Basic Geometric Properties of Triangles

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Abstract

In this work, we define angles between vectors and between three points. Building on this, we prove basic geometric properties of triangles, such as the Isosceles Triangle Theorem, the Law of Sines and the Law of Cosines, that the sum of the angles of a triangle is π, and the congruence theorems for triangles.

The definitions and proofs were developed following those by John Harrison in HOL Light. However, due to Isabelle's type class system, all definitions and theorems in the Isabelle formalisation hold for all real inner product spaces.

Contents

1 Definition of angles

theory *Angles*
imports
  HOL−Analysis.Multivariate-Analysis
begin

lemma *collinear-translate-iff*: *collinear* (((+) *a*) ' *A*) ⟷ *collinear A*
  ⟨*proof*⟩

definition *vangle* **where**
vangle u v = (*if u* = *0* ∨ *v* = *0 then pi* / *2 else arccos* (*u* · *v* / (*norm u* ∗ *norm v*)))

definition *angle* **where**
angle a b c = *vangle* (*a* − *b*) (*c* − *b*)

lemma *angle-altdef*: *angle a b c* = *arccos* ((*a* − *b*) · (*c* − *b*) / (*dist a b* ∗ *dist c b*))
⟨*proof*⟩

lemma *vangle-0-left* [*simp*]: *vangle 0 v* = *pi* / *2*
and *vangle-0-right* [*simp*]: *vangle u 0* = *pi* / *2*
⟨*proof*⟩

lemma *vangle-refl* [*simp*]: *u* ≠ *0* ⟹ *vangle u u* = *0*
⟨*proof*⟩

lemma *angle-refl* [*simp*]: *angle a a b* = *pi* / *2 angle a b b* = *pi* / *2*
⟨*proof*⟩

lemma *angle-refl-mid* [*simp*]: *a* ≠ *b* ⟹ *angle a b a* = *0*
⟨*proof*⟩

lemma *cos-vangle*: *cos* (*vangle u v*) = *u* · *v* / (*norm u* ∗ *norm v*)
⟨*proof*⟩

lemma *cos-angle*: *cos* (*angle a b c*) = (*a* − *b*) · (*c* − *b*) / (*dist a b* ∗ *dist c b*)
⟨*proof*⟩

lemma *inner-conv-angle*: (*a* − *b*) · (*c* − *b*) = *dist a b* ∗ *dist c b* ∗ *cos* (*angle a b c*)
⟨*proof*⟩

lemma *vangle-commute*: *vangle u v* = *vangle v u*
⟨*proof*⟩

lemma *angle-commute*: *angle a b c* = *angle c b a*
⟨*proof*⟩

lemma *vangle-nonneg*: *vangle u v* ≥ *0* **and** *vangle-le-pi*: *vangle u v* ≤ *pi*
⟨*proof*⟩

lemmas *vangle-bounds* = *vangle-nonneg vangle-le-pi*

lemma *angle-nonneg*: *angle a b c* ≥ *0* **and** *angle-le-pi*: *angle a b c* ≤ *pi*
⟨*proof*⟩

lemmas *angle-bounds* = *angle-nonneg angle-le-pi*

lemma *sin-vangle-nonneg*: *sin* (*vangle u v*) ≥ *0*
⟨*proof*⟩

lemma *sin-angle-nonneg*: *sin* (*angle a b c*) ≥ *0*
⟨*proof*⟩

lemma *vangle-eq-0D*:
assumes *vangle u v* = *0*
shows *norm u* $*_R$ *v* = *norm v* $*_R$ *u*
⟨*proof*⟩

lemma *vangle-eq-piD*:
assumes *vangle u v* = *pi*
shows *norm u* $*_R$ *v* + *norm v* $*_R$ *u* = *0*
⟨*proof*⟩

lemma *dist-triangle-eq*:
fixes *a b c* :: *'a* :: *real-inner*
shows (*dist a c* = *dist a b* + *dist b c*) ⟷ *dist a b* $*_R$ (*c* − *b*) + *dist b c* $*_R$ (*a* − *b*) = *0*
⟨*proof*⟩

lemma *angle-eq-pi-imp-dist-additive*:
assumes *angle a b c* = *pi*
shows *dist a c* = *dist a b* + *dist b c*
⟨*proof*⟩

lemma *orthogonal-iff-vangle*: *orthogonal u v* ⟷ *vangle u v* = *pi* / *2*
⟨*proof*⟩

lemma *cos-minus1-imp-pi*:
assumes *cos x* = −*1 x* ≥ *0 x* < *3* ∗ *pi*
shows *x* = *pi*
⟨*proof*⟩

