-int}\ (-m)$ **by** *auto*
also from m **have** $(\textit{of-int}\ (-m) :: 'a) = \textit{of-nat}\ (\textit{nat}\ (-m))$ **by** *simp-all*
finally show *?thesis* **by** (*rule that*)
qed

lemma *of-real-in-nonpos-Ints-iff*: $(\textit{of-real}\ x :: 'a :: \textit{real-algebra-1}) \in \mathbb{Z}_{\leq 0} \longleftrightarrow x \in \mathbb{Z}_{\leq 0}$
proof
assume $\textit{of-real}\ x \in (\mathbb{Z}_{\leq 0} :: 'a\ \textit{set})$
then obtain n **where** $(\textit{of-real}\ x :: 'a) = \textit{of-int}\ n\ n \leq 0$ **by** (*erule nonpos-Ints-cases*)
note ‹*of-real* $x = \textit{of-int}\ n$›
also have $\textit{of-int}\ n = \textit{of-real}\ (\textit{of-int}\ n)$ **by** (*rule of-real-of-int-eq* [*symmetric*])
finally have $x = \textit{of-int}\ n$ **by** (*subst* (*asm*) *of-real-eq-iff*)
with ‹$n \leq 0$› **show** $x \in \mathbb{Z}_{\leq 0}$ **by** (*simp add*: *nonpos-Ints-of-int*)
qed (*auto elim*!: *nonpos-Ints-cases intro*!: *nonpos-Ints-of-int*)

lemma *nonpos-Ints-altdef*: $\mathbb{Z}_{\leq 0} = \{n \in \mathbb{Z}.\ (n :: 'a :: \textit{linordered-idom}) \leq 0\}$
by (*auto intro*!: *nonpos-IntsI elim*!: *nonpos-Ints-cases*)

lemma *uminus-in-Nats-iff*: $-x \in \mathbb{N} \longleftrightarrow x \in \mathbb{Z}_{\leq 0}$
proof
assume $-x \in \mathbb{N}$
then obtain n **where** $n \geq 0\ -x = \textit{of-int}\ n$ **by** (*auto simp*: *Nats-altdef1*)
hence $-n \leq 0\ x = \textit{of-int}\ (-n)$ **by** (*simp-all add*: *eq-commute minus-equation-iff*[*of* x])
thus $x \in \mathbb{Z}_{\leq 0}$ **unfolding** *nonpos-Ints-def* **by** *blast*
next
assume $x \in \mathbb{Z}_{\leq 0}$
then obtain n **where** $n \leq 0\ x = \textit{of-int}\ n$ **by** (*auto simp*: *nonpos-Ints-def*)
hence $-n \geq 0\ -x = \textit{of-int}\ (-n)$ **by** (*simp-all add*: *eq-commute minus-equation-iff*[*of* x])
thus $-x \in \mathbb{N}$ **unfolding** *Nats-altdef1* **by** *blast*
qed

lemma *uminus-in-nonpos-Ints-iff*: $-x \in \mathbb{Z}_{\leq 0} \longleftrightarrow x \in \mathbb{N}$
using *uminus-in-Nats-iff*[*of* $-x$] **by** *simp*

lemma *nonpos-Ints-mult*: $x \in \mathbb{Z}_{\leq 0} \Longrightarrow y \in \mathbb{Z}_{\leq 0} \Longrightarrow x * y \in \mathbb{N}$
using *Nats-mult*[*of* $-x\ -y$] **by** (*simp add*: *uminus-in-Nats-iff*)

lemma *Nats-mult-nonpos-Ints*: $x \in \mathbb{N} \Longrightarrow y \in \mathbb{Z}_{\leq 0} \Longrightarrow x * y \in \mathbb{Z}_{\leq 0}$
using *Nats-mult*[*of* $x\ -y$] **by** (*simp add*: *uminus-in-Nats-iff*)

lemma *nonpos-Ints-mult-Nats*:
$x \in \mathbb{Z}_{\leq 0} \Longrightarrow y \in \mathbb{N} \Longrightarrow x * y \in \mathbb{Z}_{\leq 0}$
using *Nats-mult*[*of* $-x\ y$] **by** (*simp add*: *uminus-in-Nats-iff*)

lemma *nonpos-Ints-add*:
$x \in \mathbb{Z}_{\leq 0} \Longrightarrow y \in \mathbb{Z}_{\leq 0} \Longrightarrow x + y \in \mathbb{Z}_{\leq 0}$
using *Nats-add*[*of* $-x\ -y$] *uminus-in-Nats-iff*[*of* $y{+}x$, *simplified minus-add*]
by (*simp add*: *uminus-in-Nats-iff add.commute*)

lemma *nonpos-Ints-diff-Nats*:
$x \in \mathbb{Z}_{\leq 0} \Longrightarrow y \in \mathbb{N} \Longrightarrow x - y \in \mathbb{Z}_{\leq 0}$
using *Nats-add*[*of* $-x\ y$] *uminus-in-Nats-iff*[*of* $x{-}y$, *simplified minus-add*]
by (*simp add*: *uminus-in-Nats-iff add.commute*)

lemma *Nats-diff-nonpos-Ints*:
$x \in \mathbb{N} \Longrightarrow y \in \mathbb{Z}_{\leq 0} \Longrightarrow x - y \in \mathbb{N}$
using *Nats-add*[*of* $x\ -y$] **by** (*simp add*: *uminus-in-Nats-iff add.commute*)

lemma *plus-of-nat-eq-0-imp*: $z + \mathit{of\text{-}nat}\ n = 0 \Longrightarrow z \in \mathbb{Z}_{\leq 0}$
proof −
assume $z + \mathit{of\text{-}nat}\ n = 0$
hence *A*: $z = -\ \mathit{of\text{-}nat}\ n$ **by** (*simp add*: *eq-neg-iff-add-eq-0*)
show $z \in \mathbb{Z}_{\leq 0}$ **by** (*subst A*) *simp*
qed

50.2 Non-negative reals

definition *nonneg-Reals* :: *'a*::*real-algebra-1 set* $(\mathbb{R}_{\geq 0})$
where $\mathbb{R}_{\geq 0} = \{\mathit{of\text{-}real}\ r \mid r.\ r \geq 0\}$

lemma *nonneg-Reals-of-real-iff* [*simp*]: $\mathit{of\text{-}real}\ r \in \mathbb{R}_{\geq 0} \longleftrightarrow r \geq 0$
by (*force simp add*: *nonneg-Reals-def*)

lemma *nonneg-Reals-subset-Reals*: $\mathbb{R}_{\geq 0} \subseteq \mathbb{R}$
unfolding *nonneg-Reals-def Reals-def* **by** *blast*

lemma *nonneg-Reals-Real* [*dest*]: $x \in \mathbb{R}_{\geq 0} \Longrightarrow x \in \mathbb{R}$
unfolding *nonneg-Reals-def Reals-def* **by** *blast*

lemma *nonneg-Reals-of-nat-I* [*simp*]: $\mathit{of\text{-}nat}\ n \in \mathbb{R}_{\geq 0}$
by (*metis nonneg-Reals-of-real-iff of-nat-0-le-iff of-real-of-nat-eq*)

lemma *nonneg-Reals-cases*:
assumes $x \in \mathbb{R}_{\geq 0}$
obtains r **where** $x = \mathit{of\text{-}real}\ r\ r \geq 0$
using *assms* **unfolding** *nonneg-Reals-def* **by** (*auto elim*!: *Reals-cases*)

lemma *nonneg-Reals-zero-I* [*simp*]: $0 \in \mathbb{R}_{\geq 0}$
unfolding *nonneg-Reals-def* **by** *auto*

lemma *nonneg-Reals-one-I* [*simp*]: $1 \in \mathbb{R}_{\geq 0}$
by (*metis* (*mono-tags*, *lifting*) *nonneg-Reals-of-nat-I of-nat-1*)

lemma *nonneg-Reals-minus-one-I* [*simp*]: $-1 \notin \mathbb{R}_{\geq 0}$
by (*metis nonneg-Reals-of-real-iff le-minus-one-simps*(*3*) *of-real-1 of-real-def real-vector.scale-minus-left*)

lemma *nonneg-Reals-numeral-I* [*simp*]: *numeral w* $\in \mathbb{R}_{\geq 0}$
by (*metis* (*no-types*) *nonneg-Reals-of-nat-I of-nat-numeral*)

lemma *nonneg-Reals-minus-numeral-I* [*simp*]: $-$ *numeral w* $\notin \mathbb{R}_{\geq 0}$
using *nonneg-Reals-of-real-iff not-zero-le-neg-numeral* **by** *fastforce*

lemma *nonneg-Reals-add-I* [*simp*]: $[\![a \in \mathbb{R}_{\geq 0};\ b \in \mathbb{R}_{\geq 0}]\!] \Longrightarrow a + b \in \mathbb{R}_{\geq 0}$
apply (*simp add*: *nonneg-Reals-def*)
apply *clarify*
apply (*rename-tac r s*)
apply (*rule-tac x*=*r*+*s* **in** *exI*, *auto*)
done

lemma *nonneg-Reals-mult-I* [*simp*]: $[\![a \in \mathbb{R}_{\geq 0};\ b \in \mathbb{R}_{\geq 0}]\!] \Longrightarrow a * b \in \mathbb{R}_{\geq 0}$
unfolding *nonneg-Reals-def* **by** (*auto simp*: *of-real-def*)

lemma *nonneg-Reals-inverse-I* [*simp*]:
fixes *a* :: *'a*::*real-div-algebra*
shows $a \in \mathbb{R}_{\geq 0} \Longrightarrow$ *inverse a* $\in \mathbb{R}_{\geq 0}$
by (*simp add*: *nonneg-Reals-def image-iff*) (*metis inverse-nonnegative-iff-nonnegative of-real-inverse*)

lemma *nonneg-Reals-divide-I* [*simp*]:
fixes *a* :: *'a*::*real-div-algebra*
shows $[\![a \in \mathbb{R}_{\geq 0};\ b \in \mathbb{R}_{\geq 0}]\!] \Longrightarrow a\ /\ b \in \mathbb{R}_{\geq 0}$
by (*simp add*: *divide-inverse*)

lemma *nonneg-Reals-pow-I* [*simp*]: $a \in \mathbb{R}_{\geq 0} \Longrightarrow a\hat{}n \in \mathbb{R}_{\geq 0}$
by (*induction n*) *auto*

lemma *complex-nonneg-Reals-iff*: $z \in \mathbb{R}_{\geq 0} \longleftrightarrow Re\ z \geq 0 \wedge Im\ z = 0$
by (*auto simp*: *nonneg-Reals-def*) (*metis complex-of-real-def complex-surj*)

lemma *ii-not-nonneg-Reals* [*iff*]: $\mathrm{i} \notin \mathbb{R}_{\geq 0}$
by (*simp add*: *complex-nonneg-Reals-iff*)

50.3 Non-positive reals

definition *nonpos-Reals* :: *'a*::*real-algebra-1 set* ($\mathbb{R}_{\leq 0}$)

where $\mathbb{R}_{\leq 0} = \{\textit{of-real } r \mid r.\ r \leq 0\}$

lemma *nonpos-Reals-of-real-iff* [*simp*]: $\textit{of-real } r \in \mathbb{R}_{\leq 0} \longleftrightarrow r \leq 0$
by (*force simp add*: *nonpos-Reals-def*)

lemma *nonpos-Reals-subset-Reals*: $\mathbb{R}_{\leq 0} \subseteq \mathbb{R}$
unfolding *nonpos-Reals-def Reals-def* **by** *blast*

lemma *nonpos-Ints-subset-nonpos-Reals*: $\mathbb{Z}_{\leq 0} \subseteq \mathbb{R}_{\leq 0}$
by (*metis nonpos-Ints-cases nonpos-Ints-nonpos nonpos-Ints-of-int nonpos-Reals-of-real-iff of-real-of-int-eq subsetI*)

lemma *nonpos-Reals-of-nat-iff* [*simp*]: $\textit{of-nat } n \in \mathbb{R}_{\leq 0} \longleftrightarrow n{=}0$
by (*metis nonpos-Reals-of-real-iff of-nat-le-0-iff of-real-of-nat-eq*)

lemma *nonpos-Reals-Real* [*dest*]: $x \in \mathbb{R}_{\leq 0} \Longrightarrow x \in \mathbb{R}$
unfolding *nonpos-Reals-def Reals-def* **by** *blast*

lemma *nonpos-Reals-cases*:
assumes $x \in \mathbb{R}_{\leq 0}$
obtains r **where** $x = \textit{of-real } r\ r \leq 0$
using *assms* **unfolding** *nonpos-Reals-def* **by** (*auto elim*!: *Reals-cases*)

lemma *uminus-nonneg-Reals-iff* [*simp*]: $-x \in \mathbb{R}_{\geq 0} \longleftrightarrow x \in \mathbb{R}_{\leq 0}$
apply (*auto simp*: *nonpos-Reals-def nonneg-Reals-def*)
apply (*metis nonpos-Reals-of-real-iff minus-minus neg-le-0-iff-le of-real-minus*)
apply (*metis neg-0-le-iff-le of-real-minus*)
done

lemma *uminus-nonpos-Reals-iff* [*simp*]: $-x \in \mathbb{R}_{\leq 0} \longleftrightarrow x \in \mathbb{R}_{\geq 0}$
by (*metis* (*no-types*) *minus-minus uminus-nonneg-Reals-iff*)

lemma *nonpos-Reals-zero-I* [*simp*]: $0 \in \mathbb{R}_{\leq 0}$
unfolding *nonpos-Reals-def* **by** *force*

lemma *nonpos-Reals-one-I* [*simp*]: $1 \notin \mathbb{R}_{\leq 0}$
using *nonneg-Reals-minus-one-I uminus-nonneg-Reals-iff* **by** *blast*

lemma *nonpos-Reals-numeral-I* [*simp*]: $\textit{numeral } w \notin \mathbb{R}_{\leq 0}$
using *nonneg-Reals-minus-numeral-I uminus-nonneg-Reals-iff* **by** *blast*

lemma *nonpos-Reals-add-I* [*simp*]: $[\![a \in \mathbb{R}_{\leq 0};\ b \in \mathbb{R}_{\leq 0}]\!] \Longrightarrow a + b \in \mathbb{R}_{\leq 0}$
by (*metis nonneg-Reals-add-I add-uminus-conv-diff minus-diff-eq minus-minus uminus-nonpos-Reals-iff*)

lemma *nonpos-Reals-mult-I1*: $[\![a \in \mathbb{R}_{\geq 0};\ b \in \mathbb{R}_{\leq 0}]\!] \Longrightarrow a * b \in \mathbb{R}_{\leq 0}$
by (*metis nonneg-Reals-mult-I mult-minus-right uminus-nonneg-Reals-iff*)

lemma *nonpos-Reals-mult-I2*: $[\![a \in \mathbb{R}_{\leq 0};\ b \in \mathbb{R}_{\geq 0}]\!] \Longrightarrow a * b \in \mathbb{R}_{\leq 0}$

by (*metis nonneg-Reals-mult-I mult-minus-left uminus-nonneg-Reals-iff*)

lemma *nonpos-Reals-mult-of-nat-iff*:
fixes *a*:: *'a* :: *real-div-algebra* **shows** $a * \mathit{of\text{-}nat}\ n \in \mathbb{R}_{\leq 0} \longleftrightarrow a \in \mathbb{R}_{\leq 0} \vee n{=}0$
apply (*auto intro*: *nonpos-Reals-mult-I2*)
apply (*auto simp*: *nonpos-Reals-def*)
apply (*rule-tac x=r/n* **in** *exI*)
apply (*auto simp*: *divide-simps*)
done

lemma *nonpos-Reals-inverse-I*:
fixes *a* :: *'a*::*real-div-algebra*
shows $a \in \mathbb{R}_{\leq 0} \Longrightarrow \mathit{inverse}\ a \in \mathbb{R}_{\leq 0}$
using *nonneg-Reals-inverse-I uminus-nonneg-Reals-iff* **by** *fastforce*

lemma *nonpos-Reals-divide-I1*:
fixes *a* :: *'a*::*real-div-algebra*
shows $[\![a \in \mathbb{R}_{\geq 0};\ b \in \mathbb{R}_{\leq 0}]\!] \Longrightarrow a\ /\ b \in \mathbb{R}_{\leq 0}$
by (*simp add*: *nonpos-Reals-inverse-I nonpos-Reals-mult-I1 divide-inverse*)

lemma *nonpos-Reals-divide-I2*:
fixes *a* :: *'a*::*real-div-algebra*
shows $[\![a \in \mathbb{R}_{\leq 0};\ b \in \mathbb{R}_{\geq 0}]\!] \Longrightarrow a\ /\ b \in \mathbb{R}_{\leq 0}$
by (*metis nonneg-Reals-divide-I minus-divide-left uminus-nonneg-Reals-iff*)

lemma *nonpos-Reals-divide-of-nat-iff*:
fixes *a*:: *'a* :: *real-div-algebra* **shows** $a\ /\ \mathit{of\text{-}nat}\ n \in \mathbb{R}_{\leq 0} \longleftrightarrow a \in \mathbb{R}_{\leq 0} \vee n{=}0$
apply (*auto intro*: *nonpos-Reals-divide-I2*)
apply (*auto simp*: *nonpos-Reals-def*)
apply (*rule-tac x=r∗n* **in** *exI*)
apply (*auto simp*: *divide-simps mult-le-0-iff*)
done

lemma *nonpos-Reals-pow-I*: $[\![a \in \mathbb{R}_{\leq 0};\ \mathit{odd}\ n]\!] \Longrightarrow a\hat{}n \in \mathbb{R}_{\leq 0}$
by (*metis nonneg-Reals-pow-I power-minus-odd uminus-nonneg-Reals-iff*)

lemma *complex-nonpos-Reals-iff*: $z \in \mathbb{R}_{\leq 0} \longleftrightarrow \mathit{Re}\ z \leq 0 \wedge \mathit{Im}\ z = 0$
using *complex-is-Real-iff* **by** (*force simp add*: *nonpos-Reals-def*)

lemma *ii-not-nonpos-Reals* [*iff*]: $\mathrm{i} \notin \mathbb{R}_{\leq 0}$
by (*simp add*: *complex-nonpos-Reals-iff*)

end

51 Complex Analysis Basics

theory *Complex-Analysis-Basics*
imports *Cartesian-Euclidean-Space* ~~/*src*/*HOL*/*Library*/*Nonpos-Ints*
begin

51.1 General lemmas

lemma *nonneg-Reals-cmod-eq-Re*: $z \in \mathbb{R}_{\geq 0} \Longrightarrow norm\ z = Re\ z$
by (*simp add*: *complex-nonneg-Reals-iff cmod-eq-Re*)

lemma *has-derivative-mult-right*:
fixes *c*:: *'a* :: *real-normed-algebra*
shows ((*op* ∗ *c*) *has-derivative* (*op* ∗ *c*)) *F*
by (*rule has-derivative-mult-right* [*OF has-derivative-id*])

lemma *has-derivative-of-real*[*derivative-intros*, *simp*]:
(*f has-derivative f'*) *F* ⟹ ((λ*x*. *of-real* (*f x*)) *has-derivative* (λ*x*. *of-real* (*f' x*)))
F
using *bounded-linear.has-derivative*[*OF bounded-linear-of-real*] .

lemma *has-vector-derivative-real-complex*:
DERIV f (*of-real a*) :> *f'* ⟹ ((λ*x*. *f* (*of-real x*)) *has-vector-derivative f'*) (*at a*
within s)
using *has-derivative-compose*[*of of-real of-real a - f op* ∗ *f'*]
by (*simp add*: *scaleR-conv-of-real ac-simps has-vector-derivative-def has-field-derivative-def*)

lemma *fact-cancel*:
fixes *c* :: *'a*::*real-field*
shows *of-nat* (*Suc n*) ∗ *c* / (*fact* (*Suc n*)) = *c* / (*fact n*)
by (*simp add*: *of-nat-mult del*: *of-nat-Suc times-nat.simps*)

lemma *bilinear-times*:
fixes *c*::*'a*::*real-algebra* **shows** *bilinear* (λ*x y*::*'a*. *x*∗*y*)
by (*auto simp*: *bilinear-def distrib-left distrib-right intro*!: *linearI*)

lemma *linear-cnj*: *linear cnj*
using *bounded-linear.linear*[*OF bounded-linear-cnj*] .

lemma *tendsto-Re-upper*:
assumes ~ (*trivial-limit F*)
(*f* ⟶ *l*) *F*
eventually (λ*x*. *Re*(*f x*) ≤ *b*) *F*
shows *Re*(*l*) ≤ *b*
by (*metis assms tendsto-le* [*OF - tendsto-const*] *tendsto-Re*)

lemma *tendsto-Re-lower*:
assumes ~ (*trivial-limit F*)
(*f* ⟶ *l*) *F*
eventually (λ*x*. *b* ≤ *Re*(*f x*)) *F*
shows *b* ≤ *Re*(*l*)
by (*metis assms tendsto-le* [*OF - - tendsto-const*] *tendsto-Re*)

lemma *tendsto-Im-upper*:
assumes ~ (*trivial-limit F*)
(*f* ⟶ *l*) *F*

eventually $(\lambda x.\ Im(f\ x) \leq b)\ F$
shows $Im(l) \leq b$
by (*metis assms tendsto-le* [*OF - tendsto-const*] *tendsto-Im*)

lemma *tendsto-Im-lower*:
assumes ~ (*trivial-limit F*)
$(f \longrightarrow l)\ F$
eventually $(\lambda x.\ b \leq Im(f\ x))\ F$
shows $b \leq Im(l)$
by (*metis assms tendsto-le* [*OF - - tendsto-const*] *tendsto-Im*)

lemma *lambda-zero*: $(\lambda h{::}'a{::}mult\text{-}zero.\ 0) = op * 0$
by *auto*

lemma *lambda-one*: $(\lambda x{::}'a{::}monoid\text{-}mult.\ x) = op * 1$
by *auto*

lemma *continuous-mult-left*:
fixes $c{::}'a{::}real\text{-}normed\text{-}algebra$
shows *continuous F f* $\Longrightarrow$ *continuous F* $(\lambda x.\ c * f\ x)$
by (*rule continuous-mult* [*OF continuous-const*])

lemma *continuous-mult-right*:
fixes $c{::}'a{::}real\text{-}normed\text{-}algebra$
shows *continuous F f* $\Longrightarrow$ *continuous F* $(\lambda x.\ f\ x * c)$
by (*rule continuous-mult* [*OF - continuous-const*])

lemma *continuous-on-mult-left*:
fixes $c{::}'a{::}real\text{-}normed\text{-}algebra$
shows *continuous-on s f* $\Longrightarrow$ *continuous-on s* $(\lambda x.\ c * f\ x)$
by (*rule continuous-on-mult* [*OF continuous-on-const*])

lemma *continuous-on-mult-right*:
fixes $c{::}'a{::}real\text{-}normed\text{-}algebra$
shows *continuous-on s f* $\Longrightarrow$ *continuous-on s* $(\lambda x.\ f\ x * c)$
by (*rule continuous-on-mult* [*OF - continuous-on-const*])

lemma *uniformly-continuous-on-cmul-right* [*continuous-intros*]:
fixes *f* :: $'a{::}real\text{-}normed\text{-}vector \Rightarrow 'b{::}real\text{-}normed\text{-}algebra$
shows *uniformly-continuous-on s f* $\Longrightarrow$ *uniformly-continuous-on s* $(\lambda x.\ f\ x * c)$
using *bounded-linear.uniformly-continuous-on*[*OF bounded-linear-mult-left*] .

lemma *uniformly-continuous-on-cmul-left*[*continuous-intros*]:
fixes *f* :: $'a{::}real\text{-}normed\text{-}vector \Rightarrow 'b{::}real\text{-}normed\text{-}algebra$
assumes *uniformly-continuous-on s f*
shows *uniformly-continuous-on s* $(\lambda x.\ c * f\ x)$
by (*metis assms bounded-linear.uniformly-continuous-on bounded-linear-mult-right*)

lemma *continuous-within-norm-id* [*continuous-intros*]: *continuous* (*at x within S*)

norm
by (*rule continuous-norm* [*OF continuous-ident*])

lemma *continuous-on-norm-id* [*continuous-intros*]: *continuous-on S norm*
by (*intro continuous-on-id continuous-on-norm*)

51.2 DERIV stuff

lemma *DERIV-zero-connected-constant*:
fixes *f* :: *'a*::{*real-normed-field*,*euclidean-space*} ⇒ *'a*
assumes *connected s*
and *open s*
and *finite k*
and *continuous-on s f*
and ∀ *x*∈(*s* − *k*). *DERIV f x* :> *0*
obtains *c* **where** ⋀*x*. *x* ∈ *s* ⟹ *f*(*x*) = *c*
using *has-derivative-zero-connected-constant* [*OF assms*(*1*−*4*)] *assms*
by (*metis DERIV-const has-derivative-const Diff-iff at-within-open frechet-derivative-at has-field-derivative-def*)

lemma *DERIV-zero-constant*:
fixes *f* :: *'a*::{*real-normed-field*, *real-inner*} ⇒ *'a*
shows ⟦*convex s*;
⋀*x*. *x*∈*s* ⟹ (*f has-field-derivative 0*) (*at x within s*)⟧
⟹ ∃ *c*. ∀ *x* ∈ *s*. *f*(*x*) = *c*
by (*auto simp*: *has-field-derivative-def lambda-zero intro*: *has-derivative-zero-constant*)

lemma *DERIV-zero-unique*:
fixes *f* :: *'a*::{*real-normed-field*, *real-inner*} ⇒ *'a*
assumes *convex s*
and *d0*: ⋀*x*. *x*∈*s* ⟹ (*f has-field-derivative 0*) (*at x within s*)
and *a* ∈ *s*
and *x* ∈ *s*
shows *f x* = *f a*
by (*rule has-derivative-zero-unique* [*OF assms*(*1*) - *assms*(*4*,*3*)])
(*metis d0 has-field-derivative-imp-has-derivative lambda-zero*)

lemma *DERIV-zero-connected-unique*:
fixes *f* :: *'a*::{*real-normed-field*, *real-inner*} ⇒ *'a*
assumes *connected s*
and *open s*
and *d0*: ⋀*x*. *x*∈*s* ⟹ *DERIV f x* :> *0*
and *a* ∈ *s*
and *x* ∈ *s*
shows *f x* = *f a*
by (*rule has-derivative-zero-unique-connected* [*OF assms*(*2*,*1*) - *assms*(*5*,*4*)])
(*metis has-field-derivative-def lambda-zero d0*)

lemma *DERIV-transform-within*:

assumes (*f has-field-derivative f'*) (*at a within s*)
and *0 < d a ∈ s*
and $\bigwedge x.\ x{\in}s \Longrightarrow dist\ x\ a < d \Longrightarrow f\ x = g\ x$
shows (*g has-field-derivative f'*) (*at a within s*)
using *assms* **unfolding** *has-field-derivative-def*
by (*blast intro*: *has-derivative-transform-within*)

lemma *DERIV-transform-within-open*:
assumes *DERIV f a :> f'*
and *open s a ∈ s*
and $\bigwedge x.\ x{\in}s \Longrightarrow f\ x = g\ x$
shows *DERIV g a :> f'*
using *assms* **unfolding** *has-field-derivative-def*
by (*metis has-derivative-transform-within-open*)

lemma *DERIV-transform-at*:
assumes *DERIV f a :> f'*
and *0 < d*
and $\bigwedge x.\ dist\ x\ a < d \Longrightarrow f\ x = g\ x$
shows *DERIV g a :> f'*
by (*blast intro*: *assms DERIV-transform-within*)

lemma *DERIV-zero-UNIV-unique*:
fixes *f* :: *'a*::{*real-normed-field*, *real-inner*} ⇒ *'a*
shows $(\bigwedge x.\ DERIV\ f\ x :> 0) \Longrightarrow f\ x = f\ a$
by (*metis DERIV-zero-unique UNIV-I assms convex-UNIV*)

51.3 Some limit theorems about real part of real series etc.

lemma *sums-vec-nth* :
assumes *f sums a*
shows (*λx. f x $ i*) *sums a $ i*
using *assms* **unfolding** *sums-def*
by (*auto dest*: *tendsto-vec-nth* [**where** *i*=*i*])

lemma *summable-vec-nth* :
assumes *summable f*
shows *summable* (*λx. f x $ i*)
using *assms* **unfolding** *summable-def*
by (*blast intro*: *sums-vec-nth*)

51.4 Complex number lemmas

lemma
shows *open-halfspace-Re-lt*: *open* $\{z.\ Re(z) < b\}$
and *open-halfspace-Re-gt*: *open* $\{z.\ Re(z) > b\}$
and *closed-halfspace-Re-ge*: *closed* $\{z.\ Re(z) \geq b\}$
and *closed-halfspace-Re-le*: *closed* $\{z.\ Re(z) \leq b\}$
and *closed-halfspace-Re-eq*: *closed* $\{z.\ Re(z) = b\}$

and *open-halfspace-Im-lt*: *open* $\{z.\ Im(z) < b\}$
and *open-halfspace-Im-gt*: *open* $\{z.\ Im(z) > b\}$
and *closed-halfspace-Im-ge*: *closed* $\{z.\ Im(z) \geq b\}$
and *closed-halfspace-Im-le*: *closed* $\{z.\ Im(z) \leq b\}$
and *closed-halfspace-Im-eq*: *closed* $\{z.\ Im(z) = b\}$
by (*intro open-Collect-less closed-Collect-le closed-Collect-eq isCont-Re isCont-Im continuous-ident continuous-const*)+

lemma *closed-complex-Reals*: *closed* ($\mathbb{R}$ :: *complex set*)
proof −
have ($\mathbb{R}$ :: *complex set*) = $\{z.\ Im\ z = 0\}$
by (*auto simp*: *complex-is-Real-iff*)
then show *?thesis*
by (*metis closed-halfspace-Im-eq*)
qed

lemma *closed-Real-halfspace-Re-le*: *closed* ($\mathbb{R} \cap \{w.\ Re\ w \leq x\}$)
by (*simp add*: *closed-Int closed-complex-Reals closed-halfspace-Re-le*)

corollary *closed-nonpos-Reals-complex* [*simp*]: *closed* ($\mathbb{R}_{\leq 0}$ :: *complex set*)
proof −
have $\mathbb{R}_{\leq 0} = \mathbb{R} \cap \{z.\ Re(z) \leq 0\}$
using *complex-nonpos-Reals-iff complex-is-Real-iff* **by** *auto*
then show *?thesis*
by (*metis closed-Real-halfspace-Re-le*)
qed

lemma *closed-Real-halfspace-Re-ge*: *closed* ($\mathbb{R} \cap \{w.\ x \leq Re(w)\}$)
using *closed-halfspace-Re-ge*
by (*simp add*: *closed-Int closed-complex-Reals*)

corollary *closed-nonneg-Reals-complex* [*simp*]: *closed* ($\mathbb{R}_{\geq 0}$ :: *complex set*)
proof −
have $\mathbb{R}_{\geq 0} = \mathbb{R} \cap \{z.\ Re(z) \geq 0\}$
using *complex-nonneg-Reals-iff complex-is-Real-iff* **by** *auto*
then show *?thesis*
by (*metis closed-Real-halfspace-Re-ge*)
qed

lemma *closed-real-abs-le*: *closed* $\{w \in \mathbb{R}.\ |Re\ w| \leq r\}$
proof −
have $\{w \in \mathbb{R}.\ |Re\ w| \leq r\} = (\mathbb{R} \cap \{w.\ Re\ w \leq r\}) \cap (\mathbb{R} \cap \{w.\ Re\ w \geq -r\})$
by *auto*
then show *closed* $\{w \in \mathbb{R}.\ |Re\ w| \leq r\}$
by (*simp add*: *closed-Int closed-Real-halfspace-Re-ge closed-Real-halfspace-Re-le*)
qed

lemma *real-lim*:
fixes *l*::*complex*

assumes $(f \longrightarrow l)$ F **and** $\sim(\textit{trivial-limit}\ F)$ **and** *eventually* P F **and** $\bigwedge a.\ P\ a \Longrightarrow f\ a \in \mathbb{R}$
shows $l \in \mathbb{R}$
proof (*rule Lim-in-closed-set*[*OF closed-complex-Reals - assms*(*2*,*1*)])
show *eventually* $(\lambda x.\ f\ x \in \mathbb{R})$ F
using *assms*(*3*, *4*) **by** (*auto intro*: *eventually-mono*)
qed

lemma *real-lim-sequentially*:
fixes *l*::*complex*
shows $(f \longrightarrow l)$ *sequentially* $\Longrightarrow (\exists N.\ \forall n \geq N.\ f\ n \in \mathbb{R}) \Longrightarrow l \in \mathbb{R}$
by (*rule real-lim* [**where** *F*=*sequentially*]) (*auto simp*: *eventually-sequentially*)

lemma *real-series*:
fixes *l*::*complex*
shows *f sums l* $\Longrightarrow (\bigwedge n.\ f\ n \in \mathbb{R}) \Longrightarrow l \in \mathbb{R}$
unfolding *sums-def*
by (*metis real-lim-sequentially setsum-in-Reals*)

lemma *Lim-null-comparison-Re*:
assumes *eventually* $(\lambda x.\ \textit{norm}(f\ x) \leq \textit{Re}(g\ x))$ F $(g \longrightarrow 0)$ F **shows** $(f \longrightarrow 0)$ F
by (*rule Lim-null-comparison*[*OF assms*(*1*)] *tendsto-eq-intros assms*(*2*))+ *simp*

51.5 Holomorphic functions

definition *field-differentiable* :: [*'a* ⇒ *'a*::*real-normed-field*, *'a filter*] ⇒ *bool*
(**infixr** (*field'-differentiable*) *50*)
where *f field-differentiable F* ≡ ∃*f'*. (*f has-field-derivative f'*) *F*

lemma *field-differentiable-derivI*:
f field-differentiable (*at x*) ⟹ (*f has-field-derivative deriv f x*) (*at x*)
by (*simp add*: *field-differentiable-def DERIV-deriv-iff-has-field-derivative*)

lemma *field-differentiable-imp-continuous-at*:
f field-differentiable (*at x within s*) ⟹ *continuous* (*at x within s*) *f*
by (*metis DERIV-continuous field-differentiable-def*)

lemma *field-differentiable-within-subset*:
⟦*f field-differentiable* (*at x within s*); *t* ⊆ *s*⟧
⟹ *f field-differentiable* (*at x within t*)
by (*metis DERIV-subset field-differentiable-def*)

lemma *field-differentiable-at-within*:
⟦*f field-differentiable* (*at x*)⟧
⟹ *f field-differentiable* (*at x within s*)
unfolding *field-differentiable-def*
by (*metis DERIV-subset top-greatest*)

lemma *field-differentiable-linear* [*simp*,*derivative-intros*]: (*op* ∗ *c*) *field-differentiable F*
proof −
show *?thesis*
unfolding *field-differentiable-def has-field-derivative-def mult-commute-abs*
by (*force intro*: *has-derivative-mult-right*)
qed

lemma *field-differentiable-const* [*simp*,*derivative-intros*]: (λ*z*. *c*) *field-differentiable F*
unfolding *field-differentiable-def has-field-derivative-def*
by (*rule exI* [**where** *x*=*0*])
(*metis has-derivative-const lambda-zero*)

lemma *field-differentiable-ident* [*simp*,*derivative-intros*]: (λ*z*. *z*) *field-differentiable F*
unfolding *field-differentiable-def has-field-derivative-def*
by (*rule exI* [**where** *x*=*1*])
(*simp add*: *lambda-one* [*symmetric*])

lemma *field-differentiable-id* [*simp*,*derivative-intros*]: *id field-differentiable F*
unfolding *id-def* **by** (*rule field-differentiable-ident*)

lemma *field-differentiable-minus* [*derivative-intros*]:
f field-differentiable F ⟹ (λ*z*. − (*f z*)) *field-differentiable F*
using *assms* **unfolding** *field-differentiable-def*
by (*metis field-differentiable-minus*)

lemma *field-differentiable-add* [*derivative-intros*]:
assumes *f field-differentiable F g field-differentiable F*
shows (λ*z*. *f z* + *g z*) *field-differentiable F*
using *assms* **unfolding** *field-differentiable-def*
by (*metis field-differentiable-add*)

lemma *field-differentiable-add-const* [*simp*,*derivative-intros*]:
op + *c field-differentiable F*
by (*simp add*: *field-differentiable-add*)

lemma *field-differentiable-setsum* [*derivative-intros*]:
(⋀*i*. *i* ∈ *I* ⟹ (*f i*) *field-differentiable F*) ⟹ (λ*z*. ∑ *i*∈*I*. *f i z*) *field-differentiable F*
by (*induct I rule*: *infinite-finite-induct*)
(*auto intro*: *field-differentiable-add field-differentiable-const*)

lemma *field-differentiable-diff* [*derivative-intros*]:
assumes *f field-differentiable F g field-differentiable F*
shows (λ*z*. *f z* − *g z*) *field-differentiable F*
using *assms* **unfolding** *field-differentiable-def*
by (*metis field-differentiable-diff*)

lemma *field-differentiable-inverse* [*derivative-intros*]:
assumes *f field-differentiable* (*at a within s*) *f a* ≠ *0*
shows (λ*z*. *inverse* (*f z*)) *field-differentiable* (*at a within s*)
using *assms* **unfolding** *field-differentiable-def*
by (*metis DERIV-inverse-fun*)

lemma *field-differentiable-mult* [*derivative-intros*]:
assumes *f field-differentiable* (*at a within s*)
g field-differentiable (*at a within s*)
shows (λ*z*. *f z* ∗ *g z*) *field-differentiable* (*at a within s*)
using *assms* **unfolding** *field-differentiable-def*
by (*metis DERIV-mult* [*of f - a s g*])

lemma *field-differentiable-divide* [*derivative-intros*]:
assumes *f field-differentiable* (*at a within s*)
g field-differentiable (*at a within s*)
g a ≠ *0*
shows (λ*z*. *f z* / *g z*) *field-differentiable* (*at a within s*)
using *assms* **unfolding** *field-differentiable-def*
by (*metis DERIV-divide* [*of f - a s g*])

lemma *field-differentiable-power* [*derivative-intros*]:
assumes *f field-differentiable* (*at a within s*)
shows (λ*z*. *f z* ^ *n*) *field-differentiable* (*at a within s*)
using *assms* **unfolding** *field-differentiable-def*
by (*metis DERIV-power*)

lemma *field-differentiable-transform-within*:
0 < *d* ⟹
x ∈ *s* ⟹
(⋀*x'*. *x'* ∈ *s* ⟹ *dist x' x* < *d* ⟹ *f x'* = *g x'*) ⟹
f field-differentiable (*at x within s*)
⟹ *g field-differentiable* (*at x within s*)
unfolding *field-differentiable-def has-field-derivative-def*
by (*blast intro*: *has-derivative-transform-within*)

lemma *field-differentiable-compose-within*:
assumes *f field-differentiable* (*at a within s*)
g field-differentiable (*at* (*f a*) *within f*‘*s*)
shows (*g o f*) *field-differentiable* (*at a within s*)
using *assms* **unfolding** *field-differentiable-def*
by (*metis DERIV-image-chain*)

lemma *field-differentiable-compose*:
f field-differentiable at z ⟹ *g field-differentiable at* (*f z*)
⟹ (*g o f*) *field-differentiable at z*
by (*metis field-differentiable-at-within field-differentiable-compose-within*)

lemma *field-differentiable-within-open*:
$[\![a \in s;\ open\ s]\!] \Longrightarrow f\ field\text{-}differentiable\ at\ a\ within\ s \longleftrightarrow$
$f\ field\text{-}differentiable\ at\ a$
unfolding *field-differentiable-def*
by (*metis at-within-open*)

51.6 Caratheodory characterization

lemma *field-differentiable-caratheodory-at*:
$f\ field\text{-}differentiable\ (at\ z) \longleftrightarrow$
$(\exists g.\ (\forall w.\ f(w) - f(z) = g(w) * (w - z)) \wedge continuous\ (at\ z)\ g)$
using *CARAT-DERIV* [*of f*]
by (*simp add*: *field-differentiable-def has-field-derivative-def*)

lemma *field-differentiable-caratheodory-within*:
$f\ field\text{-}differentiable\ (at\ z\ within\ s) \longleftrightarrow$
$(\exists g.\ (\forall w.\ f(w) - f(z) = g(w) * (w - z)) \wedge continuous\ (at\ z\ within\ s)\ g)$
using *DERIV-caratheodory-within* [*of f*]
by (*simp add*: *field-differentiable-def has-field-derivative-def*)

51.7 Holomorphic

definition *holomorphic-on* :: [*complex* ⇒ *complex*, *complex set*] ⇒ *bool*
(**infixl** (*holomorphic'-on*) *50*)
where $f\ holomorphic\text{-}on\ s \equiv \forall x{\in}s.\ f\ field\text{-}differentiable\ (at\ x\ within\ s)$

named-theorems *holomorphic-intros structural introduction rules for holomorphic-on*

lemma *holomorphic-onI* [*intro?*]: $(\bigwedge x.\ x \in s \Longrightarrow f\ field\text{-}differentiable\ (at\ x\ within\ s)) \Longrightarrow f\ holomorphic\text{-}on\ s$
by (*simp add*: *holomorphic-on-def*)

lemma *holomorphic-onD* [*dest?*]: $[\![f\ holomorphic\text{-}on\ s;\ x \in s]\!] \Longrightarrow f\ field\text{-}differentiable\ (at\ x\ within\ s)$
by (*simp add*: *holomorphic-on-def*)

lemma *holomorphic-on-imp-differentiable-at*:
$[\![f\ holomorphic\text{-}on\ s;\ open\ s;\ x \in s]\!] \Longrightarrow f\ field\text{-}differentiable\ (at\ x)$
using *at-within-open holomorphic-on-def* **by** *fastforce*

lemma *holomorphic-on-empty* [*holomorphic-intros*]: $f\ holomorphic\text{-}on\ \{\}$
by (*simp add*: *holomorphic-on-def*)

lemma *holomorphic-on-open*:
$open\ s \Longrightarrow f\ holomorphic\text{-}on\ s \longleftrightarrow (\forall x \in s.\ \exists f'.\ DERIV\ f\ x :> f')$
by (*auto simp*: *holomorphic-on-def field-differentiable-def has-field-derivative-def at-within-open* [*of - s*])

lemma *holomorphic-on-imp-continuous-on*:
$f\ holomorphic\text{-}on\ s \Longrightarrow continuous\text{-}on\ s\ f$

by (*metis field-differentiable-imp-continuous-at continuous-on-eq-continuous-within holomorphic-on-def*)

lemma *holomorphic-on-subset* [*elim*]:
f holomorphic-on s $\Longrightarrow$ *t* $\subseteq$ *s* $\Longrightarrow$ *f holomorphic-on t*
unfolding *holomorphic-on-def*
by (*metis field-differentiable-within-subset subsetD*)

lemma *holomorphic-transform*: $[\![f$ *holomorphic-on s*; $\bigwedge x.\ x \in s \Longrightarrow f\ x = g\ x]\!]$ $\Longrightarrow$ *g holomorphic-on s*
by (*metis field-differentiable-transform-within linordered-field-no-ub holomorphic-on-def*)

lemma *holomorphic-cong*: *s* = *t* ==> $(\bigwedge x.\ x \in s \Longrightarrow f\,x = g\,x) \Longrightarrow f$ *holomorphic-on s* $\longleftrightarrow$ *g holomorphic-on t*
by (*metis holomorphic-transform*)

lemma *holomorphic-on-linear* [*simp*, *holomorphic-intros*]: (*op* $*$ *c*) *holomorphic-on s*
unfolding *holomorphic-on-def* **by** (*metis field-differentiable-linear*)

lemma *holomorphic-on-const* [*simp*, *holomorphic-intros*]: ($\lambda z.\ c$) *holomorphic-on s*
unfolding *holomorphic-on-def* **by** (*metis field-differentiable-const*)

lemma *holomorphic-on-ident* [*simp*, *holomorphic-intros*]: ($\lambda x.\ x$) *holomorphic-on s*
unfolding *holomorphic-on-def* **by** (*metis field-differentiable-ident*)

lemma *holomorphic-on-id* [*simp*, *holomorphic-intros*]: *id holomorphic-on s*
unfolding *id-def* **by** (*rule holomorphic-on-ident*)

lemma *holomorphic-on-compose*:
f holomorphic-on s $\Longrightarrow$ *g holomorphic-on* (*f* ‘ *s*) $\Longrightarrow$ (*g o f*) *holomorphic-on s*
using *field-differentiable-compose-within*[*of f - s g*]
by (*auto simp*: *holomorphic-on-def*)

lemma *holomorphic-on-compose-gen*:
f holomorphic-on s $\Longrightarrow$ *g holomorphic-on t* $\Longrightarrow$ *f* ‘ *s* $\subseteq$ *t* $\Longrightarrow$ (*g o f*) *holomorphic-on s*
by (*metis holomorphic-on-compose holomorphic-on-subset*)

lemma *holomorphic-on-minus* [*holomorphic-intros*]: *f holomorphic-on s* $\Longrightarrow$ ($\lambda z.$ $-(f\ z)$) *holomorphic-on s*
by (*metis field-differentiable-minus holomorphic-on-def*)

lemma *holomorphic-on-add* [*holomorphic-intros*]:
$[\![f$ *holomorphic-on s*; *g holomorphic-on s*$]\!]$ $\Longrightarrow$ ($\lambda z.$ *f z* + *g z*) *holomorphic-on s*
unfolding *holomorphic-on-def* **by** (*metis field-differentiable-add*)

lemma *holomorphic-on-diff* [*holomorphic-intros*]:
⟦*f holomorphic-on s*; *g holomorphic-on s*⟧ ⟹ (λ*z*. *f z* − *g z*) *holomorphic-on s*
unfolding *holomorphic-on-def* **by** (*metis field-differentiable-diff*)

lemma *holomorphic-on-mult* [*holomorphic-intros*]:
⟦*f holomorphic-on s*; *g holomorphic-on s*⟧ ⟹ (λ*z*. *f z* ∗ *g z*) *holomorphic-on s*
unfolding *holomorphic-on-def* **by** (*metis field-differentiable-mult*)

lemma *holomorphic-on-inverse* [*holomorphic-intros*]:
⟦*f holomorphic-on s*; ⋀*z*. *z* ∈ *s* ⟹ *f z* ≠ *0*⟧ ⟹ (λ*z*. *inverse* (*f z*)) *holomorphic-on s*
unfolding *holomorphic-on-def* **by** (*metis field-differentiable-inverse*)

lemma *holomorphic-on-divide* [*holomorphic-intros*]:
⟦*f holomorphic-on s*; *g holomorphic-on s*; ⋀*z*. *z* ∈ *s* ⟹ *g z* ≠ *0*⟧ ⟹ (λ*z*. *f z* / *g z*) *holomorphic-on s*
unfolding *holomorphic-on-def* **by** (*metis field-differentiable-divide*)

lemma *holomorphic-on-power* [*holomorphic-intros*]:
f holomorphic-on s ⟹ (λ*z*. (*f z*) ^ *n*) *holomorphic-on s*
unfolding *holomorphic-on-def* **by** (*metis field-differentiable-power*)

lemma *holomorphic-on-setsum* [*holomorphic-intros*]:
(⋀*i*. *i* ∈ *I* ⟹ (*f i*) *holomorphic-on s*) ⟹ (λ*x*. *setsum* (λ*i*. *f i x*) *I*) *holomorphic-on s*
unfolding *holomorphic-on-def* **by** (*metis field-differentiable-setsum*)

lemma *DERIV-deriv-iff-field-differentiable*:
DERIV f x :> *deriv f x* ⟷ *f field-differentiable at x*
unfolding *field-differentiable-def* **by** (*metis DERIV-imp-deriv*)

lemma *holomorphic-derivI*:
⟦*f holomorphic-on S*; *open S*; *x* ∈ *S*⟧
⟹ (*f has-field-derivative deriv f x*) (*at x within T*)
by (*metis DERIV-deriv-iff-field-differentiable at-within-open holomorphic-on-def has-field-derivative-at-within*)

lemma *complex-derivative-chain*:
f field-differentiable at x ⟹ *g field-differentiable at* (*f x*)
⟹ *deriv* (*g o f*) *x* = *deriv g* (*f x*) ∗ *deriv f x*
by (*metis DERIV-deriv-iff-field-differentiable DERIV-chain DERIV-imp-deriv*)

lemma *deriv-linear* [*simp*]: *deriv* (λ*w*. *c* ∗ *w*) = (λ*z*. *c*)
by (*metis DERIV-imp-deriv DERIV-cmult-Id*)

lemma *deriv-ident* [*simp*]: *deriv* (λ*w*. *w*) = (λ*z*. *1*)
by (*metis DERIV-imp-deriv DERIV-ident*)

lemma *deriv-id* [*simp*]: *deriv id* = (λ*z*. *1*)

by (*simp add*: *id-def*)

lemma *deriv-const* [*simp*]: *deriv* (λ*w*. *c*) = (λ*z*. *0*)
by (*metis DERIV-imp-deriv DERIV-const*)

lemma *deriv-add* [*simp*]:
⟦*f field-differentiable at z*; *g field-differentiable at z*⟧
⟹ *deriv* (λ*w*. *f w* + *g w*) *z* = *deriv f z* + *deriv g z*
unfolding *DERIV-deriv-iff-field-differentiable*[*symmetric*]
by (*auto intro*!: *DERIV-imp-deriv derivative-intros*)

lemma *deriv-diff* [*simp*]:
⟦*f field-differentiable at z*; *g field-differentiable at z*⟧
⟹ *deriv* (λ*w*. *f w* − *g w*) *z* = *deriv f z* − *deriv g z*
unfolding *DERIV-deriv-iff-field-differentiable*[*symmetric*]
by (*auto intro*!: *DERIV-imp-deriv derivative-intros*)

lemma *deriv-mult* [*simp*]:
⟦*f field-differentiable at z*; *g field-differentiable at z*⟧
⟹ *deriv* (λ*w*. *f w* ∗ *g w*) *z* = *f z* ∗ *deriv g z* + *deriv f z* ∗ *g z*
unfolding *DERIV-deriv-iff-field-differentiable*[*symmetric*]
by (*auto intro*!: *DERIV-imp-deriv derivative-eq-intros*)

lemma *deriv-cmult* [*simp*]:
f field-differentiable at z ⟹ *deriv* (λ*w*. *c* ∗ *f w*) *z* = *c* ∗ *deriv f z*
unfolding *DERIV-deriv-iff-field-differentiable*[*symmetric*]
by (*auto intro*!: *DERIV-imp-deriv derivative-eq-intros*)

lemma *deriv-cmult-right* [*simp*]:
f field-differentiable at z ⟹ *deriv* (λ*w*. *f w* ∗ *c*) *z* = *deriv f z* ∗ *c*
unfolding *DERIV-deriv-iff-field-differentiable*[*symmetric*]
by (*auto intro*!: *DERIV-imp-deriv derivative-eq-intros*)

lemma *deriv-cdivide-right* [*simp*]:
f field-differentiable at z ⟹ *deriv* (λ*w*. *f w* / *c*) *z* = *deriv f z* / *c*
unfolding *Fields.field-class.field-divide-inverse*
by (*blast intro*: *deriv-cmult-right*)

lemma *complex-derivative-transform-within-open*:
⟦*f holomorphic-on s*; *g holomorphic-on s*; *open s*; *z* ∈ *s*; ⋀*w*. *w* ∈ *s* ⟹ *f w* = *g w*⟧
⟹ *deriv f z* = *deriv g z*
unfolding *holomorphic-on-def*
by (*rule DERIV-imp-deriv*)
(*metis DERIV-deriv-iff-field-differentiable DERIV-transform-within-open at-within-open*)

lemma *deriv-compose-linear*:
f field-differentiable at (*c* ∗ *z*) ⟹ *deriv* (λ*w*. *f* (*c* ∗ *w*)) *z* = *c* ∗ *deriv f* (*c* ∗ *z*)
apply (*rule DERIV-imp-deriv*)

apply (*simp add*: *DERIV-deriv-iff-field-differentiable* [*symmetric*])
apply (*drule DERIV-chain'* [*of times c c z UNIV f deriv f* $(c * z)$, *OF DERIV-cmult-Id*])
apply (*simp add*: *algebra-simps*)
done

lemma *nonzero-deriv-nonconstant*:
assumes *df*: *DERIV f* ξ :> *df* **and** *S*: *open S* $\xi \in S$ **and** $df \neq 0$
shows $\neg$ *f constant-on S*
unfolding *constant-on-def*
by (*metis* ‹$df \neq 0$› *DERIV-transform-within-open* [*OF df S*] *DERIV-const DERIV-unique*)

lemma *holomorphic-nonconstant*:
assumes *holf*: *f holomorphic-on S* **and** *open S* $\xi \in S$ *deriv f* $\xi \neq 0$
shows $\neg$ *f constant-on S*
apply (*rule nonzero-deriv-nonconstant* [*of f deriv f* ξ ξ *S*])
using *assms*
apply (*auto simp*: *holomorphic-derivI*)
done

51.8 Analyticity on a set

definition *analytic-on* (**infixl** (*analytic'-on*) *50*)
where
f analytic-on s $\equiv \forall x \in s.\ \exists e.\ 0 < e \wedge$ *f holomorphic-on* (*ball x e*)

lemma *analytic-imp-holomorphic*: *f analytic-on s* $\Longrightarrow$ *f holomorphic-on s*
by (*simp add*: *at-within-open* [*OF - open-ball*] *analytic-on-def holomorphic-on-def*)
(*metis centre-in-ball field-differentiable-at-within*)

lemma *analytic-on-open*: *open s* $\Longrightarrow$ *f analytic-on s* $\longleftrightarrow$ *f holomorphic-on s*
apply (*auto simp*: *analytic-imp-holomorphic*)
apply (*auto simp*: *analytic-on-def holomorphic-on-def*)
by (*metis holomorphic-on-def holomorphic-on-subset open-contains-ball*)

lemma *analytic-on-imp-differentiable-at*:
f analytic-on s $\Longrightarrow$ $x \in s \Longrightarrow$ *f field-differentiable* (*at x*)
apply (*auto simp*: *analytic-on-def holomorphic-on-def*)
by (*metis Topology-Euclidean-Space.open-ball centre-in-ball field-differentiable-within-open*)

lemma *analytic-on-subset*: *f analytic-on s* $\Longrightarrow$ $t \subseteq s \Longrightarrow$ *f analytic-on t*
by (*auto simp*: *analytic-on-def*)

lemma *analytic-on-Un*: *f analytic-on* $(s \cup t) \longleftrightarrow$ *f analytic-on s* $\wedge$ *f analytic-on t*
by (*auto simp*: *analytic-on-def*)

lemma *analytic-on-Union*: *f analytic-on* $(\bigcup s) \longleftrightarrow (\forall t \in s.$ *f analytic-on t*$)$
by (*auto simp*: *analytic-on-def*)

lemma *analytic-on-UN*: *f analytic-on* $(\bigcup i{\in}I.\ s\ i) \longleftrightarrow (\forall i{\in}I.$ *f analytic-on* $(s\ i))$

by (*auto simp*: *analytic-on-def*)

lemma *analytic-on-holomorphic*:
f analytic-on s $\longleftrightarrow$ ($\exists$ *t*. *open t* $\wedge$ *s* $\subseteq$ *t* $\wedge$ *f holomorphic-on t*)
(**is** *?lhs* = *?rhs*)
proof −
have *?lhs* $\longleftrightarrow$ ($\exists$ *t*. *open t* $\wedge$ *s* $\subseteq$ *t* $\wedge$ *f analytic-on t*)
proof *safe*
assume *f analytic-on s*
then show $\exists$ *t*. *open t* $\wedge$ *s* $\subseteq$ *t* $\wedge$ *f analytic-on t*
apply (*simp add*: *analytic-on-def*)
apply (*rule exI* [**where** *x*=$\bigcup$ {*u*. *open u* $\wedge$ *f analytic-on u*}], *auto*)
apply (*metis Topology-Euclidean-Space.open-ball analytic-on-open centre-in-ball*)
by (*metis analytic-on-def*)
next
fix *t*
assume *open t s* $\subseteq$ *t f analytic-on t*
then show *f analytic-on s*
by (*metis analytic-on-subset*)
qed
also have ... $\longleftrightarrow$ *?rhs*
by (*auto simp*: *analytic-on-open*)
finally show *?thesis* **.**
qed

lemma *analytic-on-linear*: (*op* ∗ *c*) *analytic-on s*
by (*auto simp add*: *analytic-on-holomorphic holomorphic-on-linear*)

lemma *analytic-on-const*: (λ*z*. *c*) *analytic-on s*
by (*metis analytic-on-def holomorphic-on-const zero-less-one*)

lemma *analytic-on-ident*: (λ*x*. *x*) *analytic-on s*
by (*simp add*: *analytic-on-def holomorphic-on-ident gt-ex*)

lemma *analytic-on-id*: *id analytic-on s*
unfolding *id-def* **by** (*rule analytic-on-ident*)

lemma *analytic-on-compose*:
assumes *f*: *f analytic-on s*
and *g*: *g analytic-on* (*f* ‘ *s*)
shows (*g o f*) *analytic-on s*
unfolding *analytic-on-def*
proof (*intro ballI*)
fix *x*
assume *x*: *x* ∈ *s*
then obtain *e* **where** *e*: *0* < *e* **and** *fh*: *f holomorphic-on ball x e* **using** *f*
by (*metis analytic-on-def*)
obtain *e′* **where** *e′*: *0* < *e′* **and** *gh*: *g holomorphic-on ball* (*f x*) *e′* **using** *g*
by (*metis analytic-on-def g image-eqI x*)

have *isCont f x*
by (*metis analytic-on-imp-differentiable-at field-differentiable-imp-continuous-at f x*)
with *e'* **obtain** *d* **where** *d*: *0 < d* **and** *fd*: *f ' ball x d ⊆ ball (f x) e'*
by (*auto simp*: *continuous-at-ball*)
have *g ∘ f holomorphic-on ball x (min d e)*
apply (*rule holomorphic-on-compose*)
apply (*metis fh holomorphic-on-subset min.bounded-iff order-refl subset-ball*)
by (*metis fd gh holomorphic-on-subset image-mono min.cobounded1 subset-ball*)
then show ∃ *e>0*. *g ∘ f holomorphic-on ball x e*
by (*metis d e min-less-iff-conj*)
qed

lemma *analytic-on-compose-gen*:
f analytic-on s ⟹ g analytic-on t ⟹ (⋀z. z ∈ s ⟹ f z ∈ t)
⟹ g o f analytic-on s
by (*metis analytic-on-compose analytic-on-subset image-subset-iff*)

lemma *analytic-on-neg*:
f analytic-on s ⟹ (λz. −(f z)) analytic-on s
by (*metis analytic-on-holomorphic holomorphic-on-minus*)

lemma *analytic-on-add*:
assumes *f*: *f analytic-on s*
and *g*: *g analytic-on s*
shows *(λz. f z + g z) analytic-on s*
unfolding *analytic-on-def*
proof (*intro ballI*)
fix *z*
assume *z*: *z ∈ s*
then obtain *e* **where** *e*: *0 < e* **and** *fh*: *f holomorphic-on ball z e* **using** *f*
by (*metis analytic-on-def*)
obtain *e'* **where** *e'*: *0 < e'* **and** *gh*: *g holomorphic-on ball z e'* **using** *g*
by (*metis analytic-on-def g z*)
have *(λz. f z + g z) holomorphic-on ball z (min e e')*
apply (*rule holomorphic-on-add*)
apply (*metis fh holomorphic-on-subset min.bounded-iff order-refl subset-ball*)
by (*metis gh holomorphic-on-subset min.bounded-iff order-refl subset-ball*)
then show ∃ *e>0*. *(λz. f z + g z) holomorphic-on ball z e*
by (*metis e e' min-less-iff-conj*)
qed

lemma *analytic-on-diff*:
assumes *f*: *f analytic-on s*
and *g*: *g analytic-on s*
shows *(λz. f z − g z) analytic-on s*
unfolding *analytic-on-def*
proof (*intro ballI*)
fix *z*

assume z: $z \in s$
then obtain e **where** e: $0 < e$ **and** fh: f *holomorphic-on ball* z e **using** f
by (*metis analytic-on-def*)
obtain e' **where** e': $0 < e'$ **and** gh: g *holomorphic-on ball* z e' **using** g
by (*metis analytic-on-def g z*)
have $(\lambda z.\ f\ z - g\ z)$ *holomorphic-on ball* z (*min e e'*)
apply (*rule holomorphic-on-diff*)
apply (*metis fh holomorphic-on-subset min.bounded-iff order-refl subset-ball*)
by (*metis gh holomorphic-on-subset min.bounded-iff order-refl subset-ball*)
then show $\exists e{>}0.\ (\lambda z.\ f\ z - g\ z)$ *holomorphic-on ball* z e
by (*metis e e' min-less-iff-conj*)
qed

lemma *analytic-on-mult*:
assumes f: f *analytic-on* s
and g: g *analytic-on* s
shows $(\lambda z.\ f\ z * g\ z)$ *analytic-on* s
unfolding *analytic-on-def*
proof (*intro ballI*)
fix z
assume z: $z \in s$
then obtain e **where** e: $0 < e$ **and** fh: f *holomorphic-on ball* z e **using** f
by (*metis analytic-on-def*)
obtain e' **where** e': $0 < e'$ **and** gh: g *holomorphic-on ball* z e' **using** g
by (*metis analytic-on-def g z*)
have $(\lambda z.\ f\ z * g\ z)$ *holomorphic-on ball* z (*min e e'*)
apply (*rule holomorphic-on-mult*)
apply (*metis fh holomorphic-on-subset min.bounded-iff order-refl subset-ball*)
by (*metis gh holomorphic-on-subset min.bounded-iff order-refl subset-ball*)
then show $\exists e{>}0.\ (\lambda z.\ f\ z * g\ z)$ *holomorphic-on ball* z e
by (*metis e e' min-less-iff-conj*)
qed

lemma *analytic-on-inverse*:
assumes f: f *analytic-on* s
and nz: $(\bigwedge z.\ z \in s \Longrightarrow f\ z \neq 0)$
shows $(\lambda z.\ inverse\ (f\ z))$ *analytic-on* s
unfolding *analytic-on-def*
proof (*intro ballI*)
fix z
assume z: $z \in s$
then obtain e **where** e: $0 < e$ **and** fh: f *holomorphic-on ball* z e **using** f
by (*metis analytic-on-def*)
have *continuous-on* (*ball z e*) f
by (*metis fh holomorphic-on-imp-continuous-on*)
then obtain e' **where** e': $0 < e'$ **and** nz': $\bigwedge y.\ dist\ z\ y < e' \Longrightarrow f\ y \neq 0$
by (*metis Topology-Euclidean-Space.open-ball centre-in-ball continuous-on-open-avoid e z nz*)
have $(\lambda z.\ inverse\ (f\ z))$ *holomorphic-on ball* z (*min e e'*)

apply (*rule holomorphic-on-inverse*)
apply (*metis fh holomorphic-on-subset min.cobounded2 min.commute subset-ball*)
by (*metis nz' mem-ball min-less-iff-conj*)
then show $\exists e{>}0.$ (*λz. inverse* (*f z*)) *holomorphic-on ball z e*
by (*metis e e' min-less-iff-conj*)
qed

lemma *analytic-on-divide*:
assumes *f*: *f analytic-on s*
and *g*: *g analytic-on s*
and *nz*: (⋀*z*. *z* ∈ *s* ⟹ *g z* ≠ *0*)
shows (*λz*. *f z* / *g z*) *analytic-on s*
unfolding *divide-inverse*
by (*metis analytic-on-inverse analytic-on-mult f g nz*)

lemma *analytic-on-power*:
f analytic-on s ⟹ (*λz*. (*f z*) ^ *n*) *analytic-on s*
by (*induct n*) (*auto simp*: *analytic-on-const analytic-on-mult*)

lemma *analytic-on-setsum*:
(⋀*i*. *i* ∈ *I* ⟹ (*f i*) *analytic-on s*) ⟹ (*λx*. *setsum* (*λi*. *f i x*) *I*) *analytic-on s*
by (*induct I rule*: *infinite-finite-induct*) (*auto simp*: *analytic-on-const analytic-on-add*)

lemma *deriv-left-inverse*:
assumes *f holomorphic-on S* **and** *g holomorphic-on T*
and *open S* **and** *open T*
and *f* ‘ *S* ⊆ *T*
and [*simp*]: ⋀*z*. *z* ∈ *S* ⟹ *g* (*f z*) = *z*
and *w* ∈ *S*
shows *deriv f w* ∗ *deriv g* (*f w*) = *1*
proof −
have *deriv f w* ∗ *deriv g* (*f w*) = *deriv g* (*f w*) ∗ *deriv f w*
by (*simp add*: *algebra-simps*)
also have ... = *deriv* (*g o f*) *w*
using *assms*
by (*metis analytic-on-imp-differentiable-at analytic-on-open complex-derivative-chain image-subset-iff*)
also have ... = *deriv id w*
apply (*rule complex-derivative-transform-within-open* [**where** *s*=*S*])
apply (*rule assms holomorphic-on-compose-gen holomorphic-intros*)+
apply *simp*
done
also have ... = *1*
by *simp*
finally show *?thesis* **.**
qed

51.9 analyticity at a point

lemma *analytic-at-ball*:
f analytic-on $\{z\} \longleftrightarrow (\exists e.\ 0<e \wedge f$ *holomorphic-on ball* $z\ e)$
by (*metis analytic-on-def singleton-iff*)

lemma *analytic-at*:
f *analytic-on* $\{z\} \longleftrightarrow (\exists s.$ *open* $s \wedge z \in s \wedge f$ *holomorphic-on* $s)$
by (*metis analytic-on-holomorphic empty-subsetI insert-subset*)

lemma *analytic-on-analytic-at*:
f *analytic-on* $s \longleftrightarrow (\forall z \in s.\ f$ *analytic-on* $\{z\})$
by (*metis analytic-at-ball analytic-on-def*)

lemma *analytic-at-two*:
f analytic-on $\{z\} \wedge g$ *analytic-on* $\{z\} \longleftrightarrow$
$(\exists s.$ *open* $s \wedge z \in s \wedge f$ *holomorphic-on* $s \wedge g$ *holomorphic-on* $s)$
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then obtain *s t*
where *st*: *open s z* $\in$ *s f holomorphic-on s*
open t z $\in$ *t g holomorphic-on t*
by (*auto simp*: *analytic-at*)
show *?rhs*
apply (*rule-tac x*=$s \cap t$ **in** *exI*)
using *st*
apply (*auto simp*: *Diff-subset holomorphic-on-subset*)
done
next
assume *?rhs*
then show *?lhs*
by (*force simp add*: *analytic-at*)
qed

51.10 Combining theorems for derivative with "analytic at" hypotheses

lemma
assumes *f analytic-on* $\{z\}$ *g analytic-on* $\{z\}$
shows *complex-derivative-add-at*: *deriv* $(\lambda w.\ f\ w + g\ w)\ z =$ *deriv f z* + *deriv g z*
and *complex-derivative-diff-at*: *deriv* $(\lambda w.\ f\ w - g\ w)\ z =$ *deriv f z* − *deriv g z*
and *complex-derivative-mult-at*: *deriv* $(\lambda w.\ f\ w * g\ w)\ z =$
f z ∗ *deriv g z* + *deriv f z* ∗ *g z*
proof −
obtain *s* **where** *s*: *open s z* $\in$ *s f holomorphic-on s g holomorphic-on s*
using *assms* **by** (*metis analytic-at-two*)
show *deriv* $(\lambda w.\ f\ w + g\ w)\ z =$ *deriv f z* + *deriv g z*
apply (*rule DERIV-imp-deriv* [*OF DERIV-add*])

using *s*
apply (*auto simp*: *holomorphic-on-open field-differentiable-def DERIV-deriv-iff-field-differentiable*)
done
show *deriv* (λ*w*. *f w* − *g w*) *z* = *deriv f z* − *deriv g z*
apply (*rule DERIV-imp-deriv* [*OF DERIV-diff*])
using *s*
apply (*auto simp*: *holomorphic-on-open field-differentiable-def DERIV-deriv-iff-field-differentiable*)
done
show *deriv* (λ*w*. *f w* ∗ *g w*) *z* = *f z* ∗ *deriv g z* + *deriv f z* ∗ *g z*
apply (*rule DERIV-imp-deriv* [*OF DERIV-mult'*])
using *s*
apply (*auto simp*: *holomorphic-on-open field-differentiable-def DERIV-deriv-iff-field-differentiable*)
done
qed

lemma *deriv-cmult-at*:
f analytic-on {*z*} ⟹ *deriv* (λ*w*. *c* ∗ *f w*) *z* = *c* ∗ *deriv f z*
by (*auto simp*: *complex-derivative-mult-at deriv-const analytic-on-const*)

lemma *deriv-cmult-right-at*:
f analytic-on {*z*} ⟹ *deriv* (λ*w*. *f w* ∗ *c*) *z* = *deriv f z* ∗ *c*
by (*auto simp*: *complex-derivative-mult-at deriv-const analytic-on-const*)

51.11 Complex differentiation of sequences and series

lemma *has-complex-derivative-sequence*:
fixes *s* :: *complex set*
assumes *cvs*: *convex s*
and *df*: ⋀*n x*. *x* ∈ *s* ⟹ (*f n has-field-derivative f' n x*) (*at x within s*)
and *conv*: ⋀*e*. 0 < *e* ⟹ ∃*N*. ∀*n x*. *n* ≥ *N* ⟶ *x* ∈ *s* ⟶ *norm* (*f' n x* − *g' x*) ≤ *e*
and ∃*x l*. *x* ∈ *s* ∧ ((λ*n*. *f n x*) ⟶ *l*) *sequentially*
shows ∃*g*. ∀*x* ∈ *s*. ((λ*n*. *f n x*) ⟶ *g x*) *sequentially* ∧
(*g has-field-derivative* (*g' x*)) (*at x within s*)
proof −
from *assms* **obtain** *x l* **where** *x*: *x* ∈ *s* **and** *tf*: ((λ*n*. *f n x*) ⟶ *l*) *sequentially*
by *blast*
{ fix *e*::*real* **assume** *e*: *e* > 0
then obtain *N* **where** *N*: ∀*n*≥*N*. ∀*x*. *x* ∈ *s* ⟶ *cmod* (*f' n x* − *g' x*) ≤ *e*
by (*metis conv*)
have ∃*N*. ∀*n*≥*N*. ∀*x*∈*s*. ∀*h*. *cmod* (*f' n x* ∗ *h* − *g' x* ∗ *h*) ≤ *e* ∗ *cmod h*
proof (*rule exI* [*of - N*], *clarify*)
fix *n y h*
assume *N* ≤ *n y* ∈ *s*
then have *cmod* (*f' n y* − *g' y*) ≤ *e*
by (*metis N*)
then have *cmod h* ∗ *cmod* (*f' n y* − *g' y*) ≤ *cmod h* ∗ *e*
by (*auto simp*: *antisym-conv2 mult-le-cancel-left norm-triangle-ineq2*)
then show *cmod* (*f' n y* ∗ *h* − *g' y* ∗ *h*) ≤ *e* ∗ *cmod h*

```
      by (simp add: norm-mult [symmetric] field-simps)
    qed
  } note ** = this
  show ?thesis
  unfolding has-field-derivative-def
  proof (rule has-derivative-sequence [OF cvs - - x])
    show ∀ n. ∀ x∈s. (f n has-derivative (op * (f' n x))) (at x within s)
      by (metis has-field-derivative-def df)
  next show (λn. f n x) ⟶⟶ l
    by (rule tf)
  next show ∀ e>0. ∃ N. ∀ n≥N. ∀ x∈s. ∀ h. cmod (f' n x * h − g' x * h) ≤ e *
cmod h
    by (blast intro: **)
  qed
qed

lemma has-complex-derivative-series:
  fixes s :: complex set
  assumes cvs: convex s
      and df: ⋀n x. x ∈ s ⟹ (f n has-field-derivative f' n x) (at x within s)
      and conv: ⋀e. 0 < e ⟹ ∃ N. ∀ n x. n ≥ N ⟶ x ∈ s
            ⟶ cmod ((∑ i<n. f' i x) − g' x) ≤ e
      and ∃ x l. x ∈ s ∧ ((λn. f n x) sums l)
    shows ∃ g. ∀ x ∈ s. ((λn. f n x) sums g x) ∧ ((g has-field-derivative g' x) (at x
within s))
proof −
  from assms obtain x l where x: x ∈ s and sf: ((λn. f n x) sums l)
    by blast
  { fix e::real assume e: e > 0
    then obtain N where N: ∀ n x. n ≥ N ⟶ x ∈ s
          ⟶ cmod ((∑ i<n. f' i x) − g' x) ≤ e
      by (metis conv)
    have ∃ N. ∀ n≥N. ∀ x∈s. ∀ h. cmod ((∑ i<n. h * f' i x) − g' x * h) ≤ e *
cmod h
    proof (rule exI [of - N], clarify)
      fix n y h
      assume N ≤ n y ∈ s
      then have cmod ((∑ i<n. f' i y) − g' y) ≤ e
        by (metis N)
      then have cmod h * cmod ((∑ i<n. f' i y) − g' y) ≤ cmod h * e
        by (auto simp: antisym-conv2 mult-le-cancel-left norm-triangle-ineq2)
      then show cmod ((∑ i<n. h * f' i y) − g' y * h) ≤ e * cmod h
        by (simp add: norm-mult [symmetric] field-simps setsum-right-distrib)
    qed
  } note ** = this
  show ?thesis
  unfolding has-field-derivative-def
  proof (rule has-derivative-series [OF cvs - - x])
    fix n x
```

assume $x \in s$
then show $((f\ n)$ *has-derivative* $(\lambda z.\ z * f'\ n\ x))$ $(at\ x\ within\ s)$
by (*metis df has-field-derivative-def mult-commute-abs*)
next show $((\lambda n.\ f\ n\ x)$ *sums* $l)$
by (*rule sf*)
next show $\forall e{>}0.\ \exists N.\ \forall n{\geq}N.\ \forall x{\in}s.\ \forall h.\ cmod\ ((\sum i{<}n.\ h * f'\ i\ x) - g'\ x * h) \leq e * cmod\ h$
by (*blast intro*: $**$)
qed
qed

lemma *field-differentiable-series*:
fixes $f :: nat \Rightarrow complex \Rightarrow complex$
assumes *convex s open s*
assumes $\bigwedge n\ x.\ x \in s \Longrightarrow (f\ n$ *has-field-derivative* $f'\ n\ x)$ $(at\ x)$
assumes *uniformly-convergent-on s* $(\lambda n\ x.\ \sum i{<}n.\ f'\ i\ x)$
assumes $x0 \in s$ *summable* $(\lambda n.\ f\ n\ x0)$ **and** x: $x \in s$
shows *summable* $(\lambda n.\ f\ n\ x)$ **and** $(\lambda x.\ \sum n.\ f\ n\ x)$ *field-differentiable* $(at\ x)$
proof −
from *assms*(*4*) **obtain** g' **where** A: *uniform-limit s* $(\lambda n\ x.\ \sum i{<}n.\ f'\ i\ x)$ g' *sequentially*
unfolding *uniformly-convergent-on-def* **by** *blast*
from x **and** ‹*open s*› **have** s: *at x within s* = *at x* **by** (*rule at-within-open*)
have $\exists g.\ \forall x{\in}s.\ (\lambda n.\ f\ n\ x)$ *sums* $g\ x \wedge (g$ *has-field-derivative* $g'\ x)$ $(at\ x\ within\ s)$
by (*intro has-field-derivative-series*[*of s f f' g' x0*] *assms A has-field-derivative-at-within*)
then obtain g **where** g: $\bigwedge x.\ x \in s \Longrightarrow (\lambda n.\ f\ n\ x)$ *sums* $g\ x$
$\bigwedge x.\ x \in s \Longrightarrow (g$ *has-field-derivative* $g'\ x)$ $(at\ x\ within\ s)$ **by** *blast*
from g[*OF x*] **show** *summable* $(\lambda n.\ f\ n\ x)$ **by** (*auto simp*: *summable-def*)
from $g(2)$[*OF x*] **have** g': $(g$ *has-derivative op* $*$ $(g'\ x))$ $(at\ x)$
by (*simp add*: *has-field-derivative-def s*)
have $((\lambda x.\ \sum n.\ f\ n\ x)$ *has-derivative op* $*$ $(g'\ x))$ $(at\ x)$
by (*rule has-derivative-transform-within-open*[*OF g'* ‹*open s*› *x*])
(*insert g*, *auto simp*: *sums-iff*)
thus $(\lambda x.\ \sum n.\ f\ n\ x)$ *field-differentiable* $(at\ x)$ **unfolding** *differentiable-def*
by (*auto simp*: *summable-def field-differentiable-def has-field-derivative-def*)
qed

lemma *field-differentiable-series'*:
fixes $f :: nat \Rightarrow complex \Rightarrow complex$
assumes *convex s open s*
assumes $\bigwedge n\ x.\ x \in s \Longrightarrow (f\ n$ *has-field-derivative* $f'\ n\ x)$ $(at\ x)$
assumes *uniformly-convergent-on s* $(\lambda n\ x.\ \sum i{<}n.\ f'\ i\ x)$
assumes $x0 \in s$ *summable* $(\lambda n.\ f\ n\ x0)$
shows $(\lambda x.\ \sum n.\ f\ n\ x)$ *field-differentiable* $(at\ x0)$
using *field-differentiable-series*[*OF assms*, *of x0*] ‹$x0 \in s$› **by** *blast*+

51.12 Bound theorem

lemma *field-differentiable-bound*:
fixes s :: *complex set*
assumes *cvs*: *convex s*
and *df*: $\bigwedge z.\ z \in s \Longrightarrow (f$ *has-field-derivative* $f'\ z)$ *(at z within s)*
and *dn*: $\bigwedge z.\ z \in s \Longrightarrow$ *norm* $(f'\ z) \leq B$
and $x \in s\ \ y \in s$
shows $norm(f\ x - f\ y) \leq B * norm(x - y)$
apply (*rule differentiable-bound* [*OF cvs*])
apply (*rule ballI*, *erule df* [*unfolded has-field-derivative-def*])
apply (*rule ballI*, *rule onorm-le*, *simp add*: *norm-mult mult-right-mono dn*)
apply *fact*
apply *fact*
done

51.13 Inverse function theorem for complex derivatives

lemma *has-complex-derivative-inverse-basic*:
fixes f :: *complex* $\Rightarrow$ *complex*
shows *DERIV f* $(g\ y)$:> $f' \Longrightarrow$
$f' \neq 0 \Longrightarrow$
continuous $(at\ y)\ g \Longrightarrow$
open $t \Longrightarrow$
$y \in t \Longrightarrow$
$(\bigwedge z.\ z \in t \Longrightarrow f\ (g\ z) = z)$
$\Longrightarrow$ *DERIV g y* :> *inverse* (f')
unfolding *has-field-derivative-def*
apply (*rule has-derivative-inverse-basic*)
apply (*auto simp*: *bounded-linear-mult-right*)
done

lemma *has-complex-derivative-inverse-strong*:
fixes f :: *complex* $\Rightarrow$ *complex*
shows *DERIV f x* :> $f' \Longrightarrow$
$f' \neq 0 \Longrightarrow$
open s $\Longrightarrow$
$x \in s \Longrightarrow$
continuous-on s f $\Longrightarrow$
$(\bigwedge z.\ z \in s \Longrightarrow g\ (f\ z) = z)$
$\Longrightarrow$ *DERIV g* $(f\ x)$:> *inverse* (f')
unfolding *has-field-derivative-def*
apply (*rule has-derivative-inverse-strong* [*of s x f g*])
using *assms*
by *auto*

lemma *has-complex-derivative-inverse-strong-x*:
fixes f :: *complex* $\Rightarrow$ *complex*
shows *DERIV f* $(g\ y)$:> $f' \Longrightarrow$

$f' \neq 0 \Longrightarrow$
open s $\Longrightarrow$
continuous-on s f $\Longrightarrow$
$g\ y \in s \Longrightarrow f(g\ y) = y \Longrightarrow$
$(\bigwedge z.\ z \in s \Longrightarrow g\ (f\ z) = z)$
$\Longrightarrow$ *DERIV g y :> inverse* (f')
unfolding *has-field-derivative-def*
apply (*rule has-derivative-inverse-strong-x* [*of s g y f*])
using *assms*
by *auto*

51.14 Taylor on Complex Numbers

lemma *setsum-Suc-reindex*:
fixes f :: *nat* $\Rightarrow$ $'a$::*ab-group-add*
shows *setsum f* $\{0..n\}$ = *f 0* − *f* (*Suc n*) + *setsum* $(\lambda i.\ f\ (Suc\ i))\ \{0..n\}$
by (*induct n*) *auto*

lemma *complex-taylor*:
assumes *s*: *convex s*
and *f*: $\bigwedge i\ x.\ x \in s \Longrightarrow i \leq n \Longrightarrow$ (*f i has-field-derivative f* (*Suc i*) *x*) (*at x within s*)
and *B*: $\bigwedge x.\ x \in s \Longrightarrow$ *cmod* (*f* (*Suc n*) *x*) ≤ *B*
and *w*: $w \in s$
and *z*: $z \in s$
shows *cmod*(*f 0 z* − ($\sum i{\leq}n.$ *f i w* ∗ (*z*−*w*) ^ *i* / (*fact i*)))
≤ *B* ∗ *cmod*(*z* − *w*)^(*Suc n*) / *fact n*
proof −
have *wzs*: *closed-segment w z* ⊆ *s* **using** *assms*
by (*metis convex-contains-segment*)
{ fix *u*
assume *u* ∈ *closed-segment w z*
then have *u* ∈ *s*
by (*metis wzs subsetD*)
have ($\sum i{\leq}n.$ *f i u* ∗ (− *of-nat i* ∗ (*z*−*u*)^(*i* − *1*)) / (*fact i*) +
f (*Suc i*) *u* ∗ (*z*−*u*)^*i* / (*fact i*)) =
f (*Suc n*) *u* ∗ (*z*−*u*) ^ *n* / (*fact n*)
proof (*induction n*)
case *0* **show** *?case* **by** *simp*
next
case (*Suc n*)
have ($\sum i{\leq}Suc\ n.$ *f i u* ∗ (− *of-nat i* ∗ (*z*−*u*) ^ (*i* − *1*)) / (*fact i*) +
f (*Suc i*) *u* ∗ (*z*−*u*) ^ *i* / (*fact i*)) =
f (*Suc n*) *u* ∗ (*z*−*u*) ^ *n* / (*fact n*) +
f (*Suc* (*Suc n*)) *u* ∗ ((*z*−*u*) ∗ (*z*−*u*) ^ *n*) / (*fact* (*Suc n*)) −
f (*Suc n*) *u* ∗ ((*1* + *of-nat n*) ∗ (*z*−*u*) ^ *n*) / (*fact* (*Suc n*))
using *Suc* **by** *simp*
also have ... = *f* (*Suc* (*Suc n*)) *u* ∗ (*z*−*u*) ^ *Suc n* / (*fact* (*Suc n*))
proof −

```
      have (fact(Suc n)) *
          (f(Suc n) u *(z−u) ^ n / (fact n) +
            f(Suc(Suc n)) u *((z−u) *(z−u) ^ n) / (fact(Suc n)) −
            f(Suc n) u *((1 + of-nat n) *(z−u) ^ n) / (fact(Suc n))) =
         ((fact(Suc n)) *(f(Suc n) u *(z−u) ^ n)) / (fact n) +
        ((fact(Suc n)) *(f(Suc(Suc n)) u *((z−u) *(z−u) ^ n)) / (fact(Suc n)))
−
          ((fact(Suc n)) *(f(Suc n) u *(of-nat(Suc n) *(z−u) ^ n))) / (fact(Suc
n))
        by (simp add: algebra-simps del: fact.simps)
      also have ... = ((fact (Suc n)) * (f (Suc n) u * (z−u) ^ n)) / (fact n) +
                      (f (Suc (Suc n)) u * ((z−u) * (z−u) ^ n)) −
                      (f (Suc n) u * ((1 + of-nat n) * (z−u) ^ n))
        by (simp del: fact.simps)
      also have ... = (of-nat (Suc n) * (f (Suc n) u * (z−u) ^ n)) +
                      (f (Suc (Suc n)) u * ((z−u) * (z−u) ^ n)) −
                      (f (Suc n) u * ((1 + of-nat n) * (z−u) ^ n))
        by (simp only: fact.simps of-nat-mult ac-simps) simp
      also have ... = f (Suc (Suc n)) u * ((z−u) * (z−u) ^ n)
        by (simp add: algebra-simps)
      finally show ?thesis
       by (simp add: mult-left-cancel [where c = (fact (Suc n)), THEN iffD1]
del: fact.simps)
    qed
    finally show ?case .
  qed
  then have ((λv. (∑ i≤n. f i v * (z − v) ^i / (fact i)))
             has-field-derivative f (Suc n) u * (z−u) ^ n / (fact n))
            (at u within s)
    apply (intro derivative-eq-intros)
    apply (blast intro: assms ‹u ∈ s›)
    apply (rule refl)+
    apply (auto simp: field-simps)
    done
 } note sum-deriv = this
 { fix u
  assume u: u ∈ closed-segment w z
  then have us: u ∈ s
    by (metis wzs subsetD)
  have cmod (f (Suc n) u) * cmod (z − u) ^ n ≤ cmod (f (Suc n) u) * cmod
(u − z) ^ n
    by (metis norm-minus-commute order-refl)
  also have ... ≤ cmod (f (Suc n) u) * cmod (z − w) ^ n
    by (metis mult-left-mono norm-ge-zero power-mono segment-bound [OF u])
  also have ... ≤ B * cmod (z − w) ^ n
    by (metis norm-ge-zero zero-le-power mult-right-mono  B [OF us])
  finally have cmod (f (Suc n) u) * cmod (z − u) ^ n ≤ B * cmod (z − w) ^
n .
 } note cmod-bound = this
```

```
  have (∑ i≤n. f i z * (z − z) ^ i / (fact i)) = (∑ i≤n. (f i z / (fact i)) * 0 ^ i)
    by simp
  also have ... = f 0 z / (fact 0)
    by (subst setsum-zero-power) simp
  finally have cmod (f 0 z − (∑ i≤n. f i w * (z − w) ^ i / (fact i)))
                ≤ cmod ((∑ i≤n. f i w * (z − w) ^ i / (fact i)) −
                        (∑ i≤n. f i z * (z − z) ^ i / (fact i)))
    by (simp add: norm-minus-commute)
  also have ... ≤ B * cmod (z − w) ^ n / (fact n) * cmod (w − z)
    apply (rule field-differentiable-bound
      [where f' = λw. f (Suc n) w * (z − w)^n / (fact n)
        and s = closed-segment w z, OF convex-closed-segment])
    apply (auto simp: ends-in-segment DERIV-subset [OF sum-deriv wzs]
                norm-divide norm-mult norm-power divide-le-cancel cmod-bound)
    done
  also have ... ≤ B * cmod (z − w) ^ Suc n / (fact n)
    by (simp add: algebra-simps norm-minus-commute)
  finally show ?thesis .
qed
```

Something more like the traditional MVT for real components

```
lemma complex-mvt-line:
  assumes ⋀u. u ∈ closed-segment w z ⟹ (f has-field-derivative f'(u)) (at u)
    shows ∃ u. u ∈ closed-segment w z ∧ Re(f z) − Re(f w) = Re(f'(u) * (z − w))
proof −
  have twz: ⋀t. (1 − t) *R w + t *R z = w + t *R (z − w)
    by (simp add: real-vector.scale-left-diff-distrib real-vector.scale-right-diff-distrib)
  note assms[unfolded has-field-derivative-def, derivative-intros]
  show ?thesis
    apply (cut-tac mvt-simple
                  [of 0 1 Re o f o (λt. (1 − t) *R w +  t *R z)
                   λu. Re o (λh. f'((1 − u) *R w + u *R z) * h) o (λt. t *R (z −
w))])
    apply auto
    apply (rule-tac x=(1 − x) *R w + x *R z in exI)
    apply (auto simp: closed-segment-def twz) []
    apply (intro derivative-eq-intros has-derivative-at-within, simp-all)
    apply (simp add: fun-eq-iff real-vector.scale-right-diff-distrib)
    apply (force simp: twz closed-segment-def)
    done
qed

lemma complex-taylor-mvt:
  assumes ⋀i x. [[x ∈ closed-segment w z; i ≤ n]] ⟹ ((f i) has-field-derivative f
(Suc i) x) (at x)
    shows ∃ u. u ∈ closed-segment w z ∧
          Re (f 0 z) =
          Re ((∑ i = 0..n. f i w * (z − w) ^ i / (fact i)) +
             (f (Suc n) u * (z−u)^n / (fact n)) * (z − w))
```

```
proof −
  { fix u
    assume u: u ∈ closed-segment w z
    have (∑ i = 0..n.
            (f (Suc i) u * (z−u) ^ i − of-nat i * (f i u * (z−u) ^ (i − Suc 0))) /
             (fact i)) =
          f (Suc 0) u −
            (f (Suc (Suc n)) u * ((z−u) ^ Suc n) − (of-nat (Suc n)) * (z−u) ^ n
* f (Suc n) u) /
            (fact (Suc n)) +
            (∑ i = 0..n.
               (f (Suc (Suc i)) u * ((z−u) ^ Suc i) − of-nat (Suc i) * (f (Suc i) u
* (z−u) ^ i)) /
                (fact (Suc i)))
       by (subst setsum-Suc-reindex) simp
    also have ... = f (Suc 0) u −
            (f (Suc (Suc n)) u * ((z−u) ^ Suc n) − (of-nat (Suc n)) * (z−u) ^ n
* f (Suc n) u) /
            (fact (Suc n)) +
            (∑ i = 0..n.
               f (Suc (Suc i)) u * ((z−u) ^ Suc i) / (fact (Suc i)) −
               f (Suc i) u * (z−u) ^ i / (fact i))
      by (simp only: diff-divide-distrib fact-cancel ac-simps)
    also have ... = f (Suc 0) u −
            (f (Suc (Suc n)) u * (z−u) ^ Suc n − of-nat (Suc n) * (z−u) ^ n * f
(Suc n) u) /
            (fact (Suc n)) +
            f (Suc (Suc n)) u * (z−u) ^ Suc n / (fact (Suc n)) − f (Suc 0) u
      by (subst setsum-Suc-diff) auto
    also have ... = f (Suc n) u * (z−u) ^ n / (fact n)
      by (simp only: algebra-simps diff-divide-distrib fact-cancel)
    finally have (∑ i = 0..n. (f (Suc i) u * (z − u) ^ i
                          − of-nat i * (f i u * (z−u) ^ (i − Suc 0))) / (fact i)) =
                  f (Suc n) u * (z − u) ^ n / (fact n) .
    then have ((λu. ∑ i = 0..n. f i u * (z − u) ^ i / (fact i)) has-field-derivative
              f (Suc n) u * (z − u) ^ n / (fact n)) (at u)
      apply (intro derivative-eq-intros)+
      apply (force intro: u assms)
      apply (rule refl)+
      apply (auto simp: ac-simps)
      done
  }
  then show ?thesis
    apply (cut-tac complex-mvt-line [of w z λu. ∑ i = 0..n. f i u * (z−u) ^ i /
(fact i)
              λu. (f (Suc n) u * (z−u)^n / (fact n))])
    apply (auto simp add: intro: open-closed-segment)
    done
qed
```

51.15 Polynomal function extremal theorem, from HOL Light

lemma *polyfun-extremal-lemma*:
fixes $c :: nat \Rightarrow 'a$::*real-normed-div-algebra*
assumes $0 < e$
shows $\exists M.\ \forall z.\ M \leq norm(z) \longrightarrow norm\ (\sum i{\leq}n.\ c(i) * z\hat{}i) \leq e * norm(z) \hat{}\ (Suc\ n)$
proof (*induct n*)
case *0* **with** *assms*
show *?case*
apply (*rule-tac x*=*norm* (*c 0*) / *e* **in** *exI*)
apply (*auto simp*: *field-simps*)
done
next
case (*Suc n*)
obtain *M* **where** *M*: $\bigwedge z.\ M \leq norm\ z \Longrightarrow norm\ (\sum i{\leq}n.\ c\ i * z\hat{}i) \leq e * norm\ z \hat{}\ Suc\ n$
using *Suc assms* **by** *blast*
show *?case*
proof (*rule exI* [**where** *x*= *max M* (*1* + *norm*(*c*(*Suc n*)) / *e*)], *clarsimp simp del*: *power-Suc*)
fix *z*::*'a*
assume *z1*: $M \leq norm\ z$ **and** $1 + norm\ (c\ (Suc\ n)) / e \leq norm\ z$
then have *z2*: $e + norm\ (c\ (Suc\ n)) \leq e * norm\ z$
using *assms* **by** (*simp add*: *field-simps*)
have $norm\ (\sum i{\leq}n.\ c\ i * z\hat{}i) \leq e * norm\ z \hat{}\ Suc\ n$
using *M* [*OF z1*] **by** *simp*
then have $norm\ (\sum i{\leq}n.\ c\ i * z\hat{}i) + norm\ (c\ (Suc\ n) * z \hat{}\ Suc\ n) \leq e * norm\ z \hat{}\ Suc\ n + norm\ (c\ (Suc\ n) * z \hat{}\ Suc\ n)$
by *simp*
then have $norm\ ((\sum i{\leq}n.\ c\ i * z\hat{}i) + c\ (Suc\ n) * z \hat{}\ Suc\ n) \leq e * norm\ z \hat{}\ Suc\ n + norm\ (c\ (Suc\ n) * z \hat{}\ Suc\ n)$
by (*blast intro*: *norm-triangle-le elim*:)
also have $... \leq (e + norm\ (c\ (Suc\ n))) * norm\ z \hat{}\ Suc\ n$
by (*simp add*: *norm-power norm-mult algebra-simps*)
also have $... \leq (e * norm\ z) * norm\ z \hat{}\ Suc\ n$
by (*metis z2 mult.commute mult-left-mono norm-ge-zero norm-power*)
finally show $norm\ ((\sum i{\leq}n.\ c\ i * z\hat{}i) + c\ (Suc\ n) * z \hat{}\ Suc\ n) \leq e * norm\ z \hat{}\ Suc\ (Suc\ n)$
by *simp*
qed
qed

lemma *polyfun-extremal*:
fixes $c :: nat \Rightarrow 'a$::*real-normed-div-algebra*
assumes *k*: $c\ k \neq 0\ 1{\leq}k$ **and** *kn*: $k{\leq}n$
shows *eventually* $(\lambda z.\ norm\ (\sum i{\leq}n.\ c(i) * z\hat{}i) \geq B)$ *at-infinity*
using *kn*
proof (*induction n*)
case *0*

```
  then show ?case
    using k  by simp
next
  case (Suc m)
  let ?even = ?case
  show ?even
  proof (cases c (Suc m) = 0)
    case True
    then show ?even using Suc k
      by auto (metis antisym-conv less-eq-Suc-le not-le)
  next
    case False
    then obtain M where M:
          ⋀z. M ≤ norm z ⟹ norm (∑ i≤m. c i * z^i) ≤ norm (c (Suc m)) / 2
* norm z ^ Suc m
      using polyfun-extremal-lemma [of norm(c (Suc m)) / 2 c m] Suc
      by auto
    have ∃ b. ∀ z. b ≤ norm z ⟶ B ≤ norm (∑ i≤Suc m. c i * z^i)
    proof (rule exI [where x=max M (max 1 (|B| / (norm(c (Suc m)) / 2)))],
clarsimp simp del: power-Suc)
      fix z::'a
      assume z1: M ≤ norm z 1 ≤ norm z
        and |B| * 2 / norm (c (Suc m)) ≤ norm z
      then have z2: |B| ≤ norm (c (Suc m)) * norm z / 2
        using False by (simp add: field-simps)
      have nz: norm z ≤ norm z ^ Suc m
      by (metis ‹1 ≤ norm z› One-nat-def less-eq-Suc-le power-increasing power-one-right
zero-less-Suc)
      have *: ⋀y x. norm (c (Suc m)) * norm z / 2 ≤ norm y − norm x ⟹ B
≤ norm (x + y)
        by (metis abs-le-iff add.commute norm-diff-ineq order-trans z2)
      have norm z * norm (c (Suc m)) + 2 * norm (∑ i≤m. c i * z^i)
            ≤ norm (c (Suc m)) * norm z + norm (c (Suc m)) * norm z ^ Suc m
        using M [of z] Suc z1  by auto
      also have ... ≤ 2 * (norm (c (Suc m)) * norm z ^ Suc m)
        using nz by (simp add: mult-mono del: power-Suc)
      finally show B ≤ norm ((∑ i≤m. c i * z^i) + c (Suc m) * z ^ Suc m)
        using Suc.IH
        apply (auto simp: eventually-at-infinity)
        apply (rule *)
        apply (simp add: field-simps norm-mult norm-power)
        done
    qed
    then show ?even
      by (simp add: eventually-at-infinity)
  qed
qed

end
```

52 Complex Transcendental Functions

By John Harrison et al. Ported from HOL Light by L C Paulson (2015)

theory *Complex-Transcendental*
imports
Complex-Analysis-Basics
Summation
begin

definition *moebius a b c d* = (λ*z*. (*a*∗*z*+*b*) / (*c*∗*z*+*d* :: *'a* :: *field*))

lemma *moebius-inverse*:
assumes *a* ∗ *d* ≠ *b* ∗ *c c* ∗ *z* + *d* ≠ *0*
shows *moebius d* (−*b*) (−*c*) *a* (*moebius a b c d z*) = *z*
proof −
from *assms* **have** (−*c*) ∗ *moebius a b c d z* + *a* ≠ *0* **unfolding** *moebius-def*
by (*simp add*: *field-simps*)
with *assms* **show** *?thesis*
unfolding *moebius-def* **by** (*simp add*: *moebius-def divide-simps*) (*simp add*: *algebra-simps*)*?*
qed

lemma *moebius-inverse'*:
assumes *a* ∗ *d* ≠ *b* ∗ *c c* ∗ *z* − *a* ≠ *0*
shows *moebius a b c d* (*moebius d* (−*b*) (−*c*) *a z*) = *z*
using *assms moebius-inverse*[*of d a* −*b* −*c z*]
by (*auto simp*: *algebra-simps*)

lemma *cmod-add-real-less*:
assumes *Im z* ≠ *0 r*≠*0*
shows *cmod* (*z* + *r*) < *cmod z* + |*r*|
proof (*cases z*)
case (*Complex x y*)
have *r* ∗ *x* / |*r*| < *sqrt* (*x*∗*x* + *y*∗*y*)
apply (*rule real-less-rsqrt*)
using *assms*
apply (*simp add*: *Complex power2-eq-square*)
using *not-real-square-gt-zero* **by** *blast*
then show *?thesis* **using** *assms Complex*
apply (*auto simp*: *cmod-def*)
apply (*rule power2-less-imp-less*, *auto*)
apply (*simp add*: *power2-eq-square field-simps*)
done
qed

lemma *cmod-diff-real-less*: *Im z* ≠ *0* ⟹ *x*≠*0* ⟹ *cmod* (*z* − *x*) < *cmod z* + |*x*|
using *cmod-add-real-less* [*of z* −*x*]
by *simp*

lemma *cmod-square-less-1-plus*:
assumes $Im\ z = 0 \Longrightarrow |Re\ z| < 1$
shows $(cmod\ z)^2 < 1 + cmod\ (1 - z^2)$
using *assms*
apply (*cases* $Im\ z = 0 \vee Re\ z = 0$)
using *abs-square-less-1*
apply (*force simp add*: *Re-power2 Im-power2 cmod-def*)
using *cmod-diff-real-less* [*of* $1 - z^2$ *1*]
apply (*simp add*: *norm-power Im-power2*)
done

52.1 The Exponential Function is Differentiable and Continuous

lemma *field-differentiable-within-exp*: *exp field-differentiable* (*at z within s*)
using *DERIV-exp field-differentiable-at-within field-differentiable-def* **by** *blast*

lemma *continuous-within-exp*:
fixes *z*::'*a*::{*real-normed-field*,*banach*}
shows *continuous* (*at z within s*) *exp*
by (*simp add*: *continuous-at-imp-continuous-within*)

lemma *holomorphic-on-exp* [*holomorphic-intros*]: *exp holomorphic-on s*
by (*simp add*: *field-differentiable-within-exp holomorphic-on-def*)

52.2 Euler and de Moivre formulas.

The sine series times *i*

lemma *sin-ii-eq*: $(\lambda n.\ (ii * sin\text{-}coeff\ n) * z\hat{}n)\ sums\ (ii * sin\ z)$
proof −
have $(\lambda n.\ ii * sin\text{-}coeff\ n *_R z\hat{}n)\ sums\ (ii * sin\ z)$
using *sin-converges sums-mult* **by** *blast*
then show *?thesis*
by (*simp add*: *scaleR-conv-of-real field-simps*)
qed

theorem *exp-Euler*: $exp(ii * z) = cos(z) + ii * sin(z)$
proof −
have $(\lambda n.\ (cos\text{-}coeff\ n + ii * sin\text{-}coeff\ n) * z\hat{}n)$
$= (\lambda n.\ (ii * z)\ \hat{}\ n\ /_R\ (fact\ n))$
proof
fix *n*
show $(cos\text{-}coeff\ n + ii * sin\text{-}coeff\ n) * z\hat{}n = (ii * z)\ \hat{}\ n\ /_R\ (fact\ n)$
by (*auto simp*: *cos-coeff-def sin-coeff-def scaleR-conv-of-real field-simps elim*!: *evenE oddE*)
qed
also have ... *sums* (*exp* (*ii* ∗ *z*))
by (*rule exp-converges*)

finally have $(\lambda n.\ (cos\text{-}coeff\ n + ii * sin\text{-}coeff\ n) * z\hat{}n)\ sums\ (exp\ (ii * z))$ **.**
moreover have $(\lambda n.\ (cos\text{-}coeff\ n + ii * sin\text{-}coeff\ n) * z\hat{}n)\ sums\ (cos\ z + ii * sin\ z)$
using *sums-add* [*OF cos-converges* [*of z*] *sin-ii-eq* [*of z*]]
by (*simp add*: *field-simps scaleR-conv-of-real*)
ultimately show *?thesis*
using *sums-unique2* **by** *blast*
qed

corollary *exp-minus-Euler*: $exp(-(ii * z)) = cos(z) - ii * sin(z)$
using *exp-Euler* [*of* $-z$]
by *simp*

lemma *sin-exp-eq*: $sin\ z = (exp(ii * z) - exp(-(ii * z))) / (2*ii)$
by (*simp add*: *exp-Euler exp-minus-Euler*)

lemma *sin-exp-eq'*: $sin\ z = ii * (exp(-(ii * z)) - exp(ii * z)) / 2$
by (*simp add*: *exp-Euler exp-minus-Euler*)

lemma *cos-exp-eq*: $cos\ z = (exp(ii * z) + exp(-(ii * z))) / 2$
by (*simp add*: *exp-Euler exp-minus-Euler*)

52.3 Relationships between real and complex trig functions

lemma *real-sin-eq* [*simp*]:
fixes *x*::*real*
shows $Re(sin(of\text{-}real\ x)) = sin\ x$
by (*simp add*: *sin-of-real*)

lemma *real-cos-eq* [*simp*]:
fixes *x*::*real*
shows $Re(cos(of\text{-}real\ x)) = cos\ x$
by (*simp add*: *cos-of-real*)

lemma *DeMoivre*: $(cos\ z + ii * sin\ z)\ \hat{}\ n = cos(n * z) + ii * sin(n * z)$
apply (*simp add*: *exp-Euler* [*symmetric*])
by (*metis exp-of-nat-mult mult.left-commute*)

lemma *exp-cnj*:
fixes *z*::*complex*
shows $cnj\ (exp\ z) = exp\ (cnj\ z)$
proof −
have $(\lambda n.\ cnj\ (z\ \hat{}\ n\ /_R\ (fact\ n))) = (\lambda n.\ (cnj\ z)\hat{}n\ /_R\ (fact\ n))$
by *auto*
also have *...* $sums\ (exp\ (cnj\ z))$
by (*rule exp-converges*)
finally have $(\lambda n.\ cnj\ (z\ \hat{}\ n\ /_R\ (fact\ n)))\ sums\ (exp\ (cnj\ z))$ **.**
moreover have $(\lambda n.\ cnj\ (z\ \hat{}\ n\ /_R\ (fact\ n)))\ sums\ (cnj\ (exp\ z))$
by (*metis exp-converges sums-cnj*)

ultimately show *?thesis*
using *sums-unique2*
by *blast*
qed

lemma *cnj-sin*: *cnj*(*sin z*) = *sin*(*cnj z*)
by (*simp add*: *sin-exp-eq exp-cnj field-simps*)

lemma *cnj-cos*: *cnj*(*cos z*) = *cos*(*cnj z*)
by (*simp add*: *cos-exp-eq exp-cnj field-simps*)

lemma *field-differentiable-at-sin*: *sin field-differentiable at z*
using *DERIV-sin field-differentiable-def* **by** *blast*

lemma *field-differentiable-within-sin*: *sin field-differentiable* (*at z within s*)
by (*simp add*: *field-differentiable-at-sin field-differentiable-at-within*)

lemma *field-differentiable-at-cos*: *cos field-differentiable at z*
using *DERIV-cos field-differentiable-def* **by** *blast*

lemma *field-differentiable-within-cos*: *cos field-differentiable* (*at z within s*)
by (*simp add*: *field-differentiable-at-cos field-differentiable-at-within*)

lemma *holomorphic-on-sin*: *sin holomorphic-on s*
by (*simp add*: *field-differentiable-within-sin holomorphic-on-def*)

lemma *holomorphic-on-cos*: *cos holomorphic-on s*
by (*simp add*: *field-differentiable-within-cos holomorphic-on-def*)

52.4 Get a nice real/imaginary separation in Euler's formula.

lemma *Euler*: *exp*(*z*) = *of-real*(*exp*(*Re z*)) ∗
(*of-real*(*cos*(*Im z*)) + *ii* ∗ *of-real*(*sin*(*Im z*)))
by (*cases z*) (*simp add*: *exp-add exp-Euler cos-of-real exp-of-real sin-of-real*)

lemma *Re-sin*: *Re*(*sin z*) = *sin*(*Re z*) ∗ (*exp*(*Im z*) + *exp*(−(*Im z*))) / *2*
by (*simp add*: *sin-exp-eq field-simps Re-divide Im-exp*)

lemma *Im-sin*: *Im*(*sin z*) = *cos*(*Re z*) ∗ (*exp*(*Im z*) − *exp*(−(*Im z*))) / *2*
by (*simp add*: *sin-exp-eq field-simps Im-divide Re-exp*)

lemma *Re-cos*: *Re*(*cos z*) = *cos*(*Re z*) ∗ (*exp*(*Im z*) + *exp*(−(*Im z*))) / *2*
by (*simp add*: *cos-exp-eq field-simps Re-divide Re-exp*)

lemma *Im-cos*: *Im*(*cos z*) = *sin*(*Re z*) ∗ (*exp*(−(*Im z*)) − *exp*(*Im z*)) / *2*
by (*simp add*: *cos-exp-eq field-simps Im-divide Im-exp*)

lemma *Re-sin-pos*: *0* < *Re z* ⟹ *Re z* < *pi* ⟹ *Re* (*sin z*) > *0*
by (*auto simp*: *Re-sin Im-sin add-pos-pos sin-gt-zero*)

lemma *Im-sin-nonneg*: $Re\ z = 0 \Longrightarrow 0 \leq Im\ z \Longrightarrow 0 \leq Im\ (sin\ z)$
by (*simp add*: *Re-sin Im-sin algebra-simps*)

lemma *Im-sin-nonneg2*: $Re\ z = pi \Longrightarrow Im\ z \leq 0 \Longrightarrow 0 \leq Im\ (sin\ z)$
by (*simp add*: *Re-sin Im-sin algebra-simps*)

52.5 More on the Polar Representation of Complex Numbers

lemma *exp-Complex*: $exp(Complex\ r\ t) = of\text{-}real(exp\ r) * Complex\ (cos\ t)\ (sin\ t)$
by (*simp add*: *exp-add exp-Euler exp-of-real sin-of-real cos-of-real*)

lemma *exp-eq-1*: $exp\ z = 1 \longleftrightarrow Re(z) = 0 \wedge (\exists\, n{::}int.\ Im(z) = of\text{-}int\ (2 * n) * pi)$
apply *auto*
apply (*metis exp-eq-one-iff norm-exp-eq-Re norm-one*)
apply (*metis Re-exp cos-one-2pi-int mult.commute mult.left-neutral norm-exp-eq-Re norm-one one-complex.simps(1)*)
by (*metis Im-exp Re-exp complex-Re-Im-cancel-iff cos-one-2pi-int sin-double Re-complex-of-real complex-Re-numeral exp-zero mult.assoc mult.left-commute mult-eq-0-iff mult-numeral-1 numeral-One of-real-0 sin-zero-iff-int2*)

lemma *exp-eq*: $exp\ w = exp\ z \longleftrightarrow (\exists\, n{::}int.\ w = z + (of\text{-}int\ (2 * n) * pi) * ii)$
(**is** *?lhs = ?rhs*)
proof −
have $exp\ w = exp\ z \longleftrightarrow exp\ (w-z) = 1$
by (*simp add*: *exp-diff*)
also have ... $\longleftrightarrow (Re\ w = Re\ z \wedge (\exists\, n{::}int.\ Im\ w - Im\ z = of\text{-}int\ (2 * n) * pi))$
by (*simp add*: *exp-eq-1*)
also have ... $\longleftrightarrow$ *?rhs*
by (*auto simp*: *algebra-simps intro*!: *complex-eqI*)
finally show *?thesis* **.**
qed

lemma *exp-complex-eqI*: $|Im\ w - Im\ z| < 2{*}pi \Longrightarrow exp\ w = exp\ z \Longrightarrow w = z$
by (*auto simp*: *exp-eq abs-mult*)

lemma *exp-integer-2pi*:
assumes $n \in \mathbb{Z}$
shows $exp((2 * n * pi) * ii) = 1$
proof −
have $exp((2 * n * pi) * ii) = exp\ 0$
using *assms*
by (*simp only*: *Ints-def exp-eq*) *auto*
also have ... $= 1$
by *simp*
finally show *?thesis* **.**

qed

lemma *sin-cos-eq-iff*: *sin y = sin x ∧ cos y = cos x ⟷ (∃ n::int. y = x + 2 * n * pi)*
proof −
  { assume *sin y = sin x cos y = cos x*
    then have *cos (y−x) = 1*
      using *cos-add [of y −x]* **by** *simp*
    then have *∃ n::int. y−x = n * 2 * pi*
      using *cos-one-2pi-int* **by** *blast* **}**
  then show *?thesis*
  apply (*auto simp*: *sin-add cos-add*)
  apply (*metis add.commute diff-add-cancel mult.commute*)
  done
qed

lemma *exp-i-ne-1*:
  assumes *0 < x x < 2∗pi*
  shows *exp*(i ∗ *of-real x*) ≠ *1*
proof
  assume *exp* (i ∗ *of-real x*) = *1*
  then have *exp* (i ∗ *of-real x*) = *exp 0*
    by *simp*
  then obtain *n* **where** i ∗ *of-real x* = (*of-int* (*2* ∗ *n*) ∗ *pi*) ∗ i
    by (*simp only*: *Ints-def exp-eq*) *auto*
  then have *of-real x* = (*of-int* (*2* ∗ *n*) ∗ *pi*)
   by (*metis complex-i-not-zero mult.commute mult-cancel-left of-real-eq-iff real-scaleR-def scaleR-conv-of-real*)
  then have *x* = (*of-int* (*2* ∗ *n*) ∗ *pi*)
    by *simp*
  then show *False* **using** *assms*
    by (*cases n*) (*auto simp*: *zero-less-mult-iff mult-less-0-iff*)
qed

lemma *sin-eq-0*:
  fixes *z*::*complex*
  shows *sin z = 0 ⟷* (∃ *n*::*int*. *z = of-real*(*n ∗ pi*))
  by (*simp add*: *sin-exp-eq exp-eq of-real-numeral*)

lemma *cos-eq-0*:
  fixes *z*::*complex*
  shows *cos z = 0 ⟷* (∃ *n*::*int*. *z = of-real*(*n ∗ pi*) + *of-real pi*/*2*)
  using *sin-eq-0* [*of z − of-real pi*/*2*]
  by (*simp add*: *sin-diff algebra-simps*)

lemma *cos-eq-1*:
  fixes *z*::*complex*
  shows *cos z = 1 ⟷* (∃ *n*::*int*. *z = of-real*(*2 ∗ n ∗ pi*))
proof −

have *cos z = cos (2∗(z/2))*
by *simp*
also have *... = 1 − 2 ∗ sin (z/2) ^ 2*
by (*simp only*: *cos-double-sin*)
finally have [*simp*]: *cos z = 1 ⟷ sin (z/2) = 0*
by *simp*
show *?thesis*
by (*auto simp*: *sin-eq-0 of-real-numeral*)
qed

lemma *csin-eq-1*:
fixes *z*::*complex*
shows *sin z = 1 ⟷ (∃ n*::*int. z = of-real(2 ∗ n ∗ pi) + of-real pi/2)*
using *cos-eq-1* [*of z − of-real pi/2*]
by (*simp add*: *cos-diff algebra-simps*)

lemma *csin-eq-minus1*:
fixes *z*::*complex*
shows *sin z = −1 ⟷ (∃ n*::*int. z = of-real(2 ∗ n ∗ pi) + 3/2∗pi)*
(**is** *- = ?rhs*)
proof −
have *sin z = −1 ⟷ sin (−z) = 1*
by (*simp add*: *equation-minus-iff*)
also have *... ⟷ (∃ n*::*int. −z = of-real(2 ∗ n ∗ pi) + of-real pi/2)*
by (*simp only*: *csin-eq-1*)
also have *... ⟷ (∃ n*::*int. z = − of-real(2 ∗ n ∗ pi) − of-real pi/2)*
apply (*rule iff-exI*)
by (*metis* (*no-types*) *is-num-normalize*(*8*) *minus-minus of-real-def real-vector.scale-minus-left uminus-add-conv-diff*)
also have *... = ?rhs*
apply (*auto simp*: *of-real-numeral*)
apply (*rule-tac* [*2*] *x*=−(*x*+*1*) **in** *exI*)
apply (*rule-tac x*=−(*x*+*1*) **in** *exI*)
apply (*simp-all add*: *algebra-simps*)
done
finally show *?thesis* **.**
qed

lemma *ccos-eq-minus1*:
fixes *z*::*complex*
shows *cos z = −1 ⟷ (∃ n*::*int. z = of-real(2 ∗ n ∗ pi) + pi)*
using *csin-eq-1* [*of z − of-real pi/2*]
apply (*simp add*: *sin-diff*)
apply (*simp add*: *algebra-simps of-real-numeral equation-minus-iff*)
done

lemma *sin-eq-1*: *sin x = 1 ⟷ (∃ n*::*int. x = (2 ∗ n + 1 / 2) ∗ pi)*
(**is** *- = ?rhs*)
proof −

have *sin x = 1 ⟷ sin (complex-of-real x) = 1*
by (*metis of-real-1 one-complex.simps(1) real-sin-eq sin-of-real*)
also have ... ⟷ (∃ *n*::*int*. *complex-of-real x = of-real*(*2 * n * pi*) + *of-real pi/2*)
by (*simp only*: *csin-eq-1*)
also have ... ⟷ (∃ *n*::*int*. *x = of-real*(*2 * n * pi*) + *of-real pi/2*)
apply (*rule iff-exI*)
apply (*auto simp*: *algebra-simps of-real-numeral*)
apply (*rule injD* [*OF inj-of-real* [**where** *'a = complex*]])
apply (*auto simp*: *of-real-numeral*)
done
also have ... = *?rhs*
by (*auto simp*: *algebra-simps*)
finally show *?thesis* **.**
qed

lemma *sin-eq-minus1*: *sin x = −1 ⟷* (∃ *n*::*int*. *x = (2∗n + 3/2) * pi*) (**is** - = *?rhs*)
proof −
have *sin x = −1 ⟷ sin (complex-of-real x) = −1*
by (*metis Re-complex-of-real of-real-def scaleR-minus1-left sin-of-real*)
also have ... ⟷ (∃ *n*::*int*. *complex-of-real x = of-real*(*2 * n * pi*) + *3/2∗pi*)
by (*simp only*: *csin-eq-minus1*)
also have ... ⟷ (∃ *n*::*int*. *x = of-real*(*2 * n * pi*) + *3/2∗pi*)
apply (*rule iff-exI*)
apply (*auto simp*: *algebra-simps*)
apply (*rule injD* [*OF inj-of-real* [**where** *'a = complex*]], *auto*)
done
also have ... = *?rhs*
by (*auto simp*: *algebra-simps*)
finally show *?thesis* **.**
qed

lemma *cos-eq-minus1*: *cos x = −1 ⟷* (∃ *n*::*int*. *x = (2∗n + 1) * pi*)
(**is** - = *?rhs*)
proof −
have *cos x = −1 ⟷ cos (complex-of-real x) = −1*
by (*metis Re-complex-of-real of-real-def scaleR-minus1-left cos-of-real*)
also have ... ⟷ (∃ *n*::*int*. *complex-of-real x = of-real*(*2 * n * pi*) + *pi*)
by (*simp only*: *ccos-eq-minus1*)
also have ... ⟷ (∃ *n*::*int*. *x = of-real*(*2 * n * pi*) + *pi*)
apply (*rule iff-exI*)
apply (*auto simp*: *algebra-simps*)
apply (*rule injD* [*OF inj-of-real* [**where** *'a = complex*]], *auto*)
done
also have ... = *?rhs*
by (*auto simp*: *algebra-simps*)
finally show *?thesis* **.**
qed

lemma *dist-exp-ii-1*: *norm(exp(ii * of-real t) − 1) = 2 * |sin(t / 2)|*
apply (*simp add*: *exp-Euler cmod-def power2-diff sin-of-real cos-of-real algebra-simps*)
using *cos-double-sin* [*of t/2*]
apply (*simp add*: *real-sqrt-mult*)
done

lemma *sinh-complex*:
fixes *z* :: *complex*
shows (*exp z − inverse* (*exp z*)) / *2* = *−ii * sin(ii * z)*
by (*simp add*: *sin-exp-eq divide-simps exp-minus of-real-numeral*)

lemma *sin-ii-times*:
fixes *z* :: *complex*
shows *sin(ii * z) = ii * ((exp z − inverse* (*exp z*)) / *2*)
using *sinh-complex* **by** *auto*

lemma *sinh-real*:
fixes *x* :: *real*
shows *of-real*((*exp x − inverse* (*exp x*)) / *2*) = *−ii * sin*(*ii * of-real x*)
by (*simp add*: *exp-of-real sin-ii-times of-real-numeral*)

lemma *cosh-complex*:
fixes *z* :: *complex*
shows (*exp z + inverse* (*exp z*)) / *2* = *cos*(*ii * z*)
by (*simp add*: *cos-exp-eq divide-simps exp-minus of-real-numeral exp-of-real*)

lemma *cosh-real*:
fixes *x* :: *real*
shows *of-real*((*exp x + inverse* (*exp x*)) / *2*) = *cos*(*ii * of-real x*)
by (*simp add*: *cos-exp-eq divide-simps exp-minus of-real-numeral exp-of-real*)

lemmas *cos-ii-times* = *cosh-complex* [*symmetric*]

lemma *norm-cos-squared*:
norm(*cos z*) ^ *2* = *cos*(*Re z*) ^ *2* + (*exp*(*Im z*) − *inverse*(*exp*(*Im z*))) ^ *2* / *4*
apply (*cases z*)
apply (*simp add*: *cos-add cmod-power2 cos-of-real sin-of-real*)
apply (*simp add*: *cos-exp-eq sin-exp-eq exp-minus exp-of-real Re-divide Im-divide power-divide*)
apply (*simp only*: *left-diff-distrib* [*symmetric*] *power-mult-distrib*)
apply (*simp add*: *sin-squared-eq*)
apply (*simp add*: *power2-eq-square algebra-simps divide-simps*)
done

lemma *norm-sin-squared*:
norm(*sin z*) ^ *2* = (*exp*(*2 * Im z*) + *inverse*(*exp*(*2 * Im z*)) − *2 * cos*(*2 * Re z*)) / *4*
apply (*cases z*)

apply (*simp add*: *sin-add cmod-power2 cos-of-real sin-of-real cos-double-cos exp-double*)
apply (*simp add*: *cos-exp-eq sin-exp-eq exp-minus exp-of-real Re-divide Im-divide power-divide*)
apply (*simp only*: *left-diff-distrib* [*symmetric*] *power-mult-distrib*)
apply (*simp add*: *cos-squared-eq*)
apply (*simp add*: *power2-eq-square algebra-simps divide-simps*)
done

lemma *exp-uminus-Im*: *exp* (− *Im z*) ≤ *exp* (*cmod z*)
using *abs-Im-le-cmod linear order-trans* **by** *fastforce*

lemma *norm-cos-le*:
fixes *z*::*complex*
shows *norm*(*cos z*) ≤ *exp*(*norm z*)
proof −
have *Im z* ≤ *cmod z*
using *abs-Im-le-cmod abs-le-D1* **by** *auto*
with *exp-uminus-Im* **show** *?thesis*
apply (*simp add*: *cos-exp-eq norm-divide*)
apply (*rule order-trans* [*OF norm-triangle-ineq*], *simp*)
apply (*metis add-mono exp-le-cancel-iff mult-2-right*)
done
qed

lemma *norm-cos-plus1-le*:
fixes *z*::*complex*
shows *norm*(*1* + *cos z*) ≤ *2* ∗ *exp*(*norm z*)
proof −
have *mono*: ⋀*u w z*::*real*. (*1* ≤ *w* | *1* ≤ *z*) ⟹ (*w* ≤ *u* & *z* ≤ *u*) ⟹ *2* + *w* + *z* ≤ *4* ∗ *u*
by *arith*
have ∗: *Im z* ≤ *cmod z*
using *abs-Im-le-cmod abs-le-D1* **by** *auto*
have *triangle3*: ⋀*x y z*. *norm*(*x* + *y* + *z*) ≤ *norm*(*x*) + *norm*(*y*) + *norm*(*z*)
by (*simp add*: *norm-add-rule-thm*)
have *norm*(*1* + *cos z*) = *cmod* (*1* + (*exp* (i ∗ *z*) + *exp* (− (i ∗ *z*))) / *2*)
by (*simp add*: *cos-exp-eq*)
also have ... = *cmod* ((*2* + *exp* (i ∗ *z*) + *exp* (− (i ∗ *z*))) / *2*)
by (*simp add*: *field-simps*)
also have ... = *cmod* (*2* + *exp* (i ∗ *z*) + *exp* (− (i ∗ *z*))) / *2*
by (*simp add*: *norm-divide*)
finally show *?thesis*
apply (*rule ssubst*, *simp*)
apply (*rule order-trans* [*OF triangle3*], *simp*)
using *exp-uminus-Im* ∗
apply (*auto intro*: *mono*)
done
qed

52.6 Taylor series for complex exponential, sine and cosine.

declare *power-Suc* [*simp del*]

lemma *Taylor-exp*:
norm(*exp z* − (∑ *k*≤*n*. *z* ^ *k* / (*fact k*))) ≤ *exp*|*Re z*| ∗ (*norm z*) ^ (*Suc n*) / (*fact n*)
proof (*rule complex-taylor* [*of - n* λ*k*. *exp exp*|*Re z*| *0 z*, *simplified*])
show *convex* (*closed-segment 0 z*)
by (*rule convex-closed-segment* [*of 0 z*])
next
fix *k x*
assume *x* ∈ *closed-segment 0 z k* ≤ *n*
show (*exp has-field-derivative exp x*) (*at x within closed-segment 0 z*)
using *DERIV-exp DERIV-subset* **by** *blast*
next
fix *x*
assume *x* ∈ *closed-segment 0 z*
then show *Re x* ≤ |*Re z*|
apply (*auto simp*: *closed-segment-def scaleR-conv-of-real*)
by (*meson abs-ge-self abs-ge-zero linear mult-left-le-one-le mult-nonneg-nonpos order-trans*)
next
show *0* ∈ *closed-segment 0 z*
by (*auto simp*: *closed-segment-def*)
next
show *z* ∈ *closed-segment 0 z*
apply (*simp add*: *closed-segment-def scaleR-conv-of-real*)
using *of-real-1 zero-le-one* **by** *blast*
qed

lemma
assumes *0* ≤ *u u* ≤ *1*
shows *cmod-sin-le-exp*: *cmod* (*sin* (*u* ∗*R z*)) ≤ *exp* |*Im z*|
and *cmod-cos-le-exp*: *cmod* (*cos* (*u* ∗*R z*)) ≤ *exp* |*Im z*|
proof −
have *mono*: ⋀*u w z*::*real*. *w* ≤ *u* ⟹ *z* ≤ *u* ⟹ *w* + *z* ≤ *u*∗*2*
by *arith*
show *cmod* (*sin* (*u* ∗*R z*)) ≤ *exp* |*Im z*| **using** *assms*
apply (*auto simp*: *scaleR-conv-of-real norm-mult norm-power sin-exp-eq norm-divide*)
apply (*rule order-trans* [*OF norm-triangle-ineq4*])
apply (*rule mono*)
apply (*auto simp*: *abs-if mult-left-le-one-le*)
apply (*meson mult-nonneg-nonneg neg-le-0-iff-le not-le order-trans*)
apply (*meson less-eq-real-def mult-nonneg-nonpos neg-0-le-iff-le order-trans*)
done
show *cmod* (*cos* (*u* ∗*R z*)) ≤ *exp* |*Im z*| **using** *assms*
apply (*auto simp*: *scaleR-conv-of-real norm-mult norm-power cos-exp-eq norm-divide*)
apply (*rule order-trans* [*OF norm-triangle-ineq*])
apply (*rule mono*)

```
    apply (auto simp: abs-if mult-left-le-one-le)
    apply (meson mult-nonneg-nonneg neg-le-0-iff-le not-le order-trans)
    apply (meson less-eq-real-def mult-nonneg-nonpos neg-0-le-iff-le order-trans)
    done
qed

lemma Taylor-sin:
  norm(sin z − (∑ k≤n. complex-of-real (sin-coeff k) ∗ z ^ k))
   ≤ exp|Im z| ∗ (norm z) ^ (Suc n) / (fact n)
proof −
  have mono: ⋀u w z::real. w ≤ u ⟹ z ≤ u ⟹ w + z ≤ u∗2
     by arith
  have ∗: cmod (sin z −
              (∑ i≤n. (−1) ^ (i div 2) ∗ (if even i then sin 0 else cos 0) ∗ z ^ i /
(fact i)))
         ≤ exp |Im z| ∗ cmod z ^ Suc n / (fact n)
  proof (rule complex-taylor [of closed-segment 0 z n
                                λk x. (−1)^(k div 2) ∗ (if even k then sin x else cos x)
                                exp|Im z| 0 z,  simplified])
    fix k x
   show ((λx. (− 1) ^ (k div 2) ∗ (if even k then sin x else cos x)) has-field-derivative
           (− 1) ^ (Suc k div 2) ∗ (if odd k then sin x else cos x))
           (at x within closed-segment 0 z)
      apply (auto simp: power-Suc)
      apply (intro derivative-eq-intros | simp)+
      done
  next
    fix x
    assume x ∈ closed-segment 0 z
    then show cmod ((− 1) ^ (Suc n div 2) ∗ (if odd n then sin x else cos x)) ≤
exp |Im z|
      by (auto simp: closed-segment-def norm-mult norm-power cmod-sin-le-exp
cmod-cos-le-exp)
  qed
  have ∗∗: ⋀k. complex-of-real (sin-coeff k) ∗ z ^ k
         = (−1)^(k div 2) ∗ (if even k then sin 0 else cos 0) ∗ z^k / of-nat (fact
k)
    by (auto simp: sin-coeff-def elim!: oddE)
  show ?thesis
    apply (rule order-trans [OF - ∗])
    apply (simp add: ∗∗)
    done
qed

lemma Taylor-cos:
  norm(cos z − (∑ k≤n. complex-of-real (cos-coeff k) ∗ z ^ k))
   ≤ exp|Im z| ∗ (norm z) ^ Suc n / (fact n)
proof −
  have mono: ⋀u w z::real. w ≤ u ⟹ z ≤ u ⟹ w + z ≤ u∗2
```

by *arith*
have ∗: *cmod* (*cos z* −
(∑ *i*≤*n*. (−*1*) ^ (*Suc i div 2*) ∗ (*if even i then cos 0 else sin 0*) ∗ *z* ^ *i* / (*fact i*)))
≤ *exp* |*Im z*| ∗ *cmod z* ^ *Suc n* / (*fact n*)
proof (*rule complex-taylor* [*of closed-segment 0 z n* λ*k x*. (−*1*) ^(*Suc k div 2*) ∗ (*if even k then cos x else sin x*) *exp*|*Im z*| *0 z*, *simplified*])
fix *k x*
assume *x* ∈ *closed-segment 0 z k* ≤ *n*
show ((λ*x*. (− *1*) ^ (*Suc k div 2*) ∗ (*if even k then cos x else sin x*)) *has-field-derivative*
(− *1*) ^ *Suc* (*k div 2*) ∗ (*if odd k then cos x else sin x*))
(*at x within closed-segment 0 z*)
apply (*auto simp*: *power-Suc*)
apply (*intro derivative-eq-intros* | *simp*)+
done
next
fix *x*
assume *x* ∈ *closed-segment 0 z*
then show *cmod* ((− *1*) ^ *Suc* (*n div 2*) ∗ (*if odd n then cos x else sin x*)) ≤ *exp* |*Im z*|
by (*auto simp*: *closed-segment-def norm-mult norm-power cmod-sin-le-exp cmod-cos-le-exp*)
qed
have ∗∗: ⋀*k*. *complex-of-real* (*cos-coeff k*) ∗ *z* ^ *k*
= (−*1*) ^(*Suc k div 2*) ∗ (*if even k then cos 0 else sin 0*) ∗ *z*^*k* / *of-nat* (*fact k*)
by (*auto simp*: *cos-coeff-def elim*!: *evenE*)
show *?thesis*
apply (*rule order-trans* [*OF* - ∗])
apply (*simp add*: ∗∗)
done
qed

declare *power-Suc* [*simp*]

32-bit Approximation to e

lemma *e-approx-32*: |*exp*(*1*) − *5837465777* / *2147483648*| ≤ (*inverse*(*2* ^ *32*)::*real*)
using *Taylor-exp* [*of 1 14*] *exp-le*
apply (*simp add*: *setsum-left-distrib in-Reals-norm Re-exp atMost-nat-numeral fact-numeral*)
apply (*simp only*: *pos-le-divide-eq* [*symmetric*], *linarith*)
done

lemma *e-less-3*: *exp 1* < (*3*::*real*)
using *e-approx-32*
by (*simp add*: *abs-if split*: *if-split-asm*)

lemma *ln3-gt-1*: *ln 3 > (1::real)*
by (*metis e-less-3 exp-less-cancel-iff exp-ln-iff less-trans ln-exp*)

52.7 The argument of a complex number

definition *Arg* :: *complex ⇒ real* **where**
Arg z ≡ if z = 0 then 0
else THE t. 0 ≤ t ∧ t < 2∗pi ∧
z = of-real(norm z) ∗ exp(ii ∗ of-real t)

lemma *Arg-0* [*simp*]: *Arg(0) = 0*
by (*simp add*: *Arg-def*)

lemma *Arg-unique-lemma*:
assumes *z*: *z = of-real(norm z) ∗ exp(ii ∗ of-real t)*
and *z'*: *z = of-real(norm z) ∗ exp(ii ∗ of-real t')*
and *t*: *0 ≤ t t < 2∗pi*
and *t'*: *0 ≤ t' t' < 2∗pi*
and *nz*: *z ≠ 0*
shows *t' = t*
proof −
have [*dest*]: *⋀x y z::real. x≥0 ⟹ x+y < z ⟹ y<z*
by *arith*
have *of-real (cmod z) ∗ exp (*i *∗ of-real t') = of-real (cmod z) ∗ exp (*i *∗ of-real t)*
by (*metis z z'*)
then have *exp (*i *∗ of-real t') = exp (*i *∗ of-real t)*
by (*metis nz mult-left-cancel mult-zero-left z*)
then have *sin t' = sin t ∧ cos t' = cos t*
apply (*simp add*: *exp-Euler sin-of-real cos-of-real*)
by (*metis Complex-eq complex.sel*)
then obtain *n::int* **where** *n*: *t' = t + 2 ∗ n ∗ pi*
by (*auto simp*: *sin-cos-eq-iff*)
then have *n=0*
apply (*rule-tac z=n* **in** *int-cases*)
using *t t'*
apply (*auto simp*: *mult-less-0-iff algebra-simps*)
done
then show *t' = t*
by (*simp add*: *n*)
qed

lemma *Arg*: *0 ≤ Arg z & Arg z < 2∗pi & z = of-real(norm z) ∗ exp(ii ∗ of-real(Arg z))*
proof (*cases z=0*)
case *True* **then show** *?thesis*
by (*simp add*: *Arg-def*)
next
case *False*

```
  obtain t where t: 0 ≤ t t < 2*pi
          and ReIm: Re z / cmod z = cos t Im z / cmod z = sin t
    using sincos-total-2pi [OF complex-unit-circle [OF False]]
    by blast
  have z: z = of-real(norm z) * exp(ii * of-real t)
    apply (rule complex-eqI)
    using t False ReIm
    apply (auto simp: exp-Euler sin-of-real cos-of-real divide-simps)
    done
  show ?thesis
    apply (simp add: Arg-def False)
    apply (rule theI [where a=t])
    using t z False
    apply (auto intro: Arg-unique-lemma)
    done
qed

corollary
  shows Arg-ge-0: 0 ≤ Arg z
    and Arg-lt-2pi: Arg z < 2*pi
    and Arg-eq: z = of-real(norm z) * exp(ii * of-real(Arg z))
  using Arg by auto

lemma complex-norm-eq-1-exp: norm z = 1 ⟷ (∃ t. z = exp(ii * of-real t))
  using Arg [of z] by auto

lemma Arg-unique: ⟦of-real r * exp(ii * of-real a) = z; 0 < r; 0 ≤ a; a < 2*pi⟧
⟹ Arg z = a
  apply (rule Arg-unique-lemma [OF - Arg-eq])
  using Arg [of z]
  apply (auto simp: norm-mult)
  done

lemma Arg-minus: z ≠ 0 ⟹ Arg (-z) = (if Arg z < pi then Arg z + pi else Arg
z - pi)
  apply (rule Arg-unique [of norm z])
  apply (rule complex-eqI)
  using Arg-ge-0 [of z] Arg-eq [of z] Arg-lt-2pi [of z] Arg-eq [of z]
  apply auto
  apply (auto simp: Re-exp Im-exp cos-diff sin-diff cis-conv-exp [symmetric])
  apply (metis Re-rcis Im-rcis rcis-def)+
  done

lemma Arg-times-of-real [simp]: 0 < r ⟹ Arg (of-real r * z) = Arg z
  apply (cases z=0, simp)
  apply (rule Arg-unique [of r * norm z])
  using Arg
  apply auto
  done
```

lemma *Arg-times-of-real2* [*simp*]: $0 < r \Longrightarrow Arg\ (z * of\text{-}real\ r) = Arg\ z$
by (*metis Arg-times-of-real mult.commute*)

lemma *Arg-divide-of-real* [*simp*]: $0 < r \Longrightarrow Arg\ (z\ /\ of\text{-}real\ r) = Arg\ z$
by (*metis Arg-times-of-real2 less-numeral-extra*(*3*) *nonzero-eq-divide-eq of-real-eq-0-iff*)

lemma *Arg-le-pi*: $Arg\ z \leq pi \longleftrightarrow 0 \leq Im\ z$
proof (*cases z=0*)
case *True* **then show** *?thesis*
by *simp*
next
case *False*
have $0 \leq Im\ z \longleftrightarrow 0 \leq Im\ (of\text{-}real\ (cmod\ z) * exp\ (\mathrm{i} * complex\text{-}of\text{-}real\ (Arg\ z)))$
by (*metis Arg-eq*)
also have $\ldots = (0 \leq Im\ (exp\ (\mathrm{i} * complex\text{-}of\text{-}real\ (Arg\ z))))$
using *False*
by (*simp add*: *zero-le-mult-iff*)
also have $\ldots \longleftrightarrow Arg\ z \leq pi$
by (*simp add*: *Im-exp*) (*metis Arg-ge-0 Arg-lt-2pi sin-lt-zero sin-ge-zero not-le*)
finally show *?thesis*
by *blast*
qed

lemma *Arg-lt-pi*: $0 < Arg\ z \wedge Arg\ z < pi \longleftrightarrow 0 < Im\ z$
proof (*cases z=0*)
case *True* **then show** *?thesis*
by *simp*
next
case *False*
have $0 < Im\ z \longleftrightarrow 0 < Im\ (of\text{-}real\ (cmod\ z) * exp\ (\mathrm{i} * complex\text{-}of\text{-}real\ (Arg\ z)))$
by (*metis Arg-eq*)
also have $\ldots = (0 < Im\ (exp\ (\mathrm{i} * complex\text{-}of\text{-}real\ (Arg\ z))))$
using *False*
by (*simp add*: *zero-less-mult-iff*)
also have $\ldots \longleftrightarrow 0 < Arg\ z \wedge Arg\ z < pi$
using *Arg-ge-0 Arg-lt-2pi sin-le-zero sin-gt-zero*
apply (*auto simp*: *Im-exp*)
using *le-less* **apply** *fastforce*
using *not-le* **by** *blast*
finally show *?thesis*
by *blast*
qed

lemma *Arg-eq-0*: $Arg\ z = 0 \longleftrightarrow z \in \mathbb{R} \wedge 0 \leq Re\ z$
proof (*cases z=0*)
case *True* **then show** *?thesis*

by *simp*
next
case *False*
have $z \in \mathbb{R} \wedge 0 \leq Re\ z \longleftrightarrow z \in \mathbb{R} \wedge 0 \leq Re\ (of\text{-}real\ (cmod\ z) * exp\ (\mathrm{i} * complex\text{-}of\text{-}real\ (Arg\ z)))$
by (*metis Arg-eq*)
also have $\ldots \longleftrightarrow z \in \mathbb{R} \wedge 0 \leq Re\ (exp\ (\mathrm{i} * complex\text{-}of\text{-}real\ (Arg\ z)))$
using *False*
by (*simp add*: *zero-le-mult-iff*)
also have $\ldots \longleftrightarrow Arg\ z = 0$
apply (*auto simp*: *Re-exp*)
apply (*metis Arg-lt-pi Arg-ge-0 Arg-le-pi cos-pi complex-is-Real-iff leD less-linear less-minus-one-simps*(2) *minus-minus neg-less-eq-nonneg order-refl*)
using *Arg-eq* [*of z*]
apply (*auto simp*: *Reals-def*)
done
finally show *?thesis*
by *blast*
qed

corollary *Arg-gt-0*:
assumes $z \in \mathbb{R} \Longrightarrow Re\ z < 0$
shows $Arg\ z > 0$
using *Arg-eq-0 Arg-ge-0 assms dual-order.strict-iff-order* **by** *fastforce*

lemma *Arg-of-real*: $Arg(of\text{-}real\ x) = 0 \longleftrightarrow 0 \leq x$
by (*simp add*: *Arg-eq-0*)

lemma *Arg-eq-pi*: $Arg\ z = pi \longleftrightarrow z \in \mathbb{R} \wedge Re\ z < 0$
apply (*cases z=0*, *simp*)
using *Arg-eq-0* [*of* $-z$]
apply (*auto simp*: *complex-is-Real-iff Arg-minus*)
apply (*simp add*: *complex-Re-Im-cancel-iff*)
apply (*metis Arg-minus pi-gt-zero add.left-neutral minus-minus minus-zero*)
done

lemma *Arg-eq-0-pi*: $Arg\ z = 0 \vee Arg\ z = pi \longleftrightarrow z \in \mathbb{R}$
using *Arg-eq-0 Arg-eq-pi not-le* **by** *auto*

lemma *Arg-inverse*: $Arg(inverse\ z) = (if\ z \in \mathbb{R} \wedge 0 \leq Re\ z\ then\ Arg\ z\ else\ 2*pi - Arg\ z)$
apply (*cases z=0*, *simp*)
apply (*rule Arg-unique* [*of inverse* (*norm z*)])
using *Arg-ge-0* [*of z*] *Arg-lt-2pi* [*of z*] *Arg-eq* [*of z*] *Arg-eq-0* [*of z*] *exp-two-pi-i*
apply (*auto simp*: *of-real-numeral algebra-simps exp-diff divide-simps*)
done

lemma *Arg-eq-iff*:
assumes $w \neq 0\ z \neq 0$

shows *Arg w = Arg z ⟷ (∃ x. 0 < x & w = of-real x * z)*
using *assms Arg-eq [of z] Arg-eq [of w]*
apply *auto*
apply *(rule-tac x=norm w / norm z* **in** *exI)*
apply *(simp add: divide-simps)*
by *(metis mult.commute mult.left-commute)*

lemma *Arg-inverse-eq-0: Arg(inverse z) = 0 ⟷ Arg z = 0*
using *complex-is-Real-iff*
apply *(simp add: Arg-eq-0)*
apply *(auto simp: divide-simps not-sum-power2-lt-zero)*
done

lemma *Arg-divide:*
assumes *w ≠ 0 z ≠ 0 Arg w ≤ Arg z*
shows *Arg(z / w) = Arg z − Arg w*
apply *(rule Arg-unique [of norm(z / w)])*
using *assms Arg-eq [of z] Arg-eq [of w] Arg-ge-0 [of w] Arg-lt-2pi [of z]*
apply *(auto simp: exp-diff norm-divide algebra-simps divide-simps)*
done

lemma *Arg-le-div-sum:*
assumes *w ≠ 0 z ≠ 0 Arg w ≤ Arg z*
shows *Arg z = Arg w + Arg(z / w)*
by *(simp add: Arg-divide assms)*

lemma *Arg-le-div-sum-eq:*
assumes *w ≠ 0 z ≠ 0*
shows *Arg w ≤ Arg z ⟷ Arg z = Arg w + Arg(z / w)*
using *assms*
by *(auto simp: Arg-ge-0 intro: Arg-le-div-sum)*

lemma *Arg-diff:*
assumes *w ≠ 0 z ≠ 0*
shows *Arg w − Arg z = (if Arg z ≤ Arg w then Arg(w / z) else Arg(w/z) − 2∗pi)*
using *assms*
apply *(auto simp: Arg-ge-0 Arg-divide not-le)*
using *Arg-divide [of w z] Arg-inverse [of w/z]*
apply *auto*
by *(metis Arg-eq-0 less-irrefl minus-diff-eq right-minus-eq)*

lemma *Arg-add:*
assumes *w ≠ 0 z ≠ 0*
shows *Arg w + Arg z = (if Arg w + Arg z < 2∗pi then Arg(w ∗ z) else Arg(w ∗ z) + 2∗pi)*
using *assms*
using *Arg-diff [of w∗z z] Arg-le-div-sum-eq [of z w∗z]*
apply *(auto simp: Arg-ge-0 Arg-divide not-le)*

apply (*metis Arg-lt-2pi add.commute*)
apply (*metis* (*no-types*) *Arg add.commute diff-0 diff-add-cancel diff-less-eq diff-minus-eq-add not-less*)
done

lemma *Arg-times*:
assumes $w \neq 0$ $z \neq 0$
shows $Arg\ (w * z) = (if\ Arg\ w + Arg\ z < 2*pi\ then\ Arg\ w + Arg\ z$
$else\ (Arg\ w + Arg\ z) - 2*pi)$
using *Arg-add* [*OF assms*]
by *auto*

lemma *Arg-cnj*: $Arg(cnj\ z) = (if\ z \in \mathbb{R} \wedge 0 \leq Re\ z\ then\ Arg\ z\ else\ 2*pi - Arg\ z)$
apply (*cases z=0*, *simp*)
apply (*rule trans* [*of - Arg*(*inverse z*)])
apply (*simp add*: *Arg-eq-iff divide-simps complex-norm-square* [*symmetric*] *mult.commute*)
apply (*metis norm-eq-zero of-real-power zero-less-power2*)
apply (*auto simp*: *of-real-numeral Arg-inverse*)
done

lemma *Arg-real*: $z \in \mathbb{R} \Longrightarrow Arg\ z = (if\ 0 \leq Re\ z\ then\ 0\ else\ pi)$
using *Arg-eq-0 Arg-eq-0-pi*
by *auto*

lemma *Arg-exp*: $0 \leq Im\ z \Longrightarrow Im\ z < 2*pi \Longrightarrow Arg(exp\ z) = Im\ z$
by (*rule Arg-unique* [*of* $exp(Re\ z)$]) (*auto simp*: *exp-eq-polar*)

lemma *complex-split-polar*:
obtains *r a*::*real* **where** $z = complex\text{-}of\text{-}real\ r * (cos\ a + \mathrm{i} * sin\ a)\ 0 \leq r\ 0 \leq a\ a < 2*pi$
using *Arg cis.ctr cis-conv-exp* **by** *fastforce*

lemma *Re-Im-le-cmod*: $Im\ w * sin\ \varphi + Re\ w * cos\ \varphi \leq cmod\ w$
proof (*cases w rule*: *complex-split-polar*)
case (*1 r a*) **with** *sin-cos-le1* [*of a* φ] **show** *?thesis*
apply (*simp add*: *norm-mult cmod-unit-one*)
by (*metis* (*no-types*, *hide-lams*) *abs-le-D1 distrib-left mult.commute mult.left-commute mult-left-le*)
qed

52.8 Analytic properties of tangent function

lemma *cnj-tan*: $cnj(tan\ z) = tan(cnj\ z)$
by (*simp add*: *cnj-cos cnj-sin tan-def*)

lemma *field-differentiable-at-tan*: $\sim(cos\ z = 0) \Longrightarrow$ *tan field-differentiable at z*
unfolding *field-differentiable-def*
using *DERIV-tan* **by** *blast*

lemma *field-differentiable-within-tan*: ~(*cos z = 0*)
⟹ *tan field-differentiable* (*at z within s*)
using *field-differentiable-at-tan field-differentiable-at-within* **by** *blast*

lemma *continuous-within-tan*: ~(*cos z = 0*) ⟹ *continuous* (*at z within s*) *tan*
using *continuous-at-imp-continuous-within isCont-tan* **by** *blast*

lemma *continuous-on-tan* [*continuous-intros*]: (⋀*z*. *z* ∈ *s* ⟹ ~(*cos z = 0*)) ⟹ *continuous-on s tan*
by (*simp add*: *continuous-at-imp-continuous-on*)

lemma *holomorphic-on-tan*: (⋀*z*. *z* ∈ *s* ⟹ ~(*cos z = 0*)) ⟹ *tan holomorphic-on s*
by (*simp add*: *field-differentiable-within-tan holomorphic-on-def*)

52.9 Complex logarithms (the conventional principal value)

instantiation *complex* :: *ln*
begin

definition *ln-complex* :: *complex* ⇒ *complex*
where *ln-complex* ≡ λ*z*. *THE w*. *exp w = z* & −*pi* < *Im*(*w*) & *Im*(*w*) ≤ *pi*

lemma
assumes *z* ≠ *0*
shows *exp-Ln* [*simp*]: *exp*(*ln z*) = *z*
and *mpi-less-Im-Ln*: −*pi* < *Im*(*ln z*)
and *Im-Ln-le-pi*: *Im*(*ln z*) ≤ *pi*
proof −
obtain ψ **where** *z*: *z* / (*cmod z*) = *Complex* (*cos* ψ) (*sin* ψ)
using *complex-unimodular-polar* [*of z* / (*norm z*)] *assms*
by (*auto simp*: *norm-divide divide-simps*)
obtain φ **where** φ: − *pi* < φ φ ≤ *pi sin* φ = *sin* ψ *cos* φ = *cos* ψ
using *sincos-principal-value* [*of* ψ] *assms*
by (*auto simp*: *norm-divide divide-simps*)
have *exp*(*ln z*) = *z* & −*pi* < *Im*(*ln z*) & *Im*(*ln z*) ≤ *pi* **unfolding** *ln-complex-def*
apply (*rule theI* [**where** *a* = *Complex* (*ln*(*norm z*)) φ])
using *z assms* φ
apply (*auto simp*: *field-simps exp-complex-eqI exp-eq-polar cis.code*)
done
then show *exp*(*ln z*) = *z* −*pi* < *Im*(*ln z*) *Im*(*ln z*) ≤ *pi*
by *auto*
qed

lemma *Ln-exp* [*simp*]:
assumes −*pi* < *Im*(*z*) *Im*(*z*) ≤ *pi*
shows *ln*(*exp z*) = *z*
apply (*rule exp-complex-eqI*)

using *assms mpi-less-Im-Ln* [*of exp z*] *Im-Ln-le-pi* [*of exp z*]
apply *auto*
done

52.10 Relation to Real Logarithm

lemma *Ln-of-real*:
assumes $0 < z$
shows $ln(of\text{-}real\ z{::}complex) = of\text{-}real(ln\ z)$
proof −
have $ln(of\text{-}real\ (exp\ (ln\ z)){::}complex) = ln\ (exp\ (of\text{-}real\ (ln\ z)))$
by (*simp add*: *exp-of-real*)
also have $... = of\text{-}real(ln\ z)$
using *assms*
by (*subst Ln-exp*) *auto*
finally show *?thesis*
using *assms* **by** *simp*
qed

corollary *Ln-in-Reals* [*simp*]: $z \in \mathbb{R} \Longrightarrow Re\ z > 0 \Longrightarrow ln\ z \in \mathbb{R}$
by (*auto simp*: *Ln-of-real elim*: *Reals-cases*)

corollary *Im-Ln-of-real* [*simp*]: $r > 0 \Longrightarrow Im\ (ln\ (of\text{-}real\ r)) = 0$
by (*simp add*: *Ln-of-real*)

lemma *cmod-Ln-Reals* [*simp*]: $z \in \mathbb{R} \Longrightarrow 0 < Re\ z \Longrightarrow cmod\ (ln\ z) = norm\ (ln\ (Re\ z))$
using *Ln-of-real* **by** *force*

lemma *Ln-1*: $ln\ 1 = (0{::}complex)$
proof −
have $ln\ (exp\ 0) = (0{::}complex)$
by (*metis* (*mono-tags*, *hide-lams*) *Ln-of-real exp-zero ln-one of-real-0 of-real-1 zero-less-one*)
then show *?thesis*
by *simp*
qed

instance
by *intro-classes* (*rule ln-complex-def Ln-1*)

end

abbreviation $Ln :: complex \Rightarrow complex$
where $Ln \equiv ln$

lemma *Ln-eq-iff*: $w \neq 0 \Longrightarrow z \neq 0 \Longrightarrow (Ln\ w = Ln\ z \longleftrightarrow w = z)$
by (*metis exp-Ln*)

lemma *Ln-unique*: $exp(z) = w \Longrightarrow -pi < Im(z) \Longrightarrow Im(z) \leq pi \Longrightarrow Ln\ w = z$
using *Ln-exp* **by** *blast*

lemma *Re-Ln* [*simp*]: $z \neq 0 \Longrightarrow Re(Ln\ z) = ln(norm\ z)$
by (*metis exp-Ln assms ln-exp norm-exp-eq-Re*)

corollary *ln-cmod-le*:
assumes *z*: $z \neq 0$
shows $ln\ (cmod\ z) \leq cmod\ (Ln\ z)$
using *norm-exp* [*of Ln z*, *simplified exp-Ln* [*OF z*]]
by (*metis Re-Ln complex-Re-le-cmod z*)

proposition *exists-complex-root*:
fixes *z* :: *complex*
assumes $n \neq 0$ **obtains** *w* **where** $z = w\ \hat{}\ n$
apply (*cases z=0*)
using *assms* **apply** (*simp add*: *power-0-left*)
apply (*rule-tac* $w = exp(Ln\ z\ /\ n)$ **in** *that*)
apply (*auto simp*: *assms exp-of-nat-mult* [*symmetric*])
done

corollary *exists-complex-root-nonzero*:
fixes *z*::*complex*
assumes $z \neq 0\ n \neq 0$
obtains *w* **where** $w \neq 0\ z = w\ \hat{}\ n$
by (*metis exists-complex-root* [*of n z*] *assms power-0-left*)

52.11 The Unwinding Number and the Ln-product Formula

Note that in this special case the unwinding number is -1, 0 or 1.

definition *unwinding* :: *complex* ⇒ *complex* **where**
$unwinding(z) = (z - Ln(exp\ z))\ /\ (of\text{-}real(2{*}pi) * ii)$

lemma *unwinding-2pi*: $(2{*}pi) * ii * unwinding(z) = z - Ln(exp\ z)$
by (*simp add*: *unwinding-def*)

lemma *Ln-times-unwinding*:
$w \neq 0 \Longrightarrow z \neq 0 \Longrightarrow Ln(w * z) = Ln(w) + Ln(z) - (2{*}pi) * ii * unwinding(Ln\ w + Ln\ z)$
using *unwinding-2pi* **by** (*simp add*: *exp-add*)

52.12 Derivative of Ln away from the branch cut

lemma
assumes $z \notin \mathbb{R}_{\leq 0}$
shows *has-field-derivative-Ln*: (*Ln has-field-derivative inverse*(*z*)) (*at z*)
and *Im-Ln-less-pi*: $Im\ (Ln\ z) < pi$
proof −
have *znz*: $z \neq 0$

using *assms* **by** *auto*
then have *Im* (*Ln z*) ≠ *pi*
by (*metis* (*no-types*) *Im-exp Ln-in-Reals assms complex-nonpos-Reals-iff complex-is-Real-iff exp-Ln mult-zero-right not-less pi-neq-zero sin-pi znz*)
then show ∗: *Im* (*Ln z*) < *pi* **using** *assms Im-Ln-le-pi*
by (*simp add*: *le-neq-trans znz*)
have (*exp has-field-derivative z*) (*at* (*Ln z*))
by (*metis znz DERIV-exp exp-Ln*)
then show (*Ln has-field-derivative inverse*(*z*)) (*at z*)
apply (*rule has-complex-derivative-inverse-strong-x*
[**where** *s* = {*w*. −*pi* < *Im*(*w*) ∧ *Im*(*w*) < *pi*}])
using *znz* ∗
apply (*auto simp*: *Transcendental.continuous-on-exp* [*OF continuous-on-id*] *open-Collect-conj open-halfspace-Im-gt open-halfspace-Im-lt mpi-less-Im-Ln*)
done
qed

declare *has-field-derivative-Ln* [*derivative-intros*]
declare *has-field-derivative-Ln* [*THEN DERIV-chain2*, *derivative-intros*]

lemma *field-differentiable-at-Ln*: *z* ∉ $\mathbb{R}_{\leq 0}$ ⟹ *Ln field-differentiable at z*
using *field-differentiable-def has-field-derivative-Ln* **by** *blast*

lemma *field-differentiable-within-Ln*: *z* ∉ $\mathbb{R}_{\leq 0}$
⟹ *Ln field-differentiable* (*at z within s*)
using *field-differentiable-at-Ln field-differentiable-within-subset* **by** *blast*

lemma *continuous-at-Ln*: *z* ∉ $\mathbb{R}_{\leq 0}$ ⟹ *continuous* (*at z*) *Ln*
by (*simp add*: *field-differentiable-imp-continuous-at field-differentiable-within-Ln*)

lemma *isCont-Ln'* [*simp*]:
⟦*isCont f z*; *f z* ∉ $\mathbb{R}_{\leq 0}$⟧ ⟹ *isCont* (λ*x*. *Ln* (*f x*)) *z*
by (*blast intro*: *isCont-o2* [*OF* - *continuous-at-Ln*])

lemma *continuous-within-Ln*: *z* ∉ $\mathbb{R}_{\leq 0}$ ⟹ *continuous* (*at z within s*) *Ln*
using *continuous-at-Ln continuous-at-imp-continuous-within* **by** *blast*

lemma *continuous-on-Ln* [*continuous-intros*]: (⋀*z*. *z* ∈ *s* ⟹ *z* ∉ $\mathbb{R}_{\leq 0}$) ⟹ *continuous-on s Ln*
by (*simp add*: *continuous-at-imp-continuous-on continuous-within-Ln*)

lemma *holomorphic-on-Ln*: (⋀*z*. *z* ∈ *s* ⟹ *z* ∉ $\mathbb{R}_{\leq 0}$) ⟹ *Ln holomorphic-on s*
by (*simp add*: *field-differentiable-within-Ln holomorphic-on-def*)

