Undirected Graph Theory

Chelsea Edmonds

May 26, 2024

Abstract

This entry presents a general library for undirected graph theory - enabling reasoning on simple graphs and undirected graphs with loops. It primarily builds off Noschinski's basic ugraph definition [4], however generalises it in a number of ways and significantly expands on the range of basic graph theory definitions formalised. Notably, this library removes the constraint of vertices being a type synonym with the natural numbers which causes issues in more complex mathematical reasoning using graphs, such as the Balog Szemeredi Gowers theorem which this library is used for. Secondly this library also presents a locale-centric approach, enabling more concise, flexible, and reusable modelling of different types of graphs. Using this approach enables easy links to be made with more expansive formalisations of other combinatorial structures, such as incidence systems, as well as various types of formal representations of graphs. Further inspiration is also taken from Noschinski's [5] Directed Graph library for some proofs and definitions on walks, paths and cycles, however these are much simplified using the set based representation of graphs, and also extended on in this formalisation.

Contents

Acknowledgements

Chelsea Edmonds is jointly funded by the Cambridge Trust (Cambridge Australia Scholarship) and a Cambridge Department of Computer Science and Technology Premium Research Studentship. The ALEXANDRIA project is funded by the European Research Council, Advanced Grant GA 742178.

This library aims to present a general theory for undirected graphs. The formalisation approach models edges as sets with two elements, and is inspired in part by the graph theory basics defined by Lars Noschinski in [4] which are used in [2, 1]. Crucially this library makes the definition more flexible by removing the type synonym from vertices to natural numbers. This is limiting in more advanced mathematical applications, where it is common for vertices to represent elements of some other set. It additionally extends significantly on basic graph definitions.

The approach taken in this formalisation is the "locale-centric" approach for modelling different graph properties, which has been successfully used in other combinatorial structure formalisations.

1 Undirected Graph Theory Basics

This first theory focuses on the basics of graph theory (vertices, edges, degree, incidence, neighbours etc), as well as defining a number of different types of basic graphs. This theory draws inspiration from [4, 2, 1]

```
theory Undirected-Graph-Basics imports Main HOL−Library.Multiset HOL−Library.Disjoint-Sets

HOL−Library.Extended-Real Girth-Chromatic.Girth-Chromatic-Misc
begin
```

1.1 Miscellaneous Extras

Useful concepts on lists and sets

```
lemma distinct-tl-rev:
  assumes hd xs = last xs
  shows distinct (tl xs) ⟷ distinct (tl (rev xs))
  using assms
proof (induct xs)
  case Nil
  then show ?case by simp
next
  case (Cons a xs)
  then show ?case proof (cases xs = [])
    case True
    then show ?thesis by simp
  next
    case False
    then have a = last xs
      using Cons.prems by auto
    then obtain xs′ where xs = xs′ @ [last xs]
      by (metis False append-butlast-last-id)
    then have tleq: tl (rev xs) = rev (xs′)
      by (metis butlast-rev butlast-snoc rev-rev-ident)
    have distinct (tl (a # xs)) ⟷ distinct xs  by simp
```

also have ... ⟷ *distinct* (*rev xs′*) ∧ *a* ∉ *set* (*rev xs′*)
by (*metis False Nil-is-rev-conv* ‹*a* = *last xs*› *distinct.simps*(*2*) *distinct-rev hd-rev list.exhaust-sel tleq*)
finally show *distinct* (*tl* (*a* # *xs*)) ⟷ *distinct* (*tl* (*rev* (*a* # *xs*)))
using *tleq* **by** (*simp add*: *False*)
qed
qed

lemma *last-in-list-set*: *length xs* ≥ *1* ⟹ *last xs* ∈ *set* (*xs*)
using *dual-order.strict-trans1 last-in-set* **by** *blast*

lemma *last-in-list-tl-set*:
assumes *length xs* ≥ *2*
shows *last xs* ∈ *set* (*tl xs*)
using *assms* **by** (*induct xs*) *auto*

lemma *length-list-decomp-lt*: *ys* ≠ [] ⟹ *length* (*xs* @*zs*) < *length* (*xs*@*ys*@*zs*)
using *length-append* **by** *simp*

lemma *obtains-Max*:
assumes *finite A* **and** *A* ≠ {}
obtains *x* **where** *x* ∈ *A* **and** *Max A* = *x*
using *assms Max-in* **by** *blast*

lemma *obtains-MAX*:
assumes *finite A* **and** *A* ≠ {}
obtains *x* **where** *x* ∈ *A* **and** *Max* (*f* ‘ *A*) = *f x*
using *obtains-Max*
by (*metis* (*mono-tags*, *opaque-lifting*) *assms*(*1*) *assms*(*2*) *empty-is-image finite-imageI image-iff*)

lemma *obtains-Min*:
assumes *finite A* **and** *A* ≠ {}
obtains *x* **where** *x* ∈ *A* **and** *Min A* = *x*
using *assms Min-in* **by** *blast*

lemma *obtains-MIN*:
assumes *finite A* **and** *A* ≠ {}
obtains *x* **where** *x* ∈ *A* **and** *Min* (*f* ‘ *A*) = *f x*
using *obtains-Min assms empty-is-image finite-imageI image-iff*
by (*metis* (*mono-tags*, *opaque-lifting*))

1.2 Initial Set up

For convenience and readability, some functions and type synonyms are defined outside locale context

fun *mk-triangle-set* :: (*′a* × *′a* × *′a*) ⇒ *′a set*
where *mk-triangle-set* (*x*, *y*, *z*) = {*x*,*y*,*z*}

type-synonym *'a edge* = *'a set*

type-synonym *'a pregraph* = (*'a set*) × (*'a edge set*)

abbreviation *gverts* :: *'a pregraph* ⇒ *'a set* **where**
gverts H ≡ *fst H*

abbreviation *gedges* :: *'a pregraph* ⇒ *'a edge set* **where**
gedges H ≡ *snd H*

fun *mk-edge* :: *'a* × *'a* ⇒ *'a edge* **where**
mk-edge (*u*,*v*) = {*u*,*v*}

All edges is simply the set of subsets of a set S of size 2

definition *all-edges S* ≡ {*e* . *e* ⊆ *S* ∧ *card e* = *2*}

Note, this is a different definition to Noschinski's [4] ugraph which uses the *mk-edge* function unnecessarily

Basic properties of these functions

lemma *all-edges-mono*:
vs ⊆ *ws* ⟹ *all-edges vs* ⊆ *all-edges ws*
unfolding *all-edges-def* **by** *auto*

lemma *all-edges-alt*: *all-edges S* = {{*x*, *y*} | *x y* . *x* ∈ *S* ∧ *y* ∈ *S* ∧ *x* ≠ *y*}
unfolding *all-edges-def*
proof (*intro subset-antisym subsetI*)
fix *x* **assume** *x* ∈ {*e*. *e* ⊆ *S* ∧ *card e* = *2*}
then obtain *u v* **where** *x* = {*u*, *v*} **and** *card* {*u*, *v*} = *2* **and** {*u*, *v*} ⊆ *S*
by (*metis* (*mono-tags*, *lifting*) *card-2-iff mem-Collect-eq*)
then show *x* ∈ {{*x*, *y*} |*x y*. *x* ∈ *S* ∧ *y* ∈ *S* ∧ *x* ≠ *y*}
by *fastforce*
next
show ⋀*x*. *x* ∈ {{*x*, *y*} |*x y*. *x* ∈ *S* ∧ *y* ∈ *S* ∧ *x* ≠ *y*} ⟹ *x* ∈ {*e*. *e* ⊆ *S* ∧ *card e* = *2*}
by *auto*
qed

lemma *all-edges-alt-pairs*: *all-edges S* = *mk-edge* ' {*uv* ∈ *S* × *S*. *fst uv* ≠ *snd uv*}
unfolding *all-edges-alt*
proof (*intro subset-antisym*)
have *img*: *mk-edge* ' {*uv* ∈ *S* × *S*. *fst uv* ≠ *snd uv*} = {*mk-edge* (*u*, *v*) | *u v*. (*u*, *v*) ∈ *S* × *S* ∧ *u* ≠*v*}
by (*smt* (*z3*) *Collect-cong fst-conv prod.collapse setcompr-eq-image snd-conv*)
then show *mk-edge* ' {*uv* ∈ *S* × *S*. *fst uv* ≠ *snd uv*} ⊆ {{*x*, *y*} |*x y*. *x* ∈ *S* ∧ *y* ∈ *S* ∧ *x* ≠ *y*}
by *auto*

show $\{\{x, y\} \mid x\ y.\ x \in S \wedge y \in S \wedge x \neq y\} \subseteq$ *mk-edge* ‘ $\{uv \in S \times S.$ *fst uv* $\neq$ *snd uv*$\}$
using *img* **by** *simp*
qed

lemma *all-edges-subset-Pow*: *all-edges A* $\subseteq$ *Pow A*
by (*auto simp*: *all-edges-def*)

lemma *all-edges-disjoint*: $S \cap T = \{\} \Longrightarrow$ *all-edges S* $\cap$ *all-edges T* $= \{\}$
by (*auto simp add*: *all-edges-def disjoint-iff subset-eq*)

lemma *card-all-edges*: *finite A* $\Longrightarrow$ *card* (*all-edges A*) = *card A choose 2*
using *all-edges-def* **by** (*metis* (*full-types*) *n-subsets*)

lemma *finite-all-edges*: *finite S* $\Longrightarrow$ *finite* (*all-edges S*)
by (*meson all-edges-subset-Pow finite-Pow-iff finite-subset*)

lemma *in-mk-edge-img*: $(a,b) \in A \vee (b,a) \in A \Longrightarrow \{a,b\} \in$ *mk-edge* ‘ A
by (*auto intro*: *rev-image-eqI*)

thm *in-mk-edge-img*
lemma *in-mk-uedge-img-iff*: $\{a,b\} \in$ *mk-edge* ‘ $A \longleftrightarrow (a,b) \in A \vee (b,a) \in A$
by (*auto simp*: *doubleton-eq-iff intro*: *rev-image-eqI*)

lemma *inj-on-mk-edge*: $X \cap Y = \{\} \Longrightarrow$ *inj-on mk-edge* $(X \times Y)$
by (*auto simp*: *inj-on-def doubleton-eq-iff*)

definition *complete-graph* :: *'a set* $\Rightarrow$ *'a pregraph* **where**
complete-graph S $\equiv$ (*S*, *all-edges S*)

definition *all-edges-loops*:: *'a set* $\Rightarrow$ *'a edge set***where**
all-edges-loops S $\equiv$ *all-edges S* $\cup \{\{v\} \mid v.\ v \in S\}$

lemma *all-edges-loops-alt*: *all-edges-loops S* $= \{e\ .\ e \subseteq S \wedge$ (*card e* = *2* $\vee$ *card e* = *1*)$\}$
proof –
have *1*: $\{\{v\} \mid v.\ v \in S\} = \{e\ .\ e \subseteq S \wedge$ *card e* = *1*$\}$
by (*metis One-nat-def card.empty card-Suc-eq empty-iff empty-subsetI insert-subset is-singleton-altdef is-singleton-the-elem*)
have $\{e\ .\ e \subseteq S \wedge$ (*card e* = *2* $\vee$ *card e* = *1*)$\} = \{e\ .\ e \subseteq S \wedge$ *card e* = *2*$\} \cup \{e\ .\ e \subseteq S \wedge$ *card e* = *1*$\}$
by *auto*
then have $\{e\ .\ e \subseteq S \wedge$ (*card e* = *2* $\vee$ *card e* = *1*)$\}$ = *all-edges S* $\cup \{\{v\} \mid v.\ v \in S\}$
by (*simp add*: *all-edges-def 1*)
then show *?thesis* **unfolding** *all-edges-loops-def* **by** *simp*
qed

lemma *loops-disjoint*: *all-edges S* $\cap \{\{v\} \mid v.\ v \in S\} = \{\}$

unfolding *all-edges-def* **using** *card-2-iff*
by *fastforce*

lemma *all-edges-loops-ss*: *all-edges S* ⊆ *all-edges-loops S* {{*v*} | *v*. *v* ∈ *S*} ⊆ *all-edges-loops S*
by (*simp-all add*: *all-edges-loops-def*)

lemma *finite-singletons*: *finite S* ⟹ *finite* ({{*v*} | *v*. *v* ∈ *S*})
by (*auto*)

lemma *card-singletons*:
assumes *finite S* **shows** *card* {{*v*} | *v*. *v* ∈ *S*} = *card S*
using *assms*
proof (*induct S rule*: *finite-induct*)
case *empty*
then show *?case* **by** *simp*
next
case (*insert x F*)
then have *disj*: {{*x*}} ∩ {{*v*} |*v*. *v* ∈ *F*} = {} **by** *auto*
have {{*v*} |*v*. *v* ∈ *insert x F*} = ({{*x*}} ∪ {{*v*} |*v*. *v* ∈ *F*}) **by** *auto*
then have *card* {{*v*} |*v*. *v* ∈ *insert x F*} = *card* ({{*x*}} ∪ {{*v*} |*v*. *v* ∈ *F*}) **by** *simp*
also have ... = *card* {{*x*}} + *card* {{*v*} |*v*. *v* ∈ *F*} **using** *card-Un-disjoint disj assms finite-subset*
using *insert.hyps*(*1*) **by** *force*
also have ... = *1* + *card* {{*v*} |*v*. *v* ∈ *F*} **using** *is-singleton-altdef* **by** *simp*
also have ... = *1* + *card F* **using** *insert.hyps* **by** *auto*
finally show *?case* **using** *insert.hyps*(*1*) *insert.hyps*(*2*) **by** *force*
qed

lemma *finite-all-edges-loops*: *finite S* ⟹ *finite* (*all-edges-loops S*)
unfolding *all-edges-loops-def* **using** *finite-all-edges finite-singletons* **by** *auto*

lemma *card-all-edges-loops*:
assumes *finite S*
shows *card* (*all-edges-loops S*) = (*card S*) *choose 2* + *card S*
proof −
have *card* (*all-edges-loops S*) = *card* (*all-edges S* ∪ {{*v*} | *v*. *v* ∈ *S*})
by (*simp add*: *all-edges-loops-def*)
also have ... = *card* (*all-edges S*) + *card* {{*v*} | *v*. *v* ∈ *S*}
using *loops-disjoint assms card-Un-disjoint*[*of all-edges S* {{*v*} | *v*. *v* ∈ *S*}]
all-edges-loops-ss finite-all-edges-loops finite-subset **by** *fastforce*
also have ... = (*card S*) *choose 2* + *card* {{*v*} | *v*. *v* ∈ *S*} **by**(*simp add*: *card-all-edges assms*)
finally show *?thesis* **using** *assms card-singletons* **by** *auto*
qed

1.3 Graph System Locale

A generic incidence set system re-labeled to graph notation, where repeated edges are not allowed. All the definitions here do not need the "edge" size to be constrained to make sense.

locale *graph-system* =
  fixes *vertices* :: *'a set* (*V*)
  fixes *edges* :: *'a edge set* (*E*)
  assumes *wellformed*: $e \in E \Longrightarrow e \subseteq V$
begin

abbreviation *gorder* :: *nat* **where**
gorder ≡ *card* (*V*)

abbreviation *graph-size* :: *nat* **where**
graph-size ≡ *card E*

definition *vincident* :: *'a* ⇒ *'a edge* ⇒ *bool* **where**
vincident v e ≡ $v \in e$

lemma *incident-edge-in-wf*: $e \in E \Longrightarrow$ *vincident v e* $\Longrightarrow v \in V$
  using *wellformed vincident-def* **by** *auto*

definition *incident-edges* :: *'a* ⇒ *'a edge set* **where**
incident-edges v ≡{*e* . $e \in E \wedge$ *vincident v e*}

lemma *incident-edges-empty*: $\neg (v \in V) \Longrightarrow$ *incident-edges v* = {}
  using *incident-edges-def incident-edge-in-wf* **by** *auto*

lemma *finite-incident-edges*: *finite E* ⟹ *finite* (*incident-edges v*)
  by (*simp add*: *incident-edges-def*)

definition *edge-adj* :: *'a edge* ⇒ *'a edge* ⇒ *bool* **where**
edge-adj e1 e2 ≡ *e1* ∩ *e2* ≠ {} ∧ *e1* ∈ *E* ∧ *e2* ∈ *E*

lemma *edge-adj-inE*: *edge-adj e1 e2* ⟹ *e1* ∈ *E* ∧ *e2* ∈ *E*
  using *edge-adj-def* **by** *auto*

lemma *edge-adjacent-alt-def*: *e1* ∈ *E* ⟹ *e2* ∈ *E* ⟹ ∃ *x* . *x* ∈ *V* ∧ *x* ∈ *e1* ∧ *x* ∈ *e2* ⟹ *edge-adj e1 e2*
  unfolding *edge-adj-def* **by** *auto*

lemma *wellformed-alt-fst*: $\{x, y\} \in E \Longrightarrow x \in V$
  using *wellformed* **by** *auto*

lemma *wellformed-alt-snd*: $\{x, y\} \in E \Longrightarrow y \in V$
  using *wellformed* **by** *auto*
end

Simple constraints on a graph system may include finite and non-empty constraints

locale *fin-graph-system* = *graph-system* +
  assumes *finV*: *finite V*
begin

lemma *fin-edges*: *finite E*
  using *wellformed finV*
  by (*meson PowI finite-Pow-iff finite-subset subsetI*)

end

locale *ne-graph-system* = *graph-system* +
  assumes *not-empty*: $V \neq \{\}$

1.4 Undirected Graph with Loops

This formalisation models a loop by a singleton set. In this case a graph has the edge size criteria if it has edges of size 1 or 2. Notably this removes the option for an edge to be empty

locale *ulgraph* = *graph-system* +
  assumes *edge-size*: $e \in E \Longrightarrow card\ e > 0 \wedge card\ e \leq 2$

begin

lemma *alt-edge-size*: $e \in E \Longrightarrow card\ e = 1 \vee card\ e = 2$
  using *edge-size* **by** *fastforce*

definition *is-loop*:: $'a\ edge \Rightarrow bool$ **where**
$is\text{-}loop\ e \equiv card\ e = 1$

definition *is-sedge* :: $'a\ edge \Rightarrow bool$ **where**
$is\text{-}sedge\ e \equiv card\ e = 2$

lemma *is-edge-or-loop*: $e \in E \Longrightarrow is\text{-}loop\ e \vee is\text{-}sedge\ e$
  using *alt-edge-size is-loop-def is-sedge-def* **by** *simp*

lemma *edges-split-loop*: $E = \{e \in E\ .\ is\text{-}loop\ e\ \} \cup \{e \in E\ .\ is\text{-}sedge\ e\}$
  using *is-edge-or-loop* **by** *auto*

lemma *edges-split-loop-inter-empty*: $\{\} = \{e \in E\ .\ is\text{-}loop\ e\ \} \cap \{e \in E\ .\ is\text{-}sedge\ e\}$
  unfolding *is-loop-def is-sedge-def* **by** *auto*

definition *vert-adj* :: $'a \Rightarrow 'a \Rightarrow bool$ **where** — Neighbor in graph from Roth [1]
$vert\text{-}adj\ v1\ v2 \equiv \{v1,\ v2\} \in E$

lemma *vert-adj-sym*: $vert\text{-}adj\ v1\ v2 \longleftrightarrow vert\text{-}adj\ v2\ v1$

unfolding *vert-adj-def* **by** (*simp-all add*: *insert-commute*)

lemma *vert-adj-imp-inV*: *vert-adj v1 v2* $\Longrightarrow$ *v1* $\in$ *V* $\wedge$ *v2* $\in$ *V*
using *vert-adj-def wellformed* **by** *auto*

lemma *vert-adj-inc-edge-iff*: *vert-adj v1 v2* $\longleftrightarrow$ *vincident v1* $\{v1, v2\}$ $\wedge$ *vincident v2* $\{v1, v2\}$ $\wedge$ $\{v1, v2\} \in E$
unfolding *vert-adj-def vincident-def* **by** *auto*

lemma *not-vert-adj*[*simp*]: $\neg$ *vert-adj v u* $\Longrightarrow$ $\{v, u\} \notin E$
by (*simp add*: *vert-adj-def*)

definition *neighborhood* :: $'a \Rightarrow 'a$ *set* **where** — Neighbors in Roth Development [1]
neighborhood x $\equiv$ $\{v \in V$. *vert-adj x v*$\}$

lemma *neighborhood-incident*: *u* $\in$ *neighborhood v* $\longleftrightarrow$ $\{u, v\} \in$ *incident-edges v*
unfolding *neighborhood-def incident-edges-def*
by (*smt* (*verit*) *vincident-def insert-commute insert-subset mem-Collect-eq subset-insertI vert-adj-def wellformed*)

definition *neighbors-ss* :: $'a \Rightarrow 'a$ *set* $\Rightarrow 'a$ *set* **where**
neighbors-ss x Y $\equiv$ $\{y \in Y$. *vert-adj x y*$\}$

lemma *vert-adj-edge-iff2*:
assumes *v1* $\neq$ *v2*
shows *vert-adj v1 v2* $\longleftrightarrow$ ($\exists$ *e* $\in$ *E* . *vincident v1 e* $\wedge$ *vincident v2 e*)
proof (*intro iffI*)
show *vert-adj v1 v2* $\Longrightarrow$ $\exists$ *e*$\in$*E*. *vincident v1 e* $\wedge$ *vincident v2 e* **using** *vert-adj-inc-edge-iff* **by** *blast*
assume $\exists$ *e*$\in$*E*. *vincident v1 e* $\wedge$ *vincident v2 e*
then obtain *e* **where** *ein*: *e* $\in$ *E* **and** *vincident v1 e* **and** *vincident v2 e*
using *vert-adj-inc-edge-iff assms alt-edge-size* **by** *auto*
then have *e* = $\{v1, v2\}$ **using** *alt-edge-size assms*
by (*smt* (*verit*) *card-1-singletonE card-2-iff vincident-def insertE insert-commute singletonD*)
then show *vert-adj v1 v2* **using** *ein vert-adj-def*
by *simp*
qed