lemma *vangle-eqI*:
assumes *u* ≠ *0 v* ≠ *0 w* ≠ *0 x* ≠ *0*
assumes (*u* · *v*) ∗ *norm w* ∗ *norm x* = (*w* · *x*) ∗ *norm u* ∗ *norm v*
shows *vangle u v* = *vangle w x*
⟨*proof*⟩

lemma *angle-eqI*:
assumes *a* ≠ *b a* ≠ *c d* ≠ *e d* ≠ *f*
assumes ((*b*−*a*) · (*c*−*a*)) ∗ *dist d e* ∗ *dist d f* = ((*e*−*d*) · (*f*−*d*)) ∗ *dist a b* ∗

dist a c
shows *angle b a c = angle e d f*
⟨*proof*⟩

lemma *cos-vangle-eqD*: *cos (vangle u v) = cos (vangle w x) ⟹ vangle u v = vangle w x*
⟨*proof*⟩

lemma *cos-angle-eqD*: *cos (angle a b c) = cos (angle d e f) ⟹ angle a b c = angle d e f*
⟨*proof*⟩

lemma *sin-vangle-zero-iff*: *sin (vangle u v) = 0 ⟷ vangle u v ∈ {0, pi}*
⟨*proof*⟩

lemma *sin-angle-zero-iff*: *sin (angle a b c) = 0 ⟷ angle a b c ∈ {0, pi}*
⟨*proof*⟩

lemma *vangle-collinear*: *vangle u v ∈ {0, pi} ⟹ collinear {0, u, v}*
⟨*proof*⟩

lemma *angle-collinear*: *angle a b c ∈ {0, pi} ⟹ collinear {a, b, c}*
⟨*proof*⟩

lemma *not-collinear-vangle*: *¬collinear {0,u,v} ⟹ vangle u v ∈ {0<..<pi}*
⟨*proof*⟩

lemma *not-collinear-angle*: *¬collinear {a,b,c} ⟹ angle a b c ∈ {0<..<pi}*
⟨*proof*⟩

1.1 Contributions from Lukas Bulwahn

lemma *vangle-scales*:
assumes $0 < c$
shows *vangle* $(c *_R v_1)\ v_2$ = *vangle* $v_1\ v_2$
⟨*proof*⟩

lemma *vangle-inverse*:
vangle $(- v_1)\ v_2$ = *pi* − *vangle* $v_1\ v_2$
⟨*proof*⟩

lemma *orthogonal-iff-angle*:
shows *orthogonal (A − B) (C − B) ⟷ angle A B C = pi / 2*
⟨*proof*⟩

lemma *angle-inverse*:
assumes *between (A, C) B*
assumes *A ≠ B B ≠ C*
shows *angle A B D = pi − angle C B D*

⟨*proof*⟩

lemma *strictly-between-implies-angle-eq-pi*:
  assumes *between* (A, C) B
  assumes $A \neq B$ $B \neq C$
  shows *angle* A B C = *pi*
⟨*proof*⟩

end

2 Basic Properties of Triangles

theory *Triangle*
imports
  Angles
begin

We prove a number of basic geometric properties of triangles. All theorems hold in any real inner product space.

2.1 Thales' theorem

theorem *thales*:
  fixes A B C :: *'a* :: *real-inner*
  assumes *dist* B (*midpoint* A C) = *dist* A C / *2*
  shows *orthogonal* $(A - B)$ $(C - B)$
⟨*proof*⟩

2.2 Sine and cosine laws

The proof of the Law of Cosines follows trivially from the definition of the angle, the definition of the norm in vector spaces with an inner product and the bilinearity of the inner product.

lemma *cosine-law-vector*:
  norm $(u - v)$ ^ *2* = *norm* u ^ *2* + *norm* v ^ *2* − *2* ∗ *norm* u ∗ *norm* v ∗ *cos* (*vangle* u v)
  ⟨*proof*⟩

lemma *cosine-law-triangle*:
  dist b c ^ *2* = *dist* a b ^ *2* + *dist* a c ^ *2* − *2* ∗ *dist* a b ∗ *dist* a c ∗ *cos* (*angle* b a c)
  ⟨*proof*⟩

According to our definition, angles are always between 0 and π and therefore, the sign of an angle is always non-negative. We can therefore look at $\sin(\alpha)^2$, which we can express in terms of $\cos(\alpha)$ using the identity $\sin(\alpha)^2 + \cos(\alpha)^2 = 1$. The remaining proof is then a trivial consequence of the definitions.

