Young's Inequality for Increasing Functions

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Abstract

Young's inequality states that

$$ab \leq \int_0^a f(x)dx + \int_0^b f^{-1}(y)dy$$

where $a \geq 0$, $b \geq 0$ and f is strictly increasing and continuous. Its proof is formalised following the development by Cunningham and Grossman [1]. Their idea is to make the intuitive, geometric folklore proof rigorous by reasoning about step functions. The lack of the Riemann integral makes the development longer than one would like, but their argument is reproduced faithfully.

Contents

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1 Young's Inequality for Increasing Functions

From the following paper: Cunningham, F., and Nathaniel Grossman. "On Young's Inequality." The American Mathematical Monthly 78, no. 7 (1971): 781–83. https://doi.org/10.2307/2318018

theory *Youngs* **imports**
HOL−Analysis.Analysis

begin

1.1 Toward Young's inequality

lemma *strict-mono-image-endpoints*:
fixes f :: *'a::linear-continuum-topology* ⇒ *'b::linorder-topology*
assumes *strict-mono-on* {*a..b*} *f* **and** *f*: *continuous-on* {*a..b*} *f* **and** $a \leq b$
shows *f* ' {*a..b*} = {*f a..f b*}
⟨*proof*⟩

Generalisations of the type of f are not obvious

lemma *strict-mono-continuous-invD*:
fixes f :: *real* ⇒ *real*
assumes *sm*: *strict-mono-on* {*a..*} *f* **and** *contf*: *continuous-on* {*a..*} *f*
and *fim*: *f* ' {*a..*} = {*f a..*} **and** *g*: $\bigwedge x.\ x \geq a \Longrightarrow g\ (f\ x) = x$
shows *continuous-on* {*f a..*} *g*
⟨*proof*⟩

1.2 Regular divisions

Our lack of the Riemann integral forces us to construct explicitly the step functions mentioned in the text.

definition *segment* ≡ λ*n k*. {*real k* / *real n*..(*1* + *k*) / *real n*}

lemma *segment-nonempty*: *segment n k* ≠ {}
⟨*proof*⟩

lemma *segment-Suc*: *segment n* ' {..<*Suc k*} = *insert* {*k*/*n*..(*1* + *real k*) / *n*} (*segment n* ' {..<*k*})
⟨*proof*⟩

lemma *Union-segment-image*: ⋃ (*segment n* ' {..<*k*}) = (*if k=0 then* {} *else* {*0..real k*/*real n*})
⟨*proof*⟩

definition *segments* ≡ λ*n*. *segment n* ' {..<*n*}

lemma *card-segments* [*simp*]: *card* (*segments n*) = *n*
⟨*proof*⟩

lemma *segments-0* [*simp*]: *segments 0* = {}
⟨*proof*⟩

lemma *Union-segments*: $\bigcup$ (*segments n*) = (*if n=0 then* {} *else* {*0..1*})
⟨*proof*⟩

definition *regular-division* ≡ λ*a b n*. (*image* ((+) *a* ∘ (∗) (*b*−*a*))) ' (*segments n*)

lemma *translate-scale-01*:
assumes *a* ≤ *b*
shows (λ*x*. *a* + (*b* − *a*) ∗ *x*) ' {*0..1*} = {*a..b*::*real*}
⟨*proof*⟩

lemma *finite-regular-division* [*simp*]: *finite* (*regular-division a b n*)
⟨*proof*⟩

lemma *card-regular-division* [*simp*]:
assumes *a*<*b*
shows *card* (*regular-division a b n*) = *n*
⟨*proof*⟩

lemma *Union-regular-division*:
assumes *a* ≤ *b*
shows $\bigcup$(*regular-division a b n*) = (*if n=0 then* {} *else* {*a..b*})
⟨*proof*⟩

lemma *regular-division-eqI*:
assumes *K*: *K* = {*a* + (*b*−*a*)∗(*real k* / *n*) .. *a* + (*b*−*a*)∗((*1* + *real k*) / *n*)}
and *a*<*b k* < *n*
shows *K* ∈ *regular-division a b n*
⟨*proof*⟩

lemma *regular-divisionE*:
assumes *K* ∈ *regular-division a b n a*<*b*
obtains *k* **where** *k*<*n K* = {*a* + (*b*−*a*)∗(*real k* / *n*) .. *a* + (*b*−*a*)∗((*1* + *real k*) / *n*)}
⟨*proof*⟩

lemma *regular-division-division-of*:
assumes *a* < *b n*>*0*
shows (*regular-division a b n*) *division-of* {*a..b*}
⟨*proof*⟩

1.3 Special cases of Young's inequality

lemma *weighted-nesting-sum*:
fixes *g* :: *nat* ⇒ '*a*::*comm-ring-1*
shows ($\sum$ *k*<*n*. (*1* + *of-nat k*) ∗ (*g* (*Suc k*) − *g k*)) = *of-nat n* ∗ *g n* − ($\sum$ *i*<*n*. *g i*)

⟨*proof*⟩

theorem *Youngs-exact*:
fixes f :: *real* ⇒ *real*
assumes *sm*: *strict-mono-on* {*0*..} f **and** *cont*: *continuous-on* {*0*..} f **and** a: $a \geq 0$
and f: $f\ 0 = 0\ f\ a = b$
and g: $\bigwedge x.\ [\![0 \leq x;\ x \leq a]\!] \Longrightarrow g\ (f\ x) = x$
shows $a * b$ = *integral* {*0*..a} f + *integral* {*0*..b} g
⟨*proof*⟩

corollary *Youngs-strict*:
fixes f :: *real* ⇒ *real*
assumes *sm*: *strict-mono-on* {*0*..} f **and** *cont*: *continuous-on* {*0*..} f **and** $a>0$ $b \geq 0$
and f: $f\ 0 = 0\ f\ a \neq b$ **and** *fim*: f ‘ {*0*..} = {*0*..}
and g: $\bigwedge x.\ 0 \leq x \Longrightarrow g\ (f\ x) = x$
shows $a * b <$ *integral* {*0*..a} f + *integral* {*0*..b} g
⟨*proof*⟩

corollary *Youngs-inequality*:
fixes f :: *real* ⇒ *real*
assumes *sm*: *strict-mono-on* {*0*..} f **and** *cont*: *continuous-on* {*0*..} f **and** $a \geq 0$ $b \geq 0$
and f: $f\ 0 = 0$ **and** *fim*: f ‘ {*0*..} = {*0*..}
and g: $\bigwedge x.\ 0 \leq x \Longrightarrow g\ (f\ x) = x$
shows $a * b \leq$ *integral* {*0*..a} f + *integral* {*0*..b} g
⟨*proof*⟩

end

References

[1] F. Cunningham and N. Grossman. On Young's inequality. *The American Mathematical Monthly*, 78(7):781–783, 1971.