The Irrationality of $\zeta(3)$

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Abstract

This article provides a formalisation of Beukers's straightforward analytic proof [2] that $\zeta(3)$ is irrational. This was first proven by Apéry [1] (which is why this result is also often called 'Apéry's Theorem') using a more algebraic approach. This formalisation follows Filaseta's presentation of Beukers's proof [5].

Contents

1 The Irrationality of $\zeta(3)$

theory *Zeta-3-Irrational*
imports
E-Transcendental.E-Transcendental
Prime-Number-Theorem.Prime-Number-Theorem
Prime-Distribution-Elementary.PNT-Consequences
begin

hide-const (**open**) *UnivPoly.coeff* *UnivPoly.up-ring.monom*
hide-const (**open**) *Module.smult* *Coset.order*

Apéry's original proof of the irrationality of $\zeta(3)$ contained several tricky identities of sums involving binomial coefficients that are difficult to prove. I instead follow Beukers's proof, which consists of several fairly straightforward manipulations of integrals with absolutely no caveats.

Both Apéry and Beukers make use of an asymptotic upper bound on $\text{lcm}\{1 \ldots n\}$ – namely $\text{lcm}\{1 \ldots n\} \in o(c^n)$ for any $c > e$, which is a consequence of the Prime Number Theorem (which, fortunately, is available in the *Archive of Formal Proofs* [4, 3]).

I follow the concise presentation of Beukers's proof in Filaseta's lecture notes [5], which turned out to be very amenable to formalisation.

There is another earlier formalisation of the irrationality of $\zeta(3)$ by Mahboubi *et al.* [6], who followed Apéry's original proof, but were ultimately forced to find a more elementary way to prove the asymptotics of $\text{lcm}\{1 \ldots n\}$ than the Prime Number Theorem.

First, we will require some auxiliary material before we get started with the actual proof.

1.1 Auxiliary facts about polynomials

lemma *higher-pderiv-minus*: $(pderiv \;\widehat{}\widehat{}\; n)\ (-p :: 'a :: idom\ poly) = -(pderiv \;\widehat{}\widehat{}\; n)\ p$
⟨*proof*⟩

lemma *pderiv-power*: $pderiv\ (p\ \widehat{}\ n) = smult\ (of\text{-}nat\ n)\ (p\ \widehat{}\ (n - 1)) * pderiv\ p$
⟨*proof*⟩

lemma *higher-pderiv-monom*:
$k \le n \Longrightarrow (pderiv \;\widehat{}\widehat{}\; k)\ (monom\ c\ n) = monom\ (of\text{-}nat\ (pochhammer\ (n - k + 1)\ k) * c)\ (n - k)$
⟨*proof*⟩

lemma *higher-pderiv-mult*:
$(pderiv \;\widehat{}\widehat{}\; n)\ (p * q) =$
$(\sum k \le n.\ Polynomial.smult\ (of\text{-}nat\ (n\ choose\ k))\ ((pderiv \;\widehat{}\widehat{}\; k)\ p * (pderiv \;\widehat{}\widehat{}\; (n - k))\ q))$

⟨*proof*⟩

1.2 Auxiliary facts about integrals

theorem (**in** *pair-sigma-finite*) *Fubini-set-integrable*:
  fixes f :: - ⇒ -::{*banach*, *second-countable-topology*}
  assumes *f*[*measurable*]: *set-borel-measurable* ($M1 \bigotimes_M M2$) ($A \times B$) f
    and *integ1*: *set-integrable M1 A* ($\lambda x.\ \int y{\in}B.\ norm\ (f\ (x,\ y))\ \partial M2$)
    and *integ2*: *AE x*∈*A in M1*. *set-integrable M2 B* ($\lambda y.\ f\ (x,\ y)$)
  shows *set-integrable* ($M1 \bigotimes_M M2$) ($A \times B$) f
  ⟨*proof*⟩

lemma (**in** *pair-sigma-finite*) *set-integral-fst′*:
  fixes f :: - ⇒ ′*c* :: {*second-countable-topology*, *banach*}
  assumes *set-integrable* ($M1 \bigotimes_M M2$) ($A \times B$) f
  shows *set-lebesgue-integral* ($M1 \bigotimes_M M2$) ($A \times B$) f =
      ($\int x{\in}A.\ (\int y{\in}B.\ f\ (x,\ y)\ \partial M2)\ \partial M1$)
⟨*proof*⟩

lemma (**in** *pair-sigma-finite*) *set-integral-snd*:
  fixes f :: - ⇒ ′*c* :: {*second-countable-topology*, *banach*}
  assumes *set-integrable* ($M1 \bigotimes_M M2$) ($A \times B$) f
  shows *set-lebesgue-integral* ($M1 \bigotimes_M M2$) ($A \times B$) f =
      ($\int y{\in}B.\ (\int x{\in}A.\ f\ (x,\ y)\ \partial M1)\ \partial M2$)
⟨*proof*⟩