lemma *sine-law-triangle*:
  sin (*angle a b c*) ∗ *dist b c* = *sin* (*angle b a c*) ∗ *dist a c* (**is** *?A* = *?B*)
⟨*proof*⟩

The following forms of the Law of Sines/Cosines are more convenient for eliminating sines/cosines from a goal completely.

lemma *cosine-law-triangle′*:
  2 ∗ *dist a b* ∗ *dist a c* ∗ *cos* (*angle b a c*) = (*dist a b* ^ *2* + *dist a c* ^ *2* − *dist b c* ^ *2*)
  ⟨*proof*⟩

lemma *cosine-law-triangle′′*:
  cos (*angle b a c*) = (*dist a b* ^ *2* + *dist a c* ^ *2* − *dist b c* ^ *2*) / (*2* ∗ *dist a b* ∗ *dist a c*)
  ⟨*proof*⟩

lemma *sine-law-triangle′*:
  b ≠ *c* ⟹ *sin* (*angle a b c*) = *sin* (*angle b a c*) ∗ *dist a c* / *dist b c*
  ⟨*proof*⟩

lemma *sine-law-triangle′′*:
  b ≠ *c* ⟹ *sin* (*angle c b a*) = *sin* (*angle b a c*) ∗ *dist a c* / *dist b c*
  ⟨*proof*⟩

2.3 Sum of angles

context
begin

private lemma *gather-squares*: *a* ∗ (*a* ∗ *b*) = *a*^*2* ∗ (*b* :: *real*)
  ⟨*proof*⟩ **lemma** *eval-power*: *x* ^ *numeral n* = *x* ∗ *x* ^ *pred-numeral n*
  ⟨*proof*⟩

The proof that the sum of the angles in a triangle is π is somewhat more involved. Following the HOL Light proof by John Harrison, we first prove that $\cos(\alpha + \beta + \gamma) = -1$ and $\alpha + \beta + \gamma \in [0; 3\pi)$, which then implies the theorem.

The main work is proving $\cos(\alpha + \beta + \gamma)$. This is done using the addition theorems for the sine and cosine, then using the Laws of Sines to eliminate all sin terms save $\sin(\gamma)^2$, which only appears squared in the remaining goal. We then use $\sin(\gamma)^2 = 1 - \cos(\gamma)^2$ to eliminate this term and apply the law of cosines to eliminate this term as well.

The remaining goal is a non-linear equation containing only the length of the sides of the triangle. It can be shown by simple algebraic rewriting.

lemma *angle-sum-triangle*:
  assumes *a* ≠ *b* ∨ *b* ≠ *c* ∨ *a* ≠ *c*
  shows   *angle c a b* + *angle a b c* + *angle b c a* = *pi*

⟨*proof*⟩

end

2.4 Congruence Theorems

If two triangles agree on two angles at a non-degenerate side, the third angle must also be equal.

lemma *similar-triangle-aa*:
  assumes $b1 \neq c1$ $b2 \neq c2$
  assumes *angle a1 b1 c1 = angle a2 b2 c2*
  assumes *angle b1 c1 a1 = angle b2 c2 a2*
  shows *angle b1 a1 c1 = angle b2 a2 c2*
⟨*proof*⟩

A triangle is defined by its three angles and the lengths of three sides up to congruence. Two triangles are congruent if they have their angles are the same and their sides have the same length.

locale *congruent-triangle* =
  fixes *a1 b1 c1* :: *'a* :: *real-inner* **and** *a2 b2 c2* :: *'b* :: *real-inner*
  assumes *sides'*: *dist a1 b1 = dist a2 b2 dist a1 c1 = dist a2 c2 dist b1 c1 = dist b2 c2*
    and *angles'*: *angle b1 a1 c1 = angle b2 a2 c2 angle a1 b1 c1 = angle a2 b2 c2*
      angle a1 c1 b1 = angle a2 c2 b2
begin

lemma *sides*:
  dist a1 b1 = dist a2 b2 dist a1 c1 = dist a2 c2 dist b1 c1 = dist b2 c2
  dist b1 a1 = dist a2 b2 dist c1 a1 = dist a2 c2 dist c1 b1 = dist b2 c2
  dist a1 b1 = dist b2 a2 dist a1 c1 = dist c2 a2 dist b1 c1 = dist c2 b2
  dist b1 a1 = dist b2 a2 dist c1 a1 = dist c2 a2 dist c1 b1 = dist c2 b2
  ⟨*proof*⟩

lemma *angles*:
  angle b1 a1 c1 = angle b2 a2 c2 angle a1 b1 c1 = angle a2 b2 c2 angle a1 c1 b1 = angle a2 c2 b2
  angle c1 a1 b1 = angle b2 a2 c2 angle c1 b1 a1 = angle a2 b2 c2 angle b1 c1 a1 = angle a2 c2 b2
  angle b1 a1 c1 = angle c2 a2 b2 angle a1 b1 c1 = angle c2 b2 a2 angle a1 c1 b1 = angle b2 c2 a2
  angle c1 a1 b1 = angle c2 a2 b2 angle c1 b1 a1 = angle c2 b2 a2 angle b1 c1 a1 = angle b2 c2 a2
  ⟨*proof*⟩

end

lemmas *congruent-triangleD = congruent-triangle.sides congruent-triangle.angles*

