Transport via Partial Galois Connections and Equivalences

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Abstract

This entry contains the accompanying formalisation of the paper "Transport via Partial Galois Connections and Equivalences" (APLAS 2023) [2]. It contains a theoretical framework to transport programs via equivalences, subsuming the theory of Isabelle's Lifting package [1]. It also contains a prototype to automate transports using this framework in Isabelle/HOL, but this prototype is not yet ready for production. Finally, it contains a library on top of Isabelle/HOL's axioms, including various relativised concepts on orders, functions, binary relations, and Galois connections and equivalences.

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Chapter 1

HOL-Basics

theory *HOL-Basics-Base*
imports
HOL.HOL
begin

end

1.0.1 Lattice Syntax

theory *HOL-Syntax-Bundles-Lattices*
imports
HOL.Lattices
begin

open-bundle *lattice-syntax* — minimised version from theory Main
begin
notation
bot (‹⊥›) **and**
top (‹⊤›) **and**
inf (**infixl** ‹⊓› *70*) **and**
sup (**infixl** ‹⊔› *65*)
end

end

1.0.2 Lattice

theory *Predicates-Lattice*
imports
HOL-Syntax-Bundles-Lattices
HOL.Boolean-Algebras
begin

lemma *inf-predI* [*intro*]:
assumes *P x*

and $Q\ x$
shows $(P \sqcap Q)\ x$
using *assms* **by** (*intro inf1I*)

lemma *inf-predE* [*elim*]:
assumes $(P \sqcap Q)\ x$
obtains $P\ x\ Q\ x$
using *assms* **by** (*rule inf1E*)

lemma *inf-predD*:
assumes $(P \sqcap Q)\ x$
shows $P\ x$ **and** $Q\ x$
using *assms* **by** *auto*

end

1.0.3 Bounded Quantifiers

theory *Bounded-Quantifiers*
imports
HOL-Basics-Base
Predicates-Lattice
ML-Unification.ML-Unification-HOL-Setup
begin

consts *ball* :: $'a \Rightarrow ('b \Rightarrow bool) \Rightarrow bool$

open-bundle *ball-syntax-aux*
begin
notation *ball* (‹∀(_)›)
syntax
-ball :: ‹[*idts*, $'a$, *bool*] ⇒ *bool*› (‹(*2*∀ - : -./ -)› *10*)
-ball2 :: ‹[*idts*, $'a$, *bool*] ⇒ *bool*›
end

open-bundle *ball-syntax*
begin
unbundle *ball-syntax-aux*
syntax-consts
-ball -ball2 ⇌ *ball*
translations
$\forall x\ xs : P.\ Q \rightharpoonup CONST\ ball\ P\ (\lambda x.\ \text{-}ball2\ xs\ P\ Q)$
-ball2 $x\ P\ Q \rightharpoonup \forall x : P.\ Q$
$\forall x : P.\ Q \rightleftharpoons CONST\ ball\ P\ (\lambda x.\ Q)$
end

consts *bex* :: $'a \Rightarrow ('b \Rightarrow bool) \Rightarrow bool$

open-bundle *bex-syntax-aux*
begin
notation *bex* (‹∃(_)›)
syntax
-bex :: ‹[*idts*, *'a*, *bool*] ⇒ *bool*› (‹(*2*∃- : -./ -)› *10*)
-bex2 :: ‹[*idts*, *'a*, *bool*] ⇒ *bool*›
end

open-bundle *bex-syntax*
begin
unbundle *bex-syntax-aux*
syntax-consts
-bex -bex2 ⇌ *bex*
translations
∃ *x xs* : *P*. *Q* ⇀ *CONST bex P* (λ*x*. *-bex2 xs P Q*)
-bex2 x P Q ⇀ ∃ *x* : *P*. *Q*
∃ *x* : *P*. *Q* ⇌ *CONST bex P* (λ*x*. *Q*)
end

consts *bex1* :: *'a* ⇒ (*'b* ⇒ *bool*) ⇒ *bool*

open-bundle *bex1-syntax-aux*
begin
notation *bex1* (‹∃!(_)›)
syntax
-bex1 :: ‹[*idts*, *'a*, *bool*] ⇒ *bool*› (‹(*2*∃!- : -./ -)› *10*)
-bex12 :: ‹[*idts*, *'a*, *bool*] ⇒ *bool*›
end

open-bundle *bex1-syntax*
begin
unbundle *bex1-syntax-aux*
syntax-consts
-bex1 -bex12 ⇌ *bex1*
translations
∃!*x xs* : *P*. *Q* ⇀ *CONST bex1 P* (λ*x*. *-bex12 xs P Q*)
-bex12 x P Q ⇀ ∃!*x* : *P*. *Q*
∃!*x* : *P*. *Q* ⇌ *CONST bex1 P* (λ*x*. *Q*)
end

bundle *bounded-quantifier-syntax*
begin
unbundle *ball-syntax* **and** *bex-syntax* **and** *bex1-syntax*
end

definition *ball-pred P Q* ≡ ∀ *x*. *P x* ⟶ *Q x*
adhoc-overloading *ball* ⇌ *ball-pred*

definition *bex-pred* $P\ Q \equiv \exists x.\ P\ x \wedge Q\ x$
adhoc-overloading *bex* $\rightleftharpoons$ *bex-pred*

definition *bex1-pred* $P\ Q \equiv \exists! x.\ P\ x \wedge Q\ x$
adhoc-overloading *bex1* $\rightleftharpoons$ *bex1-pred*

simproc-setup *defined-ball* $(\forall x : P.\ Q\ x \longrightarrow U\ x) =$ ‹
K (*Quantifier1.rearrange-Ball* (*fn ctxt => unfold-tac ctxt* @{*thms ball-pred-def*}))
›
simproc-setup *defined-bex* $(\exists x : P.\ Q\ x \wedge U\ x) =$ ‹
K (*Quantifier1.rearrange-Bex* (*fn ctxt => unfold-tac ctxt* @{*thms bex-pred-def*}))
›

lemma *ballI* [*intro*!]:
assumes $\bigwedge x.\ P\ x \Longrightarrow Q\ x$
shows $\forall x : P.\ Q\ x$
using *assms* **unfolding** *ball-pred-def* **by** *blast*

lemma *ballE* [*elim*]:
assumes $\forall x : P.\ Q\ x$
obtains $\bigwedge x.\ P\ x \Longrightarrow Q\ x$
using *assms* **unfolding** *ball-pred-def* **by** *blast*

lemma *ballE'*:
assumes $\forall x : P.\ Q\ x$
obtains $\neg(P\ x) \mid P\ x\ Q\ x$
using *assms* **by** *blast*

lemma *ballD*: $\forall x : P.\ Q\ x \Longrightarrow P\ x \Longrightarrow Q\ x$
by *blast*

lemma *ball-cong*: $[\![P = P'; \bigwedge x.\ P'\ x \Longrightarrow Q\ x \longleftrightarrow Q'\ x]\!] \Longrightarrow (\forall x : P.\ Q\ x) \longleftrightarrow (\forall x : P'.\ Q'\ x)$
by *auto*

lemma *ball-cong-simp* [*cong*]:
$[\![P = P'; \bigwedge x.\ P'\ x =simp=> Q\ x \longleftrightarrow Q'\ x]\!] \Longrightarrow (\forall x : P.\ Q\ x) \longleftrightarrow (\forall x : P'.\ Q'\ x)$
unfolding *simp-implies-def* **by** (*rule ball-cong*)

ML ‹
structure Simpdata =
struct
open Simpdata
val mksimps-pairs = [(***const-name*** ‹*ball-pred*›, @{*thms ballD*})] @ *mksimps-pairs*
end
open Simpdata

›
declaration ‹*fn - => Simplifier.map-ss (Simplifier.set-mksimps (mksimps mk-simps-pairs))*›

lemma *atomize-ball*: $(\bigwedge x.\ P\ x \Longrightarrow Q\ x) \equiv \mathit{Trueprop}\ (\forall x : P.\ Q\ x)$
by (*simp only*: *ball-pred-def atomize-all atomize-imp*)

declare *atomize-ball*[*symmetric*, *rulify*]
declare *atomize-ball*[*symmetric*, *defn*]

lemma *bexI* [*intro*!]:

assumes $\exists x.\ Q\ x \wedge P\ x$
shows $\exists x : P.\ Q\ x$
using *assms* **unfolding** *bex-pred-def* **by** *blast*

lemma *bexE* [*elim*!]:
assumes $\exists x : P.\ Q\ x$
obtains x **where** $P\ x\ Q\ x$
using *assms* **unfolding** *bex-pred-def* **by** *blast*

lemma *bexD*:
assumes $\exists x : P.\ Q\ x$
shows $\exists x.\ P\ x \wedge Q\ x$
using *assms* **by** *blast*

lemma *bex-cong*:
$[\![P = P'; \bigwedge x.\ P'\ x \Longrightarrow Q\ x \longleftrightarrow Q'\ x]\!] \Longrightarrow (\exists x : P.\ Q\ x) \longleftrightarrow (\exists x : P'.\ Q'\ x)$
by *blast*

lemma *bex-cong-simp* [*cong*]:
$[\![P = P'; \bigwedge x.\ P'\ x =\mathit{simp}\Rightarrow Q\ x \longleftrightarrow Q'\ x]\!] \Longrightarrow (\exists x : P.\ Q\ x) \longleftrightarrow (\exists x : P'.\ Q'\ x)$
unfolding *simp-implies-def* **by** (*rule bex-cong*)

lemma *bex1I* [*intro*]:

assumes $\exists!x.\ Q\ x \wedge P\ x$
shows $\exists!x : P.\ Q\ x$
using *assms* **unfolding** *bex1-pred-def* **by** *blast*

lemma *bex1D* [*dest*!]:
assumes $\exists!x : P.\ Q\ x$
shows $\exists!x.\ P\ x \wedge Q\ x$
using *assms* **unfolding** *bex1-pred-def* **by** *blast*

lemma *bex1-cong*: $[\![P = P'; \bigwedge x.\ P\ x \Longrightarrow Q\ x \longleftrightarrow Q'\ x]\!] \Longrightarrow (\exists!x : P.\ Q\ x) \longleftrightarrow (\exists!x : P'.\ Q'\ x)$
by *blast*

lemma *bex1-cong-simp* [*cong*]:
⟦$P = P'$; $\bigwedge x.\ P\ x$ =*simp*=> $Q\ x \longleftrightarrow Q'\ x$⟧ ⟹ $(\exists!x : P.\ Q\ x) \longleftrightarrow (\exists!x : P'.\ Q'\ x)$
unfolding *simp-implies-def* **by** (*rule bex1-cong*)

lemma *ball-iff-ex-pred* [*iff*]: $(\forall x : P.\ Q) \longleftrightarrow ((\exists x.\ P\ x) \longrightarrow Q)$
by *auto*

lemma *bex-iff-ex-and* [*iff*]: $(\exists x : P.\ Q) \longleftrightarrow ((\exists x.\ P\ x) \wedge Q)$
by *blast*

lemma *ball-eq-imp-iff-imp* [*iff*]: $(\forall x : P.\ x = y \longrightarrow Q\ x) \longleftrightarrow (P\ y \longrightarrow Q\ y)$
by *blast*

lemma *ball-eq-imp-iff-imp'* [*iff*]: $(\forall x : P.\ y = x \longrightarrow Q\ x) \longleftrightarrow (P\ y \longrightarrow Q\ y)$
by *blast*

lemma *bex-eq-iff-pred* [*iff*]: $(\exists x : P.\ x = y) \longleftrightarrow P\ y$
by *blast*

lemma *bex-eq-iff-pred'* [*iff*]: $(\exists x : P.\ y = x) \longleftrightarrow P\ y$
by *blast*

lemma *bex-eq-and-iff-pred* [*iff*]: $(\exists x : P.\ x = y \wedge Q\ x) \longleftrightarrow P\ y \wedge Q\ y$
by *blast*