52.13 Quadrant-type results for Ln

lemma *cos-lt-zero-pi*: *pi*/2 < *x* ⟹ *x* < 3∗*pi*/2 ⟹ *cos x* < 0
using *cos-minus-pi cos-gt-zero-pi* [*of x*−*pi*]
by *simp*

lemma *Re-Ln-pos-lt*:
assumes $z \neq 0$
shows $|Im(Ln\ z)| < pi/2 \longleftrightarrow 0 < Re(z)$
proof −
{ fix w
assume $w = Ln\ z$
then have w: $Im\ w \leq pi - pi < Im\ w$
using *Im-Ln-le-pi* [*of z*] *mpi-less-Im-Ln* [*of z*] *assms*
by *auto*
then have $|Im\ w| < pi/2 \longleftrightarrow 0 < Re(exp\ w)$
apply (*auto simp*: *Re-exp zero-less-mult-iff cos-gt-zero-pi*)
using *cos-lt-zero-pi* [*of* $-(Im\ w)$] *cos-lt-zero-pi* [*of* $(Im\ w)$]
apply (*simp add*: *abs-if split*: *if-split-asm*)
apply (*metis* (*no-types*) *cos-minus cos-pi-half eq-divide-eq-numeral1*(*1*) *eq-numeral-simps*(*4*)
less-numeral-extra(*3*) *linorder-neqE-linordered-idom minus-mult-minus*
minus-mult-right
mult-numeral-1-right)
done
}
then show *?thesis* **using** *assms*
by *auto*
qed

lemma *Re-Ln-pos-le*:
assumes $z \neq 0$
shows $|Im(Ln\ z)| \leq pi/2 \longleftrightarrow 0 \leq Re(z)$
proof −
{ fix w
assume $w = Ln\ z$
then have w: $Im\ w \leq pi - pi < Im\ w$
using *Im-Ln-le-pi* [*of z*] *mpi-less-Im-Ln* [*of z*] *assms*
by *auto*
then have $|Im\ w| \leq pi/2 \longleftrightarrow 0 \leq Re(exp\ w)$
apply (*auto simp*: *Re-exp zero-le-mult-iff cos-ge-zero*)
using *cos-lt-zero-pi* [*of* $-$ $(Im\ w)$] *cos-lt-zero-pi* [*of* $(Im\ w)$] *not-le*
apply (*auto simp*: *abs-if split*: *if-split-asm*)
done
}
then show *?thesis* **using** *assms*
by *auto*
qed

lemma *Im-Ln-pos-lt*:
assumes $z \neq 0$
shows $0 < Im(Ln\ z) \wedge Im(Ln\ z) < pi \longleftrightarrow 0 < Im(z)$
proof −
{ fix w
assume $w = Ln\ z$

then have *w*: $Im\ w \leq pi - pi < Im\ w$
using *Im-Ln-le-pi* [*of z*] *mpi-less-Im-Ln* [*of z*] *assms*
by *auto*
then have $0 < Im\ w \wedge Im\ w < pi \longleftrightarrow 0 < Im(exp\ w)$
using *sin-gt-zero* [*of* $-$ (*Im w*)] *sin-gt-zero* [*of* (*Im w*)]
apply (*auto simp*: *Im-exp zero-less-mult-iff*)
using *less-linear* **apply** *fastforce*
using *less-linear* **apply** *fastforce*
done
}
then show *?thesis* **using** *assms*
by *auto*
qed

lemma *Im-Ln-pos-le*:
assumes $z \neq 0$
shows $0 \leq Im(Ln\ z) \wedge Im(Ln\ z) \leq pi \longleftrightarrow 0 \leq Im(z)$
proof −
{ **fix** *w*
assume $w = Ln\ z$
then have *w*: $Im\ w \leq pi - pi < Im\ w$
using *Im-Ln-le-pi* [*of z*] *mpi-less-Im-Ln* [*of z*] *assms*
by *auto*
then have $0 \leq Im\ w \wedge Im\ w \leq pi \longleftrightarrow 0 \leq Im(exp\ w)$
using *sin-ge-zero* [*of* $-$ (*Im w*)] *sin-ge-zero* [*of* (*Im w*)]
apply (*auto simp*: *Im-exp zero-le-mult-iff sin-ge-zero*)
apply (*metis not-le not-less-iff-gr-or-eq pi-not-less-zero sin-eq-0-pi*)
done }
then show *?thesis* **using** *assms*
by *auto*
qed

lemma *Re-Ln-pos-lt-imp*: $0 < Re(z) \Longrightarrow |Im(Ln\ z)| < pi/2$
by (*metis Re-Ln-pos-lt less-irrefl zero-complex.simps*(*1*))

lemma *Im-Ln-pos-lt-imp*: $0 < Im(z) \Longrightarrow 0 < Im(Ln\ z) \wedge Im(Ln\ z) < pi$
by (*metis Im-Ln-pos-lt not-le order-refl zero-complex.simps*(*2*))

A reference to the set of positive real numbers

lemma *Im-Ln-eq-0*: $z \neq 0 \Longrightarrow (Im(Ln\ z) = 0 \longleftrightarrow 0 < Re(z) \wedge Im(z) = 0)$
by (*metis Im-complex-of-real Im-exp Ln-in-Reals Re-Ln-pos-lt Re-Ln-pos-lt-imp*
Re-complex-of-real complex-is-Real-iff exp-Ln exp-of-real pi-gt-zero)

lemma *Im-Ln-eq-pi*: $z \neq 0 \Longrightarrow (Im(Ln\ z) = pi \longleftrightarrow Re(z) < 0 \wedge Im(z) = 0)$
by (*metis Im-Ln-eq-0 Im-Ln-pos-le Im-Ln-pos-lt add.left-neutral complex-eq less-eq-real-def*
mult-zero-right not-less-iff-gr-or-eq pi-ge-zero pi-neq-zero rcis-zero-arg rcis-zero-mod)

52.14 More Properties of Ln

lemma *cnj-Ln*: $z \notin \mathbb{R}_{\leq 0} \Longrightarrow cnj(Ln\ z) = Ln(cnj\ z)$
apply (*cases z=0*, *auto*)
apply (*rule exp-complex-eqI*)
apply (*auto simp*: *abs-if split*: *if-split-asm*)
using *Im-Ln-less-pi Im-Ln-le-pi* **apply** *force*
apply (*metis complex-cnj-zero-iff diff-minus-eq-add diff-strict-mono minus-less-iff mpi-less-Im-Ln mult.commute mult-2-right*)
by (*metis exp-Ln exp-cnj*)

lemma *Ln-inverse*: $z \notin \mathbb{R}_{\leq 0} \Longrightarrow Ln(inverse\ z) = -(Ln\ z)$
apply (*cases z=0*, *auto*)
apply (*rule exp-complex-eqI*)
using *mpi-less-Im-Ln* [*of z*] *mpi-less-Im-Ln* [*of inverse z*]
apply (*auto simp*: *abs-if exp-minus split*: *if-split-asm*)
apply (*metis Im-Ln-less-pi Im-Ln-le-pi add.commute add-mono-thms-linordered-field*(*3*) *inverse-nonzero-iff-nonzero mult-2*)
done

lemma *Ln-minus1* [*simp*]: $Ln(-1) = ii * pi$
apply (*rule exp-complex-eqI*)
using *Im-Ln-le-pi* [*of −1*] *mpi-less-Im-Ln* [*of −1*] *cis-conv-exp cis-pi*
apply (*auto simp*: *abs-if*)
done

lemma *Ln-ii* [*simp*]: $Ln\ ii = ii * of\text{-}real\ pi/2$
using *Ln-exp* [*of ii * (of-real pi/2)*]
unfolding *exp-Euler*
by *simp*

lemma *Ln-minus-ii* [*simp*]: $Ln(-ii) = -\ (ii * pi/2)$
proof −
have $Ln(-ii) = Ln(inverse\ ii)$ **by** *simp*
also have ... $= -\ (Ln\ ii)$ **using** *Ln-inverse* **by** *blast*
also have ... $= -\ (ii * pi/2)$ **by** *simp*
finally show *?thesis* **.**
qed

lemma *Ln-times*:
assumes $w \neq 0\ z \neq 0$
shows $Ln(w * z) =$
$(if\ Im(Ln\ w + Ln\ z) \leq -pi\ then$
$(Ln(w) + Ln(z)) + ii * of\text{-}real(2{*}pi)$
$else\ if\ Im(Ln\ w + Ln\ z) > pi\ then$
$(Ln(w) + Ln(z)) - ii * of\text{-}real(2{*}pi)$
$else\ Ln(w) + Ln(z))$
using *pi-ge-zero Im-Ln-le-pi* [*of w*] *Im-Ln-le-pi* [*of z*]
using *assms mpi-less-Im-Ln* [*of w*] *mpi-less-Im-Ln* [*of z*]
by (*auto simp*: *exp-add exp-diff sin-double cos-double exp-Euler intro*!: *Ln-unique*)

corollary *Ln-times-simple*:
$[\![w \neq 0;\ z \neq 0;\ -pi < Im(Ln\ w) + Im(Ln\ z);\ Im(Ln\ w) + Im(Ln\ z) \leq pi]\!]$
$\Longrightarrow Ln(w * z) = Ln(w) + Ln(z)$
by (*simp add*: *Ln-times*)

corollary *Ln-times-of-real*:
$[\![r > 0;\ z \neq 0]\!] \Longrightarrow Ln(of\text{-}real\ r * z) = ln\ r + Ln(z)$
using *mpi-less-Im-Ln Im-Ln-le-pi*
by (*force simp*: *Ln-times*)

corollary *Ln-divide-of-real*:
$[\![r > 0;\ z \neq 0]\!] \Longrightarrow Ln(z\ /\ of\text{-}real\ r) = Ln(z) - ln\ r$
using *Ln-times-of-real* [*of inverse r z*]
by (*simp add*: *ln-inverse Ln-of-real mult.commute divide-inverse of-real-inverse* [*symmetric*]
del: *of-real-inverse*)

lemma *Ln-minus*:
assumes $z \neq 0$
shows $Ln(-z) = (if\ Im(z) \leq 0 \wedge \sim(Re(z) < 0 \wedge Im(z) = 0)$
then $Ln(z) + ii * pi$
else $Ln(z) - ii * pi$) (**is** - = *?rhs*)
using *Im-Ln-le-pi* [*of z*] *mpi-less-Im-Ln* [*of z*] *assms*
Im-Ln-eq-pi [*of z*] *Im-Ln-pos-lt* [*of z*]
by (*fastforce simp*: *exp-add exp-diff exp-Euler intro*!: *Ln-unique*)

lemma *Ln-inverse-if*:
assumes $z \neq 0$
shows $Ln\ (inverse\ z) = (if\ z \in \mathbb{R}_{\leq 0}\ then\ -(Ln\ z) + \mathrm{i} * 2 * complex\text{-}of\text{-}real\ pi\ else\ -(Ln\ z))$
proof (*cases* $z \in \mathbb{R}_{\leq 0}$)
case *False* **then show** *?thesis*
by (*simp add*: *Ln-inverse*)
next
case *True*
then have *z*: $Im\ z = 0\ Re\ z < 0$
using *assms*
apply (*auto simp*: *complex-nonpos-Reals-iff*)
by (*metis complex-is-Real-iff le-imp-less-or-eq of-real-0 of-real-Re*)
have $Ln(inverse\ z) = Ln(-\ (inverse\ (-z)))$
by *simp*
also have $... = Ln\ (inverse\ (-z)) + \mathrm{i} * complex\text{-}of\text{-}real\ pi$
using *assms z*
apply (*simp add*: *Ln-minus*)
apply (*simp add*: *field-simps*)
done
also have $... = -\ Ln\ (-\ z) + \mathrm{i} * complex\text{-}of\text{-}real\ pi$
apply (*subst Ln-inverse*)

```
    using z by (auto simp add: complex-nonneg-Reals-iff)
  also have ... = − (Ln z) + i ∗ 2 ∗ complex-of-real pi
    apply (subst Ln-minus [OF assms])
    using assms z
    apply simp
    done
  finally show ?thesis by (simp add: True)
qed

lemma Ln-times-ii:
  assumes z ≠ 0
    shows  Ln(ii ∗ z) = (if 0 ≤ Re(z) | Im(z) < 0
                         then Ln(z) + ii ∗ of-real pi/2
                         else Ln(z) − ii ∗ of-real(3 ∗ pi/2))
  using Im-Ln-le-pi [of z] mpi-less-Im-Ln [of z] assms
        Im-Ln-eq-pi [of z] Im-Ln-pos-lt [of z] Re-Ln-pos-le [of z]
  by (auto simp: Ln-times)

lemma Ln-of-nat: 0 < n ⟹ Ln (of-nat n) = of-real (ln (of-nat n))
  by (subst of-real-of-nat-eq[symmetric], subst Ln-of-real[symmetric]) simp-all

lemma Ln-of-nat-over-of-nat:
  assumes m > 0 n > 0
  shows   Ln (of-nat m / of-nat n) = of-real (ln (of-nat m) − ln (of-nat n))
proof −
  have of-nat m / of-nat n = (of-real (of-nat m / of-nat n) :: complex) by simp
  also from assms have Ln ... = of-real (ln (of-nat m / of-nat n))
    by (simp add: Ln-of-real[symmetric])
  also from assms have ... = of-real (ln (of-nat m) − ln (of-nat n))
    by (simp add: ln-div)
  finally show ?thesis .
qed
```

52.15 Relation between Ln and Arg, and hence continuity of Arg

```
lemma Arg-Ln:
  assumes 0 < Arg z shows Arg z = Im(Ln(−z)) + pi
proof (cases z = 0)
  case True
  with assms show ?thesis
    by simp
next
  case False
  then have z / of-real(norm z) = exp(ii ∗ of-real(Arg z))
    using Arg [of z]
   by (metis abs-norm-cancel nonzero-mult-divide-cancel-left norm-of-real zero-less-norm-iff)
  then have − z / of-real(norm z) = exp (i ∗ (of-real (Arg z) − pi))
    using cis-conv-exp cis-pi
```

by (*auto simp*: *exp-diff algebra-simps*)
then have *ln* (− *z* / *of-real*(*norm z*)) = *ln* (*exp* (i ∗ (*of-real* (*Arg z*) − *pi*)))
by *simp*
also have ... = i ∗ (*of-real*(*Arg z*) − *pi*)
using *Arg* [*of z*] *assms pi-not-less-zero*
by *auto*
finally have *Arg z* = *Im* (*Ln* (− *z* / *of-real* (*cmod z*))) + *pi*
by *simp*
also have ... = *Im* (*Ln* (−*z*) − *ln* (*cmod z*)) + *pi*
by (*metis diff-0-right minus-diff-eq zero-less-norm-iff Ln-divide-of-real False*)
also have ... = *Im* (*Ln* (−*z*)) + *pi*
by *simp*
finally show *?thesis* **.**
qed

lemma *continuous-at-Arg*:
assumes *z* ∉ $\mathbb{R}_{\geq 0}$
shows *continuous* (*at z*) *Arg*
proof −
have ∗: *isCont* (λ*z*. *Im* (*Ln* (− *z*)) + *pi*) *z*
by (*rule Complex.isCont-Im isCont-Ln′ continuous-intros* | *simp add*: *assms complex-is-Real-iff*)+
have [*simp*]: ⋀*x*. ⟦*Im x* ≠ *0*⟧ ⟹ *Im* (*Ln* (− *x*)) + *pi* = *Arg x*
using *Arg-Ln Arg-gt-0 complex-is-Real-iff* **by** *auto*
consider *Re z* < *0* | *Im z* ≠ *0* **using** *assms*
using *complex-nonneg-Reals-iff not-le* **by** *blast*
then have [*simp*]: (λ*z*. *Im* (*Ln* (− *z*)) + *pi*) −*z*→ *Arg z*
using ∗ **by** (*simp add*: *isCont-def*) (*metis Arg-Ln Arg-gt-0 complex-is-Real-iff*)
show *?thesis*
apply (*simp add*: *continuous-at*)
apply (*rule Lim-transform-within-open* [**where** *s*= −$\mathbb{R}_{\geq 0}$ **and** *f* = λ*z*. *Im*(*Ln*(−*z*)) + *pi*])
apply (*auto simp add*: *not-le Arg-Ln* [*OF Arg-gt-0*] *complex-nonneg-Reals-iff closed-def* [*symmetric*])
using *assms* **apply** (*force simp add*: *complex-nonneg-Reals-iff*)
done
qed

lemma *Ln-series*:
fixes *z* :: *complex*
assumes *norm z* < *1*
shows (λ*n*. (−*1*) ^*Suc n* / *of-nat n* ∗ *z*^*n*) *sums ln* (*1* + *z*) (**is** (λ*n*. *?f n* ∗ *z*^*n*) *sums* -)
proof −
let *?F* = λ*z*. ∑ *n*. *?f n* ∗ *z*^*n* **and** *?F′* = λ*z*. ∑ *n*. *diffs ?f n* ∗ *z*^*n*
have *r*: *conv-radius ?f* = *1*
by (*intro conv-radius-ratio-limit-nonzero*[*of - 1*])
(*simp-all add*: *norm-divide LIMSEQ-Suc-n-over-n del*: *of-nat-Suc*)

```
  have ∃c. ∀z∈ball 0 1. ln (1 + z) - ?F z = c
  proof (rule has-field-derivative-zero-constant)
    fix z :: complex assume z': z ∈ ball 0 1
    hence z: norm z < 1 by (simp add: dist-0-norm)
    def t ≡ of-real (1 + norm z) / 2 :: complex
    from z have t: norm z < norm t norm t < 1 unfolding t-def
      by (simp-all add: field-simps norm-divide del: of-real-add)

    have Re (-z) ≤ norm (-z) by (rule complex-Re-le-cmod)
    also from z have ... < 1 by simp
    finally have ((λz. ln (1 + z)) has-field-derivative inverse (1+z)) (at z)
      by (auto intro!: derivative-eq-intros simp: complex-nonpos-Reals-iff)
    moreover have (?F has-field-derivative ?F' z) (at z) using t r
      by (intro termdiffs-strong[of - t] summable-in-conv-radius) simp-all
    ultimately have ((λz. ln (1 + z) - ?F z) has-field-derivative (inverse (1 +
z) - ?F' z))
                    (at z within ball 0 1)
      by (intro derivative-intros) (simp-all add: at-within-open[OF z'])
    also have (λn. of-nat n * ?f n * z ^ (n - Suc 0)) sums ?F' z using t r
       by (intro diffs-equiv termdiff-converges[OF t(1)] summable-in-conv-radius)
simp-all
    from sums-split-initial-segment[OF this, of 1]
        have (λi. (-z) ^ i) sums ?F' z by (simp add: power-minus[of z] del:
of-nat-Suc)
   hence ?F' z = inverse (1 + z) using z by (simp add: sums-iff suminf-geometric
divide-inverse)
    also have inverse (1 + z) - inverse (1 + z) = 0 by simp
    finally show ((λz. ln (1 + z) - ?F z) has-field-derivative 0) (at z within ball
0 1) .
  qed simp-all
  then obtain c where c: ⋀z. z ∈ ball 0 1 ⟹ ln (1 + z) - ?F z = c by blast
  from c[of 0] have c = 0 by (simp only: powser-zero) simp
  with c[of z] assms have ln (1 + z) = ?F z by (simp add: dist-0-norm)
  moreover have summable (λn. ?f n * z^n) using assms r
    by (intro summable-in-conv-radius) simp-all
  ultimately show ?thesis by (simp add: sums-iff)
qed

lemma Ln-approx-linear:
  fixes z :: complex
  assumes norm z < 1
  shows   norm (ln (1 + z) - z) ≤ norm z^2 / (1 - norm z)
proof -
  let ?f = λn. (-1)^Suc n / of-nat n
  from assms have (λn. ?f n * z^n) sums ln (1 + z) using Ln-series by simp
 moreover have (λn. (if n = 1 then 1 else 0) * z^n) sums z using powser-sums-if[of
1] by simp
  ultimately have (λn. (?f n - (if n = 1 then 1 else 0)) * z^n) sums (ln (1 +
z) - z)
```

by (*subst left-diff-distrib*, *intro sums-diff*) *simp-all*
from *sums-split-initial-segment*[*OF this*, *of Suc 1*]
have (λ*i*. (−(*z*^2)) ∗ *inverse* (2 + *of-nat i*) ∗ (− *z*) ^*i*) *sums* (*Ln* (1 + *z*) − *z*)
by (*simp add*: *power2-eq-square mult-ac power-minus*[*of z*] *divide-inverse*)
hence (*Ln* (1 + *z*) − *z*) = (∑ *i*. (−(*z*^2)) ∗ *inverse* (*of-nat* (*i*+2)) ∗ (−*z*) ^*i*)
by (*simp add*: *sums-iff*)
also have *A*: *summable* (λ*n*. *norm z*^2 ∗ (*inverse* (*real-of-nat* (*Suc* (*Suc n*))) ∗ *cmod z* ^ *n*))
by (*rule summable-mult*, *rule summable-comparison-test-ev*[*OF - summable-geometric*[*of norm z*]])
(*auto simp*: *assms field-simps intro*!: *always-eventually*)
hence *norm* (∑ *i*. (−(*z*^2)) ∗ *inverse* (*of-nat* (*i*+2)) ∗ (−*z*) ^*i*) ≤
(∑ *i*. *norm* (−(*z*^2) ∗ *inverse* (*of-nat* (*i*+2)) ∗ (−*z*) ^*i*))
by (*intro summable-norm*)
(*auto simp*: *norm-power norm-inverse norm-mult mult-ac simp del*: *of-nat-add of-nat-Suc*)
also have *norm* ((−*z*) ^2 ∗ (−*z*) ^*i*) ∗ *inverse* (*of-nat* (*i*+2)) ≤ *norm* ((−*z*) ^2 ∗ (−*z*) ^*i*) ∗ *1* **for** *i*
by (*intro mult-left-mono*) (*simp-all add*: *divide-simps*)
hence (∑ *i*. *norm* (−(*z*^2) ∗ *inverse* (*of-nat* (*i*+2)) ∗ (−*z*) ^*i*)) ≤
(∑ *i*. *norm* (−(*z*^2) ∗ (−*z*) ^*i*)) **using** *A assms*
apply (*simp-all only*: *norm-power norm-inverse norm-divide norm-mult*)
apply (*intro suminf-le summable-mult summable-geometric*)
apply (*auto simp*: *norm-power field-simps simp del*: *of-nat-add of-nat-Suc*)
done
also have ... = *norm z*^2 ∗ (∑ *i*. *norm z*^*i*) **using** *assms*
by (*subst suminf-mult* [*symmetric*]) (*auto intro*!: *summable-geometric simp*: *norm-mult norm-power*)
also have (∑ *i*. *norm z*^*i*) = *inverse* (*1* − *norm z*) **using** *assms*
by (*subst suminf-geometric*) (*simp-all add*: *divide-inverse*)
also have *norm z*^2 ∗ ... = *norm z*^2 / (*1* − *norm z*) **by** (*simp add*: *divide-inverse*)
finally show *?thesis* **.**
qed

Relation between Arg and arctangent in upper halfplane

lemma *Arg-arctan-upperhalf*:
assumes *0* < *Im z*
shows *Arg z* = *pi*/*2* − *arctan*(*Re z* / *Im z*)
proof (*cases z* = *0*)
case *True* **with** *assms* **show** *?thesis*
by *simp*
next
case *False*
show *?thesis*
apply (*rule Arg-unique* [*of norm z*])
using *False assms arctan* [*of Re z* / *Im z*] *pi-ge-two pi-half-less-two*
apply (*auto simp*: *exp-Euler cos-diff sin-diff*)
using *norm-complex-def* [*of z*, *symmetric*]
apply (*simp add*: *sin-of-real cos-of-real sin-arctan cos-arctan field-simps real-sqrt-divide*)

apply (*metis complex-eq mult.assoc ring-class.ring-distribs*(*2*))
done
qed

lemma *Arg-eq-Im-Ln*:
assumes $0 \leq Im\ z\ 0 < Re\ z$
shows $Arg\ z = Im\ (Ln\ z)$
proof (*cases* $z = 0 \vee Im\ z = 0$)
case *True* **then show** *?thesis*
using *assms Arg-eq-0 complex-is-Real-iff*
apply *auto*
by (*metis Arg-eq-0-pi Arg-eq-pi Im-Ln-eq-0 Im-Ln-eq-pi less-numeral-extra*(*3*) *zero-complex.simps*(*1*))
next
case *False*
then have $Arg\ z > 0$
using *Arg-gt-0 complex-is-Real-iff* **by** *blast*
then show *?thesis*
using *assms False*
by (*subst Arg-Ln*) (*auto simp*: *Ln-minus*)
qed

lemma *continuous-within-upperhalf-Arg*:
assumes $z \neq 0$
shows *continuous* (*at z within* $\{z.\ 0 \leq Im\ z\}$) *Arg*
proof (*cases* $z \in \mathbb{R}_{\geq 0}$)
case *False* **then show** *?thesis*
using *continuous-at-Arg continuous-at-imp-continuous-within* **by** *auto*
next
case *True*
then have *z*: $z \in \mathbb{R}\ 0 < Re\ z$
using *assms* **by** (*auto simp*: *complex-nonneg-Reals-iff complex-is-Real-iff complex-neq-0*)
then have [*simp*]: $Arg\ z = 0\ Im\ (Ln\ z) = 0$
by (*auto simp*: *Arg-eq-0 Im-Ln-eq-0 assms complex-is-Real-iff*)
show *?thesis*
proof (*clarsimp simp add*: *continuous-within Lim-within dist-norm*)
fix *e*::*real*
assume $0 < e$
moreover have *continuous* (*at z*) ($\lambda x.\ Im\ (Ln\ x)$)
using *z* **by** (*simp add*: *continuous-at-Ln complex-nonpos-Reals-iff*)
ultimately
obtain *d* **where** *d*: $d{>}0\ \bigwedge x.\ x \neq z \Longrightarrow cmod\ (x - z) < d \Longrightarrow |Im\ (Ln\ x)| < e$
by (*auto simp*: *continuous-within Lim-within dist-norm*)
{ fix *x*
assume $cmod\ (x - z) < Re\ z\ /\ 2$
then have $|Re\ x - Re\ z| < Re\ z\ /\ 2$
by (*metis le-less-trans abs-Re-le-cmod minus-complex.simps*(*1*))
then have $0 < Re\ x$

```
      using z by linarith
    }
    then show ∃ d>0. ∀ x. 0 ≤ Im x ⟶ x ≠ z ∧ cmod (x − z) < d ⟶ |Arg x|
< e
      apply (rule-tac x=min d (Re z / 2) in exI)
      using z d
      apply (auto simp: Arg-eq-Im-Ln)
      done
  qed
qed

lemma continuous-on-upperhalf-Arg: continuous-on ({z. 0 ≤ Im z} − {0}) Arg
  apply (auto simp: continuous-on-eq-continuous-within)
  by (metis Diff-subset continuous-within-subset continuous-within-upperhalf-Arg)

lemma open-Arg-less-Int:
  assumes 0 ≤ s t ≤ 2∗pi
    shows open ({y. s < Arg y} ∩ {y. Arg y < t})
proof −
  have 1: continuous-on (UNIV − ℝ≥0) Arg
    using continuous-at-Arg continuous-at-imp-continuous-within
    by (auto simp: continuous-on-eq-continuous-within)
  have 2: open (UNIV − ℝ≥0 :: complex set)  by (simp add: open-Diff)
  have open ({z. s < z} ∩ {z. z < t})
    using open-lessThan [of t] open-greaterThan [of s]
    by (metis greaterThan-def lessThan-def open-Int)
  moreover have {y. s < Arg y} ∩ {y. Arg y < t} ⊆ − ℝ≥0
   using assms by (auto simp: Arg-real complex-nonneg-Reals-iff complex-is-Real-iff)
  ultimately show ?thesis
    using continuous-imp-open-vimage [OF 1 2, of {z. Re z > s} ∩ {z. Re z <
t}]
    by auto
qed

lemma open-Arg-gt: open {z. t < Arg z}
proof (cases t < 0)
  case True then have {z. t < Arg z} = UNIV
    using Arg-ge-0 less-le-trans by auto
  then show ?thesis
    by simp
next
  case False then show ?thesis
    using open-Arg-less-Int [of t 2∗pi] Arg-lt-2pi
    by auto
qed

lemma closed-Arg-le: closed {z. Arg z ≤ t}
  using open-Arg-gt [of t]
  by (simp add: closed-def Set.Collect-neg-eq [symmetric] not-le)
```

52.16 Complex Powers

lemma *powr-to-1* [*simp*]: *z powr 1 = (z::complex)*
by (*simp add*: *powr-def*)

lemma *powr-nat*:
fixes *n::nat* **and** *z::complex* **shows** *z powr n = (if z = 0 then 0 else z^n)*
by (*simp add*: *exp-of-nat-mult powr-def*)

lemma *powr-add-complex*:
fixes *w::complex* **shows** *w powr (z1 + z2) = w powr z1 * w powr z2*
by (*simp add*: *powr-def algebra-simps exp-add*)

lemma *powr-minus-complex*:
fixes *w::complex* **shows** *w powr (−z) = inverse(w powr z)*
by (*simp add*: *powr-def exp-minus*)

lemma *powr-diff-complex*:
fixes *w::complex* **shows** *w powr (z1 − z2) = w powr z1 / w powr z2*
by (*simp add*: *powr-def algebra-simps exp-diff*)

lemma *norm-powr-real*: $w \in \mathbb{R} \Longrightarrow 0 < Re\ w \Longrightarrow norm(w\ powr\ z) = exp(Re\ z * ln(Re\ w))$
apply (*simp add*: *powr-def*)
using *Im-Ln-eq-0 complex-is-Real-iff norm-complex-def*
by *auto*

lemma *cnj-powr*:
assumes *Im a = 0 ⟹ Re a ≥ 0*
shows *cnj (a powr b) = cnj a powr cnj b*
proof (*cases a = 0*)
case *False*
with *assms* **have** $a \notin \mathbb{R}_{\leq 0}$ **by** (*auto simp*: *complex-eq-iff complex-nonpos-Reals-iff*)
with *False* **show** *?thesis* **by** (*simp add*: *powr-def exp-cnj cnj-Ln*)
qed *simp*

lemma *powr-real-real*:
$[\![w \in \mathbb{R};\ z \in \mathbb{R};\ 0 < Re\ w]\!] \Longrightarrow w\ powr\ z = exp(Re\ z * ln(Re\ w))$
apply (*simp add*: *powr-def*)
by (*metis complex-eq complex-is-Real-iff diff-0 diff-0-right diff-minus-eq-add exp-ln exp-not-eq-zero exp-of-real Ln-of-real mult-zero-right of-real-0 of-real-mult*)

lemma *powr-of-real*:
fixes *x::real* **and** *y::real*
shows *0 < x ⟹ of-real x powr (of-real y::complex) = of-real (x powr y)*
by (*simp add*: *powr-def*) (*metis exp-of-real of-real-mult Ln-of-real*)

lemma *norm-powr-real-mono*:
$[\![w \in \mathbb{R};\ 1 < Re\ w]\!]$

$\Longrightarrow$ *cmod*(*w powr z1*) $\leq$ *cmod*(*w powr z2*) $\longleftrightarrow$ *Re z1* $\leq$ *Re z2*
by (*auto simp*: *powr-def algebra-simps Reals-def Ln-of-real*)

lemma *powr-times-real*:
$[\![x \in \mathbb{R}; y \in \mathbb{R}; 0 \leq Re\ x; 0 \leq Re\ y]\!]$
$\Longrightarrow (x * y)\ powr\ z = x\ powr\ z * y\ powr\ z$
by (*auto simp*: *Reals-def powr-def Ln-times exp-add algebra-simps less-eq-real-def Ln-of-real*)

lemma *powr-neg-real-complex*:
shows (− *of-real x*) *powr a* = (−*1*) *powr* (*of-real* (*sgn x*) ∗ *a*) ∗ *of-real x powr* (*a* :: *complex*)
proof (*cases x* = *0*)
assume *x*: *x* ≠ *0*
hence (−*x*) *powr a* = *exp* (*a* ∗ *ln* (−*of-real x*)) **by** (*simp add*: *powr-def*)
also from *x* **have** *ln* (−*of-real x*) = *Ln* (*of-real x*) + *of-real* (*sgn x*) ∗ *pi* ∗ i
by (*simp add*: *Ln-minus Ln-of-real*)
also from *x assms* **have** *exp* (*a* ∗ ...) = *cis pi powr* (*of-real* (*sgn x*) ∗ *a*) ∗ *of-real x powr a*
by (*simp add*: *powr-def exp-add algebra-simps Ln-of-real cis-conv-exp*)
also note *cis-pi*
finally show *?thesis* **by** *simp*
qed *simp-all*

lemma *has-field-derivative-powr*:
fixes *z* :: *complex*
shows *z* ∉ $\mathbb{R}_{\leq 0}$ $\Longrightarrow$ ((λ*z*. *z powr s*) *has-field-derivative* (*s* ∗ *z powr* (*s* − *1*))) (*at z*)
apply (*cases z=0*, *auto*)
apply (*simp add*: *powr-def*)
apply (*rule DERIV-transform-at* [**where** *d* = *norm z* **and** *f* = λ*z*. *exp* (*s* ∗ *Ln z*)])
apply (*auto simp*: *dist-complex-def*)
apply (*intro derivative-eq-intros* | *simp add*: *assms*)+
apply (*simp add*: *field-simps exp-diff*)
done

declare *has-field-derivative-powr*[*THEN DERIV-chain2*, *derivative-intros*]

lemma *has-field-derivative-powr-right*:
w ≠ *0* $\Longrightarrow$ ((λ*z*. *w powr z*) *has-field-derivative Ln w* ∗ *w powr z*) (*at z*)
apply (*simp add*: *powr-def*)
apply (*intro derivative-eq-intros* | *simp add*: *assms*)+
done

lemma *field-differentiable-powr-right*:
fixes *w*::*complex*
shows

$w \neq 0 \Longrightarrow (\lambda z.\ w\ powr\ z)$ *field-differentiable* $(at\ z)$
using *field-differentiable-def has-field-derivative-powr-right* **by** *blast*

lemma *holomorphic-on-powr-right*:
f *holomorphic-on* $s \Longrightarrow w \neq 0 \Longrightarrow (\lambda z.\ w\ powr\ (f\ z))$ *holomorphic-on* s
unfolding *holomorphic-on-def field-differentiable-def*
by (*metis* (*full-types*) *DERIV-chain'* *has-field-derivative-powr-right*)

lemma *norm-powr-real-powr*:
$w \in \mathbb{R} \Longrightarrow 0 < Re\ w \Longrightarrow norm(w\ powr\ z) = Re\ w\ powr\ Re\ z$
by (*auto simp add*: *norm-powr-real powr-def Im-Ln-eq-0 complex-is-Real-iff in-Reals-norm*)

52.17 Some Limits involving Logarithms

lemma *lim-Ln-over-power*:
fixes s::*complex*
assumes $0 < Re\ s$
shows $((\lambda n.\ Ln\ n\ /\ (n\ powr\ s)) \longrightarrow 0)$ *sequentially*
proof (*simp add*: *lim-sequentially dist-norm*, *clarify*)
fix e::*real*
assume e: $0 < e$
have $\exists xo{>}0.\ \forall x{\geq}xo.\ 0 < e * 2 + (e * Re\ s * 2 - 2) * x + e * (Re\ s)^2 * x^2$
proof (*rule-tac* $x{=}2/(e * (Re\ s)^2)$ **in** *exI*, *safe*)
show $0 < 2\ /\ (e * (Re\ s)^2)$
using *e assms* **by** (*simp add*: *field-simps*)
next
fix x::*real*
assume x: $2\ /\ (e * (Re\ s)^2) \leq x$
then have $x{>}0$
using *e assms*
by (*metis less-le-trans mult-eq-0-iff mult-pos-pos pos-less-divide-eq power2-eq-square*
zero-less-numeral)
then show $0 < e * 2 + (e * Re\ s * 2 - 2) * x + e * (Re\ s)^2 * x^2$
using *e assms x*
apply (*auto simp*: *field-simps*)
apply (*rule-tac* $y = e * (x^2 * (Re\ s)^2)$ **in** *le-less-trans*)
apply (*auto simp*: *power2-eq-square field-simps add-pos-pos*)
done
qed
then have $\exists xo{>}0.\ \forall x{\geq}xo.\ x\ /\ e < 1 + (Re\ s * x) + (1/2) * (Re\ s * x)\hat{}2$
using *e* **by** (*simp add*: *field-simps*)
then have $\exists xo{>}0.\ \forall x{\geq}xo.\ x\ /\ e < exp\ (Re\ s * x)$
using *assms*
by (*force intro*: *less-le-trans* [*OF - exp-lower-taylor-quadratic*])
then have $\exists xo{>}0.\ \forall x{\geq}xo.\ x < e * exp\ (Re\ s * x)$
using *e* **by** (*auto simp*: *field-simps*)
with *e* **show** $\exists no.\ \forall n{\geq}no.\ norm\ (Ln\ (of\text{-}nat\ n)\ /\ of\text{-}nat\ n\ powr\ s) < e$
apply (*auto simp*: *norm-divide norm-powr-real divide-simps*)
apply (*rule-tac* $x{=}nat\ \lceil exp\ xo \rceil$ **in** *exI*)

apply *clarify*
apply (*drule-tac x=ln n* **in** *spec*)
apply (*auto simp*: *exp-less-mono nat-ceiling-le-eq not-le*)
apply (*metis exp-less-mono exp-ln not-le of-nat-0-less-iff*)
done
qed

lemma *lim-Ln-over-n*: $((\lambda n.\ Ln(of\text{-}nat\ n)\ /\ of\text{-}nat\ n) \longrightarrow 0)\ sequentially$
using *lim-Ln-over-power* [*of 1*]
by *simp*

lemma *Ln-Reals-eq*: $x \in \mathbb{R} \Longrightarrow Re\ x > 0 \Longrightarrow Ln\ x = of\text{-}real\ (ln\ (Re\ x))$
using *Ln-of-real* **by** *force*

lemma *powr-Reals-eq*: $x \in \mathbb{R} \Longrightarrow Re\ x > 0 \Longrightarrow x\ powr\ complex\text{-}of\text{-}real\ y = of\text{-}real\ (x\ powr\ y)$
by (*simp add*: *powr-of-real*)

lemma *lim-ln-over-power*:
fixes *s* :: *real*
assumes $0 < s$
shows $((\lambda n.\ ln\ n\ /\ (n\ powr\ s)) \longrightarrow 0)\ sequentially$
using *lim-Ln-over-power* [*of of-real s*, *THEN filterlim-sequentially-Suc* [*THEN iffD2*]] *assms*
apply (*subst filterlim-sequentially-Suc* [*symmetric*])
apply (*simp add*: *lim-sequentially dist-norm*
Ln-Reals-eq norm-powr-real-powr norm-divide)
done

lemma *lim-ln-over-n*: $((\lambda n.\ ln(real\text{-}of\text{-}nat\ n)\ /\ of\text{-}nat\ n) \longrightarrow 0)\ sequentially$
using *lim-ln-over-power* [*of 1*, *THEN filterlim-sequentially-Suc* [*THEN iffD2*]]
apply (*subst filterlim-sequentially-Suc* [*symmetric*])
apply (*simp add*: *lim-sequentially dist-norm*)
done

lemma *lim-1-over-complex-power*:
assumes $0 < Re\ s$
shows $((\lambda n.\ 1\ /\ (of\text{-}nat\ n\ powr\ s)) \longrightarrow 0)\ sequentially$
proof −
have $\forall n{>}0.\ 3 \le n \longrightarrow 1 \le ln\ (real\text{-}of\text{-}nat\ n)$
using *ln3-gt-1*
by (*force intro*: *order-trans* [*of - ln 3*] *ln3-gt-1*)
moreover have $(\lambda n.\ cmod\ (Ln\ (of\text{-}nat\ n)\ /\ of\text{-}nat\ n\ powr\ s)) \longrightarrow 0$
using *lim-Ln-over-power* [*OF assms*]
by (*metis tendsto-norm-zero-iff*)
ultimately show *?thesis*
apply (*auto intro*!: *Lim-null-comparison* [**where** $g = \lambda n.\ norm\ (Ln(of\text{-}nat\ n)\ /\ of\text{-}nat\ n\ powr\ s)$])
apply (*auto simp*: *norm-divide divide-simps eventually-sequentially*)

```
    done
qed

lemma lim-1-over-real-power:
  fixes s :: real
  assumes 0 < s
    shows ((λn. 1 / (of-nat n powr s)) ⟶ 0) sequentially
   using lim-1-over-complex-power [of of-real s, THEN filterlim-sequentially-Suc
[THEN iffD2]] assms
  apply (subst filterlim-sequentially-Suc [symmetric])
  apply (simp add: lim-sequentially dist-norm)
  apply (simp add: Ln-Reals-eq norm-powr-real-powr norm-divide)
  done

lemma lim-1-over-Ln: ((λn. 1 / Ln(of-nat n)) ⟶ 0) sequentially
proof (clarsimp simp add: lim-sequentially dist-norm norm-divide divide-simps)
  fix r::real
  assume 0 < r
  have ir: inverse (exp (inverse r)) > 0
    by simp
  obtain n where n: 1 < of-nat n * inverse (exp (inverse r))
    using ex-less-of-nat-mult [of - 1, OF ir]
    by auto
  then have exp (inverse r) < of-nat n
    by (simp add: divide-simps)
  then have ln (exp (inverse r)) < ln (of-nat n)
    by (metis exp-gt-zero less-trans ln-exp ln-less-cancel-iff)
  with ‹0 < r› have 1 < r * ln (real-of-nat n)
    by (simp add: field-simps)
  moreover have n > 0 using n
    using neq0-conv by fastforce
  ultimately show ∃no. ∀n. Ln (of-nat n) ≠ 0 ⟶ no ≤ n ⟶ 1 < r * cmod
(Ln (of-nat n))
    using n ‹0 < r›
    apply (rule-tac x=n in exI)
    apply (auto simp: divide-simps)
    apply (erule less-le-trans, auto)
    done
qed

lemma lim-1-over-ln: ((λn. 1 / ln(real-of-nat n)) ⟶ 0) sequentially
  using lim-1-over-Ln [THEN filterlim-sequentially-Suc [THEN iffD2]] assms
  apply (subst filterlim-sequentially-Suc [symmetric])
  apply (simp add: lim-sequentially dist-norm)
  apply (simp add: Ln-Reals-eq norm-powr-real-powr norm-divide)
  done
```

52.18 Relation between Square Root and exp/ln, hence its derivative

lemma *csqrt-exp-Ln*:
assumes $z \neq 0$
shows $csqrt\ z = exp(Ln(z)\ /\ 2)$
proof −
have $(exp\ (Ln\ z\ /\ 2))^2 = (exp\ (Ln\ z))$
by (*metis exp-double nonzero-mult-divide-cancel-left times-divide-eq-right zero-neq-numeral*)
also have ... $= z$
using *assms exp-Ln* **by** *blast*
finally have $csqrt\ z = csqrt\ ((exp\ (Ln\ z\ /\ 2))^2)$
by *simp*
also have ... $= exp\ (Ln\ z\ /\ 2)$
apply (*subst csqrt-square*)
using *cos-gt-zero-pi* [*of* $(Im\ (Ln\ z)\ /\ 2)$] *Im-Ln-le-pi mpi-less-Im-Ln assms*
apply (*auto simp*: *Re-exp Im-exp zero-less-mult-iff zero-le-mult-iff*, *fastforce*+)
done
finally show *?thesis* **using** *assms csqrt-square*
by *simp*
qed

lemma *csqrt-inverse*:
assumes $z \notin \mathbb{R}_{\leq 0}$
shows $csqrt\ (inverse\ z) = inverse\ (csqrt\ z)$
proof (*cases z=0*, *simp*)
assume $z \neq 0$
then show *?thesis*
using *assms csqrt-exp-Ln Ln-inverse exp-minus*
by (*simp add*: *csqrt-exp-Ln Ln-inverse exp-minus*)
qed

lemma *cnj-csqrt*:
assumes $z \notin \mathbb{R}_{\leq 0}$
shows $cnj(csqrt\ z) = csqrt(cnj\ z)$
proof (*cases z=0*, *simp*)
assume $z \neq 0$
then show *?thesis*
by (*simp add*: *assms cnj-Ln csqrt-exp-Ln exp-cnj*)
qed

lemma *has-field-derivative-csqrt*:
assumes $z \notin \mathbb{R}_{\leq 0}$
shows (*csqrt has-field-derivative inverse*$(2 * csqrt\ z)$) (*at z*)
proof −
have *z*: $z \neq 0$
using *assms* **by** *auto*
then have ∗: $inverse\ z = inverse\ (2{*}z) * 2$
by (*simp add*: *divide-simps*)
have [*simp*]: $exp\ (Ln\ z\ /\ 2) * inverse\ z = inverse\ (csqrt\ z)$

by (*simp add*: *z field-simps csqrt-exp-Ln* [*symmetric*]) (*metis power2-csqrt power2-eq-square*)
have *Im z = 0 ⟹ 0 < Re z*
using *assms complex-nonpos-Reals-iff not-less* **by** *blast*
with *z* **have** ((*λz. exp* (*Ln z / 2*)) *has-field-derivative inverse* (*2 * csqrt z*)) (*at z*)
by (*force intro*: *derivative-eq-intros * simp add*: *assms*)
then show *?thesis*
apply (*rule DERIV-transform-at*[**where** *d = norm z*])
apply (*intro z derivative-eq-intros | simp add*: *assms*)+
using *z*
apply (*metis csqrt-exp-Ln dist-0-norm less-irrefl*)
done
qed

lemma *field-differentiable-at-csqrt*:
z ∉ ℝ≤0 ⟹ csqrt field-differentiable at z
using *field-differentiable-def has-field-derivative-csqrt* **by** *blast*

lemma *field-differentiable-within-csqrt*:
z ∉ ℝ≤0 ⟹ csqrt field-differentiable (*at z within s*)
using *field-differentiable-at-csqrt field-differentiable-within-subset* **by** *blast*

lemma *continuous-at-csqrt*:
z ∉ ℝ≤0 ⟹ continuous (*at z*) *csqrt*
by (*simp add*: *field-differentiable-within-csqrt field-differentiable-imp-continuous-at*)

corollary *isCont-csqrt'* [*simp*]:
⟦*isCont f z*; *f z ∉ ℝ≤0*⟧ *⟹ isCont* (*λx. csqrt* (*f x*)) *z*
by (*blast intro*: *isCont-o2* [*OF - continuous-at-csqrt*])

lemma *continuous-within-csqrt*:
z ∉ ℝ≤0 ⟹ continuous (*at z within s*) *csqrt*
by (*simp add*: *field-differentiable-imp-continuous-at field-differentiable-within-csqrt*)

lemma *continuous-on-csqrt* [*continuous-intros*]:
(*⋀z. z ∈ s ⟹ z ∉ ℝ≤0*) *⟹ continuous-on s csqrt*
by (*simp add*: *continuous-at-imp-continuous-on continuous-within-csqrt*)

lemma *holomorphic-on-csqrt*:
(*⋀z. z ∈ s ⟹ z ∉ ℝ≤0*) *⟹ csqrt holomorphic-on s*
by (*simp add*: *field-differentiable-within-csqrt holomorphic-on-def*)

lemma *continuous-within-closed-nontrivial*:
closed s ⟹ a ∉ s ==> continuous (*at a within s*) *f*
using *open-Compl*
by (*force simp add*: *continuous-def eventually-at-topological filterlim-iff open-Collect-neg*)

lemma *continuous-within-csqrt-posreal*:

continuous (*at z within* ($\mathbb{R} \cap \{w.\ 0 \leq Re(w)\}$)) *csqrt*
proof (*cases* $z \in \mathbb{R}_{\leq 0}$)
case *True*
then have *Im z = 0 Re z < 0 ∨ z = 0*
using *cnj.code complex-cnj-zero-iff* **by** (*auto simp*: *complex-nonpos-Reals-iff*)
fastforce
then show *?thesis*
apply (*auto simp*: *continuous-within-closed-nontrivial* [*OF closed-Real-halfspace-Re-ge*])
apply (*auto simp*: *continuous-within-eps-delta norm-conv-dist* [*symmetric*])
apply (*rule-tac x=e^2* **in** *exI*)
apply (*auto simp*: *Reals-def*)
by (*metis linear not-less real-sqrt-less-iff real-sqrt-pow2-iff real-sqrt-power*)
next
case *False*
then show *?thesis* **by** (*blast intro*: *continuous-within-csqrt*)
qed

52.19 Complex arctangent

The branch cut gives standard bounds in the real case.

definition *Arctan* :: *complex* ⇒ *complex* **where**
Arctan ≡ λ*z*. (i/*2*) ∗ *Ln*((*1* − i∗*z*) / (*1* + i∗*z*))

lemma *Arctan-def-moebius*: *Arctan z* = i/*2* ∗ *Ln* (*moebius* (−i) *1* i *1 z*)
by (*simp add*: *Arctan-def moebius-def add-ac*)

lemma *Ln-conv-Arctan*:
assumes *z* ≠ −*1*
shows *Ln z* = −*2*∗i ∗ *Arctan* (*moebius 1* (− *1*) (− i) (− i) *z*)
proof −
have *Arctan* (*moebius 1* (− *1*) (− i) (− i) *z*) =
i/*2* ∗ *Ln* (*moebius* (− i) *1* i *1* (*moebius 1* (− *1*) (− i) (− i) *z*))
by (*simp add*: *Arctan-def-moebius*)
also from *assms* **have** i ∗ *z* ≠ i ∗ (−*1*) **by** (*subst mult-left-cancel*) *simp*
hence i ∗ *z* − −i ≠ *0* **by** (*simp add*: *eq-neg-iff-add-eq-0*)
from *moebius-inverse'*[*OF - this*, *of 1 1*]
have *moebius* (− i) *1* i *1* (*moebius 1* (− *1*) (− i) (− i) *z*) = *z* **by** *simp*
finally show *?thesis* **by** (*simp add*: *field-simps*)
qed

lemma *Arctan-0* [*simp*]: *Arctan 0* = *0*
by (*simp add*: *Arctan-def*)

lemma *Im-complex-div-lemma*: *Im*((*1* − i∗*z*) / (*1* + i∗*z*)) = *0* ⟷ *Re z* = *0*
by (*auto simp*: *Im-complex-div-eq-0 algebra-simps*)

lemma *Re-complex-div-lemma*: *0* < *Re*((*1* − i∗*z*) / (*1* + i∗*z*)) ⟷ *norm z* < *1*
by (*simp add*: *Re-complex-div-gt-0 algebra-simps cmod-def power2-eq-square*)

lemma *tan-Arctan*:
assumes $z^2 \neq -1$
shows [*simp*]:*tan*(*Arctan z*) = *z*
proof −
have *1* + i∗*z* ≠ *0*
by (*metis assms complex-i-mult-minus i-squared minus-unique power2-eq-square power2-minus*)
moreover
have *1* − i∗*z* ≠ *0*
by (*metis assms complex-i-mult-minus i-squared power2-eq-square power2-minus right-minus-eq*)
ultimately
show *?thesis*
by (*simp add*: *Arctan-def tan-def sin-exp-eq cos-exp-eq exp-minus csqrt-exp-Ln* [*symmetric*]
divide-simps power2-eq-square [*symmetric*])
qed

lemma *Arctan-tan* [*simp*]:
assumes |*Re z*| < *pi*/*2*
shows *Arctan*(*tan z*) = *z*
proof −
have *ge-pi2*: ⋀*n*::*int*. |*of-int* (*2*∗*n* + *1*) ∗ *pi*/*2*| ≥ *pi*/*2*
by (*case-tac n rule*: *int-cases*) (*auto simp*: *abs-mult*)
have *exp* (i∗*z*)∗*exp* (i∗*z*) = −*1* ⟷ *exp* (*2*∗i∗*z*) = −*1*
by (*metis distrib-right exp-add mult-2*)
also have ... ⟷ *exp* (*2*∗i∗*z*) = *exp* (i∗*pi*)
using *cis-conv-exp cis-pi* **by** *auto*
also have ... ⟷ *exp* (*2*∗i∗*z* − i∗*pi*) = *1*
by (*metis* (*no-types*) *diff-add-cancel diff-minus-eq-add exp-add exp-minus-inverse mult.commute*)
also have ... ⟷ *Re*(i∗*2*∗*z* − i∗*pi*) = *0* ∧ (∃ *n*::*int*. *Im*(i∗*2*∗*z* − i∗*pi*) = *of-int* (*2* ∗ *n*) ∗ *pi*)
by (*simp add*: *exp-eq-1*)
also have ... ⟷ *Im z* = *0* ∧ (∃ *n*::*int*. *2* ∗ *Re z* = *of-int* (*2*∗*n* + *1*) ∗ *pi*)
by (*simp add*: *algebra-simps*)
also have ... ⟷ *False*
using *assms ge-pi2*
apply (*auto simp*: *algebra-simps*)
by (*metis abs-mult-pos not-less of-nat-less-0-iff of-nat-numeral*)
finally have ∗: *exp* (i∗*z*)∗*exp* (i∗*z*) + *1* ≠ *0*
by (*auto simp*: *add.commute minus-unique*)
show *?thesis*
using *assms* ∗
apply (*simp add*: *Arctan-def tan-def sin-exp-eq cos-exp-eq exp-minus divide-simps*
ii-times-eq-iff power2-eq-square [*symmetric*])
apply (*rule Ln-unique*)
apply (*auto simp*: *divide-simps exp-minus*)
apply (*simp add*: *algebra-simps exp-double* [*symmetric*])

```
    done
qed

lemma
  assumes Re z = 0 ⟹ |Im z| < 1
  shows Re-Arctan-bounds: |Re(Arctan z)| < pi/2
     and has-field-derivative-Arctan: (Arctan has-field-derivative inverse(1 + z²))
(at z)
proof −
  have nz0: 1 + i∗z ≠ 0
    using assms
   by (metis abs-one complex-i-mult-minus diff-0-right diff-minus-eq-add ii.simps(1)
ii.simps(2)
            less-irrefl minus-diff-eq mult.right-neutral right-minus-eq)
  have z ≠ −i using assms
    by auto
  then have zz: 1 + z ∗ z ≠ 0
   by (metis abs-one assms i-squared ii.simps less-irrefl minus-unique square-eq-iff)
  have nz1: 1 − i∗z ≠ 0
    using assms by (force simp add: ii-times-eq-iff)
  have nz2: inverse (1 + i∗z) ≠ 0
    using assms
   by (metis Im-complex-div-lemma Re-complex-div-lemma cmod-eq-Im divide-complex-def
          less-irrefl mult-zero-right zero-complex.simps(1) zero-complex.simps(2))
  have nzi: ((1 − i∗z) ∗ inverse (1 + i∗z)) ≠ 0
    using nz1 nz2 by auto
  have Im ((1 − i∗z) / (1 + i∗z)) = 0 ⟹ 0 < Re ((1 − i∗z) / (1 + i∗z))
    apply (simp add: divide-complex-def)
    apply (simp add: divide-simps split: if-split-asm)
    using assms
    apply (auto simp: algebra-simps abs-square-less-1 [unfolded power2-eq-square])
    done
  then have ∗: ((1 − i∗z) / (1 + i∗z)) ∉ ℝ≤0
    by (auto simp add: complex-nonpos-Reals-iff)
  show |Re(Arctan z)| < pi/2
    unfolding Arctan-def divide-complex-def
    using mpi-less-Im-Ln [OF nzi]
    apply (auto simp: abs-if intro!: Im-Ln-less-pi ∗ [unfolded divide-complex-def])
    done
  show (Arctan has-field-derivative inverse(1 + z²)) (at z)
    unfolding Arctan-def scaleR-conv-of-real
    apply (rule DERIV-cong)
    apply (intro derivative-eq-intros | simp add: nz0 ∗)+
    using nz0 nz1 zz
    apply (simp add: divide-simps power2-eq-square)
    apply (auto simp: algebra-simps)
    done
qed
```

lemma *field-differentiable-at-Arctan*: $(Re\ z = 0 \Longrightarrow |Im\ z| < 1) \Longrightarrow Arctan$ *field-differentiable at z*
using *has-field-derivative-Arctan*
by (*auto simp*: *field-differentiable-def*)

lemma *field-differentiable-within-Arctan*:
$(Re\ z = 0 \Longrightarrow |Im\ z| < 1) \Longrightarrow Arctan$ *field-differentiable* (*at z within s*)
using *field-differentiable-at-Arctan field-differentiable-at-within* **by** *blast*

declare *has-field-derivative-Arctan* [*derivative-intros*]
declare *has-field-derivative-Arctan* [*THEN DERIV-chain2*, *derivative-intros*]

lemma *continuous-at-Arctan*:
$(Re\ z = 0 \Longrightarrow |Im\ z| < 1) \Longrightarrow$ *continuous* (*at z*) *Arctan*
by (*simp add*: *field-differentiable-imp-continuous-at field-differentiable-within-Arctan*)

lemma *continuous-within-Arctan*:
$(Re\ z = 0 \Longrightarrow |Im\ z| < 1) \Longrightarrow$ *continuous* (*at z within s*) *Arctan*
using *continuous-at-Arctan continuous-at-imp-continuous-within* **by** *blast*

lemma *continuous-on-Arctan* [*continuous-intros*]:
$(\bigwedge z.\ z \in s \Longrightarrow Re\ z = 0 \Longrightarrow |Im\ z| < 1) \Longrightarrow$ *continuous-on s Arctan*
by (*auto simp*: *continuous-at-imp-continuous-on continuous-within-Arctan*)

lemma *holomorphic-on-Arctan*:
$(\bigwedge z.\ z \in s \Longrightarrow Re\ z = 0 \Longrightarrow |Im\ z| < 1) \Longrightarrow Arctan$ *holomorphic-on s*
by (*simp add*: *field-differentiable-within-Arctan holomorphic-on-def*)

lemma *Arctan-series*:
assumes *z*: *norm* $(z :: complex) < 1$
defines $g \equiv \lambda n.$ *if odd n then* $-\mathrm{i}*\mathrm{i}\ \hat{}\ n$ */ n else 0*
defines $h \equiv \lambda z\ n.\ (-1)\ \hat{}\ n$ */ of-nat* $(2*n+1) * (z::complex)\ \hat{}\ (2*n+1)$
shows $(\lambda n.\ g\ n * z\ \hat{}\ n)$ *sums Arctan z*
and *h z sums Arctan z*
proof −
def $G \equiv \lambda z.\ (\sum n.\ g\ n * z\ \hat{}\ n)$
have *summable*: *summable* $(\lambda n.\ g\ n * u\ \hat{}\ n)$ **if** *norm u < 1* **for** *u*
proof (*cases u = 0*)
assume *u*: $u \neq 0$
have $(\lambda n.$ *ereal* $(norm\ (h\ u\ n)$ */ norm* $(h\ u\ (Suc\ n)))) = (\lambda n.$ *ereal* $(inverse\ (norm\ u)\ \hat{}\ 2) *$
ereal $((2 + inverse\ (real\ (Suc\ n))) / (2 - inverse\ (real\ (Suc\ n)))))$
proof
fix *n*
have *ereal* $(norm\ (h\ u\ n)$ */ norm* $(h\ u\ (Suc\ n))) =$
ereal $(inverse\ (norm\ u)\ \hat{}\ 2) *$ *ereal* $((of\text{-}nat\ (2*Suc\ n+1)$ */ of-nat* $(Suc\ n))$ /
$(of\text{-}nat\ (2*Suc\ n-1)$ */ of-nat* $(Suc\ n)))$
by (*simp add*: *h-def norm-mult norm-power norm-divide divide-simps*

```
            power2-eq-square eval-nat-numeral del: of-nat-add of-nat-Suc)
    also have of-nat (2*Suc n+1) / of-nat (Suc n) = (2::real) + inverse (real
(Suc n))
      by (auto simp: divide-simps simp del: of-nat-Suc) simp-all?
    also have of-nat (2*Suc n-1) / of-nat (Suc n) = (2::real) - inverse (real
(Suc n))
      by (auto simp: divide-simps simp del: of-nat-Suc) simp-all?
    finally show ereal (norm (h u n) / norm (h u (Suc n))) = ereal (inverse
(norm u) ^2) *
          ereal ((2 + inverse (real (Suc n))) / (2 - inverse (real (Suc n)))) .
  qed
  also have ... ⟶ ereal (inverse (norm u) ^2) * ereal ((2 + 0) / (2 - 0))
    by (intro tendsto-intros LIMSEQ-inverse-real-of-nat) simp-all
  finally have liminf (λn. ereal (cmod (h u n) / cmod (h u (Suc n)))) = inverse
(norm u) ^2
    by (intro lim-imp-Liminf) simp-all
  moreover from power-strict-mono[OF that, of 2] u have inverse (norm u) ^2
< 1
    by (simp add: divide-simps)
  ultimately have A: liminf (λn. ereal (cmod (h u n) / cmod (h u (Suc n))))
< 1 by simp
  from u have summable (h u)
    by (intro summable-norm-cancel[OF ratio-test-convergence[OF - A]])
      (auto simp: h-def norm-divide norm-mult norm-power simp del: of-nat-Suc
          intro!: mult-pos-pos divide-pos-pos always-eventually)
  thus summable (λn. g n * u^n)
    by (subst summable-mono-reindex[of λn. 2*n+1, symmetric])
      (auto simp: power-mult subseq-def g-def h-def elim!: oddE)
qed (simp add: h-def)

have ∃c. ∀u∈ball 0 1. Arctan u - G u = c
proof (rule has-field-derivative-zero-constant)
  fix u :: complex assume u ∈ ball 0 1
  hence u: norm u < 1 by (simp add: dist-0-norm)
  def K ≡ (norm u + 1) / 2
  from u and abs-Im-le-cmod[of u] have Im-u: |Im u| < 1 by linarith
  from u have K: 0 ≤ K norm u < K K < 1 by (simp-all add: K-def)
  hence (G has-field-derivative (∑n. diffs g n * u ^ n)) (at u) unfolding G-def
    by (intro termdiffs-strong[of - of-real K] summable) simp-all
  also have (λn. diffs g n * u^n) = (λn. if even n then (i*u) ^n else 0)
   by (intro ext) (simp-all del: of-nat-Suc add: g-def diffs-def power-mult-distrib)
  also have suminf ... = (∑n. (-(u^2)) ^n)
    by (subst suminf-mono-reindex[of λn. 2*n, symmetric])
      (auto elim!: evenE simp: subseq-def power-mult power-mult-distrib)
  also from u have norm u^2 < 1^2 by (intro power-strict-mono) simp-all
  hence (∑n. (-(u^2)) ^n) = inverse (1 + u^2)
    by (subst suminf-geometric) (simp-all add: norm-power inverse-eq-divide)
  finally have (G has-field-derivative inverse (1 + u²)) (at u) .
  from DERIV-diff[OF has-field-derivative-Arctan this] Im-u u
```

show ((λu. *Arctan* u − *G* u) *has-field-derivative 0*) (*at* u *within ball 0 1*)
by (*simp-all add*: *dist-0-norm at-within-open*[*OF - open-ball*])
qed *simp-all*
then obtain c **where** c: ⋀u. *norm* u < *1* ⟹ *Arctan* u − *G* u = c **by** (*auto simp*: *dist-0-norm*)
from *this*[*of 0*] **have** c = *0* **by** (*simp add*: *G-def g-def powser-zero*)
with c z **have** *Arctan* z = *G* z **by** *simp*
with *summable*[*OF* z] **show** (λn. *g* n ∗ z^n) *sums Arctan* z **unfolding** *G-def* **by** (*simp add*: *sums-iff*)
thus *h* z *sums Arctan* z **by** (*subst* (*asm*) *sums-mono-reindex*[*of* λn. *2*∗n+*1*, *symmetric*])
(*auto elim*!: *oddE simp*: *subseq-def power-mult g-def h-def*)
qed

A quickly-converging series for the logarithm, based on the arctangent.

lemma *ln-series-quadratic*:
assumes x: x > (*0*::*real*)
shows (λn. (*2*∗((x − *1*) / (x + *1*)) ^ (*2*∗n+*1*) / *of-nat* (*2*∗n+*1*))) *sums ln* x
proof −
def y ≡ *of-real* ((x−*1*)/(x+*1*)) :: *complex*
from x **have** x′: *complex-of-real* x ≠ *of-real* (−*1*) **by** (*subst of-real-eq-iff*) *auto*
from x **have** |x − *1*| < |x + *1*| **by** *linarith*
hence *norm* (*complex-of-real* (x − *1*) / *complex-of-real* (x + *1*)) < *1*
by (*simp add*: *norm-divide del*: *of-real-add of-real-diff*)
hence *norm* (i ∗ y) < *1* **unfolding** *y-def* **by** (*subst norm-mult*) *simp*
hence (λn. (−*2*∗i) ∗ ((−*1*) ^n / *of-nat* (*2*∗n+*1*) ∗ (i∗y) ^(*2*∗n+*1*))) *sums* ((−*2*∗i) ∗ *Arctan* (i∗y))
by (*intro Arctan-series sums-mult*) *simp-all*
also have (λn. (−*2*∗i) ∗ ((−*1*) ^n / *of-nat* (*2*∗n+*1*) ∗ (i∗y) ^(*2*∗n+*1*))) =
(λn. (−*2*∗i) ∗ ((−*1*) ^n ∗ (i∗y∗(−y^2) ^n)/*of-nat* (*2*∗n+*1*)))
by (*intro ext*) (*simp-all add*: *power-mult power-mult-distrib*)
also have ... = (λn. *2*∗y∗ ((−*1*) ∗ (−y^2)) ^n/*of-nat* (*2*∗n+*1*))
by (*intro ext*, *subst power-mult-distrib*) (*simp add*: *algebra-simps power-mult*)
also have ... = (λn. *2*∗y^(*2*∗n+*1*) / *of-nat* (*2*∗n+*1*))
by (*subst power-add*, *subst power-mult*) (*simp add*: *mult-ac*)
also have ... = (λn. *of-real* (*2*∗((x−*1*)/(x+*1*)) ^(*2*∗n+*1*) / *of-nat* (*2*∗n+*1*)))
by (*intro ext*) (*simp add*: *y-def*)
also have i ∗ y = (*of-real* x − *1*) / (−i ∗ (*of-real* x + *1*))
by (*subst divide-divide-eq-left* [*symmetric*]) (*simp add*: *y-def*)
also have ... = *moebius 1* (−*1*) (−i) (−i) (*of-real* x) **by** (*simp add*: *moebius-def algebra-simps*)
also from x′ **have** −*2*∗i∗*Arctan* ... = *Ln* (*of-real* x) **by** (*intro Ln-conv-Arctan* [*symmetric*]) *simp-all*
also from x **have** ... = *ln* x **by** (*rule Ln-of-real*)
finally show *?thesis* **by** (*subst* (*asm*) *sums-of-real-iff*)
qed

52.20 Real arctangent

lemma *norm-exp-ii-times* [*simp*]: *norm* (*exp*(i ∗ *of-real y*)) = *1*
by *simp*

lemma *norm-exp-imaginary*: *norm*(*exp z*) = *1* ⟹ *Re z* = *0*
by (*simp add*: *complex-norm-eq-1-exp*)

lemma *Im-Arctan-of-real* [*simp*]: *Im* (*Arctan* (*of-real x*)) = *0*
unfolding *Arctan-def divide-complex-def*
apply (*simp add*: *complex-eq-iff*)
apply (*rule norm-exp-imaginary*)
apply (*subst exp-Ln*, *auto*)
apply (*simp-all add*: *cmod-def complex-eq-iff*)
apply (*auto simp*: *divide-simps*)
apply (*metis power-one sum-power2-eq-zero-iff zero-neq-one*, *algebra*)
done

lemma *arctan-eq-Re-Arctan*: *arctan x* = *Re* (*Arctan* (*of-real x*))
proof (*rule arctan-unique*)
show − (*pi* / *2*) < *Re* (*Arctan* (*complex-of-real x*))
apply (*simp add*: *Arctan-def*)
apply (*rule Im-Ln-less-pi*)
apply (*auto simp*: *Im-complex-div-lemma complex-nonpos-Reals-iff*)
done
next
have ∗: (*1* − i∗*x*) / (*1* + i∗*x*) ≠ *0*
by (*simp add*: *divide-simps*) (*simp add*: *complex-eq-iff*)
show *Re* (*Arctan* (*complex-of-real x*)) < *pi* / *2*
using *mpi-less-Im-Ln* [*OF* ∗]
by (*simp add*: *Arctan-def*)
next
have *tan* (*Re* (*Arctan* (*of-real x*))) = *Re* (*tan* (*Arctan* (*of-real x*)))
apply (*auto simp*: *tan-def Complex.Re-divide Re-sin Re-cos Im-sin Im-cos*)
apply (*simp add*: *field-simps*)
by (*simp add*: *power2-eq-square*)
also have ... = *x*
apply (*subst tan-Arctan*, *auto*)
by (*metis diff-0-right minus-diff-eq mult-zero-left not-le of-real-1 of-real-eq-iff of-real-minus of-real-power power2-eq-square real-minus-mult-self-le zero-less-one*)
finally show *tan* (*Re* (*Arctan* (*complex-of-real x*))) = *x* .
qed

lemma *Arctan-of-real*: *Arctan* (*of-real x*) = *of-real* (*arctan x*)
unfolding *arctan-eq-Re-Arctan divide-complex-def*
by (*simp add*: *complex-eq-iff*)

lemma *Arctan-in-Reals* [*simp*]: *z* ∈ ℝ ⟹ *Arctan z* ∈ ℝ
by (*metis Reals-cases Reals-of-real Arctan-of-real*)

declare *arctan-one* [*simp*]

lemma *arctan-less-pi4-pos*: $x < 1 \Longrightarrow \arctan x < pi/4$
by (*metis arctan-less-iff arctan-one*)

lemma *arctan-less-pi4-neg*: $-1 < x \Longrightarrow -(pi/4) < \arctan x$
by (*metis arctan-less-iff arctan-minus arctan-one*)

lemma *arctan-less-pi4*: $|x| < 1 \Longrightarrow |\arctan x| < pi/4$
by (*metis abs-less-iff arctan-less-pi4-pos arctan-minus*)

lemma *arctan-le-pi4*: $|x| \leq 1 \Longrightarrow |\arctan x| \leq pi/4$
by (*metis abs-le-iff arctan-le-iff arctan-minus arctan-one*)

lemma *abs-arctan*: $|\arctan x| = \arctan |x|$
by (*simp add*: *abs-if arctan-minus*)

lemma *arctan-add-raw*:
assumes $|\arctan x + \arctan y| < pi/2$
shows $\arctan x + \arctan y = \arctan((x + y) / (1 - x * y))$
proof (*rule arctan-unique* [*symmetric*])
show *12*: $-(pi / 2) < \arctan x + \arctan y \; \arctan x + \arctan y < pi / 2$
using *assms* **by** *linarith+*
show $\tan (\arctan x + \arctan y) = (x + y) / (1 - x * y)$
using *cos-gt-zero-pi* [*OF 12*]
by (*simp add*: *arctan tan-add*)
qed

lemma *arctan-inverse*:
assumes $0 < x$
shows $\arctan(inverse\ x) = pi/2 - \arctan x$
proof −
have $\arctan(inverse\ x) = \arctan(inverse(\tan(\arctan x)))$
by (*simp add*: *arctan*)
also have $... = \arctan (\tan (pi / 2 - \arctan x))$
by (*simp add*: *tan-cot*)
also have $... = pi/2 - \arctan x$
proof −
have $0 < pi - \arctan x$
using *arctan-ubound* [*of x*] *pi-gt-zero* **by** *linarith*
with *assms* **show** *?thesis*
by (*simp add*: *Transcendental.arctan-tan*)
qed
finally show *?thesis* **.**
qed

lemma *arctan-add-small*:
assumes $|x * y| < 1$
shows $(\arctan x + \arctan y = \arctan((x + y) / (1 - x * y)))$

proof (*cases* $x = 0 \vee y = 0$)
case *True* **then show** *?thesis*
by *auto*
next
case *False*
then have $*$: $|arctan\ x| < pi\ /\ 2 - |arctan\ y|$ **using** *assms*
apply (*auto simp add*: *abs-arctan arctan-inverse* [*symmetric*] *arctan-less-iff*)
apply (*simp add*: *divide-simps abs-mult*)
done
show *?thesis*
apply (*rule arctan-add-raw*)
using $*$ **by** *linarith*
qed

lemma *abs-arctan-le*:
fixes x::*real* **shows** $|arctan\ x| \leq |x|$
proof −
{ fix w::*complex* **and** z::*complex*
assume $*$: $w \in \mathbb{R}$ $z \in \mathbb{R}$
have *cmod* (*Arctan w* − *Arctan z*) ≤ *1* $*$ *cmod* (*w*−*z*)
apply (*rule field-differentiable-bound* [*OF convex-Reals*, *of Arctan - 1*])
apply (*rule has-field-derivative-at-within* [*OF has-field-derivative-Arctan*])
apply (*force simp add*: *Reals-def*)
apply (*simp add*: *norm-divide divide-simps in-Reals-norm complex-is-Real-iff power2-eq-square*)
using $*$ **by** *auto*
}
then have *cmod* (*Arctan* (*of-real x*) − *Arctan 0*) ≤ *1* $*$ *cmod* (*of-real x* −*0*)
using *Reals-0 Reals-of-real* **by** *blast*
then show *?thesis*
by (*simp add*: *Arctan-of-real*)
qed

lemma *arctan-le-self*: $0 \leq x \Longrightarrow arctan\ x \leq x$
by (*metis abs-arctan-le abs-of-nonneg zero-le-arctan-iff*)

lemma *abs-tan-ge*: $|x| < pi/2 \Longrightarrow |x| \leq |tan\ x|$
by (*metis abs-arctan-le abs-less-iff arctan-tan minus-less-iff*)

52.21 Inverse Sine

definition *Arcsin* :: *complex* ⇒ *complex* **where**
Arcsin ≡ $\lambda z.\ -\mathrm{i} * Ln(\mathrm{i} * z + csqrt(1 - z^2))$

lemma *Arcsin-body-lemma*: $\mathrm{i} * z + csqrt(1 - z^2) \neq 0$
using *power2-csqrt* [*of* $1 - z^2$]
apply *auto*
by (*metis complex-i-mult-minus diff-add-cancel diff-minus-eq-add diff-self mult.assoc mult.left-commute numeral-One power2-csqrt power2-eq-square zero-neq-numeral*)

lemma *Arcsin-range-lemma*: $|Re\ z| < 1 \Longrightarrow 0 < Re(\mathrm{i} * z + csqrt(1 - z^2))$
using *Complex.cmod-power2* [*of z*, *symmetric*]
by (*simp add*: *real-less-rsqrt algebra-simps Re-power2 cmod-square-less-1-plus*)

lemma *Re-Arcsin*: $Re(Arcsin\ z) = Im\ (Ln\ (\mathrm{i} * z + csqrt(1 - z^2)))$
by (*simp add*: *Arcsin-def*)

lemma *Im-Arcsin*: $Im(Arcsin\ z) = -\ ln\ (cmod\ (\mathrm{i} * z + csqrt\ (1 - z^2)))$
by (*simp add*: *Arcsin-def Arcsin-body-lemma*)

lemma *one-minus-z2-notin-nonpos-Reals*:
assumes $(Im\ z = 0 \Longrightarrow |Re\ z| < 1)$
shows $1 - z^2 \notin \mathbb{R}_{\leq 0}$
using *assms*
apply (*auto simp*: *complex-nonpos-Reals-iff Re-power2 Im-power2*)
using *power2-less-0* [*of Im z*] **apply** *force*
using *abs-square-less-1 not-le* **by** *blast*

lemma *isCont-Arcsin-lemma*:
assumes *le0*: $Re\ (\mathrm{i} * z + csqrt\ (1 - z^2)) \leq 0$ **and** $(Im\ z = 0 \Longrightarrow |Re\ z| < 1)$
shows *False*
proof (*cases Im z = 0*)
case *True*
then show *?thesis*
using *assms* **by** (*fastforce simp*: *cmod-def abs-square-less-1* [*symmetric*])
next
case *False*
have *neq*: $(cmod\ z)^2 \neq 1 + cmod\ (1 - z^2)$
proof (*clarsimp simp add*: *cmod-def*)
assume $(Re\ z)^2 + (Im\ z)^2 = 1 + sqrt\ ((1 - Re\ (z^2))^2 + (Im\ (z^2))^2)$
then have $((Re\ z)^2 + (Im\ z)^2 - 1)^2 = ((1 - Re\ (z^2))^2 + (Im\ (z^2))^2)$
by *simp*
then show *False* **using** *False*
by (*simp add*: *power2-eq-square algebra-simps*)
qed
moreover have *2*: $(Im\ z)^2 = (1 + ((Im\ z)^2 + cmod\ (1 - z^2)) - (Re\ z)^2)\ /\ 2$
using *le0*
apply *simp*
apply (*drule sqrt-le-D*)
using *cmod-power2* [*of z*] *norm-triangle-ineq2* [*of z^2 1*]
apply (*simp add*: *norm-power Re-power2 norm-minus-commute* [*of 1*])
done
ultimately show *False*
by (*simp add*: *Re-power2 Im-power2 cmod-power2*)
qed

lemma *isCont-Arcsin*:
assumes $(Im\ z = 0 \Longrightarrow |Re\ z| < 1)$

shows *isCont Arcsin z*
proof −
have ∗: i ∗ *z* + *csqrt* (*1* − *z*^{2}) ∉ $\mathbb{R}_{\leq 0}$
by (*metis isCont-Arcsin-lemma assms complex-nonpos-Reals-iff*)
show *?thesis*
using *assms*
apply (*simp add*: *Arcsin-def*)
apply (*rule isCont-Ln′ isCont-csqrt′ continuous-intros*)+
apply (*simp add*: *one-minus-z2-notin-nonpos-Reals assms*)
apply (*rule* ∗)
done
qed

lemma *isCont-Arcsin′* [*simp*]:
shows *isCont f z* ⟹ (*Im* (*f z*) = *0* ⟹ |*Re* (*f z*)| < *1*) ⟹ *isCont* (λ*x*. *Arcsin* (*f x*)) *z*
by (*blast intro*: *isCont-o2* [*OF* - *isCont-Arcsin*])

lemma *sin-Arcsin* [*simp*]: *sin*(*Arcsin z*) = *z*
proof −
have i∗*z*∗*2* + *csqrt* (*1* − *z*^{2})∗*2* = *0* ⟷ (i∗*z*)∗*2* + *csqrt* (*1* − *z*^{2})∗*2* = *0*
by (*simp add*: *algebra-simps*) — Cancelling a factor of 2
moreover have ... ⟷ (i∗*z*) + *csqrt* (*1* − *z*^{2}) = *0*
by (*metis Arcsin-body-lemma distrib-right no-zero-divisors zero-neq-numeral*)
ultimately show *?thesis*
apply (*simp add*: *sin-exp-eq Arcsin-def Arcsin-body-lemma exp-minus divide-simps*)
apply (*simp add*: *algebra-simps*)
apply (*simp add*: *power2-eq-square* [*symmetric*] *algebra-simps*)
done
qed

lemma *Re-eq-pihalf-lemma*:
|*Re z*| = *pi*/*2* ⟹ *Im z* = *0* ⟹
Re ((*exp* (i∗*z*) + *inverse* (*exp* (i∗*z*))) / *2*) = *0* ∧ *0* ≤ *Im* ((*exp* (i∗*z*) + *inverse* (*exp* (i∗*z*))) / *2*)
apply (*simp add*: *cos-ii-times* [*symmetric*] *Re-cos Im-cos abs-if del*: *eq-divide-eq-numeral1*)
by (*metis cos-minus cos-pi-half*)

lemma *Re-less-pihalf-lemma*:
assumes |*Re z*| < *pi* / *2*
shows *0* < *Re* ((*exp* (i∗*z*) + *inverse* (*exp* (i∗*z*))) / *2*)
proof −
have *0* < *cos* (*Re z*) **using** *assms*
using *cos-gt-zero-pi* **by** *auto*
then show *?thesis*
by (*simp add*: *cos-ii-times* [*symmetric*] *Re-cos Im-cos add-pos-pos*)
qed

lemma *Arcsin-sin*:

assumes $|Re\ z| < pi/2 \vee (|Re\ z| = pi/2 \wedge Im\ z = 0)$
shows $Arcsin(sin\ z) = z$
proof −
have $Arcsin(sin\ z) = -\ (\mathrm{i} * Ln\ (csqrt\ (1 - (\mathrm{i} * (exp\ (\mathrm{i}*z) - inverse\ (exp\ (\mathrm{i}*z))))^2\ /\ 4) - (inverse\ (exp\ (\mathrm{i}*z)) - exp\ (\mathrm{i}*z))\ /\ 2))$
by (*simp add*: *sin-exp-eq Arcsin-def exp-minus power-divide*)
also have $... = -\ (\mathrm{i} * Ln\ (csqrt\ (((exp\ (\mathrm{i}*z) + inverse\ (exp\ (\mathrm{i}*z)))/2)^2) - (inverse\ (exp\ (\mathrm{i}*z)) - exp\ (\mathrm{i}*z))\ /\ 2))$
by (*simp add*: *field-simps power2-eq-square*)
also have $... = -\ (\mathrm{i} * Ln\ (((exp\ (\mathrm{i}*z) + inverse\ (exp\ (\mathrm{i}*z)))/2) - (inverse\ (exp\ (\mathrm{i}*z)) - exp\ (\mathrm{i}*z))\ /\ 2))$
apply (*subst csqrt-square*)
using *assms Re-eq-pihalf-lemma Re-less-pihalf-lemma*
apply *auto*
done
also have $... = \ -\ (\mathrm{i} * Ln\ (exp\ (\mathrm{i}*z)))$
by (*simp add*: *field-simps power2-eq-square*)
also have $... = z$
apply (*subst Complex-Transcendental.Ln-exp*)
using *assms*
apply (*auto simp*: *abs-if simp del*: *eq-divide-eq-numeral1 split*: *if-split-asm*)
done
finally show *?thesis* **.**
qed

lemma *Arcsin-unique*:
$[\![sin\ z = w;\ |Re\ z| < pi/2 \vee (|Re\ z| = pi/2 \wedge Im\ z = 0)]\!] \Longrightarrow Arcsin\ w = z$
by (*metis Arcsin-sin*)

lemma *Arcsin-0* [*simp*]: $Arcsin\ 0 = 0$
by (*metis Arcsin-sin norm-zero pi-half-gt-zero real-norm-def sin-zero zero-complex.simps*(*1*))

lemma *Arcsin-1* [*simp*]: $Arcsin\ 1 = pi/2$
by (*metis Arcsin-sin Im-complex-of-real Re-complex-of-real numeral-One of-real-numeral pi-half-ge-zero real-sqrt-abs real-sqrt-pow2 real-sqrt-power sin-of-real sin-pi-half*)

lemma *Arcsin-minus-1* [*simp*]: $Arcsin(-1) = -\ (pi/2)$
by (*metis Arcsin-1 Arcsin-sin Im-complex-of-real Re-complex-of-real abs-of-nonneg of-real-minus pi-half-ge-zero power2-minus real-sqrt-abs sin-Arcsin sin-minus*)

lemma *has-field-derivative-Arcsin*:
assumes $(Im\ z = 0 \Longrightarrow |Re\ z| < 1)$
shows $(Arcsin\ has\text{-}field\text{-}derivative\ inverse(cos(Arcsin\ z)))\ (at\ z)$
proof −
have $(sin\ (Arcsin\ z))^2 \neq 1$
using *assms*
apply *atomize*
apply (*auto simp*: *complex-eq-iff Re-power2 Im-power2 abs-square-eq-1*)
apply (*metis abs-minus-cancel abs-one abs-power2 numeral-One numeral-neq-neg-one*)

by (*metis abs-minus-cancel abs-one abs-power2 one-neq-neg-one*)
then have *cos* (*Arcsin z*) ≠ *0*
by (*metis diff-0-right power-zero-numeral sin-squared-eq*)
then show *?thesis*
apply (*rule has-complex-derivative-inverse-basic* [*OF DERIV-sin*])
apply (*auto intro*: *isCont-Arcsin open-ball* [*of z 1*] *assms*)
done
qed

declare *has-field-derivative-Arcsin* [*derivative-intros*]
declare *has-field-derivative-Arcsin* [*THEN DERIV-chain2*, *derivative-intros*]

lemma *field-differentiable-at-Arcsin*:
(*Im z* = *0* ⟹ |*Re z*| < *1*) ⟹ *Arcsin field-differentiable at z*
using *field-differentiable-def has-field-derivative-Arcsin* **by** *blast*

lemma *field-differentiable-within-Arcsin*:
(*Im z* = *0* ⟹ |*Re z*| < *1*) ⟹ *Arcsin field-differentiable* (*at z within s*)
using *field-differentiable-at-Arcsin field-differentiable-within-subset* **by** *blast*

lemma *continuous-within-Arcsin*:
(*Im z* = *0* ⟹ |*Re z*| < *1*) ⟹ *continuous* (*at z within s*) *Arcsin*
using *continuous-at-imp-continuous-within isCont-Arcsin* **by** *blast*

lemma *continuous-on-Arcsin* [*continuous-intros*]:
(⋀*z*. *z* ∈ *s* ⟹ *Im z* = *0* ⟹ |*Re z*| < *1*) ⟹ *continuous-on s Arcsin*
by (*simp add*: *continuous-at-imp-continuous-on*)

lemma *holomorphic-on-Arcsin*: (⋀*z*. *z* ∈ *s* ⟹ *Im z* = *0* ⟹ |*Re z*| < *1*) ⟹
Arcsin holomorphic-on s
by (*simp add*: *field-differentiable-within-Arcsin holomorphic-on-def*)

52.22 Inverse Cosine

definition *Arccos* :: *complex* ⇒ *complex* **where**
Arccos ≡ λ*z*. −i ∗ *Ln*(*z* + i ∗ *csqrt*(*1* − *z*²))

lemma *Arccos-range-lemma*: |*Re z*| < *1* ⟹ *0* < *Im*(*z* + i ∗ *csqrt*(*1* − *z*²))
using *Arcsin-range-lemma* [*of* −*z*]
by *simp*

lemma *Arccos-body-lemma*: *z* + i ∗ *csqrt*(*1* − *z*²) ≠ *0*
using *Arcsin-body-lemma* [*of z*]
by (*metis complex-i-mult-minus diff-add-cancel minus-diff-eq minus-unique mult.assoc mult.left-commute*
power2-csqrt power2-eq-square zero-neq-one)

lemma *Re-Arccos*: *Re*(*Arccos z*) = *Im* (*Ln* (*z* + i ∗ *csqrt*(*1* − *z*²)))
by (*simp add*: *Arccos-def*)

lemma *Im-Arccos*: $Im(Arccos\ z) = -\ ln\ (cmod\ (z + \mathrm{i} * csqrt\ (1 - z^2)))$
by (*simp add*: *Arccos-def Arccos-body-lemma*)

A very tricky argument to find!

lemma *isCont-Arccos-lemma*:
assumes *eq0*: $Im\ (z + \mathrm{i} * csqrt\ (1 - z^2)) = 0$ **and** $(Im\ z = 0 \Longrightarrow |Re\ z| < 1)$
shows *False*
proof (*cases* $Im\ z = 0$)
case *True*
then show *?thesis*
using *assms* **by** (*fastforce simp add*: *cmod-def abs-square-less-1* [*symmetric*])
next
case *False*
have *Imz*: $Im\ z = -\ sqrt\ ((1 + ((Im\ z)^2 + cmod\ (1 - z^2)) - (Re\ z)^2)\ /\ 2)$
using *eq0 abs-Re-le-cmod* [*of* $1-z^2$]
by (*simp add*: *Re-power2 algebra-simps*)
have $(cmod\ z)^2 - 1 \neq cmod\ (1 - z^2)$
proof (*clarsimp simp add*: *cmod-def*)
assume $(Re\ z)^2 + (Im\ z)^2 - 1 = sqrt\ ((1 - Re\ (z^2))^2 + (Im\ (z^2))^2)$
then have $((Re\ z)^2 + (Im\ z)^2 - 1)^2 = ((1 - Re\ (z^2))^2 + (Im\ (z^2))^2)$
by *simp*
then show *False* **using** *False*
by (*simp add*: *power2-eq-square algebra-simps*)
qed
moreover have $(Im\ z)^2 = ((1 + ((Im\ z)^2 + cmod\ (1 - z^2)) - (Re\ z)^2)\ /\ 2)$
apply (*subst Imz*)
using *abs-Re-le-cmod* [*of* $1-z^2$]
apply (*simp add*: *Re-power2*)
done
ultimately show *False*
by (*simp add*: *cmod-power2*)
qed

lemma *isCont-Arccos*:
assumes $(Im\ z = 0 \Longrightarrow |Re\ z| < 1)$
shows *isCont Arccos z*
proof −
have $z + \mathrm{i} * csqrt\ (1 - z^2) \notin \mathbb{R}_{\leq 0}$
by (*metis complex-nonpos-Reals-iff isCont-Arccos-lemma assms*)
with *assms* **show** *?thesis*
apply (*simp add*: *Arccos-def*)
apply (*rule isCont-Ln′ isCont-csqrt′ continuous-intros*)+
apply (*simp-all add*: *one-minus-z2-notin-nonpos-Reals assms*)
done
qed

lemma *isCont-Arccos′* [*simp*]:
shows $isCont\ f\ z \Longrightarrow (Im\ (f\ z) = 0 \Longrightarrow |Re\ (f\ z)| < 1) \Longrightarrow isCont\ (\lambda x.\ Arccos$

```
(f x)) z
  by (blast intro: isCont-o2 [OF - isCont-Arccos])

lemma cos-Arccos [simp]: cos(Arccos z) = z
proof −
  have z*2 + i * (2 * csqrt (1 − z^2)) = 0 ⟷ z*2 + i * csqrt (1 − z^2)*2 = 0
    by (simp add: algebra-simps) — Cancelling a factor of 2
  moreover have ... ⟷ z + i * csqrt (1 − z^2) = 0
    by (metis distrib-right mult-eq-0-iff zero-neq-numeral)
  ultimately show ?thesis
   apply (simp add: cos-exp-eq Arccos-def Arccos-body-lemma exp-minus field-simps)
    apply (simp add: power2-eq-square [symmetric])
    done
qed

lemma Arccos-cos:
    assumes 0 < Re z & Re z < pi ∨
            Re z = 0 & 0 ≤ Im z ∨
            Re z = pi & Im z ≤ 0
      shows Arccos(cos z) = z
proof −
  have *: ((i − (exp (i * z))^2 * i) / (2 * exp (i * z))) = sin z
    by (simp add: sin-exp-eq exp-minus field-simps power2-eq-square)
  have 1 − (exp (i * z) + inverse (exp (i * z)))^2 / 4 = ((i − (exp (i * z))^2 * i) /
(2 * exp (i * z)))^2
    by (simp add: field-simps power2-eq-square)
  then have Arccos(cos z) = − (i * Ln ((exp (i * z) + inverse (exp (i * z))) / 2
+
                        i * csqrt (((i − (exp (i * z))^2 * i) / (2 * exp (i * z)))^2)))
    by (simp add: cos-exp-eq Arccos-def exp-minus power-divide)
  also have ... = − (i * Ln ((exp (i * z) + inverse (exp (i * z))) / 2 +
                          i * ((i − (exp (i * z))^2 * i) / (2 * exp (i * z)))))
    apply (subst csqrt-square)
    using assms Re-sin-pos [of z] Im-sin-nonneg [of z] Im-sin-nonneg2 [of z]
    apply (auto simp: * Re-sin Im-sin)
    done
  also have ... =  − (i * Ln (exp (i*z)))
    by (simp add: field-simps power2-eq-square)
  also have ... = z
    using assms
    apply (subst Complex-Transcendental.Ln-exp, auto)
    done
  finally show ?thesis .
qed

lemma Arccos-unique:
    ⟦cos z = w;
      0 < Re z ∧ Re z < pi ∨
      Re z = 0 ∧ 0 ≤ Im z ∨
```

```
    Re z = pi ∧ Im z ≤ 0⟧ ⟹ Arccos w = z
  using Arccos-cos by blast

lemma Arccos-0 [simp]: Arccos 0 = pi/2
  by (rule Arccos-unique) (auto simp: of-real-numeral)

lemma Arccos-1 [simp]: Arccos 1 = 0
  by (rule Arccos-unique) auto

lemma Arccos-minus1: Arccos(−1) = pi
  by (rule Arccos-unique) auto

lemma has-field-derivative-Arccos:
  assumes (Im z = 0 ⟹ |Re z| < 1)
    shows (Arccos has-field-derivative − inverse(sin(Arccos z))) (at z)
proof −
  have (cos (Arccos z))^2 ≠ 1
    using assms
    apply atomize
    apply (auto simp: complex-eq-iff Re-power2 Im-power2 abs-square-eq-1)
   apply (metis abs-minus-cancel abs-one abs-power2 numeral-One numeral-neq-neg-one)
   apply (metis left-minus less-irrefl power-one sum-power2-gt-zero-iff zero-neq-neg-one)
    done
  then have − sin (Arccos z) ≠ 0
   by (metis cos-squared-eq diff-0-right mult-zero-left neg-0-equal-iff-equal power2-eq-square)
  then have (Arccos has-field-derivative inverse(− sin(Arccos z))) (at z)
    apply (rule has-complex-derivative-inverse-basic [OF DERIV-cos])
    apply (auto intro: isCont-Arccos open-ball [of z 1] assms)
    done
  then show ?thesis
    by simp
qed

declare has-field-derivative-Arcsin [derivative-intros]
declare has-field-derivative-Arcsin [THEN DERIV-chain2, derivative-intros]

lemma field-differentiable-at-Arccos:
    (Im z = 0 ⟹ |Re z| < 1) ⟹ Arccos field-differentiable at z
  using field-differentiable-def has-field-derivative-Arccos by blast

lemma field-differentiable-within-Arccos:
    (Im z = 0 ⟹ |Re z| < 1) ⟹ Arccos field-differentiable (at z within s)
  using field-differentiable-at-Arccos field-differentiable-within-subset by blast

lemma continuous-within-Arccos:
    (Im z = 0 ⟹ |Re z| < 1) ⟹ continuous (at z within s) Arccos
  using continuous-at-imp-continuous-within isCont-Arccos by blast

lemma continuous-on-Arccos [continuous-intros]:
```

$(\bigwedge z.\ z \in s \Longrightarrow Im\ z = 0 \Longrightarrow |Re\ z| < 1) \Longrightarrow$ *continuous-on s Arccos*
by (*simp add*: *continuous-at-imp-continuous-on*)

lemma *holomorphic-on-Arccos*: $(\bigwedge z.\ z \in s \Longrightarrow Im\ z = 0 \Longrightarrow |Re\ z| < 1) \Longrightarrow$ *Arccos holomorphic-on s*
by (*simp add*: *field-differentiable-within-Arccos holomorphic-on-def*)

52.23 Upper and Lower Bounds for Inverse Sine and Cosine

lemma *Arcsin-bounds*: $|Re\ z| < 1 \Longrightarrow |Re(Arcsin\ z)| < pi/2$
unfolding *Re-Arcsin*
by (*blast intro*: *Re-Ln-pos-lt-imp Arcsin-range-lemma*)

lemma *Arccos-bounds*: $|Re\ z| < 1 \Longrightarrow 0 < Re(Arccos\ z) \wedge Re(Arccos\ z) < pi$
unfolding *Re-Arccos*
by (*blast intro*!: *Im-Ln-pos-lt-imp Arccos-range-lemma*)

lemma *Re-Arccos-bounds*: $-pi < Re(Arccos\ z) \wedge Re(Arccos\ z) \leq pi$
unfolding *Re-Arccos*
by (*blast intro*!: *mpi-less-Im-Ln Im-Ln-le-pi Arccos-body-lemma*)

lemma *Re-Arccos-bound*: $|Re(Arccos\ z)| \leq pi$
by (*meson Re-Arccos-bounds abs-le-iff less-eq-real-def minus-less-iff*)

lemma *Re-Arcsin-bounds*: $-pi < Re(Arcsin\ z)$ & $Re(Arcsin\ z) \leq pi$
unfolding *Re-Arcsin*
by (*blast intro*!: *mpi-less-Im-Ln Im-Ln-le-pi Arcsin-body-lemma*)

lemma *Re-Arcsin-bound*: $|Re(Arcsin\ z)| \leq pi$
by (*meson Re-Arcsin-bounds abs-le-iff less-eq-real-def minus-less-iff*)

52.24 Interrelations between Arcsin and Arccos

lemma *cos-Arcsin-nonzero*:
assumes $z^2 \neq 1$ **shows** $cos(Arcsin\ z) \neq 0$
proof −
have *eq*: $(\mathrm{i} * z * (csqrt\ (1 - z^2)))^2 = z^2 * (z^2 - 1)$
by (*simp add*: *power-mult-distrib algebra-simps*)
have $\mathrm{i} * z * (csqrt\ (1 - z^2)) \neq z^2 - 1$
proof
assume $\mathrm{i} * z * (csqrt\ (1 - z^2)) = z^2 - 1$
then have $(\mathrm{i} * z * (csqrt\ (1 - z^2)))^2 = (z^2 - 1)^2$
by *simp*
then have $z^2 * (z^2 - 1) = (z^2 - 1) * (z^2 - 1)$
using *eq power2-eq-square* **by** *auto*
then show *False*
using *assms* **by** *simp*
qed
then have $1 + \mathrm{i} * z * (csqrt\ (1 - z * z)) \neq z^2$
by (*metis add-minus-cancel power2-eq-square uminus-add-conv-diff*)

```
  then have 2*(1 + i * z * (csqrt (1 - z * z))) ≠ 2*z^2
    by (metis mult-cancel-left zero-neq-numeral)
  then have (i * z + csqrt (1 - z^2))^2 ≠ -1
    using assms
    apply (auto simp: power2-sum)
    apply (simp add: power2-eq-square algebra-simps)
    done
  then show ?thesis
    apply (simp add: cos-exp-eq Arcsin-def exp-minus)
    apply (simp add: divide-simps Arcsin-body-lemma)
    apply (metis add.commute minus-unique power2-eq-square)
    done
qed

lemma sin-Arccos-nonzero:
  assumes z^2 ≠ 1 shows sin(Arccos z) ≠ 0
proof -
  have eq: (i * z * (csqrt (1 - z^2)))^2 = -(z^2) * (1 - z^2)
    by (simp add: power-mult-distrib algebra-simps)
  have i * z * (csqrt (1 - z^2)) ≠ 1 - z^2
  proof
    assume i * z * (csqrt (1 - z^2)) = 1 - z^2
    then have (i * z * (csqrt (1 - z^2)))^2 = (1 - z^2)^2
      by simp
    then have -(z^2) * (1 - z^2) = (1 - z^2)*(1 - z^2)
      using eq power2-eq-square by auto
    then have -(z^2) = (1 - z^2)
      using assms
      by (metis add.commute add.right-neutral diff-add-cancel mult-right-cancel)
    then show False
      using assms by simp
  qed
  then have z^2 + i * z * (csqrt (1 - z^2)) ≠ 1
    by (simp add: algebra-simps)
  then have 2*(z^2 + i * z * (csqrt (1 - z^2))) ≠ 2*1
    by (metis mult-cancel-left2 zero-neq-numeral)
  then have (z + i * csqrt (1 - z^2))^2 ≠ 1
    using assms
   apply (auto simp: Power.comm-semiring-1-class.power2-sum power-mult-distrib)
    apply (simp add: power2-eq-square algebra-simps)
    done
  then show ?thesis
    apply (simp add: sin-exp-eq Arccos-def exp-minus)
    apply (simp add: divide-simps Arccos-body-lemma)
    apply (simp add: power2-eq-square)
    done
qed

lemma cos-sin-csqrt:
```

assumes $0 < cos(Re\ z) \ \vee \ cos(Re\ z) = 0 \wedge Im\ z * sin(Re\ z) \leq 0$
shows $cos\ z = csqrt(1 - (sin\ z)^2)$
apply (*rule csqrt-unique* [*THEN sym*])
apply (*simp add*: *cos-squared-eq*)
using *assms*
apply (*auto simp*: *Re-cos Im-cos add-pos-pos mult-le-0-iff zero-le-mult-iff*)
done

lemma *sin-cos-csqrt*:
assumes $0 < sin(Re\ z) \ \vee \ sin(Re\ z) = 0 \wedge 0 \leq Im\ z * cos(Re\ z)$
shows $sin\ z = csqrt(1 - (cos\ z)^2)$
apply (*rule csqrt-unique* [*THEN sym*])
apply (*simp add*: *sin-squared-eq*)
using *assms*
apply (*auto simp*: *Re-sin Im-sin add-pos-pos mult-le-0-iff zero-le-mult-iff*)
done

lemma *Arcsin-Arccos-csqrt-pos*:
$(0 < Re\ z \mid Re\ z = 0 \ \& \ 0 \leq Im\ z) \Longrightarrow Arcsin\ z = Arccos(csqrt(1 - z^2))$
by (*simp add*: *Arcsin-def Arccos-def Complex.csqrt-square add.commute*)

lemma *Arccos-Arcsin-csqrt-pos*:
$(0 < Re\ z \mid Re\ z = 0 \ \& \ 0 \leq Im\ z) \Longrightarrow Arccos\ z = Arcsin(csqrt(1 - z^2))$
by (*simp add*: *Arcsin-def Arccos-def Complex.csqrt-square add.commute*)

lemma *sin-Arccos*:
$0 < Re\ z \mid Re\ z = 0 \ \& \ 0 \leq Im\ z \Longrightarrow sin(Arccos\ z) = csqrt(1 - z^2)$
by (*simp add*: *Arccos-Arcsin-csqrt-pos*)

lemma *cos-Arcsin*:
$0 < Re\ z \mid Re\ z = 0 \ \& \ 0 \leq Im\ z \Longrightarrow cos(Arcsin\ z) = csqrt(1 - z^2)$
by (*simp add*: *Arcsin-Arccos-csqrt-pos*)

52.25 Relationship with Arcsin on the Real Numbers

lemma *Im-Arcsin-of-real*:
assumes $|x| \leq 1$
shows $Im\ (Arcsin\ (of\text{-}real\ x)) = 0$
proof −
have $csqrt\ (1 - (of\text{-}real\ x)^2) = (if\ x\hat{}2 \leq 1\ then\ sqrt\ (1 - x\hat{}2)\ else\ \mathrm{i} * sqrt\ (x\hat{}2 - 1))$
by (*simp add*: *of-real-sqrt del*: *csqrt-of-real-nonneg*)
then have $cmod\ (\mathrm{i} * of\text{-}real\ x + csqrt\ (1 - (of\text{-}real\ x)^2))\ \hat{}\ 2 = 1$
using *assms abs-square-le-1*
by (*force simp add*: *Complex.cmod-power2*)
then have $cmod\ (\mathrm{i} * of\text{-}real\ x + csqrt\ (1 - (of\text{-}real\ x)^2)) = 1$
by (*simp add*: *norm-complex-def*)
then show *?thesis*
by (*simp add*: *Im-Arcsin exp-minus*)

qed

corollary *Arcsin-in-Reals* [*simp*]: $z \in \mathbb{R} \Longrightarrow |Re\ z| \leq 1 \Longrightarrow Arcsin\ z \in \mathbb{R}$
by (*metis Im-Arcsin-of-real Re-complex-of-real Reals-cases complex-is-Real-iff*)

lemma *arcsin-eq-Re-Arcsin*:
assumes $|x| \leq 1$
shows *arcsin x* = *Re* (*Arcsin* (*of-real x*))
unfolding *arcsin-def*
proof (*rule the-equality*, *safe*)
show $-(pi\ /\ 2) \leq Re\ (Arcsin\ (complex\text{-}of\text{-}real\ x))$
using *Re-Ln-pos-le* [*OF Arcsin-body-lemma*, *of of-real x*]
by (*auto simp*: *Complex.in-Reals-norm Re-Arcsin*)
next
show $Re\ (Arcsin\ (complex\text{-}of\text{-}real\ x)) \leq pi\ /\ 2$
using *Re-Ln-pos-le* [*OF Arcsin-body-lemma*, *of of-real x*]
by (*auto simp*: *Complex.in-Reals-norm Re-Arcsin*)
next
show $sin\ (Re\ (Arcsin\ (complex\text{-}of\text{-}real\ x))) = x$
using *Re-sin* [*of Arcsin* (*of-real x*)] *Arcsin-body-lemma* [*of of-real x*]
by (*simp add*: *Im-Arcsin-of-real assms*)
next
fix x'
assume $-(pi\ /\ 2) \leq x'\ x' \leq pi\ /\ 2\ x = sin\ x'$
then show $x' = Re\ (Arcsin\ (complex\text{-}of\text{-}real\ (sin\ x')))$
apply (*simp add*: *sin-of-real* [*symmetric*])
apply (*subst Arcsin-sin*)
apply (*auto simp*:)
done
qed

lemma *of-real-arcsin*: $|x| \leq 1 \Longrightarrow of\text{-}real(arcsin\ x) = Arcsin(of\text{-}real\ x)$
by (*metis Im-Arcsin-of-real add.right-neutral arcsin-eq-Re-Arcsin complex-eq mult-zero-right of-real-0*)

52.26 Relationship with Arccos on the Real Numbers

lemma *Im-Arccos-of-real*:
assumes $|x| \leq 1$
shows *Im* (*Arccos* (*of-real x*)) = *0*
proof −
have *csqrt* $(1 - (of\text{-}real\ x)^2) = (if\ x\hat{}2 \leq 1\ then\ sqrt\ (1 - x\hat{}2)\ else\ \mathrm{i} * sqrt\ (x\hat{}2 - 1))$
by (*simp add*: *of-real-sqrt del*: *csqrt-of-real-nonneg*)
then have *cmod* $(of\text{-}real\ x + \mathrm{i} * csqrt\ (1 - (of\text{-}real\ x)^2))\ \hat{}\ 2 = 1$
using *assms abs-square-le-1*
by (*force simp add*: *Complex.cmod-power2*)
then have *cmod* $(of\text{-}real\ x + \mathrm{i} * csqrt\ (1 - (of\text{-}real\ x)^2)) = 1$
by (*simp add*: *norm-complex-def*)

then show *?thesis*
by (*simp add*: *Im-Arccos exp-minus*)
qed

corollary *Arccos-in-Reals* [*simp*]: $z \in \mathbb{R} \Longrightarrow |Re\ z| \leq 1 \Longrightarrow Arccos\ z \in \mathbb{R}$
by (*metis Im-Arccos-of-real Re-complex-of-real Reals-cases complex-is-Real-iff*)

lemma *arccos-eq-Re-Arccos*:
assumes $|x| \leq 1$
shows *arccos x* = *Re* (*Arccos* (*of-real x*))
unfolding *arccos-def*
proof (*rule the-equality*, *safe*)
show $0 \leq Re\ (Arccos\ (complex\text{-}of\text{-}real\ x))$
using *Im-Ln-pos-le* [*OF Arccos-body-lemma*, *of of-real x*]
by (*auto simp*: *Complex.in-Reals-norm Re-Arccos*)
next
show $Re\ (Arccos\ (complex\text{-}of\text{-}real\ x)) \leq pi$
using *Im-Ln-pos-le* [*OF Arccos-body-lemma*, *of of-real x*]
by (*auto simp*: *Complex.in-Reals-norm Re-Arccos*)
next
show $cos\ (Re\ (Arccos\ (complex\text{-}of\text{-}real\ x))) = x$
using *Re-cos* [*of Arccos* (*of-real x*)] *Arccos-body-lemma* [*of of-real x*]
by (*simp add*: *Im-Arccos-of-real assms*)
next
fix x'
assume $0 \leq x'\ x' \leq pi\ x = cos\ x'$
then show $x' = Re\ (Arccos\ (complex\text{-}of\text{-}real\ (cos\ x')))$
apply (*simp add*: *cos-of-real* [*symmetric*])
apply (*subst Arccos-cos*)
apply (*auto simp*:)
done
qed

lemma *of-real-arccos*: $|x| \leq 1 \Longrightarrow of\text{-}real(arccos\ x) = Arccos(of\text{-}real\ x)$
by (*metis Im-Arccos-of-real add.right-neutral arccos-eq-Re-Arccos complex-eq mult-zero-right of-real-0*)

52.27 Some interrelationships among the real inverse trig functions.

lemma *arccos-arctan*:
assumes $-1 < x\ x < 1$
shows $arccos\ x = pi/2 - arctan(x\ /\ sqrt(1 - x^2))$
proof −
have $arctan(x\ /\ sqrt(1 - x^2)) - (pi/2 - arccos\ x) = 0$
proof (*rule sin-eq-0-pi*)
show $-\ pi < arctan\ (x\ /\ sqrt\ (1 - x^2)) - (pi\ /\ 2 - arccos\ x)$
using *arctan-lbound* [*of x / sqrt*$(1 - x^2)$] *arccos-bounded* [*of x*] *assms*
by (*simp add*: *algebra-simps*)

```
  next
    show arctan (x / sqrt (1 - x^2)) - (pi / 2 - arccos x) < pi
      using arctan-ubound [of x / sqrt(1 - x^2)]  arccos-bounded [of x] assms
      by (simp add: algebra-simps)
  next
    show sin (arctan (x / sqrt (1 - x^2)) - (pi / 2 - arccos x)) = 0
      using assms
      by (simp add: algebra-simps sin-diff cos-add sin-arccos sin-arctan cos-arctan
                power2-eq-square square-eq-1-iff)
  qed
  then show ?thesis
    by simp
qed

lemma arcsin-plus-arccos:
  assumes -1 ≤ x x ≤ 1
    shows arcsin x + arccos x = pi/2
proof -
  have arcsin x = pi/2 - arccos x
    apply (rule sin-inj-pi)
    using assms arcsin [OF assms] arccos [OF assms]
    apply (auto simp: algebra-simps sin-diff)
    done
  then show ?thesis
    by (simp add: algebra-simps)
qed

lemma arcsin-arccos-eq: -1 ≤ x ⟹ x ≤ 1 ⟹ arcsin x = pi/2 - arccos x
  using arcsin-plus-arccos by force

lemma arccos-arcsin-eq: -1 ≤ x ⟹ x ≤ 1 ⟹ arccos x = pi/2 - arcsin x
  using arcsin-plus-arccos by force

lemma arcsin-arctan: -1 < x ⟹ x < 1 ⟹ arcsin x = arctan(x / sqrt(1 -
x^2))
  by (simp add: arccos-arctan arcsin-arccos-eq)

lemma csqrt-1-diff-eq: csqrt (1 - (of-real x)^2) = (if x^2 ≤ 1 then sqrt (1 - x^2)
else i * sqrt (x^2 - 1))
  by ( simp add: of-real-sqrt del: csqrt-of-real-nonneg)

lemma arcsin-arccos-sqrt-pos: 0 ≤ x ⟹ x ≤ 1 ⟹ arcsin x = arccos(sqrt(1 -
x^2))
  apply (simp add: abs-square-le-1 arcsin-eq-Re-Arcsin arccos-eq-Re-Arccos)
  apply (subst Arcsin-Arccos-csqrt-pos)
  apply (auto simp: power-le-one csqrt-1-diff-eq)
  done

lemma arcsin-arccos-sqrt-neg: -1 ≤ x ⟹ x ≤ 0 ⟹ arcsin x = -arccos(sqrt(1
```

$- x^2))$
using *arcsin-arccos-sqrt-pos* [*of* $-x$]
by (*simp add*: *arcsin-minus*)

lemma *arccos-arcsin-sqrt-pos*: $0 \leq x \Longrightarrow x \leq 1 \Longrightarrow \arccos x = \arcsin(\sqrt{1 - x^2})$
apply (*simp add*: *abs-square-le-1 arcsin-eq-Re-Arcsin arccos-eq-Re-Arccos*)
apply (*subst Arccos-Arcsin-csqrt-pos*)
apply (*auto simp*: *power-le-one csqrt-1-diff-eq*)
done

lemma *arccos-arcsin-sqrt-neg*: $-1 \leq x \Longrightarrow x \leq 0 \Longrightarrow \arccos x = pi - \arcsin(\sqrt{1 - x^2})$
using *arccos-arcsin-sqrt-pos* [*of* $-x$]
by (*simp add*: *arccos-minus*)

52.28 continuity results for arcsin and arccos.

lemma *continuous-on-Arcsin-real* [*continuous-intros*]:
continuous-on $\{w \in \mathbb{R}.\ |Re\ w| \leq 1\}$ *Arcsin*
proof −
have *continuous-on* $\{w \in \mathbb{R}.\ |Re\ w| \leq 1\}$ $(\lambda x.\ complex\text{-}of\text{-}real\ (\arcsin\ (Re\ x)))$ =
continuous-on $\{w \in \mathbb{R}.\ |Re\ w| \leq 1\}$ $(\lambda x.\ complex\text{-}of\text{-}real\ (Re\ (Arcsin\ (of\text{-}real\ (Re\ x)))))$
by (*rule continuous-on-cong* [*OF refl*]) (*simp add*: *arcsin-eq-Re-Arcsin*)
also have ... = *?thesis*
by (*rule continuous-on-cong* [*OF refl*]) *simp*
finally show *?thesis*
using *continuous-on-arcsin* [*OF continuous-on-Re* [*OF continuous-on-id*], *of* $\{w \in \mathbb{R}.\ |Re\ w| \leq 1\}$]
continuous-on-of-real
by *fastforce*
qed

lemma *continuous-within-Arcsin-real*:
continuous (*at z within* $\{w \in \mathbb{R}.\ |Re\ w| \leq 1\}$) *Arcsin*
proof (*cases* $z \in \{w \in \mathbb{R}.\ |Re\ w| \leq 1\}$)
case *True* **then show** *?thesis*
using *continuous-on-Arcsin-real continuous-on-eq-continuous-within*
by *blast*
next
case *False*
with *closed-real-abs-le* [*of 1*] **show** *?thesis*
by (*rule continuous-within-closed-nontrivial*)
qed

lemma *continuous-on-Arccos-real*:
continuous-on $\{w \in \mathbb{R}.\ |Re\ w| \leq 1\}$ *Arccos*

proof −
have *continuous-on* $\{w \in \mathbb{R}.\ |Re\ w| \leq 1\}$ $(\lambda x.\ complex\text{-}of\text{-}real\ (arccos\ (Re\ x)))$
=
continuous-on $\{w \in \mathbb{R}.\ |Re\ w| \leq 1\}$ $(\lambda x.\ complex\text{-}of\text{-}real\ (Re\ (Arccos\ (of\text{-}real\ (Re\ x)))))$
by (*rule continuous-on-cong* [*OF refl*]) (*simp add*: *arccos-eq-Re-Arccos*)
also have ... = *?thesis*
by (*rule continuous-on-cong* [*OF refl*]) *simp*
finally show *?thesis*
using *continuous-on-arccos* [*OF continuous-on-Re* [*OF continuous-on-id*], *of* $\{w \in \mathbb{R}.\ |Re\ w| \leq 1\}$]
continuous-on-of-real
by *fastforce*
qed

lemma *continuous-within-Arccos-real*:
continuous (*at z within* $\{w \in \mathbb{R}.\ |Re\ w| \leq 1\}$) *Arccos*
proof (*cases* $z \in \{w \in \mathbb{R}.\ |Re\ w| \leq 1\}$)
case *True* **then show** *?thesis*
using *continuous-on-Arccos-real continuous-on-eq-continuous-within*
by *blast*
next
case *False*
with *closed-real-abs-le* [*of 1*] **show** *?thesis*
by (*rule continuous-within-closed-nontrivial*)
qed

52.29 Roots of unity

lemma *complex-root-unity*:
fixes *j*::*nat*
assumes $n \neq 0$
shows $exp(2 * of\text{-}real\ pi * \mathrm{i} * of\text{-}nat\ j\ /\ of\text{-}nat\ n)\ \hat{}\ n = 1$
proof −
have $*$: $of\text{-}nat\ j * (complex\text{-}of\text{-}real\ pi * 2) = complex\text{-}of\text{-}real\ (2 * real\ j * pi)$
by (*simp add*: *of-real-numeral*)
then show *?thesis*
apply (*simp add*: *exp-of-nat-mult* [*symmetric*] *mult-ac exp-Euler*)
apply (*simp only*: $*$ *cos-of-real sin-of-real*)
apply (*simp add*:)
done
qed

lemma *complex-root-unity-eq*:
fixes *j*::*nat* **and** *k*::*nat*
assumes $1 \leq n$
shows $(exp(2 * of\text{-}real\ pi * \mathrm{i} * of\text{-}nat\ j\ /\ of\text{-}nat\ n) = exp(2 * of\text{-}real\ pi * \mathrm{i} * of\text{-}nat\ k\ /\ of\text{-}nat\ n)$
$\longleftrightarrow j\ mod\ n = k\ mod\ n)$

proof −
have (∃ *z*::*int*. i ∗ (*of-nat j* ∗ (*of-real pi* ∗ *2*)) =
i ∗ (*of-nat k* ∗ (*of-real pi* ∗ *2*)) + i ∗ (*of-int z* ∗ (*of-nat n* ∗ (*of-real pi* ∗ *2*)))) ⟷
(∃ *z*::*int*. *of-nat j* ∗ (i ∗ (*of-real pi* ∗ *2*)) =
(*of-nat k* + *of-nat n* ∗ *of-int z*) ∗ (i ∗ (*of-real pi* ∗ *2*)))
by (*simp add*: *algebra-simps*)
also have ... ⟷ (∃ *z*::*int*. *of-nat j* = *of-nat k* + *of-nat n* ∗ (*of-int z* :: *complex*))
by *simp*
also have ... ⟷ (∃ *z*::*int*. *of-nat j* = *of-nat k* + *of-nat n* ∗ *z*)
apply (*rule HOL.iff-exI*)
apply (*auto simp*:)
using *of-int-eq-iff* **apply** *fastforce*
by (*metis of-int-add of-int-mult of-int-of-nat-eq*)
also have ... ⟷ *int j mod int n* = *int k mod int n*
by (*auto simp*: *zmod-eq-dvd-iff dvd-def algebra-simps*)
also have ... ⟷ *j mod n* = *k mod n*
by (*metis of-nat-eq-iff zmod-int*)
finally have (∃ *z*. i ∗ (*of-nat j* ∗ (*of-real pi* ∗ *2*)) =
i ∗ (*of-nat k* ∗ (*of-real pi* ∗ *2*)) + i ∗ (*of-int z* ∗ (*of-nat n* ∗ (*of-real pi* ∗ *2*)))) ⟷ *j mod n* = *k mod n* **.**
note ∗ = *this*
show *?thesis*
using *assms*
by (*simp add*: *exp-eq divide-simps mult-ac of-real-numeral* ∗)
qed

corollary *bij-betw-roots-unity*:
bij-betw (λ*j*. *exp*(*2* ∗ *of-real pi* ∗ i ∗ *of-nat j* / *of-nat n*))
{..<*n*} {*exp*(*2* ∗ *of-real pi* ∗ i ∗ *of-nat j* / *of-nat n*) | *j*. *j* < *n*}
by (*auto simp*: *bij-betw-def inj-on-def complex-root-unity-eq*)

lemma *complex-root-unity-eq-1*:
fixes *j*::*nat* **and** *k*::*nat*
assumes *1* ≤ *n*
shows *exp*(*2* ∗ *of-real pi* ∗ i ∗ *of-nat j* / *of-nat n*) = *1* ⟷ *n dvd j*
proof −
have *1* = *exp*(*2* ∗ *of-real pi* ∗ i ∗ (*of-nat n* / *of-nat n*))
using *assms* **by** *simp*
then have *exp*(*2* ∗ *of-real pi* ∗ i ∗ (*of-nat j* / *of-nat n*)) = *1* ⟷ *j mod n* = *n mod n*
using *complex-root-unity-eq* [*of n j n*] *assms*
by *simp*
then show *?thesis*
by *auto*
qed

lemma *finite-complex-roots-unity-explicit*:
finite {*exp*(*2* ∗ *of-real pi* ∗ i ∗ *of-nat j* / *of-nat n*) | *j*::*nat*. *j* < *n*}

by *simp*

lemma *card-complex-roots-unity-explicit*:
card {*exp*(*2* ∗ *of-real pi* ∗ i ∗ *of-nat j* / *of-nat n*) | *j*::*nat*. *j* < *n*} = *n*
by (*simp add*: *Finite-Set.bij-betw-same-card* [*OF bij-betw-roots-unity*, *symmetric*])

lemma *complex-roots-unity*:
assumes *1* ≤ *n*
shows {*z*::*complex*. *z*^*n* = *1*} = {*exp*(*2* ∗ *of-real pi* ∗ i ∗ *of-nat j* / *of-nat n*) | *j*::*nat*. *j* < *n*}
apply (*rule Finite-Set.card-seteq* [*symmetric*])
using *assms*
apply (*auto simp*: *card-complex-roots-unity-explicit finite-roots-unity complex-root-unity card-roots-unity*)
done

lemma *card-complex-roots-unity*: *1* ≤ *n* ⟹ *card* {*z*::*complex*. *z*^*n* = *1*} = *n*
by (*simp add*: *card-complex-roots-unity-explicit complex-roots-unity*)

lemma *complex-not-root-unity*:
1 ≤ *n* ⟹ ∃ *u*::*complex*. *norm u* = *1* ∧ *u*^*n* ≠ *1*
apply (*rule-tac x*=*exp* (*of-real pi* ∗ i ∗ *of-real* (*1* / *n*)) **in** *exI*)
apply (*auto simp*: *Re-complex-div-eq-0 exp-of-nat-mult* [*symmetric*] *mult-ac exp-Euler*)
done

end

53 Complex path integrals and Cauchy's integral theorem

By John Harrison et al. Ported from HOL Light by L C Paulson (2015)

theory *Cauchy-Integral-Thm*
imports *Complex-Transcendental Weierstrass Ordered-Euclidean-Space*
begin

53.1 Homeomorphisms of arc images

lemma *homeomorphism-arc*:
fixes *g* :: *real* ⇒ *'a*::*t2-space*
assumes *arc g*
obtains *h* **where** *homeomorphism* {*0*..*1*} (*path-image g*) *g h*
using *assms* **by** (*force simp add*: *arc-def homeomorphism-compact path-def path-image-def*)

lemma *homeomorphic-arc-image-interval*:
fixes *g* :: *real* ⇒ *'a*::*t2-space* **and** *a*::*real*
assumes *arc g a* < *b*
shows (*path-image g*) *homeomorphic* {*a*..*b*}

proof −
have (*path-image g*) *homeomorphic* {*0..1*::*real*}
by (*meson assms*(*1*) *homeomorphic-def homeomorphic-sym homeomorphism-arc*)
also have ... *homeomorphic* {*a..b*}
using *assms* **by** (*force intro*: *homeomorphic-closed-intervals-real*)
finally show *?thesis* **.**
qed

lemma *homeomorphic-arc-images*:
fixes *g* :: *real* ⇒ *'a*::*t2-space* **and** *h* :: *real* ⇒ *'b*::*t2-space*
assumes *arc g arc h*
shows (*path-image g*) *homeomorphic* (*path-image h*)
proof −
have (*path-image g*) *homeomorphic* {*0..1*::*real*}
by (*meson assms homeomorphic-def homeomorphic-sym homeomorphism-arc*)
also have ... *homeomorphic* (*path-image h*)
by (*meson assms homeomorphic-def homeomorphism-arc*)
finally show *?thesis* **.**
qed

53.2 Piecewise differentiable functions

definition *piecewise-differentiable-on*
(**infixr** *piecewise'-differentiable'-on 50*)
where *f piecewise-differentiable-on i* ≡
continuous-on i f ∧
(∃ *s*. *finite s* ∧ (∀ *x* ∈ *i* − *s*. *f differentiable* (*at x within i*)))

lemma *piecewise-differentiable-on-imp-continuous-on*:
f piecewise-differentiable-on s ⟹ *continuous-on s f*
by (*simp add*: *piecewise-differentiable-on-def*)

lemma *piecewise-differentiable-on-subset*:
f piecewise-differentiable-on s ⟹ *t* ≤ *s* ⟹ *f piecewise-differentiable-on t*
using *continuous-on-subset*
unfolding *piecewise-differentiable-on-def*
apply *safe*
apply (*blast intro*: *elim*: *continuous-on-subset*)
by (*meson Diff-iff differentiable-within-subset subsetCE*)

lemma *differentiable-on-imp-piecewise-differentiable*:
fixes *a*:: *'a*::{*linorder-topology*,*real-normed-vector*}
shows *f differentiable-on* {*a..b*} ⟹ *f piecewise-differentiable-on* {*a..b*}
apply (*simp add*: *piecewise-differentiable-on-def differentiable-imp-continuous-on*)
apply (*rule-tac x*={*a*,*b*} **in** *exI*, *simp add*: *differentiable-on-def*)
done

lemma *differentiable-imp-piecewise-differentiable*:
(⋀*x*. *x* ∈ *s* ⟹ *f differentiable* (*at x within s*))

$\Longrightarrow$ *f piecewise-differentiable-on s*
by (*auto simp*: *piecewise-differentiable-on-def differentiable-imp-continuous-on differentiable-on-def*
intro: *differentiable-within-subset*)

lemma *piecewise-differentiable-const* [*iff*]: ($\lambda x.\ z$) *piecewise-differentiable-on s*
by (*simp add*: *differentiable-imp-piecewise-differentiable*)

lemma *piecewise-differentiable-compose*:
⟦*f piecewise-differentiable-on s*; *g piecewise-differentiable-on* (*f* ‘ *s*);
$\bigwedge x.$ *finite* (*s* ∩ *f* −‘{*x*})⟧
$\Longrightarrow$ (*g o f*) *piecewise-differentiable-on s*
apply (*simp add*: *piecewise-differentiable-on-def*, *safe*)
apply (*blast intro*: *continuous-on-compose2*)
apply (*rename-tac A B*)
apply (*rule-tac x*=*A* ∪ (⋃*x*∈*B*. *s* ∩ *f* −‘{*x*}) **in** *exI*)
apply (*blast intro*: *differentiable-chain-within*)
done

lemma *piecewise-differentiable-affine*:
fixes *m*::*real*
assumes *f piecewise-differentiable-on* (($\lambda x.\ m *_R x + c$) ‘ *s*)
shows (*f o* ($\lambda x.\ m *_R x + c$)) *piecewise-differentiable-on s*
proof (*cases m* = *0*)
case *True*
then show *?thesis*
unfolding *o-def*
by (*force intro*: *differentiable-imp-piecewise-differentiable differentiable-const*)
next
case *False*
show *?thesis*
apply (*rule piecewise-differentiable-compose* [*OF differentiable-imp-piecewise-differentiable*])
apply (*rule assms derivative-intros* | *simp add*: *False vimage-def real-vector-affinity-eq*)+
done
qed

lemma *piecewise-differentiable-cases*:
fixes *c*::*real*
assumes *f piecewise-differentiable-on* {*a*..*c*}
g piecewise-differentiable-on {*c*..*b*}
a ≤ *c c* ≤ *b f c* = *g c*
shows ($\lambda x.$ *if x* ≤ *c then f x else g x*) *piecewise-differentiable-on* {*a*..*b*}
proof −
obtain *s t* **where** *st*: *finite s finite t*
$\forall x \in \{a..c\} - s.$ *f differentiable at x within* {*a*..*c*}
$\forall x \in \{c..b\} - t.$ *g differentiable at x within* {*c*..*b*}
using *assms*
by (*auto simp*: *piecewise-differentiable-on-def*)
have *finabc*: *finite* ({*a*,*b*,*c*} ∪ (*s* ∪ *t*))
by (*metis* ‹*finite s*› ‹*finite t*› *finite-Un finite-insert finite.emptyI*)

```
  have continuous-on {a..c} f continuous-on {c..b} g
    using assms piecewise-differentiable-on-def by auto
  then have continuous-on {a..b} (λx. if x ≤ c then f x else g x)
    using continuous-on-cases [OF closed-real-atLeastAtMost [of a c],
                               OF closed-real-atLeastAtMost [of c b],
                               of f g λx. x≤c]  assms
    by (force simp: ivl-disj-un-two-touch)
  moreover
  { fix x
    assume x: x ∈ {a..b} − ({a,b,c} ∪ (s ∪ t))
   have (λx. if x ≤ c then f x else g x) differentiable at x within {a..b} (is ?diff-fg)
    proof (cases x c rule: le-cases)
      case le show ?diff-fg
        apply (rule differentiable-transform-within [where d = dist x c and f =
f])
        using x le st
        apply (simp-all add: dist-real-def)
        apply (rule differentiable-at-withinI)
         apply (rule differentiable-within-open [where s = {a<..<c} − s, THEN
iffD1], simp-all)
        apply (blast intro: open-greaterThanLessThan finite-imp-closed)
        apply (force elim!: differentiable-subset)+
        done
    next
      case ge show ?diff-fg
        apply (rule differentiable-transform-within [where d = dist x c and f =
g])
        using x ge st
        apply (simp-all add: dist-real-def)
        apply (rule differentiable-at-withinI)
         apply (rule differentiable-within-open [where s = {c<..<b} − t, THEN
iffD1], simp-all)
        apply (blast intro: open-greaterThanLessThan finite-imp-closed)
        apply (force elim!: differentiable-subset)+
        done
    qed
  }
  then have ∃ s. finite s ∧
                (∀ x∈{a..b} − s. (λx. if x ≤ c then f x else g x) differentiable at x
within {a..b})
    by (meson finabc)
  ultimately show ?thesis
    by (simp add: piecewise-differentiable-on-def)
qed

lemma piecewise-differentiable-neg:
    f piecewise-differentiable-on s ⟹ (λx. −(f x)) piecewise-differentiable-on s
  by (auto simp: piecewise-differentiable-on-def continuous-on-minus)
```

lemma *piecewise-differentiable-add*:
assumes *f piecewise-differentiable-on i*
g piecewise-differentiable-on i
shows $(\lambda x.\ f\ x + g\ x)$ *piecewise-differentiable-on i*
proof −
obtain *s t* **where** *st*: *finite s finite t*
$\forall x{\in}i - s.$ *f differentiable at x within i*
$\forall x{\in}i - t.$ *g differentiable at x within i*
using *assms* **by** (*auto simp*: *piecewise-differentiable-on-def*)
then have *finite* $(s \cup t) \wedge (\forall x{\in}i - (s \cup t).$ $(\lambda x.\ f\ x + g\ x)$ *differentiable at x within i*)
by *auto*
moreover have *continuous-on i f continuous-on i g*
using *assms piecewise-differentiable-on-def* **by** *auto*
ultimately show *?thesis*
by (*auto simp*: *piecewise-differentiable-on-def continuous-on-add*)
qed

lemma *piecewise-differentiable-diff*:
$[\![f$ *piecewise-differentiable-on s*; *g piecewise-differentiable-on s*$]\!]$
$\Longrightarrow (\lambda x.\ f\ x - g\ x)$ *piecewise-differentiable-on s*
unfolding *diff-conv-add-uminus*
by (*metis piecewise-differentiable-add piecewise-differentiable-neg*)

lemma *continuous-on-joinpaths-D1*:
continuous-on $\{0..1\}$ $(g1$ *+++* $g2) \Longrightarrow$ *continuous-on* $\{0..1\}$ *g1*
apply (*rule continuous-on-eq* [*of* - $(g1$ *+++* $g2)$ *o* $(op{*}(inverse\ 2))$])
apply (*rule continuous-intros* | *simp*)+
apply (*auto elim*!: *continuous-on-subset simp*: *joinpaths-def*)
done

lemma *continuous-on-joinpaths-D2*:
$[\![$*continuous-on* $\{0..1\}$ $(g1$ *+++* $g2)$; *pathfinish g1* = *pathstart g2*$]\!] \Longrightarrow$ *continuous-on* $\{0..1\}$ *g2*
apply (*rule continuous-on-eq* [*of* - $(g1$ *+++* $g2)$ *o* $(\lambda x.\ inverse\ 2{*}x + 1/2)$])
apply (*rule continuous-intros* | *simp*)+
apply (*auto elim*!: *continuous-on-subset simp add*: *joinpaths-def pathfinish-def pathstart-def Ball-def*)
done

lemma *piecewise-differentiable-D1*:
$(g1$ *+++* $g2)$ *piecewise-differentiable-on* $\{0..1\} \Longrightarrow$ *g1 piecewise-differentiable-on* $\{0..1\}$
apply (*clarsimp simp add*: *piecewise-differentiable-on-def dest*!: *continuous-on-joinpaths-D1*)
apply (*rule-tac x*=*insert 1* $((op{*}2)\text{`}s)$ **in** *exI*)
apply *simp*
apply (*intro ballI*)
apply (*rule-tac d*=*dist* $(x/2)$ $(1/2)$ **and** $f = (g1$ *+++* $g2)$ *o* $(op{*}(inverse\ 2))$
in *differentiable-transform-within*)

apply (*auto simp*: *dist-real-def joinpaths-def*)
apply (*rule differentiable-chain-within derivative-intros* | *simp*)+
apply (*rule differentiable-subset*)
apply (*force simp*:)+
done

lemma *piecewise-differentiable-D2*:
⟦(*g1* +++ *g2*) *piecewise-differentiable-on* {*0..1*}; *pathfinish g1* = *pathstart g2*⟧
⟹ *g2 piecewise-differentiable-on* {*0..1*}
apply (*clarsimp simp add*: *piecewise-differentiable-on-def dest*!: *continuous-on-joinpaths-D2*)
apply (*rule-tac x*=*insert 0* ((λ*x*. *2*∗*x*−*1*)‘*s*) **in** *exI*)
apply *simp*
apply (*intro ballI*)
apply (*rule-tac d*=*dist* ((*x*+*1*)/*2*) (*1*/*2*) **and** *f* = (*g1* +++ *g2*) *o* (λ*x*. (*x*+*1*)/*2*)
in *differentiable-transform-within*)
apply (*auto simp*: *dist-real-def joinpaths-def abs-if field-simps split*: *if-split-asm*)
apply (*rule differentiable-chain-within derivative-intros* | *simp*)+
apply (*rule differentiable-subset*)
apply (*force simp*: *divide-simps*)+
done

53.2.1 The concept of continuously differentiable

John Harrison writes as follows:

"The usual assumption in complex analysis texts is that a path γ should be piecewise continuously differentiable, which ensures that the path integral exists at least for any continuous f, since all piecewise continuous functions are integrable. However, our notion of validity is weaker, just piecewise differentiability... [namely] continuity plus differentiability except on a nite set ... [Our] underlying theory of integration is the Kurzweil-Henstock theory. In contrast to the Riemann or Lebesgue theory (but in common with a simple notion based on antiderivatives), this can integrate all derivatives."

"Formalizing basic complex analysis." From Insight to Proof: Festschrift in Honour of Andrzej Trybulec. Studies in Logic, Grammar and Rhetoric 10.23 (2007): 151-165.

And indeed he does not assume that his derivatives are continuous, but the penalty is unreasonably difficult proofs concerning winding numbers. We need a self-contained and straightforward theorem asserting that all derivatives can be integrated before we can adopt Harrison's choice.

definition *C1-differentiable-on* :: (*real* ⇒ *'a*::*real-normed-vector*) ⇒ *real set* ⇒ *bool*
(**infix** *C1'-differentiable'-on 50*)
where
f C1-differentiable-on s ⟷
(∃ *D*. (∀ *x* ∈ *s*. (*f has-vector-derivative* (*D x*)) (*at x*)) ∧ *continuous-on s D*)

lemma *C1-differentiable-on-eq*:
  f C1-differentiable-on s $\longleftrightarrow$
   $(\forall x \in s.$ *f differentiable at x*$)$ $\wedge$ *continuous-on s* $(\lambda x.$ *vector-derivative f* $($*at x*$))$
 unfolding *C1-differentiable-on-def*
 apply *safe*
 using *differentiable-def has-vector-derivative-def* **apply** *blast*
 apply (*erule continuous-on-eq*)
 using *vector-derivative-at* **apply** *fastforce*
 using *vector-derivative-works* **apply** *fastforce*
 done

lemma *C1-differentiable-on-subset*:
 f C1-differentiable-on t $\Longrightarrow$ *s* $\subseteq$ *t* $\Longrightarrow$ *f C1-differentiable-on s*
 unfolding *C1-differentiable-on-def continuous-on-eq-continuous-within*
 by (*blast intro*: *continuous-within-subset*)

lemma *C1-differentiable-compose*:
  $[\![$*f C1-differentiable-on s*; *g C1-differentiable-on* $(f$ ' *s*$)$;
    $\bigwedge x.$ *finite* $(s \cap f -$ '$\{x\})]\!]$
    $\Longrightarrow$ $(g \circ f)$ *C1-differentiable-on s*
 apply (*simp add*: *C1-differentiable-on-eq*, *safe*)
  using *differentiable-chain-at* **apply** *blast*
 apply (*rule continuous-on-eq* [*of* - $\lambda x.$ *vector-derivative f* $($*at x*$) *_R$ vector-derivative g* $($*at* $(f\, x))$])
  apply (*rule Limits.continuous-on-scaleR*, *assumption*)
 apply (*metis* (*mono-tags*, *lifting*) *continuous-on-eq continuous-at-imp-continuous-on continuous-on-compose differentiable-imp-continuous-within o-def*)
 by (*simp add*: *vector-derivative-chain-at*)

lemma *C1-diff-imp-diff*: *f C1-differentiable-on s* $\Longrightarrow$ *f differentiable-on s*
 by (*simp add*: *C1-differentiable-on-eq differentiable-at-imp-differentiable-on*)

lemma *C1-differentiable-on-ident* [*simp*, *derivative-intros*]: $(\lambda x.\ x)$ *C1-differentiable-on s*
 by (*auto simp*: *C1-differentiable-on-eq continuous-on-const*)

lemma *C1-differentiable-on-const* [*simp*, *derivative-intros*]: $(\lambda z.\ a)$ *C1-differentiable-on s*
 by (*auto simp*: *C1-differentiable-on-eq continuous-on-const*)

lemma *C1-differentiable-on-add* [*simp*, *derivative-intros*]:
 f C1-differentiable-on s $\Longrightarrow$ *g C1-differentiable-on s* $\Longrightarrow$ $(\lambda x.\ f\, x + g\, x)$ *C1-differentiable-on s*
 unfolding *C1-differentiable-on-eq* **by** (*auto intro*: *continuous-intros*)

lemma *C1-differentiable-on-minus* [*simp*, *derivative-intros*]:
 f C1-differentiable-on s $\Longrightarrow$ $(\lambda x.\ -\, f\, x)$ *C1-differentiable-on s*
 unfolding *C1-differentiable-on-eq* **by** (*auto intro*: *continuous-intros*)

lemma *C1-differentiable-on-diff* [*simp*, *derivative-intros*]:
f C1-differentiable-on s $\Longrightarrow$ *g C1-differentiable-on s* $\Longrightarrow$ $(\lambda x.\ f\,x - g\,x)$ *C1-differentiable-on s*
unfolding *C1-differentiable-on-eq* **by** (*auto intro*: *continuous-intros*)

lemma *C1-differentiable-on-mult* [*simp*, *derivative-intros*]:
fixes *f g* :: *real* $\Rightarrow$ *'a* :: *real-normed-algebra*
shows *f C1-differentiable-on s* $\Longrightarrow$ *g C1-differentiable-on s* $\Longrightarrow$ $(\lambda x.\ f\,x * g\,x)$ *C1-differentiable-on s*
unfolding *C1-differentiable-on-eq*
by (*auto simp*: *continuous-on-add continuous-on-mult continuous-at-imp-continuous-on differentiable-imp-continuous-within*)

lemma *C1-differentiable-on-scaleR* [*simp*, *derivative-intros*]:
f C1-differentiable-on s $\Longrightarrow$ *g C1-differentiable-on s* $\Longrightarrow$ $(\lambda x.\ f\,x *_R g\,x)$ *C1-differentiable-on s*
unfolding *C1-differentiable-on-eq*
by (*rule continuous-intros* | *simp add*: *continuous-at-imp-continuous-on differentiable-imp-continuous-within*)

definition *piecewise-C1-differentiable-on*
(**infixr** *piecewise'-C1'-differentiable'-on 50*)
where *f piecewise-C1-differentiable-on i* $\equiv$
continuous-on i f $\wedge$
$(\exists s.$ *finite s* $\wedge$ *(f C1-differentiable-on* $(i - s)))$

lemma *C1-differentiable-imp-piecewise*:
f C1-differentiable-on s $\Longrightarrow$ *f piecewise-C1-differentiable-on s*
by (*auto simp*: *piecewise-C1-differentiable-on-def C1-differentiable-on-eq continuous-at-imp-continuous-on differentiable-imp-continuous-within*)

lemma *piecewise-C1-imp-differentiable*:
f piecewise-C1-differentiable-on i $\Longrightarrow$ *f piecewise-differentiable-on i*
by (*auto simp*: *piecewise-C1-differentiable-on-def piecewise-differentiable-on-def C1-differentiable-on-def differentiable-def has-vector-derivative-def intro*: *has-derivative-at-within*)

lemma *piecewise-C1-differentiable-compose*:
$[\![$*f piecewise-C1-differentiable-on s*; *g piecewise-C1-differentiable-on* $(f\ `\ s)$;
$\bigwedge x.$ *finite* $(s \cap f -` \{x\})]\!]$
$\Longrightarrow$ $(g\ o\ f)$ *piecewise-C1-differentiable-on s*
apply (*simp add*: *piecewise-C1-differentiable-on-def*, *safe*)
apply (*blast intro*: *continuous-on-compose2*)
apply (*rename-tac A B*)
apply (*rule-tac x*=$A \cup (\bigcup x \in B.\ s \cap f -` \{x\})$ **in** *exI*)
apply (*rule conjI*, *blast*)
apply (*rule C1-differentiable-compose*)
apply (*blast intro*: *C1-differentiable-on-subset*)

apply (*blast intro*: *C1-differentiable-on-subset*)
by (*simp add*: *Diff-Int-distrib2*)

lemma *piecewise-C1-differentiable-on-subset*:
f piecewise-C1-differentiable-on s $\Longrightarrow$ *t* $\subseteq$ *s* $\Longrightarrow$ *f piecewise-C1-differentiable-on t*
by (*auto simp*: *piecewise-C1-differentiable-on-def elim*!: *continuous-on-subset C1-differentiable-on-subset*)

lemma *C1-differentiable-imp-continuous-on*:
f C1-differentiable-on s $\Longrightarrow$ *continuous-on s f*
unfolding *C1-differentiable-on-eq continuous-on-eq-continuous-within*
using *differentiable-at-withinI differentiable-imp-continuous-within* **by** *blast*

lemma *C1-differentiable-on-empty* [*iff*]: *f C1-differentiable-on* {}
unfolding *C1-differentiable-on-def*
by *auto*

lemma *piecewise-C1-differentiable-affine*:
fixes *m*::*real*
assumes *f piecewise-C1-differentiable-on* ((λ*x*. *m* ∗ *x* + *c*) ‘ *s*)
shows (*f o* (λ*x*. *m* ∗$_R$ *x* + *c*)) *piecewise-C1-differentiable-on s*
proof (*cases m* = *0*)
case *True*
then show *?thesis*
unfolding *o-def* **by** (*auto simp*: *piecewise-C1-differentiable-on-def continuous-on-const*)
next
case *False*
show *?thesis*
apply (*rule piecewise-C1-differentiable-compose* [*OF C1-differentiable-imp-piecewise*])
apply (*rule assms derivative-intros* | *simp add*: *False vimage-def*)+
using *real-vector-affinity-eq* [*OF False*, **where** *c*=*c*, *unfolded scaleR-conv-of-real*]
apply *simp*
done
qed

lemma *piecewise-C1-differentiable-cases*:
fixes *c*::*real*
assumes *f piecewise-C1-differentiable-on* {*a*..*c*}
g piecewise-C1-differentiable-on {*c*..*b*}
a ≤ *c c* ≤ *b f c* = *g c*
shows (λ*x*. *if x* ≤ *c then f x else g x*) *piecewise-C1-differentiable-on* {*a*..*b*}
proof −
obtain *s t* **where** *st*: *f C1-differentiable-on* ({*a*..*c*} − *s*)
g C1-differentiable-on ({*c*..*b*} − *t*)
finite s finite t
using *assms*
by (*force simp*: *piecewise-C1-differentiable-on-def*)
then have *f-diff*: *f differentiable-on* {*a*..<*c*} − *s*
and *g-diff*: *g differentiable-on* {*c*<..*b*} − *t*

by (*simp-all add*: *C1-differentiable-on-eq differentiable-at-withinI differentiable-on-def*)
have *continuous-on* {*a*..*c*} *f continuous-on* {*c*..*b*} *g*
using *assms piecewise-C1-differentiable-on-def* **by** *auto*
then have *cab*: *continuous-on* {*a*..*b*} (λ*x*. *if x* ≤ *c then f x else g x*)
using *continuous-on-cases* [*OF closed-real-atLeastAtMost* [*of a c*],
OF closed-real-atLeastAtMost [*of c b*],
of f g λ*x*. *x*≤*c*] *assms*
by (*force simp*: *ivl-disj-un-two-touch*)
{ fix *x*
assume *x*: *x* ∈ {*a*..*b*} − *insert c* (*s* ∪ *t*)
have (λ*x*. *if x* ≤ *c then f x else g x*) *differentiable at x* (**is** *?diff-fg*)
proof (*cases x c rule*: *le-cases*)
case *le* **show** *?diff-fg*
apply (*rule differentiable-transform-within* [**where** *f*=*f* **and** *d* = *dist x c*])
using *x dist-real-def le st* **by** (*auto simp*: *C1-differentiable-on-eq*)
next
case *ge* **show** *?diff-fg*
apply (*rule differentiable-transform-within* [**where** *f*=*g* **and** *d* = *dist x c*])
using *dist-nz x dist-real-def ge st x* **by** (*auto simp*: *C1-differentiable-on-eq*)
qed
}
then have (∀ *x* ∈ {*a*..*b*} − *insert c* (*s* ∪ *t*). (λ*x*. *if x* ≤ *c then f x else g x*) *differentiable at x*)
by *auto*
moreover
{ assume *fcon*: *continuous-on* ({*a*<..<*c*} − *s*) (λ*x*. *vector-derivative f* (*at x*))
and *gcon*: *continuous-on* ({*c*<..<*b*} − *t*) (λ*x*. *vector-derivative g* (*at x*))
have *open* ({*a*<..<*c*} − *s*) *open* ({*c*<..<*b*} − *t*)
using *st* **by** (*simp-all add*: *open-Diff finite-imp-closed*)
moreover have *continuous-on* ({*a*<..<*c*} − *s*) (λ*x*. *vector-derivative* (λ*x*. *if x* ≤ *c then f x else g x*) (*at x*))
apply (*rule continuous-on-eq* [*OF fcon*])
apply (*simp add*:)
apply (*rule vector-derivative-at* [*symmetric*])
apply (*rule-tac f*=*f* **and** *d*=*dist x c* **in** *has-vector-derivative-transform-within*)
apply (*simp-all add*: *dist-norm vector-derivative-works* [*symmetric*])
apply (*metis* (*full-types*) *C1-differentiable-on-eq Diff-iff Groups.add-ac*(*2*) *add-mono-thms-linordered-field*(*5*) *atLeastAtMost-iff linorder-not-le order-less-irrefl st*(*1*))
apply *auto*
done
moreover have *continuous-on* ({*c*<..<*b*} − *t*) (λ*x*. *vector-derivative* (λ*x*. *if x* ≤ *c then f x else g x*) (*at x*))
apply (*rule continuous-on-eq* [*OF gcon*])
apply (*simp add*:)
apply (*rule vector-derivative-at* [*symmetric*])
apply (*rule-tac f*=*g* **and** *d*=*dist x c* **in** *has-vector-derivative-transform-within*)
apply (*simp-all add*: *dist-norm vector-derivative-works* [*symmetric*])
apply (*metis* (*full-types*) *C1-differentiable-on-eq Diff-iff Groups.add-ac*(*2*)

add-mono-thms-linordered-field(*5*) *atLeastAtMost-iff less-irrefl not-le st*(*2*))
apply *auto*
done
ultimately have *continuous-on* ({*a*<..<*b*} − *insert c* (*s* ∪ *t*))
(λ*x*. *vector-derivative* (λ*x*. *if x* ≤ *c then f x else g x*) (*at x*))
apply (*rule continuous-on-subset* [*OF continuous-on-open-Un*], *auto*)
done
} note ∗ = *this*
have *continuous-on* ({*a*<..<*b*} − *insert c* (*s* ∪ *t*)) (λ*x*. *vector-derivative* (λ*x*. *if x* ≤ *c then f x else g x*) (*at x*))
using *st*
by (*auto simp*: *C1-differentiable-on-eq elim*!: *continuous-on-subset intro*: ∗)
ultimately have ∃ *s*. *finite s* ∧ ((λ*x*. *if x* ≤ *c then f x else g x*) *C1-differentiable-on* {*a*..*b*} − *s*)
apply (*rule-tac x*={*a*,*b*,*c*} ∪ *s* ∪ *t* **in** *exI*)
using *st* **by** (*auto simp*: *C1-differentiable-on-eq elim*!: *continuous-on-subset*)
with *cab* **show** *?thesis*
by (*simp add*: *piecewise-C1-differentiable-on-def*)
qed

lemma *piecewise-C1-differentiable-neg*:
f piecewise-C1-differentiable-on s ⟹ (λ*x*. −(*f x*)) *piecewise-C1-differentiable-on s*
unfolding *piecewise-C1-differentiable-on-def*
by (*auto intro*!: *continuous-on-minus C1-differentiable-on-minus*)

lemma *piecewise-C1-differentiable-add*:
assumes *f piecewise-C1-differentiable-on i*
g piecewise-C1-differentiable-on i
shows (λ*x*. *f x* + *g x*) *piecewise-C1-differentiable-on i*
proof −
obtain *s t* **where** *st*: *finite s finite t*
f C1-differentiable-on (*i*−*s*)
g C1-differentiable-on (*i*−*t*)
using *assms* **by** (*auto simp*: *piecewise-C1-differentiable-on-def*)
then have *finite* (*s* ∪ *t*) ∧ (λ*x*. *f x* + *g x*) *C1-differentiable-on i* − (*s* ∪ *t*)
by (*auto intro*: *C1-differentiable-on-add elim*!: *C1-differentiable-on-subset*)
moreover have *continuous-on i f continuous-on i g*
using *assms piecewise-C1-differentiable-on-def* **by** *auto*
ultimately show *?thesis*
by (*auto simp*: *piecewise-C1-differentiable-on-def continuous-on-add*)
qed

lemma *piecewise-C1-differentiable-diff*:
⟦*f piecewise-C1-differentiable-on s*; *g piecewise-C1-differentiable-on s*⟧
⟹ (λ*x*. *f x* − *g x*) *piecewise-C1-differentiable-on s*
unfolding *diff-conv-add-uminus*
by (*metis piecewise-C1-differentiable-add piecewise-C1-differentiable-neg*)

lemma *piecewise-C1-differentiable-D1*:
fixes *g1* :: *real* ⇒ *'a*::*real-normed-field*
assumes (*g1* +++ *g2*) *piecewise-C1-differentiable-on* {*0..1*}
shows *g1 piecewise-C1-differentiable-on* {*0..1*}
proof −
obtain *s* **where** *finite s*
and *co12*: *continuous-on* ({*0..1*} − *s*) (λ*x*. *vector-derivative* (*g1* +++ *g2*) (*at x*))
and *g12D*: ∀ *x*∈{*0..1*} − *s*. *g1* +++ *g2 differentiable at x*
using *assms* **by** (*auto simp*: *piecewise-C1-differentiable-on-def C1-differentiable-on-eq*)
then have *g1D*: *g1 differentiable at x* **if** *x* ∈ {*0..1*} − *insert 1* (*op* ∗ *2* ‘ *s*) **for** *x*
apply (*rule-tac d*=*dist* (*x*/*2*) (*1*/*2*) **and** *f* = (*g1* +++ *g2*) *o* (*op*∗(*inverse 2*))
in *differentiable-transform-within*)
using *that*
apply (*simp-all add*: *dist-real-def joinpaths-def*)
apply (*rule differentiable-chain-at derivative-intros* | *force*)+
done
have [*simp*]: *vector-derivative* (*g1* ∘ *op* ∗ *2*) (*at* (*x*/*2*)) = *2* ∗*R vector-derivative* *g1* (*at x*)
if *x* ∈ {*0..1*} − *insert 1* (*op* ∗ *2* ‘ *s*) **for** *x*
apply (*subst vector-derivative-chain-at*)
using *that*
apply (*rule derivative-eq-intros g1D* | *simp*)+
done
have *continuous-on* ({*0..1*/*2*} − *insert* (*1*/*2*) *s*) (λ*x*. *vector-derivative* (*g1* +++ *g2*) (*at x*))
using *co12* **by** (*rule continuous-on-subset*) *force*
then have *coDhalf*: *continuous-on* ({*0..1*/*2*} − *insert* (*1*/*2*) *s*) (λ*x*. *vector-derivative* (*g1 o op*∗*2*) (*at x*))
apply (*rule continuous-on-eq* [*OF - vector-derivative-at*])
apply (*rule-tac f*=*g1 o op*∗*2* **and** *d*=*dist x* (*1*/*2*) **in** *has-vector-derivative-transform-within*)
apply (*simp-all add*: *dist-norm joinpaths-def vector-derivative-works* [*symmetric*])
apply (*force intro*: *g1D differentiable-chain-at*)
apply *auto*
done
have *continuous-on* ({*0..1*} − *insert 1* (*op* ∗ *2* ‘ *s*))
((λ*x*. *1*/*2* ∗ *vector-derivative* (*g1 o op*∗*2*) (*at x*)) *o* *op*∗(*1*/*2*))
apply (*rule continuous-intros*)+
using *coDhalf*
apply (*simp add*: *scaleR-conv-of-real image-set-diff image-image*)
done
then have *con-g1*: *continuous-on* ({*0..1*} − *insert 1* (*op* ∗ *2* ‘ *s*)) (λ*x*. *vector-derivative* *g1* (*at x*))
by (*rule continuous-on-eq*) (*simp add*: *scaleR-conv-of-real*)
have *continuous-on* {*0..1*} *g1*
using *continuous-on-joinpaths-D1 assms piecewise-C1-differentiable-on-def* **by** *blast*
with ‹*finite s*› **show** *?thesis*
apply (*clarsimp simp add*: *piecewise-C1-differentiable-on-def C1-differentiable-on-eq*)

```
    apply (rule-tac x=insert 1 ((op*2)`s) in exI)
    apply (simp add: g1D con-g1)
  done
qed

lemma piecewise-C1-differentiable-D2:
  fixes g2 :: real ⇒ 'a::real-normed-field
  assumes (g1 +++ g2) piecewise-C1-differentiable-on {0..1} pathfinish g1 =
pathstart g2
    shows g2 piecewise-C1-differentiable-on {0..1}
proof −
  obtain s where finite s
            and co12: continuous-on ({0..1} − s) (λx. vector-derivative (g1 +++
g2) (at x))
            and g12D: ∀x∈{0..1} − s. g1 +++ g2 differentiable at x
   using assms by (auto simp: piecewise-C1-differentiable-on-def C1-differentiable-on-eq)
  then have g2D: g2 differentiable at x if x ∈ {0..1} − insert 0 ((λx. 2*x−1) `
s) for x
     apply (rule-tac d=dist ((x+1)/2) (1/2) and f = (g1 +++ g2) o (λx.
(x+1)/2) in differentiable-transform-within)
    using that
    apply (simp-all add: dist-real-def joinpaths-def)
    apply (auto simp: dist-real-def joinpaths-def field-simps)
    apply (rule differentiable-chain-at derivative-intros | force)+
    apply (drule-tac x= (x + 1) / 2 in bspec, force simp: divide-simps)
    apply assumption
    done
   have [simp]: vector-derivative (g2 ∘ (λx. 2*x−1)) (at ((x+1)/2)) = 2 *R
vector-derivative g2 (at x)
             if x ∈ {0..1} − insert 0 ((λx. 2*x−1) ` s) for x
    using that  by (auto simp: vector-derivative-chain-at divide-simps g2D)
  have continuous-on ({1/2..1} − insert (1/2) s) (λx. vector-derivative (g1 +++
g2) (at x))
    using co12 by (rule continuous-on-subset) force
  then have coDhalf: continuous-on ({1/2..1} − insert (1/2) s) (λx. vector-derivative
(g2 o (λx. 2*x−1)) (at x))
    apply (rule continuous-on-eq [OF - vector-derivative-at])
   apply (rule-tac f=g2 o (λx. 2*x−1) and d=dist (3/4) ((x+1)/2) in has-vector-derivative-transform-within
    apply (auto simp: dist-real-def field-simps joinpaths-def vector-derivative-works
[symmetric]
                intro!: g2D differentiable-chain-at)
    done
  have [simp]: ((λx. (x + 1) / 2) ` ({0..1} − insert 0 ((λx. 2 * x − 1) ` s))) =
({1/2..1} − insert (1/2) s)
    apply (simp add: image-set-diff inj-on-def image-image)
    apply (auto simp: image-affinity-atLeastAtMost-div add-divide-distrib)
    done
  have continuous-on ({0..1} − insert 0 ((λx. 2*x−1) ` s))
                     ((λx. 1/2 * vector-derivative (g2 ∘ (λx. 2*x−1)) (at x)) o (λx.
```

(x+1)/2))
by (*rule continuous-intros* | *simp add*: *coDhalf*)+
then have *con-g2*: *continuous-on* ({*0..1*} − *insert 0* ((λ*x*. *2*∗*x*−*1*) ‘ *s*)) (λ*x*. *vector-derivative g2* (*at x*))
by (*rule continuous-on-eq*) (*simp add*: *scaleR-conv-of-real*)
have *continuous-on* {*0..1*} *g2*
using *continuous-on-joinpaths-D2 assms piecewise-C1-differentiable-on-def* **by** *blast*
with ⟨*finite s*⟩ **show** *?thesis*
apply (*clarsimp simp add*: *piecewise-C1-differentiable-on-def C1-differentiable-on-eq*)
apply (*rule-tac x*=*insert 0* ((λ*x*. *2* ∗ *x* − *1*) ‘ *s*) **in** *exI*)
apply (*simp add*: *g2D con-g2*)
done
qed

53.3 Valid paths, and their start and finish

lemma *Diff-Un-eq*: *A* − (*B* ∪ *C*) = *A* − *B* − *C*
by *blast*

definition *valid-path* :: (*real* ⇒ *'a* :: *real-normed-vector*) ⇒ *bool*
where *valid-path f* ≡ *f piecewise-C1-differentiable-on* {*0..1*::*real*}

definition *closed-path* :: (*real* ⇒ *'a* :: *real-normed-vector*) ⇒ *bool*
where *closed-path g* ≡ *g 0* = *g 1*