Incident simple edges, i.e. excluding loops

definition *incident-sedges* :: $'a \Rightarrow 'a$ *edge set* **where**
incident-sedges v $\equiv$ $\{e \in E$. *vincident v e* $\wedge$ *card e* = *2*$\}$

lemma *finite-inc-sedges*: *finite E* $\Longrightarrow$ *finite* (*incident-sedges v*)
by (*simp add*: *incident-sedges-def*)

lemma *incident-sedges-empty*[*simp*]: *v* $\notin$ *V* $\Longrightarrow$ *incident-sedges v* = $\{\}$
unfolding *incident-sedges-def* **using** *vincident-def wellformed* **by** *fastforce*

definition *has-loop* :: $'a \Rightarrow$ *bool* **where**
has-loop v $\equiv \{v\} \in E$

lemma *has-loop-in-verts*: *has-loop v* $\Longrightarrow v \in V$
using *has-loop-def wellformed* **by** *auto*

lemma *is-loop-set-alt*: $\{\{v\} \mid v$. *has-loop v*$\} = \{e \in E$. *is-loop e*$\}$
proof (*intro subset-antisym subsetI*)
fix *x* **assume** $x \in \{\{v\} \mid v.$ *has-loop v*$\}$
then obtain *v* **where** $x = \{v\}$ **and** *has-loop v*
by *blast*
then show $x \in \{e \in E.$ *is-loop e*$\}$ **using** *has-loop-def is-loop-def* **by** *auto*
next
fix *x* **assume** *a*: $x \in \{e \in E.$ *is-loop e*$\}$
then have *is-loop x* **by** *blast*
then obtain *v* **where** $x = \{v\}$ **and** $\{v\} \in E$ **using** *is-loop-def a*
by (*metis card-1-singletonE mem-Collect-eq*)
thus $x \in \{\{v\} \mid v.$ *has-loop v*$\}$ **using** *has-loop-def* **by** *simp*
qed

definition *incident-loops* :: $'a \Rightarrow 'a$ *edge set* **where**
incident-loops v $\equiv \{e \in E.\ e = \{v\}\}$

lemma *card1-incident-imp-vert*: *vincident v e* $\wedge$ *card e* $= 1 \Longrightarrow e = \{v\}$
by (*metis card-1-singletonE vincident-def singleton-iff*)

lemma *incident-loops-alt*: *incident-loops v* $= \{e \in E.$ *vincident v e* $\wedge$ *card e* $= 1\}$
unfolding *incident-loops-def* **using** *card1-incident-imp-vert vincident-def* **by** *auto*

lemma *incident-loops-simp*: *has-loop v* $\Longrightarrow$ *incident-loops v* $= \{\{v\}\}$ $\neg$ *has-loop v* $\Longrightarrow$ *incident-loops v* $= \{\}$
unfolding *incident-loops-def has-loop-def* **by** *auto*

lemma *incident-loops-union*: $\bigcup$ (*incident-loops* ‘ V) $= \{e \in E$. *is-loop e*$\}$
proof –
have $V = \{v \in V.$ *has-loop v*$\} \cup \{v \in V$. $\neg$ *has-loop v*$\}$
by *auto*
then have $\bigcup$ (*incident-loops* ‘ V) $= \bigcup$ (*incident-loops* ‘ $\{v \in V.$ *has-loop v*$\}$) $\cup$
$\bigcup$ (*incident-loops* ‘ $\{v \in V.\ \neg$ *has-loop v*$\}$) **by** *auto*
also have ... $= \bigcup$ (*incident-loops* ‘ $\{v \in V.$ *has-loop v*$\}$) **using** *incident-loops-simp*(*2*) **by** *simp*
also have ... $= \bigcup$ ($\{\{\{v\}\} \mid v$. *has-loop v*$\}$) **using** *has-loop-in-verts incident-loops-simp*(*1*) **by** *auto*
also have ... $=$ ($\{\{v\} \mid v$. *has-loop v*$\}$) **by** *auto*
finally show *?thesis* **using** *is-loop-set-alt* **by** *simp*
qed

lemma *finite-incident-loops*: *finite* (*incident-loops v*)
using *incident-loops-simp* **by** (*cases has-loop v*) *auto*

lemma *incident-loops-card*: *card* (*incident-loops v*) ≤ *1*
by (*cases has-loop v*) (*simp-all add*: *incident-loops-simp*)

lemma *incident-edges-union*: *incident-edges v* = *incident-sedges v* ∪ *incident-loops v*
unfolding *incident-edges-def incident-sedges-def incident-loops-alt* **using** *alt-edge-size*
by *auto*

lemma *incident-edges-sedges*[*simp*]: ¬ *has-loop v* ⟹ *incident-edges v* = *incident-sedges v*
using *incident-edges-union incident-loops-simp* **by** *auto*

lemma *incident-sedges-union*: ⋃ (*incident-sedges* ‘ *V*) = {*e* ∈ *E* . *is-sedge e*}
proof (*intro subset-antisym subsetI*)
fix *x* **assume** *x* ∈ ⋃ (*incident-sedges* ‘ *V*)
then obtain *v* **where** *x* ∈ *incident-sedges v* **by** *blast*
then show *x* ∈ {*e* ∈ *E*. *is-sedge e*} **using** *incident-sedges-def is-sedge-def* **by** *auto*
next
fix *x* **assume** *x* ∈ {*e* ∈ *E*. *is-sedge e*}
then have *xin*: *x* ∈ *E* **and** *c2*: *card x* = *2* **using** *is-sedge-def* **by** *auto*
then obtain *v* **where** *v* ∈ *x* **and** *vin*: *v* ∈ *V* **using** *wellformed*
by (*meson card-2-iff′ subsetD*)
then have *x* ∈ *incident-sedges v* **unfolding** *incident-sedges-def vincident-def* **using** *xin c2* **by** *auto*
then show *x* ∈ ⋃ (*incident-sedges* ‘ *V*) **using** *vin* **by** *auto*
qed

lemma *empty-not-edge*: {} ∉ *E*
using *edge-size* **by** *fastforce*

The degree definition is complicated by loops - each loop contributes two to degree. This is required for basic counting properties on the degree to hold

definition *degree* :: *′a* ⇒ *nat* **where**
degree v ≡ *card* (*incident-sedges v*) + *2* ∗ (*card* (*incident-loops v*))

lemma *degree-no-loops*[*simp*]: ¬ *has-loop v* ⟹ *degree v* = *card* (*incident-edges v*)
using *incident-edges-sedges degree-def incident-loops-simp*(*2*) **by** *auto*

lemma *degree-none*[*simp*]: ¬ *v* ∈ *V* ⟹ *degree v* = *0*
using *degree-def degree-no-loops has-loop-in-verts incident-edges-sedges incident-sedges-empty* **by** *auto*

lemma *degree0-inc-edges-empt-iff*:

assumes *finite E*
shows *degree v = 0* ⟷ *incident-edges v* = {}
proof (*intro iffI*)
assume *degree v = 0*
then have *card* (*incident-sedges v*) + *2* ∗ (*card* (*incident-loops v*)) = *0* **using** *degree-def* **by** *simp*
then have *incident-sedges v* = {} **and** *incident-loops v* = {}
using *degree-def incident-edges-union assms finite-incident-edges finite-incident-loops* **by** *auto*
thus *incident-edges v* = {} **using** *incident-edges-union* **by** *auto*
next
show *incident-edges v* = {} ⟹ *degree v = 0* **using** *incident-edges-union degree-def*
by *simp*
qed

lemma *incident-edges-neighbors-img*: *incident-edges v* = (λ *u* . {*v*, *u*}) ‘ (*neighborhood v*)
proof (*intro subset-antisym subsetI*)
fix *x* **assume** *a*: *x* ∈ *incident-edges v*
then have *xE*: *x* ∈ *E* **and** *vx*: *v* ∈ *x* **using** *incident-edges-def vincident-def* **by** *auto*
then obtain *u* **where** *x* = {*u*, *v*} **using** *alt-edge-size*
by (*smt* (*verit*, *best*) *card-1-singletonE card-2-iff insertE insert-absorb2 insert-commute singletonD*)
then have *u* ∈ *neighborhood v*
using *a neighborhood-incident* **by** *blast*
then show *x* ∈ (λ*u*. {*v*, *u*}) ‘ *neighborhood v* **using** ‹*x* = {*u*, *v*}› **by** *blast*
next
fix *x* **assume** *x* ∈ (λ*u*. {*v*, *u*}) ‘ *neighborhood v*
then obtain *u′* **where** *x* = {*v*, *u′*} **and** *u′* ∈ *neighborhood v*
by *blast*
then show *x* ∈ *incident-edges v*
by (*simp add*: *insert-commute neighborhood-incident*)
qed

lemma *card-incident-sedges-neighborhood*: *card* (*incident-edges v*) = *card* (*neighborhood v*)
proof −
have *bij-betw* (λ *u* . {*v*, *u*}) (*neighborhood v*) (*incident-edges v*)
by(*intro bij-betw-imageI inj-onI*, *simp-all add*:*incident-edges-neighbors-img*)(*metis doubleton-eq-iff*)
thus *?thesis*
by (*metis bij-betw-same-card*)
qed

lemma *degree0-neighborhood-empt-iff*:
assumes *finite E*
shows *degree v = 0* ⟷ *neighborhood v* = {}

using *degree0-inc-edges-empt-iff incident-edges-neighbors-img*
by (*simp add*: *assms*)

definition *is-isolated-vertex*:: *'a* ⇒ *bool* **where**
is-isolated-vertex v ≡ *v* ∈ *V* ∧ (∀ *u* ∈ *V* . ¬ *vert-adj u v*)

lemma *is-isolated-vertex-edge*: *is-isolated-vertex v* ⟹ (⋀ *e*. *e* ∈ *E* ⟹ ¬ (*vincident v e*))
unfolding *is-isolated-vertex-def*
by (*metis* (*full-types*) *all-not-in-conv vincident-def insert-absorb insert-iff mk-disjoint-insert*

vert-adj-def vert-adj-edge-iff2 vert-adj-imp-inV)

lemma *is-isolated-vertex-no-loop*: *is-isolated-vertex v* ⟹ ¬ *has-loop v*
unfolding *has-loop-def is-isolated-vertex-def vert-adj-def* **by** *auto*

lemma *is-isolated-vertex-degree0*: *is-isolated-vertex v* ⟹ *degree v* = *0*
proof −
assume *assm*: *is-isolated-vertex v*
then have ¬ *has-loop v* **using** *is-isolated-vertex-no-loop* **by** *simp*
then have *degree v* = *card* (*incident-edges v*) **using** *degree-no-loops* **by** *auto*
moreover have ⋀ *e*. *e* ∈ *E* ⟹ ¬ (*vincident v e*)
using *is-isolated-vertex-edge assm* **by** *auto*
then have (*incident-edges v*) = {} **unfolding** *incident-edges-def* **by** *auto*
ultimately show *degree v* = *0* **by** *simp*
qed

lemma *iso-vertex-empty-neighborhood*: *is-isolated-vertex v* ⟹ *neighborhood v* = {}
using *is-isolated-vertex-def neighborhood-def*
by (*metis* (*mono-tags*, *lifting*) *Collect-empty-eq is-isolated-vertex-edge vert-adj-inc-edge-iff*)

definition *max-degree* :: *nat* **where**
max-degree ≡ *Max* {*degree v* | *v*. *v* ∈ *V*}

definition *min-degree* :: *nat* **where**
min-degree ≡ *Min* {*degree v* | *v* . *v* ∈ *V*}

definition *is-edge-between* :: *'a set* ⇒ *'a set* ⇒ *'a edge* ⇒ *bool* **where**
is-edge-between X Y e ≡ ∃ *x y*. *e* = {*x*, *y*} ∧ *x* ∈ *X* ∧ *y* ∈ *Y*

All edges between two sets of vertices, X and Y, in a graph, G. Inspired by Szemeredi development [2] and generalised here

definition *all-edges-between* :: *'a set* ⇒ *'a set* ⇒ (*'a* × *'a*) *set* **where**
all-edges-between X Y ≡ {(*x*, *y*) . *x* ∈ *X* ∧ *y* ∈ *Y* ∧ {*x*, *y*} ∈ *E*}

lemma *all-edges-betw-D3*: (*x*, *y*) ∈ *all-edges-between X Y* ⟹ {*x*, *y*} ∈ *E*
by (*simp add*: *all-edges-between-def*)

lemma *all-edges-betw-I*: $x \in X \Longrightarrow y \in Y \Longrightarrow \{x, y\} \in E \Longrightarrow (x, y) \in$ *all-edges-between* $X\ Y$
  by (*simp add*: *all-edges-between-def*)

lemma *all-edges-between-subset*: *all-edges-between* $X\ Y \subseteq X \times Y$
  by (*auto simp*: *all-edges-between-def*)

lemma *all-edges-between-E-ss*: *mk-edge* ‘ *all-edges-between* $X\ Y \subseteq E$
  by (*auto simp add*: *all-edges-between-def*)

lemma *all-edges-between-rem-wf*: *all-edges-between* $X\ Y =$ *all-edges-between* $(X \cap V)\ (Y \cap V)$
  using *wellformed* **by** (*simp add*: *all-edges-between-def*) *blast*

lemma *all-edges-between-empty* [*simp*]:
  all-edges-between $\{\}\ Z = \{\}$ *all-edges-between* $Z\ \{\} = \{\}$
  by (*auto simp*: *all-edges-between-def*)

lemma *all-edges-between-disjnt1*: *disjnt* $X\ Y \Longrightarrow$ *disjnt* (*all-edges-between* $X\ Z$) (*all-edges-between* $Y\ Z$)
  by (*auto simp*: *all-edges-between-def disjnt-iff*)

lemma *all-edges-between-disjnt2*: *disjnt* $Y\ Z \Longrightarrow$ *disjnt* (*all-edges-between* $X\ Y$) (*all-edges-between* $X\ Z$)
  by (*auto simp*: *all-edges-between-def disjnt-iff*)

lemma *max-all-edges-between*:
  assumes *finite* X *finite* Y
  shows *card* (*all-edges-between* $X\ Y$) $\leq$ *card* X $*$ *card* Y
  by (*metis assms card-mono finite-SigmaI all-edges-between-subset card-cartesian-product*)

lemma *all-edges-between-Un1*:
  all-edges-between $(X \cup Y)\ Z =$ *all-edges-between* $X\ Z \cup$ *all-edges-between* $Y\ Z$
  by (*auto simp*: *all-edges-between-def*)

lemma *all-edges-between-Un2*:
  all-edges-between $X\ (Y \cup Z) =$ *all-edges-between* $X\ Y \cup$ *all-edges-between* $X\ Z$
  by (*auto simp*: *all-edges-between-def*)

lemma *finite-all-edges-between*:
  assumes *finite* X *finite* Y
  shows *finite* (*all-edges-between* $X\ Y$)
  by (*meson all-edges-between-subset assms finite-cartesian-product finite-subset*)

lemma *all-edges-between-Union1*:
  all-edges-between (*Union* $\mathcal{X}$) $Y = (\bigcup X \in \mathcal{X}.$ *all-edges-between* $X\ Y)$
  by (*auto simp*: *all-edges-between-def*)

lemma *all-edges-between-Union2*:
all-edges-between X (Union $\mathcal{Y}$) = ($\bigcup$ Y∈$\mathcal{Y}$. all-edges-between X Y)
by (*auto simp*: *all-edges-between-def*)

lemma *all-edges-between-disjoint1*:
assumes *disjoint R*
shows *disjoint* ((λX. *all-edges-between X Y*) ' *R*)
using *assms* **by** (*auto simp*: *all-edges-between-def disjoint-def*)

lemma *all-edges-between-disjoint2*:
assumes *disjoint R*
shows *disjoint* ((λY. *all-edges-between X Y*) ' *R*)
using *assms* **by** (*auto simp*: *all-edges-between-def disjoint-def*)

lemma *all-edges-between-disjoint-family-on1*:
assumes *disjoint R*
shows *disjoint-family-on* (λX. *all-edges-between X Y*) *R*
by (*metis* (*no-types*, *lifting*) *all-edges-between-disjnt1 assms disjnt-def disjoint-family-on-def pairwiseD*)

lemma *all-edges-between-disjoint-family-on2*:
assumes *disjoint R*
shows *disjoint-family-on* (λY. *all-edges-between X Y*) *R*
by (*metis* (*no-types*, *lifting*) *all-edges-between-disjnt2 assms disjnt-def disjoint-family-on-def pairwiseD*)

lemma *all-edges-between-mono1*:
$Y \subseteq Z \Longrightarrow$ *all-edges-between Y X* $\subseteq$ *all-edges-between Z X*
by (*auto simp*: *all-edges-between-def*)

lemma *all-edges-between-mono2*:
$Y \subseteq Z \Longrightarrow$ *all-edges-between X Y* $\subseteq$ *all-edges-between X Z*
by (*auto simp*: *all-edges-between-def*)

lemma *inj-on-mk-edge*: $X \cap Y = \{\} \Longrightarrow$ *inj-on mk-edge* (*all-edges-between X Y*)
by (*auto simp*: *inj-on-def doubleton-eq-iff all-edges-between-def*)

lemma *all-edges-between-subset-times*: *all-edges-between X Y* $\subseteq (X \cap \bigcup E) \times (Y \cap \bigcup E)$
by (*auto simp*: *all-edges-between-def*)

lemma *all-edges-betw-prod-def-neighbors*: *all-edges-between X Y* = $\{(x, y) \in X \times Y$. *vert-adj x y* $\}$
by (*auto simp*: *vert-adj-def all-edges-between-def*)

lemma *all-edges-betw-sigma-neighbor*:
all-edges-between X Y = (*SIGMA x*:*X*. *neighbors-ss x Y*)
by (*auto simp add*: *all-edges-between-def neighbors-ss-def vert-adj-def*)

```
lemma card-all-edges-betw-neighbor:
  assumes finite X finite Y
  shows card (all-edges-between X Y) = (∑ x∈X. card (neighbors-ss x Y))
  using all-edges-betw-sigma-neighbor assms by (simp add: neighbors-ss-def)

lemma all-edges-between-swap:
  all-edges-between X Y = (λ(x,y). (y,x)) ‘ (all-edges-between Y X)
  unfolding all-edges-between-def
  by (auto simp add: insert-commute image-iff split: prod.split)

lemma card-all-edges-between-commute:
  card (all-edges-between X Y) = card (all-edges-between Y X)
proof −
  have inj-on (λ(x, y). (y, x)) A for A :: (nat∗nat)set
    by (auto simp: inj-on-def)
  then show ?thesis using all-edges-between-swap [of X Y] card-image
    by (metis swap-inj-on)
qed

lemma all-edges-between-set: mk-edge ‘ all-edges-between X Y = {{x, y}| x y. x ∈
X ∧ y ∈ Y ∧ {x, y} ∈ E}
  unfolding all-edges-between-def
proof (intro subset-antisym subsetI)
  fix e assume e ∈ mk-edge ‘ {(x, y). x ∈ X ∧ y ∈ Y ∧ {x, y} ∈ E}
  then obtain x y where e = mk-edge (x, y) and x ∈ X and y ∈ Y and {x, y}
∈ E
    by blast
  then show e ∈ {{x, y} |x y. x ∈ X ∧ y ∈ Y ∧ {x, y} ∈ E}
    by auto
next
  fix e assume e ∈ {{x, y} |x y. x ∈ X ∧ y ∈ Y ∧ {x, y} ∈ E}
  then obtain x y where e ={x, y} and x ∈ X and y ∈ Y and {x, y} ∈ E
    by blast
  then have e = mk-edge (x, y)
    by auto
  then show e ∈ mk-edge ‘ {(x, y). x ∈ X ∧ y ∈ Y ∧ {x, y} ∈ E}
    using ‹x ∈ X› ‹y ∈ Y› ‹{x, y} ∈ E› by blast
qed
```

1.5 Edge Density

The edge density between two sets of vertices, X and Y, in G. This is the same definition as taken in the Szemeredi development, generalised here [2]

```
definition edge-density X Y ≡ card (all-edges-between X Y)/(card X ∗ card Y)
lemma edge-density-ge0: edge-density X Y ≥ 0
  by (auto simp: edge-density-def)

lemma edge-density-le1: edge-density X Y ≤ 1
proof (cases finite X ∧ finite Y)
```

case *True*
then show *?thesis*
using *of-nat-mono* [*OF max-all-edges-between*, *of X Y*]
by (*fastforce simp add*: *edge-density-def divide-simps*)
qed (*auto simp*: *edge-density-def*)

lemma *edge-density-zero*: $Y = \{\} \Longrightarrow$ *edge-density X Y* = *0*
by (*simp add*: *edge-density-def*)

lemma *edge-density-commute*: *edge-density X Y* = *edge-density Y X*
by (*simp add*: *edge-density-def card-all-edges-between-commute mult.commute*)

lemma *edge-density-Un*:
assumes *disjnt X1 X2 finite X1 finite X2 finite Y*
shows *edge-density* (*X1* ∪ *X2*) *Y* = (*edge-density X1 Y* ∗ *card X1* + *edge-density X2 Y* ∗ *card X2*) / (*card X1* + *card X2*)
using *assms* **unfolding** *edge-density-def*
by (*simp add*: *all-edges-between-disjnt1 all-edges-between-Un1 finite-all-edges-between card-Un-disjnt divide-simps*)

lemma *edge-density-eq0*:
assumes *all-edges-between A B* = {} **and** *X* ⊆ *A Y* ⊆ *B*
shows *edge-density X Y* = *0*
proof −
have *all-edges-between X Y* = {}
by (*metis all-edges-between-mono1 all-edges-between-mono2 assms subset-empty*)
then show *?thesis*
by (*auto simp*: *edge-density-def*)
qed

end

A number of lemmas are limited to a finite graph

locale *fin-ulgraph* = *ulgraph* + *fin-graph-system*
begin

lemma *card-is-has-loop-eq*: *card* {*e* ∈ *E* . *is-loop e*} = *card* {*v* ∈ *V* . *has-loop v*}
proof −
have ⋀ *e* . *e* ∈ *E* ⟹ *is-loop e* ⟷ (∃ *v*. *e* = {*v*}) **using** *is-loop-def*
using *is-singleton-altdef is-singleton-def* **by** *blast*
define *f* :: *'a* ⇒ *'a set* **where** *f* = (λ *v* . {*v*})
have *feq*: *f* ‘ {*v* ∈ *V* . *has-loop v*} = {{*v*} | *v* . *has-loop v*} **using** *has-loop-in-verts f-def* **by** *auto*
have *inj-on f* {*v* ∈ *V* . *has-loop v*} **by** (*simp add*: *f-def*)
then have *card* {*v* ∈ *V* . *has-loop v*} = *card* (*f* ‘ {*v* ∈ *V* . *has-loop v*})
using *card-image* **by** *fastforce*
also have ... = *card* {{*v*} | *v* . *has-loop v*} **using** *feq* **by** *simp*
finally have *card* {*v* ∈ *V* . *has-loop v*} = *card* {*e* ∈ *E* . *is-loop e*} **using** *is-loop-set-alt* **by** *simp*

thus *card* $\{e \in E$. *is-loop* $e\}$ = *card* $\{v \in V$. *has-loop* $v\}$ **by** *simp*
qed

lemma *finite-all-edges-between′*: *finite* (*all-edges-between* X Y)
using *finV wellformed*
by (*metis all-edges-between-rem-wf finite-Int finite-all-edges-between*)

lemma *card-all-edges-between*:
assumes *finite* Y
shows *card* (*all-edges-between* X Y) = ($\sum y \in Y$. *card* (*all-edges-between* X $\{y\}$))
proof −
have *all-edges-between* X Y = ($\bigcup y \in Y$. *all-edges-between* X $\{y\}$)
by (*auto simp*: *all-edges-between-def*)
moreover have *disjoint-family-on* (λy. *all-edges-between* X $\{y\}$) Y
unfolding *disjoint-family-on-def*
by (*auto simp*: *disjoint-family-on-def all-edges-between-def*)
ultimately show *?thesis*
by (*simp add*: *card-UN-disjoint′ assms finite-all-edges-between′*)
qed

end

1.6 Simple Graphs

A simple graph (or sgraph) constrains edges to size of two. This is the classic definition of an undirected graph

locale *sgraph* = *graph-system* +
assumes *two-edges*: $e \in E \Longrightarrow$ *card* $e = 2$
begin

lemma *wellformed-all-edges*: $E \subseteq$ *all-edges* V
unfolding *all-edges-def* **using** *wellformed two-edges* **by** *auto*

lemma *e-in-all-edges*: $e \in E \Longrightarrow e \in$ *all-edges* V
using *wellformed-all-edges* **by** *auto*

lemma *e-in-all-edges-ss*: $e \in E \Longrightarrow e \subseteq V' \Longrightarrow V' \subseteq V \Longrightarrow e \in$ *all-edges* V'
unfolding *all-edges-def* **using** *wellformed two-edges* **by** *auto*

lemma *singleton-not-edge*: $\{x\} \notin E$ — Suggested by Mantas Baksys
using *two-edges* **by** *fastforce*

end

It is easy to proof that *sgraph* is a sublocale of *ulgraph*. By using indirect inheritance, we avoid two unneeded cardinality conditions

sublocale *sgraph* $\subseteq$ *ulgraph* V E
by (*unfold-locales*)(*simp add*: *two-edges*)

locale *fin-sgraph* = *sgraph* + *fin-graph-system*
begin

lemma *fin-neighbourhood*: *finite* (*neighborhood x*)
  unfolding *neighborhood-def* **using** *finV* **by** *simp*

lemma *fin-all-edges*: *finite* (*all-edges V*)
  unfolding *all-edges-def* **by** (*simp add*: *finV*)

lemma *max-edges-graph*: *card E* ≤ (*card V*)^*2*
proof −
  have *card E* ≤ *card V choose 2*
    by (*metis fin-all-edges finV card-all-edges card-mono wellformed-all-edges*)
  thus *?thesis*
    by (*metis binomial-le-pow le0 neq0-conv order.trans zero-less-binomial-iff*)
qed

end

sublocale *fin-sgraph* ⊆ *fin-ulgraph*
  by (*unfold-locales*)

context *sgraph*
begin

lemma *no-loops*: *v* ∈ *V* ⟹ ¬ *has-loop v*
  using *has-loop-def two-edges* **by** *fastforce*