lemma *bex-eq-and-iff-pred'* [*iff*]: $(\exists x : P.\ y = x \wedge Q\ x) \longleftrightarrow P\ y \wedge Q\ y$
by *blast*

lemma *ball-and-iff-ball-and-ball*: $(\forall x : P.\ Q\ x \wedge U\ x) \longleftrightarrow (\forall x : P.\ Q\ x) \wedge (\forall x : P.\ U\ x)$
by *auto*

lemma *bex-or-iff-bex-or-bex*: $(\exists x : P.\ Q\ x \vee U\ x) \longleftrightarrow (\exists x : P.\ Q\ x) \vee (\exists x : P.\ U\ x)$
by *auto*

lemma *ball-or-iff-ball-or* [*iff*]: $(\forall x : P.\ Q\ x \vee U) \longleftrightarrow ((\forall x : P.\ Q\ x) \vee U)$
by *auto*

lemma *ball-or-iff-or-ball* [*iff*]: $(\forall x : P.\ Q \vee U\ x) \longleftrightarrow (Q \vee (\forall x : P.\ U\ x))$
by *auto*

lemma *ball-imp-iff-imp-ball* [*iff*]: $(\forall x : P.\ Q \longrightarrow U\ x) \longleftrightarrow (Q \longrightarrow (\forall x : P.\ U\ x))$
by *auto*

lemma *bex-and-iff-bex-and* [*iff*]: $(\exists x : P.\ Q\ x \wedge U) \longleftrightarrow ((\exists x : P.\ Q\ x) \wedge U)$

by *auto*

lemma *bex-and-iff-and-bex* [*iff*]: $(\exists\, x : P.\ Q \wedge U\ x) \longleftrightarrow (Q \wedge (\exists\, x : P.\ U\ x))$
by *auto*

lemma *ball-imp-iff-bex-imp* [*iff*]: $(\forall\, x : P.\ Q\ x \longrightarrow U) \longleftrightarrow ((\exists\, x : P.\ Q\ x) \longrightarrow U)$
by *auto*

lemma *not-ball-iff-bex-not* [*iff*]: $(\neg(\forall\, x : P.\ Q\ x)) \longleftrightarrow (\exists\, x : P.\ \neg(Q\ x))$
by *auto*

lemma *not-bex-iff-ball-not* [*iff*]: $(\neg(\exists\, x : P.\ Q\ x)) \longleftrightarrow (\forall\, x : P.\ \neg(Q\ x))$
by *auto*

lemma *bex1-iff-bex-and* [*iff*]: $(\exists!x : P.\ Q) \longleftrightarrow ((\exists!x.\ P\ x) \wedge Q)$
by *auto*

lemma *ball-bottom-eq-top* [*simp*]: $\forall_{\bot} = \top$ **by** *auto*
lemma *bex-bottom-eq-bottom* [*simp*]: $\exists_{\bot} = \bot$ **by** *fastforce*
lemma *bex1-bottom-eq-bottom* [*simp*]: $\exists!_{\bot} = \bot$ **by** *fastforce*

lemma *ball-top-eq-all* [*simp*]: $\forall_{\top} = All$ **by** *fastforce*

lemma *ball-top-eq-all-uhint* [*uhint*]:
assumes $P \equiv (\top :: 'a \Rightarrow bool)$
shows $\forall_P \equiv All$
using *assms* **by** *simp*

lemma *bex-top-eq-ex* [*simp*]: $\exists_{\top} = Ex$ **by** *fastforce*

lemma *bex-top-eq-ex-uhint* [*uhint*]:
assumes $P \equiv (\top :: 'a \Rightarrow bool)$
shows $\exists_P \equiv Ex$
using *assms* **by** *simp*

lemma *bex1-top-eq-ex1* [*simp*]: $\exists!_{\top} = Ex1$ **by** *fastforce*

lemma *bex1-top-eq-ex1-uhint* [*uhint*]:
assumes $P \equiv (\top :: 'a \Rightarrow bool)$
shows $\exists!_P \equiv Ex1$
using *assms* **by** *simp*

end

1.1 Binary Relations

1.1.1 Basic Functions

theory *Binary-Relation-Functions*
imports
Bounded-Quantifiers
ML-Unification.Unify-Resolve-Tactics
begin

Summary Basic functions on binary relations.

Domain, Codomain, and Field

definition *in-dom* $R\ x \equiv \exists\, y.\ R\ x\ y$

lemma *in-domI* [*intro*]:
assumes $R\ x\ y$
shows *in-dom* $R\ x$
using *assms* **unfolding** *in-dom-def* **by** *blast*

lemma *in-domE* [*elim*]:
assumes *in-dom* $R\ x$
obtains y **where** $R\ x\ y$
using *assms* **unfolding** *in-dom-def* **by** *blast*

lemma *in-dom-bottom-eq-bottom* [*simp*]: *in-dom* $\bot = \bot$ **by** *fastforce*
lemma *in-dom-top-eq-top* [*simp*]: *in-dom* $\top = \top$ **by** *fastforce*

lemma *in-dom-sup-eq-in-dom-sup-in-dom* [*simp*]: *in-dom* $(R \sqcup S) =$ *in-dom* $R \sqcup$ *in-dom* S **by** *fastforce*

consts *in-codom-on* :: $'a \Rightarrow 'b \Rightarrow 'c \Rightarrow$ *bool*

definition *in-codom-on-pred* $(P :: 'a \Rightarrow bool)\ R\ y \equiv \exists\, x : P.\ R\ x\ y$
adhoc-overloading *in-codom-on* $\rightleftharpoons$ *in-codom-on-pred*

lemma *in-codom-onI* [*intro*]:
assumes $R\ x\ y$
and $P\ x$
shows *in-codom-on* $P\ R\ y$
using *assms* **unfolding** *in-codom-on-pred-def* **by** *blast*

lemma *in-codom-onE* [*elim*]:
assumes *in-codom-on* $P\ R\ y$
obtains x **where** $P\ x\ R\ x\ y$
using *assms* **unfolding** *in-codom-on-pred-def* **by** *blast*

lemma *in-codom-on-pred-iff-bex-rel*: *in-codom-on P R y* $\longleftrightarrow$ $(\exists x : P.\ R\ x\ y)$ **by** *blast*

lemma *in-codom-bottom-pred-eq-bottom* [*simp*]: *in-codom-on* $\bot = \bot$ **by** *fastforce*
lemma *in-codom-bottom-rel-eq-bottom* [*simp*]: *in-codom-on P* $\bot = \bot$ **by** *fastforce*
lemma *in-codom-top-top-eq* [*simp*]: *in-codom-on* $\top\ \top = \top$ **by** *fastforce*

lemma *in-codom-on-eq-pred-eq* [*simp*]: *in-codom-on* $((=)\ P)\ R = R\ P$
by *auto*

lemma *in-codom-on-sup-pred-eq-in-codom-on-sup-in-codom-on* [*simp*]:
$\textit{in-codom-on}\ (P \sqcup Q) = \textit{in-codom-on}\ P \sqcup \textit{in-codom-on}\ Q$
by *fastforce*

lemma *in-codom-on-sup-rel-eq-in-codom-on-sup-in-codom-on* [*simp*]:
$\textit{in-codom-on}\ P\ (R \sqcup S) = \textit{in-codom-on}\ P\ R \sqcup \textit{in-codom-on}\ P\ S$
by *fastforce*

definition $\textit{in-codom} \equiv \textit{in-codom-on}\ (\top :: 'a \Rightarrow \textit{bool})$

lemma *in-codom-eq-in-codom-on-top*: *in-codom* = *in-codom-on* $\top$ **unfolding** *in-codom-def*
by *auto*

lemma *in-codom-eq-in-codom-on-top-uhint* [*uhint*]:
assumes $P \equiv (\top :: 'a \Rightarrow \textit{bool})$
shows $\textit{in-codom} \equiv \textit{in-codom-on}\ P$
using *assms* **by** (*simp add*: *in-codom-eq-in-codom-on-top*)

lemma *in-codomI* [*intro*]:
assumes *R x y*
shows *in-codom R y*
using *assms* **by** (*urule in-codom-onI*) *simp*

lemma *in-codomE* [*elim*]:
assumes *in-codom R y*
obtains *x* **where** *R x y*
using *assms* **by** (*urule* (*e*) *in-codom-onE*)

definition $\textit{in-field}\ R\ x \equiv \textit{in-dom}\ R\ x \vee \textit{in-codom}\ R\ x$

lemma *in-field-if-in-dom*:
assumes *in-dom R x*
shows *in-field R x*
unfolding *in-field-def* **using** *assms* **by** *blast*

lemma *in-field-if-in-codom*:
assumes *in-codom R x*
shows *in-field R x*
unfolding *in-field-def* **using** *assms* **by** *blast*

lemma *in-fieldE* [*elim*]:
assumes *in-field R x*
obtains (*in-dom*) *x'* **where** *R x x'* | (*in-codom*) *x'* **where** *R x' x*
using *assms* **unfolding** *in-field-def* **by** *blast*

lemma *in-fieldE'*:
assumes *in-field R x*
obtains (*in-dom*) *in-dom R x* | (*in-codom*) *in-codom R x*
using *assms* **by** *blast*

lemma *in-fieldI* [*intro*]:
assumes *R x y*
shows *in-field R x in-field R y*
using *assms* **by** (*auto intro*: *in-field-if-in-dom in-field-if-in-codom*)

lemma *in-field-iff-in-dom-or-in-codom*:
in-field R x $\longleftrightarrow$ *in-dom R x* $\vee$ *in-codom R x*
by *blast*

lemma *in-field-eq-in-dom-if-in-codom-eq-in-dom*:
assumes *in-codom R* = *in-dom R*
shows *in-field R* = *in-dom R*
using *assms* **by** (*intro ext*) (*auto elim*: *in-fieldE'*)

lemma *in-field-bottom-eq-bottom* [*simp*]: *in-field* $\bot$ = $\bot$ **by** *fastforce*
lemma *in-field-top-eq-top* [*simp*]: *in-field* $\top$ = $\top$ **by** *fastforce*

lemma *in-field-sup-eq-in-field-sup-in-field* [*simp*]: *in-field* (*R* $\sqcup$ *S*) = *in-field R* $\sqcup$ *in-field S*
by *fastforce*

Composition

consts *rel-comp* :: *'a* $\Rightarrow$ *'b* $\Rightarrow$ *'c*

open-bundle *rel-comp-syntax*
begin
notation *rel-comp* (**infixl** ‹$\circ\circ$› *55*)
end

definition *rel-comp-rel R S x y* $\equiv \exists z.$ *R x z* $\wedge$ *S z y*
adhoc-overloading *rel-comp* $\rightleftharpoons$ *rel-comp-rel*

lemma *rel-compI* [*intro*]:
assumes *R x y*
and *S y z*
shows (*R* $\circ\circ$ *S*) *x z*
using *assms* **unfolding** *rel-comp-rel-def* **by** *blast*

lemma *rel-compE* [*elim*]:
assumes $(R \circ\circ S)\ x\ y$
obtains z **where** $R\ x\ z\ S\ z\ y$
using *assms* **unfolding** *rel-comp-rel-def* **by** *blast*

lemma *rel-comp-assoc*: $R \circ\circ (S \circ\circ T) = (R \circ\circ S) \circ\circ T$
by (*intro ext*) *blast*

lemma *in-dom-if-in-dom-rel-comp*:
assumes *in-dom* $(R \circ\circ S)\ x$
shows *in-dom* $R\ x$
using *assms* **by** *blast*

lemma *in-codom-if-in-codom-rel-comp*:
assumes *in-codom* $(R \circ\circ S)\ y$
shows *in-codom* $S\ y$
using *assms* **by** *blast*

lemma *in-field-compE* [*elim*]:
assumes *in-field* $(R \circ\circ S)\ x$
obtains (*in-dom*) *in-dom* $R\ x$ | (*in-codom*) *in-codom* $S\ x$
using *assms* **by** *blast*

