Pólya's Proof of the Weighted Arithmetic–Geometric Mean Inequality

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Abstract

This article provides a formalisation of the Weighted Arithmetic–Geometric Mean Inequality: given non-negative reals $a_1, \ldots, a_n$ and non-negative weights $w_1, \ldots, w_n$ such that $w_1 + \ldots + w_n = 1$, we have

$$\prod_{i=1}^{n} a_i^{w_i} \leq \sum_{i=1}^{n} w_i a_i \ .$$

If the weights are additionally all non-zero, equality holds if and only if $a_1 = \ldots = a_n$.

As a corollary with $w_1 = \ldots = w_n = \frac{1}{n}$, the regular arithmetic–geometric mean inequality follows, namely that

$$\sqrt[n]{a_1 \ldots a_n} \leq \frac{1}{n}(a_1 + \ldots + a_n) \ .$$

I follow Pólya's elegant proof, which uses the inequality $1 + x \leq e^x$ as a starting point. Pólya claims that this proof came to him in a dream, and that it was 'the best mathematics he had ever dreamt." [1, pp. 22–26]

Contents

1 The Weighted Arithmetic–Geometric Mean Inequality

theory *Weighted-Arithmetic-Geometric-Mean*
  imports *Complex-Main*
begin

1.1 Auxiliary Facts

lemma *root-powr-inverse'*: $0 < n \Longrightarrow 0 \leq x \Longrightarrow root\ n\ x = x\ powr\ (1/n)$
  ⟨*proof*⟩

lemma *powr-sum-distrib-real-right*:
  assumes $a \neq 0$
  shows $(\prod x{\in}X.\ a\ powr\ e\ x :: real) = a\ powr\ (\sum x{\in}X.\ e\ x)$
  ⟨*proof*⟩

lemma *powr-sum-distrib-real-left*:
  assumes $\bigwedge x.\ x \in X \Longrightarrow a\ x \geq 0$
  shows $(\prod x{\in}X.\ a\ x\ powr\ e :: real) = (\prod x{\in}X.\ a\ x)\ powr\ e$
  ⟨*proof*⟩

lemma *prod-ge-pointwise-le-imp-pointwise-eq*:
  fixes $f :: 'a \Rightarrow real$
  assumes *finite X*
  assumes *ge*: $prod\ f\ X \geq prod\ g\ X$
  assumes *nonneg*: $\bigwedge x.\ x \in X \Longrightarrow f\ x \geq 0$
  assumes *pos*: $\bigwedge x.\ x \in X \Longrightarrow g\ x > 0$
  assumes *le*: $\bigwedge x.\ x \in X \Longrightarrow f\ x \leq g\ x$ **and** *x*: $x \in X$
  shows $f\ x = g\ x$
⟨*proof*⟩

lemma *powr-right-real-eq-iff*:
  assumes $a \geq (0 :: real)$
  shows $a\ powr\ x = a\ powr\ y \longleftrightarrow a = 0 \vee a = 1 \vee x = y$
  ⟨*proof*⟩

lemma *powr-left-real-eq-iff*:
  assumes $a \geq (0 :: real)\ b \geq 0\ x \neq 0$
  shows $a\ powr\ x = b\ powr\ x \longleftrightarrow a = b$
  ⟨*proof*⟩

lemma *exp-real-eq-one-plus-iff*:
  fixes $x :: real$
  shows $exp\ x = 1 + x \longleftrightarrow x = 0$
⟨*proof*⟩

1.2 The Inequality

We first prove the equality under the assumption that all the a_i and w_i are positive.

lemma *weighted-arithmetic-geometric-mean-pos*:
fixes $a\ w :: 'a \Rightarrow real$
assumes *finite X*
assumes *pos1*: $\bigwedge x.\ x \in X \Longrightarrow a\ x > 0$
assumes *pos2*: $\bigwedge x.\ x \in X \Longrightarrow w\ x > 0$
assumes *sum-weights*: $(\sum x{\in}X.\ w\ x) = 1$
shows $(\prod x{\in}X.\ a\ x\ powr\ w\ x) \leq (\sum x{\in}X.\ w\ x * a\ x)$
⟨*proof*⟩

We can now relax the positivity assumptions to non-negativity: if one of the a_i is zero, the theorem becomes trivial (note that $0^0 = 0$ by convention for the real-valued power operator (*powr*)).