Ideally, we'd redefine degree in the context of a simple graph. However, this requires a named loop locale, which complicates notation unnecessarily. This is the lemma that should always be used when unfolding the degree definition in a simple graph context

lemma *alt-degree-def*[*simp*]: *degree v* = *card* (*incident-edges v*)
  using *no-loops degree-no-loops degree-none incident-edges-empty* **by** (*cases v* ∈ *V*) *simp-all*

lemma *alt-deg-neighborhood*: *degree v* = *card* (*neighborhood v*)
using *card-incident-sedges-neighborhood* **by** *simp*

definition *degree-set* :: *'a set* ⇒ *nat* **where**
degree-set vs ≡ *card* {*e* ∈ *E*. *vs* ⊆ *e*}

definition *is-complete-n-graph*:: *nat* ⇒ *bool* **where**
is-complete-n-graph n ≡ *gorder* = *n* ∧ *E* = *all-edges V*

The complement of a graph is a basic concept

definition *is-complement* :: *'a pregraph* ⇒ *bool* **where**
is-complement G ≡ *V* = *gverts G* ∧ *gedges G* = *all-edges V* − *E*

definition *complement-edges* :: *'a edge set* **where**
complement-edges ≡ *all-edges* $V - E$

lemma *is-complement-edges*: *is-complement* $(V', E') \longleftrightarrow V = V' \wedge$ *complement-edges* $= E'$
unfolding *is-complement-def complement-edges-def* **by** *auto*

interpretation *G-comp*: *sgraph V complement-edges*
by (*unfold-locales*)(*auto simp add*: *complement-edges-def all-edges-def*)

lemma *is-complement-edge-iff*: $e \subseteq V \Longrightarrow e \in$ *complement-edges* $\longleftrightarrow e \notin E \wedge$ *card* $e = 2$
unfolding *complement-edges-def all-edges-def* **by** *auto*

end

A complete graph is a simple graph

lemma *complete-sgraph*: *sgraph S* (*all-edges S*)
unfolding *all-edges-def* **by** (*unfold-locales*) (*simp-all*)

interpretation *comp-sgraph*: *sgraph S* (*all-edges S*)
using *complete-sgraph* **by** *auto*

lemma *complete-fin-sgraph*: *finite S* ⟹ *fin-sgraph S* (*all-edges S*)
using *complete-sgraph*
by (*intro-locales*) (*auto simp add*: *sgraph.axioms*(*1*) *sgraph-def fin-graph-system-axioms-def*)

1.7 Subgraph Basics

A subgraph is defined as a graph where the vertex and edge sets are subsets of the original graph. Note that using the locale approach, we require each graph to be wellformed. This is interestingly omitted in a number of other formal definitions.

locale *subgraph* = *H*: *graph-system* V_H :: *'a set* E_H + *G*: *graph-system* V_G :: *'a set* E_G **for** V_H E_H V_G E_G +
assumes *verts-ss*: $V_H \subseteq V_G$
assumes *edges-ss*: $E_H \subseteq E_G$

lemma *is-subgraphI*[*intro*]: $V' \subseteq V \Longrightarrow E' \subseteq E \Longrightarrow$ *graph-system* V' E' ⟹ *graph-system* V E ⟹ *subgraph* V' E' V E
using *graph-system-def* **by** (*unfold-locales*)
(*auto simp add*: *graph-system.vincident-def graph-system.incident-edge-in-wf*)

context *subgraph*
begin

Note: it could also be useful to have similar rules in *ulgraph* locale etc with subgraph assumption

lemma *is-subgraph-ulgraph*:
  assumes *ulgraph* V_G E_G
  shows *ulgraph* V_H E_H
  using *assms ulgraph.edge-size*[*of* V_G E_G] *edges-ss* **by** (*unfold-locales*) *auto*

lemma *is-simp-subgraph*:
  assumes *sgraph* V_G E_G
  shows *sgraph* V_H E_H
  using *assms sgraph.two-edges edges-ss* **by** (*unfold-locales*) *auto*

lemma *is-finite-subgraph*:
  assumes *fin-graph-system* V_G E_G
  shows *fin-graph-system* V_H E_H
  using *assms verts-ss*
  by (*unfold-locales*) (*simp add*: *fin-graph-system.finV finite-subset*)

lemma (**in** *graph-system*) *subgraph-refl*: *subgraph V E V E*
  by (*simp add*: *graph-system-axioms is-subgraphI*)

lemma *subgraph-trans*:
  assumes *graph-system V E*
  assumes *graph-system* V' E'
  assumes *graph-system* V'' E''
  shows *subgraph* V'' E'' V' E' $\Longrightarrow$ *subgraph* V' E' V E $\Longrightarrow$ *subgraph* V'' E'' V E
  by (*meson assms*(*1*) *assms*(*3*) *is-subgraphI subgraph.edges-ss subgraph.verts-ss subset-trans*)

lemma *subgraph-antisym*: *subgraph* V' E' V E $\Longrightarrow$ *subgraph* V E V' E' $\Longrightarrow$ $V = V' \wedge E = E'$
  by (*simp add*: *dual-order.eq-iff subgraph.edges-ss subgraph.verts-ss*)

end

lemma (**in** *sgraph*) *subgraph-complete*: *subgraph V E V* (*all-edges V*)
proof −
  interpret *comp*: *sgraph V* (*all-edges V*)
    using *complete-sgraph* **by** *auto*
  show *?thesis* **by** (*unfold-locales*) (*simp-all add*: *wellformed-all-edges*)
qed

We are often interested in the set of subgraphs. This is still very possible using locale definitions. Interesting Note - random graphs [3] has a different definition for the well formed constraint to be added in here instead of in the main subgraph definition

definition (**in** *graph-system*) *subgraphs*:: *'a pregraph set* **where**
subgraphs $\equiv$ {*G* . *subgraph* (*gverts G*) (*gedges G*) *V E*}

Induced subgraph - really only affects edges

definition (**in** *graph-system*) *induced-edges*:: *'a set* ⇒ *'a edge set* **where**
induced-edges $V' \equiv \{e \in E.\ e \subseteq V'\}$

lemma (**in** *sgraph*) *induced-edges-alt*: *induced-edges* $V' = E \cap$ *all-edges* V'
unfolding *induced-edges-def all-edges-def* **using** *two-edges* **by** *blast*

lemma (**in** *sgraph*) *induced-edges-self*: *induced-edges* $V = E$
unfolding *induced-edges-def*
by (*simp add*: *subsetI subset-antisym wellformed*)

context *graph-system*
begin

lemma *induced-edges-ss*: $V' \subseteq V \Longrightarrow$ *induced-edges* $V' \subseteq E$
unfolding *induced-edges-def* **by** *auto*

lemma *induced-is-graph-sys*: *graph-system* V' (*induced-edges* V')
by (*unfold-locales*) (*simp add*: *induced-edges-def*)

interpretation *induced-graph*: *graph-system* V' (*induced-edges* V')
using *induced-is-graph-sys* **by** *simp*

lemma *induced-is-subgraph*: $V' \subseteq V \Longrightarrow$ *subgraph* V' (*induced-edges* V') V E
using *induced-edges-ss* **by** (*unfold-locales*) *auto*

lemma *induced-edges-union*:
assumes $VH1 \subseteq S$ $VH2 \subseteq T$
assumes *graph-system VH1 EH1 graph-system VH2 EH2*
assumes $EH1 \cup EH2 \subseteq$ (*induced-edges* $(S \cup T)$)
shows $EH1 \subseteq$ (*induced-edges* S)
proof (*intro subsetI*, *simp add*: *induced-edges-def*, *intro conjI*)
show $\bigwedge x.\ x \in EH1 \Longrightarrow x \in E$ **using** *assms*(*5*)
by (*simp add*: *induced-edges-def subset-iff*)
show $\bigwedge x.\ x \in EH1 \Longrightarrow x \subseteq S$
using *assms*(*1*) *assms*(*3*) *graph-system.wellformed* **by** *blast*
qed

lemma *induced-edges-union-subgraph-single*:
assumes $VH1 \subseteq S$ $VH2 \subseteq T$
assumes *graph-system VH1 EH1 graph-system VH2 EH2*
assumes *subgraph* $(VH1 \cup VH2)$ $(EH1 \cup EH2)$ $(S \cup T)$ (*induced-edges* $(S \cup T)$)
shows *subgraph VH1 EH1 S* (*induced-edges* S)
proof −
interpret *ug*: *subgraph* $(VH1 \cup VH2)$ $(EH1 \cup EH2)$ $(S \cup T)$ (*induced-edges* $(S \cup T)$)
using *assms*(*5*) **by** *simp*
show *subgraph VH1 EH1 S* (*induced-edges* S)

```
    using assms(3) graph-system-def
  by (unfold-locales) (blast, simp add: assms(1), meson assms induced-edges-union
ug.edges-ss)
qed

lemma induced-union-subgraph:
  assumes VH1 ⊆ S and VH2 ⊆ T
  assumes graph-system VH1 EH1 graph-system VH2 EH2
  shows subgraph VH1 EH1 S (induced-edges S) ∧ subgraph VH2 EH2 T (induced-edges
T) ⟷
    subgraph (VH1 ∪ VH2) (EH1 ∪ EH2) (S ∪ T) (induced-edges (S ∪ T))
proof (intro iffI conjI, elim conjE)
  show subgraph (VH1 ∪ VH2) (EH1 ∪ EH2) (S ∪ T) (induced-edges (S ∪ T))
      ⟹ subgraph VH1 EH1 S (induced-edges S)
    using induced-edges-union-subgraph-single assms by simp
  show subgraph (VH1 ∪ VH2) (EH1 ∪ EH2) (S ∪ T) (induced-edges (S ∪ T))
      ⟹ subgraph VH2 EH2 T (induced-edges T)
    using induced-edges-union-subgraph-single assms by (simp add: Un-commute)
  assume a1: subgraph VH1 EH1 S (induced-edges S) and a2: subgraph VH2 EH2
T (induced-edges T)
  then interpret h1: subgraph VH1 EH1 S (induced-edges S)
    by simp
  interpret h2: subgraph VH2 EH2 T (induced-edges T) using a2 by simp
  show subgraph (VH1 ∪ VH2) (EH1 ∪ EH2) (S ∪ T) (induced-edges (S ∪ T))
      using h1.H.wellformed h2.H.wellformed h1.verts-ss h2.verts-ss h1.edges-ss
h2.edges-ss
    by (unfold-locales) (auto simp add: induced-edges-def)
qed

end
end
theory Undirected-Graph-Walks imports Undirected-Graph-Basics
begin
```

2 Walks, Paths and Cycles

The definition of walks, paths, cycles, and related concepts are foundations of graph theory, yet there can be some differences in literature between definitions. This formalisation draws inspiration from Noschinski's Graph Library [5], however focuses on an undirected graph context compared to a directed graph context, and extends on some definitions, as required to formalise Balog Szemeredi Gowers theorem.

```
context ulgraph
begin
```

2.1 Walks

This definition is taken from the directed graph library, however edges are undirected

fun *walk-edges* :: *'a list* ⇒ *'a edge list* **where**
  walk-edges [] = []
| *walk-edges* [*x*] = []
| *walk-edges* (*x* # *y* # *ys*) = {*x*,*y*} # *walk-edges* (*y* # *ys*)

lemma *walk-edges-app*: *walk-edges* (*xs* @ [*y*, *x*]) = *walk-edges* (*xs* @ [*y*]) @ [{*y*, *x*}]
 by (*induct xs rule*: *walk-edges.induct*, *simp-all*)

lemma *walk-edges-tl-ss*: *set* (*walk-edges* (*tl xs*)) ⊆ *set* (*walk-edges xs*)
 by (*induct xs rule*: *walk-edges.induct*) *auto*

lemma *walk-edges-rev*: *rev* (*walk-edges xs*) = *walk-edges* (*rev xs*)
proof (*induct xs rule*: *walk-edges.induct*, *simp-all*)
 fix *x y ys* **assume** *assm*: *rev* (*walk-edges* (*y* # *ys*)) = *walk-edges* (*rev ys* @ [*y*])
 then show *walk-edges* (*rev ys* @ [*y*]) @ [{*x*, *y*}] = *walk-edges* (*rev ys* @ [*y*, *x*])
  using *walk-edges-app* **by** *fastforce*
qed

lemma *walk-edges-append-ss1*: *set* (*walk-edges* (*ys*)) ⊆ *set* (*walk-edges* (*xs*@*ys*))
proof (*induct xs rule*: *walk-edges.induct*)
 case *1*
 then show *?case* **by** *simp*
next
 case (*2 x*)
 then show *?case*
  using *walk-edges-tl-ss* **by** *fastforce*
next
 case (*3 x y ys*)
 then show *?case* **by** (*simp add*: *subset-iff*)
qed

lemma *walk-edges-append-ss2*: *set* (*walk-edges* (*xs*)) ⊆ *set* (*walk-edges* (*xs*@*ys*))
 by (*induct xs rule*: *walk-edges.induct*) *auto*

lemma *walk-edges-singleton-app*: *ys* ≠ [] ⟹ *walk-edges* ([*x*]@*ys*) = {*x*, *hd ys*} # *walk-edges ys*
 using *list.exhaust-sel walk-edges.simps*(*3*) **by** (*metis Cons-eq-appendI eq-Nil-appendI*)

lemma *walk-edges-append-union*: *xs* ≠ [] ⟹ *ys* ≠ [] ⟹
  set (*walk-edges* (*xs*@*ys*)) = *set* (*walk-edges* (*xs*)) ∪ *set* (*walk-edges ys*) ∪ {{*last xs*, *hd ys*}}
 using *walk-edges-singleton-app* **by** (*induct xs rule*: *walk-edges.induct*) *auto*

lemma *walk-edges-decomp-ss*: *set* (*walk-edges* (*xs*@[*y*]@*zs*)) ⊆ *set* (*walk-edges* (*xs*@[*y*]@*ys*@[*y*]@*zs*))
proof –
  have *half-ss*: *set* (*walk-edges* (*xs*@[*y*])) ⊆ *set* (*walk-edges* (*xs*@[*y*]@*ys*@[*y*]))
    using *walk-edges-append-ss2* **by** *fastforce*
  thus *?thesis* **proof** (*cases zs* = [])
    case *True*
    then show *?thesis* **using** *half-ss* **by** *auto*
  next
    case *False*
    then have *decomp1*: *set* (*walk-edges* (*xs*@[*y*]@*zs*)) = *set* (*walk-edges* (*xs*@[*y*])) ∪ *set* (*walk-edges* (*zs*)) ∪ {{*y*, *hd zs*}}
      using *walk-edges-append-union*
      by (*metis append-assoc append-is-Nil-conv last-snoc neq-Nil-conv*)
    have *set* (*walk-edges* (*xs*@[*y*]@*ys*@[*y*]@*zs*)) = *set* (*walk-edges* (*xs*@[*y*]@*ys*@[*y*])) ∪ *set* (*walk-edges* (*zs*)) ∪ {{*y*, *hd zs*}}
      using *walk-edges-append-union False*
        by (*metis append-assoc append-is-Nil-conv empty-iff empty-set last-snoc list.set-intros*(*1*))
    then show *?thesis* **using** *decomp1 half-ss* **by** *auto*
  qed
qed

definition *walk-length* :: *'a list* ⇒ *nat* **where**
  walk-length p ≡ *length* (*walk-edges p*)

lemma *walk-length-conv*: *walk-length p* = *length p* − *1*
  by (*induct p rule*: *walk-edges.induct*) (*auto simp*: *walk-length-def*)

lemma *walk-length-rev*: *walk-length p* = *walk-length* (*rev p*)
  using *walk-edges-rev walk-length-def*
  by (*metis length-rev*)

lemma *walk-length-app*: *xs* ≠ [] ⟹ *ys* ≠ [] ⟹ *walk-length* (*xs* @ *ys*) = *walk-length xs* + *walk-length ys* + *1*
  apply (*induct xs rule*: *walk-edges.induct*)
    apply (*simp-all add*: *walk-length-def*)
  using *walk-edges-singleton-app* **by** *force*

lemma *walk-length-app-ineq*: *walk-length* (*xs* @ *ys*) ≥ *walk-length xs* + *walk-length ys* ∧
  walk-length (*xs* @ *ys*) ≤ *walk-length xs* + *walk-length ys* + *1*
proof (*cases xs* = [] ∨ *ys* = [])
  case *True*
  then show *?thesis* **using** *walk-length-def* **by** *auto*
next
  case *False*
  then show *?thesis*
    by (*simp add*: *walk-length-app*)
qed

Note that while the trivial walk is allowed, the empty walk is not

definition *is-walk* :: *'a list* ⇒ *bool* **where**
$is\text{-}walk\ xs \equiv set\ xs \subseteq V \land set\ (walk\text{-}edges\ xs) \subseteq E \land xs \neq []$

lemma *is-walkI*: $set\ xs \subseteq V \Longrightarrow set\ (walk\text{-}edges\ xs) \subseteq E \Longrightarrow xs \neq [] \Longrightarrow is\text{-}walk\ xs$
using *is-walk-def* **by** *simp*

lemma *is-walk-wf*: $is\text{-}walk\ xs \Longrightarrow set\ xs \subseteq V$
by (*simp add*: *is-walk-def*)

lemma *is-walk-wf-hd*: $is\text{-}walk\ xs \Longrightarrow hd\ xs \in V$
using *is-walk-wf hd-in-set is-walk-def* **by** *blast*

lemma *is-walk-wf-last*: $is\text{-}walk\ xs \Longrightarrow last\ xs \in V$
using *is-walk-wf last-in-set is-walk-def* **by** *blast*

lemma *is-walk-singleton*: $u \in V \Longrightarrow is\text{-}walk\ [u]$
unfolding *is-walk-def* **using** *walk-edges.simps* **by** *simp*

lemma *is-walk-not-empty*: $is\text{-}walk\ xs \Longrightarrow xs \neq []$
unfolding *is-walk-def* **by** *simp*

lemma *is-walk-not-empty2*: $is\text{-}walk\ [] = False$
unfolding *is-walk-def* **by** *simp*

Reasoning on transformations of a walk

lemma *is-walk-rev*: $is\text{-}walk\ xs \longleftrightarrow is\text{-}walk\ (rev\ xs)$
unfolding *is-walk-def* **using** *walk-edges-rev*
by (*metis rev-is-Nil-conv set-rev*)

lemma *is-walk-tl*: $length\ xs \geq 2 \Longrightarrow is\text{-}walk\ xs \Longrightarrow is\text{-}walk\ (tl\ xs)$
using *walk-edges-tl-ss is-walk-def in-mono list.set-sel*(*2*) *tl-Nil* **by** *fastforce*