proposition (**in** *pair-sigma-finite*) *Fubini-set-integral*:
  fixes f :: - ⇒ - ⇒ - :: {*banach*, *second-countable-topology*}
  assumes *f*: *set-integrable* ($M1 \bigotimes_M M2$) ($A \times B$) (*case-prod f*)
  shows ($\int y{\in}B.\ (\int x{\in}A.\ f\ x\ y\ \partial M1)\ \partial M2$) = ($\int x{\in}A.\ (\int y{\in}B.\ f\ x\ y\ \partial M2)\ \partial M1$)
⟨*proof*⟩

lemma (**in** *pair-sigma-finite*) *nn-integral-swap*:
  assumes [*measurable*]: $f \in$ *borel-measurable* ($M1 \bigotimes_M M2$)
  shows ($\int^+ x.\ f\ x\ \partial(M1 \bigotimes_M M2)$) = ($\int^+ (y,x).\ f\ (x,y)\ \partial(M2 \bigotimes_M M1)$)
  ⟨*proof*⟩

lemma *set-integrable-bound*:
  fixes f :: ′*a* ⇒ ′*b*::{*banach*, *second-countable-topology*}
    and g :: ′*a* ⇒ ′*c*::{*banach*, *second-countable-topology*}
  shows *set-integrable M A f* ⟹ *set-borel-measurable M A g* ⟹
      (*AE x in M*. $x \in A \longrightarrow norm\ (g\ x) \le norm\ (f\ x)$) ⟹ *set-integrable M*
A g
  ⟨*proof*⟩

lemma *set-integrableI-nonneg*:
  fixes f :: ′*a* ⇒ *real*
  assumes *set-borel-measurable M A f*
  assumes *AE x in M*. $x \in A \longrightarrow 0 \le f\ x\ (\int^+ x{\in}A.\ f\ x\ \partial M) < \infty$

shows *set-integrable M A f*
⟨*proof*⟩

lemma *set-integral-eq-nn-integral*:
assumes *set-borel-measurable M A f*
assumes *set-nn-integral M A f = ennreal x x ≥ 0*
assumes *AE x in M. x ∈ A ⟶ f x ≥ 0*
shows *set-integrable M A f*
and *set-lebesgue-integral M A f = x*
⟨*proof*⟩

lemma *set-integral-0* [*simp*, *intro*]: *set-integrable M A* (*λy. 0*)
⟨*proof*⟩

lemma *set-integrable-sum*:
fixes *f* :: - ⇒ - ⇒ - :: {*banach*, *second-countable-topology*}
assumes *finite B*
assumes ⋀*x. x ∈ B* ⟹ *set-integrable M A* (*f x*)
shows *set-integrable M A* (*λy.* ∑ *x∈B. f x y*)
⟨*proof*⟩

lemma *set-integral-sum*:
fixes *f* :: - ⇒ - ⇒ - :: {*banach*, *second-countable-topology*}
assumes *finite B*
assumes ⋀*x. x ∈ B* ⟹ *set-integrable M A* (*f x*)
shows *set-lebesgue-integral M A* (*λy.* ∑ *x∈B. f x y*) = (∑ *x∈B. set-lebesgue-integral M A* (*f x*))
⟨*proof*⟩

lemma *set-nn-integral-cong*:
assumes *M = M′ A = B* ⋀*x. x ∈ space M ∩ A* ⟹ *f x = g x*
shows *set-nn-integral M A f = set-nn-integral M′ B g*
⟨*proof*⟩

lemma *set-nn-integral-mono*:
assumes ⋀*x. x ∈ space M ∩ A* ⟹ *f x ≤ g x*
shows *set-nn-integral M A f ≤ set-nn-integral M A g*
⟨*proof*⟩

lemma
fixes *f* :: *real* ⇒ *real*
assumes *a ≤ b*
assumes *deriv*: ⋀*x. a ≤ x* ⟹ *x ≤ b* ⟹ (*F has-field-derivative f x*) (*at x within* {*a..b*})
assumes *cont*: *continuous-on* {*a..b*} *f*
shows *has-bochner-integral-FTC-Icc-real*:
has-bochner-integral lborel (*λx. f x* ∗ *indicator* {*a* .. *b*} *x*) (*F b* − *F a*) (**is** *?has*)
and *integral-FTC-Icc-real*: (∫ *x. f x* ∗ *indicator* {*a* .. *b*} *x* ∂*lborel*) = *F b* − *F*

a (**is** *?eq*)
⟨*proof*⟩

lemma *integral-by-parts-integrable*:
fixes *f g F G*::*real* ⇒ *real*
assumes $a \le b$
assumes *cont-f*[*intro*]: *continuous-on* {*a..b*} *f*
assumes *cont-g*[*intro*]: *continuous-on* {*a..b*} *g*
assumes [*intro*]: ⋀*x*. *x* ∈ {*a..b*} ⟹ (*F has-field-derivative f x*) (*at x within* {*a..b*})
assumes [*intro*]: ⋀*x*. *x* ∈ {*a..b*} ⟹ (*G has-field-derivative g x*) (*at x within* {*a..b*})
shows *integrable lborel* (λ*x*.((*F x*) ∗ (*g x*) + (*f x*) ∗ (*G x*)) ∗ *indicator* {*a* .. *b*} *x*)
⟨*proof*⟩