Inverse

consts *rel-inv* :: $'a \Rightarrow 'b$

open-bundle *rel-inv-syntax*
begin
notation *rel-inv* ($\langle(\text{-}^{-1})\rangle$ [*1000*])
end

definition *rel-inv-rel* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow 'b \Rightarrow 'a \Rightarrow bool$
where *rel-inv-rel* $R\ x\ y \equiv R\ y\ x$
adhoc-overloading *rel-inv* $\rightleftharpoons$ *rel-inv-rel*

lemma *rel-invI* [*intro*]:
assumes $R\ x\ y$
shows $R^{-1}\ y\ x$
using *assms* **unfolding** *rel-inv-rel-def* **.**

lemma *rel-invD* [*dest*]:
assumes $R^{-1}\ x\ y$
shows $R\ y\ x$
using *assms* **unfolding** *rel-inv-rel-def* **.**

lemma *rel-inv-iff-rel* [*simp*]: $R^{-1}\ x\ y \longleftrightarrow R\ y\ x$
by *blast*

lemma *in-dom-rel-inv-eq-in-codom* [*simp*]: *in-dom* R^{-1} = *in-codom* R
by (*intro ext*) *blast*

lemma *in-codom-rel-inv-eq-in-dom* [*simp*]: *in-codom* R^{-1} = *in-dom* R
by (*intro ext*) *blast*

lemma *in-field-rel-inv-eq-in-field* [*simp*]: *in-field* R^{-1} = *in-field* R
by (*intro ext*) *auto*

lemma *rel-inv-comp-eq* [*simp*]: $(R \circ\circ S)^{-1} = S^{-1} \circ\circ R^{-1}$
by (*intro ext*) *blast*

lemma *rel-inv-inv-eq-self* [*simp*]: $R^{-1-1} = R$
by *blast*

lemma *rel-inv-eq-iff-eq* [*iff*]: $R^{-1} = S^{-1} \longleftrightarrow R = S$
by (*blast dest*: *fun-cong*)

lemma *rel-inv-le-rel-inv-iff* [*iff*]: $R^{-1} \leq S^{-1} \longleftrightarrow R \leq S$
by (*blast intro*: *predicate2I dest*: *predicate2D*)

lemma *rel-inv-top-eq* [*simp*]: $\top^{-1} = \top$ **by** *fastforce*
lemma *rel-inv-bottom-eq* [*simp*]: $\bot^{-1} = \bot$ **by** *fastforce*

Restrictions

consts *rel-if* :: *bool* ⇒ *'a* ⇒ *'a*

open-bundle *rel-if-syntax*
begin
notation (**output**) *rel-if* (**infixl** ‹⟶› *50*)
end

definition *rel-if-rel B R x y* ≡ *B* ⟶ *R x y*
adhoc-overloading *rel-if* ⇌ *rel-if-rel*

lemma *rel-if-eq-rel-if-pred* [*simp*]:
assumes *B*
shows (*rel-if B R*) = *R*
unfolding *rel-if-rel-def* **using** *assms* **by** *blast*

lemma *rel-if-eq-top-if-not-pred* [*simp*]:
assumes ¬*B*
shows (*rel-if B R*) = (λ- -. *True*)
unfolding *rel-if-rel-def* **using** *assms* **by** *blast*

lemma *rel-if-if-impI* [*intro*]:
assumes *B* ⟹ *R x y*

shows (*rel-if B R*) *x y*
unfolding *rel-if-rel-def* **using** *assms* **by** *blast*

lemma *rel-ifE* [*elim*]:
assumes (*rel-if B R*) *x y*
obtains $\neg B \mid B\ R\ x\ y$
using *assms* **unfolding** *rel-if-rel-def* **by** *blast*

lemma *rel-ifD*:
assumes (*rel-if B R*) *x y*
and *B*
shows *R x y*
using *assms* **by** *blast*

consts *rel-restrict-left* :: $'a \Rightarrow 'b \Rightarrow 'a$
consts *rel-restrict-right* :: $'a \Rightarrow 'b \Rightarrow 'a$
consts *rel-restrict* :: $'a \Rightarrow 'b \Rightarrow 'a$

open-bundle *rel-restrict-syntax*
begin
notation *rel-restrict-left* ($\langle$(-)$\restriction$(_)$\rangle$ [*1000*])
notation *rel-restrict-right* ($\langle$(-)$\restriction$(_)$\rangle$ [*1000*])
notation *rel-restrict* ($\langle$(-)$\restriction$(_)$\rangle$ [*1000*])
end

definition *rel-restrict-left-pred R P x y* $\equiv P\ x \wedge R\ x\ y$
adhoc-overloading *rel-restrict-left* $\rightleftharpoons$ *rel-restrict-left-pred*
definition *rel-restrict-right-pred R P x y* $\equiv P\ y \wedge R\ x\ y$
adhoc-overloading *rel-restrict-right* $\rightleftharpoons$ *rel-restrict-right-pred*
definition *rel-restrict-pred R P* $\equiv R\restriction_P\restriction_P$
adhoc-overloading *rel-restrict* $\rightleftharpoons$ *rel-restrict-pred*

lemma *rel-restrict-leftI* [*intro*]:
assumes *R x y*
and *P x*
shows $R\restriction_P\ x\ y$
using *assms* **unfolding** *rel-restrict-left-pred-def* **by** *blast*

lemma *rel-restrict-leftE* [*elim*]:
assumes $R\restriction_P\ x\ y$
obtains *P x R x y*
using *assms* **unfolding** *rel-restrict-left-pred-def* **by** *blast*

lemma *rel-restrict-left-cong*:
assumes $\bigwedge x.\ P\ x \longleftrightarrow P'\ x$
and $\bigwedge x\ y.\ P'\ x \Longrightarrow R\ x\ y \longleftrightarrow R'\ x\ y$
shows $R\restriction_P = R'\restriction_{P'}$
using *assms* **by** (*intro ext iffI*) *blast+*

lemma *rel-restrict-left-restrict-left-eq-restrict-left* [*simp*]: $R{\restriction}_P{\restriction}_P = R{\restriction}_P$
by *blast*

lemma *rel-restrict-left-top-eq* [*simp*]: $(R :: 'a \Rightarrow 'b \Rightarrow bool){\restriction}_{\top :: 'a \Rightarrow bool} = R$
by (*intro ext rel-restrict-leftI*) *auto*

lemma *rel-restrict-left-top-eq-uhint* [*uhint*]:
assumes $P \equiv (\top :: 'a \Rightarrow bool)$
shows $(R :: 'a \Rightarrow 'b \Rightarrow bool){\restriction}_P \equiv R$
using *assms* **by** *simp*

lemma *rel-restrict-left-bottom-eq* [*simp*]: $(R :: 'a \Rightarrow 'b \Rightarrow bool){\restriction}_{\bot :: 'a \Rightarrow bool} = \bot$
by (*intro ext rel-restrict-leftI*) *auto*

lemma *bottom-restrict-left-eq* [*simp*]: $\bot{\restriction}_{P :: 'a \Rightarrow bool} = \bot$
by *fastforce*

lemma *rel-restrict-left-le-self*: $(R :: 'a \Rightarrow 'b \Rightarrow bool){\restriction}_{(P :: 'a \Rightarrow bool)} \leq R$
by (*blast intro*: *predicate2I dest*: *predicate2D*)

lemma *le-rel-restrict-left-self-if-in-dom-le*:
assumes *in-dom* $R \leq P$
shows $R \leq (R :: 'a \Rightarrow 'b \Rightarrow bool){\restriction}_{(P :: 'a \Rightarrow bool)}$
using *assms* **by** (*blast intro*: *predicate2I dest*: *predicate2D predicate1D*)

corollary *rel-restrict-left-eq-self-if-in-dom-le* [*simp*]:
assumes *in-dom* $R \leq P$
shows $R{\restriction}_P = R$
using *assms rel-restrict-left-le-self le-rel-restrict-left-self-if-in-dom-le* **by** (*intro antisym*)

lemma *ex-rel-restrict-left-iff-in-codom-on* [*iff*]: $(\exists\, x.\ R{\restriction}_P\ x\ y) \longleftrightarrow$ (*in-codom-on* $P\ R\ y$) **by** *blast*

lemma *in-dom-rel-restrict-leftI* [*intro*]:
assumes $R\ x\ y$
and $P\ x$
shows *in-dom* $R{\restriction}_P\ x$
using *assms* **by** *blast*

lemma *in-codom-rel-restrict-leftI* [*intro*]:
assumes $R\ x\ y$
and $P\ x$
shows *in-codom* $R{\restriction}_P\ y$
using *assms* **by** *blast*

lemma *rel-restrict-rightI* [*intro*]:
assumes $R\ x\ y$

and $P\ y$
shows $R{\restriction}_P\ x\ y$
using *assms* **unfolding** *rel-restrict-right-pred-def* **by** *blast*

lemma *rel-restrict-rightE* [*elim*]:
assumes $R{\restriction}_P\ x\ y$
obtains $P\ y\ R\ x\ y$
using *assms* **unfolding** *rel-restrict-right-pred-def* **by** *blast*

lemma *rel-restrict-right-cong*:
assumes $\bigwedge y.\ P\ y \longleftrightarrow P'\ y$
and $\bigwedge x\ y.\ P'\ y \Longrightarrow R\ x\ y \longleftrightarrow R'\ x\ y$
shows $R{\restriction}_P = R'{\restriction}_{P'}$
using *assms* **by** (*intro ext iffI*) *blast+*

lemma *rel-restrict-right-restrict-right-eq-restrict-right* [*simp*]: $R{\restriction}_P{\restriction}_P = R{\restriction}_P$
by *blast*

lemma *rel-restrict-right-top-eq* [*simp*]: $(R :: 'a \Rightarrow 'b \Rightarrow bool){\restriction}_{\top\ ::'b\ \Rightarrow\ bool} = R$
by (*intro ext rel-restrict-rightI*) *auto*

lemma *rel-restrict-right-top-eq-uhint* [*uhint*]:
assumes $P \equiv (\top ::'b \Rightarrow bool)$
shows $(R :: 'a \Rightarrow 'b \Rightarrow bool){\restriction}_P \equiv R$
using *assms* **by** *simp*

lemma *rel-restrict-right-bottom-eq* [*simp*]: $(R :: 'a \Rightarrow 'b \Rightarrow bool){\restriction}_{\bot\ ::\ 'b\ \Rightarrow\ bool} = \bot$
by (*intro ext rel-restrict-rightI*) *auto*

lemma *bottom-restrict-right-eq* [*simp*]: $\bot{\restriction}_{P\ ::\ 'b\ \Rightarrow\ bool} = \bot$
by *fastforce*

lemma *rel-restrict-right-le-self*: $(R :: 'a \Rightarrow 'b \Rightarrow bool){\restriction}_{(P\ ::\ 'b\ \Rightarrow\ bool)} \leq R$
by (*blast intro*: *predicate2I dest*: *predicate2D*)

lemma *le-rel-restrict-right-self-if-in-codom-le*:
assumes *in-codom* $R \leq P$
shows $R \leq (R :: 'a \Rightarrow 'b \Rightarrow bool){\restriction}_{(P\ ::\ 'b\ \Rightarrow\ bool)}$
using *assms* **by** (*blast intro*: *predicate2I dest*: *predicate2D predicate1D*)

corollary *rel-restrict-right-eq-self-if-in-codom-le* [*simp*]:
assumes *in-codom* $R \leq P$
shows $R{\restriction}_P = R$
using *assms rel-restrict-right-le-self le-rel-restrict-right-self-if-in-codom-le*
by (*intro antisym*)

context
fixes $R :: 'a \Rightarrow 'b \Rightarrow bool$

begin

lemma *rel-restrict-right-eq*: $R{\restriction}_P :: 'b \Rightarrow bool = ((R^{-1}){\restriction}_P)^{-1}$
by *blast*