lemma *is-walk-append*:
assumes *is-walk xs*
assumes *is-walk ys*
assumes $last\ xs = hd\ ys$
shows $is\text{-}walk\ (xs\ @\ (tl\ ys))$
proof (*intro is-walkI subsetI*)
show $xs\ @\ tl\ ys \neq []$ **using** *is-walk-def assms* **by** *auto*
show $\bigwedge x.\ x \in set\ (xs\ @\ tl\ ys) \Longrightarrow x \in V$ **using** *assms is-walk-def is-walk-wf*
by (*metis Un-iff in-mono list-set-tl set-append*)
next
fix *x* **assume** *xin*: $x \in set\ (walk\text{-}edges\ (xs\ @\ tl\ ys))$
show $x \in E$ **proof** (*cases* $tl\ ys = []$)
case *True*
then show *?thesis* **using** *assms*(*1*) *is-walk-def xin* **by** *auto*
next

```
    case False
    then have xin2: x ∈ (set (walk-edges xs) ∪ set (walk-edges (tl ys)) ∪ {{last xs,
hd (tl ys)}})
      using walk-edges-append-union is-walk-not-empty assms xin by auto
    have 1: set (walk-edges xs) ⊆ E using assms(1) is-walk-def
      by simp
    have 2: set (walk-edges (tl ys)) ⊆ E using assms(2) is-walk-def
      by (meson dual-order.trans walk-edges-tl-ss)
    have {last xs, hd (tl ys)} ∈ E using is-walk-def assms(2) assms(3)
      by (metis False hd-Cons-tl insert-subset list.simps(15) walk-edges.simps(3))
    then show ?thesis using 1 2 xin2 by auto
  qed
qed

lemma is-walk-decomp:
  assumes is-walk (xs@[y]@ys@[y]@zs) (is is-walk ?w)
  shows is-walk (xs@[y]@zs)
proof (intro is-walkI)
  show set (xs @ [y] @ zs) ⊆ V  using assms is-walk-def by simp
  show xs @ [y] @ zs ≠ [] by simp
  show set (walk-edges (xs @ [y] @ zs)) ⊆ E
    using walk-edges-decomp-ss assms(1) is-walk-def by blast
qed

lemma is-walk-hd-tl:
  assumes is-walk (y # ys)
  assumes {x, y} ∈ E
  shows is-walk (x # y # ys)
proof (intro is-walkI)
  show set (x # y # ys) ⊆ V
    using assms by (simp add: is-walk-def wellformed-alt-fst)
  show set (walk-edges (x # y # ys)) ⊆ E
    using walk-edges.simps assms is-walk-def by simp
  show x # y # ys ≠ [] by simp
qed

lemma is-walk-drop-hd:
  assumes ys ≠ []
  assumes is-walk (y # ys)
  shows is-walk ys
proof (intro is-walkI)
  show set ys ⊆ V
    using assms is-walk-wf by fastforce
  show set (walk-edges ys) ⊆ E
    using assms is-walk-def walk-edges-tl-ss by force
  show ys ≠ [] using assms by simp
qed

lemma walk-edges-index:
```

```
  assumes i ≥ 0 i < walk-length w
  assumes is-walk w
  shows (walk-edges w) ! i ∈ E
  using assms
proof (induct w arbitrary: i rule: walk-edges.induct, simp add: is-walk-not-empty2,

    simp add: walk-length-def)
  case (3 x y ys)
  then show ?case proof (cases i = 0)
    case True
    then show ?thesis
      using 3.prems(3) is-walk-def by fastforce
  next
    case False
    have gt: 0 ≤ i −1 using False by simp
    have lt: i − 1 < walk-length (y # ys)
      using 3.prems(2) False walk-length-conv by auto
    have is-walk (y # ys)
      using 3.prems(3) is-walk-def by fastforce
    then show ?thesis using 3.hyps[of i −1]
    by (metis 3.prems(1) False gt lt le-neq-implies-less nth-Cons-pos walk-edges.simps(3))

  qed
qed

lemma is-walk-index:
  assumes i ≥ 0 Suc i < (length w)
  assumes is-walk w
  shows {w ! i, w ! (i + 1)} ∈ E
  using assms proof (induct w arbitrary: i rule: walk-edges.induct, simp, simp)
  fix x y ys i
  assume IH: ⋀j. 0 ≤ j ⟹ Suc j < length (y # ys) ⟹ is-walk (y # ys) ⟹
{(y # ys) ! j, (y # ys) ! (j + 1)} ∈ E
  assume 1:  0 ≤ i and 2: Suc i < length (x # y # ys) and 3: is-walk (x # y #
ys)
  show {(x # y # ys) ! i, (x # y # ys) ! (i + 1)} ∈ E
  proof (cases i = 0)
    case True
    then show ?thesis using 3 is-walk-def
      by simp
  next
    case False
    have is-walk (y # ys) using is-walk-def 3 by fastforce
    then show ?thesis using 2 IH[of i − 1]
      by (simp add: False nat-less-le)
  qed
qed

lemma is-walk-take:
```

```
  assumes is-walk w
  assumes n > 0
  assumes n ≤ length w
  shows is-walk (take n w)
  using assms proof (induct w arbitrary: n rule: walk-edges.induct)
  case 1
  then show ?case by simp
next
  case (2 x)
  then have n = 1 using 2 by auto
  then show ?case by (simp add: 2.prems(1))
next
  case (3 x y ys)
  then show ?case proof (cases n = 1)
    case True
    then have take n (x # y # ys) = [x]
      by simp
    then show ?thesis using is-walk-def 3.prems(1) by simp
  next
    case False
    then have ngt: n ≥ 2 using 3.prems(2) by auto
    then have tk-split1: take n (x # y # ys) = x # take (n − 1) (y # ys) using 3
      by (simp add: take-Cons')
    then have tk-split: take n (x # y # ys) = x # y # (take (n − 2) ys)
      using 3 ngt take-Cons'[of n −1 y ys]
      by (metis False diff-diff-left less-one nat-neq-iff one-add-one zero-less-diff)
    have w: is-walk (y # ys) using is-walk-tl
      using 3.prems(1) is-walk-def by force
    have n − 1 ≤ length (y # ys) using 3.prems(3) by simp
     then have w-tl: is-walk (take (n − 1) (y # ys)) using 3.hyps[of n − 1] w
3.prems ngt
      by linarith
    have {x, y} ∈ E using is-walk-def walk-edges.simps 3.prems(1) by auto
    then show ?thesis using is-walk-hd-tl[of y (take (n − 2) ys) x] tk-split
      using tk-split1 w-tl by force
  qed
qed

lemma is-walk-drop:
  assumes is-walk w
  assumes n < length w
  shows is-walk (drop n w)
  using assms proof (induct w arbitrary: n rule: walk-edges.induct)
  case 1
  then show ?case by simp
next
  case (2 x)
  then have n = 0 using 2 by auto
  then show ?case by (simp add: 2.prems(1))
```

next
  case (*3 x y ys*)
  then show *?case* **proof** (*cases* $n \geq 2$)
    case *True*
    then have *ngt*: $n \geq 2$ **using** *3.prems*(*2*) **by** *auto*
    then have *tk-split1*: *drop n* (*x* # *y* # *ys*) = *drop* (*n* − *1*) (*y* # *ys*) **using** *3*
      by (*simp add*: *drop-Cons′*)
    then have *tk-split*: *drop n* (*x* # *y* # *ys*) = (*drop* (*n* − *2*) *ys*)
      using *3 ngt drop-Cons′*[*of n* −*1 y ys*] *True*
      by (*metis Suc-1 Suc-le-eq diff-diff-left less-not-refl nat-1-add-1 zero-less-diff*)
    have *w*: *is-walk* (*y* # *ys*) **using** *is-walk-tl*
      using *3.prems*(*1*) *is-walk-def* **by** *force*
    have *n* − *1* < *length* (*y* # *ys*) **using** *3.prems*(*2*) **by** *simp*
     then have *w-tl*: *is-walk* (*drop* (*n* − *1*) (*y* # *ys*)) **using** *3.hyps*[*of n* − *1*] *w*
3.prems ngt
      by *linarith*
    have {*x*, *y*} ∈ *E* **using** *is-walk-def walk-edges.simps 3.prems*(*1*) **by** *auto*
    then show *?thesis* **using** *is-walk-hd-tl*[*of y* (*take* (*n* − *2*) *ys*) *x*] *tk-split*
      using *tk-split1 w-tl* **by** *force*
  next
    case *False*
    then have *or*: *n* = *0* ∨ *n* = *1*
      by *auto*
    have *walk*: *is-walk* (*y* # *ys*) **using** *is-walk-drop-hd 3* **by** *blast*
    have *n0*: *n* = *0* ⟹ (*drop n* (*x* # *y* # *ys*)) = (*x* # *y* # *ys*) **by** *simp*
    have *n* = *1* ⟹ (*drop n* (*x* # *y* # *ys*)) = *y* # *ys* **by** *simp*
    then show *?thesis* **using** *n0 3 walk or* **by** *auto*
  qed
qed

definition *walks* :: *′a list set* **where**
  walks ≡ {*p*. *is-walk p*}

definition *is-open-walk* :: *′a list* ⇒ *bool* **where**
is-open-walk xs ≡ *is-walk xs* ∧ *hd xs* ≠ *last xs*

lemma *is-open-walk-rev*: *is-open-walk xs* ⟷ *is-open-walk* (*rev xs*)
  unfolding *is-open-walk-def* **using** *is-walk-rev*
  by (*metis hd-rev last-rev*)

definition *is-closed-walk* :: *′a list* ⇒ *bool* **where**
is-closed-walk xs ≡ *is-walk xs* ∧ *hd xs* = *last xs*

lemma *is-closed-walk-rev*: *is-closed-walk xs* ⟷ *is-closed-walk* (*rev xs*)
  unfolding *is-closed-walk-def* **using** *is-walk-rev*
  by (*metis hd-rev last-rev*)

definition *is-trail* :: *′a list* ⇒ *bool* **where**
is-trail xs ≡ *is-walk xs* ∧ *distinct* (*walk-edges xs*)

lemma *is-trail-rev*: *is-trail xs* ⟷ *is-trail* (*rev xs*)
  unfolding *is-trail-def* **using** *is-walk-rev*
  by (*metis distinct-rev walk-edges-rev*)

2.2 Paths

There are two common definitions of a path. The first, given below, excludes the case where a path is a cycle. Note this also excludes the trivial path [x]

definition *is-path* :: *'a list* ⇒ *bool* **where**
is-path xs ≡ (*is-open-walk xs* ∧ *distinct* (*xs*))

lemma *is-path-rev*: *is-path xs* ⟷ *is-path* (*rev xs*)
  unfolding *is-path-def* **using** *is-open-walk-rev*
  by (*metis distinct-rev*)

lemma *is-path-walk*: *is-path xs* ⟹ *is-walk xs*
  unfolding *is-path-def is-open-walk-def* **by** *auto*

definition *paths* :: *'a list set* **where**
paths ≡ {*p* . *is-path p*}

lemma *paths-ss-walk*: *paths* ⊆ *walks*
  unfolding *paths-def walks-def is-path-def is-open-walk-def* **by** *auto*

A more generic definition of a path - used when a cycle is considered a path, and therefore includes the trivial path [x]

definition *is-gen-path*:: *'a list* ⇒ *bool* **where**
is-gen-path p ≡ *is-walk p* ∧ ((*distinct* (*tl p*) ∧ *hd p* = *last p*) ∨ *distinct p*)

lemma *is-path-gen-path*: *is-path p* ⟹ *is-gen-path p*
  unfolding *is-path-def is-gen-path-def is-open-walk-def* **by** (*auto simp add*: *distinct-tl*)

lemma *is-gen-path-rev*: *is-gen-path p* ⟷ *is-gen-path* (*rev p*)
  unfolding *is-gen-path-def* **using** *is-walk-rev distinct-tl-rev*
  by (*metis distinct-rev hd-rev last-rev*)

lemma *is-gen-path-distinct*: *is-gen-path p* ⟹ *hd p* ≠ *last p* ⟹ *distinct p*
  unfolding *is-gen-path-def* **by** *auto*

lemma *is-gen-path-distinct-tl*:
  assumes *is-gen-path p* **and** *hd p* = *last p*
  shows *distinct* (*tl p*)
proof (*cases length p* > *1*)
  case *True*
  then show *?thesis*
    using *assms*(*1*) *distinct-tl is-gen-path-def* **by** *auto*
next

case *False*
then show *?thesis*
using *assms*(*1*) *distinct-tl is-gen-path-def* **by** *auto*
qed

lemma *is-gen-path-trivial*: $x \in V \Longrightarrow$ *is-gen-path* [x]
unfolding *is-gen-path-def is-walk-def* **by** *simp*

definition *gen-paths* :: *'a list set* **where**
gen-paths $\equiv$ {*p* . *is-gen-path p*}

lemma *gen-paths-ss-walks*: *gen-paths* $\subseteq$ *walks*
unfolding *gen-paths-def walks-def is-gen-path-def* **by** *auto*

2.3 Cycles

Note, a cycle must be non trivial (i.e. have an edge), but as we let a loop by a cycle we broaden the definition in comparison to Noschinski [5] for a cycle to be of length greater than 1 rather than 3

definition *is-cycle* :: *'a list* $\Rightarrow$ *bool* **where**
is-cycle xs $\equiv$ *is-closed-walk xs* $\wedge$ *walk-length xs* $\geq$ *1* $\wedge$ *distinct* (*tl xs*)

lemma *is-gen-path-cycle*: *is-cycle p* $\Longrightarrow$ *is-gen-path p*
unfolding *is-cycle-def is-gen-path-def is-closed-walk-def* **by** *auto*

lemma *is-cycle-alt-gen-path*: *is-cycle xs* $\longleftrightarrow$ *is-gen-path xs* $\wedge$ *walk-length xs* $\geq$ *1* $\wedge$ *hd xs* = *last xs*
proof (*intro iffI*)
show *is-cycle xs* $\Longrightarrow$ *is-gen-path xs* $\wedge$ *1* $\leq$ *walk-length xs* $\wedge$ *hd xs* = *last xs*
using *is-gen-path-cycle is-cycle-def is-closed-walk-def*
by *auto*
show *is-gen-path xs* $\wedge$ *1* $\leq$ *walk-length xs* $\wedge$ *hd xs* = *last xs* $\Longrightarrow$ *is-cycle xs*
using *distinct-tl is-closed-walk-def is-cycle-def is-gen-path-def* **by** *blast*
qed

lemma *is-cycle-alt*: *is-cycle xs* $\longleftrightarrow$ *is-walk xs* $\wedge$ *distinct* (*tl xs*) $\wedge$ *walk-length xs* $\geq$ *1* $\wedge$ *hd xs* = *last xs*
proof (*intro iffI*)
show *is-cycle xs* $\Longrightarrow$ *is-walk xs* $\wedge$ *distinct* (*tl xs*) $\wedge$ *1* $\leq$ *walk-length xs* $\wedge$ *hd xs* = *last xs*
using *is-cycle-alt-gen-path is-cycle-def is-gen-path-def* **by** *blast*
show *is-walk xs* $\wedge$ *distinct* (*tl xs*) $\wedge$ *1* $\leq$ *walk-length xs* $\wedge$ *hd xs* = *last xs* $\Longrightarrow$ *is-cycle xs*
by (*simp add*: *is-cycle-alt-gen-path is-gen-path-def*)
qed

lemma *is-cycle-rev*: *is-cycle xs* $\longleftrightarrow$ *is-cycle* (*rev xs*)
proof –
have *len*: *1* $\leq$ *walk-length xs* $\longleftrightarrow$ *1* $\leq$ *walk-length* (*rev xs*)

by (*metis length-rev walk-edges-rev walk-length-def*)
have *hd xs* = *last xs* ⟹ *distinct* (*tl xs*) ⟷ *distinct* (*tl* (*rev xs*))
using *distinct-tl-rev* **by** *blast*
then show *?thesis* **using** *len is-cycle-def*
using *is-closed-walk-def is-closed-walk-rev* **by** *auto*
qed

lemma *cycle-tl-is-path*: *is-cycle xs* ∧ *walk-length xs* ≥ *3* ⟹ *is-path* (*tl xs*)
proof (*simp add*: *is-cycle-def is-path-def is-open-walk-def is-closed-walk-def walk-length-conv*,

elim conjE, *intro conjI*, *simp add*: *is-walk-tl*)
assume *w*: *is-walk xs* **and** *eq*: *hd xs* = *last xs* **and** *3* ≤ *length xs* − *Suc 0* **and**
dis: *distinct* (*tl xs*)
then have *len*: *4* ≤ *length xs*
by *linarith*
then have *lentl*: *3* ≤ *length* (*tl xs*) **by** *simp*
then have *lentltl*: *2* ≤ *length* (*tl* (*tl xs*)) **by** *simp*
have *last* (*tl* (*tl xs*)) = *last* (*tl xs*)
by (*metis One-nat-def Suc-1* ‹*3* ≤ *length xs* − *Suc 0*› *diff-is-0-eq′ is-walk-def is-walk-tl last-tl*
lentl not-less-eq-eq numeral-le-one-iff one-le-numeral order.trans semiring-norm(*70*) *w*)
then have *last* (*tl xs*) ∈ *set* (*tl* (*tl xs*))
using *last-in-list-tl-set lentltl* **by** (*metis last-in-set list.sel*(*2*))
moreover have *hd* (*tl xs*) ∉ *set* (*tl* (*tl xs*)) **using** *dis lentltl*
by (*metis distinct.simps*(*2*) *hd-Cons-tl list.sel*(*2*) *list.size*(*3*) *not-numeral-le-zero*)

ultimately show *hd* (*tl xs*) ≠ *last* (*tl xs*) **by** *fastforce*
qed

lemma *is-gen-path-path*:
assumes *is-gen-path p* **and** *walk-length p* > *0* **and** (¬ *is-cycle p*)
shows *is-path p*
proof (*simp add*: *is-gen-path-def is-path-def is-open-walk-def*, *intro conjI*)
show *is-walk p* **using** *is-gen-path-def assms*(*1*) **by** *simp*
show *ne*: *hd p* ≠ *last p*
using *assms*(*1*) *assms*(*2*) *assms*(*3*) *is-cycle-alt-gen-path* **by** *auto*
have ((*distinct* (*tl p*) ∧ *hd p* = *last p*) ∨ *distinct p*) **using** *is-gen-path-def assms*(*1*)
by *auto*
thus *distinct p* **using** *ne* **by** *auto*
qed

lemma *is-gen-path-options*: *is-gen-path p* ⟷ *is-cycle p* ∨ *is-path p* ∨ (∃ *v* ∈ *V* .
p = [*v*])
proof (*intro iffI*)
assume *a*: *is-gen-path p*
then have *p* ≠ [] **unfolding** *is-gen-path-def is-walk-def* **by** *auto*
then have (∀ *v* ∈ *V* . *p* ≠ [*v*]) ⟹ *walk-length p* > *0* **using** *walk-length-def*
by (*metis a is-gen-path-def is-walk-wf-hd length-greater-0-conv list.collapse*

list.distinct(*1*) *walk-edges.simps*(*3*))
then show *is-cycle p* ∨ *is-path p* ∨ (∃ *v* ∈ *V*. *p* = [*v*])
using *a is-gen-path-path* **by** *auto*
next
show *is-cycle p* ∨ *is-path p* ∨ (∃ *v* ∈ *V*. *p* = [*v*]) ⟹ *is-gen-path p*
using *is-gen-path-cycle is-path-gen-path is-gen-path-trivial* **by** *auto*
qed

definition *cycles* :: *'a list set* **where**
cycles ≡ {*p*. *is-cycle p*}

lemma *cycles-ss-gen-paths*: *cycles* ⊆ *gen-paths*
unfolding *cycles-def gen-paths-def* **using** *is-gen-path-cycle* **by** *auto*

lemma *gen-paths-ss*: *gen-paths* ⊆ *cycles* ∪ *paths* ∪ {[*v*] | *v*. *v* ∈ *V*}
unfolding *gen-paths-def cycles-def paths-def* **using** *is-gen-path-options* **by** *auto*

Walk edges are distinct in a path and cycle

lemma *distinct-edgesI*:
assumes *distinct p* **shows** *distinct* (*walk-edges p*)
proof −
from *assms* **have** *?thesis* ⋀*u*. *u* ∉ *set p* ⟹ (⋀*v*. *u* ≠ *v* ⟹ {*u*,*v*} ∉ *set* (*walk-edges p*))
by (*induct p rule*: *walk-edges.induct*) *auto*
then show *?thesis* **by** *simp*
qed

lemma *scycles-distinct-edges*:
assumes *c* ∈ *cycles 3* ≤ *walk-length c* **shows** *distinct* (*walk-edges c*)
proof −
from *assms* **have** *c-props*: *distinct* (*tl c*) *4* ≤ *length c hd c* = *last c*
by (*auto simp add*: *cycles-def is-cycle-def is-closed-walk-def walk-length-conv*)
then have {*hd c*, *hd* (*tl c*)} ∉ *set* (*walk-edges* (*tl c*))
proof (*induct c rule*: *walk-edges.induct*)
case (*3 x y ys*)
then have *hd ys* ≠ *last ys* **by** (*cases ys*) *auto*
moreover
from *3* **have** *walk-edges* (*y* # *ys*) = {*y*, *hd ys*} # *walk-edges ys*
by (*cases ys*) *auto*
moreover
{ fix *xs* **have** *set* (*walk-edges xs*) ⊆ *Pow* (*set xs*)
by (*induct xs rule*: *walk-edges.induct*) *auto* **}**
ultimately
show *?case* **using** *3* **by** *auto*
qed *simp-all*
moreover
from *assms* **have** *distinct* (*walk-edges* (*tl c*))
by (*intro distinct-edgesI*) (*simp add*: *cycles-def is-cycle-def*)
ultimately

show *?thesis* **by**(*cases c*, *simp-all*)
(*metis distinct.simps*(*1*) *distinct.simps*(*2*) *list.sel*(*1*) *list.sel*(*3*) *walk-edges.elims*)
qed

end

context *fin-ulgraph*
begin

lemma *finite-paths*: *finite paths*
proof −
have *ss*: *paths* $\subseteq$ {*xs. set xs* $\subseteq$ *V* $\wedge$ *length xs* $\leq$ (*card* (*V*))}
proof (*rule*, *simp*, *intro conjI*)
show *1*: $\bigwedge$*x. x* $\in$ *paths* $\Longrightarrow$ *set x* $\subseteq$ *V*
unfolding *paths-def is-path-def is-open-walk-def is-walk-def* **by** *simp*
fix *x* **assume** *a*: *x* $\in$ *paths*
then have *distinct x*
using *paths-def is-path-def* **by** *simp-all*
then have *eq*: *length x* = *card* (*set x*)
by (*simp add*: *distinct-card*)
then show *length x* $\leq$ *gorder* **using** *a 1*
by (*simp add*: *card-mono finV*)
qed
have *finite* {*xs. set xs* $\subseteq$ *V* $\wedge$ *length xs* $\leq$ (*card* (*V*))}
using *finV* **by** (*simp add*: *finite-lists-length-le*)
thus *?thesis* **using** *ss finite-subset* **by** *auto*
qed

lemma *finite-cycles*: *finite* (*cycles*)
proof −
have *cycles* $\subseteq$ {*xs. set xs* $\subseteq$ *V* $\wedge$ *length xs* $\leq$ *Suc* (*card* (*V*))}
proof (*rule*, *simp*)
fix *p* **assume** *p* $\in$ *cycles*
then have *distinct* (*tl p*) **and** *set p* $\subseteq$ *V*
unfolding *cycles-def walks-def is-cycle-def is-closed-walk-def is-walk-def*
by (*simp-all*)
then have *set* (*tl p*) $\subseteq$ *V*
by (*cases p*) *auto*
with *finV* **have** *card* (*set* (*tl p*)) $\leq$ *card* (*V*)
by (*rule card-mono*)
then have *length* (*p*) $\leq$ *1* + *card* (*V*)
using *distinct-card*[*OF* ‹*distinct* (*tl p*)›] **by** *auto*
then show *set p* $\subseteq$ *V* $\wedge$ *length p* $\leq$ *Suc* (*card* (*V*))
by (*simp add*: ‹*set p* $\subseteq$ *V*›)
qed
moreover
have *finite* {*xs. set xs* $\subseteq$ *V* $\wedge$ *length xs* $\leq$ *Suc* (*card* (*V*))}
using *finV* **by** (*rule finite-lists-length-le*)
ultimately

show *?thesis* **by** (*rule finite-subset*)
qed

lemma *finite-gen-paths*: *finite* (*gen-paths*)
proof –
have *finite* ({[*v*] | *v* . *v* ∈ *V*}) **using** *finV* **by** *auto*
thus *?thesis* **using** *gen-paths-ss finite-cycles finite-paths finite-subset* **by** *auto*
qed

end

end

3 Connectivity

This theory defines concepts around the connectivity of a graph and its vertices, as well as graph properties that depend on connectivity definitions, such as shortest path, radius, diameter, and eccentricity

theory *Connectivity* **imports** *Undirected-Graph-Walks*
begin

context *ulgraph*
begin

3.1 Connecting Walks and Paths

definition *connecting-walk* :: *'a* ⇒ *'a* ⇒ *'a list* ⇒ *bool* **where**
connecting-walk u v xs ≡ *is-walk xs* ∧ *hd xs*= *u* ∧ *last xs* = *v*

lemma *connecting-walk-rev*: *connecting-walk u v xs* ⟷ *connecting-walk v u* (*rev xs*)
unfolding *connecting-walk-def* **using** *is-walk-rev*
by (*auto simp add*: *hd-rev last-rev*)

lemma *connecting-walk-wf*: *connecting-walk u v xs* ⟹ *u* ∈ *V* ∧ *v* ∈ *V*
using *is-walk-wf-hd is-walk-wf-last* **by** (*auto simp add*: *connecting-walk-def*)

lemma *connecting-walk-self*: *u* ∈ *V* ⟹ *connecting-walk u u* [*u*] = *True*
unfolding *connecting-walk-def* **by** (*simp add*: *is-walk-singleton*)