lemma *integral-by-parts*:
fixes *f g F G*::*real* ⇒ *real*
assumes [*arith*]: $a \le b$
assumes *cont-f*[*intro*]: *continuous-on* {*a..b*} *f*
assumes *cont-g*[*intro*]: *continuous-on* {*a..b*} *g*
assumes [*intro*]: ⋀*x*. *x* ∈ {*a..b*} ⟹ (*F has-field-derivative f x*) (*at x within* {*a..b*})
assumes [*intro*]: ⋀*x*. *x* ∈ {*a..b*} ⟹ (*G has-field-derivative g x*) (*at x within* {*a..b*})
shows (∫ *x*. (*F x* ∗ *g x*) ∗ *indicator* {*a* .. *b*} *x* ∂*lborel*)
= *F b* ∗ *G b* − *F a* ∗ *G a* − ∫ *x*. (*f x* ∗ *G x*) ∗ *indicator* {*a* .. *b*} *x* ∂*lborel*
⟨*proof*⟩

lemma *interval-lebesgue-integral-by-parts*:
assumes $a \le b$
assumes *cont-f*[*intro*]: *continuous-on* {*a..b*} *f*
assumes *cont-g*[*intro*]: *continuous-on* {*a..b*} *g*
assumes [*intro*]: ⋀*x*. *x* ∈ {*a..b*} ⟹ (*F has-field-derivative f x*) (*at x within* {*a..b*})
assumes [*intro*]: ⋀*x*. *x* ∈ {*a..b*} ⟹ (*G has-field-derivative g x*) (*at x within* {*a..b*})
shows (*LBINT x*=*a..b*. *F x* ∗ *g x*) = *F b* ∗ *G b* − *F a* ∗ *G a* − (*LBINT x*=*a..b*. *f x* ∗ *G x*)
⟨*proof*⟩

lemma *interval-lebesgue-integral-by-parts-01*:
assumes *cont-f*[*intro*]: *continuous-on* {*0..1*} *f*
assumes *cont-g*[*intro*]: *continuous-on* {*0..1*} *g*
assumes [*intro*]: ⋀*x*. *x* ∈ {*0..1*} ⟹ (*F has-field-derivative f x*) (*at x within* {*0..1*})
assumes [*intro*]: ⋀*x*. *x* ∈ {*0..1*} ⟹ (*G has-field-derivative g x*) (*at x within* {*0..1*})
shows (*LBINT x*=*0..1*. *F x* ∗ *g x*) = *F 1* ∗ *G 1* − *F 0* ∗ *G 0* − (*LBINT x*=*0..1*.

f x ∗ G x)
  ⟨*proof*⟩

lemma *continuous-on-imp-set-integrable-cbox*:
  fixes *h* :: *'a* :: *euclidean-space* ⇒ *real*
  assumes *continuous-on* (*cbox a b*) *h*
  shows *set-integrable lborel* (*cbox a b*) *h*
⟨*proof*⟩

1.3 Shifted Legendre polynomials

The first ingredient we need to show Apéry's theorem is the *shifted Legendre polynomials*

$$P_n(X) = \frac{1}{n!}\frac{\partial^n}{\partial X^n}(X^n(1-X)^n)$$

and the auxiliary polynomials

$$Q_{n,k}(X) = \frac{\partial^k}{\partial X^k}(X^n(1-X)^n)\ .$$

Note that P_n is in fact an *integer* polynomial.

Only some very basic properties of these will be proven, since that is all we will need.

context
  fixes *n* :: *nat*
begin

definition *gen-shleg-poly* :: *nat* ⇒ *int poly* **where**
  gen-shleg-poly k = (*pderiv* ^^ *k*) ([:*0*, *1*, −*1*:] ^ *n*)

definition *shleg-poly* **where** *shleg-poly* = *gen-shleg-poly n div* [:*fact n*:]

We can easily prove the following more explicit formula for $Q_{n,k}$:

$$Q_{n,k}(X) = \sum_{i=0}^{k}(-1)^{k-1}\binom{k}{i}n^{\underline{i}}\,n^{\underline{k-i}}\,X^{n-i}\,(1-X)^{n-k+i}$$

lemma *gen-shleg-poly-altdef*:
  assumes *k* ≤ *n*
  shows *gen-shleg-poly k* =
        (∑ *i*≤*k*. *smult* ((−*1*) ^ (*k*−*i*) ∗ *of-nat* (*k choose i*) ∗
              pochhammer (*n*−*i*+*1*) *i* ∗ *pochhammer* (*n*−*k*+*i*+*1*) (*k*−*i*))
              ([:*0*, *1*:] ^ (*n* − *i*) ∗ [:*1*, −*1*:] ^ (*n*−*k*+*i*)))
⟨*proof*⟩

lemma *degree-gen-shleg-poly* [*simp*]: *degree* (*gen-shleg-poly k*) = *2* ∗ *n* − *k*
  ⟨*proof*⟩

lemma *gen-shleg-poly-n*: *gen-shleg-poly n* = *smult* (*fact n*) *shleg-poly*
⟨*proof*⟩

lemma *degree-shleg-poly* [*simp*]: *degree shleg-poly* = *n*
⟨*proof*⟩

lemma *pderiv-gen-shleg-poly* [*simp*]: *pderiv* (*gen-shleg-poly k*) = *gen-shleg-poly* (*Suc k*)
⟨*proof*⟩