53.3.1 In particular, all results for paths apply

lemma *valid-path-imp-path*: *valid-path g* ⟹ *path g*
by (*simp add*: *path-def piecewise-C1-differentiable-on-def valid-path-def*)

lemma *connected-valid-path-image*: *valid-path g* ⟹ *connected*(*path-image g*)
by (*metis connected-path-image valid-path-imp-path*)

lemma *compact-valid-path-image*: *valid-path g* ⟹ *compact*(*path-image g*)
by (*metis compact-path-image valid-path-imp-path*)

lemma *bounded-valid-path-image*: *valid-path g* ⟹ *bounded*(*path-image g*)
by (*metis bounded-path-image valid-path-imp-path*)

lemma *closed-valid-path-image*: *valid-path g* ⟹ *closed*(*path-image g*)
by (*metis closed-path-image valid-path-imp-path*)

proposition *valid-path-compose*:
assumes *valid-path g*
and *der*: ⋀*x*. *x* ∈ *path-image g* ⟹ ∃*f'*. (*f has-field-derivative f'*) (*at x*)
and *con*: *continuous-on* (*path-image g*) (*deriv f*)
shows *valid-path* (*f o g*)
proof −
obtain *s* **where** *finite s* **and** *g-diff*: *g C1-differentiable-on* {*0..1*} − *s*

using ‹*valid-path g*› **unfolding** *valid-path-def piecewise-C1-differentiable-on-def*
by *auto*
have *f* ∘ *g differentiable at t* **when** *t*∈{*0..1*} − *s* **for** *t*
proof (*rule differentiable-chain-at*)
show *g differentiable at t* **using** ‹*valid-path g*›
by (*meson C1-differentiable-on-eq* ‹*g C1-differentiable-on* {*0..1*} − *s*› *that*)
next
have *g t*∈*path-image g* **using** *that DiffD1 image-eqI path-image-def* **by** *metis*
then obtain *f′* **where** (*f has-field-derivative f′*) (*at* (*g t*))
using *der* **by** *auto*
then have (*f has-derivative op* ∗ *f′*) (*at* (*g t*))
using *has-field-derivative-imp-has-derivative*[*of f f′ at* (*g t*)] **by** *auto*
then show *f differentiable at* (*g t*) **using** *differentiableI* **by** *auto*
qed
moreover have *continuous-on* ({*0..1*} − *s*) (λ*x*. *vector-derivative* (*f* ∘ *g*) (*at*
x))
proof (*rule continuous-on-eq* [**where** *f* = λ*x*. *vector-derivative g* (*at x*) ∗ *deriv*
f (*g x*)],
rule continuous-intros)
show *continuous-on* ({*0..1*} − *s*) (λ*x*. *vector-derivative g* (*at x*))
using *g-diff C1-differentiable-on-eq* **by** *auto*
next
have *continuous-on* {*0..1*} (λ*x*. *deriv f* (*g x*))
using *continuous-on-compose*[*OF - con*[*unfolded path-image-def*],*unfolded*
comp-def]
‹*valid-path g*› *piecewise-C1-differentiable-on-def valid-path-def*
by *blast*
then show *continuous-on* ({*0..1*} − *s*) (λ*x*. *deriv f* (*g x*))
using *continuous-on-subset* **by** *blast*
next
show *vector-derivative g* (*at t*) ∗ *deriv f* (*g t*) = *vector-derivative* (*f* ∘ *g*) (*at*
t)
when *t* ∈ {*0..1*} − *s* **for** *t*
proof (*rule vector-derivative-chain-at-general*[*symmetric*])
show *g differentiable at t* **by** (*meson C1-differentiable-on-eq g-diff that*)
next
have *g t*∈*path-image g* **using** *that DiffD1 image-eqI path-image-def* **by**
metis
then obtain *f′* **where** (*f has-field-derivative f′*) (*at* (*g t*))
using *der* **by** *auto*
then show ∃ *g′*. (*f has-field-derivative g′*) (*at* (*g t*)) **by** *auto*
qed
qed
ultimately have *f o g C1-differentiable-on* {*0..1*} − *s*
using *C1-differentiable-on-eq* **by** *blast*
moreover have *path* (*f o g*)
proof −
have *isCont f x* **when** *x*∈*path-image g* **for** *x*
proof −

obtain *f'* **where** (*f has-field-derivative f'*) (*at x*)
using *der*[*rule-format*] ‹*x*∈*path-image g*› **by** *auto*
thus *?thesis* **using** *DERIV-isCont* **by** *auto*
qed
then have *continuous-on* (*path-image g*) *f* **using** *continuous-at-imp-continuous-on* **by** *auto*
then show *?thesis* **using** *path-continuous-image* ‹*valid-path g*› *valid-path-imp-path* **by** *auto*
qed
ultimately show *?thesis* **unfolding** *valid-path-def piecewise-C1-differentiable-on-def path-def*
using ‹*finite s*› **by** *auto*
qed

53.4 Contour Integrals along a path

This definition is for complex numbers only, and does not generalise to line integrals in a vector field

piecewise differentiable function on [0,1]

definition *has-contour-integral* :: (*complex* ⇒ *complex*) ⇒ *complex* ⇒ (*real* ⇒ *complex*) ⇒ *bool*
(**infixr** *has'-contour'-integral 50*)
where (*f has-contour-integral i*) *g* ≡
((λ*x*. *f*(*g x*) ∗ *vector-derivative g* (*at x within* {*0..1*}))
has-integral i) {*0..1*}

definition *contour-integrable-on*
(**infixr** *contour'-integrable'-on 50*)
where *f contour-integrable-on g* ≡ ∃ *i*. (*f has-contour-integral i*) *g*

definition *contour-integral*
where *contour-integral g f* ≡ @*i*. (*f has-contour-integral i*) *g* ∨ ~ *f contour-integrable-on g* ∧ *i=0*

lemma *not-integrable-contour-integral*: ~ *f contour-integrable-on g* ⟹ *contour-integral g f* = *0*
unfolding *contour-integrable-on-def contour-integral-def* **by** *blast*

lemma *contour-integral-unique*: (*f has-contour-integral i*) *g* ⟹ *contour-integral g f* = *i*
apply (*simp add*: *contour-integral-def has-contour-integral-def contour-integrable-on-def*)
using *has-integral-unique* **by** *blast*

corollary *has-contour-integral-eqpath*:
⟦(*f has-contour-integral y*) *p*; *f contour-integrable-on* γ;
contour-integral p f = *contour-integral* γ *f*⟧
⟹ (*f has-contour-integral y*) γ
using *contour-integrable-on-def contour-integral-unique* **by** *auto*

lemma *has-contour-integral-integral*:
f contour-integrable-on i $\Longrightarrow$ (*f has-contour-integral* (*contour-integral i f*)) *i*
by (*metis contour-integral-unique contour-integrable-on-def*)

lemma *has-contour-integral-unique*:
(*f has-contour-integral i*) *g* $\Longrightarrow$ (*f has-contour-integral j*) *g* $\Longrightarrow$ *i* = *j*
using *has-integral-unique*
by (*auto simp*: *has-contour-integral-def*)

lemma *has-contour-integral-integrable*: (*f has-contour-integral i*) *g* $\Longrightarrow$ *f contour-integrable-on g*
using *contour-integrable-on-def* **by** *blast*

lemma *vector-derivative-within-interior*:
⟦*x* ∈ *interior s*; *NO-MATCH UNIV s*⟧
$\Longrightarrow$ *vector-derivative f* (*at x within s*) = *vector-derivative f* (*at x*)
apply (*simp add*: *vector-derivative-def has-vector-derivative-def has-derivative-def netlimit-within-interior*)
apply (*subst lim-within-interior*, *auto*)
done

lemma *has-integral-localized-vector-derivative*:
((λ*x*. *f* (*g x*) ∗ *vector-derivative g* (*at x within* {*a*..*b*})) *has-integral i*) {*a*..*b*} ⟷
((λ*x*. *f* (*g x*) ∗ *vector-derivative g* (*at x*)) *has-integral i*) {*a*..*b*}
proof −
have {*a*..*b*} − {*a*,*b*} = *interior* {*a*..*b*}
by (*simp add*: *atLeastAtMost-diff-ends*)
show *?thesis*
apply (*rule has-integral-spike-eq* [*of* {*a*,*b*}])
apply (*auto simp*: *vector-derivative-within-interior*)
done
qed

lemma *integrable-on-localized-vector-derivative*:
(λ*x*. *f* (*g x*) ∗ *vector-derivative g* (*at x within* {*a*..*b*})) *integrable-on* {*a*..*b*} ⟷
(λ*x*. *f* (*g x*) ∗ *vector-derivative g* (*at x*)) *integrable-on* {*a*..*b*}
by (*simp add*: *integrable-on-def has-integral-localized-vector-derivative*)

lemma *has-contour-integral*:
(*f has-contour-integral i*) *g* ⟷
((λ*x*. *f* (*g x*) ∗ *vector-derivative g* (*at x*)) *has-integral i*) {*0*..*1*}
by (*simp add*: *has-integral-localized-vector-derivative has-contour-integral-def*)

lemma *contour-integrable-on*:
f contour-integrable-on g ⟷

$(\lambda t.\ f(g\ t) * \textit{vector-derivative}\ g\ (\textit{at}\ t))\ \textit{integrable-on}\ \{0..1\}$
by (*simp add*: *has-contour-integral integrable-on-def contour-integrable-on-def*)

53.5 Reversing a path

lemma *valid-path-imp-reverse*:
assumes *valid-path g*
shows *valid-path*(*reversepath g*)
proof −
obtain *s* **where** *finite s g C1-differentiable-on* ($\{0..1\} - s$)
using *assms* **by** (*auto simp*: *valid-path-def piecewise-C1-differentiable-on-def*)
then have *finite* (*op* − *1* ‘ *s*) (*reversepath g C1-differentiable-on* ($\{0..1\} -$ *op* − *1* ‘ *s*))
apply (*auto simp*: *reversepath-def*)
apply (*rule C1-differentiable-compose* [*of* λx::*real*. *1*−*x* - *g*, *unfolded o-def*])
apply (*auto simp*: *C1-differentiable-on-eq*)
apply (*rule continuous-intros*, *force*)
apply (*force elim*!: *continuous-on-subset*)
apply (*simp add*: *finite-vimageI inj-on-def*)
done
then show *?thesis* **using** *assms*
by (*auto simp*: *valid-path-def piecewise-C1-differentiable-on-def path-def* [*symmetric*])
qed

lemma *valid-path-reversepath* [*simp*]: *valid-path*(*reversepath g*) $\longleftrightarrow$ *valid-path g*
using *valid-path-imp-reverse* **by** *force*

lemma *has-contour-integral-reversepath*:
assumes *valid-path g* (*f has-contour-integral i*) *g*
shows (*f has-contour-integral* (−*i*)) (*reversepath g*)
proof −
{ fix *s x*
assume *xs*: *g C1-differentiable-on* ($\{0..1\} - s$) $x \notin$ *op* − *1* ‘ *s* $0 \le x\ x \le 1$
have *vector-derivative* ($\lambda x.\ g\ (1 - x)$) (*at x within* $\{0..1\}$) =
− *vector-derivative g* (*at* (*1* − *x*) *within* $\{0..1\}$)
proof −
obtain f' **where** f': (*g has-vector-derivative* f') (*at* (*1* − *x*))
using *xs*
by (*force simp*: *has-vector-derivative-def C1-differentiable-on-def*)
have (*g o* ($\lambda x.\ 1 - x$) *has-vector-derivative* $-1 *_R f'$) (*at x*)
apply (*rule vector-diff-chain-within*)
apply (*intro vector-diff-chain-within derivative-eq-intros* | *simp*)+
apply (*rule has-vector-derivative-at-within* [*OF* f'])
done
then have mf': (($\lambda x.\ g\ (1 - x)$) *has-vector-derivative* $-f'$) (*at x*)
by (*simp add*: *o-def*)
show *?thesis*
using *xs*
by (*auto simp*: *vector-derivative-at-within-ivl* [*OF* mf'] *vector-derivative-at-within-ivl*

[*OF f'*])
qed
} note $* = this$
have *01*: $\{0..1::real\} = cbox\ 0\ 1$
by *simp*
show *?thesis* **using** *assms*
apply (*auto simp*: *has-contour-integral-def*)
apply (*drule has-integral-affinity01* [**where** $m= -1$ **and** $c=1$])
apply (*auto simp*: *reversepath-def valid-path-def piecewise-C1-differentiable-on-def*)
apply (*drule has-integral-neg*)
apply (*rule-tac* $s = (\lambda x.\ 1 - x)$ ' *s* **in** *has-integral-spike-finite*)
apply (*auto simp*: $*$)
done
qed

lemma *contour-integrable-reversepath*:
valid-path g $\Longrightarrow$ *f contour-integrable-on g* $\Longrightarrow$ *f contour-integrable-on* (*reversepath g*)
using *has-contour-integral-reversepath contour-integrable-on-def* **by** *blast*

lemma *contour-integrable-reversepath-eq*:
valid-path g $\Longrightarrow$ (*f contour-integrable-on* (*reversepath g*) $\longleftrightarrow$ *f contour-integrable-on g*)
using *contour-integrable-reversepath valid-path-reversepath* **by** *fastforce*

lemma *contour-integral-reversepath*:
assumes *valid-path g*
shows *contour-integral* (*reversepath g*) *f* $= -$ (*contour-integral g f*)
proof (*cases f contour-integrable-on g*)
case *True* **then show** *?thesis*
by (*simp add*: *assms contour-integral-unique has-contour-integral-integral has-contour-integral-reversepath*)
next
case *False* **then have** $\sim$ *f contour-integrable-on* (*reversepath g*)
by (*simp add*: *assms contour-integrable-reversepath-eq*)
with *False* **show** *?thesis* **by** (*simp add*: *not-integrable-contour-integral*)
qed

53.6 Joining two paths together

lemma *valid-path-join*:
assumes *valid-path g1 valid-path g2 pathfinish g1* $=$ *pathstart g2*
shows *valid-path*(*g1* $+\!\!+\!\!+$ *g2*)
proof $-$
have *g1 1* $=$ *g2 0*
using *assms* **by** (*auto simp*: *pathfinish-def pathstart-def*)
moreover have (*g1 o* $(\lambda x.\ 2*x)$) *piecewise-C1-differentiable-on* $\{0..1/2\}$
apply (*rule piecewise-C1-differentiable-compose*)
using *assms*
apply (*auto simp*: *valid-path-def piecewise-C1-differentiable-on-def continuous-on-joinpaths*)

apply (*rule continuous-intros* | *simp*)+
apply (*force intro*: *finite-vimageI* [**where** *h* = *op∗2*] *inj-onI*)
done
moreover have (*g2 o* (λ*x*. *2∗x−1*)) *piecewise-C1-differentiable-on* {*1/2..1*}
apply (*rule piecewise-C1-differentiable-compose*)
using *assms* **unfolding** *valid-path-def piecewise-C1-differentiable-on-def*
by (*auto intro*!: *continuous-intros finite-vimageI* [**where** *h* = (λ*x*. *2∗x − 1*)] *inj-onI*
simp: *image-affinity-atLeastAtMost-diff continuous-on-joinpaths*)
ultimately show *?thesis*
apply (*simp only*: *valid-path-def continuous-on-joinpaths joinpaths-def*)
apply (*rule piecewise-C1-differentiable-cases*)
apply (*auto simp*: *o-def*)
done
qed

lemma *valid-path-join-D1*:
fixes *g1* :: *real* ⇒ *'a*::*real-normed-field*
shows *valid-path* (*g1 +++ g2*) ⟹ *valid-path g1*
unfolding *valid-path-def*
by (*rule piecewise-C1-differentiable-D1*)

lemma *valid-path-join-D2*:
fixes *g2* :: *real* ⇒ *'a*::*real-normed-field*
shows ⟦*valid-path* (*g1 +++ g2*); *pathfinish g1* = *pathstart g2*⟧ ⟹ *valid-path g2*
unfolding *valid-path-def*
by (*rule piecewise-C1-differentiable-D2*)

lemma *valid-path-join-eq* [*simp*]:
fixes *g2* :: *real* ⇒ *'a*::*real-normed-field*
shows *pathfinish g1* = *pathstart g2* ⟹ (*valid-path*(*g1 +++ g2*) ⟷ *valid-path g1* ∧ *valid-path g2*)
using *valid-path-join-D1 valid-path-join-D2 valid-path-join* **by** *blast*

lemma *has-contour-integral-join*:
assumes (*f has-contour-integral i1*) *g1* (*f has-contour-integral i2*) *g2*
valid-path g1 valid-path g2
shows (*f has-contour-integral* (*i1* + *i2*)) (*g1 +++ g2*)
proof −
obtain *s1 s2*
where *s1*: *finite s1* ∀ *x*∈{*0..1*} − *s1*. *g1 differentiable at x*
and *s2*: *finite s2* ∀ *x*∈{*0..1*} − *s2*. *g2 differentiable at x*
using *assms*
by (*auto simp*: *valid-path-def piecewise-C1-differentiable-on-def C1-differentiable-on-eq*)
have *1*: ((λ*x*. *f* (*g1 x*) ∗ *vector-derivative g1* (*at x*)) *has-integral i1*) {*0..1*}
and *2*: ((λ*x*. *f* (*g2 x*) ∗ *vector-derivative g2* (*at x*)) *has-integral i2*) {*0..1*}
using *assms*
by (*auto simp*: *has-contour-integral*)
have *i1*: ((λ*x*. (*2∗f* (*g1* (*2∗x*))) ∗ *vector-derivative g1* (*at* (*2∗x*))) *has-integral*

i1) {*0..1/2*}
and *i2*: ((λ*x*. (*2*∗*f* (*g2* (*2*∗*x* − *1*))) ∗ *vector-derivative g2* (*at* (*2*∗*x* − *1*))) *has-integral i2*) {*1/2..1*}
using *has-integral-affinity01* [*OF 1*, **where** *m*= *2* **and** *c*=*0*, *THEN has-integral-cmul* [**where** *c*=*2*]]
has-integral-affinity01 [*OF 2*, **where** *m*= *2* **and** *c*=−*1*, *THEN has-integral-cmul* [**where** *c*=*2*]]
by (*simp-all only*: *image-affinity-atLeastAtMost-div-diff*, *simp-all add*: *scaleR-conv-of-real mult-ac*)
have *g1*: ⟦*0* ≤ *z*; *z*∗*2* < *1*; *z*∗*2* ∉ *s1*⟧ ⟹
vector-derivative (λ*x*. *if x*∗*2* ≤ *1 then g1* (*2*∗*x*) *else g2* (*2*∗*x* − *1*)) (*at z*) =
2 ∗*R vector-derivative g1* (*at* (*z*∗*2*)) **for** *z*
apply (*rule vector-derivative-at* [*OF has-vector-derivative-transform-within* [**where** *f* = (λ*x*. *g1*(*2*∗*x*)) **and** *d* = |*z* − *1/2*|]])
apply (*simp-all add*: *dist-real-def abs-if split*: *if-split-asm*)
apply (*rule vector-diff-chain-at* [*of* λ*x*. *2*∗*x 2* - *g1*, *simplified o-def*])
apply (*simp add*: *has-vector-derivative-def has-derivative-def bounded-linear-mult-left*)
using *s1*
apply (*auto simp*: *algebra-simps vector-derivative-works*)
done
have *g2*: ⟦*1* < *z*∗*2*; *z* ≤ *1*; *z*∗*2* − *1* ∉ *s2*⟧ ⟹
vector-derivative (λ*x*. *if x*∗*2* ≤ *1 then g1* (*2*∗*x*) *else g2* (*2*∗*x* − *1*)) (*at z*) =
2 ∗*R vector-derivative g2* (*at* (*z*∗*2* − *1*)) **for** *z*
apply (*rule vector-derivative-at* [*OF has-vector-derivative-transform-within* [**where** *f* = (λ*x*. *g2* (*2*∗*x* − *1*)) **and** *d* = |*z* − *1/2*|]])
apply (*simp-all add*: *dist-real-def abs-if split*: *if-split-asm*)
apply (*rule vector-diff-chain-at* [*of* λ*x*. *2*∗*x* − *1 2* - *g2*, *simplified o-def*])
apply (*simp add*: *has-vector-derivative-def has-derivative-def bounded-linear-mult-left*)
using *s2*
apply (*auto simp*: *algebra-simps vector-derivative-works*)
done
have ((λ*x*. *f* ((*g1* +++ *g2*) *x*) ∗ *vector-derivative* (*g1* +++ *g2*) (*at x*)) *has-integral i1*) {*0..1/2*}
apply (*rule has-integral-spike-finite* [*OF - - i1*, *of insert* (*1/2*) (*op*∗*2* −‘ *s1*)])
using *s1*
apply (*force intro*: *finite-vimageI* [**where** *h* = *op*∗*2*] *inj-onI*)
apply (*clarsimp simp add*: *joinpaths-def scaleR-conv-of-real mult-ac g1*)
done
moreover have ((λ*x*. *f* ((*g1* +++ *g2*) *x*) ∗ *vector-derivative* (*g1* +++ *g2*) (*at x*)) *has-integral i2*) {*1/2..1*}
apply (*rule has-integral-spike-finite* [*OF - - i2*, *of insert* (*1/2*) ((λ*x*. *2*∗*x*−*1*) −‘ *s2*)])
using *s2*
apply (*force intro*: *finite-vimageI* [**where** *h* = λ*x*. *2*∗*x*−*1*] *inj-onI*)
apply (*clarsimp simp add*: *joinpaths-def scaleR-conv-of-real mult-ac g2*)
done
ultimately

show *?thesis*
apply (*simp add*: *has-contour-integral*)
apply (*rule has-integral-combine* [**where** *c* = *1/2*], *auto*)
done
qed

lemma *contour-integrable-joinI*:
assumes *f contour-integrable-on g1 f contour-integrable-on g2*
valid-path g1 valid-path g2
shows *f contour-integrable-on* (*g1 +++ g2*)
using *assms*
by (*meson has-contour-integral-join contour-integrable-on-def*)

lemma *contour-integrable-joinD1*:
assumes *f contour-integrable-on* (*g1 +++ g2*) *valid-path g1*
shows *f contour-integrable-on g1*
proof −
obtain *s1*
where *s1*: *finite s1* ∀ *x*∈{*0..1*} − *s1*. *g1 differentiable at x*
using *assms* **by** (*auto simp*: *valid-path-def piecewise-C1-differentiable-on-def C1-differentiable-on-eq*)
have (λ*x*. *f* ((*g1 +++ g2*) (*x/2*)) ∗ *vector-derivative* (*g1 +++ g2*) (*at* (*x/2*))) *integrable-on* {*0..1*}
using *assms*
apply (*auto simp*: *contour-integrable-on*)
apply (*drule integrable-on-subcbox* [**where** *a=0* **and** *b=1/2*])
apply (*auto intro*: *integrable-affinity* [*of - 0 1/2*::*real 1/2 0*, *simplified*])
done
then have ∗: (λ*x*. (*f* ((*g1 +++ g2*) (*x/2*))/*2*) ∗ *vector-derivative* (*g1 +++ g2*) (*at* (*x/2*))) *integrable-on* {*0..1*}
by (*auto dest*: *integrable-cmul* [**where** *c=1/2*] *simp*: *scaleR-conv-of-real*)
have *g1*: ⟦*0* < *z*; *z* < *1*; *z* ∉ *s1*⟧ ⟹
vector-derivative (λ*x*. *if x*∗*2* ≤ *1 then g1* (*2*∗*x*) *else g2* (*2*∗*x* − *1*)) (*at* (*z/2*)) =
2 ∗$_R$ *vector-derivative g1* (*at z*) **for** *z*
apply (*rule vector-derivative-at* [*OF has-vector-derivative-transform-within* [**where** *f* = (λ*x*. *g1*(*2*∗*x*)) **and** *d* = |(*z*−*1*)/*2*|]])
apply (*simp-all add*: *field-simps dist-real-def abs-if split*: *if-split-asm*)
apply (*rule vector-diff-chain-at* [*of* λ*x*. *x*∗*2 2 - g1*, *simplified o-def*])
using *s1*
apply (*auto simp*: *vector-derivative-works has-vector-derivative-def has-derivative-def bounded-linear-mult-left*)
done
show *?thesis*
using *s1*
apply (*auto simp*: *contour-integrable-on*)
apply (*rule integrable-spike-finite* [*of* {*0*,*1*} ∪ *s1*, *OF - - ∗*])
apply (*auto simp*: *joinpaths-def scaleR-conv-of-real g1*)
done

qed

lemma *contour-integrable-joinD2*:
assumes *f contour-integrable-on* (*g1 +++ g2*) *valid-path g2*
shows *f contour-integrable-on g2*
proof −
obtain *s2*
where *s2*: *finite s2* ∀ *x*∈{*0..1*} − *s2*. *g2 differentiable at x*
using *assms* **by** (*auto simp*: *valid-path-def piecewise-C1-differentiable-on-def C1-differentiable-on-eq*)
have (λ*x*. *f* ((*g1 +++ g2*) (*x*/*2* + *1*/*2*)) ∗ *vector-derivative* (*g1 +++ g2*) (*at* (*x*/*2* + *1*/*2*))) *integrable-on* {*0..1*}
using *assms*
apply (*auto simp*: *contour-integrable-on*)
apply (*drule integrable-on-subcbox* [**where** *a*=*1*/*2* **and** *b*=*1*], *auto*)
apply (*drule integrable-affinity* [*of* - *1*/*2*::*real 1 1*/*2 1*/*2*, *simplified*])
apply (*simp add*: *image-affinity-atLeastAtMost-diff*)
done
then have ∗: (λ*x*. (*f* ((*g1 +++ g2*) (*x*/*2* + *1*/*2*))/*2*) ∗ *vector-derivative* (*g1 +++ g2*) (*at* (*x*/*2* + *1*/*2*)))
integrable-on {*0..1*}
by (*auto dest*: *integrable-cmul* [**where** *c*=*1*/*2*] *simp*: *scaleR-conv-of-real*)
have *g2*: ⟦*0* < *z*; *z* < *1*; *z* ∉ *s2*⟧ ⟹
vector-derivative (λ*x*. *if x*∗*2* ≤ *1 then g1* (*2*∗*x*) *else g2* (*2*∗*x* − *1*)) (*at* (*z*/*2*+*1*/*2*)) =
2 ∗*R* *vector-derivative g2* (*at z*) **for** *z*
apply (*rule vector-derivative-at* [*OF has-vector-derivative-transform-within* [**where** *f* = (λ*x*. *g2*(*2*∗*x*−*1*)) **and** *d* = |*z*/*2*|]])
apply (*simp-all add*: *field-simps dist-real-def abs-if split*: *if-split-asm*)
apply (*rule vector-diff-chain-at* [*of* λ*x*. *x*∗*2*−*1 2* - *g2*, *simplified o-def*])
using *s2*
apply (*auto simp*: *has-vector-derivative-def has-derivative-def bounded-linear-mult-left vector-derivative-works add-divide-distrib*)
done
show *?thesis*
using *s2*
apply (*auto simp*: *contour-integrable-on*)
apply (*rule integrable-spike-finite* [*of* {*0*,*1*} ∪ *s2*, *OF* - - ∗])
apply (*auto simp*: *joinpaths-def scaleR-conv-of-real g2*)
done
qed

lemma *contour-integrable-join* [*simp*]:
shows
⟦*valid-path g1*; *valid-path g2*⟧
⟹ *f contour-integrable-on* (*g1 +++ g2*) ⟷ *f contour-integrable-on g1* ∧ *f contour-integrable-on g2*
using *contour-integrable-joinD1 contour-integrable-joinD2 contour-integrable-joinI*
by *blast*

lemma *contour-integral-join* [*simp*]:
shows
⟦*f contour-integrable-on g1*; *f contour-integrable-on g2*; *valid-path g1*; *valid-path g2*⟧
⟹ *contour-integral* (*g1* +++ *g2*) *f* = *contour-integral g1 f* + *contour-integral g2 f*
by (*simp add*: *has-contour-integral-integral has-contour-integral-join contour-integral-unique*)

53.7 Shifting the starting point of a (closed) path

lemma *shiftpath-alt-def*: *shiftpath a f* = (λ*x*. *if x* ≤ *1*−*a then f* (*a* + *x*) *else f* (*a* + *x* − *1*))
by (*auto simp*: *shiftpath-def*)

lemma *valid-path-shiftpath* [*intro*]:
assumes *valid-path g pathfinish g* = *pathstart g a* ∈ {*0..1*}
shows *valid-path*(*shiftpath a g*)
using *assms*
apply (*auto simp*: *valid-path-def shiftpath-alt-def*)
apply (*rule piecewise-C1-differentiable-cases*)
apply (*auto simp*: *algebra-simps*)
apply (*rule piecewise-C1-differentiable-affine* [*of g 1 a*, *simplified o-def scaleR-one*])
apply (*auto simp*: *pathfinish-def pathstart-def elim*: *piecewise-C1-differentiable-on-subset*)
apply (*rule piecewise-C1-differentiable-affine* [*of g 1 a*−*1*, *simplified o-def scaleR-one algebra-simps*])
apply (*auto simp*: *pathfinish-def pathstart-def elim*: *piecewise-C1-differentiable-on-subset*)
done

lemma *has-contour-integral-shiftpath*:
assumes *f*: (*f has-contour-integral i*) *g valid-path g*
and *a*: *a* ∈ {*0..1*}
shows (*f has-contour-integral i*) (*shiftpath a g*)
proof −
obtain *s*
where *s*: *finite s* **and** *g*: ∀*x*∈{*0..1*} − *s*. *g differentiable at x*
using *assms* **by** (*auto simp*: *valid-path-def piecewise-C1-differentiable-on-def C1-differentiable-on-eq*)
have ∗: ((λ*x*. *f* (*g x*) ∗ *vector-derivative g* (*at x*)) *has-integral i*) {*0..1*}
using *assms* **by** (*auto simp*: *has-contour-integral*)
then have *i*: *i* = *integral* {*a..1*} (λ*x*. *f* (*g x*) ∗ *vector-derivative g* (*at x*)) +
integral {*0..a*} (λ*x*. *f* (*g x*) ∗ *vector-derivative g* (*at x*))
apply (*rule has-integral-unique*)
apply (*subst add.commute*)
apply (*subst Integration.integral-combine*)
using *assms* ∗ *integral-unique* **by** *auto*
{ fix *x*
have *0* ≤ *x* ⟹ *x* + *a* < *1* ⟹ *x* ∉ (λ*x*. *x* − *a*) ‘ *s* ⟹
vector-derivative (*shiftpath a g*) (*at x*) = *vector-derivative g* (*at* (*x* + *a*))

unfolding *shiftpath-def*
apply (*rule vector-derivative-at* [*OF has-vector-derivative-transform-within* [**where** $f = (\lambda x.\ g(a+x))$ **and** $d = dist(1-a)\ x$]])
apply (*auto simp*: *field-simps dist-real-def abs-if split*: *if-split-asm*)
apply (*rule vector-diff-chain-at* [*of* $\lambda x.\ x+a$ *1 - g*, *simplified o-def scaleR-one*])
apply (*intro derivative-eq-intros* | *simp*)+
using *g*
apply (*drule-tac* $x=x+a$ **in** *bspec*)
using *a* **apply** (*auto simp*: *has-vector-derivative-def vector-derivative-works image-def add.commute*)
done
} note *vd1* = *this*
{ fix *x*
have $1 < x + a \Longrightarrow x \leq 1 \Longrightarrow x \notin (\lambda x.\ x - a + 1)\ `\ s \Longrightarrow$
vector-derivative (*shiftpath a g*) (*at x*) = *vector-derivative g* (*at* $(x + a - 1)$)
unfolding *shiftpath-def*
apply (*rule vector-derivative-at* [*OF has-vector-derivative-transform-within* [**where** $f = (\lambda x.\ g(a+x-1))$ **and** $d = dist\ (1-a)\ x$]])
apply (*auto simp*: *field-simps dist-real-def abs-if split*: *if-split-asm*)
apply (*rule vector-diff-chain-at* [*of* $\lambda x.\ x+a-1$ *1 - g*, *simplified o-def scaleR-one*])
apply (*intro derivative-eq-intros* | *simp*)+
using *g*
apply (*drule-tac* $x=x+a-1$ **in** *bspec*)
using *a* **apply** (*auto simp*: *has-vector-derivative-def vector-derivative-works image-def add.commute*)
done
} note *vd2* = *this*
have *va1*: $(\lambda x.\ f\ (g\ x) * vector\text{-}derivative\ g\ (at\ x))$ *integrable-on* $(\{a..1\})$
using $* a$ **by** (*fastforce intro*: *integrable-subinterval-real*)
have *v0a*: $(\lambda x.\ f\ (g\ x) * vector\text{-}derivative\ g\ (at\ x))$ *integrable-on* $(\{0..a\})$
apply (*rule integrable-subinterval-real*)
using $* a$ **by** *auto*
have $((\lambda x.\ f\ (shiftpath\ a\ g\ x) * vector\text{-}derivative\ (shiftpath\ a\ g)\ (at\ x))$
has-integral integral $\{a..1\}$ $(\lambda x.\ f\ (g\ x) * vector\text{-}derivative\ g\ (at\ x)))$ $\{0..1 - a\}$
apply (*rule has-integral-spike-finite*
[**where** $s = \{1-a\} \cup (\lambda x.\ x-a)\ `\ s$ **and** $f = \lambda x.\ f(g(a+x)) * vector\text{-}derivative\ g\ (at(a+x))$])
using *s* **apply** *blast*
using *a* **apply** (*auto simp*: *algebra-simps vd1*)
apply (*force simp*: *shiftpath-def add.commute*)
using *has-integral-affinity* [**where** *m=1* **and** *c=a*, *simplified*, *OF integrable-integral* [*OF va1*]]
apply (*simp add*: *image-affinity-atLeastAtMost-diff* [**where** *m=1* **and** *c=a*, *simplified*] *add.commute*)
done
moreover
have $((\lambda x.\ f\ (shiftpath\ a\ g\ x) * vector\text{-}derivative\ (shiftpath\ a\ g)\ (at\ x))$

```
      has-integral  integral {0..a} (λx. f (g x) * vector-derivative g (at x)))  {1
− a..1}
   apply (rule has-integral-spike-finite
          [where s = {1−a} ∪ (λx. x−a+1) ' s and f = λx. f(g(a+x−1)) *
vector-derivative g (at(a+x−1))])
    using s apply blast
   using a apply (auto simp: algebra-simps vd2)
   apply (force simp: shiftpath-def add.commute)
  using has-integral-affinity [where m=1 and c=a−1, simplified, OF integrable-integral
[OF v0a]]
    apply (simp add: image-affinity-atLeastAtMost [where m=1 and c=1−a,
simplified])
   apply (simp add: algebra-simps)
   done
 ultimately show ?thesis
   using a
   by (auto simp: i has-contour-integral intro: has-integral-combine [where c =
1−a])
qed

lemma has-contour-integral-shiftpath-D:
 assumes (f has-contour-integral i) (shiftpath a g)
       valid-path g pathfinish g = pathstart g a ∈ {0..1}
   shows (f has-contour-integral i) g
proof −
 obtain s
   where s: finite s and g: ∀x∈{0..1} − s. g differentiable at x
    using assms by (auto simp: valid-path-def piecewise-C1-differentiable-on-def
C1-differentiable-on-eq)
 { fix x
   assume x: 0 < x x < 1 x ∉ s
   then have gx: g differentiable at x
     using g by auto
   have vector-derivative g (at x within {0..1}) =
        vector-derivative (shiftpath (1 − a) (shiftpath a g)) (at x within {0..1})
     apply (rule vector-derivative-at-within-ivl
              [OF has-vector-derivative-transform-within-open
                    [where f = (shiftpath (1 − a) (shiftpath a g)) and s =
{0<..<1}−s]])
     using s g assms x
     apply (auto simp: finite-imp-closed open-Diff shiftpath-shiftpath
                       vector-derivative-within-interior vector-derivative-works
[symmetric])
     apply (rule differentiable-transform-within [OF gx, of min x (1−x)])
     apply (auto simp: dist-real-def shiftpath-shiftpath abs-if split: if-split-asm)
     done
 } note vd = this
 have fi: (f has-contour-integral i) (shiftpath (1 − a) (shiftpath a g))
   using assms  by (auto intro!: has-contour-integral-shiftpath)
```

show *?thesis*
apply (*simp add*: *has-contour-integral-def*)
apply (*rule has-integral-spike-finite* [*of* {*0*,*1*} ∪ *s*, *OF - - fi* [*unfolded has-contour-integral-def*]])
using *s assms vd*
apply (*auto simp*: *Path-Connected.shiftpath-shiftpath*)
done
qed

lemma *has-contour-integral-shiftpath-eq*:
assumes *valid-path g pathfinish g* = *pathstart g a* ∈ {*0..1*}
shows (*f has-contour-integral i*) (*shiftpath a g*) ⟷ (*f has-contour-integral i*) *g*
using *assms has-contour-integral-shiftpath has-contour-integral-shiftpath-D* **by** *blast*

lemma *contour-integrable-on-shiftpath-eq*:
assumes *valid-path g pathfinish g* = *pathstart g a* ∈ {*0..1*}
shows *f contour-integrable-on* (*shiftpath a g*) ⟷ *f contour-integrable-on g*
using *assms contour-integrable-on-def has-contour-integral-shiftpath-eq* **by** *auto*

lemma *contour-integral-shiftpath*:
assumes *valid-path g pathfinish g* = *pathstart g a* ∈ {*0..1*}
shows *contour-integral* (*shiftpath a g*) *f* = *contour-integral g f*
using *assms*
by (*simp add*: *contour-integral-def contour-integrable-on-def has-contour-integral-shiftpath-eq*)

53.8 More about straight-line paths

lemma *has-vector-derivative-linepath-within*:
(*linepath a b has-vector-derivative* (*b* − *a*)) (*at x within s*)
apply (*simp add*: *linepath-def has-vector-derivative-def algebra-simps*)
apply (*rule derivative-eq-intros* | *simp*)+
done

lemma *vector-derivative-linepath-within*:
x ∈ {*0..1*} ⟹ *vector-derivative* (*linepath a b*) (*at x within* {*0..1*}) = *b* − *a*
apply (*rule vector-derivative-within-closed-interval* [*of 0 1*::*real*, *simplified*])
apply (*auto simp*: *has-vector-derivative-linepath-within*)
done

lemma *vector-derivative-linepath-at* [*simp*]: *vector-derivative* (*linepath a b*) (*at x*)
= *b* − *a*
by (*simp add*: *has-vector-derivative-linepath-within vector-derivative-at*)

lemma *valid-path-linepath* [*iff*]: *valid-path* (*linepath a b*)
apply (*simp add*: *valid-path-def piecewise-C1-differentiable-on-def C1-differentiable-on-eq continuous-on-linepath*)
apply (*rule-tac x*={} **in** *exI*)
apply (*simp add*: *differentiable-on-def differentiable-def*)
using *has-vector-derivative-def has-vector-derivative-linepath-within*

apply (*fastforce simp add*: *continuous-on-eq-continuous-within*)
done

lemma *has-contour-integral-linepath*:
shows (*f has-contour-integral i*) (*linepath a b*) ⟷
((λ*x*. *f*(*linepath a b x*) ∗ (*b* − *a*)) *has-integral i*) {*0..1*}
by (*simp add*: *has-contour-integral vector-derivative-linepath-at*)

lemma *linepath-in-path*:
shows *x* ∈ {*0..1*} ⟹ *linepath a b x* ∈ *closed-segment a b*
by (*auto simp*: *segment linepath-def*)

lemma *linepath-image-01*: *linepath a b* ‘ {*0..1*} = *closed-segment a b*
by (*auto simp*: *segment linepath-def*)

lemma *linepath-in-convex-hull*:
fixes *x*::*real*
assumes *a*: *a* ∈ *convex hull s*
and *b*: *b* ∈ *convex hull s*
and *x*: *0*≤*x x*≤*1*
shows *linepath a b x* ∈ *convex hull s*
apply (*rule closed-segment-subset-convex-hull* [*OF a b*, *THEN subsetD*])
using *x*
apply (*auto simp*: *linepath-image-01* [*symmetric*])
done

lemma *Re-linepath*: *Re*(*linepath* (*of-real a*) (*of-real b*) *x*) = (*1* − *x*)∗*a* + *x*∗*b*
by (*simp add*: *linepath-def*)

lemma *Im-linepath*: *Im*(*linepath* (*of-real a*) (*of-real b*) *x*) = *0*
by (*simp add*: *linepath-def*)

lemma *has-contour-integral-trivial* [*iff*]: (*f has-contour-integral 0*) (*linepath a a*)
by (*simp add*: *has-contour-integral-linepath*)

lemma *contour-integral-trivial* [*simp*]: *contour-integral* (*linepath a a*) *f* = *0*
using *has-contour-integral-trivial contour-integral-unique* **by** *blast*

53.9 Relation to subpath construction

lemma *valid-path-subpath*:
fixes *g* :: *real* ⇒ *'a* :: *real-normed-vector*
assumes *valid-path g u* ∈ {*0..1*} *v* ∈ {*0..1*}
shows *valid-path*(*subpath u v g*)
proof (*cases v*=*u*)
case *True*
then show *?thesis*
unfolding *valid-path-def subpath-def*
by (*force intro*: *C1-differentiable-on-const C1-differentiable-imp-piecewise*)

next
case *False*
have (*g o* (λ*x*. ((*v*−*u*) ∗ *x* + *u*))) *piecewise-C1-differentiable-on* {*0..1*}
apply (*rule piecewise-C1-differentiable-compose*)
apply (*simp add*: *C1-differentiable-imp-piecewise*)
apply (*simp add*: *image-affinity-atLeastAtMost*)
using *assms False*
apply (*auto simp*: *algebra-simps valid-path-def piecewise-C1-differentiable-on-subset*)
apply (*subst Int-commute*)
apply (*auto simp*: *inj-on-def algebra-simps crossproduct-eq finite-vimage-IntI*)
done
then show *?thesis*
by (*auto simp*: *o-def valid-path-def subpath-def*)
qed

lemma *has-contour-integral-subpath-refl* [*iff*]: (*f has-contour-integral 0*) (*subpath u u g*)
by (*simp add*: *has-contour-integral subpath-def*)

lemma *contour-integrable-subpath-refl* [*iff*]: *f contour-integrable-on* (*subpath u u g*)
using *has-contour-integral-subpath-refl contour-integrable-on-def* **by** *blast*

lemma *contour-integral-subpath-refl* [*simp*]: *contour-integral* (*subpath u u g*) *f* = *0*
by (*simp add*: *has-contour-integral-subpath-refl contour-integral-unique*)

lemma *has-contour-integral-subpath*:
assumes *f*: *f contour-integrable-on g* **and** *g*: *valid-path g*
and *uv*: *u* ∈ {*0..1*} *v* ∈ {*0..1*} *u* ≤ *v*
shows (*f has-contour-integral integral* {*u..v*} (λ*x*. *f*(*g x*) ∗ *vector-derivative g* (*at x*)))
(*subpath u v g*)
proof (*cases v*=*u*)
case *True*
then show *?thesis*
using *f* **by** (*simp add*: *contour-integrable-on-def subpath-def has-contour-integral*)
next
case *False*
obtain *s* **where** *s*: ⋀*x*. *x* ∈ {*0..1*} − *s* ⟹ *g differentiable at x* **and** *fs*: *finite s*
using *g* **unfolding** *piecewise-C1-differentiable-on-def C1-differentiable-on-eq valid-path-def* **by** *blast*
have ∗: ((λ*x*. *f* (*g* ((*v* − *u*) ∗ *x* + *u*)) ∗ *vector-derivative g* (*at* ((*v* − *u*) ∗ *x* + *u*)))
has-integral (*1* / (*v* − *u*)) ∗ *integral* {*u..v*} (λ*t*. *f* (*g t*) ∗ *vector-derivative g* (*at t*)))
{*0..1*}
using *f uv*
apply (*simp add*: *contour-integrable-on subpath-def has-contour-integral*)

apply (*drule integrable-on-subcbox* [**where** *a*=*u* **and** *b*=*v*, *simplified*])
apply (*simp-all add*: *has-integral-integral*)
apply (*drule has-integral-affinity* [**where** *m*=*v*−*u* **and** *c*=*u*, *simplified*])
apply (*simp-all add*: *False image-affinity-atLeastAtMost-div-diff scaleR-conv-of-real*)
apply (*simp add*: *divide-simps False*)
done
{ fix *x*
have $x \in \{0..1\} \Longrightarrow$
$x \notin (\lambda t.\ (v-u) *_R\ t + u) -` s \Longrightarrow$
vector-derivative $(\lambda x.\ g\ ((v-u) * x + u))\ (at\ x) = (v-u) *_R$ *vector-derivative* $g\ (at\ ((v-u) * x + u))$
apply (*rule vector-derivative-at* [*OF vector-diff-chain-at* [*simplified o-def*]])
apply (*intro derivative-eq-intros* | *simp*)+
apply (*cut-tac s* [*of* $(v - u) * x + u$])
using *uv mult-left-le* [*of x v*−*u*]
apply (*auto simp*: *vector-derivative-works*)
done
} note *vd* = *this*
show *?thesis*
apply (*cut-tac has-integral-cmul* [*OF* ∗, **where** *c* = *v*−*u*])
using *fs assms*
apply (*simp add*: *False subpath-def has-contour-integral*)
apply (*rule-tac s* = $(\lambda t.\ ((v-u) *_R\ t + u)) -` s$ **in** *has-integral-spike-finite*)
apply (*auto simp*: *inj-on-def False finite-vimageI vd scaleR-conv-of-real*)
done
qed

lemma *contour-integrable-subpath*:
assumes *f contour-integrable-on g valid-path g u* ∈ {*0..1*} *v* ∈ {*0..1*}
shows *f contour-integrable-on* (*subpath u v g*)
apply (*cases u v rule*: *linorder-class.le-cases*)
apply (*metis contour-integrable-on-def has-contour-integral-subpath* [*OF assms*])
apply (*subst reversepath-subpath* [*symmetric*])
apply (*rule contour-integrable-reversepath*)
using *assms* **apply** (*blast intro*: *valid-path-subpath*)
apply (*simp add*: *contour-integrable-on-def*)
using *assms* **apply** (*blast intro*: *has-contour-integral-subpath*)
done

lemma *has-integral-integrable-integral*: (*f has-integral i*) *s* ⟷ *f integrable-on s* ∧ *integral s f* = *i*
by *blast*

lemma *has-integral-contour-integral-subpath*:
assumes *f contour-integrable-on g valid-path g u* ∈ {*0..1*} *v* ∈ {*0..1*} *u* ≤ *v*
shows (((λ*x*. *f*(*g x*) ∗ *vector-derivative g* (*at x*)))
has-integral contour-integral (*subpath u v g*) *f*) {*u..v*}
using *assms*
apply (*auto simp*: *has-integral-integrable-integral*)

apply (*rule integrable-on-subcbox* [**where** *a*=*u* **and** *b*=*v* **and** *s* = {*0..1*}, *simplified*])
apply (*auto simp*: *contour-integral-unique* [*OF has-contour-integral-subpath*] *contour-integrable-on*)
done

lemma *contour-integral-subcontour-integral*:
assumes *f contour-integrable-on g valid-path g u* ∈ {*0..1*} *v* ∈ {*0..1*} *u* ≤ *v*
shows *contour-integral* (*subpath u v g*) *f* =
integral {*u..v*} (λ*x*. *f*(*g x*) ∗ *vector-derivative g* (*at x*))
using *assms has-contour-integral-subpath contour-integral-unique* **by** *blast*

lemma *contour-integral-subpath-combine-less*:
assumes *f contour-integrable-on g valid-path g u* ∈ {*0..1*} *v* ∈ {*0..1*} *w* ∈ {*0..1*}
u<*v v*<*w*
shows *contour-integral* (*subpath u v g*) *f* + *contour-integral* (*subpath v w g*) *f* =
contour-integral (*subpath u w g*) *f*
using *assms* **apply** (*auto simp*: *contour-integral-subcontour-integral*)
apply (*rule integral-combine*, *auto*)
apply (*rule integrable-on-subcbox* [**where** *a*=*u* **and** *b*=*w* **and** *s* = {*0..1*}, *simplified*])
apply (*auto simp*: *contour-integrable-on*)
done

lemma *contour-integral-subpath-combine*:
assumes *f contour-integrable-on g valid-path g u* ∈ {*0..1*} *v* ∈ {*0..1*} *w* ∈ {*0..1*}
shows *contour-integral* (*subpath u v g*) *f* + *contour-integral* (*subpath v w g*) *f* =
contour-integral (*subpath u w g*) *f*
proof (*cases u*≠*v* ∧ *v*≠*w* ∧ *u*≠*w*)
case *True*
have ∗: *subpath v u g* = *reversepath*(*subpath u v g*) ∧
subpath w u g = *reversepath*(*subpath u w g*) ∧
subpath w v g = *reversepath*(*subpath v w g*)
by (*auto simp*: *reversepath-subpath*)
have *u* < *v* ∧ *v* < *w* ∨
u < *w* ∧ *w* < *v* ∨
v < *u* ∧ *u* < *w* ∨
v < *w* ∧ *w* < *u* ∨
w < *u* ∧ *u* < *v* ∨
w < *v* ∧ *v* < *u*
using *True assms* **by** *linarith*
with *assms* **show** *?thesis*
using *contour-integral-subpath-combine-less* [*of f g u v w*]
contour-integral-subpath-combine-less [*of f g u w v*]
contour-integral-subpath-combine-less [*of f g v u w*]
contour-integral-subpath-combine-less [*of f g v w u*]
contour-integral-subpath-combine-less [*of f g w u v*]
contour-integral-subpath-combine-less [*of f g w v u*]

apply *simp*
apply (*elim disjE*)
apply (*auto simp*: ∗ *contour-integral-reversepath contour-integrable-subpath valid-path-reversepath valid-path-subpath algebra-simps*)
done
next
case *False*
then show *?thesis*
apply (*auto simp*: *contour-integral-subpath-refl*)
using *assms*
by (*metis eq-neg-iff-add-eq-0 contour-integrable-subpath contour-integral-reversepath reversepath-subpath valid-path-subpath*)
qed

lemma *contour-integral-integral*:
contour-integral g f = *integral* {*0..1*} (λ*x*. *f* (*g x*) ∗ *vector-derivative g* (*at x*))
by (*simp add*: *contour-integral-def integral-def has-contour-integral contour-integrable-on*)

Cauchy's theorem where there's a primitive

lemma *contour-integral-primitive-lemma*:
fixes *f* :: *complex* ⇒ *complex* **and** *g* :: *real* ⇒ *complex*
assumes *a* ≤ *b*
and ⋀*x*. *x* ∈ *s* ⟹ (*f has-field-derivative f* ′ *x*) (*at x within s*)
and *g piecewise-differentiable-on* {*a..b*} ⋀*x*. *x* ∈ {*a..b*} ⟹ *g x* ∈ *s*
shows ((λ*x*. *f* ′(*g x*) ∗ *vector-derivative g* (*at x within* {*a..b*}))
has-integral (*f*(*g b*) − *f*(*g a*))) {*a..b*}
proof −
obtain *k* **where** *k*: *finite k* ∀ *x*∈{*a..b*} − *k*. *g differentiable* (*at x within* {*a..b*})
and *cg*: *continuous-on* {*a..b*} *g*
using *assms* **by** (*auto simp*: *piecewise-differentiable-on-def*)
have *cfg*: *continuous-on* {*a..b*} (λ*x*. *f* (*g x*))
apply (*rule continuous-on-compose* [*OF cg*, *unfolded o-def*])
using *assms*
apply (*metis field-differentiable-def field-differentiable-imp-continuous-at continuous-on-eq-continuous-within continuous-on-subset image-subset-iff*)
done
{ fix *x*::*real*
assume *a*: *a* < *x* **and** *b*: *x* < *b* **and** *xk*: *x* ∉ *k*
then have *g differentiable at x within* {*a..b*}
using *k* **by** (*simp add*: *differentiable-at-withinI*)
then have (*g has-vector-derivative vector-derivative g* (*at x within* {*a..b*})) (*at x within* {*a..b*})
by (*simp add*: *vector-derivative-works has-field-derivative-def scaleR-conv-of-real*)
then have *gdiff*: (*g has-derivative* (λ*u*. *u* ∗ *vector-derivative g* (*at x within* {*a..b*}))) (*at x within* {*a..b*})
by (*simp add*: *has-vector-derivative-def scaleR-conv-of-real*)
have (*f has-field-derivative* (*f* ′ (*g x*))) (*at* (*g x*) *within g* ‘ {*a..b*})
using *assms* **by** (*metis a atLeastAtMost-iff b DERIV-subset image-subset-iff less-eq-real-def*)

```
    then have fdiff: (f has-derivative op * (f' (g x))) (at (g x) within g ' {a..b})
      by (simp add: has-field-derivative-def)
     have ((λx. f (g x)) has-vector-derivative f' (g x) * vector-derivative g (at x
within {a..b})) (at x within {a..b})
      using diff-chain-within [OF gdiff fdiff]
      by (simp add: has-vector-derivative-def scaleR-conv-of-real o-def mult-ac)
  } note * = this
  show ?thesis
    apply (rule fundamental-theorem-of-calculus-interior-strong)
    using k assms cfg *
    apply (auto simp: at-within-closed-interval)
    done
qed

lemma contour-integral-primitive:
  assumes ⋀x. x ∈ s ⟹ (f has-field-derivative f' x) (at x within s)
      and valid-path g path-image g ⊆ s
    shows (f' has-contour-integral (f(pathfinish g) − f(pathstart g))) g
  using assms
 apply (simp add: valid-path-def path-image-def pathfinish-def pathstart-def has-contour-integral-def )
 apply (auto intro!: piecewise-C1-imp-differentiable contour-integral-primitive-lemma
[of 0 1 s])
  done

corollary Cauchy-theorem-primitive:
  assumes ⋀x. x ∈ s ⟹ (f has-field-derivative f' x) (at x within s)
      and valid-path g  path-image g ⊆ s pathfinish g = pathstart g
    shows (f' has-contour-integral 0) g
  using assms
  by (metis diff-self contour-integral-primitive)
```

Existence of path integral for continuous function

```
lemma contour-integrable-continuous-linepath:
  assumes continuous-on (closed-segment a b) f
  shows f contour-integrable-on (linepath a b)
proof −
  have continuous-on {0..1} ((λx. f x * (b − a)) o linepath a b)
     apply (rule continuous-on-compose [OF continuous-on-linepath], simp add:
linepath-image-01)
    apply (rule continuous-intros | simp add: assms)+
    done
  then show ?thesis
   apply (simp add: contour-integrable-on-def has-contour-integral-def integrable-on-def
[symmetric])
    apply (rule integrable-continuous [of 0 1::real, simplified])
    apply (rule continuous-on-eq [where f = λx. f(linepath a b x)*(b − a)])
    apply (auto simp: vector-derivative-linepath-within)
    done
qed
```

lemma *has-field-der-id*: $((\lambda x.\ x^2\ /\ 2)$ *has-field-derivative* $x)$ $(at\ x)$
by (*rule has-derivative-imp-has-field-derivative*)
(*rule derivative-intros* | *simp*)+

lemma *contour-integral-id* [*simp*]: *contour-integral* (*linepath a b*) $(\lambda y.\ y) = (b\hat{}2 - a\hat{}2)/2$
apply (*rule contour-integral-unique*)
using *contour-integral-primitive* [*of UNIV* $\lambda x.\ x\hat{}2/2$ $\lambda x.\ x$ *linepath a b*]
apply (*auto simp*: *field-simps has-field-der-id*)
done

lemma *contour-integrable-on-const* [*iff*]: $(\lambda x.\ c)$ *contour-integrable-on* (*linepath a b*)
by (*simp add*: *continuous-on-const contour-integrable-continuous-linepath*)

lemma *contour-integrable-on-id* [*iff*]: $(\lambda x.\ x)$ *contour-integrable-on* (*linepath a b*)
by (*simp add*: *continuous-on-id contour-integrable-continuous-linepath*)

53.10 Arithmetical combining theorems

lemma *has-contour-integral-neg*:
(*f has-contour-integral i*) *g* $\Longrightarrow$ $((\lambda x.\ -(f\ x))$ *has-contour-integral* $(-i))$ *g*
by (*simp add*: *has-integral-neg has-contour-integral-def*)

lemma *has-contour-integral-add*:
⟦(*f1 has-contour-integral i1*) *g*; (*f2 has-contour-integral i2*) *g*⟧
$\Longrightarrow$ $((\lambda x.\ f1\ x + f2\ x)$ *has-contour-integral* $(i1 + i2))$ *g*
by (*simp add*: *has-integral-add has-contour-integral-def algebra-simps*)

lemma *has-contour-integral-diff*:
⟦(*f1 has-contour-integral i1*) *g*; (*f2 has-contour-integral i2*) *g*⟧
$\Longrightarrow$ $((\lambda x.\ f1\ x - f2\ x)$ *has-contour-integral* $(i1 - i2))$ *g*
by (*simp add*: *has-integral-sub has-contour-integral-def algebra-simps*)

lemma *has-contour-integral-lmul*:
(*f has-contour-integral i*) *g* $\Longrightarrow$ $((\lambda x.\ c * (f\ x))$ *has-contour-integral* $(c*i))$ *g*
apply (*simp add*: *has-contour-integral-def*)
apply (*drule has-integral-mult-right*)
apply (*simp add*: *algebra-simps*)
done

lemma *has-contour-integral-rmul*:
(*f has-contour-integral i*) *g* $\Longrightarrow$ $((\lambda x.\ (f\ x) * c)$ *has-contour-integral* $(i*c))$ *g*
apply (*drule has-contour-integral-lmul*)
apply (*simp add*: *mult.commute*)
done

lemma *has-contour-integral-div*:

(*f has-contour-integral i*) *g* ⟹ ((λ*x*. *f x*/*c*) *has-contour-integral* (*i*/*c*)) *g*
by (*simp add*: *field-class.field-divide-inverse*) (*metis has-contour-integral-rmul*)

lemma *has-contour-integral-eq*:
⟦(*f has-contour-integral y*) *p*; ⋀*x*. *x* ∈ *path-image p* ⟹ *f x* = *g x*⟧ ⟹ (*g has-contour-integral y*) *p*
apply (*simp add*: *path-image-def has-contour-integral-def*)
by (*metis* (*no-types*, *lifting*) *image-eqI has-integral-eq*)

lemma *has-contour-integral-bound-linepath*:
assumes (*f has-contour-integral i*) (*linepath a b*)
0 ≤ *B* ⋀*x*. *x* ∈ *closed-segment a b* ⟹ *norm*(*f x*) ≤ *B*
shows *norm i* ≤ *B* ∗ *norm*(*b* − *a*)
proof −
{ fix *x*::*real*
assume *x*: *0* ≤ *x x* ≤ *1*
have *norm* (*f* (*linepath a b x*)) ∗
norm (*vector-derivative* (*linepath a b*) (*at x within* {*0..1*})) ≤ *B* ∗ *norm* (*b* − *a*)
by (*auto intro*: *mult-mono simp*: *assms linepath-in-path of-real-linepath vector-derivative-linepath-within x*)
} note ∗ = *this*
have *norm i* ≤ (*B* ∗ *norm* (*b* − *a*)) ∗ *content* (*cbox 0* (*1*::*real*))
apply (*rule has-integral-bound*
[*of* - λ*x*. *f* (*linepath a b x*) ∗ *vector-derivative* (*linepath a b*) (*at x within* {*0..1*})])
using *assms* ∗ **unfolding** *has-contour-integral-def*
apply (*auto simp*: *norm-mult*)
done
then show *?thesis*
by (*auto simp*: *content-real*)
qed

lemma *has-contour-integral-const-linepath*: ((λ*x*. *c*) *has-contour-integral c*∗(*b* − *a*))(*linepath a b*)
unfolding *has-contour-integral-linepath*
by (*metis content-real diff-0-right has-integral-const-real lambda-one of-real-1 scaleR-conv-of-real zero-le-one*)

lemma *has-contour-integral-0*: ((λ*x*. *0*) *has-contour-integral 0*) *g*
by (*simp add*: *has-contour-integral-def*)

lemma *has-contour-integral-is-0*:
(⋀*z*. *z* ∈ *path-image g* ⟹ *f z* = *0*) ⟹ (*f has-contour-integral 0*) *g*
by (*rule has-contour-integral-eq* [*OF has-contour-integral-0*]) *auto*

lemma *has-contour-integral-setsum*:

⟦*finite s*; ⋀*a*. *a* ∈ *s* ⟹ (*f a has-contour-integral i a*) *p*⟧
⟹ ((λ*x*. *setsum* (λ*a*. *f a x*) *s*) *has-contour-integral setsum i s*) *p*
by (*induction s rule*: *finite-induct*) (*auto simp*: *has-contour-integral-0 has-contour-integral-add*)

53.11 Operations on path integrals

lemma *contour-integral-const-linepath* [*simp*]: *contour-integral* (*linepath a b*) (λ*x*. *c*) = *c*∗(*b* − *a*)
by (*rule contour-integral-unique* [*OF has-contour-integral-const-linepath*])

lemma *contour-integral-neg*:
f contour-integrable-on g ⟹ *contour-integral g* (λ*x*. −(*f x*)) = −(*contour-integral g f*)
by (*simp add*: *contour-integral-unique has-contour-integral-integral has-contour-integral-neg*)

lemma *contour-integral-add*:
f1 contour-integrable-on g ⟹ *f2 contour-integrable-on g* ⟹ *contour-integral g* (λ*x*. *f1 x* + *f2 x*) =
contour-integral g f1 + *contour-integral g f2*
by (*simp add*: *contour-integral-unique has-contour-integral-integral has-contour-integral-add*)

lemma *contour-integral-diff*:
f1 contour-integrable-on g ⟹ *f2 contour-integrable-on g* ⟹ *contour-integral g* (λ*x*. *f1 x* − *f2 x*) =
contour-integral g f1 − *contour-integral g f2*
by (*simp add*: *contour-integral-unique has-contour-integral-integral has-contour-integral-diff*)

lemma *contour-integral-lmul*:
shows *f contour-integrable-on g*
⟹ *contour-integral g* (λ*x*. *c* ∗ *f x*) = *c*∗*contour-integral g f*
by (*simp add*: *contour-integral-unique has-contour-integral-integral has-contour-integral-lmul*)

lemma *contour-integral-rmul*:
shows *f contour-integrable-on g*
⟹ *contour-integral g* (λ*x*. *f x* ∗ *c*) = *contour-integral g f* ∗ *c*
by (*simp add*: *contour-integral-unique has-contour-integral-integral has-contour-integral-rmul*)

lemma *contour-integral-div*:
shows *f contour-integrable-on g*
⟹ *contour-integral g* (λ*x*. *f x* / *c*) = *contour-integral g f* / *c*
by (*simp add*: *contour-integral-unique has-contour-integral-integral has-contour-integral-div*)

lemma *contour-integral-eq*:
(⋀*x*. *x* ∈ *path-image p* ⟹ *f x* = *g x*) ⟹ *contour-integral p f* = *contour-integral p g*
apply (*simp add*: *contour-integral-def*)
using *has-contour-integral-eq*
by (*metis contour-integral-unique has-contour-integral-integrable has-contour-integral-integral*)

lemma *contour-integral-eq-0*:
(⋀*z*. *z* ∈ *path-image g* ⟹ *f z* = *0*) ⟹ *contour-integral g f* = *0*
by (*simp add*: *has-contour-integral-is-0 contour-integral-unique*)

lemma *contour-integral-bound-linepath*:
shows
⟦*f contour-integrable-on* (*linepath a b*);
0 ≤ *B*; ⋀*x*. *x* ∈ *closed-segment a b* ⟹ *norm*(*f x*) ≤ *B*⟧
⟹ *norm*(*contour-integral* (*linepath a b*) *f*) ≤ *B*∗*norm*(*b* − *a*)
apply (*rule has-contour-integral-bound-linepath* [*of f*])
apply (*auto simp*: *has-contour-integral-integral*)
done

lemma *contour-integral-0* [*simp*]: *contour-integral g* (λ*x*. *0*) = *0*
by (*simp add*: *contour-integral-unique has-contour-integral-0*)

lemma *contour-integral-setsum*:
⟦*finite s*; ⋀*a*. *a* ∈ *s* ⟹ (*f a*) *contour-integrable-on p*⟧
⟹ *contour-integral p* (λ*x*. *setsum* (λ*a*. *f a x*) *s*) = *setsum* (λ*a*. *contour-integral p* (*f a*)) *s*
by (*auto simp*: *contour-integral-unique has-contour-integral-setsum has-contour-integral-integral*)

lemma *contour-integrable-eq*:
⟦*f contour-integrable-on p*; ⋀*x*. *x* ∈ *path-image p* ⟹ *f x* = *g x*⟧ ⟹ *g contour-integrable-on p*
unfolding *contour-integrable-on-def*
by (*metis has-contour-integral-eq*)

53.12 Arithmetic theorems for path integrability

lemma *contour-integrable-neg*:
f contour-integrable-on g ⟹ (λ*x*. −(*f x*)) *contour-integrable-on g*
using *has-contour-integral-neg contour-integrable-on-def* **by** *blast*

lemma *contour-integrable-add*:
⟦*f1 contour-integrable-on g*; *f2 contour-integrable-on g*⟧ ⟹ (λ*x*. *f1 x* + *f2 x*) *contour-integrable-on g*
using *has-contour-integral-add contour-integrable-on-def*
by *fastforce*

lemma *contour-integrable-diff*:
⟦*f1 contour-integrable-on g*; *f2 contour-integrable-on g*⟧ ⟹ (λ*x*. *f1 x* − *f2 x*) *contour-integrable-on g*
using *has-contour-integral-diff contour-integrable-on-def*
by *fastforce*

lemma *contour-integrable-lmul*:
f contour-integrable-on g ⟹ (λ*x*. *c* ∗ *f x*) *contour-integrable-on g*
using *has-contour-integral-lmul contour-integrable-on-def*

by *fastforce*

lemma *contour-integrable-rmul*:
f contour-integrable-on g $\Longrightarrow$ $(\lambda x.\ f\ x * c)$ *contour-integrable-on g*
using *has-contour-integral-rmul contour-integrable-on-def*
by *fastforce*

lemma *contour-integrable-div*:
f contour-integrable-on g $\Longrightarrow$ $(\lambda x.\ f\ x\ /\ c)$ *contour-integrable-on g*
using *has-contour-integral-div contour-integrable-on-def*
by *fastforce*

lemma *contour-integrable-setsum*:
$[\![$*finite s*; $\bigwedge a.\ a \in s \Longrightarrow (f\ a)$ *contour-integrable-on p*$]\!]$
$\Longrightarrow (\lambda x.\ setsum\ (\lambda a.\ f\ a\ x)\ s)$ *contour-integrable-on p*
unfolding *contour-integrable-on-def*
by (*metis has-contour-integral-setsum*)

53.13 Reversing a path integral

lemma *has-contour-integral-reverse-linepath*:
(*f has-contour-integral i*) (*linepath a b*)
$\Longrightarrow$ (*f has-contour-integral* $(-i)$) (*linepath b a*)
using *has-contour-integral-reversepath valid-path-linepath* **by** *fastforce*

lemma *contour-integral-reverse-linepath*:
continuous-on (*closed-segment a b*) *f*
$\Longrightarrow$ *contour-integral* (*linepath a b*) *f* $= -$ (*contour-integral*(*linepath b a*) *f*)
apply (*rule contour-integral-unique*)
apply (*rule has-contour-integral-reverse-linepath*)
by (*simp add*: *closed-segment-commute contour-integrable-continuous-linepath has-contour-integral-integral*)

lemma *has-contour-integral-split*:
assumes *f*: (*f has-contour-integral i*) (*linepath a c*) (*f has-contour-integral j*)
(*linepath c b*)
and *k*: $0 \le k\ k \le 1$
and *c*: $c - a = k *_R (b - a)$
shows (*f has-contour-integral* $(i + j)$) (*linepath a b*)
proof (*cases* $k = 0 \vee k = 1$)
case *True*
then show *?thesis*
using *assms*
apply *auto*
apply (*metis add.left-neutral has-contour-integral-trivial has-contour-integral-unique*)
apply (*metis add.right-neutral has-contour-integral-trivial has-contour-integral-unique*)
done

```
next
  case False
  then have k: 0 < k k < 1 complex-of-real k ≠ 1
    using assms apply auto
    using of-real-eq-iff by fastforce
  have c': c = k *R (b − a) + a
    by (metis diff-add-cancel c)
  have bc: (b − c) = (1 − k) *R (b − a)
    by (simp add: algebra-simps c')
  { assume *: ((λx. f ((1 − x) *R a + x *R c) * (c − a)) has-integral i) {0..1}
    have **: ⋀x. ((k − x) / k) *R a + (x / k) *R c = (1 − x) *R a + x *R b
      using False
      apply (simp add: c' algebra-simps)
      apply (simp add: real-vector.scale-left-distrib [symmetric] divide-simps)
      done
    have ((λx. f ((1 − x) *R a + x *R b) * (b − a)) has-integral i) {0..k}
      using * k
      apply −
      apply (drule has-integral-affinity [of - - 0 1::real inverse k 0, simplified])
    apply (simp-all add: divide-simps mult.commute [of - k] image-affinity-atLeastAtMost
** c)
      apply (drule Integration.has-integral-cmul [where c = inverse k])
      apply (simp add: Integration.has-integral-cmul)
      done
  } note fi = this
  { assume *: ((λx. f ((1 − x) *R c + x *R b) * (b − c)) has-integral j) {0..1}
    have **: ⋀x. (((1 − x) / (1 − k)) *R c + ((x − k) / (1 − k)) *R b) = ((1
− x) *R a + x *R b)
      using k
      apply (simp add: c' field-simps)
      apply (simp add: scaleR-conv-of-real divide-simps)
      apply (simp add: field-simps)
      done
    have ((λx. f ((1 − x) *R a + x *R b) * (b − a)) has-integral j) {k..1}
      using * k
      apply −
      apply (drule has-integral-affinity [of - - 0 1::real inverse(1 − k) −(k/(1 −
k)), simplified])
    apply (simp-all add: divide-simps mult.commute [of - 1−k] image-affinity-atLeastAtMost
** bc)
      apply (drule Integration.has-integral-cmul [where k = (1 − k) *R j and c
= inverse (1 − k)])
      apply (simp add: Integration.has-integral-cmul)
      done
  } note fj = this
  show ?thesis
    using f k
    apply (simp add: has-contour-integral-linepath)
    apply (simp add: linepath-def)
```

apply (*rule has-integral-combine* [*OF - - fi fj*], *simp-all*)
done
qed

lemma *continuous-on-closed-segment-transform*:
assumes *f*: *continuous-on* (*closed-segment a b*) *f*
and *k*: *0* ≤ *k k* ≤ *1*
and *c*: *c* − *a* = *k* ∗*R* (*b* − *a*)
shows *continuous-on* (*closed-segment a c*) *f*
proof −
have *c′*: *c* = (*1* − *k*) ∗*R* *a* + *k* ∗*R* *b*
using *c* **by** (*simp add*: *algebra-simps*)
show *continuous-on* (*closed-segment a c*) *f*
apply (*rule continuous-on-subset* [*OF f*])
apply (*simp add*: *segment-convex-hull*)
apply (*rule convex-hull-subset*)
using *assms*
apply (*auto simp*: *hull-inc c′ Convex.convexD-alt*)
done
qed

lemma *contour-integral-split*:
assumes *f*: *continuous-on* (*closed-segment a b*) *f*
and *k*: *0* ≤ *k k* ≤ *1*
and *c*: *c* − *a* = *k* ∗*R* (*b* − *a*)
shows *contour-integral*(*linepath a b*) *f* = *contour-integral*(*linepath a c*) *f* +
contour-integral(*linepath c b*) *f*
proof −
have *c′*: *c* = (*1* − *k*) ∗*R* *a* + *k* ∗*R* *b*
using *c* **by** (*simp add*: *algebra-simps*)
have ∗: *continuous-on* (*closed-segment a c*) *f continuous-on* (*closed-segment c b*)
f
apply (*rule-tac* [!] *continuous-on-subset* [*OF f*])
apply (*simp-all add*: *segment-convex-hull*)
apply (*rule-tac* [!] *convex-hull-subset*)
using *assms*
apply (*auto simp*: *hull-inc c′ Convex.convexD-alt*)
done
show *?thesis*
apply (*rule contour-integral-unique*)
apply (*rule has-contour-integral-split* [*OF has-contour-integral-integral has-contour-integral-integral*
k c])
apply (*rule contour-integrable-continuous-linepath* ∗)+
done
qed

lemma *contour-integral-split-linepath*:
assumes *f*: *continuous-on* (*closed-segment a b*) *f*
and *c*: *c* ∈ *closed-segment a b*

shows *contour-integral*(*linepath a b*) *f* = *contour-integral*(*linepath a c*) *f* + *contour-integral*(*linepath c b*) *f*
using *c*
by (*auto simp*: *closed-segment-def algebra-simps intro*!: *contour-integral-split* [*OF f*])

lemma *has-contour-integral-midpoint*:
assumes (*f has-contour-integral i*) (*linepath a* (*midpoint a b*))
(*f has-contour-integral j*) (*linepath* (*midpoint a b*) *b*)
shows (*f has-contour-integral* (*i* + *j*)) (*linepath a b*)
apply (*rule has-contour-integral-split* [**where** *c* = *midpoint a b* **and** *k* = *1/2*])
using *assms*
apply (*auto simp*: *midpoint-def algebra-simps scaleR-conv-of-real*)
done

lemma *contour-integral-midpoint*:
continuous-on (*closed-segment a b*) *f*
⟹ *contour-integral* (*linepath a b*) *f* =
contour-integral (*linepath a* (*midpoint a b*)) *f* + *contour-integral* (*linepath* (*midpoint a b*) *b*) *f*
apply (*rule contour-integral-split* [**where** *c* = *midpoint a b* **and** *k* = *1/2*])
using *assms*
apply (*auto simp*: *midpoint-def algebra-simps scaleR-conv-of-real*)
done

A couple of special case lemmas that are useful below

lemma *triangle-linear-has-chain-integral*:
((λ*x*. *m*∗*x* + *d*) *has-contour-integral 0*) (*linepath a b* +++ *linepath b c* +++ *linepath c a*)
apply (*rule Cauchy-theorem-primitive* [*of UNIV* λ*x*. *m/2* ∗ *x*^*2* + *d*∗*x*])
apply (*auto intro*!: *derivative-eq-intros*)
done

lemma *has-chain-integral-chain-integral3*:
(*f has-contour-integral i*) (*linepath a b* +++ *linepath b c* +++ *linepath c d*)
⟹ *contour-integral* (*linepath a b*) *f* + *contour-integral* (*linepath b c*) *f* + *contour-integral* (*linepath c d*) *f* = *i*
apply (*subst contour-integral-unique* [*symmetric*], *assumption*)
apply (*drule has-contour-integral-integrable*)
apply (*simp add*: *valid-path-join*)
done

lemma *has-chain-integral-chain-integral4*:
(*f has-contour-integral i*) (*linepath a b* +++ *linepath b c* +++ *linepath c d* +++ *linepath d e*)
⟹ *contour-integral* (*linepath a b*) *f* + *contour-integral* (*linepath b c*) *f* + *contour-integral* (*linepath c d*) *f* + *contour-integral* (*linepath d e*) *f* = *i*

apply (*subst contour-integral-unique* [*symmetric*], *assumption*)
apply (*drule has-contour-integral-integrable*)
apply (*simp add*: *valid-path-join*)
done

53.14 Reversing the order in a double path integral

The condition is stronger than needed but it's often true in typical situations

lemma *fst-im-cbox* [*simp*]: *cbox c d* ≠ {} ⟹ (*fst* ‘ *cbox* (*a*,*c*) (*b*,*d*)) = *cbox a b*
by (*auto simp*: *cbox-Pair-eq*)

lemma *snd-im-cbox* [*simp*]: *cbox a b* ≠ {} ⟹ (*snd* ‘ *cbox* (*a*,*c*) (*b*,*d*)) = *cbox c d*
by (*auto simp*: *cbox-Pair-eq*)

lemma *contour-integral-swap*:
assumes *fcon*: *continuous-on* (*path-image g* × *path-image h*) (λ(*y1*,*y2*). *f y1 y2*)
and *vp*: *valid-path g valid-path h*
and *gvcon*: *continuous-on* {*0..1*} (λ*t*. *vector-derivative g* (*at t*))
and *hvcon*: *continuous-on* {*0..1*} (λ*t*. *vector-derivative h* (*at t*))
shows *contour-integral g* (λ*w*. *contour-integral h* (*f w*)) =
contour-integral h (λ*z*. *contour-integral g* (λ*w*. *f w z*))
proof −
have *gcon*: *continuous-on* {*0..1*} *g* **and** *hcon*: *continuous-on* {*0..1*} *h*
using *assms* **by** (*auto simp*: *valid-path-def piecewise-C1-differentiable-on-def*)
have *fgh1*: ⋀*x*. (λ*t*. *f* (*g x*) (*h t*)) = (λ(*y1*,*y2*). *f y1 y2*) *o* (λ*t*. (*g x*, *h t*))
by (*rule ext*) *simp*
have *fgh2*: ⋀*x*. (λ*t*. *f* (*g t*) (*h x*)) = (λ(*y1*,*y2*). *f y1 y2*) *o* (λ*t*. (*g t*, *h x*))
by (*rule ext*) *simp*
have *fcon-im1*: ⋀*x*. *0* ≤ *x* ⟹ *x* ≤ *1* ⟹ *continuous-on* ((λ*t*. (*g x*, *h t*)) ‘ {*0..1*}) (λ(*x*, *y*). *f x y*)
by (*rule continuous-on-subset* [*OF fcon*]) (*auto simp*: *path-image-def*)
have *fcon-im2*: ⋀*x*. *0* ≤ *x* ⟹ *x* ≤ *1* ⟹ *continuous-on* ((λ*t*. (*g t*, *h x*)) ‘ {*0..1*}) (λ(*x*, *y*). *f x y*)
by (*rule continuous-on-subset* [*OF fcon*]) (*auto simp*: *path-image-def*)
have *vdg*: ⋀*y*. *y* ∈ {*0..1*} ⟹ (λ*x*. *f* (*g x*) (*h y*) ∗ *vector-derivative g* (*at x*)) *integrable-on* {*0..1*}
apply (*rule integrable-continuous-real*)
apply (*rule continuous-on-mult* [*OF - gvcon*])
apply (*subst fgh2*)
apply (*rule fcon-im2 gcon continuous-intros* | *simp*)+
done
have (λ*z*. *vector-derivative g* (*at* (*fst z*))) = (λ*x*. *vector-derivative g* (*at x*)) *o fst*
by *auto*
then have *gvcon'*: *continuous-on* (*cbox* (*0*, *0*) (*1*, *1*::*real*)) (λ*x*. *vector-derivative g* (*at* (*fst x*)))
apply (*rule ssubst*)
apply (*rule continuous-intros* | *simp add*: *gvcon*)+
done

have ($\lambda z.$ *vector-derivative h* (*at* (*snd z*))) = ($\lambda x.$ *vector-derivative h* (*at x*)) *o snd*
by *auto*
then have *hvcon'*: *continuous-on* (*cbox* (*0*, *0*) (*1*::*real*, *1*)) ($\lambda x.$ *vector-derivative h* (*at* (*snd x*)))
apply (*rule ssubst*)
apply (*rule continuous-intros* | *simp add*: *hvcon*)+
done
have ($\lambda x.$ *f* (*g* (*fst x*)) (*h* (*snd x*))) = (λ(*y1*,*y2*). *f y1 y2*) *o* ($\lambda w.$ ((*g o fst*) *w*, (*h o snd*) *w*))
by *auto*
then have *fgh*: *continuous-on* (*cbox* (*0*, *0*) (*1*, *1*)) ($\lambda x.$ *f* (*g* (*fst x*)) (*h* (*snd x*)))
apply (*rule ssubst*)
apply (*rule gcon hcon continuous-intros* | *simp*)+
apply (*auto simp*: *path-image-def intro*: *continuous-on-subset* [*OF fcon*])
done
have *integral* {*0..1*} ($\lambda x.$ *contour-integral h* (*f* (*g x*)) ∗ *vector-derivative g* (*at x*)) =
integral {*0..1*} ($\lambda x.$ *contour-integral h* ($\lambda y.$ *f* (*g x*) *y* ∗ *vector-derivative g* (*at x*)))
apply (*rule integral-cong* [*OF contour-integral-rmul* [*symmetric*]])
apply (*clarsimp simp*: *contour-integrable-on*)
apply (*rule integrable-continuous-real*)
apply (*rule continuous-on-mult* [*OF - hvcon*])
apply (*subst fgh1*)
apply (*rule fcon-im1 hcon continuous-intros* | *simp*)+
done
also have ... = *integral* {*0..1*}
($\lambda y.$ *contour-integral g* ($\lambda x.$ *f x* (*h y*) ∗ *vector-derivative h* (*at y*)))
apply (*simp only*: *contour-integral-integral*)
apply (*subst integral-swap-continuous* [**where** *'a* = *real* **and** *'b* = *real*, *of 0 0 1 1*, *simplified*])
apply (*rule fgh gvcon' hvcon' continuous-intros* | *simp add*: *split-def*)+
unfolding *integral-mult-left* [*symmetric*]
apply (*simp only*: *mult-ac*)
done
also have ... = *contour-integral h* ($\lambda z.$ *contour-integral g* ($\lambda w.$ *f w z*))
apply (*simp add*: *contour-integral-integral*)
apply (*rule integral-cong*)
unfolding *integral-mult-left* [*symmetric*]
apply (*simp add*: *algebra-simps*)
done
finally show *?thesis*
by (*simp add*: *contour-integral-integral*)
qed

53.15 The key quadrisection step

lemma *norm-sum-half*:

assumes *norm*(*a* + *b*) >= *e*
shows *norm a* >= *e*/*2* ∨ *norm b* >= *e*/*2*
proof −
have *e* ≤ *norm* (− *a* − *b*)
by (*simp add*: *add.commute assms norm-minus-commute*)
thus *?thesis*
using *norm-triangle-ineq4 order-trans* **by** *fastforce*
qed

lemma *norm-sum-lemma*:
assumes *e* ≤ *norm* (*a* + *b* + *c* + *d*)
shows *e* / *4* ≤ *norm a* ∨ *e* / *4* ≤ *norm b* ∨ *e* / *4* ≤ *norm c* ∨ *e* / *4* ≤ *norm d*
proof −
have *e* ≤ *norm* ((*a* + *b*) + (*c* + *d*)) **using** *assms*
by (*simp add*: *algebra-simps*)
then show *?thesis*
by (*auto dest*!: *norm-sum-half*)
qed

lemma *Cauchy-theorem-quadrisection*:
assumes *f*: *continuous-on* (*convex hull* {*a*,*b*,*c*}) *f*
and *dist*: *dist a b* ≤ *K dist b c* ≤ *K dist c a* ≤ *K*
and *e*: *e* ∗ *K*^*2* ≤
norm (*contour-integral*(*linepath a b*) *f* + *contour-integral*(*linepath b c*) *f* + *contour-integral*(*linepath c a*) *f*)
shows ∃ *a*′ *b*′ *c*′.
a′ ∈ *convex hull* {*a*,*b*,*c*} ∧ *b*′ ∈ *convex hull* {*a*,*b*,*c*} ∧ *c*′ ∈ *convex hull* {*a*,*b*,*c*} ∧
dist a′ *b*′ ≤ *K*/*2* ∧ *dist b*′ *c*′ ≤ *K*/*2* ∧ *dist c*′ *a*′ ≤ *K*/*2* ∧
e ∗ (*K*/*2*) ^*2* ≤ *norm*(*contour-integral*(*linepath a*′ *b*′) *f* + *contour-integral*(*linepath b*′ *c*′) *f* + *contour-integral*(*linepath c*′ *a*′) *f*)
proof −
note *divide-le-eq-numeral1* [*simp del*]
def *a*′ ≡ *midpoint b c*
def *b*′ ≡ *midpoint c a*
def *c*′ ≡ *midpoint a b*
have *fabc*: *continuous-on* (*closed-segment a b*) *f continuous-on* (*closed-segment b c*) *f continuous-on* (*closed-segment c a*) *f*
using *f continuous-on-subset segments-subset-convex-hull* **by** *metis*+
have *fcont*′: *continuous-on* (*closed-segment c*′ *b*′) *f*
continuous-on (*closed-segment a*′ *c*′) *f*
continuous-on (*closed-segment b*′ *a*′) *f*
unfolding *a*′*-def b*′*-def c*′*-def*
apply (*rule continuous-on-subset* [*OF f*],
metis midpoints-in-convex-hull convex-hull-subset hull-subset insert-subset segment-convex-hull)+
done
let *?pathint* = λ*x y*. *contour-integral*(*linepath x y*) *f*

have $*$: *?pathint a b + ?pathint b c + ?pathint c a =*
(?pathint a c′ + ?pathint c′ b′ + ?pathint b′ a) +
(?pathint a′ c′ + ?pathint c′ b + ?pathint b a′) +
(?pathint a′ c + ?pathint c b′ + ?pathint b′ a′) +
(?pathint a′ b′ + ?pathint b′ c′ + ?pathint c′ a′)
apply (*simp add*: *fcont′ contour-integral-reverse-linepath*)
apply (*simp add*: *a′-def b′-def c′-def contour-integral-midpoint fabc*)
done
have [*simp*]: $\bigwedge x\ y.$ *cmod* $(x * 2 - y * 2) =$ *cmod* $(x - y) * 2$
by (*metis left-diff-distrib mult.commute norm-mult-numeral1*)
have [*simp*]: $\bigwedge x\ y.$ *cmod* $(x - y) =$ *cmod* $(y - x)$
by (*simp add*: *norm-minus-commute*)
consider $e * K^2$ / 4 $\leq$ *cmod (?pathint a c′ + ?pathint c′ b′ + ?pathint b′ a)* |
$e * K^2$ / 4 $\leq$ *cmod (?pathint a′ c′ + ?pathint c′ b + ?pathint b a′)* |
$e * K^2$ / 4 $\leq$ *cmod (?pathint a′ c + ?pathint c b′ + ?pathint b′ a′)* |
$e * K^2$ / 4 $\leq$ *cmod (?pathint a′ b′ + ?pathint b′ c′ + ?pathint c′ a′)*
using *assms*
apply (*simp only*: $*$)
apply (*blast intro*: *that dest*!: *norm-sum-lemma*)
done
then show *?thesis*
proof *cases*
case *1* **then show** *?thesis*
apply (*rule-tac x=a* **in** *exI*)
apply (*rule exI* [**where** *x=c′*])
apply (*rule exI* [**where** *x=b′*])
using *assms*
apply (*auto simp*: *a′-def c′-def b′-def midpoints-in-convex-hull hull-subset* [*THEN subsetD*])
apply (*auto simp*: *midpoint-def dist-norm scaleR-conv-of-real divide-simps*)
done
next
case *2* **then show** *?thesis*
apply (*rule-tac x=a′* **in** *exI*)
apply (*rule exI* [**where** *x=c′*])
apply (*rule exI* [**where** *x=b*])
using *assms*
apply (*auto simp*: *a′-def c′-def b′-def midpoints-in-convex-hull hull-subset* [*THEN subsetD*])
apply (*auto simp*: *midpoint-def dist-norm scaleR-conv-of-real divide-simps*)
done
next
case *3* **then show** *?thesis*
apply (*rule-tac x=a′* **in** *exI*)
apply (*rule exI* [**where** *x=c*])
apply (*rule exI* [**where** *x=b′*])
using *assms*
apply (*auto simp*: *a′-def c′-def b′-def midpoints-in-convex-hull hull-subset* [*THEN subsetD*])

apply (*auto simp*: *midpoint-def dist-norm scaleR-conv-of-real divide-simps*)
done
next
case *4* **then show** *?thesis*
apply (*rule-tac* $x{=}a'$ **in** *exI*)
apply (*rule exI* [**where** $x{=}b'$])
apply (*rule exI* [**where** $x{=}c'$])
using *assms*
apply (*auto simp*: *a'-def c'-def b'-def midpoints-in-convex-hull hull-subset* [*THEN subsetD*])
apply (*auto simp*: *midpoint-def dist-norm scaleR-conv-of-real divide-simps*)
done
qed
qed

53.16 Cauchy's theorem for triangles

lemma *triangle-points-closer*:
fixes *a*::*complex*
shows $[\![x \in \textit{convex hull } \{a,b,c\};\ y \in \textit{convex hull } \{a,b,c\}]\!]$
$\Longrightarrow \textit{norm}(x - y) \le \textit{norm}(a - b) \vee$
$\textit{norm}(x - y) \le \textit{norm}(b - c) \vee$
$\textit{norm}(x - y) \le \textit{norm}(c - a)$
using *simplex-extremal-le* [*of* $\{a,b,c\}$]
by (*auto simp*: *norm-minus-commute*)

lemma *holomorphic-point-small-triangle*:
assumes *x*: $x \in s$
and *f*: *continuous-on s f*
and *cd*: *f field-differentiable* (*at x within s*)
and *e*: $0 < e$
shows $\exists k{>}0.\ \forall a\ b\ c.\ \textit{dist}\ a\ b \le k \wedge \textit{dist}\ b\ c \le k \wedge \textit{dist}\ c\ a \le k \wedge$
$x \in \textit{convex hull } \{a,b,c\} \wedge \textit{convex hull } \{a,b,c\} \subseteq s$
$\longrightarrow \textit{norm}(\textit{contour-integral}(\textit{linepath}\ a\ b)\ f + \textit{contour-integral}(\textit{linepath}\ b\ c)\ f +$
$\textit{contour-integral}(\textit{linepath}\ c\ a)\ f)$
$\le e*(\textit{dist}\ a\ b + \textit{dist}\ b\ c + \textit{dist}\ c\ a)\hat{}2$
(**is** $\exists k{>}0.\ \forall a\ b\ c.$ - $\longrightarrow$ *?normle a b c*)
proof −
have *le-of-3*: $\bigwedge a\ x\ y\ z.\ [\![0 \le x*y;\ 0 \le x*z;\ 0 \le y*z;\ a \le (e*(x + y + z))*x + (e*(x + y + z))*y + (e*(x + y + z))*z]\!]$
$\Longrightarrow a \le e*(x + y + z)\hat{}2$
by (*simp add*: *algebra-simps power2-eq-square*)
have *disj-le*: $[\![x \le a \vee x \le b \vee x \le c;\ 0 \le a;\ 0 \le b;\ 0 \le c]\!] \Longrightarrow x \le a + b + c$
for *x*::*real* **and** *a b c*
by *linarith*
have *fabc*: *f contour-integrable-on linepath a b f contour-integrable-on linepath b c f contour-integrable-on linepath c a*
if *convex hull* $\{a,\ b,\ c\} \subseteq s$ **for** *a b c*

using *segments-subset-convex-hull that*
by (*metis continuous-on-subset f contour-integrable-continuous-linepath*)+
note *path-bound* = *has-contour-integral-bound-linepath* [*simplified norm-minus-commute*, *OF has-contour-integral-integral*]
{ fix f' *a b c d*
assume *d*: $0 < d$
and f': $\bigwedge y.\ [\![cmod\ (y - x) \le d;\ y \in s]\!] \Longrightarrow cmod\ (f\ y - f\ x - f' * (y - x)) \le e * cmod\ (y - x)$
and *le*: $cmod\ (a - b) \le d\ cmod\ (b - c) \le d\ cmod\ (c - a) \le d$
and *xc*: $x \in convex\ hull\ \{a, b, c\}$
and *s*: $convex\ hull\ \{a, b, c\} \subseteq s$
have *pa*: $contour\text{-}integral\ (linepath\ a\ b)\ f + contour\text{-}integral\ (linepath\ b\ c)\ f + contour\text{-}integral\ (linepath\ c\ a)\ f =$
$contour\text{-}integral\ (linepath\ a\ b)\ (\lambda y.\ f\ y - f\ x - f'(y - x)) +$
$contour\text{-}integral\ (linepath\ b\ c)\ (\lambda y.\ f\ y - f\ x - f'(y - x)) +$
$contour\text{-}integral\ (linepath\ c\ a)\ (\lambda y.\ f\ y - f\ x - f'(y - x))$
apply (*simp add*: *contour-integral-diff contour-integral-lmul contour-integrable-lmul contour-integrable-diff fabc* [*OF s*])
apply (*simp add*: *field-simps*)
done
{ fix *y*
assume *yc*: $y \in convex\ hull\ \{a,b,c\}$
have $cmod\ (f\ y - f\ x - f' * (y - x)) \le e * norm(y - x)$
apply (*rule f'*)
apply (*metis triangle-points-closer* [*OF xc yc*] *le norm-minus-commute order-trans*)
using *s yc* **by** *blast*
also have ... $\le e * (cmod\ (a - b) + cmod\ (b - c) + cmod\ (c - a))$
by (*simp add*: *yc e xc disj-le* [*OF triangle-points-closer*])
finally have $cmod\ (f\ y - f\ x - f' * (y - x)) \le e * (cmod\ (a - b) + cmod\ (b - c) + cmod\ (c - a))$ **.**
} **note** *cm-le* = *this*
have *?normle a b c*
apply (*simp add*: *dist-norm pa*)
apply (*rule le-of-3*)
using f' *xc s e*
apply *simp-all*
apply (*intro norm-triangle-le add-mono path-bound*)
apply (*simp-all add*: *contour-integral-diff contour-integral-lmul contour-integrable-lmul contour-integrable-diff fabc*)
apply (*blast intro*: *cm-le elim*: *dest*: *segments-subset-convex-hull* [*THEN subsetD*])+
done
} **note** $*$ = *this*
show *?thesis*
using *cd e*
apply (*simp add*: *field-differentiable-def has-field-derivative-def has-derivative-within-alt approachable-lt-le2 Ball-def*)
apply (*clarify dest*!: *spec mp*)

```
    using ∗
    apply (simp add: dist-norm, blast)
    done
qed



locale Chain =
  fixes x0 At Follows
  assumes At0: At x0 0
      and AtSuc: ⋀x n. At x n ⟹ ∃x'. At x' (Suc n) ∧ Follows x' x
begin
  primrec f where
    f 0 = x0
  | f (Suc n) = (SOME x. At x (Suc n) ∧ Follows x (f n))

  lemma At: At (f n) n
  proof (induct n)
    case 0 show ?case
      by (simp add: At0)
  next
    case (Suc n) show ?case
      by (metis (no-types, lifting) AtSuc [OF Suc] f.simps(2) someI-ex)
  qed

  lemma Follows: Follows (f(Suc n)) (f n)
    by (metis (no-types, lifting) AtSuc [OF At [of n]] f.simps(2) someI-ex)

  declare f.simps(2) [simp del]
end

lemma Chain3:
  assumes At0: At x0 y0 z0 0
      and AtSuc: ⋀x y z n. At x y z n ⟹ ∃x' y' z'. At x' y' z' (Suc n) ∧ Follows
x' y' z' x y z
  obtains f g h where
    f 0 = x0 g 0 = y0 h 0 = z0
                  ⋀n. At (f n) (g n) (h n) n
                  ⋀n. Follows (f(Suc n)) (g(Suc n)) (h(Suc n)) (f n) (g n) (h n)
proof −
  interpret three: Chain (x0,y0,z0) λ(x,y,z). At x y z λ(x',y',z'). λ(x,y,z). Follows
x' y' z' x y z
    apply unfold-locales
    using At0 AtSuc by auto
  show ?thesis
  apply (rule that [of λn. fst (three.f n)  λn. fst (snd (three.f n)) λn. snd (snd
(three.f n))])
  apply simp-all
  using three.At three.Follows
```

```
  apply (simp-all add: split-beta')
  done
qed

lemma Cauchy-theorem-triangle:
  assumes f holomorphic-on (convex hull {a,b,c})
    shows (f has-contour-integral 0) (linepath a b +++ linepath b c +++ linepath
c a)
proof −
  have contf: continuous-on (convex hull {a,b,c}) f
    by (metis assms holomorphic-on-imp-continuous-on)
  let ?pathint = λx y. contour-integral(linepath x y) f
  { fix y::complex
    assume fy: (f has-contour-integral y) (linepath a b +++ linepath b c +++
linepath c a)
       and ynz: y ≠ 0
    def K ≡ 1 + max (dist a b) (max (dist b c) (dist c a))
    def e ≡ norm y / K^2
    have K1: K ≥ 1  by (simp add: K-def max.coboundedI1)
    then have K: K > 0 by linarith
    have [iff]: dist a b ≤ K dist b c ≤ K dist c a ≤ K
      by (simp-all add: K-def)
    have e: e > 0
      unfolding e-def using ynz K1 by simp
    def At ≡ λx y z n. convex hull {x,y,z} ⊆ convex hull {a,b,c} ∧
                       dist x y ≤ K/2^n ∧ dist y z ≤ K/2^n ∧ dist z x ≤ K/2^n ∧
                     norm(?pathint x y + ?pathint y z + ?pathint z x) ≥ e*(K/2^n)^2
    have At0: At a b c 0
      using fy
      by (simp add: At-def e-def has-chain-integral-chain-integral3)
    { fix x y z n
      assume At: At x y z n
      then have contf': continuous-on (convex hull {x,y,z}) f
        using contf At-def continuous-on-subset by blast
       have ∃x' y' z'. At x' y' z' (Suc n) ∧ convex hull {x',y',z'} ⊆ convex hull
{x,y,z}
        using At
        apply (simp add: At-def)
        using  Cauchy-theorem-quadrisection [OF contf', of K/2^n e]
        apply clarsimp
        apply (rule-tac x=a' in exI)
        apply (rule-tac x=b' in exI)
        apply (rule-tac x=c' in exI)
        apply (simp add: algebra-simps)
        apply (meson convex-hull-subset empty-subsetI insert-subset subsetCE)
        done
    } note AtSuc = this
    obtain fa fb fc
      where f0 [simp]: fa 0 = a fb 0 = b fc 0 = c
```

and *cosb*: $\bigwedge n.$ *convex hull* $\{fa\ n, fb\ n, fc\ n\} \subseteq$ *convex hull* $\{a,b,c\}$
and *dist*: $\bigwedge n.$ *dist* $(fa\ n)\ (fb\ n) \le K/2\hat{}n$
$\bigwedge n.$ *dist* $(fb\ n)\ (fc\ n) \le K/2\hat{}n$
$\bigwedge n.$ *dist* $(fc\ n)\ (fa\ n) \le K/2\hat{}n$
and *no*: $\bigwedge n.$ *norm*(*?pathint* $(fa\ n)\ (fb\ n) +$
?pathint $(fb\ n)\ (fc\ n) +$
?pathint $(fc\ n)\ (fa\ n)) \ge e * (K/2\hat{}n)\hat{}2$
and *conv-le*: $\bigwedge n.$ *convex hull* $\{fa(Suc\ n), fb(Suc\ n), fc(Suc\ n)\} \subseteq$ *convex hull* $\{fa\ n, fb\ n, fc\ n\}$
apply (*rule Chain3* [*of At*, *OF At0 AtSuc*])
apply (*auto simp*: *At-def*)
done
have $\exists x.\ \forall n.\ x \in$ *convex hull* $\{fa\ n, fb\ n, fc\ n\}$
apply (*rule bounded-closed-nest*)
apply (*simp-all add*: *compact-imp-closed finite-imp-compact-convex-hull finite-imp-bounded-convex-hull*)
apply (*rule allI*)
apply (*rule transitive-stepwise-le*)
apply (*auto simp*: *conv-le*)
done
then obtain x **where** x: $\bigwedge n.\ x \in$ *convex hull* $\{fa\ n, fb\ n, fc\ n\}$ **by** *auto*
then have *xin*: $x \in$ *convex hull* $\{a,b,c\}$
using *assms f0* **by** *blast*
then have *fx*: *f field-differentiable at x within* (*convex hull* $\{a,b,c\}$)
using *assms holomorphic-on-def* **by** *blast*
{ fix $k\ n$
assume k: $0 < k$
and *le*:
$\bigwedge x'\ y'\ z'.$
$[\![dist\ x'\ y' \le k;\ dist\ y'\ z' \le k;\ dist\ z'\ x' \le k;$
$x \in$ *convex hull* $\{x',y',z'\}$;
convex hull $\{x',y',z'\} \subseteq$ *convex hull* $\{a,b,c\}]\!]$
$\Longrightarrow$
cmod (*?pathint* $x'\ y' +$ *?pathint* $y'\ z' +$ *?pathint* $z'\ x') * 10$
$\le e * (dist\ x'\ y' + dist\ y'\ z' + dist\ z'\ x')^2$
and *Kk*: $K\ /\ k < 2\ \hat{}\ n$
have $K\ /\ 2\ \hat{}\ n < k$ **using** *Kk k*
by (*auto simp*: *field-simps*)
then have *DD*: *dist* $(fa\ n)\ (fb\ n) \le k$ *dist* $(fb\ n)\ (fc\ n) \le k$ *dist* $(fc\ n)\ (fa\ n) \le k$
using *dist* [*of n*] k
by *linarith*+
have *dle*: $(dist\ (fa\ n)\ (fb\ n) + dist\ (fb\ n)\ (fc\ n) + dist\ (fc\ n)\ (fa\ n))^2$
$\le (3 * K\ /\ 2\ \hat{}\ n)^2$
using *dist* [*of n*] *e K*
by (*simp add*: *abs-le-square-iff* [*symmetric*])
have *less10*: $\bigwedge x\ y$::*real*. $0 < x \Longrightarrow y \le 9*x \Longrightarrow y < x*10$
by *linarith*
have $e * (dist\ (fa\ n)\ (fb\ n) + dist\ (fb\ n)\ (fc\ n) + dist\ (fc\ n)\ (fa\ n))^2 \le e * (3 * K\ /\ 2\ \hat{}\ n)^2$

```
      using ynz dle e mult-le-cancel-left-pos by blast
    also have ... <
      cmod (?pathint (fa n) (fb n) + ?pathint (fb n) (fc n) + ?pathint (fc n) (fa
n)) * 10
      using no [of n] e K
      apply (simp add: e-def field-simps)
      apply (simp only: zero-less-norm-iff [symmetric])
      done
    finally have False
      using le [OF DD x cosb] by auto
  } then
  have ?thesis
    using holomorphic-point-small-triangle [OF xin contf fx, of e/10] e
    apply clarsimp
    apply (rule-tac y1=K/k in exE [OF real-arch-pow[of 2]])
    apply force+
    done
 }
 moreover have f contour-integrable-on (linepath a b +++ linepath b c +++
linepath c a)
  by simp (meson contf continuous-on-subset contour-integrable-continuous-linepath
segments-subset-convex-hull(1)
                segments-subset-convex-hull(3) segments-subset-convex-hull(5))
 ultimately show ?thesis
  using has-contour-integral-integral by fastforce
qed
```

53.17 Version needing function holomorphic in interior only

```
lemma Cauchy-theorem-flat-lemma:
 assumes f: continuous-on (convex hull {a,b,c}) f
    and c: c - a = k *R (b - a)
    and k: 0 ≤ k
  shows contour-integral (linepath a b) f + contour-integral (linepath b c) f +
        contour-integral (linepath c a) f = 0
proof -
 have fabc: continuous-on (closed-segment a b) f continuous-on (closed-segment b
c) f continuous-on (closed-segment c a) f
  using f continuous-on-subset segments-subset-convex-hull by metis+
 show ?thesis
 proof (cases k ≤ 1)
  case True show ?thesis
  by (simp add: contour-integral-split [OF fabc(1) k True c] contour-integral-reverse-linepath
fabc)
 next
  case False then show ?thesis
    using fabc c
    apply (subst contour-integral-split [of a c f 1/k b, symmetric])
    apply (metis closed-segment-commute fabc(3))
```

apply (*auto simp*: *k contour-integral-reverse-linepath*)
done
qed
qed

lemma *Cauchy-theorem-flat*:
assumes *f*: *continuous-on* (*convex hull* {*a*,*b*,*c*}) *f*
and *c*: *c* − *a* = *k* ∗*R* (*b* − *a*)
shows *contour-integral* (*linepath a b*) *f* +
contour-integral (*linepath b c*) *f* +
contour-integral (*linepath c a*) *f* = *0*
proof (*cases 0* ≤ *k*)
case *True* **with** *assms* **show** *?thesis*
by (*blast intro*: *Cauchy-theorem-flat-lemma*)
next
case *False*
have *continuous-on* (*closed-segment a b*) *f continuous-on* (*closed-segment b c*) *f continuous-on* (*closed-segment c a*) *f*
using *f continuous-on-subset segments-subset-convex-hull* **by** *metis+*
moreover have *contour-integral* (*linepath b a*) *f* + *contour-integral* (*linepath a c*) *f* +
contour-integral (*linepath c b*) *f* = *0*
apply (*rule Cauchy-theorem-flat-lemma* [*of b a c f* 1−*k*])
using *False c*
apply (*auto simp*: *f insert-commute scaleR-conv-of-real algebra-simps*)
done
ultimately show *?thesis*
apply (*auto simp*: *contour-integral-reverse-linepath*)
using *add-eq-0-iff* **by** *force*
qed

lemma *Cauchy-theorem-triangle-interior*:
assumes *contf*: *continuous-on* (*convex hull* {*a*,*b*,*c*}) *f*
and *holf*: *f holomorphic-on interior* (*convex hull* {*a*,*b*,*c*})
shows (*f has-contour-integral 0*) (*linepath a b* +++ *linepath b c* +++ *linepath c a*)
proof −
have *fabc*: *continuous-on* (*closed-segment a b*) *f continuous-on* (*closed-segment b c*) *f continuous-on* (*closed-segment c a*) *f*
using *contf continuous-on-subset segments-subset-convex-hull* **by** *metis+*
have *bounded* (*f* ‘ (*convex hull* {*a*,*b*,*c*}))
by (*simp add*: *compact-continuous-image compact-convex-hull compact-imp-bounded contf*)
then obtain *B* **where** *0* < *B* **and** *Bnf*: ⋀*x*. *x* ∈ *convex hull* {*a*,*b*,*c*} ⟹ *norm* (*f x*) ≤ *B*
by (*auto simp*: *dest*!: *bounded-pos* [*THEN iffD1*])
have *bounded* (*convex hull* {*a*,*b*,*c*})
by (*simp add*: *bounded-convex-hull*)

then obtain C **where** C: $0 < C$ **and** Cno: $\bigwedge y.\ y \in$ *convex hull* $\{a,b,c\} \Longrightarrow$ *norm* $y < C$
using *bounded-pos-less* **by** *blast*
then have *diff-2C*: $norm(x - y) \leq 2{*}C$
if x: $x \in$ *convex hull* $\{a,\ b,\ c\}$ **and** y: $y \in$ *convex hull* $\{a,\ b,\ c\}$ **for** $x\ y$
proof −
have *cmod* $x \leq C$
using x **by** (*meson Cno not-le not-less-iff-gr-or-eq*)
hence *cmod* $(x - y) \leq C + C$
using y **by** (*meson Cno add-mono-thms-linordered-field*(*4*) *less-eq-real-def norm-triangle-ineq4 order-trans*)
thus *cmod* $(x - y) \leq 2 * C$
by (*metis mult-2*)
qed
have *contf'*: *continuous-on* (*convex hull* $\{b,a,c\}$) f
using *contf* **by** (*simp add*: *insert-commute*)
{ fix y::*complex*
assume *fy*: (f *has-contour-integral* y) (*linepath* $a\ b$ +++ *linepath* $b\ c$ +++ *linepath* $c\ a$)
and *ynz*: $y \neq 0$
have *pi-eq-y*: *contour-integral* (*linepath* $a\ b$) f + *contour-integral* (*linepath* $b\ c$) f + *contour-integral* (*linepath* $c\ a$) $f = y$
by (*rule has-chain-integral-chain-integral3* [*OF fy*])
have *?thesis*
proof (*cases* $c{=}a \vee a{=}b \vee b{=}c$)
case *True* **then show** *?thesis*
using *Cauchy-theorem-flat* [*OF contf*, *of 0*]
using *has-chain-integral-chain-integral3* [*OF fy*] *ynz*
by (*force simp*: *fabc contour-integral-reverse-linepath*)
next
case *False*
then have *car3*: *card* $\{a,\ b,\ c\} = Suc\ (DIM(complex))$
by *auto*
{ assume *interior*(*convex hull* $\{a,b,c\}$) = $\{\}$
then have *collinear*$\{a,b,c\}$
using *interior-convex-hull-eq-empty* [*OF car3*]
by (*simp add*: *collinear-3-eq-affine-dependent*)
then have *False*
using *False*
apply (*clarsimp simp add*: *collinear-3 collinear-lemma*)
apply (*drule Cauchy-theorem-flat* [*OF contf'*])
using *pi-eq-y ynz*
apply (*simp add*: *fabc add-eq-0-iff contour-integral-reverse-linepath*)
done
}
then obtain d **where** d: $d \in$ *interior* (*convex hull* $\{a,\ b,\ c\}$)
by *blast*
{ fix *d1*
assume *d1-pos*: $0 < d1$

and *d1*: $\bigwedge x\ x'.\ [\![x \in \textit{convex hull}\ \{a,\ b,\ c\};\ x' \in \textit{convex hull}\ \{a,\ b,\ c\};\ \textit{cmod}\ (x' - x) < \textit{d1}]\!]$
$\Longrightarrow \textit{cmod}\ (f\ x' - f\ x) < \textit{cmod}\ y\ /\ (24 * C)$
def *e* $\equiv \textit{min}\ 1\ (\textit{min}\ (\textit{d1}/(4*C))\ ((\textit{norm}\ y\ /\ 24\ /\ C)\ /\ B))$
def *shrink* $\equiv \lambda x.\ x - e *_R (x - d)$
let *?pathint* $= \lambda x\ y.\ \textit{contour-integral}(\textit{linepath}\ x\ y)\ f$
have *e*: $0 < e\ e \le 1\ e \le \textit{d1}\ /\ (4 * C)\ e \le \textit{cmod}\ y\ /\ 24\ /\ C\ /\ B$
using *d1-pos* ‹*C*>0› ‹*B*>0› *ynz* **by** (*simp-all add*: *e-def*)
then have *eCB*: $24 * e * C * B \le \textit{cmod}\ y$
using ‹*C*>0› ‹*B*>0› **by** (*simp add*: *field-simps*)
have *e-le-d1*: $e * (4 * C) \le \textit{d1}$
using *e* ‹*C*>0› **by** (*simp add*: *field-simps*)
have *shrink a* $\in \textit{interior}(\textit{convex hull}\ \{a,b,c\})$
shrink b $\in \textit{interior}(\textit{convex hull}\ \{a,b,c\})$
shrink c $\in \textit{interior}(\textit{convex hull}\ \{a,b,c\})$
using *d e* **by** (*auto simp*: *hull-inc mem-interior-convex-shrink shrink-def*)
then have *fhp0*: (*f has-contour-integral 0*)
(*linepath* (*shrink a*) (*shrink b*) +++ *linepath* (*shrink b*) (*shrink c*) +++ *linepath* (*shrink c*) (*shrink a*))
by (*simp add*: *Cauchy-theorem-triangle holomorphic-on-subset* [*OF holf*] *hull-minimal convex-interior*)
then have *f-0-shrink*: *?pathint* (*shrink a*) (*shrink b*) + *?pathint* (*shrink b*) (*shrink c*) + *?pathint* (*shrink c*) (*shrink a*) = *0*
by (*simp add*: *has-chain-integral-chain-integral3*)
have *fpi-abc*: *f contour-integrable-on linepath* (*shrink a*) (*shrink b*)
f contour-integrable-on linepath (*shrink b*) (*shrink c*)
f contour-integrable-on linepath (*shrink c*) (*shrink a*)
using *fhp0* **by** (*auto simp*: *valid-path-join dest*: *has-contour-integral-integrable*)
have *cmod-shr*: $\bigwedge x\ y.\ \textit{cmod}\ (\textit{shrink}\ y - \textit{shrink}\ x - (y - x)) = e * \textit{cmod}\ (x - y)$
using *e* **by** (*simp add*: *shrink-def real-vector.scale-right-diff-distrib* [*symmetric*])
have *sh-eq*: $\bigwedge a\ b\ d{::}\textit{complex}.\ (b - e *_R (b - d)) - (a - e *_R (a - d)) - (b - a) = e *_R (a - b)$
by (*simp add*: *algebra-simps*)
have *cmod y* / (*24* * *C*) ≤ *cmod y* / *cmod* (*b* − *a*) / *12*
using *False* ‹*C*>0› *diff-2C* [*of b a*] *ynz*
by (*auto simp*: *divide-simps hull-inc*)
have *less-C*: $[\![u \in \textit{convex hull}\ \{a,\ b,\ c\};\ 0 \le x;\ x \le 1]\!] \Longrightarrow x * \textit{cmod}\ u < C$ **for** *x u*
apply (*cases x*=*0*, *simp add*: ‹*0*<*C*›)
using *Cno* [*of u*] *mult-left-le-one-le* [*of cmod u x*] *le-less-trans norm-ge-zero* **by** *blast*
{ fix *u v*
assume *uv*: $u \in \textit{convex hull}\ \{a,\ b,\ c\}\ v \in \textit{convex hull}\ \{a,\ b,\ c\}\ u \neq v$
and *fpi-uv*: *f contour-integrable-on linepath* (*shrink u*) (*shrink v*)
have *shr-uv*: *shrink u* $\in \textit{interior}(\textit{convex hull}\ \{a,b,c\})$
shrink v $\in \textit{interior}(\textit{convex hull}\ \{a,b,c\})$
using *d e uv*

by (*auto simp*: *hull-inc mem-interior-convex-shrink shrink-def*)
have *cmod-fuv*: $\bigwedge x.\ 0{\le}x \Longrightarrow x{\le}1 \Longrightarrow cmod\ (f\ (linepath\ (shrink\ u)\ (shrink\ v)\ x)) \le B$
using *shr-uv* **by** (*blast intro*: *Bnf linepath-in-convex-hull interior-subset* [*THEN subsetD*])
have *By-uv*: $B * (12 * (e * cmod\ (u - v))) \le cmod\ y$
apply (*rule order-trans* [*OF - eCB*])
using *e* ‹*B*>0› *diff-2C* [*of u v*] *uv*
by (*auto simp*: *field-simps*)
{ fix *x*::*real* **assume** *x*: $0{\le}x\ x{\le}1$
have *cmod-less-4C*: $cmod\ ((1 - x) *_R u - (1 - x) *_R d) + cmod\ (x *_R v - x *_R d) < (C{+}C) + (C{+}C)$
apply (*rule add-strict-mono*; *rule norm-triangle-half-l* [*of - 0*])
using *uv x d interior-subset*
apply (*auto simp*: *hull-inc intro*!: *less-C*)
done
have *ll*: $linepath\ (shrink\ u)\ (shrink\ v)\ x - linepath\ u\ v\ x = -e * ((1 - x) *_R (u - d) + x *_R (v - d))$
by (*simp add*: *linepath-def shrink-def algebra-simps scaleR-conv-of-real*)
have *cmod-less-dt*: $cmod\ (linepath\ (shrink\ u)\ (shrink\ v)\ x - linepath\ u\ v\ x) < d1$
using ‹*e*>0›
apply (*simp add*: *ll norm-mult scaleR-diff-right*)
apply (*rule less-le-trans* [*OF - e-le-d1*])
using *cmod-less-4C*
apply (*force intro*: *norm-triangle-lt*)
done
have *cmod* $(f\ (linepath\ (shrink\ u)\ (shrink\ v)\ x) - f\ (linepath\ u\ v\ x)) < cmod\ y\ /\ (24 * C)$
using *x uv shr-uv cmod-less-dt*
by (*auto simp*: *hull-inc intro*: *d1 interior-subset* [*THEN subsetD*] *linepath-in-convex-hull*)
also have ... $\le cmod\ y\ /\ cmod\ (v - u)\ /\ 12$
using *False uv* ‹*C*>0› *diff-2C* [*of v u*] *ynz*
by (*auto simp*: *divide-simps hull-inc*)
finally have *cmod* $(f\ (linepath\ (shrink\ u)\ (shrink\ v)\ x) - f\ (linepath\ u\ v\ x)) \le cmod\ y\ /\ cmod\ (v - u)\ /\ 12$
by *simp*
then have *cmod-12-le*: $cmod\ (v - u) * cmod\ (f\ (linepath\ (shrink\ u)\ (shrink\ v)\ x) - f\ (linepath\ u\ v\ x)) * 12 \le cmod\ y$
using *uv False* **by** (*auto simp*: *field-simps*)
have $cmod\ (f\ (linepath\ (shrink\ u)\ (shrink\ v)\ x)) * cmod\ (shrink\ v - shrink\ u - (v - u)) +$
$cmod\ (v - u) * cmod\ (f\ (linepath\ (shrink\ u)\ (shrink\ v)\ x) - f\ (linepath\ u\ v\ x))$
$\le cmod\ y\ /\ 6$
apply (*rule order-trans* [*of - B*∗((*norm y / 24 / C / B*)∗2∗*C*) + (2∗*C*)∗(*norm y* /24 / *C*)])
apply (*rule add-mono* [*OF mult-mono*])

using *By-uv e ‹0 < B› ‹0 < C› x ynz*
apply (*simp-all add*: *cmod-fuv cmod-shr cmod-12-le hull-inc*)
apply (*simp add*: *field-simps*)
done
} note *cmod-diff-le = this*
have *f-uv*: *continuous-on* (*closed-segment u v*) *f*
by (*blast intro*: *uv continuous-on-subset* [*OF contf closed-segment-subset-convex-hull*])
have ∗∗: ⋀*f' x' f x*::*complex*. *f'∗x' − f∗x = f'∗*(*x' − x*) + *x∗*(*f' − f*)
by (*simp add*: *algebra-simps*)
have *norm* (*?pathint* (*shrink u*) (*shrink v*) − *?pathint u v*) ≤ *norm y / 6*
apply (*rule order-trans*)
apply (*rule has-integral-bound*
[*of B∗*(*norm y /24/C/B*)*2∗C* + (*2∗C*)∗(*norm y*/*24*/*C*)
λ*x*. *f*(*linepath* (*shrink u*) (*shrink v*) *x*) ∗ (*shrink v* − *shrink u*)
− *f*(*linepath u v x*)∗(*v* − *u*)
- 0 1])
using *ynz ‹0 < B› ‹0 < C›*
apply (*simp-all del*: *le-divide-eq-numeral1*)
apply (*simp add*: *has-integral-sub has-contour-integral-linepath* [*symmetric*]
has-contour-integral-integral
fpi-uv f-uv contour-integrable-continuous-linepath, *clarify*)
apply (*simp only*: ∗∗)
apply (*simp add*: *norm-triangle-le norm-mult cmod-diff-le del*: *le-divide-eq-numeral1*)
done
} note ∗ = *this*
have *norm* (*?pathint* (*shrink a*) (*shrink b*) − *?pathint a b*) ≤ *norm y / 6*
using *False fpi-abc* **by** (*rule-tac* ∗) (*auto simp*: *hull-inc*)
moreover
have *norm* (*?pathint* (*shrink b*) (*shrink c*) − *?pathint b c*) ≤ *norm y / 6*
using *False fpi-abc* **by** (*rule-tac* ∗) (*auto simp*: *hull-inc*)
moreover
have *norm* (*?pathint* (*shrink c*) (*shrink a*) − *?pathint c a*) ≤ *norm y / 6*
using *False fpi-abc* **by** (*rule-tac* ∗) (*auto simp*: *hull-inc*)
ultimately
have *norm*((*?pathint* (*shrink a*) (*shrink b*) − *?pathint a b*) +
(*?pathint* (*shrink b*) (*shrink c*) − *?pathint b c*) + (*?pathint* (*shrink c*) (*shrink a*) − *?pathint c a*))
≤ *norm y / 6* + *norm y / 6* + *norm y / 6*
by (*metis norm-triangle-le add-mono*)
also have ... = *norm y / 2*
by *simp*
finally have *norm*((*?pathint* (*shrink a*) (*shrink b*) + *?pathint* (*shrink b*) (*shrink c*) + *?pathint* (*shrink c*) (*shrink a*)) −
(*?pathint a b* + *?pathint b c* + *?pathint c a*))
≤ *norm y / 2*
by (*simp add*: *algebra-simps*)
then
have *norm*(*?pathint a b* + *?pathint b c* + *?pathint c a*) ≤ *norm y / 2*
by (*simp add*: *f-0-shrink*) (*metis* (*mono-tags*) *add.commute minus-add-distrib*

norm-minus-cancel uminus-add-conv-diff)
then have *False*
using *pi-eq-y ynz* **by** *auto*
}
moreover have *uniformly-continuous-on* (*convex hull* {*a*,*b*,*c*}) *f*
by (*simp add*: *contf compact-convex-hull compact-uniformly-continuous*)
ultimately have *False*
unfolding *uniformly-continuous-on-def*
by (*force simp*: *ynz* ‹*0* < *C*› *dist-norm*)
then show *?thesis* **..**
qed
}
moreover have *f contour-integrable-on* (*linepath a b +++ linepath b c +++ linepath c a*)
using *fabc contour-integrable-continuous-linepath* **by** *auto*
ultimately show *?thesis*
using *has-contour-integral-integral* **by** *fastforce*
qed

53.18 Version allowing finite number of exceptional points

lemma *Cauchy-theorem-triangle-cofinite*:
assumes *continuous-on* (*convex hull* {*a*,*b*,*c*}) *f*
and *finite s*
and (⋀*x*. *x* ∈ *interior*(*convex hull* {*a*,*b*,*c*}) − *s* ⟹ *f field-differentiable* (*at x*))
shows (*f has-contour-integral 0*) (*linepath a b +++ linepath b c +++ linepath c a*)
using *assms*
proof (*induction card s arbitrary*: *a b c s rule*: *less-induct*)
case (*less s a b c*)
show *?case*
proof (*cases s*={})
case *True* **with** *less* **show** *?thesis*
by (*fastforce simp*: *holomorphic-on-def field-differentiable-at-within*
Cauchy-theorem-triangle-interior)
next
case *False*
then obtain *d s'* **where** *d*: *s* = *insert d s' d* ∉ *s'*
by (*meson Set.set-insert all-not-in-conv*)
then show *?thesis*
proof (*cases d* ∈ *convex hull* {*a*,*b*,*c*})
case *False*
show (*f has-contour-integral 0*) (*linepath a b +++ linepath b c +++ linepath c a*)
apply (*rule less.hyps* [*of s'*])
using *False d* ‹*finite s*› *interior-subset*
apply (*auto intro*!: *less.prems*)
done

next
case *True*
have ∗: *convex hull* {*a*, *b*, *d*} ⊆ *convex hull* {*a*, *b*, *c*}
by (*meson True hull-subset insert-subset convex-hull-subset*)
have *abd*: (*f has-contour-integral 0*) (*linepath a b +++ linepath b d +++ linepath d a*)
apply (*rule less.hyps* [*of s'*])
using *True d* ‹*finite s*› *not-in-interior-convex-hull-3*
apply (*auto intro*!: *less.prems continuous-on-subset* [*OF - ∗*])
apply (*metis ∗ insert-absorb insert-subset interior-mono*)
done
have ∗: *convex hull* {*b*, *c*, *d*} ⊆ *convex hull* {*a*, *b*, *c*}
by (*meson True hull-subset insert-subset convex-hull-subset*)
have *bcd*: (*f has-contour-integral 0*) (*linepath b c +++ linepath c d +++ linepath d b*)
apply (*rule less.hyps* [*of s'*])
using *True d* ‹*finite s*› *not-in-interior-convex-hull-3*
apply (*auto intro*!: *less.prems continuous-on-subset* [*OF - ∗*])
apply (*metis ∗ insert-absorb insert-subset interior-mono*)
done
have ∗: *convex hull* {*c*, *a*, *d*} ⊆ *convex hull* {*a*, *b*, *c*}
by (*meson True hull-subset insert-subset convex-hull-subset*)
have *cad*: (*f has-contour-integral 0*) (*linepath c a +++ linepath a d +++ linepath d c*)
apply (*rule less.hyps* [*of s'*])
using *True d* ‹*finite s*› *not-in-interior-convex-hull-3*
apply (*auto intro*!: *less.prems continuous-on-subset* [*OF - ∗*])
apply (*metis ∗ insert-absorb insert-subset interior-mono*)
done
have *f contour-integrable-on linepath a b*
using *less.prems*
by (*metis continuous-on-subset insert-commute contour-integrable-continuous-linepath segments-subset-convex-hull*(*3*))
moreover have *f contour-integrable-on linepath b c*
using *less.prems*
by (*metis continuous-on-subset contour-integrable-continuous-linepath segments-subset-convex-hull*(*3*))
moreover have *f contour-integrable-on linepath c a*
using *less.prems*
by (*metis continuous-on-subset insert-commute contour-integrable-continuous-linepath segments-subset-convex-hull*(*3*))
ultimately have *fpi*: *f contour-integrable-on* (*linepath a b +++ linepath b c +++ linepath c a*)
by *auto*
{ fix *y*::*complex*
assume *fy*: (*f has-contour-integral y*) (*linepath a b +++ linepath b c +++ linepath c a*)
and *ynz*: *y* ≠ *0*
have *cont-ad*: *continuous-on* (*closed-segment a d*) *f*
by (*meson ∗ continuous-on-subset less.prems*(*1*) *segments-subset-convex-hull*(*3*))

have *cont-bd*: *continuous-on* (*closed-segment b d*) *f*
by (*meson True closed-segment-subset-convex-hull continuous-on-subset hull-subset insert-subset less.prems*(*1*))
have *cont-cd*: *continuous-on* (*closed-segment c d*) *f*
by (*meson* ∗ *continuous-on-subset less.prems*(*1*) *segments-subset-convex-hull*(*2*))
have *contour-integral* (*linepath a b*) *f* = − (*contour-integral* (*linepath b d*) *f* + (*contour-integral* (*linepath d a*) *f*))
contour-integral (*linepath b c*) *f* = − (*contour-integral* (*linepath c d*) *f* + (*contour-integral* (*linepath d b*) *f*))
contour-integral (*linepath c a*) *f* = − (*contour-integral* (*linepath a d*) *f* + *contour-integral* (*linepath d c*) *f*)
using *has-chain-integral-chain-integral3* [*OF abd*]
has-chain-integral-chain-integral3 [*OF bcd*]
has-chain-integral-chain-integral3 [*OF cad*]
by (*simp-all add*: *algebra-simps add-eq-0-iff*)
then have *?thesis*
using *cont-ad cont-bd cont-cd fy has-chain-integral-chain-integral3 contour-integral-reverse-linepath*
by *fastforce*
}
then show *?thesis*
using *fpi contour-integrable-on-def* **by** *blast*
qed
qed
qed

53.19 Cauchy's theorem for an open starlike set

lemma *starlike-convex-subset*:
assumes *s*: *a* ∈ *s closed-segment b c* ⊆ *s* **and** *subs*: ⋀*x*. *x* ∈ *s* ⟹ *closed-segment a x* ⊆ *s*
shows *convex hull* {*a*,*b*,*c*} ⊆ *s*
using *s*
apply (*clarsimp simp add*: *convex-hull-insert* [*of* {*b*,*c*} *a*] *segment-convex-hull*)
apply (*meson subs convexD convex-closed-segment ends-in-segment*(*1*) *ends-in-segment*(*2*) *subsetCE*)
done

lemma *triangle-contour-integrals-starlike-primitive*:
assumes *contf*: *continuous-on s f*
and *s*: *a* ∈ *s open s*
and *x*: *x* ∈ *s*
and *subs*: ⋀*y*. *y* ∈ *s* ⟹ *closed-segment a y* ⊆ *s*
and *zer*: ⋀*b c*. *closed-segment b c* ⊆ *s*
⟹ *contour-integral* (*linepath a b*) *f* + *contour-integral* (*linepath b c*) *f* +
contour-integral (*linepath c a*) *f* = *0*
shows ((*λx*. *contour-integral*(*linepath a x*) *f*) *has-field-derivative f x*) (*at x*)
proof −
let *?pathint* = *λx y*. *contour-integral*(*linepath x y*) *f*

```
  { fix e y
    assume e: 0 < e and bxe: ball x e ⊆ s and close: cmod (y − x) < e
    have y: y ∈ s
      using bxe close  by (force simp: dist-norm norm-minus-commute)
    have cont-ayf: continuous-on (closed-segment a y) f
      using contf continuous-on-subset subs y by blast
    have xys: closed-segment x y ⊆ s
      apply (rule order-trans [OF - bxe])
      using close
     by (auto simp: dist-norm ball-def norm-minus-commute dest: segment-bound)
    have ?pathint a y − ?pathint a x = ?pathint x y
    using zer [OF xys] contour-integral-reverse-linepath [OF cont-ayf] add-eq-0-iff
by force
  } note [simp] = this
  { fix e::real
    assume e: 0 < e
    have cont-atx: continuous (at x) f
      using x s contf continuous-on-eq-continuous-at by blast
    then obtain d1 where d1: d1>0 and d1-less: ⋀y. cmod (y − x) < d1 ⟹
cmod (f y − f x) < e/2
      unfolding continuous-at Lim-at dist-norm  using e
      by (drule-tac x=e/2 in spec) force
    obtain d2 where d2: d2>0 ball x d2 ⊆ s using ‹open s› x
      by (auto simp: open-contains-ball)
    have dpos: min d1 d2 > 0 using d1 d2 by simp
    { fix y
      assume yx: y ≠ x and close: cmod (y − x) < min d1 d2
      have y: y ∈ s
        using d2 close  by (force simp: dist-norm norm-minus-commute)
      have fxy: f contour-integrable-on linepath x y
        apply (rule contour-integrable-continuous-linepath)
        apply (rule continuous-on-subset [OF contf])
        using close d2
       apply (auto simp: dist-norm norm-minus-commute dest!: segment-bound(1))
        done
      then obtain i where i: (f has-contour-integral i) (linepath x y)
        by (auto simp: contour-integrable-on-def)
     then have ((λw. f w − f x) has-contour-integral (i − f x ∗ (y − x))) (linepath
x y)
       by (rule has-contour-integral-diff [OF - has-contour-integral-const-linepath])
      then have cmod (i − f x ∗ (y − x)) ≤ e / 2 ∗ cmod (y − x)
        apply (rule has-contour-integral-bound-linepath [where B = e/2])
        using e apply simp
        apply (rule d1-less [THEN less-imp-le])
        using close segment-bound
        apply force
        done
      also have ... < e ∗ cmod (y − x)
        by (simp add: e yx)
```

```
    finally have cmod (?pathint x y − f x ∗ (y−x)) / cmod (y−x) < e
      using i yx by (simp add: contour-integral-unique divide-less-eq)
   }
   then have ∃ d>0. ∀ y. y ≠ x ∧ cmod (y−x) < d ⟶ cmod (?pathint x y − f
x ∗ (y−x)) / cmod (y−x) < e
    using dpos by blast
 }
 then have ∗: (λy. (?pathint x y − f x ∗ (y − x)) /R cmod (y − x)) −x→ 0
  by (simp add: Lim-at dist-norm inverse-eq-divide)
 show ?thesis
  apply (simp add: has-field-derivative-def has-derivative-at bounded-linear-mult-right)
   apply (rule Lim-transform [OF ∗ Lim-eventually])
   apply (simp add: inverse-eq-divide [symmetric] eventually-at)
   using ‹open s› x
   apply (force simp: dist-norm open-contains-ball)
   done
qed



lemma holomorphic-starlike-primitive:
 fixes f :: complex ⇒ complex
 assumes contf: continuous-on s f
    and s: starlike s and os: open s
    and k: finite k
    and fcd: ⋀x. x ∈ s − k ⟹ f field-differentiable at x
  shows ∃ g. ∀ x ∈ s. (g has-field-derivative f x) (at x)
proof −
 obtain a where a: a∈s and a-cs: ⋀x. x∈s ⟹ closed-segment a x ⊆ s
  using s by (auto simp: starlike-def)
 { fix x b c
  assume x ∈ s closed-segment b c ⊆ s
  then have abcs: convex hull {a, b, c} ⊆ s
   by (simp add: a a-cs starlike-convex-subset)
  then have ∗: continuous-on (convex hull {a, b, c}) f
   by (simp add: continuous-on-subset [OF contf])
  have (f has-contour-integral 0) (linepath a b +++ linepath b c +++ linepath
c a)
   apply (rule Cauchy-theorem-triangle-cofinite [OF - k])
   using abcs apply (simp add: continuous-on-subset [OF contf])
   using ∗ abcs interior-subset apply (auto intro: fcd)
   done
 } note 0 = this
 show ?thesis
  apply (intro exI ballI)
   apply (rule triangle-contour-integrals-starlike-primitive [OF contf a os], as-
sumption)
  apply (metis a-cs)
  apply (metis has-chain-integral-chain-integral3 0)
```

done
qed

lemma *Cauchy-theorem-starlike*:
⟦*open s*; *starlike s*; *finite k*; *continuous-on s f*;
⋀*x*. *x* ∈ *s* − *k* ⟹ *f field-differentiable at x*;
valid-path g; *path-image g* ⊆ *s*; *pathfinish g* = *pathstart g*⟧
⟹ (*f has-contour-integral 0*) *g*
by (*metis holomorphic-starlike-primitive Cauchy-theorem-primitive at-within-open*)

lemma *Cauchy-theorem-starlike-simple*:
⟦*open s*; *starlike s*; *f holomorphic-on s*; *valid-path g*; *path-image g* ⊆ *s*; *pathfinish g* = *pathstart g*⟧
⟹ (*f has-contour-integral 0*) *g*
apply (*rule Cauchy-theorem-starlike* [*OF - - finite.emptyI*])
apply (*simp-all add*: *holomorphic-on-imp-continuous-on*)
apply (*metis at-within-open holomorphic-on-def*)
done

53.20 Cauchy's theorem for a convex set

For a convex set we can avoid assuming openness and boundary analyticity

lemma *triangle-contour-integrals-convex-primitive*:
assumes *contf*: *continuous-on s f*
and *s*: *a* ∈ *s convex s*
and *x*: *x* ∈ *s*
and *zer*: ⋀*b c*. ⟦*b* ∈ *s*; *c* ∈ *s*⟧
⟹ *contour-integral* (*linepath a b*) *f* + *contour-integral* (*linepath b c*) *f* +
contour-integral (*linepath c a*) *f* = *0*
shows ((λ*x*. *contour-integral*(*linepath a x*) *f*) *has-field-derivative f x*) (*at x within s*)
proof −
let *?pathint* = λ*x y*. *contour-integral*(*linepath x y*) *f*
{ fix *y*
assume *y*: *y* ∈ *s*
have *cont-ayf*: *continuous-on* (*closed-segment a y*) *f*
using *s y* **by** (*meson contf continuous-on-subset convex-contains-segment*)
have *xys*: *closed-segment x y* ⊆ *s*
using *convex-contains-segment s x y* **by** *auto*
have *?pathint a y* − *?pathint a x* = *?pathint x y*
using *zer* [*OF x y*] *contour-integral-reverse-linepath* [*OF cont-ayf*] *add-eq-0-iff*
by *force*
} note [*simp*] = *this*
{ fix *e*::*real*
assume *e*: *0* < *e*
have *cont-atx*: *continuous* (*at x within s*) *f*
using *x s contf* **by** (*simp add*: *continuous-on-eq-continuous-within*)
then obtain *d1* **where** *d1*: *d1*>*0* **and** *d1-less*: ⋀*y*. ⟦*y* ∈ *s*; *cmod* (*y* − *x*) <

```
d1⟧ ⟹ cmod (f y − f x) < e/2
      unfolding continuous-within Lim-within dist-norm using e
      by (drule-tac x=e/2 in spec) force
    { fix y
      assume yx: y ≠ x and close: cmod (y − x) < d1 and y: y ∈ s
      have fxy: f contour-integrable-on linepath x y
        using convex-contains-segment s x y
        by (blast intro!: contour-integrable-continuous-linepath continuous-on-subset
[OF contf])
      then obtain i where i: (f has-contour-integral i) (linepath x y)
        by (auto simp: contour-integrable-on-def)
     then have ((λw. f w − f x) has-contour-integral (i − f x ∗ (y − x))) (linepath
x y)
       by (rule has-contour-integral-diff [OF - has-contour-integral-const-linepath])
      then have cmod (i − f x ∗ (y − x)) ≤ e / 2 ∗ cmod (y − x)
        apply (rule has-contour-integral-bound-linepath [where B = e/2])
        using e apply simp
        apply (rule d1-less [THEN less-imp-le])
        using convex-contains-segment s(2) x y apply blast
        using close segment-bound(1) apply fastforce
        done
      also have ... < e ∗ cmod (y − x)
        by (simp add: e yx)
      finally have cmod (?pathint x y − f x ∗ (y−x)) / cmod (y−x) < e
        using i yx  by (simp add: contour-integral-unique divide-less-eq)
    }
    then have ∃ d>0. ∀ y∈s. y ≠ x ∧ cmod (y−x) < d ⟶ cmod (?pathint x y
− f x ∗ (y−x)) / cmod (y−x) < e
      using d1 by blast
  }
  then have ∗: ((λy. (contour-integral (linepath x y) f − f x ∗ (y − x)) /R cmod
(y − x)) ⟶ 0) (at x within s)
    by (simp add: Lim-within dist-norm inverse-eq-divide)
  show ?thesis
   apply (simp add: has-field-derivative-def has-derivative-within bounded-linear-mult-right)
    apply (rule Lim-transform [OF ∗ Lim-eventually])
    using linordered-field-no-ub
    apply (force simp: inverse-eq-divide [symmetric] eventually-at)
    done
qed

lemma contour-integral-convex-primitive:
  ⟦convex s; continuous-on s f;
    ⋀a b c. ⟦a ∈ s; b ∈ s; c ∈ s⟧ ⟹ (f has-contour-integral 0) (linepath a b +++
linepath b c +++ linepath c a)⟧
        ⟹ ∃ g. ∀ x ∈ s. (g has-field-derivative f x) (at x within s)
  apply (cases s={})
  apply (simp-all add: ex-in-conv [symmetric])
 apply (blast intro: triangle-contour-integrals-convex-primitive has-chain-integral-chain-integral3)
```

done

lemma *holomorphic-convex-primitive*:
fixes $f :: complex \Rightarrow complex$
shows
$[\![convex\ s;\ finite\ k;\ continuous\text{-}on\ s\ f;$
$\bigwedge x.\ x \in interior\ s - k \Longrightarrow f\ field\text{-}differentiable\ at\ x]\!]$
$\Longrightarrow \exists g.\ \forall x \in s.\ (g\ has\text{-}field\text{-}derivative\ f\ x)\ (at\ x\ within\ s)$
apply (*rule contour-integral-convex-primitive* [*OF - - Cauchy-theorem-triangle-cofinite*])
prefer *3*
apply (*erule continuous-on-subset*)
apply (*simp add*: *subset-hull continuous-on-subset*, *assumption*+)
by (*metis Diff-iff convex-contains-segment insert-absorb insert-subset interior-mono segment-convex-hull subset-hull*)

lemma *Cauchy-theorem-convex*:
$[\![continuous\text{-}on\ s\ f;\ convex\ s;\ finite\ k;$
$\bigwedge x.\ x \in interior\ s - k \Longrightarrow f\ field\text{-}differentiable\ at\ x;$
$valid\text{-}path\ g;\ path\text{-}image\ g \subseteq s;$
$pathfinish\ g = pathstart\ g]\!] \Longrightarrow (f\ has\text{-}contour\text{-}integral\ 0)\ g$
by (*metis holomorphic-convex-primitive Cauchy-theorem-primitive*)

lemma *Cauchy-theorem-convex-simple*:
$[\![f\ holomorphic\text{-}on\ s;\ convex\ s;$
$valid\text{-}path\ g;\ path\text{-}image\ g \subseteq s;$
$pathfinish\ g = pathstart\ g]\!] \Longrightarrow (f\ has\text{-}contour\text{-}integral\ 0)\ g$
apply (*rule Cauchy-theorem-convex*)
apply (*simp-all add*: *holomorphic-on-imp-continuous-on*)
apply (*rule finite.emptyI*)
using *at-within-interior holomorphic-on-def interior-subset* **by** *fastforce*

In particular for a disc

lemma *Cauchy-theorem-disc*:
$[\![finite\ k;\ continuous\text{-}on\ (cball\ a\ e)\ f;$
$\bigwedge x.\ x \in ball\ a\ e - k \Longrightarrow f\ field\text{-}differentiable\ at\ x;$
$valid\text{-}path\ g;\ path\text{-}image\ g \subseteq cball\ a\ e;$
$pathfinish\ g = pathstart\ g]\!] \Longrightarrow (f\ has\text{-}contour\text{-}integral\ 0)\ g$
apply (*rule Cauchy-theorem-convex*)
apply (*auto simp*: *convex-cball interior-cball*)
done

lemma *Cauchy-theorem-disc-simple*:
$[\![f\ holomorphic\text{-}on\ (ball\ a\ e);\ valid\text{-}path\ g;\ path\text{-}image\ g \subseteq ball\ a\ e;$
$pathfinish\ g = pathstart\ g]\!] \Longrightarrow (f\ has\text{-}contour\text{-}integral\ 0)\ g$
by (*simp add*: *Cauchy-theorem-convex-simple*)

53.21 Generalize integrability to local primitives

lemma *contour-integral-local-primitive-lemma*:

fixes f :: *complex⇒complex*
shows
$[\![g$ *piecewise-differentiable-on* $\{a..b\};$
$\bigwedge x.\ x \in s \Longrightarrow (f$ *has-field-derivative* $f'\ x)$ *(at x within s)*;
$\bigwedge x.\ x \in \{a..b\} \Longrightarrow g\ x \in s]\!]$
$\Longrightarrow (\lambda x.\ f'\ (g\ x) * $ *vector-derivative* g *(at x within* $\{a..b\}))$
integrable-on $\{a..b\}$
apply *(cases cbox a b = {}, force)*
apply *(simp add: integrable-on-def)*
apply *(rule exI)*
apply *(rule contour-integral-primitive-lemma, assumption+)*
using *atLeastAtMost-iff* **by** *blast*

lemma *contour-integral-local-primitive-any*:
fixes f :: *complex ⇒ complex*
assumes *gpd*: g *piecewise-differentiable-on* $\{a..b\}$
and *dh*: $\bigwedge x.\ x \in s$
$\Longrightarrow \exists\, d\ h.\ 0 < d \wedge$
$(\forall\, y.\ norm(y - x) < d \longrightarrow (h$ *has-field-derivative* $f\ y)$ *(at y within s))*
and *gs*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow g\ x \in s$
shows $(\lambda x.\ f(g\ x) * $ *vector-derivative* g *(at* $x))$ *integrable-on* $\{a..b\}$
proof −
{ fix x
assume *x*: $a \le x\ x \le b$
obtain $d\ h$ **where** *d*: $0 < d$
and *h*: $(\bigwedge y.\ norm(y - g\ x) < d \Longrightarrow (h$ *has-field-derivative* $f\ y)$ *(at y within s))*
using x *gs dh* **by** *(metis atLeastAtMost-iff)*
have *continuous-on* $\{a..b\}$ g **using** *gpd piecewise-differentiable-on-def* **by** *blast*
then obtain e **where** *e*: $e>0$ **and** *lessd*: $\bigwedge x'.\ x' \in \{a..b\} \Longrightarrow |x' - x| < e \Longrightarrow$ *cmod* $(g\ x' - g\ x) < d$
using $x\ d$
apply *(auto simp: dist-norm continuous-on-iff)*
apply *(drule-tac x=x* **in** *bspec)*
using x **apply** *simp*
apply *(drule-tac x=d* **in** *spec, auto)*
done
have $\exists\, d>0.\ \forall\, u\ v.\ u \le x \wedge x \le v \wedge \{u..v\} \subseteq ball\ x\ d \wedge (u \le v \longrightarrow a \le u \wedge v \le b) \longrightarrow$
$(\lambda x.\ f\ (g\ x) * $ *vector-derivative* g *(at* $x))$ *integrable-on* $\{u..v\}$
apply *(rule-tac x=e* **in** *exI)*
using e
apply *(simp add: integrable-on-localized-vector-derivative [symmetric], clarify)*
apply *(rule-tac* $f = h$ **and** $s = g$ ‘ $\{u..v\}$ **in** *contour-integral-local-primitive-lemma)*
apply *(meson atLeastatMost-subset-iff gpd piecewise-differentiable-on-subset)*
apply *(force simp: ball-def dist-norm intro: lessd gs DERIV-subset [OF h], force)*
done

} then
show *?thesis*
by (*force simp*: *intro*!: *integrable-on-little-subintervals* [*of a b*, *simplified*])
qed

lemma *contour-integral-local-primitive*:
fixes *f* :: *complex* ⇒ *complex*
assumes *g*: *valid-path g path-image g* ⊆ *s*
and *dh*: ⋀*x*. *x* ∈ *s*
⟹ ∃ *d h*. 0 < *d* ∧
(∀ *y*. *norm*(*y* − *x*) < *d* ⟶ (*h has-field-derivative f y*) (*at y within s*))
shows *f contour-integrable-on g*
using *g*
apply (*simp add*: *valid-path-def path-image-def contour-integrable-on-def has-contour-integral-def has-integral-localized-vector-derivative integrable-on-def* [*symmetric*])
using *contour-integral-local-primitive-any* [*OF - dh*]
by (*meson image-subset-iff piecewise-C1-imp-differentiable*)

In particular if a function is holomorphic

lemma *contour-integrable-holomorphic*:
assumes *contf*: *continuous-on s f*
and *os*: *open s*
and *k*: *finite k*
and *g*: *valid-path g path-image g* ⊆ *s*
and *fcd*: ⋀*x*. *x* ∈ *s* − *k* ⟹ *f field-differentiable at x*
shows *f contour-integrable-on g*
proof −
{ fix *z*
assume *z*: *z* ∈ *s*
obtain *d* **where** *d*: *d*>0 *ball z d* ⊆ *s* **using** ‹*open s*› *z*
by (*auto simp*: *open-contains-ball*)
then have *contfb*: *continuous-on* (*ball z d*) *f*
using *contf continuous-on-subset* **by** *blast*
obtain *h* **where** ∀ *y*∈*ball z d*. (*h has-field-derivative f y*) (*at y within ball z d*)
using *holomorphic-convex-primitive* [*OF convex-ball k contfb fcd*] *d*
interior-subset **by** *force*
then have ∀ *y*∈*ball z d*. (*h has-field-derivative f y*) (*at y within s*)
by (*metis Topology-Euclidean-Space.open-ball at-within-open d*(2) *os subsetCE*)
then have ∃ *h*. (∀ *y*. *cmod* (*y* − *z*) < *d* ⟶ (*h has-field-derivative f y*) (*at y within s*))
by (*force simp*: *dist-norm norm-minus-commute*)
then have ∃ *d h*. 0 < *d* ∧ (∀ *y*. *cmod* (*y* − *z*) < *d* ⟶ (*h has-field-derivative f y*) (*at y within s*))
using *d* **by** *blast*
}
then show *?thesis*
by (*rule contour-integral-local-primitive* [*OF g*])

qed

lemma *contour-integrable-holomorphic-simple*:
assumes *fh*: *f holomorphic-on s*
and *os*: *open s*
and *g*: *valid-path g path-image g* $\subseteq$ *s*
shows *f contour-integrable-on g*
apply (*rule contour-integrable-holomorphic* [*OF - os Finite-Set.finite.emptyI g*])
apply (*simp add*: *fh holomorphic-on-imp-continuous-on*)
using *fh* **by** (*simp add*: *field-differentiable-def holomorphic-on-open os*)

lemma *continuous-on-inversediff*:
fixes *z*:: *'a*::*real-normed-field* **shows** *z* $\notin$ *s* $\Longrightarrow$ *continuous-on s* ($\lambda w.$ *1 / (w* $-$ *z)*)
by (*rule continuous-intros* | *force*)+

corollary *contour-integrable-inversediff*:
⟦*valid-path g*; *z* $\notin$ *path-image g*⟧ $\Longrightarrow$ ($\lambda w.$ *1 / (w*$-$*z)*) *contour-integrable-on g*
apply (*rule contour-integrable-holomorphic-simple* [*of - UNIV*$-\{z\}$])
apply (*auto simp*: *holomorphic-on-open open-delete intro*!: *derivative-eq-intros*)
done

Key fact that path integral is the same for a "nearby" path. This is the main lemma for the homotopy form of Cauchy's theorem and is also useful if we want "without loss of generality" to assume some nice properties of a path (e.g. smoothness). It can also be used to define the integrals of analytic functions over arbitrary continuous paths. This is just done for winding numbers now.