lemma *rel-inv-restrict-right-rel-inv-eq-restrict-left* [*simp*]: $((R^{-1}){\restriction}_P :: 'a \Rightarrow bool)^{-1}$
$= R{\restriction}_P$
by *blast*

lemma *rel-restrict-right-iff-restrict-left*: $R{\restriction}_P :: 'b \Rightarrow bool\ x\ y \longleftrightarrow (R^{-1}){\restriction}_P\ y\ x$
unfolding *rel-restrict-right-eq* **by** *simp*

end

lemma *rel-inv-rel-restrict-left-inv-rel-restrict-left-eq*:
fixes $R :: 'a \Rightarrow 'b \Rightarrow bool$ **and** $P :: 'a \Rightarrow bool$ **and** $Q :: 'b \Rightarrow bool$
shows $(((R{\restriction}_P)^{-1}){\restriction}_Q)^{-1} = (((R^{-1}){\restriction}_Q)^{-1}){\restriction}_P$
by (*intro ext iffI rel-restrict-leftI rel-invI*) *auto*

lemma *rel-restrict-left-right-eq-restrict-right-left*:
fixes $R :: 'a \Rightarrow 'b \Rightarrow bool$ **and** $P :: 'a \Rightarrow bool$ **and** $Q :: 'b \Rightarrow bool$
shows $R{\restriction}_P{\restriction}_Q = R{\restriction}_Q{\restriction}_P$
unfolding *rel-restrict-right-eq*
by (*fact rel-inv-rel-restrict-left-inv-rel-restrict-left-eq*)

lemma *rel-restrictI* [*intro*]:
assumes $R{\restriction}_P{\restriction}_P\ x\ y$
shows $R{\uparrow}_P\ x\ y$
using *assms* **unfolding** *rel-restrict-pred-def* **by** *blast*

lemma *rel-restrictD* [*dest*!]:
assumes $R{\uparrow}_P\ x\ y$
shows $R{\restriction}_P{\restriction}_P\ x\ y$
using *assms* **unfolding** *rel-restrict-pred-def* **by** *blast*

lemma *rel-restrict-eq-restrict-left-right* [*simp*]: $R{\uparrow}_P = R{\restriction}_P{\restriction}_P$
by (*intro ext*) *auto*

lemma *rel-restrict-cong*:
assumes $\bigwedge y.\ P\ y \longleftrightarrow P'\ y$
and $\bigwedge x\ y.\ P'\ x \Longrightarrow P'\ y \Longrightarrow R\ x\ y \longleftrightarrow R'\ x\ y$
shows $R{\uparrow}_P = R'{\uparrow}_{P'}$
using *assms* **by** (*intro ext*) *fast*

lemma *rel-restrict-restrict-eq-restrict* [*simp*]: $R{\uparrow}_P{\uparrow}_P = R{\uparrow}_P$
by *blast*

lemma *rel-restrict-top-eq* [*simp*]: $(R :: 'a \Rightarrow 'a \Rightarrow bool){\uparrow}_\top :: 'a \Rightarrow bool = R$
by (*intro ext rel-restrictI*) *auto*

lemma *rel-restrict-top-eq-uhint* [*uhint*]:
assumes $P \equiv (\top :: 'a \Rightarrow bool)$
shows $(R :: 'a \Rightarrow 'a \Rightarrow bool)\restriction_P \equiv R$
using *assms* **by** *simp*

lemma *rel-restrict-bottom-eq* [*simp*]: $(R :: 'a \Rightarrow 'a \Rightarrow bool)\restriction_{\bot :: 'a \Rightarrow bool} = \bot$
by (*intro ext rel-restrictI*) *auto*

lemma *bottom-restrict-eq* [*simp*]: $\bot\restriction_{P :: 'a \Rightarrow bool} = \bot$
by *fastforce*

lemma *rel-restrict-le-self*: $(R :: 'a \Rightarrow 'a \Rightarrow bool)\restriction_{(P :: 'a \Rightarrow bool)} \leq R$
by (*auto intro*: *predicate2I dest*: *predicate2D*)

lemma *le-rel-restrict-self-if-in-field-le*:
assumes *in-field* $R \leq P$
shows $R \leq (R :: 'a \Rightarrow 'a \Rightarrow bool)\restriction_{(P :: 'a \Rightarrow bool)}$
using *assms* **by** (*blast intro*: *predicate2I dest*: *predicate2D predicate1D*)

corollary *rel-restrict-eq-self-if-in-field-le* [*simp*]:
assumes *in-field* $R \leq P$
shows $R\restriction_P = R$
using *assms rel-restrict-le-self le-rel-restrict-self-if-in-field-le*
by (*intro antisym*)

Mappers

definition *rel-bimap f g* $(R :: 'a \Rightarrow 'b \Rightarrow bool)$ *x y* $\equiv R\ (f\ x)\ (g\ y)$

lemma *rel-bimap-eq* [*simp*]: *rel-bimap f g R x y* $= R\ (f\ x)\ (g\ y)$
unfolding *rel-bimap-def* **by** *simp*

definition *rel-map f R* $\equiv$ *rel-bimap f f R*

lemma *rel-bimap-self-eq-rel-map* [*simp*]: *rel-bimap f f R* $=$ *rel-map f R*
unfolding *rel-map-def* **by** *simp*

lemma *rel-map-eq* [*simp*]: *rel-map f R x y* $= R\ (f\ x)\ (f\ y)$
by (*simp only*: *rel-bimap-self-eq-rel-map*[*symmetric*] *rel-bimap-eq*)

end

1.1.2 Order

theory *Binary-Relations-Order-Base*
imports
Binary-Relation-Functions
HOL.Orderings

begin

lemma *le-relI* [*intro*]:
assumes $\bigwedge x\ y.\ R\ x\ y \Longrightarrow S\ x\ y$
shows $R \leq S$
using *assms* **by** (*rule predicate2I*)

lemma *le-relD* [*dest*]:
assumes $R \leq S$
and $R\ x\ y$
shows $S\ x\ y$
using *assms* **by** (*rule predicate2D*)

lemma *le-relE*:
assumes $R \leq S$
obtains $\bigwedge x\ y.\ \ R\ x\ y \Longrightarrow S\ x\ y$
using *assms* **by** *blast*

end

1.2 Functions

1.2.1 Basic Functions

theory *Functions-Base*
imports
Binary-Relation-Functions
begin

definition $id\ x \equiv x$

lemma *id-eq-self* [*simp*]: $id\ x = x$
unfolding *id-def* **..**

lemma *rel-map-id-eq-self* [*simp*]: *rel-map id R* = *R* **by** (*intro ext*) *auto*

consts *comp* :: $'a \Rightarrow 'b \Rightarrow 'c$

open-bundle *comp-syntax*
begin
notation *comp* (**infixl** ‹○› *55*)
end

definition *comp-fun f g x* ≡ *f* (*g x*)
adhoc-overloading *comp* ⇌ *comp-fun*

lemma *comp-eq* [*simp*]: (*f* ∘ *g*) *x* = *f* (*g x*)
unfolding *comp-fun-def* **..**

lemma *id-comp-eq* [*simp*]: $id \circ f = f$
by (*rule ext*) *simp*

lemma *comp-id-eq* [*simp*]: $f \circ id = f$
by (*rule ext*) *simp*

lemma *bottom-comp-eq* [*simp*]: $\bot \circ f = \bot$ **by** *auto*
lemma *top-comp-eq* [*simp*]: $\top \circ f = \top$ **by** *auto*

definition *dep-fun-map f g h x* $\equiv g\ x\ (f\ x)\ (h\ (f\ x))$

definition *fun-map f g h* $\equiv$ *dep-fun-map f* $(\lambda\text{-}\ \text{-}.\ g)\ h$

open-bundle *dep-fun-map-syntax*
begin
notation *fun-map* (**infixr** $\langle\rightsquigarrow\rangle$ *40*)
syntax
-dep-fun-map :: *idt* $\Rightarrow$ $('a \Rightarrow 'b) \Rightarrow ('c \Rightarrow 'd) \Rightarrow ('b \Rightarrow 'c) \Rightarrow$
$('a \Rightarrow 'd)$ $(\langle'(\text{-}/ : / \text{-}')\rightsquigarrow (\text{-})\rangle\ [41, 41, 40]\ 40)$
syntax-consts *-dep-fun-map* $\rightleftharpoons$ *dep-fun-map*
translations
$(x : f) \rightsquigarrow g \rightleftharpoons CONST\ dep\text{-}fun\text{-}map\ f\ (\lambda x.\ g)$
end

lemma *fun-map-eq-dep-fun-map*: $(f \rightsquigarrow g) = ((\text{-} :\ f) \rightsquigarrow (\lambda\text{-}.\ g))$
unfolding *fun-map-def* **by** *simp*

lemma *fun-map-eq-dep-fun-map-uhint* [*uhint*]:
assumes $f \equiv f'$
and $g' \equiv (\lambda\text{-}\ \text{-}.\ g)$
shows $(f \rightsquigarrow g) = ((x :\ f') \rightsquigarrow g'\ x)$
using *assms* **by** (*simp add*: *fun-map-eq-dep-fun-map*)

lemma *dep-fun-map-eq* [*simp*]: $((x : f) \rightsquigarrow g\ x)\ h\ x = g\ x\ (f\ x)\ (h\ (f\ x))$
unfolding *dep-fun-map-def* **..**

lemma *fun-map-eq-comp* [*simp*]: $(f \rightsquigarrow g)\ h = g \circ h \circ f$
by (*intro ext*) (*urule dep-fun-map-eq*)

lemma *fun-map-eq* [*simp*]: $(f \rightsquigarrow g)\ h\ x = g\ (h\ (f\ x))$
unfolding *fun-map-eq-comp* **by** *simp*

lemma *fun-map-id-eq-comp* [*simp*]: *fun-map id* $= (\circ)$
by (*intro ext*) *simp*

lemma *fun-map-id-eq-comp'* [*simp*]: $(f \rightsquigarrow id)\ h = h \circ f$
by (*intro ext*) *simp*

consts *has-inverse-on* :: $'a \Rightarrow 'b \Rightarrow 'c \Rightarrow$ *bool*

definition *has-inverse-on-pred* (P :: $'a \Rightarrow$ *bool*) $\equiv$ *in-codom-on* $P \circ ((\circ)\ (=))$
adhoc-overloading *has-inverse-on* $\rightleftharpoons$ *has-inverse-on-pred*

lemma *has-inverse-on-pred-eq-in-codom-on*: *has-inverse-on* P = *in-codom-on* $P \circ ((\circ)\ (=))$
  unfolding *has-inverse-on-pred-def* **by** *auto*

lemma *has-inverse-onI* [*intro*]:
  assumes $P\ x$
  shows *has-inverse-on* $P\ f\ (f\ x)$
  using *assms* **unfolding** *has-inverse-on-pred-def* **by** *auto*

lemma *has-inverse-on-if-pred-if-eq*:
  assumes $y = f\ x$
  and $P\ x$
  shows *has-inverse-on* $P\ f\ y$
  using *assms* **by** *auto*

lemma *has-inverse-onE* [*elim*]:
  assumes *has-inverse-on* $P\ f\ y$
  obtains x **where** $P\ x\ y = f\ x$
  using *assms* **unfolding** *has-inverse-on-pred-def* **by** *auto*

context
  notes *has-inverse-on-pred-eq-in-codom-on*[*simp*]
begin

lemma *has-inverse-on-bottom-eq* [*simp*]: *has-inverse-on* $\bot = \bot$
  by (*urule in-codom-bottom-pred-eq-bottom*)

lemma *has-inverse-on-eq-pred-eq-eq-app* [*simp*]: *has-inverse-on* $((=)\ P)\ f = (=)\ (f\ P)$
  by (*urule in-codom-on-eq-pred-eq*)

lemma *has-inverse-on-sup-eq-has-inverse-on-sup-has-inverse-on* [*simp*]:
  has-inverse-on $(P \sqcup Q)$ = *has-inverse-on* $P \sqcup$ *has-inverse-on* Q
 supply *ext*[*simp*] **by** (*urule in-codom-on-sup-pred-eq-in-codom-on-sup-in-codom-on*)