We define two definitions of connecting paths. The first uses the *gen-path* definition, which allows for trivial paths and cycles, the second uses the stricter definition of a path which requires it to be an open walk

definition *connecting-path* :: *'a* ⇒ *'a* ⇒ *'a list* ⇒ *bool* **where**
connecting-path u v xs ≡ *is-gen-path xs* ∧ *hd xs* = *u* ∧ *last xs* = *v*

definition *connecting-path-str* :: *'a* ⇒ *'a* ⇒ *'a list* ⇒ *bool* **where**
connecting-path-str u v xs ≡ *is-path xs* ∧ *hd xs* = *u* ∧ *last xs* = *v*

lemma *connecting-path-rev*: *connecting-path u v xs* ⟷ *connecting-path v u* (*rev xs*)
unfolding *connecting-path-def* **using** *is-gen-path-rev*
by (*auto simp add*: *hd-rev last-rev*)

lemma *connecting-path-walk*: *connecting-path u v xs* ⟹ *connecting-walk u v xs*
unfolding *connecting-path-def connecting-walk-def* **using** *is-gen-path-def* **by** *auto*

lemma *connecting-path-str-gen*: *connecting-path-str u v xs* ⟹ *connecting-path u v xs*
unfolding *connecting-path-def connecting-path-str-def is-gen-path-def is-path-def*
by (*simp add*: *is-open-walk-def*)

lemma *connecting-path-gen-str*: *connecting-path u v xs* ⟹ (¬ *is-cycle xs*) ⟹ *walk-length xs* > *0* ⟹ *connecting-path-str u v xs*
unfolding *connecting-path-def connecting-path-str-def* **using** *is-gen-path-path* **by** *auto*

lemma *connecting-path-alt-def*: *connecting-path u v xs* ⟷ *connecting-walk u v xs* ∧ *is-gen-path xs*
proof −
have *is-gen-path xs* ⟹ *is-walk xs*
by (*simp add*: *is-gen-path-def*)
then have (*is-walk xs* ∧ *hd xs* = *u* ∧ *last xs* = *v*) ∧ *is-gen-path xs* ⟷ (*hd xs* = *u* ∧ *last xs* = *v*) ∧ *is-gen-path xs*
by *blast*
thus *?thesis*
by (*auto simp add*: *connecting-path-def connecting-walk-def*)
qed

lemma *connecting-path-length-bound*: *u* ≠ *v* ⟹ *connecting-path u v p* ⟹ *walk-length p* ≥ *1*
using *walk-length-def*
by (*metis connecting-path-def is-gen-path-def is-walk-not-empty2 last-ConsL le-refl length-0-conv*
less-one list.exhaust-sel nat-less-le nat-neq-iff neq-Nil-conv walk-edges.simps(*3*))

lemma *connecting-path-self*: *u* ∈ *V* ⟹ *connecting-path u u* [*u*] = *True*
unfolding *connecting-path-alt-def* **using** *connecting-walk-self*
by (*simp add*: *is-gen-path-def is-walk-singleton*)

lemma *connecting-path-singleton*: *connecting-path u v xs* ⟹ *length xs* = *1* ⟹ *u* = *v*
by (*metis cancel-comm-monoid-add-class.diff-cancel connecting-path-def fact-1 fact-nonzero*
last-rev length-0-conv neq-Nil-conv singleton-rev-conv walk-edges.simps(*3*) *walk-length-conv walk-length-def*)

```
lemma connecting-walk-path:
  assumes connecting-walk u v xs
  shows ∃ ys . connecting-path u v ys ∧ walk-length ys ≤ walk-length xs
proof (cases u = v)
  case True
  then show ?thesis
    using assms connecting-path-self connecting-walk-wf
    by (metis bot-nat-0.extremum list.size(3) walk-edges.simps(2) walk-length-def)
next
  case False
  then have walk-length xs ≠ 0 using assms connecting-walk-def is-walk-def
   by (metis last-ConsL length-0-conv list.distinct(1) list.exhaust-sel walk-edges.simps(3)
walk-length-def)
  then show ?thesis using assms False proof (induct walk-length xs arbitrary: xs
rule: less-induct)
    fix xs assume IH: (⋀xsa. walk-length xsa < walk-length xs ⟹ walk-length xsa
≠ 0 ⟹
     connecting-walk u v xsa ⟹ u ≠ v ⟹ ∃ys. connecting-path u v ys ∧ walk-length
ys ≤ walk-length xsa)
    assume assm: connecting-walk u v xs and ne: u ≠ v and n0: walk-length xs ≠
0
    then show ∃ys. connecting-path u v ys ∧ walk-length ys ≤ walk-length xs
    proof (cases walk-length xs ≤ 1) — Base Cases
      case True
      then have walk-length xs = 1
        using n0 by auto
      then show ?thesis using ne assm cancel-comm-monoid-add-class.diff-cancel
connecting-path-alt-def connecting-walk-def
               distinct-length-2-or-more distinct-singleton hd-Cons-tl is-gen-path-def
is-walk-def last-ConsL
            last-ConsR length-0-conv length-tl walk-length-conv
        by (metis True)
    next
      case False
      then show ?thesis
      proof (cases distinct xs)
        case True
        then show ?thesis
          using assm connecting-path-alt-def connecting-walk-def is-gen-path-def by
auto
      next
        case False
        then obtain ws ys zs y where xs-decomp: xs = ws@[y]@ys@[y]@zs using
not-distinct-decomp
          by blast
        let ?rs = ws@[y]@zs
        have hd: hd ?rs = u using xs-decomp assm connecting-walk-def
          by (metis hd-append list.distinct(1))
```

have *lst*: *last ?rs* = *v* **using** *xs-decomp assm connecting-walk-def* **by** *simp*
have *wl*: *walk-length ?rs* ≠ *0* **using** *hd lst ne walk-length-conv* **by** *auto*
have *set ?rs* ⊆ *V* **using** *assm connecting-walk-def is-walk-def xs-decomp* **by** *auto*
have *cw*: *connecting-walk u v ?rs* **unfolding** *connecting-walk-def is-walk-decomp*
using *assm connecting-walk-def hd is-walk-decomp lst xs-decomp* **by** *blast*
have *ys*@[*y*] ≠ []**by** *simp*
then have *length ?rs* < *length xs* **using** *xs-decomp length-list-decomp-lt* **by** *auto*
have *walk-length ?rs* < *walk-length xs* **using** *walk-length-conv xs-decomp* **by** *force*
then show *?thesis* **using** *IH*[*of ?rs*] **using** *cw ne wl le-trans less-or-eq-imp-le* **by** *blast*
qed
qed
qed
qed

lemma *connecting-walk-split*:
assumes *connecting-walk u v xs* **assumes** *connecting-walk v z ys*
shows *connecting-walk u z* (*xs* @ (*tl ys*))
using *connecting-walk-def is-walk-append*
by (*metis append.right-neutral assms*(*1*) *assms*(*2*) *connecting-walk-self connecting-walk-wf hd-append2 is-walk-not-empty last-appendR last-tl list.collapse*)

lemma *connecting-path-split*:
assumes *connecting-path u v xs connecting-path v z ys*
obtains *p* **where** *connecting-path u z p* **and** *walk-length p* ≤ *walk-length* (*xs* @ (*tl ys*))
using *connecting-walk-split connecting-walk-path connecting-path-walk assms*(*1*) *assms*(*2*) **by** *blast*

lemma *connecting-path-split-length*:
assumes *connecting-path u v xs connecting-path v z ys*
obtains *p* **where** *connecting-path u z p* **and** *walk-length p* ≤ *walk-length xs* + *walk-length ys*
proof −
have *connecting-walk u z* (*xs* @ (*tl ys*))
using *connecting-walk-split assms connecting-path-walk* **by** *blast*
have *walk-length* (*xs* @ (*tl ys*)) ≤ *walk-length xs* + *walk-length ys*
using *walk-length-app-ineq*
by (*simp add*: *le-diff-conv walk-length-conv*)
thus *?thesis* **using** *connecting-path-split*
by (*metis* (*full-types*) *assms*(*1*) *assms*(*2*) *dual-order.trans that*)
qed

3.2 Vertex Connectivity

Two vertices are defined to be connected if there exists a connecting path. Note that the more general version of a connecting path is again used as a vertex should be considered as connected to itself

definition *vert-connected* :: *'a* ⇒ *'a* ⇒ *bool* **where**
vert-connected u v ≡ ∃ *xs* . *connecting-path u v xs*

lemma *vert-connected-rev*: *vert-connected u v* ⟷ *vert-connected v u*
unfolding *vert-connected-def* **using** *connecting-path-rev* **by** *auto*

lemma *vert-connected-id*: *u* ∈ *V* ⟹ *vert-connected u u* = *True*
unfolding *vert-connected-def* **using** *connecting-path-self* **by** *auto*

lemma *vert-connected-trans*: *vert-connected u v* ⟹ *vert-connected v z* ⟹ *vert-connected u z*
unfolding *vert-connected-def* **using** *connecting-path-split*
by *meson*

lemma *vert-connected-wf*: *vert-connected u v* ⟹ *u* ∈ *V* ∧ *v* ∈ *V*
using *vert-connected-def connecting-path-walk connecting-walk-wf* **by** *blast*

definition *vert-connected-n* :: *'a* ⇒ *'a* ⇒ *nat* ⇒ *bool* **where**
vert-connected-n u v n ≡ ∃ *p*. *connecting-path u v p* ∧ *walk-length p* = *n*

lemma *vert-connected-n-imp*: *vert-connected-n u v n* ⟹ *vert-connected u v*
by (*auto simp add*: *vert-connected-def vert-connected-n-def*)

lemma *vert-connected-n-rev*: *vert-connected-n u v n* ⟷ *vert-connected-n v u n*
unfolding *vert-connected-n-def* **using** *walk-length-rev*
by (*metis connecting-path-rev*)

definition *connecting-paths* :: *'a* ⇒ *'a* ⇒ *'a list set* **where**
connecting-paths u v ≡ {*xs* . *connecting-path u v xs*}

lemma *connecting-paths-self*: *u* ∈ *V* ⟹ [*u*] ∈ *connecting-paths u u*
unfolding *connecting-paths-def* **using** *connecting-path-self* **by** *auto*

lemma *connecting-paths-empty-iff*: *vert-connected u v* ⟷ *connecting-paths u v* ≠ {}
unfolding *connecting-paths-def vert-connected-def* **by** *auto*

lemma *elem-connecting-paths*: *p* ∈ *connecting-paths u v* ⟹ *connecting-path u v p*
using *connecting-paths-def* **by** *blast*

lemma *connecting-paths-ss-gen*: *connecting-paths u v* ⊆ *gen-paths*
unfolding *connecting-paths-def gen-paths-def connecting-path-def* **by** *auto*

lemma *connecting-paths-sym*: *xs* ∈ *connecting-paths u v* ⟷ *rev xs* ∈ *connect-*

ing-paths v u
unfolding *connecting-paths-def* **using** *connecting-path-rev* **by** *simp*

A set is considered to be connected, if all the vertices within that set are pairwise connected

definition *is-connected-set* :: *'a set* ⇒ *bool* **where**
is-connected-set $V' \equiv (\forall\ u\ v\ .\ u \in V' \longrightarrow v \in V' \longrightarrow$ *vert-connected u v*)

lemma *is-connected-set-empty*: *is-connected-set* {}
unfolding *is-connected-set-def* **by** *simp*

lemma *is-connected-set-singleton*: $x \in V \Longrightarrow$ *is-connected-set* $\{x\}$
unfolding *is-connected-set-def* **by** (*auto simp add*: *vert-connected-id*)

lemma *is-connected-set-wf*: *is-connected-set* $V' \Longrightarrow V' \subseteq V$
unfolding *is-connected-set-def*
by (*meson connecting-path-walk connecting-walk-wf subsetI vert-connected-def*)

lemma *is-connected-setD*: *is-connected-set* $V' \Longrightarrow u \in V' \Longrightarrow v \in V' \Longrightarrow$ *vert-connected u v*
by (*simp add*: *is-connected-set-def*)

lemma *not-connected-set*: $\neg$ *is-connected-set* $V' \Longrightarrow u \in V' \Longrightarrow \exists\ v \in V'\ .\ \neg$ *vert-connected u v*
using *is-connected-setD* **by** (*meson is-connected-set-def vert-connected-rev vert-connected-trans*)

3.3 Graph Properties on Connectivity

The shortest path is defined to be the infinum of the set of connecting path walk lengths. Drawing inspiration from [4], we use the infinum and enats as this enables more natural reasoning in a non-finite setting, while also being useful for proofs of a more probabilistic or analysis nature

definition *shortest-path* :: *'a* ⇒ *'a* ⇒ *enat* **where**
shortest-path u v ≡ *INF p*∈ *connecting-paths u v*. *enat* (*walk-length p*)

lemma *shortest-path-walk-length*: *shortest-path u v* = $n \Longrightarrow p \in$ *connecting-paths u v* $\Longrightarrow$ *walk-length p* $\geq n$
using *shortest-path-def INF-lower*[*of p connecting-paths u v* λ *p* . *enat* (*walk-length p*)]
by *auto*

lemma *shortest-path-lte*: $\bigwedge$ *p* . *p* ∈ *connecting-paths u v* $\Longrightarrow$ *shortest-path u v* ≤ *walk-length p*
unfolding *shortest-path-def* **by** (*simp add*: *Inf-lower*)

lemma *shortest-path-obtains*:
assumes *shortest-path u v* = *n*
assumes $n \neq$ *top*

obtains *p* **where** *p* ∈ *connecting-paths u v* **and** *walk-length p* = *n*
using *enat-in-INF shortest-path-def* **by** (*metis assms*(*1*) *assms*(*2*) *the-enat.simps*)

lemma *shortest-path-intro*:
assumes *n* ≠ *top*
assumes (∃ *p* ∈ *connecting-paths u v* . *walk-length p* = *n*)
assumes (⋀ *p*. *p* ∈ *connecting-paths u v* ⟹ *n* ≤ *walk-length p*)
shows *shortest-path u v* = *n*
proof (*rule ccontr*)
assume *a*: *shortest-path u v* ≠ *enat n*
then have *shortest-path u v* < *n*
by (*metis antisym-conv2 assms*(*2*) *shortest-path-lte*)
then have ∃ *p* ∈ *connecting-paths u v* .*walk-length p* < *n*
using *shortest-path-def* **by** (*simp add*: *INF-less-iff*)
thus *False* **using** *assms*(*3*)
using *le-antisym less-imp-le-nat* **by** *blast*
qed

lemma *shortest-path-self*:
assumes *u* ∈ *V*
shows *shortest-path u u* = *0*
proof −
have [*u*] ∈ *connecting-paths u u*
using *connecting-paths-self* **by** (*simp add*: *assms*)
then have *walk-length* [*u*] = *0*
using *walk-length-def walk-edges.simps* **by** *auto*
thus *?thesis* **using** *shortest-path-def*
by (*metis* ‹[*u*] ∈ *connecting-paths u u*› *le-zero-eq shortest-path-lte zero-enat-def*)

qed

lemma *connecting-paths-sym-length*: *i* ∈ *connecting-paths u v* ⟹ ∃ *j*∈*connecting-paths v u*. (*walk-length j*) = (*walk-length i*)
using *connecting-paths-sym* **by** (*metis walk-length-rev*)

lemma *shortest-path-sym*: *shortest-path u v* = *shortest-path v u*
unfolding *shortest-path-def*
by (*intro INF-eq*)(*metis add.right-neutral le-iff-add connecting-paths-sym-length*)+

lemma *shortest-path-inf*: ¬ *vert-connected u v* ⟹ *shortest-path u v* = ∞
using *connecting-paths-empty-iff shortest-path-def* **by** (*simp add*: *top-enat-def*)

lemma *shortest-path-not-inf*:
assumes *vert-connected u v*
shows *shortest-path u v* ≠ ∞
proof −
have ⋀ *p*. *connecting-path u v p* ⟹ *enat* (*walk-length p*) ≠ ∞

using *connecting-path-def is-gen-path-def* **by** *auto*
thus *?thesis* **unfolding** *shortest-path-def connecting-paths-def*
by (*metis assms connecting-paths-def infinity-ileE mem-Collect-eq shortest-path-def shortest-path-lte vert-connected-def*)
qed

lemma *shortest-path-obtains2*:
assumes *vert-connected u v*
obtains *p* **where** $p \in$ *connecting-paths u v* **and** *walk-length p* = *shortest-path u v*
proof −
have *connecting-paths u v* $\neq \{\}$ **using** *assms connecting-paths-empty-iff* **by** *auto*
have *shortest-path u v* $\neq \infty$ **using** *assms shortest-path-not-inf* **by** *simp*
thus *?thesis* **using** *shortest-path-def enat-in-INF*
by (*metis that top-enat-def*)
qed

lemma *shortest-path-split*: *shortest-path x y* $\leq$ *shortest-path x z* + *shortest-path z y*
proof (*cases vert-connected x y* $\wedge$ *vert-connected x z*)
case *True*
show *?thesis*
proof (*rule ccontr*)
assume ¬ *shortest-path x y* $\leq$ *shortest-path x z* + *shortest-path z y*
then have *c*: *shortest-path x y* > *shortest-path x z* + *shortest-path z y* **by** *simp*
have *vert-connected z y* **using** *True vert-connected-trans vert-connected-rev* **by** *blast*
then obtain *p1 p2* **where** *connecting-path x z p1* **and** *connecting-path z y p2* **and**
s1: *shortest-path x z* = *walk-length p1* **and** *s2*: *shortest-path z y* = *walk-length p2*
using *True shortest-path-obtains2 connecting-paths-def elem-connecting-paths* **by** *metis*
then obtain *p3* **where** *cp*: *connecting-path x y p3* **and** *walk-length p1* + *walk-length p2* $\geq$ *walk-length p3*
using *connecting-path-split-length* **by** *blast*
then have *shortest-path x z* + *shortest-path z y* $\geq$ *walk-length p3* **using** *s1 s2* **by** *simp*
then have *lt*: *shortest-path x y* > *walk-length p3* **using** *c* **by** *auto*
have *p3* $\in$ *connecting-paths x y* **using** *cp connecting-paths-def* **by** *auto*
then show *False* **using** *shortest-path-def shortest-path-obtains2*
by (*metis True enat-ord-simps*(*1*) *enat-ord-simps*(*2*) *le-Suc-ex lt not-add-less1 shortest-path-lte*)
qed
next
case *False*
then show *?thesis*
by (*metis enat-ord-code*(*3*) *plus-enat-simps*(*2*) *plus-enat-simps*(*3*) *shortest-path-inf vert-connected-trans*)

qed

lemma *shortest-path-invalid-v*: $v \notin V \lor u \notin V \Longrightarrow$ *shortest-path* $u\ v = \infty$
using *shortest-path-inf vert-connected-wf* **by** *blast*

lemma *shortest-path-lb*:
assumes $u \neq v$
assumes *vert-connected* $u\ v$
shows *shortest-path* $u\ v > 0$
proof −
have $\bigwedge p$. *connecting-path* $u\ v\ p \Longrightarrow$ *enat* (*walk-length* p) > 0
using *connecting-path-length-bound assms* **by** *fastforce*
thus *?thesis* **unfolding** *shortest-path-def*
by (*metis elem-connecting-paths shortest-path-def shortest-path-obtains2 assms*(*2*))
qed

Eccentricity of a vertex v is the furthest distance between it and a (different) vertex

definition *eccentricity* :: $'a \Rightarrow$ *enat* **where**
eccentricity $v \equiv$ *SUP* $u \in V - \{v\}$. *shortest-path* $v\ u$

lemma *eccentricity-empty-vertices*: $V = \{\} \Longrightarrow$ *eccentricity* $v = 0$
$V = \{v\} \Longrightarrow$ *eccentricity* $v = 0$
unfolding *eccentricity-def* **using** *bot-enat-def* **by** *simp-all*

lemma *eccentricity-bot-iff*: *eccentricity* $v = 0 \longleftrightarrow V = \{\} \lor V = \{v\}$
proof (*intro iffI*)
assume *a*: *eccentricity* $v = 0$
show $V = \{\} \lor V = \{v\}$
proof (*rule ccontr*, *simp*)
assume *a2*: $V \neq \{\} \land V \neq \{v\}$
have *eq0*: $\forall\ u \in V - \{v\}$. *shortest-path* $v\ u = 0$
using *SUP-bot-conv*(*1*)[*of* λ *u*. *shortest-path* $v\ u\ V - \{v\}$] *a eccentricity-def*
bot-enat-def **by** *simp*
have *nc*: $\forall\ u \in V - \{v\}$. $\neg$ *vert-connected* $v\ u \longrightarrow$ *shortest-path* $v\ u = \infty$
using *shortest-path-inf* **by** *simp*
have $\forall\ u \in V - \{v\}$. *vert-connected* $v\ u \longrightarrow$ *shortest-path* $v\ u > 0$
using *shortest-path-lb* **by** *auto*
then show *False* **using** *eq0 a2 nc*
by *auto*
qed
next
show $V = \{\} \lor V = \{v\} \Longrightarrow$ *eccentricity* $v = 0$ **using** *eccentricity-empty-vertices*
by *auto*
qed

lemma *eccentricity-invalid-v*:
assumes $v \notin V$
assumes $V \neq \{\}$

shows *eccentricity v* = ∞
proof –
have ⋀ *u. shortest-path v u* = ∞ **using** *assms shortest-path-invalid-v* **by** *blast*
have *V* − {*v*} = *V* **using** *assms* **by** *simp*
then have *eccentricity v* = (*SUP u* ∈ *V* . *shortest-path v u*) **by** (*simp add*: *eccentricity-def*)
thus *?thesis* **using** *eccentricity-def shortest-path-invalid-v assms* **by** *simp*
qed

lemma *eccentricity-gt-shortest-path*:
assumes *u* ∈ *V*
shows *eccentricity v* ≥ *shortest-path v u*
proof (*cases u* ∈ *V* − {*v*})
case *True*
then show *?thesis* **unfolding** *eccentricity-def* **by** (*simp add*: *SUP-upper*)
next
case *f1*: *False*
then have *u* = *v* **using** *assms* **by** *auto*
then have *shortest-path u v* = *0* **using** *shortest-path-self assms* **by** *auto*
then show *?thesis* **by** (*simp add*: ‹*u* = *v*›)
qed

lemma *eccentricity-disconnected-graph*:
assumes ¬ *is-connected-set V*
assumes *v* ∈ *V*
shows *eccentricity v* = ∞
proof –
obtain *u* **where** *uin*: *u* ∈ *V* **and** *nvc*: ¬ *vert-connected v u*
using *not-connected-set assms* **by** *auto*
then have *u* ≠ *v* **using** *vert-connected-id* **by** *auto*
then have *u* ∈ *V* − {*v*} **using** *uin* **by** *simp*
moreover have *shortest-path v u* = ∞ **using** *nvc shortest-path-inf* **by** *auto*
thus *?thesis* **using** *eccentricity-gt-shortest-path*
by (*metis enat-ord-simps*(*5*) *uin*)
qed

The diameter is the largest distance between any two vertices

definition *diameter* :: *enat* **where**
diameter ≡ *SUP v* ∈ *V* . *eccentricity v*

lemma *diameter-gt-eccentricity*: *v* ∈ *V* ⟹ *diameter* ≥ *eccentricity v*
using *diameter-def* **by** (*simp add*: *SUP-upper*)

lemma *diameter-disconnected-graph*:
assumes ¬ *is-connected-set V*
shows *diameter* = ∞
unfolding *diameter-def* **using** *eccentricity-disconnected-graph*
by (*metis SUP-eq-const assms is-connected-set-empty*)

lemma *diameter-empty*: $V = \{\} \Longrightarrow$ *diameter* $=$ *0*
unfolding *diameter-def* **using** *Sup-empty bot-enat-def* **by** *simp*

lemma *diameter-singleton*: $V = \{v\} \Longrightarrow$ *diameter* $=$ *eccentricity v*
unfolding *diameter-def* **by** *simp*