A technical definition to avoid duplication of similar proofs, for paths joined at the ends versus looping paths

definition *linked-paths* :: *bool* $\Rightarrow$ (*real* $\Rightarrow$ *'a*) $\Rightarrow$ (*real* $\Rightarrow$ *'a*::*topological-space*) $\Rightarrow$ *bool*
where *linked-paths atends g h* ==
(*if atends then pathstart h* = *pathstart g* $\wedge$ *pathfinish h* = *pathfinish g*
else pathfinish g = *pathstart g* $\wedge$ *pathfinish h* = *pathstart h*)

This formulation covers two cases: g and h share their start and end points; g and h both loop upon themselves.

lemma *contour-integral-nearby*:
assumes *os*: *open s* **and** *p*: *path p path-image p* $\subseteq$ *s*
shows
$\exists$ *d*. *0* $<$ *d* $\wedge$
($\forall$ *g h*. *valid-path g* $\wedge$ *valid-path h* $\wedge$
($\forall$ *t* $\in$ $\{$*0..1*$\}$. *norm(g t* $-$ *p t)* $<$ *d* $\wedge$ *norm(h t* $-$ *p t)* $<$ *d*) $\wedge$
linked-paths atends g h
$\longrightarrow$ *path-image g* $\subseteq$ *s* $\wedge$ *path-image h* $\subseteq$ *s* $\wedge$
($\forall$ *f*. *f holomorphic-on s* $\longrightarrow$ *contour-integral h f* = *contour-integral g f*))

proof −
have $\forall z.\ \exists e.\ z \in$ *path-image* $p \longrightarrow 0 < e \wedge$ *ball* $z\ e \subseteq s$
using *open-contains-ball os p*(*2*) **by** *blast*
then obtain *ee* **where** *ee*: $\bigwedge z.\ z \in$ *path-image* $p \Longrightarrow 0 <$ *ee* $z \wedge$ *ball* z (*ee* z) $\subseteq s$
by *metis*
def *cover* $\equiv (\lambda z.$ *ball* z (*ee* z/*3*)) ‘ (*path-image* p)
have *compact* (*path-image* p)
by (*metis p*(*1*) *compact-path-image*)
moreover have *path-image* $p \subseteq (\bigcup c \in$ *path-image* $p.$ *ball* c (*ee* c / *3*))
using *ee* **by** *auto*
ultimately have $\exists D \subseteq$ *cover*. *finite* $D \wedge$ *path-image* $p \subseteq \bigcup D$
by (*simp add*: *compact-eq-heine-borel cover-def*)
then obtain D **where** D: $D \subseteq$ *cover finite* D *path-image* $p \subseteq \bigcup D$
by *blast*
then obtain k **where** k: $k \subseteq \{0..1\}$ *finite* k **and** *D-eq*: $D = ((\lambda z.$ *ball* z (*ee* z / *3*)) $\circ$ $p)$ ‘ k
apply (*simp add*: *cover-def path-image-def image-comp*)
apply (*blast dest*!: *finite-subset-image* [*OF* ‹*finite D*›])
done
then have *kne*: $k \neq \{\}$
using D **by** *auto*
have *pi*: $\bigwedge i.\ i \in k \Longrightarrow p\ i \in$ *path-image* p
using k **by** (*auto simp*: *path-image-def*)
then have *eepi*: $\bigwedge i.\ i \in k \Longrightarrow 0 <$ *ee*$((p\ i))$
by (*metis ee*)
def $e \equiv$ *Min*((*ee o p*) ‘ k)
have *fin-eep*: *finite* ((*ee o p*) ‘ k)
using k **by** *blast*
have *enz*: $0 < e$
using *ee* k **by** (*simp add*: *kne e-def Min-gr-iff* [*OF fin-eep*] *eepi*)
have *uniformly-continuous-on* $\{0..1\}$ p
using p **by** (*simp add*: *path-def compact-uniformly-continuous*)
then obtain d::*real* **where** d: d>*0*
and *de*: $\bigwedge x\ x'.\ |x' - x| < d \Longrightarrow x \in \{0..1\} \Longrightarrow x' \in \{0..1\} \Longrightarrow$ *cmod* $(p\ x' - p\ x) < e/3$
unfolding *uniformly-continuous-on-def dist-norm real-norm-def*
by (*metis divide-pos-pos enz zero-less-numeral*)
then obtain N::*nat* **where** N: N>*0 inverse* $N < d$
using *real-arch-inverse* [*of d*] **by** *auto*
{ fix *g h*
assume *g*: *valid-path g* **and** *gp*: $\forall t \in \{0..1\}.$ *cmod* $(g\ t - p\ t) <$ *e* / *3*
and *h*: *valid-path h* **and** *hp*: $\forall t \in \{0..1\}.$ *cmod* $(h\ t - p\ t) <$ *e* / *3*
and *joins*: *linked-paths atends g h*
{ fix t::*real*
assume t: *0* $\leq t\ t \leq$ *1*
then obtain u **where** u: $u \in k$ **and** *ptu*: $p\ t \in$ *ball*$(p\ u)$ (*ee*$(p\ u)$ / *3*)
using ‹*path-image* $p \subseteq \bigcup D$› *D-eq* **by** (*force simp*: *path-image-def*)
then have *ele*: $e \leq$ *ee* $(p\ u)$ **using** *fin-eep*

```
      by (simp add: e-def)
    have cmod (g t - p t) < e / 3 cmod (h t - p t) < e / 3
      using gp hp t by auto
    with ele have cmod (g t - p t) < ee (p u) / 3
                  cmod (h t - p t) < ee (p u) / 3
      by linarith+
    then have g t ∈ ball(p u) (ee(p u))  h t ∈ ball(p u) (ee(p u))
      using norm-diff-triangle-ineq [of g t p t p t p u]
            norm-diff-triangle-ineq [of h t p t p t p u] ptu eepi u
      by (force simp: dist-norm ball-def norm-minus-commute)+
    then have g t ∈ s h t ∈ s using ee u k
      by (auto simp: path-image-def ball-def)
  }
  then have ghs: path-image g ⊆ s path-image h ⊆ s
    by (auto simp: path-image-def)
  moreover
  { fix f
    assume fhols: f holomorphic-on s
    then have fpa: f contour-integrable-on g  f contour-integrable-on h
    using g ghs h holomorphic-on-imp-continuous-on os contour-integrable-holomorphic-simple
      by blast+
    have contf: continuous-on s f
      by (simp add: fhols holomorphic-on-imp-continuous-on)
    { fix z
      assume z: z ∈ path-image p
      have f holomorphic-on ball z (ee z)
        using fhols ee z holomorphic-on-subset by blast
      then have ∃ff. (∀ w ∈ ball z (ee z). (ff has-field-derivative f w) (at w))
        using holomorphic-convex-primitive [of ball z (ee z) {} f, simplified]
     by (metis open-ball at-within-open holomorphic-on-def holomorphic-on-imp-continuous-on
mem-ball)
    }
    then obtain ff where ff:
          ⋀z w. ⟦z ∈ path-image p; w ∈ ball z (ee z)⟧ ⟹ (ff z has-field-derivative
f w) (at w)
      by metis
    { fix n
      assume n: n ≤ N
    then have contour-integral(subpath 0 (n/N) h) f - contour-integral(subpath
0 (n/N) g) f =
          contour-integral(linepath (g(n/N)) (h(n/N))) f - contour-integral(linepath
(g 0) (h 0)) f
      proof (induct n)
        case 0 show ?case by simp
      next
        case (Suc n)
        obtain t where t: t ∈ k and p (n/N) ∈ ball(p t) (ee(p t) / 3)
          using ‹path-image p ⊆ ⋃ D› [THEN subsetD, where c=p (n/N)] D-eq
N Suc.prems
```

```
      by (force simp: path-image-def)
    then have ptu: cmod (p t - p (n/N)) < ee (p t) / 3
      by (simp add: dist-norm)
    have e3le: e/3 ≤ ee (p t) / 3  using fin-eep t
      by (simp add: e-def)
    { fix x
      assume x: n/N ≤ x x ≤ (1 + n)/N
      then have nN01: 0 ≤ n/N (1 + n)/N ≤ 1
        using Suc.prems by auto
      then have x01: 0 ≤ x x ≤ 1
        using x by linarith+
      have cmod (p t - p x)  < ee (p t) / 3 + e/3
        apply (rule norm-diff-triangle-less [OF ptu de])
        using x N x01 Suc.prems
        apply (auto simp: field-simps)
        done
      then have ptx: cmod (p t - p x) < 2*ee (p t)/3
        using e3le eepi [OF t] by simp
      have cmod (p t - g x) < 2*ee (p t)/3 + e/3
        apply (rule norm-diff-triangle-less [OF ptx])
        using gp x01 by (simp add: norm-minus-commute)
      also have ... ≤ ee (p t)
        using e3le eepi [OF t] by simp
      finally have gg: cmod (p t - g x) < ee (p t) .
      have cmod (p t - h x) < 2*ee (p t)/3 + e/3
        apply (rule norm-diff-triangle-less [OF ptx])
        using hp x01 by (simp add: norm-minus-commute)
      also have ... ≤ ee (p t)
        using e3le eepi [OF t] by simp
      finally have cmod (p t - g x) < ee (p t)
                   cmod (p t - h x) < ee (p t)
        using gg by auto
    } note ptgh-ee = this
    have pi-hgn: path-image (linepath (h (n/N)) (g (n/N))) ⊆ ball (p t) (ee
(p t))
      using ptgh-ee [of n/N] Suc.prems
       by (auto simp: field-simps dist-norm dest: segment-furthest-le [where
y=p t])
    then have gh-ns: closed-segment (g (n/N)) (h (n/N)) ⊆ s
      using ‹N>0› Suc.prems
      apply (simp add: path-image-join field-simps closed-segment-commute)
      apply (erule order-trans)
      apply (simp add: ee pi t)
      done
    have pi-ghn': path-image (linepath (g ((1 + n) / N)) (h ((1 + n) / N)))
          ⊆ ball (p t) (ee (p t))
      using ptgh-ee [of (1+n)/N] Suc.prems
       by (auto simp: field-simps dist-norm dest: segment-furthest-le [where
y=p t])
```

```
        then have gh-n's: closed-segment (g ((1 + n) / N)) (h ((1 + n) / N))
⊆ s
          using ‹N>0› Suc.prems ee pi t
          by (auto simp: Path-Connected.path-image-join field-simps)
        have pi-subset-ball:
            path-image (subpath (n/N) ((1+n) / N) g +++ linepath (g ((1+n)
/ N)) (h ((1+n) / N)) +++
                        subpath ((1+n) / N) (n/N) h +++ linepath (h (n/N)) (g
(n/N)))
              ⊆ ball (p t) (ee (p t))
          apply (intro subset-path-image-join pi-hgn pi-ghn')
          using ‹N>0› Suc.prems
       apply (auto simp: path-image-subpath dist-norm field-simps closed-segment-eq-real-ivl
ptgh-ee)
          done
        have pi0: (f has-contour-integral 0)
                  (subpath (n/ N) ((Suc n)/N) g +++ linepath(g ((Suc n) / N))
(h((Suc n) / N)) +++
                       subpath ((Suc n) / N) (n/N) h +++ linepath(h (n/N)) (g
(n/N)))
          apply (rule Cauchy-theorem-primitive [of ball(p t) (ee(p t)) ff (p t) f])
          apply (metis ff open-ball at-within-open pi t)
          apply (intro valid-path-join)
           using Suc.prems pi-subset-ball apply (simp-all add: valid-path-subpath
g h)
          done
        have fpa1: f contour-integrable-on subpath (real n / real N) (real (Suc n)
/ real N) g
          using Suc.prems by (simp add: contour-integrable-subpath g fpa)
        have fpa2: f contour-integrable-on linepath (g (real (Suc n) / real N)) (h
(real (Suc n) / real N))
          using gh-n's
       by (auto intro!: contour-integrable-continuous-linepath continuous-on-subset
[OF contf])
        have fpa3: f contour-integrable-on linepath (h (real n / real N)) (g (real n
/ real N))
          using gh-ns
       by (auto simp: closed-segment-commute intro!: contour-integrable-continuous-linepath
continuous-on-subset [OF contf])
        have eq0: contour-integral (subpath (n/N) ((Suc n) / real N) g) f +
                  contour-integral (linepath (g ((Suc n) / N)) (h ((Suc n) / N))) f
+
                  contour-integral (subpath ((Suc n) / N) (n/N) h) f +
                  contour-integral (linepath (h (n/N)) (g (n/N))) f = 0
          using contour-integral-unique [OF pi0] Suc.prems
          by (simp add: g h fpa valid-path-subpath contour-integrable-subpath
                   fpa1 fpa2 fpa3 algebra-simps del: of-nat-Suc)
        have *: ⋀hn he hn' gn gd gn' hgn ghn gh0 ghn'.
                ⟦hn − gn = ghn − gh0;
```

```
                gd + ghn' + he + hgn = (0::complex);
                 hn − he = hn'; gn + gd = gn'; hgn = −ghn⟧ ⟹ hn' − gn' =
ghn' − gh0
        by (auto simp: algebra-simps)
      have contour-integral (subpath 0 (n/N) h) f − contour-integral (subpath
((Suc n) / N) (n/N) h) f =
            contour-integral (subpath 0 (n/N) h) f + contour-integral (subpath
(n/N) ((Suc n) / N) h) f
        unfolding reversepath-subpath [symmetric, of ((Suc n) / N)]
          using Suc.prems by (simp add: h fpa contour-integral-reversepath
valid-path-subpath contour-integrable-subpath)
      also have ... = contour-integral (subpath 0 ((Suc n) / N) h) f
        using Suc.prems by (simp add: contour-integral-subpath-combine h fpa)
      finally have pi0-eq:
            contour-integral (subpath 0 (n/N) h) f − contour-integral (subpath
((Suc n) / N) (n/N) h) f =
          contour-integral (subpath 0 ((Suc n) / N) h) f .
      show ?case
        apply (rule ∗ [OF Suc.hyps eq0 pi0-eq])
        using Suc.prems
        apply (simp-all add: g h fpa contour-integral-subpath-combine
          contour-integral-reversepath [symmetric] contour-integrable-continuous-linepath
              continuous-on-subset [OF contf gh-ns])
        done
    qed
    } note ind = this
    have contour-integral h f = contour-integral g f
      using ind [OF order-refl] N joins
     by (simp add: linked-paths-def pathstart-def pathfinish-def split: if-split-asm)
  }
  ultimately
   have path-image g ⊆ s ∧ path-image h ⊆ s ∧ (∀f. f holomorphic-on s ⟶
contour-integral h f = contour-integral g f)
    by metis
 } note ∗ = this
 show ?thesis
  apply (rule-tac x=e/3 in exI)
  apply (rule conjI)
  using enz apply simp
  apply (clarsimp simp only: ball-conj-distrib)
  apply (rule ∗; assumption)
  done
qed


lemma
 assumes open s path p path-image p ⊆ s
  shows contour-integral-nearby-ends:
    ∃d. 0 < d ∧
```

$(\forall\, g\ h.\ \textit{valid-path}\ g \wedge \textit{valid-path}\ h \wedge$
$(\forall\, t \in \{0..1\}.\ \textit{norm}(g\ t - p\ t) < d \wedge \textit{norm}(h\ t - p\ t) < d) \wedge$
$\textit{pathstart}\ h = \textit{pathstart}\ g \wedge \textit{pathfinish}\ h = \textit{pathfinish}\ g$
$\longrightarrow \textit{path-image}\ g \subseteq s \wedge$
$\textit{path-image}\ h \subseteq s \wedge$
$(\forall f.\ f\ \textit{holomorphic-on}\ s$
$\longrightarrow \textit{contour-integral}\ h\ f = \textit{contour-integral}\ g\ f))$
and *contour-integral-nearby-loops*:
$\exists\, d.\ 0 < d \wedge$
$(\forall\, g\ h.\ \textit{valid-path}\ g \wedge \textit{valid-path}\ h \wedge$
$(\forall\, t \in \{0..1\}.\ \textit{norm}(g\ t - p\ t) < d \wedge \textit{norm}(h\ t - p\ t) < d) \wedge$
$\textit{pathfinish}\ g = \textit{pathstart}\ g \wedge \textit{pathfinish}\ h = \textit{pathstart}\ h$
$\longrightarrow \textit{path-image}\ g \subseteq s \wedge$
$\textit{path-image}\ h \subseteq s \wedge$
$(\forall f.\ f\ \textit{holomorphic-on}\ s$
$\longrightarrow \textit{contour-integral}\ h\ f = \textit{contour-integral}\ g\ f))$
using *contour-integral-nearby* [*OF assms*, **where** *atends=True*]
using *contour-integral-nearby* [*OF assms*, **where** *atends=False*]
unfolding *linked-paths-def* **by** *simp-all*

corollary *differentiable-polynomial-function*:
fixes p :: *real* $\Rightarrow$ *'a::euclidean-space*
shows *polynomial-function* $p \Longrightarrow p$ *differentiable-on* s
by (*meson has-vector-derivative-polynomial-function differentiable-at-imp-differentiable-on differentiable-def has-vector-derivative-def*)

lemma *C1-differentiable-polynomial-function*:
fixes p :: *real* $\Rightarrow$ *'a::euclidean-space*
shows *polynomial-function* $p \Longrightarrow p$ *C1-differentiable-on* s
by (*metis continuous-on-polymonial-function C1-differentiable-on-def has-vector-derivative-polynomial-functi*

lemma *valid-path-polynomial-function*:
fixes p :: *real* $\Rightarrow$ *'a::euclidean-space*
shows *polynomial-function* $p \Longrightarrow$ *valid-path* p
by (*force simp*: *valid-path-def piecewise-C1-differentiable-on-def continuous-on-polymonial-function C1-differentiable-polynomial-function*)

lemma *valid-path-subpath-trivial* [*simp*]:
fixes g :: *real* $\Rightarrow$ *'a::euclidean-space*
shows $z \neq g\ x \Longrightarrow$ *valid-path* (*subpath x x g*)
by (*simp add*: *subpath-def valid-path-polynomial-function*)

lemma *contour-integral-bound-exists*:
assumes *s*: *open s*
and *g*: *valid-path g*
and *pag*: *path-image g* $\subseteq s$
shows $\exists\, L.\ 0 < L \wedge$
$(\forall f\ B.\ f\ \textit{holomorphic-on}\ s \wedge (\forall\, z \in s.\ \textit{norm}(f\ z) \leq B)$
$\longrightarrow \textit{norm}(\textit{contour-integral}\ g\ f) \leq L{*}B)$

proof −
have *path g* **using** *g*
by (*simp add*: *valid-path-imp-path*)
then obtain *d*::*real* **and** *p*
where *d*: *0 < d*
and *p*: *polynomial-function p path-image p ⊆ s*
and *pi*: ⋀*f*. *f holomorphic-on s* ⟹ *contour-integral g f = contour-integral p f*
using *contour-integral-nearby-ends* [*OF s* ‹*path g*› *pag*]
apply *clarify*
apply (*drule-tac x*=*g* **in** *spec*)
apply (*simp only*: *assms*)
apply (*force simp*: *valid-path-polynomial-function dest*: *path-approx-polynomial-function*)
done
then obtain *p′* **where** *p′*: *polynomial-function p′*
⋀*x*. (*p has-vector-derivative* (*p′ x*)) (*at x*)
using *has-vector-derivative-polynomial-function* **by** *force*
then have *bounded*(*p′* ‘ {*0..1*})
using *continuous-on-polymonial-function*
by (*force simp*: *intro*!: *compact-imp-bounded compact-continuous-image*)
then obtain *L* **where** *L*: *L>0* **and** *nop′*: ⋀*x*. *x* ∈ {*0..1*} ⟹ *norm* (*p′ x*) ≤ *L*
by (*force simp*: *bounded-pos*)
{ fix *f B*
assume *f*: *f holomorphic-on s*
and *B*: ⋀*z*. *z*∈*s* ⟹ *cmod* (*f z*) ≤ *B*
then have *f contour-integrable-on p* ∧ *valid-path p*
using *p s*
by (*blast intro*: *valid-path-polynomial-function contour-integrable-holomorphic-simple holomorphic-on-imp-continuous-on*)
moreover have ⋀*x*. *x* ∈ {*0..1*} ⟹ *cmod* (*vector-derivative p* (*at x*)) ∗ *cmod* (*f* (*p x*)) ≤ *L* ∗ *B*
apply (*rule mult-mono*)
apply (*subst Derivative.vector-derivative-at*; *force intro*: *p′ nop′*)
using *L B p*
apply (*auto simp*: *path-image-def image-subset-iff*)
done
ultimately have *cmod* (*contour-integral g f*) ≤ *L* ∗ *B*
apply (*simp add*: *pi* [*OF f*])
apply (*simp add*: *contour-integral-integral*)
apply (*rule order-trans* [*OF integral-norm-bound-integral*])
apply (*auto simp*: *mult.commute integral-norm-bound-integral contour-integrable-on* [*symmetric*] *norm-mult*)
done
} then
show *?thesis*
by (*force simp*: *L contour-integral-integral*)
qed

53.22 Constancy of a function from a connected set into a finite, disconnected or discrete set

Still missing: versions for a set that is smaller than R, or countable.

lemma *continuous-disconnected-range-constant*:
assumes *s*: *connected s*
and *conf*: *continuous-on s f*
and *fim*: f ‘ $s \subseteq t$
and *cct*: $\bigwedge y.\ y \in t \Longrightarrow$ *connected-component-set t y* $= \{y\}$
shows $\exists a.\ \forall x \in s.\ f\ x = a$
proof (*cases* $s = \{\}$)
case *True* **then show** *?thesis* **by** *force*
next
case *False*
{ fix x **assume** $x \in s$
then have f ‘ $s \subseteq \{f\ x\}$
by (*metis connected-continuous-image conf connected-component-maximal fim image-subset-iff rev-image-eqI s cct*)
}
with *False* **show** *?thesis*
by *blast*
qed

lemma *discrete-subset-disconnected*:
fixes s :: *'a::topological-space set*
fixes t :: *'b::real-normed-vector set*
assumes *conf*: *continuous-on s f*
and *no*: $\bigwedge x.\ x \in s \Longrightarrow \exists e{>}0.\ \forall y.\ y \in s \wedge f\ y \neq f\ x \longrightarrow e \leq$ *norm* $(f\ y - f\ x)$
shows f ‘ $s \subseteq \{y.$ *connected-component-set* $(f$ ‘ $s)\ y = \{y\}\}$
proof −
{ fix x **assume** x: $x \in s$
then obtain e **where** $e{>}0$ **and** *ele*: $\bigwedge y.\ [\![y \in s; f\ y \neq f\ x]\!] \Longrightarrow e \leq$ *norm* $(f\ y - f\ x)$
using *conf no* [*OF x*] **by** *auto*
then have *e2*: $0 \leq e\ /\ 2$
by *simp*
have $f\ y = f\ x$ **if** $y \in s$ **and** *ccs*: $f\ y \in$ *connected-component-set* $(f$ ‘ $s)\ (f\ x)$ **for** y
apply (*rule ccontr*)
using *connected-closed* [*of connected-component-set* $(f$ ‘ $s)\ (f\ x)$] ‹$e{>}0$›
apply (*simp add*: *del*: *ex-simps*)
apply (*drule spec* [**where** x=*cball* $(f\ x)\ (e\ /\ 2)$])
apply (*drule spec* [**where** x=− *ball*$(f\ x)\ e$])
apply (*auto simp*: *dist-norm open-closed* [*symmetric*] *simp del*: *le-divide-eq-numeral1 dest*!: *connected-component-in*)
apply (*metis diff-self e2 ele norm-minus-commute norm-zero not-less*)
using *centre-in-cball connected-component-refl-eq e2 x* **apply** *blast*
using *ccs*

```
    apply (force simp: cball-def dist-norm norm-minus-commute dest: ele [OF ‹y
∈ s›])
     done
   moreover have connected-component-set (f ‘ s) (f x) ⊆ f ‘ s
     by (auto simp: connected-component-in)
   ultimately have connected-component-set (f ‘ s) (f x) = {f x}
     by (auto simp: x)
 }
 with assms show ?thesis
   by blast
qed

lemma finite-implies-discrete:
 fixes s :: 'a::topological-space set
 assumes finite (f ‘ s)
 shows (∀ x ∈ s. ∃ e>0. ∀ y. y ∈ s ∧ f y ≠ f x ⟶ e ≤ norm (f y − f x))
proof −
 have ∃ e>0. ∀ y. y ∈ s ∧ f y ≠ f x ⟶ e ≤ norm (f y − f x) if x ∈ s for x
 proof (cases f ‘ s − {f x} = {})
   case True
   with zero-less-numeral show ?thesis
     by (fastforce simp add: Set.image-subset-iff cong: conj-cong)
 next
   case False
   then obtain z where z: z ∈ s f z ≠ f x
     by blast
   have finn: finite {norm (z − f x) |z. z ∈ f ‘ s − {f x}}
     using assms by simp
   then have ∗: 0 < Inf{norm(z − f x) | z. z ∈ f ‘ s − {f x}}
     apply (rule finite-imp-less-Inf)
     using z apply force+
     done
   show ?thesis
     by (force intro!: ∗ cInf-le-finite [OF finn])
 qed
 with assms show ?thesis
   by blast
qed
```

This proof requires the existence of two separate values of the range type.

```
lemma finite-range-constant-imp-connected:
 assumes ⋀f::'a::topological-space ⇒ 'b::real-normed-algebra-1.
           ⟦continuous-on s f; finite(f ‘ s)⟧ ⟹ ∃ a. ∀ x ∈ s. f x = a
   shows connected s
proof −
 { fix t u
   assume clt: closedin (subtopology euclidean s) t
     and clu: closedin (subtopology euclidean s) u
     and tue: t ∩ u = {} and tus: t ∪ u = s
```

```
    have conif: continuous-on s (λx. if x ∈ t then 0 else 1)
      apply (subst tus [symmetric])
      apply (rule continuous-on-cases-local)
      using clt clu tue
      apply (auto simp: tus continuous-on-const)
      done
    have fi: finite ((λx. if x ∈ t then 0 else 1) ‘ s)
      by (rule finite-subset [of - {0,1}]) auto
    have t = {} ∨ u = {}
      using assms [OF conif fi] tus [symmetric]
      by (auto simp: Ball-def) (metis IntI empty-iff one-neq-zero tue)
  }
  then show ?thesis
    by (simp add: connected-closedin-eq)
qed

lemma continuous-disconnected-range-constant-eq:
      (connected s ⟷
          (∀f::'a::topological-space ⇒ 'b::real-normed-algebra-1.
           ∀t. continuous-on s f ∧ f ‘ s ⊆ t ∧ (∀y ∈ t. connected-component-set t
y = {y})
            ⟶ (∃a::'b. ∀x ∈ s. f x = a))) (is ?thesis1)
  and continuous-discrete-range-constant-eq:
      (connected s ⟷
        (∀f::'a::topological-space ⇒ 'b::real-normed-algebra-1.
         continuous-on s f ∧
        (∀x ∈ s. ∃e. 0 < e ∧ (∀y. y ∈ s ∧ (f y ≠ f x) ⟶ e ≤ norm(f y - f x)))
         ⟶ (∃a::'b. ∀x ∈ s. f x = a))) (is ?thesis2)
  and continuous-finite-range-constant-eq:
      (connected s ⟷
        (∀f::'a::topological-space ⇒ 'b::real-normed-algebra-1.
         continuous-on s f ∧ finite (f ‘ s)
         ⟶ (∃a::'b. ∀x ∈ s. f x = a))) (is ?thesis3)
proof −
  have ∗: ⋀s t u v. ⟦s ⟹ t; t ⟹ u; u ⟹ v; v ⟹ s⟧
    ⟹ (s ⟷ t) ∧ (s ⟷ u) ∧ (s ⟷ v)
    by blast
  have ?thesis1 ∧ ?thesis2 ∧ ?thesis3
    apply (rule ∗)
    using continuous-disconnected-range-constant apply metis
    apply clarify
    apply (frule discrete-subset-disconnected; blast)
    apply (blast dest: finite-implies-discrete)
    apply (blast intro!: finite-range-constant-imp-connected)
    done
  then show ?thesis1 ?thesis2 ?thesis3
    by blast+
qed
```

lemma *continuous-discrete-range-constant*:
fixes f :: *'a::topological-space* ⇒ *'b::real-normed-algebra-1*
assumes s: *connected s*
and *continuous-on s f*
and $\bigwedge x.\ x \in s \Longrightarrow \exists e{>}0.\ \forall y.\ y \in s \wedge f\ y \neq f\ x \longrightarrow e \leq norm\ (f\ y - f\ x)$
shows $\exists a.\ \forall x \in s.\ f\ x = a$
using *continuous-discrete-range-constant-eq* [*THEN iffD1*, *OF s*] *assms*
by *blast*

lemma *continuous-finite-range-constant*:
fixes f :: *'a::topological-space* ⇒ *'b::real-normed-algebra-1*
assumes *connected s*
and *continuous-on s f*
and *finite* (f ‘ s)
shows $\exists a.\ \forall x \in s.\ f\ x = a$
using *assms continuous-finite-range-constant-eq*
by *blast*

We can treat even non-rectifiable paths as having a "length" for bounds on analytic functions in open sets.

53.23 Winding Numbers

definition *winding-number*:: [*real* ⇒ *complex*, *complex*] ⇒ *complex* **where**
winding-number $\gamma\ z \equiv$
@$n.\ \forall e > 0.\ \exists p.$ *valid-path* $p \wedge z \notin$ *path-image* $p \wedge$
pathstart p = *pathstart* $\gamma \wedge$
pathfinish p = *pathfinish* $\gamma \wedge$
$(\forall t \in \{0..1\}.\ norm(\gamma\ t - p\ t) < e) \wedge$
contour-integral p $(\lambda w.\ 1/(w - z)) = 2 * pi * ii * n$

lemma *winding-number*:
assumes *path* $\gamma\ z \notin$ *path-image* $\gamma\ 0 < e$
shows $\exists p.$ *valid-path* $p \wedge z \notin$ *path-image* $p \wedge$
pathstart p = *pathstart* $\gamma \wedge$
pathfinish p = *pathfinish* $\gamma \wedge$
$(\forall t \in \{0..1\}.\ norm\ (\gamma\ t - p\ t) < e) \wedge$
contour-integral p $(\lambda w.\ 1/(w - z)) = 2 * pi * ii *$ *winding-number* γ
z
proof −
have *path-image* $\gamma \subseteq UNIV - \{z\}$
using *assms* **by** *blast*
then obtain d
where d: $d{>}0$
and *pi-eq*: $\bigwedge h1\ h2.$ *valid-path h1* $\wedge$ *valid-path h2* $\wedge$
$(\forall t{\in}\{0..1\}.\ cmod\ (h1\ t - \gamma\ t) < d \wedge cmod\ (h2\ t - \gamma\ t) < d) \wedge$
pathstart h2 = *pathstart h1* $\wedge$ *pathfinish h2* = *pathfinish h1* $\longrightarrow$
path-image h1 $\subseteq UNIV - \{z\} \wedge$ *path-image h2* $\subseteq UNIV - \{z\}$
$\wedge$

```
                (∀f. f holomorphic-on UNIV − {z} ⟶ contour-integral h2 f =
contour-integral h1 f)
    using contour-integral-nearby-ends [of UNIV − {z} γ] assms by (auto simp:
open-delete)
  then obtain h where h: polynomial-function h ∧ pathstart h = pathstart γ ∧
pathfinish h = pathfinish γ ∧
                        (∀t ∈ {0..1}. norm(h t − γ t) < d/2)
    using path-approx-polynomial-function [OF ‹path γ›, of d/2] d by auto
  def nn ≡ 1/(2* pi*ii) * contour-integral h (λw. 1/(w − z))
  have ∃n. ∀e > 0. ∃p. valid-path p ∧ z ∉ path-image p ∧
                       pathstart p = pathstart γ ∧  pathfinish p = pathfinish γ ∧
                       (∀t ∈ {0..1}. norm(γ t − p t) < e) ∧
                       contour-integral p (λw. 1/(w − z)) = 2 * pi * ii * n
                  (is ∃n. ∀e > 0. ?PP e n)
    proof (rule-tac x=nn in exI, clarify)
      fix e::real
      assume e: e>0
      obtain p where p: polynomial-function p ∧
            pathstart p = pathstart γ ∧ pathfinish p = pathfinish γ ∧ (∀t∈{0..1}.
cmod (p t − γ t) < min e (d / 2))
        using path-approx-polynomial-function [OF ‹path γ›, of min e (d/2)] d
‹0<e› by auto
      have (λw. 1 / (w − z)) holomorphic-on UNIV − {z}
        by (auto simp: intro!: holomorphic-intros)
      then show ?PP e nn
        apply (rule-tac x=p in exI)
        using pi-eq [of h p] h p d
          apply (auto simp: valid-path-polynomial-function norm-minus-commute
nn-def)
        done
    qed
  then show ?thesis
    unfolding winding-number-def
    apply (rule someI2-ex)
    apply (blast intro: ‹0<e›)
    done
qed

lemma winding-number-unique:
  assumes γ: path γ z ∉ path-image γ
      and pi:
        ⋀e. e>0 ⟹ ∃p. valid-path p ∧ z ∉ path-image p ∧
                       pathstart p = pathstart γ ∧ pathfinish p = pathfinish γ ∧
                       (∀t ∈ {0..1}. norm (γ t − p t) < e) ∧
                       contour-integral p (λw. 1/(w − z)) = 2 * pi * ii * n
   shows winding-number γ z = n
proof −
  have path-image γ ⊆ UNIV − {z}
    using assms by blast
```

then obtain *e*
where *e*: *e>0*
and *pi-eq*: $\bigwedge h1\ h2\ f.\ [\![valid\text{-}path\ h1;\ valid\text{-}path\ h2;$
$(\forall t{\in}\{0..1\}.\ cmod\ (h1\ t - \gamma\ t) < e \wedge cmod\ (h2\ t - \gamma\ t) < e);$
$pathstart\ h2 = pathstart\ h1;\ pathfinish\ h2 = pathfinish\ h1;\ f$ *holomorphic-on UNIV* $- \{z\}]\!] \Longrightarrow$
contour-integral h2 f = contour-integral h1 f
using *contour-integral-nearby-ends* [*of UNIV* $- \{z\}\ \gamma$] *assms* **by** (*auto simp*: *open-delete*)
obtain *p* **where** *p*:
valid-path p $\wedge$ $z \notin$ *path-image p* $\wedge$
pathstart p = pathstart γ $\wedge$ *pathfinish p = pathfinish* γ $\wedge$
$(\forall t \in \{0..1\}.\ norm\ (\gamma\ t - p\ t) < e) \wedge$
contour-integral p $(\lambda w.\ 1/(w - z)) = 2 * pi * ii * n$
using *pi* [*OF e*] **by** *blast*
obtain *q* **where** *q*:
valid-path q $\wedge$ $z \notin$ *path-image q* $\wedge$
pathstart q = pathstart γ $\wedge$ *pathfinish q = pathfinish* γ $\wedge$
$(\forall t{\in}\{0..1\}.\ cmod\ (\gamma\ t - q\ t) < e) \wedge$ *contour-integral q* $(\lambda w.\ 1\ /\ (w - z))$
$=$ *complex-of-real* $(2 * pi) *$ i $*$ *winding-number* $\gamma\ z$
using *winding-number* [*OF* γ *e*] **by** *blast*
have $2 *$ *complex-of-real pi* $*$ i $* n =$ *contour-integral p* $(\lambda w.\ 1\ /\ (w - z))$
using *p* **by** *auto*
also have $... =$ *contour-integral q* $(\lambda w.\ 1\ /\ (w - z))$
apply (*rule pi-eq*)
using *p q*
by (*auto simp*: *valid-path-polynomial-function norm-minus-commute intro*!: *holomorphic-intros*)
also have $... = 2 *$ *complex-of-real pi* $*$ i $*$ *winding-number* $\gamma\ z$
using *q* **by** *auto*
finally have $2 *$ *complex-of-real pi* $*$ i $* n = 2 *$ *complex-of-real pi* $*$ i $*$ *winding-number* $\gamma\ z$ **.**
then show *?thesis*
by *simp*
qed

lemma *winding-number-unique-loop*:
assumes γ: *path* γ $z \notin$ *path-image* γ
and *loop*: *pathfinish* γ $=$ *pathstart* γ
and *pi*:
$\bigwedge e.\ e>0 \Longrightarrow \exists p.$ *valid-path p* $\wedge$ $z \notin$ *path-image p* $\wedge$
pathfinish p = pathstart p $\wedge$
$(\forall t \in \{0..1\}.\ norm\ (\gamma\ t - p\ t) < e) \wedge$
contour-integral p $(\lambda w.\ 1/(w - z)) = 2 * pi * ii * n$
shows *winding-number* $\gamma\ z = n$
proof $-$
have *path-image* $\gamma \subseteq$ *UNIV* $- \{z\}$
using *assms* **by** *blast*
then obtain *e*

where *e*: *e>0*
and *pi-eq*: $\bigwedge h1\ h2\ f.\ [\![valid\text{-}path\ h1;\ valid\text{-}path\ h2;$
$(\forall t{\in}\{0..1\}.\ cmod\ (h1\ t - \gamma\ t) < e \wedge cmod\ (h2\ t - \gamma\ t) < e);$
$pathfinish\ h1 = pathstart\ h1;\ pathfinish\ h2 = pathstart\ h2;\ f\ holomorphic\text{-}on\ UNIV - \{z\}]\!] \Longrightarrow$
$contour\text{-}integral\ h2\ f = contour\text{-}integral\ h1\ f$
using *contour-integral-nearby-loops* [*of UNIV* $-$ $\{z\}$ γ] *assms* **by** (*auto simp*: *open-delete*)
obtain *p* **where** *p*:
$valid\text{-}path\ p \wedge z \notin path\text{-}image\ p \wedge$
$pathfinish\ p = pathstart\ p \wedge$
$(\forall t \in \{0..1\}.\ norm\ (\gamma\ t - p\ t) < e) \wedge$
$contour\text{-}integral\ p\ (\lambda w.\ 1/(w - z)) = 2 * pi * ii * n$
using *pi* [*OF e*] **by** *blast*
obtain *q* **where** *q*:
$valid\text{-}path\ q \wedge z \notin path\text{-}image\ q \wedge$
$pathstart\ q = pathstart\ \gamma \wedge pathfinish\ q = pathfinish\ \gamma \wedge$
$(\forall t{\in}\{0..1\}.\ cmod\ (\gamma\ t - q\ t) < e) \wedge contour\text{-}integral\ q\ (\lambda w.\ 1\ /\ (w - z)) = complex\text{-}of\text{-}real\ (2 * pi) * \mathrm{i} * winding\text{-}number\ \gamma\ z$
using *winding-number* [*OF* γ *e*] **by** *blast*
have $2 * complex\text{-}of\text{-}real\ pi * \mathrm{i} * n = contour\text{-}integral\ p\ (\lambda w.\ 1\ /\ (w - z))$
using *p* **by** *auto*
also have ... = $contour\text{-}integral\ q\ (\lambda w.\ 1\ /\ (w - z))$
apply (*rule pi-eq*)
using *p q loop*
by (*auto simp*: *valid-path-polynomial-function norm-minus-commute intro*!: *holomorphic-intros*)
also have ... = $2 * complex\text{-}of\text{-}real\ pi * \mathrm{i} * winding\text{-}number\ \gamma\ z$
using *q* **by** *auto*
finally have $2 * complex\text{-}of\text{-}real\ pi * \mathrm{i} * n = 2 * complex\text{-}of\text{-}real\ pi * \mathrm{i} * winding\text{-}number\ \gamma\ z$ **.**
then show *?thesis*
by *simp*
qed

lemma *winding-number-valid-path*:
assumes $valid\text{-}path\ \gamma\ z \notin path\text{-}image\ \gamma$
shows $winding\text{-}number\ \gamma\ z = 1/(2{*}pi{*}ii) * contour\text{-}integral\ \gamma\ (\lambda w.\ 1/(w - z))$
using *assms* **by** (*auto simp*: *valid-path-imp-path intro*!: *winding-number-unique*)

lemma *has-contour-integral-winding-number*:
assumes γ: $valid\text{-}path\ \gamma\ z \notin path\text{-}image\ \gamma$
shows $((\lambda w.\ 1/(w - z))\ has\text{-}contour\text{-}integral\ (2{*}pi{*}ii{*}winding\text{-}number\ \gamma\ z))\ \gamma$
by (*simp add*: *winding-number-valid-path has-contour-integral-integral contour-integrable-inversediff assms*)

lemma *winding-number-trivial* [*simp*]: $z \neq a \Longrightarrow winding\text{-}number(linepath\ a\ a)\ z$

= 0
by (*simp add*: *winding-number-valid-path*)

lemma *winding-number-subpath-trivial* [*simp*]: *z ≠ g x ⟹ winding-number* (*subpath x x g*) *z = 0*
by (*simp add*: *path-image-subpath winding-number-valid-path*)

lemma *winding-number-join*:
assumes *g1*: *path g1 z ∉ path-image g1*
and *g2*: *path g2 z ∉ path-image g2*
and *pathfinish g1 = pathstart g2*
shows *winding-number*(*g1 +++ g2*) *z = winding-number g1 z + winding-number g2 z*
apply (*rule winding-number-unique*)
using *assms* **apply** (*simp-all add*: *not-in-path-image-join*)
apply (*frule winding-number* [*OF g2*])
apply (*frule winding-number* [*OF g1*], *clarify*)
apply (*rename-tac p2 p1*)
apply (*rule-tac x=p1+++p2* **in** *exI*)
apply (*simp add*: *not-in-path-image-join contour-integrable-inversediff algebra-simps*)
apply (*auto simp*: *joinpaths-def*)
done

lemma *winding-number-reversepath*:
assumes *path γ z ∉ path-image γ*
shows *winding-number*(*reversepath γ*) *z = − winding-number γ z*)
apply (*rule winding-number-unique*)
using *assms*
apply *simp-all*
apply (*frule winding-number* [*OF assms*], *clarify*)
apply (*rule-tac x=reversepath p* **in** *exI*)
apply (*simp add*: *contour-integral-reversepath contour-integrable-inversediff valid-path-imp-reverse*)
apply (*auto simp*: *reversepath-def*)
done

lemma *winding-number-shiftpath*:
assumes *γ*: *path γ z ∉ path-image γ*
and *pathfinish γ = pathstart γ a ∈ {0..1}*
shows *winding-number*(*shiftpath a γ*) *z = winding-number γ z*
apply (*rule winding-number-unique-loop*)
using *assms*
apply (*simp-all add*: *path-shiftpath path-image-shiftpath pathfinish-shiftpath pathstart-shiftpath*)
apply (*frule winding-number* [*OF γ*], *clarify*)
apply (*rule-tac x=shiftpath a p* **in** *exI*)
apply (*simp add*: *contour-integral-shiftpath path-image-shiftpath pathfinish-shiftpath pathstart-shiftpath valid-path-shiftpath*)
apply (*auto simp*: *shiftpath-def*)
done

lemma *winding-number-split-linepath*:
assumes $c \in$ *closed-segment a b* $z \notin$ *closed-segment a b*
shows *winding-number*(*linepath a b*) $z =$ *winding-number*(*linepath a c*) $z +$ *winding-number*(*linepath c b*) z
proof −
have $z \notin$ *closed-segment a c* $z \notin$ *closed-segment c b*
using *assms* **apply** (*meson convex-contains-segment convex-segment ends-in-segment*(*1*) *subsetCE*)
using *assms* **by** (*meson convex-contains-segment convex-segment ends-in-segment*(*2*) *subsetCE*)
then show *?thesis*
using *assms*
by (*simp add*: *winding-number-valid-path contour-integral-split-linepath* [*symmetric*] *continuous-on-inversediff field-simps*)
qed

lemma *winding-number-cong*:
$(\bigwedge t.\ [\![0 \le t; t \le 1]\!] \Longrightarrow p\ t = q\ t) \Longrightarrow$ *winding-number p z* = *winding-number q z*
by (*simp add*: *winding-number-def pathstart-def pathfinish-def*)

lemma *winding-number-offset*: *winding-number p z* = *winding-number* ($\lambda w.$ *p w* − *z*) *0*
apply (*simp add*: *winding-number-def contour-integral-integral path-image-def valid-path-def pathstart-def pathfinish-def*)
apply (*intro ext arg-cong* [**where** *f* = *Eps*] *arg-cong* [**where** *f* = *All*] *imp-cong refl*, *safe*)
apply (*rename-tac g*)
apply (*rule-tac x*=$\lambda t.$ *g t* − *z* **in** *exI*)
apply (*force simp*: *vector-derivative-def has-vector-derivative-diff-const piecewise-C1-differentiable-diff C1-differentiable-imp-piecewise*)
apply (*rename-tac g*)
apply (*rule-tac x*=$\lambda t.$ *g t* + *z* **in** *exI*)
apply (*simp add*: *piecewise-C1-differentiable-add vector-derivative-def has-vector-derivative-add-const C1-differentiable-imp-piecewise*)
apply (*force simp*: *algebra-simps*)
done

lemma *winding-number-join-pos-combined*:
[*valid-path* $\gamma 1$; $z \notin$ *path-image* $\gamma 1$; $0 <$ *Re*(*winding-number* $\gamma 1$ z);
valid-path $\gamma 2$; $z \notin$ *path-image* $\gamma 2$; $0 <$ *Re*(*winding-number* $\gamma 2$ z); *pathfinish* $\gamma 1$ = *pathstart* $\gamma 2$]
$\Longrightarrow$ *valid-path*($\gamma 1$ +++ $\gamma 2$) $\wedge$ $z \notin$ *path-image*($\gamma 1$ +++ $\gamma 2$) $\wedge$ $0 <$ *Re*(*winding-number*($\gamma 1$ +++ $\gamma 2$) z)
by (*simp add*: *valid-path-join path-image-join winding-number-join valid-path-imp-path*)

lemma *Re-winding-number*:
⟦*valid-path* γ; $z \notin$ *path-image* γ⟧
$\Longrightarrow$ *Re*(*winding-number* γ z) = *Im*(*contour-integral* γ ($\lambda w.\ 1/(w - z)$)) / (*2***pi*)
by (*simp add*: *winding-number-valid-path field-simps Re-divide power2-eq-square*)

lemma *winding-number-pos-le*:
assumes γ: *valid-path* γ $z \notin$ *path-image* γ
and *ge*: $\bigwedge x.$ ⟦$0 < x$; $x < 1$⟧ $\Longrightarrow$ $0 \le$ *Im* (*vector-derivative* γ (*at* x) * *cnj*(γ $x - z$))
shows $0 \le$ *Re*(*winding-number* γ z)
proof −
have *: $0 \le$ *Im* (*vector-derivative* γ (*at* x) / (γ $x - z$)) **if** x: $0 < x$ $x < 1$ **for** x
using *ge* **by** (*simp add*: *Complex.Im-divide algebra-simps* x)
show *?thesis*
apply (*simp add*: *Re-winding-number* [*OF* γ] *field-simps*)
apply (*rule has-integral-component-nonneg*
[*of ii* $\lambda x.$ *if* $x \in \{0<..<1\}$
then $1/(\gamma\ x - z)$ * *vector-derivative* γ (*at* x) *else* 0, *simplified*])
prefer *3* **apply** (*force simp*: *)
apply (*simp add*: *Basis-complex-def*)
apply (*rule has-integral-spike-interior* [*of 0 1 -* $\lambda x.$ $1/(\gamma\ x - z)$ * *vector-derivative* γ (*at* x)])
apply *simp*
apply (*simp only*: *box-real*)
apply (*subst has-contour-integral* [*symmetric*])
using γ
apply (*simp add*: *contour-integrable-inversediff has-contour-integral-integral*)
done
qed

lemma *winding-number-pos-lt-lemma*:
assumes γ: *valid-path* γ $z \notin$ *path-image* γ
and e: $0 < e$
and *ge*: $\bigwedge x.$ ⟦$0 < x$; $x < 1$⟧ $\Longrightarrow$ $e \le$ *Im* (*vector-derivative* γ (*at* x) / (γ $x - z$))
shows $0 <$ *Re*(*winding-number* γ z)
proof −
have $e \le$ *Im* (*contour-integral* γ ($\lambda w.$ 1 / ($w - z$)))
apply (*rule has-integral-component-le*
[*of ii* $\lambda x.$ *ii***e ii***e* $\{0..1\}$
$\lambda x.$ *if* $x \in \{0<..<1\}$ *then* $1/(\gamma\ x - z)$ * *vector-derivative* γ (*at* x) *else ii***e*
contour-integral γ ($\lambda w.$ $1/(w - z)$), *simplified*])
using e
apply (*simp-all add*: *Basis-complex-def*)
using *has-integral-const-real* [*of - 0 1*] **apply** *force*
apply (*rule has-integral-spike-interior* [*of 0 1 -* $\lambda x.$ $1/(\gamma\ x - z)$ * *vector-derivative*

γ (*at x*), *simplified box-real*])
apply *simp*
apply (*subst has-contour-integral* [*symmetric*])
using γ
apply (*simp-all add*: *contour-integrable-inversediff has-contour-integral-integral ge*)
done
with *e* **show** *?thesis*
by (*simp add*: *Re-winding-number* [*OF* γ] *field-simps*)
qed

lemma *winding-number-pos-lt*:
assumes γ: *valid-path* γ *z* $\notin$ *path-image* γ
and *e*: *0* < *e*
and *ge*: $\bigwedge x.\ [\![0 < x; x < 1]\!] \Longrightarrow e \leq Im$ (*vector-derivative* γ (*at x*) $*$ *cnj*(γ *x* − *z*))
shows *0* < *Re* (*winding-number* γ *z*)
proof −
have *bm*: *bounded* ((λ*w*. *w* − *z*) ‘ (*path-image* γ))
using *bounded-translation* [*of* - −*z*] γ **by** (*simp add*: *bounded-valid-path-image*)
then obtain *B* **where** *B*: *B* > *0* **and** *Bno*: $\bigwedge x.\ x \in$ (λ*w*. *w* − *z*) ‘ (*path-image* γ) $\Longrightarrow$ *norm x* ≤ *B*
using *bounded-pos* [*THEN iffD1*, *OF bm*] **by** *blast*
{ fix *x*::*real* **assume** *x*: *0* < *x x* < *1*
then have *B2*: *cmod* (γ *x* − *z*) ^ *2* ≤ *B* ^ *2* **using** *Bno* [*of* γ *x* − *z*]
by (*simp add*: *path-image-def power2-eq-square mult-mono'*)
with *x* **have** γ *x* ≠ *z* **using** γ
using *path-image-def* **by** *fastforce*
then have *e* / *B*^{2} ≤ *Im* (*vector-derivative* γ (*at x*) $*$ *cnj* (γ *x* − *z*)) / (*cmod* (γ *x* − *z*))^2
using *B ge* [*OF x*] *B2 e*
apply (*rule-tac y*=*e* / (*cmod* (γ *x* − *z*))^2 **in** *order-trans*)
apply (*auto simp*: *divide-left-mono divide-right-mono*)
done
then have *e* / *B*^{2} ≤ *Im* (*vector-derivative* γ (*at x*) / (γ *x* − *z*))
by (*simp add*: *Im-divide-Reals complex-div-cnj* [*of* - γ *x* − *z* **for** *x*] *del*: *complex-cnj-diff times-complex.sel*)
} note $*$ = *this*
show *?thesis*
using *e B* **by** (*simp add*: $*$ *winding-number-pos-lt-lemma* [*OF* γ, *of e*/*B* ^ *2*])
qed

53.24 The winding number is an integer

Proof from the book Complex Analysis by Lars V. Ahlfors, Chapter 4, section 2.1, Also on page 134 of Serge Lang's book with the name title, etc.

lemma *exp-fg*:
fixes *z*::*complex*
assumes *g*: (*g has-vector-derivative g'*) (*at x within s*)

```
    and f: (f has-vector-derivative (g' / (g x − z))) (at x within s)
    and z: g x ≠ z
  shows ((λx. exp(−f x) ∗ (g x − z)) has-vector-derivative 0) (at x within s)
proof −
  have ∗: (exp o (λx. (− f x)) has-vector-derivative − (g' / (g x − z)) ∗ exp (− f
x)) (at x within s)
    using assms unfolding has-vector-derivative-def scaleR-conv-of-real
    by (auto intro!: derivative-eq-intros)
  show ?thesis
    apply (rule has-vector-derivative-eq-rhs)
    apply (rule bounded-bilinear.has-vector-derivative [OF bounded-bilinear-mult])
    using z
    apply (auto simp: intro!: derivative-eq-intros ∗ [unfolded o-def] g)
    done
qed

lemma winding-number-exp-integral:
  fixes z::complex
  assumes γ: γ piecewise-C1-differentiable-on {a..b}
    and ab: a ≤ b
    and z: z ∉ γ ' {a..b}
  shows (λx. vector-derivative γ (at x) / (γ x − z)) integrable-on {a..b}
        (is ?thesis1)
        exp (− (integral {a..b} (λx. vector-derivative γ (at x) / (γ x − z)))) ∗ (γ
b − z) = γ a − z
        (is ?thesis2)
proof −
  let ?Dγ = λx. vector-derivative γ (at x)
  have [simp]: ⋀x. a ≤ x ⟹ x ≤ b ⟹ γ x ≠ z
    using z by force
  have cong: continuous-on {a..b} γ
    using γ by (simp add: piecewise-C1-differentiable-on-def)
  obtain k where fink: finite k and g-C1-diff: γ C1-differentiable-on ({a..b} −
k)
    using γ by (force simp: piecewise-C1-differentiable-on-def)
  have o: open ({a<..<b} − k)
    using ‹finite k› by (simp add: finite-imp-closed open-Diff)
  moreover have {a<..<b} − k ⊆ {a..b} − k
    by force
  ultimately have g-diff-at: ⋀x. ⟦x ∉ k; x ∈ {a<..<b}⟧ ⟹ γ differentiable at x
   by (metis Diff-iff differentiable-on-subset C1-diff-imp-diff [OF g-C1-diff] differentiable-on-def
differentiable-within-open)
  { fix w
    assume w ≠ z
    have continuous-on (ball w (cmod (w − z))) (λw. 1 / (w − z))
      by (auto simp: dist-norm intro!: continuous-intros)
    moreover have ⋀x. cmod (w − x) < cmod (w − z) ⟹ ∃f'. ((λw. 1 / (w
− z)) has-field-derivative f') (at x)
      by (auto simp: intro!: derivative-eq-intros)
```

ultimately have $\exists h.\ \forall y.\ norm(y - w) < norm(w - z) \longrightarrow (h\ has\text{-}field\text{-}derivative\ 1/(y - z))\ (at\ y)$
using *holomorphic-convex-primitive* [*of ball w* (*norm*(*w* − *z*)) {} $\lambda w.\ 1/(w - z)$]
by (*simp add*: *field-differentiable-def Ball-def dist-norm at-within-open-NO-MATCH norm-minus-commute*)
}
then obtain *h* **where** *h*: $\bigwedge w\ y.\ w \neq z \Longrightarrow norm(y - w) < norm(w - z) \Longrightarrow (h\ w\ has\text{-}field\text{-}derivative\ 1/(y - z))\ (at\ y)$
by *meson*
have *exy*: $\exists y.\ ((\lambda x.\ inverse\ (\gamma\ x - z) * ?D\gamma\ x)\ has\text{-}integral\ y)\ \{a..b\}$
unfolding *integrable-on-def* [*symmetric*]
apply (*rule contour-integral-local-primitive-any* [*OF piecewise-C1-imp-differentiable* [*OF* γ], *of* $-\{z\}$])
apply (*rename-tac w*)
apply (*rule-tac x*=*norm*(*w* − *z*) **in** *exI*)
apply (*simp-all add*: *inverse-eq-divide*)
apply (*metis has-field-derivative-at-within h*)
done
have *vg-int*: $(\lambda x.\ ?D\gamma\ x\ /\ (\gamma\ x - z))\ integrable\text{-}on\ \{a..b\}$
unfolding *box-real* [*symmetric*] *divide-inverse-commute*
by (*auto intro*!: *exy integrable-subinterval simp add*: *integrable-on-def ab*)
with *ab* **show** *?thesis1*
by (*simp add*: *divide-inverse-commute integral-def integrable-on-def*)
{ fix *t*
assume *t*: $t \in \{a..b\}$
have *cball*: *continuous-on* (*ball* (γ *t*) (*dist* (γ *t*) *z*)) ($\lambda x.\ inverse\ (x - z)$)
using *z* **by** (*auto intro*!: *continuous-intros simp*: *dist-norm*)
have *icd*: $\bigwedge x.\ cmod\ (\gamma\ t - x) < cmod\ (\gamma\ t - z) \Longrightarrow (\lambda w.\ inverse\ (w - z))$ *field-differentiable at x*
unfolding *field-differentiable-def* **by** (*force simp*: *intro*!: *derivative-eq-intros*)
obtain *h* **where** *h*: $\bigwedge x.\ cmod\ (\gamma\ t - x) < cmod\ (\gamma\ t - z) \Longrightarrow$
$(h\ has\text{-}field\text{-}derivative\ inverse\ (x - z))\ (at\ x\ within\ \{y.\ cmod\ (\gamma\ t - y) < cmod\ (\gamma\ t - z)\})$
using *holomorphic-convex-primitive* [**where** $f = \lambda w.\ inverse(w - z)$, *OF convex-ball finite.emptyI cball icd*]
by *simp* (*auto simp*: *ball-def dist-norm that*)
{ fix *x D*
assume *x*: $x \notin k\ a < x\ x < b$
then have $x \in interior\ (\{a..b\} - k)$
using *open-subset-interior* [*OF o*] **by** *fastforce*
then have *con*: *isCont* ($\lambda x.\ ?D\gamma\ x$) *x*
using *g-C1-diff x* **by** (*auto simp*: *C1-differentiable-on-eq intro*: *continuous-on-interior*)
then have *con-vd*: *continuous* (*at x within* $\{a..b\}$) ($\lambda x.\ ?D\gamma\ x$)
by (*rule continuous-at-imp-continuous-within*)
have *gdx*: γ *differentiable at x*
using *x* **by** (*simp add*: *g-diff-at*)
have $((\lambda c.\ exp\ (-\ integral\ \{a..c\}\ (\lambda x.\ vector\text{-}derivative\ \gamma\ (at\ x)\ /\ (\gamma\ x - z))) * (\gamma\ c - z))\ has\text{-}derivative\ (\lambda h.\ 0))$

(*at x within* {*a..b*})
using *x gdx t*
apply (*clarsimp simp add*: *differentiable-iff-scaleR*)
apply (*rule exp-fg* [*unfolded has-vector-derivative-def*, *simplified*], *blast intro*: *has-derivative-at-within*)
apply (*simp-all add*: *has-vector-derivative-def* [*symmetric*])
apply (*rule has-vector-derivative-eq-rhs* [*OF integral-has-vector-derivative-continuous-at*])
apply (*rule con-vd continuous-intros cong vg-int* | *simp add*: *continuous-at-imp-continuous-within has-vector-derivative-continuous vector-derivative-at*)+
done
} **note** ∗ = *this*
have *exp* (− (*integral* {*a..t*} (λ*x*. *?Dγ x* / (*γ x* − *z*)))) ∗ (*γ t* − *z*) =*γ a* − *z*
apply (*rule has-derivative-zero-unique-strong-interval* [*of* {*a*,*b*} ∪ *k a b*])
using *t*
apply (*auto intro*!: ∗ *continuous-intros fink cong indefinite-integral-continuous* [*OF vg-int*] *simp add*: *ab*)+
done
}
with *ab* **show** *?thesis2*
by (*simp add*: *divide-inverse-commute integral-def*)
qed

corollary *winding-number-exp-2pi*:
⟦*path p*; *z* ∉ *path-image p*⟧
⟹ *pathfinish p* − *z* = *exp* (2 ∗ *pi* ∗ *ii* ∗ *winding-number p z*) ∗ (*pathstart p* − *z*)
using *winding-number* [*of p z 1*] **unfolding** *valid-path-def path-image-def pathstart-def pathfinish-def*
by (*force dest*: *winding-number-exp-integral*(*2*) [*of - 0 1 z*] *simp*: *field-simps contour-integral-integral exp-minus*)

53.25 The version with complex integers and equality

lemma *integer-winding-number-eq*:
assumes *γ*: *path γ* **and** *z*: *z* ∉ *path-image γ*
shows *winding-number γ z* ∈ ℤ ⟷ *pathfinish γ* = *pathstart γ*
proof −
have ∗: ⋀*i*::*complex*. ⋀*g0 g1*. ⟦*i* ≠ *0*; *g0* ≠ *z*; (*g1* − *z*) / *i* = *g0* − *z*⟧ ⟹ (*i* = *1* ⟷ *g1* = *g0*)
by (*simp add*: *field-simps*) *algebra*
obtain *p* **where** *p*: *valid-path p z* ∉ *path-image p*
pathstart p = *pathstart γ pathfinish p* = *pathfinish γ*
contour-integral p (λ*w*. *1* / (*w* − *z*)) = *complex-of-real* (*2* ∗ *pi*) ∗ i ∗ *winding-number γ z*
using *winding-number* [*OF assms*, *of 1*] **by** *auto*
have [*simp*]: (*winding-number γ z* ∈ ℤ) = (*exp* (*contour-integral p* (λ*w*. *1* / (*w* − *z*))) = *1*)
using *p* **by** (*simp add*: *exp-eq-1 complex-is-Int-iff*)
have *winding-number p z* ∈ ℤ ⟷ *pathfinish p* = *pathstart p*

using *p z*
apply (*simp add*: *winding-number-valid-path valid-path-def path-image-def pathstart-def pathfinish-def*)
using *winding-number-exp-integral*(*2*) [*of p 0 1 z*]
apply (*simp add*: *field-simps contour-integral-integral exp-minus*)
apply (*rule* ∗)
apply (*auto simp*: *path-image-def field-simps*)
done
then show *?thesis* **using** *p*
by (*auto simp*: *winding-number-valid-path*)
qed

theorem *integer-winding-number*:
⟦*path γ*; *pathfinish γ* = *pathstart γ*; *z* ∉ *path-image γ*⟧ ⟹ *winding-number γ z* ∈ ℤ
by (*metis integer-winding-number-eq*)

If the winding number's magnitude is at least one, then the path must contain points in every direction.*) We can thus bound the winding number of a path that doesn't intersect a given ray.

lemma *winding-number-pos-meets*:
fixes *z*::*complex*
assumes *γ*: *valid-path γ* **and** *z*: *z* ∉ *path-image γ* **and** *1*: *Re* (*winding-number γ z*) ≥ *1*
and *w*: *w* ≠ *z*
shows ∃ *a*::*real*. *0* < *a* ∧ *z* + *a*∗(*w* − *z*) ∈ *path-image γ*
proof −
have [*simp*]: ⋀*x*. *0* ≤ *x* ⟹ *x* ≤ *1* ⟹ *γ x* ≠ *z*
using *z* **by** (*auto simp*: *path-image-def*)
have [*simp*]: *z* ∉ *γ* ‘ {*0*..*1*}
using *path-image-def z* **by** *auto*
have *gpd*: *γ piecewise-C1-differentiable-on* {*0*..*1*}
using *γ valid-path-def* **by** *blast*
def *r* ≡ (*w* − *z*) / (*γ 0* − *z*)
have [*simp*]: *r* ≠ *0*
using *w z* **by** (*auto simp*: *r-def*)
have *Arg r* ≤ *2*∗*pi*
by (*simp add*: *Arg less-eq-real-def*)
also have ... ≤ *Im* (*integral* {*0*..*1*} (λ*x*. *vector-derivative γ* (*at x*) / (*γ x* − *z*)))
using *1*
apply (*simp add*: *winding-number-valid-path* [*OF γ z*] *Cauchy-Integral-Thm.contour-integral-integral*)
apply (*simp add*: *Complex.Re-divide field-simps power2-eq-square*)
done
finally have *Arg r* ≤ *Im* (*integral* {*0*..*1*} (λ*x*. *vector-derivative γ* (*at x*) / (*γ x* − *z*))) .
then have ∃ *t*. *t* ∈ {*0*..*1*} ∧ *Im*(*integral* {*0*..*t*} (λ*x*. *vector-derivative γ* (*at x*)/(*γ x* − *z*))) = *Arg r*
apply (*simp add*:)
apply (*rule Topological-Spaces.IVT′*)

apply (*simp-all add*: *Complex-Transcendental.Arg-ge-0*)
apply (*intro continuous-intros indefinite-integral-continuous winding-number-exp-integral* [*OF gpd*]; *simp*)
done
then obtain *t* **where** *t*: $t \in \{0..1\}$
and *eqArg*: *Im* (*integral* $\{0..t\}$ ($\lambda x.$ *vector-derivative* γ (*at x*)/(γ *x* − *z*))) = *Arg r*
by *blast*
def $i \equiv$ *integral* $\{0..t\}$ ($\lambda x.$ *vector-derivative* γ (*at x*) / (γ *x* − *z*))
have *iArg*: *Arg r* = *Im i*
using *eqArg* **by** (*simp add*: *i-def*)
have *gpdt*: γ *piecewise-C1-differentiable-on* $\{0..t\}$
by (*metis atLeastAtMost-iff atLeastatMost-subset-iff order-refl piecewise-C1-differentiable-on-subset gpd t*)
have *exp* (− *i*) ∗ (γ *t* − *z*) = γ *0* − *z*
unfolding *i-def*
apply (*rule winding-number-exp-integral* [*OF gpdt*])
using *t z* **unfolding** *path-image-def*
apply *force+*
done
then have ∗: γ *t* − *z* = *exp i* ∗ (γ *0* − *z*)
by (*simp add*: *exp-minus field-simps*)
then have (*w* − *z*) = *r* ∗ (γ *0* − *z*)
by (*simp add*: *r-def*)
then have *z* + *complex-of-real* (*exp* (*Re i*)) ∗ (*w* − *z*) / *complex-of-real* (*cmod r*) = γ *t*
apply (*simp add*:)
apply (*subst Complex-Transcendental.Arg-eq* [*of r*])
apply (*simp add*: *iArg*)
using ∗
apply (*simp add*: *exp-eq-polar field-simps*)
done
with *t* **show** *?thesis*
by (*rule-tac x*=*exp*(*Re i*) / *norm r* **in** *exI*) (*auto simp*: *path-image-def*)
qed

lemma *winding-number-big-meets*:
fixes *z*::*complex*
assumes γ: *valid-path* γ **and** *z*: *z* ∉ *path-image* γ **and** |*Re* (*winding-number* γ *z*)| ≥ *1*
and *w*: *w* ≠ *z*
shows ∃ *a*::*real*. *0* < *a* ∧ *z* + *a*∗(*w* − *z*) ∈ *path-image* γ
proof −
{ assume *Re* (*winding-number* γ *z*) ≤ − *1*
then have *Re* (*winding-number* (*reversepath* γ) *z*) ≥ *1*
by (*simp add*: γ *valid-path-imp-path winding-number-reversepath z*)
moreover have *valid-path* (*reversepath* γ)
using γ *valid-path-imp-reverse* **by** *auto*
moreover have *z* ∉ *path-image* (*reversepath* γ)

by (*simp add*: *z*)
ultimately have ∃ *a*::*real*. *0* < *a* ∧ *z* + *a*∗(*w* − *z*) ∈ *path-image* (*reversepath* *γ*)
using *winding-number-pos-meets w* **by** *blast*
then have *?thesis*
by *simp*
}
then show *?thesis*
using *assms*
by (*simp add*: *Groups.abs-if-class.abs-if winding-number-pos-meets split*: *if-split-asm*)
qed

lemma *winding-number-less-1*:
fixes *z*::*complex*
shows
⟦*valid-path* *γ*; *z* ∉ *path-image* *γ*; *w* ≠ *z*;
⋀*a*::*real*. *0* < *a* ⟹ *z* + *a*∗(*w* − *z*) ∉ *path-image* *γ*⟧
⟹ |*Re*(*winding-number* *γ* *z*)| < *1*
by (*auto simp*: *not-less dest*: *winding-number-big-meets*)

One way of proving that WN=1 for a loop.

lemma *winding-number-eq-1*:
fixes *z*::*complex*
assumes *γ*: *valid-path* *γ* **and** *z*: *z* ∉ *path-image* *γ* **and** *loop*: *pathfinish* *γ* = *pathstart* *γ*
and *0*: *0* < *Re*(*winding-number* *γ* *z*) **and** *2*: *Re*(*winding-number* *γ* *z*) < *2*
shows *winding-number* *γ* *z* = *1*
proof −
have *winding-number* *γ* *z* ∈ *Ints*
by (*simp add*: *γ integer-winding-number loop valid-path-imp-path z*)
then show *?thesis*
using *0 2* **by** (*auto simp*: *Ints-def*)
qed

53.26 Continuity of winding number and invariance on connected sets.

lemma *continuous-at-winding-number*:
fixes *z*::*complex*
assumes *γ*: *path* *γ* **and** *z*: *z* ∉ *path-image* *γ*
shows *continuous* (*at z*) (*winding-number* *γ*)
proof −
obtain *e* **where** *e*>*0* **and** *cbg*: *cball z e* ⊆ − *path-image* *γ*
using *open-contains-cball* [*of* − *path-image* *γ*] *z*
by (*force simp*: *closed-def* [*symmetric*] *closed-path-image* [*OF* *γ*])
then have *ppag*: *path-image* *γ* ⊆ − *cball z* (*e*/*2*)
by (*force simp*: *cball-def dist-norm*)
have *oc*: *open* (− *cball z* (*e* / *2*))
by (*simp add*: *closed-def* [*symmetric*])

```
 obtain d where d>0 and pi-eq:
   ⋀h1 h2. ⟦valid-path h1; valid-path h2;
            (∀t∈{0..1}. cmod (h1 t − γ t) < d ∧ cmod (h2 t − γ t) < d);
            pathstart h2 = pathstart h1; pathfinish h2 = pathfinish h1⟧
           ⟹
            path-image h1 ⊆ − cball z (e / 2) ∧
            path-image h2 ⊆ − cball z (e / 2) ∧
            (∀f. f holomorphic-on − cball z (e / 2) ⟶ contour-integral h2 f =
contour-integral h1 f)
   using contour-integral-nearby-ends [OF oc γ ppag] by metis
 obtain p where p: valid-path p z ∉ path-image p
                   pathstart p = pathstart γ ∧ pathfinish p = pathfinish γ
           and pg: ⋀t. t∈{0..1} ⟹ cmod (γ t − p t) < min d e / 2
            and pi: contour-integral p (λx. 1 / (x − z)) = complex-of-real (2 *
pi) * i * winding-number γ z
   using winding-number [OF γ z, of min d e / 2] ‹d>0› ‹e>0› by auto
 { fix w
   assume d2: cmod (w − z) < d/2 and e2: cmod (w − z) < e/2
   then have wnotp: w ∉ path-image p
     using cbg ‹d>0› ‹e>0›
     apply (simp add: path-image-def cball-def dist-norm, clarify)
     apply (frule pg)
     apply (drule-tac c=γ x in subsetD)
    apply (auto simp: less-eq-real-def norm-minus-commute norm-triangle-half-l)
     done
   have wnotg: w ∉ path-image γ
     using cbg e2 ‹e>0› by (force simp: dist-norm norm-minus-commute)
   { fix k::real
     assume k: k>0
     then obtain q where q: valid-path q w ∉ path-image q
                         pathstart q = pathstart γ ∧ pathfinish q = pathfinish γ
                  and qg: ⋀t. t ∈ {0..1} ⟹ cmod (γ t − q t) < min k (min d e)
/ 2
                  and qi: contour-integral q (λu. 1 / (u − w)) = complex-of-real (2
* pi) * i * winding-number γ w
       using winding-number [OF γ wnotg, of min k (min d e) / 2] ‹d>0› ‹e>0›
k
       by (force simp: min-divide-distrib-right)
     have contour-integral p (λu. 1 / (u − w)) = contour-integral q (λu. 1 / (u
− w))
      apply (rule pi-eq [OF ‹valid-path q› ‹valid-path p›, THEN conjunct2, THEN
conjunct2, rule-format])
       apply (frule pg)
       apply (frule qg)
       using p q ‹d>0› e2
     apply (auto simp: dist-norm norm-minus-commute intro!: holomorphic-intros)
       done
     then have contour-integral p (λx. 1 / (x − w)) = complex-of-real (2 * pi)
* i * winding-number γ w
```

by (*simp add*: *pi qi*)
} note *pip* = *this*
have *path p*
using *p* **by** (*simp add*: *valid-path-imp-path*)
then have *winding-number p w* = *winding-number* γ *w*
apply (*rule winding-number-unique* [*OF - wnotp*])
apply (*rule-tac x*=*p* **in** *exI*)
apply (*simp add*: *p wnotp min-divide-distrib-right pip*)
done
} note *wnwn* = *this*
obtain *pe* **where** *pe*>*0* **and** *cbp*: *cball z* (*3 / 4 * pe*) ⊆ − *path-image p*
using *p open-contains-cball* [*of* − *path-image p*]
by (*force simp*: *closed-def* [*symmetric*] *closed-path-image* [*OF valid-path-imp-path*])
obtain *L*
where *L*>*0*
and *L*: ⋀*f B*. ⟦*f holomorphic-on* − *cball z* (*3 / 4 * pe*);
∀ *z* ∈ − *cball z* (*3 / 4 * pe*). *cmod* (*f z*) ≤ *B*⟧ ⟹
cmod (*contour-integral p f*) ≤ *L * B*
using *contour-integral-bound-exists* [*of* − *cball z* (*3/4***pe*) *p*] *cbp* ‹*valid-path p*› **by** *force*
{ fix *e*::*real* **and** *w*::*complex*
assume *e*: *0* < *e* **and** *w*: *cmod* (*w* − *z*) < *pe/4 cmod* (*w* − *z*) < *e * pe*^{2} */* (*8 * L*)
then have [*simp*]: *w* ∉ *path-image p*
using *cbp p*(*2*) ‹*0* < *pe*›
by (*force simp*: *dist-norm norm-minus-commute path-image-def cball-def*)
have [*simp*]: *contour-integral p* (λ*x*. *1*/(*x* − *w*)) − *contour-integral p* (λ*x*. *1*/(*x* − *z*)) =
contour-integral p (λ*x*. *1*/(*x* − *w*) − *1*/(*x* − *z*))
by (*simp add*: *p contour-integrable-inversediff contour-integral-diff*)
{ fix *x*
assume *pe*: *3/4 * pe* < *cmod* (*z* − *x*)
have *cmod* (*w* − *x*) < *pe/4* + *cmod* (*z* − *x*)
by (*meson add-less-cancel-right norm-diff-triangle-le order-refl order-trans-rules*(*21*) *w*(*1*))
then have *wx*: *cmod* (*w* − *x*) < *4/3 * cmod* (*z* − *x*) **using** *pe* **by** *simp*
have *cmod* (*z* − *x*) ≤ *cmod* (*z* − *w*) + *cmod* (*w* − *x*)
using *norm-diff-triangle-le* **by** *blast*
also have ... < *pe/4* + *cmod* (*w* − *x*)
using *w* **by** (*simp add*: *norm-minus-commute*)
finally have *pe/2* < *cmod* (*w* − *x*)
using *pe* **by** *auto*
then have (*pe/2*) ^ *2* < *cmod* (*w* − *x*) ^ *2*
apply (*rule power-strict-mono*)
using ‹*pe*>*0*› **by** *auto*
then have *pe2*: *pe* ^ *2* < *4 * cmod* (*w* − *x*) ^ *2*
by (*simp add*: *power-divide*)
have *8 * L * cmod* (*w* − *z*) < *e * pe*^{2}
using *w* ‹*L*>*0*› **by** (*simp add*: *field-simps*)

also have ... $< e * 4 * cmod\ (w - x) * cmod\ (w - x)$
using *pe2* ‹*e>0*› **by** (*simp add*: *power2-eq-square*)
also have ... $< e * 4 * cmod\ (w - x) * (4/3 * cmod\ (z - x))$
using *wx*
apply (*rule mult-strict-left-mono*)
using *pe2 e not-less-iff-gr-or-eq* **by** *fastforce*
finally have $L * cmod\ (w - z) < 2/3 * e * cmod\ (w - x) * cmod\ (z - x)$
by *simp*
also have ... $\leq e * cmod\ (w - x) * cmod\ (z - x)$
using *e* **by** *simp*
finally have *Lwz*: $L * cmod\ (w - z) < e * cmod\ (w - x) * cmod\ (z - x)$ **.**
have $L * cmod\ (1\ /\ (x - w) - 1\ /\ (x - z)) \leq e$
apply (*cases* $x{=}z \vee x{=}w$)
using *pe* ‹*pe>0*› *w* ‹*L>0*›
apply (*force simp*: *norm-minus-commute*)
using *wx w*(*2*) ‹*L>0*› *pe pe2 Lwz*
apply (*auto simp*: *divide-simps mult-less-0-iff norm-minus-commute norm-divide norm-mult power2-eq-square*)
done
} note *L-cmod-le* = *this*
have ∗: $cmod\ (contour\text{-}integral\ p\ (\lambda x.\ 1\ /\ (x - w) - 1\ /\ (x - z))) \leq L * (e * pe^2\ /\ L\ /\ 4 * (inverse\ (pe\ /\ 2))^2)$
apply (*rule L*)
using ‹*pe>0*› *w*
apply (*force simp*: *dist-norm norm-minus-commute intro*!: *holomorphic-intros*)
using ‹*pe>0*› *w* ‹*L>0*›
apply (*auto simp*: *cball-def dist-norm field-simps L-cmod-le simp del*: *less-divide-eq-numeral1 le-divide-eq-numeral1*)
done
have $cmod\ (contour\text{-}integral\ p\ (\lambda x.\ 1\ /\ (x - w)) - contour\text{-}integral\ p\ (\lambda x.\ 1\ /\ (x - z))) < 2{*}e$
apply (*simp add*:)
apply (*rule le-less-trans* [*OF* ∗])
using ‹*L>0*› *e*
apply (*force simp*: *field-simps*)
done
then have $cmod\ (winding\text{-}number\ p\ w - winding\text{-}number\ p\ z) < e$
using *pi-ge-two e*
by (*force simp*: *winding-number-valid-path p field-simps norm-divide norm-mult intro*: *less-le-trans*)
} note *cmod-wn-diff* = *this*
then have *isCont* (*winding-number p*) *z*
apply (*simp add*: *continuous-at-eps-delta*, *clarify*)
apply (*rule-tac x=min* (*pe/4*) (*e/2∗pe^2/L/4*) **in** *exI*)
using ‹*pe>0*› ‹*L>0*›
apply (*simp add*: *dist-norm cmod-wn-diff*)
done
then show *?thesis*
apply (*rule continuous-transform-within* [**where** *d* = *min d e / 2*])

apply (*auto simp*: ‹*d*>0› ‹*e*>0› *dist-norm wnwn*)
done
qed

corollary *continuous-on-winding-number*:
path $\gamma \Longrightarrow$ *continuous-on* (− *path-image* γ) (λw. *winding-number* γ *w*)
by (*simp add*: *continuous-at-imp-continuous-on continuous-at-winding-number*)

53.27 The winding number is constant on a connected region

lemma *winding-number-constant*:
assumes γ: *path* γ **and** *loop*: *pathfinish* γ = *pathstart* γ **and** *cs*: *connected s* **and** *sg*: *s* ∩ *path-image* γ = {}
shows ∃ *k*. ∀ *z* ∈ *s*. *winding-number* γ *z* = *k*
proof −
{ fix *y z*
assume *ne*: *winding-number* γ *y* ≠ *winding-number* γ *z*
assume *y* ∈ *s z* ∈ *s*
then have *winding-number* γ *y* ∈ ℤ *winding-number* γ *z* ∈ ℤ
using *integer-winding-number* [*OF* γ *loop*] *sg* ‹*y* ∈ *s*› **by** *auto*
with *ne* **have** *1* ≤ *cmod* (*winding-number* γ *y* − *winding-number* γ *z*)
by (*auto simp*: *Ints-def of-int-diff* [*symmetric*] *simp del*: *of-int-diff*)
} note ∗ = *this*
show *?thesis*
apply (*rule continuous-discrete-range-constant* [*OF cs*])
using *continuous-on-winding-number* [*OF* γ] *sg*
apply (*metis Diff-Compl Diff-eq-empty-iff continuous-on-subset*)
apply (*rule-tac x=1* **in** *exI*)
apply (*auto simp*: ∗)
done
qed

lemma *winding-number-eq*:
⟦*path* γ; *pathfinish* γ = *pathstart* γ; *w* ∈ *s*; *z* ∈ *s*; *connected s*; *s* ∩ *path-image* γ = {}⟧
$\Longrightarrow$ *winding-number* γ *w* = *winding-number* γ *z*
using *winding-number-constant* **by** *fastforce*

lemma *open-winding-number-levelsets*:
assumes γ: *path* γ **and** *loop*: *pathfinish* γ = *pathstart* γ
shows *open* {*z*. *z* ∉ *path-image* γ ∧ *winding-number* γ *z* = *k*}
proof −
have *op*: *open* (− *path-image* γ)
by (*simp add*: *closed-path-image* γ *open-Compl*)
{ fix *z* **assume** *z*: *z* ∉ *path-image* γ **and** *k*: *k* = *winding-number* γ *z*
obtain *e* **where** *e*: *e*>0 *ball z e* ⊆ − *path-image* γ
using *open-contains-ball* [*of* − *path-image* γ] *op z*
by *blast*
have ∃ *e*>0. ∀ *y*. *dist y z* < *e* ⟶ *y* ∉ *path-image* γ ∧ *winding-number* γ *y* =

winding-number γ *z*
apply (*rule-tac x=e* **in** *exI*)
using *e* **apply** (*simp add*: *dist-norm ball-def norm-minus-commute*)
apply (*auto simp*: *dist-norm norm-minus-commute intro*!: *winding-number-eq* [*OF assms*, **where** $s = ball\ z\ e$])
done
} then
show *?thesis*
by (*auto simp*: *open-dist*)
qed

53.28 Winding number is zero "outside" a curve, in various senses

lemma *winding-number-zero-in-outside*:
assumes γ: *path* γ **and** *loop*: *pathfinish* γ = *pathstart* γ **and** *z*: $z \in outside$ (*path-image* γ)
shows *winding-number* γ *z* = *0*
proof −
obtain *B*::*real* **where** $0 < B$ **and** *B*: *path-image* $\gamma \subseteq$ *ball 0 B*
using *bounded-subset-ballD* [*OF bounded-path-image* [*OF* γ]] **by** *auto*
obtain *w*::*complex* **where** *w*: $w \notin$ *ball 0* (B + *1*)
by (*metis abs-of-nonneg le-less less-irrefl mem-ball-0 norm-of-real*)
have − *ball 0* (B + *1*) $\subseteq$ *outside* (*path-image* γ)
apply (*rule outside-subset-convex*)
using *B subset-ball* **by** *auto*
then have *wout*: $w \in$ *outside* (*path-image* γ)
using *w* **by** *blast*
moreover obtain *k* **where** $\bigwedge z.\ z \in$ *outside* (*path-image* γ) $\Longrightarrow$ *winding-number* γ *z* = *k*
using *winding-number-constant* [*OF* γ *loop*, *of outside*(*path-image* γ)] *connected-outside*
by (*metis DIM-complex bounded-path-image dual-order.refl* γ *outside-no-overlap*)
ultimately have *winding-number* γ *z* = *winding-number* γ *w*
using *z* **by** *blast*
also have ... = *0*
proof −
have *wnot*: $w \notin$ *path-image* γ **using** *wout* **by** (*simp add*: *outside-def*)
{ fix *e*::*real* **assume** *0*<*e*
obtain *p* **where** *p*: *polynomial-function p pathstart p* = *pathstart* γ *pathfinish p* = *pathfinish* γ
and *pg1*: $(\bigwedge t.\ [\![0 \le t; t \le 1]\!] \Longrightarrow cmod\ (p\ t - \gamma\ t) < 1)$
and *pge*: $(\bigwedge t.\ [\![0 \le t; t \le 1]\!] \Longrightarrow cmod\ (p\ t - \gamma\ t) < e)$
using *path-approx-polynomial-function* [*OF* γ, *of min 1 e*] ‹*e*>*0*› **by** *force*
have *pip*: *path-image p* $\subseteq$ *ball 0* (B + *1*)
using *B*
apply (*clarsimp simp add*: *path-image-def dist-norm ball-def*)
apply (*frule* (*1*) *pg1*)
apply (*fastforce dest*: *norm-add-less*)
done

then have $w \notin$ *path-image p* **using** *w* **by** *blast*
then have $\exists p.$ *valid-path p* $\wedge$ $w \notin$ *path-image p* $\wedge$
pathstart p = pathstart γ $\wedge$ *pathfinish p = pathfinish* γ $\wedge$
$(\forall t \in \{0..1\}.$ *cmod* $(\gamma\ t - p\ t) < e) \wedge$ *contour-integral p* $(\lambda wa.\ 1$
/ $(wa - w)) = 0$
apply (*rule-tac x=p* **in** *exI*)
apply (*simp add*: *p valid-path-polynomial-function*)
apply (*intro conjI*)
using *pge* **apply** (*simp add*: *norm-minus-commute*)
apply (*rule contour-integral-unique* [*OF Cauchy-theorem-convex-simple* [*OF*
- convex-ball [*of 0 B+1*]]])
apply (*rule holomorphic-intros* | *simp add*: *dist-norm*)+
using *mem-ball-0 w* **apply** *blast*
using *p* **apply** (*simp-all add*: *valid-path-polynomial-function loop pip*)
done
}
then show *?thesis*
by (*auto intro*: *winding-number-unique* [*OF* γ] *simp add*: *wnot*)
qed
finally show *?thesis* **.**
qed

lemma *winding-number-zero-outside*:
$[\![$*path* γ; *convex s*; *pathfinish* γ = *pathstart* γ; $z \notin s$; *path-image* $\gamma \subseteq s]\!] \Longrightarrow$
winding-number $\gamma\ z = 0$
by (*meson convex-in-outside outside-mono subsetCE winding-number-zero-in-outside*)

lemma *winding-number-zero-at-infinity*:
assumes γ: *path* γ **and** *loop*: *pathfinish* γ = *pathstart* γ
shows $\exists B.\ \forall z.\ B \leq$ *norm* $z \longrightarrow$ *winding-number* $\gamma\ z = 0$
proof −
obtain *B*::*real* **where** $0 < B$ **and** *B*: *path-image* $\gamma \subseteq$ *ball 0 B*
using *bounded-subset-ballD* [*OF bounded-path-image* [*OF* γ]] **by** *auto*
then show *?thesis*
apply (*rule-tac x=B+1* **in** *exI*, *clarify*)
apply (*rule winding-number-zero-outside* [*OF* γ *convex-cball* [*of 0 B*] *loop*])
apply (*meson less-add-one mem-cball-0 not-le order-trans*)
using *ball-subset-cball* **by** *blast*
qed

lemma *winding-number-zero-point*:
$[\![$*path* γ; *convex s*; *pathfinish* γ = *pathstart* γ; *open s*; *path-image* $\gamma \subseteq s]\!]$
$\Longrightarrow \exists z.\ z \in s \wedge$ *winding-number* $\gamma\ z = 0$
using *outside-compact-in-open* [*of path-image* γ *s*] *path-image-nonempty winding-number-zero-in-outside*
by (*fastforce simp add*: *compact-path-image*)

If a path winds round a set, it winds rounds its inside.

lemma *winding-number-around-inside*:
assumes γ: *path* γ **and** *loop*: *pathfinish* γ = *pathstart* γ

and *cls*: *closed s* **and** *cos*: *connected s* **and** *s-disj*: $s \cap$ *path-image* $\gamma = \{\}$
and z: $z \in s$ **and** *wn-nz*: *winding-number* γ $z \neq 0$ **and** w: $w \in s \cup$ *inside s*
shows *winding-number* γ w = *winding-number* γ z
proof −
have *ssb*: $s \subseteq$ *inside*(*path-image* γ)
proof
fix x :: *complex*
assume $x \in s$
hence $x \notin$ *path-image* γ
by (*meson disjoint-iff-not-equal s-disj*)
thus $x \in$ *inside* (*path-image* γ)
using ‹$x \in s$› **by** (*metis* (*no-types*) *ComplI UnE cos* γ *loop s-disj union-with-outside winding-number-eq winding-number-zero-in-outside wn-nz z*)
qed
show *?thesis*
apply (*rule winding-number-eq* [*OF* γ *loop w*])
using z **apply** *blast*
apply (*simp add*: *cls connected-with-inside cos*)
apply (*simp add*: *Int-Un-distrib2 s-disj*, *safe*)
by (*meson ssb inside-inside-compact-connected* [*OF cls*, *of path-image* γ] *compact-path-image connected-path-image contra-subsetD disjoint-iff-not-equal* γ *inside-no-overlap*)
qed

Bounding a WN by 1/2 for a path and point in opposite halfspaces.

lemma *winding-number-subpath-continuous*:
assumes γ: *valid-path* γ **and** z: $z \notin$ *path-image* γ
shows *continuous-on* {*0..1*} (λx. *winding-number*(*subpath 0 x* γ) z)
proof −
have ∗: *integral* {*0..x*} (λt. *vector-derivative* γ (*at t*) / (γ t − z)) / (*2* ∗ *of-real pi* ∗ i) =
winding-number (*subpath 0 x* γ) z
if x: $0 \leq x$ $x \leq 1$ **for** x
proof −
have *integral* {*0..x*} (λt. *vector-derivative* γ (*at t*) / (γ t − z)) / (*2* ∗ *of-real pi* ∗ i) =
1 / (*2*∗*pi*∗*ii*) ∗ *contour-integral* (*subpath 0 x* γ) (λw. *1*/(w − z))
using *assms x*
apply (*simp add*: *contour-integral-subcontour-integral* [*OF contour-integrable-inversediff*])
done
also have ... = *winding-number* (*subpath 0 x* γ) z
apply (*subst winding-number-valid-path*)
using *assms x*
apply (*simp-all add*: *path-image-subpath valid-path-subpath*)
by (*force simp*: *path-image-def*)
finally show *?thesis* **.**
qed
show *?thesis*
apply (*rule continuous-on-eq*
[**where** $f = \lambda x$. *1* / (*2*∗*pi*∗*ii*) ∗

integral $\{0..x\}$ $(\lambda t.\ 1/(\gamma\ t - z) * \textit{vector-derivative}\ \gamma$ $(\textit{at}\ t))]$)
apply (*rule continuous-intros*)+
apply (*rule indefinite-integral-continuous*)
apply (*rule contour-integrable-inversediff* [*OF assms*, *unfolded contour-integrable-on*])
using *assms*
apply (*simp add*: ∗)
done
qed

lemma *winding-number-ivt-pos*:
assumes γ: *valid-path* γ **and** z: $z \notin$ *path-image* γ **and** $0 \le w\ w \le Re(\textit{winding-number}$ $\gamma\ z)$
shows $\exists t \in \{0..1\}.\ Re(\textit{winding-number}(\textit{subpath}\ 0\ t\ \gamma)\ z) = w$
apply (*rule ivt-increasing-component-on-1* [*of 0 1*, **where** *?k* = *1*::*complex*, *simplified complex-inner-1-right*])
apply (*simp add*:)
apply (*rule winding-number-subpath-continuous* [*OF* $\gamma\ z$])
using *assms*
apply (*auto simp*: *path-image-def image-def*)
done

lemma *winding-number-ivt-neg*:
assumes γ: *valid-path* γ **and** z: $z \notin$ *path-image* γ **and** $Re(\textit{winding-number}\ \gamma$ $z) \le w\ w \le 0$
shows $\exists t \in \{0..1\}.\ Re(\textit{winding-number}(\textit{subpath}\ 0\ t\ \gamma)\ z) = w$
apply (*rule ivt-decreasing-component-on-1* [*of 0 1*, **where** *?k* = *1*::*complex*, *simplified complex-inner-1-right*])
apply (*simp add*:)
apply (*rule winding-number-subpath-continuous* [*OF* $\gamma\ z$])
using *assms*
apply (*auto simp*: *path-image-def image-def*)
done

lemma *winding-number-ivt-abs*:
assumes γ: *valid-path* γ **and** z: $z \notin$ *path-image* γ **and** $0 \le w\ w \le |Re(\textit{winding-number}$ $\gamma\ z)|$
shows $\exists t \in \{0..1\}.\ |Re\ (\textit{winding-number}\ (\textit{subpath}\ 0\ t\ \gamma)\ z)| = w$
using *assms winding-number-ivt-pos* [*of* $\gamma\ z\ w$] *winding-number-ivt-neg* [*of* $\gamma\ z$ $-w$]
by *force*

lemma *winding-number-lt-half-lemma*:
assumes γ: *valid-path* γ **and** z: $z \notin$ *path-image* γ **and** *az*: $a \cdot z \le b$ **and** *pag*: *path-image* $\gamma \subseteq \{w.\ a \cdot w > b\}$
shows $Re(\textit{winding-number}\ \gamma\ z) < 1/2$
proof −
{ assume $Re(\textit{winding-number}\ \gamma\ z) \ge 1/2$
then obtain *t*::*real* **where** *t*: $0 \le t\ t \le 1$ **and** *sub12*: *Re* (*winding-number*

```
(subpath 0 t γ) z) = 1/2
      using winding-number-ivt-pos [OF γ z, of 1/2] by auto
    have gt: γ t − z = − (of-real (exp (− (2 ∗ pi ∗ Im (winding-number (subpath
0 t γ) z)))) ∗ (γ 0 − z))
      using winding-number-exp-2pi [of subpath 0 t γ z]
      apply (simp add: t γ valid-path-imp-path)
     using closed-segment-eq-real-ivl path-image-def t z by (fastforce simp: path-image-subpath
Euler sub12)
    have b < a · γ 0
    proof −
      have γ 0 ∈ {c. b < a · c}
           by (metis (no-types) pag atLeastAtMost-iff image-subset-iff order-refl
path-image-def zero-le-one)
      thus ?thesis
        by blast
    qed
    moreover have b < a · γ t
    proof −
      have γ t ∈ {c. b < a · c}
        by (metis (no-types) pag atLeastAtMost-iff image-subset-iff path-image-def
t)
      thus ?thesis
        by blast
    qed
    ultimately have 0 < a · (γ 0 − z) 0 < a · (γ t − z) using az
      by (simp add: inner-diff-right)+
    then have False
      by (simp add: gt inner-mult-right mult-less-0-iff)
  }
  then show ?thesis by force
qed

lemma winding-number-lt-half:
  assumes valid-path γ a · z ≤ b path-image γ ⊆ {w. a · w > b}
    shows |Re (winding-number γ z)| < 1/2
proof −
  have z ∉ path-image γ using assms by auto
  with assms show ?thesis
   apply (simp add: winding-number-lt-half-lemma abs-if del: less-divide-eq-numeral1)
   apply (metis complex-inner-1-right winding-number-lt-half-lemma [OF valid-path-imp-reverse,
of γ z a b]
                winding-number-reversepath valid-path-imp-path inner-minus-left
path-image-reversepath)
    done
qed

lemma winding-number-le-half:
  assumes γ: valid-path γ and z: z ∉ path-image γ
     and anz: a ≠ 0 and azb: a · z ≤ b and pag: path-image γ ⊆ {w. a · w ≥ b}
```

shows $|Re\ (winding\text{-}number\ \gamma\ z)| \leq 1/2$
proof −
{ assume *wnz-12*: $|Re\ (winding\text{-}number\ \gamma\ z)| > 1/2$
have $isCont\ (winding\text{-}number\ \gamma)\ z$
by (*metis continuous-at-winding-number valid-path-imp-path* $\gamma\ z$)
then obtain d **where** $d{>}0$ **and** d: $\bigwedge x'.\ dist\ x'\ z < d \Longrightarrow dist\ (winding\text{-}number\ \gamma\ x')\ (winding\text{-}number\ \gamma\ z) < |Re(winding\text{-}number\ \gamma\ z)| - 1/2$
using *continuous-at-eps-delta wnz-12 diff-gt-0-iff-gt* **by** *blast*
def $z' \equiv z - (d\ /\ (2 * cmod\ a)) *_R a$
have $*$: $a \cdot z' \leq b - d\ /\ 3 * cmod\ a$
unfolding *z'-def inner-mult-right' divide-inverse*
apply (*simp add*: *divide-simps algebra-simps dot-square-norm power2-eq-square anz*)
apply (*metis* ‹$0 < d$› *add-increasing azb less-eq-real-def mult-nonneg-nonneg mult-right-mono norm-ge-zero norm-numeral*)
done
have $cmod\ (winding\text{-}number\ \gamma\ z' - winding\text{-}number\ \gamma\ z) < |Re\ (winding\text{-}number\ \gamma\ z)| - 1/2$
using d [*of z'*] *anz* ‹$d{>}0$› **by** (*simp add*: *dist-norm z'-def*)
then have $1/2 < |Re\ (winding\text{-}number\ \gamma\ z)| - cmod\ (winding\text{-}number\ \gamma\ z' - winding\text{-}number\ \gamma\ z)$
by *simp*
then have $1/2 < |Re\ (winding\text{-}number\ \gamma\ z)| - |Re\ (winding\text{-}number\ \gamma\ z') - Re\ (winding\text{-}number\ \gamma\ z)|$
using *abs-Re-le-cmod* [*of winding-number* $\gamma\ z' - winding\text{-}number\ \gamma\ z$] **by** *simp*
then have *wnz-12'*: $|Re\ (winding\text{-}number\ \gamma\ z')| > 1/2$
by *linarith*
moreover have $|Re\ (winding\text{-}number\ \gamma\ z')| < 1/2$
apply (*rule winding-number-lt-half* [*OF* γ $*$])
using *azb* ‹$d{>}0$› *pag*
apply (*auto simp*: *add-strict-increasing anz divide-simps algebra-simps dest*!: *subsetD*)
done
ultimately have *False*
by *simp*
}
then show *?thesis* **by** *force*
qed

lemma *winding-number-lt-half-linepath*: $z \notin closed\text{-}segment\ a\ b \Longrightarrow |Re\ (winding\text{-}number\ (linepath\ a\ b)\ z)| < 1/2$
using *separating-hyperplane-closed-point* [*of closed-segment a b z*]
apply *auto*
apply (*simp add*: *closed-segment-def*)
apply (*drule less-imp-le*)
apply (*frule winding-number-lt-half* [*OF valid-path-linepath* [*of a b*]])
apply (*auto simp*: *segment*)
done

Positivity of WN for a linepath.

lemma *winding-number-linepath-pos-lt*:
  assumes *0 < Im ((b − a) ∗ cnj (b − z))*
    shows *0 < Re(winding-number(linepath a b) z)*
proof −
  have *z*: *z ∉ path-image (linepath a b)*
    using *assms*
    by (*simp add*: *closed-segment-def*) (*force simp*: *algebra-simps*)
  show *?thesis*
    apply (*rule winding-number-pos-lt* [*OF valid-path-linepath z assms*])
    apply (*simp add*: *linepath-def algebra-simps*)
    done
qed

53.29 Cauchy's integral formula, again for a convex enclosing set.

lemma *Cauchy-integral-formula-weak*:
  assumes *s*: *convex s* **and** *finite k* **and** *conf*: *continuous-on s f*
    and *fcd*: (⋀*x*. *x ∈ interior s − k ⟹ f field-differentiable at x*)
    and *z*: *z ∈ interior s − k* **and** *vpg*: *valid-path γ*
    and *pasz*: *path-image γ ⊆ s − {z}* **and** *loop*: *pathfinish γ = pathstart γ*
   shows ((*λw*. *f w / (w − z)*) *has-contour-integral* (*2∗pi ∗ ii ∗ winding-number γ z ∗ f z*)) *γ*
proof −
  obtain *f′* **where** *f′*: (*f has-field-derivative f′*) (*at z*)
    using *fcd* [*OF z*] **by** (*auto simp*: *field-differentiable-def*)
  have *pas*: *path-image γ ⊆ s* **and** *znotin*: *z ∉ path-image γ* **using** *pasz* **by** *blast+*
  have *c*: *continuous* (*at x within s*) (*λw*. *if w = z then f′ else (f w − f z) / (w − z)*) **if** *x ∈ s* **for** *x*
  proof (*cases x = z*)
    case *True* **then show** *?thesis*
      apply (*simp add*: *continuous-within*)
      apply (*rule Lim-transform-away-within* [*of - z+1 - λw::complex*. (*f w − f z*)/(*w − z*)])
      using *has-field-derivative-at-within DERIV-within-iff f′*
      apply (*fastforce simp add*:)+
      done
  next
    case *False*
    then have *dxz*: *dist x z > 0* **by** *auto*
    have *cf*: *continuous* (*at x within s*) *f*
      using *conf continuous-on-eq-continuous-within that* **by** *blast*
    have *continuous* (*at x within s*) (*λw*. (*f w − f z*) / (*w − z*))
      by (*rule cf continuous-intros* | *simp add*: *False*)+
    then show *?thesis*
      apply (*rule continuous-transform-within* [*OF - dxz that*, *of λw::complex*. (*f w − f z*)/(*w − z*)])
      apply (*force simp*: *dist-commute*)

```
    done
  qed
  have fink': finite (insert z k) using ‹finite k› by blast
  have *: ((λw. if w = z then f' else (f w − f z) / (w − z)) has-contour-integral
0) γ
    apply (rule Cauchy-theorem-convex [OF - s fink' - vpg pas loop])
    using c apply (force simp: continuous-on-eq-continuous-within)
    apply (rename-tac w)
   apply (rule-tac d=dist w z and f = λw. (f w − f z)/(w − z) in field-differentiable-transform-within)
    apply (simp-all add: dist-pos-lt dist-commute)
    apply (metis less-irrefl)
    apply (rule derivative-intros fcd | simp)+
    done
  show ?thesis
    apply (rule has-contour-integral-eq)
   using znotin has-contour-integral-add [OF has-contour-integral-lmul [OF has-contour-integral-winding-numb
[OF vpg znotin], of f z] *]
    apply (auto simp: mult-ac divide-simps)
    done
qed

theorem Cauchy-integral-formula-convex-simple:
    ⟦convex s; f holomorphic-on s; z ∈ interior s; valid-path γ; path-image γ ⊆ s −
{z};
      pathfinish γ = pathstart γ⟧
    ⟹ ((λw. f w / (w − z)) has-contour-integral (2*pi * ii * winding-number γ
z * f z)) γ
  apply (rule Cauchy-integral-formula-weak [where k = {}])
  using holomorphic-on-imp-continuous-on
  by auto (metis at-within-interior holomorphic-on-def interiorE subsetCE)
```

53.30 Homotopy forms of Cauchy's theorem

```
proposition Cauchy-theorem-homotopic:
    assumes hom: if atends then homotopic-paths s g h else homotopic-loops s g h
        and open s and f: f holomorphic-on s
        and vpg: valid-path g and vph: valid-path h
    shows contour-integral g f = contour-integral h f
proof −
  have pathsf: linked-paths atends g h
     using hom  by (auto simp: linked-paths-def homotopic-paths-imp-pathstart
homotopic-paths-imp-pathfinish homotopic-loops-imp-loop)
  obtain k :: real × real ⇒ complex
    where contk: continuous-on ({0..1} × {0..1}) k
      and ks: k ‘ ({0..1} × {0..1}) ⊆ s
      and k [simp]: ∀x. k (0, x) = g x ∀x. k (1, x) = h x
      and ksf: ∀t∈{0..1}. linked-paths atends g (λx. k (t, x))
    using hom pathsf by (auto simp: linked-paths-def homotopic-paths-def homotopic-loops-def
homotopic-with-def split: if-split-asm)
```

have *ucontk*: *uniformly-continuous-on* $(\{0..1\} \times \{0..1\})$ *k*
by (*blast intro*: *compact-Times compact-uniformly-continuous* [*OF contk*])
{ fix *t*::*real* **assume** *t*: $t \in \{0..1\}$
have *pak*: *path* $(k \circ (\lambda u.\ (t,\ u)))$
unfolding *path-def*
apply (*rule continuous-intros continuous-on-subset* [*OF contk*])+
using *t* **by** *force*
have *pik*: *path-image* $(k \circ Pair\ t) \subseteq s$
using *ks t* **by** (*auto simp*: *path-image-def*)
obtain *e* **where** $e>0$ **and** *e*:
$\bigwedge g\ h.\ [\![valid\text{-}path\ g;\ valid\text{-}path\ h;$
$\forall u{\in}\{0..1\}.\ cmod\ (g\ u - (k \circ Pair\ t)\ u) < e \wedge cmod\ (h\ u - (k \circ Pair\ t)\ u) < e;$
$linked\text{-}paths\ atends\ g\ h]\!] \Longrightarrow contour\text{-}integral\ h\ f = contour\text{-}integral\ g\ f$
using *contour-integral-nearby* [*OF* ‹*open s*› *pak pik*, *of atends*] *f* **by** *metis*
obtain *d* **where** $d>0$ **and** *d*:
$\bigwedge x\ x'.\ [\![x \in \{0..1\} \times \{0..1\};\ x' \in \{0..1\} \times \{0..1\};\ norm\ (x'-x) < d]\!] \Longrightarrow norm\ (k\ x' - k\ x) < e/4$
by (*rule uniformly-continuous-onE* [*OF ucontk*, *of* $e/4$]) (*auto simp*: *dist-norm* ‹$e>0$›)
{ fix *t1 t2*
assume *t1*: $0 \le t1\ t1 \le 1$ **and** *t2*: $0 \le t2\ t2 \le 1$ **and** *ltd*: $|t1 - t| < d\ |t2 - t| < d$
have *no2*: $\bigwedge g1\ k1\ kt.\ [\![norm(g1 - k1) < e/4;\ norm(k1 - kt) < e/4]\!] \Longrightarrow norm(g1 - kt) < e$
using ‹$e > 0$›
apply (*rule-tac* $y = k1$ **in** *norm-triangle-half-l*)
apply (*auto simp*: *norm-minus-commute intro*: *order-less-trans*)
done
have $\exists\, d{>}0.\ \forall\, g1\ g2.\ valid\text{-}path\ g1 \wedge valid\text{-}path\ g2 \wedge$
$(\forall\, u{\in}\{0..1\}.\ cmod\ (g1\ u - k\ (t1,\ u)) < d \wedge cmod\ (g2\ u - k\ (t2,\ u)) < d) \wedge$
$linked\text{-}paths\ atends\ g1\ g2 \longrightarrow$
$contour\text{-}integral\ g2\ f = contour\text{-}integral\ g1\ f$
apply (*rule-tac* $x{=}e/4$ **in** *exI*)
using *t t1 t2 ltd* ‹$e > 0$›
apply (*auto intro*!: *e simp*: *d no2 simp del*: *less-divide-eq-numeral1*)
done
}
then have $\exists\, e.\ 0 < e \wedge$
$(\forall\, t1\ t2.\ t1 \in \{0..1\} \wedge t2 \in \{0..1\} \wedge |t1 - t| < e \wedge |t2 - t| < e$
$\longrightarrow (\exists\, d.\ 0 < d \wedge$
$(\forall\, g1\ g2.\ valid\text{-}path\ g1 \wedge valid\text{-}path\ g2 \wedge$
$(\forall\, u \in \{0..1\}.$
$norm(g1\ u - k((t1,u))) < d \wedge norm(g2\ u - k((t2,u))) < d) \wedge$
$linked\text{-}paths\ atends\ g1\ g2$
$\longrightarrow contour\text{-}integral\ g2\ f = contour\text{-}integral\ g1\ f)))$

```
    by (rule-tac x=d in exI) (simp add: ‹d > 0›)
  }
  then obtain ee where ee:
      ⋀t. t ∈ {0..1} ⟹ ee t > 0 ∧
        (∀ t1 t2. t1 ∈ {0..1} ⟶ t2 ∈ {0..1} ⟶ |t1 − t| < ee t ⟶ |t2 − t| <
ee t
          ⟶ (∃ d. 0 < d ∧
              (∀ g1 g2. valid-path g1 ∧ valid-path g2 ∧
                (∀ u ∈ {0..1}.
                   norm(g1 u − k((t1,u))) < d ∧ norm(g2 u − k((t2,u))) < d) ∧
                   linked-paths atends g1 g2
                   ⟶ contour-integral g2 f = contour-integral g1 f)))
    by metis
  note ee-rule = ee [THEN conjunct2, rule-format]
  def C ≡ (λt. ball t (ee t / 3)) ‘ {0..1}
  have ∀ t ∈ C. open t by (simp add: C-def)
  moreover have {0..1} ⊆ ⋃ C
    using ee [THEN conjunct1] by (auto simp: C-def dist-norm)
  ultimately obtain C' where C': C' ⊆ C finite C' and C'01: {0..1} ⊆ ⋃ C'
    by (rule compactE [OF compact-interval])
  def kk ≡ {t ∈ {0..1}. ball t (ee t / 3) ∈ C'}
  have kk01: kk ⊆ {0..1} by (auto simp: kk-def)
  def e ≡ Min (ee ‘ kk)
  have C'-eq: C' = (λt. ball t (ee t / 3)) ‘ kk
    using C' by (auto simp: kk-def C-def)
  have ee-pos[simp]: ⋀t. t ∈ {0..1} ⟹ ee t > 0
    by (simp add: kk-def ee)
  moreover have finite kk
     using ‹finite C'› kk01 by (force simp: C'-eq inj-on-def ball-eq-ball-iff dest:
ee-pos finite-imageD)
  moreover have kk ≠ {} using ‹{0..1} ⊆ ⋃ C'› C'-eq by force
  ultimately have e > 0
    using finite-less-Inf-iff [of ee ‘ kk 0] kk01 by (force simp: e-def)
  then obtain N::nat where N > 0 and N: 1/N < e/3
    by (meson divide-pos-pos nat-approx-posE zero-less-Suc zero-less-numeral)
  have e-le-ee: ⋀i. i ∈ kk ⟹ e ≤ ee i
    using ‹finite kk› by (simp add: e-def Min-le-iff [of ee ‘ kk])
  have plus: ∃ t ∈ kk. x ∈ ball t (ee t / 3) if x ∈ {0..1} for x
    using C' subsetD [OF C'01 that] unfolding C'-eq by blast
  have [OF order-refl]:
      ∃ d. 0 < d ∧ (∀ j. valid-path j ∧ (∀ u ∈ {0..1}. norm(j u − k (n/N, u)) <
d) ∧ linked-paths atends g j
                    ⟶ contour-integral j f = contour-integral g f)
       if n ≤ N for n
  using that
  proof (induct n)
    case 0 show ?case using ee-rule [of 0 0 0]
      apply clarsimp
      apply (rule-tac x=d in exI, safe)
```

by (*metis diff-self vpg norm-zero*)
next
case (*Suc n*)
then have *N01*: $n/N \in \{0..1\}$ $(Suc\ n)/N \in \{0..1\}$ **by** *auto*
then obtain *t* **where** *t*: $t \in kk\ n/N \in ball\ t\ (ee\ t\ /\ 3)$
using *plus* [*of* n/N] **by** *blast*
then have *nN-less*: $|n/N - t| < ee\ t$
by (*simp add*: *dist-norm del*: *less-divide-eq-numeral1*)
have *n'N-less*: $|real\ (Suc\ n)\ /\ real\ N - t| < ee\ t$
using *t N* ‹$N > 0$› *e-le-ee* [*of t*]
by (*simp add*: *dist-norm add-divide-distrib abs-diff-less-iff del*: *less-divide-eq-numeral1*) (*simp add*: *field-simps*)
have *t01*: $t \in \{0..1\}$ **using** ‹$kk \subseteq \{0..1\}$› ‹$t \in kk$› **by** *blast*
obtain *d1* **where** $d1 > 0$ **and** *d1*:
$\bigwedge g1\ g2.\ [\![valid\text{-}path\ g1;\ valid\text{-}path\ g2;$
$\forall u{\in}\{0..1\}.\ cmod\ (g1\ u - k\ (n/N,\ u)) < d1 \wedge cmod\ (g2\ u - k\ ((Suc\ n)\ /\ N,\ u)) < d1;$
$linked\text{-}paths\ atends\ g1\ g2]\!]$
$\Longrightarrow contour\text{-}integral\ g2\ f = contour\text{-}integral\ g1\ f$
using *ee* [*THEN conjunct2*, *rule-format*, *OF t01 N01 nN-less n'N-less*] **by** *fastforce*
have $n \le N$ **using** *Suc.prems* **by** *auto*
with *Suc.hyps*
obtain *d2* **where** $d2 > 0$
and *d2*: $\bigwedge j.\ [\![valid\text{-}path\ j;\ \forall u{\in}\{0..1\}.\ cmod\ (j\ u - k\ (n/N,\ u)) < d2;$ $linked\text{-}paths\ atends\ g\ j]\!]$
$\Longrightarrow contour\text{-}integral\ j\ f = contour\text{-}integral\ g\ f$
by *auto*
have *continuous-on* $\{0..1\}$ $(k\ o\ (\lambda u.\ (n/N,\ u)))$
apply (*rule continuous-intros continuous-on-subset* [*OF contk*])+
using *N01* **by** *auto*
then have *pkn*: *path* $(\lambda u.\ k\ (n/N,\ u))$
by (*simp add*: *path-def*)
have *min12*: $min\ d1\ d2 > 0$ **by** (*simp add*: ‹$0 < d1$› ‹$0 < d2$›)
obtain *p* **where** *polynomial-function p*
and *psf*: $pathstart\ p = pathstart\ (\lambda u.\ k\ (n/N,\ u))$
$pathfinish\ p = pathfinish\ (\lambda u.\ k\ (n/N,\ u))$
and *pk-le*: $\bigwedge t.\ t{\in}\{0..1\} \Longrightarrow cmod\ (p\ t - k\ (n/N,\ t)) < min\ d1\ d2$
using *path-approx-polynomial-function* [*OF pkn min12*] **by** *blast*
then have *vpp*: *valid-path p* **using** *valid-path-polynomial-function* **by** *blast*
have *lpa*: *linked-paths atends g p*
by (*metis* (*mono-tags*, *lifting*) *N01*(*1*) *ksf linked-paths-def pathfinish-def pathstart-def psf*)
show *?case*
apply (*rule-tac x*=*min d1 d2* **in** *exI*)
apply (*simp add*: ‹$0 < d1$› ‹$0 < d2$›, *clarify*)
apply (*rule-tac s*=*contour-integral p f* **in** *trans*)
using *pk-le N01*(*1*) *ksf pathfinish-def pathstart-def*
apply (*force intro*!: *vpp d1 simp add*: *linked-paths-def psf ksf*)

using *pk-le N01* **apply** (*force intro*!: *vpp d2 lpa simp add*: *linked-paths-def psf ksf*)
done
qed
then obtain *d* **where** $0 < d$
$\bigwedge j.\ \textit{valid-path}\ j \wedge (\forall u \in \{0..1\}.\ \textit{norm}(j\ u - k\ (1,u)) < d) \wedge$
linked-paths atends g j
$\Longrightarrow$ *contour-integral j f* = *contour-integral g f*
using ‹*N*>*0*› **by** *auto*
then have *linked-paths atends g h* $\Longrightarrow$ *contour-integral h f* = *contour-integral g f*
using ‹*N*>*0*› *vph* **by** *fastforce*
then show *?thesis*
by (*simp add*: *pathsf*)
qed

proposition *Cauchy-theorem-homotopic-paths*:
assumes *hom*: *homotopic-paths s g h*
and *open s* **and** *f*: *f holomorphic-on s*
and *vpg*: *valid-path g* **and** *vph*: *valid-path h*
shows *contour-integral g f* = *contour-integral h f*
using *Cauchy-theorem-homotopic* [*of True s g h*] *assms* **by** *simp*

proposition *Cauchy-theorem-homotopic-loops*:
assumes *hom*: *homotopic-loops s g h*
and *open s* **and** *f*: *f holomorphic-on s*
and *vpg*: *valid-path g* **and** *vph*: *valid-path h*
shows *contour-integral g f* = *contour-integral h f*
using *Cauchy-theorem-homotopic* [*of False s g h*] *assms* **by** *simp*

lemma *has-contour-integral-newpath*:
⟦(*f has-contour-integral y*) *h*; *f contour-integrable-on g*; *contour-integral g f* = *contour-integral h f*⟧
$\Longrightarrow$ (*f has-contour-integral y*) *g*
using *has-contour-integral-integral contour-integral-unique* **by** *auto*

lemma *Cauchy-theorem-null-homotopic*:
⟦*f holomorphic-on s*; *open s*; *valid-path g*; *homotopic-loops s g* (*linepath a a*)⟧
$\Longrightarrow$ (*f has-contour-integral 0*) *g*
apply (*rule has-contour-integral-newpath* [**where** *h* = *linepath a a*], *simp*)
using *contour-integrable-holomorphic-simple*
apply (*blast dest*: *holomorphic-on-imp-continuous-on homotopic-loops-imp-subset*)
by (*simp add*: *Cauchy-theorem-homotopic-loops*)

53.31 More winding number properties

including the fact that it's +-1 inside a simple closed curve.

lemma *winding-number-homotopic-paths*:
assumes *homotopic-paths* (−{*z*}) *g h*

shows *winding-number g z = winding-number h z*
proof −
have *path g path h* **using** *homotopic-paths-imp-path* [*OF assms*] **by** *auto*
moreover have *pag*: *z* ∉ *path-image g* **and** *pah*: *z* ∉ *path-image h*
using *homotopic-paths-imp-subset* [*OF assms*] **by** *auto*
ultimately obtain *d e* **where** *d* > *0 e* > *0*
and *d*: ⋀*p*. ⟦*path p*; *pathstart p* = *pathstart g*; *pathfinish p* = *pathfinish g*; ∀ *t*∈{*0..1*}. *norm* (*p t* − *g t*) < *d*⟧
⟹ *homotopic-paths* (−{*z*}) *g p*
and *e*: ⋀*q*. ⟦*path q*; *pathstart q* = *pathstart h*; *pathfinish q* = *pathfinish h*; ∀ *t*∈{*0..1*}. *norm* (*q t* − *h t*) < *e*⟧
⟹ *homotopic-paths* (−{*z*}) *h q*
using *homotopic-nearby-paths* [*of g* −{*z*}] *homotopic-nearby-paths* [*of h* −{*z*}]
by *force*
obtain *p* **where** *p*:
valid-path p z ∉ *path-image p*
pathstart p = *pathstart g pathfinish p* = *pathfinish g*
and *gp-less*:∀ *t*∈{*0..1*}. *cmod* (*g t* − *p t*) < *d*
and *pap*: *contour-integral p* (λ*w*. *1* / (*w* − *z*)) = *complex-of-real* (*2* ∗ *pi*) ∗ i ∗ *winding-number g z*
using *winding-number* [*OF* ‹*path g*› *pag* ‹*0* < *d*›] **by** *blast*
obtain *q* **where** *q*:
valid-path q z ∉ *path-image q*
pathstart q = *pathstart h pathfinish q* = *pathfinish h*
and *hq-less*: ∀ *t*∈{*0..1*}. *cmod* (*h t* − *q t*) < *e*
and *paq*: *contour-integral q* (λ*w*. *1* / (*w* − *z*)) = *complex-of-real* (*2* ∗ *pi*) ∗ i ∗ *winding-number h z*
using *winding-number* [*OF* ‹*path h*› *pah* ‹*0* < *e*›] **by** *blast*
have *gp*: *homotopic-paths* (− {*z*}) *g p*
by (*simp add*: *d p valid-path-imp-path norm-minus-commute gp-less*)
have *hq*: *homotopic-paths* (− {*z*}) *h q*
by (*simp add*: *e q valid-path-imp-path norm-minus-commute hq-less*)
have *contour-integral p* (λ*w*. *1*/(*w* − *z*)) = *contour-integral q* (λ*w*. *1*/(*w* − *z*))
apply (*rule Cauchy-theorem-homotopic-paths* [*of* −{*z*}])
apply (*blast intro*: *homotopic-paths-trans homotopic-paths-sym gp hq assms*)
apply (*auto intro*!: *holomorphic-intros simp*: *p q*)
done
then show *?thesis*
by (*simp add*: *pap paq*)
qed

lemma *winding-number-homotopic-loops*:
assumes *homotopic-loops* (−{*z*}) *g h*
shows *winding-number g z* = *winding-number h z*
proof −
have *path g path h* **using** *homotopic-loops-imp-path* [*OF assms*] **by** *auto*
moreover have *pag*: *z* ∉ *path-image g* **and** *pah*: *z* ∉ *path-image h*
using *homotopic-loops-imp-subset* [*OF assms*] **by** *auto*
moreover have *gloop*: *pathfinish g* = *pathstart g* **and** *hloop*: *pathfinish h* =

pathstart h
using *homotopic-loops-imp-loop* [*OF assms*] **by** *auto*
ultimately obtain *d e* **where** *d* > *0 e* > *0*
and *d*: ⋀*p*. ⟦*path p*; *pathfinish p* = *pathstart p*; ∀ *t*∈{*0..1*}. *norm* (*p t* − *g t*) < *d*⟧
⟹ *homotopic-loops* (−{*z*}) *g p*
and *e*: ⋀*q*. ⟦*path q*; *pathfinish q* = *pathstart q*; ∀ *t*∈{*0..1*}. *norm* (*q t* − *h t*) < *e*⟧
⟹ *homotopic-loops* (−{*z*}) *h q*
using *homotopic-nearby-loops* [*of g* −{*z*}] *homotopic-nearby-loops* [*of h* −{*z*}]
by *force*
obtain *p* **where** *p*:
valid-path p z ∉ *path-image p*
pathstart p = *pathstart g pathfinish p* = *pathfinish g*
and *gp-less*:∀ *t*∈{*0..1*}. *cmod* (*g t* − *p t*) < *d*
and *pap*: *contour-integral p* (λ*w*. *1* / (*w* − *z*)) = *complex-of-real* (*2* ∗ *pi*) ∗ i ∗ *winding-number g z*
using *winding-number* [*OF* ‹*path g*› *pag* ‹*0* < *d*›] **by** *blast*
obtain *q* **where** *q*:
valid-path q z ∉ *path-image q*
pathstart q = *pathstart h pathfinish q* = *pathfinish h*
and *hq-less*: ∀ *t*∈{*0..1*}. *cmod* (*h t* − *q t*) < *e*
and *paq*: *contour-integral q* (λ*w*. *1* / (*w* − *z*)) = *complex-of-real* (*2* ∗ *pi*) ∗ i ∗ *winding-number h z*
using *winding-number* [*OF* ‹*path h*› *pah* ‹*0* < *e*›] **by** *blast*
have *gp*: *homotopic-loops* (− {*z*}) *g p*
by (*simp add*: *gloop d gp-less norm-minus-commute p valid-path-imp-path*)
have *hq*: *homotopic-loops* (− {*z*}) *h q*
by (*simp add*: *e hloop hq-less norm-minus-commute q valid-path-imp-path*)
have *contour-integral p* (λ*w*. *1*/(*w* − *z*)) = *contour-integral q* (λ*w*. *1*/(*w* − *z*))
apply (*rule Cauchy-theorem-homotopic-loops* [*of* −{*z*}])
apply (*blast intro*: *homotopic-loops-trans homotopic-loops-sym gp hq assms*)
apply (*auto intro*!: *holomorphic-intros simp*: *p q*)
done
then show *?thesis*
by (*simp add*: *pap paq*)
qed

lemma *winding-number-paths-linear-eq*:
⟦*path g*; *path h*; *pathstart h* = *pathstart g*; *pathfinish h* = *pathfinish g*;
⋀*t*. *t* ∈ {*0..1*} ⟹ *z* ∉ *closed-segment* (*g t*) (*h t*)⟧
⟹ *winding-number h z* = *winding-number g z*
by (*blast intro*: *sym homotopic-paths-linear winding-number-homotopic-paths elim*:
)

lemma *winding-number-loops-linear-eq*:
⟦*path g*; *path h*; *pathfinish g* = *pathstart g*; *pathfinish h* = *pathstart h*;
⋀*t*. *t* ∈ {*0..1*} ⟹ *z* ∉ *closed-segment* (*g t*) (*h t*)⟧
⟹ *winding-number h z* = *winding-number g z*

by (*blast intro*: *sym homotopic-loops-linear winding-number-homotopic-loops elim*:
)

lemma *winding-number-nearby-paths-eq*:
⟦*path g*; *path h*;
pathstart h = *pathstart g*; *pathfinish h* = *pathfinish g*;
⋀*t*. *t* ∈ {*0..1*} ⟹ *norm*(*h t* − *g t*) < *norm*(*g t* − *z*)⟧
⟹ *winding-number h z* = *winding-number g z*
by (*metis segment-bound*(*2*) *norm-minus-commute not-le winding-number-paths-linear-eq*)

lemma *winding-number-nearby-loops-eq*:
⟦*path g*; *path h*;
pathfinish g = *pathstart g*;
pathfinish h = *pathstart h*;
⋀*t*. *t* ∈ {*0..1*} ⟹ *norm*(*h t* − *g t*) < *norm*(*g t* − *z*)⟧
⟹ *winding-number h z* = *winding-number g z*
by (*metis segment-bound*(*2*) *norm-minus-commute not-le winding-number-loops-linear-eq*)

proposition *winding-number-subpath-combine*:
⟦*path g*; *z* ∉ *path-image g*;
u ∈ {*0..1*}; *v* ∈ {*0..1*}; *w* ∈ {*0..1*}⟧
⟹ *winding-number* (*subpath u v g*) *z* + *winding-number* (*subpath v w g*) *z* =
winding-number (*subpath u w g*) *z*
apply (*rule trans* [*OF winding-number-join* [*THEN sym*]
winding-number-homotopic-paths [*OF homotopic-join-subpaths*]])
apply (*auto dest*: *path-image-subpath-subset*)
done

53.32 Partial circle path

definition *part-circlepath* :: [*complex*, *real*, *real*, *real*, *real*] ⇒ *complex*
where *part-circlepath z r s t* ≡ λ*x*. *z* + *of-real r* ∗ *exp* (*ii* ∗ *of-real* (*linepath s t x*))

lemma *pathstart-part-circlepath* [*simp*]:
pathstart(*part-circlepath z r s t*) = *z* + *r*∗*exp*(*ii* ∗ *s*)
by (*metis part-circlepath-def pathstart-def pathstart-linepath*)

lemma *pathfinish-part-circlepath* [*simp*]:
pathfinish(*part-circlepath z r s t*) = *z* + *r*∗*exp*(*ii*∗*t*)
by (*metis part-circlepath-def pathfinish-def pathfinish-linepath*)

proposition *has-vector-derivative-part-circlepath* [*derivative-intros*]:
((*part-circlepath z r s t*) *has-vector-derivative*
(*ii* ∗ *r* ∗ (*of-real t* − *of-real s*) ∗ *exp*(*ii* ∗ *linepath s t x*)))
(*at x within X*)
apply (*simp add*: *part-circlepath-def linepath-def scaleR-conv-of-real*)
apply (*rule has-vector-derivative-real-complex*)

apply (*rule derivative-eq-intros* | *simp*)+
done

corollary *vector-derivative-part-circlepath*:
vector-derivative (*part-circlepath z r s t*) (*at x*) =
ii ∗ *r* ∗ (*of-real t* − *of-real s*) ∗ *exp*(*ii* ∗ *linepath s t x*)
using *has-vector-derivative-part-circlepath vector-derivative-at* **by** *blast*

corollary *vector-derivative-part-circlepath01*:
⟦0 ≤ *x*; *x* ≤ 1⟧
⟹ *vector-derivative* (*part-circlepath z r s t*) (*at x within* {0..1}) =
ii ∗ *r* ∗ (*of-real t* − *of-real s*) ∗ *exp*(*ii* ∗ *linepath s t x*)
using *has-vector-derivative-part-circlepath*
by (*auto simp*: *vector-derivative-at-within-ivl*)

lemma *valid-path-part-circlepath* [*simp*]: *valid-path* (*part-circlepath z r s t*)
apply (*simp add*: *valid-path-def*)
apply (*rule C1-differentiable-imp-piecewise*)
apply (*auto simp*: *C1-differentiable-on-eq vector-derivative-works vector-derivative-part-circlepath has-vector-derivative-part-circlepath*
intro!: *continuous-intros*)
done

lemma *path-part-circlepath* [*simp*]: *path* (*part-circlepath z r s t*)
by (*simp add*: *valid-path-imp-path*)

proposition *path-image-part-circlepath*:
assumes *s* ≤ *t*
shows *path-image* (*part-circlepath z r s t*) = {*z* + *r* ∗ *exp*(*ii* ∗ *of-real x*) | *x*. *s* ≤ *x* ∧ *x* ≤ *t*}
proof −
{ fix *z*::*real*
assume 0 ≤ *z z* ≤ 1
with ‹*s* ≤ *t*› **have** ∃*x*. (*exp* (i ∗ *linepath s t z*) = *exp* (i ∗ *of-real x*)) ∧ *s* ≤ *x* ∧ *x* ≤ *t*
apply (*rule-tac x*=(1 − *z*) ∗ *s* + *z* ∗ *t* **in** *exI*)
apply (*simp add*: *linepath-def scaleR-conv-of-real algebra-simps*)
apply (*rule conjI*)
using *mult-right-mono* **apply** *blast*
using *affine-ineq* **by** (*metis mult.commute*)
}
moreover
{ fix *z*
assume *s* ≤ *z z* ≤ *t*
then have *z* + *of-real r* ∗ *exp* (i ∗ *of-real z*) ∈ (λ*x*. *z* + *of-real r* ∗ *exp* (i ∗ *linepath s t x*)) ‘ {0..1}
apply (*rule-tac x*=(*z* − *s*)/(*t* − *s*) **in** *image-eqI*)
apply (*simp add*: *linepath-def scaleR-conv-of-real divide-simps exp-eq*)
apply (*auto simp*: *algebra-simps divide-simps*)

```
    done
  }
  ultimately show ?thesis
    by (fastforce simp add: path-image-def part-circlepath-def)
qed

corollary path-image-part-circlepath-subset:
    ⟦s ≤ t; 0 ≤ r⟧ ⟹ path-image(part-circlepath z r s t) ⊆ sphere z r
by (auto simp: path-image-part-circlepath sphere-def dist-norm algebra-simps norm-mult)

proposition in-path-image-part-circlepath:
  assumes w ∈ path-image(part-circlepath z r s t) s ≤ t 0 ≤ r
    shows norm(w − z) = r
proof −
  have w ∈ {c. dist z c = r}
   by (metis (no-types) path-image-part-circlepath-subset sphere-def subset-eq assms)
  thus ?thesis
    by (simp add: dist-norm norm-minus-commute)
qed

proposition finite-bounded-log: finite {z::complex. norm z ≤ b ∧ exp z = w}
proof (cases w = 0)
  case True then show ?thesis by auto
next
  case False
  have ∗: finite {x. cmod (complex-of-real (2 ∗ real-of-int x ∗ pi) ∗ i) ≤ b + cmod
(Ln w)}
    apply (simp add: norm-mult finite-int-iff-bounded-le)
    apply (rule-tac x=⌊(b + cmod (Ln w)) / (2∗pi)⌋ in exI)
    apply (auto simp: divide-simps le-floor-iff)
    done
  have [simp]: ⋀P f. {z. P z ∧ (∃ n. z = f n)} = f ‘ {n. P (f n)}
    by blast
  show ?thesis
    apply (subst exp-Ln [OF False, symmetric])
    apply (simp add: exp-eq)
    using norm-add-leD apply (fastforce intro: finite-subset [OF - ∗])
    done
qed

lemma finite-bounded-log2:
  fixes a::complex
    assumes a ≠ 0
    shows finite {z. norm z ≤ b ∧ exp(a∗z) = w}
proof −
  have ∗: finite ((λz. z / a) ‘ {z. cmod z ≤ b ∗ cmod a ∧ exp z = w})
    by (rule finite-imageI [OF finite-bounded-log])
  show ?thesis
    by (rule finite-subset [OF - ∗]) (force simp: assms norm-mult)
```

qed

proposition *has-contour-integral-bound-part-circlepath-strong*:
assumes *fi*: (*f has-contour-integral i*) (*part-circlepath z r s t*)
and *finite k* **and** *le*: *0 ≤ B 0 < r s ≤ t*
and *B*: ⋀*x*. *x ∈ path-image*(*part-circlepath z r s t*) − *k* ⟹ *norm*(*f x*) ≤ *B*
shows *cmod i* ≤ *B* ∗ *r* ∗ (*t* − *s*)
proof −
consider *s* = *t* | *s* < *t* **using** ‹*s* ≤ *t*› **by** *linarith*
then show *?thesis*
proof *cases*
case *1* **with** *fi* [*unfolded has-contour-integral*]
have *i* = *0* **by** (*simp add*: *vector-derivative-part-circlepath*)
with *assms* **show** *?thesis* **by** *simp*
next
case *2*
have [*simp*]: |*r*| = *r* **using** ‹*r* > *0*› **by** *linarith*
have [*simp*]: *cmod* (*complex-of-real t* − *complex-of-real s*) = *t*−*s*
by (*metis 2 abs-of-pos diff-gt-0-iff-gt norm-of-real of-real-diff*)
have *finite* (*part-circlepath z r s t* −‘ {*y*} ∩ {*0..1*}) **if** *y* ∈ *k* **for** *y*
proof −
def *w* ≡ (*y* − *z*)/*of-real r* / *exp*(*ii* ∗ *of-real s*)
have *fin*: *finite* (*of-real* −‘ {*z*. *cmod z* ≤ *1* ∧ *exp* (i ∗ *complex-of-real* (*t* − *s*) ∗ *z*) = *w*})
apply (*rule finite-vimageI* [*OF finite-bounded-log2*])
using ‹*s* < *t*› **apply** (*auto simp*: *inj-of-real*)
done
show *?thesis*
apply (*simp add*: *part-circlepath-def linepath-def vimage-def*)
apply (*rule finite-subset* [*OF - fin*])
using *le*
apply (*auto simp*: *w-def algebra-simps scaleR-conv-of-real exp-add exp-diff*)
done
qed
then have *fin01*: *finite* ((*part-circlepath z r s t*) −‘ *k* ∩ {*0..1*})
by (*rule finite-finite-vimage-IntI* [*OF* ‹*finite k*›])
have ∗∗: ((λ*x*. *if* (*part-circlepath z r s t x*) ∈ *k then 0*
else f(*part-circlepath z r s t x*) ∗
vector-derivative (*part-circlepath z r s t*) (*at x*)) *has-integral i*) {*0..1*}
apply (*rule has-integral-spike*
[**where** *f* = λ*x*. *f*(*part-circlepath z r s t x*) ∗ *vector-derivative* (*part-circlepath z r s t*) (*at x*)])
apply (*rule negligible-finite* [*OF fin01*])
using *fi has-contour-integral*
apply *auto*
done
have ∗: ⋀*x*. ⟦*0 ≤ x*; *x ≤ 1*; *part-circlepath z r s t x* ∉ *k*⟧ ⟹ *cmod* (*f* (*part-circlepath z r s t x*)) ≤ *B*

by (*auto intro*!: *B* [*unfolded path-image-def image-def*, *simplified*])
show *?thesis*
apply (*rule has-integral-bound* [**where** *'a=real*, *simplified*, *OF - ∗∗*, *simplified*])
using *assms* **apply** *force*
apply (*simp add*: *norm-mult vector-derivative-part-circlepath*)
using *le ∗ 2* ‹*r > 0*› **by** *auto*
qed
qed

corollary *has-contour-integral-bound-part-circlepath*:
⟦(*f has-contour-integral i*) (*part-circlepath z r s t*);
0 ≤ B; *0 < r*; *s ≤ t*;
⋀*x*. *x ∈ path-image*(*part-circlepath z r s t*) ⟹ *norm*(*f x*) *≤ B*⟧
⟹ *norm i ≤ B∗r∗*(*t − s*)
by (*auto intro*: *has-contour-integral-bound-part-circlepath-strong*)

proposition *contour-integrable-continuous-part-circlepath*:
continuous-on (*path-image* (*part-circlepath z r s t*)) *f*
⟹ *f contour-integrable-on* (*part-circlepath z r s t*)
apply (*simp add*: *contour-integrable-on has-contour-integral-def vector-derivative-part-circlepath path-image-def*)
apply (*rule integrable-continuous-real*)
apply (*fast intro*: *path-part-circlepath* [*unfolded path-def*] *continuous-intros continuous-on-compose2* [**where** *g=f*, *OF - - order-refl*])
done

proposition *winding-number-part-circlepath-pos-less*:
assumes *s < t* **and** *no*: *norm*(*w − z*) *< r*
shows *0 < Re* (*winding-number*(*part-circlepath z r s t*) *w*)
proof −
have *0 < r* **by** (*meson no norm-not-less-zero not-le order.strict-trans2*)
note *valid-path-part-circlepath*
moreover have *w ∉ path-image* (*part-circlepath z r s t*)
using *assms* **by** (*auto simp*: *path-image-def image-def part-circlepath-def norm-mult linepath-def*)
moreover have *0 < r ∗* (*t − s*) *∗* (*r − cmod* (*w − z*))
using *assms* **by** (*metis* ‹*0 < r*› *diff-gt-0-iff-gt mult-pos-pos*)
ultimately show *?thesis*
apply (*rule winding-number-pos-lt* [**where** *e* = *r∗*(*t − s*)*∗*(*r − norm*(*w − z*))])
apply (*simp add*: *vector-derivative-part-circlepath right-diff-distrib* [*symmetric*] *mult-ac*)
apply (*rule mult-left-mono*)+
using *Re-Im-le-cmod* [*of w−z linepath s t x* **for** *x*]
apply (*simp add*: *exp-Euler cos-of-real sin-of-real part-circlepath-def algebra-simps cos-squared-eq* [*unfolded power2-eq-square*])
using *assms* ‹*0 < r*› **by** *auto*
qed

proposition *simple-path-part-circlepath*:

simple-path(*part-circlepath z r s t*) ⟷ (*r* ≠ *0* ∧ *s* ≠ *t* ∧ |*s* − *t*| ≤ *2*∗*pi*)
proof (*cases r* = *0* ∨ *s* = *t*)
case *True*
then show *?thesis*
apply (*rule disjE*)
apply (*force simp*: *part-circlepath-def simple-path-def intro*: *bexI* [**where** *x* = *1/4*] *bexI* [**where** *x* = *1/3*])+
done
next
case *False* **then have** *r* ≠ *0 s* ≠ *t* **by** *auto*
have ∗: ⋀*x y z s t*. *ii*∗((*1* − *x*) ∗ *s* + *x* ∗ *t*) = *ii*∗(((*1* − *y*) ∗ *s* + *y* ∗ *t*)) + *z* ⟷ *ii*∗(*x* − *y*) ∗ (*t* − *s*) = *z*
by (*simp add*: *algebra-simps*)
have *abs01*: ⋀*x y*::*real*. *0* ≤ *x* ∧ *x* ≤ *1* ∧ *0* ≤ *y* ∧ *y* ≤ *1*
⟹ (*x* = *y* ∨ *x* = *0* ∧ *y* = *1* ∨ *x* = *1* ∧ *y* = *0* ⟷ |*x* − *y*| ∈ {*0*,*1*})
by *auto*
have *abs-away*: ⋀*P*. (∀ *x*∈{*0*..*1*}. ∀ *y*∈{*0*..*1*}. *P* |*x* − *y*|) ⟷ (∀ *x*::*real*. *0* ≤ *x* ∧ *x* ≤ *1* ⟶ *P x*)
by *force*
have ∗∗: ⋀*x y*. (∃ *n*. (*complex-of-real x* − *of-real y*) ∗ (*of-real t* − *of-real s*) = *2* ∗ (*of-int n* ∗ *of-real pi*)) ⟷
(∃ *n*. |*x* − *y*| ∗ (*t* − *s*) = *2* ∗ (*of-int n* ∗ *pi*))
by (*force simp*: *algebra-simps abs-if dest*: *arg-cong* [**where** *f*=*Re*] *arg-cong* [**where** *f*=*complex-of-real*]
intro: *exI* [**where** *x* = −*n* **for** *n*])
have *1*: ∀ *x*. *0* ≤ *x* ∧ *x* ≤ *1* ⟶ (∃ *n*. *x* ∗ (*t* − *s*) = *2* ∗ (*real-of-int n* ∗ *pi*)) ⟶ *x* = *0* ∨ *x* = *1* ⟹ |*s* − *t*| ≤ *2* ∗ *pi*
apply (*rule ccontr*)
apply (*drule-tac x*=*2*∗*pi* / |*t* − *s*| **in** *spec*)
using *False*
apply (*simp add*: *abs-minus-commute divide-simps*)
apply (*frule-tac x*=*1* **in** *spec*)
apply (*drule-tac x*=−*1* **in** *spec*)
apply (*simp add*:)
done
have *2*: |*s* − *t*| = |*2* ∗ (*real-of-int n* ∗ *pi*) / *x*| **if** *x* ≠ *0 x* ∗ (*t* − *s*) = *2* ∗ (*real-of-int n* ∗ *pi*) **for** *x n*
proof −
have *t*−*s* = *2* ∗ (*real-of-int n* ∗ *pi*)/*x*
using *that* **by** (*simp add*: *field-simps*)
then show *?thesis* **by** (*metis abs-minus-commute*)
qed
show *?thesis* **using** *False*
apply (*simp add*: *simple-path-def path-part-circlepath*)
apply (*simp add*: *part-circlepath-def linepath-def exp-eq* ∗ ∗∗ *abs01* *del*: *Set.insert-iff*)
apply (*subst abs-away*)
apply (*auto simp*: *1*)

apply (*rule ccontr*)
apply (*auto simp*: *2 divide-simps abs-mult dest*: *of-int-leD*)
done
qed

proposition *arc-part-circlepath*:
assumes $r \neq 0$ $s \neq t$ $|s - t| < 2*pi$
shows *arc* (*part-circlepath z r s t*)
proof −
have ∗: $x = y$ **if** *eq*: i ∗ (*linepath s t x*) = i ∗ (*linepath s t y*) + *2* ∗ *of-int n* ∗ *complex-of-real pi* ∗ i
and *x*: $x \in \{0..1\}$ **and** *y*: $y \in \{0..1\}$ **for** *x y n*
proof −
have (*linepath s t x*) = (*linepath s t y*) + *2* ∗ *of-int n* ∗ *complex-of-real pi*
by (*metis add-divide-eq-iff complex-i-not-zero mult.commute nonzero-mult-divide-cancel-left eq*)
then have $s*y + t*x = s*x + (t*y + \textit{of-int } n * (pi * 2))$
by (*force simp*: *algebra-simps linepath-def dest*: *arg-cong* [**where** *f*=*Re*])
then have *st*: $x \neq y \Longrightarrow (s-t) = (\textit{of-int } n * (pi * 2)) / (y-x))$
by (*force simp*: *field-simps*)
show *?thesis*
apply (*rule ccontr*)
using *assms x y*
apply (*simp add*: *st abs-mult field-simps*)
using *st*
apply (*auto simp*: *dest*: *of-int-lessD*)
done
qed
show *?thesis*
using *assms*
apply (*simp add*: *arc-def*)
apply (*simp add*: *part-circlepath-def inj-on-def exp-eq*)
apply (*blast intro*: ∗)
done
qed

53.33 Special case of one complete circle

definition *circlepath* :: [*complex*, *real*, *real*] ⇒ *complex*
where *circlepath z r* ≡ *part-circlepath z r 0* (*2∗pi*)

lemma *circlepath*: *circlepath z r* = (λ*x*. *z* + *r* ∗ *exp*(*2* ∗ *of-real pi* ∗ *ii* ∗ *of-real x*))
by (*simp add*: *circlepath-def part-circlepath-def linepath-def algebra-simps*)

lemma *pathstart-circlepath* [*simp*]: *pathstart* (*circlepath z r*) = *z* + *r*
by (*simp add*: *circlepath-def*)

lemma *pathfinish-circlepath* [*simp*]: *pathfinish* (*circlepath z r*) = *z* + *r*

by (*simp add*: *circlepath-def*) (*metis exp-two-pi-i mult.commute*)

lemma *circlepath-minus*: *circlepath z* (−*r*) *x* = *circlepath z r* (*x* + *1/2*)
proof −
have *z* + *of-real r* ∗ *exp* (*2* ∗ *pi* ∗ i ∗ (*x* + *1* / *2*)) =
z + *of-real r* ∗ *exp* (*2* ∗ *pi* ∗ i ∗ *x* + *pi* ∗ i)
by (*simp add*: *divide-simps*) (*simp add*: *algebra-simps*)
also have ... = *z* − *r* ∗ *exp* (*2* ∗ *pi* ∗ i ∗ *x*)
by (*simp add*: *exp-add*)
finally show *?thesis*
by (*simp add*: *circlepath path-image-def sphere-def dist-norm*)
qed

lemma *circlepath-add1*: *circlepath z r* (*x*+*1*) = *circlepath z r x*
using *circlepath-minus* [*of z r x*+*1/2*] *circlepath-minus* [*of z* −*r x*]
by (*simp add*: *add.commute*)

lemma *circlepath-add-half*: *circlepath z r* (*x* + *1/2*) = *circlepath z r* (*x* − *1/2*)
using *circlepath-add1* [*of z r x*−*1/2*]
by (*simp add*: *add.commute*)

lemma *path-image-circlepath-minus-subset*:
path-image (*circlepath z* (−*r*)) ⊆ *path-image* (*circlepath z r*)
apply (*simp add*: *path-image-def image-def circlepath-minus*, *clarify*)
apply (*case-tac xa* ≤ *1/2*, *force*)
apply (*force simp add*: *circlepath-add-half*)+
done

lemma *path-image-circlepath-minus*: *path-image* (*circlepath z* (−*r*)) = *path-image* (*circlepath z r*)
using *path-image-circlepath-minus-subset* **by** *fastforce*

proposition *has-vector-derivative-circlepath* [*derivative-intros*]:
((*circlepath z r*) *has-vector-derivative* (*2* ∗ *pi* ∗ *ii* ∗ *r* ∗ *exp* (*2* ∗ *of-real pi* ∗ *ii* ∗ *of-real x*)))
(*at x within X*)
apply (*simp add*: *circlepath-def scaleR-conv-of-real*)
apply (*rule derivative-eq-intros*)
apply (*simp add*: *algebra-simps*)
done

corollary *vector-derivative-circlepath*:
vector-derivative (*circlepath z r*) (*at x*) =
2 ∗ *pi* ∗ *ii* ∗ *r* ∗ *exp*(*2* ∗ *of-real pi* ∗ *ii* ∗ *x*)
using *has-vector-derivative-circlepath vector-derivative-at* **by** *blast*

corollary *vector-derivative-circlepath01*:
⟦*0* ≤ *x*; *x* ≤ *1*⟧
⟹ *vector-derivative* (*circlepath z r*) (*at x within* {*0*..*1*}) =

```
        2 * pi * ii * r * exp(2 * of-real pi * ii * x)
  using has-vector-derivative-circlepath
  by (auto simp: vector-derivative-at-within-ivl)

lemma valid-path-circlepath [simp]: valid-path (circlepath z r)
  by (simp add: circlepath-def)

lemma path-circlepath [simp]: path (circlepath z r)
  by (simp add: valid-path-imp-path)

lemma path-image-circlepath-nonneg:
  assumes 0 ≤ r shows path-image (circlepath z r) = sphere z r
proof −
  have ∗: x ∈ (λu. z + (cmod (x − z)) * exp (i * (of-real u * (of-real pi * 2)))) ‘
{0..1} for x
  proof (cases x = z)
    case True then show ?thesis by force
  next
    case False
    def w ≡ x − z
    then have w ≠ 0 by (simp add: False)
     have ∗∗: ⋀t. ⟦Re w = cos t * cmod w; Im w = sin t * cmod w⟧ ⟹ w =
of-real (cmod w) * exp (i * t)
      using cis-conv-exp complex-eq-iff by auto
    show ?thesis
      apply (rule sincos-total-2pi [of Re(w/of-real(norm w)) Im(w/of-real(norm
w))])
      apply (simp add: divide-simps ‹w ≠ 0› cmod-power2 [symmetric])
      apply (rule-tac x=t / (2∗pi) in image-eqI)
      apply (simp add: divide-simps ‹w ≠ 0›)
      using False ∗∗
      apply (auto simp: w-def)
      done
  qed
  show ?thesis
    unfolding circlepath path-image-def sphere-def dist-norm
    by (force simp: assms algebra-simps norm-mult norm-minus-commute intro: ∗)
qed

proposition path-image-circlepath [simp]:
    path-image (circlepath z r) = sphere z |r|
  using path-image-circlepath-minus
  by (force simp add: path-image-circlepath-nonneg abs-if)

lemma has-contour-integral-bound-circlepath-strong:
      ⟦(f has-contour-integral i) (circlepath z r);
        finite k; 0 ≤ B; 0 < r;
        ⋀x. ⟦norm(x − z) = r; x ∉ k⟧ ⟹ norm(f x) ≤ B⟧
        ⟹ norm i ≤ B∗(2∗pi∗r)
```

unfolding *circlepath-def*
by (*auto simp*: *algebra-simps in-path-image-part-circlepath dest*!: *has-contour-integral-bound-part-circlepath-st*

corollary *has-contour-integral-bound-circlepath*:
⟦(*f has-contour-integral i*) (*circlepath z r*);
0 ≤ *B*; *0* < *r*; ⋀*x*. *norm*(*x* − *z*) = *r* ⟹ *norm*(*f x*) ≤ *B*⟧
⟹ *norm i* ≤ *B*∗(*2*∗*pi*∗*r*)
by (*auto intro*: *has-contour-integral-bound-circlepath-strong*)

proposition *contour-integrable-continuous-circlepath*:
continuous-on (*path-image* (*circlepath z r*)) *f*
⟹ *f contour-integrable-on* (*circlepath z r*)
by (*simp add*: *circlepath-def contour-integrable-continuous-part-circlepath*)

lemma *simple-path-circlepath*: *simple-path*(*circlepath z r*) ⟷ (*r* ≠ *0*)
by (*simp add*: *circlepath-def simple-path-part-circlepath*)

lemma *notin-path-image-circlepath* [*simp*]: *cmod* (*w* − *z*) < *r* ⟹ *w* ∉ *path-image* (*circlepath z r*)
by (*simp add*: *sphere-def dist-norm norm-minus-commute*)

proposition *contour-integral-circlepath*:
0 < *r* ⟹ *contour-integral* (*circlepath z r*) (λ*w*. *1* / (*w* − *z*)) = *2* ∗ *complex-of-real pi* ∗ i
apply (*rule contour-integral-unique*)
apply (*simp add*: *has-contour-integral-def*)
apply (*subst has-integral-cong*)
apply (*simp add*: *vector-derivative-circlepath01*)
using *has-integral-const-real* [*of - 0 1*]
apply (*force simp*: *circlepath*)
done

lemma *winding-number-circlepath-centre*: *0* < *r* ⟹ *winding-number* (*circlepath z r*) *z* = *1*
apply (*rule winding-number-unique-loop*)
apply (*simp-all add*: *sphere-def valid-path-imp-path*)
apply (*rule-tac x*=*circlepath z r* **in** *exI*)
apply (*simp add*: *sphere-def contour-integral-circlepath*)
done

proposition *winding-number-circlepath*:
assumes *norm*(*w* − *z*) < *r* **shows** *winding-number*(*circlepath z r*) *w* = *1*
proof (*cases w* = *z*)
case *True* **then show** *?thesis*
using *assms winding-number-circlepath-centre* **by** *auto*
next
case *False*
have [*simp*]: *r* > *0*
using *assms le-less-trans norm-ge-zero* **by** *blast*

def $r' \equiv norm(w - z)$
have $r' < r$
by (*simp add*: *assms r'-def*)
have *disjo*: *cball z r'* ∩ *sphere z r* = {}
using ‹$r' < r$› **by** (*force simp*: *cball-def sphere-def*)
have *winding-number*(*circlepath z r*) *w* = *winding-number*(*circlepath z r*) *z*
apply (*rule winding-number-around-inside* [**where** *s* = *cball z r'*])
apply (*simp-all add*: *disjo order.strict-implies-order winding-number-circlepath-centre*)
apply (*simp-all add*: *False r'-def dist-norm norm-minus-commute*)
done
also have ... = *1*
by (*simp add*: *winding-number-circlepath-centre*)
finally show *?thesis* **.**
qed

Hence the Cauchy formula for points inside a circle.

theorem *Cauchy-integral-circlepath*:
assumes *continuous-on* (*cball z r*) *f f holomorphic-on* (*ball z r*) *norm*(*w* − *z*) < *r*
shows ((λ*u*. *f u*/(*u* − *w*)) *has-contour-integral* (*2* ∗ *of-real pi* ∗ *ii* ∗ *f w*)) (*circlepath z r*)
proof −
have *r* > *0*
using *assms le-less-trans norm-ge-zero* **by** *blast*
have ((λ*u*. *f u* / (*u* − *w*)) *has-contour-integral* (*2* ∗ *pi*) ∗ i ∗ *winding-number* (*circlepath z r*) *w* ∗ *f w*) (*circlepath z r*)
apply (*rule Cauchy-integral-formula-weak* [**where** *s* = *cball z r* **and** *k* = {}])
using *assms* ‹*r* > *0*›
apply (*simp-all add*: *dist-norm norm-minus-commute*)
apply (*metis at-within-interior dist-norm holomorphic-on-def interior-ball mem-ball norm-minus-commute*)
apply (*simp add*: *cball-def sphere-def dist-norm*, *clarify*)
apply (*simp add*:)
by (*metis dist-commute dist-norm less-irrefl*)
then show *?thesis*
by (*simp add*: *winding-number-circlepath assms*)
qed

corollary *Cauchy-integral-circlepath-simple*:
assumes *f holomorphic-on cball z r norm*(*w* − *z*) < *r*
shows ((λ*u*. *f u*/(*u* − *w*)) *has-contour-integral* (*2* ∗ *of-real pi* ∗ *ii* ∗ *f w*)) (*circlepath z r*)
using *assms* **by** (*force simp*: *holomorphic-on-imp-continuous-on holomorphic-on-subset Cauchy-integral-circlepath*)

lemma *no-bounded-connected-component-imp-winding-number-zero*:
assumes *g*: *path g path-image g* ⊆ *s pathfinish g* = *pathstart g z* ∉ *s*

and *nb*: $\bigwedge z.$ *bounded* (*connected-component-set* (− *s*) *z*) ⟶ $z \in s$
shows *winding-number g z* = *0*
apply (*rule winding-number-zero-in-outside*)
apply (*simp-all add*: *assms*)
by (*metis nb* [*of z*] ‹*path-image g* ⊆ *s*› ‹*z* ∉ *s*› *contra-subsetD mem-Collect-eq outside outside-mono*)

lemma *no-bounded-path-component-imp-winding-number-zero*:
assumes *g*: *path g path-image g* ⊆ *s pathfinish g* = *pathstart g z* ∉ *s*
and *nb*: $\bigwedge z.$ *bounded* (*path-component-set* (− *s*) *z*) ⟶ $z \in s$
shows *winding-number g z* = *0*
apply (*rule no-bounded-connected-component-imp-winding-number-zero* [*OF g*])
by (*simp add*: *bounded-subset nb path-component-subset-connected-component*)

53.34 Uniform convergence of path integral

Uniform convergence when the derivative of the path is bounded, and in particular for the special case of a circle.

proposition *contour-integral-uniform-limit*:
assumes *ev-fint*: *eventually* (λ*n*::′*a*. (*f n*) *contour-integrable-on* γ) *F*
and *ev-no*: $\bigwedge e.$ *0* < *e* ⟹ *eventually* (λ*n*. ∀ *x* ∈ *path-image* γ. *norm*(*f n x* − *l x*) < *e*) *F*
and *noleB*: $\bigwedge t.$ *t* ∈ {*0..1*} ⟹ *norm* (*vector-derivative* γ (*at t*)) ≤ *B*
and γ: *valid-path* γ
and [*simp*]: ~ (*trivial-limit F*)
shows *l contour-integrable-on* γ ((λ*n*. *contour-integral* γ (*f n*)) ⟶ *contour-integral* γ *l*) *F*
proof −
have *0* ≤ *B* **by** (*meson noleB* [*of 0*] *atLeastAtMost-iff norm-ge-zero order-refl order-trans zero-le-one*)
{ fix *e*::*real*
assume *0* < *e*
then have *eB*: *0* < *e* / (|*B*| + *1*) **by** *simp*
obtain *a* **where** *fga*: $\bigwedge x.$ *x* ∈ {*0..1*} ⟹ *cmod* (*f a* (γ *x*) − *l* (γ *x*)) < *e* / (|*B*| + *1*)
and *inta*: (λ*t*. *f a* (γ *t*) ∗ *vector-derivative* γ (*at t*)) *integrable-on* {*0..1*}
using *eventually-happens* [*OF eventually-conj* [*OF ev-no* [*OF eB*] *ev-fint*]]
by (*fastforce simp*: *contour-integrable-on path-image-def*)
have *Ble*: *B* ∗ *e* / (|*B*| + *1*) ≤ *e*
using ‹*0* ≤ *B*› ‹*0* < *e*› **by** (*simp add*: *divide-simps*)
have ∃ *h*. (∀ *x*∈{*0..1*}. *cmod* (*l* (γ *x*) ∗ *vector-derivative* γ (*at x*) − *h x*) ≤ *e*) ∧ *h integrable-on* {*0..1*}
apply (*rule-tac x*=λ*x*. *f* (*a*::′*a*) (γ *x*) ∗ *vector-derivative* γ (*at x*) **in** *exI*)
apply (*intro inta conjI ballI*)
apply (*rule order-trans* [*OF* - *Ble*])
apply (*frule noleB*)
apply (*frule fga*)
using ‹*0* ≤ *B*› ‹*0* < *e*›

apply (*simp add*: *norm-mult left-diff-distrib* [*symmetric*] *norm-minus-commute divide-simps*)
apply (*drule* (*1*) *mult-mono* [*OF less-imp-le*])
apply (*simp-all add*: *mult-ac*)
done
}
then show *lintg*: *l contour-integrable-on* γ
apply (*simp add*: *contour-integrable-on*)
apply (*blast intro*: *integrable-uniform-limit-real*)
done
{ fix *e*::*real*
def $B' \equiv B+1$
have *B'*: $B' > 0$ $B' > B$ **using** ‹$0 \le B$› **by** (*auto simp*: *B'-def*)
assume $0 < e$
then have *ev-no'*: $\forall_F\ n$ *in F*. $\forall x \in$ *path-image* γ. $2 * cmod\ (f\ n\ x - l\ x) < e / B'$
using *ev-no* [*of e / B' / 2*] *B'* **by** (*simp add*: *field-simps*)
have *ie*: *integral* $\{0..1::real\}$ $(\lambda x.\ e / 2) < e$ **using** ‹$0 < e$› **by** *simp*
have $*$: $cmod\ (f\ x\ (\gamma\ t) * vector\text{-}derivative\ \gamma\ (at\ t) - l\ (\gamma\ t) * vector\text{-}derivative\ \gamma\ (at\ t)) \le e / 2$
if *t*: $t \in \{0..1\}$ **and** *leB'*: $2 * cmod\ (f\ x\ (\gamma\ t) - l\ (\gamma\ t)) < e / B'$ **for** *x t*
proof −
have $2 * cmod\ (f\ x\ (\gamma\ t) - l\ (\gamma\ t)) * cmod\ (vector\text{-}derivative\ \gamma\ (at\ t)) \le e * (B/\ B')$
using *mult-mono* [*OF less-imp-le* [*OF leB'*] *noleB*] *B'* ‹$0 < e$› *t* **by** *auto*
also have ... $< e$
by (*simp add*: *B'* ‹$0 < e$› *mult-imp-div-pos-less*)
finally have $2 * cmod\ (f\ x\ (\gamma\ t) - l\ (\gamma\ t)) * cmod\ (vector\text{-}derivative\ \gamma\ (at\ t)) < e$ **.**
then show *?thesis*
by (*simp add*: *left-diff-distrib* [*symmetric*] *norm-mult*)
qed
have $\forall_F\ x$ *in F*. *dist* (*contour-integral* γ (*f x*)) (*contour-integral* γ *l*) $< e$
apply (*rule eventually-mono* [*OF eventually-conj* [*OF ev-no' ev-fint*]])
apply (*simp add*: *dist-norm contour-integrable-on path-image-def contour-integral-integral*)
apply (*simp add*: *lintg integral-diff* [*symmetric*] *contour-integrable-on* [*symmetric*], *clarify*)
apply (*rule le-less-trans* [*OF integral-norm-bound-integral ie*])
apply (*simp add*: *lintg integrable-diff contour-integrable-on* [*symmetric*])
apply (*blast intro*: $*$)+
done
}
then show $((\lambda n.\ contour\text{-}integral\ \gamma\ (f\ n)) \longrightarrow contour\text{-}integral\ \gamma\ l)\ F$
by (*rule tendstoI*)
qed

proposition *contour-integral-uniform-limit-circlepath*:
assumes *ev-fint*: *eventually* $(\lambda n::'a.\ (f\ n)$ *contour-integrable-on* $(circlepath\ z\ r))$ *F*

and *ev-no*: $\bigwedge e.\ 0 < e \Longrightarrow$ *eventually* $(\lambda n.\ \forall x \in$ *path-image* $($*circlepath* $z\ r).$ *norm*$(f\ n\ x - l\ x) < e)\ F$
and [*simp*]: $\sim$ $($*trivial-limit* $F)\ 0 < r$
shows l *contour-integrable-on* $($*circlepath* $z\ r)$ $((\lambda n.$ *contour-integral* $($*circlepath* $z\ r)\ (f\ n)) \longrightarrow$ *contour-integral* $($*circlepath* $z\ r)\ l)\ F$
by (*auto simp*: *vector-derivative-circlepath norm-mult intro*: *contour-integral-uniform-limit assms*)

53.35 General stepping result for derivative formulas.

lemma *sum-sqs-eq*:
fixes x::$'a$::*idom* **shows** $x * x + y * y = x * (y * 2) \Longrightarrow y = x$
by *algebra*

proposition *Cauchy-next-derivative*:
assumes *continuous-on* $($*path-image* $\gamma)\ f'$
and *leB*: $\bigwedge t.\ t \in \{0..1\} \Longrightarrow$ *norm* $($*vector-derivative* $\gamma\ ($*at* $t)) \leq B$
and *int*: $\bigwedge w.\ w \in s -$ *path-image* $\gamma \Longrightarrow ((\lambda u.\ f'\ u\ /\ (u - w)\ \hat{}\ k)$ *has-contour-integral* $f\ w)\ \gamma$
and k: $k \neq 0$
and *open* s
and γ: *valid-path* γ
and w: $w \in s -$ *path-image* γ
shows $(\lambda u.\ f'\ u\ /\ (u - w)\ \hat{}\ ($*Suc* $k))$ *contour-integrable-on* γ
and $(f$ *has-field-derivative* $(k *$ *contour-integral* $\gamma\ (\lambda u.\ f'\ u/(u - w)\ \hat{}\ ($*Suc* $k))))$
$($*at* $w)$ (**is** *?thes2*)
proof −
have *open* $(s -$ *path-image* $\gamma)$ **using** ‹*open s*› *closed-valid-path-image* γ **by** *blast*
then obtain d **where** d>0 **and** d: *ball* $w\ d \subseteq s -$ *path-image* γ **using** w
using *open-contains-ball* **by** *blast*
have [*simp*]: $\bigwedge n.$ *cmod* $(1 +$ *of-nat* $n) = 1 +$ *of-nat* n
by (*metis norm-of-nat of-nat-Suc*)
have *1*: $\forall_F\ n$ *in at* $w.\ (\lambda x.\ f'\ x * ($*inverse* $(x - n)\ \hat{}\ k -$ *inverse* $(x - w)\ \hat{}\ k)$ / $(n - w)$ / *of-nat* $k)$
contour-integrable-on γ
apply (*simp add*: *eventually-at*)
apply (*rule-tac* x=d **in** *exI*)
apply (*simp add*: ‹$d > 0$› *dist-norm field-simps*, *clarify*)
apply (*rule contour-integrable-div* [*OF contour-integrable-diff*])
using *int* w d
apply (*force simp*: *dist-norm norm-minus-commute intro*!: *has-contour-integral-integrable*)+
done
have *bim-g*: *bounded* $($*image* f' $($*path-image* $\gamma))$
by (*simp add*: *compact-imp-bounded compact-continuous-image compact-valid-path-image assms*)
then obtain C **where** $C > 0$ **and** C: $\bigwedge x.\ [\![0 \leq x; x \leq 1]\!] \Longrightarrow$ *cmod* $(f'\ (\gamma\ x)) \leq C$
by (*force simp*: *bounded-pos path-image-def*)

```
  have twom: ∀F n in at w.
            ∀x∈path-image γ.
              cmod ((inverse (x − n) ^ k − inverse (x − w) ^ k) / (n − w) / k
− inverse (x − w) ^ Suc k) < e
        if 0 < e for e
  proof −
    have ∗: cmod ((inverse (x − u) ^ k − inverse (x − w) ^ k) / ((u − w) ∗ k)
− inverse (x − w) ^ Suc k)  < e
          if x: x ∈ path-image γ and u ≠ w and uwd: cmod (u − w) < d/2
             and uw-less: cmod (u − w) < e ∗ (d / 2) ^ (k+2) / (1 + real k)
          for u x
     proof −
       def ff ≡ λn::nat. λw. if n = 0 then inverse(x − w)^k
                            else if n = 1 then k / (x − w)^(Suc k)
                            else (k ∗ of-real(Suc k)) / (x − w)^(k + 2)
       have km1: ⋀z::complex. z ≠ 0 ⟹ z ^ (k − Suc 0) = z ^ k / z
         by (simp add: field-simps) (metis Suc-pred ‹k ≠ 0› neq0-conv power-Suc)
       have ff1: (ff i has-field-derivative ff (Suc i) z) (at z within ball w (d / 2))
             if z ∈ ball w (d / 2) i ≤ 1 for i z
       proof −
         have z ∉ path-image γ
           using ‹x ∈ path-image γ› d that ball-divide-subset-numeral by blast
         then have xz[simp]: x ≠ z using ‹x ∈ path-image γ› by blast
         then have neq: x ∗ x + z ∗ z ≠ x ∗ (z ∗ 2)
           by (blast intro: dest!: sum-sqs-eq)
         with xz have ⋀v. v ≠ 0 ⟹ (x ∗ x + z ∗ z) ∗ v ≠ (x ∗ (z ∗ 2) ∗ v) by
auto
        then have neqq: ⋀v. v ≠ 0 ⟹ x ∗ (x ∗ v) + z ∗ (z ∗ v) ≠ x ∗ (z ∗ (2 ∗
v))
           by (simp add: algebra-simps)
         show ?thesis using ‹i ≤ 1›
           apply (simp add: ff-def dist-norm Nat.le-Suc-eq km1, safe)
            apply (rule derivative-eq-intros | simp add: km1 | simp add: field-simps
neq neqq)+
           done
       qed
       { fix a::real and b::real assume ab: a > 0 b > 0
         then have k ∗ (1 + real k) ∗ (1 / a) ≤ k ∗ (1 + real k) ∗ (4 / b) ⟷ b
≤ 4 ∗ a
           apply (subst mult-le-cancel-left-pos)
           using ‹k ≠ 0›
           apply (auto simp: divide-simps)
           done
        with ab have real k ∗ (1 + real k) / a ≤ (real k ∗ 4 + real k ∗ real k ∗ 4)
/ b ⟷ b ≤ 4 ∗ a
           by (simp add: field-simps)
       } note canc = this
       have ff2: cmod (ff (Suc 1) v) ≤ real (k ∗ (k + 1)) / (d / 2) ^ (k + 2)
             if v ∈ ball w (d / 2) for v
```

proof −
have $d/2 \le cmod\ (x - v)$ **using** *d x that*
apply (*simp add*: *dist-norm path-image-def ball-def not-less* [*symmetric*] *del*: *divide-const-simps*, *clarify*)
apply (*drule subsetD*)
prefer *2* **apply** *blast*
apply (*metis norm-minus-commute norm-triangle-half-r CollectI*)
done
then have $d \le cmod\ (x - v) * 2$
by (*simp add*: *divide-simps*)
then have *dpow-le*: $d\ \hat{}\ (k+2) \le (cmod\ (x - v) * 2)\ \hat{}\ (k+2)$
using ‹$0 < d$› *order-less-imp-le power-mono* **by** *blast*
have $x \neq v$ **using** *that*
using ‹$x \in$ *path-image* γ› *ball-divide-subset-numeral d* **by** *fastforce*
then show *?thesis*
using ‹$d > 0$›
apply (*simp add*: *ff-def norm-mult norm-divide norm-power dist-norm canc*)
using *dpow-le*
apply (*simp add*: *algebra-simps divide-simps mult-less-0-iff*)
done
qed
have *ub*: $u \in ball\ w\ (d\ /\ 2)$
using *uwd* **by** (*simp add*: *dist-commute dist-norm*)
have $cmod\ (inverse\ (x - u)\ \hat{}\ k - (inverse\ (x - w)\ \hat{}\ k + of\text{-}nat\ k * (u - w) / ((x - w) * (x - w)\ \hat{}\ k)))$
$\le (real\ k * 4 + real\ k * real\ k * 4) * (cmod\ (u - w) * cmod\ (u - w)) / (d * (d * (d\ /\ 2)\ \hat{}\ k))$
using *complex-taylor* [*OF - ff1 ff2 - ub*, *of w*, *simplified*]
by (*simp add*: *ff-def* ‹$0 < d$›)
then have $cmod\ (inverse\ (x - u)\ \hat{}\ k - (inverse\ (x - w)\ \hat{}\ k + of\text{-}nat\ k * (u - w) / ((x - w) * (x - w)\ \hat{}\ k)))$
$\le (cmod\ (u - w) * real\ k) * (1 + real\ k) * cmod\ (u - w) / (d\ /\ 2)\ \hat{}\ (k+2)$
by (*simp add*: *field-simps*)
then have $cmod\ (inverse\ (x - u)\ \hat{}\ k - (inverse\ (x - w)\ \hat{}\ k + of\text{-}nat\ k * (u - w) / ((x - w) * (x - w)\ \hat{}\ k)))$
$/\ (cmod\ (u - w) * real\ k)$
$\le (1 + real\ k) * cmod\ (u - w) / (d\ /\ 2)\ \hat{}\ (k+2)$
using ‹$k \neq 0$› ‹$u \neq w$› **by** (*simp add*: *mult-ac zero-less-mult-iff pos-divide-le-eq*)
also have *...* $< e$
using *uw-less* ‹$0 < d$› **by** (*simp add*: *mult-ac divide-simps*)
finally have *e*: $cmod\ (inverse\ (x-u)\hat{}k - (inverse\ (x-w)\hat{}k + of\text{-}nat\ k * (u-w) / ((x-w) * (x-w)\hat{}k)))$
$/\ cmod\ ((u - w) * real\ k) \quad < \quad e$
by (*simp add*: *norm-mult*)
have $x \neq u$
using *uwd* ‹$0 < d$› *x d* **by** (*force simp*: *dist-norm ball-def norm-minus-commute*)
show *?thesis*
apply (*rule le-less-trans* [*OF - e*])

```
      using ‹k ≠ 0› ‹x ≠ u› ‹u ≠ w›
      apply (simp add: field-simps norm-divide [symmetric])
      done
  qed
  show ?thesis
    unfolding eventually-at
    apply (rule-tac x = min (d/2) ((e*(d/2) ^ (k + 2))/(Suc k)) in exI)
    apply (force simp: ‹d > 0› dist-norm that simp del: power-Suc intro: *)
    done
 qed
 have 2: ∀F n in at w.
          ∀x∈path-image γ.
           cmod (f' x * (inverse (x − n) ^ k − inverse (x − w) ^ k) / (n − w)
/ of-nat k − f' x / (x − w) ^ Suc k) < e
        if 0 < e for e
 proof −
   have *: cmod (f' (γ x) * (inverse (γ x − u) ^ k − inverse (γ x − w) ^ k) /
((u − w) * k) −
                    f' (γ x) / ((γ x − w) * (γ x − w) ^ k)) < e
          if ec: cmod ((inverse (γ x − u) ^ k − inverse (γ x − w) ^ k) / ((u −
w) * k) −
                      inverse (γ x − w) * inverse (γ x − w) ^ k) < e / C
             and x: 0 ≤ x x ≤ 1
          for u x
   proof (cases (f' (γ x)) = 0)
     case True then show ?thesis by (simp add: ‹0 < e›)
   next
     case False
     have cmod (f' (γ x) * (inverse (γ x − u) ^ k − inverse (γ x − w) ^ k) /
((u − w) * k) −
                    f' (γ x) / ((γ x − w) * (γ x − w) ^ k)) =
          cmod (f' (γ x) * ((inverse (γ x − u) ^ k − inverse (γ x − w) ^ k) / ((u
− w) * k) −
                            inverse (γ x − w) * inverse (γ x − w) ^ k))
       by (simp add: field-simps)
     also have ... = cmod (f' (γ x)) *
                     cmod ((inverse (γ x − u) ^ k − inverse (γ x − w) ^ k) / ((u
− w) * k) −
                            inverse (γ x − w) * inverse (γ x − w) ^ k)
       by (simp add: norm-mult)
     also have ... < cmod (f' (γ x)) * (e/C)
       apply (rule mult-strict-left-mono [OF ec])
       using False by simp
     also have ... ≤ e using C
      by (metis False ‹0 < e› frac-le less-eq-real-def mult.commute pos-le-divide-eq
x zero-less-norm-iff)
     finally show ?thesis .
   qed
   show ?thesis
```

```
    using twom [OF divide-pos-pos [OF that ‹C > 0›]] unfolding path-image-def
     by (force intro: * elim: eventually-mono)
  qed
  show (λu. f' u / (u − w) ^ (Suc k)) contour-integrable-on γ
    by (rule contour-integral-uniform-limit [OF 1 2 leB γ]) auto
  have *: (λn. contour-integral γ (λx. f' x * (inverse (x − n) ^ k − inverse (x −
w) ^ k) / (n − w) / k))
          −w→ contour-integral γ (λu. f' u / (u − w) ^ (Suc k))
    by (rule contour-integral-uniform-limit [OF 1 2 leB γ]) auto
  have **: contour-integral γ (λx. f' x * (inverse (x − u) ^ k − inverse (x − w)
^ k) / ((u − w) * k)) =
             (f u − f w) / (u − w) / k
          if dist u w < d for u
    apply (rule contour-integral-unique)
    apply (simp add: diff-divide-distrib algebra-simps)
    apply (rule has-contour-integral-diff; rule has-contour-integral-div; simp add:
field-simps; rule int)
    apply (metis contra-subsetD d dist-commute mem-ball that)
    apply (rule w)
    done
  show ?thes2
    apply (simp add: DERIV-within-iff del: power-Suc)
    apply (rule Lim-transform-within [OF tendsto-mult-left [OF *] ‹0 < d› ])
    apply (simp add: ‹k ≠ 0› **)
    done
qed

corollary Cauchy-next-derivative-circlepath:
  assumes contf: continuous-on (path-image (circlepath z r)) f
      and int: ⋀w. w ∈ ball z r ⟹ ((λu. f u / (u − w) ^k) has-contour-integral g
w) (circlepath z r)
      and k: k ≠ 0
      and w: w ∈ ball z r
    shows (λu. f u / (u − w) ^(Suc k)) contour-integrable-on (circlepath z r)
          (is ?thes1)
      and (g has-field-derivative (k * contour-integral (circlepath z r) (λu. f u/(u
− w) ^(Suc k)))) (at w)
          (is ?thes2)
proof −
  have r > 0 using w
    using ball-eq-empty by fastforce
  have wim: w ∈ ball z r − path-image (circlepath z r)
    using w by (auto simp: dist-norm)
  show ?thes1 ?thes2
    by (rule Cauchy-next-derivative [OF contf - int k open-ball valid-path-circlepath
wim, where B = 2 * pi * |r|];
        auto simp: vector-derivative-circlepath norm-mult)+
qed
```