The following functions are the interpretation of the shifted Legendre polynomials and the auxiliary polynomials as a function from reals to reals.

definition *Gen-Shleg* :: *nat* ⇒ *real* ⇒ *real*
where *Gen-Shleg k x* = *poly* (*of-int-poly* (*gen-shleg-poly k*)) *x*

definition *Shleg* :: *real* ⇒ *real* **where** *Shleg* = *poly* (*of-int-poly shleg-poly*)

lemma *Gen-Shleg-altdef*:
assumes $k \leq n$
shows *Gen-Shleg k x* = (∑ *i*≤*k*. (−*1*)^(*k*−*i*) ∗ *of-nat* (*k choose i*) ∗ *of-int* (*pochhammer* (*n*−*i*+*1*) *i* ∗ *pochhammer* (*n*−*k*+*i*+*1*) (*k*−*i*)) ∗ *x* ^ (*n* − *i*) ∗ (*1* − *x*) ^ (*n*−*k*+*i*))
⟨*proof*⟩

lemma *Gen-Shleg-0* [*simp*]: *k* < *n* ⟹ *Gen-Shleg k 0* = *0*
⟨*proof*⟩

lemma *Gen-Shleg-1* [*simp*]: *k* < *n* ⟹ *Gen-Shleg k 1* = *0*
⟨*proof*⟩

lemma *Gen-Shleg-n-0* [*simp*]: *Gen-Shleg n 0* = *fact n*
⟨*proof*⟩

lemma *Gen-Shleg-n-1* [*simp*]: *Gen-Shleg n 1* = (−*1*) ^ *n* ∗ *fact n*
⟨*proof*⟩

lemma *Shleg-altdef*: *Shleg x* = *Gen-Shleg n x* / *fact n*
⟨*proof*⟩

lemma *Shleg-0* [*simp*]: *Shleg 0* = *1* **and** *Shleg-1* [*simp*]: *Shleg 1* = (−*1*) ^ *n*
⟨*proof*⟩

lemma *Gen-Shleg-0-left*: *Gen-Shleg 0 x* = *x* ^ *n* ∗ (*1* − *x*) ^ *n*
⟨*proof*⟩

lemma *has-field-derivative-Gen-Shleg*:
(*Gen-Shleg k has-field-derivative Gen-Shleg* (*Suc k*) *x*) (*at x*)

⟨*proof*⟩

lemma *continuous-on-Gen-Shleg*: *continuous-on A* (*Gen-Shleg k*)
⟨*proof*⟩

lemma *continuous-on-Gen-Shleg′* [*continuous-intros*]:
continuous-on A f ⟹ *continuous-on A* (*λx. Gen-Shleg k* (*f x*))
⟨*proof*⟩

lemma *continuous-on-Shleg*: *continuous-on A Shleg*
⟨*proof*⟩

lemma *continuous-on-Shleg′* [*continuous-intros*]:
continuous-on A f ⟹ *continuous-on A* (*λx. Shleg* (*f x*))
⟨*proof*⟩

lemma *measurable-Gen-Shleg* [*measurable*]: *Gen-Shleg n* ∈ *borel-measurable borel*
⟨*proof*⟩

lemma *measurable-Shleg* [*measurable*]: *Shleg* ∈ *borel-measurable borel*
⟨*proof*⟩

end

1.4 Auxiliary facts about the ζ function

lemma *Re-zeta-ge-1*:
assumes *x* > *1*
shows *Re* (*zeta* (*of-real x*)) ≥ *1*
⟨*proof*⟩

lemma *sums-zeta-of-nat-offset*:
fixes *r* :: *nat*
assumes *n*: *n* > *1*
shows (*λk. 1* / (*r* + *k* + *1*) ^ *n*) *sums* (*zeta* (*of-nat n*) − (∑*k*=*1*..*r*. *1* / *k* ^ *n*))
⟨*proof*⟩

lemma *sums-Re-zeta-of-nat-offset*:
fixes *r* :: *nat*
assumes *n*: *n* > *1*
shows (*λk. 1* / (*r* + *k* + *1*) ^ *n*) *sums* (*Re* (*zeta* (*of-nat n*)) − (∑*k*=*1*..*r*. *1* / *k* ^ *n*))
⟨*proof*⟩

1.5 Divisor of a sum of rationals

A finite sum of rationals of the form $\frac{a_1}{b_1} + \ldots + \frac{a_n}{b_n}$ can be brought into the form $\frac{c}{d}$, where d is the LCM of the b_i (or some integer multiple thereof).

lemma *sum-rationals-common-divisor*:
  fixes $f\ g :: {}'a \Rightarrow int$
  assumes *finite A*
  assumes $\bigwedge x.\ x \in A \Longrightarrow g\ x \neq 0$
  shows $\exists c.\ (\sum x{\in}A.\ f\ x\ /\ g\ x) = real\text{-}of\text{-}int\ c\ /\ (LCM\ x{\in}A.\ g\ x)$
  ⟨*proof*⟩

lemma *sum-rationals-common-divisor'*:
  fixes $f\ g :: {}'a \Rightarrow int$
  assumes *finite A*
  assumes $\bigwedge x.\ x \in A \Longrightarrow g\ x \neq 0$ **and** $(\bigwedge x.\ x \in A \Longrightarrow g\ x\ dvd\ d)$ **and** $d \neq 0$
  shows $\exists c.\ (\sum x{\in}A.\ f\ x\ /\ g\ x) = real\text{-}of\text{-}int\ c\ /\ real\text{-}of\text{-}int\ d$
⟨*proof*⟩