The radius is the smallest "shortest" distance between any two vertices

definition *radius* :: *enat* **where**
radius $\equiv$ *INF* $v \in V$. *eccentricity v*

lemma *radius-lt-eccentricity*: $v \in V \Longrightarrow$ *radius* $\leq$ *eccentricity v*
using *radius-def* **by** (*simp add*: *INF-lower*)

lemma *radius-disconnected-graph*: $\neg$ *is-connected-set V* $\Longrightarrow$ *radius* $= \infty$
unfolding *radius-def* **using** *eccentricity-disconnected-graph*
by (*metis INF-eq-const is-connected-set-empty*)

lemma *radius-empty*: $V = \{\} \Longrightarrow$ *radius* $= \infty$
unfolding *radius-def* **using** *Inf-empty top-enat-def* **by** *simp*

lemma *radius-singleton*: $V = \{v\} \Longrightarrow$ *radius* $=$ *eccentricity v*
unfolding *radius-def* **by** *simp*

The centre of the graph is all vertices whose eccentricity equals the radius

definition *centre* :: *'a set* **where**
centre $\equiv \{v \in V.$ *eccentricity v* $=$ *radius* $\}$

lemma *centre-disconnected-graph*: $\neg$ *is-connected-set V* $\Longrightarrow$ *centre* $= V$
unfolding *centre-def* **using** *radius-disconnected-graph eccentricity-disconnected-graph*
by *auto*

end

lemma (**in** *fin-ulgraph*) *fin-connecting-paths*: *finite* (*connecting-paths u v*)
using *connecting-paths-ss-gen finite-gen-paths finite-subset* **by** *fastforce*

3.4 We define a connected graph as a non-empty graph (the empty set is not usually considered connected by convention), where the vertex set is connected

locale *connected-ulgraph* $=$ *ulgraph* $+$ *ne-graph-system* $+$
assumes *connected*: *is-connected-set V*
begin

lemma *vertices-connected*: $u \in V \Longrightarrow v \in V \Longrightarrow$ *vert-connected u v*
using *is-connected-set-def connected* **by** *auto*

lemma *vertices-connected-path*: $u \in V \Longrightarrow v \in V \Longrightarrow \exists$ *p. connecting-path u v p*
using *vertices-connected* **by** (*simp add*: *vert-connected-def*)

lemma *connecting-paths-not-empty*: *u* ∈ *V* ⟹ *v* ∈ *V* ⟹ *connecting-paths u v* ≠ {}
  using *connected not-empty connecting-paths-empty-iff is-connected-setD* **by** *blast*

lemma *min-shortest-path*: **assumes** *u* ∈ *V v* ∈ *V u* ≠ *v*
  shows *shortest-path u v* > *0*
  using *shortest-path-lb assms vertices-connected* **by** *auto*

The eccentricity, diameter, radius, and centre definitions tend to be only used in a connected context, as otherwise they are the INF/SUP value. In these contexts, we can obtain the vertex responsible

lemma *eccentricity-obtains-inf*:
  assumes *V* ≠ {*v*}
  shows *eccentricity v* = ∞ ∨ (∃ *u* ∈ (*V* − {*v*}) . *shortest-path v u* = *eccentricity v*)
proof (*cases finite* ((λ *u*. *shortest-path v u*) ‘ (*V* − {*v*})))
  case *True*
  then have *e*: *eccentricity v* = *Max* ((λ *u*. *shortest-path v u*) ‘ (*V* − {*v*})) **unfolding** *eccentricity-def* **using** *Sup-enat-def*
    using *assms not-empty* **by** *auto*
  have (*V* − {*v*}) ≠ {} **using** *assms not-empty* **by** *auto*
  then have ((λ *u*. *shortest-path v u*) ‘ (*V* − {*v*})) ≠ {} **by** *simp*
  then obtain *n* **where** *n* ∈ ((λ *u*. *shortest-path v u*) ‘ (*V* − {*v*})) **and** *n* = *eccentricity v*
    using *Max-in e True* **by** *auto*
  then obtain *u* **where** *u* ∈ (*V* − {*v*}) **and** *shortest-path v u* = *eccentricity v*
    by *blast*
  then show *?thesis* **by** *auto*
next
  case *False*
  then have *eccentricity v* = ∞ **unfolding** *eccentricity-def* **using** *Sup-enat-def*
    by (*metis* (*mono-tags*, *lifting*) *cSup-singleton empty-iff finite-insert insert-iff*)
  then show *?thesis* **by** *simp*
qed

lemma *diameter-obtains*: *diameter* = ∞ ∨ (∃ *v* ∈ *V* . *eccentricity v* = *diameter*)
proof (*cases is-singleton V*)
  case *True*
  then obtain *v* **where** *V* = {*v*}
    using *is-singletonE* **by** *auto*
  then show *?thesis* **using** *diameter-singleton*
    by *simp*
next
  case *f1*: *False*
  then show *?thesis* **proof** (*cases finite* ((λ *v*. *eccentricity v*) ‘ *V*))
    case *True*
    then have *diameter* = *Max* ((λ *v*. *eccentricity v*) ‘ *V*) **unfolding** *diameter-def*

using *Sup-enat-def not-empty*
by *simp*
then obtain *n* **where** $n \in ((\lambda\ v.\ eccentricity\ v)\ `\ V)$ **and** *diameter* = *n* **using** *Max-in True*
using *not-empty* **by** *auto*
then obtain *u* **where** $u \in V$ **and** *eccentricity u* = *diameter*
by *fastforce*
then show *?thesis* **by** *auto*
next
case *False*
then have *diameter* = ∞ **unfolding** *diameter-def* **using** *Sup-enat-def* **by** *auto*
then show *?thesis* **by** *simp*
qed
qed

lemma *radius-diameter-singleton-eq*: **assumes** *card V* = *1* **shows** *radius* = *diameter*
proof −
obtain *v* **where** $V = \{v\}$ **using** *assms card-1-singletonE* **by** *auto*
thus *?thesis* **unfolding** *radius-def diameter-def* **by** *auto*
qed

end

locale *fin-connected-ulgraph* = *connected-ulgraph* + *fin-ulgraph*
begin

In a finite context the supremum/infinum are equivalent to the Max/Min of the sets respectively. This can make reasoning easier

lemma *shortest-path-Min-alt*:
assumes $u \in V$ $v \in V$
shows *shortest-path u v* = *Min* ((λ *p*. *enat* (*walk-length p*)) ‘ (*connecting-paths u v*)) (**is** *shortest-path u v* = *Min ?A*)
proof −
have *ne*: *?A* $\neq$ {}
using *connecting-paths-not-empty assms* **by** *auto*
have *finite* (*connecting-paths u v*)
by (*simp add*: *fin-connecting-paths*)
then have *fin*: *finite ?A*
by *simp*
have *shortest-path u v* = *Inf ?A* **unfolding** *shortest-path-def* **by** *simp*
thus *?thesis* **using** *Min-Inf ne*
by (*metis fin*)
qed

lemma *eccentricity-Max-alt*:
assumes $v \in V$
assumes $V \neq \{v\}$
shows *eccentricity v* = *Max* ((λ *u*. *shortest-path v u*) ‘ (*V* − {*v*}))

unfolding *eccentricity-def* **using** *assms Sup-enat-def finV not-empty*
by *auto*

lemma *diameter-Max-alt*: *diameter* = *Max* ((λ *v*. *eccentricity v*) ‘ *V*)
unfolding *diameter-def* **using** *Sup-enat-def finV not-empty* **by** *auto*

lemma *radius-Min-alt*: *radius* = *Min* ((λ *v*. *eccentricity v*) ‘ *V*)
unfolding *radius-def* **using** *Min-Inf finV not-empty*
by (*metis* (*no-types*, *opaque-lifting*) *empty-is-image finite-imageI*)

lemma *eccentricity-obtains*:
assumes *v* ∈ *V*
assumes *V* ≠ {*v*}
obtains *u* **where** *u* ∈ *V* **and** *u* ≠ *v* **and** *shortest-path u v* = *eccentricity v*
proof −
have *ni*: ⋀ *u*. *u* ∈ *V* − {*v*} ⟹ *u* ≠ *v* ∧ *u* ∈ *V* **by** *auto*
have *ne*: *V* − {*v*} ≠ {} **using** *assms not-empty* **by** *auto*
have *eccentricity v* = *Max* ((λ *u*. *shortest-path v u*) ‘ (*V* − {*v*})) **using** *eccentricity-Max-alt assms* **by** *simp*
then obtain *u* **where** *ui*: *u* ∈ *V* − {*v*} **and** *eq*: *shortest-path v u* = *eccentricity v*
using *obtains-MAX assms finV ne* **by** (*metis finite-Diff*)
then have *neq*: *u* ≠ *v* **by** *blast*
have *uin*: *u* ∈ *V* **using** *ui* **by** *auto*
thus *?thesis* **using** *neq eq that*[*of u*] *shortest-path-sym* **by** *simp*
qed

lemma *radius-obtains*:
obtains *v* **where** *v* ∈ *V* **and** *radius* = *eccentricity v*
proof −
have *radius* = *Min* ((λ *v*. *eccentricity v*) ‘ *V*) **using** *radius-Min-alt* **by** *simp*
then obtain *v* **where** *v* ∈ *V* **and** *radius* = *eccentricity v*
using *obtains-MIN*[*of V* (λ *v* . *eccentricity v*)] *not-empty finV* **by** *auto*
thus *?thesis*
by (*simp add*: *that*)
qed

lemma *radius-obtains-path-vertices*:
assumes *card V* ≥ *2*
obtains *u v* **where** *u* ∈ *V* **and** *v* ∈ *V* **and** *u* ≠ *v* **and** *radius* = *shortest-path u v*
proof −
obtain *v* **where** *vin*: *v* ∈ *V* **and** *e*: *radius* = *eccentricity v*
using *radius-obtains* **by** *blast*
then have *V* ≠ {*v*} **using** *assms* **by** *auto*
then obtain *u* **where** *u* ∈ *V* **and** *u* ≠ *v* **and** *shortest-path u v* = *radius*
using *eccentricity-obtains vin e* **by** *auto*
thus *?thesis* **using** *vin*
by (*simp add*: *that*)

qed

lemma *diameter-obtains*:
obtains *v* **where** $v \in V$ **and** *diameter* = *eccentricity v*
proof −
have *diameter* = *Max* ((λ *v*. *eccentricity v*) ‘ *V*) **using** *diameter-Max-alt* **by** *simp*
then obtain *v* **where** $v \in V$ **and** *diameter* = *eccentricity v*
using *obtains-MAX*[*of V* (λ *v* . *eccentricity v*)] *not-empty finV* **by** *auto*
thus *?thesis*
by (*simp add*: *that*)
qed

lemma *diameter-obtains-path-vertices*:
assumes *card* $V \geq 2$
obtains *u v* **where** $u \in V$ **and** $v \in V$ **and** $u \neq v$ **and** *diameter* = *shortest-path u v*
proof −
obtain *v* **where** *vin*: $v \in V$ **and** *e*: *diameter* = *eccentricity v*
using *diameter-obtains* **by** *blast*
then have $V \neq \{v\}$ **using** *assms* **by** *auto*
then obtain *u* **where** $u \in V$ **and** $u \neq v$ **and** *shortest-path u v* = *diameter*
using *eccentricity-obtains vin e* **by** *auto*
thus *?thesis* **using** *vin*
by (*simp add*: *that*)
qed

lemma *radius-diameter-bounds*:
shows *radius* ≤ *diameter diameter* ≤ *2* ∗ *radius*
proof −
show *radius* ≤ *diameter* **unfolding** *radius-def diameter-def*
by (*simp add*: *INF-le-SUP not-empty*)
next
show *diameter* ≤ *2* ∗ *radius*
proof (*cases card* $V \geq 2$)
case *True*
then obtain *x y* **where** *xin*: $x \in V$ **and** *yin*: $y \in V$ **and** *d*: *shortest-path x y* = *diameter*
using *diameter-obtains-path-vertices* **by** *metis*
obtain *z* **where** *zin*: $z \in V$ **and** *e*: *eccentricity z* = *radius* **using** *radius-obtains* **by** *metis*
have *shortest-path x z* ≤ *eccentricity z*
using *eccentricity-gt-shortest-path xin shortest-path-sym* **by** *simp*
have *shortest-path x y* ≤ *shortest-path x z* + *shortest-path z y* **using** *shortest-path-split* **by** *simp*
also have ... ≤ *eccentricity z* + *eccentricity z*
using *eccentricity-gt-shortest-path shortest-path-sym zin xin yin* **by** (*simp add*: *add-mono*)
also have ... ≤ *radius* + *radius* **using** *e* **by** *simp*

finally show *?thesis* **using** *d* **by** (*simp add*: *mult-2*)
next
case *False*
have *card V* ≠ *0* **using** *not-empty finV* **by** *auto*
then have *card V* = *1* **using** *False* **by** *simp*
then show *?thesis* **using** *radius-diameter-singleton-eq* **by** (*simp add*: *mult-2*)
qed
qed

end

We define various subclasses of the general connected graph, using the functor locale pattern

locale *connected-sgraph* = *sgraph* + *ne-graph-system* +
assumes *connected*: *is-connected-set V*

sublocale *connected-sgraph* ⊆ *connected-ulgraph*
by (*unfold-locales*) (*simp add*: *connected*)

locale *fin-connected-sgraph* = *connected-sgraph* + *fin-sgraph*

sublocale *fin-connected-sgraph* ⊆ *fin-connected-ulgraph*
by (*unfold-locales*)

end
theory *Girth-Independence* **imports** *Connectivity*
begin

4 Girth and Independence

We translate and extend on a number of definitions and lemmas on girth and independence from Noschinski's ugraph representation [4].

context *sgraph*
begin

definition *girth* :: *enat* **where**
girth ≡ *INF p*∈ *cycles*. *enat* (*walk-length p*)

lemma *girth-acyclic*: *cycles* = {} ⟹ *girth* = ∞
unfolding *girth-def* **using** *top-enat-def* **by** *simp*

lemma *girth-lte*: *c* ∈ *cycles* ⟹ *girth* ≤ *walk-length c*
using *girth-def INF-lower* **by** *auto*

lemma *girth-obtains*:
assumes *girth* ≠ *top*
obtains *c* **where** *c* ∈ *cycles* **and** *walk-length c* = *girth*
using *enat-in-INF girth-def assms* **by** (*metis* (*full-types*) *the-enat.simps*)

lemma *girthI*:
  assumes $c' \in$ *cycles*
  assumes $\bigwedge$ *c* . $c \in$ *cycles* $\Longrightarrow$ *walk-length* $c' \leq$ *walk-length c*
  shows *girth* = *walk-length* c'
proof (*rule ccontr*)
  assume *girth* $\neq$ *walk-length* c'
  then have *girth* < *walk-length* c'
    using *assms girth-lte* **by** *fastforce*
  then obtain *c* **where** $c \in$ *cycles* **and** *walk-length c* < *walk-length* c'
   using *girth-def* **by** (*metis enat-ord-simps*(*2*) *girth-obtains infinity-ilessE top-enat-def*)

  thus *False* **using** *assms*(*2*) *less-imp-le-nat le-antisym*
    by *fastforce*
qed

lemma (**in** *fin-sgraph*) *girth-min-alt*:
  assumes *cycles* $\neq$ {}
  shows *girth* = *Min* ((λ *c* . *enat* (*walk-length c*)) ' *cycles*) (**is** *girth* = *Min ?A*)
  unfolding *girth-def* **using** *finite-cycles assms Min-Inf*
 by (*metis* (*full-types*) *INF-le-SUP bot-enat-def ccInf-empty ccSup-empty enat-ord-code*(*5*) *finite-imageI top-enat-def zero-enat-def*)

definition *is-independent-set* :: *'a set* ⇒ *bool* **where**
is-independent-set vs ≡ *vs* ⊆ *V* ∧ (*all-edges vs*) ∩ *E* = {}

A More mathematical way of thinking about it

lemma *is-independent-alt*: *is-independent-set vs* ⟷ *vs* ⊆ *V* ∧ (∀ *v* ∈ *vs*. ∀ *u* ∈ *vs*. ¬ *vert-adj v u*)
  unfolding *is-independent-set-def*
proof (*auto*)
  fix *v u* **assume** *ss*: *vs* ⊆ *V* **and** *inter*: *all-edges vs* ∩ *E* = {} **and** *vin*: *v* ∈ *vs* **and** *uin*: *u* ∈ *vs* **and** *adj*: *vert-adj v u*
  then have *inE*: {*v*, *u*} ∈ *E* **using** *vert-adj-def* **by** *simp*
  then have *imp*: {*v*, *u*} ∈ *all-edges vs* **using** *vin uin e-in-all-edges-ss vin uin*
    by (*simp add*: *ss*)
  then show *False*
    using *inE inter* **by** *blast*
next
  fix *x* **assume** *vs* ⊆ *V* ∀ *v*∈*vs*. ∀ *u*∈*vs*. ¬ *vert-adj v u* *x* ∈ *all-edges vs x* ∈ *E*
  then have ⋀ *u v*. {*u*, *v*} ⊆ *vs* ⟹ {*u*, *v*} ∉ *E* **by** (*simp add*: *vert-adj-def*)
  then have ⋀ *x* . *x* ⊆ *vs* ⟹ *card x* = *2* ⟹ *x* ∉ *E* **by** (*metis card-2-iff*)
  then show *False* **using** *all-edges-def*
    by (*metis* (*mono-tags*, *lifting*) ‹*x* ∈ *E*› ‹*x* ∈ *all-edges vs*› *mem-Collect-eq*)
qed

lemma *singleton-independent-set*: *v* ∈ *V* ⟹ *is-independent-set* {*v*}
  by (*metis empty-subsetI insert-absorb2 insert-subset is-independent-alt*
    singletonD singleton-not-edge vert-adj-def)

definition *independent-sets* :: *'a set set* **where**
independent-sets ≡ {*vs*. *is-independent-set vs*}

definition *independence-number* :: *enat* **where**
independence-number ≡ *SUP vs* ∈ *independent-sets*. *enat* (*card vs*)

abbreviation α ≡ *independence-number*

lemma *independent-sets-mono*:
vs ∈ *independent-sets* ⟹ *us* ⊆ *vs* ⟹ *us* ∈ *independent-sets*
using *Int-mono*[*OF all-edges-mono*, *of us vs E E*]
unfolding *independent-sets-def is-independent-set-def* **by** *auto*

lemma *le-independence-iff*:
assumes *0* < *k*
shows *k* ≤ α ⟷ *k* ∈ *card* ‘ *independent-sets* (**is** *?L* ⟷ *?R*)
proof
assume *?L*
then obtain *vs* **where** *vs* ∈ *independent-sets* **and** *klt*: *k* ≤ *card vs*
using *assms* **unfolding** *independence-number-def enat-le-Sup-iff* **by** *auto*
moreover
obtain *us* **where** *us* ⊆ *vs* **and** *k* = *card us*
using *card-Ex-subset klt* **by** *auto*
ultimately
have *us* ∈ *independent-sets* **by** (*auto intro*: *independent-sets-mono*)
then show *?R* **using** ‹*k* = *card us*› **by** *auto*
qed (*auto intro*: *SUP-upper simp*: *independence-number-def*)

lemma *zero-less-independence*:
assumes *V* ≠ {}
shows *0* < α
proof −
from *assms* **obtain** *a* **where** *a* ∈ *V* **by** *auto*
then have *0* < *enat* (*card* {*a*}) {*a*} ∈ *independent-sets*
using *independent-sets-def is-independent-set-def all-edges-def singleton-independent-set*
by *simp-all*
then show *?thesis* **unfolding** *independence-number-def less-SUP-iff* **..**
qed

end

context *fin-sgraph*
begin
lemma *fin-independent-sets*: *finite* (*independent-sets*)
unfolding *independent-sets-def is-independent-set-def* **using** *finV* **by** *auto*

lemma *independence-le-card*:
shows α ≤ *card V*

proof −
{ **fix** *x* **assume** $x \in$ *independent-sets*
then have $x \subseteq V$ **by** (*auto simp*: *independent-sets-def is-independent-set-def*)
}
with *finV* **show** *?thesis* **unfolding** *independence-number-def*
by (*intro SUP-least*) (*auto intro*: *card-mono*)
qed

lemma *independence-fin*: $\alpha \neq \infty$
using *independence-le-card* **by** (*cases* α) *auto*

lemma *independence-max-alt*: $V \neq \{\} \Longrightarrow \alpha = Max\ ((\lambda\ vs\ .\ enat\ (card\ vs))$ ‘ *independent-sets*)
unfolding *independence-number-def* **using** *Sup-enat-def zero-less-independence*
by (*metis i0-less independence-fin independence-number-def*)

lemma *independent-sets-ne*:
assumes $V \neq \{\}$
shows *independent-sets* $\neq \{\}$
proof −
from *assms* **obtain** *a* **where** $a \in V$ **by** *auto*
then have $\{a\} \in$ *independent-sets* **using** *independent-sets-def singleton-independent-set* **by** *simp*
thus *?thesis* **by** *blast*
qed

lemma *independence-obtains*:
assumes $V \neq \{\}$
obtains *vs* **where** *is-independent-set vs* **and** *card vs* $= \alpha$
proof −
have $\alpha = Max\ ((\lambda\ vs\ .\ enat\ (card\ vs))$ ‘ *independent-sets*) **using** *independence-max-alt assms* **by** *simp*
then obtain *vs* **where** $vs \in$ *independent-sets* **and** *enat* (*card vs*) $= \alpha$
using *obtains-MIN*[*of independent-sets* λ *vs* . *enat* (*card vs*)] *assms fin-independent-sets independent-sets-ne*
by (*metis* (*no-types*, *lifting*) *Max-in finite-imageI imageE image-is-empty*)
thus *?thesis* **using** *independent-sets-def that* **by** *simp*
qed
end
end

5 Triangles in Graph

Triangles are an important tool in graph theory. This theory presents a number of basic definitions/lemmas which are useful for general reasoning using triangles. The definitions and lemmas in this theory are adapted from previous less general work in [2] and [1]

theory *Graph-Triangles* **imports** *Undirected-Graph-Basics*

HOL−Combinatorics.Multiset-Permutations
begin

Triangles don't make as much sense in a loop context, hence we restrict this to simple graphs

context *sgraph*
begin

definition *triangle-in-graph* :: $'a \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$ **where**
$triangle\text{-}in\text{-}graph\ x\ y\ z \equiv (\{x,y\} \in E) \wedge (\{y,z\} \in E) \wedge (\{x,z\} \in E)$

lemma *triangle-in-graph-edge-empty*: $E = \{\} \Longrightarrow \neg\ triangle\text{-}in\text{-}graph\ x\ y\ z$
using *triangle-in-graph-def* **by** *auto*

definition *triangle-triples* **where**
$triangle\text{-}triples\ X\ Y\ Z \equiv \{(x,y,z) \in X \times Y \times Z.\ triangle\text{-}in\text{-}graph\ x\ y\ z\ \}$

definition
unique-triangles
$\equiv \forall\, e \in E.\ \exists!T.\ \exists\, x\ y\ z.\ T = \{x,y,z\} \wedge triangle\text{-}in\text{-}graph\ x\ y\ z\ \wedge e \subseteq T$

definition *triangle-set* :: $'a\ set\ set$
where $triangle\text{-}set \equiv \{\ \{x,y,z\} \mid x\ y\ z.\ triangle\text{-}in\text{-}graph\ x\ y\ z\}$