In particular, the first derivative formula.

proposition *Cauchy-derivative-integral-circlepath*:
assumes *contf*: *continuous-on* (*cball z r*) *f*
and *holf*: *f holomorphic-on ball z r*
and *w*: *w* ∈ *ball z r*
shows (λ*u*. *f u*/(*u* − *w*) ^ *2*) *contour-integrable-on* (*circlepath z r*)
(**is** *?thes1*)
and (*f has-field-derivative* (*1* / (*2* ∗ *of-real pi* ∗ *ii*) ∗ *contour-integral*(*circlepath z r*) (λ*u*. *f u* / (*u* − *w*) ^ *2*))) (*at w*)
(**is** *?thes2*)
proof −
have [*simp*]: *r* ≥ *0* **using** *w*
using *ball-eq-empty* **by** *fastforce*
have *f*: *continuous-on* (*path-image* (*circlepath z r*)) *f*
by (*rule continuous-on-subset* [*OF contf*]) (*force simp add*: *cball-def sphere-def*)
have *int*: ⋀*w*. *dist z w* < *r* ⟹
((λ*u*. *f u* / (*u* − *w*)) *has-contour-integral* (λ*x*. *2* ∗ *of-real pi* ∗ *ii* ∗ *f x*) *w*) (*circlepath z r*)
by (*rule Cauchy-integral-circlepath* [*OF contf holf*]) (*simp add*: *dist-norm norm-minus-commute*)
show *?thes1*
apply (*simp add*: *power2-eq-square*)
apply (*rule Cauchy-next-derivative-circlepath* [*OF f - - w*, **where** *k=1*, *simplified*])
apply (*blast intro*: *int*)
done
have ((λ*x*. *2* ∗ *of-real pi* ∗ i ∗ *f x*) *has-field-derivative contour-integral* (*circlepath z r*) (λ*u*. *f u* / (*u* − *w*) ^ *2*)) (*at w*)
apply (*simp add*: *power2-eq-square*)
apply (*rule Cauchy-next-derivative-circlepath* [*OF f - - w*, **where** *k=1* **and** *g* = λ*x*. *2* ∗ *of-real pi* ∗ *ii* ∗ *f x*, *simplified*])
apply (*blast intro*: *int*)
done
then have *fder*: (*f has-field-derivative contour-integral* (*circlepath z r*) (λ*u*. *f u* / (*u* − *w*) ^ *2*) / (*2* ∗ *of-real pi* ∗ i)) (*at w*)
by (*rule DERIV-cdivide* [**where** *f* = λ*x*. *2* ∗ *of-real pi* ∗ i ∗ *f x* **and** *c* = *2* ∗ *of-real pi* ∗ i, *simplified*])
show *?thes2*
by *simp* (*rule fder*)
qed

53.36 Existence of all higher derivatives.

proposition *derivative-is-holomorphic*:
assumes *open s*
and *fder*: ⋀*z*. *z* ∈ *s* ⟹ (*f has-field-derivative f* ′ *z*) (*at z*)
shows *f* ′ *holomorphic-on s*
proof −
have ∗: ∃*h*. (*f* ′ *has-field-derivative h*) (*at z*) **if** *z* ∈ *s* **for** *z*
proof −

```
    obtain r where r > 0 and r: cball z r ⊆ s
      using open-contains-cball ‹z ∈ s› ‹open s› by blast
    then have holf-cball: f holomorphic-on cball z r
      apply (simp add: holomorphic-on-def)
      using field-differentiable-at-within field-differentiable-def fder by blast
    then have continuous-on (path-image (circlepath z r)) f
    using ‹r > 0› by (force elim: holomorphic-on-subset [THEN holomorphic-on-imp-continuous-on])
     then have contfpi: continuous-on (path-image (circlepath z r)) (λx. 1/(2 *
of-real pi*ii) * f x)
      by (auto intro: continuous-intros)+
    have contf-cball: continuous-on (cball z r) f using holf-cball
      by (simp add: holomorphic-on-imp-continuous-on holomorphic-on-subset)
    have holf-ball: f holomorphic-on ball z r using holf-cball
      using ball-subset-cball holomorphic-on-subset by blast
    { fix w assume w: w ∈ ball z r
      have intf: (λu. f u / (u − w)^2) contour-integrable-on circlepath z r
          by (blast intro: w Cauchy-derivative-integral-circlepath [OF contf-cball
holf-ball])
       have fder': (f has-field-derivative 1 / (2 * of-real pi * i) * contour-integral
(circlepath z r) (λu. f u / (u − w)^2))
                (at w)
          by (blast intro: w Cauchy-derivative-integral-circlepath [OF contf-cball
holf-ball])
      have f'-eq: f' w = contour-integral (circlepath z r) (λu. f u / (u − w)^2) / (2
* of-real pi * i)
        using fder' ball-subset-cball r w by (force intro: DERIV-unique [OF fder])
      have ((λu. f u / (u − w)^2 / (2 * of-real pi * i)) has-contour-integral
              contour-integral (circlepath z r) (λu. f u / (u − w)^2) / (2 * of-real pi
* i))
              (circlepath z r)
      by (rule Cauchy-Integral-Thm.has-contour-integral-div [OF has-contour-integral-integral
[OF intf]])
      then have ((λu. f u / (2 * of-real pi * i * (u − w)^2)) has-contour-integral
              contour-integral (circlepath z r) (λu. f u / (u − w)^2) / (2 * of-real pi
* i))
              (circlepath z r)
        by (simp add: algebra-simps)
      then have ((λu. f u / (2 * of-real pi * i * (u − w)^2)) has-contour-integral f'
w) (circlepath z r)
        by (simp add: f'-eq)
    } note * = this
    show ?thesis
      apply (rule exI)
     apply (rule Cauchy-next-derivative-circlepath [OF contfpi, of 2 f', simplified])
      apply (simp-all add: ‹0 < r› * dist-norm)
      done
  qed
  show ?thesis
    by (simp add: holomorphic-on-open [OF ‹open s›] *)
```

qed

lemma *holomorphic-deriv* [*holomorphic-intros*]:
  ⟦*f holomorphic-on s*; *open s*⟧ ⟹ (*deriv f*) *holomorphic-on s*
by (*metis DERIV-deriv-iff-field-differentiable at-within-open derivative-is-holomorphic holomorphic-on-def*)

lemma *analytic-deriv*: *f analytic-on s* ⟹ (*deriv f*) *analytic-on s*
  using *analytic-on-holomorphic holomorphic-deriv* **by** *auto*

lemma *holomorphic-higher-deriv* [*holomorphic-intros*]: ⟦*f holomorphic-on s*; *open s*⟧ ⟹ (*deriv* ^^ *n*) *f holomorphic-on s*
  by (*induction n*) (*auto simp*: *holomorphic-deriv*)

lemma *analytic-higher-deriv*: *f analytic-on s* ⟹ (*deriv* ^^ *n*) *f analytic-on s*
  unfolding *analytic-on-def* **using** *holomorphic-higher-deriv* **by** *blast*

lemma *has-field-derivative-higher-deriv*:
  ⟦*f holomorphic-on s*; *open s*; *x* ∈ *s*⟧
  ⟹ ((*deriv* ^^ *n*) *f has-field-derivative* (*deriv* ^^ (*Suc n*)) *f x*) (*at x*)
by (*metis* (*no-types*, *hide-lams*) *DERIV-deriv-iff-field-differentiable at-within-open comp-apply*
    funpow.simps(*2*) *holomorphic-higher-deriv holomorphic-on-def*)

lemma *valid-path-compose-holomorphic*:
  assumes *valid-path g* **and** *holo*:*f holomorphic-on s* **and** *open s path-image g* ⊆ *s*
  shows *valid-path* (*f o g*)
proof (*rule valid-path-compose*[*OF* ‹*valid-path g*›])
  fix *x* **assume** *x* ∈ *path-image g*
  then show ∃*f'*. (*f has-field-derivative f'*) (*at x*)
    using *holo holomorphic-on-open*[*OF* ‹*open s*›] ‹*path-image g* ⊆ *s*› **by** *auto*
next
  have *deriv f holomorphic-on s*
    using *holomorphic-deriv holo* ‹*open s*› **by** *auto*
  then show *continuous-on* (*path-image g*) (*deriv f*)
    using *assms*(*4*) *holomorphic-on-imp-continuous-on holomorphic-on-subset* **by** *auto*
qed

53.37 Morera's theorem.

lemma *Morera-local-triangle-ball*:
  assumes ⋀*z*. *z* ∈ *s*
    ⟹ ∃*e a*. *0* < *e* ∧ *z* ∈ *ball a e* ∧ *continuous-on* (*ball a e*) *f* ∧
      (∀ *b c*. *closed-segment b c* ⊆ *ball a e*
        ⟶ *contour-integral* (*linepath a b*) *f* +
          contour-integral (*linepath b c*) *f* +
          contour-integral (*linepath c a*) *f* = *0*)
  shows *f analytic-on s*

proof −
{ fix z **assume** $z \in s$
with *assms* **obtain** e a **where**
$0 < e$ **and** z: $z \in$ *ball a e* **and** *contf*: *continuous-on* (*ball a e*) f
and 0: $\bigwedge b\ c$. *closed-segment b c* $\subseteq$ *ball a e*
$\Longrightarrow$ *contour-integral* (*linepath a b*) f +
contour-integral (*linepath b c*) f +
contour-integral (*linepath c a*) $f = 0$
by *fastforce*
have *az*: *dist a z* $< e$ **using** *mem-ball z* **by** *blast*
have *sb-ball*: *ball z* (e − *dist a z*) $\subseteq$ *ball a e*
by (*simp add*: *dist-commute ball-subset-ball-iff*)
have $\exists\, e{>}0$. *f holomorphic-on ball z e*
apply (*rule-tac* $x{=}e$ − *dist a z* **in** *exI*)
apply (*simp add*: *az*)
apply (*rule holomorphic-on-subset* [*OF* - *sb-ball*])
apply (*rule derivative-is-holomorphic*[*OF open-ball*])
apply (*rule triangle-contour-integrals-starlike-primitive* [*OF contf* - *open-ball*, *of a*])
apply (*simp-all add*: *0* ‹*0* < *e*›)
apply (*meson* ‹*0* < *e*› *centre-in-ball convex-ball convex-contains-segment mem-ball*)
done
}
then show *?thesis*
by (*simp add*: *analytic-on-def*)
qed

lemma *Morera-local-triangle*:
assumes $\bigwedge z$. $z \in s$
$\Longrightarrow \exists\, t$. *open t* $\wedge$ $z \in t$ $\wedge$ *continuous-on t f* $\wedge$
(∀ *a b c*. *convex hull* {*a*,*b*,*c*} $\subseteq$ *t*
$\longrightarrow$ *contour-integral* (*linepath a b*) f +
contour-integral (*linepath b c*) f +
contour-integral (*linepath c a*) $f = 0$)
shows *f analytic-on s*
proof −
{ fix z **assume** $z \in s$
with *assms* **obtain** t **where**
open t **and** z: $z \in t$ **and** *contf*: *continuous-on t f*
and 0: $\bigwedge a\ b\ c$. *convex hull* {*a*,*b*,*c*} $\subseteq$ *t*
$\Longrightarrow$ *contour-integral* (*linepath a b*) f +
contour-integral (*linepath b c*) f +
contour-integral (*linepath c a*) $f = 0$
by *force*
then obtain e **where** $e{>}0$ **and** e: *ball z e* $\subseteq$ *t*
using *open-contains-ball* **by** *blast*
have [*simp*]: *continuous-on* (*ball z e*) f **using** *contf*
using *continuous-on-subset e* **by** *blast*

have $\exists e\ a.\ 0 < e \wedge$
$z \in ball\ a\ e \wedge$
continuous-on $(ball\ a\ e)\ f \wedge$
$(\forall b\ c.\ closed\text{-}segment\ b\ c \subseteq ball\ a\ e \longrightarrow$
contour-integral $(linepath\ a\ b)\ f$ + *contour-integral* $(linepath\ b\ c)\ f$ + *contour-integral* $(linepath\ c\ a)\ f = 0)$
apply (*rule-tac x=e* **in** *exI*)
apply (*rule-tac x=z* **in** *exI*)
apply (*simp add*: ‹$e > 0$›, *clarify*)
apply (*rule 0*)
apply (*meson z ‹0 < e› centre-in-ball closed-segment-subset convex-ball dual-order.trans e starlike-convex-subset*)
done
}
then show *?thesis*
by (*simp add*: *Morera-local-triangle-ball*)
qed

proposition *Morera-triangle*:
⟦*continuous-on s f*; *open s*;
$\bigwedge a\ b\ c.$ *convex hull* $\{a,b,c\} \subseteq s$
$\longrightarrow$ *contour-integral* $(linepath\ a\ b)\ f$ +
contour-integral $(linepath\ b\ c)\ f$ +
contour-integral $(linepath\ c\ a)\ f = 0$⟧
$\Longrightarrow$ *f analytic-on s*
using *Morera-local-triangle* **by** *blast*

53.38 Combining theorems for higher derivatives including Leibniz rule.

lemma *higher-deriv-linear* [*simp*]:
$(deriv\ \hat{}\hat{}\ n)\ (\lambda w.\ c*w) = (\lambda z.\ if\ n = 0\ then\ c*z\ else\ if\ n = 1\ then\ c\ else\ 0)$
by (*induction n*) (*auto simp*: *deriv-const deriv-linear*)

lemma *higher-deriv-const* [*simp*]: $(deriv\ \hat{}\hat{}\ n)\ (\lambda w.\ c) = (\lambda w.\ if\ n=0\ then\ c\ else\ 0)$
by (*induction n*) (*auto simp*: *deriv-const*)

lemma *higher-deriv-ident* [*simp*]:
$(deriv\ \hat{}\hat{}\ n)\ (\lambda w.\ w)\ z = (if\ n = 0\ then\ z\ else\ if\ n = 1\ then\ 1\ else\ 0)$
apply (*induction n*, *simp*)
apply (*metis higher-deriv-linear lambda-one*)
done

corollary *higher-deriv-id* [*simp*]:
$(deriv\ \hat{}\hat{}\ n)\ id\ z = (if\ n = 0\ then\ z\ else\ if\ n = 1\ then\ 1\ else\ 0)$
by (*simp add*: *id-def*)

lemma *has-complex-derivative-funpow-1*:

```
    ⟦(f has-field-derivative 1) (at z); f z = z⟧ ⟹ (f^^n has-field-derivative 1) (at
z)
  apply (induction n)
  apply auto
  apply (metis DERIV-ident DERIV-transform-at id-apply zero-less-one)
  by (metis DERIV-chain comp-funpow comp-id funpow-swap1 mult.right-neutral)

proposition higher-deriv-uminus:
  assumes f holomorphic-on s open s and z: z ∈ s
    shows (deriv ^^ n) (λw. −(f w)) z = − ((deriv ^^ n) f z)
using z
proof (induction n arbitrary: z)
  case 0 then show ?case by simp
next
  case (Suc n z)
  have ∗: ((deriv ^^ n) f has-field-derivative deriv ((deriv ^^ n) f) z) (at z)
    using Suc.prems assms has-field-derivative-higher-deriv by auto
  show ?case
    apply simp
    apply (rule DERIV-imp-deriv)
    apply (rule DERIV-transform-within-open [of λw. −((deriv ^^ n) f w)])
    apply (rule derivative-eq-intros | rule ∗ refl assms Suc)+
    apply (simp add: Suc)
    done
qed

proposition higher-deriv-add:
  fixes z::complex
  assumes f holomorphic-on s g holomorphic-on s open s and z: z ∈ s
    shows (deriv ^^ n) (λw. f w + g w) z = (deriv ^^ n) f z + (deriv ^^ n) g z
using z
proof (induction n arbitrary: z)
  case 0 then show ?case by simp
next
  case (Suc n z)
  have ∗: ((deriv ^^ n) f has-field-derivative deriv ((deriv ^^ n) f) z) (at z)
        ((deriv ^^ n) g has-field-derivative deriv ((deriv ^^ n) g) z) (at z)
    using Suc.prems assms has-field-derivative-higher-deriv by auto
  show ?case
    apply simp
    apply (rule DERIV-imp-deriv)
    apply (rule DERIV-transform-within-open [of λw. (deriv ^^ n) f w + (deriv
^^ n) g w])
    apply (rule derivative-eq-intros | rule ∗ refl assms Suc)+
    apply (simp add: Suc)
    done
qed

corollary higher-deriv-diff:
```

fixes *z*::*complex*
assumes *f holomorphic-on s g holomorphic-on s open s* **and** *z*: *z* ∈ *s*
shows (*deriv* ^^ *n*) (λ*w*. *f w* − *g w*) *z* = (*deriv* ^^ *n*) *f z* − (*deriv* ^^ *n*) *g z*
apply (*simp only*: *Groups.group-add-class.diff-conv-add-uminus higher-deriv-add*)
apply (*subst higher-deriv-add*)
using *assms holomorphic-on-minus* **apply** (*auto simp*: *higher-deriv-uminus*)
done

lemma *bb*: *Suc n choose k* = (*n choose k*) + (*if k* = *0 then 0 else* (*n choose* (*k* − *1*)))
by (*simp add*: *Binomial.binomial.simps*)

proposition *higher-deriv-mult*:
fixes *z*::*complex*
assumes *f holomorphic-on s g holomorphic-on s open s* **and** *z*: *z* ∈ *s*
shows (*deriv* ^^ *n*) (λ*w*. *f w* ∗ *g w*) *z* =
(∑ *i* = *0*..*n*. *of-nat* (*n choose i*) ∗ (*deriv* ^^ *i*) *f z* ∗ (*deriv* ^^ (*n* − *i*)) *g z*)
using *z*
proof (*induction n arbitrary*: *z*)
case *0* **then show** *?case* **by** *simp*
next
case (*Suc n z*)
have ∗: ⋀*n*. ((*deriv* ^^ *n*) *f has-field-derivative deriv* ((*deriv* ^^ *n*) *f*) *z*) (*at z*)
⋀*n*. ((*deriv* ^^ *n*) *g has-field-derivative deriv* ((*deriv* ^^ *n*) *g*) *z*) (*at z*)
using *Suc.prems assms has-field-derivative-higher-deriv* **by** *auto*
have *sumeq*: (∑ *i* = *0*..*n*.
of-nat (*n choose i*) ∗ (*deriv* ((*deriv* ^^ *i*) *f*) *z* ∗ (*deriv* ^^ (*n* − *i*)) *g z* + *deriv* ((*deriv* ^^ (*n* − *i*)) *g*) *z* ∗ (*deriv* ^^ *i*) *f z*)) =
g z ∗ *deriv* ((*deriv* ^^ *n*) *f*) *z* + (∑ *i* = *0*..*n*. (*deriv* ^^ *i*) *f z* ∗ (*of-nat* (*Suc n choose i*) ∗ (*deriv* ^^ (*Suc n* − *i*)) *g z*))
apply (*simp add*: *bb distrib-right algebra-simps setsum.distrib*)
apply (*subst* (*4*) *setsum-Suc-reindex*)
apply (*auto simp*: *algebra-simps Suc-diff-le intro*: *setsum.cong*)
done
show *?case*
apply (*simp only*: *funpow.simps o-apply*)
apply (*rule DERIV-imp-deriv*)
apply (*rule DERIV-transform-within-open*
[*of* λ*w*. (∑ *i* = *0*..*n*. *of-nat* (*n choose i*) ∗ (*deriv* ^^ *i*) *f w* ∗ (*deriv* ^^ (*n* − *i*)) *g w*)])
apply (*simp add*: *algebra-simps*)
apply (*rule DERIV-cong* [*OF DERIV-setsum*])
apply (*rule DERIV-cmult*)
apply (*auto simp*: *intro*: *DERIV-mult* ∗ *sumeq* ‹*open s*› *Suc.prems Suc.IH* [*symmetric*])
done
qed

proposition *higher-deriv-transform-within-open*:
fixes *z*::*complex*
assumes *f holomorphic-on s g holomorphic-on s open s* **and** *z*: *z* ∈ *s*
and *fg*: ⋀*w*. *w* ∈ *s* ⟹ *f w* = *g w*
shows (*deriv* ^^ *i*) *f z* = (*deriv* ^^ *i*) *g z*
using *z*
by (*induction i arbitrary*: *z*)
(*auto simp*: *fg intro*: *complex-derivative-transform-within-open holomorphic-higher-deriv assms*)

proposition *higher-deriv-compose-linear*:
fixes *z*::*complex*
assumes *f*: *f holomorphic-on t* **and** *s*: *open s* **and** *t*: *open t* **and** *z*: *z* ∈ *s*
and *fg*: ⋀*w*. *w* ∈ *s* ⟹ *u* ∗ *w* ∈ *t*
shows (*deriv* ^^ *n*) (λ*w*. *f* (*u* ∗ *w*)) *z* = *u*^*n* ∗ (*deriv* ^^ *n*) *f* (*u* ∗ *z*)
using *z*
proof (*induction n arbitrary*: *z*)
case *0* **then show** *?case* **by** *simp*
next
case (*Suc n z*)
have *holo0*: *f holomorphic-on op* ∗ *u* ‘ *s*
by (*meson fg f holomorphic-on-subset image-subset-iff*)
have *holo1*: (λ*w*. *f* (*u* ∗ *w*)) *holomorphic-on s*
apply (*rule holomorphic-on-compose* [**where** *g*=*f*, *unfolded o-def*])
apply (*rule holo0 holomorphic-intros*)+
done
have *holo2*: (λ*z*. *u* ^ *n* ∗ (*deriv* ^^ *n*) *f* (*u* ∗ *z*)) *holomorphic-on s*
apply (*rule holomorphic-intros*)+
apply (*rule holomorphic-on-compose* [**where** *g*=(*deriv* ^^ *n*) *f*, *unfolded o-def*])
apply (*rule holomorphic-intros*)
apply (*rule holomorphic-on-subset* [**where** *s*=*t*])
apply (*rule holomorphic-intros assms*)+
apply (*blast intro*: *fg*)
done
have *deriv* ((*deriv* ^^ *n*) (λ*w*. *f* (*u* ∗ *w*))) *z* = *deriv* (λ*z*. *u*^*n* ∗ (*deriv* ^^ *n*) *f* (*u*∗*z*)) *z*
apply (*rule complex-derivative-transform-within-open* [*OF - holo2 s Suc.prems*])
apply (*rule holomorphic-higher-deriv* [*OF holo1 s*])
apply (*simp add*: *Suc.IH*)
done
also have ... = *u*^*n* ∗ *deriv* (λ*z*. (*deriv* ^^ *n*) *f* (*u* ∗ *z*)) *z*
apply (*rule deriv-cmult*)
apply (*rule analytic-on-imp-differentiable-at* [*OF - Suc.prems*])
apply (*rule analytic-on-compose-gen* [**where** *g*=(*deriv* ^^ *n*) *f* **and** *t*=*t*, *unfolded o-def*])
apply (*simp add*: *analytic-on-linear*)
apply (*simp add*: *analytic-on-open f holomorphic-higher-deriv t*)

```
    apply (blast intro: fg)
    done
  also have ... = u * u ^ n * deriv ((deriv ^^ n) f) (u * z)
      apply (subst complex-derivative-chain [where g = (deriv ^^ n) f and f =
op*u, unfolded o-def])
      apply (rule derivative-intros)
      using Suc.prems field-differentiable-def f fg has-field-derivative-higher-deriv t
apply blast
      apply (simp add: deriv-linear)
      done
  finally show ?case
    by simp
qed

lemma higher-deriv-add-at:
  assumes f analytic-on {z} g analytic-on {z}
    shows (deriv ^^ n) (λw. f w + g w) z = (deriv ^^ n) f z + (deriv ^^ n) g z
proof −
  have f analytic-on {z} ∧ g analytic-on {z}
    using assms by blast
  with higher-deriv-add show ?thesis
    by (auto simp: analytic-at-two)
qed

lemma higher-deriv-diff-at:
  assumes f analytic-on {z} g analytic-on {z}
    shows (deriv ^^ n) (λw. f w − g w) z = (deriv ^^ n) f z − (deriv ^^ n) g z
proof −
  have f analytic-on {z} ∧ g analytic-on {z}
    using assms by blast
  with higher-deriv-diff show ?thesis
    by (auto simp: analytic-at-two)
qed

lemma higher-deriv-uminus-at:
   f analytic-on {z} ⟹ (deriv ^^ n) (λw. −(f w)) z = − ((deriv ^^ n) f z)
  using higher-deriv-uminus
    by (auto simp: analytic-at)

lemma higher-deriv-mult-at:
  assumes f analytic-on {z} g analytic-on {z}
    shows (deriv ^^ n) (λw. f w * g w) z =
          (∑ i = 0..n. of-nat (n choose i) * (deriv ^^ i) f z * (deriv ^^ (n − i)) g
z)
proof −
  have f analytic-on {z} ∧ g analytic-on {z}
    using assms by blast
  with higher-deriv-mult show ?thesis
    by (auto simp: analytic-at-two)
```

qed

Nonexistence of isolated singularities and a stronger integral formula.

proposition *no-isolated-singularity*:
fixes *z*::*complex*
assumes *f*: *continuous-on s f* **and** *holf*: *f holomorphic-on* (*s* − *k*) **and** *s*: *open s* **and** *k*: *finite k*
shows *f holomorphic-on s*
proof −
{ fix *z*
assume *z* ∈ *s* **and** *cdf*: ⋀*x*. *x*∈*s* − *k* ⟹ *f field-differentiable at x*
have *f field-differentiable at z*
proof (*cases z* ∈ *k*)
case *False* **then show** *?thesis* **by** (*blast intro*: *cdf* ‹*z* ∈ *s*›)
next
case *True*
with *finite-set-avoid* [*OF k*, *of z*]
obtain *d* **where** *d*>*0* **and** *d*: ⋀*x*. ⟦*x*∈*k*; *x* ≠ *z*⟧ ⟹ *d* ≤ *dist z x*
by *blast*
obtain *e* **where** *e*>*0* **and** *e*: *ball z e* ⊆ *s*
using *s* ‹*z* ∈ *s*› **by** (*force simp add*: *open-contains-ball*)
have *fde*: *continuous-on* (*ball z* (*min d e*)) *f*
by (*metis Int-iff ball-min-Int continuous-on-subset e f subsetI*)
have ∃ *g*. ∀ *w* ∈ *ball z* (*min d e*). (*g has-field-derivative f w*) (*at w within ball z* (*min d e*))
apply (*rule contour-integral-convex-primitive* [*OF convex-ball fde*])
apply (*rule Cauchy-theorem-triangle-cofinite* [*OF - k*])
apply (*metis continuous-on-subset* [*OF fde*] *closed-segment-subset convex-ball starlike-convex-subset*)
apply (*rule cdf*)
apply (*metis Diff-iff Int-iff ball-min-Int bot-least contra-subsetD convex-ball e insert-subset*
interior-mono interior-subset subset-hull)
done
then have *f holomorphic-on ball z* (*min d e*)
by (*metis open-ball at-within-open derivative-is-holomorphic*)
then show *?thesis*
unfolding *holomorphic-on-def*
by (*metis open-ball* ‹*0* < *d*› ‹*0* < *e*› *at-within-open centre-in-ball min-less-iff-conj*)
qed
}
with *holf s k* **show** *?thesis*
by (*simp add*: *holomorphic-on-open open-Diff finite-imp-closed field-differentiable-def* [*symmetric*])
qed

proposition *Cauchy-integral-formula-convex*:
assumes *s*: *convex s* **and** *k*: *finite k* **and** *contf*: *continuous-on s f*
and *fcd*: (⋀*x*. *x* ∈ *interior s* − *k* ⟹ *f field-differentiable at x*)

and *z*: $z \in$ *interior s* **and** *vpg*: *valid-path* γ
and *pasz*: *path-image* $\gamma \subseteq s - \{z\}$ **and** *loop*: *pathfinish* γ = *pathstart* γ
shows $((\lambda w.\ f\ w\ /\ (w - z))$ *has-contour-integral* $(2{*}pi * ii * winding\text{-}number\ \gamma\ z * f\ z))\ \gamma$
apply (*rule Cauchy-integral-formula-weak* [*OF s finite.emptyI contf*])
apply (*simp add*: *holomorphic-on-open* [*symmetric*] *field-differentiable-def*)
using *no-isolated-singularity* [**where** *s* = *interior s*]
apply (*metis k contf fcd holomorphic-on-open field-differentiable-def continuous-on-subset has-field-derivative-at-within holomorphic-on-def interior-subset open-interior*)
using *assms*
apply *auto*
done

Formula for higher derivatives.

proposition *Cauchy-has-contour-integral-higher-derivative-circlepath*:
assumes *contf*: *continuous-on* (*cball z r*) *f*
and *holf*: *f holomorphic-on ball z r*
and *w*: $w \in$ *ball z r*
shows $((\lambda u.\ f\ u\ /\ (u - w)\ \hat{}\ (Suc\ k))$ *has-contour-integral* $((2 * pi * ii)\ /\ (fact\ k) * (deriv\ \hat{}\hat{}\ k)\ f\ w))$
(*circlepath z r*)
using *w*
proof (*induction k arbitrary*: *w*)
case *0* **then show** *?case*
using *assms* **by** (*auto simp*: *Cauchy-integral-circlepath dist-commute dist-norm*)
next
case (*Suc k*)
have [*simp*]: *r* > *0* **using** *w*
using *ball-eq-empty* **by** *fastforce*
have *f*: *continuous-on* (*path-image* (*circlepath z r*)) *f*
by (*rule continuous-on-subset* [*OF contf*]) (*force simp add*: *cball-def sphere-def less-imp-le*)
obtain *X* **where** *X*: $((\lambda u.\ f\ u\ /\ (u - w)\ \hat{}\ Suc\ (Suc\ k))$ *has-contour-integral X*) (*circlepath z r*)
using *Cauchy-next-derivative-circlepath*(*1*) [*OF f Suc.IH - Suc.prems*]
by (*auto simp*: *contour-integrable-on-def*)
then have *con*: *contour-integral* (*circlepath z r*) $((\lambda u.\ f\ u\ /\ (u - w)\ \hat{}\ Suc\ (Suc\ k))) = X$
by (*rule contour-integral-unique*)
have $\bigwedge n.\ ((deriv\ \hat{}\hat{}\ n)\ f$ *has-field-derivative deriv* $((deriv\ \hat{}\hat{}\ n)\ f)\ w)$ (*at w*)
using *Suc.prems assms has-field-derivative-higher-deriv* **by** *auto*
then have *dnf-diff*: $\bigwedge n.$ (*deriv* $\hat{}\hat{}$ *n*) *f field-differentiable* (*at w*)
by (*force simp add*: *field-differentiable-def*)
have *deriv* $(\lambda w.$ *complex-of-real* $(2 * pi) *$ i $/$ (*fact k*) $*$ (*deriv* $\hat{}\hat{}$ *k*) *f w*) *w* =
of-nat (*Suc k*) $*$ *contour-integral* (*circlepath z r*) $(\lambda u.\ f\ u\ /\ (u - w)\ \hat{}\ Suc\ (Suc\ k))$
by (*force intro*!: *DERIV-imp-deriv Cauchy-next-derivative-circlepath* [*OF f Suc.IH - Suc.prems*])

also have ... = *of-nat* (*Suc k*) ∗ *X*
by (*simp only*: *con*)
finally have *deriv* (λ*w*. ((2 ∗ *pi*) ∗ i / (*fact k*)) ∗ (*deriv* ^^ *k*) *f w*) *w* = *of-nat* (*Suc k*) ∗ *X* **.**
then have ((2 ∗ *pi*) ∗ i / (*fact k*)) ∗ *deriv* (λ*w*. (*deriv* ^^ *k*) *f w*) *w* = *of-nat* (*Suc k*) ∗ *X*
by (*metis deriv-cmult dnf-diff*)
then have *deriv* (λ*w*. (*deriv* ^^ *k*) *f w*) *w* = *of-nat* (*Suc k*) ∗ *X* / ((2 ∗ *pi*) ∗ i / (*fact k*))
by (*simp add*: *field-simps*)
then show *?case*
using *of-nat-eq-0-iff X* **by** *fastforce*
qed

proposition *Cauchy-higher-derivative-integral-circlepath*:
assumes *contf*: *continuous-on* (*cball z r*) *f*
and *holf*: *f holomorphic-on ball z r*
and *w*: *w* ∈ *ball z r*
shows (λ*u*. *f u* / (*u* − *w*) ^ (*Suc k*)) *contour-integrable-on* (*circlepath z r*)
(**is** *?thes1*)
and (*deriv* ^^ *k*) *f w* = (*fact k*) / (2 ∗ *pi* ∗ *ii*) ∗ *contour-integral*(*circlepath z r*) (λ*u*. *f u*/(*u* − *w*) ^ (*Suc k*))
(**is** *?thes2*)
proof −
have ∗: ((λ*u*. *f u* / (*u* − *w*) ^ *Suc k*) *has-contour-integral* (2 ∗ *pi*) ∗ i / (*fact k*) ∗ (*deriv* ^^ *k*) *f w*)
(*circlepath z r*)
using *Cauchy-has-contour-integral-higher-derivative-circlepath* [*OF assms*]
by *simp*
show *?thes1* **using** ∗
using *contour-integrable-on-def* **by** *blast*
show *?thes2*
unfolding *contour-integral-unique* [*OF* ∗] **by** (*simp add*: *divide-simps*)
qed

corollary *Cauchy-contour-integral-circlepath*:
assumes *continuous-on* (*cball z r*) *f f holomorphic-on ball z r w* ∈ *ball z r*
shows *contour-integral*(*circlepath z r*) (λ*u*. *f u*/(*u* − *w*) ^ (*Suc k*)) = (2 ∗ *pi* ∗ *ii*) ∗ (*deriv* ^^ *k*) *f w* / (*fact k*)
by (*simp add*: *Cauchy-higher-derivative-integral-circlepath* [*OF assms*])

corollary *Cauchy-contour-integral-circlepath-2*:
assumes *continuous-on* (*cball z r*) *f f holomorphic-on ball z r w* ∈ *ball z r*
shows *contour-integral*(*circlepath z r*) (λ*u*. *f u*/(*u* − *w*) ^ 2) = (2 ∗ *pi* ∗ *ii*) ∗ *deriv f w*
using *Cauchy-contour-integral-circlepath* [*OF assms*, *of 1*]
by (*simp add*: *power2-eq-square*)

53.39 A holomorphic function is analytic, i.e. has local power series.

theorem *holomorphic-power-series*:
assumes *holf*: *f holomorphic-on ball z r*
and *w*: *w* ∈ *ball z r*
shows ((λ*n*. (*deriv* ^^ *n*) *f z* / (*fact n*) ∗ (*w* − *z*) ^ *n*) *sums f w*)
proof −
have *fh'*: *f holomorphic-on cball z* ((*r* + *dist w z*) / *2*)
apply (*rule holomorphic-on-subset* [*OF holf*])
apply (*clarsimp simp del*: *divide-const-simps*)
apply (*metis add.commute dist-commute le-less-trans mem-ball real-gt-half-sum w*)
done
— Replacing *r* and the original (weak) premises
obtain *r* **where** *0* < *r* **and** *holfc*: *f holomorphic-on cball z r* **and** *w*: *w* ∈ *ball z r*
apply (*rule that* [*of* (*r* + *dist w z*) / *2*])
apply (*simp-all add*: *fh'*)
apply (*metis add-0-iff ball-eq-empty dist-nz dist-self empty-iff not-less pos-add-strict w*)
apply (*metis add-less-cancel-right dist-commute mem-ball mult-2-right w*)
done
then have *holf*: *f holomorphic-on ball z r* **and** *contf*: *continuous-on* (*cball z r*) *f*
using *ball-subset-cball holomorphic-on-subset* **apply** *blast*
by (*simp add*: *holfc holomorphic-on-imp-continuous-on*)
have *cint*: ⋀*k*. (λ*u*. *f u* / (*u* − *z*) ^ *Suc k*) *contour-integrable-on circlepath z r*
apply (*rule Cauchy-higher-derivative-integral-circlepath* [*OF contf holf*])
apply (*simp add*: ‹*0* < *r*›)
done
obtain *B* **where** *0* < *B* **and** *B*: ⋀*u*. *u* ∈ *cball z r* ⟹ *norm*(*f u*) ≤ *B*
by (*metis* (*no-types*) *bounded-pos compact-cball compact-continuous-image compact-imp-bounded contf image-eqI*)
obtain *k* **where** *k*: *0* < *k k* ≤ *r* **and** *wz-eq*: *norm*(*w* − *z*) = *r* − *k*
and *kle*: ⋀*u*. *norm*(*u* − *z*) = *r* ⟹ *k* ≤ *norm*(*u* − *w*)
apply (*rule-tac k* = *r* − *dist z w* **in** *that*)
using *w*
apply (*auto simp*: *dist-norm norm-minus-commute*)
by (*metis add-diff-eq diff-add-cancel norm-diff-ineq norm-minus-commute*)
have ∗: ∀ *F n in sequentially*. ∀ *x*∈*path-image* (*circlepath z r*).
norm ((∑ *k*<*n*. (*w* − *z*) ^ *k* ∗ (*f x* / (*x* − *z*) ^ *Suc k*)) − *f x* / (*x* − *w*)) < *e*
if *0* < *e* **for** *e*
proof −
have *rr*: *0* ≤ (*r* − *k*) / *r* (*r* − *k*) / *r* < *1* **using** *k* **by** *auto*
obtain *n* **where** *n*: ((*r* − *k*) / *r*) ^ *n* < *e* / *B* ∗ *k*
using *real-arch-pow-inv* [*of e*/*B*∗*k* (*r* − *k*)/*r*] ‹*0* < *e*› ‹*0* < *B*› *k* **by** *force*
have *norm* ((∑ *k*<*N*. (*w* − *z*) ^ *k* ∗ *f u* / (*u* − *z*) ^ *Suc k*) − *f u* / (*u* − *w*)) < *e*
if *n* ≤ *N* **and** *r*: *r* = *dist z u* **for** *N u*

proof −
have N: $((r - k) / r) \hat{} N < e / B * k$
apply (*rule le-less-trans* [*OF power-decreasing n*])
using ‹$n \leq N$› k **by** *auto*
have u [*simp*]: $(u \neq z) \wedge (u \neq w)$
using ‹$0 < r$› r w **by** *auto*
have *wzu-not1*: $(w - z) / (u - z) \neq 1$
by (*metis* (*no-types*) *dist-norm divide-eq-1-iff less-irrefl mem-ball norm-minus-commute r w*)
have *norm* $((\sum k{<}N.\ (w - z) \hat{} k * f\ u / (u - z) \hat{} Suc\ k) * (u - w) - f\ u)$
$=$ *norm* $((\sum k{<}N.\ (((w - z) / (u - z)) \hat{} k)) * f\ u * (u - w) / (u - z) - f\ u)$
unfolding *setsum-left-distrib setsum-divide-distrib power-divide* **by** (*simp add*: *algebra-simps*)
also have ... $=$ *norm* $((((w - z) / (u - z)) \hat{} N - 1) * (u - w) / (((w - z) / (u - z) - 1) * (u - z)) - 1) * norm\ (f\ u)$
using ‹$0 < B$›
apply (*auto simp*: *geometric-sum* [*OF wzu-not1*])
apply (*simp add*: *field-simps norm-mult* [*symmetric*])
done
also have ... $=$ *norm* $((u{-}z) \hat{} N * (w - u) - ((w - z) \hat{} N - (u{-}z) \hat{} N) * (u{-}w)) / (r \hat{} N * norm\ (u{-}w)) * norm\ (f\ u)$
using ‹$0 < r$› r **by** (*simp add*: *divide-simps norm-mult norm-divide norm-power dist-norm norm-minus-commute*)
also have ... $=$ *norm* $((w - z) \hat{} N * (w - u)) / (r \hat{} N * norm\ (u - w)) * norm\ (f\ u)$
by (*simp add*: *algebra-simps*)
also have ... $=$ *norm* $(w - z) \hat{} N * norm\ (f\ u) / r \hat{} N$
by (*simp add*: *norm-mult norm-power norm-minus-commute*)
also have ... $\leq (((r - k)/r) \hat{} N) * B$
using ‹$0 < r$› w k
apply (*simp add*: *divide-simps*)
apply (*rule mult-mono* [*OF power-mono*])
apply (*auto simp*: *norm-divide wz-eq norm-power dist-norm norm-minus-commute B r*)
done
also have ... $< e * k$
using ‹$0 < B$› N **by** (*simp add*: *divide-simps*)
also have ... $\leq e * norm\ (u - w)$
using r *kle* ‹$0 < e$› **by** (*simp add*: *dist-commute dist-norm*)
finally show *?thesis*
by (*simp add*: *divide-simps norm-divide del*: *power-Suc*)
qed
with ‹$0 < r$› **show** *?thesis*
by (*auto simp*: *mult-ac less-imp-le eventually-sequentially Ball-def*)
qed
have *eq*: $\forall_F\ x$ *in sequentially*.
contour-integral (*circlepath z r*) $(\lambda u.\ \sum k{<}x.\ (w - z) \hat{} k * (f\ u / (u - z) \hat{} Suc\ k)) =$

($\sum k<x$. *contour-integral* (*circlepath z r*) (λu. *f u* / (*u* − *z*) ^ *Suc k*) ∗ (*w* − *z*) ^ *k*)
apply (*rule eventuallyI*)
apply (*subst contour-integral-setsum*, *simp*)
using *contour-integrable-lmul* [*OF cint*, *of* (*w* − *z*) ^ *a* **for** *a*] **apply** (*simp add*: *field-simps*)
apply (*simp only*: *contour-integral-lmul cint algebra-simps*)
done
have (λk. *contour-integral* (*circlepath z r*) (λu. *f u*/(*u* − *z*)^(*Suc k*)) ∗ (*w* − *z*)^*k*)
sums contour-integral (*circlepath z r*) (λu. *f u*/(*u* − *w*))
unfolding *sums-def*
apply (*rule Lim-transform-eventually* [*OF eq*])
apply (*rule contour-integral-uniform-limit-circlepath* [*OF eventuallyI* ∗])
apply (*rule contour-integrable-setsum*, *simp*)
apply (*rule contour-integrable-lmul*)
apply (*rule Cauchy-higher-derivative-integral-circlepath* [*OF contf holf*])
using ‹*0* < *r*›
apply *auto*
done
then have (λk. *contour-integral* (*circlepath z r*) (λu. *f u*/(*u* − *z*)^(*Suc k*)) ∗ (*w* − *z*)^*k*)
sums (*2* ∗ *of-real pi* ∗ *ii* ∗ *f w*)
using *w* **by** (*auto simp*: *dist-commute dist-norm contour-integral-unique* [*OF Cauchy-integral-circlepath-simple* [*OF holfc*]])
then have (λk. *contour-integral* (*circlepath z r*) (λu. *f u* / (*u* − *z*) ^ *Suc k*) ∗ (*w* − *z*)^*k* / (i ∗ (*of-real pi* ∗ *2*)))
sums ((*2* ∗ *of-real pi* ∗ *ii* ∗ *f w*) / (i ∗ (*complex-of-real pi* ∗ *2*)))
by (*rule sums-divide*)
then have (λn. (*w* − *z*) ^ *n* ∗ *contour-integral* (*circlepath z r*) (λu. *f u* / (*u* − *z*) ^ *Suc n*) / (i ∗ (*of-real pi* ∗ *2*)))
sums f w
by (*simp add*: *field-simps*)
then show *?thesis*
by (*simp add*: *field-simps* ‹*0* < *r*› *Cauchy-higher-derivative-integral-circlepath* [*OF contf holf*])
qed

53.40 The Liouville theorem and the Fundamental Theorem of Algebra.

These weak Liouville versions don't even need the derivative formula.

lemma *Liouville-weak-0*:
assumes *holf*: *f holomorphic-on UNIV* **and** *inf*: (*f* ⟶ *0*) *at-infinity*
shows *f z* = *0*
proof (*rule ccontr*)
assume *fz*: *f z* ≠ *0*
with *inf* [*unfolded Lim-at-infinity*, *rule-format*, *of norm*(*f z*)/*2*]
obtain *B* **where** *B*: $\bigwedge x$. *B* ≤ *cmod x* ⟹ *norm* (*f x*) ∗ *2* < *cmod* (*f z*)

```
    by (auto simp: dist-norm)
  def R ≡ 1 + |B| + norm z
  have R > 0 unfolding R-def
  proof −
    have 0 ≤ cmod z + |B|
      by (metis (full-types) add-nonneg-nonneg norm-ge-zero real-norm-def)
    then show 0 < 1 + |B| + cmod z
      by linarith
  qed
  have ∗: ((λu. f u / (u − z)) has-contour-integral 2 ∗ complex-of-real pi ∗ i ∗ f
z) (circlepath z R)
    apply (rule Cauchy-integral-circlepath)
   using ‹R > 0› apply (auto intro: holomorphic-on-subset [OF holf] holomorphic-on-imp-continuous-on)+
    done
  have cmod (x − z) = R ⟹ cmod (f x) ∗ 2 ≤ cmod (f z) for x
    apply (simp add: R-def)
    apply (rule less-imp-le)
    apply (rule B)
    using norm-triangle-ineq4 [of x z]
    apply (auto simp:)
    done
  with ‹R > 0› fz show False
    using has-contour-integral-bound-circlepath [OF ∗, of norm(f z)/2/R]
    by (auto simp: norm-mult norm-divide divide-simps)
qed

proposition Liouville-weak:
  assumes f holomorphic-on UNIV and (f ⟶ l) at-infinity
    shows f z = l
  using Liouville-weak-0 [of λz. f z − l]
  by (simp add: assms holomorphic-on-const holomorphic-on-diff LIM-zero)


proposition Liouville-weak-inverse:
  assumes f holomorphic-on UNIV and unbounded: ⋀B. eventually (λx. norm (f
x) ≥ B) at-infinity
    obtains z where f z = 0
proof −
  { assume f: ⋀z. f z ≠ 0
    have 1: (λx. 1 / f x) holomorphic-on UNIV
      by (simp add: holomorphic-on-divide holomorphic-on-const assms f)
    have 2: ((λx. 1 / f x) ⟶ 0) at-infinity
      apply (rule tendstoI [OF eventually-mono])
      apply (rule-tac B=2/e in unbounded)
      apply (simp add: dist-norm norm-divide divide-simps mult-ac)
      done
    have False
      using Liouville-weak-0 [OF 1 2] f by simp
  }
```

then show *?thesis*
using *that* **by** *blast*
qed

In particular we get the Fundamental Theorem of Algebra.

theorem *fundamental-theorem-of-algebra*:
fixes $a :: nat \Rightarrow complex$
assumes $a\ 0 = 0 \vee (\exists i \in \{1..n\}.\ a\ i \neq 0)$
obtains z **where** $(\sum i{\leq}n.\ a\ i * z\hat{}i) = 0$
using *assms*
proof (*elim disjE bexE*)
assume $a\ 0 = 0$ **then show** *?thesis*
by (*auto simp*: *that* [*of 0*])
next
fix i
assume i: $i \in \{1..n\}$ **and** nz: $a\ i \neq 0$
have *1*: $(\lambda z.\ \sum i{\leq}n.\ a\ i * z\hat{}i)$ *holomorphic-on UNIV*
by (*rule holomorphic-intros*)+
show *?thesis*
apply (*rule Liouville-weak-inverse* [*OF 1*])
apply (*rule polyfun-extremal*)
apply (*rule nz*)
using *i that*
apply (*auto simp*:)
done
qed

53.41 Weierstrass convergence theorem.

proposition *holomorphic-uniform-limit*:
assumes *cont*: *eventually* $(\lambda n.$ *continuous-on* $(cball\ z\ r)\ (f\ n) \wedge (f\ n)$ *holomorphic-on ball z r*$)$ *F*
and *lim*: $\bigwedge e.\ 0 < e \Longrightarrow$ *eventually* $(\lambda n.\ \forall x \in cball\ z\ r.\ norm(f\ n\ x - g\ x) < e)$ *F*
and *F*: $\sim$ *trivial-limit F*
obtains *continuous-on* $(cball\ z\ r)\ g$ *g holomorphic-on ball z r*
proof (*cases r 0*::*real rule*: *linorder-cases*)
case *less* **then show** *?thesis* **by** (*force simp add*: *ball-empty less-imp-le continuous-on-def holomorphic-on-def intro*: *that*)
next
case *equal* **then show** *?thesis*
by (*force simp add*: *holomorphic-on-def continuous-on-sing intro*: *that*)
next
case *greater*
have *contg*: *continuous-on* $(cball\ z\ r)\ g$
using *cont*
by (*fastforce simp*: *eventually-conj-iff dist-norm intro*: *eventually-mono* [*OF lim*] *continuous-uniform-limit* [*OF F*])
have *1*: *continuous-on* $(path\text{-}image\ (circlepath\ z\ r))\ (\lambda x.\ 1\ /\ (2 *$ *complex-of-real*

```
pi * i) * g x)
    apply (rule continuous-intros continuous-on-subset [OF contg])+
    using ‹0 < r› by auto
  have 2: ((λu. 1 / (2 * of-real pi * i) * g u / (u - w) ^ 1) has-contour-integral
g w) (circlepath z r)
       if w: w ∈ ball z r for w
  proof -
    def d ≡ (r - norm(w - z))
     have 0 < d  d ≤ r using w by (auto simp: norm-minus-commute d-def
dist-norm)
    have dle: ⋀u. cmod (z - u) = r ⟹ d ≤ cmod (u - w)
     unfolding d-def by (metis add-diff-eq diff-add-cancel norm-diff-ineq norm-minus-commute)
    have ev-int: ∀F n in F. (λu. f n u / (u - w)) contour-integrable-on circlepath
z r
      apply (rule eventually-mono [OF cont])
      using w
      apply (auto intro: Cauchy-higher-derivative-integral-circlepath [where k=0,
simplified])
      done
    have ev-less: ∀F n in F. ∀x∈path-image (circlepath z r). cmod (f n x / (x -
w) - g x / (x - w)) < e
        if e > 0 for e
      using greater ‹0 < d› ‹0 < e›
      apply (simp add: norm-divide diff-divide-distrib [symmetric] divide-simps)
      apply (rule-tac e1=e * d in eventually-mono [OF lim])
      apply (force simp: dist-norm intro: dle mult-left-mono less-le-trans)+
      done
    have g-cint: (λu. g u/(u - w)) contour-integrable-on circlepath z r
       by (rule contour-integral-uniform-limit-circlepath [OF ev-int ev-less F ‹0 <
r›])
   have cif-tends-cig: ((λn. contour-integral(circlepath z r) (λu. f n u / (u - w)))
⟶ contour-integral(circlepath z r) (λu. g u/(u - w))) F
      by (rule contour-integral-uniform-limit-circlepath [OF ev-int ev-less F ‹0 <
r›])
     have f-tends-cig: ((λn. 2 * of-real pi * ii * f n w) ⟶ contour-integral
(circlepath z r) (λu. g u / (u - w))) F
        apply (rule Lim-transform-eventually [where f = λn. contour-integral
(circlepath z r) (λu. f n u/(u - w))])
      apply (rule eventually-mono [OF cont])
      apply (rule contour-integral-unique [OF Cauchy-integral-circlepath])
      using w
      apply (auto simp: norm-minus-commute dist-norm cif-tends-cig)
      done
    have ((λn. 2 * of-real pi * i * f n w) ⟶ 2 * of-real pi * i * g w) F
      apply (rule tendsto-mult-left [OF tendstoI])
      apply (rule eventually-mono [OF lim], assumption)
      using w
      apply (force simp add: dist-norm)
      done
```

then have $((\lambda u.\ g\ u\ /\ (u - w))$ *has-contour-integral* $2 * \textit{of-real}\ \textit{pi} * \mathrm{i} * g\ w)$ $(\textit{circlepath}\ z\ r)$
using *has-contour-integral-integral* [*OF g-cint*] *tendsto-unique* [*OF F f-tends-cig*] w
by (*force simp add*: *dist-norm*)
then have $((\lambda u.\ g\ u\ /\ (2 * \textit{of-real}\ \textit{pi} * \mathrm{i} * (u - w)))$ *has-contour-integral* $g\ w)$ $(\textit{circlepath}\ z\ r)$
using *has-contour-integral-div* [**where** $c = 2 * \textit{of-real}\ \textit{pi} * \mathrm{i}$]
by (*force simp add*: *field-simps*)
then show *?thesis*
by (*simp add*: *dist-norm*)
qed
show *?thesis*
using *Cauchy-next-derivative-circlepath*(*2*) [*OF 1 2*, *simplified*]
by (*fastforce simp add*: *holomorphic-on-open contg intro*: *that*)
qed

Version showing that the limit is the limit of the derivatives.

proposition *has-complex-derivative-uniform-limit*:
fixes *z*::*complex*
assumes *cont*: *eventually* $(\lambda n.$ *continuous-on* $(\textit{cball}\ z\ r)\ (f\ n) \wedge$
$(\forall w \in \textit{ball}\ z\ r.\ ((f\ n)$ *has-field-derivative* $(f'\ n\ w))\ (\textit{at}\ w)))\ F$
and *lim*: $\bigwedge e.\ 0 < e \Longrightarrow$ *eventually* $(\lambda n.\ \forall x \in \textit{cball}\ z\ r.\ \textit{norm}(f\ n\ x - g\ x) < e)\ F$
and *F*: $\sim$ *trivial-limit F* **and** $0 < r$
obtains g' **where**
continuous-on $(\textit{cball}\ z\ r)\ g$
$\bigwedge w.\ w \in \textit{ball}\ z\ r \Longrightarrow (g$ *has-field-derivative* $(g'\ w))\ (\textit{at}\ w) \wedge ((\lambda n.\ f'\ n\ w) \longrightarrow g'\ w)\ F$
proof −
let *?conint* = *contour-integral* $(\textit{circlepath}\ z\ r)$
have *g*: *continuous-on* $(\textit{cball}\ z\ r)\ g$ *g holomorphic-on ball z r*
by (*rule holomorphic-uniform-limit* [*OF eventually-mono* [*OF cont*] *lim F*];
auto simp: *holomorphic-on-open field-differentiable-def*)+
then obtain g' **where** g': $\bigwedge x.\ x \in \textit{ball}\ z\ r \Longrightarrow (g$ *has-field-derivative* $g'\ x)\ (\textit{at}\ x)$
using *DERIV-deriv-iff-has-field-derivative*
by (*fastforce simp add*: *holomorphic-on-open*)
then have *derg*: $\bigwedge x.\ x \in \textit{ball}\ z\ r \Longrightarrow \textit{deriv}\ g\ x = g'\ x$
by (*simp add*: *DERIV-imp-deriv*)
have *tends-f'n-g'*: $((\lambda n.\ f'\ n\ w) \longrightarrow g'\ w)\ F$ **if** *w*: $w \in \textit{ball}\ z\ r$ **for** w
proof −
have *eq-f'*: *?conint* $(\lambda x.\ f\ n\ x\ /\ (x - w)^2) -$ *?conint* $(\lambda x.\ g\ x\ /\ (x - w)^2) = (f'\ n\ w - g'\ w) * (2 * \textit{of-real}\ \textit{pi} * \mathrm{i})$
if *cont-fn*: *continuous-on* $(\textit{cball}\ z\ r)\ (f\ n)$
and *fnd*: $\bigwedge w.\ w \in \textit{ball}\ z\ r \Longrightarrow (f\ n$ *has-field-derivative* $f'\ n\ w)\ (\textit{at}\ w)$
for n
proof −

have *hol-fn*: *f n holomorphic-on ball z r*
using *fnd* **by** (*force simp add*: *holomorphic-on-open*)
have (*f n has-field-derivative 1 / (2 * of-real pi * i) * ?conint* (λ*u*. *f n u / (u − w)*²)) (*at w*)
by (*rule Cauchy-derivative-integral-circlepath* [*OF cont-fn hol-fn w*])
then have *f'*: *f' n w = 1 / (2 * of-real pi * i) * ?conint* (λ*u*. *f n u / (u − w)*²)
using *DERIV-unique* [*OF fnd*] *w* **by** *blast*
show *?thesis*
by (*simp add*: *f' Cauchy-contour-integral-circlepath-2* [*OF g w*] *derg* [*OF w*] *divide-simps*)
qed
def *d* ≡ (*r − norm*(*w − z*)) ^2
have *d > 0*
using *w* **by** (*simp add*: *dist-commute dist-norm d-def*)
have *dle*: ⋀*y*. *r = cmod* (*z − y*) ⟹ *d* ≤ *cmod* ((*y − w*)²)
apply (*simp add*: *d-def norm-power*)
apply (*rule power-mono*)
apply (*metis add-diff-eq diff-add-cancel norm-diff-ineq norm-minus-commute*)
apply (*metis diff-ge-0-iff-ge dist-commute dist-norm less-eq-real-def mem-ball w*)
done
have *1*: ∀*F n in F*. (λ*x*. *f n x / (x − w)*²) *contour-integrable-on circlepath z r*
by (*force simp add*: *holomorphic-on-open intro*: *w Cauchy-derivative-integral-circlepath eventually-mono* [*OF cont*])
have *2*: *0 < e* ⟹ ∀*F n in F*. ∀*x* ∈ *path-image* (*circlepath z r*). *cmod* (*f n x / (x − w)*² − *g x / (x − w)*²) < *e* **for** *e*
using ‹*r > 0*›
apply (*simp add*: *diff-divide-distrib* [*symmetric*] *norm-divide divide-simps sphere-def*)
apply (*rule eventually-mono* [*OF lim*, *of e*d*])
apply (*simp add*: ‹*0 < d*›)
apply (*force simp add*: *dist-norm dle intro*: *less-le-trans*)
done
have ((λ*n*. *contour-integral* (*circlepath z r*) (λ*x*. *f n x / (x − w)*²)) ⟶ *contour-integral* (*circlepath z r*) ((λ*x*. *g x / (x − w)*²))) *F*
by (*rule Cauchy-Integral-Thm.contour-integral-uniform-limit-circlepath* [*OF 1 2 F* ‹*0 < r*›])
then have *tendsto-0*: ((λ*n*. *1 / (2 * of-real pi * i) * (?conint* (λ*x*. *f n x / (x − w)*²) − *?conint* (λ*x*. *g x / (x − w)*²))) ⟶ *0*) *F*
using *Lim-null* **by** (*force intro*!: *tendsto-mult-right-zero*)
have ((λ*n*. *f' n w − g' w*) ⟶ *0*) *F*
apply (*rule Lim-transform-eventually* [*OF - tendsto-0*])
apply (*force simp add*: *divide-simps intro*: *eq-f' eventually-mono* [*OF cont*])
done
then show *?thesis* **using** *Lim-null* **by** *blast*
qed
obtain *g'* **where** ⋀*w*. *w* ∈ *ball z r* ⟹ (*g has-field-derivative* (*g' w*)) (*at w*) ∧ ((λ*n*. *f' n w*) ⟶ *g' w*) *F*

by (*blast intro*: *tends-f'n-g' g'*)
then show *?thesis* **using** *g*
using *that* **by** *blast*
qed

53.42 Some more simple/convenient versions for applications.

lemma *holomorphic-uniform-sequence*:
assumes *s*: *open s*
and *hol-fn*: $\bigwedge n.$ (*f n*) *holomorphic-on s*
and *to-g*: $\bigwedge x.$ *x* ∈ *s*
$\Longrightarrow \exists d.$ *0* < *d* ∧ *cball x d* ⊆ *s* ∧
(∀ *e*. *0* < *e* ⟶ *eventually* (λ*n*. ∀ *y* ∈ *cball x d*. *norm*(*f n y* − *g y*) < *e*) *sequentially*)
shows *g holomorphic-on s*
proof −
have ∃*f'*. (*g has-field-derivative f'*) (*at z*) **if** *z* ∈ *s* **for** *z*
proof −
obtain *r* **where** *0* < *r* **and** *r*: *cball z r* ⊆ *s*
and *fg*: ∀ *e*. *0* < *e* ⟶ *eventually* (λ*n*. ∀ *y* ∈ *cball z r*. *norm*(*f n y* − *g y*) < *e*) *sequentially*
using *to-g* [*OF* ‹*z* ∈ *s*›] **by** *blast*
have ∗: ∀ *F n in sequentially*. *continuous-on* (*cball z r*) (*f n*) ∧ *f n holomorphic-on ball z r*
apply (*intro eventuallyI conjI*)
using *hol-fn holomorphic-on-imp-continuous-on holomorphic-on-subset r* **apply** *blast*
apply (*metis hol-fn holomorphic-on-subset interior-cball interior-subset r*)
done
show *?thesis*
apply (*rule holomorphic-uniform-limit* [*OF* ∗])
using ‹*0* < *r*› *centre-in-ball fg*
apply (*auto simp*: *holomorphic-on-open*)
done
qed
with *s* **show** *?thesis*
by (*simp add*: *holomorphic-on-open*)
qed

lemma *has-complex-derivative-uniform-sequence*:
fixes *s* :: *complex set*
assumes *s*: *open s*
and *hfd*: $\bigwedge n\ x.$ *x* ∈ *s* ⟹ ((*f n*) *has-field-derivative f' n x*) (*at x*)
and *to-g*: $\bigwedge x.$ *x* ∈ *s*
⟹ ∃ *d*. *0* < *d* ∧ *cball x d* ⊆ *s* ∧
(∀ *e*. *0* < *e* ⟶ *eventually* (λ*n*. ∀ *y* ∈ *cball x d*. *norm*(*f n y* − *g y*) < *e*) *sequentially*)
shows ∃ *g'*. ∀ *x* ∈ *s*. (*g has-field-derivative g' x*) (*at x*) ∧ ((λ*n*. *f' n x*) ⟶ *g' x*) *sequentially*

proof −
have *y*: ∃ *y*. (*g has-field-derivative y*) (*at z*) ∧ (λ*n*. *f* ′ *n z*) ⟶ *y* **if** *z* ∈ *s* **for** *z*
proof −
obtain *r* **where** *0* < *r* **and** *r*: *cball z r* ⊆ *s*
and *fg*: ∀ *e*. *0* < *e* ⟶ *eventually* (λ*n*. ∀ *y* ∈ *cball z r*. *norm*(*f n y* − *g y*) < *e*) *sequentially*
using *to-g* [*OF* ‹*z* ∈ *s*›] **by** *blast*
have ∗: ∀ *F n in sequentially*. *continuous-on* (*cball z r*) (*f n*) ∧
(∀ *w* ∈ *ball z r*. ((*f n*) *has-field-derivative* (*f* ′ *n w*)) (*at w*))
apply (*intro eventuallyI conjI*)
apply (*meson hfd holomorphic-on-imp-continuous-on holomorphic-on-open holomorphic-on-subset r s*)
using *ball-subset-cball hfd r* **apply** *blast*
done
show *?thesis*
apply (*rule has-complex-derivative-uniform-limit* [*OF* ∗, *of g*])
using ‹*0* < *r*› *centre-in-ball fg*
apply *force*+
done
qed
show *?thesis*
by (*rule bchoice*) (*blast intro*: *y*)
qed

53.43 On analytic functions defined by a series.

lemma *series-and-derivative-comparison*:
fixes *s* :: *complex set*
assumes *s*: *open s*
and *h*: *summable h*
and *hfd*: ⋀*n x*. *x* ∈ *s* ⟹ (*f n has-field-derivative f* ′ *n x*) (*at x*)
and *to-g*: ⋀*n x*. ⟦*N* ≤ *n*; *x* ∈ *s*⟧ ⟹ *norm*(*f n x*) ≤ *h n*
obtains *g g* ′ **where** ∀ *x* ∈ *s*. ((λ*n*. *f n x*) *sums g x*) ∧ ((λ*n*. *f* ′ *n x*) *sums g* ′ *x*) ∧ (*g has-field-derivative g* ′ *x*) (*at x*)
proof −
obtain *g* **where** *g*: ⋀*e*. *e*>*0* ⟹ ∃ *N*. ∀ *n x*. *N* ≤ *n* ∧ *x* ∈ *s* ⟶ *dist* (∑ *n*<*n*. *f n x*) (*g x*) < *e*
using *series-comparison-uniform* [*OF h to-g*, *of N s*] **by** *force*
have ∗: ∃ *d*>*0*. *cball x d* ⊆ *s* ∧ (∀ *e*>*0*. ∀ *F n in sequentially*. ∀ *y*∈*cball x d*. *cmod* ((∑ *a*<*n*. *f a y*) − *g y*) < *e*)
if *x* ∈ *s* **for** *x*
proof −
obtain *d* **where** *d*>*0* **and** *d*: *cball x d* ⊆ *s*
using *open-contains-cball* [*of s*] ‹*x* ∈ *s*› *s* **by** *blast*
then show *?thesis*
apply (*rule-tac x*=*d* **in** *exI*)
apply (*auto simp*: *dist-norm eventually-sequentially*)

apply (*metis g contra-subsetD dist-norm*)
done
qed
have ($\forall x{\in}s.\ (\lambda n.\ \sum i{<}n.\ f\ i\ x) \longrightarrow g\ x$)
using *g* **by** (*force simp add*: *lim-sequentially*)
moreover have $\exists g'.\ \forall x{\in}s.$ (*g has-field-derivative g' x*) (*at x*) $\wedge$ $(\lambda n.\ \sum i{<}n.\ f'\ i\ x) \longrightarrow g'\ x$
by (*rule has-complex-derivative-uniform-sequence* [*OF s*]) (*auto intro*: $*$ *hfd DERIV-setsum*)+
ultimately show *?thesis*
by (*force simp add*: *sums-def conj-commute intro*: *that*)
qed

A version where we only have local uniform/comparative convergence.

lemma *series-and-derivative-comparison-local*:
fixes *s* :: *complex set*
assumes *s*: *open s*
and *hfd*: $\bigwedge n\ x.\ x \in s \Longrightarrow$ (*f n has-field-derivative f' n x*) (*at x*)
and *to-g*: $\bigwedge x.\ x \in s \Longrightarrow$
$\exists d\ h\ N.\ 0 < d \wedge$ *summable h* $\wedge\ (\forall n\ y.\ N \le n \wedge y \in$ *ball x d* $\longrightarrow$ *norm*$(f\ n\ y) \le h\ n$)
shows $\exists g\ g'.\ \forall x \in s.$ $((\lambda n.\ f\ n\ x)$ *sums g x*$) \wedge ((\lambda n.\ f'\ n\ x)$ *sums g' x*$) \wedge$ (*g has-field-derivative g' x*) (*at x*)
proof −
have $\exists y.\ (\lambda n.\ f\ n\ z)$ *sums* $(\sum n.\ f\ n\ z) \wedge (\lambda n.\ f'\ n\ z)$ *sums y* $\wedge\ ((\lambda x.\ \sum n.\ f\ n\ x)$ *has-field-derivative y*) (*at z*)
if $z \in s$ **for** *z*
proof −
obtain *d h N* **where** $0 < d$ *summable h* **and** *le-h*: $\bigwedge n\ y.\ [\![N \le n;\ y \in$ *ball z d*$]\!] \Longrightarrow$ *norm*$(f\ n\ y) \le h\ n$
using *to-g* ‹$z \in s$› **by** *meson*
then obtain *r* **where** $r{>}0$ **and** *r*: *ball z r* $\subseteq$ *ball z d* $\cap$ *s* **using** ‹$z \in s$› *s*
by (*metis Int-iff open-ball centre-in-ball open-Int open-contains-ball-eq*)
have *1*: *open* (*ball z d* $\cap$ *s*)
by (*simp add*: *open-Int s*)
have *2*: $\bigwedge n\ x.\ x \in$ *ball z d* $\cap$ *s* $\Longrightarrow$ (*f n has-field-derivative f' n x*) (*at x*)
by (*auto simp*: *hfd*)
obtain *g g'* **where** *gg'*: $\forall x \in$ *ball z d* $\cap$ *s*. $((\lambda n.\ f\ n\ x)$ *sums g x*$) \wedge$
$((\lambda n.\ f'\ n\ x)$ *sums g' x*$) \wedge$ (*g has-field-derivative g' x*) (*at x*)
by (*auto intro*: *le-h series-and-derivative-comparison* [*OF 1* ‹*summable h*› *hfd*])
then have $(\lambda n.\ f'\ n\ z)$ *sums g' z*
by (*meson* ‹$0 < r$› *centre-in-ball contra-subsetD r*)
moreover have $(\lambda n.\ f\ n\ z)$ *sums* $(\sum n.\ f\ n\ z)$
by (*metis summable-comparison-test' summable-sums centre-in-ball* ‹$0 < d$› ‹*summable h*› *le-h*)
moreover have $((\lambda x.\ \sum n.\ f\ n\ x)$ *has-field-derivative g' z*) (*at z*)
apply (*rule-tac f=g* **in** *DERIV-transform-at* [*OF -* ‹$0 < r$›])

apply (*simp add*: *gg'* ‹*z* ∈ *s*› ‹0 < *d*›)
apply (*metis* (*full-types*) *contra-subsetD dist-commute gg' mem-ball r sums-unique*)
done
ultimately show *?thesis* **by** *auto*
qed
then show *?thesis*
by (*rule-tac x*=λ*x*. *suminf* (λ*n*. *f n x*) **in** *exI*) *meson*
qed

Sometimes convenient to compare with a complex series of positive reals. (?)

lemma *series-and-derivative-comparison-complex*:
fixes *s* :: *complex set*
assumes *s*: *open s*
and *hfd*: ⋀*n x*. *x* ∈ *s* ⟹ (*f n has-field-derivative f' n x*) (*at x*)
and *to-g*: ⋀*x*. *x* ∈ *s* ⟹
∃ *d h N*. 0 < *d* ∧ *summable h* ∧ *range h* ⊆ *nonneg-Reals* ∧ (∀ *n y*. *N* ≤ *n* ∧ *y* ∈ *ball x d* ⟶ *cmod*(*f n y*) ≤ *cmod* (*h n*))
shows ∃ *g g'*. ∀ *x* ∈ *s*. ((λ*n*. *f n x*) *sums g x*) ∧ ((λ*n*. *f' n x*) *sums g' x*) ∧ (*g has-field-derivative g' x*) (*at x*)
apply (*rule series-and-derivative-comparison-local* [*OF s hfd*], *assumption*)
apply (*frule to-g*)
apply (*erule ex-forward*)
apply (*erule exE*)
apply (*rule-tac x*=*Re o h* **in** *exI*)
apply (*erule ex-forward*)
apply (*simp add*: *summable-Re o-def*)
apply (*elim conjE all-forward*)
apply (*simp add*: *nonneg-Reals-cmod-eq-Re image-subset-iff*)
done

In particular, a power series is analytic inside circle of convergence.

lemma *power-series-and-derivative-0*:
fixes *a* :: *nat* ⇒ *complex* **and** *r*::*real*
assumes *summable* (λ*n*. *a n* ∗ *r*^*n*)
shows ∃ *g g'*. ∀ *z*. *cmod z* < *r* ⟶
((λ*n*. *a n* ∗ *z*^*n*) *sums g z*) ∧ ((λ*n*. *of-nat n* ∗ *a n* ∗ *z*^(*n* − 1)) *sums g' z*) ∧ (*g has-field-derivative g' z*) (*at z*)
proof (*cases* 0 < *r*)
case *True*
have *der*: ⋀*n z*. ((λ*x*. *a n* ∗ *x* ^ *n*) *has-field-derivative of-nat n* ∗ *a n* ∗ *z* ^ (*n* − 1)) (*at z*)
by (*rule derivative-eq-intros* | *simp*)+
have *y-le*: ⟦*cmod* (*z* − *y*) ∗ 2 < *r* − *cmod z*⟧ ⟹ *cmod y* ≤ *cmod* (*of-real r* + *of-real* (*cmod z*)) / 2 **for** *z y*
using ‹*r* > 0›
apply (*auto simp*: *algebra-simps norm-mult norm-divide norm-power of-real-add* [*symmetric*] *simp del*: *of-real-add*)
using *norm-triangle-ineq2* [*of y z*]

apply (*simp only*: *diff-le-eq norm-minus-commute mult-2*)
done
have *summable* (λ*n*. *a n* ∗ *complex-of-real r* ^ *n*)
using *assms* ‹*r* > *0*› **by** *simp*
moreover have ⋀*z*. *cmod z* < *r* ⟹ *cmod* ((*of-real r* + *of-real* (*cmod z*)) / *2*) < *cmod* (*of-real r*)
using ‹*r* > *0*›
by (*simp add*: *of-real-add* [*symmetric*] *del*: *of-real-add*)
ultimately have *sum*: ⋀*z*. *cmod z* < *r* ⟹ *summable* (λ*n*. *of-real* (*cmod* (*a n*)) ∗ ((*of-real r* + *complex-of-real* (*cmod z*)) / *2*) ^ *n*)
by (*rule power-series-conv-imp-absconv-weak*)
have ∃ *g g'*. ∀ *z* ∈ *ball 0 r*. (λ*n*. (*a n*) ∗ *z* ^ *n*) *sums g z* ∧
(λ*n*. *of-nat n* ∗ (*a n*) ∗ *z* ^ (*n* − *1*)) *sums g' z* ∧ (*g has-field-derivative g' z*) (*at z*)
apply (*rule series-and-derivative-comparison-complex* [*OF open-ball der*])
apply (*rule-tac x*=(*r* − *norm z*)/*2* **in** *exI*)
apply (*simp add*: *dist-norm*)
apply (*rule-tac x*=λ*n*. *of-real*(*norm*(*a n*)∗((*r* + *norm z*)/*2*) ^*n*) **in** *exI*)
using ‹*r* > *0*›
apply (*auto simp*: *sum nonneg-Reals-divide-I*)
apply (*rule-tac x*=*0* **in** *exI*)
apply (*force simp*: *norm-mult norm-divide norm-power intro*!: *mult-left-mono power-mono y-le*)
done
then show *?thesis*
by (*simp add*: *dist-0-norm ball-def*)
next
case *False* **then show** *?thesis*
apply (*simp add*: *not-less*)
using *less-le-trans norm-not-less-zero* **by** *blast*
qed

proposition *power-series-and-derivative*:
fixes *a* :: *nat* ⇒ *complex* **and** *r*::*real*
assumes *summable* (λ*n*. *a n* ∗ *r* ^ *n*)
obtains *g g'* **where** ∀ *z* ∈ *ball w r*.
((λ*n*. *a n* ∗ (*z* − *w*) ^ *n*) *sums g z*) ∧ ((λ*n*. *of-nat n* ∗ *a n* ∗ (*z* − *w*) ^ (*n* − *1*)) *sums g' z*) ∧
(*g has-field-derivative g' z*) (*at z*)
using *power-series-and-derivative-0* [*OF assms*]
apply *clarify*
apply (*rule-tac g*=(λ*z*. *g*(*z* − *w*)) **in** *that*)
using *DERIV-shift* [**where** *z*=−*w*]
apply (*auto simp*: *norm-minus-commute Ball-def dist-norm*)
done

proposition *power-series-holomorphic*:
assumes ⋀*w*. *w* ∈ *ball z r* ⟹ ((λ*n*. *a n*∗(*w* − *z*) ^*n*) *sums f w*)
shows *f holomorphic-on ball z r*

proof −
have $\exists f'.$ (*f has-field-derivative f'*) (*at w*) **if** *w*: *dist z w* < *r* **for** *w*
proof −
have *inb*: *z* + *complex-of-real* ((*dist z w* + *r*) / *2*) ∈ *ball z r*
proof −
have *wz*: *cmod* (*w* − *z*) < *r* **using** *w*
by (*auto simp*: *divide-simps dist-norm norm-minus-commute*)
then have *0* ≤ *r*
by (*meson less-eq-real-def norm-ge-zero order-trans*)
show *?thesis*
using *w* **by** (*simp add*: *dist-norm* ‹*0*≤*r*› *of-real-add* [*symmetric*] *del*: *of-real-add*)
qed
have *sum*: *summable* (λ*n*. *a n* ∗ *of-real* (((*cmod* (*z* − *w*) + *r*) / *2*) ^ *n*))
using *assms* [*OF inb*] **by** (*force simp add*: *summable-def dist-norm*)
obtain *g g'* **where** *gg'*: ⋀*u*. *u* ∈ *ball z* ((*cmod* (*z* − *w*) + *r*) / *2*) ⟹
(λ*n*. *a n* ∗ (*u* − *z*) ^ *n*) *sums g u* ∧
(λ*n*. *of-nat n* ∗ *a n* ∗ (*u* − *z*) ^ (*n* − *1*)) *sums g' u* ∧ (*g has-field-derivative g' u*) (*at u*)
by (*rule power-series-and-derivative* [*OF sum*, *of z*]) *fastforce*
have [*simp*]: *g u* = *f u* **if** *cmod* (*u* − *w*) < (*r* − *cmod* (*z* − *w*)) / *2* **for** *u*
proof −
have *less*: *cmod* (*z* − *u*) ∗ *2* < *cmod* (*z* − *w*) + *r*
using *that dist-triangle2* [*of z u w*]
by (*simp add*: *dist-norm* [*symmetric*] *algebra-simps*)
show *?thesis*
apply (*rule sums-unique2* [*of* λ*n*. *a n*∗(*u* − *z*) ^*n*])
using *gg'* [*of u*] *less w*
apply (*auto simp*: *assms dist-norm*)
done
qed
show *?thesis*
apply (*rule-tac x*=*g' w* **in** *exI*)
apply (*rule DERIV-transform-at* [**where** *f*=*g* **and** *d*=(*r* − *norm*(*z* − *w*))/*2*])
using *w gg'* [*of w*]
apply (*auto simp*: *dist-norm*)
done
qed
then show *?thesis* **by** (*simp add*: *holomorphic-on-open*)
qed

corollary *holomorphic-iff-power-series*:
f holomorphic-on ball z r ⟷
(∀ *w* ∈ *ball z r*. (λ*n*. (*deriv* ^^ *n*) *f z* / (*fact n*) ∗ (*w* − *z*) ^*n*) *sums f w*)
apply (*intro iffI ballI*)
using *holomorphic-power-series* **apply** *force*
apply (*rule power-series-holomorphic* [**where** *a* = λ*n*. (*deriv* ^^ *n*) *f z* / (*fact n*)])
apply *force*

done

corollary *power-series-analytic*:
(⋀*w*. *w* ∈ *ball z r* ⟹ (λ*n*. *a n*∗(*w* − *z*) ^ *n*) *sums f w*) ⟹ *f analytic-on ball z r*
by (*force simp add*: *analytic-on-open intro*!: *power-series-holomorphic*)

corollary *analytic-iff-power-series*:
f analytic-on ball z r ⟷
(∀ *w* ∈ *ball z r*. (λ*n*. (*deriv* ^^ *n*) *f z* / (*fact n*) ∗ (*w* − *z*) ^ *n*) *sums f w*)
by (*simp add*: *analytic-on-open holomorphic-iff-power-series*)

53.44 Equality between holomorphic functions, on open ball then connected set.

lemma *holomorphic-fun-eq-on-ball*:
⟦*f holomorphic-on ball z r*; *g holomorphic-on ball z r*;
w ∈ *ball z r*;
⋀*n*. (*deriv* ^^ *n*) *f z* = (*deriv* ^^ *n*) *g z*⟧
⟹ *f w* = *g w*
apply (*rule sums-unique2* [*of* λ*n*. (*deriv* ^^ *n*) *f z* / (*fact n*) ∗ (*w* − *z*) ^ *n*])
apply (*auto simp*: *holomorphic-iff-power-series*)
done

lemma *holomorphic-fun-eq-0-on-ball*:
⟦*f holomorphic-on ball z r*; *w* ∈ *ball z r*;
⋀*n*. (*deriv* ^^ *n*) *f z* = 0⟧
⟹ *f w* = 0
apply (*rule sums-unique2* [*of* λ*n*. (*deriv* ^^ *n*) *f z* / (*fact n*) ∗ (*w* − *z*) ^ *n*])
apply (*auto simp*: *holomorphic-iff-power-series*)
done

lemma *holomorphic-fun-eq-0-on-connected*:
assumes *holf*: *f holomorphic-on s* **and** *open s*
and *cons*: *connected s*
and *der*: ⋀*n*. (*deriv* ^^ *n*) *f z* = 0
and *z* ∈ *s w* ∈ *s*
shows *f w* = 0
proof −
have ∗: ⋀*x e*. ⟦ ∀ *xa*. (*deriv* ^^ *xa*) *f x* = 0; *ball x e* ⊆ *s*⟧
⟹ *ball x e* ⊆ (⋂ *n*. {*w* ∈ *s*. (*deriv* ^^ *n*) *f w* = 0})
apply *auto*
apply (*rule holomorphic-fun-eq-0-on-ball* [*OF holomorphic-higher-deriv*])
apply (*rule holomorphic-on-subset* [*OF holf*], *simp-all*)
by (*metis funpow-add o-apply*)
have *1*: *openin* (*subtopology euclidean s*) (⋂ *n*. {*w* ∈ *s*. (*deriv* ^^ *n*) *f w* = 0})
apply (*rule open-subset*, *force*)
using ‹*open s*›
apply (*simp add*: *open-contains-ball Ball-def*)

apply (*erule all-forward*)
using ∗ **by** *auto blast+*
have *2*: *closedin* (*subtopology euclidean s*) (⋂ *n*. {*w* ∈ *s*. (*deriv* ^^ *n*) *f w* = *0*})
using *assms*
by (*auto intro*: *continuous-closedin-preimage-constant holomorphic-on-imp-continuous-on holomorphic-higher-deriv*)
obtain *e* **where** *e>0* **and** *e*: *ball w e* ⊆ *s* **using** *openE* [*OF* ‹*open s*› ‹*w* ∈ *s*›] **.**
then have *holfb*: *f holomorphic-on ball w e*
using *holf holomorphic-on-subset* **by** *blast*
have *3*: (⋂ *n*. {*w* ∈ *s*. (*deriv* ^^ *n*) *f w* = *0*}) = *s* ⟹ *f w* = *0*
using ‹*e>0*› *e* **by** (*force intro*: *holomorphic-fun-eq-0-on-ball* [*OF holfb*])
show *?thesis*
using *cons der* ‹*z* ∈ *s*›
apply (*simp add*: *connected-clopen*)
apply (*drule-tac x*=⋂ *n*. {*w* ∈ *s*. (*deriv* ^^ *n*) *f w* = *0*} **in** *spec*)
apply (*auto simp*: *1 2 3*)
done
qed

lemma *holomorphic-fun-eq-on-connected*:
assumes *f holomorphic-on s g holomorphic-on s* **and** *open s connected s*
and ⋀*n*. (*deriv* ^^ *n*) *f z* = (*deriv* ^^ *n*) *g z*
and *z* ∈ *s w* ∈ *s*
shows *f w* = *g w*
apply (*rule holomorphic-fun-eq-0-on-connected* [*of* λ*x*. *f x* − *g x s z*, *simplified*])
apply (*intro assms holomorphic-intros*)
using *assms* **apply** *simp-all*
apply (*subst higher-deriv-diff*, *auto*)
done

lemma *holomorphic-fun-eq-const-on-connected*:
assumes *holf*: *f holomorphic-on s* **and** *open s*
and *cons*: *connected s*
and *der*: ⋀*n*. *0* < *n* ⟹ (*deriv* ^^ *n*) *f z* = *0*
and *z* ∈ *s w* ∈ *s*
shows *f w* = *f z*
apply (*rule holomorphic-fun-eq-0-on-connected* [*of* λ*w*. *f w* − *f z s z*, *simplified*])
apply (*intro assms holomorphic-intros*)
using *assms* **apply** *simp-all*
apply (*subst higher-deriv-diff*)
apply (*intro holomorphic-intros* | *simp*)+
done

53.45 Some basic lemmas about poles/singularities.

lemma *pole-lemma*:
assumes *holf*: *f holomorphic-on s* **and** *a*: *a* ∈ *interior s*
shows (λ*z*. *if z* = *a then deriv f a*
else (*f z* − *f a*) / (*z* − *a*)) *holomorphic-on s* (**is** *?F holomorphic-on*

```
s)
proof −
  have F1: ?F field-differentiable (at u within s) if u ∈ s u ≠ a for u
  proof −
    have fcd: f field-differentiable at u within s
      using holf holomorphic-on-def by (simp add: ‹u ∈ s›)
    have cd: (λz. (f z − f a) / (z − a)) field-differentiable at u within s
      by (rule fcd derivative-intros | simp add: that)+
    have 0 < dist a u using that dist-nz by blast
    then show ?thesis
      by (rule field-differentiable-transform-within [OF - - - cd]) (auto simp: ‹u ∈
s›)
  qed
  have F2: ?F field-differentiable at a if 0 < e ball a e ⊆ s for e
  proof −
    have holfb: f holomorphic-on ball a e
      by (rule holomorphic-on-subset [OF holf ‹ball a e ⊆ s›])
    have 2: ?F holomorphic-on ball a e − {a}
      apply (rule holomorphic-on-subset [where s = s − {a}])
      apply (simp add: holomorphic-on-def field-differentiable-def [symmetric])
      using mem-ball that
      apply (auto intro: F1 field-differentiable-within-subset)
      done
    have isCont (λz. if z = a then deriv f a else (f z − f a) / (z − a)) x
          if dist a x < e for x
    proof (cases x=a)
      case True then show ?thesis
      using holfb ‹0 < e›
      apply (simp add: holomorphic-on-open field-differentiable-def [symmetric])
      apply (drule-tac x=a in bspec)
    apply (auto simp: DERIV-deriv-iff-field-differentiable [symmetric] continuous-at
DERIV-iff2
                elim: rev-iffD1 [OF - LIM-equal])
      done
    next
      case False with 2 that show ?thesis
     by (force simp: holomorphic-on-open open-Diff field-differentiable-def [symmetric]
field-differentiable-imp-continuous-at)
    qed
    then have 1: continuous-on (ball a e) ?F
      by (clarsimp simp:  continuous-on-eq-continuous-at)
    have ?F holomorphic-on ball a e
      by (auto intro: no-isolated-singularity [OF 1 2])
    with that show ?thesis
      by (simp add: holomorphic-on-open field-differentiable-def [symmetric]
                    field-differentiable-at-within)
  qed
  show ?thesis
  proof
```

```
    fix x assume x ∈ s show ?F field-differentiable at x within s
    proof (cases x=a)
      case True then show ?thesis
      using a by (auto simp: mem-interior intro: field-differentiable-at-within F2)
    next
      case False with F1 ‹x ∈ s›
      show ?thesis by blast
    qed
  qed
qed

proposition pole-theorem:
  assumes holg: g holomorphic-on s and a: a ∈ interior s
      and eq: ⋀z. z ∈ s − {a} ⟹ g z = (z − a) ∗ f z
    shows (λz. if z = a then deriv g a
                else f z − g a/(z − a)) holomorphic-on s
  using pole-lemma [OF holg a]
  by (rule holomorphic-transform) (simp add: eq divide-simps)

lemma pole-lemma-open:
  assumes f holomorphic-on s open s
    shows (λz. if z = a then deriv f a else (f z − f a)/(z − a)) holomorphic-on s
proof (cases a ∈ s)
  case True with assms interior-eq pole-lemma
    show ?thesis by fastforce
next
  case False with assms show ?thesis
    apply (simp add: holomorphic-on-def field-differentiable-def [symmetric], clar-
ify)
    apply (rule field-differentiable-transform-within [where f = λz. (f z − f a)/(z
− a) and d = 1])
    apply (rule derivative-intros | force)+
    done
qed

proposition pole-theorem-open:
  assumes holg: g holomorphic-on s and s: open s
      and eq: ⋀z. z ∈ s − {a} ⟹ g z = (z − a) ∗ f z
    shows (λz. if z = a then deriv g a
                else f z − g a/(z − a)) holomorphic-on s
  using pole-lemma-open [OF holg s]
  by (rule holomorphic-transform) (auto simp: eq divide-simps)

proposition pole-theorem-0:
  assumes holg: g holomorphic-on s and a: a ∈ interior s
      and eq: ⋀z. z ∈ s − {a} ⟹ g z = (z − a) ∗ f z
      and [simp]: f a = deriv g a g a = 0
    shows f holomorphic-on s
  using pole-theorem [OF holg a eq]
```

by (*rule holomorphic-transform*) (*auto simp*: *eq divide-simps*)

proposition *pole-theorem-open-0*:
assumes *holg*: *g holomorphic-on s* **and** *s*: *open s*
and *eq*: $\bigwedge z.\ z \in s - \{a\} \Longrightarrow g\ z = (z - a) * f\ z$
and [*simp*]: *f a* = *deriv g a g a* = *0*
shows *f holomorphic-on s*
using *pole-theorem-open* [*OF holg s eq*]
by (*rule holomorphic-transform*) (*auto simp*: *eq divide-simps*)

lemma *pole-theorem-analytic*:
assumes *g*: *g analytic-on s*
and *eq*: $\bigwedge z.\ z \in s$
$\Longrightarrow \exists d.\ 0 < d \wedge (\forall w \in ball\ z\ d - \{a\}.\ g\ w = (w - a) * f\ w)$
shows ($\lambda z.$ *if z* = *a then deriv g a*
else f z − *g a*/(*z* − *a*)) *analytic-on s*
using *g*
apply (*simp add*: *analytic-on-def Ball-def*)
apply (*safe elim*!: *all-forward dest*!: *eq*)
apply (*rule-tac x*=*min d e* **in** *exI*, *simp*)
apply (*rule pole-theorem-open*)
apply (*auto simp*: *holomorphic-on-subset subset-ball*)
done

lemma *pole-theorem-analytic-0*:
assumes *g*: *g analytic-on s*
and *eq*: $\bigwedge z.\ z \in s \Longrightarrow \exists d.\ 0 < d \wedge (\forall w \in ball\ z\ d - \{a\}.\ g\ w = (w - a) * f\ w)$
and [*simp*]: *f a* = *deriv g a g a* = *0*
shows *f analytic-on s*
proof −
have [*simp*]: ($\lambda z.$ *if z* = *a then deriv g a else f z* − *g a* / (*z* − *a*)) = *f*
by *auto*
show *?thesis*
using *pole-theorem-analytic* [*OF g eq*] **by** *simp*
qed

lemma *pole-theorem-analytic-open-superset*:
assumes *g*: *g analytic-on s* **and** *s* ⊆ *t open t*
and *eq*: $\bigwedge z.\ z \in t - \{a\} \Longrightarrow g\ z = (z - a) * f\ z$
shows ($\lambda z.$ *if z* = *a then deriv g a*
else f z − *g a*/(*z* − *a*)) *analytic-on s*
apply (*rule pole-theorem-analytic* [*OF g*])
apply (*rule openE* [*OF* ‹*open t*›])
using *assms eq* **by** *auto*

lemma *pole-theorem-analytic-open-superset-0*:
assumes *g*: *g analytic-on s s* ⊆ *t open t* $\bigwedge z.\ z \in t - \{a\} \Longrightarrow g\ z = (z - a) * f\ z$

and [*simp*]: *f a = deriv g a g a = 0*
shows *f analytic-on s*
proof −
have [*simp*]: (λ*z*. *if z = a then deriv g a else f z − g a / (z − a)*) = *f*
by *auto*
have (λ*z*. *if z = a then deriv g a else f z − g a/(z − a)*) *analytic-on s*
by (*rule pole-theorem-analytic-open-superset* [*OF g*])
then show *?thesis* **by** *simp*
qed

53.46 General, homology form of Cauchy's theorem.

Proof is based on Dixon's, as presented in Lang's "Complex Analysis" book (page 147).

lemma *contour-integral-continuous-on-linepath-2D*:
assumes *open u* **and** *cont-dw*: ⋀*w*. *w* ∈ *u* ⟹ *F w contour-integrable-on* (*linepath a b*)
and *cond-uu*: *continuous-on* (*u* × *u*) (λ(*x*,*y*). *F x y*)
and *abu*: *closed-segment a b* ⊆ *u*
shows *continuous-on u* (λ*w*. *contour-integral* (*linepath a b*) (*F w*))
proof −
have ∗: ∃ *d*>*0*. ∀ *x′*∈*u*. *dist x′ w* < *d* ⟶
dist (*contour-integral* (*linepath a b*) (*F x′*))
(*contour-integral* (*linepath a b*) (*F w*)) ≤ ε
if *w* ∈ *u 0* < ε *a* ≠ *b* **for** *w* ε
proof −
obtain δ **where** δ>*0* **and** δ: *cball w* δ ⊆ *u* **using** *open-contains-cball* ‹*open u*› ‹*w* ∈ *u*› **by** *force*
let *?TZ* = {(*t*,*z*) |*t z*. *t* ∈ *cball w* δ ∧ *z* ∈ *closed-segment a b*}
have *uniformly-continuous-on ?TZ* (λ(*x*,*y*). *F x y*)
apply (*rule compact-uniformly-continuous*)
apply (*rule continuous-on-subset*[*OF cond-uu*])
using *abu* δ
apply (*auto simp*: *compact-Times simp del*: *mem-cball*)
done
then obtain η **where** η>*0*
and η: ⋀*x x′*. ⟦*x*∈*?TZ*; *x′*∈*?TZ*; *dist x′ x* < η⟧ ⟹
dist ((λ(*x*,*y*). *F x y*) *x′*) ((λ(*x*,*y*). *F x y*) *x*) < ε/*norm*(*b* − *a*)
apply (*rule uniformly-continuous-onE* [**where** *e* = ε/*norm*(*b* − *a*)])
using ‹*0* < ε› ‹*a* ≠ *b*› **by** *auto*
have η: ⟦*norm* (*w* − *x1*) ≤ δ; *x2* ∈ *closed-segment a b*;
norm (*w* − *x1′*) ≤ δ; *x2′* ∈ *closed-segment a b*; *norm* ((*x1′*, *x2′*) − (*x1*, *x2*)) < η⟧
⟹ *norm* (*F x1′ x2′* − *F x1 x2*) ≤ ε / *cmod* (*b* − *a*)
for *x1 x2 x1′ x2′*
using η [*of* (*x1*,*x2*) (*x1′*,*x2′*)] **by** (*force simp add*: *dist-norm*)
have *le-ee*: *cmod* (*contour-integral* (*linepath a b*) (λ*x*. *F x′ x* − *F w x*)) ≤ ε
if *x′* ∈ *u cmod* (*x′* − *w*) < δ *cmod* (*x′* − *w*) < η **for** *x′*
proof −

have *cmod* (*contour-integral* (*linepath a b*) ($\lambda x.$ *F x' x − F w x*)) ≤ *ε*/*norm*(*b − a*) ∗ *norm*(*b − a*)
apply (*rule has-contour-integral-bound-linepath* [*OF has-contour-integral-integral - η*])
apply (*rule contour-integrable-diff* [*OF cont-dw cont-dw*])
using ‹*0 < ε*› ‹*a ≠ b*› ‹*0 < δ*› ‹*w ∈ u*› *that*
apply (*auto simp*: *norm-minus-commute*)
done
also have ... = *ε* **using** ‹*a ≠ b*› **by** *simp*
finally show *?thesis* **.**
qed
show *?thesis*
apply (*rule-tac x=min δ η* **in** *exI*)
using ‹*0 < δ*› ‹*0 < η*›
apply (*auto simp*: *dist-norm contour-integral-diff* [*OF cont-dw cont-dw*, *symmetric*] ‹*w ∈ u*› *intro*: *le-ee*)
done
qed
show *?thesis*
apply (*rule continuous-onI*)
apply (*cases a=b*)
apply (*auto intro*: ∗)
done
qed

This version has *polynomial-function* γ as an additional assumption.

lemma *Cauchy-integral-formula-global-weak*:
assumes *u*: *open u* **and** *holf*: *f holomorphic-on u*
and *z*: *z ∈ u* **and** γ: *polynomial-function* γ
and *pasz*: *path-image* γ ⊆ *u − {z}* **and** *loop*: *pathfinish* γ = *pathstart* γ
and *zero*: ⋀*w*. *w ∉ u* ⟹ *winding-number* γ *w* = *0*
shows ((*λw*. *f w* / (*w − z*)) *has-contour-integral* (*2∗pi ∗ ii ∗ winding-number* γ *z ∗ f z*)) γ
proof −
obtain γ' **where** *pfγ'*: *polynomial-function* γ' **and** γ': ⋀*x*. (γ *has-vector-derivative* (γ' *x*)) (*at x*)
using *has-vector-derivative-polynomial-function* [*OF* γ] **by** *blast*
then have *bounded*(*path-image* γ')
by (*simp add*: *path-image-def compact-imp-bounded compact-continuous-image continuous-on-polymonial-function*)
then obtain *B* **where** *B>0* **and** *B*: ⋀*x*. *x ∈ path-image* γ' ⟹ *norm x* ≤ *B*
using *bounded-pos* **by** *force*
def *d* ≡ *λz w*. *if w = z then deriv f z else* (*f w − f z*)/(*w − z*)
def *v* ≡ {*w*. *w ∉ path-image* γ ∧ *winding-number* γ *w* = *0*}
have *path* γ *valid-path* γ **using** γ
by (*auto simp*: *path-polynomial-function valid-path-polynomial-function*)
then have *ov*: *open v*
by (*simp add*: *v-def open-winding-number-levelsets loop*)
have *uv-Un*: *u* ∪ *v* = *UNIV*

```
    using pasz zero by (auto simp: v-def)
  have conf: continuous-on u f
    by (metis holf holomorphic-on-imp-continuous-on)
  have hol-d: (d y) holomorphic-on u if y ∈ u for y
  proof −
    have ∗: (λc. if c = y then deriv f y else (f c − f y) / (c − y)) holomorphic-on
u
      by (simp add: holf pole-lemma-open u)
    then have isCont (λx. if x = y then deriv f y else (f x − f y) / (x − y)) y
     using at-within-open field-differentiable-imp-continuous-at holomorphic-on-def
that u by fastforce
    then have continuous-on u (d y)
      apply (simp add: d-def continuous-on-eq-continuous-at u, clarify)
      using ∗ holomorphic-on-def
     by (meson field-differentiable-within-open field-differentiable-imp-continuous-at
u)
    moreover have d y holomorphic-on u − {y}
     apply (simp add: d-def holomorphic-on-open u open-delete field-differentiable-def
[symmetric])
      apply (intro ballI)
      apply (rename-tac w)
     apply (rule-tac d=dist w y and f = λw. (f w − f y)/(w − y) in field-differentiable-transform-within)
      apply (auto simp: dist-pos-lt dist-commute intro!: derivative-intros)
      using analytic-on-imp-differentiable-at analytic-on-open holf u apply blast
      done
    ultimately show ?thesis
      by (rule no-isolated-singularity) (auto simp: u)
  qed
   have cint-fxy: (λx. (f x − f y) / (x − y)) contour-integrable-on γ if y ∉
path-image γ for y
    apply (rule contour-integrable-holomorphic-simple [where s = u−{y}])
    using ‹valid-path γ› pasz
    apply (auto simp: u open-delete)
     apply (rule continuous-intros holomorphic-intros continuous-on-subset [OF
conf] holomorphic-on-subset [OF holf] |
                force simp add: that)+
    done
  def h ≡ λz. if z ∈ u then contour-integral γ (d z) else contour-integral γ (λw. f
w/(w − z))
  have U: ⋀z. z ∈ u ⟹ ((d z) has-contour-integral h z) γ
    apply (simp add: h-def)
   apply (rule has-contour-integral-integral [OF contour-integrable-holomorphic-simple
[where s=u]])
    using u pasz ‹valid-path γ›
    apply (auto intro: holomorphic-on-imp-continuous-on hol-d)
    done
  have V: ((λw. f w / (w − z)) has-contour-integral h z) γ if z: z ∈ v for z
  proof −
    have 0: 0 = (f z) ∗ 2 ∗ of-real (2 ∗ pi) ∗ i ∗ winding-number γ z
```

using *v-def z* **by** *auto*
then have $((\lambda x.\ 1\ /\ (x - z))\ has\text{-}contour\text{-}integral\ 0)\ \gamma$
using *z v-def has-contour-integral-winding-number* [*OF* ‹*valid-path γ*›] **by** *fastforce*
then have $((\lambda x.\ f\ z * (1\ /\ (x - z)))\ has\text{-}contour\text{-}integral\ 0)\ \gamma$
using *has-contour-integral-lmul* **by** *fastforce*
then have $((\lambda x.\ f\ z\ /\ (x - z))\ has\text{-}contour\text{-}integral\ 0)\ \gamma$
by (*simp add*: *divide-simps*)
moreover have $((\lambda x.\ (f\ x - f\ z)\ /\ (x - z))\ has\text{-}contour\text{-}integral\ contour\text{-}integral\ \gamma\ (d\ z))\ \gamma$
using *z*
apply (*auto simp*: *v-def*)
apply (*metis* (*no-types*, *lifting*) *contour-integrable-eq d-def has-contour-integral-eq has-contour-integral-integral cint-fxy*)
done
ultimately have ∗: $((\lambda x.\ f\ z\ /\ (x - z) + (f\ x - f\ z)\ /\ (x - z))\ has\text{-}contour\text{-}integral\ (0 + contour\text{-}integral\ \gamma\ (d\ z)))\ \gamma$
by (*rule has-contour-integral-add*)
have $((\lambda w.\ f\ w\ /\ (w - z))\ has\text{-}contour\text{-}integral\ contour\text{-}integral\ \gamma\ (d\ z))\ \gamma$
if $z \in u$
using ∗ **by** (*auto simp*: *divide-simps has-contour-integral-eq*)
moreover have $((\lambda w.\ f\ w\ /\ (w - z))\ has\text{-}contour\text{-}integral\ contour\text{-}integral\ \gamma\ (\lambda w.\ f\ w\ /\ (w - z)))\ \gamma$
if $z \notin u$
apply (*rule has-contour-integral-integral* [*OF contour-integrable-holomorphic-simple* [**where** *s*=*u*]])
using *u pasz* ‹*valid-path γ*› *that*
apply (*auto intro*: *holomorphic-on-imp-continuous-on hol-d*)
apply (*rule continuous-intros conf holomorphic-intros holf* | *force*)+
done
ultimately show *?thesis*
using *z* **by** (*simp add*: *h-def*)
qed
have *znot*: $z \notin path\text{-}image\ \gamma$
using *pasz* **by** *blast*
obtain *d0* **where** *d0*>*0* **and** *d0*: $\bigwedge x\ y.\ x \in path\text{-}image\ \gamma \Longrightarrow y \in -\ u \Longrightarrow d0 \leq dist\ x\ y$
using *separate-compact-closed* [*of path-image γ* −*u*] *pasz u*
by (*fastforce simp add*: ‹*path γ*› *compact-path-image*)
obtain *dd* **where** $0 < dd$ **and** *dd*: $\{y + k \mid y\ k.\ y \in path\text{-}image\ \gamma \wedge k \in ball\ 0\ dd\} \subseteq u$
apply (*rule that* [*of d0*/*2*])
using ‹*0* < *d0*›
apply (*auto simp*: *dist-norm dest*: *d0*)
done
have $\bigwedge x\ x'.\ [\![x \in path\text{-}image\ \gamma;\ dist\ x\ x' * 2 < dd]\!] \Longrightarrow \exists y\ k.\ x' = y + k \wedge y \in path\text{-}image\ \gamma \wedge dist\ 0\ k * 2 \leq dd$
apply (*rule-tac x*=*x* **in** *exI*)
apply (*rule-tac x*=*x'*−*x* **in** *exI*)

apply (*force simp add*: *dist-norm*)
done
then have *1*: *path-image* $\gamma \subseteq$ *interior* $\{y + k \mid y\ k.\ y \in$ *path-image* $\gamma \wedge k \in$ *cball 0* (*dd / 2*)$\}$
apply (*clarsimp simp add*: *mem-interior*)
using ‹*0 < dd*›
apply (*rule-tac x=dd/2* **in** *exI*, *auto*)
done
obtain *t* **where** *compact t* **and** *subt*: *path-image* $\gamma \subseteq$ *interior t* **and** *t*: $t \subseteq u$
apply (*rule that* [*OF - 1*])
apply (*fastforce simp add*: ‹*valid-path* γ› *compact-valid-path-image intro*!: *compact-sums*)
apply (*rule order-trans* [*OF - dd*])
using ‹*0 < dd*› **by** *fastforce*
obtain *L* **where** *L>0*
and *L*: $\bigwedge f\ B.\ [\![f$ *holomorphic-on interior t*; $\bigwedge z.\ z \in$ *interior t* $\Longrightarrow$ *cmod* (*f z*) $\leq B]\!] \Longrightarrow$
cmod (*contour-integral* γ *f*) $\leq L * B$
using *contour-integral-bound-exists* [*OF open-interior* ‹*valid-path* γ› *subt*]
by *blast*
have *bounded*(*f ‘ t*)
by (*meson* ‹*compact t*› *compact-continuous-image compact-imp-bounded conf continuous-on-subset t*)
then obtain *D* **where** *D>0* **and** *D*: $\bigwedge x.\ x \in t \Longrightarrow$ *norm* (*f x*) $\leq D$
by (*auto simp*: *bounded-pos*)
obtain *C* **where** *C>0* **and** *C*: $\bigwedge x.\ x \in t \Longrightarrow$ *norm x* $\leq C$
using ‹*compact t*› *bounded-pos compact-imp-bounded* **by** *force*
have *dist* (*h y*) *0* $\leq e$ **if** *0* $< e$ **and** *le*: $D * L$ / $e + C \leq$ *cmod y* **for** *e y*
proof −
have $D * L$ / $e > 0$ **using** ‹*D>0*› ‹*L>0*› ‹*e>0*› **by** *simp*
with *le* **have** *ybig*: *norm y* $> C$ **by** *force*
with *C* **have** $y \notin t$ **by** *force*
then have *ynot*: $y \notin$ *path-image* γ
using *subt interior-subset* **by** *blast*
have [*simp*]: *winding-number* γ *y* = *0*
apply (*rule winding-number-zero-outside* [*of - cball 0 C*])
using *ybig interior-subset subt*
apply (*force simp add*: *loop* ‹*path* γ› *dist-norm intro*!: *C*)+
done
have [*simp*]: *h y* = *contour-integral* γ $(\lambda w.\ f\ w/(w - y))$
by (*rule contour-integral-unique* [*symmetric*]) (*simp add*: *v-def ynot V*)
have *holint*: $(\lambda w.\ f\ w$ / $(w - y))$ *holomorphic-on interior t*
apply (*rule holomorphic-on-divide*)
using *holf holomorphic-on-subset interior-subset t* **apply** *blast*
apply (*rule holomorphic-intros*)+
using ‹$y \notin t$› *interior-subset* **by** *auto*
have *leD*: *cmod* (*f z* / $(z - y)) \leq D * (e$ / *L* / *D*) **if** *z*: $z \in$ *interior t* **for** *z*
proof −
have $D * L$ / $e +$ *cmod z* $\leq$ *cmod y*
using *le C* [*of z*] *z* **using** *interior-subset* **by** *force*

```
      then have DL2: D * L / e ≤ cmod (z - y)
        using norm-triangle-ineq2 [of y z] by (simp add: norm-minus-commute)
      have cmod (f z / (z - y)) = cmod (f z) * inverse (cmod (z - y))
       by (simp add: norm-mult norm-inverse Fields.field-class.field-divide-inverse)
      also have ... ≤ D * (e / L / D)
        apply (rule mult-mono)
        using that D interior-subset apply blast
        using ‹L>0› ‹e>0› ‹D>0› DL2
        apply (auto simp: norm-divide divide-simps algebra-simps)
        done
      finally show ?thesis .
    qed
    have dist (h y) 0 = cmod (contour-integral γ (λw. f w / (w - y)))
      by (simp add: dist-norm)
    also have ... ≤ L * (D * (e / L / D))
      by (rule L [OF holint leD])
    also have ... = e
      using ‹L>0› ‹0 < D› by auto
    finally show ?thesis .
  qed
  then have (h ⟶ 0) at-infinity
    by (meson Lim-at-infinityI)
  moreover have h holomorphic-on UNIV
  proof −
    have con-ff: continuous (at (x,z)) (λ(x,y). (f y − f x) / (y − x))
                 if x ∈ u z ∈ u x ≠ z for x z
      using that conf
      apply (simp add: split-def continuous-on-eq-continuous-at u)
       apply (simp | rule continuous-intros continuous-within-compose2 [where
g=f])+
      done
    have con-fstsnd: continuous-on UNIV (λx. (fst x − snd x) ::complex)
      by (rule continuous-intros)+
    have open-uu-Id: open (u × u − Id)
      apply (rule open-Diff)
      apply (simp add: open-Times u)
       using continuous-closed-preimage-constant [OF con-fstsnd closed-UNIV, of
0]
      apply (auto simp: Id-fstsnd-eq algebra-simps)
      done
    have con-derf: continuous (at z) (deriv f) if z ∈ u for z
      apply (rule continuous-on-interior [of u])
      apply (simp add: holf holomorphic-deriv holomorphic-on-imp-continuous-on
u)
      by (simp add: interior-open that u)
    have tendsto-f': ((λ(x,y). if y = x then deriv f (x)
                               else (f (y) − f (x)) / (y − x)) ⟶ deriv f x)
                     (at (x, x) within u × u) if x ∈ u for x
    proof (rule Lim-withinI)
```

fix *e*::*real* **assume** *0 < e*
obtain *k1* **where** *k1>0* **and** *k1*: $\bigwedge x'$. *norm* $(x' - x) \leq k1 \Longrightarrow$ *norm* (*deriv f x' − deriv f x*) *< e*
using ‹*0 < e*› *continuous-within-E* [*OF con-derf* [*OF* ‹*x* ∈ *u*›]]
by (*metis UNIV-I dist-norm*)
obtain *k2* **where** *k2>0* **and** *k2*: *ball x k2* ⊆ *u* **by** (*blast intro*: *openE* [*OF u*] ‹*x* ∈ *u*›)
have *neq*: *norm* $((f\,z' - f\,x') / (z' - x') -$ *deriv f x*$) \leq e$
if $z' \neq x'$ **and** *less-k1*: *norm* $(x'-x,\ z'-x) < k1$ **and** *less-k2*: *norm* $(x'-x,\ z'-x) < k2$
for $x'\ z'$
proof −
have *cs-less*: $w \in$ *closed-segment* $x'\ z' \Longrightarrow$ *cmod* $(w - x) \leq$ *norm* $(x'-x,\ z'-x)$ **for** *w*
apply (*drule segment-furthest-le* [**where** *y*=*x*])
by (*metis* (*no-types*) *dist-commute dist-norm norm-fst-le norm-snd-le order-trans*)
have *derf-le*: $w \in$ *closed-segment* $x'\ z' \Longrightarrow z' \neq x' \Longrightarrow$ *cmod* (*deriv f w − deriv f x*) $\leq e$ **for** *w*
by (*blast intro*: *cs-less less-k1 k1* [*unfolded divide-const-simps dist-norm*] *less-imp-le le-less-trans*)
have *f-has-der*: $\bigwedge x.\ x \in u \Longrightarrow$ (*f has-field-derivative deriv f x*) (*at x within u*)
by (*metis DERIV-deriv-iff-field-differentiable at-within-open holf holomorphic-on-def u*)
have *closed-segment* $x'\ z' \subseteq u$
by (*rule order-trans* [*OF - k2*]) (*simp add*: *cs-less le-less-trans* [*OF - less-k2*] *dist-complex-def norm-minus-commute subset-iff*)
then have *cint-derf*: (*deriv f has-contour-integral f z' − f x'*) (*linepath x' z'*)
using *contour-integral-primitive* [*OF f-has-der valid-path-linepath*] *pasz* **by** *simp*
then have ∗: $((\lambda x.$ *deriv f x* $/ (z' - x'))$ *has-contour-integral* $(f\,z' - f\,x') / (z' - x')$) (*linepath x' z'*)
by (*rule has-contour-integral-div*)
have *norm* $((f\,z' - f\,x') / (z' - x') -$ *deriv f x*$) \leq e/norm(z' - x') *$ *norm*$(z' - x')$
apply (*rule has-contour-integral-bound-linepath* [*OF has-contour-integral-diff* [*OF* ∗]])
using *has-contour-integral-div* [**where** $c = z' - x'$, *OF has-contour-integral-const-linepath* [*of deriv f x z' x'*]]
‹$e > 0$› ‹$z' \neq x'$›
apply (*auto simp*: *norm-divide divide-simps derf-le*)
done
also have ... ≤ *e* **using** ‹*0 < e*› **by** *simp*
finally show *?thesis* **.**
qed
show ∃ *d>0*. ∀ *xa*∈*u* × *u*.
0 < dist xa $(x,\ x) \wedge$ *dist xa* $(x,\ x) < d \longrightarrow$

dist (*case xa of* $(x, y) \Rightarrow$ *if* $y = x$ *then deriv f x else* $(f\ y - f\ x)$ / $(y - x)$) (*deriv f x*) $\leq e$
apply (*rule-tac x=min k1 k2* **in** *exI*)
using ‹*k1>0*› ‹*k2>0*› ‹*e>0*›
apply (*force simp*: *dist-norm neq intro*: *dual-order.strict-trans2 k1 less-imp-le norm-fst-le*)
done
qed
have *con-pa-f*: *continuous-on* (*path-image* γ) *f*
by (*meson holf holomorphic-on-imp-continuous-on holomorphic-on-subset interior-subset subt t*)
have *le-B*: $\bigwedge t.\ t \in \{0..1\} \Longrightarrow$ *cmod* (*vector-derivative* γ (*at t*)) $\leq B$
apply (*rule B*)
using γ' **using** *path-image-def vector-derivative-at* **by** *fastforce*
have *f-has-cint*: $\bigwedge w.\ w \in v -$ *path-image* $\gamma \Longrightarrow ((\lambda u.\ f\ u\ /\ (u - w)\ \hat{}\ 1)$ *has-contour-integral h w*) γ
by (*simp add*: *V*)
have *cond-uu*: *continuous-on* $(u \times u)$ $(\lambda(x,y).\ d\ x\ y)$
apply (*simp add*: *continuous-on-eq-continuous-within d-def continuous-within tendsto-f'*)
apply (*simp add*: *tendsto-within-open-NO-MATCH open-Times u*, *clarify*)
apply (*rule Lim-transform-within-open* [*OF - open-uu-Id*, **where** $f = (\lambda(x,y).$ $(f\ y - f\ x)\ /\ (y - x))$])
using *con-ff*
apply (*auto simp*: *continuous-within*)
done
have *hol-dw*: $(\lambda z.\ d\ z\ w)$ *holomorphic-on u* **if** $w \in u$ **for** *w*
proof −
have *continuous-on u* $((\lambda(x,y).\ d\ x\ y)\ o\ (\lambda z.\ (w,z)))$
by (*rule continuous-on-compose continuous-intros continuous-on-subset* [*OF cond-uu*] | *force intro*: *that*)+
then have ∗: *continuous-on u* $(\lambda z.$ *if* $w = z$ *then deriv f z else* $(f\ w - f\ z)$ / $(w - z))$
by (*rule rev-iffD1* [*OF - continuous-on-cong* [*OF refl*]]) (*simp add*: *d-def field-simps*)
have ∗∗: $\bigwedge x.\ [\![x \in u;\ x \neq w]\!] \Longrightarrow (\lambda z.$ *if* $w = z$ *then deriv f z else* $(f\ w - f\ z)$ / $(w - z))$ *field-differentiable at x*
apply (*rule-tac* $f = \lambda x.\ (f\ w - f\ x)/(w - x)$ **and** $d =$ *dist x w* **in** *field-differentiable-transform-within*)
apply (*rule u derivative-intros holomorphic-on-imp-differentiable-at* [*OF holf*] | *force simp add*: *dist-commute*)+
done
show *?thesis*
unfolding *d-def*
apply (*rule no-isolated-singularity* [*OF* ∗ - *u*, **where** $k = \{w\}$])
apply (*auto simp*: *field-differentiable-def* [*symmetric*] *holomorphic-on-open open-Diff u* ∗∗)
done
qed

{ fix *a b*
assume *abu*: *closed-segment a b* ⊆ *u*
then have ⋀*w*. *w* ∈ *u* ⟹ (λ*z*. *d z w*) *contour-integrable-on* (*linepath a b*)
by (*metis hol-dw continuous-on-subset contour-integrable-continuous-linepath holomorphic-on-imp-continuous-on*)
then have *cont-cint-d*: *continuous-on u* (λ*w*. *contour-integral* (*linepath a b*) (λ*z*. *d z w*))
apply (*rule contour-integral-continuous-on-linepath-2D* [*OF* ‹*open u*› - - *abu*])
apply (*auto simp*: *intro*: *continuous-on-swap-args cond-uu*)
done
have *cont-cint-dγ*: *continuous-on* {*0..1*} ((λ*w*. *contour-integral* (*linepath a b*) (λ*z*. *d z w*)) *o* γ)
apply (*rule continuous-on-compose*)
using ‹*path* γ› *path-def pasz*
apply (*auto intro*!: *continuous-on-subset* [*OF cont-cint-d*])
apply (*force simp add*: *path-image-def*)
done
have *cint-cint*: (λ*w*. *contour-integral* (*linepath a b*) (λ*z*. *d z w*)) *contour-integrable-on* γ
apply (*simp add*: *contour-integrable-on*)
apply (*rule integrable-continuous-real*)
apply (*rule continuous-on-mult* [*OF cont-cint-dγ* [*unfolded o-def*]])
using *pfγ′*
by (*simp add*: *continuous-on-polymonial-function vector-derivative-at* [*OF γ′*])
have *contour-integral* (*linepath a b*) *h* = *contour-integral* (*linepath a b*) (λ*z*. *contour-integral* γ (*d z*))
using *abu* **by** (*force simp add*: *h-def intro*: *contour-integral-eq*)
also have ... = *contour-integral* γ (λ*w*. *contour-integral* (*linepath a b*) (λ*z*. *d z w*))
apply (*rule contour-integral-swap*)
apply (*rule continuous-on-subset* [*OF cond-uu*])
using *abu pasz* ‹*valid-path* γ›
apply (*auto intro*!: *continuous-intros*)
by (*metis γ′ continuous-on-eq path-def path-polynomial-function pfγ′ vector-derivative-at*)
finally have *cint-h-eq*:
contour-integral (*linepath a b*) *h* =
contour-integral γ (λ*w*. *contour-integral* (*linepath a b*) (λ*z*. *d z w*))
.
note *cint-cint cint-h-eq*
} note *cint-h* = *this*
have *conthu*: *continuous-on u h*
proof (*simp add*: *continuous-on-sequentially*, *clarify*)
fix *a x*
assume *x*: *x* ∈ *u* **and** *au*: ∀ *n*. *a n* ∈ *u* **and** *ax*: *a* ⟶ *x*
then have *A1*: ∀ *F n in sequentially*. *d* (*a n*) *contour-integrable-on* γ
by (*meson U contour-integrable-on-def eventuallyI*)

obtain *dd* **where** *dd>0* **and** *dd*: *cball x dd* ⊆ *u* **using** *open-contains-cball u x* **by** *force*
have *A2*: ∀ *F n in sequentially*. ∀ *xa*∈*path-image γ*. *cmod* (*d* (*a n*) *xa* − *d x xa*) < *ee* **if** *0* < *ee* **for** *ee*
proof −
let *?ddpa* = {(*w*,*z*) |*w z*. *w* ∈ *cball x dd* ∧ *z* ∈ *path-image γ*}
have *uniformly-continuous-on ?ddpa* (λ(*x*,*y*). *d x y*)
apply (*rule compact-uniformly-continuous* [*OF continuous-on-subset*[*OF cond-uu*]])
using *dd pasz* ‹*valid-path γ*›
apply (*auto simp*: *compact-Times compact-valid-path-image simp del*: *mem-cball*)
done
then obtain *kk* **where** *kk>0*
and *kk*: ⋀*x x'*. ⟦*x*∈*?ddpa*; *x'*∈*?ddpa*; *dist x' x* < *kk*⟧ ⟹
dist ((λ(*x*,*y*). *d x y*) *x'*) ((λ(*x*,*y*). *d x y*) *x*) < *ee*
apply (*rule uniformly-continuous-onE* [**where** *e* = *ee*])
using ‹*0* < *ee*› **by** *auto*
have *kk*: ⟦*norm* (*w* − *x*) ≤ *dd*; *z* ∈ *path-image γ*; *norm* ((*w*, *z*) − (*x*, *z*)) < *kk*⟧ ⟹ *norm* (*d w z* − *d x z*) < *ee*
for *w z*
using ‹*dd>0*› *kk* [*of* (*x*,*z*) (*w*,*z*)] **by** (*force simp add*: *norm-minus-commute dist-norm*)
show *?thesis*
using *ax* **unfolding** *lim-sequentially eventually-sequentially*
apply (*drule-tac x*=*min dd kk* **in** *spec*)
using ‹*dd* > *0*› ‹*kk* > *0*›
apply (*fastforce simp*: *kk dist-norm*)
done
qed
have *tendsto-hx*: (λ*n*. *contour-integral γ* (*d* (*a n*))) ⟶ *h x*
apply (*simp add*: *contour-integral-unique* [*OF U*, *symmetric*] *x*)
apply (*rule contour-integral-uniform-limit* [*OF A1 A2 le-B*])
apply (*auto simp*: ‹*valid-path γ*›)
done
then show (*h* ∘ *a*) ⟶ *h x*
by (*simp add*: *h-def x au o-def*)
qed
show *?thesis*
proof (*simp add*: *holomorphic-on-open field-differentiable-def* [*symmetric*], *clarify*)
fix *z0*
consider *z0* ∈ *v* | *z0* ∈ *u* **using** *uv-Un* **by** *blast*
then show *h field-differentiable at z0*
proof *cases*
assume *z0* ∈ *v* **then show** *?thesis*
using *Cauchy-next-derivative* [*OF con-pa-f le-B f-has-cint - ov*] *V f-has-cint* ‹*valid-path γ*›
by (*auto simp*: *field-differentiable-def v-def*)

```
    next
      assume z0 ∈ u then
      obtain e where e>0 and e: ball z0 e ⊆ u by (blast intro: openE [OF u])
      have ∗: contour-integral (linepath a b) h + contour-integral (linepath b c) h
+ contour-integral (linepath c a) h = 0
            if abc-subset: convex hull {a, b, c} ⊆ ball z0 e  for a b c
      proof −
        have ∗: ⋀x1 x2 z. z ∈ u ⟹ closed-segment x1 x2 ⊆ u ⟹ (λw. d w z)
contour-integrable-on linepath x1 x2
          using  hol-dw holomorphic-on-imp-continuous-on u
          by (auto intro!: contour-integrable-holomorphic-simple)
        have abc: closed-segment a b ⊆ u  closed-segment b c ⊆ u  closed-segment
c a ⊆ u
          using that e segments-subset-convex-hull by fastforce+
        have eq0: ⋀w. w ∈ u ⟹ contour-integral (linepath a b +++ linepath b
c +++ linepath c a) (λz. d z w) = 0
          apply (rule contour-integral-unique [OF Cauchy-theorem-triangle])
          apply (rule holomorphic-on-subset [OF hol-dw])
          using e abc-subset by auto
        have contour-integral γ
               (λx. contour-integral (linepath a b) (λz. d z x) +
                    (contour-integral (linepath b c) (λz. d z x) +
                     contour-integral (linepath c a) (λz. d z x))) = 0
          apply (rule contour-integral-eq-0)
          using abc pasz u
          apply (subst contour-integral-join [symmetric], auto intro: eq0 ∗)+
          done
        then show ?thesis
            by (simp add: cint-h abc contour-integrable-add contour-integral-add
[symmetric] add-ac)
      qed
      show ?thesis
        using e ‹e > 0›
          by (auto intro!: holomorphic-on-imp-differentiable-at [OF - open-ball]
analytic-imp-holomorphic
                       Morera-triangle continuous-on-subset [OF conthu] ∗)
    qed
  qed
 qed
 ultimately have [simp]: h z = 0 for z
   by (meson Liouville-weak)
  have ((λw. 1 / (w − z)) has-contour-integral complex-of-real (2 ∗ pi) ∗ i ∗
winding-number γ z) γ
   by (rule has-contour-integral-winding-number [OF ‹valid-path γ› znot])
 then have ((λw. f z ∗ (1 / (w − z))) has-contour-integral complex-of-real (2 ∗
pi) ∗ i ∗ winding-number γ z ∗ f z) γ
   by (metis mult.commute has-contour-integral-lmul)
 then have 1: ((λw. f z / (w − z)) has-contour-integral complex-of-real (2 ∗ pi)
∗ i ∗ winding-number γ z ∗ f z) γ
```

by (*simp add*: *divide-simps*)
moreover have *2*: ((λ*w*. (*f w* − *f z*) / (*w* − *z*)) *has-contour-integral 0*) γ
using *U* [*OF z*] *pasz d-def* **by** (*force elim*: *has-contour-integral-eq* [**where** *g* = λ*w*. (*f w* − *f z*)/(*w* − *z*)])
show *?thesis*
using *has-contour-integral-add* [*OF 1 2*] **by** (*simp add*: *diff-divide-distrib*)
qed

theorem *Cauchy-integral-formula-global*:
assumes *s*: *open s* **and** *holf*: *f holomorphic-on s*
and *z*: *z* ∈ *s* **and** *vpg*: *valid-path* γ
and *pasz*: *path-image* γ ⊆ *s* − {*z*} **and** *loop*: *pathfinish* γ = *pathstart* γ
and *zero*: ⋀*w*. *w* ∉ *s* ⟹ *winding-number* γ *w* = *0*
shows ((λ*w*. *f w* / (*w* − *z*)) *has-contour-integral* (*2*∗*pi* ∗ *ii* ∗ *winding-number* γ *z* ∗ *f z*)) γ
proof −
have *path* γ **using** *vpg* **by** (*blast intro*: *valid-path-imp-path*)
have *hols*: (λ*w*. *f w* / (*w* − *z*)) *holomorphic-on s* − {*z*} (λ*w*. *1* / (*w* − *z*)) *holomorphic-on s* − {*z*}
by (*rule holomorphic-intros holomorphic-on-subset* [*OF holf*] | *force*)+
then have *cint-fw*: (λ*w*. *f w* / (*w* − *z*)) *contour-integrable-on* γ
by (*meson contour-integrable-holomorphic-simple holomorphic-on-imp-continuous-on open-delete s vpg pasz*)
obtain *d* **where** *d>0*
and *d*: ⋀*g h*. ⟦*valid-path g*; *valid-path h*; ∀ *t*∈{*0..1*}. *cmod* (*g t* − γ *t*) < *d* ∧ *cmod* (*h t* − γ *t*) < *d*;
pathstart h = *pathstart g* ∧ *pathfinish h* = *pathfinish g*⟧
⟹ *path-image h* ⊆ *s* − {*z*} ∧ (∀*f*. *f holomorphic-on s* − {*z*} ⟶ *contour-integral h f* = *contour-integral g f*)
using *contour-integral-nearby-ends* [*OF* - ‹*path* γ› *pasz*] *s* **by** (*simp add*: *open-Diff*) *metis*
obtain *p* **where** *polyp*: *polynomial-function p*
and *ps*: *pathstart p* = *pathstart* γ **and** *pf*: *pathfinish p* = *pathfinish* γ **and** *led*: ∀ *t*∈{*0..1*}. *cmod* (*p t* − γ *t*) < *d*
using *path-approx-polynomial-function* [*OF* ‹*path* γ› ‹*d* > *0*›] **by** *blast*
then have *ploop*: *pathfinish p* = *pathstart p* **using** *loop* **by** *auto*
have *vpp*: *valid-path p* **using** *polyp valid-path-polynomial-function* **by** *blast*
have [*simp*]: *z* ∉ *path-image* γ **using** *pasz* **by** *blast*
have *paps*: *path-image p* ⊆ *s* − {*z*} **and** *cint-eq*: (⋀*f*. *f holomorphic-on s* − {*z*} ⟹ *contour-integral p f* = *contour-integral* γ *f*)
using *pf ps led d* [*OF vpg vpp*] ‹*d* > *0*› **by** *auto*
have *wn-eq*: *winding-number p z* = *winding-number* γ *z*
using *vpp paps*
by (*simp add*: *subset-Diff-insert vpg valid-path-polynomial-function winding-number-valid-path cint-eq hols*)
have *winding-number p w* = *winding-number* γ *w* **if** *w* ∉ *s* **for** *w*
proof −
have *hol*: (λ*v*. *1* / (*v* − *w*)) *holomorphic-on s* − {*z*}

using *that* **by** (*force intro*: *holomorphic-intros holomorphic-on-subset* [*OF holf*])
have $w \notin$ *path-image p* $w \notin$ *path-image* γ **using** *paps pasz that* **by** *auto*
then show *?thesis*
using *vpp vpg* **by** (*simp add*: *subset-Diff-insert valid-path-polynomial-function winding-number-valid-path cint-eq* [*OF hol*])
qed
then have *wn0*: $\bigwedge w.\ w \notin s \Longrightarrow$ *winding-number p* $w = 0$
by (*simp add*: *zero*)
show *?thesis*
using *Cauchy-integral-formula-global-weak* [*OF s holf z polyp paps ploop wn0*] *hols*
by (*metis wn-eq cint-eq has-contour-integral-eqpath cint-fw cint-eq*)
qed

theorem *Cauchy-theorem-global*:
assumes *s*: *open s* **and** *holf*: *f holomorphic-on s*
and *vpg*: *valid-path* γ **and** *loop*: *pathfinish* γ = *pathstart* γ
and *pas*: *path-image* $\gamma \subseteq s$
and *zero*: $\bigwedge w.\ w \notin s \Longrightarrow$ *winding-number* γ $w = 0$
shows (*f has-contour-integral 0*) γ
proof −
obtain *z* **where** $z \in s$ **and** *znot*: $z \notin$ *path-image* γ
proof −
have *compact* (*path-image* γ)
using *compact-valid-path-image vpg* **by** *blast*
then have *path-image* $\gamma \neq s$
by (*metis* (*no-types*) *compact-open path-image-nonempty s*)
with *pas* **show** *?thesis* **by** (*blast intro*: *that*)
qed
then have *pasz*: *path-image* $\gamma \subseteq s - \{z\}$ **using** *pas* **by** *blast*
have *hol*: $(\lambda w.\ (w - z) * f\ w)$ *holomorphic-on s*
by (*rule holomorphic-intros holf*)+
show *?thesis*
using *Cauchy-integral-formula-global* [*OF s hol* ‹$z \in s$› *vpg pasz loop zero*]
by (*auto simp*: *znot elim*!: *has-contour-integral-eq*)
qed

corollary *Cauchy-theorem-global-outside*:
assumes *open s f holomorphic-on s valid-path* γ *pathfinish* γ = *pathstart* γ *path-image* $\gamma \subseteq s$
$\bigwedge w.\ w \notin s \Longrightarrow w \in$ *outside*(*path-image* γ)
shows (*f has-contour-integral 0*) γ
by (*metis Cauchy-theorem-global assms winding-number-zero-in-outside valid-path-imp-path*)

end

54 Conformal Mappings. Consequences of Cauchy's integral theorem.

By John Harrison et al. Ported from HOL Light by L C Paulson (2016)

Also Cauchy's residue theorem by Wenda Li (2016)

theory *Conformal-Mappings*
imports ~~/*src/HOL/Multivariate-Analysis/Cauchy-Integral-Thm*

begin

54.1 Cauchy's inequality and more versions of Liouville

lemma *Cauchy-higher-deriv-bound*:
  assumes *holf*: *f holomorphic-on* (*ball z r*)
    and *contf*: *continuous-on* (*cball z r*) *f*
    and *0 < r* **and** *0 < n*
    and *fin* : $\bigwedge w.\ w \in ball\ z\ r \Longrightarrow f\ w \in ball\ y\ B0$
   shows *norm* ((*deriv* ^^ *n*) *f z*) ≤ (*fact n*) * *B0* / *r*^*n*
proof −
 have *0 < B0* **using** ‹*0 < r*› *fin* [*of z*]
  by (*metis ball-eq-empty ex-in-conv fin not-less*)
 have *le-B0*: $\bigwedge w.\ cmod\ (w - z) \leq r \Longrightarrow cmod\ (f\ w - y) \leq B0$
  apply (*rule continuous-on-closure-norm-le* [*of ball z r* λ*w. f w − y*])
  apply (*auto simp*: ‹*0 < r*› *dist-norm norm-minus-commute*)
  apply (*rule continuous-intros contf*)+
  using *fin* **apply** (*simp add*: *dist-commute dist-norm less-eq-real-def*)
  done
 have (*deriv* ^^ *n*) *f z* = (*deriv* ^^ *n*) (λ*w. f w*) *z* − (*deriv* ^^ *n*) (λ*w. y*) *z*
  using ‹*0 < n*› **by** *simp*
 also have ... = (*deriv* ^^ *n*) (λ*w. f w − y*) *z*
  by (*rule higher-deriv-diff* [*OF holf*, *symmetric*]) (*auto simp*: ‹*0 < r*›)
 finally have (*deriv* ^^ *n*) *f z* = (*deriv* ^^ *n*) (λ*w. f w − y*) *z* **.**
 have *contf'*: *continuous-on* (*cball z r*) (λ*u. f u − y*)
  by (*rule contf continuous-intros*)+
 have *holf'*: (λ*u.* (*f u − y*)) *holomorphic-on* (*ball z r*)
  by (*simp add*: *holf holomorphic-on-diff*)
 def *a* ≡ (*2 * pi*)/(*fact n*)
 have *0 < a* **by** (*simp add*: *a-def*)
 have *B0*/*r*^(*Suc n*)*2 * *pi* * *r* = *a*∗((*fact n*)∗*B0*/*r*^*n*)
  using ‹*0 < r*› **by** (*simp add*: *a-def divide-simps*)
 have *der-dif*: (*deriv* ^^ *n*) (λ*w. f w − y*) *z* = (*deriv* ^^ *n*) *f z*
  using ‹*0 < r*› ‹*0 < n*›
  by (*auto simp*: *higher-deriv-diff* [*OF holf holomorphic-on-const*])
 have *norm* ((*2 * of-real pi * ii*)/(*fact n*) * (*deriv* ^^ *n*) (λ*w. f w − y*) *z*)
     ≤ (*B0*/*r*^(*Suc n*)) * (*2 * pi * r*)
   apply (*rule has-contour-integral-bound-circlepath* [*of* (λ*u.* (*f u − y*)/(*u − z*)^(*Suc n*)) - *z*])
  using *Cauchy-has-contour-integral-higher-derivative-circlepath* [*OF contf' holf'*]

using ‹*0 < B0*› ‹*0 < r*›
apply (*auto simp*: *norm-divide norm-mult norm-power divide-simps le-B0*)
done
then show *?thesis*
using ‹*0 < r*›
by (*auto simp*: *norm-divide norm-mult norm-power field-simps der-dif le-B0*)
qed

proposition *Cauchy-inequality*:
assumes *holf*: *f holomorphic-on* (*ball ξ r*)
and *contf*: *continuous-on* (*cball ξ r*) *f*
and *0 < r*
and *nof*: ⋀*x*. *norm*(*ξ*−*x*) = *r* ⟹ *norm*(*f x*) ≤ *B*
shows *norm* ((*deriv* ^^ *n*) *f ξ*) ≤ (*fact n*) ∗ *B* / *r*^*n*
proof −
obtain *x* **where** *norm* (*ξ*−*x*) = *r*
by (*metis abs-of-nonneg add-diff-cancel-left' ‹0 < r› diff-add-cancel dual-order.strict-implies-order norm-of-real*)
then have *0* ≤ *B*
by (*metis nof norm-not-less-zero not-le order-trans*)
have ((λ*u*. *f u* / (*u* − *ξ*) ^ *Suc n*) *has-contour-integral* (*2* ∗ *pi*) ∗ i / *fact n* ∗ (*deriv* ^^ *n*) *f ξ*)
(*circlepath ξ r*)
apply (*rule Cauchy-has-contour-integral-higher-derivative-circlepath* [*OF contf holf*])
using ‹*0 < r*› **by** *simp*
then have *norm* ((*2* ∗ *pi* ∗ *ii*)/(*fact n*) ∗ (*deriv* ^^ *n*) *f ξ*) ≤ (*B* / *r*^(*Suc n*)) ∗ (*2* ∗ *pi* ∗ *r*)
apply (*rule has-contour-integral-bound-circlepath*)
using ‹*0 ≤ B*› ‹*0 < r*›
apply (*simp-all add*: *norm-divide norm-power nof frac-le norm-minus-commute del*: *power-Suc*)
done
then show *?thesis* **using** ‹*0 < r*›
by (*simp add*: *norm-divide norm-mult field-simps*)
qed

proposition *Liouville-polynomial*:
assumes *holf*: *f holomorphic-on UNIV*
and *nof*: ⋀*z*. *A* ≤ *norm z* ⟹ *norm*(*f z*) ≤ *B* ∗ *norm z* ^ *n*
shows *f ξ* = (∑ *k*≤*n*. (*deriv*^^*k*) *f 0* / *fact k* ∗ *ξ* ^ *k*)
proof (*cases rule*: *le-less-linear* [*THEN disjE*])
assume *B* ≤ *0*
then have ⋀*z*. *A* ≤ *norm z* ⟹ *norm*(*f z*) = *0*
by (*metis nof less-le-trans zero-less-mult-iff neqE norm-not-less-zero norm-power not-le*)
then have *f0*: (*f* ⟶ *0*) *at-infinity*
using *Lim-at-infinity* **by** *force*
then have [*simp*]: *f* = (λ*w*. *0*)

```
    using Liouville-weak [OF holf, of 0]
    by (simp add: eventually-at-infinity f0) meson
  show ?thesis by simp
next
  assume 0 < B
  have ((λk. (deriv ^^ k) f 0 / (fact k) * (ξ - 0) ^k) sums f ξ)
    apply (rule holomorphic-power-series [where r = norm ξ + 1])
    using holf holomorphic-on-subset apply auto
    done
  then have sumsf: ((λk. (deriv ^^ k) f 0 / (fact k) * ξ ^k) sums f ξ) by simp
  have (deriv ^^ k) f 0 / fact k * ξ ^ k = 0 if k>n for k
  proof (cases (deriv ^^ k) f 0 = 0)
    case True then show ?thesis by simp
  next
    case False
      def w ≡ complex-of-real (fact k * B / cmod ((deriv ^^ k) f 0) + (|A| + 1))
      have 1 ≤ abs (fact k * B / cmod ((deriv ^^ k) f 0) + (|A| + 1))
        using ‹0 < B› by simp
      then have wge1: 1 ≤ norm w
        by (metis norm-of-real w-def)
      then have w ≠ 0 by auto
      have kB: 0 < fact k * B
        using ‹0 < B› by simp
      then have 0 ≤ fact k * B / cmod ((deriv ^^ k) f 0)
        by simp
      then have wgeA: A ≤ cmod w
        by (simp only: w-def norm-of-real)
      have fact k * B / cmod ((deriv ^^ k) f 0) < abs (fact k * B / cmod ((deriv
^^ k) f 0) + (|A| + 1))
        using ‹0 < B› by simp
      then have wge: fact k * B / cmod ((deriv ^^ k) f 0) < norm w
        by (metis norm-of-real w-def)
      then have fact k * B / norm w < cmod ((deriv ^^ k) f 0)
        using False by (simp add: divide-simps mult.commute split: if-split-asm)
      also have ... ≤ fact k * (B * norm w ^ n) / norm w ^ k
        apply (rule Cauchy-inequality)
          using holf holomorphic-on-subset apply force
        using holf holomorphic-on-imp-continuous-on holomorphic-on-subset apply
blast
         using ‹w ≠ 0› apply (simp add:)
         by (metis nof wgeA dist-0-norm dist-norm)
      also have ... = fact k * (B * 1 / cmod w ^ (k-n))
        apply (simp only: mult-cancel-left times-divide-eq-right [symmetric])
      using ‹k>n› ‹w ≠ 0› ‹0 < B› apply (simp add: divide-simps semiring-normalization-rules)
        done
      also have ... = fact k * B / cmod w ^ (k-n)
        by simp
      finally have fact k * B / cmod w < fact k * B / cmod w ^ (k - n) .
      then have 1 / cmod w < 1 / cmod w ^ (k - n)
```

by (*metis kB divide-inverse inverse-eq-divide mult-less-cancel-left-pos*)
then have *cmod w ^ (k − n) < cmod w*
by (*metis frac-le le-less-trans norm-ge-zero norm-one not-less order-refl wge1 zero-less-one*)
with *self-le-power* [*OF wge1*] **have** *False*
by (*meson diff-is-0-eq not-gr0 not-le that*)
then show *?thesis* **by** *blast*
qed
then have (*deriv ^^ (k + Suc n)*) *f 0 / fact (k + Suc n) * ξ ^ (k + Suc n) = 0* **for** *k*
using *not-less-eq* **by** *blast*
then have (*λi. (deriv ^^ (i + Suc n)) f 0 / fact (i + Suc n) * ξ ^ (i + Suc n)*) *sums 0*
by (*rule sums-0*)
with *sums-split-initial-segment* [*OF sumsf*, **where** *n = Suc n*]
show *?thesis*
using *atLeast0AtMost lessThan-Suc-atMost sums-unique2* **by** *fastforce*
qed

Every bounded entire function is a constant function.

theorem *Liouville-theorem*:
assumes *holf*: *f holomorphic-on UNIV*
and *bf*: *bounded* (*range f*)
obtains *c* **where** *⋀z. f z = c*
proof −
obtain *B* **where** *⋀z. cmod (f z) ≤ B*
by (*meson bf bounded-pos rangeI*)
then show *?thesis*
using *Liouville-polynomial* [*OF holf*, *of 0 B 0*, *simplified*] *that* **by** *blast*
qed

A holomorphic function f has only isolated zeros unless f is 0.

proposition *powser-0-nonzero*:
fixes *a :: nat ⇒ 'a*::{*real-normed-field*,*banach*}
assumes *r*: *0 < r*
and *sm*: *⋀x. norm (x − ξ) < r ⟹ (λn. a n * (x − ξ) ^ n) sums (f x)*
and [*simp*]: *f ξ = 0*
and *m0*: *a m ≠ 0* **and** *m>0*
obtains *s* **where** *0 < s* **and** *⋀z. z ∈ cball ξ s − {ξ} ⟹ f z ≠ 0*
proof −
have *r ≤ conv-radius a*
using *sm sums-summable* **by** (*auto simp*: *le-conv-radius-iff* [**where** *ξ=ξ*])
obtain *m* **where** *am*: *a m ≠ 0* **and** *az* [*simp*]: (*⋀n. n<m ⟹ a n = 0*)
apply (*rule-tac m = LEAST n. a n ≠ 0* **in** *that*)
using *m0*
apply (*rule LeastI2*)
apply (*fastforce intro*: *dest*!: *not-less-Least*)+
done
def *b ≡ λi. a (i+m) / a m*

def $g \equiv \lambda x.\ suminf\ (\lambda i.\ b\ i * (x - \xi)\ \hat{}\ i)$
have [*simp*]: $b\ 0 = 1$
by (*simp add*: *am b-def*)
{ fix $x::'a$
assume $norm\ (x - \xi) < r$
then have $(\lambda n.\ (a\ m * (x - \xi)\ \hat{}\ m) * (b\ n * (x - \xi)\ \hat{}\ n))\ sums\ (f\ x)$
using *am az sm sums-zero-iff-shift* [*of m* $(\lambda n.\ a\ n * (x - \xi)\ \hat{}\ n)$ *f x*]
by (*simp add*: *b-def monoid-mult-class.power-add algebra-simps*)
then have $x \neq \xi \Longrightarrow (\lambda n.\ b\ n * (x - \xi)\ \hat{}\ n)\ sums\ (f\ x\ /\ (a\ m * (x - \xi)\ \hat{}\ m))$
using *am* **by** (*simp add*: *sums-mult-D*)
} note *bsums* = *this*
then have $norm\ (x - \xi) < r \Longrightarrow summable\ (\lambda n.\ b\ n * (x - \xi)\ \hat{}\ n)$ **for** x
using *sums-summable* **by** (*cases x=ξ*) *auto*
then have $r \leq conv\text{-}radius\ b$
by (*simp add*: *le-conv-radius-iff* [**where** *ξ=ξ*])
then have $r/2 < conv\text{-}radius\ b$
using *not-le order-trans r* **by** *fastforce*
then have *continuous-on* $(cball\ \xi\ (r/2))\ g$
using *powser-continuous-suminf* [*of r/2 b ξ*] **by** (*simp add*: *g-def*)
then obtain s **where** $s{>}0\ \bigwedge x.\ [\![norm\ (x - \xi) \leq s;\ norm\ (x - \xi) \leq r/2]\!] \Longrightarrow dist\ (g\ x)\ (g\ \xi) < 1/2$
apply (*rule continuous-onE* [**where** *x=ξ* **and** *e = 1/2*])
using *r* **apply** (*auto simp*: *norm-minus-commute dist-norm*)
done
moreover have $g\ \xi = 1$
by (*simp add*: *g-def*)
ultimately have *gnz*: $\bigwedge x.\ [\![norm\ (x - \xi) \leq s;\ norm\ (x - \xi) \leq r/2]\!] \Longrightarrow (g\ x) \neq 0$
by *fastforce*
have $f\ x \neq 0$ **if** $x \neq \xi\ norm\ (x - \xi) \leq s\ norm\ (x - \xi) \leq r/2$ **for** x
using *bsums* [*of x*] *that gnz* [*of x*]
apply (*auto simp*: *g-def*)
using *r sums-iff* **by** *fastforce*
then show *?thesis*
apply (*rule-tac s=min s* $(r/2)$ **in** *that*)
using ‹$0 < r$› ‹$0 < s$› **by** (*auto simp*: *dist-commute dist-norm*)
qed

proposition *isolated-zeros*:
assumes *holf*: *f holomorphic-on S*
and *open S connected S ξ ∈ S f ξ = 0 β ∈ S f β ≠ 0*
obtains r **where** $0 < r\ ball\ \xi\ r \subseteq S\ \bigwedge z.\ z \in ball\ \xi\ r - \{\xi\} \Longrightarrow f\ z \neq 0$
proof −
obtain r **where** $0 < r$ **and** r: $ball\ \xi\ r \subseteq S$
using ‹*open S*› ‹$\xi \in S$› *open-contains-ball-eq* **by** *blast*
have *powf*: $((\lambda n.\ (deriv\ \hat{}\hat{}\ n)\ f\ \xi\ /\ (fact\ n) * (z - \xi)\ \hat{}\ n)\ sums\ f\ z)$ **if** $z \in ball\ \xi\ r$ **for** z
apply (*rule holomorphic-power-series* [*OF - that*])
apply (*rule holomorphic-on-subset* [*OF holf r*])

done
obtain *m* **where** *m*: (*deriv* ^^ *m*) *f* ξ / (*fact m*) ≠ *0*
using *holomorphic-fun-eq-0-on-connected* [*OF holf* ‹*open S*› ‹*connected S*› - ‹ξ ∈ *S*› ‹β ∈ *S*›] ‹*f* β ≠ *0*›
by *auto*
then have *m* ≠ *0* **using** *assms*(*5*) *funpow-0* **by** *fastforce*
obtain *s* **where** *0* < *s* **and** *s*: ⋀*z*. *z* ∈ *cball* ξ *s* − {ξ} ⟹ *f z* ≠ *0*
apply (*rule powser-0-nonzero* [*OF* ‹*0* < *r*› *powf* ‹*f* ξ = *0*› *m*])
using ‹*m* ≠ *0*› **by** (*auto simp*: *dist-commute dist-norm*)
have *0* < *min r s* **by** (*simp add*: ‹*0* < *r*› ‹*0* < *s*›)
then show *?thesis*
apply (*rule that*)
using *r s* **by** *auto*
qed

proposition *analytic-continuation*:
assumes *holf*: *f holomorphic-on S*
and *S*: *open S connected S*
and *U* ⊆ *S* ξ ∈ *S*
and ξ *islimpt U*
and *fU0* [*simp*]: ⋀*z*. *z* ∈ *U* ⟹ *f z* = *0*
and *w* ∈ *S*
shows *f w* = *0*
proof −
obtain *e* **where** *0* < *e* **and** *e*: *cball* ξ *e* ⊆ *S*
using ‹*open S*› ‹ξ ∈ *S*› *open-contains-cball-eq* **by** *blast*
def *T* ≡ *cball* ξ *e* ∩ *U*
have *contf*: *continuous-on* (*closure T*) *f*
by (*metis T-def closed-cball closure-minimal e holf holomorphic-on-imp-continuous-on holomorphic-on-subset inf*.*cobounded1*)
have *fT0* [*simp*]: ⋀*x*. *x* ∈ *T* ⟹ *f x* = *0*
by (*simp add*: *T-def*)
have ⋀*r*. ⟦∀ *e*>*0*. ∃ *x*′∈*U*. *x*′ ≠ ξ ∧ *dist x*′ ξ < *e*; *0* < *r*⟧ ⟹ ∃ *x*′∈*cball* ξ *e* ∩ *U*. *x*′ ≠ ξ ∧ *dist x*′ ξ < *r*
by (*metis* ‹*0* < *e*› *IntI dist-commute less-eq-real-def mem-cball min-less-iff-conj*)
then have ξ *islimpt T* **using** ‹ξ *islimpt U*›
by (*auto simp*: *T-def islimpt-approachable*)
then have ξ ∈ *closure T*
by (*simp add*: *closure-def*)
then have *f* ξ = *0*
by (*auto simp*: *continuous-constant-on-closure* [*OF contf*])
show *?thesis*
apply (*rule ccontr*)
apply (*rule isolated-zeros* [*OF holf* ‹*open S*› ‹*connected S*› ‹ξ ∈ *S*› ‹*f* ξ = *0*› ‹*w* ∈ *S*›], *assumption*)
by (*metis open-ball* ‹ξ *islimpt T*› *centre-in-ball fT0 insertE insert-Diff islimptE*)
qed

54.2 Open mapping theorem

lemma *holomorphic-contract-to-zero*:
assumes *contf*: *continuous-on* (*cball* ξ *r*) *f*
and *holf*: *f holomorphic-on ball* ξ *r*
and $0 < r$
and *norm-less*: $\bigwedge z.\ norm(\xi - z) = r \Longrightarrow norm(f\ \xi) < norm(f\ z)$
obtains *z* **where** $z \in ball\ \xi\ r\ f\ z = 0$
proof −
{ assume *fnz*: $\bigwedge w.\ w \in ball\ \xi\ r \Longrightarrow f\ w \neq 0$
then have $0 < norm\ (f\ \xi)$
by (*simp add*: ‹$0 < r$›)
have *fnz'*: $\bigwedge w.\ w \in cball\ \xi\ r \Longrightarrow f\ w \neq 0$
by (*metis norm-less dist-norm fnz less-eq-real-def mem-ball mem-cball norm-not-less-zero norm-zero*)
have *frontier*(*cball* ξ *r*) ≠ {}
using ‹$0 < r$› **by** *simp*
def $g \equiv \lambda z.\ inverse\ (f\ z)$
have *contg*: *continuous-on* (*cball* ξ *r*) *g*
unfolding *g-def* **using** *contf continuous-on-inverse fnz'* **by** *blast*
have *holg*: *g holomorphic-on ball* ξ *r*
unfolding *g-def* **using** *fnz holf holomorphic-on-inverse* **by** *blast*
have *frontier* (*cball* ξ *r*) ⊆ *cball* ξ *r*
by (*simp add*: *subset-iff*)
then have *contf'*: *continuous-on* (*frontier* (*cball* ξ *r*)) *f*
and *contg'*: *continuous-on* (*frontier* (*cball* ξ *r*)) *g*
by (*blast intro*: *contf contg continuous-on-subset*)+
have *froc*: *frontier*(*cball* ξ *r*) ≠ {}
using ‹$0 < r$› **by** *simp*
moreover have *continuous-on* (*frontier* (*cball* ξ *r*)) (*norm o f*)
using *contf' continuous-on-compose continuous-on-norm-id* **by** *blast*
ultimately obtain *w* **where** *w*: $w \in frontier(cball\ \xi\ r)$
and *now*: $\bigwedge x.\ x \in frontier(cball\ \xi\ r) \Longrightarrow norm\ (f\ w) \leq norm\ (f\ x)$
apply (*rule bexE* [*OF continuous-attains-inf* [*OF compact-frontier* [*OF compact-cball*]]])
apply (*simp add*:)
done
then have *fw*: $0 < norm\ (f\ w)$
by (*simp add*: *fnz'*)
have *continuous-on* (*frontier* (*cball* ξ *r*)) (*norm o g*)
using *contg' continuous-on-compose continuous-on-norm-id* **by** *blast*
then obtain *v* **where** *v*: $v \in frontier(cball\ \xi\ r)$
and *nov*: $\bigwedge x.\ x \in frontier(cball\ \xi\ r) \Longrightarrow norm\ (g\ v) \geq norm\ (g\ x)$
apply (*rule bexE* [*OF continuous-attains-sup* [*OF compact-frontier* [*OF compact-cball*] *froc*]])
apply (*simp add*:)
done
then have *fv*: $0 < norm\ (f\ v)$
by (*simp add*: *fnz'*)

```
    have norm ((deriv ^^ 0) g ξ) ≤ fact 0 * norm (g v) / r ^ 0
      by (rule Cauchy-inequality [OF holg contg ‹0 < r›]) (simp add: dist-norm
nov)
    then have cmod (g ξ) ≤ norm (g v)
      by simp
    with w have wr: norm (ξ − w) = r and nfw: norm (f w) ≤ norm (f ξ)
      apply (simp-all add: dist-norm)
      by (metis ‹0 < cmod (f ξ)› g-def less-imp-inverse-less norm-inverse not-le
now order-trans v)
    with fw have False
      using norm-less by force
  }
  with that show ?thesis by blast
qed


theorem open-mapping-thm:
  assumes holf: f holomorphic-on S
      and S: open S connected S
      and open U U ⊆ S
      and fne: ~ f constant-on S
    shows open (f ‘ U)
proof −
  have *: open (f ‘ U)
          if U ≠ {} and U: open U connected U and f holomorphic-on U and
fneU: ⋀x. ∃ y ∈ U. f y ≠ x
          for U
  proof (clarsimp simp: open-contains-ball)
    fix ξ assume ξ: ξ ∈ U
    show ∃ e>0. ball (f ξ) e ⊆ f ‘ U
    proof −
      have hol: (λz. f z − f ξ) holomorphic-on U
        by (rule holomorphic-intros that)+
      obtain s where 0 < s and sbU: ball ξ s ⊆ U
                and sne: ⋀z. z ∈ ball ξ s − {ξ} ⟹ (λz. f z − f ξ) z ≠ 0
        using isolated-zeros [OF hol U ξ]  by (metis fneU right-minus-eq)
      obtain r where 0 < r and r: cball ξ r ⊆ ball ξ s
        apply (rule-tac r=s/2 in that)
        using ‹0 < s› by auto
      have cball ξ r ⊆ U
        using sbU r by blast
      then have frsbU: frontier (cball ξ r) ⊆ U
        using Diff-subset frontier-def order-trans by fastforce
      then have cof: compact (frontier(cball ξ r))
        by blast
      have frne: frontier (cball ξ r) ≠ {}
        using ‹0 < r› by auto
      have contfr: continuous-on (frontier (cball ξ r)) (λz. norm (f z − f ξ))
      apply (rule continuous-on-compose2 [OF Complex-Analysis-Basics.continuous-on-norm-id])
```

```
        using hol frsbU holomorphic-on-imp-continuous-on holomorphic-on-subset
by blast+
      obtain w where norm (ξ − w) = r
                and w: (⋀z. norm (ξ − z) = r ⟹ norm (f w − f ξ) ≤ norm(f z
− f ξ))
        apply (rule bexE [OF continuous-attains-inf [OF cof frne contfr]])
        apply (simp add: dist-norm)
        done
      moreover def ε ≡ norm (f w − f ξ) / 3
      ultimately have 0 < ε
        using ‹0 < r› dist-complex-def r sne by auto
      have ball (f ξ) ε ⊆ f ‘ U
      proof
        fix γ
        assume γ: γ ∈ ball (f ξ) ε
        have ∗: cmod (γ − f ξ) < cmod (γ − f z) if cmod (ξ − z) = r for z
        proof −
          have lt: cmod (f w − f ξ) / 3 < cmod (γ − f z)
            using w [OF that] γ
            using dist-triangle2 [of f ξ γ  f z] dist-triangle2 [of f ξ f z γ]
            by (simp add: ε-def dist-norm norm-minus-commute)
          show ?thesis
            by (metis ε-def dist-commute dist-norm less-trans lt mem-ball γ)
        qed
        have continuous-on (cball ξ r) (λz. γ − f z)
          apply (rule continuous-intros)+
          using ‹cball ξ r ⊆ U› ‹f holomorphic-on U›
        apply (blast intro: continuous-on-subset holomorphic-on-imp-continuous-on)
          done
        moreover have (λz. γ − f z) holomorphic-on ball ξ r
          apply (rule holomorphic-intros)+
         apply (metis ‹cball ξ r ⊆ U› ‹f holomorphic-on U› holomorphic-on-subset
interior-cball interior-subset)
          done
        ultimately obtain z where z ∈ ball ξ r γ − f z = 0
          apply (rule holomorphic-contract-to-zero)
          apply (blast intro!: ‹0 < r› ∗)+
          done
        then show γ ∈ f ‘ U
          using ‹cball ξ r ⊆ U› by fastforce
      qed
      then show ?thesis using ‹0 < ε› by blast
    qed
  qed
  have open (f ‘ X) if X ∈ components U for X
  proof −
    have holfU: f holomorphic-on U
      using ‹U ⊆ S› holf holomorphic-on-subset by blast
    have X ≠ {}
```

```
    using that by (simp add: in-components-nonempty)
  moreover have open X
    using that ‹open U› open-components by auto
  moreover have connected X
    using that in-components-maximal by blast
  moreover have f holomorphic-on X
    by (meson that holfU holomorphic-on-subset in-components-maximal)
  moreover have ∃ y∈X. f y ≠ x for x
  proof (rule ccontr)
    assume not: ¬ (∃ y∈X. f y ≠ x)
    have X ⊆ S
      using ‹U ⊆ S› in-components-subset that by blast
    obtain w where w: w ∈ X using ‹X ≠ {}› by blast
    have wis: w islimpt X
      using w ‹open X› interior-eq by auto
    have hol: (λz. f z − x) holomorphic-on S
      by (simp add: holf holomorphic-on-diff)
    with fne [unfolded constant-on-def] analytic-continuation [OF hol S ‹X ⊆ S›
- wis]
         not ‹X ⊆ S› w
    show False by auto
  qed
  ultimately show ?thesis
    by (rule *)
 qed
 then have open (f ‘ ⋃components U)
  by (metis (no-types, lifting) imageE image-Union open-Union)
 then show ?thesis
  by force
qed
```