1.6 The first double integral

We shall now investigate the double integral

$$I_1 := \int_0^1 \int_0^1 -\frac{\ln(xy)}{1-xy}\, x^r y^s\, \mathrm{d}x\, \mathrm{d}y\ .$$

Since everything is non-negative for now, we can work over the extended non-negative real numbers and the issues of integrability or summability do not arise at all.

definition *beukers-nn-integral1* :: $nat \Rightarrow nat \Rightarrow ennreal$ **where**
  beukers-nn-integral1 r s =
    $(\int^+ (x,y){\in}\{0{<}..{<}1\}{\times}\{0{<}..{<}1\}.\ ennreal\ (-ln\ (x{*}y)\ /\ (1 - x{*}y) * x\hat{\ }r * y\hat{\ }s)\ \partial lborel)$

definition *beukers-integral1* :: $nat \Rightarrow nat \Rightarrow real$ **where**
  $beukers\text{-}integral1\ r\ s = (\int (x,y){\in}\{0{<}..{<}1\}{\times}\{0{<}..{<}1\}.\ (-ln\ (x{*}y)\ /\ (1 - x{*}y) * x\hat{\ }r * y\hat{\ }s)\ \partial lborel)$

lemma
  fixes $x\ y\ z :: real$
  assumes *xyz*: $x \in \{0{<}..{<}1\}\ y \in \{0{<}..{<}1\}\ z \in \{0..1\}$
  shows *beukers-denom-ineq*: $(1 - x * y) * z < 1$ **and** *beukers-denom-neq*: $(1 - x * y) * z \neq 1$
⟨*proof*⟩

We first evaluate the improper integral

$$\int_0^1 -\ln x \cdot x^e\, \mathrm{d}x = \frac{1}{(e+1)^2}\ .$$

for any $e > -1$.

lemma *integral-0-1-ln-times-powr*:
  assumes $e > -1$

shows (*LBINT x=0..1. −ln x ∗ x powr e*) = *1* / (*e* + *1*)^2
and *interval-lebesgue-integrable lborel 0 1* (*λx. −ln x ∗ x powr e*)
⟨*proof*⟩

lemma *interval-lebesgue-integral-lborel-01-cong*:
assumes ⋀*x. x* ∈ {*0<..<1*} ⟹ *f x* = *g x*
shows *interval-lebesgue-integral lborel 0 1 f* =
interval-lebesgue-integral lborel 0 1 g
⟨*proof*⟩

lemma *nn-integral-0-1-ln-times-powr*:
assumes *e* > −*1*
shows (∫$^+$*y*∈{*0<..<1*}. *ennreal* (−*ln y* ∗ *y powr e*) ∂*lborel*) = *ennreal* (*1* / (*e* + *1*)^2)
⟨*proof*⟩

lemma *nn-integral-0-1-ln-times-power*:
(∫$^+$*y*∈{*0<..<1*}. *ennreal* (−*ln y* ∗ *y* ^ *n*) ∂*lborel*) = *ennreal* (*1* / (*n* + *1*)^2)
⟨*proof*⟩

Next, we also evaluate the more trivial integral

$$\int_0^1 x^n \, \mathrm{d}x \ .$$

lemma *nn-integral-0-1-power*:
(∫$^+$*y*∈{*0<..<1*}. *ennreal* (*y* ^ *n*) ∂*lborel*) = *ennreal* (*1* / (*n* + *1*))
⟨*proof*⟩

I_1 can alternatively be written as the triple integral

$$\int_0^1 \int_0^1 \int_0^1 \frac{x^r y^s}{1-(1-xy)w} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}w \ .$$

lemma *beukers-nn-integral1-altdef*:
beukers-nn-integral1 r s =
(∫$^+$(*w*,*x*,*y*)∈{*0<..<1*}×{*0<..<1*}×{*0<..<1*}.
ennreal (*1* / (*1*−(*1*−*x*∗*y*)∗*w*) ∗ *x*^*r* ∗ *y*^*s*) ∂*lborel*)
⟨*proof*⟩

context
fixes *r s* :: *nat* **and** *I1 I2′* :: *real* **and** *I2* :: *ennreal* **and** *D* :: (*real* × *real* × *real*) *set*
assumes *rs*: *s* ≤ *r*
defines *D* ≡ {*0<..<1*}×{*0<..<1*}×{*0<..<1*}
begin

By unfolding the geometric series, pulling the summation out and evaluating the integrals, we find that

$$I_1 = \sum_{k=0}^{\infty} \frac{1}{(k+r+1)^2(k+s+1)} + \frac{1}{(k+r+1)(k+s+1)^2} \ .$$

lemma *beukers-nn-integral1-series*:
beukers-nn-integral1 r s = ($\sum$ *k. ennreal* (*1*/((*k*+*r*+*1*) ^*2*∗(*k*+*s*+*1*)) + *1*/((*k*+*r*+*1*)∗(*k*+*s*+*1*) ^*2*)))
⟨*proof*⟩