end

definition *has-inverse* $\equiv$ *has-inverse-on* $\top$

lemma *has-inverse-eq-has-inverse-on-top*: *has-inverse* = *has-inverse-on* $\top$
  unfolding *has-inverse-def* **by** *simp*

lemma *has-inverse-eq-has-inverse-on-top-uhint* [*uhint*]:
  assumes $P \equiv \top$

shows *has-inverse* ≡ *has-inverse-on P*
using *assms* **by** (*simp add*: *has-inverse-eq-has-inverse-on-top*)

lemma *has-inverse-iff-has-inverse-on-top*: *has-inverse f y* ⟷ *has-inverse-on* ⊤ *f y*
by (*simp add*: *has-inverse-eq-has-inverse-on-top*)

lemma *has-inverseI* [*intro*]:
assumes *y* = *f x*
shows *has-inverse f y*
using *assms* **by** (*urule has-inverse-on-if-pred-if-eq*) *simp*

lemma *has-inverseE* [*elim*]:
assumes *has-inverse f y*
obtains *x* **where** *y* = *f x*
using *assms* **by** (*urule* (*e*) *has-inverse-onE*)

consts *image* :: *'a* ⇒ *'b* ⇒ *'c*

definition *image-pred* (*f* :: *'a* ⇒ *'b*) (*P* :: *'a* ⇒ *bool*) ≡ *has-inverse-on P f*
adhoc-overloading *image* ⇌ *image-pred*

lemma *image-pred-eq-has-inverse-on* [*simp*]: *image f P* = *has-inverse-on P f*
unfolding *image-pred-def* **by** *simp*

lemma *image-pred-eq-has-inverse-on-uhint* [*uhint*]:
assumes *f* ≡ *f'*
and *P* ≡ *P'*
shows *image f P* ≡ *has-inverse-on P' f'*
using *assms* **by** *simp*

definition *equaliser f g* ≡ λ*x*. *f x* = *g x*

lemma *equaliserI* [*intro*]:
assumes *f x* = *g x*
shows *equaliser f g x*
using *assms* **unfolding** *equaliser-def* **by** *auto*

lemma *equaliserD* [*dest*]:
assumes *equaliser f g x*
shows *f x* = *g x*
using *assms* **unfolding** *equaliser-def* **by** *auto*

end

1.2.2 Relators

theory *Function-Relators*
imports

Binary-Relation-Functions
Functions-Base
Predicates-Lattice
ML-Unification.ML-Unification-HOL-Setup
ML-Unification.Unify-Resolve-Tactics
begin

Summary Introduces the concept of function relators. The slogan of function relators is "related functions map related inputs to related outputs".

consts *Dep-Fun-Rel* :: $'a \Rightarrow 'b \Rightarrow 'c \Rightarrow 'd \Rightarrow bool$
consts *Fun-Rel* :: $'a \Rightarrow 'b \Rightarrow 'c \Rightarrow 'd \Rightarrow bool$

open-bundle *Dep-Fun-Rel-syntax*
begin
notation *Fun-Rel* (**infixr** $\langle\Rrightarrow\rangle$ *50*)
syntax
-Dep-Fun-Rel-rel :: *idt* $\Rightarrow$ *idt* $\Rightarrow 'a \Rightarrow 'b \Rightarrow 'c \Rightarrow 'd \Rightarrow bool$
($\langle'(-/\ -/\ ::/\ -'\rangle \Rrightarrow (-)\rangle$ [*51*, *51*, *50*, *50*] *50*)
-Dep-Fun-Rel-rel-if :: *idt* $\Rightarrow$ *idt* $\Rightarrow 'a \Rightarrow$ *bool* $\Rightarrow 'b \Rightarrow 'c \Rightarrow 'd \Rightarrow$ *bool*
($\langle'(-/\ -/\ ::/\ -/\ |/\ -'\rangle \Rrightarrow (-)\rangle$ [*51*, *51*, *50*, *50*, *50*] *50*)
-Dep-Fun-Rel-pred :: *idt* $\Rightarrow 'a \Rightarrow 'b \Rightarrow 'c \Rightarrow 'd \Rightarrow bool$
($\langle'(-/\ :/\ -'\rangle \Rrightarrow (-)\rangle$ [*51*, *50*, *50*] *50*)
-Dep-Fun-Rel-pred-if :: *idt* $\Rightarrow 'a \Rightarrow$ *bool* $\Rightarrow 'b \Rightarrow 'c \Rightarrow 'd \Rightarrow bool$
($\langle'(-/\ :/\ -/\ |/\ -'\rangle \Rrightarrow (-)\rangle$ [*51*, *50*, *50*, *50*] *50*)
syntax-consts
-Dep-Fun-Rel-rel -Dep-Fun-Rel-rel-if -Dep-Fun-Rel-pred -Dep-Fun-Rel-pred-if $\rightleftharpoons$ *Dep-Fun-Rel*
translations
$(x\ y :: R) \Rrightarrow S \rightleftharpoons$ *CONST Dep-Fun-Rel* $R\ (\lambda x\ y.\ S)$
$(x\ y :: R \mid B) \Rrightarrow S \rightleftharpoons$ *CONST Dep-Fun-Rel* $R\ (\lambda x\ y.$ *CONST rel-if* $B\ S)$
$(x : P) \Rrightarrow R \rightleftharpoons$ *CONST Dep-Fun-Rel* $P\ (\lambda x.\ R)$
$(x : P \mid B) \Rrightarrow R \rightleftharpoons$ *CONST Dep-Fun-Rel* $P\ (\lambda x.$ *CONST rel-if* $B\ R)$
end

definition *Dep-Fun-Rel-rel* $R\ S\ f\ g \equiv \forall x\ y.\ R\ x\ y \longrightarrow S\ x\ y\ (f\ x)\ (g\ y)$
adhoc-overloading *Dep-Fun-Rel* $\rightleftharpoons$ *Dep-Fun-Rel-rel*

definition *Fun-Rel-rel* $(R :: 'a \Rightarrow 'b \Rightarrow bool)\ (S :: 'c \Rightarrow 'd \Rightarrow bool) \equiv$
$((\text{-}\ \text{-} :: R) \Rrightarrow S) :: ('a \Rightarrow 'c) \Rightarrow ('b \Rightarrow 'd) \Rightarrow bool$
adhoc-overloading *Fun-Rel* $\rightleftharpoons$ *Fun-Rel-rel*

definition *Dep-Fun-Rel-pred* $P\ R\ f\ g \equiv \forall x.\ P\ x \longrightarrow R\ x\ (f\ x)\ (g\ x)$
adhoc-overloading *Dep-Fun-Rel* $\rightleftharpoons$ *Dep-Fun-Rel-pred*

definition *Fun-Rel-pred* $(P :: 'a \Rightarrow bool)\ (R :: 'b \Rightarrow 'c \Rightarrow bool) \equiv$
$((\text{-} : P) \Rrightarrow R) :: ('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'c) \Rightarrow bool$
adhoc-overloading *Fun-Rel* $\rightleftharpoons$ *Fun-Rel-pred*

lemma *Fun-Rel-rel-eq-Dep-Fun-Rel-rel*:

((R :: 'a ⇒ 'b ⇒ bool) ⇛ (S :: 'c ⇒ 'd ⇒ bool)) = ((- - :: R) ⇛ S)
unfolding *Fun-Rel-rel-def* **by** *simp*

lemma *Fun-Rel-rel-eq-Dep-Fun-Rel-rel-uhint* [*uhint*]:
assumes $R \equiv R'$
and $S' \equiv (\lambda(\text{-} :: 'a)\ (\text{-} :: 'b).\ S)$
shows ((R :: 'a ⇒ 'b ⇒ bool) ⇛ (S :: 'c ⇒ 'd ⇒ bool)) = ((x y :: R′) ⇛ S′ x y)
using *assms* **by** (*simp add*: *Fun-Rel-rel-eq-Dep-Fun-Rel-rel*)

lemma *Fun-Rel-rel-iff-Dep-Fun-Rel-rel*:
((R :: 'a ⇒ 'b ⇒ bool) ⇛ (S :: 'c ⇒ 'd ⇒ bool)) f g ⟷ ((- - :: R) ⇛ S) f g
by (*simp add*: *Fun-Rel-rel-eq-Dep-Fun-Rel-rel*)

lemma *Fun-Rel-pred-eq-Dep-Fun-Rel-pred*:
((P :: 'a ⇒ bool) ⇛ (R :: 'b ⇒ 'c ⇒ bool)) = ((- : P) ⇛ R)
unfolding *Fun-Rel-pred-def* **by** *simp*

lemma *Fun-Rel-pred-eq-Dep-Fun-Rel-pred-uhint* [*uhint*]:
assumes $P \equiv P'$
and $R' \equiv (\lambda(\text{-} :: 'a).\ R)$
shows ((P :: 'a ⇒ bool) ⇛ (R :: 'b ⇒ 'c ⇒ bool)) = ((x : P′) ⇛ R′ x)
using *assms* **by** (*simp add*: *Fun-Rel-pred-eq-Dep-Fun-Rel-pred*)

lemma *Fun-Rel-pred-iff-Dep-Fun-Rel-pred*:
((P :: 'a ⇒ bool) ⇛ (R :: 'b ⇒ 'c ⇒ bool)) f g ⟷ ((- : P) ⇛ R) (f :: 'a ⇒ 'b) (g :: 'a ⇒ 'c)
by (*simp add*: *Fun-Rel-pred-eq-Dep-Fun-Rel-pred*)

lemma *Dep-Fun-Rel-relI* [*intro*]:
assumes $\bigwedge x\ y.\ R\ x\ y \Longrightarrow S\ x\ y\ (f\ x)\ (g\ y)$
shows ((x y :: R) ⇛ S x y) f g
unfolding *Dep-Fun-Rel-rel-def* **using** *assms* **by** *blast*

lemma *Dep-Fun-Rel-relD* [*dest*]:
assumes ((x y :: R) ⇛ S x y) f g
and $R\ x\ y$
shows $S\ x\ y\ (f\ x)\ (g\ y)$
using *assms* **unfolding** *Dep-Fun-Rel-rel-def* **by** *blast*

lemma *Dep-Fun-Rel-relE*:
assumes ((x y :: R) ⇛ S x y) f g
obtains $\bigwedge x\ y.\ R\ x\ y \Longrightarrow S\ x\ y\ (f\ x)\ (g\ y)$
using *assms* **unfolding** *Dep-Fun-Rel-rel-def* **by** *blast*

lemma *Dep-Fun-Rel-rel-cong* [*cong*]:
assumes $R = R'$
and $\bigwedge x\ y.\ R'\ x\ y \Longrightarrow S\ x\ y = S'\ x\ y$
shows ((x y :: R) ⇛ S x y) = ((x y :: R′) ⇛ S′ x y)
using *assms* **by** *force*

lemma *Fun-Rel-relI* [*intro*]:
assumes $\bigwedge x\ y.\ R\ x\ y \Longrightarrow S\ (f\ x)\ (g\ y)$
shows $(R \Rightarrow S)\ f\ g$
using *assms* **by** (*urule Dep-Fun-Rel-relI*)

lemma *Fun-Rel-relD* [*dest*]:
assumes $(R \Rightarrow S)\ f\ g$
and $R\ x\ y$
shows $S\ (f\ x)\ (g\ y)$
using *assms* **by** (*urule Dep-Fun-Rel-relD*)

lemma *Fun-Rel-relE*:
assumes $((x\ y :: R) \Rightarrow S\ x\ y)\ f\ g$
obtains $\bigwedge x\ y.\ R\ x\ y \Longrightarrow S\ x\ y\ (f\ x)\ (g\ y)$
using *assms* **by** (*urule* (*e*) *Dep-Fun-Rel-relE*)