No need for S to be connected. But the nonconstant condition is stronger.

```
corollary open-mapping-thm2:
 assumes holf: f holomorphic-on S
    and S: open S
    and open U U ⊆ S
    and fnc: ⋀X. ⟦open X; X ⊆ S; X ≠ {}⟧ ⟹ ~ f constant-on X
  shows open (f ‘ U)
proof −
 have S = ⋃(components S) by simp
 with ‹U ⊆ S› have U = (⋃ C ∈ components S. C ∩ U) by auto
 then have f ‘ U = (⋃ C ∈ components S. f ‘ (C ∩ U))
  using image-UN by fastforce
 moreover
 { fix C assume C ∈ components S
  with S ‹C ∈ components S› open-components in-components-connected
  have C: open C connected C by auto
  have C ⊆ S
    by (metis ‹C ∈ components S› in-components-maximal)
```

```
    have nf: ¬ f constant-on C
      apply (rule fnc)
      using C ‹C ⊆ S› ‹C ∈ components S› in-components-nonempty by auto
    have f holomorphic-on C
      by (metis holf holomorphic-on-subset ‹C ⊆ S›)
    then have open (f ‘ (C ∩ U))
      apply (rule open-mapping-thm [OF - C - - nf])
      apply (simp add: C ‹open U› open-Int, blast)
      done
  } ultimately show ?thesis
    by force
qed

corollary open-mapping-thm3:
  assumes holf: f holomorphic-on S
      and open S and injf: inj-on f S
    shows  open (f ‘ S)
apply (rule open-mapping-thm2 [OF holf])
using assms
apply (simp-all add:)
using injective-not-constant subset-inj-on by blast
```

54.3 Maximum Modulus Principle

If f is holomorphic, then its norm (modulus) cannot exhibit a true local maximum that is properly within the domain of f.

```
proposition maximum-modulus-principle:
  assumes holf: f holomorphic-on S
      and S: open S connected S
      and open U U ⊆ S ξ ∈ U
      and no: ⋀z. z ∈ U ⟹ norm(f z) ≤ norm(f ξ)
    shows f constant-on S
proof (rule ccontr)
  assume ¬ f constant-on S
  then have open (f ‘ U)
    using open-mapping-thm assms by blast
  moreover have ~ open (f ‘ U)
  proof −
    have ∃t. cmod (f ξ − t) < e ∧ t ∉ f ‘ U if 0 < e for e
      apply (rule-tac x=if 0 < Re(f ξ) then f ξ + (e/2) else f ξ − (e/2) in exI)
      using that
      apply (simp add: dist-norm)
      apply (fastforce simp: cmod-Re-le-iff dest!: no dest: sym)
      done
    then show ?thesis
      unfolding open-contains-ball by (metis ‹ξ ∈ U› contra-subsetD dist-norm
imageI mem-ball)
  qed
  ultimately show False
```

by *blast*
qed

proposition *maximum-modulus-frontier*:
assumes *holf*: *f holomorphic-on* (*interior S*)
and *contf*: *continuous-on* (*closure S*) *f*
and *bos*: *bounded S*
and *leB*: $\bigwedge z.\ z \in \mathit{frontier}\ S \Longrightarrow \mathit{norm}(f\ z) \leq B$
and $\xi \in S$
shows $\mathit{norm}(f\ \xi) \leq B$
proof −
have *compact* (*closure S*) **using** *bos*
by (*simp add*: *bounded-closure compact-eq-bounded-closed*)
moreover have *continuous-on* (*closure S*) (*cmod* ∘ *f*)
using *contf continuous-on-compose continuous-on-norm-id* **by** *blast*
ultimately obtain *z* **where** *zin*: $z \in \mathit{closure}\ S$ **and** *z*: $\bigwedge y.\ y \in \mathit{closure}\ S \Longrightarrow (\mathit{cmod} \circ f)\ y \leq (\mathit{cmod} \circ f)\ z$
using *continuous-attains-sup* [*of closure S norm o f*] ‹$\xi \in S$› **by** *auto*
then consider $z \in \mathit{frontier}\ S \mid z \in \mathit{interior}\ S$ **using** *frontier-def* **by** *auto*
then have $\mathit{norm}(f\ z) \leq B$
proof *cases*
case *1* **then show** *?thesis* **using** *leB* **by** *blast*
next
case *2*
have *zin*: $z \in \mathit{connected\text{-}component\text{-}set}\ (\mathit{interior}\ S)\ z$
by (*simp add*: *2*)
have *f constant-on* (*connected-component-set* (*interior S*) *z*)
apply (*rule maximum-modulus-principle* [*OF - - - - - zin*])
apply (*metis connected-component-subset holf holomorphic-on-subset*)
apply (*simp-all add*: *open-connected-component*)
by (*metis closure-subset comp-eq-dest-lhs interior-subset subsetCE z connected-component-in*)
then obtain *c* **where** *c*: $\bigwedge w.\ w \in \mathit{connected\text{-}component\text{-}set}\ (\mathit{interior}\ S)\ z \Longrightarrow f\ w = c$
by (*auto simp*: *constant-on-def*)
have *f* ‘ *closure*(*connected-component-set* (*interior S*) *z*) ⊆ {*c*}
apply (*rule image-closure-subset*)
apply (*meson closure-mono connected-component-subset contf continuous-on-subset interior-subset*)
using *c*
apply *auto*
done
then have *cc*: $\bigwedge w.\ w \in \mathit{closure}(\mathit{connected\text{-}component\text{-}set}\ (\mathit{interior}\ S)\ z) \Longrightarrow f\ w = c$ **by** *blast*
have *frontier*(*connected-component-set* (*interior S*) *z*) ≠ {}
apply (*simp add*: *frontier-eq-empty*)
by (*metis 2 bos bounded-interior connected-component-eq-UNIV connected-component-refl not-bounded-UNIV*)
then obtain *w* **where** *w*: $w \in \mathit{frontier}(\mathit{connected\text{-}component\text{-}set}\ (\mathit{interior}\ S)$

z)
by *auto*
then have *norm* (*f z*) = *norm* (*f w*) **by** (*simp add*: *2 c cc frontier-def*)
also have ... ≤ *B*
apply (*rule leB*)
using *w*
using *frontier-interior-subset frontier-of-connected-component-subset* **by** *blast*
finally show *?thesis* **.**
qed
then show *?thesis*
using *z* ‹ξ ∈ *S*› *closure-subset* **by** *fastforce*
qed

corollary *maximum-real-frontier*:
assumes *holf*: *f holomorphic-on* (*interior S*)
and *contf*: *continuous-on* (*closure S*) *f*
and *bos*: *bounded S*
and *leB*: ⋀*z*. *z* ∈ *frontier S* ⟹ *Re*(*f z*) ≤ *B*
and ξ ∈ *S*
shows *Re*(*f* ξ) ≤ *B*
using *maximum-modulus-frontier* [*of exp o f S exp B*]
Transcendental.continuous-on-exp holomorphic-on-compose holomorphic-on-exp assms
by *auto*

54.4 Factoring out a zero according to its order

lemma *holomorphic-factor-order-of-zero*:
assumes *holf*: *f holomorphic-on S*
and *os*: *open S*
and ξ ∈ *S* 0 < *n*
and *dnz*: (*deriv* ^^ *n*) *f* ξ ≠ *0*
and *dfz*: ⋀*i*. ⟦*0* < *i*; *i* < *n*⟧ ⟹ (*deriv* ^^ *i*) *f* ξ = *0*
obtains *g r* **where** *0* < *r*
g holomorphic-on ball ξ *r*
⋀*w*. *w* ∈ *ball* ξ *r* ⟹ *f w* − *f* ξ = (*w* − ξ) ^ *n* ∗ *g w*
⋀*w*. *w* ∈ *ball* ξ *r* ⟹ *g w* ≠ *0*
proof −
obtain *r* **where** *r*>*0* **and** *r*: *ball* ξ *r* ⊆ *S* **using** *assms* **by** (*blast elim*!: *openE*)
then have *holfb*: *f holomorphic-on ball* ξ *r*
using *holf holomorphic-on-subset* **by** *blast*
def *g* ≡ λ*w*. *suminf* (λ*i*. (*deriv* ^^ (*i* + *n*)) *f* ξ / (*fact*(*i* + *n*)) ∗ (*w* − ξ) ^ *i*)
have *sumsg*: (λ*i*. (*deriv* ^^ (*i* + *n*)) *f* ξ / (*fact*(*i* + *n*)) ∗ (*w* − ξ) ^ *i*) *sums g w*
and *feq*: *f w* − *f* ξ = (*w* − ξ) ^ *n* ∗ *g w*
if *w*: *w* ∈ *ball* ξ *r* **for** *w*
proof −
def *powf* ≡ (λ*i*. (*deriv* ^^ *i*) *f* ξ/(*fact i*) ∗ (*w* − ξ) ^ *i*)
have *sing*: {..<*n*} − {*i*. *powf i* = *0*} = (*if f* ξ = *0 then* {} *else* {*0*})
unfolding *powf-def* **using** ‹*0* < *n*› *dfz* **by** (*auto simp*: *dfz*; *metis funpow-0*

```
not-gr0)
    have powf sums f w
      unfolding powf-def by (rule holomorphic-power-series [OF holfb w])
    moreover have (∑ i<n. powf i) = f ξ
        apply (subst Groups-Big.comm-monoid-add-class.setsum.setdiff-irrelevant
[symmetric])
      apply (simp add:)
      apply (simp only: dfz sing)
      apply (simp add: powf-def)
      done
    ultimately have fsums: (λi. powf (i+n)) sums (f w − f ξ)
      using w sums-iff-shift' by metis
    then have ∗: summable (λi. (w − ξ) ^ n ∗ ((deriv ^^ (i + n)) f ξ ∗ (w − ξ)
^ i / fact (i + n)))
      unfolding powf-def using sums-summable
      by (auto simp: power-add mult-ac)
    have summable (λi. (deriv ^^ (i + n)) f ξ ∗ (w − ξ) ^ i / fact (i + n))
    proof (cases w=ξ)
      case False then show ?thesis
        using summable-mult [OF ∗, of 1 / (w − ξ) ^ n] by (simp add:)
    next
      case True then show ?thesis
        by (auto simp: Power.semiring-1-class.power-0-left intro!: summable-finite
[of {0}]
                 split: if-split-asm)
    qed
    then show sumsg: (λi. (deriv ^^ (i + n)) f ξ / (fact(i + n)) ∗ (w − ξ)^i)
sums g w
      by (simp add: summable-sums-iff g-def)
    show f w − f ξ = (w − ξ)^n ∗ g w
      apply (rule sums-unique2)
      apply (rule fsums [unfolded powf-def])
      using sums-mult [OF sumsg, of (w − ξ) ^ n]
      by (auto simp: power-add mult-ac)
  qed
  then have holg: g holomorphic-on ball ξ r
    by (meson sumsg power-series-holomorphic)
  then have contg: continuous-on (ball ξ r) g
    by (blast intro: holomorphic-on-imp-continuous-on)
  have g ξ ≠ 0
    using dnz unfolding g-def
    by (subst suminf-finite [of {0}]) auto
  obtain d where 0 < d and d: ⋀w. w ∈ ball ξ d ⟹ g w ≠ 0
    apply (rule exE [OF continuous-on-avoid [OF contg - ‹g ξ ≠ 0›]])
    using ‹0 < r›
    apply force
    by (metis ‹0 < r› less-trans mem-ball not-less-iff-gr-or-eq)
  show ?thesis
    apply (rule that [where g=g and r =min r d])
```

```
    using ‹0 < r› ‹0 < d› holg
    apply (auto simp: feq holomorphic-on-subset subset-ball d)
    done
qed


lemma holomorphic-factor-order-of-zero-strong:
  assumes holf: f holomorphic-on S open S  ξ ∈ S 0 < n
      and (deriv ^^ n) f ξ ≠ 0
      and ⋀i. ⟦0 < i; i < n⟧ ⟹ (deriv ^^ i) f ξ = 0
   obtains g r where 0 < r
                     g holomorphic-on ball ξ r
                     ⋀w. w ∈ ball ξ r ⟹ f w − f ξ = ((w − ξ) ∗ g w) ^ n
                     ⋀w. w ∈ ball ξ r ⟹ g w ≠ 0
proof −
  obtain g r where 0 < r
                and holg: g holomorphic-on ball ξ r
                and feq: ⋀w. w ∈ ball ξ r ⟹ f w − f ξ = (w − ξ) ^n ∗ g w
                and gne: ⋀w. w ∈ ball ξ r ⟹ g w ≠ 0
    by (auto intro: holomorphic-factor-order-of-zero [OF assms])
  have con: continuous-on (ball ξ r) (λz. deriv g z / g z)
   by (rule continuous-intros) (auto simp: gne holg holomorphic-deriv holomorphic-on-imp-continuous-on)
  have cd: ⋀x. dist ξ x < r ⟹ (λz. deriv g z / g z) field-differentiable at x
    apply (rule derivative-intros)+
   using holg mem-ball apply (blast intro: holomorphic-deriv holomorphic-on-imp-differentiable-at)
   apply (metis Topology-Euclidean-Space.open-ball at-within-open holg holomorphic-on-def
mem-ball)
    using gne mem-ball by blast
  obtain h where h: ⋀x. x ∈ ball ξ r ⟹ (h has-field-derivative deriv g x / g x)
(at x)
    apply (rule exE [OF holomorphic-convex-primitive [of ball ξ r {} λz. deriv g
z / g z]])
    apply (auto simp: con cd)
    apply (metis open-ball at-within-open mem-ball)
    done
  then have continuous-on (ball ξ r) h
    by (metis open-ball holomorphic-on-imp-continuous-on holomorphic-on-open)
  then have con: continuous-on (ball ξ r) (λx. exp (h x) / g x)
   by (auto intro!: continuous-intros simp add: holg holomorphic-on-imp-continuous-on
gne)
  have 0: dist ξ x < r ⟹ ((λx. exp (h x) / g x) has-field-derivative 0) (at x) for
x
    apply (rule h derivative-eq-intros | simp)+
    apply (rule DERIV-deriv-iff-field-differentiable [THEN iffD2])
    using holg apply (auto simp: holomorphic-on-imp-differentiable-at gne h)
    done
  obtain c where c: ⋀x. x ∈ ball ξ r ⟹ exp (h x) / g x = c
    by (rule DERIV-zero-connected-constant [of ball ξ r {} λx. exp(h x) / g x])
(auto simp: con 0)
```

```
  have hol: (λz. exp ((Ln (inverse c) + h z) / of-nat n)) holomorphic-on ball ξ r
    apply (rule holomorphic-on-compose [unfolded o-def, where g = exp])
    apply (rule holomorphic-intros)+
    using h holomorphic-on-open apply blast
    apply (rule holomorphic-intros)+
    using ‹0 < n› apply (simp add:)
    apply (rule holomorphic-intros)+
    done
  show ?thesis
    apply (rule that [where g=λz. exp((Ln(inverse c) + h z)/n) and r =r])
    using ‹0 < r› ‹0 < n›
    apply (auto simp: feq power-mult-distrib exp-divide-power-eq c [symmetric])
    apply (rule hol)
    apply (simp add: Transcendental.exp-add gne)
    done
qed


lemma
  fixes k :: 'a::wellorder
  assumes a-def: a == LEAST x. P x and P: P k
  shows def-LeastI: P a and def-Least-le: a ≤ k
unfolding a-def
by (rule LeastI Least-le; rule P)+

lemma holomorphic-factor-zero-nonconstant:
  assumes holf: f holomorphic-on S and S: open S connected S
      and ξ ∈ S f ξ = 0
      and nonconst: ⋀c. ∃z ∈ S. f z ≠ c
   obtains g r n
      where 0 < n  0 < r  ball ξ r ⊆ S
            g holomorphic-on ball ξ r
            ⋀w. w ∈ ball ξ r ⟹ f w = (w - ξ) ^ n * g w
            ⋀w. w ∈ ball ξ r ⟹ g w ≠ 0
proof (cases ∀n>0. (deriv ^^ n) f ξ = 0)
  case True then show ?thesis
   using holomorphic-fun-eq-const-on-connected [OF holf S - ‹ξ ∈ S›] nonconst by
auto
next
  case False
  then obtain n0 where n0 > 0 and n0: (deriv ^^ n0) f ξ ≠ 0 by blast
  obtain r0 where r0 > 0 ball ξ r0 ⊆ S using S openE ‹ξ ∈ S› by auto
  def n ≡ LEAST n. (deriv ^^ n) f ξ ≠ 0
  have n-ne: (deriv ^^ n) f ξ ≠ 0
    by (rule def-LeastI [OF n-def]) (rule n0)
  then have 0 < n using ‹f ξ = 0›
    using funpow-0 by fastforce
  have n-min: ⋀k. k < n ⟹ (deriv ^^ k) f ξ = 0
    using def-Least-le [OF n-def] not-le by blast
```

then obtain *g r1*
where $0 < r1$ *g holomorphic-on ball* ξ *r1*
$\bigwedge w.\ w \in ball\ \xi\ r1 \Longrightarrow f\ w = (w - \xi)\ \hat{}\ n * g\ w$
$\bigwedge w.\ w \in ball\ \xi\ r1 \Longrightarrow g\ w \neq 0$
by (*auto intro*: *holomorphic-factor-order-of-zero* [*OF holf* ‹*open S*› ‹$\xi \in S$› ‹$n > 0$› *n-ne*] *simp*: ‹$f\ \xi = 0$›)
then show *?thesis*
apply (*rule-tac g=g* **and** *r=min r0 r1* **and** *n=n* **in** *that*)
using ‹$0 < n$› ‹$0 < r0$› ‹$0 < r1$› ‹*ball* ξ *r0* $\subseteq S$›
apply (*auto simp*: *subset-ball intro*: *holomorphic-on-subset*)
done
qed

lemma *holomorphic-lower-bound-difference*:
assumes *holf*: *f holomorphic-on S* **and** *S*: *open S connected S*
and $\xi \in S$ **and** $\varphi \in S$
and *fne*: $f\ \varphi \neq f\ \xi$
obtains *k n r*
where $0 < k$ $0 < r$
ball ξ *r* $\subseteq S$
$\bigwedge w.\ w \in ball\ \xi\ r \Longrightarrow k * norm(w - \xi)\ \hat{}n \leq norm(f\ w - f\ \xi)$
proof −
def $n \equiv LEAST\ n.\ 0 < n \wedge (deriv\ \hat{}\hat{}\ n)\ f\ \xi \neq 0$
obtain *n0* **where** $0 < n0$ **and** *n0*: $(deriv\ \hat{}\hat{}\ n0)\ f\ \xi \neq 0$
using *fne holomorphic-fun-eq-const-on-connected* [*OF holf S*] ‹$\xi \in S$› ‹$\varphi \in S$›
by *blast*
then have $0 < n$ **and** *n-ne*: $(deriv\ \hat{}\hat{}\ n)\ f\ \xi \neq 0$
unfolding *n-def* **by** (*metis* (*mono-tags*, *lifting*) *LeastI*)+
have *n-min*: $\bigwedge k.\ [\![0 < k; k < n]\!] \Longrightarrow (deriv\ \hat{}\hat{}\ k)\ f\ \xi = 0$
unfolding *n-def* **by** (*blast dest*: *not-less-Least*)
then obtain *g r*
where $0 < r$ **and** *holg*: *g holomorphic-on ball* ξ *r*
and *fne*: $\bigwedge w.\ w \in ball\ \xi\ r \Longrightarrow f\ w - f\ \xi = (w - \xi)\ \hat{}\ n * g\ w$
and *gnz*: $\bigwedge w.\ w \in ball\ \xi\ r \Longrightarrow g\ w \neq 0$
by (*auto intro*: *holomorphic-factor-order-of-zero* [*OF holf* ‹*open S*› ‹$\xi \in S$› ‹$n > 0$› *n-ne*])
obtain *e* **where** $e{>}0$ **and** *e*: *ball* ξ *e* $\subseteq S$ **using** *assms* **by** (*blast elim*!: *openE*)
then have *holfb*: *f holomorphic-on ball* ξ *e*
using *holf holomorphic-on-subset* **by** *blast*
def $d \equiv (min\ e\ r)\ /\ 2$
have $0 < d$ **using** ‹$0 < r$› ‹$0 < e$› **by** (*simp add*: *d-def*)
have $d < r$
using ‹$0 < r$› **by** (*auto simp*: *d-def*)
then have *cbb*: *cball* ξ *d* $\subseteq$ *ball* ξ *r*
by (*auto simp*: *cball-subset-ball-iff*)
then have *g holomorphic-on cball* ξ *d*
by (*rule holomorphic-on-subset* [*OF holg*])
then have *closed* (*g* ‘ *cball* ξ *d*)

by (*simp add*: *compact-imp-closed compact-continuous-image holomorphic-on-imp-continuous-on*)
moreover have *g ' cball ξ d ≠ {}*
using ‹*0 < d*› **by** *auto*
ultimately obtain *x* **where** *x*: *x ∈ g ' cball ξ d* **and** ⋀*y*. *y ∈ g ' cball ξ d ⟹ dist 0 x ≤ dist 0 y*
by (*rule distance-attains-inf*) *blast*
then have *leg*: ⋀*w*. *w ∈ cball ξ d ⟹ norm x ≤ norm* (*g w*)
by *auto*
have *ball ξ d ⊆ cball ξ d* **by** *auto*
also have *... ⊆ ball ξ e* **using** ‹*0 < d*› *d-def* **by** *auto*
also have *... ⊆ S* **by** (*rule e*)
finally have *dS*: *ball ξ d ⊆ S* **.**
moreover have *x ≠ 0* **using** *gnz x* ‹*d < r*› **by** *auto*
ultimately show *?thesis*
apply (*rule-tac k=norm x* **and** *n=n* **and** *r=d* **in** *that*)
using ‹*d < r*› *leg*
apply (*auto simp*: ‹*0 < d*› *fne norm-mult norm-power algebra-simps mult-right-mono*)
done
qed

lemma
assumes *holf*: *f holomorphic-on* (*S − {ξ}*) **and** *ξ*: *ξ ∈ interior S*
shows *holomorphic-on-extend-lim*:
(∃*g*. *g holomorphic-on S ∧* (∀*z ∈ S − {ξ}*. *g z = f z*)) ⟷
((λ*z*. (*z − ξ*) ∗ *f z*) ⟶ *0*) (*at ξ*)
(**is** *?P = ?Q*)
and *holomorphic-on-extend-bounded*:
(∃*g*. *g holomorphic-on S ∧* (∀*z ∈ S − {ξ}*. *g z = f z*)) ⟷
(∃*B*. *eventually* (λ*z*. *norm*(*f z*) ≤ *B*) (*at ξ*))
(**is** *?P = ?R*)
proof −
obtain *δ* **where** *0 < δ* **and** *δ*: *ball ξ δ ⊆ S*
using *ξ mem-interior* **by** *blast*
have *?R* **if** *holg*: *g holomorphic-on S* **and** *gf*: ⋀*z*. *z ∈ S − {ξ} ⟹ g z = f z*
for *g*
proof −
have ∗: ∀ *F z in at ξ*. *dist* (*g z*) (*g ξ*) *< 1 ⟶ cmod* (*f z*) ≤ *cmod* (*g ξ*) + *1*
apply (*simp add*: *eventually-at*)
apply (*rule-tac x=δ* **in** *exI*)
using *δ* ‹*0 < δ*›
apply (*clarsimp simp*:)
apply (*drule-tac c=x* **in** *subsetD*)
apply (*simp add*: *dist-commute*)
by (*metis DiffI add.commute diff-le-eq dist-norm gf le-less-trans less-eq-real-def norm-triangle-ineq2 singletonD*)
have *continuous-on* (*interior S*) *g*
by (*meson continuous-on-subset holg holomorphic-on-imp-continuous-on interior-subset*)
then have ⋀*x*. *x ∈ interior S ⟹* (*g* ⟶ *g x*) (*at x*)
using *continuous-on-interior continuous-within holg holomorphic-on-imp-continuous-on*

```
by blast
    then have (g ⟶ g ξ) (at ξ)
      by (simp add: ξ)
    then show ?thesis
      apply (rule-tac x=norm(g ξ) + 1 in exI)
      apply (rule eventually-mp [OF ∗ tendstoD [where e=1]], auto)
      done
  qed
  moreover have ?Q if ∀ F z in at ξ. cmod (f z) ≤ B for B
    by (rule lim-null-mult-right-bounded [OF - that]) (simp add: LIM-zero)
  moreover have ?P if (λz. (z − ξ) ∗ f z) −ξ→ 0
  proof −
    def h ≡ λz. (z − ξ) ^2 ∗ f z
    have h0: (h has-field-derivative 0) (at ξ)
      apply (simp add: h-def Derivative.DERIV-within-iff )
      apply (rule Lim-transform-within [OF that, of 1])
      apply (auto simp: divide-simps power2-eq-square)
      done
    have holh: h holomorphic-on S
    proof (simp add: holomorphic-on-def, clarify)
      fix z assume z ∈ S
      show h field-differentiable at z within S
      proof (cases z = ξ)
        case True then show ?thesis
          using field-differentiable-at-within field-differentiable-def h0 by blast
      next
        case False
        then have f field-differentiable at z within S
          using holomorphic-onD [OF holf, of z] ‹z ∈ S›
          unfolding field-differentiable-def DERIV-within-iff
           by (force intro: exI [where x=dist ξ z] elim: Lim-transform-within-set
[unfolded eventually-at])
        then show ?thesis
          by (simp add: h-def power2-eq-square derivative-intros)
      qed
    qed
    def g ≡ λz. if z = ξ then deriv h ξ else (h z − h ξ) / (z − ξ)
    have holg: g holomorphic-on S
      unfolding g-def by (rule pole-lemma [OF holh ξ])
    show ?thesis
       apply (rule-tac x=λz. if z = ξ then deriv g ξ else (g z − g ξ)/(z − ξ) in
exI)
      apply (rule conjI)
      apply (rule pole-lemma [OF holg ξ])
      apply (auto simp: g-def power2-eq-square divide-simps)
      using h0 apply (simp add: h0 DERIV-imp-deriv h-def power2-eq-square)
      done
  qed
  ultimately show ?P = ?Q and ?P = ?R
```

```
   by meson+
qed


proposition pole-at-infinity:
 assumes holf: f holomorphic-on UNIV and lim: ((inverse o f) ⟶ l) at-infinity
 obtains a n where ⋀z. f z = (∑ i≤n. a i * z^i)
proof (cases l = 0)
 case False
 with tendsto-inverse [OF lim] show ?thesis
  apply (rule-tac a=(λn. inverse l) and n=0 in that)
  apply (simp add: Liouville-weak [OF holf, of inverse l])
  done
next
 case True
 then have [simp]: l = 0 .
 show ?thesis
 proof (cases ∃ r. 0 < r ∧ (∀ z ∈ ball 0 r - {0}. f(inverse z) ≠ 0))
  case True
   then obtain r where 0 < r and r: ⋀z. z ∈ ball 0 r - {0} ⟹ f(inverse
z) ≠ 0
        by auto
   have 1: inverse ∘ f ∘ inverse holomorphic-on ball 0 r - {0}
    by (rule holomorphic-on-compose holomorphic-intros holomorphic-on-subset
[OF holf] | force simp: r)+
   have 2: 0 ∈ interior (ball 0 r)
    using ‹0 < r› by simp
   have ∃ B. 0<B ∧ eventually (λz. cmod ((inverse ∘ f ∘ inverse) z) ≤ B) (at
0)
    apply (rule exI [where x=1])
    apply (simp add:)
    using tendstoD [OF lim [unfolded lim-at-infinity-0] zero-less-one]
    apply (rule eventually-mono)
    apply (simp add: dist-norm)
    done
   with holomorphic-on-extend-bounded [OF 1 2]
   obtain g where holg: g holomorphic-on ball 0 r
          and geq: ⋀z. z ∈ ball 0 r - {0} ⟹ g z = (inverse ∘ f ∘ inverse) z
    by meson
   have ifi0: (inverse ∘ f ∘ inverse) -0→ 0
    using ‹l = 0› lim lim-at-infinity-0 by blast
   have g2g0: g -0→ g 0
    using ‹0 < r› centre-in-ball continuous-at continuous-on-eq-continuous-at
holg
    by (blast intro: holomorphic-on-imp-continuous-on)
   have g2g1: g -0→ 0
    apply (rule Lim-transform-within-open [OF ifi0 open-ball [of 0 r]])
    using ‹0 < r› by (auto simp: geq)
   have [simp]: g 0 = 0
```

```
      by (rule tendsto-unique [OF - g2g0 g2g1]) simp
    have ball 0 r - {0::complex} ≠ {}
      using ‹0 < r›
      apply (clarsimp simp: ball-def dist-norm)
      apply (drule-tac c=of-real r/2 in subsetD, auto)
      done
    then obtain w::complex where w ≠ 0 and w: norm w < r by force
    then have g w ≠ 0 by (simp add: geq r)
    obtain B n e where 0 < B 0 < e e ≤ r
                and leg: ⋀w. norm w < e ⟹ B * cmod w ^ n ≤ cmod (g w)
    apply (rule holomorphic-lower-bound-difference [OF holg open-ball connected-ball,
of 0 w])
      using ‹0 < r› w ‹g w ≠ 0› by (auto simp: ball-subset-ball-iff)
    have cmod (f z) ≤ cmod z ^ n / B if 2/e ≤ cmod z for z
    proof -
      have ize: inverse z ∈ ball 0 e - {0} using that ‹0 < e›
        by (auto simp: norm-divide divide-simps algebra-simps)
      then have [simp]: z ≠ 0 and izr: inverse z ∈ ball 0 r - {0} using ‹e ≤
r›
        by auto
      then have [simp]: f z ≠ 0
        using r [of inverse z] by simp
      have [simp]: f z = inverse (g (inverse z))
        using izr geq [of inverse z] by simp
      show ?thesis using ize leg [of inverse z] ‹0 < B› ‹0 < e›
        by (simp add: divide-simps norm-divide algebra-simps)
    qed
    then show ?thesis
      apply (rule-tac a = λk. (deriv ^^ k) f 0 / (fact k) and n=n in that)
      apply (rule-tac A = 2/e and B = 1/B in Liouville-polynomial [OF holf])
      apply (simp add:)
      done
  next
    case False
    then have fi0: ⋀r. r > 0 ⟹ ∃z∈ball 0 r - {0}. f (inverse z) = 0
      by simp
    have fz0: f z = 0 if 0 < r and lt1: ⋀x. x ≠ 0 ⟹ cmod x < r ⟹ inverse
(cmod (f (inverse x))) < 1
            for z r
    proof -
      have f0: (f ⟶ 0) at-infinity
      proof -
       have DIM-complex[intro]: 2 ≤ DIM(complex) — should not be necessary!
         by simp
       have continuous-on (inverse ‘ (ball 0 r - {0})) f
          using continuous-on-subset holf holomorphic-on-imp-continuous-on by
blast
       then have connected ((f ∘ inverse) ‘ (ball 0 r - {0}))
         apply (intro connected-continuous-image continuous-intros)
```

apply (*force intro*: *connected-punctured-ball*)+
done
then have ⟦$w \neq 0$; *cmod* $w < r$⟧ ⟹ f (*inverse* w) = *0* **for** w
apply (*rule disjE* [*OF connected-closedD* [**where** $A = \{0\}$ **and** $B = -$ *ball 0 1*]], *auto*)
apply (*metis* (*mono-tags*, *hide-lams*) *not-less-iff-gr-or-eq one-less-inverse lt1 zero-less-norm-iff*)
using *False* ‹*0* < *r*› **apply** *fastforce*
by (*metis* (*no-types*, *hide-lams*) *Compl-iff IntI comp-apply empty-iff image-eqI insert-Diff-single insert-iff mem-ball-0 not-less-iff-gr-or-eq one-less-inverse that*(*2*) *zero-less-norm-iff*)
then show *?thesis*
apply (*simp add*: *lim-at-infinity-0*)
apply (*rule Lim-eventually*)
apply (*simp add*: *eventually-at*)
apply (*rule-tac x*=*r* **in** *exI*)
apply (*simp add*: ‹*0* < *r*› *dist-norm*)
done
qed
obtain w **where** $w \in$ *ball 0 r* − {*0*} **and** f (*inverse* w) = *0*
using *False* ‹*0* < *r*› **by** *blast*
then show *?thesis*
by (*auto simp*: *f0 Liouville-weak* [*OF holf*, *of 0*])
qed
show *?thesis*
apply (*rule that* [*of* λ*n*. *0 0*])
using *lim* [*unfolded lim-at-infinity-0*]
apply (*simp add*: *Lim-at dist-norm norm-inverse*)
apply (*drule-tac x*=*1* **in** *spec*)
using *fz0* **apply** *auto*
done
qed
qed

54.5 Entire proper functions are precisely the non-trivial polynomials

proposition *proper-map-polyfun*:
fixes c :: *nat* ⇒ 'a::{*real-normed-div-algebra*,*heine-borel*}
assumes *closed* S **and** *compact* K **and** c: $c\ i \neq 0\ 1 \leq i\ i \leq n$
shows *compact* ($S \cap \{z.\ (\sum i{\leq}n.\ c\ i * z\hat{}i) \in K\}$)
proof −
obtain B **where** $B > 0$ **and** B: ⋀x. $x \in K$ ⟹ *norm* $x \leq B$
by (*metis compact-imp-bounded* ‹*compact K*› *bounded-pos*)
have ∗: *norm* $x \leq b$
if ⋀x. $b \leq$ *norm* x ⟹ $B + 1 \leq$ *norm* ($\sum i{\leq}n.\ c\ i * x\ \hat{}\ i$) ($\sum i{\leq}n.\ c\ i * x\ \hat{}\ i$) ∈ K **for** $b\ x$
proof −
have *norm* ($\sum i{\leq}n.\ c\ i * x\ \hat{}\ i$) ≤ B

```
      using B that by blast
    moreover have ¬ B + 1 ≤ B
      by simp
    ultimately show norm x ≤ b
      using that by (metis (no-types) less-eq-real-def not-less order-trans)
  qed
  have bounded {z. (∑ i≤n. c i * z ^ i) ∈ K}
    using polyfun-extremal [where c=c and B=B+1, OF c]
    by (auto simp: bounded-pos eventually-at-infinity-pos *)
  moreover have closed {z. (∑ i≤n. c i * z ^ i) ∈ K}
    apply (rule allI continuous-closed-preimage-univ continuous-intros)+
    using ‹compact K› compact-eq-bounded-closed by blast
  ultimately show ?thesis
    using closed-Int-compact [OF ‹closed S›] compact-eq-bounded-closed by blast
qed

corollary proper-map-polyfun-univ:
    fixes c :: nat ⇒ 'a::{real-normed-div-algebra,heine-borel}
  assumes compact K c i ≠ 0 1 ≤ i i ≤ n
    shows compact ({z. (∑ i≤n. c i * z^i) ∈ K})
using proper-map-polyfun [of UNIV K c i n] assms by simp


proposition proper-map-polyfun-eq:
  assumes f holomorphic-on UNIV
    shows (∀ k. compact k ⟶ compact {z. f z ∈ k}) ⟷
          (∃ c n. 0 < n ∧ (c n ≠ 0) ∧ f = (λz. ∑ i≤n. c i * z^i))
         (is ?lhs = ?rhs)
proof
  assume compf [rule-format]: ?lhs
  have 2: ∃ k. 0 < k ∧ a k ≠ 0 ∧ f = (λz. ∑ i ≤ k. a i * z ^ i)
       if ⋀z. f z = (∑ i≤n. a i * z ^ i) for a n
  proof (cases ∀ i≤n. 0<i ⟶ a i = 0)
    case True
    then have [simp]: ⋀z. f z = a 0
      by (simp add: that setsum-atMost-shift)
    have False using compf [of {a 0}] by simp
    then show ?thesis ..
  next
    case False
    then obtain k where k: 0 < k k≤n a k ≠ 0 by force
    def m ≡ GREATEST k. k≤n ∧ a k ≠ 0
    have m: m≤n ∧ a m ≠ 0
      unfolding m-def
      apply (rule GreatestI [where b = Suc n])
      using k apply auto
      done
    have [simp]: a i = 0 if m < i i ≤ n for i
      using Greatest-le [where b = Suc n and P = λk. k≤n ∧ a k ≠ 0]
```

using *m-def not-le that* **by** *auto*
have $k \leq m$
unfolding *m-def*
apply (*rule Greatest-le* [**where** $b = Suc\ n$])
using *k* **apply** *auto*
done
with *k m* **show** *?thesis*
by (*rule-tac x=m* **in** *exI*) (*auto simp*: *that comm-monoid-add-class.setsum.mono-neutral-right*)
qed
have $((inverse \circ f) \longrightarrow 0)$ *at-infinity*
proof (*rule Lim-at-infinityI*)
fix *e*::*real* **assume** $0 < e$
with *compf* [*of cball 0* (*inverse e*)]
show $\exists B.\ \forall x.\ B \leq cmod\ x \longrightarrow dist\ ((inverse \circ f)\ x)\ 0 \leq e$
apply (*simp add*:)
apply (*clarsimp simp add*: *compact-eq-bounded-closed bounded-pos norm-inverse*)
apply (*rule-tac x=b+1* **in** *exI*)
apply (*metis inverse-inverse-eq less-add-same-cancel2 less-imp-inverse-less add.commute not-le not-less-iff-gr-or-eq order-trans zero-less-one*)
done
qed
then show *?rhs*
apply (*rule pole-at-infinity* [*OF assms*])
using *2* **apply** *blast*
done
next
assume *?rhs*
then obtain *c n* **where** $0 < n\ c\ n \neq 0\ f = (\lambda z.\ \sum i{\leq}n.\ c\ i * z\ \hat{}\ i)$ **by** *blast*
then have *compact* $\{z.\ f\ z \in k\}$ **if** *compact k* **for** *k*
by (*auto intro*: *proper-map-polyfun-univ* [*OF that*])
then show *?lhs* **by** *blast*
qed

54.6 Relating invertibility and nonvanishing of derivative

proposition *has-complex-derivative-locally-injective*:
assumes *holf*: *f holomorphic-on S*
and *S*: $\xi \in S$ *open S*
and *dnz*: *deriv f* $\xi \neq 0$
obtains *r* **where** $r > 0$ *ball* $\xi\ r \subseteq S$ *inj-on f* (*ball* $\xi\ r$)
proof −
have ∗: $\exists d{>}0.\ \forall x.\ dist\ \xi\ x < d \longrightarrow onorm\ (\lambda v.\ deriv\ f\ x * v - deriv\ f\ \xi * v) < e$ **if** $e > 0$ **for** *e*
proof −
have *contdf*: *continuous-on S* (*deriv f*)
by (*simp add*: *holf holomorphic-deriv holomorphic-on-imp-continuous-on* ‹*open S*›)
obtain δ **where** $\delta{>}0$ **and** δ: $\bigwedge x.\ [\![x \in S;\ dist\ x\ \xi \leq \delta]\!] \Longrightarrow cmod\ (deriv\ f\ x - deriv\ f\ \xi) \leq e/2$

using *continuous-onE* [*OF contdf* ‹$\xi \in S$›, *of e/2*] ‹$0 < e$›
by (*metis dist-complex-def half-gt-zero less-imp-le*)
obtain ε **where** $\varepsilon{>}0$ *ball* ξ $\varepsilon \subseteq S$
by (*metis openE* [*OF* ‹*open S*› ‹$\xi \in S$›])
with ‹$\delta{>}0$› **have** $\exists \delta{>}0.\ \forall x.\ dist\ \xi\ x < \delta \longrightarrow onorm\ (\lambda v.\ deriv\ f\ x * v - deriv\ f\ \xi * v) \leq e/2$
apply (*rule-tac x=min* δ ε **in** *exI*)
apply (*intro conjI allI impI Operator-Norm.onorm-le*)
apply (*simp add*:)
apply (*simp only*: *Rings.ring-class.left-diff-distrib* [*symmetric*] *norm-mult*)
apply (*rule mult-right-mono* [*OF* δ])
apply (*auto simp*: *dist-commute Rings.ordered-semiring-class.mult-right-mono* δ)
done
with ‹$e{>}0$› **show** *?thesis* **by** *force*
qed
have *inj* (*op* $*$ (*deriv f* ξ))
using *dnz* **by** *simp*
then obtain g' **where** g': *linear* g' $g' \circ$ *op* $*$ (*deriv f* ξ) $=$ *id*
using *linear-injective-left-inverse* [*of op* $*$ (*deriv f* ξ)]
by (*auto simp*: *linear-times*)
show *?thesis*
apply (*rule has-derivative-locally-injective* [*OF S*, **where** $f{=}f$ **and** $f' = \lambda z\ h.\ deriv\ f\ z * h$ **and** $g' = g'$])
using g' $*$
apply (*simp-all add*: *linear-conv-bounded-linear that*)
using *DERIV-deriv-iff-field-differentiable has-field-derivative-imp-has-derivative holf*
holomorphic-on-imp-differentiable-at ‹*open S*› **apply** *blast*
done
qed

proposition *has-complex-derivative-locally-invertible*:
assumes *holf*: *f holomorphic-on S*
and *S*: $\xi \in S$ *open S*
and *dnz*: *deriv f* $\xi \neq 0$
obtains *r* **where** $r > 0$ *ball* ξ *r* $\subseteq S$ *open* (*f* ‘ (*ball* ξ *r*)) *inj-on f* (*ball* ξ *r*)
proof −
obtain *r* **where** $r > 0$ *ball* ξ *r* $\subseteq S$ *inj-on f* (*ball* ξ *r*)
by (*blast intro*: *that has-complex-derivative-locally-injective* [*OF assms*])
then have ξ: $\xi \in$ *ball* ξ *r* **by** *simp*
then have *nc*: $\sim$ *f constant-on ball* ξ *r*
using ‹*inj-on f* (*ball* ξ *r*)› *injective-not-constant* **by** *fastforce*
have *holf'*: *f holomorphic-on ball* ξ *r*
using ‹*ball* ξ *r* $\subseteq S$› *holf holomorphic-on-subset* **by** *blast*
have *open* (*f* ‘ *ball* ξ *r*)
apply (*rule open-mapping-thm* [*OF holf'*])
using *nc* **apply** *auto*

```
    done
  then show ?thesis
    using ‹0 < r› ‹ball ξ r ⊆ S› ‹inj-on f (ball ξ r)› that  by blast
qed


proposition holomorphic-injective-imp-regular:
  assumes holf: f holomorphic-on S
      and open S and injf: inj-on f S
      and ξ ∈ S
    shows deriv f ξ ≠ 0
proof −
  obtain r where r>0 and r: ball ξ r ⊆ S using assms by (blast elim!: openE)
  have holf': f holomorphic-on ball ξ r
    using ‹ball ξ r ⊆ S› holf holomorphic-on-subset by blast
  show ?thesis
  proof (cases ∀ n>0. (deriv ^^ n) f ξ = 0)
    case True
    have fcon: f w = f ξ if w ∈ ball ξ r for w
      apply (rule holomorphic-fun-eq-const-on-connected [OF holf'])
      using True ‹0 < r› that by auto
    have False
      using fcon [of ξ + r/2] ‹0 < r› r injf unfolding inj-on-def
     by (metis ‹ξ ∈ S› contra-subsetD dist-commute fcon mem-ball perfect-choose-dist)
    then show ?thesis ..
  next
    case False
    then obtain n0 where n0: n0 > 0 ∧ (deriv ^^ n0) f ξ ≠ 0 by blast
    def n ≡ LEAST n. n > 0 ∧ (deriv ^^ n) f ξ ≠ 0
    have n-ne: n > 0 (deriv ^^ n) f ξ ≠ 0
      using def-LeastI [OF n-def n0] by auto
    have n-min: ⋀k. 0 < k ⟹ k < n ⟹ (deriv ^^ k) f ξ = 0
      using def-Least-le [OF n-def] not-le by auto
    obtain g δ where 0 < δ
          and holg: g holomorphic-on ball ξ δ
          and fd: ⋀w. w ∈ ball ξ δ ⟹ f w − f ξ = ((w − ξ) ∗ g w) ^ n
          and gnz: ⋀w. w ∈ ball ξ δ ⟹ g w ≠ 0
      apply (rule holomorphic-factor-order-of-zero-strong [OF holf ‹open S› ‹ξ ∈
S› n-ne])
      apply (blast intro: n-min)+
      done
    show ?thesis
    proof (cases n=1)
      case True
      with n-ne show ?thesis by auto
    next
      case False
      have holgw: (λw. (w − ξ) ∗ g w) holomorphic-on ball ξ (min r δ)
        apply (rule holomorphic-intros)+
```

using *holg* **by** (*simp add*: *holomorphic-on-subset subset-ball*)
have *gd*: $\bigwedge w.\ dist\ \xi\ w < \delta \Longrightarrow (g\ has\text{-}field\text{-}derivative\ deriv\ g\ w)\ (at\ w)$
using *holg*
by (*simp add*: *DERIV-deriv-iff-field-differentiable holomorphic-on-def at-within-open-NO-MATCH*)
have $*$: $\bigwedge w.\ w \in ball\ \xi\ (min\ r\ \delta) \Longrightarrow ((\lambda w.\ (w - \xi) * g\ w)\ has\text{-}field\text{-}derivative\ ((w - \xi) * deriv\ g\ w + g\ w))\ (at\ w)$
by (*rule gd derivative-eq-intros* | *simp*)+
have [*simp*]: $deriv\ (\lambda w.\ (w - \xi) * g\ w)\ \xi \neq 0$
using $*$ [*of* ξ] ‹$0 < \delta$› ‹$0 < r$› **by** (*simp add*: *DERIV-imp-deriv gnz*)
obtain *T* **where** $\xi \in T$ *open T* **and** *Tsb*: $T \subseteq ball\ \xi\ (min\ r\ \delta)$ **and** *oimT*: *open* $((\lambda w.\ (w - \xi) * g\ w)\ `\ T)$
apply (*rule has-complex-derivative-locally-invertible* [*OF holgw*, *of* ξ])
using ‹$0 < r$› ‹$0 < \delta$›
apply (*simp-all add*:)
by (*meson Topology-Euclidean-Space.open-ball centre-in-ball*)
def $U \equiv (\lambda w.\ (w - \xi) * g\ w)\ `\ T$
have *open U* **by** (*metis oimT U-def*)
have $0 \in U$
apply (*auto simp*: *U-def*)
apply (*rule image-eqI* [**where** $x = \xi$])
apply (*auto simp*: ‹$\xi \in T$›)
done
then obtain ε **where** $\varepsilon{>}0$ **and** ε: *cball* $0\ \varepsilon \subseteq U$
using ‹*open U*› *open-contains-cball* **by** *blast*
then have $\varepsilon * exp(2 * of\text{-}real\ pi * ii * (0/n)) \in cball\ 0\ \varepsilon$
$\varepsilon * exp(2 * of\text{-}real\ pi * ii * (1/n)) \in cball\ 0\ \varepsilon$
by (*auto simp*: *norm-mult*)
with ε **have** $\varepsilon * exp(2 * of\text{-}real\ pi * ii * (0/n)) \in U$
$\varepsilon * exp(2 * of\text{-}real\ pi * ii * (1/n)) \in U$ **by** *blast*+
then obtain *y0 y1* **where** $y0 \in T$ **and** *y0*: $(y0 - \xi) * g\ y0 = \varepsilon * exp(2 * of\text{-}real\ pi * ii * (0/n))$
and $y1 \in T$ **and** *y1*: $(y1 - \xi) * g\ y1 = \varepsilon * exp(2 * of\text{-}real\ pi * ii * (1/n))$
by (*auto simp*: *U-def*)
then have *y0* $\in$ *ball* $\xi\ \delta$ *y1* $\in$ *ball* $\xi\ \delta$ **using** *Tsb* **by** *auto*
moreover have $y0 \neq y1$
using *y0 y1* ‹$\varepsilon > 0$› *complex-root-unity-eq-1* [*of n 1*] ‹$n > 0$› *False* **by** *auto*
moreover have $T \subseteq S$
by (*meson Tsb min.cobounded1 order-trans r subset-ball*)
ultimately have *False*
using *inj-onD* [*OF injf*, *of y0 y1*] ‹*y0* $\in$ *T*› ‹*y1* $\in$ *T*›
using *fd* [*of y0*] *fd* [*of y1*] *complex-root-unity* [*of n 1*]
apply (*simp add*: *y0 y1 power-mult-distrib*)
apply (*force simp*: *algebra-simps*)
done
then show *?thesis* **..**
qed

```
  qed
qed
```

Hence a nice clean inverse function theorem

```
proposition holomorphic-has-inverse:
  assumes holf: f holomorphic-on S
      and open S and injf: inj-on f S
  obtains g where g holomorphic-on (f ‘ S)
                  ⋀z. z ∈ S ⟹ deriv f z * deriv g (f z) = 1
                  ⋀z. z ∈ S ⟹ g(f z) = z
proof −
  have ofs: open (f ‘ S)
    by (rule open-mapping-thm3 [OF assms])
  have contf: continuous-on S f
    by (simp add: holf holomorphic-on-imp-continuous-on)
  have ∗: (the-inv-into S f has-field-derivative inverse (deriv f z)) (at (f z)) if z ∈
S for z
  proof −
    have 1: (f has-field-derivative deriv f z) (at z)
    using DERIV-deriv-iff-field-differentiable ‹z ∈ S› ‹open S› holf holomorphic-on-imp-differentiable-at
      by blast
    have 2: deriv f z ≠ 0
      using ‹z ∈ S› ‹open S› holf holomorphic-injective-imp-regular injf by blast
    show ?thesis
      apply (rule has-complex-derivative-inverse-strong [OF 1 2 ‹open S› ‹z ∈ S›])
       apply (simp add: holf holomorphic-on-imp-continuous-on)
      by (simp add: injf the-inv-into-f-f)
  qed
  show ?thesis
    proof
      show the-inv-into S f holomorphic-on f ‘ S
        by (simp add: holomorphic-on-open ofs) (blast intro: ∗)
    next
      fix z assume z ∈ S
      have deriv f z ≠ 0
        using ‹z ∈ S› ‹open S› holf holomorphic-injective-imp-regular injf by blast
      then show deriv f z * deriv (the-inv-into S f) (f z) = 1
        using ∗ [OF ‹z ∈ S›] by (simp add: DERIV-imp-deriv)
    next
      fix z assume z ∈ S
      show the-inv-into S f (f z) = z
        by (simp add: ‹z ∈ S› injf the-inv-into-f-f)
  qed
qed
```

54.7 The Schwarz Lemma

```
lemma Schwarz1:
  assumes holf: f holomorphic-on S
```

```
    and contf: continuous-on (closure S) f
    and S: open S connected S
    and boS: bounded S
    and S ≠ {}
  obtains w where w ∈ frontier S
                ⋀z. z ∈ closure S ⟹ norm (f z) ≤ norm (f w)
proof −
  have connf: continuous-on (closure S) (norm o f)
    using contf continuous-on-compose continuous-on-norm-id by blast
  have coc: compact (closure S)
    by (simp add: ‹bounded S› bounded-closure compact-eq-bounded-closed)
  then obtain x where x: x ∈ closure S and xmax: ⋀z. z ∈ closure S ⟹ norm(f
z) ≤ norm(f x)
    apply (rule bexE [OF continuous-attains-sup [OF - - connf]])
    using ‹S ≠ {}› apply auto
    done
  then show ?thesis
  proof (cases x ∈ frontier S)
    case True
    then show ?thesis using that xmax by blast
  next
    case False
    then have x ∈ S
      using ‹open S› frontier-def interior-eq x by auto
    then have f constant-on S
      apply (rule maximum-modulus-principle [OF holf S ‹open S› order-refl])
      using closure-subset apply (blast intro: xmax)
      done
    then have f constant-on (closure S)
      by (rule constant-on-closureI [OF - contf])
    then obtain c where c: ⋀x. x ∈ closure S ⟹ f x = c
      by (meson constant-on-def)
    obtain w where w ∈ frontier S
    by (metis coc all-not-in-conv assms(6) closure-UNIV frontier-eq-empty not-compact-UNIV)
    then show ?thesis
      by (simp add: c frontier-def that)
  qed
qed

lemma Schwarz2:
 ⟦f holomorphic-on ball 0 r;
    0 < s; ball w s ⊆ ball 0 r;
    ⋀z. norm (w−z) < s ⟹ norm(f z) ≤ norm(f w)⟧
    ⟹ f constant-on ball 0 r
by (rule maximum-modulus-principle [where U = ball w s and ξ = w]) (simp-all
add: dist-norm)

lemma Schwarz3:
  assumes holf: f holomorphic-on (ball 0 r) and [simp]: f 0 = 0
```

obtains h **where** h *holomorphic-on* (*ball 0 r*) **and** $\bigwedge z.$ *norm* $z < r \Longrightarrow f\ z = z * (h\ z)$ **and** *deriv f 0* = *h 0*
proof −
def $h \equiv \lambda z.$ *if z = 0 then deriv f 0 else f z / z*
have *d0*: *deriv f 0* = *h 0*
by (*simp add*: *h-def*)
moreover have *h holomorphic-on* (*ball 0 r*)
by (*rule pole-theorem-open-0* [*OF holf*, *of 0*]) (*auto simp*: *h-def*)
moreover have *norm z < r* $\Longrightarrow$ *f z* = *z* ∗ *h z* **for** *z*
by (*simp add*: *h-def*)
ultimately show *?thesis*
using *that* **by** *blast*
qed

proposition *Schwarz-Lemma*:
assumes *holf*: *f holomorphic-on* (*ball 0 1*) **and** [*simp*]: *f 0* = *0*
and *no*: $\bigwedge z.$ *norm z < 1* $\Longrightarrow$ *norm* (*f z*) < *1*
and ξ: *norm* ξ < *1*
shows *norm* (*f* ξ) ≤ *norm* ξ **and** *norm*(*deriv f 0*) ≤ *1*
and ((∃ *z*. *norm z* < *1* ∧ *z* ≠ *0* ∧ *norm*(*f z*) = *norm z*) ∨ *norm*(*deriv f 0*) = *1*)
$\Longrightarrow$ ∃ α. (∀ *z*. *norm z* < *1* ⟶ *f z* = α ∗ *z*) ∧ *norm* α = *1* (**is** *?P* $\Longrightarrow$ *?Q*)
proof −
obtain *h* **where** *holh*: *h holomorphic-on* (*ball 0 1*)
and *fz-eq*: $\bigwedge z.$ *norm z < 1* $\Longrightarrow$ *f z* = *z* ∗ (*h z*) **and** *df0*: *deriv f 0* = *h 0*
by (*rule Schwarz3* [*OF holf*]) *auto*
have *noh-le*: *norm* (*h z*) ≤ *1* **if** *z*: *norm z < 1* **for** *z*
proof −
have *norm* (*h z*) < *a* **if** *a*: *1* < *a* **for** *a*
proof −
have *max* (*inverse a*) (*norm z*) < *1*
using *z a* **by** (*simp-all add*: *inverse-less-1-iff*)
then obtain *r* **where** *r*: *max* (*inverse a*) (*norm z*) < *r* **and** *r* < *1*
using *Rats-dense-in-real* **by** *blast*
then have *nzr*: *norm z* < *r* **and** *ira*: *inverse r* < *a*
using *z a less-imp-inverse-less* **by** *force*+
then have *0* < *r*
by (*meson norm-not-less-zero not-le order.strict-trans2*)
have *holh'*: *h holomorphic-on ball 0 r*
by (*meson holh* ‹*r* < *1*› *holomorphic-on-subset less-eq-real-def subset-ball*)
have *conth'*: *continuous-on* (*cball 0 r*) *h*
by (*meson* ‹*r* < *1*› *dual-order.trans holh holomorphic-on-imp-continuous-on holomorphic-on-subset mem-ball-0 mem-cball-0 not-less subsetI*)
obtain *w* **where** *w*: *norm w* = *r* **and** *lenw*: $\bigwedge z.$ *norm z < r* $\Longrightarrow$ *norm*(*h z*) ≤ *norm*(*h w*)
apply (*rule Schwarz1* [*OF holh'*]) **using** *conth'* ‹*0* < *r*› **by** *auto*
have *h w* = *f w* / *w* **using** *fz-eq* ‹*r* < *1*› *nzr w* **by** *auto*

```
    then have cmod (h z) < inverse r
      by (metis ‹0 < r› ‹r < 1› divide-strict-right-mono inverse-eq-divide
            le-less-trans lenw no norm-divide nzr w)
    then show ?thesis using ira by linarith
   qed
   then show norm (h z) ≤ 1
    using not-le by blast
  qed
  show cmod (f ξ) ≤ cmod ξ
  proof (cases ξ = 0)
   case True then show ?thesis by auto
  next
   case False
   then show ?thesis
    by (simp add: noh-le fz-eq ξ mult-left-le norm-mult)
  qed
  show no-df0: norm(deriv f 0) ≤ 1
   by (simp add: ‹⋀z. cmod z < 1 ⟹ cmod (h z) ≤ 1› df0)
  show ?Q if ?P
  using that
  proof
   assume ∃ z. cmod z < 1 ∧ z ≠ 0 ∧ cmod (f z) = cmod z
   then obtain γ where γ: cmod γ < 1 γ ≠ 0 cmod (f γ) = cmod γ by blast
   then have [simp]: norm (h γ) = 1
    by (simp add: fz-eq norm-mult)
   have ball γ (1 − cmod γ) ⊆ ball 0 1
    by (simp add: ball-subset-ball-iff)
   moreover have ⋀z. cmod (γ − z) < 1 − cmod γ ⟹ cmod (h z) ≤ cmod (h
γ)
    apply (simp add: algebra-simps)
   by (metis add-diff-cancel-left' diff-diff-eq2 le-less-trans noh-le norm-triangle-ineq4)
   ultimately obtain c where c: ⋀z. norm z < 1 ⟹ h z = c
    using Schwarz2 [OF holh, of 1 − norm γ γ, unfolded constant-on-def] γ by
auto
   moreover then have norm c = 1
    using γ by force
   ultimately show ?thesis
    using fz-eq by auto
  next
   assume [simp]: cmod (deriv f 0) = 1
   then obtain c where c: ⋀z. norm z < 1 ⟹ h z = c
     using Schwarz2 [OF holh zero-less-one, of 0, unfolded constant-on-def] df0
noh-le
    by auto
   moreover have norm c = 1 using df0 c by auto
   ultimately show ?thesis
    using fz-eq by auto
  qed
qed
```

54.8 The Schwarz reflection principle

lemma *hol-pal-lem0*:
assumes $d \cdot a \leq k$ $k \leq d \cdot b$
obtains c **where**
$c \in$ *closed-segment a b* $d \cdot c = k$
$\bigwedge z.\ z \in$ *closed-segment a c* $\Longrightarrow d \cdot z \leq k$
$\bigwedge z.\ z \in$ *closed-segment c b* $\Longrightarrow k \leq d \cdot z$
proof −
obtain c **where** *cin*: $c \in$ *closed-segment a b* **and** *keq*: $k = d \cdot c$
using *connected-ivt-hyperplane* [*of closed-segment a b a b d k*]
by (*auto simp*: *assms*)
have *closed-segment a c* $\subseteq \{z.\ d \cdot z \leq k\}$ *closed-segment c b* $\subseteq \{z.\ k \leq d \cdot z\}$
unfolding *segment-convex-hull* **using** *assms keq*
by (*auto simp*: *convex-halfspace-le convex-halfspace-ge hull-minimal*)
then show *?thesis* **using** *cin that* **by** *fastforce*
qed

lemma *hol-pal-lem1*:
assumes *convex S open S*
and *abc*: $a \in S$ $b \in S$ $c \in S$
$d \neq 0$ **and** *lek*: $d \cdot a \leq k$ $d \cdot b \leq k$ $d \cdot c \leq k$
and *holf1*: *f holomorphic-on* $\{z.\ z \in S \wedge d \cdot z < k\}$
and *contf*: *continuous-on S f*
shows *contour-integral* (*linepath a b*) *f* +
contour-integral (*linepath b c*) *f* +
contour-integral (*linepath c a*) *f* = *0*
proof −
have *interior* (*convex hull* $\{a, b, c\}$) $\subseteq$ *interior*$(S \cap \{x.\ d \cdot x \leq k\})$
apply (*rule interior-mono*)
apply (*rule hull-minimal*)
apply (*simp add*: *abc lek*)
apply (*rule convex-Int* [*OF* ‹*convex S*› *convex-halfspace-le*])
done
also have ... $\subseteq \{z \in S.\ d \cdot z < k\}$
by (*force simp*: *interior-open* [*OF* ‹*open S*›] ‹$d \neq 0$›)
finally have $*$: *interior* (*convex hull* $\{a, b, c\}$) $\subseteq \{z \in S.\ d \cdot z < k\}$ **.**
have *continuous-on* (*convex hull* $\{a,b,c\}$) *f*
using ‹*convex S*› *contf abc continuous-on-subset subset-hull*
by *fastforce*
moreover have *f holomorphic-on interior* (*convex hull* $\{a,b,c\}$)
by (*rule holomorphic-on-subset* [*OF holf1* $*$])
ultimately show *?thesis*
using *Cauchy-theorem-triangle-interior has-chain-integral-chain-integral3*
by *blast*
qed

lemma *hol-pal-lem2*:
assumes *S*: *convex S open S*
and *abc*: $a \in S$ $b \in S$ $c \in S$

```
    and d ≠ 0 and lek: d · a ≤ k d · b ≤ k
    and holf1: f holomorphic-on {z. z ∈ S ∧ d · z < k}
    and holf2: f holomorphic-on {z. z ∈ S ∧ k < d · z}
    and contf: continuous-on S f
  shows contour-integral (linepath a b) f +
        contour-integral (linepath b c) f +
        contour-integral (linepath c a) f = 0
proof (cases d · c ≤ k)
  case True show ?thesis
    by (rule hol-pal-lem1 [OF S abc ‹d ≠ 0› lek True holf1 contf])
next
  case False
  then have d · c > k by force
  obtain a' where a': a' ∈ closed-segment b c and d · a' = k
    and ba': ⋀z. z ∈ closed-segment b a' ⟹ d · z ≤ k
    and a'c: ⋀z. z ∈ closed-segment a' c ⟹ k ≤ d · z
    apply (rule hol-pal-lem0 [of d b k c, OF ‹d · b ≤ k›])
    using False by auto
  obtain b' where b': b' ∈ closed-segment a c and d · b' = k
    and ab': ⋀z. z ∈ closed-segment a b' ⟹ d · z ≤ k
    and b'c: ⋀z. z ∈ closed-segment b' c ⟹ k ≤ d · z
    apply (rule hol-pal-lem0 [of d a k c, OF ‹d · a ≤ k›])
    using False by auto
  have a'b': a' ∈ S ∧ b' ∈ S
    using a' abc b' convex-contains-segment ‹convex S› by auto
  have continuous-on (closed-segment c a) f
    by (meson abc contf continuous-on-subset convex-contains-segment ‹convex S›)
  then have 1: contour-integral (linepath c a) f =
              contour-integral (linepath c b') f + contour-integral (linepath b' a) f
    apply (rule contour-integral-split-linepath)
    using b' by (simp add: closed-segment-commute)
  have continuous-on (closed-segment b c) f
    by (meson abc contf continuous-on-subset convex-contains-segment ‹convex S›)
  then have 2: contour-integral (linepath b c) f =
              contour-integral (linepath b a') f + contour-integral (linepath a' c) f
    by (rule contour-integral-split-linepath [OF - a'])
  have 3: contour-integral (reversepath (linepath b' a')) f =
              - contour-integral (linepath b' a') f
    by (rule contour-integral-reversepath [OF valid-path-linepath])
  have fcd-le: f field-differentiable at x
            if x ∈ interior S ∧ x ∈ interior {x. d · x ≤ k} for x
  proof -
    have f holomorphic-on S ∩ {c. d · c < k}
      by (metis (no-types) Collect-conj-eq Collect-mem-eq holf1)
    then have ∃ C D. x ∈ interior C ∩ interior D ∧ f holomorphic-on interior C
∩ interior D
      using that
    by (metis Collect-mem-eq Int-Collect ‹d ≠ 0› interior-halfspace-le interior-open
‹open S›)
```

then show *f field-differentiable at x*
by (*metis at-within-interior holomorphic-on-def interior-Int interior-interior*)
qed
have *ab-le*: $\bigwedge x.\ x \in$ *closed-segment a b* $\Longrightarrow d \cdot x \leq k$
proof −
fix x :: *complex*
assume $x \in$ *closed-segment a b*
then have $\bigwedge C.\ x \in C \vee b \notin C \vee a \notin C \vee \neg$ *convex C*
by (*meson contra-subsetD convex-contains-segment*)
then show $d \cdot x \leq k$
by (*metis lek convex-halfspace-le mem-Collect-eq*)
qed
have *continuous-on* $(S \cap \{x.\ d \cdot x \leq k\})$ *f* **using** *contf*
by (*simp add*: *continuous-on-subset*)
then have (*f has-contour-integral 0*)
(*linepath a b +++ linepath b a' +++ linepath a' b' +++ linepath b' a*)
apply (*rule Cauchy-theorem-convex* [**where** $k = \{\}$])
apply (*simp-all add*: *path-image-join convex-Int convex-halfspace-le* ‹*convex S*› *fcd-le ab-le*
closed-segment-subset abc a'b' ba')
by (*metis* ‹$d \cdot a' = k$› ‹$d \cdot b' = k$› *convex-contains-segment convex-halfspace-le lek*(*1*) *mem-Collect-eq order-refl*)
then have *4*: *contour-integral* (*linepath a b*) *f* +
contour-integral (*linepath b a'*) *f* +
contour-integral (*linepath a' b'*) *f* +
contour-integral (*linepath b' a*) *f* = *0*
by (*rule has-chain-integral-chain-integral4*)
have *fcd-ge*: *f field-differentiable at x*
if $x \in$ *interior S* $\wedge$ $x \in$ *interior* $\{x.\ k \leq d \cdot x\}$ **for** x
proof −
have *f2*: *f holomorphic-on S* $\cap$ $\{c.\ k < d \cdot c\}$
by (*metis* (*full-types*) *Collect-conj-eq Collect-mem-eq holf2*)
have *f3*: *interior S* = *S*
by (*simp add*: *interior-open* ‹*open S*›)
then have $x \in S \cap$ *interior* $\{c.\ k \leq d \cdot c\}$
using *that* **by** *simp*
then show *f field-differentiable at x*
using *f3 f2* **unfolding** *holomorphic-on-def*
by (*metis* (*no-types*) ‹$d \neq 0$› *at-within-interior interior-Int interior-halfspace-ge interior-interior*)
qed
have *continuous-on* $(S \cap \{x.\ k \leq d \cdot x\})$ *f* **using** *contf*
by (*simp add*: *continuous-on-subset*)
then have (*f has-contour-integral 0*) (*linepath a' c +++ linepath c b' +++ linepath b' a'*)
apply (*rule Cauchy-theorem-convex* [**where** $k = \{\}$])
apply (*simp-all add*: *path-image-join convex-Int convex-halfspace-ge* ‹*convex S*›
fcd-ge closed-segment-subset abc a'b' a'c)
by (*metis* ‹$d \cdot a' = k$› *b'c closed-segment-commute convex-contains-segment*

```
        convex-halfspace-ge ends-in-segment(2) mem-Collect-eq order-refl)
  then have 5: contour-integral (linepath a' c) f + contour-integral (linepath c
b') f + contour-integral (linepath b' a') f = 0
    by (rule has-chain-integral-chain-integral3)
  show ?thesis
    using 1 2 3 4 5 by (metis add.assoc eq-neg-iff-add-eq-0 reversepath-linepath)
qed

lemma hol-pal-lem3:
  assumes S: convex S open S
      and abc: a ∈ S b ∈ S c ∈ S
      and d ≠ 0 and lek: d • a ≤ k
      and holf1: f holomorphic-on {z. z ∈ S ∧ d • z < k}
      and holf2: f holomorphic-on {z. z ∈ S ∧ k < d • z}
      and contf: continuous-on S f
    shows contour-integral (linepath a b) f +
          contour-integral (linepath b c) f +
          contour-integral (linepath c a) f = 0
proof (cases d • b ≤ k)
  case True show ?thesis
    by (rule hol-pal-lem2 [OF S abc ‹d ≠ 0› lek True holf1 holf2 contf])
next
  case False
  show ?thesis
  proof (cases d • c ≤ k)
    case True
    have contour-integral (linepath c a) f +
         contour-integral (linepath a b) f +
         contour-integral (linepath b c) f = 0
      by (rule hol-pal-lem2 [OF S ‹c ∈ S› ‹a ∈ S› ‹b ∈ S› ‹d ≠ 0› ‹d • c ≤ k› lek
holf1 holf2 contf])
    then show ?thesis
      by (simp add: algebra-simps)
  next
    case False
    have contour-integral (linepath b c) f +
         contour-integral (linepath c a) f +
         contour-integral (linepath a b) f = 0
      apply (rule hol-pal-lem2 [OF S ‹b ∈ S› ‹c ∈ S› ‹a ∈ S›, of −d −k])
      using ‹d ≠ 0› ‹¬ d • b ≤ k› False by (simp-all add: holf1 holf2 contf)
    then show ?thesis
      by (simp add: algebra-simps)
  qed
qed

lemma hol-pal-lem4:
  assumes S: convex S open S
      and abc: a ∈ S b ∈ S c ∈ S and d ≠ 0
      and holf1: f holomorphic-on {z. z ∈ S ∧ d • z < k}
```

```
    and holf2: f holomorphic-on {z. z ∈ S ∧ k < d · z}
    and contf: continuous-on S f
  shows contour-integral (linepath a b) f +
        contour-integral (linepath b c) f +
        contour-integral (linepath c a) f = 0
proof (cases d · a ≤ k)
 case True show ?thesis
  by (rule hol-pal-lem3 [OF S abc ‹d ≠ 0› True holf1 holf2 contf])
next
 case False
 show ?thesis
  apply (rule hol-pal-lem3 [OF S abc, of −d −k])
  using ‹d ≠ 0› False by (simp-all add: holf1 holf2 contf)
qed

proposition holomorphic-on-paste-across-line:
 assumes S: open S and d ≠ 0
    and holf1: f holomorphic-on (S ∩ {z. d · z < k})
    and holf2: f holomorphic-on (S ∩ {z. k < d · z})
    and contf: continuous-on S f
  shows f holomorphic-on S
proof −
 have ∗: ∃ t. open t ∧ p ∈ t ∧ continuous-on t f ∧
           (∀ a b c. convex hull {a, b, c} ⊆ t ⟶
                  contour-integral (linepath a b) f +
                  contour-integral (linepath b c) f +
                  contour-integral (linepath c a) f = 0)
       if p ∈ S for p
 proof −
  obtain e where e>0 and e: ball p e ⊆ S
   using ‹p ∈ S› openE S by blast
  then have continuous-on (ball p e) f
   using contf continuous-on-subset by blast
  moreover have f holomorphic-on {z. dist p z < e ∧ d · z < k}
   apply (rule holomorphic-on-subset [OF holf1])
   using e by auto
  moreover have f holomorphic-on {z. dist p z < e ∧ k < d · z}
   apply (rule holomorphic-on-subset [OF holf2])
   using e by auto
  ultimately show ?thesis
   apply (rule-tac x=ball p e in exI)
   using ‹e > 0› e ‹d ≠ 0›
   apply (simp add:, clarify)
   apply (rule hol-pal-lem4 [of ball p e - - - d - k])
   apply (auto simp: subset-hull)
   done
 qed
 show ?thesis
  by (blast intro: ∗ Morera-local-triangle analytic-imp-holomorphic)
```

qed

proposition *Schwarz-reflection*:
assumes *open S* **and** *cnjs*: *cnj ' S ⊆ S*
and *holf*: *f holomorphic-on* (*S ∩ {z. 0 < Im z}*)
and *contf*: *continuous-on* (*S ∩ {z. 0 ≤ Im z}*) *f*
and *f*: *⋀z. ⟦z ∈ S; z ∈ ℝ⟧ ⟹ (f z) ∈ ℝ*
shows (*λz. if 0 ≤ Im z then f z else cnj*(*f*(*cnj z*))) *holomorphic-on S*
proof −
have *1*: (*λz. if 0 ≤ Im z then f z else cnj* (*f* (*cnj z*))) *holomorphic-on* (*S ∩ {z. 0 < Im z}*)
by (*force intro*: *iffD1* [*OF holomorphic-cong* [*OF refl*] *holf*])
have *cont-cfc*: *continuous-on* (*S ∩ {z. Im z ≤ 0}*) (*cnj o f o cnj*)
apply (*intro continuous-intros continuous-on-compose continuous-on-subset* [*OF contf*])
using *cnjs* **apply** *auto*
done
have *cnj ∘ f ∘ cnj field-differentiable at x within S ∩ {z. Im z < 0}*
if *x ∈ S Im x < 0 f field-differentiable at* (*cnj x*) *within S ∩ {z. 0 < Im z}*
for *x*
using *that*
apply (*simp add*: *field-differentiable-def Derivative.DERIV-within-iff Lim-within dist-norm*, *clarify*)
apply (*rule-tac x=cnj f ′* **in** *exI*)
apply (*elim all-forward ex-forward conj-forward imp-forward asm-rl*, *clarify*)
apply (*drule-tac x=cnj xa* **in** *bspec*)
using *cnjs* **apply** *force*
apply (*metis complex-cnj-cnj complex-cnj-diff complex-cnj-divide complex-mod-cnj*)
done
then have *hol-cfc*: (*cnj o f o cnj*) *holomorphic-on* (*S ∩ {z. Im z < 0}*)
using *holf cnjs*
by (*force simp*: *holomorphic-on-def*)
have *2*: (*λz. if 0 ≤ Im z then f z else cnj* (*f* (*cnj z*))) *holomorphic-on* (*S ∩ {z. Im z < 0}*)
apply (*rule iffD1* [*OF holomorphic-cong* [*OF refl*]])
using *hol-cfc* **by** *auto*
have [*simp*]: (*S ∩ {z. 0 ≤ Im z}*) *∪* (*S ∩ {z. Im z ≤ 0}*) = *S*
by *force*
have *continuous-on* ((*S ∩ {z. 0 ≤ Im z}*) *∪* (*S ∩ {z. Im z ≤ 0}*))
(*λz. if 0 ≤ Im z then f z else cnj* (*f* (*cnj z*)))
apply (*rule continuous-on-cases-local*)
using *cont-cfc contf*
apply (*simp-all add*: *closedin-closed-Int closed-halfspace-Im-le closed-halfspace-Im-ge*)
using *f Reals-cnj-iff complex-is-Real-iff* **apply** *auto*
done
then have *3*: *continuous-on S* (*λz. if 0 ≤ Im z then f z else cnj* (*f* (*cnj z*)))
by *force*
show *?thesis*
apply (*rule holomorphic-on-paste-across-line* [*OF ‹open S›*, *of* −*ii* - *0*])

```
    using 1 2 3
    apply auto
    done
qed
```

54.9 Bloch's theorem

```
lemma Bloch-lemma-0:
  assumes holf: f holomorphic-on cball 0 r and 0 < r
      and [simp]: f 0 = 0
      and le: ⋀z. norm z < r ⟹ norm(deriv f z) ≤ 2 * norm(deriv f 0)
    shows ball 0 ((3 - 2 * sqrt 2) * r * norm(deriv f 0)) ⊆ f ‘ ball 0 r
proof -
  have sqrt 2 < 3/2
    by (rule real-less-lsqrt) (auto simp: power2-eq-square)
  then have sq3: 0 < 3 - 2 * sqrt 2 by simp
  show ?thesis
  proof (cases deriv f 0 = 0)
    case True then show ?thesis by simp
  next
    case False
    def C ≡ 2 * norm(deriv f 0)
    have 0 < C using False by (simp add: C-def)
    have holf': f holomorphic-on ball 0 r using holf
      using ball-subset-cball holomorphic-on-subset by blast
    then have holdf': deriv f holomorphic-on ball 0 r
      by (rule holomorphic-deriv [OF - open-ball])
    have Le1: norm(deriv f z - deriv f 0) ≤ norm z / (r - norm z) * C
                if norm z < r for z
    proof -
      have T1: norm(deriv f z - deriv f 0) ≤ norm z / (R - norm z) * C
              if R: norm z < R R < r for R
      proof -
        have 0 < R using R
          by (metis less-trans norm-zero zero-less-norm-iff)
        have df-le: ⋀x. norm x < r ⟹ norm (deriv f x) ≤ C
          using le by (simp add: C-def)
        have hol-df: deriv f holomorphic-on cball 0 R
          apply (rule holomorphic-on-subset) using R holdf' by auto
        have *: ((λw. deriv f w / (w - z)) has-contour-integral 2 * pi * i * deriv f
z) (circlepath 0 R)
                  if norm z < R for z
          using ‹0 < R› that Cauchy-integral-formula-convex-simple [OF convex-cball
hol-df, of - circlepath 0 R]
          by (force simp: winding-number-circlepath)
        have **: ((λx. deriv f x / (x - z) - deriv f x / x) has-contour-integral
                    of-real (2 * pi) * i * (deriv f z - deriv f 0))
                  (circlepath 0 R)
          using has-contour-integral-diff [OF * [of z] * [of 0]] ‹0 < R› that
```

```
      by (simp add: algebra-simps)
    have [simp]: ⋀x. norm x = R ⟹ x ≠ z  using that(1) by blast
    have norm (deriv f x / (x − z) − deriv f x / x)
              ≤ C * norm z / (R * (R − norm z))
            if norm x = R for x
    proof −
      have [simp]: norm (deriv f x * x − deriv f x * (x − z)) =
                   norm (deriv f x) * norm z
        by (simp add: norm-mult right-diff-distrib')
      show ?thesis
        using ‹0 < R› ‹0 < C› R that
        apply (simp add: norm-mult norm-divide divide-simps)
       using df-le norm-triangle-ineq2 ‹0 < C› apply (auto intro!: mult-mono)
        done
    qed
    then show ?thesis
      using has-contour-integral-bound-circlepath
            [OF **, of C * norm z/(R*(R − norm z))]
          ‹0 < R› ‹0 < C› R
      apply (simp add: norm-mult norm-divide)
      apply (simp add: divide-simps mult.commute)
      done
  qed
  obtain r' where r': norm z < r' r' < r
    using Rats-dense-in-real [of norm z r] ‹norm z < r› by blast
  then have [simp]: closure {r'<..<r} = {r'..r} by simp
  show ?thesis
    apply (rule continuous-ge-on-closure
           [where f = λr. norm z / (r − norm z) * C and s = {r'<..<r},
            OF - - T1])
    apply (intro continuous-intros)
    using that r'
    apply (auto simp: not-le)
    done
qed
have *: (norm z − norm z^2/(r − norm z)) * norm(deriv f 0) ≤ norm(f z)
        if r: norm z < r for z
proof −
  have 1: ⋀x. x ∈ ball 0 r ⟹
        ((λz. f z − deriv f 0 * z) has-field-derivative deriv f x − deriv f 0)
         (at x within ball 0 r)
    by (rule derivative-eq-intros holomorphic-derivI holf' | simp)+
  have 2: closed-segment 0 z ⊆ ball 0 r
    by (metis ‹0 < r› convex-ball convex-contains-segment dist-self mem-ball
mem-ball-0 that)
  have 3: (λt. (norm z)^2 * t / (r − norm z) * C) integrable-on {0..1}
    apply (rule integrable-on-cmult-right [where 'b=real, simplified])
    apply (rule integrable-on-cdivide [where 'b=real, simplified])
    apply (rule integrable-on-cmult-left [where 'b=real, simplified])
```

apply (*rule ident-integrable-on*)
done
have *4*: *norm* (*deriv f* (*x* $*_R$ *z*) − *deriv f 0*) ∗ *norm z* ≤ *norm z* ∗ *norm z* ∗ *x* ∗ *C* / (*r* − *norm z*)
if *x*: *0* ≤ *x x* ≤ *1* **for** *x*
proof −
have [*simp*]: *x* ∗ *norm z* < *r*
using *r x* **by** (*meson le-less-trans mult-le-cancel-right2 norm-not-less-zero*)
have *norm* (*deriv f* (*x* $*_R$ *z*) − *deriv f 0*) ≤ *norm* (*x* $*_R$ *z*) / (*r* − *norm* (*x* $*_R$ *z*)) ∗ *C*
apply (*rule Le1*) **using** *r x* ‹*0* < *r*› **by** *simp*
also have ... ≤ *norm* (*x* $*_R$ *z*) / (*r* − *norm z*) ∗ *C*
using *r x* ‹*0* < *r*›
apply (*simp add*: *divide-simps*)
by (*simp add*: ‹*0* < *C*› *mult.assoc mult-left-le-one-le ordered-comm-semiring-class.comm-mult-left-mono*)
finally have *norm* (*deriv f* (*x* $*_R$ *z*) − *deriv f 0*) ∗ *norm z* ≤ *norm* (*x* $*_R$ *z*) / (*r* − *norm z*) ∗ *C* ∗ *norm z*
by (*rule mult-right-mono*) *simp*
with *x* **show** *?thesis* **by** (*simp add*: *algebra-simps*)
qed
have *le-norm*: *abc* ≤ *norm d* − *e* ⟹ *norm*(*f* − *d*) ≤ *e* ⟹ *abc* ≤ *norm f* **for** *abc d e* **and** *f*::*complex*
by (*metis add-diff-cancel-left′ add-diff-eq diff-left-mono norm-diff-ineq order-trans*)
have *norm* (*integral* {*0..1*} (λ*x*. (*deriv f* (*x* $*_R$ *z*) − *deriv f 0*) ∗ *z*))
≤ *integral* {*0..1*} (λ*t*. (*norm z*)^2 ∗ *t* / (*r* − *norm z*) ∗ *C*)
apply (*rule integral-norm-bound-integral*)
using *contour-integral-primitive* [*OF 1*, *of linepath 0 z*] *2*
apply (*simp add*: *has-contour-integral-linepath has-integral-integrable-integral*)
apply (*rule 3*)
apply (*simp add*: *norm-mult power2-eq-square 4*)
done
then have *int-le*: *norm* (*f z* − *deriv f 0* ∗ *z*) ≤ (*norm z*)^2 ∗ *norm*(*deriv f 0*) / ((*r* − *norm z*))
using *contour-integral-primitive* [*OF 1*, *of linepath 0 z*] *2*
apply (*simp add*: *has-contour-integral-linepath has-integral-integrable-integral C-def*)
done
show *?thesis*
apply (*rule le-norm* [*OF - int-le*])
using ‹*norm z* < *r*›
apply (*simp add*: *power2-eq-square divide-simps C-def norm-mult*)
proof −
have *norm z* ∗ (*norm* (*deriv f 0*) ∗ (*r* − *norm z* − *norm z*)) ≤ *norm z* ∗ (*norm* (*deriv f 0*) ∗ (*r* − *norm z*) − *norm* (*deriv f 0*) ∗ *norm z*)
by (*simp add*: *linordered-field-class.sign-simps*(*38*))
then show (*norm z* ∗ (*r* − *norm z*) − *norm z* ∗ *norm z*) ∗ *norm* (*deriv f 0*) ≤ *norm* (*deriv f 0*) ∗ *norm z* ∗ (*r* − *norm z*) − *norm z* ∗ *norm z* ∗ *norm* (*deriv f 0*)

```
            by (simp add: linordered-field-class.sign-simps(38) mult.commute
mult.left-commute)
        qed
    qed
    have sq201 [simp]: 0 < (1 − sqrt 2 / 2) (1 − sqrt 2 / 2) < 1
      by (auto simp: sqrt2-less-2)
    have 1: continuous-on (closure (ball 0 ((1 − sqrt 2 / 2) ∗ r))) f
     apply (rule continuous-on-subset [OF holomorphic-on-imp-continuous-on [OF
holf]])
      apply (subst closure-ball)
      using ⟨0 < r⟩ mult-pos-pos sq201
      apply (auto simp: cball-subset-cball-iff)
      done
    have 2: open (f ‘ interior (ball 0 ((1 − sqrt 2 / 2) ∗ r)))
      apply (rule open-mapping-thm [OF holf′ open-ball connected-ball], force)
      using ⟨0 < r⟩ mult-pos-pos sq201 apply (simp add: ball-subset-ball-iff)
      using False ⟨0 < r⟩ centre-in-ball holf′ holomorphic-nonconstant by blast
    have ball 0 ((3 − 2 ∗ sqrt 2) ∗ r ∗ norm (deriv f 0)) =
          ball (f 0) ((3 − 2 ∗ sqrt 2) ∗ r ∗ norm (deriv f 0))
      by simp
    also have ... ⊆ f ‘ ball 0 ((1 − sqrt 2 / 2) ∗ r)
    proof −
      have 3: (3 − 2 ∗ sqrt 2) ∗ r ∗ norm (deriv f 0) ≤ norm (f z)
          if norm z = (1 − sqrt 2 / 2) ∗ r for z
        apply (rule order-trans [OF - ∗])
        using ⟨0 < r⟩
        apply (simp-all add: field-simps power2-eq-square that)
        apply (simp add: mult.assoc [symmetric])
        done
      show ?thesis
        apply (rule ball-subset-open-map-image [OF 1 2 - bounded-ball])
        using ⟨0 < r⟩ sq201 3 apply simp-all
        using C-def ⟨0 < C⟩ sq3 apply force
        done
     qed
    also have ... ⊆ f ‘ ball 0 r
      apply (rule image-subsetI [OF imageI], simp)
      apply (erule less-le-trans)
      using ⟨0 < r⟩ apply (auto simp: field-simps)
      done
    finally show ?thesis .
  qed
qed

lemma Bloch-lemma:
  assumes holf: f holomorphic-on cball a r and 0 < r
      and le: ⋀z. z ∈ ball a r ⟹ norm(deriv f z) ≤ 2 ∗ norm(deriv f a)
    shows ball (f a) ((3 − 2 ∗ sqrt 2) ∗ r ∗ norm(deriv f a)) ⊆ f ‘ ball a r
proof −
```

have *fz*: $(\lambda z.\ f\ (a + z)) = f\ o\ (\lambda z.\ (a + z))$
by (*simp add*: *o-def*)
have *hol0*: $(\lambda z.\ f\ (a + z))$ *holomorphic-on cball 0 r*
unfolding *fz* **by** (*intro holomorphic-intros holf holomorphic-on-compose* | *simp*)+
then have [*simp*]: $\bigwedge x.$ *norm* $x < r \Longrightarrow (\lambda z.\ f\ (a + z))$ *field-differentiable at x*
by (*metis Topology-Euclidean-Space.open-ball at-within-open ball-subset-cball diff-0 dist-norm holomorphic-on-def holomorphic-on-subset mem-ball norm-minus-cancel*)
have [*simp*]: $\bigwedge z.$ *norm* $z < r \Longrightarrow f$ *field-differentiable at* $(a + z)$
by (*metis holf open-ball add-diff-cancel-left' dist-complex-def holomorphic-on-imp-differentiable-at holomorphic-on-subset interior-cball interior-subset mem-ball norm-minus-commute*)
then have [*simp*]: *f field-differentiable at a*
by (*metis add.comm-neutral* ‹*0 < r*› *norm-eq-zero*)
have *hol1*: $(\lambda z.\ f\ (a + z) - f\ a)$ *holomorphic-on cball 0 r*
by (*intro holomorphic-intros hol0*)
then have *ball 0* $((3 - 2 * sqrt\ 2) * r * norm\ (deriv\ (\lambda z.\ f\ (a + z) - f\ a)\ 0))$
$\subseteq (\lambda z.\ f\ (a + z) - f\ a)$ ‘ *ball 0 r*
apply (*rule Bloch-lemma-0*)
apply (*simp-all add*: ‹*0 < r*›)
apply (*simp add*: *fz complex-derivative-chain*)
apply (*simp add*: *dist-norm le*)
done
then show *?thesis*
apply *clarify*
apply (*drule-tac c*=$x - f\ a$ **in** *subsetD*)
apply (*force simp*: *fz* ‹*0 < r*› *dist-norm complex-derivative-chain field-differentiable-compose*)+
done
qed

proposition *Bloch-unit*:
assumes *holf*: *f holomorphic-on ball a 1* **and** [*simp*]: *deriv f a = 1*
obtains *b r* **where** *1/12 < r ball b r* $\subseteq f$ ‘ (*ball a 1*)
proof −
def *r* ≡ *249/256*::*real*
have *0 < r r < 1* **by** (*auto simp*: *r-def*)
def *g* ≡ $\lambda z.$ *deriv f z * of-real*$(r - norm(z - a))$
have *deriv f holomorphic-on ball a 1*
by (*rule holomorphic-deriv* [*OF holf open-ball*])
then have *continuous-on* (*ball a 1*) (*deriv f*)
using *holomorphic-on-imp-continuous-on* **by** *blast*
then have *continuous-on* (*cball a r*) (*deriv f*)
by (*rule continuous-on-subset*) (*simp add*: *cball-subset-ball-iff* ‹*r < 1*›)
then have *continuous-on* (*cball a r*) *g*
by (*simp add*: *g-def continuous-intros*)
then have *1*: *compact* (*g* ‘ *cball a r*)
by (*rule compact-continuous-image* [*OF - compact-cball*])
have *2*: *g* ‘ *cball a r* ≠ {}
using ‹*r > 0*› **by** *auto*
obtain *p* **where** *pr*: *p* ∈ *cball a r*

```
        and pge: ⋀y. y ∈ cball a r ⟹ norm (g y) ≤ norm (g p)
    using distance-attains-sup [OF 1 2, of 0] by force
  def t ≡ (r − norm(p − a)) / 2
  have norm (p − a) ≠ r
    using pge [of a] ‹r > 0› by (auto simp: g-def norm-mult)
  then have norm (p − a) < r using pr
    by (simp add: norm-minus-commute dist-norm)
  then have 0 < t
    by (simp add: t-def)
  have cpt: cball p t ⊆ ball a r
    using ‹0 < t› by (simp add: cball-subset-ball-iff dist-norm t-def field-simps)
  have gen-le-dfp: norm (deriv f y) * (r − norm (y − a)) / (r − norm (p − a))
≤ norm (deriv f p)
        if y ∈ cball a r for y
  proof −
    have [simp]: norm (y − a) ≤ r
      using that by (simp add: dist-norm norm-minus-commute)
    have norm (g y) ≤ norm (g p)
      using pge [OF that] by simp
    then have norm (deriv f y) * abs (r − norm (y − a)) ≤ norm (deriv f p) *
abs (r − norm (p − a))
      by (simp only: dist-norm g-def norm-mult norm-of-real)
    with that ‹norm (p − a) < r› show ?thesis
      by (simp add: dist-norm divide-simps)
  qed
  have le-norm-dfp: r / (r − norm (p − a)) ≤ norm (deriv f p)
    using gen-le-dfp [of a] ‹r > 0› by auto
  have 1: f holomorphic-on cball p t
    apply (rule holomorphic-on-subset [OF holf])
    using cpt ‹r < 1› order-subst1 subset-ball by auto
  have 2: norm (deriv f z) ≤ 2 * norm (deriv f p) if z ∈ ball p t for z
  proof −
    have z: z ∈ cball a r
      by (meson ball-subset-cball subsetD cpt that)
    then have norm(z − a) < r
    by (metis ball-subset-cball contra-subsetD cpt dist-norm mem-ball norm-minus-commute
that)
    have norm (deriv f z) * (r − norm (z − a)) / (r − norm (p − a)) ≤ norm
(deriv f p)
      using gen-le-dfp [OF z] by simp
    with ‹norm (z − a) < r› ‹norm (p − a) < r›
    have norm (deriv f z) ≤ (r − norm (p − a)) / (r − norm (z − a)) * norm
(deriv f p)
      by (simp add: field-simps)
    also have ... ≤ 2 * norm (deriv f p)
      apply (rule mult-right-mono)
      using that ‹norm (p − a) < r› ‹norm(z − a) < r›
      apply (simp-all add: field-simps t-def dist-norm [symmetric])
      using dist-triangle3 [of z a p] by linarith
```

```
    finally show ?thesis .
  qed
  have sqrt2: sqrt 2 < 2113/1494
    by (rule real-less-lsqrt) (auto simp: power2-eq-square)
  then have sq3: 0 < 3 − 2 ∗ sqrt 2 by simp
  have 1 / 12 / ((3 − 2 ∗ sqrt 2) / 2) < r
    using sq3 sqrt2 by (auto simp: field-simps r-def)
  also have ... ≤ cmod (deriv f p) ∗ (r − cmod (p − a))
    using ‹norm (p − a) < r› le-norm-dfp   by (simp add: pos-divide-le-eq)
  finally have 1 / 12 < cmod (deriv f p) ∗ (r − cmod (p − a)) ∗ ((3 − 2 ∗ sqrt
2) / 2)
    using pos-divide-less-eq half-gt-zero-iff sq3 by blast
  then have ∗∗: 1 / 12 < (3 − 2 ∗ sqrt 2) ∗ t ∗ norm (deriv f p)
    using sq3 by (simp add: mult.commute t-def)
  have ball (f p) ((3 − 2 ∗ sqrt 2) ∗ t ∗ norm (deriv f p)) ⊆ f ‘ ball p t
    by (rule Bloch-lemma [OF 1 ‹0 < t› 2])
  also have ... ⊆ f ‘ ball a 1
    apply (rule image-mono)
    apply (rule order-trans [OF ball-subset-cball])
    apply (rule order-trans [OF cpt])
    using ‹0 < t› ‹r < 1› apply (simp add: ball-subset-ball-iff dist-norm)
    done
  finally have ball (f p) ((3 − 2 ∗ sqrt 2) ∗ t ∗ norm (deriv f p)) ⊆ f ‘ ball a 1 .
  with ∗∗ show ?thesis
    by (rule that)
qed


theorem Bloch:
  assumes holf: f holomorphic-on ball a r and 0 < r
      and r′: r′ ≤ r ∗ norm (deriv f a) / 12
  obtains b where ball b r′ ⊆ f ‘ (ball a r)
proof (cases deriv f a = 0)
  case True with r′ show ?thesis
    using ball-eq-empty that by fastforce
next
  case False
    def C ≡ deriv f a
    have 0 < norm C using False by (simp add: C-def)
    have dfa: f field-differentiable at a
      apply (rule holomorphic-on-imp-differentiable-at [OF holf])
      using ‹0 < r› by auto
    have fo: (λz. f (a + of-real r ∗ z)) = f o (λz. (a + of-real r ∗ z))
      by (simp add: o-def)
    have holf′: f holomorphic-on (λz. a + complex-of-real r ∗ z) ‘ ball 0 1
      apply (rule holomorphic-on-subset [OF holf])
      using ‹0 < r› apply (force simp: dist-norm norm-mult)
      done
    have 1: (λz. f (a + r ∗ z) / (C ∗ r)) holomorphic-on ball 0 1
```

```
      apply (rule holomorphic-intros holomorphic-on-compose holf' | simp add:
fo)+
    using ‹0 < r› by (simp add: C-def False)
  have ((λz. f (a + of-real r * z) / (C * of-real r)) has-field-derivative
      (deriv f (a + of-real r * z) / C)) (at z)
     if norm z < 1 for z
  proof −
  have *: ((λx. f (a + of-real r * x)) has-field-derivative
        (deriv f (a + of-real r * z) * of-real r)) (at z)
    apply (simp add: fo)
    apply (rule DERIV-chain [OF field-differentiable-derivI])
    apply (rule holomorphic-on-imp-differentiable-at [OF holf], simp)
    using ‹0 < r› apply (simp add: dist-norm norm-mult that)
    apply (rule derivative-eq-intros | simp)+
    done
   show ?thesis
    apply (rule derivative-eq-intros * | simp)+
    using ‹0 < r› by (auto simp: C-def False)
  qed
  have 2: deriv (λz. f (a + of-real r * z) / (C * of-real r)) 0 = 1
   apply (subst deriv-cdivide-right)
   apply (simp add: field-differentiable-def fo)
   apply (rule exI)
   apply (rule DERIV-chain [OF field-differentiable-derivI])
   apply (simp add: dfa)
   apply (rule derivative-eq-intros | simp add: C-def False fo)+
   using ‹0 < r›
   apply (simp add: C-def False fo)
   apply (simp add: derivative-intros dfa complex-derivative-chain)
   done
  have sb1: op * (C * r) ‘ (λz. f (a + of-real r * z) / (C * r)) ‘ ball 0 1
        ⊆ f ‘ ball a r
   using ‹0 < r› by (auto simp: dist-norm norm-mult C-def False)
  have sb2: ball (C * r * b) r' ⊆ op * (C * r) ‘ ball b t
        if 1 / 12 < t for b t
  proof −
   have *: r * cmod (deriv f a) / 12 ≤ r * (t * cmod (deriv f a))
   using that ‹0 < r› less-eq-real-def mult.commute mult.right-neutral mult-left-mono
norm-ge-zero times-divide-eq-right
     by auto
   show ?thesis
    apply clarify
    apply (rule-tac x=x / (C * r) in image-eqI)
    using ‹0 < r›
   apply (simp-all add: dist-norm norm-mult norm-divide C-def False field-simps)
    apply (erule less-le-trans)
    apply (rule order-trans [OF r' *])
    done
  qed
```

```
    show ?thesis
      apply (rule Bloch-unit [OF 1 2])
      apply (rename-tac t)
      apply (rule-tac b=(C * of-real r) * b in that)
      apply (drule image-mono [where f = λz. (C * of-real r) * z])
      using sb1 sb2
      apply force
      done
qed

corollary Bloch-general:
  assumes holf: f holomorphic-on s and a ∈ s
      and tle: ⋀z. z ∈ frontier s ⟹ t ≤ dist a z
      and rle: r ≤ t * norm(deriv f a) / 12
  obtains b where ball b r ⊆ f ‘ s
proof −
  consider r ≤ 0 | 0 < t * norm(deriv f a) / 12 using rle by force
  then show ?thesis
  proof cases
    case 1 then show ?thesis
      by (simp add: Topology-Euclidean-Space.ball-empty that)
  next
    case 2
    show ?thesis
    proof (cases deriv f a = 0)
      case True then show ?thesis
        using rle by (simp add: Topology-Euclidean-Space.ball-empty that)
    next
      case False
      then have t > 0
        using 2 by (force simp: zero-less-mult-iff)
      have ~ ball a t ⊆ s ⟹ ball a t ∩ frontier s ≠ {}
        apply (rule connected-Int-frontier [of ball a t s], simp-all)
        using ‹0 < t› ‹a ∈ s› centre-in-ball apply blast
        done
      with tle have *: ball a t ⊆ s by fastforce
      then have 1: f holomorphic-on ball a t
        using holf using holomorphic-on-subset by blast
      show ?thesis
        apply (rule Bloch [OF 1 ‹t > 0› rle])
        apply (rule-tac b=b in that)
        using * apply force
        done
    qed
  qed
qed
```

54.10 Cauchy's residue theorem

Wenda Li (2016)

declare *valid-path-imp-path* [*simp*]

lemma *holomorphic-factor-zero-unique*:
fixes f::*complex* ⇒ *complex* **and** z::*complex* **and** r::*real*
assumes $r>0$
and *asm*:∀ w∈*ball z r*. $f\ w = (w-z)\hat{}n * g\ w \wedge g\ w\neq 0 \wedge f\ w = (w-z)\hat{}m * h\ w \wedge h\ w\neq 0$
and *g-holo*:*g holomorphic-on ball z r* **and** *h-holo*:*h holomorphic-on ball z r*
shows $n=m$
proof −
have $n>m \Longrightarrow$ *False*
proof −
assume $n>m$
have $(h \longrightarrow 0)$ (*at z within ball z r*)
proof (*rule Lim-transform-within*[*OF* - ‹$r>0$›, **where** $f=\lambda w.\ (w-z)\ \hat{}\ (n-m) * g\ w$])
have ∀ w∈*ball z r*. $w\neq z \longrightarrow h\ w = (w-z)\hat{}(n-m) * g\ w$ **using** ‹$n>m$› *asm*
by (*auto simp add*:*field-simps power-diff*)
then show $[\![x' \in ball\ z\ r;\ 0 < dist\ x'\ z; dist\ x'\ z < r]\!] \Longrightarrow (x'-z)\ \hat{}\ (n-m) * g\ x' = h\ x'$ **for** x' **by** *auto*
next
def $F\equiv$*at z within ball z r*
and $f'\equiv\lambda x.\ (x-z)\ \hat{}\ (n-m)$
have $f'\ z=0$ **using** ‹$n>m$› **unfolding** *f'-def* **by** *auto*
moreover have *continuous F f'* **unfolding** *f'-def F-def*
by (*intro continuous-intros*)
ultimately have $(f' \longrightarrow 0)\ F$ **unfolding** *F-def*
by (*simp add*: *continuous-within*)
moreover have $(g \longrightarrow g\ z)\ F$
using *holomorphic-on-imp-continuous-on*[*OF g-holo*,*unfolded continuous-on-def*] ‹$r>0$›
unfolding *F-def* **by** *auto*
ultimately show $((\lambda w.\ f'\ w * g\ w) \longrightarrow 0)\ F$ **using** *tendsto-mult* **by** *fastforce*
qed
moreover have $(h \longrightarrow h\ z)$ (*at z within ball z r*)
using *holomorphic-on-imp-continuous-on*[*OF h-holo*]
by (*auto simp add*:*continuous-on-def* ‹$r>0$›)
moreover have *at z within ball z r* ≠ *bot* **using** ‹$r>0$›
by (*auto simp add*:*trivial-limit-within islimpt-ball*)
ultimately have $h\ z=0$ **by** (*auto intro*: *tendsto-unique*)
thus *False* **using** *asm* ‹$r>0$› **by** *auto*
qed
moreover have $m>n \Longrightarrow$ *False*
proof −

```
    assume m>n
    have (g ⟶ 0) (at z within ball z r)
     proof (rule Lim-transform-within[OF - ‹r>0›, where f=λw. (w − z) ^ (m
− n) * h w])
       have ∀w∈ball z r. w≠z ⟶ g w = (w−z)^(m−n) * h w using ‹m>n›
asm
        by (auto simp add:field-simps power-diff)
       then show ⟦x' ∈ ball z r; 0 < dist x' z;dist x' z < r⟧
        ⟹ (x' − z) ^ (m − n) * h x' = g x' for x' by auto
     next
      def F≡at z within ball z r
       and f'≡λx. (x − z) ^ (m−n)
      have f' z=0 using ‹m>n› unfolding f'-def by auto
      moreover have continuous F f' unfolding f'-def F-def
       by (intro continuous-intros)
      ultimately have (f' ⟶ 0) F unfolding F-def
       by (simp add: continuous-within)
      moreover have (h ⟶ h z) F
     using holomorphic-on-imp-continuous-on[OF h-holo,unfolded continuous-on-def]
‹r>0›
       unfolding F-def by auto
      ultimately show ((λw. f' w * h w) ⟶ 0) F using tendsto-mult by
fastforce
     qed
    moreover have (g ⟶ g z) (at z within ball z r)
     using holomorphic-on-imp-continuous-on[OF g-holo]
     by (auto simp add:continuous-on-def ‹r>0›)
    moreover have at z within ball z r ≠ bot using ‹r>0›
     by (auto simp add:trivial-limit-within islimpt-ball)
    ultimately have g z=0 by (auto intro: tendsto-unique)
    thus False using asm ‹r>0› by auto
   qed
  ultimately show n=m by fastforce
qed

lemma holomorphic-factor-zero-Ex1:
  assumes open s connected s z ∈ s and
      holo:f holomorphic-on s
      and f z = 0 and ∃w∈s. f w ≠ 0
  shows ∃!n. ∃g r. 0 < n ∧ 0 < r ∧
            g holomorphic-on cball z r
            ∧ (∀w∈cball z r. f w = (w−z)^n * g w ∧ g w≠0)
proof (rule ex-ex1I)
  obtain g r n where 0 < n 0 < r ball z r ⊆ s and
      g:g holomorphic-on ball z r
      ⋀w. w ∈ ball z r ⟹ f w = (w − z) ^ n * g w
      ⋀w. w ∈ ball z r ⟹ g w ≠ 0
   using holomorphic-factor-zero-nonconstant[OF holo ‹open s› ‹connected s› ‹z∈s›
‹f z=0›]
```

```
    by (metis assms(3) assms(5) assms(6))
  def r'≡r/2
  have cball z r' ⊆ ball z r unfolding r'-def by (simp add: ‹0 < r› cball-subset-ball-iff)
  hence cball z r' ⊆ s g holomorphic-on cball z r'
      (∀ w∈cball z r'. f w = (w − z) ^ n ∗ g w ∧ g w ≠ 0)
    using g ‹ball z r ⊆ s› by auto
  moreover have r'>0 unfolding r'-def using ‹0<r› by auto
  ultimately show ∃ n g r. 0 < n ∧ 0 < r ∧ g holomorphic-on cball z r
        ∧ (∀ w∈cball z r. f w = (w − z) ^ n ∗ g w ∧ g w ≠ 0)
    apply (intro exI[of - n] exI[of - g] exI[of - r'])
    by (simp add:‹0 < n›)
next
  fix m n
  def fac≡λn g r. ∀ w∈cball z r. f w = (w − z) ^ n ∗ g w ∧ g w ≠ 0
  assume n-asm:∃ g r1. 0 < n ∧ 0 < r1 ∧ g holomorphic-on cball z r1 ∧ fac n g
r1
     and m-asm:∃ h r2. 0 < m ∧ 0 < r2 ∧ h holomorphic-on cball z r2 ∧ fac m
h r2
  obtain g r1 where 0 < n 0 < r1 and g-holo: g holomorphic-on cball z r1
    and fac n g r1 using n-asm by auto
  obtain h r2 where 0 < m 0 < r2 and h-holo: h holomorphic-on cball z r2
    and fac m h r2 using m-asm by auto
  def r≡min r1 r2
  have r>0 using ‹r1>0› ‹r2>0› unfolding r-def by auto
  moreover have ∀ w∈ball z r. f w = (w−z) ^n ∗ g w ∧ g w≠0 ∧ f w = (w −
z) ^m ∗ h w ∧ h w≠0
    using ‹fac m h r2› ‹fac n g r1›   unfolding fac-def r-def
    by fastforce
  ultimately show m=n using g-holo h-holo
   apply (elim holomorphic-factor-zero-unique[of r z f n g m h,symmetric,rotated])
    by (auto simp add:r-def)
qed

lemma finite-ball-avoid:
  assumes open s finite pts
  shows ∀ p∈s. ∃ e>0. ∀ w∈ball p e. w∈s ∧ (w≠p ⟶ w∉pts)
proof
  fix p assume p∈s
  then obtain e1 where 0 < e1 and e1-b:ball p e1 ⊆ s
    using open-contains-ball-eq[OF ‹open s›] by auto
  obtain e2 where 0<e2 and ∀ x∈pts. x ≠ p ⟶ e2 ≤ dist p x
    using finite-set-avoid[OF ‹finite pts›,of p] by auto
  hence ∀ w∈ball p (min e1 e2). w∈s ∧ (w≠p ⟶ w∉pts) using e1-b by auto
  thus ∃ e>0. ∀ w∈ball p e. w ∈ s ∧ (w ≠ p ⟶ w ∉ pts) using ‹e2>0› ‹e1>0›
    apply (rule-tac x=min e1 e2 in exI)
    by auto
qed

lemma finite-cball-avoid:
```

fixes $s :: 'a :: euclidean\text{-}space\ set$
assumes *open s finite pts*
shows $\forall p{\in}s.\ \exists e{>}0.\ \forall w{\in}cball\ p\ e.\ w{\in}s \wedge (w{\neq}p \longrightarrow w{\notin}pts)$
proof
fix p **assume** $p{\in}s$
then obtain *e1* **where** $e1{>}0$ **and** $e1{:}\ \forall w{\in}ball\ p\ e1.\ w{\in}s \wedge (w{\neq}p \longrightarrow w{\notin}pts)$

using *finite-ball-avoid*[*OF assms*] **by** *auto*
def $e2{\equiv}e1/2$
have $e2{>}0$ **and** $e2{<}e1$ **unfolding** *e2-def* **using** ‹$e1{>}0$› **by** *auto*
then have *cball p e2* $\subseteq$ *ball p e1* **by** (*subst cball-subset-ball-iff*,*auto*)
then show $\exists e{>}0.\ \forall w{\in}cball\ p\ e.\ w \in s \wedge (w \neq p \longrightarrow w \notin pts)$ **using** ‹$e2{>}0$› *e1* **by** *auto*
qed

lemma *get-integrable-path*:
assumes *open s connected* $(s{-}pts)$ *finite pts f holomorphic-on* $(s{-}pts)$ $a{\in}s{-}pts$ $b{\in}s{-}pts$
obtains *g* **where** *valid-path g pathstart* $g = a$ *pathfinish* $g = b$
path-image $g \subseteq s{-}pts$ *f contour-integrable-on g* **using** *assms*
proof (*induct arbitrary*:*s thesis a rule*:*finite-induct*[*OF* ‹*finite pts*›]) **print-cases**
case *1*
obtain *g* **where** *valid-path g path-image* $g \subseteq s$ *pathstart* $g = a$ *pathfinish* $g = b$
using *connected-open-polynomial-connected*[*OF* ‹*open s*›,*of a b*] ‹*connected* $(s - \{\})$›
valid-path-polynomial-function 1.prems(*6*) *1.prems*(*7*) **by** *auto*
moreover have *f contour-integrable-on g*
using *contour-integrable-holomorphic-simple*[*OF* - ‹*open s*› ‹*valid-path g*› ‹*path-image* $g \subseteq s$›,*of f*]
‹*f holomorphic-on* $s - \{\}$›
by *auto*
ultimately show *?case* **using** *1*(*1*)[*of g*] **by** *auto*
next
case *idt*:(*2 p pts*)
obtain *e* **where** $e{>}0$ **and** $e{:}\forall w{\in}ball\ a\ e.\ w \in s \wedge (w \neq a \longrightarrow w \notin insert\ p\ pts)$
using *finite-ball-avoid*[*OF* ‹*open s*› ‹*finite* (*insert p pts*)›,*rule-format*,*of a*]
‹$a \in s - insert\ p\ pts$›
by *auto*
def $a'{\equiv}a{+}e/2$
have $a'{\in}s{-}\{p\} - pts$ **using** *e*[*rule-format*,*of* $a{+}e/2$] ‹$e{>}0$›
by (*auto simp add*:*dist-complex-def a'-def*)
then obtain g' **where** g'[*simp*]:*valid-path* g' *pathstart* $g' = a'$ *pathfinish* $g' = b$
path-image $g' \subseteq s - \{p\} - pts$ *f contour-integrable-on* g'
using *idt.hyps*(*3*)[*of* a' $s{-}\{p\}$] *idt.prems idt.hyps*(*1*)
by (*metis Diff-insert2 open-delete*)
def $g{\equiv}linepath\ a\ a' +{+}{+} g'$
have *valid-path g* **unfolding** *g-def* **by** (*auto intro*: *valid-path-join*)
moreover have *pathstart* $g = a$ **and** *pathfinish* $g = b$ **unfolding** *g-def* **by** *auto*

moreover have *path-image g ⊆ s − insert p pts* **unfolding** *g-def*
proof (*rule subset-path-image-join*)
have *closed-segment a a′ ⊆ ball a e* **using** ‹*e>0*›
by (*auto dest*!:*segment-bound1 simp*:*a′-def dist-complex-def norm-minus-commute*)
then show *path-image* (*linepath a a′*) ⊆ *s − insert p pts* **using** *e idt*(*9*)
by *auto*
next
show *path-image g′ ⊆ s − insert p pts* **using** *g′*(*4*) **by** *blast*
qed
moreover have *f contour-integrable-on g*
proof −
have *closed-segment a a′ ⊆ ball a e* **using** ‹*e>0*›
by (*auto dest*!:*segment-bound1 simp*:*a′-def dist-complex-def norm-minus-commute*)
then have *continuous-on* (*closed-segment a a′*) *f*
using *e idt.prems*(*6*) *holomorphic-on-imp-continuous-on*[*OF idt.prems*(*5*)]
apply (*elim continuous-on-subset*)
by *auto*
then have *f contour-integrable-on linepath a a′*
using *contour-integrable-continuous-linepath* **by** *auto*
then show *?thesis* **unfolding** *g-def*
apply (*rule contour-integrable-joinI*)
by (*auto simp add*: ‹*e>0*›)
qed
ultimately show *?case* **using** *idt.prems*(*1*)[*of g*] **by** *auto*
qed

lemma *Cauchy-theorem-aux*:
assumes *open s connected* (*s−pts*) *finite pts pts ⊆ s f holomorphic-on s−pts*
valid-path g pathfinish g = pathstart g path-image g ⊆ s−pts
∀ z. (*z ∉ s*) *⟶ winding-number g z = 0*
∀ p∈s. h p>0 ∧ (*∀ w∈cball p* (*h p*). *w∈s ∧* (*w≠p ⟶ w ∉ pts*))
shows *contour-integral g f* = (*∑ p∈pts. winding-number g p ∗ contour-integral* (*circlepath p* (*h p*)) *f*)
using *assms*
proof (*induct arbitrary*:*s g rule*:*finite-induct*[*OF* ‹*finite pts*›])
case *1*
then show *?case* **by** (*simp add*: *Cauchy-theorem-global contour-integral-unique*)
next
case (*2 p pts*)
note *fin*[*simp*] = ‹*finite* (*insert p pts*)›
and *connected* = ‹*connected* (*s − insert p pts*)›
and *valid*[*simp*] = ‹*valid-path g*›
and *g-loop*[*simp*] = ‹*pathfinish g = pathstart g*›
and *holo*[*simp*]= ‹*f holomorphic-on s − insert p pts*›
and *path-img* = ‹*path-image g ⊆ s − insert p pts*›
and *winding* = ‹*∀ z. z ∉ s ⟶ winding-number g z = 0*›
and *h* = ‹*∀ pa∈s. 0 < h pa ∧* (*∀ w∈cball pa* (*h pa*). *w ∈ s ∧* (*w ≠ pa ⟶ w ∉ insert p pts*))›

have *h p>0* **and** *p∈s*
and *h-p*: ∀ *w∈cball p* (*h p*). *w* ∈ *s* ∧ (*w* ≠ *p* ⟶ *w* ∉ *insert p pts*)
using *h* ‹*insert p pts* ⊆ *s*› **by** *auto*
obtain *pg* **where** *pg*[*simp*]: *valid-path pg pathstart pg* = *pathstart g pathfinish pg*=*p*+*h p*
path-image pg ⊆ *s*−*insert p pts f contour-integrable-on pg*
proof −
have *p* + *h p*∈*cball p* (*h p*) **using** *h*[*rule-format*,*of p*]
by (*simp add*: ‹*p* ∈ *s*› *dist-norm*)
then have *p* + *h p* ∈ *s* − *insert p pts* **using** *h*[*rule-format*,*of p*] ‹*insert p pts* ⊆ *s*›
by *fastforce*
moreover have *pathstart g* ∈ *s* − *insert p pts* **using** *path-img* **by** *auto*
ultimately show *?thesis*
using *get-integrable-path*[*OF* ‹*open s*› *connected fin holo*,*of pathstart g p*+*h p*] *that*
by *blast*
qed
obtain *n*::*int* **where** *n*=*winding-number g p*
using *integer-winding-number*[*OF - g-loop*,*of p*] *valid path-img*
by (*metis DiffD2 Ints-cases insertI1 subset-eq valid-path-imp-path*)
def *p-circ*≡*circlepath p* (*h p*) **and** *p-circ-pt*≡*linepath* (*p*+*h p*) (*p*+*h p*)
def *n-circ*≡λ*n*. (*op* +++ *p-circ* ^^ *n*) *p-circ-pt*
def *cp*≡*if n*≥*0 then reversepath* (*n-circ* (*nat n*)) *else n-circ* (*nat* (− *n*))
have *n-circ*:*valid-path* (*n-circ k*)
winding-number (*n-circ k*) *p* = *k*
pathstart (*n-circ k*) = *p* + *h p pathfinish* (*n-circ k*) = *p* + *h p*
path-image (*n-circ k*) = (*if k*=*0 then* {*p* + *h p*} *else sphere p* (*h p*))
p ∉ *path-image* (*n-circ k*)
⋀*p*′. *p*′∉*s* − *pts* ⟹ *winding-number* (*n-circ k*) *p*′=*0* ∧ *p*′∉*path-image* (*n-circ k*)
f contour-integrable-on (*n-circ k*)
contour-integral (*n-circ k*) *f* = *k* ∗ *contour-integral p-circ f*
for *k*
proof (*induct k*)
case *0*
show *valid-path* (*n-circ 0*)
and *path-image* (*n-circ 0*) = (*if 0*=*0 then* {*p* + *h p*} *else sphere p* (*h p*))
and *winding-number* (*n-circ 0*) *p* = *of-nat 0*
and *pathstart* (*n-circ 0*) = *p* + *h p*
and *pathfinish* (*n-circ 0*) = *p* + *h p*
and *p* ∉ *path-image* (*n-circ 0*)
unfolding *n-circ-def p-circ-pt-def* **using** ‹*h p* > *0*›
by (*auto simp add*: *dist-norm*)
show *winding-number* (*n-circ 0*) *p*′=*0* ∧ *p*′∉*path-image* (*n-circ 0*) **when** *p*′∉*s*− *pts* **for** *p*′
unfolding *n-circ-def p-circ-pt-def*
apply (*auto intro*!:*winding-number-trivial*)
by (*metis Diff-iff pathfinish-in-path-image pg*(*3*) *pg*(*4*) *subsetCE subset-insertI*

```
that)+
    show f contour-integrable-on (n-circ 0)
      unfolding n-circ-def p-circ-pt-def
    by (auto intro!:contour-integrable-continuous-linepath simp add:continuous-on-sing)
    show contour-integral (n-circ 0) f = of-nat 0 * contour-integral p-circ f
      unfolding n-circ-def p-circ-pt-def by auto
  next
    case (Suc k)
    have n-Suc:n-circ (Suc k) = p-circ +++ n-circ k unfolding n-circ-def by
auto
   have pcirc:p ∉ path-image p-circ valid-path p-circ pathfinish p-circ = pathstart
(n-circ k)
      using Suc(3) unfolding p-circ-def using ‹h p > 0› by (auto simp add:
p-circ-def)
    have pcirc-image:path-image p-circ ⊆ s − insert p pts
      proof −
       have path-image p-circ ⊆ cball p (h p) using ‹0 < h p› p-circ-def by auto
        then show ?thesis using h-p pcirc(1) by auto
      qed
    have pcirc-integrable:f contour-integrable-on p-circ
      by (auto simp add:p-circ-def intro!: pcirc-image[unfolded p-circ-def]
       contour-integrable-continuous-circlepath holomorphic-on-imp-continuous-on

        holomorphic-on-subset[OF holo])
    show valid-path (n-circ (Suc k))
     using valid-path-join[OF pcirc(2) Suc(1) pcirc(3)] unfolding n-circ-def by
auto
    show path-image (n-circ (Suc k))
       = (if Suc k = 0 then {p + complex-of-real (h p)} else sphere p (h p))
      proof −
        have path-image p-circ = sphere p (h p)
          unfolding p-circ-def using ‹0 < h p› by auto
        then show ?thesis unfolding n-Suc using Suc.hyps(5) ‹h p>0›
          by (auto simp add: path-image-join[OF pcirc(3)] dist-norm)
      qed
    then show p ∉ path-image (n-circ (Suc k)) using ‹h p>0› by auto
    show winding-number (n-circ (Suc k)) p = of-nat (Suc k)
      proof −
        have winding-number p-circ p = 1
          by (simp add: ‹h p > 0› p-circ-def winding-number-circlepath-centre)
       moreover have p ∉ path-image (n-circ k) using Suc(5) ‹h p>0› by auto
        then have winding-number (p-circ +++ n-circ k) p
           = winding-number p-circ p + winding-number (n-circ k) p
          using valid-path-imp-path Suc.hyps(1) Suc.hyps(2) pcirc
          apply (intro winding-number-join)
          by auto
        ultimately show ?thesis using Suc(2) unfolding n-circ-def
          by auto
      qed
```

```
      show pathstart (n-circ (Suc k)) = p + h p
        by (simp add: n-circ-def p-circ-def)
      show pathfinish (n-circ (Suc k)) = p + h p
        using Suc(4) unfolding n-circ-def by auto
      show winding-number (n-circ (Suc k)) p'=0 ∧  p'∉path-image (n-circ (Suc
k)) when p'∉s−pts for p'
        proof −
            have  p' ∉ path-image p-circ using ‹p ∈ s› h p-circ-def that using
pcirc-image by blast
          moreover have p' ∉ path-image (n-circ k)
            using Suc.hyps(7) that by blast
          moreover have winding-number p-circ p' = 0
            proof −
              have path-image p-circ ⊆ cball p (h p)
                using h unfolding p-circ-def using ‹p ∈ s› by fastforce
               moreover have p'∉cball p (h p) using ‹p ∈ s› h that 2.hyps(2) by
fastforce
              ultimately show ?thesis unfolding p-circ-def
                apply (intro winding-number-zero-outside)
                by auto
            qed
          ultimately show ?thesis
            unfolding n-Suc
            apply (subst winding-number-join)
              by (auto simp: pcirc Suc that not-in-path-image-join Suc.hyps(7)[OF
that])
        qed
      show f contour-integrable-on (n-circ (Suc k))
        unfolding n-Suc
      by (rule contour-integrable-joinI[OF pcirc-integrable Suc(8) pcirc(2) Suc(1)])
      show contour-integral (n-circ (Suc k)) f = (Suc k) ∗  contour-integral p-circ
f
        unfolding n-Suc
       by (auto simp add:contour-integral-join[OF pcirc-integrable Suc(8) pcirc(2)
Suc(1)]
          Suc(9) algebra-simps)
    qed
  have cp[simp]:pathstart cp = p + h p  pathfinish cp = p + h p
        valid-path cp path-image cp ⊆ s − insert p pts
        winding-number cp p = − n
        ⋀p'. p'∉s − pts ⟹ winding-number cp p'=0 ∧ p' ∉ path-image cp
        f contour-integrable-on cp
        contour-integral cp f = − n ∗ contour-integral p-circ f
    proof −
      show pathstart cp = p + h p and pathfinish cp = p + h p and valid-path cp
        using n-circ unfolding cp-def by auto
    next
      have sphere p (h p) ⊆  s − insert p pts
        using h[rule-format,of p] ‹insert p pts ⊆ s› by force
```

```
    moreover have p + complex-of-real (h p) ∈ s − insert p pts
      using pg(3) pg(4) by (metis pathfinish-in-path-image subsetCE)
    ultimately show path-image cp ⊆ s − insert p pts unfolding cp-def
      using n-circ(5) by auto
  next
    show winding-number cp p = − n
     unfolding cp-def using winding-number-reversepath n-circ ‹h p>0› by auto
  next
    show winding-number cp p'=0 ∧ p' ∉ path-image cp when p'∉s − pts for
p'
      unfolding cp-def
      apply (auto)
      apply (subst winding-number-reversepath)
      by (auto simp add: n-circ(7)[OF that] n-circ(1))
  next
    show f contour-integrable-on cp unfolding cp-def
      using contour-integrable-reversepath-eq n-circ(1,8) by auto
  next
    show contour-integral cp f = − n * contour-integral p-circ f
     unfolding cp-def using contour-integral-reversepath[OF n-circ(1)] n-circ(9)
      by auto
  qed
 def g'≡g +++ pg +++ cp +++ (reversepath pg)
 have contour-integral g' f = (∑ p∈pts. winding-number g' p * contour-integral
(circlepath p (h p)) f)
  proof (rule 2.hyps(3)[of s−{p} g',OF - - ‹finite pts› ])
    show connected (s − {p} − pts) using connected by (metis Diff-insert2)
    show open (s − {p}) using ‹open s› by auto
    show  pts ⊆ s − {p} using ‹insert p pts ⊆ s› ‹ p ∉ pts› by blast
      show f holomorphic-on s − {p} − pts using holo ‹p ∉ pts› by (metis
Diff-insert2)
    show valid-path g'
      unfolding g'-def cp-def using n-circ valid pg g-loop
      by (auto intro!:valid-path-join )
    show pathfinish g' = pathstart g'
      unfolding g'-def cp-def using pg(2) by simp
    show path-image g' ⊆ s − {p} − pts
      proof −
        def s'≡s − {p} − pts
        have s':s' = s−insert p pts unfolding s'-def by auto
        then show ?thesis using path-img pg(4) cp(4)
          unfolding g'-def
          apply (fold s'-def s')
          apply (intro subset-path-image-join)
          by auto
      qed
    note path-join-imp[simp]
    show ∀ z. z ∉ s − {p} ⟶ winding-number g' z = 0
      proof clarify
```

fix z **assume** $z{:}z \notin s - \{p\}$
have *winding-number* $(g +\!\!+\!\!+ pg +\!\!+\!\!+ cp +\!\!+\!\!+ reversepath\ pg)\ z =$ *winding-number* $g\ z$
$+$ *winding-number* $(pg +\!\!+\!\!+ cp +\!\!+\!\!+ (reversepath\ pg))\ z$
proof (*rule winding-number-join*)
show *path* g **using** ‹*valid-path* g› **by** *simp*
show $z \notin$ *path-image* g **using** z *path-img* **by** *auto*
show *path* $(pg +\!\!+\!\!+ cp +\!\!+\!\!+ reversepath\ pg)$ **using** $pg(3)$ cp **by** *auto*
next
have *path-image* $(pg +\!\!+\!\!+ cp +\!\!+\!\!+ reversepath\ pg) \subseteq s -$ *insert* p *pts*
using $pg(4)$ $cp(4)$ **by** (*auto simp*:*subset-path-image-join*)
then show $z \notin$ *path-image* $(pg +\!\!+\!\!+ cp +\!\!+\!\!+ reversepath\ pg)$ **using** z **by** *auto*
next
show *pathfinish* $g =$ *pathstart* $(pg +\!\!+\!\!+ cp +\!\!+\!\!+ reversepath\ pg)$ **using** *g-loop* **by** *auto*
qed
also have ... $=$ *winding-number* $g\ z +$ (*winding-number* $pg\ z$
$+$ *winding-number* $(cp +\!\!+\!\!+ (reversepath\ pg))\ z$)
proof (*subst add-left-cancel*,*rule winding-number-join*)
show *path* pg **and** *path* $(cp +\!\!+\!\!+ reversepath\ pg)$
and *pathfinish* $pg =$ *pathstart* $(cp +\!\!+\!\!+ reversepath\ pg)$ **by** *auto*
show $z \notin$ *path-image* pg **using** $pg(4)$ z **by** *blast*
show $z \notin$ *path-image* $(cp +\!\!+\!\!+ reversepath\ pg)$ **using** z
by (*metis Diff-iff* ‹$z \notin$ *path-image* pg› *contra-subsetD* $cp(4)$ *insertI1*
not-in-path-image-join path-image-reversepath singletonD)
qed
also have ... $=$ *winding-number* $g\ z +$ (*winding-number* $pg\ z$
$+$ (*winding-number* $cp\ z +$ *winding-number* $(reversepath\ pg)\ z$))
apply (*auto intro*!:*winding-number-join*)
apply (*metis Diff-iff contra-subsetD* $cp(4)$ *insertI1 singletonD* z)
by (*metis Diff-insert2 Diff-subset contra-subsetD* $pg(4)$ z)
also have ... $=$ *winding-number* $g\ z +$ *winding-number* $cp\ z$
apply (*subst winding-number-reversepath*)
apply *auto*
by (*metis Diff-iff contra-subsetD insertI1* $pg(4)$ *singletonD* z)
finally have *winding-number* $g'\ z =$ *winding-number* $g\ z +$ *winding-number* $cp\ z$
unfolding g'*-def* **.**
moreover have *winding-number* $g\ z +$ *winding-number* $cp\ z = 0$
using *winding* z ‹n=*winding-number* $g\ p$› **by** *auto*
ultimately show *winding-number* $g'\ z = 0$ **unfolding** g'*-def* **by** *auto*
qed
show $\forall\, pa{\in}s - \{p\}.\ 0 < h\ pa \wedge (\forall\, w{\in}cball\ pa\ (h\ pa).\ w \in s - \{p\} \wedge (w \neq pa \longrightarrow w \notin pts))$
using h **by** *fastforce*
qed
moreover have *contour-integral* $g'\ f =$ *contour-integral* $g\ f$
$-$ *winding-number* $g\ p *$ *contour-integral p-circ* f

```
    proof −
      have contour-integral g' f =  contour-integral g f
        + contour-integral (pg +++ cp +++ reversepath pg) f
        unfolding g'-def
        apply (subst Cauchy-Integral-Thm.contour-integral-join)
        by (auto simp add:open-Diff[OF ‹open s›,OF finite-imp-closed[OF fin]]
          intro!: contour-integrable-holomorphic-simple[OF holo - - path-img]
          contour-integrable-reversepath)
      also have ... = contour-integral g f + contour-integral pg f
          + contour-integral (cp +++ reversepath pg) f
        apply (subst Cauchy-Integral-Thm.contour-integral-join)
        by (auto simp add:contour-integrable-reversepath)
      also have ... = contour-integral g f + contour-integral pg f
          + contour-integral cp f + contour-integral (reversepath pg) f
        apply (subst Cauchy-Integral-Thm.contour-integral-join)
        by (auto simp add:contour-integrable-reversepath)
      also have ... = contour-integral g f + contour-integral cp f
        using contour-integral-reversepath
        by (auto simp add:contour-integrable-reversepath)
      also have ... = contour-integral g f − winding-number g p ∗ contour-integral
p-circ f
        using ‹n=winding-number g p› by auto
      finally show ?thesis .
    qed
  moreover have winding-number g' p' = winding-number g p' when p'∈pts for
p'
    proof −
      have [simp]: p' ∉ path-image g p' ∉ path-image pg p'∉path-image cp
        using 2.prems(8) that
        apply blast
        apply (metis Diff-iff Diff-insert2 contra-subsetD pg(4) that)
        by (meson DiffD2 cp(4) set-rev-mp subset-insertI that)
      have winding-number g' p' = winding-number g p'
         + winding-number (pg +++ cp +++ reversepath pg) p' unfolding g'-def
        apply (subst winding-number-join)
        apply simp-all
        apply (intro not-in-path-image-join)
        by auto
      also have ... = winding-number g p' + winding-number pg p'
          + winding-number (cp +++ reversepath pg) p'
        apply (subst winding-number-join)
        apply simp-all
        apply (intro not-in-path-image-join)
        by auto
     also have ... = winding-number g p' + winding-number pg p'+ winding-number
cp p'
          + winding-number (reversepath pg) p'
        apply (subst winding-number-join)
        by simp-all
```

```
    also have ... = winding-number g p' + winding-number cp p'
      apply (subst winding-number-reversepath)
      by auto
    also have ... = winding-number g p' using that by auto
    finally show ?thesis .
  qed
 ultimately show ?case unfolding p-circ-def
  apply (subst (asm) setsum.cong[OF refl,
      of pts - λp. winding-number g p * contour-integral (circlepath p (h p)) f])
  by (auto simp add:setsum.insert[OF ‹finite pts› ‹p∉pts›] algebra-simps)
qed


lemma Cauchy-theorem-singularities:
 assumes open s connected (s−pts) finite pts and
        holo:f holomorphic-on s−pts and
        valid-path g and
        loop:pathfinish g = pathstart g and
        path-image g ⊆ s−pts and
        homo:∀ z. (z ∉ s) ⟶ winding-number g z = 0 and
        avoid:∀ p∈s. h p>0 ∧ (∀ w∈cball p (h p). w∈s ∧ (w≠p ⟶ w ∉ pts))
 shows contour-integral g f = (∑ p∈pts. winding-number g p * contour-integral
(circlepath p (h p)) f)
   (is ?L=?R)
proof −
 def circ≡λp. winding-number g p * contour-integral (circlepath p (h p)) f
 def pts1≡pts ∩ s
 def pts2≡pts − pts1
 have pts=pts1 ∪ pts2 pts1 ∩ pts2 = {} pts2 ∩ s={} pts1⊆s
  unfolding pts1-def pts2-def by auto
 have contour-integral g f = (∑ p∈pts1. circ p) unfolding circ-def
  proof (rule Cauchy-theorem-aux[OF ‹open s› - - ‹pts1⊆s› - ‹valid-path g› loop
- homo])
   show connected (s − pts1) by (metis Diff-Int2 Int-absorb assms(2) pts1-def)
    show finite pts1 using ‹pts = pts1 ∪ pts2› assms(3) by auto
   show f holomorphic-on s − pts1 by (metis Diff-Int2 Int-absorb holo pts1-def)
    show path-image g ⊆ s − pts1 using assms(7) pts1-def by auto
    show ∀ p∈s. 0 < h p ∧ (∀ w∈cball p (h p). w ∈ s ∧ (w ≠ p ⟶ w ∉ pts1))
      by (simp add: avoid pts1-def)
  qed
 moreover have setsum circ pts2=0
  proof −
   have winding-number g p=0 when p∈pts2 for p
     using ‹pts2 ∩ s={}› that homo[rule-format,of p] by auto
    thus ?thesis unfolding circ-def
     apply (intro setsum.neutral)
     by auto
  qed
 moreover have ?R=setsum circ pts1 + setsum circ pts2
```

unfolding *circ-def*
using *setsum.union-disjoint*[*OF - - ‹pts1 ∩ pts2 = {}›*] *‹finite pts› ‹pts=pts1 ∪ pts2›*
by *blast*
ultimately show *?thesis*
apply (*fold circ-def*)
by *auto*
qed

definition *zorder*::(*complex ⇒ complex*) ⇒ *complex ⇒ nat* **where**
zorder f z = (*THE n. n>0 ∧* (∃ *h r. r>0 ∧ h holomorphic-on cball z r*
∧ (∀ *w∈cball z r. f w = h w ∗* (*w−z*) ^*n ∧ h w ≠0*)))

definition *zer-poly*::[*complex ⇒ complex*,*complex*]⇒*complex ⇒ complex* **where**
zer-poly f z = (*SOME h.* ∃ *r . r>0 ∧ h holomorphic-on cball z r*
∧ (∀ *w∈cball z r. f w = h w ∗* (*w−z*) ^(*zorder f z*) *∧ h w ≠0*))

definition *porder*::(*complex ⇒ complex*) ⇒ *complex ⇒ nat* **where**
porder f z = *zorder* (*inverse o f*) *z*

definition *pol-poly*::[*complex ⇒ complex*,*complex*]⇒*complex ⇒ complex* **where**
pol-poly f z = *inverse o zer-poly* (*inverse o f*) *z*

definition *residue*::(*complex ⇒ complex*) ⇒ *complex ⇒ complex* **where**
residue f z = (*let n=porder f z*;*h=pol-poly f z in* (*deriv* ^^ (*n − 1*)) *h z / fact* (*n−1*))

definition *is-pole*:: (*complex ⇒ complex*) ⇒ *complex ⇒bool* **where**
is-pole f p ≡ *isCont* (*inverse o f*) *p ∧ f p = 0*

lemma *zorder-exist*:
fixes *f*::*complex ⇒ complex* **and** *z*::*complex*
defines *n*≡*zorder f z* **and** *h*≡*zer-poly f z*
assumes *open s connected s z∈s*
and *holo*: *f holomorphic-on s*
and *f z=0* ∃ *w∈s. f w≠0*
shows ∃ *r. n>0 ∧ r>0 ∧ cball z r* ⊆ *s ∧ h holomorphic-on cball z r*
∧ (∀ *w∈cball z r. f w = h w ∗* (*w−z*) ^*n ∧ h w ≠0*)
proof −
def *P*≡*λh r n. r>0 ∧ h holomorphic-on cball z r*
∧ (∀ *w∈cball z r.* (*f w = h w ∗* (*w−z*) ^*n*) *∧ h w ≠0*)
have (∃!*n. n>0 ∧* (∃ *h r. P h r n*))
proof −

have $\exists!n.\ \exists h\ r.\ n{>}0 \wedge P\ h\ r\ n$
using *holomorphic-factor-zero-Ex1*[*OF* ‹*open s*› ‹*connected s*› ‹*z*∈*s*› *holo* ‹*f z*=*0*›
‹∃ *w*∈*s*. *f w*≠*0*›] **unfolding** *P-def*
apply (*subst mult.commute*)
by *auto*
thus *?thesis* **by** *auto*
qed
moreover have *n*:*n*=(*THE n*. *n*>*0* ∧ (∃ *h r*. *P h r n*))
unfolding *n-def zorder-def P-def* **by** *simp*
ultimately have *n*>*0* ∧ (∃ *h r*. *P h r n*)
apply (*drule-tac theI'*)
by *simp*
then have *n*>*0* **and** ∃ *h r*. *P h r n* **by** *auto*
moreover have *h*=(*SOME h*. ∃ *r*. *P h r n*)
unfolding *h-def P-def zer-poly-def*[*of f z*,*folded n-def P-def*] **by** *simp*
ultimately have ∃ *r*. *P h r n*
apply (*drule-tac someI-ex*)
by *simp*
then obtain *r1* **where** *P h r1 n* **by** *auto*
obtain *r2* **where** *r2*>*0 cball z r2* ⊆ *s*
using *assms*(*3*) *assms*(*5*) *open-contains-cball-eq* **by** *blast*
def *r3*≡*min r1 r2*
have *P h r3 n* **using** ‹*P h r1 n*› ‹*r2*>*0*› **unfolding** *P-def r3-def*
by *auto*
moreover have *cball z r3* ⊆ *s* **using** ‹*cball z r2* ⊆ *s*› **unfolding** *r3-def* **by** *auto*
ultimately show *?thesis* **using** ‹*n*>*0*› **unfolding** *P-def* **by** *auto*
qed

lemma *porder-exist*:
fixes *f*::*complex* ⇒ *complex* **and** *z*::*complex*
defines *n*≡*porder f z* **and** *h*≡*pol-poly f z*
assumes *open s connected s z*∈*s*
and *holo*:(*inverse* ∘ *f*) *holomorphic-on s*
and *f z*=*0* ∃ *w*∈*s*. *f w*≠*0*
shows ∃ *r*. *n*>*0* ∧ *r*>*0* ∧ *cball z r* ⊆ *s* ∧ *h holomorphic-on cball z r*
∧ (∀ *w*∈*cball z r*. *f w* = *h w* / (*w*−*z*) ^ *n* ∧ *h w* ≠*0*)
proof −
def *zo*≡*zorder* (*inverse* ∘ *f*) *z* **and** *zp*≡*zer-poly* (*inverse* ∘ *f*) *z*
obtain *r* **where** *0* < *zo* *0* < *r cball z r* ⊆ *s* **and** *zp-holo*: *zp holomorphic-on*
cball z r **and**
zp-fac: ∀ *w*∈*cball z r*. (*inverse* ∘ *f*) *w* = *zp w* ∗ (*w* − *z*) ^ *zo* ∧ *zp w* ≠ *0*
using *zorder-exist*[*OF* ‹*open s*› ‹*connected s*› ‹*z*∈*s*› *holo*,*folded zo-def zp-def*]
‹*f z*=*0*› ‹∃ *w*∈*s*. *f w*≠*0*›
by *auto*
have *n*:*n*=*zo* **and** *h*:*h*=*inverse o zp*
unfolding *n-def zo-def porder-def h-def zp-def pol-poly-def* **by** *simp-all*
have *h holomorphic-on cball z r*
using *zp-holo zp-fac holomorphic-on-inverse* **unfolding** *h comp-def* **by** *blast*

```
  moreover have ∀ w∈cball z r. f w  = h w / (w−z) ^n ∧ h w ≠0
    using zp-fac unfolding h n comp-def
   by (metis divide-inverse-commute field-class.field-inverse-zero inverse-inverse-eq

      inverse-mult-distrib mult.commute)
  moreover have 0 < n unfolding n using ‹zo>0› by simp
  ultimately show ?thesis using ‹0 < r› ‹cball z r ⊆ s› by auto
qed

lemma base-residue:
  fixes f::complex ⇒ complex and z::complex
  assumes  open s connected s z∈s
    and holo: (inverse o f) holomorphic-on s
    and f z=0
    and non-c:∃ w∈s. f w≠0
  shows ∃ r>0. cball z r ⊆ s
    ∧ (f has-contour-integral complex-of-real (2 * pi) * i * residue f z) (circlepath
z r)
proof −
  def n≡porder f z and h≡pol-poly f z
  obtain r where n>0 0 < r and
      r-b:cball z r ⊆ s and
      h-holo:h holomorphic-on cball z r
      and h:(∀ w∈cball z r. f w = h w / (w − z) ^ n ∧ h w ≠ 0)
    using porder-exist[OF ‹open s› ‹connected s› ‹z∈s› holo ‹f z=0› non-c]
    unfolding n-def h-def by auto
  def c≡complex-of-real (2 * pi) * i
  have residue f z =  (deriv ^^ (n − 1)) h z / fact (n−1)
    unfolding residue-def
    apply (fold n-def h-def)
    by simp
  then have ((λu. h u / (u − z) ^ n) has-contour-integral c * residue f z)
        (circlepath z r)
    using Cauchy-has-contour-integral-higher-derivative-circlepath[of z r h z n −1
      ,unfolded Suc-diff-1[OF ‹n>0›],folded c-def] h-holo r-b
   by (auto simp add:‹r>0› holomorphic-on-imp-continuous-on holomorphic-on-subset)
  then have (f has-contour-integral c * residue f z) (circlepath z r)
    proof (elim has-contour-integral-eq)
      fix x assume x ∈ path-image (circlepath z r)
      hence x∈cball z r using ‹0 < r› by auto
      then show h x / (x − z) ^ n = f x using h by auto
    qed
  then show ?thesis using ‹r>0› ‹cball z r ⊆ s› unfolding c-def by auto
qed

theorem residue-theorem:
  assumes open s connected (s−poles) and
        holo:f holomorphic-on s−poles and
        valid-path γ and
```

```
        loop:pathfinish γ = pathstart γ and
        path-image γ ⊆ s−poles and
        homo:∀ z. (z ∉ s) ⟶ winding-number γ z = 0 and
        finite {p. f p=0} and
        poles:∀ p∈poles. is-pole f p
  shows contour-integral γ f = 2 * pi * i *(∑ p∈poles. winding-number γ p *
residue f p)
proof −
  def pts≡{p. f p=0}
  def c≡2 * complex-of-real pi * i
  def contour≡λp e. (f has-contour-integral c * residue f p) (circlepath p e)
  def avoid≡λp e. ∀ w∈cball p e. w ∈ s ∧ (w ≠ p ⟶ w ∉ pts)
  have poles ⊆ pts and finite pts
    using poles ‹finite {p. f p=0}› unfolding pts-def is-pole-def by auto
  have ∃ e>0. avoid p e ∧ (p∈poles ⟶ contour p e)
      when p∈s for p
    proof −
      obtain e1 where e:e1>0 and e:avoid p e1
        using finite-cball-avoid[OF ‹open s› ‹finite pts›] ‹p∈s› unfolding avoid-def
by auto
      have ∃ e2>0. cball p e2 ⊆ ball p e1 ∧ contour p e2
        when p∈poles unfolding c-def contour-def
        proof (rule base-residue[of ball p e1 p f,simplified,OF ‹e1>0›])
          show inverse ∘ f holomorphic-on ball p e1
            proof −
              def f'≡inverse o f
              have f holomorphic-on ball p e1 − {p}
                using holo e ‹poles ⊆ pts› unfolding avoid-def
                apply (elim holomorphic-on-subset)
                by auto
             then have f'-holo:f' holomorphic-on ball p e1 − {p} unfolding f'-def
comp-def
                apply (elim holomorphic-on-inverse)
                using e pts-def  ball-subset-cball unfolding avoid-def by blast
              moreover have isCont f' p using ‹p∈poles› poles unfolding f'-def
is-pole-def by auto
              ultimately show f' holomorphic-on ball p e1
                apply (elim no-isolated-singularity[rotated])
                     apply (auto simp add:continuous-on-eq-continuous-at[of ball p
e1,simplified])
                using field-differentiable-imp-continuous-at f'-holo
                  holomorphic-on-imp-differentiable-at by fastforce
            qed
        next
          show f p = 0 using ‹p∈poles› poles unfolding is-pole-def by auto
        next
          def p'≡p+e1/2
           have p'∈ball p e1 and p'≠p using ‹e1>0› unfolding p'-def by (auto
simp add:dist-norm)
```

```
        then show ∃ w∈ball p e1. f w ≠ 0 using e unfolding avoid-def
          apply (rule-tac x=p' in bexI)
          by (auto simp add:pts-def)
       qed
      then obtain e2 where e2:p∈poles ⟶ e2>0 ∧ cball p e2 ⊆ ball p e1 ∧
contour p e2 by auto
      def e3≡if p∈poles then e2 else e1
      have avoid p e3
        using e2 e that  avoid-def e3-def by auto
      moreover have e3>0 using ‹e1>0› e2 unfolding e3-def by auto
      moreover have  p∈poles ⟶ contour p e3 using e2 unfolding e3-def by
auto
      ultimately show ?thesis by auto
    qed
  then obtain h where h:∀ p∈s. h p>0 ∧ (avoid p (h p) ∧ (p∈poles ⟶ contour
p (h p)))
    by metis
  def cont≡λp. contour-integral (circlepath p (h p)) f
  have contour-integral γ f = (∑ p∈poles. winding-number γ p * cont p)
    unfolding cont-def
    proof (rule Cauchy-theorem-singularities[OF ‹open s› ‹connected (s−poles)› -
holo ‹valid-path γ›
        loop ‹path-image γ ⊆ s−poles› homo])
    show finite poles using ‹poles ⊆ pts› and ‹finite pts› by (simp add: finite-subset)
    next
     show ∀ p∈s. 0 < h p ∧ (∀ w∈cball p (h p). w ∈ s ∧ (w ≠ p ⟶ w ∉ poles))
       using  ‹poles ⊆ pts› h unfolding avoid-def by blast
    qed
  also have ... = (∑ p∈poles. c * (winding-number γ p * residue f p))
    proof (rule setsum.cong[of poles poles,simplified])
     fix p assume p ∈ poles
     show winding-number γ p * cont p = c * (winding-number γ p * residue f p)
       proof (cases p∈s)
         assume p ∈ s
         then have cont p=c*residue f p
           unfolding cont-def
           apply (intro contour-integral-unique)
           using h[unfolded contour-def] ‹p ∈ poles› by blast
         then show ?thesis by auto
       next
         assume p∉s
         then have winding-number γ p=0 using homo by auto
         then show ?thesis by auto
       qed
    qed
  also have ... = c * (∑ p∈poles. winding-number γ p * residue f p)
    apply (subst setsum-right-distrib)
    by simp
  finally show ?thesis unfolding c-def by auto
```