Remembering that $\zeta(3) = \sum k^{-3}$, it is easy to see that if $r = s$, this sum is simply

$$2\left(\zeta(3) - \sum_{k=1}^{r} \frac{1}{k^3}\right) \ .$$

lemma *beukers-nn-integral1-same*:
assumes *r* = *s*
shows *beukers-nn-integral1 r s* = *ennreal* (*2* ∗ (*Re* (*zeta 3*) − ($\sum$ *k*=*1*..*r*. *1* / *k* ^ *3*)))
and *2* ∗ (*Re* (*zeta 3*) − ($\sum$ *k*=*1*..*r*. *1* / *k* ^ *3*)) ≥ *0*
⟨*proof*⟩

lemma *beukers-integral1-same*:
assumes *r* = *s*
shows *beukers-integral1 r s* = *2* ∗ (*Re* (*zeta 3*) − ($\sum$ *k*=*1*..*r*. *1* / *k* ^ *3*))
⟨*proof*⟩

In contrast, for $r > s$, we find that

$$I_1 = \frac{1}{r-s} \sum_{k=s+1}^{r} \frac{1}{k^2} \ .$$

lemma *beukers-nn-integral1-different*:
assumes *r* > *s*
shows *beukers-nn-integral1 r s* = *ennreal* (($\sum$ *k*∈{*s*<..*r*}. *1* / *k* ^ *2*) / (*r* − *s*))
⟨*proof*⟩

lemma *beukers-integral1-different*:
assumes *r* > *s*
shows *beukers-integral1 r s* = ($\sum$ *k*∈{*s*<..*r*}. *1* / *k* ^ *2*) / (*r* − *s*)
⟨*proof*⟩

end

It is also easy to see that if we exchange r and s, nothing changes.

lemma *beukers-nn-integral1-swap*:
beukers-nn-integral1 r s = *beukers-nn-integral1 s r*
⟨*proof*⟩

lemma *beukers-nn-integral1-finite*: *beukers-nn-integral1 r s* < ∞
⟨*proof*⟩

lemma *beukers-integral1-integrable*:
set-integrable lborel ({*0*<..<*1*}×{*0*<..<*1*})
(λ(*x*,*y*). (−*ln* (*x*∗*y*) / (*1* − *x*∗*y*) ∗ *x*^*r* ∗ *y*^*s* :: *real*))
⟨*proof*⟩

lemma *beukers-integral1-integrable′*:
set-integrable lborel ({*0*<..<*1*}×{*0*<..<*1*}×{*0*<..<*1*})
(λ(*z*,*x*,*y*). (*x*^*r* ∗ *y*^*s* / (*1* − (*1* − *x*∗*y*) ∗ *z*) :: *real*))
⟨*proof*⟩

lemma *beukers-integral1-conv-nn-integral*:
beukers-integral1 r s = *enn2real* (*beukers-nn-integral1 r s*)
⟨*proof*⟩

lemma *beukers-integral1-swap*: *beukers-integral1 r s* = *beukers-integral1 s r*
⟨*proof*⟩

1.7 The second double integral

context
fixes *n* :: *nat*
fixes *D* :: (*real* × *real*) *set* **and** *D′* :: (*real* × *real* × *real*) *set*
fixes *P* :: *real* ⇒ *real* **and** *Q* :: *nat* ⇒ *real* ⇒ *real*
defines *D* ≡ {*0*<..<*1*} × {*0*<..<*1*} **and** *D′* ≡ {*0*<..<*1*} × {*0*<..<*1*} × {*0*<..<*1*}
defines *Q* ≡ *Gen-Shleg n* **and** *P* ≡ *Shleg n*
begin

The next integral to consider is the following variant of I_1:

$$I_2 := \int_0^1 \int_0^1 -\frac{\ln(xy)}{1-xy} P_n(x) P_n(y)\,\mathrm{d}x\,\mathrm{d}y\ .$$

definition *beukers-integral2* :: *real* **where**
beukers-integral2 = (∫ (*x*,*y*)∈*D*. (−*ln* (*x*∗*y*) / (*1*−*x*∗*y*) ∗ *P x* ∗ *P y*) ∂*lborel*)

I_2 is simply a sum of integrals of type I_1, so using our results for I_1, we can write I_2 in the form $A\zeta(3) + \frac{B}{\mathrm{lcm}\{1...n\}^3}$ where A and B are integers and $A > 0$:

lemma *beukers-integral2-conv-int-combination*:
obtains *A B* :: *int* **where** *A* > *0* **and**
beukers-integral2 = *of-int A* ∗ *Re* (*zeta 3*) + *of-int B* / *of-nat* (*Lcm* {*1*..*n*} ^ *3*)
⟨*proof*⟩

lemma *beukers-integral2-integrable*:
set-integrable lborel D (λ(*x*,*y*). −*ln* (*x*∗*y*) / (*1* − *x*∗*y*) ∗ *P x* ∗ *P y*)
⟨*proof*⟩