lemma *Dep-Fun-Rel-predI* [*intro*]:
assumes $\bigwedge x.\ P\ x \Longrightarrow R\ x\ (f\ x)\ (g\ x)$
shows $((x : P) \Rightarrow R\ x)\ f\ g$
unfolding *Dep-Fun-Rel-pred-def* **using** *assms* **by** *blast*

lemma *Dep-Fun-Rel-predD* [*dest*]:
assumes $((x : P) \Rightarrow R\ x)\ f\ g$
and $P\ x$
shows $R\ x\ (f\ x)\ (g\ x)$
using *assms* **unfolding** *Dep-Fun-Rel-pred-def* **by** *blast*

lemma *Dep-Fun-Rel-predE*:
assumes $((x : P) \Rightarrow R\ x)\ f\ g$
obtains $\bigwedge x.\ P\ x \Longrightarrow R\ x\ (f\ x)\ (g\ x)$
using *assms* **unfolding** *Dep-Fun-Rel-pred-def* **by** *blast*

lemma *Dep-Fun-Rel-pred-cong* [*cong*]:
assumes $P = P'$
and $\bigwedge x.\ P'\ x \Longrightarrow R\ x = R'\ x$
shows $((x : P) \Rightarrow R\ x) = ((x : P') \Rightarrow R'\ x)$
using *assms* **by** *force*

lemma *Fun-Rel-predI* [*intro*]:
assumes $\bigwedge x.\ P\ x \Longrightarrow R\ (f\ x)\ (g\ x)$
shows $(P \Rightarrow R)\ f\ g$
using *assms* **by** (*urule Dep-Fun-Rel-predI*)

lemma *Fun-Rel-predD* [*dest*]:
assumes $(P \Rightarrow R)\ f\ g$
and $P\ x$
shows $R\ (f\ x)\ (g\ x)$
using *assms* **by** (*urule Dep-Fun-Rel-predD*)

lemma *Fun-Rel-predE*:
assumes $(P \Rightarrow R)\ f\ g$
obtains $\bigwedge x.\ P\ x \Longrightarrow R\ (f\ x)\ (g\ x)$
using *assms* **by** (*urule* (*e*) *Dep-Fun-Rel-predE*)

lemma *rel-inv-Dep-Fun-Rel-rel-eq* [*simp*]:
fixes $R :: 'a \Rightarrow 'b \Rightarrow bool$ **and** $S :: 'a \Rightarrow 'b \Rightarrow 'c \Rightarrow 'd \Rightarrow bool$
shows $((x\ y :: R) \Rrightarrow S\ x\ y)^{-1} = ((y\ x :: R^{-1}) \Rrightarrow (S\ x\ y)^{-1})$
by (*intro ext*) *auto*

lemma *rel-inv-rel-inv-Dep-Fun-Rel-rel-iff-Dep-Fun-Rel-rel* [*iff*]:
$((x\ y :: R^{-1}) \Rrightarrow (S\ x\ y)^{-1})\ f\ g \longleftrightarrow ((x\ y :: R) \Rrightarrow (S\ y\ x))\ g\ f$
by (*simp flip*: *rel-inv-Dep-Fun-Rel-rel-eq*)

lemma *rel-inv-Dep-Fun-Rel-pred-eq* [*simp*]:
fixes $P :: 'a \Rightarrow bool$ **and** $R :: 'a \Rightarrow 'b \Rightarrow 'c \Rightarrow bool$
shows $((x : P) \Rrightarrow R\ x)^{-1} = ((x : P) \Rrightarrow (R\ x)^{-1})$
by (*intro ext*) *auto*

lemma *rel-inv-Dep-Fun-Rel-pred-iff-Dep-Fun-Rel-pred* [*iff*]:
$((x : P) \Rrightarrow (R\ x)^{-1})\ f\ g \longleftrightarrow ((x : P) \Rrightarrow (R\ x))\ g\ f$
by (*simp flip*: *rel-inv-Dep-Fun-Rel-pred-eq*)

lemma *Dep-Fun-Rel-pred-eq-Dep-Fun-Rel-rel*:
fixes $P :: 'a \Rightarrow bool$ **and** $R :: 'a \Rightarrow 'b \Rightarrow 'c \Rightarrow bool$
shows $((x : P) \Rrightarrow R\ x) = ((x\ (\text{-} :: 'a) :: (((\sqcap)\ P) \circ (=))) \Rrightarrow R\ x)$
by (*intro ext*) (*auto intro*!: *Dep-Fun-Rel-predI Dep-Fun-Rel-relI*)

lemma *Fun-Rel-eq-eq-eq* [*simp*]: $((=) \Rrightarrow (=)) = (=)$
by (*intro ext*) *auto*

Composition **lemma** *Dep-Fun-Rel-rel-compI*:
assumes $((x\ y :: R) \Rrightarrow S\ x\ y)\ f\ g$
and $\bigwedge x\ y.\ R\ x\ y \Longrightarrow ((x'\ y' :: T\ x\ y) \Rrightarrow U\ x\ y\ x'\ y')\ f'\ g'$
and $\bigwedge x\ y.\ R\ x\ y \Longrightarrow S\ x\ y\ (f\ x)\ (g\ y) \Longrightarrow T\ x\ y\ (f\ x)\ (g\ y)$
shows $((x\ y :: R) \Rrightarrow U\ x\ y\ (f\ x)\ (g\ y))\ (f' \circ f)\ (g' \circ g)$
using *assms* **by** (*intro Dep-Fun-Rel-relI*) (*auto 6 0*)

corollary *Dep-Fun-Rel-rel-compI'*:
assumes $((x\ y :: R) \Rrightarrow S\ x\ y)\ f\ g$
and $\bigwedge x\ y.\ R\ x\ y \Longrightarrow ((x'\ y' :: S\ x\ y) \Rrightarrow T\ x\ y\ x'\ y')\ f'\ g'$
shows $((x\ y :: R) \Rrightarrow T\ x\ y\ (f\ x)\ (g\ y))\ (f' \circ f)\ (g' \circ g)$
using *assms* **by** (*intro Dep-Fun-Rel-rel-compI*)

lemma *Dep-Fun-Rel-pred-comp-Dep-Fun-Rel-rel-compI*:
assumes $((x : P) \Rrightarrow R\ x)\ f\ g$
and $\bigwedge x.\ P\ x \Longrightarrow ((x'\ y' :: S\ x) \Rrightarrow T\ x\ x'\ y')\ f'\ g'$
and $\bigwedge x.\ P\ x \Longrightarrow R\ x\ (f\ x)\ (g\ x) \Longrightarrow S\ x\ (f\ x)\ (g\ x)$

shows $((x : P) \Rrightarrow T\ x\ (f\ x)\ (g\ x))\ (f' \circ f)\ (g' \circ g)$
using *assms* **by** (*intro Dep-Fun-Rel-predI*) (*auto 6 0*)

corollary *Dep-Fun-Rel-pred-comp-Dep-Fun-Rel-rel-compI'*:
assumes $((x : P) \Rrightarrow R\ x)\ f\ g$
and $\bigwedge x.\ P\ x \Longrightarrow ((x'\ y' :: R\ x) \Rrightarrow S\ x\ x'\ y')\ f'\ g'$
shows $((x : P) \Rrightarrow S\ x\ (f\ x)\ (g\ x))\ (f' \circ f)\ (g' \circ g)$
using *assms* **by** (*intro Dep-Fun-Rel-pred-comp-Dep-Fun-Rel-rel-compI*)

Restrictions **lemma** *Dep-Fun-Rel-restrict-left-restrict-left-eq*:
assumes $\bigwedge f\ x\ y.\ Q\ f \Longrightarrow R\ x\ y \Longrightarrow P\ x\ y\ (f\ x)$
shows $((x\ y :: R) \Rrightarrow (S\ x\ y)\restriction_{P\ x\ y})\restriction_Q = ((x\ y :: R) \Rrightarrow S\ x\ y)\restriction_Q$
using *assms* **by** (*intro ext iffI rel-restrict-leftI Dep-Fun-Rel-relI*)
(*auto dest*!: *Dep-Fun-Rel-relD*)

lemma *Dep-Fun-Rel-restrict-right-restrict-right-eq*:
assumes $\bigwedge f\ x\ y.\ Q\ f \Longrightarrow R\ x\ y \Longrightarrow P\ x\ y\ (f\ y)$
shows $((x\ y :: R) \Rrightarrow (S\ x\ y)\restriction_{P\ x\ y})\restriction_Q = (((x\ y :: R) \Rrightarrow S\ x\ y)\restriction_Q)$
unfolding *rel-restrict-right-eq*
using *assms*
Dep-Fun-Rel-restrict-left-restrict-left-eq[**where** *?R*=R^{-1} **and** *?S*=$\lambda y\ x.\ (S\ x\ y)^{-1}$]
by *simp*

end

1.2.3 Functions on Predicates

theory *Predicate-Functions*
imports
Functions-Base
begin

definition *pred-map f* $(P :: 'a \Rightarrow bool)\ x \equiv P\ (f\ x)$

lemma *pred-map-eq* [*simp*]: *pred-map f P x* $= P\ (f\ x)$
unfolding *pred-map-def* **by** *simp*

lemma *comp-eq-pred-map* [*simp*]: $P \circ f =$ *pred-map f P*
by (*intro ext*) *simp*

consts *pred-if* :: $bool \Rightarrow 'a \Rightarrow 'a$

open-bundle *pred-if-syntax*
begin
notation (**output**) *pred-if* (**infixl** $\langle\longrightarrow\rangle$ *50*)
end

definition *pred-if-pred B P x* ≡ *B* ⟶ *P x*
adhoc-overloading *pred-if* ⇌ *pred-if-pred*

lemma *pred-if-eq-pred-if-pred* [*simp*]:
assumes *B*
shows (*pred-if B P*) = *P*
unfolding *pred-if-pred-def* **using** *assms* **by** *blast*

lemma *pred-if-eq-top-if-not-pred* [*simp*]:
assumes ¬*B*
shows (*pred-if B P*) = (λ-. *True*)
unfolding *pred-if-pred-def* **using** *assms* **by** *blast*

lemma *pred-if-if-impI* [*intro*]:
assumes *B* ⟹ *P x*
shows (*pred-if B P*) *x*
unfolding *pred-if-pred-def* **using** *assms* **by** *blast*

lemma *pred-ifE* [*elim*]:
assumes (*pred-if B P*) *x*
obtains ¬*B* | *B P x*
using *assms* **unfolding** *pred-if-pred-def* **by** *blast*

lemma *pred-ifD*:
assumes (*pred-if B P*) *x*
and *B*
shows *P x*
using *assms* **by** *blast*

end

1.2.4 Orders

theory *Predicates-Order*
imports
HOL.Orderings
begin

lemma *le-predI* [*intro*]:
assumes ⋀*x*. *P x* ⟹ *Q x*
shows *P* ≤ *Q*
using *assms* **by** (*rule predicate1I*)

lemma *le-predD* [*dest*]:
assumes *P* ≤ *Q*
and *P x*
shows *Q x*
using *assms* **by** (*rule predicate1D*)

lemma *le-predE*:
assumes $P \leq Q$
and $P\ x$
obtains $Q\ x$
using *assms* **by** *blast*

declare *le-boolD*[*dest*] **and** *le-boolI*[*intro*]

end

Monotonicity

theory *Functions-Monotone*
imports
Binary-Relations-Order-Base
Function-Relators
Predicate-Functions
Predicates-Order
begin

Summary Introduces the concept of monotone functions. A function is monotone if it is related to itself - see *Dep-Fun-Rel*.