Given two triangles that agree on a subset of its side lengths and angles that are sufficient to define a triangle uniquely up to congruence, one can conclude that they must also agree on all remaining quantities, i.e. that they are congruent.

The following four congruence theorems state what constitutes such a uniquely-defining subset of quantities. Each theorem states in its name which quantities are required and in which order (clockwise or counter-clockwise): an “s” stands for a side, an “a” stands for an angle.

The lemma “congruent-triangleI-sas, for example, requires that two adjacent sides and the angle inbetween are the same in both triangles.

lemma *congruent-triangleI-sss*:
  fixes *a1 b1 c1* :: *'a* :: *real-inner* **and** *a2 b2 c2* :: *'b* :: *real-inner*
  assumes *dist a1 b1* = *dist a2 b2*
  assumes *dist b1 c1* = *dist b2 c2*
  assumes *dist a1 c1* = *dist a2 c2*
  shows *congruent-triangle a1 b1 c1 a2 b2 c2*
⟨*proof*⟩

lemmas *congruent-triangle-sss* = *congruent-triangleD*[*OF congruent-triangleI-sss*]

lemma *congruent-triangleI-sas*:
  assumes *dist a1 b1* = *dist a2 b2*
  assumes *dist b1 c1* = *dist b2 c2*
  assumes *angle a1 b1 c1* = *angle a2 b2 c2*
  shows *congruent-triangle a1 b1 c1 a2 b2 c2*
⟨*proof*⟩

lemmas *congruent-triangle-sas* = *congruent-triangleD*[*OF congruent-triangleI-sas*]

lemma *congruent-triangleI-aas*:
  assumes *angle a1 b1 c1* = *angle a2 b2 c2*
  assumes *angle b1 c1 a1* = *angle b2 c2 a2*
  assumes *dist a1 b1* = *dist a2 b2*
  assumes ¬*collinear* {*a1*,*b1*,*c1*}
  shows *congruent-triangle a1 b1 c1 a2 b2 c2*
⟨*proof*⟩

lemmas *congruent-triangle-aas* = *congruent-triangleD*[*OF congruent-triangleI-aas*]

lemma *congruent-triangleI-asa*:
  assumes *angle a1 b1 c1* = *angle a2 b2 c2*
  assumes *dist a1 b1* = *dist a2 b2*
  assumes *angle b1 a1 c1* = *angle b2 a2 c2*
  assumes ¬*collinear* {*a1*, *b1*, *c1*}
  shows *congruent-triangle a1 b1 c1 a2 b2 c2*
⟨*proof*⟩

lemmas *congruent-triangle-asa* = *congruent-triangleD*[*OF congruent-triangleI-asa*]

2.5 Isosceles Triangle Theorem

We now prove the Isosceles Triangle Theorem: in a triangle where two sides have the same length, the two angles that are adjacent to only one of the two sides must be equal.

lemma *isosceles-triangle*:
  assumes *dist a c* = *dist b c*
  shows *angle b a c* = *angle a b c*
  ⟨*proof*⟩

For the non-degenerate case (i.e. the three points are not collinear), We also prove the converse.

lemma *isosceles-triangle-converse*:
  assumes *angle a b c* = *angle b a c* ¬*collinear* {*a*,*b*,*c*}
  shows *dist a c* = *dist b c*
  ⟨*proof*⟩

2.6 Contributions by Lukas Bulwahn

lemma *Pythagoras*:
  fixes *A B C* :: *'a* :: *real-inner*
  assumes *orthogonal* (*A* − *C*) (*B* − *C*)
  shows (*dist B C*) ^ *2* + (*dist C A*) ^ *2* = (*dist A B*) ^ *2*
⟨*proof*⟩

lemma *isosceles-triangle-orthogonal-on-midpoint*:
  fixes *A B C* :: *'a* :: *euclidean-space*
  assumes *dist C A* = *dist C B*
  shows *orthogonal* (*C* − *midpoint A B*) (*A* − *midpoint A B*)
⟨*proof*⟩

end