5.1 Preliminaries on Triangles in Graphs

lemma *card-triangle-triples-rotate*: *card* (*triangle-triples X Y Z*) = *card* (*triangle-triples Y Z X*)
proof −
have $triangle\text{-}triples\ Y\ Z\ X = (\lambda(x,y,z).\ (y,z,x))$ ' $triangle\text{-}triples\ X\ Y\ Z$
by (*auto simp*: *triangle-triples-def case-prod-unfold image-iff insert-commute triangle-in-graph-def*)
moreover have $inj\text{-}on\ (\lambda(x,\ y,\ z).\ (y,\ z,\ x))\ (triangle\text{-}triples\ X\ Y\ Z)$
by (*auto simp*: *inj-on-def*)
ultimately show *?thesis*
by (*simp add*: *card-image*)
qed

lemma *triangle-commu1*:
assumes *triangle-in-graph x y z*
shows *triangle-in-graph y x z*
using *assms triangle-in-graph-def* **by** (*auto simp add*: *insert-commute*)

lemma *triangle-vertices-distinct1*:
assumes *tri*: *triangle-in-graph x y z*
shows $x \neq y$
proof (*rule ccontr*)
assume *a*: $\neg\ x \neq y$
have $card\ \{x,\ y\} = 2$ **using** *tri triangle-in-graph-def*

using *wellformed* **by** (*simp add*: *two-edges*)
thus *False* **using** *a* **by** *simp*
qed

lemma *triangle-vertices-distinct2*:
assumes *triangle-in-graph x y z*
shows $y \neq z$
by (*metis assms triangle-vertices-distinct1 triangle-in-graph-def*)

lemma *triangle-vertices-distinct3*:
assumes *triangle-in-graph x y z*
shows $z \neq x$
by (*metis assms triangle-vertices-distinct1 triangle-in-graph-def*)

lemma *triangle-in-graph-edge-point*: *triangle-in-graph x y z* $\longleftrightarrow \{y, z\} \in E \wedge$ *vert-adj x y* $\wedge$ *vert-adj x z*
by (*auto simp add*: *triangle-in-graph-def vert-adj-def*)

lemma *edge-vertices-not-equal*:
assumes $\{x,y\} \in E$
shows $x \neq y$
using *assms two-edges* **by** *fastforce*

lemma *edge-btw-vertices-not-equal*:
assumes $(x, y) \in$ *all-edges-between X Y*
shows $x \neq y$
using *edge-vertices-not-equal all-edges-between-def*
by (*metis all-edges-betw-D3 assms*)

lemma *mk-triangle-from-ss-edges*:
assumes $(x, y) \in$ *all-edges-between X Y* **and** $(x, z) \in$ *all-edges-between X Z* **and** $(y, z) \in$ *all-edges-between Y Z*
shows (*triangle-in-graph x y z*)
by (*meson all-edges-betw-D3 assms triangle-in-graph-def*)

lemma *triangle-in-graph-verts*:
assumes *triangle-in-graph x y z*
shows $x \in V$ $y \in V$ $z \in V$
proof −
show $x \in V$ **using** *triangle-in-graph-def wellformed-alt-fst assms* **by** *blast*
show $y \in V$ **using** *triangle-in-graph-def wellformed-alt-snd assms* **by** *blast*
show $z \in V$ **using** *triangle-in-graph-def wellformed-alt-snd assms* **by** *blast*
qed

lemma *convert-triangle-rep-ss*:
assumes $X \subseteq V$ **and** $Y \subseteq V$ **and** $Z \subseteq V$
shows *mk-triangle-set* ‘ $\{(x, y, z) \in X \times Y \times Z$. (*triangle-in-graph x y z*)$\} \subseteq$ *triangle-set*
by (*auto simp add*: *subsetI triangle-set-def*) (*auto*)

```
lemma (in fin-sgraph) finite-triangle-set: finite (triangle-set)
proof −
  have triangle-set ⊆ Pow V
  using insert-iff wellformed triangle-in-graph-def triangle-set-def by auto
  then show ?thesis
    by (meson finV finite-Pow-iff infinite-super)
qed

lemma card-triangle-3:
  assumes t ∈ triangle-set
  shows card t = 3
 using assms by (auto simp: triangle-set-def edge-vertices-not-equal triangle-in-graph-def)

lemma triangle-set-power-set-ss: triangle-set ⊆ Pow V
  by (auto simp add: triangle-set-def triangle-in-graph-def wellformed-alt-fst well-
formed-alt-snd)

lemma triangle-in-graph-ss:
  assumes E′ ⊆ E
  assumes sgraph.triangle-in-graph E′ x y z
  shows triangle-in-graph x y z
proof −
  interpret gnew: sgraph V E′
    apply (unfold-locales)
    using assms wellformed two-edges by auto
  have {x, y} ∈ E using assms gnew.triangle-in-graph-def by auto
  have {y, z} ∈ E using assms gnew.triangle-in-graph-def by auto
  have {x, z} ∈ E using assms gnew.triangle-in-graph-def by auto
  thus ?thesis
    by (simp add: ‹{x, y} ∈ E› ‹{y, z} ∈ E› triangle-in-graph-def)
qed

lemma triangle-set-graph-edge-ss:
  assumes E′ ⊆ E
  shows (sgraph.triangle-set E′) ⊆ (triangle-set)
proof (intro subsetI)
  interpret gnew: sgraph V E′
    using assms wellformed two-edges by (unfold-locales) auto
  fix t assume t ∈ gnew.triangle-set
  then obtain x y z where t = {x,y,z} and gnew.triangle-in-graph x y z
    using gnew.triangle-set-def assms mem-Collect-eq by auto
  then have triangle-in-graph x y z using assms triangle-in-graph-ss by simp
  thus t ∈ triangle-set using triangle-set-def assms
    using ‹t = {x,y,z}› by auto
qed

lemma (in fin-sgraph) triangle-set-graph-edge-ss-bound:
  assumes E′ ⊆ E
```

shows *card* (*triangle-set*) ≥ *card* (*sgraph.triangle-set E′*)
using *triangle-set-graph-edge-ss finite-triangle-set*
by (*simp add*: *assms card-mono*)

end

locale *triangle-free-graph* = *sgraph* +
assumes *tri-free*: ¬(∃ *x y z*. *triangle-in-graph x y z*)

lemma *triangle-free-graph-empty*: *E* = {} ⟹ *triangle-free-graph V E*
apply (*unfold-locales*, *simp-all*)
using *sgraph.triangle-in-graph-edge-empty*
by (*metis Int-absorb all-edges-disjoint complete-sgraph*)

context *fin-sgraph*
begin

Converting between ordered and unordered triples for reasoning on cardinality

lemma *card-convert-triangle-rep*:
assumes *X* ⊆ *V* **and** *Y* ⊆ *V* **and** *Z* ⊆ *V*
shows *card* (*triangle-set*) ≥ *1/6* ∗ *card* {(*x*, *y*, *z*) ∈ *X* × *Y* × *Z* . (*triangle-in-graph x y z*)}
(**is** - ≥ *1/6* ∗ *card ?TT*)
proof −
define *tofl* **where** *tofl* ≡ λ*l*::′*a list*. (*hd l*, *hd*(*tl l*), *hd*(*tl*(*tl l*)))
have *in-tofl*: (*x*, *y*, *z*) ∈ *tofl* ‘ *permutations-of-set* {*x*,*y*,*z*} **if** *x*≠*y y*≠*z x*≠*z* **for** *x y z*
proof −
have *distinct*[*x*,*y*,*z*]
using *that* **by** *simp*
then show *?thesis*
unfolding *tofl-def image-iff*
by (*smt* (*verit*, *best*) *list.sel*(*1*) *list.sel*(*3*) *list.simps*(*15*) *permutations-of-setI set-empty*)
qed
have *?TT* ⊆ {(*x*, *y*, *z*). (*triangle-in-graph x y z*)}
by *auto*
also have ... ⊆ (⋃ *t* ∈ *triangle-set*. *tofl* ‘ *permutations-of-set t*)
proof (*clarsimp simp*: *triangle-set-def*)
fix *u v w*
assume *t*: *triangle-in-graph u v w*
then have (*u*, *v*, *w*) ∈ *tofl* ‘ *permutations-of-set* {*u*,*v*,*w*}
by (*metis in-tofl triangle-commu1 triangle-vertices-distinct1 triangle-vertices-distinct2*)
with *t* **show** ∃ *t*. (∃ *x y z*. *t* = {*x*, *y*, *z*} ∧ *triangle-in-graph x y z*) ∧ (*u*, *v*, *w*) ∈ *tofl* ‘ *permutations-of-set t*
by *blast*
qed
finally have *?TT* ⊆ (⋃ *t* ∈ *triangle-set*. *tofl* ‘ *permutations-of-set t*) **.**

```
  then have card ?TT ≤ card(⋃ t ∈ triangle-set. tofl ‘ permutations-of-set t)
    by (intro card-mono finite-UN-I finite-triangle-set) (auto simp: assms)
  also have . . . ≤ (∑ t ∈ triangle-set. card (tofl ‘ permutations-of-set t))
    using card-UN-le finV finite-triangle-set wellformed by blast
  also have . . . ≤ (∑ t ∈ triangle-set. card (permutations-of-set t))
    by (meson card-image-le finite-permutations-of-set sum-mono)
  also have . . . ≤ (∑ t ∈ triangle-set. fact 3)
   by(rule sum-mono) (metis card.infinite card-permutations-of-set card-triangle-3
eq-refl nat.simps(3) numeral-3-eq-3)
  also have . . . = 6 ∗ card (triangle-set)
    by (simp add: eval-nat-numeral)
  finally have card ?TT ≤ 6 ∗ card (triangle-set) .
  then show ?thesis
    by (simp add: divide-simps)
qed

lemma card-convert-triangle-rep-bound:
  fixes t :: real
  assumes card {(x, y, z) ∈ X × Y × Z . (triangle-in-graph x y z)} ≥ t
  assumes X ⊆ V and Y ⊆ V and Z ⊆ V
  shows card (triangle-set) ≥ 1/6 ∗t
proof −
  define t′ where t′ ≡ card {(x, y, z) ∈ X × Y × Z . (triangle-in-graph x y z)}
  have t′ ≥ t using assms t′-def by simp
  then have tgt: 1/6 ∗ t′ ≥ 1/6 ∗ t by simp
   have card (triangle-set) ≥ 1/6 ∗t′ using t′-def card-convert-triangle-rep assms
by simp
  thus ?thesis using tgt by linarith
qed
end
end
theory Bipartite-Graphs imports Undirected-Graph-Walks
begin
```

6 Bipartite Graphs

An introductory library for reasoning on bipartite graphs.

6.1 Bipartite Set Up

All ”edges”, i.e. pairs, between any two sets

```
definition all-bi-edges :: ′a set ⇒ ′a set ⇒ ′a edge set where
all-bi-edges X Y ≡ mk-edge ‘ (X × Y)

lemma all-bi-edges-alt:
  assumes X ∩ Y = {}
  shows all-bi-edges X Y = {e . card e = 2 ∧ e ∩ X ≠ {} ∧ e ∩ Y ≠ {}}
  unfolding all-bi-edges-def
```

proof (*intro subset-antisym subsetI*)
  fix *e* **assume** *e* ∈ *mk-edge* ‘ (*X* × *Y*)
  then obtain *v1 v2* **where** *e* = { *v1*, *v2*} **and** *v1* ∈ *X* **and** *v2* ∈ *Y*
    by *auto*
  then show *e* ∈ {*e*. *card e* = *2* ∧ *e* ∩ *X* ≠ {} ∧ *e* ∩ *Y* ≠ {}} **using** *assms*
    using *card-2-iff* **by** *blast*
next
  fix *e′* **assume** *assm*: *e′* ∈ {*e*. *card e* = *2* ∧ *e* ∩ *X* ≠ {} ∧ *e* ∩ *Y* ≠ {}}
  then obtain *v1* **where** *v1in*: *v1* ∈ *e′* **and** *v1* ∈ *X*
    by *blast*
  moreover obtain *v2* **where** *v2in*: *v2* ∈ *e′* **and** *v2* ∈ *Y* **using** *assm* **by** *blast*
  then have *ne*: *v1* ≠ *v2*
    using *assms calculation*(*2*) **by** *blast*
  have *card e′* = *2* **using** *assm* **by** *blast*
  have {*v1*, *v2*} ⊆ *e′* **using** *v1in v2in* **by** *blast*
  then have *e′* = {*v1*, *v2*} **using** *assm v1in v2in*
    by (*metis* (*no-types*, *opaque-lifting*) ‹*card e′* = *2*› *card-2-iff′ insertCI ne subsetI subset-antisym*)
  then show *e′* ∈ *mk-edge* ‘ (*X* × *Y*)
    by (*simp add*: ‹*v2* ∈ *Y*› *calculation*(*2*) *in-mk-edge-img*)
qed

lemma *all-bi-edges-alt2*: *all-bi-edges X Y* = {{*x*, *y*} | *x y*. *x* ∈ *X* ∧ *y* ∈ *Y* }
  unfolding *all-bi-edges-def*
proof (*intro subset-antisym subsetI*)
  fix *x* **assume** *x* ∈ *mk-edge* ‘ (*X* × *Y*)
  then obtain *a b* **where** (*a*, *b*) ∈ (*X* × *Y*) **and** *xeq*: *x* = *mk-edge* (*a*, *b*) **by** *blast*
  then show *x* ∈ {{*x*, *y*} |*x y*. *x* ∈ *X* ∧ *y* ∈ *Y*}
    by *auto*
next
  fix *x* **assume** *x* ∈ {{*x*, *y*} |*x y*. *x* ∈ *X* ∧ *y* ∈ *Y*}
  then obtain *a b* **where** *xeq*: *x* = {*a*, *b*} **and** *a* ∈ *X* **and** *b* ∈ *Y*
    by *blast*
  then have (*a*, *b*) ∈ (*X* × *Y*) **by** *auto*
  then show *x* ∈ *mk-edge* ‘ (*X* × *Y*) **using** *in-mk-edge-img xeq* **by** *metis*
qed

lemma *all-bi-edges-wf*: *e* ∈ *all-bi-edges X Y* ⟹ *e* ⊆ *X* ∪ *Y*
  by (*auto simp add*: *all-bi-edges-alt2*)

lemma *all-bi-edges-2*: *X* ∩ *Y* = {} ⟹ *e* ∈ *all-bi-edges X Y* ⟹ *card e* = *2*
  using *card-2-iff* **by** (*auto simp add*: *all-bi-edges-alt2*)

lemma *all-bi-edges-main*: *X* ∩ *Y* = {} ⟹ *all-bi-edges X Y* ⊆ *all-edges* (*X* ∪ *Y*)
  unfolding *all-edges-def* **using** *all-bi-edges-wf all-bi-edges-2* **by** *blast*

lemma *all-bi-edges-finite*: *finite X* ⟹ *finite Y* ⟹ *finite* (*all-bi-edges X Y*)
  by (*simp add*: *all-bi-edges-def*)

lemma *all-bi-edges-not-ssX*: $X \cap Y = \{\} \Longrightarrow e \in$ *all-bi-edges* $X\ Y \Longrightarrow \neg\ e \subseteq X$
by (*auto simp add*: *all-bi-edges-alt*)

lemma *all-bi-edges-sym*: *all-bi-edges* $X\ Y =$ *all-bi-edges* $Y\ X$
by (*auto simp add*: *all-bi-edges-alt2*)

lemma *all-bi-edges-not-ssY*: $X \cap Y = \{\} \Longrightarrow e \in$ *all-bi-edges* $X\ Y \Longrightarrow \neg\ e \subseteq Y$
by (*auto simp add*: *all-bi-edges-alt*)

lemma *card-all-bi-edges*:
assumes *finite X finite Y*
assumes $X \cap Y = \{\}$
shows *card* (*all-bi-edges X Y*) = *card X* $*$ *card Y*
proof −
have *card* (*all-bi-edges X Y*) = *card* $(X \times Y)$
unfolding *all-bi-edges-def* **using** *inj-on-mk-edge assms card-image* **by** *blast*
thus *?thesis* **using** *card-cartesian-product* **by** *auto*
qed

lemma (**in** *sgraph*) *all-edges-between-bi-subset*: *mk-edge* ‘ *all-edges-between* $X\ Y \subseteq$ *all-bi-edges X Y*
by (*auto simp*: *all-edges-between-def all-bi-edges-def*)

6.2 Bipartite Graph Locale

For reasoning purposes, it is useful to explicitly label the two sets of vertices as X and Y. These are parameters in the locale

locale *bipartite-graph* = *graph-system* +
fixes *X Y* :: *'a set*
assumes *partition*: *partition-on* $V\ \{X, Y\}$
assumes *ne*: $X \neq Y$
assumes *edge-betw*: $e \in E \Longrightarrow e \in$ *all-bi-edges X Y*
begin

lemma *part-intersect-empty*: $X \cap Y = \{\}$
using *partition-onD2 partition disjointD ne*
by *blast*

lemma *X-not-empty*: $X \neq \{\}$
using *partition partition-onD3* **by** *auto*

lemma *Y-not-empty*: $Y \neq \{\}$
using *partition partition-onD3* **by** *auto*

lemma *XY-union*: $X \cup Y = V$
using *partition partition-onD1* **by** *auto*

lemma *card-edges-two*: $e \in E \Longrightarrow$ *card* $e = 2$
using *edge-betw all-bi-edges-alt part-intersect-empty* **by** *auto*

```
lemma partitions-ss: X ⊆ V Y ⊆ V
  using XY-union by auto

end
```

By definition, we say an edge must be between X and Y, i.e. contains two vertices

```
sublocale bipartite-graph ⊆ sgraph
  using card-edges-two by (unfold-locales)

context bipartite-graph
begin

abbreviation density ≡ edge-density X Y

lemma bipartite-sym: bipartite-graph V E Y X
  using partition ne edge-betw all-bi-edges-sym
  by (unfold-locales) (auto simp add: insert-commute)

lemma X-verts-not-adj:
  assumes x1 ∈ X x2 ∈ X
  shows ¬ vert-adj x1 x2
proof (rule ccontr, simp add: vert-adj-def)
  assume {x1, x2} ∈ E
  then have ¬ {x1, x2} ⊆ X
    using all-bi-edges-not-ssX edge-betw part-intersect-empty by auto
  then show False using assms by auto
qed

lemma Y-verts-not-adj:
  assumes y1 ∈ Y y2 ∈ Y
  shows ¬ vert-adj y1 y2
proof −
  interpret sym: bipartite-graph V E Y X using bipartite-sym by simp
  show ?thesis using sym.X-verts-not-adj
    by (simp add: assms(1) assms(2))
qed

lemma X-vert-adj-Y: x ∈X =⇒ vert-adj x y =⇒ y ∈ Y
  using X-verts-not-adj XY-union vert-adj-imp-inV by blast

lemma Y-vert-adj-X: y ∈Y =⇒ vert-adj y x =⇒ x ∈ X
  using Y-verts-not-adj XY-union vert-adj-imp-inV by blast

lemma neighbors-ss-eq-neighborhoodX: v ∈ X =⇒ neighborhood v = neighbors-ss
v Y
  unfolding neighborhood-def neighbors-ss-def
  by(auto simp add: X-vert-adj-Y vert-adj-imp-inV)
```

lemma *neighbors-ss-eq-neighborhoodY*: $v \in Y \Longrightarrow$ *neighborhood* v = *neighbors-ss* v X
  unfolding *neighborhood-def neighbors-ss-def*
  by(*auto simp add*: *Y-vert-adj-X vert-adj-imp-inV*)

lemma *neighborhood-subset-oppX*: $v \in X \Longrightarrow$ *neighborhood* $v \subseteq Y$
  using *neighbors-ss-eq-neighborhoodX neighbors-ss-def* **by** *auto*

lemma *neighborhood-subset-oppY*: $v \in Y \Longrightarrow$ *neighborhood* $v \subseteq X$
  using *neighbors-ss-eq-neighborhoodY neighbors-ss-def* **by** *auto*

lemma *degree-neighbors-ssX*: $v \in X \Longrightarrow$ *degree* v = *card* (*neighbors-ss* v Y)
  using *neighbors-ss-eq-neighborhoodX alt-deg-neighborhood* **by** *auto*

lemma *degree-neighbors-ssY*: $v \in Y \Longrightarrow$ *degree* v = *card* (*neighbors-ss* v X)
  using *neighbors-ss-eq-neighborhoodY alt-deg-neighborhood* **by** *auto*

definition *is-bicomplete*:: *bool* **where**
is-bicomplete $\equiv$ E = *all-bi-edges* X Y

lemma *edge-betw-indiv*:
  assumes $e \in E$
  obtains x y **where** $x \in X \land y \in Y \land e = \{x, y\}$
proof −
  have $e \in \{\{x, y\} \mid x\ y.\ x \in X \land y \in Y\}$
    using *edge-betw all-bi-edges-alt2 assms* **by** *blast*
  thus *?thesis*
    using *that* **by** *auto*
qed

lemma *edges-between-equals-edge-set*: *mk-edge* ‘ (*all-edges-between* X Y) = E
  by (*simp add*: *all-edges-between-set*, *intro subset-antisym subsetI*, *auto*) (*metis edge-betw-indiv*)

Lemmas for reasoning on walks and paths in a bipartite graph

lemma *walk-alternates*:
  assumes *is-walk* w
  assumes *Suc* i < *length* w $i \geq 0$
  shows $w\ !\ i \in X \longleftrightarrow w\ !\ (i + 1) \in Y$
proof −
  have $\{w\ !\ i, w\ !\ (i + 1)\} \in E$ **using** *is-walk-index assms* **by** *auto*
  then show *?thesis*
    using *X-vert-adj-Y not-vert-adj Y-vert-adj-X vert-adj-sym* **by** *blast*
qed