declare *le-funI*[*intro*]
declare *le-funE*[*elim*]

consts *dep-mono-wrt* :: $'a \Rightarrow 'b \Rightarrow 'c$
consts *mono-wrt* :: $'a \Rightarrow 'b \Rightarrow 'c$

open-bundle *dep-mono-wrt-syntax*
begin
notation *mono-wrt* (**infixr** ‹⇒› *50*)
syntax
-dep-mono-wrt-rel :: *idt* ⇒ *idt* ⇒ $'a \Rightarrow 'b \Rightarrow 'c$
(‹$'$(-/ -/ ::/ -$'$) ⇒ (-)› [*51*, *51*, *50*, *50*] *50*)
-dep-mono-wrt-rel-if :: *idt* ⇒ *idt* ⇒ $'a$ ⇒ *bool* ⇒ $'b \Rightarrow 'c$
(‹$'$(-/ -/ ::/ -/ |/ -$'$) ⇒ (-)› [*51*, *51*, *50*, *50*, *50*] *50*)
-dep-mono-wrt-pred :: *idt* ⇒ $'a \Rightarrow 'b \Rightarrow 'c$ (‹$'$(-/ :/ -$'$) ⇒ (-)› [*51*, *50*, *50*] *50*)
-dep-mono-wrt-pred-if :: *idt* ⇒ $'a$ ⇒ *bool* ⇒ $'b \Rightarrow 'c$
(‹$'$(-/ :/ -/ |/ -$'$) ⇒ (-)› [*51*, *50*, *50*, *50*] *50*)
syntax-consts
-dep-mono-wrt-rel -dep-mono-wrt-rel-if -dep-mono-wrt-pred -dep-mono-wrt-pred-if ⇌ *dep-mono-wrt*
translations
(*x y* :: *R*) ⇒ *S* ⇌ *CONST dep-mono-wrt R* (λ*x y*. *S*)
(*x y* :: *R* | *B*) ⇒ *S* ⇌ *CONST dep-mono-wrt R* (λ*x y*. *CONST rel-if B S*)
(*x* : *P*) ⇒ *Q* ⇌ *CONST dep-mono-wrt P* (λ*x*. *Q*)
(*x* : *P* | *B*) ⇒ *Q* ⇌ *CONST dep-mono-wrt P* (λ*x*. *CONST pred-if B Q*)
end

definition *dep-mono-wrt-rel* $(R :: 'a \Rightarrow 'a \Rightarrow bool)$
$(S :: 'a \Rightarrow 'a \Rightarrow 'b \Rightarrow 'b \Rightarrow bool)$ $(f :: 'a \Rightarrow 'b) \equiv ((x\ y :: R) \Rrightarrow S\ x\ y)\ f\ f$
adhoc-overloading *dep-mono-wrt* $\rightleftharpoons$ *dep-mono-wrt-rel*

definition *mono-wrt-rel* $(R :: 'a \Rightarrow 'a \Rightarrow bool)$ $(S :: 'b \Rightarrow 'b \Rightarrow bool) \equiv$
$((- - :: R) \Rightarrow S) :: ('a \Rightarrow 'b) \Rightarrow bool$
adhoc-overloading *mono-wrt* $\rightleftharpoons$ *mono-wrt-rel*

definition *dep-mono-wrt-pred* $(P :: 'a \Rightarrow bool)$ $(Q :: 'a \Rightarrow 'b \Rightarrow bool)$ $(f :: 'a \Rightarrow 'b) \equiv$
$((x : P) \Rrightarrow (\lambda(- :: 'b).\ Q\ x))\ f\ f$
adhoc-overloading *dep-mono-wrt* $\rightleftharpoons$ *dep-mono-wrt-pred*

definition *mono-wrt-pred* $(P :: 'a \Rightarrow bool)$ $(Q :: 'b \Rightarrow bool) \equiv$
$(((- :: 'a) : P) \Rightarrow Q) :: ('a \Rightarrow 'b) \Rightarrow bool$
adhoc-overloading *mono-wrt* $\rightleftharpoons$ *mono-wrt-pred*

lemma *mono-wrt-rel-eq-dep-mono-wrt-rel*:
$((R :: 'a \Rightarrow 'a \Rightarrow bool) \Rightarrow (S :: 'b \Rightarrow 'b \Rightarrow bool)) = ((- - :: R) \Rightarrow S)$
unfolding *mono-wrt-rel-def* **by** *simp*

lemma *mono-wrt-rel-eq-dep-mono-wrt-rel-uhint* [*uhint*]:
assumes $R \equiv R'$
and $\bigwedge x\ y.\ S'\ x\ y \equiv S$
shows $((R :: 'a \Rightarrow 'a \Rightarrow bool) \Rightarrow (S :: 'b \Rightarrow 'b \Rightarrow bool)) = ((x\ y :: R') \Rightarrow S'\ x\ y)$
using *assms* **by** (*simp add*: *mono-wrt-rel-eq-dep-mono-wrt-rel*)

lemma *mono-wrt-rel-iff-dep-mono-wrt-rel*:
$((R :: 'a \Rightarrow 'a \Rightarrow bool) \Rightarrow (S :: 'b \Rightarrow 'b \Rightarrow bool))\ f \longleftrightarrow$
dep-mono-wrt R $(\lambda(- :: 'a)$ $(- :: 'a).\ S)$ $(f :: 'a \Rightarrow 'b)$
by (*simp add*: *mono-wrt-rel-eq-dep-mono-wrt-rel*)

lemma *mono-wrt-pred-eq-dep-mono-wrt-pred*:
$((P :: 'a \Rightarrow bool) \Rightarrow (Q :: 'b \Rightarrow bool)) = (((- :: 'a) : P) \Rightarrow Q)$
unfolding *mono-wrt-pred-def* **by** *simp*

lemma *mono-wrt-pred-eq-dep-mono-wrt-pred-uhint* [*uhint*]:
assumes $P \equiv P'$
and $\bigwedge x.\ Q \equiv Q'\ x$
shows $((P :: 'a \Rightarrow bool) \Rightarrow (Q :: 'b \Rightarrow bool)) = (((x : P') \Rightarrow Q'\ x) :: ('a \Rightarrow 'b) \Rightarrow bool)$
using *assms* **by** (*simp add*: *mono-wrt-pred-eq-dep-mono-wrt-pred*)

lemma *mono-wrt-pred-iff-dep-mono-wrt-pred*:
$((P :: 'a \Rightarrow bool) \Rightarrow (Q :: 'b \Rightarrow bool))\ f \longleftrightarrow (((- :: 'a) : P) \Rightarrow Q)$ $(f :: 'a \Rightarrow 'b)$
by (*simp add*: *mono-wrt-pred-eq-dep-mono-wrt-pred*)

lemma *dep-mono-wrt-relI* [*intro*]:

assumes $\bigwedge x\ y.\ R\ x\ y \Longrightarrow S\ x\ y\ (f\ x)\ (f\ y)$
shows $((x\ y :: R) \Rightarrow S\ x\ y)\ f$
using *assms* **unfolding** *dep-mono-wrt-rel-def* **by** *blast*

lemma *dep-mono-wrt-relE*:
assumes $((x\ y :: R) \Rightarrow S\ x\ y)\ f$
obtains $\bigwedge x\ y.\ R\ x\ y \Longrightarrow S\ x\ y\ (f\ x)\ (f\ y)$
using *assms* **unfolding** *dep-mono-wrt-rel-def* **by** *blast*

lemma *dep-mono-wrt-relD* [*dest*]:
assumes $((x\ y :: R) \Rightarrow S\ x\ y)\ f$
and $R\ x\ y$
shows $S\ x\ y\ (f\ x)\ (f\ y)$
using *assms* **unfolding** *dep-mono-wrt-rel-def* **by** *blast*

lemma *dep-mono-wrt-rel-cong* [*cong*]:
assumes $R = R'$
and $\bigwedge x\ y.\ R'\ x\ y \Longrightarrow S\ x\ y = S'\ x\ y$
shows $((x\ y :: R) \Rightarrow S\ x\ y) = ((x\ y :: R') \Rightarrow S'\ x\ y)$
using *assms* **by** *force*

lemma *mono-wrt-relI* [*intro*]:
assumes $\bigwedge x\ y.\ R\ x\ y \Longrightarrow S\ (f\ x)\ (f\ y)$
shows $(R \Rightarrow S)\ f$
using *assms* **by** (*urule dep-mono-wrt-relI*)

lemma *mono-wrt-relE*:
assumes $(R \Rightarrow S)\ f$
obtains $\bigwedge x\ y.\ R\ x\ y \Longrightarrow S\ (f\ x)\ (f\ y)$
using *assms* **by** (*urule* (*e*) *dep-mono-wrt-relE*)

lemma *mono-wrt-relD* [*dest*]:
assumes $(R \Rightarrow S)\ f$
and $R\ x\ y$
shows $S\ (f\ x)\ (f\ y)$
using *assms* **by** (*urule dep-mono-wrt-relD*)

lemma *dep-mono-wrt-predI* [*intro*]:
assumes $\bigwedge x.\ P\ x \Longrightarrow Q\ x\ (f\ x)$
shows $((x : P) \Rightarrow Q\ x)\ f$
using *assms* **unfolding** *dep-mono-wrt-pred-def* **by** *blast*

lemma *dep-mono-wrt-predE*:
assumes $((x : P) \Rightarrow Q\ x)\ f$
obtains $\bigwedge x.\ P\ x \Longrightarrow Q\ x\ (f\ x)$
using *assms* **unfolding** *dep-mono-wrt-pred-def* **by** *blast*

lemma *dep-mono-wrt-predD* [*dest*]:
assumes $((x : P) \Rightarrow Q\ x)\ f$

and $P\ x$
shows $Q\ x\ (f\ x)$
using *assms* **unfolding** *dep-mono-wrt-pred-def* **by** *blast*

lemma *dep-mono-wrt-pred-cong* [*cong*]:
assumes $P = P'$
and $\bigwedge x.\ P'\ x \Longrightarrow Q\ x = Q'\ x$
shows $((x : P) \Rightarrow Q\ x) = ((x : P') \Rightarrow Q'\ x)$
using *assms* **by** *force*

lemma *dep-mono-wrt-pred-codom-iff-cong*:
assumes $P = P'$
and $\bigwedge x.\ P'\ x \Longrightarrow Q\ x\ (f\ x) \longleftrightarrow Q'\ x\ (f'\ x)$
shows $((x : P) \Rightarrow Q\ x)\ f \longleftrightarrow ((x : P') \Rightarrow Q'\ x)\ f'$
using *assms* **by** *force*

lemma *mono-wrt-predI* [*intro*]:
assumes $\bigwedge x.\ P\ x \Longrightarrow Q\ (f\ x)$
shows $(P \Rightarrow Q)\ f$
using *assms* **by** (*urule dep-mono-wrt-predI*)

lemma *mono-wrt-predE*:
assumes $(P \Rightarrow Q)\ f$
obtains $\bigwedge x.\ P\ x \Longrightarrow Q\ (f\ x)$
using *assms* **by** (*urule* (*e*) *dep-mono-wrt-predE*)

lemma *mono-wrt-predD* [*dest*]:
assumes $(P \Rightarrow Q)\ f$
and $P\ x$
shows $Q\ (f\ x)$
using *assms* **by** (*urule dep-mono-wrt-predD*)

context
fixes $R :: 'a \Rightarrow 'a \Rightarrow bool$ **and** $S :: 'a \Rightarrow 'a \Rightarrow 'b \Rightarrow 'b \Rightarrow bool$ **and** $f :: 'a \Rightarrow 'b$
and $P :: 'a \Rightarrow bool$ **and** $Q :: 'a \Rightarrow 'b \Rightarrow bool$
begin

lemma *dep-mono-wrt-rel-if-Dep-Fun-Rel-rel-self*:
assumes $((x\ y :: R) \Rrightarrow S\ x\ y)\ f\ f$
shows $((x\ y :: R) \Rightarrow S\ x\ y)\ f$
using *assms* **by** *blast*

lemma *dep-mono-wrt-pred-if-Dep-Fun-Rel-pred-self*:
assumes $((x : P) \Rrightarrow (\lambda\text{-}.\ Q\ x))\ f\ f$
shows $((x : P) \Rightarrow Q\ x)\ f$
using *assms* **by** *blast*