qed

theorem *argument-principle*:
fixes *f*::*complex* ⇒ *complex* **and** *poles s*:: *complex set*
defines *pts*≡{*p*. *f p*=*0*}
defines *zeros*≡*pts* − *poles*
assumes *open s* **and**
connected:*connected* (*s*− *pts*) **and**
f-holo:*f holomorphic-on s*−*poles* **and**
h-holo:*h holomorphic-on s* **and**
valid-path g **and**
loop:*pathfinish g* = *pathstart g* **and**
path-img:*path-image g* ⊆ *s* − *pts* **and**
homo:∀ *z*. (*z* ∉ *s*) ⟶ *winding-number g z* = *0* **and**
finite:*finite pts* **and**
poles:∀ *p*∈*poles*. *is-pole f p*
shows *contour-integral g* (λ*x*. *deriv f x* ∗ *h x* / *f x*) = *2* ∗ *pi* ∗ i ∗
((∑ *p*∈*zeros*. *winding-number g p* ∗ *h p* ∗ *zorder f p*)
− (∑ *p*∈*poles*. *winding-number g p* ∗ *h p* ∗ *porder f p*))
(**is** *?L*=*?R*)
proof −
def *c*≡*2* ∗ *complex-of-real pi* ∗ i
def *ff*≡(λ*x*. *deriv f x* ∗ *h x* / *f x*)
def *cont-pole*≡λ*ff p e*. (*ff has-contour-integral* − *c* ∗ *porder f p* ∗ *h p*) (*circlepath p e*)
def *cont-zero*≡λ*ff p e*. (*ff has-contour-integral c* ∗ *zorder f p* ∗ *h p*) (*circlepath p e*)
def *avoid*≡λ*p e*. ∀ *w*∈*cball p e*. *w* ∈ *s* ∧ (*w* ≠ *p* ⟶ *w* ∉ *pts*)
have *poles* ⊆ *pts* **and** *zeros* ⊆ *pts* **and** *finite zeros* **and** *pts*=*zeros* ∪ *poles*
using *poles* ‹*finite pts*› **unfolding** *pts-def zeros-def is-pole-def* **by** *auto*
have ∃ *e*>*0*. *avoid p e* ∧ (*p*∈*poles* ⟶ *cont-pole ff p e*) ∧ (*p*∈*zeros* ⟶ *cont-zero ff p e*)
when *p*∈*s* **for** *p*
proof −
obtain *e1* **where** *e*:*e1*>*0* **and** *e*:*avoid p e1*
using *finite-cball-avoid*[*OF* ‹*open s*› ‹*finite pts*›] ‹*p*∈*s*› **unfolding** *avoid-def* **by** *auto*
have ∃ *e2*>*0*. *cball p e2* ⊆ *ball p e1* ∧ *cont-pole ff p e2*
when *p*∈*poles*
proof −
def *po*≡*porder f p*
def *pp*≡*pol-poly f p*
def *f'*≡λ*w*. *pp w* / (*w* − *p*) ^ *po*
def *ff'*≡(λ*x*. *deriv f' x* ∗ *h x* / *f' x*)
have *inverse* ∘ *f holomorphic-on ball p e1*
proof −
have *f holomorphic-on ball p e1* − {*p*}
using *f-holo e* ‹*poles* ⊆ *pts*› **unfolding** *avoid-def*
apply (*elim holomorphic-on-subset*)

```
            by auto
          then have inv-holo:(inverse o f) holomorphic-on ball p e1 - {p}
            unfolding comp-def
            apply (elim holomorphic-on-inverse)
            using e pts-def  ball-subset-cball unfolding avoid-def by blast
          moreover have isCont (inverse o f) p
            using ‹p∈poles› poles unfolding is-pole-def by auto
          ultimately show (inverse o f) holomorphic-on ball p e1
            apply (elim no-isolated-singularity[rotated])
                apply (auto simp add:continuous-on-eq-continuous-at[of ball p
e1,simplified])
            using field-differentiable-imp-continuous-at inv-holo
              holomorphic-on-imp-differentiable-at unfolding comp-def
              by fastforce
        qed
      moreover have f p = 0 using ‹p∈poles› poles unfolding is-pole-def by
auto
      moreover have ∃w∈ball p e1. f w ≠ 0
        proof -
          def p'≡p+e1/2
         have p'∈ball p e1 and p'≠p using ‹e1>0› unfolding p'-def by (auto
simp add:dist-norm)
          then show ∃w∈ball p e1. f w ≠ 0 using e unfolding avoid-def
            apply (rule-tac x=p' in bexI)
            by (auto simp add:pts-def)
        qed
      ultimately obtain r where
          0 < po r>0
          cball p r ⊆ ball p e1 and
          pp-holo:pp holomorphic-on cball p r and
          pp-po:(∀w∈cball p r. f w = pp w / (w - p) ^ po ∧ pp w ≠ 0)
           using porder-exist[of ball p e1 p f,simplified,OF ‹e1>0›] unfolding
po-def pp-def
        by auto
      def e2≡r/2
      have e2>0 using ‹r>0› unfolding e2-def by auto
      def anal≡λw. deriv pp w * h w / pp w and prin≡λw. - of-nat po * h w
/ (w - p)
        have ((λw.  prin w + anal w) has-contour-integral - c * po * h p)
(circlepath p e2)
        proof (rule  has-contour-integral-add[of - - - - 0,simplified])
          have ball p r ⊆ s
            using ‹cball p r ⊆ ball p e1› avoid-def ball-subset-cball e by blast
          then have cball p e2 ⊆ s
            using ‹r>0› unfolding e2-def by auto
          then have (λw. - of-nat po * h w) holomorphic-on cball p e2
            using h-holo
            by (auto intro!: holomorphic-intros)
        then show (prin has-contour-integral - c * of-nat po * h p ) (circlepath
```

```
p e2)
            using Cauchy-integral-circlepath-simple[folded c-def, of λw. − of-nat
po ∗ h w]
              ‹e2>0›
             unfolding prin-def
             by (auto simp add: mult.assoc)
           have anal holomorphic-on ball p r unfolding anal-def
             using pp-holo h-holo pp-po ‹ball p r ⊆ s›
             by (auto intro!: holomorphic-intros)
           then show (anal has-contour-integral 0) (circlepath p e2)
             using e2-def ‹r>0›
             by (auto elim!: Cauchy-theorem-disc-simple)
         qed
       then have cont-pole ff′ p e2 unfolding cont-pole-def po-def
         proof (elim has-contour-integral-eq)
           fix w assume w ∈ path-image (circlepath p e2)
            then have w∈ball p r and w≠p unfolding e2-def using ‹r>0› by
auto
           def wp≡w−p
           have wp≠0 and pp w ≠0
             unfolding wp-def using ‹w≠p› ‹w∈ball p r› pp-po by auto
           moreover have der-f′:deriv f′ w = − po ∗ pp w / (w−p) ^(po+1) +
deriv pp w / (w−p) ^po
             proof (rule DERIV-imp-deriv)
              def der ≡ − po ∗ pp w / (w−p) ^(po+1) + deriv pp w / (w−p) ^po
               have po:po = Suc (po − Suc 0) using ‹po>0› by auto
               have (pp has-field-derivative (deriv pp w)) (at w)
                 using DERIV-deriv-iff-field-differentiable ‹w∈ball p r›
                       holomorphic-on-imp-differentiable-at[of - ball p r]
                       holomorphic-on-subset [OF pp-holo ball-subset-cball]
                 by (metis open-ball)
               then show (f′ has-field-derivative der) (at w)
                 using ‹w≠p› ‹po>0› unfolding der-def f′-def
                 apply (auto intro!: derivative-eq-intros simp add:field-simps)
                 apply (subst (4) po)
                 apply (subst power-Suc)
                 by (auto simp add:field-simps)
             qed
           ultimately show prin w + anal w = ff′ w
             unfolding ff′-def prin-def anal-def
             apply simp
             apply (unfold f′-def)
             apply (fold wp-def)
             by (auto simp add:field-simps)
         qed
       then have cont-pole ff p e2 unfolding cont-pole-def
         proof (elim has-contour-integral-eq)
           fix w assume w ∈ path-image (circlepath p e2)
            then have w∈ball p r and w≠p unfolding e2-def using ‹r>0› by
```

```
auto
          have deriv f' w =  deriv f w
          proof (rule complex-derivative-transform-within-open[where s=ball p
r − {p}])
              show f' holomorphic-on ball p r − {p} unfolding f'-def using
pp-holo
            by (auto intro!: holomorphic-intros)
          next
            have ball p e1 − {p} ⊆ s − poles
              using avoid-def ball-subset-cball e ‹poles ⊆ pts› by auto
            then have ball p r − {p} ⊆ s − poles using ‹cball p r ⊆ ball p e1›
              using ball-subset-cball by blast
            then show f holomorphic-on ball p r − {p} using f-holo
              by auto
          next
            show open (ball p r − {p}) by auto
          next
            show w ∈ ball p r − {p} using ‹w∈ball p r› ‹w≠p› by auto
          next
            fix x assume x ∈ ball p r − {p}
            then show f' x = f x
              using pp-po unfolding f'-def by auto
          qed
          moreover have  f' w  =  f w
           using ‹w ∈ ball p r› ball-subset-cball subset-iff pp-po unfolding f'-def
by auto
          ultimately show ff' w = ff w
            unfolding ff'-def ff-def by simp
        qed
        moreover have cball p e2 ⊆ ball p e1
          using ‹0 < r› ‹cball p r ⊆ ball p e1› e2-def by auto
        ultimately show ?thesis using ‹e2>0› by auto
      qed
     then obtain e2 where e2:p∈poles ⟶ e2>0 ∧ cball p e2 ⊆ ball p e1 ∧
cont-pole ff p e2
      by auto
    have ∃ e3>0. cball p e3 ⊆ ball p e1 ∧ cont-zero ff p e3
      when p∈zeros
      proof −
        def zo≡zorder f p
        def zp≡zer-poly f p
        def f'≡λw. zp w * (w − p) ^ zo
        def ff'≡(λx. deriv f' x * h x / f' x)
        have f holomorphic-on ball p e1
          proof −
            have ball p e1 ⊆ s − poles
              using ‹poles ⊆ pts› avoid-def ball-subset-cball e that zeros-def by
fastforce
            thus ?thesis using f-holo by auto
```

```
          qed
        moreover have f p = 0 using ‹p∈zeros›
          using DiffD1 mem-Collect-eq pts-def zeros-def by blast
        moreover have ∃ w∈ball p e1. f w ≠ 0
          proof −
            def p'≡p+e1/2
           have p'∈ball p e1 and p'≠p using ‹e1>0› unfolding p'-def by (auto
simp add:dist-norm)
            then show ∃ w∈ball p e1. f w ≠ 0 using e unfolding avoid-def
              apply (rule-tac x=p' in bexI)
              by (auto simp add:pts-def)
          qed
        ultimately obtain r where
            0 < zo r>0
            cball p r ⊆ ball p e1 and
            pp-holo:zp holomorphic-on cball p r and
            pp-po:(∀ w∈cball p r. f w = zp w * (w − p) ^ zo ∧ zp w ≠ 0)
             using zorder-exist[of ball p e1 p f,simplified,OF ‹e1>0›] unfolding
zo-def zp-def
          by auto
        def e2≡r/2
        have e2>0 using ‹r>0› unfolding e2-def by auto
         def anal≡λw. deriv zp w * h w / zp w and prin≡λw. of-nat zo * h w /
(w − p)
       have ((λw. prin w + anal w) has-contour-integral c * zo * h p) (circlepath
p e2)
          proof (rule has-contour-integral-add[of - - - - 0,simplified])
            have ball p r ⊆ s
              using ‹cball p r ⊆ ball p e1› avoid-def ball-subset-cball e by blast
            then have cball p e2 ⊆ s
              using ‹r>0› unfolding e2-def by auto
            then have (λw. of-nat zo * h w) holomorphic-on cball p e2
              using h-holo
              by (auto intro!: holomorphic-intros)
            then show (prin has-contour-integral c * of-nat zo * h p ) (circlepath
p e2)
             using Cauchy-integral-circlepath-simple[folded c-def, of λw. of-nat zo
* h w]
                ‹e2>0›
              unfolding prin-def
              by (auto simp add: mult.assoc)
            have anal holomorphic-on ball p r unfolding anal-def
              using pp-holo h-holo pp-po ‹ball p r ⊆ s›
              by (auto intro!: holomorphic-intros)
            then show (anal has-contour-integral 0) (circlepath p e2)
              using e2-def ‹r>0›
              by (auto elim!: Cauchy-theorem-disc-simple)
          qed
        then have cont-zero ff' p e2 unfolding cont-zero-def zo-def
```

```
          proof (elim has-contour-integral-eq)
            fix w assume w ∈ path-image (circlepath p e2)
             then have w∈ball p r and w≠p unfolding e2-def using ‹r>0› by
auto
            def wp≡w−p
            have wp≠0 and zp w ≠0
              unfolding wp-def using ‹w≠p› ‹w∈ball p r› pp-po by auto
              moreover have der-f′:deriv f′ w = zo ∗ zp w ∗ (w−p) ^(zo−1) +
deriv zp w ∗ (w−p) ^zo
              proof (rule DERIV-imp-deriv)
                def der≡zo ∗ zp w ∗ (w−p) ^(zo−1) + deriv zp w ∗ (w−p) ^zo
                have po:zo = Suc (zo − Suc 0)  using ‹zo>0› by auto
                have (zp has-field-derivative (deriv zp w)) (at w)
                  using DERIV-deriv-iff-field-differentiable ‹w∈ball p r›
                        holomorphic-on-imp-differentiable-at[of - ball p r]
                        holomorphic-on-subset [OF pp-holo ball-subset-cball]
                  by (metis open-ball)
                then show (f′ has-field-derivative  der) (at w)
                  using ‹w≠p› ‹zo>0› unfolding der-def f′-def
                  by (auto intro!: derivative-eq-intros simp add:field-simps)
              qed
            ultimately show prin w + anal w = ff′ w
              unfolding ff′-def prin-def anal-def
              apply simp
              apply (unfold f′-def)
              apply (fold wp-def)
              apply (auto simp add:field-simps)
              by (metis Suc-diff-Suc minus-nat.diff-0 power-Suc)
          qed
        then have cont-zero ff p e2 unfolding cont-zero-def
          proof (elim has-contour-integral-eq)
            fix w assume w ∈ path-image (circlepath p e2)
             then have w∈ball p r and w≠p unfolding e2-def using ‹r>0› by
auto
            have deriv f′ w =  deriv f w
              proof (rule complex-derivative-transform-within-open[where s=ball
p r − {p}])
                show f′ holomorphic-on ball p r − {p} unfolding f′-def using
pp-holo
                  by (auto intro!: holomorphic-intros)
              next
                have ball p e1 − {p} ⊆ s − poles
                  using avoid-def ball-subset-cball e ‹poles ⊆ pts› by auto
               then have ball p r − {p} ⊆ s − poles using ‹cball p r ⊆ ball p e1›
                  using ball-subset-cball by blast
                then show f holomorphic-on ball p r − {p} using f-holo
                  by auto
              next
                show open (ball p r − {p}) by auto
```

```
            next
              show w ∈ ball p r − {p} using ‹w∈ball p r› ‹w≠p› by auto
            next
              fix x assume x ∈ ball p r − {p}
              then show f' x = f x
                using pp-po unfolding f'-def by auto
            qed
          moreover have f' w = f w
           using ‹w ∈ ball p r› ball-subset-cball subset-iff pp-po unfolding f'-def
by auto
          ultimately show ff' w = ff w
            unfolding ff'-def ff-def by simp
        qed
       moreover have cball p e2 ⊆ ball p e1
        using ‹0 < r› ‹cball p r ⊆ ball p e1› e2-def by auto
       ultimately show ?thesis using ‹e2>0› by auto
      qed
     then obtain e3 where e3:p∈zeros ⟶ e3>0 ∧ cball p e3 ⊆ ball p e1 ∧
cont-zero ff p e3
      by auto
    def e4≡if p∈poles then e2 else if p∈zeros then e3 else e1
    have e4>0 using e2 e3 ‹e1>0› unfolding e4-def by auto
   moreover have avoid p e4 using e2 e3 ‹e1>0› e unfolding e4-def avoid-def
by auto
    moreover have p∈poles ⟶ cont-pole ff p e4 and p∈zeros ⟶ cont-zero ff
p e4
      by (auto simp add: e2 e3 e4-def pts-def zeros-def)
    ultimately show ?thesis by auto
  qed
 then obtain get-e where get-e:∀ p∈s. get-e p>0 ∧ avoid p (get-e p)
    ∧ (p∈poles ⟶ cont-pole ff p (get-e p)) ∧ (p∈zeros ⟶ cont-zero ff p (get-e
p))
  by metis
 def cont≡λp. contour-integral (circlepath p (get-e p)) ff
 def w≡λp. winding-number g p
 have contour-integral g ff = (∑ p∈pts. w p * cont p)
  unfolding cont-def w-def
 proof (rule Cauchy-theorem-singularities[OF ‹open s› connected finite - ‹valid-path
g› loop
      path-img homo])
   have open (s − pts) using open-Diff[OF - finite-imp-closed[OF finite]] ‹open
s› by auto
    then show ff holomorphic-on s − pts unfolding ff-def using f-holo ‹poles
⊆ pts› h-holo
      by (auto intro!: holomorphic-intros simp add:pts-def)
  next
    show ∀ p∈s. 0 < get-e p ∧ (∀ w∈cball p (get-e p). w ∈ s ∧ (w ≠ p ⟶ w ∉
pts))
      using get-e using avoid-def by blast
```

```
  qed
 also have ... = (∑ p∈zeros. w p * cont p) + (∑ p∈poles. w p * cont p)
  using ‹finite pts› unfolding ‹pts=zeros ∪ poles›
  apply (subst setsum.union-disjoint)
  by (auto simp add:zeros-def)
 also have ... = c * ((∑ p∈zeros. w p *  h p * zorder f p) - (∑ p∈poles. w p *
h p * porder f p))
  proof -
   have (∑ p∈zeros. w p * cont p) = (∑ p∈zeros. c * w p *  h p * zorder f p)
    proof (rule setsum.cong[of zeros zeros,simplified])
     fix p assume p ∈ zeros
     show w p * cont p = c * w p * h p * (zorder f p)
      proof (cases p∈s)
       assume p ∈ s
       have cont p = c * h p * (zorder f p) unfolding cont-def
        apply (rule contour-integral-unique)
        using get-e ‹p∈s› ‹p∈zeros› unfolding cont-zero-def
        by (metis mult.assoc mult.commute)
       thus ?thesis by auto
      next
       assume p∉s
       then have w p=0 using homo unfolding w-def by auto
       then show ?thesis by auto
      qed
    qed
   then have (∑ p∈zeros. w p * cont p) = c * (∑ p∈zeros.  w p *  h p * zorder
f p)
    apply (subst setsum-right-distrib)
    by (simp add:algebra-simps)
   moreover have (∑ p∈poles. w p * cont p) = (∑ p∈poles.  - c * w p *  h p
* porder f p)
    proof (rule setsum.cong[of poles poles,simplified])
     fix p assume p ∈ poles
     show w p * cont p = - c * w p * h p * (porder f p)
      proof (cases p∈s)
       assume p ∈ s
       have cont p = - c * h p * (porder f p) unfolding cont-def
        apply (rule contour-integral-unique)
        using get-e ‹p∈s› ‹p∈poles› unfolding cont-pole-def
        by (metis mult.assoc mult.commute)
       thus ?thesis by auto
      next
       assume p∉s
       then have w p=0 using homo unfolding w-def by auto
       then show ?thesis by auto
      qed
    qed
   then have (∑ p∈poles. w p * cont p) = - c * (∑ p∈poles. w p *  h p *
porder f p)
```

```
      apply (subst setsum-right-distrib)
      by (simp add:algebra-simps)
    ultimately show ?thesis by (simp add: right-diff-distrib)
  qed
 finally show ?thesis unfolding w-def ff-def c-def by auto
qed

lemma holomorphic-imp-diff-on:
 assumes f-holo:f holomorphic-on s and open s
 shows f differentiable-on s
proof (rule differentiable-at-imp-differentiable-on)
 fix x assume x∈s
 obtain f' where (f has-field-derivative f') (at x)
  using holomorphic-on-imp-differentiable-at[OF f-holo ‹open s› ‹x∈s›]
  unfolding field-differentiable-def by auto
 then have (f has-derivative op * f') (at x)
  using has-field-derivative-imp-has-derivative[of f f' at x] by auto
 then show f differentiable at x using differentiableI by auto
qed

theorem Rouche-theorem:
 fixes f g::complex ⇒ complex and s:: complex set
 defines fg≡(λp. f p+ g p)
 defines zeros-fg≡{p. fg p =0} and zeros-f≡{p. f p=0}
 assumes
  open s and
  connected-fg:connected (s− zeros-fg) and
  connected-f:connected (s− zeros-f) and
  finite zeros-fg and
  finite zeros-f and
  f-holo:f holomorphic-on s and
  g-holo:g holomorphic-on s and
  valid-path γ and
  loop:pathfinish γ = pathstart γ and
  path-img:path-image γ ⊆ s and
  path-less:∀ z∈path-image γ. cmod(f z) > cmod(g z) and
  homo:∀ z. (z ∉ s) ⟶ winding-number γ z = 0
 shows (∑ p∈zeros-fg. winding-number γ p * zorder fg p)
       = (∑ p∈zeros-f. winding-number γ p * zorder f p)
proof −
 have path-fg:path-image γ ⊆ s − zeros-fg
  proof −
   have False when z∈path-image γ and f z + g z=0 for z
    proof −
     have cmod (f z) > cmod (g z) using ‹z∈path-image γ› path-less by auto
       moreover have f z = − g z  using ‹f z + g z =0› by (simp add:
eq-neg-iff-add-eq-0)
     then have cmod (f z) = cmod (g z) by auto
     ultimately show False by auto
```

```
      qed
    then show ?thesis unfolding zeros-fg-def fg-def using path-img by auto
  qed
 have path-f:path-image γ ⊆ s − zeros-f
  proof −
    have False when z∈path-image γ and f z =0 for z
     proof −
      have cmod (g z) < cmod (f z)  using ‹z∈path-image γ› path-less by auto
       then have cmod (g z) < 0 using ‹f z=0› by auto
       then show False by auto
     qed
    then show ?thesis unfolding zeros-f-def using path-img by auto
  qed
 def w≡λp. winding-number γ p
 def c≡2 ∗ complex-of-real pi ∗ i
 def h≡λp. g p / f p + 1
 obtain spikes
   where finite spikes and spikes: ∀ x∈{0..1} − spikes. γ differentiable at x
   using ‹valid-path γ›
  by (auto simp: valid-path-def piecewise-C1-differentiable-on-def C1-differentiable-on-eq)
 have h-contour:((λx. deriv h x / h x) has-contour-integral 0) γ
  proof −
    have outside-img:0 ∈ outside (path-image (h o γ))
     proof −
       have h p ∈ ball 1 1 when p∈path-image γ for p
        proof −
          have cmod (g p/f p) <1 using path-less[rule-format,OF that]
            apply (cases cmod (f p) = 0)
            by (auto simp add: norm-divide)
         then show ?thesis unfolding h-def by (auto simp add:dist-complex-def)
        qed
       then have path-image (h o γ) ⊆ ball 1 1
        by (simp add: image-subset-iff path-image-compose)
       moreover have  (0::complex) ∉ ball 1 1 by (simp add: dist-norm)
       ultimately show ?thesis
         using  convex-in-outside[of ball 1 1 0::complex]
         apply (drule-tac outside-mono)
         by auto
     qed
    have valid-h:valid-path (h ∘ γ)
     proof (rule valid-path-compose-holomorphic[OF ‹valid-path γ› - - path-f])
       show h holomorphic-on s − zeros-f
         unfolding h-def using f-holo g-holo
         by (auto intro!: holomorphic-intros simp add:zeros-f-def)
     next
       show open (s − zeros-f) using ‹finite zeros-f› ‹open s› finite-imp-closed
         by auto
     qed
    have ((λz. 1/z) has-contour-integral 0) (h ∘ γ)
```

```
      proof −
      have 0 ∉ path-image (h ∘ γ) using outside-img by (simp add: outside-def)
       then have ((λz. 1/z) has-contour-integral c ∗ winding-number (h ∘ γ) 0)
(h ∘ γ)
        using has-contour-integral-winding-number[of h o γ 0,simplified] valid-h
         unfolding c-def by auto
       moreover have winding-number (h o γ) 0 = 0
        proof −
          have 0 ∈ outside (path-image (h ∘ γ)) using outside-img .
          moreover have path (h o γ)
            using valid-h by auto
          moreover have pathfinish (h o γ) = pathstart (h o γ)
            by (simp add: loop pathfinish-compose pathstart-compose)
            ultimately show ?thesis using winding-number-zero-in-outside by
auto
        qed
       ultimately show ?thesis by auto
      qed
    moreover have vector-derivative (h ∘ γ) (at x) = vector-derivative γ (at x)
∗ deriv h (γ x)
        when x∈{0..1} − spikes for x
      proof (rule vector-derivative-chain-at-general)
        show γ differentiable at x using that ‹valid-path γ› spikes by auto
      next
        def der≡λp. (deriv g p ∗ f p − g p ∗ deriv f p)/(f p ∗ f p)
        def t≡γ x
        have f t≠0 unfolding zeros-f-def t-def
          by (metis DiffD1 image-eqI norm-not-less-zero norm-zero path-defs(4)
path-less that)
        moreover have t∈s
          using contra-subsetD path-image-def path-fg t-def that by fastforce
        ultimately have (h has-field-derivative der t) (at t)
          unfolding h-def der-def using g-holo f-holo
          apply (auto intro!: derivative-eq-intros)
          by (auto simp add: DERIV-deriv-iff-field-differentiable
            holomorphic-on-imp-differentiable-at[OF - ‹open s›])
        then show ∃ g'. (h has-field-derivative g') (at (γ x)) unfolding t-def by
auto
      qed
    then have (op / 1 has-contour-integral 0) (h ∘ γ)
      = ((λx. deriv h x / h x) has-contour-integral 0) γ
      unfolding has-contour-integral
      apply (intro has-integral-spike-eq[OF negligible-finite, OF ‹finite spikes›])
      by auto
    ultimately show ?thesis by auto
  qed
 then have contour-integral γ (λx. deriv h x / h x) = 0
  using contour-integral-unique by simp
 moreover have contour-integral γ (λx. deriv fg x / fg x) = contour-integral γ
```

```
(λx. deriv f x / f x)
     + contour-integral γ (λp. deriv h p / h p)
   proof −
     have (λp. deriv f p / f p) contour-integrable-on γ
        proof (rule contour-integrable-holomorphic-simple[OF - - ‹valid-path γ›
path-f])
       show open (s − zeros-f) using finite-imp-closed[OF ‹finite zeros-f›] ‹open
s›
         by auto
        then show (λp. deriv f p / f p) holomorphic-on s − zeros-f
         using f-holo
         by (auto intro!: holomorphic-intros simp add:zeros-f-def)
       qed
     moreover have (λp. deriv h p / h p) contour-integrable-on γ
      using h-contour
      by (simp add: has-contour-integral-integrable)
    ultimately have contour-integral γ (λx. deriv f x / f x + deriv h x / h x) =
       contour-integral γ (λp. deriv f p / f p) + contour-integral γ (λp. deriv h p
/ h p)
      using contour-integral-add[of (λp. deriv f p / f p) γ (λp. deriv h p / h p) ]
      by auto
     moreover have deriv fg p / fg p =  deriv f p / f p + deriv h p / h p
       when p∈ path-image γ for p
      proof −
       have fg p≠0 and f p≠0 using path-f path-fg that unfolding zeros-f-def
zeros-fg-def
         by auto
        have h p≠0
         proof (rule ccontr)
          assume ¬ h p ≠ 0
         then have g p / f p= −1 unfolding h-def by (simp add: add-eq-0-iff2)
          then have cmod (g p/f p) = 1 by auto
           moreover have cmod (g p/f p) <1 using path-less[rule-format,OF
that]
            apply (cases cmod (f p) = 0)
            by (auto simp add: norm-divide)
           ultimately show False by auto
         qed
        have der-fg:deriv fg p =  deriv f p + deriv g p unfolding fg-def
         using f-holo g-holo holomorphic-on-imp-differentiable-at[OF - ‹open s›]
path-img that
         by (auto intro!: deriv-add)
        have der-h:deriv h p = (deriv g p ∗ f p − g p ∗ deriv f p)/(f p ∗ f p)
         proof −
          def der≡λp. (deriv g p ∗ f p − g p ∗ deriv f p)/(f p ∗ f p)
          have p∈s using path-img that by auto
          then have (h has-field-derivative der p) (at p)
           unfolding h-def der-def using g-holo f-holo ‹f p≠0›
           apply (auto intro!: derivative-eq-intros)
```

by (*auto simp add*: *DERIV-deriv-iff-field-differentiable holomorphic-on-imp-differentiable-at*[*OF* - ‹*open s*›])
then show *?thesis* **unfolding** *der-def* **using** *DERIV-imp-deriv* **by** *auto*
qed
show *?thesis*
apply (*simp only*:*der-fg der-h*)
apply (*auto simp add*:*field-simps* ‹*h p*≠*0*› ‹*f p*≠*0*› ‹*fg p*≠*0*›)
by (*auto simp add*:*field-simps h-def* ‹*f p*≠*0*› *fg-def*)
qed
then have *contour-integral* γ ($\lambda p.$ *deriv fg p / fg p*)
= *contour-integral* γ ($\lambda p.$ *deriv f p / f p + deriv h p / h p*)
by (*elim contour-integral-eq*)
ultimately show *?thesis* **by** *auto*
qed
moreover have *contour-integral* γ ($\lambda x.$ *deriv fg x / fg x*) = *c* $*$ ($\sum$ *p*∈*zeros-fg. w p* $*$ *zorder fg p*)
unfolding *c-def zeros-fg-def w-def*
proof (*rule argument-principle*[*OF* ‹*open s*› - - - ‹*valid-path* γ› *loop - homo*, *of* - {} λ-. *1*,*simplified*])
show *connected* (*s* − {*p. fg p* = *0*}) **using** *connected-fg* **unfolding** *zeros-fg-def* .
show *fg holomorphic-on s* **unfolding** *fg-def* **using** *f-holo g-holo holomorphic-on-add* **by** *auto*
show *path-image* γ ⊆ *s* − {*p. fg p* = *0*} **using** *path-fg* **unfolding** *zeros-fg-def* .
show *finite* {*p. fg p* = *0*} **using** ‹*finite zeros-fg*› **unfolding** *zeros-fg-def* .
qed
moreover have *contour-integral* γ ($\lambda x.$ *deriv f x / f x*) = *c* $*$ ($\sum$ *p*∈*zeros-f. w p* $*$ *zorder f p*)
unfolding *c-def zeros-f-def w-def*
proof (*rule argument-principle*[*OF* ‹*open s*› - - - ‹*valid-path* γ› *loop - homo*, *of* - {} λ-. *1*,*simplified*])
show *connected* (*s* − {*p. f p* = *0*}) **using** *connected-f* **unfolding** *zeros-f-def* .
show *f holomorphic-on s* **using** *f-holo g-holo holomorphic-on-add* **by** *auto*
show *path-image* γ ⊆ *s* − {*p. f p* = *0*} **using** *path-f* **unfolding** *zeros-f-def* .
show *finite* {*p. f p* = *0*} **using** ‹*finite zeros-f*› **unfolding** *zeros-f-def* .
qed
ultimately have *c* $*$ ($\sum$ *p*∈*zeros-fg. w p* $*$ (*zorder fg p*)) = *c*$*$ ($\sum$ *p*∈*zeros-f. w p* $*$ (*zorder f p*))
by *auto*
then show *?thesis* **unfolding** *c-def* **using** *w-def* **by** *auto*
qed

end

55 Generalised Binomial Theorem

The proof of the Generalised Binomial Theorem and related results. We prove the generalised binomial theorem for complex numbers, following the proof at: https://proofwiki.org/wiki/Binomial_Theorem/General_Binomial_Theorem

theory *Generalised-Binomial-Theorem*
imports
Complex-Main
Complex-Transcendental
Summation
begin

lemma *gbinomial-ratio-limit*:
fixes $a :: 'a :: real\text{-}normed\text{-}field$
assumes $a \notin \mathbb{N}$
shows $(\lambda n.\ (a\ gchoose\ n)\ /\ (a\ gchoose\ Suc\ n)) \longrightarrow -1$
proof (*rule Lim-transform-eventually*)
let $?f = \lambda n.\ inverse\ (a\ /\ of\text{-}nat\ (Suc\ n) - of\text{-}nat\ n\ /\ of\text{-}nat\ (Suc\ n))$
from *eventually-gt-at-top*[*of* 0::*nat*]
show *eventually* $(\lambda n.\ ?f\ n = (a\ gchoose\ n)\ /(a\ gchoose\ Suc\ n))\ sequentially$
proof *eventually-elim*
fix $n :: nat$ **assume** *n*: $n > 0$
let $?P = \prod i = 0..n - 1.\ a - of\text{-}nat\ i$
from *n* **have** $(a\ gchoose\ n)\ /\ (a\ gchoose\ Suc\ n) = (of\text{-}nat\ (Suc\ n) :: 'a) *$
$(?P\ /\ (\prod i = 0..n.\ a - of\text{-}nat\ i))$ **by** (*simp add*: *gbinomial-def*)
also from *n* **have** $(\prod i = 0..n.\ a - of\text{-}nat\ i) = ?P * (a - of\text{-}nat\ n)$
by (*cases n*) (*simp-all add*: *setprod-nat-ivl-Suc*)
also have $?P\ /\ \ldots = (?P\ /\ ?P)\ /\ (a - of\text{-}nat\ n)$ **by** (*rule divide-divide-eq-left*[*symmetric*])
also from *assms* **have** $?P\ /\ ?P = 1$ **by** *auto*
also have $of\text{-}nat\ (Suc\ n) * (1\ /\ (a - of\text{-}nat\ n)) =$
$inverse\ (inverse\ (of\text{-}nat\ (Suc\ n)) * (a - of\text{-}nat\ n))$ **by** (*simp add*: *field-simps*)
also have $inverse\ (of\text{-}nat\ (Suc\ n)) * (a - of\text{-}nat\ n) = a\ /\ of\text{-}nat\ (Suc\ n) - of\text{-}nat\ n\ /\ of\text{-}nat\ (Suc\ n)$
by (*simp add*: *field-simps del*: *of-nat-Suc*)
finally show $?f\ n = (a\ gchoose\ n)\ /\ (a\ gchoose\ Suc\ n)$ **by** *simp*
qed

have $(\lambda n.\ norm\ a\ /\ (of\text{-}nat\ (Suc\ n))) \longrightarrow 0$
unfolding *divide-inverse*
by (*intro tendsto-mult-right-zero LIMSEQ-inverse-real-of-nat*)
hence $(\lambda n.\ a\ /\ of\text{-}nat\ (Suc\ n)) \longrightarrow 0$
by (*subst tendsto-norm-zero-iff*[*symmetric*]) (*simp add*: *norm-divide del*: *of-nat-Suc*)
hence $?f \longrightarrow inverse\ (0 - 1)$
by (*intro tendsto-inverse tendsto-diff LIMSEQ-n-over-Suc-n*) *simp-all*
thus $?f \longrightarrow -1$ **by** *simp*
qed

lemma *conv-radius-gchoose*:
fixes $a :: 'a :: \{real\text{-}normed\text{-}field, banach\}$
shows *conv-radius* $(\lambda n.\ a\ gchoose\ n) = (if\ a \in \mathbb{N}\ then\ \infty\ else\ 1)$
proof $(cases\ a \in \mathbb{N})$
assume *a*: $a \in \mathbb{N}$
have *eventually* $(\lambda n.\ (a\ gchoose\ n) = 0)$ *sequentially*
using *eventually-gt-at-top*[*of nat* $\lfloor norm\ a \rfloor$]
by *eventually-elim* (*insert a*, *auto elim*!: *Nats-cases simp*: *binomial-gbinomial*[*symmetric*])
from *conv-radius-cong*[*OF this*] *a* **show** *?thesis* **by** *simp*
next
assume *a*: $a \notin \mathbb{N}$
from *tendsto-norm*[*OF gbinomial-ratio-limit*[*OF this*]]
have *conv-radius* $(\lambda n.\ a\ gchoose\ n) = 1$
by (*intro conv-radius-ratio-limit-nonzero*[*of - 1*]) (*simp-all add*: *norm-divide*)
with *a* **show** *?thesis* **by** *simp*
qed

lemma *gen-binomial-complex*:
fixes $z :: complex$
assumes $norm\ z < 1$
shows $(\lambda n.\ (a\ gchoose\ n) * z\hat{}n)\ sums\ (1 + z)\ powr\ a$
proof −
def $K \equiv 1 - (1 - norm\ z)\ /\ 2$
from *assms* **have** *K*: $K > 0\ K < 1\ norm\ z < K$
unfolding *K-def* **by** (*auto simp*: *field-simps intro*!: *add-pos-nonneg*)
let *?f* $= \lambda n.\ a\ gchoose\ n$ **and** *?f'* $= diffs\ (\lambda n.\ a\ gchoose\ n)$
have *summable-strong*: $summable\ (\lambda n.\ ?f\ n * z\ \hat{}\ n)$ **if** $norm\ z < 1$ **for** *z* **using** *that*
by (*intro summable-in-conv-radius*) (*simp-all add*: *conv-radius-gchoose*)
with *K* **have** *summable*: $summable\ (\lambda n.\ ?f\ n * z\ \hat{}\ n)$ **if** $norm\ z < K$ **for** *z* **using** *that* **by** *auto*
hence *summable'*: $summable\ (\lambda n.\ ?f'\ n * z\ \hat{}\ n)$ **if** $norm\ z < K$ **for** *z* **using** *that*
by (*intro termdiff-converges*[*of - K*]) *simp-all*

def $f \equiv \lambda z.\ \sum n.\ ?f\ n * z\ \hat{}\ n$ **and** $f' \equiv \lambda z.\ \sum n.\ ?f'\ n * z\ \hat{}\ n$
{
fix *z* :: *complex* **assume** *z*: $norm\ z < K$
from *summable-mult2*[*OF summable'*[*OF z*], *of z*]
have *summable1*: $summable\ (\lambda n.\ ?f'\ n * z\ \hat{}\ Suc\ n)$ **by** (*simp add*: *mult-ac*)
hence *summable2*: $summable\ (\lambda n.\ of\text{-}nat\ n * ?f\ n * z\hat{}n)$
unfolding *diffs-def* **by** (*subst* (*asm*) *summable-Suc-iff*)

have $(1 + z) * f'\ z = (\sum n.\ ?f'\ n * z\hat{}n) + (\sum n.\ ?f'\ n * z\hat{}Suc\ n)$
unfolding *f'-def* **using** *summable' z* **by** (*simp add*: *algebra-simps suminf-mult*)
also have $(\sum n.\ ?f'\ n * z\hat{}n) = (\sum n.\ of\text{-}nat\ (Suc\ n) * ?f\ (Suc\ n) * z\hat{}n)$
by (*intro suminf-cong*) (*simp add*: *diffs-def*)
also have $(\sum n.\ ?f'\ n * z\hat{}Suc\ n) = (\sum n.\ of\text{-}nat\ n * ?f\ n * z\ \hat{}\ n)$

using *summable1 suminf-split-initial-segment*[*OF summable1*] **unfolding** *diffs-def*
by (*subst suminf-split-head*, *subst* (*asm*) *summable-Suc-iff*) *simp-all*
also have $(\sum n.\ of\text{-}nat\ (Suc\ n) * ?f\ (Suc\ n) * z\hat{}n) + (\sum n.\ of\text{-}nat\ n * ?f\ n * z\hat{}n) =$
$(\sum n.\ a * ?f\ n * z\hat{}n)$
by (*subst gbinomial-mult-1*, *subst suminf-add*)
(*insert summable'*[*OF z*] *summable2*,
simp-all add: *summable-powser-split-head algebra-simps diffs-def*)
also have $\ldots = a * f\ z$ **unfolding** *f-def*
by (*subst suminf-mult*[*symmetric*]) (*simp-all add*: *summable*[*OF z*] *mult-ac*)
finally have $a * f\ z = (1 + z) * f'\ z$ **by** *simp*
} note *deriv* = *this*

have [*derivative-intros*]: (*f has-field-derivative f' z*) (*at z*) **if** *norm z* < *of-real K* **for** *z*
unfolding *f-def f'-def* **using** *K that*
by (*intro termdiffs-strong*[*of ?f K z*] *summable-strong*) *simp-all*
have $f\ 0 = (\sum n.\ if\ n = 0\ then\ 1\ else\ 0)$ **unfolding** *f-def* **by** (*intro suminf-cong*) *simp*
also have $\ldots = 1$ **using** *sums-single*[*of 0* λ*-. 1*::*complex*] **unfolding** *sums-iff* **by** *simp*
finally have [*simp*]: $f\ 0 = 1$ **.**

have $\exists c.\ \forall z{\in}ball\ 0\ K.\ f\ z * (1 + z)\ powr\ (-a) = c$
proof (*rule has-field-derivative-zero-constant*)
fix *z* :: *complex* **assume** *z'*: $z \in ball\ 0\ K$
hence *z*: *norm z* < *K* **by** (*simp add*: *dist-0-norm*)
with *K* **have** *nz*: $1 + z \neq 0$ **by** (*auto dest*!: *minus-unique*)
from *z K* **have** *norm z* < *1* **by** *simp*
hence $(1 + z) \notin \mathbb{R}_{\leq 0}$ **by** (*cases z*) (*auto simp*: *complex-nonpos-Reals-iff*)
hence $((\lambda z.\ f\ z * (1 + z)\ powr\ (-a))\ has\text{-}field\text{-}derivative$
$f'\ z * (1 + z)\ powr\ (-a) - a * f\ z * (1 + z)\ powr\ (-a-1))\ (at\ z)$ **using** *z*
by (*auto intro*!: *derivative-eq-intros*)
also from *z* **have** $a * f\ z = (1 + z) * f'\ z$ **by** (*rule deriv*)
finally show $((\lambda z.\ f\ z * (1 + z)\ powr\ (-a))\ has\text{-}field\text{-}derivative\ 0)\ (at\ z\ within\ ball\ 0\ K)$
using *nz* **by** (*simp add*: *field-simps powr-diff-complex at-within-open*[*OF z']*)
qed *simp-all*
then obtain *c* **where** *c*: $\bigwedge z.\ z \in ball\ 0\ K \Longrightarrow f\ z * (1 + z)\ powr\ (-a) = c$ **by** *blast*
from *c*[*of 0*] **and** *K* **have** *c* = *1* **by** *simp*
with *c*[*of z*] **have** *f z* = (*1* + *z*) *powr a* **using** *K*
by (*simp add*: *powr-minus-complex field-simps dist-complex-def*)
with *summable K* **show** *?thesis* **unfolding** *f-def* **by** (*simp add*: *sums-iff*)
qed

lemma *gen-binomial-complex'*:

```
  fixes x y :: real and a :: complex
  assumes |x| < |y|
  shows   (λn. (a gchoose n) * of-real x^n * of-real y powr (a - of-nat n)) sums
            of-real (x + y) powr a (is ?P x y)
proof -
  {
    fix x y :: real assume xy: |x| < |y| y ≥ 0
    hence y > 0 by simp
    note xy = xy this
    from xy have (λn. (a gchoose n) * of-real (x / y) ^ n) sums (1 + of-real (x
/ y)) powr a
        by (intro gen-binomial-complex) (simp add: norm-divide)
    hence (λn. (a gchoose n) * of-real (x / y) ^ n * y powr a) sums
              ((1 + of-real (x / y)) powr a * y powr a)
      by (rule sums-mult2)
    also have (1 + complex-of-real (x / y)) = complex-of-real (1 + x/y) by simp
    also from xy have ... powr a * of-real y powr a = (... * y) powr a
      by (subst powr-times-real[symmetric]) (simp-all add: field-simps)
    also from xy have complex-of-real (1 + x / y) * complex-of-real y = of-real
(x + y)
      by (simp add: field-simps)
    finally have ?P x y using xy by (simp add: field-simps powr-diff-complex
powr-nat)
  } note A = this

  show ?thesis
  proof (cases y < 0)
    assume y: y < 0
    with assms have xy: x + y < 0 by simp
    with assms have |-x| < |-y| -y ≥ 0 by simp-all
    note A[OF this]
    also have complex-of-real (-x + -y) = - complex-of-real (x + y) by simp
    also from xy assms have ... powr a = (-1) powr -a * of-real (x + y) powr a
      by (subst powr-neg-real-complex) (simp add: abs-real-def split: if-split-asm)
    also {
      fix n :: nat
      from y have (a gchoose n) * of-real (-x) ^ n * of-real (-y) powr (a - of-nat
n) =
                    (a gchoose n) * (-of-real x / -of-real y) ^ n * (- of-real y)
powr a
        by (subst power-divide) (simp add: powr-diff-complex powr-nat)
      also from y have (- of-real y) powr a = (-1) powr -a * of-real y powr a
        by (subst powr-neg-real-complex) simp
      also have -complex-of-real x / -complex-of-real y = complex-of-real x /
complex-of-real y
        by simp
      also have ... ^ n = of-real x ^ n / of-real y ^ n by (simp add: power-divide)
      also have (a gchoose n) * ... * ((-1) powr -a * of-real y powr a) =
                  (-1) powr -a * ((a gchoose n) * of-real x ^ n * of-real y powr (a
```

```
− n))
      by (simp add: algebra-simps powr-diff-complex powr-nat)
    finally have (a gchoose n) * of-real (− x) ^ n * of-real (− y) powr (a −
of-nat n) =
                  (−1) powr −a * ((a gchoose n) * of-real x ^ n * of-real y powr
(a − of-nat n)) .
  }
  note sums-cong[OF this]
  finally show ?thesis by (simp add: sums-mult-iff)
 qed (insert A[of x y] assms, simp-all add: not-less)
qed

lemma gen-binomial-complex'':
 fixes x y :: real and a :: complex
 assumes |y| < |x|
 shows   (λn. (a gchoose n) * of-real x powr (a − of-nat n) * of-real y ^ n) sums

           of-real (x + y) powr a
 using gen-binomial-complex'[OF assms] by (simp add: mult-ac add.commute)

lemma gen-binomial-real:
 fixes z :: real
 assumes |z| < 1
 shows   (λn. (a gchoose n) * z^n) sums (1 + z) powr a
proof −
 from assms have norm (of-real z :: complex) < 1 by simp
 from gen-binomial-complex[OF this]
  have (λn. (of-real a gchoose n :: complex) * of-real z ^ n) sums
         (of-real (1 + z)) powr (of-real a) by simp
 also have (of-real (1 + z) :: complex) powr (of-real a) = of-real ((1 + z) powr
a)
  using assms by (subst powr-of-real) simp-all
 also have (of-real a gchoose n :: complex) = of-real (a gchoose n) for n
  by (simp add: gbinomial-def)
 hence (λn. (of-real a gchoose n :: complex) * of-real z ^ n) =
       (λn. of-real ((a gchoose n) * z ^ n)) by (intro ext) simp
 finally show ?thesis by (simp only: sums-of-real-iff)
qed

lemma gen-binomial-real':
 fixes x y a :: real
 assumes |x| < y
 shows   (λn. (a gchoose n) * x^n * y powr (a − of-nat n)) sums (x + y) powr a
proof −
 from assms have y > 0 by simp
 note xy = this assms
 from assms have |x / y| < 1 by simp
 hence (λn. (a gchoose n) * (x / y) ^ n) sums (1 + x / y) powr a
  by (rule gen-binomial-real)
```

hence $(\lambda n.\ (a\ gchoose\ n) * (x\ /\ y)\ \hat{}\ n * y\ powr\ a)\ sums\ ((1 + x\ /\ y)\ powr\ a * y\ powr\ a)$
by *(rule sums-mult2)*
with *xy* **show** *?thesis*
by *(simp add: field-simps powr-divide powr-divide2[symmetric] powr-realpow)*
qed

lemma *one-plus-neg-powr-powser*:
fixes *z s* :: *complex*
assumes *norm* $(z :: complex) < 1$
shows $(\lambda n.\ (-1)\hat{}n * ((s + n - 1)\ gchoose\ n) * z\hat{}n)\ sums\ (1 + z)\ powr\ (-s)$
using *gen-binomial-complex[OF assms, of* $-s$*]* **by** *(simp add: gbinomial-minus)*

lemma *gen-binomial-real''*:
fixes *x y a* :: *real*
assumes $|y| < x$
shows $(\lambda n.\ (a\ gchoose\ n) * x\ powr\ (a - of\text{-}nat\ n) * y\hat{}n)\ sums\ (x + y)\ powr\ a$
using *gen-binomial-real'[OF assms]* **by** *(simp add: mult-ac add.commute)*

lemma *sqrt-series'*:
$|z| < a \Longrightarrow (\lambda n.\ ((1/2)\ gchoose\ n) * a\ powr\ (1/2 - real\text{-}of\text{-}nat\ n) * z\ \hat{}\ n)\ sums$

$sqrt\ (a + z :: real)$
using *gen-binomial-real''[of z a 1/2]* **by** *(simp add: powr-half-sqrt)*

lemma *sqrt-series*:
$|z| < 1 \Longrightarrow (\lambda n.\ ((1/2)\ gchoose\ n) * z\ \hat{}\ n)\ sums\ sqrt\ (1 + z)$
using *gen-binomial-real[of z 1/2]* **by** *(simp add: powr-half-sqrt)*

end

56 Integral Test for Summability

theory *Integral-Test*
imports *Integration*
begin

The integral test for summability. We show here that for a decreasing non-negative function, the infinite sum over that function evaluated at the natural numbers converges iff the corresponding integral converges.

As a useful side result, we also provide some results on the difference between the integral and the partial sum. (This is useful e.g. for the definition of the Euler-Mascheroni constant)

locale *antimono-fun-sum-integral-diff* =
fixes *f* :: $real \Rightarrow real$
assumes *dec*: $\bigwedge x\ y.\ x \geq 0 \Longrightarrow x \leq y \Longrightarrow f\ x \geq f\ y$
assumes *nonneg*: $\bigwedge x.\ x \geq 0 \Longrightarrow f\ x \geq 0$
assumes *cont*: *continuous-on* $\{0..\}\ f$

begin

definition *sum-integral-diff-series n* = (∑ *k*≤*n*. *f* (*of-nat k*)) − (*integral* {*0*..*of-nat n*} *f*)

lemma *sum-integral-diff-series-nonneg*:
sum-integral-diff-series n ≥ *0*
proof −
note *int* = *integrable-continuous-real*[*OF continuous-on-subset*[*OF cont*]]
let *?int* = λ*a b*. *integral* {*of-nat a*..*of-nat b*} *f*
have −*sum-integral-diff-series n* = *?int 0 n* − (∑ *k*≤*n*. *f* (*of-nat k*))
by (*simp add*: *sum-integral-diff-series-def*)
also have *?int 0 n* = (∑ *k*<*n*. *?int k* (*Suc k*))
proof (*induction n*)
case (*Suc n*)
have *?int 0* (*Suc n*) = *?int 0 n* + *?int n* (*Suc n*)
by (*intro integral-combine*[*symmetric*] *int*) *simp-all*
with *Suc* **show** *?case* **by** *simp*
qed *simp-all*
also have ... ≤ (∑ *k*<*n*. *integral* {*of-nat k*..*of-nat* (*Suc k*)} (λ-::*real*. *f* (*of-nat k*)))
by (*intro setsum-mono integral-le int*) (*auto intro*: *dec*)
also have ... = (∑ *k*<*n*. *f* (*of-nat k*)) **by** *simp*
also have ... − (∑ *k*≤*n*. *f* (*of-nat k*)) = −(∑ *k*∈{..*n*} − {..<*n*}. *f* (*of-nat k*))
by (*subst setsum-diff*) *auto*
also have ... ≤ *0* **by** (*auto intro*!: *setsum-nonneg nonneg*)
finally show *sum-integral-diff-series n* ≥ *0* **by** *simp*
qed

lemma *sum-integral-diff-series-antimono*:
assumes *m* ≤ *n*
shows *sum-integral-diff-series m* ≥ *sum-integral-diff-series n*
proof −
let *?int* = λ*a b*. *integral* {*of-nat a*..*of-nat b*} *f*
note *int* = *integrable-continuous-real*[*OF continuous-on-subset*[*OF cont*]]
have *d-mono*: *sum-integral-diff-series* (*Suc n*) ≤ *sum-integral-diff-series n* **for** *n*
proof −
fix *n* :: *nat*
have *sum-integral-diff-series* (*Suc n*) − *sum-integral-diff-series n* =
f (*of-nat* (*Suc n*)) + (*?int 0 n* − *?int 0* (*Suc n*))
unfolding *sum-integral-diff-series-def* **by** (*simp add*: *algebra-simps*)
also have *?int 0 n* − *?int 0* (*Suc n*) = −*?int n* (*Suc n*)
by (*subst integral-combine* [*symmetric*, *of of-nat 0 of-nat n of-nat* (*Suc n*)])
(*auto intro*!: *int simp*: *algebra-simps*)
also have *?int n* (*Suc n*) ≥ *integral* {*of-nat n*..*of-nat* (*Suc n*)} (λ-::*real*. *f* (*of-nat* (*Suc n*)))
by (*intro integral-le int*) (*auto intro*: *dec*)
hence *f* (*of-nat* (*Suc n*)) + −*?int n* (*Suc n*) ≤ *0* **by** (*simp add*: *algebra-simps*)
finally show *sum-integral-diff-series* (*Suc n*) ≤ *sum-integral-diff-series n* **by**

simp
qed
with *assms* **show** *?thesis*
by (*induction rule*: *inc-induct*) (*auto intro*: *order.trans*[*OF - d-mono*])
qed

lemma *sum-integral-diff-series-Bseq*: *Bseq sum-integral-diff-series*
proof −
from *sum-integral-diff-series-nonneg* **and** *sum-integral-diff-series-antimono*
have *norm* (*sum-integral-diff-series n*) ≤ *sum-integral-diff-series 0* **for** *n* **by**
simp
thus *Bseq sum-integral-diff-series* **by** (*rule BseqI'*)
qed

lemma *sum-integral-diff-series-monoseq*: *monoseq sum-integral-diff-series*
using *sum-integral-diff-series-antimono* **unfolding** *monoseq-def* **by** *blast*

lemma *sum-integral-diff-series-convergent*: *convergent sum-integral-diff-series*
using *sum-integral-diff-series-Bseq sum-integral-diff-series-monoseq*
by (*blast intro*!: *Bseq-monoseq-convergent*)

lemma *integral-test*:
summable (λ*n*. *f* (*of-nat n*)) ⟷ *convergent* (λ*n*. *integral* {*0..of-nat n*} *f*)
proof −
have *summable* (λ*n*. *f* (*of-nat n*)) ⟷ *convergent* (λ*n*. ∑ *k*≤*n*. *f* (*of-nat k*))
by (*simp add*: *summable-iff-convergent'*)
also have ... ⟷ *convergent* (λ*n*. *integral* {*0..of-nat n*} *f*)
proof
assume *convergent* (λ*n*. ∑ *k*≤*n*. *f* (*of-nat k*))
from *convergent-diff*[*OF this sum-integral-diff-series-convergent*]
show *convergent* (λ*n*. *integral* {*0..of-nat n*} *f*)
unfolding *sum-integral-diff-series-def* **by** *simp*
next
assume *convergent* (λ*n*. *integral* {*0..of-nat n*} *f*)
from *convergent-add*[*OF this sum-integral-diff-series-convergent*]
show *convergent* (λ*n*. ∑ *k*≤*n*. *f* (*of-nat k*)) **unfolding** *sum-integral-diff-series-def*
by *simp*
qed
finally show *?thesis* **by** *simp*
qed

end

end

57 Harmonic Numbers

theory *Harmonic-Numbers*
imports

Complex-Transcendental
Summation
Integral-Test
begin

The definition of the Harmonic Numbers and the Euler-Mascheroni constant. Also provides a reasonably accurate approximation of *ln 2* and the Euler-Mascheroni constant.

lemma *ln-2-less-1*: *ln 2* < (*1*::*real*)
proof −
have *2* < *5*/(*2*::*real*) **by** *simp*
also have *5*/*2* ≤ *exp* (*1*::*real*) **using** *exp-lower-taylor-quadratic*[*of 1*, *simplified*]
by *simp*
finally have *exp* (*ln 2*) < *exp* (*1*::*real*) **by** *simp*
thus *ln 2* < (*1*::*real*) **by** (*subst* (*asm*) *exp-less-cancel-iff*) *simp*
qed

lemma *setsum-Suc-diff'*:
fixes *f* :: *nat* ⇒ 'a::*ab-group-add*
assumes *m* ≤ *n*
shows (∑ *i* = *m*..<*n*. *f* (*Suc i*) − *f i*) = *f n* − *f m*
using *assms* **by** (*induct n*) (*auto simp*: *le-Suc-eq*)

57.1 The Harmonic numbers

definition *harm* :: *nat* ⇒ 'a :: *real-normed-field* **where**
harm n = (∑ *k*=*1*..*n*. *inverse* (*of-nat k*))

lemma *harm-altdef*: *harm n* = (∑ *k*<*n*. *inverse* (*of-nat* (*Suc k*)))
unfolding *harm-def* **by** (*induction n*) *simp-all*

lemma *harm-Suc*: *harm* (*Suc n*) = *harm n* + *inverse* (*of-nat* (*Suc n*))
by (*simp add*: *harm-def*)

lemma *harm-nonneg*: *harm n* ≥ (*0* :: 'a :: {*real-normed-field*,*linordered-field*})
unfolding *harm-def* **by** (*intro setsum-nonneg*) *simp-all*

lemma *harm-pos*: *n* > *0* ⟹ *harm n* > (*0* :: 'a :: {*real-normed-field*,*linordered-field*})
unfolding *harm-def* **by** (*intro setsum-pos*) *simp-all*

lemma *of-real-harm*: *of-real* (*harm n*) = *harm n*
unfolding *harm-def* **by** *simp*

lemma *norm-harm*: *norm* (*harm n*) = *harm n*
by (*subst of-real-harm* [*symmetric*]) (*simp add*: *harm-nonneg*)

lemma *harm-expand*:
harm 0 = *0*
harm (*Suc 0*) = *1*

harm (*numeral n*) = *harm* (*pred-numeral n*) + *inverse* (*numeral n*)
proof −
have *numeral n* = *Suc* (*pred-numeral n*) **by** *simp*
also have *harm* ... = *harm* (*pred-numeral n*) + *inverse* (*numeral n*)
by (*subst harm-Suc*, *subst numeral-eq-Suc*[*symmetric*]) *simp*
finally show *harm* (*numeral n*) = *harm* (*pred-numeral n*) + *inverse* (*numeral n*) **.**
qed (*simp-all add*: *harm-def*)

lemma *not-convergent-harm*: ¬*convergent* (*harm* :: *nat* ⇒ *'a* :: *real-normed-field*)
proof −
have *convergent* (λ*n*. *norm* (*harm n* :: *'a*)) ⟷
convergent (*harm* :: *nat* ⇒ *real*) **by** (*simp add*: *norm-harm*)
also have ... ⟷ *convergent* (λ*n*. ∑ *k*=*Suc 0*..*Suc n*. *inverse* (*of-nat k*) :: *real*)
unfolding *harm-def*[*abs-def*] **by** (*subst convergent-Suc-iff*) *simp-all*
also have ... ⟷ *convergent* (λ*n*. ∑ *k*≤*n*. *inverse* (*of-nat* (*Suc k*)) :: *real*)
by (*subst setsum-shift-bounds-cl-Suc-ivl*) (*simp add*: *atLeast0AtMost*)
also have ... ⟷ *summable* (λ*n*. *inverse* (*of-nat n*) :: *real*)
by (*subst summable-Suc-iff* [*symmetric*]) (*simp add*: *summable-iff-convergent'*)
also have ¬... **by** (*rule not-summable-harmonic*)
finally show *?thesis* **by** (*blast dest*: *convergent-norm*)
qed

lemma *harm-pos-iff* [*simp*]: *harm n* > (*0* :: *'a* :: {*real-normed-field*,*linordered-field*})
⟷ *n* > *0*
by (*rule iffI*, *cases n*, *simp add*: *harm-expand*, *simp*, *rule harm-pos*)

lemma *ln-diff-le-inverse*:
assumes *x* ≥ (*1*::*real*)
shows *ln* (*x* + *1*) − *ln x* < *1* / *x*
proof −
from *assms* **have** ∃ *z*>*x*. *z* < *x* + *1* ∧ *ln* (*x* + *1*) − *ln x* = (*x* + *1* − *x*) ∗ *inverse z*
by (*intro MVT2*) (*auto intro*!: *derivative-eq-intros simp*: *field-simps*)
then obtain *z* **where** *z*: *z* > *x z* < *x* + *1 ln* (*x* + *1*) − *ln x* = *inverse z* **by** *auto*
have *ln* (*x* + *1*) − *ln x* = *inverse z* **by** *fact*
also from *z*(*1*,*2*) *assms* **have** ... < *1* / *x* **by** (*simp add*: *field-simps*)
finally show *?thesis* **.**
qed

lemma *ln-le-harm*: *ln* (*real n* + *1*) ≤ (*harm n* :: *real*)
proof (*induction n*)
fix *n* **assume** *IH*: *ln* (*real n* + *1*) ≤ *harm n*
have *ln* (*real* (*Suc n*) + *1*) = *ln* (*real n* + *1*) + (*ln* (*real n* + *2*) − *ln* (*real n* + *1*)) **by** *simp*
also have (*ln* (*real n* + *2*) − *ln* (*real n* + *1*)) ≤ *1* / *real* (*Suc n*)
using *ln-diff-le-inverse*[*of real n* + *1*] **by** (*simp add*: *add-ac*)
also note *IH*

also have *harm n + 1 / real (Suc n) = harm (Suc n)* **by** (*simp add*: *harm-Suc field-simps*)
finally show *ln (real (Suc n) + 1) ≤ harm (Suc n)* **by** − *simp*
qed (*simp-all add*: *harm-def*)

57.2 The Euler–Mascheroni constant

The limit of the difference between the partial harmonic sum and the natural logarithm (approximately 0.577216). This value occurs e.g. in the definition of the Gamma function.

definition *euler-mascheroni* :: *'a* :: *real-normed-algebra-1* **where**
euler-mascheroni = of-real (lim (λn. harm n − ln (of-nat n)))

lemma *of-real-euler-mascheroni* [*simp*]: *of-real euler-mascheroni = euler-mascheroni*
by (*simp add*: *euler-mascheroni-def*)

interpretation *euler-mascheroni*: *antimono-fun-sum-integral-diff λx. inverse (x + 1)*
by *unfold-locales* (*auto intro*!: *continuous-intros*)

lemma *euler-mascheroni-sum-integral-diff-series*:
euler-mascheroni.sum-integral-diff-series n = harm (Suc n) − ln (of-nat (Suc n))
proof −
have *harm (Suc n) = (∑k=0..n. inverse (of-nat k + 1) :: real)* **unfolding** *harm-def*
unfolding *One-nat-def* **by** (*subst setsum-shift-bounds-cl-Suc-ivl*) (*simp add*: *add-ac*)
moreover have ((*λx. inverse (x + 1) :: real*) *has-integral ln (of-nat n + 1) − ln (0 + 1)*)
{0..of-nat n}
by (*intro fundamental-theorem-of-calculus*)
(*auto intro*!: *derivative-eq-intros simp*: *divide-inverse*
has-field-derivative-iff-has-vector-derivative[*symmetric*])
hence *integral {0..of-nat n} (λx. inverse (x + 1) :: real) = ln (of-nat (Suc n))*
by (*auto dest*!: *integral-unique*)
ultimately show *?thesis*
by (*simp add*: *euler-mascheroni.sum-integral-diff-series-def atLeast0AtMost*)
qed

lemma *euler-mascheroni-sequence-decreasing*:
m > 0 ⟹ m ≤ n ⟹ harm n − ln (of-nat n) ≤ harm m − ln (of-nat m :: real)
by (*cases m*, *simp*, *cases n*, *simp*, *hypsubst*,
subst (*1 2*) *euler-mascheroni-sum-integral-diff-series* [*symmetric*],
rule euler-mascheroni.sum-integral-diff-series-antimono, *simp*)

lemma *euler-mascheroni-sequence-nonneg*:
n > 0 ⟹ harm n − ln (of-nat n) ≥ (0::real)
by (*cases n*, *simp*, *hypsubst*, *subst euler-mascheroni-sum-integral-diff-series* [*symmetric*],

rule euler-mascheroni.sum-integral-diff-series-nonneg)

lemma *euler-mascheroni-convergent*: *convergent* (λ*n*. *harm n* − *ln* (*of-nat n*) :: *real*)
proof −
have *A*: (λ*n*. *harm* (*Suc n*) − *ln* (*of-nat* (*Suc n*))) =
euler-mascheroni.sum-integral-diff-series
by (*subst euler-mascheroni-sum-integral-diff-series* [*symmetric*]) (*rule refl*)
have *convergent* (λ*n*. *harm* (*Suc n*) − *ln* (*of-nat* (*Suc n*) :: *real*))
by (*subst A*) (*fact euler-mascheroni.sum-integral-diff-series-convergent*)
thus *?thesis* **by** (*subst* (*asm*) *convergent-Suc-iff*)
qed

lemma *euler-mascheroni-LIMSEQ*:
(λ*n*. *harm n* − *ln* (*of-nat n*) :: *real*) ⟶ *euler-mascheroni*
unfolding *euler-mascheroni-def*
by (*simp add*: *convergent-LIMSEQ-iff* [*symmetric*] *euler-mascheroni-convergent*)

lemma *euler-mascheroni-LIMSEQ-of-real*:
(λ*n*. *of-real* (*harm n* − *ln* (*of-nat n*))) ⟶
(*euler-mascheroni* :: *'a* :: {*real-normed-algebra-1*, *topological-space*})
proof −
have (λ*n*. *of-real* (*harm n* − *ln* (*of-nat n*))) ⟶ (*of-real* (*euler-mascheroni*) :: *'a*)
by (*intro tendsto-of-real euler-mascheroni-LIMSEQ*)
thus *?thesis* **by** *simp*
qed

lemma *euler-mascheroni-sum*:
(λ*n*. *inverse* (*of-nat* (*n*+*1*)) + *ln* (*of-nat* (*n*+*1*)) − *ln* (*of-nat* (*n*+*2*)) :: *real*)
sums euler-mascheroni
using *sums-add*[*OF telescope-sums*[*OF LIMSEQ-Suc*[*OF euler-mascheroni-LIMSEQ*]]
telescope-sums'[*OF LIMSEQ-inverse-real-of-nat*]]
by (*simp-all add*: *harm-def algebra-simps*)

lemma *alternating-harmonic-series-sums*: (λ*k*. (−*1*) ^ *k* / *real-of-nat* (*Suc k*)) *sums ln 2*
proof −
let *?f* = λ*n*. *harm n* − *ln* (*real-of-nat n*)
let *?g* = λ*n*. *if even n then 0 else* (*2*::*real*)
let *?em* = λ*n*. *harm n* − *ln* (*real-of-nat n*)
have *eventually* (λ*n*. *?em* (*2*∗*n*) − *?em n* + *ln 2* = (∑ *k*<*2*∗*n*. (−*1*) ^ *k* / *real-of-nat* (*Suc k*))) *at-top*
using *eventually-gt-at-top*[*of 0*::*nat*]
proof *eventually-elim*
fix *n* :: *nat* **assume** *n*: *n* > *0*
have (∑ *k*<*2*∗*n*. (−*1*) ^ *k* / *real-of-nat* (*Suc k*)) =
(∑ *k*<*2*∗*n*. ((−*1*) ^ *k* + *?g k*) / *of-nat* (*Suc k*)) − (∑ *k*<*2*∗*n*. *?g k* / *of-nat* (*Suc k*))

by (*simp add*: *setsum.distrib algebra-simps divide-inverse*)
also have $(\sum k{<}2{*}n.\ ((-1)\ \hat{}\ k + ?g\ k)\ /\ real\text{-}of\text{-}nat\ (Suc\ k)) = harm\ (2{*}n)$
unfolding *harm-altdef* **by** (*intro setsum.cong*) (*auto simp*: *field-simps*)
also have $(\sum k{<}2{*}n.\ ?g\ k\ /\ real\text{-}of\text{-}nat\ (Suc\ k)) = (\sum k|k{<}2{*}n \wedge odd\ k.\ ?g\ k\ /\ of\text{-}nat\ (Suc\ k))$
by (*intro setsum.mono-neutral-right*) *auto*
also have $\ldots = (\sum k|k{<}2{*}n \wedge odd\ k.\ 2\ /\ (real\text{-}of\text{-}nat\ (Suc\ k)))$
by (*intro setsum.cong*) *auto*
also have $(\sum k|k{<}2{*}n \wedge odd\ k.\ 2\ /\ (real\text{-}of\text{-}nat\ (Suc\ k))) = harm\ n$
unfolding *harm-altdef*
by (*intro setsum.reindex-cong*[*of* $\lambda n.\ 2{*}n{+}1$]) (*auto simp*: *inj-on-def field-simps elim*!: *oddE*)
also have $harm\ (2{*}n) - harm\ n = ?em\ (2{*}n) - ?em\ n + ln\ 2$ **using** *n*
by (*simp-all add*: *algebra-simps ln-mult*)
finally show $?em\ (2{*}n) - ?em\ n + ln\ 2 = (\sum k{<}2{*}n.\ (-1)\ \hat{}\ k\ /\ real\text{-}of\text{-}nat\ (Suc\ k))$ **..**
qed
moreover have $(\lambda n.\ ?em\ (2{*}n) - ?em\ n + ln\ (2{::}real)) \longrightarrow euler\text{-}mascheroni - euler\text{-}mascheroni + ln\ 2$
by (*intro tendsto-intros euler-mascheroni-LIMSEQ filterlim-compose*[*OF euler-mascheroni-LIMSEQ*] *filterlim-subseq*) (*auto simp*: *subseq-def*)
hence $(\lambda n.\ ?em\ (2{*}n) - ?em\ n + ln\ (2{::}real)) \longrightarrow ln\ 2$ **by** *simp*
ultimately have $(\lambda n.\ (\sum k{<}2{*}n.\ (-1)\ \hat{}\ k\ /\ real\text{-}of\text{-}nat\ (Suc\ k))) \longrightarrow ln\ 2$
by (*rule Lim-transform-eventually*)

moreover have *summable* $(\lambda k.\ (-1)\ \hat{}\ k * inverse\ (real\text{-}of\text{-}nat\ (Suc\ k)))$
using *LIMSEQ-inverse-real-of-nat*
by (*intro summable-Leibniz*(1) *decseq-imp-monoseq decseq-SucI*) *simp-all*
hence *A*: $(\lambda n.\ \sum k{<}n.\ (-1)\ \hat{}\ k\ /\ real\text{-}of\text{-}nat\ (Suc\ k)) \longrightarrow (\sum k.\ (-1)\ \hat{}\ k\ /\ real\text{-}of\text{-}nat\ (Suc\ k))$
by (*simp add*: *summable-sums-iff divide-inverse sums-def*)
from *filterlim-compose*[*OF this filterlim-subseq*[*of op* $*$ (2::*nat*)]]
have $(\lambda n.\ \sum k{<}2{*}n.\ (-1)\ \hat{}\ k\ /\ real\text{-}of\text{-}nat\ (Suc\ k)) \longrightarrow (\sum k.\ (-1)\ \hat{}\ k\ /\ real\text{-}of\text{-}nat\ (Suc\ k))$
by (*simp add*: *subseq-def*)
ultimately have $(\sum k.\ (-\ 1)\ \hat{}\ k\ /\ real\text{-}of\text{-}nat\ (Suc\ k)) = ln\ 2$ **by** (*intro LIMSEQ-unique*)
with *A* **show** *?thesis* **by** (*simp add*: *sums-def*)
qed

lemma *alternating-harmonic-series-sums'*:
$(\lambda k.\ inverse\ (real\text{-}of\text{-}nat\ (2{*}k{+}1)) - inverse\ (real\text{-}of\text{-}nat\ (2{*}k{+}2)))\ sums\ ln\ 2$
unfolding *sums-def*
proof (*rule Lim-transform-eventually*)
show $(\lambda n.\ \sum k{<}2{*}n.\ (-1)\ \hat{}\ k\ /\ (real\text{-}of\text{-}nat\ (Suc\ k))) \longrightarrow ln\ 2$
using *alternating-harmonic-series-sums* **unfolding** *sums-def*
by (*rule filterlim-compose*) (*rule mult-nat-left-at-top*, *simp*)
show *eventually* $(\lambda n.\ (\sum k{<}2{*}n.\ (-1)\ \hat{}\ k\ /\ (real\text{-}of\text{-}nat\ (Suc\ k))) = (\sum k{<}n.\ inverse\ (real\text{-}of\text{-}nat\ (2{*}k{+}1)) - inverse\ (real\text{-}of\text{-}nat\ (2{*}k{+}2))))$

sequentially
proof (*intro always-eventually allI*)
fix *n* :: *nat*
show ($\sum k{<}2{*}n.\ (-1)\ \hat{}\ k\ /\ (\textit{real-of-nat}\ (\textit{Suc}\ k))) =$
$(\sum k{<}n.\ \textit{inverse}\ (\textit{real-of-nat}\ (2{*}k{+}1)) - \textit{inverse}\ (\textit{real-of-nat}\ (2{*}k{+}2)))$
by (*induction n*) (*simp-all add*: *inverse-eq-divide*)
qed
qed

57.3 Bounds on the Euler–Mascheroni constant

lemma *ln-inverse-approx-le*:
assumes (x::*real*) $>$ *0* *a* $>$ *0*
shows *ln* (x + *a*) − *ln x* ≤ *a* ∗ (*inverse x* + *inverse* (x + *a*))/*2* (**is** - ≤ *?A*)
proof −
def *f′* ≡ (*inverse* (x + *a*) − *inverse x*)/*a*
have *f′-nonpos*: *f′* ≤ *0* **using** *assms* **by** (*simp add*: *f′-def divide-simps*)
let *?f* = λ*t*. (t − *x*) ∗ *f′* + *inverse x*
let *?F* = λ*t*. (t − *x*) ^ *2* ∗ *f′* / *2* + *t* ∗ *inverse x*
have *diff*: ∀ *t*∈{*x*..*x*+*a*}. (*?F has-vector-derivative ?f t*)
(*at t within* {*x*..*x*+*a*}) **using** *assms*
by (*auto intro*!: *derivative-eq-intros*
simp: *has-field-derivative-iff-has-vector-derivative*[*symmetric*])
from *assms* **have** (*?f has-integral* (*?F* (*x*+*a*) − *?F x*)) {*x*..*x*+*a*}
by (*intro fundamental-theorem-of-calculus*[*OF - diff*])
(*auto simp*: *has-field-derivative-iff-has-vector-derivative*[*symmetric*] *field-simps*
intro!: *derivative-eq-intros*)
also have *?F* (*x*+*a*) − *?F x* = (*a*∗*2* + *f′*∗*a*^*2*∗*x*) / (*2*∗*x*) **using** *assms* **by** (*simp add*: *field-simps*)
also have *f′*∗*a* ^ *2* = − (*a* ^ *2*) / (*x*∗(x + *a*)) **using** *assms*
by (*simp add*: *divide-simps f′-def power2-eq-square*)
also have (*a*∗*2* + − *a*^*2*/(*x*∗(*x*+*a*))∗*x*) / (*2*∗*x*) = *?A* **using** *assms*
by (*simp add*: *divide-simps power2-eq-square*) (*simp add*: *algebra-simps*)
finally have *int1*: ((λ*t*. (t − *x*) ∗ *f′* + *inverse x*) *has-integral ?A*) {*x*..*x* + *a*} **.**

from *assms* **have** *int2*: (*inverse has-integral* (*ln* (x + *a*) − *ln x*)) {*x*..*x*+*a*}
by (*intro fundamental-theorem-of-calculus*)
(*auto simp*: *has-field-derivative-iff-has-vector-derivative*[*symmetric*] *divide-simps*
intro!: *derivative-eq-intros*)
hence *ln* (x + *a*) − *ln x* = *integral* {*x*..*x*+*a*} *inverse* **by** (*simp add*: *integral-unique*)
also have *ineq*: ∀ *xa*∈{*x*..*x* + *a*}. *inverse xa* ≤ (*xa* − *x*) ∗ *f′* + *inverse x*
proof
fix t **assume** t': t ∈ {*x*..*x*+*a*}
with *assms* **have** t: *0* ≤ (t − *x*) / *a* (t − *x*) / *a* ≤ *1* **by** *simp-all*
have *inverse* t = *inverse* ((*1* − (t − *x*) / *a*) ∗$_R$ *x* + ((t − *x*) / *a*) ∗$_R$ (x + *a*)) (**is** - = *?A*)
using *assms* t' **by** (*simp add*: *field-simps*)
also from *assms* **have** *convex-on* {*x*..*x*+*a*} *inverse* **by** (*intro convex-on-inverse*) *auto*

```
    from convex-onD-Icc[OF this - t] assms
      have ?A ≤ (1 - (t - x) / a) * inverse x + (t - x) / a * inverse (x + a)
by simp
    also have ... = (t - x) * f' + inverse x using assms
      by (simp add: f'-def divide-simps) (simp add: f'-def field-simps)
    finally show inverse t ≤ (t - x) * f' + inverse x .
  qed
  hence integral {x..x+a} inverse ≤ integral {x..x+a} ?f using f'-nonpos assms
    by (intro integral-le has-integral-integrable[OF int1] has-integral-integrable[OF
int2] ineq)
  also have ... = ?A using int1 by (rule integral-unique)
  finally show ?thesis .
qed

lemma ln-inverse-approx-ge:
  assumes (x::real) > 0 x < y
  shows   ln y - ln x ≥ 2 * (y - x) / (x + y) (is - ≥ ?A)
proof -
  def m ≡ (x+y)/2
  def f' ≡ -inverse (m^2)
  from assms have m: m > 0 by (simp add: m-def)
  let ?F = λt. (t - m)^2 * f' / 2 + t / m
  from assms have ((λt. (t - m) * f' + inverse m) has-integral (?F y - ?F x))
{x..y}
    by (intro fundamental-theorem-of-calculus)
     (auto simp: has-field-derivative-iff-has-vector-derivative[symmetric] divide-simps
           intro!: derivative-eq-intros)
  also from m have ?F y - ?F x = ((y - m)^2 - (x - m)^2) * f' / 2 + (y -
x) / m
    by (simp add: field-simps)
  also have ((y - m)^2 - (x - m)^2) = 0 by (simp add: m-def power2-eq-square
field-simps)
  also have 0 * f' / 2 + (y - x) / m = ?A by (simp add: m-def)
  finally have int1: ((λt. (t - m) * f' + inverse m) has-integral ?A) {x..y} .

  from assms have int2: (inverse has-integral (ln y - ln x)) {x..y}
    by (intro fundamental-theorem-of-calculus)
     (auto simp: has-field-derivative-iff-has-vector-derivative[symmetric] divide-simps
           intro!: derivative-eq-intros)
  hence ln y - ln x = integral {x..y} inverse by (simp add: integral-unique)
  also have ineq: ∀xa∈{x..y}. inverse xa ≥ (xa - m) * f' + inverse m
  proof
    fix t assume t: t ∈ {x..y}
    from t assms have inverse t - inverse m ≥ f' * (t - m)
      by (intro convex-on-imp-above-tangent[of {0<..}] convex-on-inverse)
       (auto simp: m-def interior-open f'-def power2-eq-square intro!: derivative-eq-intros)
    thus (t - m) * f' + inverse m ≤ inverse t by (simp add: algebra-simps)
  qed
  hence integral {x..y} inverse ≥ integral {x..y} (λt. (t - m) * f' + inverse m)
```

using *int1 int2* **by** (*intro integral-le has-integral-integrable*)
also have *integral* {*x..y*} (λ*t*. (*t* − *m*) ∗ *f* ′ + *inverse m*) = *?A*
using *integral-unique*[*OF int1*] **by** *simp*
finally show *?thesis* **.**
qed

lemma *euler-mascheroni-lower*:
euler-mascheroni ≥ *harm* (*Suc n*) − *ln* (*real-of-nat* (*n* + *2*)) + *1*/*real-of-nat* (*2* ∗ (*n* + *2*))
and *euler-mascheroni-upper*:
euler-mascheroni ≤ *harm* (*Suc n*) − *ln* (*real-of-nat* (*n* + *2*)) + *1*/*real-of-nat* (*2* ∗ (*n* + *1*))
proof −
def *D* ≡ λ*n*. *inverse* (*of-nat* (*n*+*1*)) + *ln* (*of-nat* (*n*+*1*)) − *ln* (*of-nat* (*n*+*2*)) :: *real*
let *?g* = λ*n*. *ln* (*of-nat* (*n*+*2*)) − *ln* (*of-nat* (*n*+*1*)) − *inverse* (*of-nat* (*n*+*1*)) :: *real*
def *inv* ≡ λ*n*. *inverse* (*real-of-nat n*)
fix *n* :: *nat*
note *summable* = *sums-summable*[*OF euler-mascheroni-sum*, *folded D-def*]
have *sums*: (λ*k*. (*inv* (*Suc* (*k* + (*n*+*1*))) − *inv* (*Suc* (*Suc k* + (*n*+*1*))))/*2*) *sums* ((*inv* (*Suc* (*0* + (*n*+*1*))) − *0*)/*2*)
unfolding *inv-def*
by (*intro sums-divide telescope-sums' LIMSEQ-ignore-initial-segment LIMSEQ-inverse-real-of-nat*)
have *sums'*: (λ*k*. (*inv* (*Suc* (*k* + *n*)) − *inv* (*Suc* (*Suc k* + *n*)))/*2*) *sums* ((*inv* (*Suc* (*0* + *n*)) − *0*)/*2*)
unfolding *inv-def*
by (*intro sums-divide telescope-sums' LIMSEQ-ignore-initial-segment LIMSEQ-inverse-real-of-nat*)
from *euler-mascheroni-sum* **have** *euler-mascheroni* = (∑ *k*. *D k*)
by (*simp add*: *sums-iff D-def*)
also have ... = (∑ *k*. *D* (*k* + *Suc n*)) + (∑ *k*≤*n*. *D k*)
by (*subst suminf-split-initial-segment*[*OF summable*, *of Suc n*], *subst lessThan-Suc-atMost*) *simp*
finally have *sum*: (∑ *k*≤*n*. *D k*) − *euler-mascheroni* = −(∑ *k*. *D* (*k* + *Suc n*)) **by** *simp*

note *sum*
also have ... ≤ −(∑ *k*. (*inv* (*k* + *Suc n* + *1*) − *inv* (*k* + *Suc n* + *2*)) / *2*)
proof (*intro le-imp-neg-le suminf-le allI summable-ignore-initial-segment*[*OF summable*])
fix *k* ′ :: *nat*
def *k* ≡ *k* ′ + *Suc n*
hence *k*: *k* > *0* **by** (*simp add*: *k-def*)
have *real-of-nat* (*k*+*1*) > *0* **by** (*simp add*: *k-def*)
with *ln-inverse-approx-le*[*OF this zero-less-one*]
have *ln* (*of-nat k* + *2*) − *ln* (*of-nat k* + *1*) ≤ (*inv* (*k*+*1*) + *inv* (*k*+*2*))/*2*
by (*simp add*: *inv-def add-ac*)
hence (*inv* (*k*+*1*) − *inv* (*k*+*2*))/*2* ≤ *inv* (*k*+*1*) + *ln* (*of-nat* (*k*+*1*)) − *ln* (*of-nat* (*k*+*2*))

by (*simp add*: *field-simps*)
also have ... = *D k* **unfolding** *D-def inv-def* **..**
finally show *D* (*k'* + *Suc n*) ≥ (*inv* (*k'* + *Suc n* + *1*) − *inv* (*k'* + *Suc n* + *2*)) / *2*
by (*simp add*: *k-def*)
from *sums-summable*[*OF sums*]
show *summable* (λ*k*. (*inv* (*k* + *Suc n* + *1*) − *inv* (*k* + *Suc n* + *2*))/*2*) **by** *simp*
qed
also from *sums* **have** ... = −*inv* (*n*+*2*) / *2* **by** (*simp add*: *sums-iff*)
finally have *euler-mascheroni* ≥ (∑ *k*≤*n*. *D k*) + *1* / (*of-nat* (*2* ∗ (*n*+*2*)))
by (*simp add*: *inv-def field-simps*)
also have (∑ *k*≤*n*. *D k*) = *harm* (*Suc n*) − (∑ *k*≤*n*. *ln* (*real-of-nat* (*Suc k*+*1*)) − *ln* (*of-nat* (*k*+*1*)))
unfolding *harm-altdef D-def* **by** (*subst lessThan-Suc-atMost*) (*simp add*: *setsum.distrib setsum-subtractf*)
also have (∑ *k*≤*n*. *ln* (*real-of-nat* (*Suc k*+*1*)) − *ln* (*of-nat* (*k*+*1*))) = *ln* (*of-nat* (*n*+*2*))
by (*subst atLeast0AtMost* [*symmetric*], *subst setsum-Suc-diff*) *simp-all*
finally show *euler-mascheroni* ≥ *harm* (*Suc n*) − *ln* (*real-of-nat* (*n* + *2*)) + *1*/*real-of-nat* (*2* ∗ (*n* + *2*))
by *simp*

note *sum*
also have −(∑ *k*. *D* (*k* + *Suc n*)) ≥ −(∑ *k*. (*inv* (*Suc* (*k* + *n*)) − *inv* (*Suc* (*Suc k* + *n*)))/*2*)
proof (*intro le-imp-neg-le suminf-le allI summable-ignore-initial-segment*[*OF summable*])
fix *k'* :: *nat*
def *k* ≡ *k'* + *Suc n*
hence *k*: *k* > *0* **by** (*simp add*: *k-def*)
have *real-of-nat* (*k*+*1*) > *0* **by** (*simp add*: *k-def*)
from *ln-inverse-approx-ge*[*of of-nat k* + *1 of-nat k* + *2*]
have *2* / (*2* ∗ *real-of-nat k* + *3*) ≤ *ln* (*of-nat* (*k*+*2*)) − *ln* (*real-of-nat* (*k*+*1*))
by (*simp add*: *add-ac*)
hence *D k* ≤ *1* / *real-of-nat* (*k*+*1*) − *2* / (*2* ∗ *real-of-nat k* + *3*)
by (*simp add*: *D-def inverse-eq-divide inv-def*)
also have ... = *inv* ((*k*+*1*)∗(*2*∗*k*+*3*)) **unfolding** *inv-def* **by** (*simp add*: *field-simps*)
also have ... ≤ *inv* (*2*∗*k*∗(*k*+*1*)) **unfolding** *inv-def* **using** *k*
by (*intro le-imp-inverse-le*)
(*simp add*: *algebra-simps*, *simp del*: *of-nat-add*)
also have ... = (*inv k* − *inv* (*k*+*1*))/*2* **unfolding** *inv-def* **using** *k*
by (*simp add*: *divide-simps del*: *of-nat-mult*) (*simp add*: *algebra-simps*)
finally show *D k* ≤ (*inv* (*Suc* (*k'* + *n*)) − *inv* (*Suc* (*Suc k'* + *n*)))/*2* **unfolding** *k-def* **by** *simp*
next
from *sums-summable*[*OF sums'*]
show *summable* (λ*k*. (*inv* (*Suc* (*k* + *n*)) − *inv* (*Suc* (*Suc k* + *n*)))/*2*) **by** *simp*

qed
also from *sums'* **have** $(\sum k.\ (inv\ (Suc\ (k + n)) - inv\ (Suc\ (Suc\ k + n)))/2) = inv\ (n+1)/2$
by (*simp add*: *sums-iff*)
finally have *euler-mascheroni* $\leq (\sum k{\leq}n.\ D\ k) + 1\ /\ of\text{-}nat\ (2 * (n+1))$
by (*simp add*: *inv-def field-simps*)
also have $(\sum k{\leq}n.\ D\ k) = harm\ (Suc\ n) - (\sum k{\leq}n.\ ln\ (real\text{-}of\text{-}nat\ (Suc\ k+1)) - ln\ (of\text{-}nat\ (k+1)))$
unfolding *harm-altdef D-def* **by** (*subst lessThan-Suc-atMost*) (*simp add*: *setsum.distrib setsum-subtractf*)
also have $(\sum k{\leq}n.\ ln\ (real\text{-}of\text{-}nat\ (Suc\ k+1)) - ln\ (of\text{-}nat\ (k+1))) = ln\ (of\text{-}nat\ (n+2))$
by (*subst atLeast0AtMost* [*symmetric*], *subst setsum-Suc-diff*) *simp-all*
finally show *euler-mascheroni* $\leq harm\ (Suc\ n) - ln\ (real\text{-}of\text{-}nat\ (n + 2)) + 1/real\text{-}of\text{-}nat\ (2 * (n + 1))$
by *simp*
qed

lemma *euler-mascheroni-pos*: *euler-mascheroni* $> (0::real)$
using *euler-mascheroni-lower*[*of 0*] *ln-2-less-1* **by** (*simp add*: *harm-def*)

context
begin

private lemma *ln-approx-aux*:
fixes $n :: nat$ **and** $x :: real$
defines $y \equiv (x-1)/(x+1)$
assumes x: $x > 0\ x \neq 1$
shows $inverse\ (2*y\hat{}(2*n+1)) * (ln\ x - (\sum k{<}n.\ 2*y\hat{}(2*k+1)\ /\ of\text{-}nat\ (2*k+1))) \in$
$\{0..(1\ /\ (1 - y\hat{}2)\ /\ of\text{-}nat\ (2*n+1))\}$
proof −
from x **have** *norm-y*: $norm\ y < 1$ **unfolding** *y-def* **by** *simp*
from *power-strict-mono*[*OF this*, *of 2*] **have** *norm-y'*: $norm\ y\hat{}2 < 1$ **by** *simp*

let *?f* $= \lambda k.\ 2 * y\ \hat{}\ (2*k+1)\ /\ of\text{-}nat\ (2*k+1)$
note *sums* = *ln-series-quadratic*[*OF x*(*1*)]
def $c \equiv inverse\ (2*y\hat{}(2*n+1))$
let *?d* $= c * (ln\ x - (\sum k{<}n.\ ?f\ k))$
have $\forall k.\ y^2\ \hat{}k\ /\ of\text{-}nat\ (2*(k+n)+1) \leq y^2\ \hat{}\ k\ /\ of\text{-}nat\ (2*n+1)$
by (*intro allI divide-left-mono mult-right-mono mult-pos-pos zero-le-power*[*of y*^*2*]) *simp-all*
moreover {
have $(\lambda k.\ ?f\ (k + n))\ sums\ (ln\ x - (\sum k{<}n.\ ?f\ k))$
using *sums-split-initial-segment*[*OF sums*] **by** (*simp add*: *y-def*)
hence $(\lambda k.\ c * ?f\ (k + n))\ sums\ ?d$ **by** (*rule sums-mult*)
also have $(\lambda k.\ c * (2*y\hat{}(2*(k+n)+1)\ /\ of\text{-}nat\ (2*(k+n)+1))) =$
$(\lambda k.\ (c * (2*y\hat{}(2*n+1))) * ((y\hat{}2)\hat{}k\ /\ of\text{-}nat\ (2*(k+n)+1)))$
by (*simp only*: *ring-distribs power-add power-mult*) (*simp add*: *mult-ac*)

```
    also from x have c * (2*y^(2*n+1)) = 1 by (simp add: c-def y-def)
    finally have (λk. (y^2) ^k / of-nat (2*(k+n)+1)) sums ?d by simp
  } note sums' = this
  moreover from norm-y' have (λk. (y^2) ^k / of-nat (2*n+1)) sums (1 / (1
- y^2) / of-nat (2*n+1))
    by (intro sums-divide geometric-sums) (simp-all add: norm-power)
  ultimately have ?d ≤ (1 / (1 - y^2) / of-nat (2*n+1)) by (rule sums-le)
  moreover have c * (ln x - (∑k<n. 2 * y ^ (2 * k + 1) / real-of-nat (2 * k
+ 1))) ≥ 0
    by (intro sums-le[OF - sums-zero sums'[]]) simp-all
  ultimately show ?thesis unfolding c-def by simp
qed

lemma
  fixes n :: nat and x :: real
  defines y ≡ (x-1)/(x+1)
  defines approx ≡ (∑k<n. 2*y^(2*k+1) / of-nat (2*k+1))
  defines d ≡ y^(2*n+1) / (1 - y^2) / of-nat (2*n+1)
  assumes x: x > 1
  shows   ln-approx-bounds: ln x ∈ {approx..approx + 2*d}
  and     ln-approx-abs:    abs (ln x - (approx + d)) ≤ d
proof -
  def c ≡ 2*y^(2*n+1)
  from x have c-pos: c > 0 unfolding c-def y-def
    by (intro mult-pos-pos zero-less-power) simp-all
  have A: inverse c * (ln x - (∑k<n. 2*y^(2*k+1) / of-nat (2*k+1))) ∈
            {0.. (1 / (1 - y^2) / of-nat (2*n+1))} using assms unfolding y-def
c-def
    by (intro ln-approx-aux) simp-all
  hence inverse c * (ln x - (∑k<n. 2*y^(2*k+1)/of-nat (2*k+1))) ≤ (1 /
(1-y^2) / of-nat (2*n+1))
    by simp
  hence (ln x - (∑k<n. 2*y^(2*k+1) / of-nat (2*k+1))) / c ≤ (1 / (1 - y^2)
/ of-nat (2*n+1))
    by (auto simp add: divide-simps)
  with c-pos have ln x ≤ c / (1 - y^2) / of-nat (2*n+1) + approx
    by (subst (asm) pos-divide-le-eq) (simp-all add: mult-ac approx-def)
  moreover {
    from A c-pos have 0 ≤ c * (inverse c * (ln x - (∑k<n. 2*y^(2*k+1) /
of-nat (2*k+1))))
      by (intro mult-nonneg-nonneg[of c]) simp-all
    also have ... = (c * inverse c) * (ln x - (∑k<n. 2*y^(2*k+1) / of-nat
(2*k+1)))
      by (simp add: mult-ac)
    also from c-pos have c * inverse c = 1 by simp
    finally have ln x ≥ approx by (simp add: approx-def)
  }
  ultimately show ln x ∈ {approx..approx + 2*d} by (simp add: c-def d-def)
  thus abs (ln x - (approx + d)) ≤ d by auto
```

qed

end

lemma *euler-mascheroni-bounds*:
fixes *n* :: *nat* **assumes** $n \geq 1$ **defines** $t \equiv harm\ n - ln\ (of\text{-}nat\ (Suc\ n))$ :: *real*
shows *euler-mascheroni* $\in \{t + inverse\ (of\text{-}nat\ (2*(n+1)))..t + inverse\ (of\text{-}nat\ (2*n))\}$
using *assms euler-mascheroni-upper*[*of n−1*] *euler-mascheroni-lower*[*of n−1*]
unfolding *t-def* **by** (*cases n*) (*simp-all add*: *harm-Suc t-def inverse-eq-divide*)

lemma *euler-mascheroni-bounds'*:
fixes *n* :: *nat* **assumes** $n \geq 1$ *ln* (*real-of-nat* (*Suc n*)) $\in \{l<..<u\}$
shows *euler-mascheroni* $\in$
$\{harm\ n - u + inverse\ (of\text{-}nat\ (2*(n+1)))<..<harm\ n - l + inverse\ (of\text{-}nat\ (2*n))\}$
using *euler-mascheroni-bounds*[*OF assms*(*1*)] *assms*(*2*) **by** *auto*

Approximation of *ln* (*2*::*'a*). The lower bound is accurate to about 0.03; the upper bound is accurate to about 0.0015.

lemma *ln2-ge-two-thirds*: $2/3 \leq ln\ (2::real)$
and *ln2-le-25-over-36*: $ln\ (2::real) \leq 25/36$
using *ln-approx-bounds*[*of 2 1*, *simplified*, *simplified eval-nat-numeral*, *simplified*]
by *simp-all*

Approximation of the Euler–Mascheroni constant. The lower bound is accurate to about 0.0015; the upper bound is accurate to about 0.015.

lemma *euler-mascheroni-gt-19-over-33*: (*euler-mascheroni* :: *real*) $> 19/33$ (**is** *?th1*)
and *euler-mascheroni-less-13-over-22*: (*euler-mascheroni* :: *real*) $< 13/22$ (**is** *?th2*)
proof −
have $ln\ (real\ (Suc\ 7)) = 3 * ln\ 2$ **by** (*simp add*: *ln-powr* [*symmetric*] *powr-numeral*)
also from *ln-approx-bounds*[*of 2 3*] **have** $\ldots \in \{3*307/443<..<3*4615/6658\}$
by (*simp add*: *eval-nat-numeral*)
finally have $ln\ (real\ (Suc\ 7)) \in \ldots$.
from *euler-mascheroni-bounds'*[*OF - this*] **have** *?th1* ∧ *?th2* **by** (*simp-all add*: *harm-expand*)
thus *?th1 ?th2* **by** *blast*+
qed

end

58 Periodic Functions

theory *Periodic-Fun*
imports *Complex-Main*
begin

A locale for periodic functions. The idea is that one proves $f(x + p) = f(x)$ for some period p and gets derived results like $f(x - p) = f(x)$ and $f(x + 2p) = f(x)$ for free.

g and *gm* are "plus/minus k periods" functions. *g1* and *gn1* are "plus/minus one period" functions. This is useful e.g. if the period is one; the lemmas one gets are then $f\ (x + (1::'b)) = f\ x$ instead of $f\ (x + (1::'b) * (1::'b)) = f\ x$ etc.

locale *periodic-fun* =
  fixes $f :: ('a :: \{ring\text{-}1\}) \Rightarrow 'b$ **and** $g\ gm :: 'a \Rightarrow 'a \Rightarrow 'a$ **and** $g1\ gn1 :: 'a \Rightarrow 'a$
  assumes *plus-1*: $f\ (g1\ x) = f\ x$
  assumes *periodic-arg-plus-0*: $g\ x\ 0 = x$
  assumes *periodic-arg-plus-distrib*: $g\ x\ (of\text{-}int\ (m + n)) = g\ (g\ x\ (of\text{-}int\ n))\ (of\text{-}int\ m)$
  assumes *plus-1-eq*: $g\ x\ 1 = g1\ x$ **and** *minus-1-eq*: $g\ x\ (-1) = gn1\ x$
    and *minus-eq*: $g\ x\ (-y) = gm\ x\ y$
begin

lemma *plus-of-nat*: $f\ (g\ x\ (of\text{-}nat\ n)) = f\ x$
  by (*induction n*) (*insert periodic-arg-plus-distrib*[*of - 1 int n* **for** *n*],
    simp-all add: *plus-1 periodic-arg-plus-0 plus-1-eq*)

lemma *minus-of-nat*: $f\ (gm\ x\ (of\text{-}nat\ n)) = f\ x$
proof −
  have $f\ (g\ x\ (-\ of\text{-}nat\ n)) = f\ (g\ (g\ x\ (-\ of\text{-}nat\ n))\ (of\text{-}nat\ n))$
    by (*rule plus-of-nat*[*symmetric*])
  also have $\ldots = f\ (g\ (g\ x\ (of\text{-}int\ (-\ of\text{-}nat\ n)))\ (of\text{-}int\ (of\text{-}nat\ n)))$ **by** *simp*
  also have $\ldots = f\ x$
    by (*subst periodic-arg-plus-distrib* [*symmetric*]) (*simp add*: *periodic-arg-plus-0*)
  finally show *?thesis* **by** (*simp add*: *minus-eq*)
qed

lemma *plus-of-int*: $f\ (g\ x\ (of\text{-}int\ n)) = f\ x$
  by (*induction n*) (*simp-all add*: *plus-of-nat minus-of-nat minus-eq del*: *of-nat-Suc*)

lemma *minus-of-int*: $f\ (gm\ x\ (of\text{-}int\ n)) = f\ x$
  using *plus-of-int*[*of x of-int* $(-n)$] **by** (*simp add*: *minus-eq*)

lemma *plus-numeral*: $f\ (g\ x\ (numeral\ n)) = f\ x$
  by (*subst of-nat-numeral*[*symmetric*], *subst plus-of-nat*) (*rule refl*)

lemma *minus-numeral*: $f\ (gm\ x\ (numeral\ n)) = f\ x$
  by (*subst of-nat-numeral*[*symmetric*], *subst minus-of-nat*) (*rule refl*)

lemma *minus-1*: $f\ (gn1\ x) = f\ x$
  using *minus-of-nat*[*of x 1*] **by** (*simp add*: *minus-1-eq minus-eq*[*symmetric*])

lemmas *periodic-simps* = *plus-of-nat minus-of-nat plus-of-int minus-of-int*
    plus-numeral minus-numeral plus-1 minus-1

end

Specialised case of the *periodic-fun* locale for periods that are not 1. Gives lemmas $f\ (x - period) = f\ x$ etc.

locale *periodic-fun-simple* =
  fixes $f :: ('a :: \{ring\text{-}1\}) \Rightarrow 'b$ **and** $period :: 'a$
  assumes *plus-period*: $f\ (x + period) = f\ x$
begin
sublocale *periodic-fun* $f\ \lambda z\ x.\ z + x * period\ \lambda z\ x.\ z - x * period$
  $\lambda z.\ z + period\ \lambda z.\ z - period$
  by *standard* (*simp-all add*: *ring-distribs plus-period*)
end

Specialised case of the *periodic-fun* locale for period 1. Gives lemmas $f\ (x - (1::'b)) = f\ x$ etc.

locale *periodic-fun-simple'* =
  fixes $f :: ('a :: \{ring\text{-}1\}) \Rightarrow 'b$
  assumes *plus-period*: $f\ (x + 1) = f\ x$
begin
sublocale *periodic-fun* $f\ \lambda z\ x.\ z + x\ \lambda z\ x.\ z - x\ \lambda z.\ z + 1\ \lambda z.\ z - 1$
  by *standard* (*simp-all add*: *ring-distribs plus-period*)

lemma *of-nat*: $f\ (of\text{-}nat\ n) = f\ 0$ **using** *plus-of-nat*[*of 0 n*] **by** *simp*
lemma *uminus-of-nat*: $f\ (-of\text{-}nat\ n) = f\ 0$ **using** *minus-of-nat*[*of 0 n*] **by** *simp*
lemma *of-int*: $f\ (of\text{-}int\ n) = f\ 0$ **using** *plus-of-int*[*of 0 n*] **by** *simp*
lemma *uminus-of-int*: $f\ (-of\text{-}int\ n) = f\ 0$ **using** *minus-of-int*[*of 0 n*] **by** *simp*
lemma *of-numeral*: $f\ (numeral\ n) = f\ 0$ **using** *plus-numeral*[*of 0 n*] **by** *simp*
lemma *of-neg-numeral*: $f\ (-numeral\ n) = f\ 0$ **using** *minus-numeral*[*of 0 n*] **by** *simp*
lemma *of-1*: $f\ 1 = f\ 0$ **using** *plus-of-nat*[*of 0 1*] **by** *simp*
lemma *of-neg-1*: $f\ (-1) = f\ 0$ **using** *minus-of-nat*[*of 0 1*] **by** *simp*

lemmas *periodic-simps'* =
  of-nat uminus-of-nat of-int uminus-of-int of-numeral of-neg-numeral of-1 of-neg-1

end

lemma *sin-plus-pi*: $sin\ ((z :: 'a :: \{real\text{-}normed\text{-}field, banach\}) + of\text{-}real\ pi) = -\ sin\ z$
  by (*simp add*: *sin-add*)

lemma *cos-plus-pi*: $cos\ ((z :: 'a :: \{real\text{-}normed\text{-}field, banach\}) + of\text{-}real\ pi) = -\ cos\ z$
  by (*simp add*: *cos-add*)

interpretation *sin*: *periodic-fun-simple sin* $2 * of\text{-}real\ pi :: 'a :: \{real\text{-}normed\text{-}field, banach\}$
proof
  fix $z :: 'a$

have *sin* (*z* + 2 ∗ *of-real pi*) = *sin* (*z* + *of-real pi* + *of-real pi*) **by** (*simp add*: *ac-simps*)
also have ... = *sin z* **by** (*simp only*: *sin-plus-pi*) *simp*
finally show *sin* (*z* + 2 ∗ *of-real pi*) = *sin z* **.**
qed

interpretation *cos*: *periodic-fun-simple cos* 2 ∗ *of-real pi* :: 'a :: {*real-normed-field*,*banach*}
proof
fix *z* :: 'a
have *cos* (*z* + 2 ∗ *of-real pi*) = *cos* (*z* + *of-real pi* + *of-real pi*) **by** (*simp add*: *ac-simps*)
also have ... = *cos z* **by** (*simp only*: *cos-plus-pi*) *simp*
finally show *cos* (*z* + 2 ∗ *of-real pi*) = *cos z* **.**
qed

interpretation *tan*: *periodic-fun-simple tan* 2 ∗ *of-real pi* :: 'a :: {*real-normed-field*,*banach*}
by *standard* (*simp only*: *tan-def* [*abs-def*] *sin.plus-1 cos.plus-1*)

interpretation *cot*: *periodic-fun-simple cot* 2 ∗ *of-real pi* :: 'a :: {*real-normed-field*,*banach*}
by *standard* (*simp only*: *cot-def* [*abs-def*] *sin.plus-1 cos.plus-1*)

end

59 The Gamma Function

theory *Gamma*
imports
Complex-Transcendental
Summation
Harmonic-Numbers
~~/*src*/*HOL*/*Library*/*Nonpos-Ints*
~~/*src*/*HOL*/*Library*/*Periodic-Fun*
begin

Several equivalent definitions of the Gamma function and its most important properties. Also contains the definition and some properties of the log-Gamma function and the Digamma function and the other Polygamma functions.

Based on the Gamma function, we also prove the Weierstra product form of the sin function and, based on this, the solution of the Basel problem (the sum over all *1 / real* (n^2).

lemma *pochhammer-eq-0-imp-nonpos-Int*:
pochhammer (*x*::'*a*::*field-char-0*) *n* = *0* ⟹ *x* ∈ $\mathbb{Z}_{\leq 0}$
by (*auto simp*: *pochhammer-eq-0-iff*)

lemma *closed-nonpos-Ints* [*simp*]: *closed* ($\mathbb{Z}_{\leq 0}$ :: 'a :: *real-normed-algebra-1 set*)
proof −
have $\mathbb{Z}_{\leq 0}$ = (*of-int* ‘ {*n*. *n* ≤ *0*} :: 'a *set*)

by (*auto elim*!: *nonpos-Ints-cases intro*!: *nonpos-Ints-of-int*)
also have *closed* ... **by** (*rule closed-of-int-image*)
finally show *?thesis* **.**
qed

lemma *plus-one-in-nonpos-Ints-imp*: $z + 1 \in \mathbb{Z}_{\leq 0} \Longrightarrow z \in \mathbb{Z}_{\leq 0}$
using *nonpos-Ints-diff-Nats*[*of z+1 1*] **by** *simp-all*

lemma *fraction-not-in-ints*:
assumes $\neg(n\ dvd\ m)$ $n \neq 0$
shows *of-int m / of-int n* $\notin$ ($\mathbb{Z}$ :: *'a* :: {*division-ring*,*ring-char-0*} *set*)
proof
assume *of-int m / (of-int n :: 'a)* $\in \mathbb{Z}$
then obtain *k* **where** *of-int m / of-int n = (of-int k :: 'a)* **by** (*elim Ints-cases*)
with *assms* **have** *of-int m = (of-int (k * n) :: 'a)* **by** (*auto simp add*: *divide-simps*)
hence *m = k * n* **by** (*subst* (*asm*) *of-int-eq-iff*)
hence *n dvd m* **by** *simp*
with *assms(1)* **show** *False* **by** *contradiction*
qed

lemma *not-in-Ints-imp-not-in-nonpos-Ints*: $z \notin \mathbb{Z} \Longrightarrow z \notin \mathbb{Z}_{\leq 0}$
by (*auto simp*: *Ints-def nonpos-Ints-def*)

lemma *double-in-nonpos-Ints-imp*:
assumes *2 * (z :: 'a :: field-char-0)* $\in \mathbb{Z}_{\leq 0}$
shows $z \in \mathbb{Z}_{\leq 0} \vee z + 1/2 \in \mathbb{Z}_{\leq 0}$
proof−
from *assms* **obtain** *k* **where** *k*: *2 * z = − of-nat k* **by** (*elim nonpos-Ints-cases'*)
thus *?thesis* **by** (*cases even k*) (*auto elim*!: *evenE oddE simp*: *field-simps*)
qed

lemma *sin-series*: $(\lambda n.\ ((-1)\hat{}n\ /\ fact\ (2*n+1)) *_R z\hat{}(2*n+1))$ *sums sin z*
proof −
from *sin-converges*[*of z*] **have** $(\lambda n.\ sin\text{-}coeff\ n *_R z\hat{}n)$ *sums sin z* **.**
also have $(\lambda n.\ sin\text{-}coeff\ n *_R z\hat{}n)$ *sums sin z* $\longleftrightarrow$
$(\lambda n.\ ((-1)\hat{}n\ /\ fact\ (2*n+1)) *_R z\hat{}(2*n+1))$ *sums sin z*
by (*subst sums-mono-reindex*[*of* $\lambda n.\ 2*n+1$, *symmetric*])
(*auto simp*: *sin-coeff-def subseq-def ac-simps elim*!: *oddE*)
finally show *?thesis* **.**
qed

lemma *cos-series*: $(\lambda n.\ ((-1)\hat{}n\ /\ fact\ (2*n)) *_R z\hat{}(2*n))$ *sums cos z*
proof −
from *cos-converges*[*of z*] **have** $(\lambda n.\ cos\text{-}coeff\ n *_R z\hat{}n)$ *sums cos z* **.**
also have $(\lambda n.\ cos\text{-}coeff\ n *_R z\hat{}n)$ *sums cos z* $\longleftrightarrow$
$(\lambda n.\ ((-1)\hat{}n\ /\ fact\ (2*n)) *_R z\hat{}(2*n))$ *sums cos z*
by (*subst sums-mono-reindex*[*of* $\lambda n.\ 2*n$, *symmetric*])
(*auto simp*: *cos-coeff-def subseq-def ac-simps elim*!: *evenE*)

finally show *?thesis* **.**
qed

lemma *sin-z-over-z-series*:
fixes $z :: 'a :: \{real\text{-}normed\text{-}field, banach\}$
assumes $z \neq 0$
shows $(\lambda n.\ (-1)\hat{}n\ /\ fact\ (2*n+1) * z\hat{}(2*n))\ sums\ (sin\ z\ /\ z)$
proof −
from *sin-series*[*of z*] **have** $(\lambda n.\ z * ((-1)\hat{}n\ /\ fact\ (2*n+1)) * z\hat{}(2*n))\ sums\ sin\ z$
by (*simp add*: *field-simps scaleR-conv-of-real*)
from *sums-mult*[*OF this*, *of inverse z*] **and** *assms* **show** *?thesis*
by (*simp add*: *field-simps*)
qed

lemma *sin-z-over-z-series'*:
fixes $z :: 'a :: \{real\text{-}normed\text{-}field, banach\}$
assumes $z \neq 0$
shows $(\lambda n.\ sin\text{-}coeff\ (n+1) *_R z\hat{}n)\ sums\ (sin\ z\ /\ z)$
proof −
from *sums-split-initial-segment*[*OF sin-converges*[*of z*], *of 1*]
have $(\lambda n.\ z * (sin\text{-}coeff\ (n+1) *_R z\ \hat{}\ n))\ sums\ sin\ z$ **by** *simp*
from *sums-mult*[*OF this*, *of inverse z*] *assms* **show** *?thesis* **by** (*simp add*: *field-simps*)
qed

lemma *has-field-derivative-sin-z-over-z*:
fixes $A :: 'a :: \{real\text{-}normed\text{-}field, banach\}\ set$
shows $((\lambda z.\ if\ z = 0\ then\ 1\ else\ sin\ z\ /\ z)\ has\text{-}field\text{-}derivative\ 0)\ (at\ 0\ within\ A)$
(**is** (*?f has-field-derivative ?f'*) -)
proof (*rule has-field-derivative-at-within*)
have $((\lambda z::'a.\ \sum n.\ of\text{-}real\ (sin\text{-}coeff\ (n+1)) * z\hat{}n)$
$has\text{-}field\text{-}derivative\ (\sum n.\ diffs\ (\lambda n.\ of\text{-}real\ (sin\text{-}coeff\ (n+1)))\ n * 0\hat{}n))$
$(at\ 0)$
proof (*rule termdiffs-strong*)
from *summable-ignore-initial-segment*[*OF sums-summable*[*OF sin-converges*[*of* $1::'a$]], *of 1*]
show $summable\ (\lambda n.\ of\text{-}real\ (sin\text{-}coeff\ (n+1)) * (1::'a)\hat{}n)$ **by** (*simp add*: *of-real-def*)
qed *simp*
also have $(\lambda z::'a.\ \sum n.\ of\text{-}real\ (sin\text{-}coeff\ (n+1)) * z\hat{}n) = ?f$
proof
fix *z*
show $(\sum n.\ of\text{-}real\ (sin\text{-}coeff\ (n+1)) * z\hat{}n)\ = ?f\ z$
by (*cases z = 0*) (*insert sin-z-over-z-series'*[*of z*],
simp-all add: *scaleR-conv-of-real sums-iff powser-zero sin-coeff-def*)
qed
also have $(\sum n.\ diffs\ (\lambda n.\ of\text{-}real\ (sin\text{-}coeff\ (n\ +\ 1)))\ n * (0::'a)\ \hat{}\ n) =$
$diffs\ (\lambda n.\ of\text{-}real\ (sin\text{-}coeff\ (Suc\ n)))\ 0$ **by** (*simp add*: *powser-zero*)

also have ... = *0* **by** (*simp add*: *sin-coeff-def diffs-def*)
finally show ((λz::'a. *if z = 0 then 1 else sin z / z*) *has-field-derivative 0*) (*at 0*) **.**
qed

lemma *round-Re-minimises-norm*:
norm ((*z*::*complex*) − *of-int m*) ≥ *norm* (*z* − *of-int* (*round* (*Re z*)))
proof −
let *?n* = *round* (*Re z*)
have *norm* (*z* − *of-int ?n*) = *sqrt* ((*Re z* − *of-int ?n*)^2 + (*Im z*)^2)
by (*simp add*: *cmod-def*)
also have |*Re z* − *of-int ?n*| ≤ |*Re z* − *of-int m*| **by** (*rule round-diff-minimal*)
hence *sqrt* ((*Re z* − *of-int ?n*)^2 + (*Im z*)^2) ≤ *sqrt* ((*Re z* − *of-int m*)^2 + (*Im z*)^2)
by (*intro real-sqrt-le-mono add-mono*) (*simp-all add*: *abs-le-square-iff*)
also have ... = *norm* (*z* − *of-int m*) **by** (*simp add*: *cmod-def*)
finally show *?thesis* **.**
qed

lemma *Re-pos-in-ball*:
assumes *Re z* > *0 t* ∈ *ball z* (*Re z*/*2*)
shows *Re t* > *0*
proof −
have *Re* (*z* − *t*) ≤ *norm* (*z* − *t*) **by** (*rule complex-Re-le-cmod*)
also from *assms* **have** ... < *Re z* / *2* **by** (*simp add*: *dist-complex-def*)
finally show *Re t* > *0* **using** *assms* **by** *simp*
qed

lemma *no-nonpos-Int-in-ball-complex*:
assumes *Re z* > *0 t* ∈ *ball z* (*Re z*/*2*)
shows *t* ∉ $\mathbb{Z}_{\leq 0}$
using *Re-pos-in-ball*[*OF assms*] **by** (*force elim*!: *nonpos-Ints-cases*)

lemma *no-nonpos-Int-in-ball*:
assumes *t* ∈ *ball z* (*dist z* (*round* (*Re z*)))
shows *t* ∉ $\mathbb{Z}_{\leq 0}$
proof
assume *t* ∈ $\mathbb{Z}_{\leq 0}$
then obtain *n* **where** *t* = *of-int n* **by** (*auto elim*!: *nonpos-Ints-cases*)
have *dist z* (*of-int n*) ≤ *dist z t* + *dist t* (*of-int n*) **by** (*rule dist-triangle*)
also from *assms* **have** *dist z t* < *dist z* (*round* (*Re z*)) **by** *simp*
also have ... ≤ *dist z* (*of-int n*)
using *round-Re-minimises-norm*[*of z*] **by** (*simp add*: *dist-complex-def*)
finally have *dist t* (*of-int n*) > *0* **by** *simp*
with ‹*t* = *of-int n*› **show** *False* **by** *simp*
qed

lemma *no-nonpos-Int-in-ball'*:
assumes (*z* :: 'a :: {*euclidean-space*,*real-normed-algebra-1*}) ∉ $\mathbb{Z}_{\leq 0}$

obtains d **where** $d > 0$ $\bigwedge t.\ t \in \mathit{ball}\ z\ d \Longrightarrow t \notin \mathbb{Z}_{\leq 0}$
proof (*rule that*)
from *assms* **show** *setdist* $\{z\}$ $\mathbb{Z}_{\leq 0} > 0$ **by** (*subst setdist-gt-0-compact-closed*) *auto*
next
fix t **assume** $t \in \mathit{ball}\ z\ (\mathit{setdist}\ \{z\}\ \mathbb{Z}_{\leq 0})$
thus $t \notin \mathbb{Z}_{\leq 0}$ **using** *setdist-le-dist*[*of* z $\{z\}$ t $\mathbb{Z}_{\leq 0}$] **by** *force*
qed

lemma *no-nonpos-Real-in-ball*:
assumes z: $z \notin \mathbb{R}_{\leq 0}$ **and** t: $t \in \mathit{ball}\ z\ (\mathit{if}\ \mathit{Im}\ z = 0\ \mathit{then}\ \mathit{Re}\ z\ /\ 2\ \mathit{else}\ \mathit{abs}\ (\mathit{Im}\ z)\ /\ 2)$
shows $t \notin \mathbb{R}_{\leq 0}$
using z
proof (*cases* $\mathit{Im}\ z = 0$)
assume A: $\mathit{Im}\ z = 0$
with z **have** $\mathit{Re}\ z > 0$ **by** (*force simp add*: *complex-nonpos-Reals-iff*)
with t A *Re-pos-in-ball*[*of* z t] **show** *?thesis* **by** (*force simp add*: *complex-nonpos-Reals-iff*)
next
assume A: $\mathit{Im}\ z \neq 0$
have $\mathit{abs}\ (\mathit{Im}\ z) - \mathit{abs}\ (\mathit{Im}\ t) \leq \mathit{abs}\ (\mathit{Im}\ z - \mathit{Im}\ t)$ **by** *linarith*
also have $\ldots = \mathit{abs}\ (\mathit{Im}\ (z - t))$ **by** *simp*
also have $\ldots \leq \mathit{norm}\ (z - t)$ **by** (*rule abs-Im-le-cmod*)
also from A t **have** $\ldots \leq \mathit{abs}\ (\mathit{Im}\ z)\ /\ 2$ **by** (*simp add*: *dist-complex-def*)
finally have $\mathit{abs}\ (\mathit{Im}\ t) > 0$ **using** A **by** *simp*
thus *?thesis* **by** (*force simp add*: *complex-nonpos-Reals-iff*)
qed

59.1 Definitions

We define the Gamma function by first defining its multiplicative inverse *Gamma-inv*. This is more convenient because *Gamma-inv* is entire, which makes proofs of its properties more convenient because one does not have to watch out for discontinuities. (e.g. *Gamma-inv* fulfils $\mathit{rGamma}\ z = z * \mathit{rGamma}\ (z + (1::'a))$ everywhere, whereas *Gamma* does not fulfil the analogous equation on the non-positive integers)

We define the Gamma function (resp. its inverse) in the Euler form. This form has the advantage that it is a relatively simple limit that converges everywhere. The limit at the poles is 0 (due to division by 0). The functional equation $\mathit{Gamma}\ (z + (1::'a)) = z * \mathit{Gamma}\ z$ follows immediately from the definition.

definition *Gamma-series* :: $('a :: \{\mathit{banach}, \mathit{real\text{-}normed\text{-}field}\}) \Rightarrow \mathit{nat} \Rightarrow 'a$ **where**
$\mathit{Gamma\text{-}series}\ z\ n = \mathit{fact}\ n * \mathit{exp}\ (z * \mathit{of\text{-}real}\ (\mathit{ln}\ (\mathit{of\text{-}nat}\ n))) / \mathit{pochhammer}\ z\ (n+1)$

definition *Gamma-series'* :: $('a :: \{\mathit{banach}, \mathit{real\text{-}normed\text{-}field}\}) \Rightarrow \mathit{nat} \Rightarrow 'a$ **where**
$\mathit{Gamma\text{-}series'}\ z\ n = \mathit{fact}\ (n - 1) * \mathit{exp}\ (z * \mathit{of\text{-}real}\ (\mathit{ln}\ (\mathit{of\text{-}nat}\ n))) / \mathit{pochham}$-

mer z n

definition *rGamma-series* :: ('*a* :: {*banach*,*real-normed-field*}) ⇒ *nat* ⇒ '*a* **where**
rGamma-series z n = *pochhammer z* (*n*+*1*) / (*fact n* ∗ *exp* (*z* ∗ *of-real* (*ln* (*of-nat n*))))

lemma *Gamma-series-altdef*: *Gamma-series z n* = *inverse* (*rGamma-series z n*)
and *rGamma-series-altdef*: *rGamma-series z n* = *inverse* (*Gamma-series z n*)
unfolding *Gamma-series-def rGamma-series-def* **by** *simp-all*

lemma *rGamma-series-minus-of-nat*:
eventually (λ*n*. *rGamma-series* (− *of-nat k*) *n* = *0*) *sequentially*
using *eventually-ge-at-top*[*of k*]
by *eventually-elim* (*auto simp*: *rGamma-series-def pochhammer-of-nat-eq-0-iff*)

lemma *Gamma-series-minus-of-nat*:
eventually (λ*n*. *Gamma-series* (− *of-nat k*) *n* = *0*) *sequentially*
using *eventually-ge-at-top*[*of k*]
by *eventually-elim* (*auto simp*: *Gamma-series-def pochhammer-of-nat-eq-0-iff*)

lemma *Gamma-series'-minus-of-nat*:
eventually (λ*n*. *Gamma-series'* (− *of-nat k*) *n* = *0*) *sequentially*
using *eventually-gt-at-top*[*of k*]
by *eventually-elim* (*auto simp*: *Gamma-series'-def pochhammer-of-nat-eq-0-iff*)

lemma *rGamma-series-nonpos-Ints-LIMSEQ*: *z* ∈ ℤ≤0 ⟹ *rGamma-series z* ⟶ *0*
by (*elim nonpos-Ints-cases'*, *hypsubst*, *subst tendsto-cong*, *rule rGamma-series-minus-of-nat*, *simp*)

lemma *Gamma-series-nonpos-Ints-LIMSEQ*: *z* ∈ ℤ≤0 ⟹ *Gamma-series z* ⟶ *0*
by (*elim nonpos-Ints-cases'*, *hypsubst*, *subst tendsto-cong*, *rule Gamma-series-minus-of-nat*, *simp*)

lemma *Gamma-series'-nonpos-Ints-LIMSEQ*: *z* ∈ ℤ≤0 ⟹ *Gamma-series' z* ⟶ *0*
by (*elim nonpos-Ints-cases'*, *hypsubst*, *subst tendsto-cong*, *rule Gamma-series'-minus-of-nat*, *simp*)

lemma *Gamma-series-Gamma-series'*:
assumes *z*: *z* ∉ ℤ≤0
shows (λ*n*. *Gamma-series' z n* / *Gamma-series z n*) ⟶ *1*
proof (*rule Lim-transform-eventually*)
from *eventually-gt-at-top*[*of 0*::*nat*]
show *eventually* (λ*n*. *z* / *of-nat n* + *1* = *Gamma-series' z n* / *Gamma-series z n*) *sequentially*
proof *eventually-elim*
fix *n* :: *nat* **assume** *n*: *n* > *0*

from *n z* **have** *Gamma-series' z n / Gamma-series z n = (z + of-nat n) / of-nat n*
by (*cases n*, *simp*)
(*auto simp add*: *Gamma-series-def Gamma-series'-def pochhammer-rec'*
dest: *pochhammer-eq-0-imp-nonpos-Int plus-of-nat-eq-0-imp*)
also from *n* **have** *... = z / of-nat n + 1* **by** (*simp add*: *divide-simps*)
finally show *z / of-nat n + 1 = Gamma-series' z n / Gamma-series z n* **..**
qed
have (λ*x*. *z / of-nat x*) ⟶ *0*
by (*rule tendsto-norm-zero-cancel*)
(*insert tendsto-mult*[*OF tendsto-const*[*of norm z*] *lim-inverse-n*],
simp add: *norm-divide inverse-eq-divide*)
from *tendsto-add*[*OF this tendsto-const*[*of 1*]] **show** (λ*n*. *z / of-nat n + 1*) ⟶ *1* **by** *simp*
qed

59.2 Convergence of the Euler series form

We now show that the series that defines the Gamma function in the Euler form converges and that the function defined by it is continuous on the complex halfspace with positive real part.

We do this by showing that the logarithm of the Euler series is continuous and converges locally uniformly, which means that the log-Gamma function defined by its limit is also continuous.

This will later allow us to lift holomorphicity and continuity from the log-Gamma function to the inverse of the Gamma function, and from that to the Gamma function itself.

definition *ln-Gamma-series* :: ('*a* :: {*banach*,*real-normed-field*,*ln*}) ⇒ *nat* ⇒ '*a* **where**
*ln-Gamma-series z n = z * ln (of-nat n) − ln z − (∑k=1..n. ln (z / of-nat k + 1))*

definition *ln-Gamma-series'* :: ('*a* :: {*banach*,*real-normed-field*,*ln*}) ⇒ *nat* ⇒ '*a* **where**
ln-Gamma-series' z n =
*− euler-mascheroni*z − ln z + (∑k=1..n. z / of-nat n − ln (z / of-nat k + 1))*

definition *ln-Gamma* :: ('*a* :: {*banach*,*real-normed-field*,*ln*}) ⇒ '*a* **where**
ln-Gamma z = lim (ln-Gamma-series z)

We now show that the log-Gamma series converges locally uniformly for all complex numbers except the non-positive integers. We do this by proving that the series is locally Cauchy, adapting this proof: http://math.stackexchange.com/questions/88715 of-gammaz-lim-n-to-infty-fracnzn-prod-m-0nzm

context
begin

private lemma *ln-Gamma-series-complex-converges-aux*:
fixes *z* :: *complex* **and** *k* :: *nat*
assumes *z*: $z \neq 0$ **and** *k*: $of\text{-}nat\ k \geq 2 * norm\ z\ k \geq 2$
shows $norm\ (z * ln\ (1 - 1/of\text{-}nat\ k) + ln\ (z/of\text{-}nat\ k + 1)) \leq 2*(norm\ z + norm\ z\hat{}2)\ /\ of\text{-}nat\ k\hat{}2$
proof −
let $?k = of\text{-}nat\ k$ :: *complex* **and** $?z = norm\ z$
have $z * ln\ (1 - 1/?k) + ln\ (z/?k+1) = z*(ln\ (1 - 1/?k :: complex) + 1/?k) + (ln\ (1+z/?k) - z/?k)$
by (*simp add*: *algebra-simps*)
also have $norm\ ... \leq ?z * norm\ (ln\ (1-1/?k) + 1/?k :: complex) + norm\ (ln\ (1+z/?k) - z/?k)$
by (*subst norm-mult* [*symmetric*], *rule norm-triangle-ineq*)
also have $norm\ (Ln\ (1 + -1/?k) - (-1/?k)) \leq (norm\ (-1/?k))^2\ /\ (1 - norm(-1/?k))$
using *k* **by** (*intro Ln-approx-linear*) (*simp add*: *norm-divide*)
hence $?z * norm\ (ln\ (1-1/?k) + 1/?k) \leq ?z * ((norm\ (1/?k))\hat{}2\ /\ (1 - norm\ (1/?k)))$
by (*intro mult-left-mono*) *simp-all*
also have $... \leq (?z * (of\text{-}nat\ k\ /\ (of\text{-}nat\ k - 1)))\ /\ of\text{-}nat\ k\hat{}2$ **using** *k*
by (*simp add*: *field-simps power2-eq-square norm-divide*)
also have $... \leq (?z * 2)\ /\ of\text{-}nat\ k\hat{}2$ **using** *k*
by (*intro divide-right-mono mult-left-mono*) (*simp-all add*: *field-simps*)
also have $norm\ (ln\ (1+z/?k) - z/?k) \leq norm\ (z/?k)\hat{}2\ /\ (1 - norm\ (z/?k))$ **using** *k*
by (*intro Ln-approx-linear*) (*simp add*: *norm-divide*)
hence $norm\ (ln\ (1+z/?k) - z/?k) \leq ?z\hat{}2\ /\ of\text{-}nat\ k\hat{}2\ /\ (1 - ?z\ /\ of\text{-}nat\ k)$
by (*simp add*: *field-simps norm-divide*)
also have $... \leq (?z\hat{}2 * (of\text{-}nat\ k\ /\ (of\text{-}nat\ k - ?z)))\ /\ of\text{-}nat\ k\hat{}2$ **using** *k*
by (*simp add*: *field-simps power2-eq-square*)
also have $... \leq (?z\hat{}2 * 2)\ /\ of\text{-}nat\ k\hat{}2$ **using** *k*
by (*intro divide-right-mono mult-left-mono*) (*simp-all add*: *field-simps*)
also note *add-divide-distrib* [*symmetric*]
finally show *?thesis* **by** (*simp only*: *distrib-left mult.commute*)
qed

lemma *ln-Gamma-series-complex-converges*:
assumes *z*: $z \notin \mathbb{Z}_{\leq 0}$
assumes *d*: $d > 0\ \bigwedge n.\ n \in \mathbb{Z}_{\leq 0} \Longrightarrow norm\ (z - of\text{-}int\ n) > d$
shows *uniformly-convergent-on* (*ball z d*) (λ*n z*. *ln-Gamma-series z n* :: *complex*)
proof (*intro Cauchy-uniformly-convergent uniformly-Cauchy-onI'*)
fix *e* :: *real* **assume** *e*: $e > 0$
def $e'' \equiv SUP\ t{:}ball\ z\ d.\ norm\ t + norm\ t\hat{}2$
def $e' \equiv e\ /\ (2*e'')$
have *bounded* ((λ*t*. *norm t* + *norm t*^2) ‘ *cball z d*)
by (*intro compact-imp-bounded compact-continuous-image*) (*auto intro*!: *continuous-intros*)
hence *bounded* ((λ*t*. *norm t* + *norm t*^2) ‘ *ball z d*) **by** (*rule bounded-subset*)
auto

hence *bdd*: *bdd-above* ((λ*t*. *norm t* + *norm t*^*2*) ‘ *ball z d*) **by** (*rule bounded-imp-bdd-above*)

with *z d*(*1*) *d*(*2*)[*of* −*1*] **have** *e''-pos*: *e''* > *0* **unfolding** *e''-def*
by (*subst less-cSUP-iff*) (*auto intro*!: *add-pos-nonneg bexI*[*of* - *z*])
have *e''*: *norm t* + *norm t*^*2* ≤ *e''* **if** *t* ∈ *ball z d* **for** *t* **unfolding** *e''-def* **using** *that*
by (*rule cSUP-upper*[*OF* - *bdd*])
from *e z e''-pos* **have** *e'*: *e'* > *0* **unfolding** *e'-def*
by (*intro divide-pos-pos mult-pos-pos add-pos-pos*) (*simp-all add*: *field-simps*)

have *summable* (λ*k*. *inverse* ((*real-of-nat k*) ^*2*))
by (*rule inverse-power-summable*) *simp*
from *summable-partial-sum-bound*[*OF this e'*] **guess** *M* **. note** *M* = *this*

def *N* ≡ *max 2* (*max* (*nat* ⌈*2* ∗ (*norm z* + *d*)⌉) *M*)
{
from *d* **have** ⌈*2* ∗ (*cmod z* + *d*)⌉ ≥ ⌈*0*::*real*⌉
by (*intro ceiling-mono mult-nonneg-nonneg add-nonneg-nonneg*) *simp-all*
hence *2* ∗ (*norm z* + *d*) ≤ *of-nat* (*nat* ⌈*2* ∗ (*norm z* + *d*)⌉) **unfolding** *N-def*
by (*simp-all add*: *le-of-int-ceiling*)
also have ... ≤ *of-nat N* **unfolding** *N-def*
by (*subst of-nat-le-iff*) (*rule max.coboundedI2*, *rule max.cobounded1*)
finally have *of-nat N* ≥ *2* ∗ (*norm z* + *d*) **.**
moreover have *N* ≥ *2 N* ≥ *M* **unfolding** *N-def* **by** *simp-all*
moreover have (∑ *k*=*m*..*n*. *1*/(*of-nat k*)^2) < *e'* **if** *m* ≥ *N* **for** *m n*
using *M*[*OF order.trans*[*OF* ‹*N* ≥ *M*› *that*]] **unfolding** *real-norm-def*
by (*subst* (*asm*) *abs-of-nonneg*) (*auto intro*: *setsum-nonneg simp*: *divide-simps*)
moreover note *calculation*
} **note** *N* = *this*

show ∃ *M*. ∀ *t*∈*ball z d*. ∀ *m*≥*M*. ∀ *n*>*m*. *dist* (*ln-Gamma-series t m*) (*ln-Gamma-series t n*) < *e*
unfolding *dist-complex-def*
proof (*intro exI*[*of* - *N*] *ballI allI impI*)
fix *t m n* **assume** *t*: *t* ∈ *ball z d* **and** *mn*: *m* ≥ *N n* > *m*
from *d*(*2*)[*of 0*] *t* **have** *0* < *dist z 0* − *dist z t* **by** (*simp add*: *field-simps dist-complex-def*)
also have *dist z 0* − *dist z t* ≤ *dist 0 t* **using** *dist-triangle*[*of 0 z t*]
by (*simp add*: *dist-commute*)
finally have *t-nz*: *t* ≠ *0* **by** *auto*

have *norm t* ≤ *norm z* + *norm* (*t* − *z*) **by** (*rule norm-triangle-sub*)
also from *t* **have** *norm* (*t* − *z*) < *d* **by** (*simp add*: *dist-complex-def norm-minus-commute*)
also have *2* ∗ (*norm z* + *d*) ≤ *of-nat N* **by** (*rule N*)
also have *N* ≤ *m* **by** (*rule mn*)
finally have *norm-t*: *2* ∗ *norm t* < *of-nat m* **by** *simp*

have *ln-Gamma-series t m* − *ln-Gamma-series t n* =
(−(*t* ∗ *Ln* (*of-nat n*)) − (−(*t* ∗ *Ln* (*of-nat m*)))) +

```
          ((∑k=1..n. Ln (t / of-nat k + 1)) - (∑k=1..m. Ln (t / of-nat k +
1)))
    by (simp add: ln-Gamma-series-def algebra-simps)
  also have (∑k=1..n. Ln (t / of-nat k + 1)) - (∑k=1..m. Ln (t / of-nat k
+ 1)) =
              (∑k∈{1..n}-{1..m}. Ln (t / of-nat k + 1)) using mn
    by (simp add: setsum-diff)
  also from mn have {1..n}-{1..m} = {Suc m..n} by fastforce
  also have -(t * Ln (of-nat n)) - (-(t * Ln (of-nat m))) =
                (∑k = Suc m..n. t * Ln (of-nat (k - 1)) - t * Ln (of-nat k))
using mn
    by (subst setsum-telescope'' [symmetric]) simp-all
  also have ... = (∑k = Suc m..n. t * Ln (of-nat (k - 1) / of-nat k)) using
mn N
    by (intro setsum-cong-Suc)
       (simp-all del: of-nat-Suc add: field-simps Ln-of-nat Ln-of-nat-over-of-nat)
  also have of-nat (k - 1) / of-nat k = 1 - 1 / (of-nat k :: complex) if k ∈
{Suc m..n} for k
  using that of-nat-eq-0-iff[of Suc i for i] by (cases k) (simp-all add: divide-simps)
  hence (∑k = Suc m..n. t * Ln (of-nat (k - 1) / of-nat k)) =
          (∑k = Suc m..n. t * Ln (1 - 1 / of-nat k)) using mn N
    by (intro setsum.cong) simp-all
  also note setsum.distrib [symmetric]
  also have norm (∑k=Suc m..n. t * Ln (1 - 1/of-nat k) + Ln (t/of-nat k
+ 1)) ≤
      (∑k=Suc m..n. 2 * (norm t + (norm t)^2) / (real-of-nat k)^2) using t-nz
N(2) mn norm-t
  by (intro order.trans[OF norm-setsum setsum-mono[OF ln-Gamma-series-complex-converges-aux]])
simp-all
  also have ... ≤ 2 * (norm t + norm t^2) * (∑k=Suc m..n. 1 / (of-nat k)^2)
    by (simp add: setsum-right-distrib)
  also have ... < 2 * (norm t + norm t^2) * e' using mn z t-nz
    by (intro mult-strict-left-mono N mult-pos-pos add-pos-pos) simp-all
  also from e''-pos have ... = e * ((cmod t + (cmod t)^2) / e'')
    by (simp add: e'-def field-simps power2-eq-square)
  also from e''[OF t] e''-pos e
    have ... ≤ e * 1 by (intro mult-left-mono) (simp-all add: field-simps)
  finally show norm (ln-Gamma-series t m - ln-Gamma-series t n) < e by
simp
 qed
qed

end

lemma ln-Gamma-series-complex-converges':
 assumes z: (z :: complex) ∉ ℤ≤0
 shows ∃d>0. uniformly-convergent-on (ball z d) (λn z. ln-Gamma-series z n)
proof -
 def d' ≡ Re z
```

def $d \equiv$ *if* $d' > 0$ *then* $d' / 2$ *else norm* $(z -$ *of-int* $(\textit{round}\ d'))\ /\ 2$
have *of-int* $(\textit{round}\ d') \in \mathbb{Z}_{\leq 0}$ **if** $d' \leq 0$ **using** *that*
by (*intro nonpos-Ints-of-int*) (*simp-all add*: *round-def*)
with *assms* **have** *d-pos*: $d > 0$ **unfolding** *d-def* **by** (*force simp*: *not-less*)

have $d <$ *cmod* $(z -$ *of-int* $n)$ **if** $n \in \mathbb{Z}_{\leq 0}$ **for** n
proof (*cases Re* $z > 0$)
case *True*
from *nonpos-Ints-nonpos*[*OF that*] **have** n: $n \leq 0$ **by** *simp*
from *True* **have** $d =$ *Re* $z/2$ **by** (*simp add*: *d-def d'-def*)
also from n *True* **have** $\ldots <$ *Re* $(z -$ *of-int* $n)$ **by** *simp*
also have $\ldots \leq$ *norm* $(z -$ *of-int* $n)$ **by** (*rule complex-Re-le-cmod*)
finally show *?thesis* **.**
next
case *False*
with *assms nonpos-Ints-of-int*[*of round* (*Re z*)]
have $z \neq$ *of-int* $(\textit{round}\ d')$ **by** (*auto simp*: *not-less*)
with *False* **have** $d <$ *norm* $(z -$ *of-int* $(\textit{round}\ d'))$ **by** (*simp add*: *d-def d'-def*)
also have $\ldots \leq$ *norm* $(z -$ *of-int* $n)$ **unfolding** *d'-def* **by** (*rule round-Re-minimises-norm*)
finally show *?thesis* **.**
qed
hence *conv*: *uniformly-convergent-on* (*ball z d*) $(\lambda n\ z.$ *ln-Gamma-series z n*)
by (*intro ln-Gamma-series-complex-converges d-pos z*) *simp-all*
from *d-pos conv* **show** *?thesis* **by** *blast*
qed

lemma *ln-Gamma-series-complex-converges''*: $(z ::$ *complex*$) \notin \mathbb{Z}_{\leq 0} \Longrightarrow$ *convergent* (*ln-Gamma-series z*)
by (*drule ln-Gamma-series-complex-converges'*) (*auto intro*: *uniformly-convergent-imp-convergent*)

lemma *ln-Gamma-complex-LIMSEQ*: $(z ::$ *complex*$) \notin \mathbb{Z}_{\leq 0} \Longrightarrow$ *ln-Gamma-series z* $\longrightarrow$ *ln-Gamma z*
using *ln-Gamma-series-complex-converges''* **by** (*simp add*: *convergent-LIMSEQ-iff ln-Gamma-def*)

lemma *exp-ln-Gamma-series-complex*:
assumes $n > 0$ $z \notin \mathbb{Z}_{\leq 0}$
shows *exp* (*ln-Gamma-series z n* :: *complex*) = *Gamma-series z n*
proof −
from *assms* **have** $z \neq 0$ **by** (*intro notI*) *auto*
with *assms* **have** *exp* (*ln-Gamma-series z n*) =
(*of-nat n*) *powr z* / (*z* ∗ (∏ *k*=*1*..*n*. *exp* (*Ln* (*z* / *of-nat k* + *1*))))
unfolding *ln-Gamma-series-def powr-def* **by** (*simp add*: *exp-diff exp-setsum*)
also from *assms* **have** (∏ *k*=*1*..*n*. *exp* (*Ln* (*z* / *of-nat k* + *1*))) = (∏ *k*=*1*..*n*. *z* / *of-nat k* + *1*)
by (*intro setprod.cong*[*OF refl*], *subst exp-Ln*) (*auto simp*: *field-simps plus-of-nat-eq-0-imp*)
also have ... = (∏ *k*=*1*..*n*. *z* + *k*) / *fact n* **unfolding** *fact-altdef*
by (*subst setprod-dividef* [*symmetric*]) (*simp-all add*: *field-simps*)
also from *assms* **have** *z* ∗ ... = (∏ *k*=*0*..*n*. *z* + *k*) / *fact n*

by (*cases n*) (*simp-all add*: *setprod-nat-ivl-1-Suc*)
also have $(\prod k{=}0..n.\ z + k) = pochhammer\ z\ (Suc\ n)$ **unfolding** *pochhammer-def*
by *simp*
also have *of-nat n powr z / (pochhammer z (Suc n) / fact n) = Gamma-series z n*
unfolding *Gamma-series-def* **using** *assms* **by** (*simp add*: *divide-simps powr-def Ln-of-nat*)
finally show *?thesis* **.**
qed

lemma *ln-Gamma-series'-aux*:
assumes $(z{::}complex) \notin \mathbb{Z}_{\leq 0}$
shows $(\lambda k.\ z\ /\ of\text{-}nat\ (Suc\ k) - ln\ (1 + z\ /\ of\text{-}nat\ (Suc\ k)))\ sums$
$(ln\text{-}Gamma\ z + euler\text{-}mascheroni * z + ln\ z)$ (**is** *?f sums ?s*)
unfolding *sums-def*
proof (*rule Lim-transform*)
show $(\lambda n.\ ln\text{-}Gamma\text{-}series\ z\ n + of\text{-}real\ (harm\ n - ln\ (of\text{-}nat\ n)) * z + ln\ z) \longrightarrow ?s$
(**is** *?g* $\longrightarrow$ -)
by (*intro tendsto-intros ln-Gamma-complex-LIMSEQ euler-mascheroni-LIMSEQ-of-real assms*)

have *A*: *eventually* $(\lambda n.\ (\sum k{<}n.\ ?f\ k) - ?g\ n = 0)$ *sequentially*
using *eventually-gt-at-top*[*of 0*::*nat*]
proof *eventually-elim*
fix *n* :: *nat* **assume** *n*: $n > 0$
have $(\sum k{<}n.\ ?f\ k) = (\sum k{=}1..n.\ z\ /\ of\text{-}nat\ k - ln\ (1 + z\ /\ of\text{-}nat\ k))$
by (*subst atLeast0LessThan* [*symmetric*], *subst setsum-shift-bounds-Suc-ivl* [*symmetric*],
subst atLeastLessThanSuc-atLeastAtMost) *simp-all*
also have $\ldots = z * of\text{-}real\ (harm\ n) - (\sum k{=}1..n.\ ln\ (1 + z\ /\ of\text{-}nat\ k))$
by (*simp add*: *harm-def setsum-subtractf setsum-right-distrib divide-inverse*)
also from *n* **have** $\ldots - ?g\ n = 0$
by (*simp add*: *ln-Gamma-series-def setsum-subtractf algebra-simps Ln-of-nat*)
finally show $(\sum k{<}n.\ ?f\ k) - ?g\ n = 0$ **.**
qed
show $(\lambda n.\ (\sum k{<}n.\ ?f\ k) - ?g\ n) \longrightarrow 0$ **by** (*subst tendsto-cong*[*OF A*]) *simp-all*
qed

lemma *uniformly-summable-deriv-ln-Gamma*:
assumes *z*: $(z :: 'a :: \{real\text{-}normed\text{-}field, banach\}) \neq 0$ **and** *d*: $d > 0\ d \leq norm\ z / 2$
shows *uniformly-convergent-on* $(ball\ z\ d)$
$(\lambda k\ z.\ \sum i{<}k.\ inverse\ (of\text{-}nat\ (Suc\ i)) - inverse\ (z + of\text{-}nat\ (Suc\ i)))$
(**is** *uniformly-convergent-on* - $(\lambda k\ z.\ \sum i{<}k.\ ?f\ i\ z)$)
proof (*rule weierstrass-m-test'-ev*)

{
fix t **assume** t: $t \in ball\ z\ d$
have $norm\ z = norm\ (t + (z - t))$ **by** *simp*
have $norm\ (t + (z - t)) \leq norm\ t + norm\ (z - t)$ **by** (*rule norm-triangle-ineq*)
also from $t\ d$ **have** $norm\ (z - t) < norm\ z\ /\ 2$ **by** (*simp add*: *dist-norm*)
finally have A: $norm\ t > norm\ z\ /\ 2$ **using** z **by** (*simp add*: *field-simps*)

have $norm\ t = norm\ (z + (t - z))$ **by** *simp*
also have $\ldots \leq norm\ z + norm\ (t - z)$ **by** (*rule norm-triangle-ineq*)
also from $t\ d$ **have** $norm\ (t - z) \leq norm\ z\ /\ 2$ **by** (*simp add*: *dist-norm norm-minus-commute*)
also from z **have** $\ldots < norm\ z$ **by** *simp*
finally have B: $norm\ t < 2 * norm\ z$ **by** *simp*
note $A\ B$
} **note** $ball = this$

show $eventually\ (\lambda n.\ \forall t \in ball\ z\ d.\ norm\ (?f\ n\ t) \leq 4 * norm\ z * inverse\ (of\text{-}nat\ (Suc\ n)\ \hat{}\ 2))\ sequentially$
using *eventually-gt-at-top* **apply** *eventually-elim*
proof *safe*
fix $t :: 'a$ **assume** t: $t \in ball\ z\ d$
from $z\ ball[OF\ t]$ **have** *t-nz*: $t \neq 0$ **by** *auto*
fix $n :: nat$ **assume** n: $n > nat\ \lceil 4 * norm\ z \rceil$
from $ball[OF\ t]$ *t-nz* **have** $4 * norm\ z > 2 * norm\ t$ **by** *simp*
also from n **have** $\ldots < of\text{-}nat\ n$ **by** *linarith*
finally have n: $of\text{-}nat\ n > 2 * norm\ t$ **.**
hence $of\text{-}nat\ n > norm\ t$ **by** *simp*
hence t': $t \neq -of\text{-}nat\ (Suc\ n)$ **by** (*intro notI*) (*simp del*: *of-nat-Suc*)

with *t-nz* **have** $?f\ n\ t = 1\ /\ (of\text{-}nat\ (Suc\ n) * (1 + of\text{-}nat\ (Suc\ n)/t))$
by (*simp add*: *divide-simps eq-neg-iff-add-eq-0 del*: *of-nat-Suc*)
also have $norm\ \ldots = inverse\ (of\text{-}nat\ (Suc\ n)) * inverse\ (norm\ (of\text{-}nat\ (Suc\ n)/t + 1))$
by (*simp add*: *norm-divide norm-mult divide-simps add-ac del*: *of-nat-Suc*)
also {
from z *t-nz* $ball[OF\ t]$ **have** $of\text{-}nat\ (Suc\ n)\ /\ (4 * norm\ z) \leq of\text{-}nat\ (Suc\ n)\ /\ (2 * norm\ t)$
by (*intro divide-left-mono mult-pos-pos*) *simp-all*
also have $\ldots < norm\ (of\text{-}nat\ (Suc\ n)\ /\ t) - norm\ (1 :: 'a)$
using *t-nz n* **by** (*simp add*: *field-simps norm-divide del*: *of-nat-Suc*)
also have $\ldots \leq norm\ (of\text{-}nat\ (Suc\ n)/t + 1)$ **by** (*rule norm-diff-ineq*)
finally have $inverse\ (norm\ (of\text{-}nat\ (Suc\ n)/t + 1)) \leq 4 * norm\ z\ /\ of\text{-}nat\ (Suc\ n)$
using z **by** (*simp add*: *divide-simps norm-divide mult-ac del*: *of-nat-Suc*)
}
also have $inverse\ (real\text{-}of\text{-}nat\ (Suc\ n)) * (4 * norm\ z\ /\ real\text{-}of\text{-}nat\ (Suc\ n)) = 4 * norm\ z * inverse\ (of\text{-}nat\ (Suc\ n)\ \hat{}\ 2)$
by (*simp add*: *divide-simps power2-eq-square del*: *of-nat-Suc*)
finally show $norm\ (?f\ n\ t) \leq 4 * norm\ z * inverse\ (of\text{-}nat\ (Suc\ n)\ \hat{}\ 2)$

```
    by (simp del: of-nat-Suc)
  qed
next
  show summable (λn. 4 * norm z * inverse ((of-nat (Suc n)) ^2))
   by (subst summable-Suc-iff) (simp add: summable-mult inverse-power-summable)
qed

lemma summable-deriv-ln-Gamma:
  z ≠ (0 :: 'a :: {real-normed-field,banach}) ⟹
    summable (λn. inverse (of-nat (Suc n)) − inverse (z + of-nat (Suc n)))
  unfolding summable-iff-convergent
  by (rule uniformly-convergent-imp-convergent,
      rule uniformly-summable-deriv-ln-Gamma[of z norm z/2]) simp-all


definition Polygamma :: nat ⇒ ('a :: {real-normed-field,banach}) ⇒ 'a where
  Polygamma n z = (if n = 0 then
      (∑k. inverse (of-nat (Suc k)) − inverse (z + of-nat k)) − euler-mascheroni
else
      (−1) ^Suc n * fact n * (∑k. inverse ((z + of-nat k) ^Suc n)))

abbreviation Digamma :: ('a :: {real-normed-field,banach}) ⇒ 'a where
  Digamma ≡ Polygamma 0

lemma Digamma-def:
   Digamma z = (∑k. inverse (of-nat (Suc k)) − inverse (z + of-nat k)) −
euler-mascheroni
  by (simp add: Polygamma-def)


lemma summable-Digamma:
  assumes (z :: 'a :: {real-normed-field,banach}) ≠ 0
  shows   summable (λn. inverse (of-nat (Suc n)) − inverse (z + of-nat n))
proof −
  have sums: (λn. inverse (z + of-nat (Suc n)) − inverse (z + of-nat n)) sums
                    (0 − inverse (z + of-nat 0))
    by (intro telescope-sums filterlim-compose[OF tendsto-inverse-0]
          tendsto-add-filterlim-at-infinity[OF tendsto-const] tendsto-of-nat)
 from summable-add[OF summable-deriv-ln-Gamma[OF assms] sums-summable[OF
sums]]
    show summable (λn. inverse (of-nat (Suc n)) − inverse (z + of-nat n)) by
simp
qed

lemma summable-offset:
  assumes summable (λn. f (n + k) :: 'a :: real-normed-vector)
  shows   summable f
proof −
 from assms have convergent (λm. ∑n<m. f (n + k)) by (simp add: summable-iff-convergent)
```

hence *convergent* $(\lambda m.\ (\sum n{<}k.\ f\ n) + (\sum n{<}m.\ f\ (n + k)))$
by (*intro convergent-add convergent-const*)
also have $(\lambda m.\ (\sum n{<}k.\ f\ n) + (\sum n{<}m.\ f\ (n + k))) = (\lambda m.\ \sum n{<}m{+}k.\ f\ n)$
proof
fix m :: *nat*
have $\{..{<}m{+}k\} = \{..{<}k\} \cup \{k..{<}m{+}k\}$ **by** *auto*
also have $(\sum n{\in}\ldots.\ f\ n) = (\sum n{<}k.\ f\ n) + (\sum n{=}k..{<}m{+}k.\ f\ n)$
by (*rule setsum.union-disjoint*) *auto*
also have $(\sum n{=}k..{<}m{+}k.\ f\ n) = (\sum n{=}0..{<}m{+}k{-}k.\ f\ (n + k))$
by (*intro setsum.reindex-cong*[*of* $\lambda n.\ n + k$])
(*simp*, *subst image-add-atLeastLessThan*, *auto*)
finally show $(\sum n{<}k.\ f\ n) + (\sum n{<}m.\ f\ (n + k)) = (\sum n{<}m{+}k.\ f\ n)$ **by** (*simp add*: *atLeast0LessThan*)
qed
finally have $(\lambda a.\ setsum\ f\ \{..{<}a\}) \longrightarrow lim\ (\lambda m.\ setsum\ f\ \{..{<}m + k\})$
by (*auto simp*: *convergent-LIMSEQ-iff dest*: *LIMSEQ-offset*)
thus *?thesis* **by** (*auto simp*: *summable-iff-convergent convergent-def*)
qed

lemma *Polygamma-converges*:
fixes z :: $'a$ :: {*real-normed-field*,*banach*}
assumes z: $z \neq 0$ **and** n: $n \geq 2$
shows *uniformly-convergent-on* (*ball z d*) $(\lambda k\ z.\ \sum i{<}k.\ inverse\ ((z + of\text{-}nat\ i)\ \hat{}\ n))$
proof (*rule weierstrass-m-test'-ev*)
def $e \equiv (1 + d\ /\ norm\ z)$
def $m \equiv nat\ \lceil norm\ z * e \rceil$
{
fix t **assume** t: $t \in ball\ z\ d$
have $norm\ t = norm\ (z + (t - z))$ **by** *simp*
also have $\ldots \leq norm\ z + norm\ (t - z)$ **by** (*rule norm-triangle-ineq*)
also from t **have** $norm\ (t - z) < d$ **by** (*simp add*: *dist-norm norm-minus-commute*)
finally have $norm\ t < norm\ z * e$ **using** z **by** (*simp add*: *divide-simps e-def*)
} note $ball = this$

show *eventually* $(\lambda k.\ \forall t{\in}ball\ z\ d.\ norm\ (inverse\ ((t + of\text{-}nat\ k)\ \hat{}\ n)) \leq inverse\ (of\text{-}nat\ (k - m)\ \hat{}\ n))\ sequentially$
using *eventually-gt-at-top*[*of m*] **apply** *eventually-elim*
proof (*intro ballI*)
fix k :: *nat* **and** t :: $'a$ **assume** k: $k > m$ **and** t: $t \in ball\ z\ d$
from k **have** $real\text{-}of\text{-}nat\ (k - m) = of\text{-}nat\ k - of\text{-}nat\ m$ **by** (*simp add*: *of-nat-diff*)
also have $\ldots \leq norm\ (of\text{-}nat\ k :: 'a) - norm\ z * e$
unfolding *m-def* **by** (*subst norm-of-nat*) *linarith*
also from *ball*[*OF t*] **have** $\ldots \leq norm\ (of\text{-}nat\ k :: 'a) - norm\ t$ **by** *simp*
also have $\ldots \leq norm\ (of\text{-}nat\ k + t)$ **by** (*rule norm-diff-ineq*)
finally have $inverse\ ((norm\ (t + of\text{-}nat\ k))\ \hat{}\ n) \leq inverse\ (real\text{-}of\text{-}nat\ (k - m)\ \hat{}\ n)$ **using** k n
by (*intro le-imp-inverse-le power-mono*) (*simp-all add*: *add-ac del*: *of-nat-Suc*)

```
    thus norm (inverse ((t + of-nat k) ^ n)) ≤ inverse (of-nat (k - m) ^ n)
      by (simp add: norm-inverse norm-power power-inverse)
  qed

  have summable (λk. inverse ((real-of-nat k) ^ n))
    using inverse-power-summable[of n] n by simp
  hence summable (λk. inverse ((real-of-nat (k + m - m)) ^ n)) by simp
  thus summable (λk. inverse ((real-of-nat (k - m)) ^ n)) by (rule summable-offset)
qed

lemma Polygamma-converges':
  fixes z :: 'a :: {real-normed-field,banach}
  assumes z: z ≠ 0 and n: n ≥ 2
  shows summable (λk. inverse ((z + of-nat k) ^ n))
  using uniformly-convergent-imp-convergent[OF Polygamma-converges[OF assms,
of 1], of z]
  by (simp add: summable-iff-convergent)

lemma has-field-derivative-ln-Gamma-complex [derivative-intros]:
  fixes z :: complex
  assumes z: z ∉ ℝ≤0
  shows   (ln-Gamma has-field-derivative Digamma z) (at z)
proof -
  have not-nonpos-Int [simp]: t ∉ ℤ≤0 if Re t > 0 for t
    using that by (auto elim!: nonpos-Ints-cases')
  from z have z': z ∉ ℤ≤0 and z'': z ≠ 0 using nonpos-Ints-subset-nonpos-Reals
nonpos-Reals-zero-I
    by blast+
  let ?f' = λz k. inverse (of-nat (Suc k)) - inverse (z + of-nat (Suc k))
  let ?f = λz k. z / of-nat (Suc k) - ln (1 + z / of-nat (Suc k)) and ?F' = λz.
∑n. ?f' z n
  def d ≡ min (norm z/2) (if Im z = 0 then Re z / 2 else abs (Im z) / 2)
  from z have d: d > 0 norm z/2 ≥ d by (auto simp add: complex-nonpos-Reals-iff
d-def)
  have ball: Im t = 0 ⟶ Re t > 0 if dist z t < d for t
    using no-nonpos-Real-in-ball[OF z, of t] that unfolding d-def by (force simp
add: complex-nonpos-Reals-iff)
  have sums: (λn. inverse (z + of-nat (Suc n)) - inverse (z + of-nat n)) sums
                  (0 - inverse (z + of-nat 0))
    by (intro telescope-sums filterlim-compose[OF tendsto-inverse-0]
          tendsto-add-filterlim-at-infinity[OF tendsto-const] tendsto-of-nat)

  have ((λz. ∑n. ?f z n) has-field-derivative ?F' z) (at z)
    using d z ln-Gamma-series'-aux[OF z']
  apply (intro has-field-derivative-series'(2)[of ball z d - - z] uniformly-summable-deriv-ln-Gamma)
    apply (auto intro!: derivative-eq-intros add-pos-pos mult-pos-pos dest!: ball
          simp: field-simps sums-iff nonpos-Reals-divide-of-nat-iff
          simp del: of-nat-Suc)
    apply (auto simp add: complex-nonpos-Reals-iff)
```

done
with z **have** $((\lambda z.\ (\sum k.\ ?f\ z\ k) - \textit{euler-mascheroni} * z - \textit{Ln}\ z)\ \textit{has-field-derivative}$
$?F'\ z - \textit{euler-mascheroni} - \textit{inverse}\ z)\ (\textit{at}\ z)$
by (*force intro*!: *derivative-eq-intros simp*: *Digamma-def*)
also have $?F'\ z - \textit{euler-mascheroni} - \textit{inverse}\ z = (?F'\ z + -\textit{inverse}\ z) - \textit{euler-mascheroni}$ **by** *simp*
also from *sums* **have** $-\textit{inverse}\ z = (\sum n.\ \textit{inverse}\ (z + \textit{of-nat}\ (\textit{Suc}\ n)) - \textit{inverse}\ (z + \textit{of-nat}\ n))$
by (*simp add*: *sums-iff*)
also from *sums summable-deriv-ln-Gamma*[*OF z'*']
have $?F'\ z + \ldots = (\sum n.\ \textit{inverse}\ (\textit{of-nat}\ (\textit{Suc}\ n)) - \textit{inverse}\ (z + \textit{of-nat}\ n))$
by (*subst suminf-add*) (*simp-all add*: *add-ac sums-iff*)
also have $\ldots - \textit{euler-mascheroni} = \textit{Digamma}\ z$ **by** (*simp add*: *Digamma-def*)
finally have $((\lambda z.\ (\sum k.\ ?f\ z\ k) - \textit{euler-mascheroni} * z - \textit{Ln}\ z)$
$\textit{has-field-derivative Digamma}\ z)\ (\textit{at}\ z)$ **.**
moreover from *eventually-nhds-ball*[*OF d*(*1*), *of z*]
have $\textit{eventually}\ (\lambda z.\ \textit{ln-Gamma}\ z = (\sum k.\ ?f\ z\ k) - \textit{euler-mascheroni} * z - \textit{Ln}\ z)\ (\textit{nhds}\ z)$
proof *eventually-elim*
fix t **assume** $t \in \textit{ball}\ z\ d$
hence $t \notin \mathbb{Z}_{\leq 0}$ **by** (*auto dest*!: *ball elim*!: *nonpos-Ints-cases*)
from *ln-Gamma-series'-aux*[*OF this*]
show $\textit{ln-Gamma}\ t = (\sum k.\ ?f\ t\ k) - \textit{euler-mascheroni} * t - \textit{Ln}\ t$ **by** (*simp add*: *sums-iff*)
qed
ultimately show *?thesis* **by** (*subst DERIV-cong-ev*[*OF refl - refl*])
qed

declare *has-field-derivative-ln-Gamma-complex*[*THEN DERIV-chain2*, *derivative-intros*]

lemma *Digamma-1* [*simp*]: $\textit{Digamma}\ (1 :: 'a :: \{\textit{real-normed-field}, \textit{banach}\}) = -\textit{euler-mascheroni}$
by (*simp add*: *Digamma-def*)

lemma *Digamma-plus1*:
assumes $z \neq 0$
shows $\textit{Digamma}\ (z+1) = \textit{Digamma}\ z + 1/z$
proof −
have *sums*: $(\lambda k.\ \textit{inverse}\ (z + \textit{of-nat}\ k) - \textit{inverse}\ (z + \textit{of-nat}\ (\textit{Suc}\ k)))$
$\textit{sums}\ (\textit{inverse}\ (z + \textit{of-nat}\ 0) - 0)$
by (*intro telescope-sums'*[*OF filterlim-compose*[*OF tendsto-inverse-0*]]
tendsto-add-filterlim-at-infinity[*OF tendsto-const*] *tendsto-of-nat*)
have $\textit{Digamma}\ (z+1) = (\sum k.\ \textit{inverse}\ (\textit{of-nat}\ (\textit{Suc}\ k)) - \textit{inverse}\ (z + \textit{of-nat}\ (\textit{Suc}\ k))) -$
$\textit{euler-mascheroni}\ (\textbf{is}\ \text{-} = \textit{suminf}\ ?f - \text{-})$ **by** (*simp add*: *Digamma-def add-ac*)
also have $\textit{suminf}\ ?f = (\sum k.\ \textit{inverse}\ (\textit{of-nat}\ (\textit{Suc}\ k)) - \textit{inverse}\ (z + \textit{of-nat}\ k))$
+