A useful reasoning pattern to mimic ”wlog” statements for properties that are symmetric is to interpret the symmetric bipartite graph and then directly apply the lemma proven earlier

lemma *walk-alternates-sym*:
assumes *is-walk w*
assumes *Suc i* $<$ *length w i* $\geq$ *0*
shows *w* ! *i* $\in$ *Y* $\longleftrightarrow$ *w* ! (*i* + *1*) $\in$ *X*
proof –
interpret *sym*: *bipartite-graph V E Y X* **using** *bipartite-sym* **by** *simp*
show *?thesis* **using** *sym.walk-alternates assms* **by** *simp*
qed

lemma *walk-length-even*:
assumes *is-walk w*
assumes *hd w* $\in$ *X* **and** *last w* $\in$ *X*
shows *even* (*walk-length w*)
using *assms*
proof (*induct length w arbitrary*: *w rule*: *nat-induct2*)
case *0*
then show *?case* **by** (*auto simp add*: *is-walk-def*)
next
case *1*
then have *walk-length w* = *0* **using** *walk-length-conv* **by** *auto*
then show *?case* **by** *simp*
next
case (*step n*)
then show *?case* **proof** (*cases n* = *0*)
case *True*
then have *length w* = *2* **using** *step* **by** *simp*
then have *hd w* $\in$ *X* $\Longrightarrow$ *last w* $\in$ *Y* **using** *walk-alternates hd-conv-nth last-conv-nth*
by (*metis add-0 add-diff-cancel-right′ less-2-cases-iff list.size*(*3*) *nat-1-add-1 step.prems*(*1*)
zero-le zero-neq-numeral)
then show *?thesis*
using *part-intersect-empty step.prems*(*2*) *step.prems*(*3*) **by** *blast*
next
case *False*
have *IH*: ($\bigwedge$*w*. *n* = *length w* $\Longrightarrow$ *is-walk w* $\Longrightarrow$ *hd w* $\in$ *X* $\Longrightarrow$ *last w* $\in$ *X* $\Longrightarrow$ *even* (*walk-length w*))
using *step* **by** *simp*
obtain *w1 w2* **where** *weq*: *w* = *w1*@*w2* **and** *w1*: *w1* = *take n w* **and** *w2*: *w2* = *drop n w*
by *simp*
then have *ne*: *w1* $\neq$ [] **using** *False is-walk-not-empty2 step.prems*(*1*) **by** *fastforce*

then have *w1-walk*: *is-walk w1* **using** *w1 is-walk-take False*
by (*metis nat-le-linear neq0-conv step.prems*(*1*) *take-all*)
have *hdw1*: *hd w1* $\in$ *X* **using** *step ne weq* **by** *auto*
then have *w1n*: *length w1* = *n* **using** *step length-take w1* **by** *auto*
then have *length w2* = *2* **using** *step length-drop*
by (*simp add*: *w2*)

```
    have last w = w ! (n + 1) using step last-conv-nth is-walk-not-empty
      by (metis add.left-commute diff-add-inverse nat-1-add-1)
    then have w ! n ∈ Y using step by (simp add: walk-alternates-sym)
    then have w ! (n − 1) ∈ X using False walk-alternates step by simp
    then have last w1 ∈ X using step last-conv-nth[of w1] ne w1n
      by (metis last-list-update list-update-id take-update-swap w1)
    then have even (walk-length w1) using w1-walk w1n hdw1 IH[of w1] by simp
    then have even (walk-length w1 + 2) by simp
    then show ?thesis using walk-length-conv weq step
      by (simp add: False w1n)
  qed
qed

lemma walk-length-even-sym:
  assumes is-walk w
  assumes hd w ∈ Y
  assumes last w ∈ Y
  shows even (walk-length w)
proof −
  interpret sym: bipartite-graph V E Y X using bipartite-sym by simp
  show ?thesis using sym.walk-length-even assms by auto
qed

lemma walk-length-odd:
  assumes is-walk w
  assumes hd w ∈ X and last w ∈ Y
  shows odd (walk-length w)
  using assms
proof (cases length w ≥ 2)
  case True
  then have hdin: hd (tl w) ∈ Y using walk-alternates hd-conv-nth
   by (metis (mono-tags, lifting) Suc-1 Suc-less-eq2 assms(1) assms(2) is-walk-not-empty2
is-walk-tl
      le-neq-implies-less le-numeral-extra(3) length-greater-0-conv less-Suc-eq nth-tl

       numeral-1-eq-Suc-0 numerals(1) plus-nat.add-0)
  have w: is-walk (tl w) using assms True is-walk-tl by auto
  have last: last (tl w) ∈ Y using assms(3) by (simp add: is-walk-not-empty last-tl
w)
  then have ev: even (walk-length (tl w)) using hdin w  walk-length-even-sym[of
tl w] by auto
  then have walk-length w = walk-length (tl w) + 1 using True walk-length-conv
by auto
  then show ?thesis using ev by simp
next
  case False
  have length w ≠ 0 using is-walk-not-empty assms by simp
  then have length w = 1 using False by linarith
  then have hd w = last w
```

using ‹*length w* ≠ *0*› *hd-conv-nth last-conv-nth* **by** *fastforce*
then have *hd w* ∈ *X* ⟹ *last w* ∉ *Y* **using** *part-intersect-empty* **by** *auto*
then show *?thesis* **using** *assms* **by** *simp*
qed

lemma *walk-length-odd-sym*:
assumes *is-walk w*
assumes *hd w* ∈ *Y* **and** *last w* ∈ *X*
shows *odd* (*walk-length w*)
proof −
interpret *sym*: *bipartite-graph V E Y X* **using** *bipartite-sym* **by** *simp*
show *?thesis* **using** *assms sym.walk-length-odd* **by** *simp*
qed

lemma *walk-length-even-iff*:
assumes *is-walk w*
shows *even* (*walk-length w*) ⟷ (*hd w* ∈ *X* ∧ *last w* ∈ *X*) ∨ (*hd w* ∈ *Y* ∧ *last w* ∈ *Y*)
proof (*intro iffI*)
assume *ev*: *even* (*walk-length w*)
show *hd w* ∈ *X* ∧ *last w* ∈ *X* ∨ *hd w* ∈ *Y* ∧ *last w* ∈ *Y*
proof (*rule ccontr*)
assume ¬ ((*hd w* ∈ *X* ∧ *last w* ∈ *X*) ∨ (*hd w* ∈ *Y* ∧ *last w* ∈ *Y*))
then have (*hd w* ∉ *X* ∨ *last w* ∉ *X*) ∧ (*hd w* ∉ *Y* ∨ *last w* ∉ *Y*) **by** *simp*
then have (*hd w* ∈ *Y* ∨ *last w* ∈ *Y*) ∧ (*hd w* ∈ *X* ∨ *last w* ∈ *X*) **using** *part-intersect-empty*
using *XY-union assms is-walk-wf-hd is-walk-wf-last* **by** *auto*
then have *split*: (*hd w* ∈ *X* ∧ *last w* ∈ *Y*) ∨ (*hd w* ∈ *Y* ∧ *last w* ∈ *X*)
using *part-intersect-empty* **by** *auto*
have *o1*: (*hd w* ∈ *X* ∧ *last w* ∈ *Y*) ⟹ *odd* (*walk-length w*) **using** *walk-length-odd assms* **by** *auto*
have (*hd w* ∈ *Y* ∧ *last w* ∈ *X*) ⟹ *odd* (*walk-length w*) **using** *walk-length-odd-sym assms* **by** *auto*
then show *False* **using** *split ev o1* **by** *auto*
qed
next
show (*hd w* ∈ *X* ∧ *last w* ∈ *X*) ∨ (*hd w* ∈ *Y* ∧ *last w* ∈ *Y*) ⟹ *even* (*walk-length w*)
using *walk-length-even walk-length-even-sym assms* **by** *auto*
qed

lemma *walk-length-odd-iff*:
assumes *is-walk w*
shows *odd* (*walk-length w*) ⟷ (*hd w* ∈ *X* ∧ *last w* ∈ *Y*) ∨ (*hd w* ∈ *Y* ∧ *last w* ∈ *X*)
proof (*intro iffI*)
assume *o*: *odd* (*walk-length w*)
show (*hd w* ∈ *X* ∧ *last w* ∈ *Y*) ∨ (*hd w* ∈ *Y* ∧ *last w* ∈ *X*)
proof (*rule ccontr*)

assume ¬ ((*hd w* ∈ *X* ∧ *last w* ∈ *Y*) ∨ (*hd w* ∈ *Y* ∧ *last w* ∈ *X*))
then have (*hd w* ∉ *X* ∨ *last w* ∉ *Y*) ∧ (*hd w* ∉ *Y* ∨ *last w* ∉ *X*) **by** *simp*
then have (*hd w* ∈ *Y* ∨ *last w* ∈ *X*) ∧ (*hd w* ∈ *X* ∨ *last w* ∈ *Y*) **using** *part-intersect-empty*
using *XY-union assms is-walk-wf-hd is-walk-wf-last* **by** *auto*
then have *split*: (*hd w* ∈ *X* ∧ *last w* ∈ *X*) ∨ (*hd w* ∈ *Y* ∧ *last w* ∈ *Y*)
using *part-intersect-empty* **by** *auto*
have *e1*: (*hd w* ∈ *X* ∧ *last w* ∈ *X*) ⟹ *even* (*walk-length w*) **using** *walk-length-even assms* **by** *auto*
have (*hd w* ∈ *Y* ∧ *last w* ∈ *Y*) ⟹ *even* (*walk-length w*) **using** *walk-length-even-sym assms* **by** *auto*
then show *False* **using** *split o e1* **by** *auto*
qed
next
show (*hd w* ∈ *X* ∧ *last w* ∈ *Y*) ∨ (*hd w* ∈ *Y* ∧ *last w* ∈ *X*) ⟹ *odd* (*walk-length w*)
using *walk-length-odd walk-length-odd-sym assms* **by** *auto*
qed

Classic basic theorem that a bipartite graph must not have any cycles with an odd length

lemma *no-odd-cycles*:
assumes *is-walk w*
assumes *odd* (*walk-length w*)
shows ¬ *is-cycle w*
proof −
have (*hd w* ∈ *X* ∧ *last w* ∈ *Y*) ∨ (*hd w* ∈ *Y* ∧ *last w* ∈ *X*) **using** *assms walk-length-odd-iff* **by** *auto*
then have *hd w* ≠ *last w* **using** *part-intersect-empty* **by** *auto*
thus *?thesis* **using** *is-cycle-def is-closed-walk-def* **by** *simp*
qed

end

A few properties rely on cardinality definitions that require the vertex sets to be finite

locale *fin-bipartite-graph* = *bipartite-graph* + *fin-graph-system*
begin

lemma *fin-bipartite-sym*: *fin-bipartite-graph V E Y X*
by (*intro-locales*) (*simp add*: *bipartite-sym bipartite-graph.axioms*(*2*))

lemma *partitions-finite*: *finite X finite Y*
using *partitions-ss finite-subset finV* **by** *auto*

lemma *card-edges-between-set*: *card* (*all-edges-between X Y*) = *card E*
proof −
have *card* (*all-edges-between X Y*) = *card* (*mk-edge* ‘ (*all-edges-between X Y*))
using *inj-on-mk-edge* **using** *partitions-finite card-image*

by (*metis inj-on-mk-edge part-intersect-empty*)
then show *?thesis* **by** (*simp add*: *edges-between-equals-edge-set*)
qed

lemma *density-simp*: *density* = *card* (*E*) / ((*card X*) ∗ (*card Y*))
unfolding *edge-density-def* **using** *card-edges-between-set* **by** *auto*

lemma *edge-size-degree-sumY*: *card E* = ($\sum$ *y* ∈ *Y* . *degree y*)
proof −
have ($\sum$ *y* ∈ *Y* . *degree y*) = ($\sum$ *y* ∈ *Y* . *card*(*neighbors-ss y X*))
using *degree-neighbors-ssY* **by** (*simp*)
also have ... = *card* (*all-edges-between X Y*)
using *card-all-edges-betw-neighbor*
by (*metis card-all-edges-between-commute partitions-finite*(*1*) *partitions-finite*(*2*))

finally show *?thesis*
by (*simp add*: *card-edges-between-set*)
qed

lemma *edge-size-degree-sumX*: *card E* = ($\sum$ *y* ∈ *X* . *degree y*)
proof −
interpret *sym*: *fin-bipartite-graph V E Y X*
using *fin-bipartite-sym* **by** *simp*
show *?thesis* **using** *sym.edge-size-degree-sumY* **by** *simp*
qed

end
end

7 Graph Theory Inheritance

This theory aims to demonstrate the use of locales to transfer theorems between different graph/combinatorial structure representations

theory *Graph-Theory-Relations* **imports** *Undirected-Graph-Basics Bipartite-Graphs*

Design-Theory.Block-Designs Design-Theory.Group-Divisible-Designs
begin

7.1 Design Inheritance

A graph is a type of incidence system, and more specifically a type of combinatorial design. This section demonstrates the correspondence between designs and graphs

sublocale *graph-system* ⊆ *inc*: *incidence-system V mset-set E*
by (*unfold-locales*) (*metis wellformed elem-mset-set ex-in-conv infinite-set-mset-mset-set*)

sublocale *fin-graph-system* ⊆ *finc*: *finite-incidence-system V mset-set E*
using *finV* **by** *unfold-locales*

sublocale *fin-ulgraph* ⊆ *d*: *design V mset-set E*
using *edge-size empty-not-edge fin-edges* **by** *unfold-locales auto*

sublocale *fin-ulgraph* ⊆ *d*: *simple-design V mset-set E*
by *unfold-locales* (*simp add*: *fin-edges*)

locale *graph-has-edges* = *graph-system* +
assumes *edges-nempty*: *E* ≠ {}

locale *fin-sgraph-wedges* = *fin-sgraph* + *graph-has-edges*

The simple graph definition of degree overlaps with the definition of a point replication number

sublocale *fin-sgraph-wedges* ⊆ *bd*: *block-design V mset-set E 2*
rewrites *point-replication-number* (*mset-set E*) *x* = *degree x*
and *points-index* (*mset-set E*) *vs* = *degree-set vs*
proof (*unfold-locales*)
show *inc*.b ≠ *0* **by** (*simp add*: *edges-nempty fin-edges*)
show ⋀*bl*. *bl* ∈# *mset-set E* ⟹ *card bl* = *2* **by** (*simp add*: *fin-edges two-edges*)
show *mset-set E index vs* = *degree-set vs*
unfolding *degree-set-def points-index-def* **by** (*simp add*: *fin-edges*)
next
have *size* {#*b* ∈# (*mset-set E*) . *x* ∈ *b*#} = *card* (*incident-edges x*)
unfolding *incident-edges-def vincident-def*
by (*simp add*: *fin-edges*)
then show *mset-set E rep x* = *degree x* **using** *alt-degree-def point-replication-number-def*
by *metis*
qed

locale *fin-bipartite-graph-wedges* = *fin-bipartite-graph* + *fin-sgraph-wedges*

sublocale *fin-bipartite-graph-wedges* ⊆ *group-design V mset-set E* {*X*, *Y*}
by *unfold-locales* (*simp-all add*: *partition ne*)

7.2 Adjacency Relation Definition

Another common formal representation of graphs is as a vertex set and an adjacency relation This is a useful representation in some contexts - we use locales to enable the transfer of results between the two representations, specifically the mutual sublocales approach

locale *graph-rel* =
fixes *vertices* :: *'a set* (*V*)
fixes *adj-rel* :: *'a rel*
assumes *wf*: ⋀ *u v*. (*u*, *v*) ∈ *adj-rel* ⟹ *u* ∈ *V* ∧ *v* ∈ *V*
begin

abbreviation *adj u v* ≡ (*u*, *v*) ∈ *adj-rel*

lemma *wf-alt*: *adj u v* ⟹ (*u*, *v*) ∈ *V* × *V*
  using *wf* **by** *blast*

end

locale *ulgraph-rel* = *graph-rel* +
  assumes *sym-adj*: *sym adj-rel*
begin

This definition makes sense in the context of an undirected graph

definition *edge-set*:: *'a edge set* **where**
edge-set ≡ {{*u*, *v*} | *u v*. *adj u v*}

lemma *obtain-edge-pair-adj*:
  assumes *e* ∈ *edge-set*
  obtains *u v* **where** *e* = {*u*, *v*} **and** *adj u v*
  using *assms edge-set-def mem-Collect-eq*
  by *fastforce*

lemma *adj-to-edge-set-card*:
  assumes *e* ∈ *edge-set*
  shows *card e* = *1* ∨ *card e* = *2*
proof −
  obtain *u v* **where** *e* = {*u*, *v*} **and** *adj u v* **using** *obtain-edge-pair-adj assms* **by** *blast*
  then show *?thesis* **by** (*cases u* = *v*, *simp-all*)
qed

lemma *adj-to-edge-set-card-lim*:
  assumes *e* ∈ *edge-set*
  shows *card e* > *0* ∧ *card e* ≤ *2*
proof −
  obtain *u v* **where** *e* = {*u*, *v*} **and** *adj u v* **using** *obtain-edge-pair-adj assms* **by** *blast*
  then show *?thesis* **by** (*cases u* = *v*, *simp-all*)
qed

lemma *edge-set-wf*: *e* ∈ *edge-set* ⟹ *e* ⊆ *V*
  using *obtain-edge-pair-adj wf* **by** (*metis insert-iff singletonD subsetI*)

lemma *is-graph-system*: *graph-system V edge-set*
  by (*unfold-locales*) (*simp add*: *edge-set-wf*)

lemma *sym-alt*: *adj u v* ⟷ *adj v u*
  using *sym-adj* **by** (*meson symE*)

lemma *is-ulgraph*: *ulgraph V edge-set*
  using *ulgraph-axioms-def is-graph-system adj-to-edge-set-card-lim*
  by (*intro-locales*) *auto*

end

context *ulgraph*
begin

definition *adj-relation* :: *'a rel* **where**
adj-relation ≡ {(*u*, *v*) | *u v* . *vert-adj u v*}

lemma *adj-relation-wf*: (*u*, *v*) ∈ *adj-relation* ⟹ {*u*, *v*} ⊆ *V*
  unfolding *adj-relation-def* **using** *vert-adj-imp-inV* **by** *auto*

lemma *adj-relation-sym*: *sym adj-relation*
  unfolding *adj-relation-def sym-def* **using** *vert-adj-sym* **by** *auto*

lemma *is-ulgraph-rel*: *ulgraph-rel V adj-relation*
  using *adj-relation-wf adj-relation-sym* **by** (*unfold-locales*) *auto*

Temporary interpretation - mutual sublocale setup

interpretation *ulgraph-rel V adj-relation* **by** (*rule is-ulgraph-rel*)

lemma *vert-adj-rel-iff*:
  assumes *u* ∈ *V v* ∈ *V*
  shows *vert-adj u v* ⟷ *adj u v*
  using *adj-relation-def* **by** *auto*

lemma *edges-rel-is*: *E* = *edge-set*
proof −
  have *E* = {{*u*, *v*} | *u v* . *vert-adj u v*}
  proof (*intro subset-antisym subsetI*)
    show ⋀*x*. *x* ∈ {{*u*, *v*} |*u v*. *vert-adj u v*} ⟹ *x* ∈ *E*
      using *vert-adj-def* **by** *fastforce*
  next
    fix *x* **assume** *x* ∈ *E*
    then have *x* ⊆ *V* **and** *card x* > *0* **and** *card x* ≤ *2* **using** *wellformed edge-size*
by *auto*
    then obtain *u v* **where** *x* = {*u*, *v*} **and** {*u*, *v*} ∈ *E*
      by (*metis* ‹*x* ∈ *E*› *alt-edge-size card-1-singletonE card-2-iff insert-absorb2*)
    then show *x* ∈ {{*u*, *v*} |*u v*. *vert-adj u v*} **unfolding** *vert-adj-def* **by** *blast*
  qed
  then have *E* = {{*u*, *v*} | *u v* . *adj u v*} **using** *vert-adj-rel-iff Collect-cong*
    by (*smt* (*verit*) *local.wf vert-adj-imp-inV*)
  thus *?thesis* **using** *edge-set-def* **by** *simp*
qed

end

context *ulgraph-rel*
begin

Temporary interpretation - mutual sublocale setup

interpretation *ulgraph V edge-set* **by** (*rule is-ulgraph*)

lemma *rel-vert-adj-iff*: *vert-adj u v* ⟷ *adj u v*
proof (*intro iffI*)
  assume *vert-adj u v*
  then have {*u*, *v*} ∈ *edge-set* **by** (*simp add*: *vert-adj-def*)
  then show *adj u v* **using** *edge-set-def*
    by (*metis* (*no-types*, *lifting*) *doubleton-eq-iff obtain-edge-pair-adj sym-alt*)
next
  assume *adj u v*
  then have {*u*, *v*} ∈ *edge-set* **using** *edge-set-def* **by** *auto*
  then show *vert-adj u v* **by** (*simp add*: *vert-adj-def*)
qed

lemma *rel-item-is*: (*u*, *v*) ∈ *adj-rel* ⟷ (*u*, *v*) ∈ *adj-relation*
  unfolding *adj-relation-def* **using** *rel-vert-adj-iff* **by** *auto*

lemma *rel-edges-is*: *adj-rel* = *adj-relation*
  using *rel-item-is* **by** *auto*

end

sublocale *ulgraph-rel* ⊆ *ulgraph V edge-set*
  rewrites *ulgraph.adj-relation edge-set* = *adj-rel*
  using *local.is-ulgraph rel-edges-is* **by** *simp-all*

sublocale *ulgraph* ⊆ *ulgraph-rel V adj-relation*
  rewrites *ulgraph-rel.edge-set adj-relation* = *E*
  using *is-ulgraph-rel edges-rel-is* **by** *simp-all*

locale *sgraph-rel* = *ulgraph-rel* +
  assumes *irrefl-adj*: *irrefl adj-rel*
begin

lemma *irrefl-alt*: *adj u v* ⟹ *u* ≠ *v*
  using *irrefl-adj irrefl-def* **by** *fastforce*

lemma *edge-is-card2*:
  assumes *e* ∈ *edge-set*
  shows *card e* = *2*
proof –
  obtain *u v* **where** *eq*: *e* = {*u*, *v*} **and** *adj u v* **using** *assms edge-set-def* **by** *blast*
  then have *u* ≠ *v* **using** *irrefl-alt* **by** *simp*

thus *?thesis* **using** *eq* **by** *simp*
qed

lemma *is-sgraph*: *sgraph V edge-set*
using *is-graph-system edge-is-card2 sgraph-axioms-def* **by** (*intro-locales*) *auto*

end

context *sgraph*
begin

lemma *is-rel-irrefl-alt*:
assumes (u, v) ∈ *adj-relation*
shows $u \neq v$
proof −
have *vert-adj u v* **using** *adj-relation-def assms* **by** *blast*
then have {u, v} ∈ E **using** *vert-adj-def* **by** *simp*
then have *card* {u, v} = *2* **using** *two-edges* **by** *simp*
thus *?thesis* **by** *auto*
qed

lemma *is-rel-irrefl*: *irrefl adj-relation*
using *irrefl-def is-rel-irrefl-alt* **by** *auto*

lemma *is-sgraph-rel*: *sgraph-rel V adj-relation*
by (*unfold-locales*) (*simp add*: *is-rel-irrefl*)

end

sublocale *sgraph-rel* ⊆ *sgraph V edge-set*
rewrites *ulgraph.adj-relation edge-set* = *adj-rel*
using *is-sgraph rel-edges-is* **by** *simp-all*

sublocale *sgraph* ⊆ *sgraph-rel V adj-relation*
rewrites *ulgraph-rel.edge-set adj-relation* = E
using *is-sgraph-rel edges-rel-is* **by** *simp-all*

end
theory *Undirected-Graphs-Root* **imports**
Undirected-Graph-Basics
Undirected-Graph-Walks
Connectivity
Girth-Independence
Graph-Triangles
Bipartite-Graphs
Graph-Theory-Relations
begin
end

References

[1] C. Edmonds, A. Koutsoukou-Argyraki, and L. C. Paulson. Roth's Theorem on Arithmetic Progressions. *Archive of Formal Proofs*, Dec. 2021.

[2] C. Edmonds, A. Koutsoukou-Argyraki, and L. C. Paulson. Szemerédi's Regularity Lemma. *Archive of Formal Proofs*, Nov. 2021.

[3] L. Hupel. Properties of random graphs – subgraph containment. *Archive of Formal Proofs*, February 2014. https://isa-afp.org/entries/Random_Graph_Subgraph_Threshold.html, Formal proof development.

[4] L. Noschinski. Proof Pearl: A Probabilistic Proof for the Girth-Chromatic Number Theorem. In *Interactive Theorem Proving. ITP 2012.*, volume 7406 of *Lecture Notes in Computer Science*. Springer Berlin Heidelberg, 2012.

[5] L. Noschinski. A Graph Library for Isabelle. *Mathematics in Computer Science*, 9(1):23–39, Mar. 2015. http://link.springer.com/10.1007/s11786-014-0183-z.