Otherwise, we can simply remove all the indices that have weight 0 and apply the above auxiliary version of the theorem.

theorem *weighted-arithmetic-geometric-mean*:
fixes $a\ w :: 'a \Rightarrow real$
assumes *finite X*
assumes *nonneg1*: $\bigwedge x.\ x \in X \Longrightarrow a\ x \geq 0$
assumes *nonneg2*: $\bigwedge x.\ x \in X \Longrightarrow w\ x \geq 0$
assumes *sum-weights*: $(\sum x{\in}X.\ w\ x) = 1$
shows $(\prod x{\in}X.\ a\ x\ powr\ w\ x) \leq (\sum x{\in}X.\ w\ x * a\ x)$
⟨*proof*⟩

We can derive the regular arithmetic/geometric mean inequality from this by simply setting all the weights to $\frac{1}{n}$:

corollary *arithmetic-geometric-mean*:
fixes $a :: 'a \Rightarrow real$
assumes *finite X*
defines $n \equiv card\ X$
assumes *nonneg*: $\bigwedge x.\ x \in X \Longrightarrow a\ x \geq 0$
shows $root\ n\ (\prod x{\in}X.\ a\ x) \leq (\sum x{\in}X.\ a\ x)\ /\ n$
⟨*proof*⟩

1.3 The Equality Case

Next, we show that weighted arithmetic and geometric mean are equal if and only if all the a_i are equal.

We first prove the more difficult direction as a lemmas and again first assume positivity of all a_i and w_i and will relax this somewhat later.

lemma *weighted-arithmetic-geometric-mean-eq-iff-pos*:
fixes $a\ w :: 'a \Rightarrow real$
assumes *finite X*

assumes *pos1*: $\bigwedge x.\ x \in X \Longrightarrow a\ x > 0$
assumes *pos2*: $\bigwedge x.\ x \in X \Longrightarrow w\ x > 0$
assumes *sum-weights*: $(\sum x{\in}X.\ w\ x) = 1$
assumes *eq*: $(\prod x{\in}X.\ a\ x\ powr\ w\ x) = (\sum x{\in}X.\ w\ x * a\ x)$
shows $\forall x{\in}X.\ \forall y{\in}X.\ a\ x = a\ y$
⟨*proof*⟩

We can now show the full theorem and relax the positivity condition on the a_i to non-negativity. This is possible because if some a_i is zero and the two means coincide, then the product is obviously 0, but the sum can only be 0 if *all* the a_i are 0.

theorem *weighted-arithmetic-geometric-mean-eq-iff*:
fixes $a\ w :: 'a \Rightarrow real$
assumes *finite X*
assumes *nonneg1*: $\bigwedge x.\ x \in X \Longrightarrow a\ x \geq 0$
assumes *pos2*: $\bigwedge x.\ x \in X \Longrightarrow w\ x > 0$
assumes *sum-weights*: $(\sum x{\in}X.\ w\ x) = 1$
shows $(\prod x{\in}X.\ a\ x\ powr\ w\ x) = (\sum x{\in}X.\ w\ x * a\ x) \longleftrightarrow X \neq \{\} \wedge (\forall x{\in}X.\ \forall y{\in}X.\ a\ x = a\ y)$
⟨*proof*⟩

Again, we derive a version for the unweighted arithmetic/geometric mean.

corollary *arithmetic-geometric-mean-eq-iff*:
fixes $a :: 'a \Rightarrow real$
assumes *finite X*
defines $n \equiv card\ X$
assumes *nonneg*: $\bigwedge x.\ x \in X \Longrightarrow a\ x \geq 0$
shows $root\ n\ (\prod x{\in}X.\ a\ x) = (\sum x{\in}X.\ a\ x)\ /\ n \longleftrightarrow (\forall x{\in}X.\ \forall y{\in}X.\ a\ x = a\ y)$
⟨*proof*⟩

1.4 The Binary Version

For convenience, we also derive versions for only two numbers:

corollary *weighted-arithmetic-geometric-mean-binary*:
fixes $w1\ w2\ x1\ x2 :: real$
assumes $x1 \geq 0\ x2 \geq 0\ w1 \geq 0\ w2 \geq 0\ w1 + w2 = 1$
shows $x1\ powr\ w1 * x2\ powr\ w2 \leq w1 * x1 + w2 * x2$
⟨*proof*⟩

corollary *weighted-arithmetic-geometric-mean-eq-iff-binary*:
fixes $w1\ w2\ x1\ x2 :: real$
assumes $x1 \geq 0\ x2 \geq 0\ w1 > 0\ w2 > 0\ w1 + w2 = 1$
shows $x1\ powr\ w1 * x2\ powr\ w2 = w1 * x1 + w2 * x2 \longleftrightarrow x1 = x2$
⟨*proof*⟩

corollary *arithmetic-geometric-mean-binary*:
fixes $x1\ x2 :: real$

assumes *x1* $\geq$ *0* *x2* $\geq$ *0*
shows *sqrt* (*x1* $*$ *x2*) $\leq$ (*x1* + *x2*) / *2*
⟨*proof*⟩

corollary *arithmetic-geometric-mean-eq-iff-binary*:
fixes *x1 x2* :: *real*
assumes *x1* $\geq$ *0* *x2* $\geq$ *0*
shows *sqrt* (*x1* $*$ *x2*) = (*x1* + *x2*) / *2* $\longleftrightarrow$ *x1* = *x2*
⟨*proof*⟩

end

References

[1] J. M. Steele. *The Cauchy–Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities.* Cambridge University Press, 2004.