$(\sum k.\ inverse\ (z + of\text{-}nat\ k) - inverse\ (z + of\text{-}nat\ (Suc\ k)))$
using *summable-Digamma*[*OF assms*] *sums* **by** (*subst suminf-add*) (*simp-all add*: *add-ac sums-iff*)
also have $(\sum k.\ inverse\ (z + of\text{-}nat\ k) - inverse\ (z + of\text{-}nat\ (Suc\ k))) = 1/z$
using *sums* **by** (*simp add*: *sums-iff inverse-eq-divide*)
finally show *?thesis* **by** (*simp add*: *Digamma-def*[*of z*])
qed

lemma *Polygamma-plus1*:
assumes $z \neq 0$
shows *Polygamma n* $(z + 1) =$ *Polygamma n z* $+ (-1)\hat{}n *$ *fact n* $/ (z\ \hat{}$ *Suc n*)
proof (*cases n* $= 0$)
assume *n*: $n \neq 0$
let *?f* $= \lambda k.$ *inverse* $((z +$ *of-nat k*$)\ \hat{}$ *Suc n*)
have *Polygamma n* $(z + 1) = (-1)\ \hat{}$ *Suc n* $*$ *fact n* $* (\sum k.$ *?f* $(k+1))$
using *n* **by** (*simp add*: *Polygamma-def add-ac*)
also have $(\sum k.$ *?f* $(k+1)) + (\sum k<1.$ *?f k*$) = (\sum k.$ *?f k*$)$
using *Polygamma-converges'*[*OF assms*, *of Suc n*] *n*
by (*subst suminf-split-initial-segment* [*symmetric*]) *simp-all*
hence $(\sum k.$ *?f* $(k+1)) = (\sum k.$ *?f k*$) -$ *inverse* $(z\ \hat{}$ *Suc n*$)$ **by** (*simp add*: *algebra-simps*)
also have $(-1)\ \hat{}$ *Suc n* $*$ *fact n* $* ((\sum k.$ *?f k*$) -$ *inverse* $(z\ \hat{}$ *Suc n*$)) =$
Polygamma n z $+ (-1)\hat{}n *$ *fact n* $/ (z\ \hat{}$ *Suc n*$)$ **using** *n*
by (*simp add*: *inverse-eq-divide algebra-simps Polygamma-def*)
finally show *?thesis* **.**
qed (*insert assms*, *simp add*: *Digamma-plus1 inverse-eq-divide*)

lemma *Digamma-of-nat*:
Digamma $(of\text{-}nat\ (Suc\ n) :: 'a :: \{real\text{-}normed\text{-}field, banach\}) =$ *harm n* $-$ *euler-mascheroni*
proof (*induction n*)
case (*Suc n*)
have *Digamma* $(of\text{-}nat\ (Suc\ (Suc\ n)) :: 'a) =$ *Digamma* $(of\text{-}nat\ (Suc\ n) + 1)$ **by** *simp*
also have $\ldots =$ *Digamma* $(of\text{-}nat\ (Suc\ n)) +$ *inverse* $(of\text{-}nat\ (Suc\ n))$
by (*subst Digamma-plus1*) (*simp-all add*: *inverse-eq-divide del*: *of-nat-Suc*)
also have *Digamma* $(of\text{-}nat\ (Suc\ n) :: 'a) =$ *harm n* $-$ *euler-mascheroni* **by** (*rule Suc*)
also have $\ldots +$ *inverse* $(of\text{-}nat\ (Suc\ n)) =$ *harm* $(Suc\ n) -$ *euler-mascheroni*
by (*simp add*: *harm-Suc*)
finally show *?case* **.**
qed (*simp add*: *harm-def*)

lemma *Digamma-numeral*: *Digamma* $(numeral\ n) =$ *harm* $(pred\text{-}numeral\ n) -$ *euler-mascheroni*
by (*subst of-nat-numeral*[*symmetric*], *subst numeral-eq-Suc*, *subst Digamma-of-nat*) (*rule refl*)

lemma *Polygamma-of-real*: $x \neq 0 \Longrightarrow$ *Polygamma n* $(of\text{-}real\ x) =$ *of-real* $($*Polygamma*

n x)
unfolding *Polygamma-def* **using** *summable-Digamma*[*of x*] *Polygamma-converges'*[*of x Suc n*]
by (*simp-all add*: *suminf-of-real*)

lemma *Polygamma-Real*: $z \in \mathbb{R} \Longrightarrow z \neq 0 \Longrightarrow$ *Polygamma n z* $\in \mathbb{R}$
by (*elim Reals-cases*, *hypsubst*, *subst Polygamma-of-real*) *simp-all*

lemma *Digamma-half-integer*:
Digamma (*of-nat n + 1/2* :: *'a* :: {*real-normed-field*,*banach*}) =
$(\sum k<n.$ *2 / (of-nat (2*k+1))*$)$ *− euler-mascheroni − of-real (2 * ln 2*)
proof (*induction n*)
case *0*
have *Digamma (1/2* :: *'a) = of-real (Digamma (1/2))* **by** (*simp add*: *Polygamma-of-real* [*symmetric*])
also have *Digamma (1/2*::*real) =*
$(\sum k.$ *inverse (of-nat (Suc k)) − inverse (of-nat k + 1/2))* − *euler-mascheroni*
by (*simp add*: *Digamma-def add-ac*)
also have $(\sum k.$ *inverse (of-nat (Suc k)* :: *real) − inverse (of-nat k + 1/2)) =*
$(\sum k.$ *inverse (1/2) * (inverse (2 * of-nat (Suc k)) − inverse (2 * of-nat k + 1)))*
by (*simp-all add*: *add-ac inverse-mult-distrib*[*symmetric*] *ring-distribs del*: *inverse-divide*)
also have *... = − 2 * ln 2* **using** *sums-minus*[*OF alternating-harmonic-series-sums'*]
by (*subst suminf-mult*) (*simp-all add*: *algebra-simps sums-iff*)
finally show *?case* **by** *simp*
next
case (*Suc n*)
have *nz*: *2 * of-nat n + (1*:: *'a) ≠ 0*
using *of-nat-neq-0*[*of 2*n*] **by** (*simp only*: *of-nat-Suc*) (*simp add*: *add-ac*)
hence *nz'*: *of-nat n + (1/2*::*'a) ≠ 0* **by** (*simp add*: *field-simps*)
have *Digamma (of-nat (Suc n) + 1/2* :: *'a) = Digamma (of-nat n + 1/2 + 1)* **by** *simp*
also from *nz'* **have** *... = Digamma (of-nat n + 1 / 2) + 1 / (of-nat n + 1 / 2)*
by (*rule Digamma-plus1*)
also from *nz nz'* **have** *1 / (of-nat n + 1 / 2* :: *'a) = 2 / (2 * of-nat n + 1)*
by (*subst divide-eq-eq*) *simp-all*
also note *Suc*
finally show *?case* **by** (*simp add*: *add-ac*)
qed

lemma *Digamma-one-half*: *Digamma (1/2) = − euler-mascheroni − of-real (2 * ln 2)*
using *Digamma-half-integer*[*of 0*] **by** *simp*

lemma *Digamma-real-three-halves-pos*: *Digamma (3/2* :: *real) > 0*
proof −
have *−Digamma (3/2* :: *real) = −Digamma (of-nat 1 + 1/2)* **by** *simp*

also have ... = *2 * ln 2 + euler-mascheroni − 2* **by** (*subst Digamma-half-integer*) *simp*
also note *euler-mascheroni-less-13-over-22*
also note *ln2-le-25-over-36*
finally show *?thesis* **by** *simp*
qed

lemma *has-field-derivative-Polygamma* [*derivative-intros*]:
fixes *z* :: *'a* :: {*real-normed-field*,*euclidean-space*}
assumes *z*: *z* ∉ $\mathbb{Z}_{\leq 0}$
shows (*Polygamma n has-field-derivative Polygamma* (*Suc n*) *z*) (*at z within A*)
proof (*rule has-field-derivative-at-within*, *cases n = 0*)
assume *n*: *n = 0*
let *?f* = *λk z. inverse* (*of-nat* (*Suc k*)) − *inverse* (*z + of-nat k*)
let *?F* = *λz.* ∑ *k. ?f k z* **and** *?f'* = *λk z. inverse* ((*z + of-nat k*)²)
from *no-nonpos-Int-in-ball'*[*OF z*] **guess** *d* **. note** *d = this*
from *z* **have** *summable*: *summable* (*λk. inverse* (*of-nat* (*Suc k*)) − *inverse* (*z + of-nat k*))
by (*intro summable-Digamma*) *force*
from *z* **have** *conv*: *uniformly-convergent-on* (*ball z d*) (*λk z.* ∑ *i*<*k. inverse* ((*z + of-nat i*)²))
by (*intro Polygamma-converges*) *auto*
with *d* **have** *summable* (*λk. inverse* ((*z + of-nat k*)²)) **unfolding** *summable-iff-convergent*
by (*auto dest*!: *uniformly-convergent-imp-convergent simp*: *summable-iff-convergent*)

have (*?F has-field-derivative* (∑ *k. ?f' k z*)) (*at z*)
proof (*rule has-field-derivative-series'*[*of ball z d - - z*])
fix *k* :: *nat* **and** *t* :: *'a* **assume** *t*: *t* ∈ *ball z d*
from *t d*(*2*)[*of t*] **show** ((*λz. ?f k z*) *has-field-derivative ?f' k t*) (*at t within ball z d*)
by (*auto intro*!: *derivative-eq-intros simp*: *power2-eq-square simp del*: *of-nat-Suc*
dest!: *plus-of-nat-eq-0-imp elim*!: *nonpos-Ints-cases*)
qed (*insert d*(*1*) *summable conv*, (*assumption*|*simp*)+)
with *z* **show** (*Polygamma n has-field-derivative Polygamma* (*Suc n*) *z*) (*at z*)
unfolding *Digamma-def* [*abs-def*] *Polygamma-def* [*abs-def*] **using** *n*
by (*force simp*: *power2-eq-square intro*!: *derivative-eq-intros*)
next
assume *n*: *n ≠ 0*
from *z* **have** *z'*: *z ≠ 0* **by** *auto*
from *no-nonpos-Int-in-ball'*[*OF z*] **guess** *d* **. note** *d = this*
def *n'* ≡ *Suc n*
from *n* **have** *n'*: *n' ≥ 2* **by** (*simp add*: *n'-def*)
have ((*λz.* ∑ *k. inverse* ((*z + of-nat k*) ^ *n'*)) *has-field-derivative*
(∑ *k.* − *of-nat n'* * *inverse* ((*z + of-nat k*) ^ (*n'*+*1*)))) (*at z*)
proof (*rule has-field-derivative-series'*[*of ball z d - - z*])
fix *k* :: *nat* **and** *t* :: *'a* **assume** *t*: *t* ∈ *ball z d*
with *d* **have** *t'*: *t* ∉ $\mathbb{Z}_{\leq 0}$ *t ≠ 0* **by** *auto*

show $((\lambda a.\ inverse\ ((a + of\text{-}nat\ k)\ \hat{}\ n'))\ has\text{-}field\text{-}derivative$
$-\ of\text{-}nat\ n' * inverse\ ((t + of\text{-}nat\ k)\ \hat{}\ (n'+1)))\ (at\ t\ within\ ball\ z\ d)$
using t'
by (*fastforce intro*!: *derivative-eq-intros simp*: *divide-simps power-diff dest*: *plus-of-nat-eq-0-imp*)
next
have *uniformly-convergent-on* (*ball z d*)
$(\lambda k\ z.\ (-\ of\text{-}nat\ n' :: 'a) * (\sum i<k.\ inverse\ ((z + of\text{-}nat\ i)\ \hat{}\ (n'+1))))$
using $z'\ n$ **by** (*intro uniformly-convergent-mult Polygamma-converges*) (*simp-all add*: *n'-def*)
thus *uniformly-convergent-on* (*ball z d*)
$(\lambda k\ z.\ \sum i<k.\ -\ of\text{-}nat\ n' * inverse\ ((z + of\text{-}nat\ i :: 'a)\ \hat{}\ (n'+1)))$
by (*subst* (*asm*) *setsum-right-distrib*) *simp*
qed (*insert Polygamma-converges'*[*OF z' n'*] *d*, *simp-all*)
also have $(\sum k.\ -\ of\text{-}nat\ n' * inverse\ ((z + of\text{-}nat\ k)\ \hat{}\ (n' + 1))) =$
$(-\ of\text{-}nat\ n') * (\sum k.\ inverse\ ((z + of\text{-}nat\ k)\ \hat{}\ (n' + 1)))$
using *Polygamma-converges'*[*OF z'*, *of n'+1*] *n'* **by** (*subst suminf-mult*) *simp-all*
finally have $((\lambda z.\ \sum k.\ inverse\ ((z + of\text{-}nat\ k)\ \hat{}\ n'))\ has\text{-}field\text{-}derivative$
$-\ of\text{-}nat\ n' * (\sum k.\ inverse\ ((z + of\text{-}nat\ k)\ \hat{}\ (n' + 1))))\ (at\ z)$ **.**
from *DERIV-cmult*[*OF this*, *of* $(-1)\ \hat{}\ Suc\ n * fact\ n :: 'a$]
show (*Polygamma n has-field-derivative Polygamma* (*Suc n*) *z*) (*at z*)
unfolding *n'-def Polygamma-def*[*abs-def*] **using** *n* **by** (*simp add*: *algebra-simps*)
qed

declare *has-field-derivative-Polygamma*[*THEN DERIV-chain2*, *derivative-intros*]

lemma *isCont-Polygamma* [*continuous-intros*]:
fixes $f :: - \Rightarrow 'a :: \{real\text{-}normed\text{-}field, euclidean\text{-}space\}$
shows $isCont\ f\ z \Longrightarrow f\ z \notin \mathbb{Z}_{\leq 0} \Longrightarrow isCont\ (\lambda x.\ Polygamma\ n\ (f\ x))\ z$
by (*rule isCont-o2*[*OF - DERIV-isCont*[*OF has-field-derivative-Polygamma*]])

lemma *continuous-on-Polygamma*:
$A \cap \mathbb{Z}_{\leq 0} = \{\} \Longrightarrow continuous\text{-}on\ A\ (Polygamma\ n :: - \Rightarrow 'a :: \{real\text{-}normed\text{-}field, euclidean\text{-}space\})$
by (*intro continuous-at-imp-continuous-on isCont-Polygamma*[*OF continuous-ident*] *ballI*) *blast*

lemma *isCont-ln-Gamma-complex* [*continuous-intros*]:
fixes $f :: 'a :: t2\text{-}space \Rightarrow complex$
shows $isCont\ f\ z \Longrightarrow f\ z \notin \mathbb{R}_{\leq 0} \Longrightarrow isCont\ (\lambda z.\ ln\text{-}Gamma\ (f\ z))\ z$
by (*rule isCont-o2*[*OF - DERIV-isCont*[*OF has-field-derivative-ln-Gamma-complex*]])

lemma *continuous-on-ln-Gamma-complex* [*continuous-intros*]:
fixes $A :: complex\ set$
shows $A \cap \mathbb{R}_{\leq 0} = \{\} \Longrightarrow continuous\text{-}on\ A\ ln\text{-}Gamma$
by (*intro continuous-at-imp-continuous-on ballI isCont-ln-Gamma-complex*[*OF continuous-ident*])
fastforce

We define a type class that captures all the fundamental properties of the

inverse of the Gamma function and defines the Gamma function upon that. This allows us to instantiate the type class both for the reals and for the complex numbers with a minimal amount of proof duplication.

class *Gamma* = *real-normed-field* + *complete-space* +
  fixes *rGamma* :: $'a \Rightarrow 'a$
  assumes *rGamma-eq-zero-iff-aux*: *rGamma z* = *0* $\longleftrightarrow$ ($\exists$ *n*. *z* = − *of-nat n*)
  assumes *differentiable-rGamma-aux1*:
    ($\bigwedge$*n*. *z* ≠ − *of-nat n*) $\Longrightarrow$
     let d = (*THE d*. (λ*n*. $\sum$ *k*<*n*. *inverse* (*of-nat* (*Suc k*)) − *inverse* (*z* + *of-nat* *k*))
                $\longrightarrow$ *d*) − *scaleR euler-mascheroni 1*
     in filterlim (λ*y*. (*rGamma y* − *rGamma z* + *rGamma z* ∗ *d* ∗ (*y* − *z*)) /$_R$
                norm (*y* − *z*)) (*nhds 0*) (*at z*)
  assumes *differentiable-rGamma-aux2*:
    let z = − *of-nat n*
    in filterlim (λ*y*. (*rGamma y* − *rGamma z* − (−*1*) ^ *n* ∗ (*setprod of-nat* {*1*..*n*}) ∗ (*y* − *z*)) /$_R$
                norm (*y* − *z*)) (*nhds 0*) (*at z*)
  assumes *rGamma-series-aux*: ($\bigwedge$*n*. *z* ≠ − *of-nat n*) $\Longrightarrow$
          let fact' = (λ*n*. *setprod of-nat* {*1*..*n*});
              exp = (λ*x*. *THE e*. (λ*n*. $\sum$ *k*<*n*. *x* ^ *k* /$_R$ *fact k*) $\longrightarrow$ *e*);
              pochhammer' = (λ*a n*. ($\prod$ *n* = *0*..*n*. *a* + *of-nat n*))
           in filterlim (λ*n*. *pochhammer' z n* / (*fact' n* ∗ *exp* (*z* ∗ (*ln* (*of-nat n*) ∗$_R$ *1*))))
                (*nhds* (*rGamma z*)) *sequentially*
begin
subclass *banach* **..**
end

definition *Gamma z* = *inverse* (*rGamma z*)

59.3 Basic properties

lemma *Gamma-nonpos-Int*: *z* ∈ $\mathbb{Z}_{\leq 0}$ $\Longrightarrow$ *Gamma z* = *0*
  and *rGamma-nonpos-Int*: *z* ∈ $\mathbb{Z}_{\leq 0}$ $\Longrightarrow$ *rGamma z* = *0*
  using *rGamma-eq-zero-iff-aux*[*of z*] **unfolding** *Gamma-def* **by** (*auto elim*!: *nonpos-Ints-cases'*)

lemma *Gamma-nonzero*: *z* ∉ $\mathbb{Z}_{\leq 0}$ $\Longrightarrow$ *Gamma z* ≠ *0*
  and *rGamma-nonzero*: *z* ∉ $\mathbb{Z}_{\leq 0}$ $\Longrightarrow$ *rGamma z* ≠ *0*
  using *rGamma-eq-zero-iff-aux*[*of z*] **unfolding** *Gamma-def* **by** (*auto elim*!: *nonpos-Ints-cases'*)

lemma *Gamma-eq-zero-iff*: *Gamma z* = *0* $\longleftrightarrow$ *z* ∈ $\mathbb{Z}_{\leq 0}$
  and *rGamma-eq-zero-iff*: *rGamma z* = *0* $\longleftrightarrow$ *z* ∈ $\mathbb{Z}_{\leq 0}$
  using *rGamma-eq-zero-iff-aux*[*of z*] **unfolding** *Gamma-def* **by** (*auto elim*!: *nonpos-Ints-cases'*)

lemma *rGamma-inverse-Gamma*: *rGamma z* = *inverse* (*Gamma z*)
  unfolding *Gamma-def* **by** *simp*

lemma *rGamma-series-LIMSEQ* [*tendsto-intros*]:

rGamma-series z ⟶ *rGamma z*
proof (*cases z* ∈ $\mathbb{Z}_{\leq 0}$)
case *False*
hence *z* ≠ − *of-nat n* **for** *n* **by** *auto*
from *rGamma-series-aux*[*OF this*] **show** *?thesis*
by (*simp add*: *rGamma-series-def*[*abs-def*] *fact-altdef pochhammer-Suc-setprod exp-def of-real-def*[*symmetric*] *suminf-def sums-def*[*abs-def*])
qed (*insert rGamma-eq-zero-iff*[*of z*], *simp-all add*: *rGamma-series-nonpos-Ints-LIMSEQ*)

lemma *Gamma-series-LIMSEQ* [*tendsto-intros*]:
Gamma-series z ⟶ *Gamma z*
proof (*cases z* ∈ $\mathbb{Z}_{\leq 0}$)
case *False*
hence (λ*n*. *inverse* (*rGamma-series z n*)) ⟶ *inverse* (*rGamma z*)
by (*intro tendsto-intros*) (*simp-all add*: *rGamma-eq-zero-iff*)
also have (λ*n*. *inverse* (*rGamma-series z n*)) = *Gamma-series z*
by (*simp add*: *rGamma-series-def Gamma-series-def*[*abs-def*])
finally show *?thesis* **by** (*simp add*: *Gamma-def*)
qed (*insert Gamma-eq-zero-iff*[*of z*], *simp-all add*: *Gamma-series-nonpos-Ints-LIMSEQ*)

lemma *Gamma-altdef*: *Gamma z* = *lim* (*Gamma-series z*)
using *Gamma-series-LIMSEQ*[*of z*] **by** (*simp add*: *limI*)

lemma *rGamma-1* [*simp*]: *rGamma 1* = *1*
proof −
have *A*: *eventually* (λ*n*. *rGamma-series 1 n* = *of-nat* (*Suc n*) / *of-nat n*) *sequentially*
using *eventually-gt-at-top*[*of 0*::*nat*]
by (*force elim*!: *eventually-mono simp*: *rGamma-series-def exp-of-real pochhammer-fact divide-simps pochhammer-rec*′ *dest*!: *pochhammer-eq-0-imp-nonpos-Int*)
have *rGamma-series 1* ⟶ *1* **by** (*subst tendsto-cong*[*OF A*]) (*rule LIMSEQ-Suc-n-over-n*)
moreover have *rGamma-series 1* ⟶ *rGamma 1* **by** (*rule tendsto-intros*)
ultimately show *?thesis* **by** (*intro LIMSEQ-unique*)
qed

lemma *rGamma-plus1*: *z* ∗ *rGamma* (*z* + *1*) = *rGamma z*
proof −
let *?f* = λ*n*. (*z* + *1*) ∗ *inverse* (*of-nat n*) + *1*
have *eventually* (λ*n*. *?f n* ∗ *rGamma-series z n* = *z* ∗ *rGamma-series* (*z* + *1*) *n*) *sequentially*
using *eventually-gt-at-top*[*of 0*::*nat*]
proof *eventually-elim*
fix *n* :: *nat* **assume** *n*: *n* > *0*
hence *z* ∗ *rGamma-series* (*z* + *1*) *n* = *inverse* (*of-nat n*) ∗ *pochhammer z* (*Suc* (*Suc n*)) / (*fact n* ∗ *exp* (*z* ∗ *of-real* (*ln* (*of-nat n*))))
by (*subst pochhammer-rec*) (*simp add*: *rGamma-series-def field-simps exp-add exp-of-real*)
also from *n* **have** ... = *?f n* ∗ *rGamma-series z n*
by (*subst pochhammer-rec*′) (*simp-all add*: *divide-simps rGamma-series-def*

add-ac)
finally show *?f n * rGamma-series z n = z * rGamma-series (z + 1) n* **..**
qed
moreover have $(\lambda n.$ *?f n * rGamma-series z n*$) \longrightarrow$ *((z+1) * 0 + 1) * rGamma z*
by (*intro tendsto-intros lim-inverse-n*)
hence $(\lambda n.$ *?f n * rGamma-series z n*$) \longrightarrow$ *rGamma z* **by** *simp*
ultimately have $(\lambda n.$ *z * rGamma-series (z + 1) n*$) \longrightarrow$ *rGamma z*
by (*rule Lim-transform-eventually*)
moreover have $(\lambda n.$ *z * rGamma-series (z + 1) n*$) \longrightarrow$ *z * rGamma (z + 1)*
by (*intro tendsto-intros*)
ultimately show *z * rGamma (z + 1) = rGamma z* **using** *LIMSEQ-unique* **by** *blast*
qed

lemma *pochhammer-rGamma*: *rGamma z = pochhammer z n * rGamma (z + of-nat n)*
proof (*induction n arbitrary*: *z*)
case (*Suc n z*)
have *rGamma z = pochhammer z n * rGamma (z + of-nat n)* **by** (*rule Suc.IH*)
also note *rGamma-plus1* [*symmetric*]
finally show *?case* **by** (*simp add*: *add-ac pochhammer-rec'*)
qed *simp-all*

lemma *Gamma-plus1*: $z \notin \mathbb{Z}_{\leq 0} \Longrightarrow$ *Gamma (z + 1) = z * Gamma z*
using *rGamma-plus1*[*of z*] **by** (*simp add*: *rGamma-inverse-Gamma field-simps Gamma-eq-zero-iff*)

lemma *pochhammer-Gamma*: $z \notin \mathbb{Z}_{\leq 0} \Longrightarrow$ *pochhammer z n = Gamma (z + of-nat n) / Gamma z*
using *pochhammer-rGamma*[*of z*]
by (*simp add*: *rGamma-inverse-Gamma Gamma-eq-zero-iff field-simps*)

lemma *Gamma-0* [*simp*]: *Gamma 0 = 0*
and *rGamma-0* [*simp*]: *rGamma 0 = 0*
and *Gamma-neg-1* [*simp*]: *Gamma (− 1) = 0*
and *rGamma-neg-1* [*simp*]: *rGamma (− 1) = 0*
and *Gamma-neg-numeral* [*simp*]: *Gamma (− numeral n) = 0*
and *rGamma-neg-numeral* [*simp*]: *rGamma (− numeral n) = 0*
and *Gamma-neg-of-nat* [*simp*]: *Gamma (− of-nat m) = 0*
and *rGamma-neg-of-nat* [*simp*]: *rGamma (− of-nat m) = 0*
by (*simp-all add*: *rGamma-eq-zero-iff Gamma-eq-zero-iff*)

lemma *Gamma-1* [*simp*]: *Gamma 1 = 1* **unfolding** *Gamma-def* **by** *simp*

lemma *Gamma-fact*: *Gamma (of-nat (Suc n)) = fact n*
by (*simp add*: *pochhammer-fact pochhammer-Gamma of-nat-in-nonpos-Ints-iff*)

lemma *Gamma-numeral*: *Gamma* (*numeral n*) = *fact* (*pred-numeral n*)
by (*subst of-nat-numeral*[*symmetric*], *subst numeral-eq-Suc*, *subst Gamma-fact*) (*rule refl*)

lemma *Gamma-of-int*: *Gamma* (*of-int n*) = (*if n* > *0 then fact* (*nat* (*n* − *1*)) *else 0*)
proof (*cases n* > *0*)
case *True*
hence *Gamma* (*of-int n*) = *Gamma* (*of-nat* (*Suc* (*nat* (*n* − *1*)))) **by** (*subst of-nat-Suc*) *simp-all*
with *True* **show** *?thesis* **by** (*subst* (*asm*) *Gamma-fact*) *simp*
qed (*simp-all add*: *Gamma-eq-zero-iff nonpos-Ints-of-int*)

lemma *rGamma-of-int*: *rGamma* (*of-int n*) = (*if n* > *0 then inverse* (*fact* (*nat* (*n* − *1*))) *else 0*)
by (*simp add*: *Gamma-of-int rGamma-inverse-Gamma*)

lemma *Gamma-seriesI*:
assumes (λ*n*. *g n* / *Gamma-series z n*) ⟶ *1*
shows *g* ⟶ *Gamma z*
proof (*rule Lim-transform-eventually*)
have *1*/*2* > (*0*::*real*) **by** *simp*
from *tendstoD*[*OF assms*, *OF this*]
show *eventually* (λ*n*. *g n* / *Gamma-series z n* ∗ *Gamma-series z n* = *g n*) *sequentially*
by (*force elim*!: *eventually-mono simp*: *dist-real-def dist-0-norm*)
from *assms* **have** (λ*n*. *g n* / *Gamma-series z n* ∗ *Gamma-series z n*) ⟶ *1* ∗ *Gamma z*
by (*intro tendsto-intros*)
thus (λ*n*. *g n* / *Gamma-series z n* ∗ *Gamma-series z n*) ⟶ *Gamma z* **by** *simp*
qed

lemma *Gamma-seriesI′*:
assumes *f* ⟶ *rGamma z*
assumes (λ*n*. *g n* ∗ *f n*) ⟶ *1*
assumes *z* ∉ $\mathbb{Z}_{\leq 0}$
shows *g* ⟶ *Gamma z*
proof (*rule Lim-transform-eventually*)
have *1*/*2* > (*0*::*real*) **by** *simp*
from *tendstoD*[*OF assms*(*2*), *OF this*] **show** *eventually* (λ*n*. *g n* ∗ *f n* / *f n* = *g n*) *sequentially*
by (*force elim*!: *eventually-mono simp*: *dist-real-def dist-0-norm*)
from *assms* **have** (λ*n*. *g n* ∗ *f n* / *f n*) ⟶ *1* / *rGamma z*
by (*intro tendsto-divide assms*) (*simp-all add*: *rGamma-eq-zero-iff*)
thus (λ*n*. *g n* ∗ *f n* / *f n*) ⟶ *Gamma z* **by** (*simp add*: *Gamma-def divide-inverse*)
qed

lemma *Gamma-series'-LIMSEQ*: *Gamma-series'* $z \longrightarrow$ *Gamma* z
by (*cases* $z \in \mathbb{Z}_{\leq 0}$) (*simp-all add*: *Gamma-nonpos-Int Gamma-seriesI*[*OF Gamma-series-Gamma-series'*]
Gamma-series'-nonpos-Ints-LIMSEQ[*of z*])

59.4 Differentiability

lemma *has-field-derivative-rGamma-no-nonpos-int*:
assumes $z \notin \mathbb{Z}_{\leq 0}$
shows (*rGamma has-field-derivative* $-$*rGamma* $z *$ *Digamma* z) (*at* z *within* A)
proof (*rule has-field-derivative-at-within*)
from *assms* **have** $z \neq -$ *of-nat* n **for** n **by** *auto*
from *differentiable-rGamma-aux1*[*OF this*]
show (*rGamma has-field-derivative* $-$*rGamma* $z *$ *Digamma* z) (*at* z)
unfolding *Digamma-def suminf-def sums-def*[*abs-def*]
has-field-derivative-def has-derivative-def netlimit-at
by (*simp add*: *Let-def bounded-linear-mult-right mult-ac of-real-def* [*symmetric*])
qed

lemma *has-field-derivative-rGamma-nonpos-int*:
(*rGamma has-field-derivative* $(-1)\hat{}n *$ *fact* n) (*at* $(-$ *of-nat* $n)$ *within* A)
apply (*rule has-field-derivative-at-within*)
using *differentiable-rGamma-aux2*[*of n*]
unfolding *Let-def has-field-derivative-def has-derivative-def netlimit-at*
by (*simp only*: *bounded-linear-mult-right mult-ac of-real-def* [*symmetric*] *fact-altdef*)

lemma *has-field-derivative-rGamma* [*derivative-intros*]:
(*rGamma has-field-derivative* (*if* $z \in \mathbb{Z}_{\leq 0}$ *then* $(-1)\hat{}$(*nat* $\lfloor$*norm* $z\rfloor$) $*$ *fact* (*nat* $\lfloor$*norm* $z\rfloor$)
else $-$*rGamma* $z *$ *Digamma* z)) (*at* z *within* A)
using *has-field-derivative-rGamma-no-nonpos-int*[*of z A*]
has-field-derivative-rGamma-nonpos-int[*of nat* $\lfloor$*norm* $z\rfloor$ A]
by (*auto elim*!: *nonpos-Ints-cases'*)

declare *has-field-derivative-rGamma-no-nonpos-int* [*THEN DERIV-chain2*, *derivative-intros*]
declare *has-field-derivative-rGamma* [*THEN DERIV-chain2*, *derivative-intros*]
declare *has-field-derivative-rGamma-nonpos-int* [*derivative-intros*]
declare *has-field-derivative-rGamma-no-nonpos-int* [*derivative-intros*]
declare *has-field-derivative-rGamma* [*derivative-intros*]

lemma *has-field-derivative-Gamma* [*derivative-intros*]:
$z \notin \mathbb{Z}_{\leq 0} \Longrightarrow$ (*Gamma has-field-derivative Gamma* $z *$ *Digamma* z) (*at* z *within* A)
unfolding *Gamma-def* [*abs-def*]
by (*fastforce intro*!: *derivative-eq-intros simp*: *rGamma-eq-zero-iff*)

declare *has-field-derivative-Gamma*[*THEN DERIV-chain2*, *derivative-intros*]

hide-fact *rGamma-eq-zero-iff-aux differentiable-rGamma-aux1 differentiable-rGamma-aux2 differentiable-rGamma-aux2 rGamma-series-aux Gamma-class.rGamma-eq-zero-iff-aux*

59.5 Continuity

lemma *continuous-on-rGamma* [*continuous-intros*]: *continuous-on A rGamma*
by (*rule DERIV-continuous-on has-field-derivative-rGamma*)+

lemma *continuous-on-Gamma* [*continuous-intros*]: $A \cap \mathbb{Z}_{\leq 0} = \{\} \Longrightarrow$ *continuous-on A Gamma*
by (*rule DERIV-continuous-on has-field-derivative-Gamma*)+ *blast*

lemma *isCont-rGamma* [*continuous-intros*]:
isCont f z $\Longrightarrow$ *isCont* ($\lambda x.$ *rGamma* (*f x*)) *z*
by (*rule isCont-o2*[*OF - DERIV-isCont*[*OF has-field-derivative-rGamma*]])

lemma *isCont-Gamma* [*continuous-intros*]:
isCont f z $\Longrightarrow$ *f z* $\notin \mathbb{Z}_{\leq 0} \Longrightarrow$ *isCont* ($\lambda x.$ *Gamma* (*f x*)) *z*
by (*rule isCont-o2*[*OF - DERIV-isCont*[*OF has-field-derivative-Gamma*]])

The complex Gamma function

instantiation *complex* :: *Gamma*
begin

definition *rGamma-complex* :: *complex* $\Rightarrow$ *complex* **where**
rGamma-complex z = *lim* (*rGamma-series z*)

lemma *rGamma-series-complex-converges*:
convergent (*rGamma-series* (*z* :: *complex*)) (**is** *?thesis1*)
and *rGamma-complex-altdef*:
rGamma z = (*if z* $\in \mathbb{Z}_{\leq 0}$ *then 0 else exp* (−*ln-Gamma z*)) (**is** *?thesis2*)
proof −
have *?thesis1* ∧ *?thesis2*
proof (*cases z* $\in \mathbb{Z}_{\leq 0}$)
case *False*
have *rGamma-series z* $\longrightarrow$ *exp* (− *ln-Gamma z*)
proof (*rule Lim-transform-eventually*)
from *ln-Gamma-series-complex-converges'*[*OF False*] **guess** *d* **by** (*elim exE conjE*)
from *this*(*1*) *uniformly-convergent-imp-convergent*[*OF this*(*2*), *of z*]
have *ln-Gamma-series z* $\longrightarrow$ *lim* (*ln-Gamma-series z*) **by** (*simp add*: *convergent-LIMSEQ-iff*)
thus ($\lambda n.$ *exp* (−*ln-Gamma-series z n*)) $\longrightarrow$ *exp* (− *ln-Gamma z*)
unfolding *convergent-def ln-Gamma-def* **by** (*intro tendsto-exp tendsto-minus*)
from *eventually-gt-at-top*[*of 0*::*nat*] *exp-ln-Gamma-series-complex False*
show *eventually* ($\lambda n.$ *exp* (−*ln-Gamma-series z n*) = *rGamma-series z n*) *sequentially*
by (*force elim*!: *eventually-mono simp*: *exp-minus Gamma-series-def rGamma-series-def*)

```
    qed
    with False show ?thesis
      by (auto simp: convergent-def rGamma-complex-def intro!: limI)
  next
    case True
    then obtain k where z = - of-nat k by (erule nonpos-Ints-cases')
    also have rGamma-series ... ⟶ 0
        by (subst tendsto-cong[OF rGamma-series-minus-of-nat]) (simp-all add:
convergent-const)
    finally show ?thesis using True
      by (auto simp: rGamma-complex-def convergent-def intro!: limI)
  qed
  thus ?thesis1 ?thesis2 by blast+
qed

context
begin


private lemma rGamma-complex-plus1: z * rGamma (z + 1) = rGamma (z ::
complex)
proof -
  let ?f = λn. (z + 1) * inverse (of-nat n) + 1
  have eventually (λn. ?f n * rGamma-series z n = z * rGamma-series (z + 1)
n) sequentially
    using eventually-gt-at-top[of 0::nat]
  proof eventually-elim
    fix n :: nat assume n: n > 0
    hence z * rGamma-series (z + 1) n = inverse (of-nat n) *
          pochhammer z (Suc (Suc n)) / (fact n * exp (z * of-real (ln (of-nat n))))
      by (subst pochhammer-rec) (simp add: rGamma-series-def field-simps exp-add
exp-of-real)
    also from n have ... = ?f n * rGamma-series z n
      by (subst pochhammer-rec') (simp-all add: divide-simps rGamma-series-def
add-ac)
    finally show ?f n * rGamma-series z n = z * rGamma-series (z + 1) n ..
  qed
  moreover have (λn. ?f n * rGamma-series z n) ⟶ ((z+1) * 0 + 1) *
rGamma z
    using rGamma-series-complex-converges
    by (intro tendsto-intros lim-inverse-n)
       (simp-all add: convergent-LIMSEQ-iff rGamma-complex-def)
  hence (λn. ?f n * rGamma-series z n) ⟶ rGamma z by simp
  ultimately have (λn. z * rGamma-series (z + 1) n) ⟶ rGamma z
    by (rule Lim-transform-eventually)
  moreover have (λn. z * rGamma-series (z + 1) n) ⟶ z * rGamma (z +
1)
    using rGamma-series-complex-converges
   by (auto intro!: tendsto-mult simp: rGamma-complex-def convergent-LIMSEQ-iff)
```

ultimately show $z * rGamma\ (z + 1) = rGamma\ z$ **using** *LIMSEQ-unique*
by *blast*
qed

private lemma *has-field-derivative-rGamma-complex-no-nonpos-Int*:
assumes $(z :: complex) \notin \mathbb{Z}_{\leq 0}$
shows $(rGamma\ has\text{-}field\text{-}derivative\ -\ rGamma\ z * Digamma\ z)\ (at\ z)$
proof −
have *diff*: $(rGamma\ has\text{-}field\text{-}derivative\ -\ rGamma\ z * Digamma\ z)\ (at\ z)$ **if** $Re\ z > 0$ **for** z
proof (*subst DERIV-cong-ev*[*OF refl - refl*])
from *that* **have** *eventually* $(\lambda t.\ t \in ball\ z\ (Re\ z/2))\ (nhds\ z)$
by (*intro eventually-nhds-in-nhd*) *simp-all*
thus *eventually* $(\lambda t.\ rGamma\ t = exp\ (-\ ln\text{-}Gamma\ t))\ (nhds\ z)$
using *no-nonpos-Int-in-ball-complex*[*OF that*]
by (*auto elim*!: *eventually-mono simp*: *rGamma-complex-altdef*)
next
have $z \notin \mathbb{R}_{\leq 0}$ **using** *that* **by** (*simp add*: *complex-nonpos-Reals-iff*)
with *that* **show** $((\lambda t.\ exp\ (-\ ln\text{-}Gamma\ t))\ has\text{-}field\text{-}derivative\ (-rGamma\ z * Digamma\ z))\ (at\ z)$
by (*force elim*!: *nonpos-Ints-cases intro*!: *derivative-eq-intros simp*: *rGamma-complex-altdef*)
qed

from *assms* **show** $(rGamma\ has\text{-}field\text{-}derivative\ -\ rGamma\ z * Digamma\ z)\ (at\ z)$
proof (*induction nat* $\lfloor 1 - Re\ z \rfloor$ *arbitrary*: z)
case (*Suc n z*)
from *Suc.prems* **have** *z*: $z \neq 0$ **by** *auto*
from *Suc.hyps* **have** $n = nat\ \lfloor -\ Re\ z \rfloor$ **by** *linarith*
hence *A*: $n = nat\ \lfloor 1 - Re\ (z + 1) \rfloor$ **by** *simp*
from *Suc.prems* **have** *B*: $z + 1 \notin \mathbb{Z}_{\leq 0}$ **by** (*force dest*: *plus-one-in-nonpos-Ints-imp*)

have $((\lambda z.\ z * (rGamma \circ (\lambda z.\ z + 1))\ z)\ has\text{-}field\text{-}derivative$
$-rGamma\ (z + 1) * (Digamma\ (z + 1) * z - 1))\ (at\ z)$
by (*rule derivative-eq-intros DERIV-chain Suc refl A B*)+ (*simp add*: *algebra-simps*)
also have $(\lambda z.\ z * (rGamma \circ (\lambda z.\ z + 1 :: complex))\ z) = rGamma$
by (*simp add*: *rGamma-complex-plus1*)
also from *z* **have** $Digamma\ (z + 1) * z - 1 = z * Digamma\ z$
by (*subst Digamma-plus1*) (*simp-all add*: *field-simps*)
also have $-rGamma\ (z + 1) * (z * Digamma\ z) = -rGamma\ z * Digamma\ z$
by (*simp add*: *rGamma-complex-plus1*[*of z*, *symmetric*])
finally show *?case* **.**
qed (*intro diff*, *simp*)
qed

private lemma *rGamma-complex-1*: $rGamma\ (1 :: complex) = 1$
proof −
have *A*: *eventually* $(\lambda n.\ rGamma\text{-}series\ 1\ n = of\text{-}nat\ (Suc\ n)\ /\ of\text{-}nat\ n)$

sequentially
using *eventually-gt-at-top*[*of 0*::*nat*]
by (*force elim*!: *eventually-mono simp*: *rGamma-series-def exp-of-real pochhammer-fact divide-simps pochhammer-rec' dest*!: *pochhammer-eq-0-imp-nonpos-Int*)
have *rGamma-series 1* $\longrightarrow$ *1* **by** (*subst tendsto-cong*[*OF A*]) (*rule LIMSEQ-Suc-n-over-n*)
thus *rGamma 1* = (*1* :: *complex*) **unfolding** *rGamma-complex-def* **by** (*rule limI*)
qed

private lemma *has-field-derivative-rGamma-complex-nonpos-Int*:
(*rGamma has-field-derivative* (−*1*) ^*n* ∗ *fact n*) (*at* (− *of-nat n* :: *complex*))
proof (*induction n*)
case *0*
have *A*: (*0*::*complex*) + *1* ∉ $\mathbb{Z}_{\leq 0}$ **by** *simp*
have ((λ*z*. *z* ∗ (*rGamma* ∘ (λ*z*. *z* + *1* :: *complex*)) *z*) *has-field-derivative 1*) (*at 0*)
by (*rule derivative-eq-intros DERIV-chain refl has-field-derivative-rGamma-complex-no-nonpos-Int A*)+ (*simp add*: *rGamma-complex-1*)
thus *?case* **by** (*simp add*: *rGamma-complex-plus1*)
next
case (*Suc n*)
hence *A*: (*rGamma has-field-derivative* (−*1*) ^*n* ∗ *fact n*) (*at* (− *of-nat* (*Suc n*) + *1* :: *complex*)) **by** *simp*
have ((λ*z*. *z* ∗ (*rGamma* ∘ (λ*z*. *z* + *1* :: *complex*)) *z*) *has-field-derivative* (− *1*) ^ *Suc n* ∗ *fact* (*Suc n*)) (*at* (− *of-nat* (*Suc n*)))
by (*rule derivative-eq-intros refl A DERIV-chain*)+ (*simp add*: *algebra-simps rGamma-complex-altdef*)
thus *?case* **by** (*simp add*: *rGamma-complex-plus1*)
qed

instance proof
fix *z* :: *complex* **show** (*rGamma z* = *0*) ⟷ (∃ *n*. *z* = − *of-nat n*)
by (*auto simp*: *rGamma-complex-altdef elim*!: *nonpos-Ints-cases'*)
next
fix *z* :: *complex* **assume** ⋀*n*. *z* ≠ − *of-nat n*
hence *z* ∉ $\mathbb{Z}_{\leq 0}$ **by** (*auto elim*!: *nonpos-Ints-cases'*)
from *has-field-derivative-rGamma-complex-no-nonpos-Int*[*OF this*]
show *let d* = (*THE d*. (λ*n*. ∑*k*<*n*. *inverse* (*of-nat* (*Suc k*)) − *inverse* (*z* + *of-nat k*)) ⟶ *d*) − *euler-mascheroni* ∗$_R$ *1 in* (λ*y*. (*rGamma y* − *rGamma z* + *rGamma z* ∗ *d* ∗ (*y* − *z*)) /$_R$ *cmod* (*y* − *z*)) −*z*→ *0*
by (*simp add*: *has-field-derivative-def has-derivative-def Digamma-def sums-def*[*abs-def*] *netlimit-at of-real-def*[*symmetric*] *suminf-def*)
next
fix *n* :: *nat*
from *has-field-derivative-rGamma-complex-nonpos-Int*[*of n*]

show *let z = − of-nat n in (λy. (rGamma y − rGamma z − (− 1) ^ n * setprod of-nat {1..n} *
(y − z)) /R cmod (y − z)) −z→ 0*
by (*simp add*: *has-field-derivative-def has-derivative-def fact-altdef netlimit-at Let-def*)
next
fix *z* :: *complex*
from *rGamma-series-complex-converges*[*of z*] **have** *rGamma-series z ⟶ rGamma z*
by (*simp add*: *convergent-LIMSEQ-iff rGamma-complex-def*)
thus *let fact′ = λn. setprod of-nat {1..n};*
exp = λx. THE e. (λn. ∑k<n. x ^ k /R fact k) ⟶ e;
pochhammer′ = λa n. ∏n = 0..n. a + of-nat n
*in (λn. pochhammer′ z n / (fact′ n * exp (z * ln (real-of-nat n) *R 1))) ⟶ rGamma z*
by (*simp add*: *fact-altdef pochhammer-Suc-setprod rGamma-series-def* [*abs-def*] *exp-def*
of-real-def [*symmetric*] *suminf-def sums-def* [*abs-def*])
qed

end
end

lemma *Gamma-complex-altdef*:
Gamma z = (if z ∈ ℤ≤0 then 0 else exp (ln-Gamma (z :: complex)))
unfolding *Gamma-def rGamma-complex-altdef* **by** (*simp add*: *exp-minus*)

lemma *cnj-rGamma*: *cnj (rGamma z) = rGamma (cnj z)*
proof −
have *rGamma-series (cnj z) = (λn. cnj (rGamma-series z n))*
by (*intro ext*) (*simp-all add*: *rGamma-series-def exp-cnj*)
also have ... ⟶ *cnj (rGamma z)* **by** (*intro tendsto-cnj tendsto-intros*)
finally show *?thesis* **unfolding** *rGamma-complex-def* **by** (*intro sym*[*OF limI*])
qed

lemma *cnj-Gamma*: *cnj (Gamma z) = Gamma (cnj z)*
unfolding *Gamma-def* **by** (*simp add*: *cnj-rGamma*)

lemma *Gamma-complex-real*:
z ∈ ℝ ⟹ Gamma z ∈ (ℝ :: complex set) **and** *rGamma-complex-real*: *z ∈ ℝ ⟹ rGamma z ∈ ℝ*
by (*simp-all add*: *Reals-cnj-iff cnj-Gamma cnj-rGamma*)

lemma *field-differentiable-rGamma*: *rGamma field-differentiable (at z within A)*
using *has-field-derivative-rGamma*[*of z*] **unfolding** *field-differentiable-def* **by** *blast*

lemma *holomorphic-on-rGamma*: *rGamma holomorphic-on A*

unfolding *holomorphic-on-def* **by** (*auto intro*!: *field-differentiable-rGamma*)

lemma *analytic-on-rGamma*: *rGamma analytic-on A*
unfolding *analytic-on-def* **by** (*auto intro*!: *exI*[*of - 1*] *holomorphic-on-rGamma*)

lemma *field-differentiable-Gamma*: $z \notin \mathbb{Z}_{\leq 0} \Longrightarrow$ *Gamma field-differentiable* (*at z within A*)
using *has-field-derivative-Gamma*[*of z*] **unfolding** *field-differentiable-def* **by** *auto*

lemma *holomorphic-on-Gamma*: $A \cap \mathbb{Z}_{\leq 0} = \{\} \Longrightarrow$ *Gamma holomorphic-on A*
unfolding *holomorphic-on-def* **by** (*auto intro*!: *field-differentiable-Gamma*)

lemma *analytic-on-Gamma*: $A \cap \mathbb{Z}_{\leq 0} = \{\} \Longrightarrow$ *Gamma analytic-on A*
by (*rule analytic-on-subset*[*of - UNIV* $-$ $\mathbb{Z}_{\leq 0}$], *subst analytic-on-open*)
(*auto intro*!: *holomorphic-on-Gamma*)

lemma *has-field-derivative-rGamma-complex'* [*derivative-intros*]:
(*rGamma has-field-derivative* (*if z* $\in \mathbb{Z}_{\leq 0}$ *then* (-1) ^(*nat* $\lfloor -Re\ z \rfloor$) $*$ *fact* (*nat* $\lfloor -Re\ z \rfloor$) *else*
$-$*rGamma z* $*$ *Digamma z*)) (*at z within A*)
using *has-field-derivative-rGamma*[*of z*] **by** (*auto elim*!: *nonpos-Ints-cases'*)

declare *has-field-derivative-rGamma-complex'*[*THEN DERIV-chain2*, *derivative-intros*]

lemma *field-differentiable-Polygamma*:
fixes *z*::*complex*
shows
$z \notin \mathbb{Z}_{\leq 0} \Longrightarrow$ *Polygamma n field-differentiable* (*at z within A*)
using *has-field-derivative-Polygamma*[*of z n*] **unfolding** *field-differentiable-def*
by *auto*

lemma *holomorphic-on-Polygamma*: $A \cap \mathbb{Z}_{\leq 0} = \{\} \Longrightarrow$ *Polygamma n holomorphic-on A*
unfolding *holomorphic-on-def* **by** (*auto intro*!: *field-differentiable-Polygamma*)

lemma *analytic-on-Polygamma*: $A \cap \mathbb{Z}_{\leq 0} = \{\} \Longrightarrow$ *Polygamma n analytic-on A*
by (*rule analytic-on-subset*[*of - UNIV* $-$ $\mathbb{Z}_{\leq 0}$], *subst analytic-on-open*)
(*auto intro*!: *holomorphic-on-Polygamma*)

The real Gamma function

lemma *rGamma-series-real*:
eventually (λn. *rGamma-series x n* = *Re* (*rGamma-series* (*of-real x*) *n*)) *sequentially*
using *eventually-gt-at-top*[*of 0* :: *nat*]
proof *eventually-elim*
fix *n* :: *nat* **assume** *n*: *n* > *0*
have *Re* (*rGamma-series* (*of-real x*) *n*) =
Re (*of-real* (*pochhammer x* (*Suc n*)) / (*fact n* $*$ *exp* (*of-real* (*x* $*$ *ln*

(*real-of-nat n*)))))
using *n* **by** (*simp add*: *rGamma-series-def powr-def Ln-of-nat pochhammer-of-real*)
also from *n* **have** ... = *Re* (*of-real* ((*pochhammer x* (*Suc n*)) /
(*fact n* * (*exp* (*x* * *ln* (*real-of-nat n*))))))
by (*subst exp-of-real*) *simp*
also from *n* **have** ... = *rGamma-series x n*
by (*subst Re-complex-of-real*) (*simp add*: *rGamma-series-def powr-def*)
finally show *rGamma-series x n* = *Re* (*rGamma-series* (*of-real x*) *n*) **..**
qed

instantiation *real* :: *Gamma*
begin

definition *rGamma-real x* = *Re* (*rGamma* (*of-real x* :: *complex*))

instance proof
fix *x* :: *real*
have *rGamma x* = *Re* (*rGamma* (*of-real x*)) **by** (*simp add*: *rGamma-real-def*)
also have *of-real* ... = *rGamma* (*of-real x* :: *complex*)
by (*intro of-real-Re rGamma-complex-real*) *simp-all*
also have ... = *0* ⟷ *x* ∈ $\mathbb{Z}_{\leq 0}$ **by** (*simp add*: *rGamma-eq-zero-iff of-real-in-nonpos-Ints-iff*)
also have ... ⟷ (∃ *n*. *x* = − *of-nat n*) **by** (*auto elim*!: *nonpos-Ints-cases'*)
finally show (*rGamma x*) = *0* ⟷ (∃ *n*. *x* = − *real-of-nat n*) **by** *simp*
next
fix *x* :: *real* **assume** ⋀*n*. *x* ≠ − *of-nat n*
hence *complex-of-real x* ∉ $\mathbb{Z}_{\leq 0}$
by (*subst of-real-in-nonpos-Ints-iff*) (*auto elim*!: *nonpos-Ints-cases'*)
moreover from *this* **have** *x* ≠ *0* **by** *auto*
ultimately have (*rGamma has-field-derivative* − *rGamma x* * *Digamma x*) (*at x*)
by (*fastforce intro*!: *derivative-eq-intros has-vector-derivative-real-complex*
simp: *Polygamma-of-real rGamma-real-def* [*abs-def*])
thus *let d* = (*THE d*. (λ*n*. ∑*k*<*n*. *inverse* (*of-nat* (*Suc k*)) − *inverse* (*x* + *of-nat k*))
⟶ *d*) − *euler-mascheroni* *$_R$ *1 in* (λ*y*. (*rGamma y* − *rGamma x* +
rGamma x * *d* * (*y* − *x*)) /$_R$ *norm* (*y* − *x*)) −*x*→ *0*
by (*simp add*: *has-field-derivative-def has-derivative-def Digamma-def sums-def* [*abs-def*]
netlimit-at of-real-def[*symmetric*] *suminf-def*)
next
fix *n* :: *nat*
have (*rGamma has-field-derivative* (−*1*) ^ *n* * *fact n*) (*at* (− *of-nat n* :: *real*))
by (*fastforce intro*!: *derivative-eq-intros has-vector-derivative-real-complex*
simp: *Polygamma-of-real rGamma-real-def* [*abs-def*])
thus *let x* = − *of-nat n in* (λ*y*. (*rGamma y* − *rGamma x* − (− *1*) ^ *n* * *setprod of-nat* {*1*..*n*} *
(*y* − *x*)) /$_R$ *norm* (*y* − *x*)) −*x*::*real*→ *0*
by (*simp add*: *has-field-derivative-def has-derivative-def fact-altdef netlimit-at*

Let-def)
next
fix x :: *real*
have *rGamma-series x* ⟶ *rGamma x*
proof (*rule Lim-transform-eventually*)
show (λ*n*. *Re* (*rGamma-series* (*of-real x*) *n*)) ⟶ *rGamma x* **unfolding** *rGamma-real-def*
by (*intro tendsto-intros*)
qed (*insert rGamma-series-real*, *simp add*: *eq-commute*)
thus *let fact′* = λ*n*. *setprod of-nat* {*1*..*n*};
exp = λ*x*. *THE e*. (λ*n*. ∑ *k*<*n*. *x* ^ *k* /$_R$ *fact k*) ⟶ *e*;
pochhammer′ = λ*a n*. ∏ *n* = *0*..*n*. *a* + *of-nat n*
in (λ*n*. *pochhammer′ x n* / (*fact′ n* ∗ *exp* (*x* ∗ *ln* (*real-of-nat n*) ∗$_R$ *1*)))
⟶ *rGamma x*
by (*simp add*: *fact-altdef pochhammer-Suc-setprod rGamma-series-def* [*abs-def*] *exp-def*
of-real-def [*symmetric*] *suminf-def sums-def* [*abs-def*])
qed

end

lemma *rGamma-complex-of-real*: *rGamma* (*complex-of-real x*) = *complex-of-real* (*rGamma x*)
unfolding *rGamma-real-def* **using** *rGamma-complex-real* **by** *simp*

lemma *Gamma-complex-of-real*: *Gamma* (*complex-of-real x*) = *complex-of-real* (*Gamma x*)
unfolding *Gamma-def* **by** (*simp add*: *rGamma-complex-of-real*)

lemma *rGamma-real-altdef*: *rGamma x* = *lim* (*rGamma-series* (*x* :: *real*))
by (*rule sym*, *rule limI*, *rule tendsto-intros*)

lemma *Gamma-real-altdef1*: *Gamma x* = *lim* (*Gamma-series* (*x* :: *real*))
by (*rule sym*, *rule limI*, *rule tendsto-intros*)

lemma *Gamma-real-altdef2*: *Gamma x* = *Re* (*Gamma* (*of-real x*))
using *rGamma-complex-real*[*OF Reals-of-real*[*of x*]]
by (*elim Reals-cases*)
(*simp only*: *Gamma-def rGamma-real-def of-real-inverse*[*symmetric*] *Re-complex-of-real*)

lemma *ln-Gamma-series-complex-of-real*:
x > *0* ⟹ *n* > *0* ⟹ *ln-Gamma-series* (*complex-of-real x*) *n* = *of-real* (*ln-Gamma-series x n*)
proof −
assume *xn*: *x* > *0 n* > *0*
have *Ln* (*complex-of-real x* / *of-nat k* + *1*) = *of-real* (*ln* (*x* / *of-nat k* + *1*)) **if** *k* ≥ *1* **for** *k*
using *that xn* **by** (*subst Ln-of-real* [*symmetric*]) (*auto intro*!: *add-nonneg-pos*

simp: *field-simps*)
with *xn* **show** *?thesis* **by** (*simp add*: *ln-Gamma-series-def Ln-of-nat Ln-of-real*)
qed

lemma *ln-Gamma-real-converges*:
assumes (*x*::*real*) > *0*
shows *convergent* (*ln-Gamma-series x*)
proof −
have (λ*n*. *ln-Gamma-series* (*complex-of-real x*) *n*) ⟶ *ln-Gamma* (*of-real x*)
using *assms*
by (*intro ln-Gamma-complex-LIMSEQ*) (*auto simp*: *of-real-in-nonpos-Ints-iff*)
moreover from *eventually-gt-at-top*[*of 0*::*nat*]
have *eventually* (λ*n*. *complex-of-real* (*ln-Gamma-series x n*) =
ln-Gamma-series (*complex-of-real x*) *n*) *sequentially*
by *eventually-elim* (*simp add*: *ln-Gamma-series-complex-of-real assms*)
ultimately have (λ*n*. *complex-of-real* (*ln-Gamma-series x n*)) ⟶ *ln-Gamma*
(*of-real x*)
by (*subst tendsto-cong*) *assumption*+
from *tendsto-Re*[*OF this*] **show** *?thesis* **by** (*auto simp*: *convergent-def*)
qed

lemma *ln-Gamma-real-LIMSEQ*: (*x*::*real*) > *0* ⟹ *ln-Gamma-series x* ⟶
ln-Gamma x
using *ln-Gamma-real-converges*[*of x*] **unfolding** *ln-Gamma-def* **by** (*simp add*:
convergent-LIMSEQ-iff)

lemma *ln-Gamma-complex-of-real*: *x* > *0* ⟹ *ln-Gamma* (*complex-of-real x*) =
of-real (*ln-Gamma x*)
proof (*unfold ln-Gamma-def*, *rule limI*, *rule Lim-transform-eventually*)
assume *x*: *x* > *0*
show *eventually* (λ*n*. *of-real* (*ln-Gamma-series x n*) =
ln-Gamma-series (*complex-of-real x*) *n*) *sequentially*
using *eventually-gt-at-top*[*of 0*::*nat*]
by *eventually-elim* (*simp add*: *ln-Gamma-series-complex-of-real x*)
qed (*intro tendsto-of-real*, *insert ln-Gamma-real-LIMSEQ*[*of x*], *simp add*: *ln-Gamma-def*)

lemma *Gamma-real-pos-exp*: *x* > (*0* :: *real*) ⟹ *Gamma x* = *exp* (*ln-Gamma x*)
by (*auto simp*: *Gamma-real-altdef2 Gamma-complex-altdef of-real-in-nonpos-Ints-iff*
ln-Gamma-complex-of-real exp-of-real)

lemma *ln-Gamma-real-pos*: *x* > *0* ⟹ *ln-Gamma x* = *ln* (*Gamma x* :: *real*)
unfolding *Gamma-real-pos-exp* **by** *simp*

lemma *Gamma-real-pos*: *x* > (*0*::*real*) ⟹ *Gamma x* > *0*
by (*simp add*: *Gamma-real-pos-exp*)

lemma *has-field-derivative-ln-Gamma-real* [*derivative-intros*]:
assumes *x*: *x* > (*0*::*real*)
shows (*ln-Gamma has-field-derivative Digamma x*) (*at x*)

proof (*subst DERIV-cong-ev*[*OF refl - refl*])
from *assms* **show** ((*Re* ∘ *ln-Gamma* ∘ *complex-of-real*) *has-field-derivative Digamma x*) (*at x*)
by (*auto intro*!: *derivative-eq-intros has-vector-derivative-real-complex*
simp: *Polygamma-of-real o-def*)
from *eventually-nhds-in-nhd*[*of x* {*0*<..}] *assms*
show *eventually* (λ*y*. *ln-Gamma y* = (*Re* ∘ *ln-Gamma* ∘ *of-real*) *y*) (*nhds x*)
by (*auto elim*!: *eventually-mono simp*: *ln-Gamma-complex-of-real interior-open*)
qed

declare *has-field-derivative-ln-Gamma-real*[*THEN DERIV-chain2*, *derivative-intros*]

lemma *has-field-derivative-rGamma-real'* [*derivative-intros*]:
(*rGamma has-field-derivative* (*if x* ∈ $\mathbb{Z}_{\leq 0}$ *then* (−*1*) ^ (*nat* ⌊−*x*⌋) ∗ *fact* (*nat* ⌊−*x*⌋) *else*
−*rGamma x* ∗ *Digamma x*)) (*at x within A*)
using *has-field-derivative-rGamma*[*of x*] **by** (*force elim*!: *nonpos-Ints-cases'*)

declare *has-field-derivative-rGamma-real'*[*THEN DERIV-chain2*, *derivative-intros*]

lemma *Polygamma-real-odd-pos*:
assumes (*x*::*real*) ∉ $\mathbb{Z}_{\leq 0}$ *odd n*
shows *Polygamma n x* > *0*
proof −
from *assms* **have** *x* ≠ *0* **by** *auto*
with *assms* **show** *?thesis*
unfolding *Polygamma-def* **using** *Polygamma-converges'*[*of x Suc n*]
by (*auto simp*: *zero-less-power-eq simp del*: *power-Suc*
dest: *plus-of-nat-eq-0-imp intro*!: *mult-pos-pos suminf-pos*)
qed

lemma *Polygamma-real-even-neg*:
assumes (*x*::*real*) > *0 n* > *0 even n*
shows *Polygamma n x* < *0*
using *assms* **unfolding** *Polygamma-def* **using** *Polygamma-converges'*[*of x Suc n*]
by (*auto intro*!: *mult-pos-pos suminf-pos*)

lemma *Polygamma-real-strict-mono*:
assumes *x* > *0 x* < (*y*::*real*) *even n*
shows *Polygamma n x* < *Polygamma n y*
proof −
have ∃ξ. *x* < ξ ∧ ξ < *y* ∧ *Polygamma n y* − *Polygamma n x* = (*y* − *x*) ∗ *Polygamma* (*Suc n*) ξ
using *assms* **by** (*intro MVT2 derivative-intros impI allI*) (*auto elim*!: *nonpos-Ints-cases*)
then guess ξ **by** (*elim exE conjE*) **note** ξ = *this*
note ξ(*3*)
also from ξ(*1*,*2*) *assms* **have** (*y* − *x*) ∗ *Polygamma* (*Suc n*) ξ > *0*

by (*intro mult-pos-pos Polygamma-real-odd-pos*) (*auto elim*!: *nonpos-Ints-cases*)
finally show *?thesis* **by** *simp*
qed

lemma *Polygamma-real-strict-antimono*:
assumes $x > 0$ $x < (y::real)$ *odd n*
shows *Polygamma n x > Polygamma n y*
proof −
have $\exists\xi.\ x < \xi \wedge \xi < y \wedge$ *Polygamma n y − Polygamma n x* $= (y - x) *$ *Polygamma* (*Suc n*) ξ
using *assms* **by** (*intro MVT2 derivative-intros impI allI*) (*auto elim*!: *nonpos-Ints-cases*)
then guess ξ **by** (*elim exE conjE*) **note** ξ = *this*
note $\xi(3)$
also from $\xi(1,2)$ *assms* **have** $(y - x) *$ *Polygamma* (*Suc n*) $\xi < 0$
by (*intro mult-pos-neg Polygamma-real-even-neg*) *simp-all*
finally show *?thesis* **by** *simp*
qed

lemma *Polygamma-real-mono*:
assumes $x > 0$ $x \leq (y::real)$ *even n*
shows *Polygamma n x ≤ Polygamma n y*
using *Polygamma-real-strict-mono*[*OF assms*(*1*) - *assms*(*3*), *of y*] *assms*(*2*)
by (*cases x = y*) *simp-all*

lemma *Digamma-real-ge-three-halves-pos*:
assumes $x \geq 3/2$
shows *Digamma* (*x* :: *real*) > *0*
proof −
have *0 < Digamma* (*3/2* :: *real*) **by** (*fact Digamma-real-three-halves-pos*)
also from *assms* **have** ... ≤ *Digamma x* **by** (*intro Polygamma-real-mono*) *simp-all*
finally show *?thesis* **.**
qed

lemma *ln-Gamma-real-strict-mono*:
assumes $x \geq 3/2$ $x < y$
shows *ln-Gamma* (*x* :: *real*) < *ln-Gamma y*
proof −
have $\exists\xi.\ x < \xi \wedge \xi < y \wedge$ *ln-Gamma y − ln-Gamma x* $= (y - x) *$ *Digamma* ξ
using *assms* **by** (*intro MVT2 derivative-intros impI allI*) (*auto elim*!: *nonpos-Ints-cases*)
then guess ξ **by** (*elim exE conjE*) **note** ξ = *this*
note $\xi(3)$
also from $\xi(1,2)$ *assms* **have** $(y - x) *$ *Digamma* $\xi > 0$
by (*intro mult-pos-pos Digamma-real-ge-three-halves-pos*) *simp-all*
finally show *?thesis* **by** *simp*
qed

lemma *Gamma-real-strict-mono*:

assumes $x \geq 3/2$ $x < y$
shows $Gamma\ (x :: real) < Gamma\ y$
proof −
from *Gamma-real-pos-exp*[*of* x] *assms* **have** $Gamma\ x = exp\ (ln\text{-}Gamma\ x)$ **by** *simp*
also have $\ldots < exp\ (ln\text{-}Gamma\ y)$ **by** (*intro exp-less-mono ln-Gamma-real-strict-mono assms*)
also from *Gamma-real-pos-exp*[*of* y] *assms* **have** $\ldots = Gamma\ y$ **by** *simp*
finally show *?thesis* **.**
qed

lemma *log-convex-Gamma-real*: *convex-on* $\{0<..\}$ $(ln \circ Gamma :: real \Rightarrow real)$
by (*rule convex-on-realI*[*of* - - *Digamma*])
(*auto intro*!: *derivative-eq-intros Polygamma-real-mono Gamma-real-pos*
simp: *o-def Gamma-eq-zero-iff elim*!: *nonpos-Ints-cases'*)

59.6 Beta function

definition *Beta* **where** $Beta\ a\ b = Gamma\ a * Gamma\ b\ /\ Gamma\ (a + b)$

lemma *Beta-altdef*: $Beta\ a\ b = Gamma\ a * Gamma\ b * rGamma\ (a + b)$
by (*simp add*: *inverse-eq-divide Beta-def Gamma-def*)

lemma *Beta-commute*: $Beta\ a\ b = Beta\ b\ a$
unfolding *Beta-def* **by** (*simp add*: *ac-simps*)

lemma *has-field-derivative-Beta1* [*derivative-intros*]:
assumes $x \notin \mathbb{Z}_{\leq 0}$ $x + y \notin \mathbb{Z}_{\leq 0}$
shows $((\lambda x.\ Beta\ x\ y)\ has\text{-}field\text{-}derivative\ (Beta\ x\ y * (Digamma\ x - Digamma\ (x + y))))$
$(at\ x\ within\ A)$ **unfolding** *Beta-altdef*
by (*rule DERIV-cong*, (*rule derivative-intros assms*)+) (*simp add*: *algebra-simps*)

lemma *has-field-derivative-Beta2* [*derivative-intros*]:
assumes $y \notin \mathbb{Z}_{\leq 0}$ $x + y \notin \mathbb{Z}_{\leq 0}$
shows $((\lambda y.\ Beta\ x\ y)\ has\text{-}field\text{-}derivative\ (Beta\ x\ y * (Digamma\ y - Digamma\ (x + y))))$
$(at\ y\ within\ A)$
using *has-field-derivative-Beta1*[*of* y x A] *assms* **by** (*simp add*: *Beta-commute add-ac*)

lemma *Beta-plus1-plus1*:
assumes $x \notin \mathbb{Z}_{\leq 0}$ $y \notin \mathbb{Z}_{\leq 0}$
shows $Beta\ (x + 1)\ y + Beta\ x\ (y + 1) = Beta\ x\ y$
proof −
have $Beta\ (x + 1)\ y + Beta\ x\ (y + 1) =$
$(Gamma\ (x + 1) * Gamma\ y + Gamma\ x * Gamma\ (y + 1)) * rGamma\ ((x + y) + 1)$
by (*simp add*: *Beta-altdef add-divide-distrib algebra-simps*)

also have ... = (*Gamma x* ∗ *Gamma y*) ∗ ((*x* + *y*) ∗ *rGamma* ((*x* + *y*) + *1*))
by (*subst assms*[*THEN Gamma-plus1*])+ (*simp add*: *algebra-simps*)
also from *assms* **have** ... = *Beta x y* **unfolding** *Beta-altdef* **by** (*subst rGamma-plus1*) *simp*
finally show *?thesis* **.**
qed

lemma *Beta-plus1-left*:
assumes *x* ∉ ℤ≤0 *y* ∉ ℤ≤0
shows (*x* + *y*) ∗ *Beta* (*x* + *1*) *y* = *x* ∗ *Beta x y*
proof −
have (*x* + *y*) ∗ *Beta* (*x* + *1*) *y* = *Gamma* (*x* + *1*) ∗ *Gamma y* ∗ ((*x* + *y*) ∗ *rGamma* ((*x* + *y*) + *1*))
unfolding *Beta-altdef* **by** (*simp only*: *ac-simps*)
also have ... = *x* ∗ *Beta x y* **unfolding** *Beta-altdef*
by (*subst assms*[*THEN Gamma-plus1*] *rGamma-plus1*)+ (*simp only*: *ac-simps*)
finally show *?thesis* **.**
qed

lemma *Beta-plus1-right*:
assumes *x* ∉ ℤ≤0 *y* ∉ ℤ≤0
shows (*x* + *y*) ∗ *Beta x* (*y* + *1*) = *y* ∗ *Beta x y*
using *Beta-plus1-left*[*of y x*] *assms* **by** (*simp add*: *Beta-commute add.commute*)

lemma *Gamma-Gamma-Beta*:
assumes *x* ∉ ℤ≤0 *y* ∉ ℤ≤0 *x* + *y* ∉ ℤ≤0
shows *Gamma x* ∗ *Gamma y* = *Beta x y* ∗ *Gamma* (*x* + *y*)
unfolding *Beta-altdef* **using** *assms Gamma-eq-zero-iff*[*of x*+*y*]
by (*simp add*: *rGamma-inverse-Gamma*)

59.7 Legendre duplication theorem

context
begin

private lemma *Gamma-legendre-duplication-aux*:
fixes *z* :: *'a* :: *Gamma*
assumes *z* ∉ ℤ≤0 *z* + *1/2* ∉ ℤ≤0
shows *Gamma z* ∗ *Gamma* (*z* + *1/2*) = *exp* ((*1* − *2*∗*z*) ∗ *of-real* (*ln 2*)) ∗ *Gamma* (*1/2*) ∗ *Gamma* (*2*∗*z*)
proof −
let *?powr* = λ*b a*. *exp* (*a* ∗ *of-real* (*ln* (*of-nat b*)))
let *?h* = λ*n*. (*fact* (*n*−*1*))^2 / *fact* (*2*∗*n*−*1*) ∗ *of-nat* (*2*^(*2*∗*n*)) ∗
exp (*1/2* ∗ *of-real* (*ln* (*real-of-nat n*)))
{
fix *z* :: *'a* **assume** *z*: *z* ∉ ℤ≤0 *z* + *1/2* ∉ ℤ≤0
let *?g* = λ*n*. *?powr 2* (*2*∗*z*) ∗ *Gamma-series' z n* ∗ *Gamma-series'* (*z* + *1/2*) *n* /
Gamma-series' (*2*∗*z*) (*2*∗*n*)

have *eventually* (λ*n*. *?g n* = *?h n*) *sequentially* **using** *eventually-gt-at-top*
proof *eventually-elim*
fix *n* :: *nat* **assume** *n*: *n* > *0*
let *?f* = *fact* (*n* − *1*) :: *'a* **and** *?f'* = *fact* (*2*∗*n* − *1*) :: *'a*
have *A*: *exp t* ∗ *exp t* = *exp* (*2*∗*t* :: *'a*) **for** *t* **by** (*subst exp-add* [*symmetric*]) *simp*
have *A*: *Gamma-series' z n* ∗ *Gamma-series'* (*z* + *1/2*) *n* = *?f*^*2* ∗ *?powr n* (*2*∗*z* + *1/2*) /
(*pochhammer z n* ∗ *pochhammer* (*z* + *1/2*) *n*)
by (*simp add*: *Gamma-series'-def exp-add ring-distribs power2-eq-square A mult-ac*)
have *B*: *Gamma-series'* (*2*∗*z*) (*2*∗*n*) =
?f' ∗ *?powr 2* (*2*∗*z*) ∗ *?powr n* (*2*∗*z*) /
(*of-nat* (*2*^(*2*∗*n*)) ∗ *pochhammer z n* ∗ *pochhammer* (*z*+*1/2*) *n*) **using** *n*
by (*simp add*: *Gamma-series'-def ln-mult exp-add ring-distribs pochhammer-double*)
from *z* **have** *pochhammer z n* ≠ *0* **by** (*auto dest*: *pochhammer-eq-0-imp-nonpos-Int*)
moreover from *z* **have** *pochhammer* (*z* + *1/2*) *n* ≠ *0* **by** (*auto dest*: *pochhammer-eq-0-imp-nonpos-Int*)
ultimately have *?powr 2* (*2*∗*z*) ∗ (*Gamma-series' z n* ∗ *Gamma-series'* (*z* + *1/2*) *n*) / *Gamma-series'* (*2*∗*z*) (*2*∗*n*) =
?f^*2* / *?f'* ∗ *of-nat* (*2*^(*2*∗*n*)) ∗ (*?powr n* ((*4*∗*z* + *1*)/*2*) ∗ *?powr n* (−*2*∗*z*))
using *n* **unfolding** *A B* **by** (*simp add*: *divide-simps exp-minus*)
also have *?powr n* ((*4*∗*z* + *1*)/*2*) ∗ *?powr n* (−*2*∗*z*) = *?powr n* (*1/2*)
by (*simp add*: *algebra-simps exp-add*[*symmetric*] *add-divide-distrib*)
finally show *?g n* = *?h n* **by** (*simp only*: *mult-ac*)
qed

moreover from *z double-in-nonpos-Ints-imp*[*of z*] **have** *2* ∗ *z* ∉ $\mathbb{Z}_{\leq 0}$ **by** *auto*
hence *?g* ⟶ *?powr 2* (*2*∗*z*) ∗ *Gamma z* ∗ *Gamma* (*z*+*1/2*) / *Gamma* (*2*∗*z*)
using *LIMSEQ-subseq-LIMSEQ*[*OF Gamma-series'-LIMSEQ*, *of op*∗*2 2*∗*z*]
by (*intro tendsto-intros Gamma-series'-LIMSEQ*)
(*simp-all add*: *o-def subseq-def Gamma-eq-zero-iff*)
ultimately have *?h* ⟶ *?powr 2* (*2*∗*z*) ∗ *Gamma z* ∗ *Gamma* (*z*+*1/2*) / *Gamma* (*2*∗*z*)
by (*rule Lim-transform-eventually*)
} **note** *lim* = *this*

from *assms double-in-nonpos-Ints-imp*[*of z*] **have** *z'*: *2* ∗ *z* ∉ $\mathbb{Z}_{\leq 0}$ **by** *auto*
from *fraction-not-in-ints*[*of 2 1*] **have** (*1/2* :: *'a*) ∉ $\mathbb{Z}_{\leq 0}$
by (*intro not-in-Ints-imp-not-in-nonpos-Ints*) *simp-all*
with *lim*[*of 1/2* :: *'a*] **have** *?h* ⟶ *2* ∗ *Gamma* (*1* / *2* :: *'a*) **by** (*simp add*: *exp-of-real*)
from *LIMSEQ-unique*[*OF this lim*[*OF assms*]] *z'* **show** *?thesis*
by (*simp add*: *divide-simps Gamma-eq-zero-iff ring-distribs exp-diff exp-of-real ac-simps*)
qed

private lemma *Gamma-reflection-aux*:
defines $h \equiv \lambda z{::}complex.\ if\ z \in \mathbb{Z}\ then\ 0\ else$
$(of\text{-}real\ pi * cot\ (of\text{-}real\ pi{*}z) + Digamma\ z - Digamma\ (1 - z))$
defines $a \equiv complex\text{-}of\text{-}real\ pi$
obtains h' **where** *continuous-on UNIV* h' $\bigwedge z.\ (h\ has\text{-}field\text{-}derivative\ (h'\ z))\ (at\ z)$
proof −
def $f \equiv \lambda n.\ a * of\text{-}real\ (cos\text{-}coeff\ (n{+}1) - sin\text{-}coeff\ (n{+}2))$
def $F \equiv \lambda z.\ if\ z = 0\ then\ 0\ else\ (cos\ (a{*}z) - sin\ (a{*}z)/(a{*}z))\ /\ z$
def $g \equiv \lambda n.\ complex\text{-}of\text{-}real\ (sin\text{-}coeff\ (n{+}1))$
def $G \equiv \lambda z.\ if\ z = 0\ then\ 1\ else\ sin\ (a{*}z)/(a{*}z)$
have *a-nz*: $a \neq 0$ **unfolding** *a-def* **by** *simp*

have $(\lambda n.\ f\ n * (a{*}z)\ \hat{}\ n)\ sums\ (F\ z) \wedge (\lambda n.\ g\ n * (a{*}z)\ \hat{}\ n)\ sums\ (G\ z)$
if $abs\ (Re\ z) < 1$ **for** z
proof (*cases* $z = 0$; *rule conjI*)
assume $z \neq 0$
note $z = this\ that$

from z **have** *sin-nz*: $sin\ (a{*}z) \neq 0$ **unfolding** *a-def* **by** (*auto simp*: *sin-eq-0*)
have $(\lambda n.\ of\text{-}real\ (sin\text{-}coeff\ n) * (a{*}z)\ \hat{}\ n)\ sums\ (sin\ (a{*}z))$ **using** *sin-converges*[*of* $a{*}z$]
by (*simp add*: *scaleR-conv-of-real*)
from *sums-split-initial-segment*[*OF this*, *of 1*]
have $(\lambda n.\ (a{*}z) * of\text{-}real\ (sin\text{-}coeff\ (n{+}1)) * (a{*}z)\ \hat{}\ n)\ sums\ (sin\ (a{*}z))$ **by** (*simp add*: *mult-ac*)
from *sums-mult*[*OF this*, *of inverse* $(a{*}z)$] z *a-nz*
have *A*: $(\lambda n.\ g\ n * (a{*}z)\ \hat{}\ n)\ sums\ (sin\ (a{*}z)/(a{*}z))$
by (*simp add*: *field-simps g-def*)
with z **show** $(\lambda n.\ g\ n * (a{*}z)\ \hat{}\ n)\ sums\ (G\ z)$ **by** (*simp add*: *G-def*)
from *A* z *a-nz sin-nz* **have** *g-nz*: $(\sum n.\ g\ n * (a{*}z)\ \hat{}\ n) \neq 0$ **by** (*simp add*: *sums-iff g-def*)

have [*simp*]: $sin\text{-}coeff\ (Suc\ 0) = 1$ **by** (*simp add*: *sin-coeff-def*)
from *sums-split-initial-segment*[*OF sums-diff*[*OF cos-converges*[*of* $a{*}z$] *A*], *of 1*]
have $(\lambda n.\ z * f\ n * (a{*}z)\ \hat{}\ n)\ sums\ (cos\ (a{*}z) - sin\ (a{*}z)\ /\ (a{*}z))$
by (*simp add*: *mult-ac scaleR-conv-of-real ring-distribs f-def g-def*)
from *sums-mult*[*OF this*, *of inverse z*] z *assms*
show $(\lambda n.\ f\ n * (a{*}z)\ \hat{}\ n)\ sums\ (F\ z)$ **by** (*simp add*: *divide-simps mult-ac f-def F-def*)
next
assume z: $z = 0$
have $(\lambda n.\ f\ n * (a * z)\ \hat{}\ n)\ sums\ f\ 0$ **using** *powser-sums-zero*[*of f*] z **by** *simp*
with z **show** $(\lambda n.\ f\ n * (a * z)\ \hat{}\ n)\ sums\ (F\ z)$
by (*simp add*: *f-def F-def sin-coeff-def cos-coeff-def*)
have $(\lambda n.\ g\ n * (a * z)\ \hat{}\ n)\ sums\ g\ 0$ **using** *powser-sums-zero*[*of g*] z **by** *simp*
with z **show** $(\lambda n.\ g\ n * (a * z)\ \hat{}\ n)\ sums\ (G\ z)$

```
    by (simp add: g-def G-def sin-coeff-def cos-coeff-def)
  qed
  note sums = conjunct1[OF this] conjunct2[OF this]

  def h2 ≡ λz. (∑ n. f n * (a*z) ^ n) / (∑ n. g n * (a*z) ^ n) +
          Digamma (1 + z) − Digamma (1 − z)
  def POWSER ≡ λf z. (∑ n. f n * (z ^ n :: complex))
  def POWSER' ≡ λf z. (∑ n. diffs f n * (z ^ n :: complex))

  def h2' ≡ λz. a * (POWSER g (a*z) * POWSER' f (a*z) − POWSER f (a*z)
* POWSER' g (a*z)) /
                (POWSER g (a*z)) ^ 2 + Polygamma 1 (1 + z) + Polygamma 1
(1 − z)

  have h-eq: h t = h2 t if abs (Re t) < 1 for t
  proof −
   from that have t: t ∈ ℤ ⟷ t = 0 by (auto elim!: Ints-cases simp: dist-0-norm)
    hence h t = a*cot (a*t) − 1/t + Digamma (1 + t) − Digamma (1 − t)
     unfolding h-def using Digamma-plus1[of t] by (force simp: field-simps a-def)
    also have a*cot (a*t) − 1/t = (F t) / (G t)
      using t by (auto simp add: divide-simps sin-eq-0 cot-def a-def F-def G-def)
    also have ... = (∑ n. f n * (a*t) ^ n) / (∑ n. g n * (a*t) ^ n)
      using sums[of t] that by (simp add: sums-iff dist-0-norm)
    finally show h t = h2 t by (simp only: h2-def)
  qed

  let ?A = {z. abs (Re z) < 1}
  have open ({z. Re z < 1} ∩ {z. Re z > −1})
    using open-halfspace-Re-gt open-halfspace-Re-lt by auto
  also have ({z. Re z < 1} ∩ {z. Re z > −1}) = {z. abs (Re z) < 1} by auto
  finally have open-A: open ?A .
  hence [simp]: interior ?A = ?A by (simp add: interior-open)

  have summable-f: summable (λn. f n * z ^ n) for z
    by (rule powser-inside, rule sums-summable, rule sums[of ı * of-real (norm z
+ 1) / a])
       (simp-all add: norm-mult a-def del: of-real-add)
  have summable-g: summable (λn. g n * z ^ n) for z
    by (rule powser-inside, rule sums-summable, rule sums[of ı * of-real (norm z
+ 1) / a])
       (simp-all add: norm-mult a-def del: of-real-add)
   have summable-fg': summable (λn. diffs f n * z ^ n) summable (λn. diffs g n *
z ^ n) for z
    by (intro termdiff-converges-all summable-f summable-g)+
  have (POWSER f has-field-derivative (POWSER' f z)) (at z)
            (POWSER g has-field-derivative (POWSER' g z)) (at z) for z
    unfolding POWSER-def POWSER'-def
    by (intro termdiffs-strong-converges-everywhere summable-f summable-g)+
  note derivs = this[THEN DERIV-chain2[OF - DERIV-cmult[OF DERIV-ident]],
```

unfolded POWSER-def]
have *isCont* (*POWSER f*) *z isCont* (*POWSER g*) *z isCont* (*POWSER' f*) *z isCont* (*POWSER' g*) *z*
for *z* **unfolding** *POWSER-def POWSER'-def*
by (*intro isCont-powser-converges-everywhere summable-f summable-g summable-fg'*)+
note *cont* = *this*[*THEN isCont-o2*[*rotated*], *unfolded POWSER-def POWSER'-def*]

{
fix *z* :: *complex* **assume** *z*: *abs* (*Re z*) < *1*
def *d* ≡ i ∗ *of-real* (*norm z* + *1*)
have *d*: *abs* (*Re d*) < *1 norm z* < *norm d* **by** (*simp-all add*: *d-def norm-mult del*: *of-real-add*)
have *eventually* (λ*z*. *h z* = *h2 z*) (*nhds z*)
using *eventually-nhds-in-nhd*[*of z ?A*] **using** *h-eq z*
by (*auto elim*!: *eventually-mono simp*: *dist-0-norm*)

moreover from *sums*(*2*)[*OF z*] *z* **have** *nz*: (∑ *n*. *g n* ∗ (*a* ∗ *z*) ^ *n*) ≠ *0*
unfolding *G-def* **by** (*auto simp*: *sums-iff sin-eq-0 a-def*)
have *A*: *z* ∈ ℤ ⟷ *z* = *0* **using** *z* **by** (*auto elim*!: *Ints-cases*)
have *no-int*: *1* + *z* ∈ ℤ ⟷ *z* = *0* **using** *z Ints-diff*[*of 1*+*z 1*] *A*
by (*auto elim*!: *nonpos-Ints-cases*)
have *no-int'*: *1* − *z* ∈ ℤ ⟷ *z* = *0* **using** *z Ints-diff*[*of 1 1*−*z*] *A*
by (*auto elim*!: *nonpos-Ints-cases*)
from *no-int no-int'* **have** *no-int*: *1* − *z* ∉ ℤ≤0 *1* + *z* ∉ ℤ≤0 **by** *auto*
have (*h2 has-field-derivative h2' z*) (*at z*) **unfolding** *h2-def*
by (*rule DERIV-cong*, (*rule derivative-intros refl derivs*[*unfolded POWSER-def*] *nz no-int*)+)
(*auto simp*: *h2'-def POWSER-def field-simps power2-eq-square*)
ultimately have *deriv*: (*h has-field-derivative h2' z*) (*at z*)
by (*subst DERIV-cong-ev*[*OF refl - refl*])

from *sums*(*2*)[*OF z*] *z* **have** (∑ *n*. *g n* ∗ (*a* ∗ *z*) ^ *n*) ≠ *0*
unfolding *G-def* **by** (*auto simp*: *sums-iff a-def sin-eq-0*)
hence *isCont h2' z* **using** *no-int* **unfolding** *h2'-def*[*abs-def*] *POWSER-def POWSER'-def*
by (*intro continuous-intros cont*
continuous-on-compose2[*OF - continuous-on-Polygamma*[*of* {*z*. *Re z* > *0*}]]) *auto*
note *deriv* **and** *this*
} **note** *A* = *this*

interpret *h*: *periodic-fun-simple' h*
proof
fix *z* :: *complex*
show *h* (*z* + *1*) = *h z*
proof (*cases z* ∈ ℤ)
assume *z*: *z* ∉ ℤ
hence *A*: *z* + *1* ∉ ℤ *z* ≠ *0* **using** *Ints-diff*[*of z+1 1*] **by** *auto*
hence *Digamma* (*z* + *1*) − *Digamma* (−*z*) = *Digamma z* − *Digamma* (−*z*

```
+ 1)
      by (subst (1 2) Digamma-plus1) simp-all
    with A z show h (z + 1) = h z
      by (simp add: h-def sin-plus-pi cos-plus-pi ring-distribs cot-def)
  qed (simp add: h-def)
 qed

 have h2'-eq: h2' (z − 1) = h2' z if z: Re z > 0 Re z < 1 for z
 proof −
  have ((λz. h (z − 1)) has-field-derivative h2' (z − 1)) (at z)
   by (rule DERIV-cong, rule DERIV-chain'[OF - A(1)])
     (insert z, auto intro!: derivative-eq-intros)
  hence (h has-field-derivative h2' (z − 1)) (at z) by (subst (asm) h.minus-1)
   moreover from z have (h has-field-derivative h2' z) (at z) by (intro A)
simp-all
  ultimately show h2' (z − 1) = h2' z by (rule DERIV-unique)
 qed

 def h2'' ≡ λz. h2' (z − of-int ⌊Re z⌋)
 have deriv: (h has-field-derivative h2'' z) (at z) for z
 proof −
  fix z :: complex
  have B: |Re z − real-of-int ⌊Re z⌋| < 1 by linarith
  have ((λt. h (t − of-int ⌊Re z⌋)) has-field-derivative h2'' z) (at z)
   unfolding h2''-def by (rule DERIV-cong, rule DERIV-chain'[OF - A(1)])
                         (insert B, auto intro!: derivative-intros)
  thus (h has-field-derivative h2'' z) (at z) by (simp add: h.minus-of-int)
 qed

 have cont: continuous-on UNIV h2''
 proof (intro continuous-at-imp-continuous-on ballI)
  fix z :: complex
  def r ≡ ⌊Re z⌋
  def A ≡ {t. of-int r − 1 < Re t ∧ Re t < of-int r + 1}
  have continuous-on A (λt. h2' (t − of-int r)) unfolding A-def
  by (intro continuous-at-imp-continuous-on isCont-o2[OF - A(2)] ballI continuous-intros)
      (simp-all add: abs-real-def)
  moreover have h2'' t = h2' (t − of-int r) if t: t ∈ A for t
  proof (cases Re t ≥ of-int r)
   case True
    from t have of-int r − 1 < Re t Re t < of-int r + 1 by (simp-all add:
A-def)
    with True have ⌊Re t⌋ = ⌊Re z⌋ unfolding r-def by linarith
    thus ?thesis by (auto simp: r-def h2''-def)
  next
   case False
    from t have t: of-int r − 1 < Re t Re t < of-int r + 1 by (simp-all add:
A-def)
    with False have t': ⌊Re t⌋ = ⌊Re z⌋ − 1 unfolding r-def by linarith
```

moreover from *t False* **have** *h2′ (t − of-int r + 1 − 1) = h2′ (t − of-int r + 1)*
by (*intro h2′-eq*) *simp-all*
ultimately show *?thesis* **by** (*auto simp*: *r-def h2′′-def algebra-simps t′*)
qed
ultimately have *continuous-on A h2′′* **by** (*subst continuous-on-cong*[*OF refl*])
moreover {
have *open* ({*t*. *of-int r − 1 < Re t*} ∩ {*t*. *of-int r + 1 > Re t*})
by (*intro open-Int open-halfspace-Re-gt open-halfspace-Re-lt*)
also have {*t*. *of-int r − 1 < Re t*} ∩ {*t*. *of-int r + 1 > Re t*} = *A*
unfolding *A-def* **by** *blast*
finally have *open A* **.**
}
ultimately have *C*: *isCont h2′′ t* **if** *t* ∈ *A* **for** *t* **using** *that*
by (*subst* (*asm*) *continuous-on-eq-continuous-at*) *auto*
have *of-int r − 1 < Re z Re z < of-int r + 1* **unfolding** *r-def* **by** *linarith*+
thus *isCont h2′′ z* **by** (*intro C*) (*simp-all add*: *A-def*)
qed

from *that*[*OF cont deriv*] **show** *?thesis* **.**
qed

lemma *Gamma-reflection-complex*:
fixes *z* :: *complex*
shows *Gamma z * Gamma (1 − z) = of-real pi / sin (of-real pi * z)*
proof −
let *?g* = *λz*::*complex*. *Gamma z * Gamma (1 − z) * sin (of-real pi * z)*
def *g* ≡ *λz*::*complex*. *if z* ∈ ℤ *then of-real pi else ?g z*
let *?h* = *λz*::*complex*. (*of-real pi * cot (of-real pi*z) + Digamma z − Digamma (1 − z)*)
def *h* ≡ *λz*::*complex*. *if z* ∈ ℤ *then 0 else ?h z*

— *g* is periodic with period 1.
interpret *g*: *periodic-fun-simple′ g*
proof
fix *z* :: *complex*
show *g (z + 1) = g z*
proof (*cases z* ∈ ℤ)
case *False*
hence *z * g z = z * Beta z (− z + 1) * sin (of-real pi * z)* **by** (*simp add*: *g-def Beta-def*)
also have *z * Beta z (− z + 1) = (z + 1 + −z) * Beta (z + 1) (− z + 1)*
using *False Ints-diff*[*of 1 1 − z*] *nonpos-Ints-subset-Ints*
by (*subst Beta-plus1-left* [*symmetric*]) *auto*
also have *... * sin (of-real pi * z) = z * (Beta (z + 1) (−z) * sin (of-real pi * (z + 1)))*
using *False Ints-diff*[*of z+1 1*] *Ints-minus*[*of −z*] *nonpos-Ints-subset-Ints*
by (*subst Beta-plus1-right*) (*auto simp*: *ring-distribs sin-plus-pi*)
also from *False* **have** *Beta (z + 1) (−z) * sin (of-real pi * (z + 1)) = g (z*

```
+ 1)
      using Ints-diff[of z+1 1] by (auto simp: g-def Beta-def)
    finally show g (z + 1) = g z using False by (subst (asm) mult-left-cancel)
auto
  qed (simp add: g-def)
 qed

 — g is entire.
 have g-g': (g has-field-derivative (h z * g z)) (at z) for z :: complex
 proof (cases z ∈ ℤ)
  let ?h' = λz. Beta z (1 − z) * ((Digamma z − Digamma (1 − z)) * sin (z *
of-real pi) +
                    of-real pi * cos (z * of-real pi))
  case False
  from False have eventually (λt. t ∈ UNIV − ℤ) (nhds z)
    by (intro eventually-nhds-in-open) (auto simp: open-Diff)
  hence eventually (λt. g t = ?g t) (nhds z) by eventually-elim (simp add: g-def)
  moreover {
    from False Ints-diff[of 1 1−z] have 1 − z ∉ ℤ by auto
    hence (?g has-field-derivative ?h' z) (at z) using nonpos-Ints-subset-Ints
      by (auto intro!: derivative-eq-intros simp: algebra-simps Beta-def)
    also from False have sin (of-real pi * z) ≠ 0 by (subst sin-eq-0) auto
    hence ?h' z = h z * g z
    using False unfolding g-def h-def cot-def by (simp add: field-simps Beta-def)
    finally have (?g has-field-derivative (h z * g z)) (at z) .
  }
  ultimately show ?thesis by (subst DERIV-cong-ev[OF refl - refl])
 next
  case True
  then obtain n where z: z = of-int n by (auto elim!: Ints-cases)
  let ?t = (λz::complex. if z = 0 then 1 else sin z / z) ∘ (λz. of-real pi * z)
  have deriv-0: (g has-field-derivative 0) (at 0)
  proof (subst DERIV-cong-ev[OF refl - refl])
    show eventually (λz. g z = of-real pi * Gamma (1 + z) * Gamma (1 − z)
* ?t z) (nhds 0)
      using eventually-nhds-ball[OF zero-less-one, of 0::complex]
    proof eventually-elim
      fix z :: complex assume z: z ∈ ball 0 1
      show g z = of-real pi * Gamma (1 + z) * Gamma (1 − z) * ?t z
      proof (cases z = 0)
        assume z': z ≠ 0
          with z have z'': z ∉ ℤ≤0 z ∉ ℤ by (auto elim!: Ints-cases simp:
dist-0-norm)
        from Gamma-plus1[OF this(1)] have Gamma z = Gamma (z + 1) / z
by simp
        with z'' z' show ?thesis by (simp add: g-def ac-simps)
      qed (simp add: g-def)
    qed
    have (?t has-field-derivative (0 * of-real pi)) (at 0)
```

using *has-field-derivative-sin-z-over-z*[*of UNIV* :: *complex set*]
by (*intro DERIV-chain*) *simp-all*
thus ((λ*z*. *of-real pi* ∗ *Gamma* (*1* + *z*) ∗ *Gamma* (*1* − *z*) ∗ *?t z*) *has-field-derivative* *0*) (*at 0*)
by (*auto intro*!: *derivative-eq-intros simp*: *o-def*)
qed

have ((*g* ∘ (λ*x*. *x* − *of-int n*)) *has-field-derivative 0* ∗ *1*) (*at* (*of-int n*))
using *deriv-0* **by** (*intro DERIV-chain*) (*auto intro*!: *derivative-eq-intros*)
also have *g* ∘ (λ*x*. *x* − *of-int n*) = *g* **by** (*intro ext*) (*simp add*: *g.minus-of-int*)
finally show (*g has-field-derivative* (*h z* ∗ *g z*)) (*at z*) **by** (*simp add*: *z h-def*)
qed

have *g-eq*: *g* (*z*/*2*) ∗ *g* ((*z*+*1*)/*2*) = *Gamma* (*1*/*2*) ^*2* ∗ *g z* **if** *Re z* > −*1 Re z* < *2* **for** *z*
proof (*cases z* ∈ ℤ)
case *True*
with *that* **have** *z* = *0* ∨ *z* = *1* **by** (*force elim*!: *Ints-cases*)
moreover have *g 0* ∗ *g* (*1*/*2*) = *Gamma* (*1*/*2*) ^*2* ∗ *g 0*
using *fraction-not-in-ints*[**where** *'a* = *complex*, *of 2 1*] **by** (*simp add*: *g-def power2-eq-square*)
moreover have *g* (*1*/*2*) ∗ *g 1* = *Gamma* (*1*/*2*) ^*2* ∗ *g 1*
using *fraction-not-in-ints*[**where** *'a* = *complex*, *of 2 1*]
by (*simp add*: *g-def power2-eq-square Beta-def algebra-simps*)
ultimately show *?thesis* **by** *force*
next
case *False*
hence *z*: *z*/*2* ∉ ℤ (*z*+*1*)/*2* ∉ ℤ **using** *Ints-diff*[*of z*+*1 1*] **by** (*auto elim*!: *Ints-cases*)
hence *z'*: *z*/*2* ∉ ℤ≤0 (*z*+*1*)/*2* ∉ ℤ≤0 **by** (*auto elim*!: *nonpos-Ints-cases*)
from *z* **have** *1*−*z*/*2* ∉ ℤ *1*−((*z*+*1*)/*2*) ∉ ℤ
using *Ints-diff*[*of 1 1*−*z*/*2*] *Ints-diff*[*of 1 1*−((*z*+*1*)/*2*)] **by** *auto*
hence *z''*: *1*−*z*/*2* ∉ ℤ≤0 *1*−((*z*+*1*)/*2*) ∉ ℤ≤0 **by** (*auto elim*!: *nonpos-Ints-cases*)
from *z* **have** *g* (*z*/*2*) ∗ *g* ((*z*+*1*)/*2*) =
(*Gamma* (*z*/*2*) ∗ *Gamma* ((*z*+*1*)/*2*)) ∗ (*Gamma* (*1*−*z*/*2*) ∗ *Gamma* (*1*−((*z*+*1*)/*2*))) ∗
(*sin* (*of-real pi* ∗ *z*/*2*) ∗ *sin* (*of-real pi* ∗ (*z*+*1*)/*2*))
by (*simp add*: *g-def*)
also from *z' Gamma-legendre-duplication-aux*[*of z*/*2*]
have *Gamma* (*z*/*2*) ∗ *Gamma* ((*z*+*1*)/*2*) = *exp* ((*1*−*z*) ∗ *of-real* (*ln 2*)) ∗ *Gamma* (*1*/*2*) ∗ *Gamma z*
by (*simp add*: *add-divide-distrib*)
also from *z''* *Gamma-legendre-duplication-aux*[*of 1*−(*z*+*1*)/*2*]
have *Gamma* (*1*−*z*/*2*) ∗ *Gamma* (*1*−(*z*+*1*)/*2*) =
Gamma (*1*−*z*) ∗ *Gamma* (*1*/*2*) ∗ *exp* (*z* ∗ *of-real* (*ln 2*))
by (*simp add*: *add-divide-distrib ac-simps*)
finally have *g* (*z*/*2*) ∗ *g* ((*z*+*1*)/*2*) = *Gamma* (*1*/*2*) ^*2* ∗ (*Gamma z* ∗ *Gamma* (*1*−*z*) ∗
(*2* ∗ (*sin* (*of-real pi*∗*z*/*2*) ∗ *sin* (*of-real pi*∗(*z*+*1*)/*2*))))

```
    by (simp add: add-ac power2-eq-square exp-add ring-distribs exp-diff exp-of-real)
    also have sin (of-real pi*(z+1)/2) = cos (of-real pi*z/2)
      using cos-sin-eq[of − of-real pi * z/2, symmetric]
      by (simp add: ring-distribs add-divide-distrib ac-simps)
    also have 2 * (sin (of-real pi*z/2) * cos (of-real pi*z/2)) = sin (of-real pi *
z)
      by (subst sin-times-cos) (simp add: field-simps)
    also have Gamma z * Gamma (1 − z) * sin (complex-of-real pi * z) = g z
      using ‹z ∉ ℤ› by (simp add: g-def)
    finally show ?thesis .
  qed
  have g-eq: g (z/2) * g ((z+1)/2) = Gamma (1/2) ^2 * g z for z
  proof −
    def r ≡ ⌊Re z / 2⌋
    have Gamma (1/2) ^2 * g z = Gamma (1/2) ^2 * g (z − of-int (2*r)) by
(simp only: g.minus-of-int)
    also have of-int (2*r) = 2 * of-int r by simp
    also have Re z − 2 * of-int r > −1 Re z − 2 * of-int r < 2 unfolding r-def
by linarith+
    hence Gamma (1/2) ^2 * g (z − 2 * of-int r) =
                g ((z − 2 * of-int r)/2) * g ((z − 2 * of-int r + 1)/2)
      unfolding r-def by (intro g-eq[symmetric]) simp-all
    also have (z − 2 * of-int r) / 2 = z/2 − of-int r by simp
    also have g ... = g (z/2) by (rule g.minus-of-int)
    also have (z − 2 * of-int r + 1) / 2 = (z + 1)/2 − of-int r by simp
    also have g ... = g ((z+1)/2) by (rule g.minus-of-int)
    finally show ?thesis ..
  qed

  have g-nz [simp]: g z ≠ 0 for z :: complex
  unfolding g-def using Ints-diff[of 1 1 − z]
    by (auto simp: Gamma-eq-zero-iff sin-eq-0 dest!: nonpos-Ints-Int)

  have h-eq: h z = (h (z/2) + h ((z+1)/2)) / 2 for z
  proof −
    have ((λt. g (t/2) * g ((t+1)/2)) has-field-derivative
                  (g (z/2) * g ((z+1)/2)) * ((h (z/2) + h ((z+1)/2)) / 2)) (at
z)
      by (auto intro!: derivative-eq-intros g-g'[THEN DERIV-chain2] simp: field-simps)
    hence ((λt. Gamma (1/2) ^2 * g t) has-field-derivative
           Gamma (1/2) ^2 * g z * ((h (z/2) + h ((z+1)/2)) / 2)) (at z)
      by (subst (1 2) g-eq[symmetric]) simp
    from DERIV-cmult[OF this, of inverse ((Gamma (1/2)) ^2)]
      have (g has-field-derivative (g z * ((h (z/2) + h ((z+1)/2))/2))) (at z)
      using fraction-not-in-ints[where 'a = complex, of 2 1]
    by (simp add: divide-simps Gamma-eq-zero-iff not-in-Ints-imp-not-in-nonpos-Ints)
    moreover have (g has-field-derivative (g z * h z)) (at z)
      using g-g'[of z] by (simp add: ac-simps)
    ultimately have g z * h z = g z * ((h (z/2) + h ((z+1)/2))/2)
```

```
    by (intro DERIV-unique)
  thus h z = (h (z/2) + h ((z+1)/2)) / 2 by simp
 qed

 obtain h' where h'-cont: continuous-on UNIV h' and
              h-h': ⋀z. (h has-field-derivative h' z) (at z)
   unfolding h-def by (erule Gamma-reflection-aux)

 have h'-eq: h' z = (h' (z/2) + h' ((z+1)/2)) / 4 for z
 proof −
  have ((λt. (h (t/2) + h ((t+1)/2)) / 2) has-field-derivative
                ((h' (z/2) + h' ((z+1)/2)) / 4)) (at z)
    by (fastforce intro!: derivative-eq-intros h-h'[THEN DERIV-chain2])
  hence (h has-field-derivative ((h' (z/2) + h' ((z+1)/2))/4)) (at z)
    by (subst (asm) h-eq[symmetric])
   from h-h' and this show h' z = (h' (z/2) + h' ((z+1)/2)) / 4 by (rule
DERIV-unique)
 qed

 have h'-zero: h' z = 0 for z
 proof −
  def m ≡ max 1 |Re z|
  def B ≡ {t. abs (Re t) ≤ m ∧ abs (Im t) ≤ abs (Im z)}
  have closed ({t. Re t ≥ −m} ∩ {t. Re t ≤ m} ∩
              {t. Im t ≥ −|Im z|} ∩ {t. Im t ≤ |Im z|})
   (is closed ?B) by (intro closed-Int closed-halfspace-Re-ge closed-halfspace-Re-le
                          closed-halfspace-Im-ge closed-halfspace-Im-le)
  also have ?B = B unfolding B-def by fastforce
  finally have closed B .
  moreover have bounded B unfolding bounded-iff
  proof (intro ballI exI)
   fix t assume t: t ∈ B
   have norm t ≤ |Re t| + |Im t| by (rule cmod-le)
   also from t have |Re t| ≤ m unfolding B-def by blast
   also from t have |Im t| ≤ |Im z| unfolding B-def by blast
   finally show norm t ≤ m + |Im z| by − simp
  qed
   ultimately have compact: compact B by (subst compact-eq-bounded-closed)
blast

  def M ≡ SUP z:B. norm (h' z)
  have compact (h' ‘ B)
    by (intro compact-continuous-image continuous-on-subset[OF h'-cont] com-
pact) blast+
  hence bdd: bdd-above ((λz. norm (h' z)) ‘ B)
   using bdd-above-norm[of h' ‘ B] by (simp add: image-comp o-def compact-imp-bounded)
  have norm (h' z) ≤ M unfolding M-def by (intro cSUP-upper bdd) (simp-all
add: B-def m-def)
  also have M ≤ M/2
```

```
    proof (subst M-def, subst cSUP-le-iff)
      have z ∈ B unfolding B-def m-def by simp
      thus B ≠ {} by auto
    next
      show ∀ z∈B. norm (h' z) ≤ M/2
      proof
        fix t :: complex assume t: t ∈ B
         from h'-eq[of t] t have h' t = (h' (t/2) + h' ((t+1)/2)) / 4 by (simp
add: dist-0-norm)
        also have norm ... = norm (h' (t/2) + h' ((t+1)/2)) / 4 by simp
        also have norm (h' (t/2) + h' ((t+1)/2)) ≤ norm (h' (t/2)) + norm (h'
((t+1)/2))
          by (rule norm-triangle-ineq)
        also from t have abs (Re ((t + 1)/2)) ≤ m unfolding m-def B-def by
auto
        with t have t/2 ∈ B (t+1)/2 ∈ B unfolding B-def by auto
       hence norm (h' (t/2)) + norm (h' ((t+1)/2)) ≤ M + M unfolding M-def
          by (intro add-mono cSUP-upper bdd) (auto simp: B-def)
        also have (M + M) / 4 = M / 2 by simp
        finally show norm (h' t) ≤ M/2 by − simp-all
      qed
    qed (insert bdd, auto simp: cball-eq-empty)
    hence M ≤ 0 by simp
    finally show h' z = 0 by simp
  qed
  have h-h'-2: (h has-field-derivative 0) (at z) for z
    using h-h'[of z] h'-zero[of z] by simp

  have g-real: g z ∈ ℝ if z ∈ ℝ for z
    unfolding g-def using that by (auto intro!: Reals-mult Gamma-complex-real)
  have h-real: h z ∈ ℝ if z ∈ ℝ for z
     unfolding h-def using that by (auto intro!: Reals-mult Reals-add Reals-diff
Polygamma-Real)
  have g-nz: g z ≠ 0 for z unfolding g-def using Ints-diff[of 1 1−z]
    by (auto simp: Gamma-eq-zero-iff sin-eq-0)

  from h'-zero h-h'-2 have ∃ c. ∀ z∈UNIV. h z = c
    by (intro has-field-derivative-zero-constant) (simp-all add: dist-0-norm)
  then obtain c where c: ⋀z. h z = c by auto
  have ∃ u. u ∈ closed-segment 0 1 ∧ Re (g 1) − Re (g 0) = Re (h u ∗ g u ∗ (1
− 0))
    by (intro complex-mvt-line g-g')
    find-theorems name:deriv Reals
  then guess u by (elim exE conjE) note u = this
  from u(1) have u': u ∈ ℝ unfolding closed-segment-def
    by (auto simp: scaleR-conv-of-real)
  from u' g-real[of u] g-nz[of u] have Re (g u) ≠ 0 by (auto elim!: Reals-cases)
  with u(2) c[of u] g-real[of u] g-nz[of u] u'
    have Re c = 0 by (simp add: complex-is-Real-iff g.of-1)
```

with *h-real*[*of 0*] *c*[*of 0*] **have** $c = 0$ **by** (*auto elim*!: *Reals-cases*)
with *c* **have** *A*: $h\ z * g\ z = 0$ **for** *z* **by** *simp*
hence (*g has-field-derivative 0*) (*at z*) **for** *z* **using** *g-g'*[*of z*] **by** *simp*
hence $\exists c'.\ \forall z{\in}UNIV.\ g\ z = c'$ **by** (*intro has-field-derivative-zero-constant*) *simp-all*
then obtain c' **where** *c*: $\bigwedge z.\ g\ z = c'$ **by** (*force simp*: *dist-0-norm*)
moreover from *this*[*of 0*] **have** $c' = pi$ **unfolding** *g-def* **by** *simp*
ultimately have $g\ z = pi$ **by** *simp*

show *?thesis*
proof (*cases* $z \in \mathbb{Z}$)
case *False*
with ‹$g\ z = pi$› **show** *?thesis* **by** (*auto simp*: *g-def divide-simps*)
next
case *True*
then obtain *n* **where** *n*: $z =$ *of-int n* **by** (*elim Ints-cases*)
with *sin-eq-0*[*of of-real pi* $*$ *z*] **have** *sin* (*of-real pi* $*$ *z*) $= 0$ **by** *force*
moreover have *of-int* $(1 - n) \in \mathbb{Z}_{\leq 0}$ **if** $n > 0$ **using** *that* **by** (*intro nonpos-Ints-of-int*) *simp*
ultimately show *?thesis* **using** *n*
by (*cases* $n \leq 0$) (*auto simp*: *Gamma-eq-zero-iff nonpos-Ints-of-int*)
qed
qed

lemma *rGamma-reflection-complex*:
rGamma z $*$ *rGamma* $(1 - z ::$ *complex*$)$ $=$ *sin* (*of-real pi* $*$ *z*) / *of-real pi*
using *Gamma-reflection-complex*[*of z*]
by (*simp add*: *Gamma-def divide-simps split*: *if-split-asm*)

lemma *rGamma-reflection-complex'*:
rGamma z $*$ *rGamma* $(-z ::$ *complex*$)$ $= -z *$ *sin* (*of-real pi* $*$ *z*) / *of-real pi*
proof −
have *rGamma z* $*$ *rGamma* $(-z) = -z *$ (*rGamma z* $*$ *rGamma* $(1 - z)$)
using *rGamma-plus1*[*of* $-z$, *symmetric*] **by** *simp*
also have *rGamma z* $*$ *rGamma* $(1 - z) =$ *sin* (*of-real pi* $*$ *z*) / *of-real pi*
by (*rule rGamma-reflection-complex*)
finally show *?thesis* **by** *simp*
qed

lemma *Gamma-reflection-complex'*:
Gamma z $*$ *Gamma* $(-z ::$ *complex*$)$ $=$ $-$ *of-real pi* / $(z *$ *sin* (*of-real pi* $*$ *z*))
using *rGamma-reflection-complex'*[*of z*] **by** (*force simp add*: *Gamma-def divide-simps mult-ac*)

lemma *Gamma-one-half-real*: *Gamma* $(1/2 ::$ *real*$)$ $=$ *sqrt pi*
proof −
from *Gamma-reflection-complex*[*of* $1/2$] *fraction-not-in-ints*[**where** *'a* $=$ *com-*

```
plex, of 2 1]
   have Gamma (1/2 :: complex) ^ 2 = of-real pi by (simp add: power2-eq-square)
  hence of-real pi = Gamma (complex-of-real (1/2)) ^ 2 by simp
  also have ... = of-real ((Gamma (1/2)) ^ 2) by (subst Gamma-complex-of-real)
simp-all
  finally have Gamma (1/2) ^ 2 = pi by (subst (asm) of-real-eq-iff) simp-all
  moreover have Gamma (1/2 :: real) ≥ 0 using Gamma-real-pos[of 1/2] by
simp
  ultimately show ?thesis by (rule real-sqrt-unique [symmetric])
qed

lemma Gamma-one-half-complex: Gamma (1/2 :: complex) = of-real (sqrt pi)
proof −
  have Gamma (1/2 :: complex) = Gamma (of-real (1/2)) by simp
 also have ... = of-real (sqrt pi) by (simp only: Gamma-complex-of-real Gamma-one-half-real)
  finally show ?thesis .
qed

lemma Gamma-legendre-duplication:
  fixes z :: complex
  assumes z ∉ ℤ≤0 z + 1/2 ∉ ℤ≤0
  shows Gamma z * Gamma (z + 1/2) =
           exp ((1 − 2*z) * of-real (ln 2)) * of-real (sqrt pi) * Gamma (2*z)
 using Gamma-legendre-duplication-aux[OF assms] by (simp add: Gamma-one-half-complex)

end
```

59.8 Limits and residues

The inverse of the Gamma function has simple zeros:

```
lemma rGamma-zeros:
  (λz. rGamma z / (z + of-nat n)) − (− of-nat n) → ((−1) ^ n * fact n :: 'a ::
Gamma)
proof (subst tendsto-cong)
  let ?f = λz. pochhammer z n * rGamma (z + of-nat (Suc n)) :: 'a
  from eventually-at-ball'[OF zero-less-one, of − of-nat n :: 'a UNIV]
    show eventually (λz. rGamma z / (z + of-nat n) = ?f z) (at (− of-nat n))
    by (subst pochhammer-rGamma[of - Suc n])
     (auto elim!: eventually-mono simp: divide-simps pochhammer-rec' eq-neg-iff-add-eq-0)
  have isCont ?f (− of-nat n) by (intro continuous-intros)
  thus ?f − (− of-nat n) → (− 1) ^ n * fact n unfolding isCont-def
    by (simp add: pochhammer-same)
qed
```

The simple zeros of the inverse of the Gamma function correspond to simple poles of the Gamma function, and their residues can easily be computed from the limit we have just proven:

```
lemma Gamma-poles: filterlim Gamma at-infinity (at (− of-nat n :: 'a :: Gamma))
```

proof −
from *eventually-at-ball'*[*OF zero-less-one*, *of* − *of-nat n* :: *'a UNIV*]
have *eventually* (λ*z*. *rGamma z* ≠ (*0* :: *'a*)) (*at* (− *of-nat n*))
by (*auto elim*!: *eventually-mono nonpos-Ints-cases'*
simp: *rGamma-eq-zero-iff dist-of-nat dist-minus*)
with *isCont-rGamma*[*of* − *of-nat n* :: *'a*, *OF continuous-ident*]
have *filterlim* (λ*z*. *inverse* (*rGamma z*) :: *'a*) *at-infinity* (*at* (− *of-nat n*))
unfolding *isCont-def* **by** (*intro filterlim-compose*[*OF filterlim-inverse-at-infinity*])
(*simp-all add*: *filterlim-at*)
moreover have (λ*z*. *inverse* (*rGamma z*) :: *'a*) = *Gamma*
by (*intro ext*) (*simp add*: *rGamma-inverse-Gamma*)
ultimately show *?thesis* **by** (*simp only*:)
qed

lemma *Gamma-residues*:
(λ*z*. *Gamma z* ∗ (*z* + *of-nat n*)) − (− *of-nat n*) → ((−*1*) ^ *n* / *fact n* :: *'a* :: *Gamma*)
proof (*subst tendsto-cong*)
let *?c* = (− *1*) ^ *n* / *fact n* :: *'a*
from *eventually-at-ball'*[*OF zero-less-one*, *of* − *of-nat n* :: *'a UNIV*]
show *eventually* (λ*z*. *Gamma z* ∗ (*z* + *of-nat n*) = *inverse* (*rGamma z* / (*z* + *of-nat n*)))
(*at* (− *of-nat n*))
by (*auto elim*!: *eventually-mono simp*: *divide-simps rGamma-inverse-Gamma*)
have (λ*z*. *inverse* (*rGamma z* / (*z* + *of-nat n*))) − (− *of-nat n*) →
inverse ((− *1*) ^ *n* ∗ *fact n* :: *'a*)
by (*intro tendsto-intros rGamma-zeros*) *simp-all*
also have *inverse* ((− *1*) ^ *n* ∗ *fact n*) = *?c*
by (*simp-all add*: *field-simps power-mult-distrib* [*symmetric*] *del*: *power-mult-distrib*)
finally show (λ*z*. *inverse* (*rGamma z* / (*z* + *of-nat n*))) − (− *of-nat n*) → *?c* .
qed

59.9 Alternative definitions

59.9.1 Variant of the Euler form

definition *Gamma-series-euler'* **where**
Gamma-series-euler' z n =
inverse z ∗ (∏ *k*=*1*..*n*. *exp* (*z* ∗ *of-real* (*ln* (*1* + *inverse* (*of-nat k*)))) / (*1* + *z* / *of-nat k*))

context
begin
private lemma *Gamma-euler'-aux1*:
fixes *z* :: *'a* :: {*real-normed-field*,*banach*}
assumes *n*: *n* > *0*
shows *exp* (*z* ∗ *of-real* (*ln* (*of-nat n* + *1*))) = (∏ *k*=*1*..*n*. *exp* (*z* ∗ *of-real* (*ln* (*1* + *1* / *of-nat k*))))
proof −
have (∏ *k*=*1*..*n*. *exp* (*z* ∗ *of-real* (*ln* (*1* + *1* / *of-nat k*)))) =

```
        exp (z * of-real (∑k = 1..n. ln (1 + 1 / real-of-nat k)))
    by (subst exp-setsum [symmetric]) (simp-all add: setsum-right-distrib)
  also have (∑k=1..n. ln (1 + 1 / of-nat k) :: real) = ln (∏k=1..n. 1 + 1 /
real-of-nat k)
    by (subst ln-setprod [symmetric]) (auto intro!: add-pos-nonneg)
  also have (∏k=1..n. 1 + 1 / of-nat k :: real) = (∏k=1..n. (of-nat k + 1) /
of-nat k)
    by (intro setprod.cong) (simp-all add: divide-simps)
  also have (∏k=1..n. (of-nat k + 1) / of-nat k :: real) = of-nat n + 1
    by (induction n) (simp-all add: setprod-nat-ivl-Suc' divide-simps)
  finally show ?thesis ..
qed

lemma Gamma-series-euler':
  assumes z: (z :: 'a :: Gamma) ∉ ℤ≤0
  shows (λn. Gamma-series-euler' z n) ⟶ Gamma z
proof (rule Gamma-seriesI, rule Lim-transform-eventually)
  let ?f = λn. fact n * exp (z * of-real (ln (of-nat n + 1))) / pochhammer z (n
+ 1)
  let ?r = λn. ?f n / Gamma-series z n
  let ?r' = λn. exp (z * of-real (ln (of-nat (Suc n) / of-nat n)))
  from z have z': z ≠ 0 by auto

  have eventually (λn. ?r' n = ?r n) sequentially using eventually-gt-at-top[of
0::nat]
    using z by (auto simp: divide-simps Gamma-series-def ring-distribs exp-diff
ln-div add-ac
                  elim!: eventually-mono dest: pochhammer-eq-0-imp-nonpos-Int)
  moreover have ?r' ⟶ exp (z * of-real (ln 1))
    by (intro tendsto-intros LIMSEQ-Suc-n-over-n) simp-all
  ultimately show ?r ⟶ 1 by (force dest!: Lim-transform-eventually)

  from eventually-gt-at-top[of 0::nat]
    show eventually (λn. ?r n = Gamma-series-euler' z n / Gamma-series z n)
sequentially
  proof eventually-elim
    fix n :: nat assume n: n > 0
    from n z' have Gamma-series-euler' z n =
      exp (z * of-real (ln (of-nat n + 1))) / (z * (∏k=1..n. (1 + z / of-nat k)))
      by (subst Gamma-euler'-aux1)
         (simp-all add: Gamma-series-euler'-def setprod.distrib
                        setprod-inverse[symmetric] divide-inverse)
    also have (∏k=1..n. (1 + z / of-nat k)) = pochhammer (z + 1) n / fact n
    by (cases n) (simp-all add: pochhammer-def fact-altdef setprod-shift-bounds-cl-Suc-ivl
                                setprod-divide[symmetric] divide-simps add-ac)
   also have z * ... = pochhammer z (Suc n) / fact n by (simp add: pochhammer-rec)
    finally show ?r n = Gamma-series-euler' z n / Gamma-series z n by simp
  qed
qed
```

end

59.9.2 Weierstrass form

definition *Gamma-series-weierstrass* :: *'a* :: {*banach*,*real-normed-field*} ⇒ *nat* ⇒ *'a* **where**
Gamma-series-weierstrass z n =
exp (−*euler-mascheroni* ∗ *z*) / *z* ∗ (∏ *k*=*1*..*n*. *exp* (*z* / *of-nat k*) / (*1* + *z* / *of-nat k*))

definition *rGamma-series-weierstrass* :: *'a* :: {*banach*,*real-normed-field*} ⇒ *nat* ⇒ *'a* **where**
rGamma-series-weierstrass z n =
exp (*euler-mascheroni* ∗ *z*) ∗ *z* ∗ (∏ *k*=*1*..*n*. (*1* + *z* / *of-nat k*) ∗ *exp* (−*z* / *of-nat k*))

lemma *Gamma-series-weierstrass-nonpos-Ints*:
eventually (λ*k*. *Gamma-series-weierstrass* (− *of-nat n*) *k* = *0*) *sequentially*
using *eventually-ge-at-top*[*of n*] **by** *eventually-elim* (*auto simp*: *Gamma-series-weierstrass-def*)

lemma *rGamma-series-weierstrass-nonpos-Ints*:
eventually (λ*k*. *rGamma-series-weierstrass* (− *of-nat n*) *k* = *0*) *sequentially*
using *eventually-ge-at-top*[*of n*] **by** *eventually-elim* (*auto simp*: *rGamma-series-weierstrass-def*)

lemma *Gamma-weierstrass-complex*: *Gamma-series-weierstrass z* ⟶ *Gamma* (*z* :: *complex*)
proof (*cases z* ∈ ℤ≤0)
case *True*
then obtain *n* **where** *z* = − *of-nat n* **by** (*elim nonpos-Ints-cases'*)
also from *True* **have** *Gamma-series-weierstrass* ... ⟶ *Gamma z*
by (*simp add*: *tendsto-cong*[*OF Gamma-series-weierstrass-nonpos-Ints*] *Gamma-nonpos-Int*)
finally show *?thesis* **.**
next
case *False*
hence *z*: *z* ≠ *0* **by** *auto*
let *?f* = (λ*x*. ∏ *x* = *Suc 0*..*x*. *exp* (*z* / *of-nat x*) / (*1* + *z* / *of-nat x*))
have *A*: *exp* (*ln* (*1* + *z* / *of-nat n*)) = (*1* + *z* / *of-nat n*) **if** *n* ≥ *1* **for** *n* :: *nat*
using *False that* **by** (*subst exp-Ln*) (*auto simp*: *field-simps dest*!: *plus-of-nat-eq-0-imp*)
have (λ*n*. ∑ *k*=*1*..*n*. *z* / *of-nat k* − *ln* (*1* + *z* / *of-nat k*)) ⟶ *ln-Gamma z* + *euler-mascheroni* ∗ *z* + *ln z*
using *ln-Gamma-series'-aux*[*OF False*]
by (*simp only*: *atLeastLessThanSuc-atLeastAtMost* [*symmetric*] *One-nat-def*
setsum-shift-bounds-Suc-ivl sums-def atLeast0LessThan)
from *tendsto-exp*[*OF this*] *False z* **have** *?f* ⟶ *z* ∗ *exp* (*euler-mascheroni* ∗ *z*) ∗ *Gamma z*
by (*simp add*: *exp-add exp-setsum exp-diff mult-ac Gamma-complex-altdef A*)
from *tendsto-mult*[*OF tendsto-const*[*of exp* (−*euler-mascheroni* ∗ *z*) / *z*] *this*] *z*
show *Gamma-series-weierstrass z* ⟶ *Gamma z*

by (*simp add*: *exp-minus divide-simps Gamma-series-weierstrass-def* [*abs-def*])
qed

lemma *tendsto-complex-of-real-iff*: ((λ*x*. *complex-of-real* (*f x*)) ⟶ *of-real c*) *F* = (*f* ⟶ *c*) *F*
by (*rule tendsto-of-real-iff*)

lemma *Gamma-weierstrass-real*: *Gamma-series-weierstrass x* ⟶ *Gamma* (*x* :: *real*)
using *Gamma-weierstrass-complex*[*of of-real x*] **unfolding** *Gamma-series-weierstrass-def*[*abs-def*]
by (*subst tendsto-complex-of-real-iff* [*symmetric*])
(*simp-all add*: *exp-of-real*[*symmetric*] *Gamma-complex-of-real*)

lemma *rGamma-weierstrass-complex*: *rGamma-series-weierstrass z* ⟶ *rGamma* (*z* :: *complex*)
proof (*cases z* ∈ ℤ≤0)
case *True*
then obtain *n* **where** *z* = − *of-nat n* **by** (*elim nonpos-Ints-cases'*)
also from *True* **have** *rGamma-series-weierstrass* ... ⟶ *rGamma z*
by (*simp add*: *tendsto-cong*[*OF rGamma-series-weierstrass-nonpos-Ints*] *rGamma-nonpos-Int*)
finally show *?thesis* **.**
next
case *False*
have *rGamma-series-weierstrass z* = (λ*n*. *inverse* (*Gamma-series-weierstrass z n*))
by (*simp add*: *rGamma-series-weierstrass-def*[*abs-def*] *Gamma-series-weierstrass-def*
exp-minus divide-inverse setprod-inversef[*symmetric*] *mult-ac*)
also from *False* **have** ... ⟶ *inverse* (*Gamma z*)
by (*intro tendsto-intros Gamma-weierstrass-complex*) (*simp add*: *Gamma-eq-zero-iff*)
finally show *?thesis* **by** (*simp add*: *Gamma-def*)
qed

59.9.3 Binomial coefficient form

lemma *Gamma-binomial*:
(λ*n*. ((*z* + *of-nat n*) *gchoose n*) ∗ *exp* (−*z* ∗ *of-real* (*ln* (*of-nat n*)))) ⟶ *rGamma* (*z*+*1*)
proof (*cases z* = *0*)
case *False*
show *?thesis*
proof (*rule Lim-transform-eventually*)
let *?powr* = λ*a b*. *exp* (*b* ∗ *of-real* (*ln* (*of-nat a*)))
show *eventually* (λ*n*. *rGamma-series z n* / *z* =
((*z* + *of-nat n*) *gchoose n*) ∗ *?powr n* (−*z*)) *sequentially*
proof (*intro always-eventually allI*)
fix *n* :: *nat*
from *False* **have** ((*z* + *of-nat n*) *gchoose n*) = *pochhammer z* (*Suc n*) / *z* / *fact n*
by (*simp add*: *gbinomial-pochhammer' pochhammer-rec*)

also have *pochhammer z (Suc n) / z / fact n * ?powr n (−z) = rGamma-series z n / z*
by (*simp add*: *rGamma-series-def divide-simps exp-minus*)
finally show *rGamma-series z n / z = ((z + of-nat n) gchoose n) * ?powr n (−z)* **..**
qed

from *False* **have** (λ*n*. *rGamma-series z n / z*) ⟶ *rGamma z / z* **by** (*intro tendsto-intros*)
also from *False* **have** *rGamma z / z = rGamma (z + 1)* **using** *rGamma-plus1*[*of z*]
by (*simp add*: *field-simps*)
finally show (λ*n*. *rGamma-series z n / z*) ⟶ *rGamma (z+1)* **.**
qed
qed (*simp-all add*: *binomial-gbinomial* [*symmetric*])

lemma *fact-binomial-limit*:
(λ*n*. *of-nat ((k + n) choose n) / of-nat (n ^ k) :: 'a :: Gamma*) ⟶ *1 / fact k*
proof (*rule Lim-transform-eventually*)
have (λ*n*. *of-nat ((k + n) choose n) / of-real (exp (of-nat k * ln (real-of-nat n))))*
⟶ *1 / Gamma (of-nat (Suc k) :: 'a)* (**is** *?f* ⟶ -)
using *Gamma-binomial*[*of of-nat k :: 'a*]
by (*simp add*: *binomial-gbinomial add-ac Gamma-def divide-simps exp-of-real* [*symmetric*] *exp-minus*)
also have *Gamma (of-nat (Suc k)) = fact k* **by** (*rule Gamma-fact*)
finally show *?f* ⟶ *1 / fact k* **.**

show *eventually* (λ*n*. *?f n = of-nat ((k + n) choose n) / of-nat (n ^ k)*) *sequentially*
using *eventually-gt-at-top*[*of 0*::*nat*]
proof *eventually-elim*
fix *n* :: *nat* **assume** *n*: *n > 0*
from *n* **have** *exp (real-of-nat k * ln (real-of-nat n)) = real-of-nat (n^k)*
by (*simp add*: *exp-of-nat-mult*)
thus *?f n = of-nat ((k + n) choose n) / of-nat (n ^ k)* **by** *simp*
qed
qed

lemma *binomial-asymptotic*:
(λ*n*. *of-nat ((k + n) choose n) / (of-nat (n ^ k) / fact k) :: 'a :: Gamma*) ⟶ *1*
using *tendsto-mult*[*OF fact-binomial-limit*[*of k*] *tendsto-const*[*of fact k :: 'a*]] **by** *simp*

59.10 The Weierstra product formula for the sine

lemma *sin-product-formula-complex*:

fixes *z* :: *complex*
shows (λ*n*. *of-real pi* ∗ *z* ∗ (∏ *k*=*1*..*n*. *1* − *z*^*2* / *of-nat k*^*2*)) ⟶ *sin* (*of-real pi* ∗ *z*)
proof −
let *?f* = *rGamma-series-weierstrass*
have (λ*n*. (− *of-real pi* ∗ *inverse z*) ∗ (*?f z n* ∗ *?f* (− *z*) *n*))
⟶ (− *of-real pi* ∗ *inverse z*) ∗ (*rGamma z* ∗ *rGamma* (− *z*))
by (*intro tendsto-intros rGamma-weierstrass-complex*)
also have (λ*n*. (− *of-real pi* ∗ *inverse z*) ∗ (*?f z n* ∗ *?f* (−*z*) *n*)) =
(λ*n*. *of-real pi* ∗ *z* ∗ (∏ *k*=*1*..*n*. *1* − *z*^*2* / *of-nat k* ^ *2*))
proof
fix *n* :: *nat*
have (− *of-real pi* ∗ *inverse z*) ∗ (*?f z n* ∗ *?f* (−*z*) *n*) =
of-real pi ∗ *z* ∗ (∏ *k*=*1*..*n*. (*of-nat k* − *z*) ∗ (*of-nat k* + *z*) / *of-nat k* ^ *2*)
by (*simp add*: *rGamma-series-weierstrass-def mult-ac exp-minus*
divide-simps setprod.distrib[*symmetric*] *power2-eq-square*)
also have (∏ *k*=*1*..*n*. (*of-nat k* − *z*) ∗ (*of-nat k* + *z*) / *of-nat k* ^ *2*) =
(∏ *k*=*1*..*n*. *1* − *z*^*2* / *of-nat k* ^ *2*)
by (*intro setprod.cong*) (*simp-all add*: *power2-eq-square field-simps*)
finally show (− *of-real pi* ∗ *inverse z*) ∗ (*?f z n* ∗ *?f* (−*z*) *n*) = *of-real pi* ∗ *z* ∗ ...
by (*simp add*: *divide-simps*)
qed
also have (− *of-real pi* ∗ *inverse z*) ∗ (*rGamma z* ∗ *rGamma* (− *z*)) = *sin* (*of-real pi* ∗ *z*)
by (*subst rGamma-reflection-complex′*) (*simp add*: *divide-simps*)
finally show *?thesis* **.**
qed

lemma *sin-product-formula-real*:
(λ*n*. *pi* ∗ (*x*::*real*) ∗ (∏ *k*=*1*..*n*. *1* − *x*^*2* / *of-nat k*^*2*)) ⟶ *sin* (*pi* ∗ *x*)
proof −
from *sin-product-formula-complex*[*of of-real x*]
have (λ*n*. *of-real pi* ∗ *of-real x* ∗ (∏ *k*=*1*..*n*. *1* − (*of-real x*)^*2* / (*of-nat k*)^*2*))
⟶ *sin* (*of-real pi* ∗ *of-real x* :: *complex*) (**is** *?f* ⟶ *?y*) **.**
also have *?f* = (λ*n*. *of-real* (*pi* ∗ *x* ∗ (∏ *k*=*1*..*n*. *1* − *x*^*2* / (*of-nat k*^*2*)))) **by** *simp*
also have *?y* = *of-real* (*sin* (*pi* ∗ *x*)) **by** (*simp only*: *sin-of-real* [*symmetric*] *of-real-mult*)
finally show *?thesis* **by** (*subst* (*asm*) *tendsto-of-real-iff*)
qed

lemma *sin-product-formula-real′*:
assumes *x* ≠ (*0*::*real*)
shows (λ*n*. (∏ *k*=*1*..*n*. *1* − *x*^*2* / *of-nat k*^*2*)) ⟶ *sin* (*pi* ∗ *x*) / (*pi* ∗ *x*)
using *tendsto-divide*[*OF sin-product-formula-real*[*of x*] *tendsto-const*[*of pi* ∗ *x*]] *assms*
by *simp*

59.11 The Solution to the Basel problem

theorem *inverse-squares-sums*: $(\lambda n.\ 1\ /\ (n + 1)^2)$ *sums* $(pi^2\ /\ 6)$
proof −
def $P \equiv \lambda x\ n.\ (\prod k{=}1..n.\ 1 - x\hat{}2\ /$ *of-nat* $k\hat{}2 :: real)$
def $K \equiv \sum n.$ *inverse* (*real-of-nat* (*Suc* n))$\hat{}2$
def $f \equiv \lambda x. \sum n.\ P\ x\ n\ /$ *of-nat* (*Suc* n)$\hat{}2$
def $g \equiv \lambda x.\ (1 - sin\ (pi * x)\ /\ (pi * x))$

have *sums*: $(\lambda n.\ P\ x\ n\ /$ *of-nat* (*Suc* n)$\hat{}2)$ *sums* (*if* $x = 0$ *then* K *else* $g\ x\ /\ x\hat{}2$) **for** x
proof (*cases* $x = 0$)
assume x: $x = 0$
have *summable* $(\lambda n.$ *inverse* ((*real-of-nat* (*Suc* n))^2))
using *inverse-power-summable*[*of 2*] **by** (*subst summable-Suc-iff*) *simp*
thus *?thesis* **by** (*simp add*: *x g-def P-def K-def inverse-eq-divide power-divide summable-sums*)
next
assume x: $x \neq 0$
have $(\lambda n.\ P\ x\ n - P\ x$ (*Suc* n)) *sums* $(P\ x\ 0 - sin\ (pi * x)\ /\ (pi * x))$
unfolding *P-def* **using** x **by** (*intro telescope-sums' sin-product-formula-real'*)
also have $(\lambda n.\ P\ x\ n - P\ x$ (*Suc* n)) = $(\lambda n.\ (x\hat{}2\ /$ *of-nat* (*Suc* n)$\hat{}2) * P\ x\ n)$
unfolding *P-def* **by** (*simp add*: *setprod-nat-ivl-Suc' algebra-simps*)
also have $P\ x\ 0 = 1$ **by** (*simp add*: *P-def*)
finally have $(\lambda n.\ x^2\ /$ (*of-nat* (*Suc* n))$^2 * P\ x\ n)$ *sums* $(1 - sin\ (pi * x)\ /\ (pi * x))$ **.**
from *sums-divide*[*OF this*, *of* $x\hat{}2$] x **show** *?thesis* **unfolding** *g-def* **by** *simp*
qed

have *continuous-on* (*ball 0 1*) f
proof (*rule uniform-limit-theorem*; (*intro always-eventually allI*)*?*)
show *uniform-limit* (*ball 0 1*) $(\lambda n\ x.\ \sum k{<}n.\ P\ x\ k\ /$ *of-nat* (*Suc* k)$\hat{}2)\ f$ *sequentially*
proof (*unfold f-def*, *rule weierstrass-m-test*)
fix $n :: nat$ **and** $x :: real$ **assume** x: $x \in$ *ball 0 1*
{
fix $k :: nat$ **assume** k: $k \geq 1$
from x **have** $x\hat{}2 < 1$ **by** (*auto simp*: *dist-0-norm abs-square-less-1*)
also from k **have** $\ldots \leq$ *of-nat* $k\hat{}2$ **by** *simp*
finally have $(1 - x\hat{}2\ /$ *of-nat* $k\hat{}2) \in \{0..1\}$ **using** k
by (*simp-all add*: *field-simps del*: *of-nat-Suc*)
}
hence $(\prod k{=}1..n.$ *abs* $(1 - x\hat{}2\ /$ *of-nat* $k\hat{}2)) \leq (\prod k{=}1..n.\ 1)$ **by** (*intro setprod-mono*) *simp*
thus *norm* $(P\ x\ n\ /$ (*of-nat* (*Suc* n)$\hat{}2)) \leq 1\ /$ *of-nat* (*Suc* n)$\hat{}2$
unfolding *P-def* **by** (*simp add*: *field-simps abs-setprod del*: *of-nat-Suc*)
qed (*subst summable-Suc-iff*, *insert inverse-power-summable*[*of 2*], *simp add*: *inverse-eq-divide*)
qed (*auto simp*: *P-def intro*!: *continuous-intros*)

```
  hence isCont f 0 by (subst (asm) continuous-on-eq-continuous-at) simp-all
  hence (f − 0 → f 0) by (simp add: isCont-def)
  also have f 0 = K unfolding f-def P-def K-def by (simp add: inverse-eq-divide
power-divide)
  finally have f − 0 → K .

  moreover have f − 0 → pi^2 / 6
  proof (rule Lim-transform-eventually)
    def f' ≡ λx. ∑ n. − sin-coeff (n+3) * pi ^ (n+2) * x^n
    have eventually (λx. x ≠ (0::real)) (at 0)
      by (auto simp add: eventually-at intro!: exI[of - 1])
    thus eventually (λx. f' x = f x) (at 0)
    proof eventually-elim
      fix x :: real assume x: x ≠ 0
        have sin-coeff 1 = (1 :: real) sin-coeff 2 = (0::real) by (simp-all add:
sin-coeff-def)
      with sums-split-initial-segment[OF sums-minus[OF sin-converges], of 3 pi*x]
      have (λn. − (sin-coeff (n+3) * (pi*x)^(n+3))) sums (pi * x − sin (pi*x))
        by (simp add: eval-nat-numeral)
      from sums-divide[OF this, of x^3 * pi] x
        have (λn. − (sin-coeff (n+3) * pi^(n+2) * x^n)) sums ((1 − sin (pi*x)
/ (pi*x)) / x^2)
        by (simp add: divide-simps eval-nat-numeral power-mult-distrib mult-ac)
      with x have (λn. − (sin-coeff (n+3) * pi^(n+2) * x^n)) sums (g x / x^2)
        by (simp add: g-def)
      hence f' x = g x / x^2 by (simp add: sums-iff f'-def)
      also have ... = f x using sums[of x] x by (simp add: sums-iff g-def f-def)
      finally show f' x = f x .
    qed

    have isCont f' 0 unfolding f'-def
    proof (intro isCont-powser-converges-everywhere)
      fix x :: real show summable (λn. −sin-coeff (n+3) * pi^(n+2) * x^n)
      proof (cases x = 0)
        assume x: x ≠ 0
      from summable-divide[OF sums-summable[OF sums-split-initial-segment[OF
            sin-converges[of pi*x]], of 3], of −pi*x^3] x
          show ?thesis by (simp add: mult-ac power-mult-distrib divide-simps
eval-nat-numeral)
      qed (simp only: summable-0-powser)
    qed
    hence f' − 0 → f' 0 by (simp add: isCont-def)
    also have f' 0 = pi * pi / fact 3 unfolding f'-def
      by (subst powser-zero) (simp add: sin-coeff-def)
    finally show f' − 0 → pi^2 / 6 by (simp add: eval-nat-numeral)
  qed

  ultimately have K = pi^2 / 6 by (rule LIM-unique)
  moreover from inverse-power-summable[of 2]
```

have *summable* ($\lambda n.$ (*inverse* (*real-of-nat* (*Suc n*)))^2)
by (*subst summable-Suc-iff*) (*simp add*: *power-inverse*)
ultimately show *?thesis* **unfolding** *K-def*
by (*auto simp add*: *sums-iff power-divide inverse-eq-divide*)
qed

end
theory *Multivariate-Analysis*
imports
Fashoda
Extended-Real-Limits
Determinants
Ordered-Euclidean-Space
Bounded-Continuous-Function
Weierstrass
Conformal-Mappings
Generalised-Binomial-Theorem
Gamma
begin

end

60 polynomial functions: extremal behaviour and root counts

theory *PolyRoots*
imports *Complex-Main*
begin

60.1 Geometric progressions

lemma *setsum-gp-basic*:
fixes x :: *'a*::{*comm-ring*,*monoid-mult*}
shows $(1 - x) * (\sum i{\leq}n.\ x\hat{}i) = 1 - x\hat{}Suc\ n$
by (*simp only*: *one-diff-power-eq* [*of Suc n x*] *lessThan-Suc-atMost*)

lemma *setsum-gp0*:
fixes x :: *'a*::{*comm-ring*,*division-ring*}
shows $(\sum i{\leq}n.\ x\hat{}i) = (if\ x = 1\ then\ of\text{-}nat(n + 1)\ else\ (1 - x\hat{}Suc\ n)\ /\ (1 - x))$
using *setsum-gp-basic*[*of x n*]
by (*simp add*: *mult.commute divide-simps*)

lemma *setsum-power-add*:
fixes x :: *'a*::{*comm-ring*,*monoid-mult*}
shows $(\sum i{\in}I.\ x\hat{}(m+i)) = x\hat{}m * (\sum i{\in}I.\ x\hat{}i)$
by (*simp add*: *setsum-right-distrib power-add*)

lemma *setsum-power-shift*:
fixes x :: *'a*::{*comm-ring*,*monoid-mult*}
assumes $m \leq n$
shows $(\sum i{=}m..n.\ x\hat{}i) = x\hat{}m * (\sum i{\leq}n{-}m.\ x\hat{}i)$
proof −
have $(\sum i{=}m..n.\ x\hat{}i) = x\hat{}m * (\sum i{=}m..n.\ x\hat{}(i{-}m))$
by (*simp add*: *setsum-right-distrib power-add* [*symmetric*])
also have $(\sum i{=}m..n.\ x\hat{}(i{-}m)) = (\sum i{\leq}n{-}m.\ x\hat{}i)$
using ‹$m \leq n$› **by** (*intro setsum.reindex-bij-witness*[**where** $j{=}\lambda i.\ i - m$ **and** $i{=}\lambda i.\ i + m$]) *auto*
finally show *?thesis* **.**
qed

lemma *setsum-gp-multiplied*:
fixes x :: *'a*::{*comm-ring*,*monoid-mult*}
assumes $m \leq n$
shows $(1 - x) * (\sum i{=}m..n.\ x\hat{}i) = x\hat{}m - x\hat{}Suc\ n$
proof −
have $(1 - x) * (\sum i{=}m..n.\ x\hat{}i) = x\hat{}m * (1 - x) * (\sum i{\leq}n{-}m.\ x\hat{}i)$
by (*metis mult.assoc mult.commute assms setsum-power-shift*)
also have ... $= x\hat{}m * (1 - x\hat{}Suc(n{-}m))$
by (*metis mult.assoc setsum-gp-basic*)
also have ... $= x\hat{}m - x\hat{}Suc\ n$
using *assms*
by (*simp add*: *algebra-simps*) (*metis le-add-diff-inverse power-add*)
finally show *?thesis* **.**
qed

lemma *setsum-gp*:
fixes x :: *'a*::{*comm-ring*,*division-ring*}
shows $(\sum i{=}m..n.\ x\hat{}i) =$
(if $n < m$ *then 0*
else if $x = 1$ *then of-nat*$((n + 1) - m)$
else $(x\hat{}m - x\hat{}Suc\ n) / (1 - x)$*)*
using *setsum-gp-multiplied* [*of m n x*]
apply *auto*
by (*metis eq-iff-diff-eq-0 mult.commute nonzero-divide-eq-eq*)

lemma *setsum-gp-offset*:
fixes x :: *'a*::{*comm-ring*,*division-ring*}
shows $(\sum i{=}m..m{+}n.\ x\hat{}i) =$
(if $x = 1$ *then of-nat* $n + 1$ *else* $x\hat{}m * (1 - x\hat{}Suc\ n) / (1 - x)$*)*
using *setsum-gp* [*of x m m+n*]
by (*auto simp*: *power-add algebra-simps*)

lemma *setsum-gp-strict*:
fixes x :: *'a*::{*comm-ring*,*division-ring*}
shows $(\sum i{<}n.\ x\hat{}i) = (\textit{if } x = 1 \textit{ then of-nat } n \textit{ else } (1 - x\hat{}n) / (1 - x))$

by (*induct n*) (*auto simp*: *algebra-simps divide-simps*)

60.2 Basics about polynomial functions: extremal behaviour and root counts.

lemma *sub-polyfun*:
fixes $x :: 'a::\{comm\text{-}ring,monoid\text{-}mult\}$
shows $(\sum i{\le}n.\ a\ i * x\hat{}i) - (\sum i{\le}n.\ a\ i * y\hat{}i) =$
$(x - y) * (\sum j{<}n.\ \sum k{=}\ Suc\ j..n.\ a\ k * y\hat{}(k - Suc\ j) * x\hat{}j)$
proof −
have $(\sum i{\le}n.\ a\ i * x\hat{}i) - (\sum i{\le}n.\ a\ i * y\hat{}i) =$
$(\sum i{\le}n.\ a\ i * (x\hat{}i - y\hat{}i))$
by (*simp add*: *algebra-simps setsum-subtractf* [*symmetric*])
also have ... $= (\sum i{\le}n.\ a\ i * (x - y) * (\sum j{<}i.\ y\hat{}(i - Suc\ j) * x\hat{}j))$
by (*simp add*: *power-diff-sumr2 ac-simps*)
also have ... $= (x - y) * (\sum i{\le}n.\ (\sum j{<}i.\ a\ i * y\hat{}(i - Suc\ j) * x\hat{}j))$
by (*simp add*: *setsum-right-distrib ac-simps*)
also have ... $= (x - y) * (\sum j{<}n.\ (\sum i{=}Suc\ j..n.\ a\ i * y\hat{}(i - Suc\ j) * x\hat{}j))$
by (*simp add*: *nested-setsum-swap'*)
finally show *?thesis* **.**
qed

lemma *sub-polyfun-alt*:
fixes $x :: 'a::\{comm\text{-}ring,monoid\text{-}mult\}$
shows $(\sum i{\le}n.\ a\ i * x\hat{}i) - (\sum i{\le}n.\ a\ i * y\hat{}i) =$
$(x - y) * (\sum j{<}n.\ \sum k{<}n{-}j.\ a\ (j{+}k{+}1) * y\hat{}k * x\hat{}j)$
proof −
{ fix *j*
have $(\sum k = Suc\ j..n.\ a\ k * y\hat{}(k - Suc\ j) * x\hat{}j) =$
$(\sum k < n - j.\ a\ (Suc\ (j + k)) * y\hat{}k * x\hat{}j)$
by (*rule setsum.reindex-bij-witness*[**where** $i{=}\lambda i.\ i + Suc\ j$ **and** $j{=}\lambda i.\ i -$ *Suc j*]) *auto* **}**
then show *?thesis*
by (*simp add*: *sub-polyfun*)
qed

lemma *polyfun-linear-factor*:
fixes $a :: 'a::\{comm\text{-}ring,monoid\text{-}mult\}$
shows $\exists b.\ \forall z.\ (\sum i{\le}n.\ c\ i * z\hat{}i) =$
$(z{-}a) * (\sum i{<}n.\ b\ i * z\hat{}i) + (\sum i{\le}n.\ c\ i * a\hat{}i)$
proof −
{ fix *z*
have $(\sum i{\le}n.\ c\ i * z\hat{}i) - (\sum i{\le}n.\ c\ i * a\hat{}i) =$
$(z - a) * (\sum j{<}n.\ (\sum k = Suc\ j..n.\ c\ k * a\hat{}(k - Suc\ j)) * z\hat{}j)$
by (*simp add*: *sub-polyfun setsum-left-distrib*)
then have $(\sum i{\le}n.\ c\ i * z\hat{}i) =$
$(z - a) * (\sum j{<}n.\ (\sum k = Suc\ j..n.\ c\ k * a\hat{}(k - Suc\ j)) * z\hat{}j)$
$+ (\sum i{\le}n.\ c\ i * a\hat{}i)$
by (*simp add*: *algebra-simps*) **}**

then show *?thesis*
by (*intro exI allI*)
qed

lemma *polyfun-linear-factor-root*:
fixes $a :: 'a::\{comm\text{-}ring,monoid\text{-}mult\}$
assumes $(\sum i{\le}n.\ c\ i * a\hat{}i) = 0$
shows $\exists b.\ \forall z.\ (\sum i{\le}n.\ c\ i * z\hat{}i) = (z-a) * (\sum i{<}n.\ b\ i * z\hat{}i)$
using *polyfun-linear-factor* [*of c n a*] *assms*
by *simp*

lemma *adhoc-norm-triangle*: $a + norm(y) \le b ==> norm(x) \le a ==> norm(x + y) \le b$
by (*metis norm-triangle-mono order.trans order-refl*)

lemma *polyfun-extremal-lemma*:
fixes $c :: nat \Rightarrow 'a::real\text{-}normed\text{-}div\text{-}algebra$
assumes $e > 0$
shows $\exists M.\ \forall z.\ M \le norm\ z \longrightarrow norm(\sum i{\le}n.\ c\ i * z\hat{}i) \le e * norm(z)\ \hat{}\ Suc\ n$
proof (*induction n*)
case *0*
show *?case*
by (*rule exI* [**where** *x=norm (c 0) / e*]) (*auto simp*: *mult.commute pos-divide-le-eq assms*)
next
case (*Suc n*)
then obtain *M* **where** *M*: $\forall z.\ M \le norm\ z \longrightarrow norm\ (\sum i{\le}n.\ c\ i * z\hat{}i) \le e * norm\ z\ \hat{}\ Suc\ n$ **..**
show *?case*
proof (*rule exI* [**where** *x=max 1 (max M ((e + norm(c(Suc n))) / e))*], *clarify*)
fix *z::'a*
assume *max 1 (max M ((e + norm (c (Suc n))) / e)) ≤ norm z*
then have *norm1*: *0 < norm z M ≤ norm z (e + norm (c (Suc n))) / e ≤ norm z*
by *auto*
then have *norm2*: *(e + norm (c (Suc n))) ≤ e * norm z (norm z * norm z ^ n) > 0*
apply (*metis assms less-divide-eq mult.commute not-le*)
using *norm1* **apply** (*metis mult-pos-pos zero-less-power*)
done
have *e * (norm z * norm z ^ n) + norm (c (Suc n) * (z * z ^ n)) =*
*(e + norm (c (Suc n))) * (norm z * norm z ^ n)*
by (*simp add*: *norm-mult norm-power algebra-simps*)
also have *... ≤ (e * norm z) * (norm z * norm z ^ n)*
using *norm2* **by** (*metis real-mult-le-cancel-iff1*)
also have *... = e * (norm z * (norm z * norm z ^ n))*
by (*simp add*: *algebra-simps*)
finally have *e * (norm z * norm z ^ n) + norm (c (Suc n) * (z * z ^ n))*

$\leq e * (norm\ z * (norm\ z * norm\ z\ \hat{}\ n))$ **.**
then show $norm\ (\sum i{\leq}Suc\ n.\ c\ i * z\hat{}i) \leq e * norm\ z\ \hat{}\ Suc\ (Suc\ n)$ **using** *M norm1*
by (*drule-tac* $x{=}z$ **in** *spec*) (*auto simp*: *intro*!: *adhoc-norm-triangle*)
qed
qed

lemma *norm-lemma-xy*: **assumes** $|b| + 1 \leq norm(y) - a\ norm(x) \leq a$ **shows** $b \leq norm(x + y)$
proof −
have $b \leq norm\ y - norm\ x$
using *assms* **by** *linarith*
then show *?thesis*
by (*metis* (*no-types*) *add.commute norm-diff-ineq order-trans*)
qed

lemma *polyfun-extremal*:
fixes $c :: nat \Rightarrow 'a{::}real\text{-}normed\text{-}div\text{-}algebra$
assumes $\exists k.\ k \neq 0 \wedge k \leq n \wedge c\ k \neq 0$
shows *eventually* $(\lambda z.\ norm(\sum i{\leq}n.\ c\ i * z\hat{}i) \geq B)$ *at-infinity*
using *assms*
proof (*induction n*)
case *0* **then show** *?case*
by *simp*
next
case (*Suc n*)
show *?case*
proof (*cases c* (*Suc n*) = *0*)
case *True*
with *Suc* **show** *?thesis*
by *auto* (*metis diff-is-0-eq diffs0-imp-equal less-Suc-eq-le not-less-eq*)
next
case *False*
with *polyfun-extremal-lemma* [*of norm*(*c* (*Suc n*)) / *2 c n*]
obtain *M* **where** $M{:}\ \bigwedge z.\ M \leq norm\ z \Longrightarrow$
$norm\ (\sum i{\leq}n.\ c\ i * z\hat{}i) \leq norm\ (c\ (Suc\ n))\ /\ 2 * norm\ z\ \hat{}\ Suc\ n$
by *auto*
show *?thesis*
unfolding *eventually-at-infinity*
proof (*rule exI* [**where** $x{=}max\ M\ (max\ 1\ ((|B| + 1)\ /\ (norm\ (c\ (Suc\ n))\ /\ 2)))$], *clarsimp*)
fix $z{::}'a$
assume *les*: $M \leq norm\ z\ \ 1 \leq norm\ z\ \ (|B| * 2 + 2)\ /\ norm\ (c\ (Suc\ n)) \leq norm\ z$
then have $|B| * 2 + 2 \leq norm\ z * norm\ (c\ (Suc\ n))$
by (*metis False pos-divide-le-eq zero-less-norm-iff*)
then have $|B| * 2 + 2 \leq norm\ z\ \hat{}\ (Suc\ n) * norm\ (c\ (Suc\ n))$
by (*metis* ‹$1 \leq norm\ z$› *order.trans mult-right-mono norm-ge-zero self-le-power zero-less-Suc*)

then show $B \leq norm\ ((\sum i{\leq}n.\ c\ i * z\hat{}i) + c\ (Suc\ n) * (z * z\ \hat{}\ n))$ **using** *M les*
apply *auto*
apply (*rule norm-lemma-xy* [**where** $a = norm\ (c\ (Suc\ n)) * norm\ z\ \hat{}\ (Suc\ n)\ /\ 2$])
apply (*simp-all add*: *norm-mult norm-power*)
done
qed
qed
qed

lemma *polyfun-rootbound*:
fixes $c :: nat \Rightarrow 'a{::}\{comm\text{-}ring,real\text{-}normed\text{-}div\text{-}algebra\}$
assumes $\exists k.\ k \leq n \wedge c\ k \neq 0$
shows $finite\ \{z.\ (\sum i{\leq}n.\ c\ i * z\hat{}i) = 0\} \wedge card\ \{z.\ (\sum i{\leq}n.\ c\ i * z\hat{}i) = 0\} \leq n$
using *assms*
proof (*induction n arbitrary*: *c*)
case (*Suc n*) **show** *?case*
proof (*cases* $\{z.\ (\sum i{\leq}Suc\ n.\ c\ i * z\hat{}i) = 0\} = \{\}$)
case *False*
then obtain *a* **where** *a*: $(\sum i{\leq}Suc\ n.\ c\ i * a\hat{}i) = 0$
by *auto*
from *polyfun-linear-factor-root* [*OF this*]
obtain *b* **where** $\bigwedge z.\ (\sum i{\leq}Suc\ n.\ c\ i * z\hat{}i) = (z - a) * (\sum i{<}\ Suc\ n.\ b\ i * z\hat{}i)$
by *auto*
then have *b*: $\bigwedge z.\ (\sum i{\leq}Suc\ n.\ c\ i * z\hat{}i) = (z - a) * (\sum i{\leq}n.\ b\ i * z\hat{}i)$
by (*metis lessThan-Suc-atMost*)
then have *ins-ab*: $\{z.\ (\sum i{\leq}Suc\ n.\ c\ i * z\hat{}i) = 0\} = insert\ a\ \{z.\ (\sum i{\leq}n.\ b\ i * z\hat{}i) = 0\}$
by *auto*
have *c0*: $c\ 0 = -\ (a * b\ 0)$ **using** *b* [*of 0*]
by *simp*
then have *extr-prem*: $\sim\ (\exists k{\leq}n.\ b\ k \neq 0) \Longrightarrow \exists k.\ k \neq 0 \wedge k \leq Suc\ n \wedge c\ k \neq 0$
by (*metis Suc.prems le0 minus-zero mult-zero-right*)
have $\exists k{\leq}n.\ b\ k \neq 0$
apply (*rule ccontr*)
using *polyfun-extremal* [*OF extr-prem*, *of 1*]
apply (*auto simp*: *eventually-at-infinity b simp del*: *setsum-atMost-Suc*)
apply (*drule-tac x=of-real ba* **in** *spec*, *simp*)
done
then show *?thesis* **using** *Suc.IH* [*of b*] *ins-ab*
by (*auto simp*: *card-insert-if*)
qed *simp*
qed *simp*

corollary

fixes $c :: nat \Rightarrow 'a::\{comm\text{-}ring,real\text{-}normed\text{-}div\text{-}algebra\}$
assumes $\exists k.\ k \leq n \wedge c\ k \neq 0$
shows *polyfun-rootbound-finite*: *finite* $\{z.\ (\sum i{\leq}n.\ c\ i * z\hat{}i) = 0\}$
and *polyfun-rootbound-card*: *card* $\{z.\ (\sum i{\leq}n.\ c\ i * z\hat{}i) = 0\} \leq n$
using *polyfun-rootbound* [*OF assms*] **by** *auto*

lemma *polyfun-finite-roots*:
fixes $c :: nat \Rightarrow 'a::\{comm\text{-}ring,real\text{-}normed\text{-}div\text{-}algebra\}$
shows *finite* $\{z.\ (\sum i{\leq}n.\ c\ i * z\hat{}i) = 0\} \longleftrightarrow (\exists k.\ k \leq n \wedge c\ k \neq 0)$
proof (*cases* $\exists k{\leq}n.\ c\ k \neq 0$)
case *True* **then show** *?thesis*
by (*blast intro*: *polyfun-rootbound-finite*)
next
case *False* **then show** *?thesis*
by (*auto simp*: *infinite-UNIV-char-0*)
qed

lemma *polyfun-eq-0*:
fixes $c :: nat \Rightarrow 'a::\{comm\text{-}ring,real\text{-}normed\text{-}div\text{-}algebra\}$
shows $(\forall z.\ (\sum i{\leq}n.\ c\ i * z\hat{}i) = 0) \longleftrightarrow (\forall k.\ k \leq n \longrightarrow c\ k = 0)$
proof (*cases* $(\forall z.\ (\sum i{\leq}n.\ c\ i * z\hat{}i) = 0)$)
case *True*
then have ~ *finite* $\{z.\ (\sum i{\leq}n.\ c\ i * z\hat{}i) = 0\}$
by (*simp add*: *infinite-UNIV-char-0*)
with *True* **show** *?thesis*
by (*metis* (*poly-guards-query*) *polyfun-rootbound-finite*)
next
case *False*
then show *?thesis*
by *auto*
qed

lemma *polyfun-eq-const*:
fixes $c :: nat \Rightarrow 'a::\{comm\text{-}ring,real\text{-}normed\text{-}div\text{-}algebra\}$
shows $(\forall z.\ (\sum i{\leq}n.\ c\ i * z\hat{}i) = k) \longleftrightarrow c\ 0 = k \wedge (\forall k.\ k \neq 0 \wedge k \leq n \longrightarrow c\ k = 0)$
proof −
{fix z
have $(\sum i{\leq}n.\ c\ i * z\hat{}i) = (\sum i{\leq}n.\ (\textit{if } i = 0 \textit{ then } c\ 0 - k \textit{ else } c\ i) * z\hat{}i) + k$
by (*induct n*) *auto*
} then
have $(\forall z.\ (\sum i{\leq}n.\ c\ i * z\hat{}i) = k) \longleftrightarrow (\forall z.\ (\sum i{\leq}n.\ (\textit{if } i = 0 \textit{ then } c\ 0 - k \textit{ else } c\ i) * z\hat{}i) = 0)$
by *auto*
also have ... $\longleftrightarrow$ $c\ 0 = k \wedge (\forall k.\ k \neq 0 \wedge k \leq n \longrightarrow c\ k = 0)$
by (*auto simp*: *polyfun-eq-0*)
finally show *?thesis* **.**
qed

end