1.8 The triple integral

Lastly, we turn to the triple integral

$$I_3 := \int_0^1 \int_0^1 \int_0^1 \frac{(x(1-x)y(1-y)w(1-w))^n}{(1-(1-xy)w)^{n+1}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}w\ .$$

definition *beukers-nn-integral3* :: *ennreal* **where**
beukers-nn-integral3 =
(∫⁺(*w*,*x*,*y*)∈*D*′. ((*x*∗(*1*−*x*)∗*y*∗(*1*−*y*)∗*w*∗(*1*−*w*))^*n* / (*1*−(*1*−*x*∗*y*)∗*w*)^(*n*+*1*))
∂*lborel*)

definition *beukers-integral3* :: *real* **where**
beukers-integral3 =
(∫ (*w*,*x*,*y*)∈*D*′. ((*x*∗(*1*−*x*)∗*y*∗(*1*−*y*)∗*w*∗(*1*−*w*))^*n* / (*1*−(*1*−*x*∗*y*)∗*w*)^(*n*+*1*))
∂*lborel*)

We first prove the following bound (which is a consequence of the arithmetic–geometric mean inequality) that will help us bound the triple integral.

lemma *beukers-integral3-integrand-bound*:
fixes *x y z* :: *real*
assumes *xyz*: *x* ∈ {*0*<..<*1*} *y* ∈ {*0*<..<*1*} *z* ∈ {*0*<..<*1*}
shows (*x*∗(*1*−*x*)∗*y*∗(*1*−*y*)∗*z*∗(*1*−*z*)) / (*1*−(*1*−*x*∗*y*)∗*z*) ≤ *1* / *27* (**is** *?lhs* ≤ -)
⟨*proof*⟩

Connecting the above bound with our results of I_1, it is easy to see that $I_3 \leq 2 \cdot 27^{-n} \cdot \zeta(3)$:

lemma *beukers-nn-integral3-le*:
beukers-nn-integral3 ≤ *ennreal* (*2* ∗ (*1* / *27*) ^ *n* ∗ *Re* (*zeta 3*))
⟨*proof*⟩

lemma *beukers-nn-integral3-finite*: *beukers-nn-integral3* < ∞
⟨*proof*⟩

lemma *beukers-integral3-integrable*:
set-integrable lborel D′ (λ(*w*,*x*,*y*). (*x*∗(*1*−*x*)∗*y*∗(*1*−*y*)∗*w*∗(*1*−*w*))^*n* / (*1*−(*1*−*x*∗*y*)∗*w*)^(*n*+*1*))
⟨*proof*⟩

lemma *beukers-integral3-conv-nn-integral*:
beukers-integral3 = *enn2real beukers-nn-integral3*
⟨*proof*⟩

lemma *beukers-integral3-le*: *beukers-integral3* ≤ *2* ∗ (*1* / *27*) ^ *n* ∗ *Re* (*zeta 3*)
⟨*proof*⟩

It is also easy to see that $I_3 > 0$.

lemma *beukers-nn-integral3-pos*: *beukers-nn-integral3 > 0*
⟨*proof*⟩

lemma *beukers-integral3-pos*: *beukers-integral3 > 0*
⟨*proof*⟩

1.9 Connecting the double and triple integral

In this section, we will prove the most technically involved part, namely that $I_2 = I_3$. I will not go into detail about how this works – the reader is advised to simply look at Filaseta's presentation of the proof.

The basic idea is to integrate by parts n times with respect to y to eliminate the factor $P(y)$, then change variables $z = \frac{1-w}{1-(1-xy)w}$, and then apply the same integration by parts n times to x to eliminate $P(x)$.

The first expand

$$-\frac{\ln(xy)}{1-xy} = \int_0^1 \frac{1}{1-(1-xy)z}\,\mathrm{d}z\ .$$

lemma *beukers-aux-ln-conv-integral*:
  fixes *x y* :: *real*
  assumes *xy*: *x* ∈ {*0*<..<*1*} *y* ∈ {*0*<..<*1*}
  shows −*ln* (*x*∗*y*) / (*1*−*x*∗*y*) = (*LBINT z=0..1. 1* / (*1*−(*1*−*x*∗*y*)∗*z*))
⟨*proof*⟩

The first part we shall show is the integration by parts.

lemma *beukers-aux-by-parts-aux*:
  assumes *xz*: *x* ∈ {*0*<..<*1*} *z* ∈ {*0*<..<*1*} **and** *k* ≤ *n*
  shows (*LBINT y=0..1. Q n y* ∗ (*1*/(*1*−(*1*−*x*∗*y*)∗*z*))) =
      (*LBINT y=0..1. Q* (*n*−*k*) *y* ∗ (*fact k* ∗ (*x*∗*z*)^*k* / (*1*−(*1*−*x*∗*y*)∗*z*) ^(*k*+*1*)))
  ⟨*proof*⟩

lemma *beukers-aux-by-parts*:
  assumes *xz*: *x* ∈ {*0*<..<*1*} *z* ∈ {*0*<..<*1*}
  shows (*LBINT y=0..1. P y* / (*1*−(*1*−*x*∗*y*)∗*z*)) =
      (*LBINT y=0..1.* (*x*∗*y*∗*z*)^*n* ∗ (*1*−*y*)^*n* / (*1*−(*1*−*x*∗*y*)∗*z*) ^(*n*+*1*))
⟨*proof*⟩

We then get the following by applying the integration by parts to y:

lemma *beukers-aux-integral-transform1*:
  fixes *z* :: *real*
  assumes *z*: *z* ∈ {*0*<..<*1*}
  shows (*LBINT* (*x*,*y*):*D. P x* ∗ *P y* / (*1*−(*1*−*x*∗*y*)∗*z*)) =
      (*LBINT* (*x*,*y*):*D. P x* ∗ (*x*∗*y*∗*z*)^*n* ∗ (*1*−*y*)^*n* / (*1*−(*1*−*x*∗*y*)∗*z*)^(*n*+*1*))
⟨*proof*⟩

We then change variables for z:

lemma *beukers-aux-integral-transform2*:
  assumes *xy*: *x* ∈ {*0*<..<*1*} *y* ∈ {*0*<..<*1*}
  shows (*LBINT z=0..1*. (*x*∗*y*∗*z*)^*n* ∗ (*1*−*y*)^*n* / (*1*−(*1*−*x*∗*y*)∗*z*)^(*n*+*1*)) =
      (*LBINT w=0..1*. (*1*−*w*)^*n* ∗ (*1*−*y*)^*n* / (*1*−(*1*−*x*∗*y*)∗*w*))
⟨*proof*⟩

Lastly, we apply the same integration by parts to x:

lemma *beukers-aux-integral-transform3*:
  assumes *w*: *w* ∈ {*0*<..<*1*}
  shows (*LBINT* (*x*,*y*):*D*. *P x* ∗ (*1*−*y*)^*n* / (*1*−(*1*−*x*∗*y*)∗*w*)) =
      (*LBINT* (*x*,*y*):*D*. (*1*−*y*)^*n* ∗ (*x*∗*y*∗*w*)^*n* ∗ (*1*−*x*)^*n* / (*1*−(*1*−*x*∗*y*)∗*w*)^(*n*+*1*))
⟨*proof*⟩

We need to show the existence of some of these triple integrals.

lemma *beukers-aux-integrable1*:
  set-integrable lborel (({*0*<..<*1*} × {*0*<..<*1*}) × {*0*<..<*1*})
    (λ((*x*,*y*),*z*). *P x* ∗ *P y* / (*1*−(*1*−*x*∗*y*)∗*z*))
⟨*proof*⟩

lemma *beukers-aux-integrable2*:
  set-integrable lborel D′ (λ(*z*,*x*,*y*). *P x* ∗ (*x*∗*y*∗*z*)^*n* ∗ (*1*−*y*)^*n* / (*1*−(*1*−*x*∗*y*)∗*z*)
^(*n*+*1*))
⟨*proof*⟩

lemma *beukers-aux-integrable3*:
  set-integrable lborel D′ (λ(*w*,*x*,*y*). *P x* ∗ (*1*−*w*)^*n* ∗ (*1*−*y*)^*n* / (*1*−(*1*−*x*∗*y*)∗*w*))
⟨*proof*⟩

Now we only need to put all of these results together:

lemma *beukers-integral2-conv-3*: *beukers-integral2* = *beukers-integral3*
⟨*proof*⟩

1.10 The main result

Combining all of the results so far, we can derive the key inequalities

$$0 < A\zeta(3) + B < 2\zeta(3) \cdot 27^{-n} \cdot \mathrm{lcm}\{1 \ldots n\}^3$$

for integers A, B with $A > 0$.

lemma *zeta-3-linear-combination-bounds*:
  obtains *A B* :: *int*
  where *A* > *0*
    A ∗ *Re* (*zeta 3*) + *B* ∈ {*0* <.. *2* ∗ *Re* (*zeta 3*) ∗ *Lcm* {*1*..*n*} ^ *3* / *27* ^ *n*}
⟨*proof*⟩

If $\zeta(3) = \frac{a}{b}$ for some integers a and b, we can thus derive the inequality $2b\zeta(3) \cdot 27^{-n} \cdot \mathrm{lcm}\{1 \ldots n\}^3 \geq 1$ for any natural number n.

lemma *beukers-key-inequality*:
fixes a :: *int* **and** b :: *nat*
assumes $b > 0$ **and** *ab*: *Re* (*zeta* 3) $= a / b$
shows $2 * b * Re\ (zeta\ 3) * Lcm\ \{1..n\}\ \hat{}\ 3\ /\ 27\ \hat{}\ n \geq 1$
⟨*proof*⟩

end

lemma *smallo-power*: $n > 0 \Longrightarrow f \in o[F](g) \Longrightarrow (\lambda x.\ f\ x\ \hat{}\ n) \in o[F](\lambda x.\ g\ x\ \hat{}\ n)$
⟨*proof*⟩

This is now a contradiction, since $\text{lcm}\{1 \ldots n\} \in o(3^n)$ by the Prime Number Theorem – hence the main result.

theorem *zeta-3-irrational*: *zeta* $3 \notin \mathbb{Q}$
⟨*proof*⟩

end

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