lemma *Dep-Fun-Rel-rel-self-if-dep-mono-wrt-rel*:
assumes $((x\ y :: R) \Rightarrow S\ x\ y)\ f$

shows $((x\ y :: R) \Rrightarrow S\ x\ y)\ f\ f$
using *assms* **by** *blast*

lemma *Dep-Fun-Rel-pred-self-if-dep-mono-wrt-pred*:
assumes $((x : P) \Rightarrow Q\ x)\ f$
shows $((x : P) \Rrightarrow (\lambda\text{-}.\ Q\ x))\ f\ f$
using *assms* **by** *blast*

corollary *Dep-Fun-Rel-rel-self-iff-dep-mono-wrt-rel*:
$((x\ y :: R) \Rrightarrow S\ x\ y)\ f\ f \longleftrightarrow ((x\ y :: R) \Rightarrow S\ x\ y)\ f$
using *dep-mono-wrt-rel-if-Dep-Fun-Rel-rel-self*
Dep-Fun-Rel-rel-self-if-dep-mono-wrt-rel **by** *blast*

corollary *Dep-Fun-Rel-pred-self-iff-dep-mono-wrt-pred*:
$((x : P) \Rrightarrow (\lambda(\text{-} :: 'b).\ Q\ x))\ f\ f \longleftrightarrow ((x : P) \Rightarrow Q\ x)\ f$
using *dep-mono-wrt-pred-if-Dep-Fun-Rel-pred-self*
Dep-Fun-Rel-pred-self-if-dep-mono-wrt-pred **by** *blast*

lemma *dep-mono-wrt-rel-eq-Dep-Fun-Rel-rel* [*uhint*]:
assumes $R \equiv R'$
and $\bigwedge x\ y.\ S\ x\ y \equiv S'\ x\ y$
and $f \equiv f'$
shows $((x\ y :: R) \Rightarrow S\ x\ y)\ f \equiv ((x\ y :: R') \Rrightarrow S'\ x\ y)\ f'\ f'$
using *assms* **unfolding** *Dep-Fun-Rel-rel-self-iff-dep-mono-wrt-rel atomize-eq* **by** *auto*

lemma *dep-mono-wrt-pred-eq-Dep-Fun-Rel-pred* [*uhint*]:
assumes $P \equiv P'$
and $\bigwedge x\ y.\ Q\ x\ y \equiv Q'\ x\ y\ y$
and $f \equiv f'$
shows $((x : P) \Rightarrow Q\ x)\ f \equiv ((x : P') \Rrightarrow Q'\ x)\ f'\ f'$
using *assms* **unfolding** *Dep-Fun-Rel-pred-self-iff-dep-mono-wrt-pred atomize-eq* **by** *auto*

lemma *dep-mono-wrt-rel-inv-eq* [*simp*]:
$((y\ x :: R^{-1}) \Rightarrow (S\ x\ y)^{-1}) = ((x\ y :: R) \Rightarrow S\ x\ y)$
by (*intro ext*) *force*

lemma *in-dom-if-rel-if-dep-mono-wrt-rel*:
assumes $((x\ y :: R) \Rightarrow S\ x\ y)\ f$
and $R\ x\ y$
shows *in-dom* $(S\ x\ y)\ (f\ x)$
using *assms* **by** (*intro in-domI*) *blast*

end

context
fixes $R :: 'a \Rightarrow 'a \Rightarrow bool$ **and** $f :: 'a \Rightarrow 'b$
begin

lemma *mono-wrt-rel-eq-Fun-Rel-rel* [*uhint*]:
assumes $R \equiv R'$
and $S \equiv S'$
and $f \equiv f'$
shows $(R \Rightarrow S)\ f \equiv (R' \Rrightarrow S')\ f'\ f'$
using *assms* **by** (*urule dep-mono-wrt-rel-eq-Dep-Fun-Rel-rel*)

lemma *mono-wrt-pred-eq-Fun-Rel-pred* [*uhint*]:
assumes $P :: \text{-} \Rightarrow \textit{bool} \equiv P'$
and $\bigwedge y.\ Q\ y \equiv Q'\ y\ y$
and $f \equiv f'$
shows $(P \Rightarrow Q)\ f \equiv (P' \Rrightarrow Q')\ f'\ f'$
using *assms* **by** (*urule dep-mono-wrt-pred-eq-Dep-Fun-Rel-pred*)

corollary *in-dom-if-in-dom-if-mono-wrt-rel*:
assumes $(R \Rightarrow S)\ f$
shows $(\textit{in-dom}\ R \Rightarrow \textit{in-dom}\ S)\ f$
using *assms in-dom-if-rel-if-dep-mono-wrt-rel* **by** *fast*

lemma *in-codom-if-rel-if-dep-mono-wrt-rel*:
assumes $((x\ y :: R) \Rightarrow S\ x\ y)\ f$
and $R\ x\ y$
shows $\textit{in-codom}\ (S\ x\ y)\ (f\ y)$
using *assms* **by** (*intro in-codomI*) *blast*

corollary *in-codom-if-in-codom-if-mono-wrt-rel*:
assumes $(R \Rightarrow S)\ f$
shows $(\textit{in-codom}\ R \Rightarrow \textit{in-codom}\ S)\ f$
using *assms in-dom-if-rel-if-dep-mono-wrt-rel*
by *fast*

corollary *in-field-if-in-field-if-mono-wrt-rel*:
assumes $(R \Rightarrow S)\ f$
shows $(\textit{in-field}\ R \Rightarrow \textit{in-field}\ S)\ f$
using *assms* **by** *fast*

lemma *le-rel-map-if-mono-wrt-rel*:
assumes $(R \Rightarrow S)\ f$
shows $R \leq \textit{rel-map}\ f\ S$
using *assms* **by** (*intro le-relI*) *auto*

lemma *le-pred-map-if-mono-wrt-pred*:
assumes $(P \Rightarrow Q)\ f$
shows $P \leq \textit{pred-map}\ f\ Q$
using *assms* **by** (*intro le-predI*) *auto*

lemma *mono-wrt-rel-if-le-rel-map*:
assumes $R \leq \textit{rel-map}\ f\ S$

shows $(R \Rightarrow S)\ f$
using *assms* **by** (*intro mono-wrt-relI*) *auto*

lemma *mono-wrt-pred-if-le-pred-map*:
assumes $P \leq$ *pred-map* $f\ Q$
shows $(P \Rightarrow Q)\ f$
using *assms* **by** (*intro mono-wrt-predI*) *auto*

corollary *mono-wrt-rel-iff-le-rel-map*: $(R \Rightarrow S)\ f \longleftrightarrow R \leq$ *rel-map* $f\ S$
using *mono-wrt-rel-if-le-rel-map le-rel-map-if-mono-wrt-rel* **by** *auto*

corollary *mono-wrt-pred-iff-le-pred-map*: $(P \Rightarrow Q)\ f \longleftrightarrow P \leq$ *pred-map* $f\ Q$
using *mono-wrt-pred-if-le-pred-map le-pred-map-if-mono-wrt-pred* **by** *auto*

end

lemma *dep-mono-comp-iff-dep-mono-if-all-app-eq*:
assumes $\bigwedge x.\ A\ x \Longrightarrow f\ (m\ x) = g\ x$
shows $((x : (A :: - \Rightarrow bool)) \Rightarrow B\ x)\ (f \circ m) \longleftrightarrow ((x : A) \Rightarrow B\ x)\ g$
using *assms* **by** *fastforce*

lemma *dep-mono-pred-map-comp-iff-dep-mono-if-all-app-eq*:
assumes $\bigwedge x\ y.\ A\ x \Longrightarrow B\ x\ y \Longrightarrow f\ x\ (m\ y) = g\ x\ y$
shows $((x : A) \Rightarrow$ *pred-map* $(\lambda f.\ f \circ m)\ ((y : B\ x) \Rightarrow C\ x\ y))\ f$
$\longleftrightarrow ((x : A) \Rightarrow (y : (B\ x :: - \Rightarrow bool)) \Rightarrow C\ x\ y)\ g$
using *assms* **by** (*simp cong*: *dep-mono-wrt-pred-codom-iff-cong*)

definition *mono* $:: (('a :: ord) \Rightarrow ('b :: ord)) \Rightarrow bool$
$\equiv (((\leq) :: 'a \Rightarrow 'a \Rightarrow bool) \Rightarrow ((\leq) :: 'b \Rightarrow 'b \Rightarrow bool))$

lemma *mono-eq-mono-wrt-le* [*simp*]: (*mono* $:: (('a :: ord) \Rightarrow ('b :: ord)) \Rightarrow bool$)
=
$(((\leq) :: 'a \Rightarrow 'a \Rightarrow bool) \Rightarrow ((\leq) :: 'b \Rightarrow 'b \Rightarrow bool))$
unfolding *mono-def* **by** *simp*

lemma *mono-eq-mono-wrt-le-uhint* [*uhint*]:
assumes $R \equiv (\leq) :: 'a \Rightarrow 'a \Rightarrow bool$
and $S \equiv (\leq) :: 'b \Rightarrow 'b \Rightarrow bool$
shows *mono* $:: (('a :: ord) \Rightarrow ('b :: ord)) \Rightarrow bool \equiv (R \Rightarrow S)$
using *assms* **by** *simp*

lemma *mono-iff-mono-wrt-le* [*iff*]: *mono* $f \longleftrightarrow ((\leq) \Rightarrow (\leq))\ f$ **by** *simp*

lemma *monoI* [*intro*]:
assumes $\bigwedge x\ y.\ x \leq y \Longrightarrow f\ x \leq f\ y$
shows *mono* f
using *assms* **by** (*urule mono-wrt-relI*)

lemma *monoE* [*elim*]:

assumes *mono f*
obtains $\bigwedge x\ y.\ x \leq y \Longrightarrow f\ x \leq f\ y$
using *assms* **by** (*urule* (*e*) *mono-wrt-relE*)

lemma *monoD*:
assumes *mono f*
and $x \leq y$
shows $f\ x \leq f\ y$
using *assms* **by** (*urule mono-wrt-relD*)

definition *antimono* :: $(('a :: ord) \Rightarrow ('b :: ord)) \Rightarrow bool$
$\equiv (((\leq) :: 'a \Rightarrow 'a \Rightarrow bool) \Rightarrow ((\geq) :: 'b \Rightarrow 'b \Rightarrow bool))$

lemma *antimono-eq-mono-wrt-le-ge* [*simp*]: (*antimono* :: $(('a :: ord) \Rightarrow ('b :: ord))$
$\Rightarrow bool) =$
$(((\leq) :: 'a \Rightarrow 'a \Rightarrow bool) \Rightarrow ((\geq) :: 'b \Rightarrow 'b \Rightarrow bool))$
unfolding *antimono-def* **by** *simp*

lemma *antimono-eq-mono-wrt-le-ge-uhint* [*uhint*]:
assumes $R \equiv (\leq) :: 'a \Rightarrow 'a \Rightarrow bool$
and $S \equiv (\geq) :: 'b \Rightarrow 'b \Rightarrow bool$
shows *antimono* :: $(('a :: ord) \Rightarrow ('b :: ord)) \Rightarrow bool \equiv (R \Rightarrow S)$
using *assms* **by** *simp*

lemma *antimono-iff-mono-wrt-le-ge* [*iff*]: *antimono* $f \longleftrightarrow ((\leq) \Rightarrow (\geq))\ f$ **by** *simp*

lemma *antimonoI* [*intro*]:
assumes $\bigwedge x\ y.\ x \leq y \Longrightarrow f\ x \geq f\ y$
shows *antimono f*
by (*urule mono-wrt-relI*) (*urule assms*)

lemma *antimonoE* [*elim*]:
assumes *antimono f*
obtains $\bigwedge x\ y.\ x \leq y \Longrightarrow f\ x \geq f\ y$
using *assms* **by** (*urule* (*e*) *mono-wrt-relE*)

lemma *antimonoD*:
assumes *antimono f*
and $x \leq y$
shows $f\ x \geq f\ y$
using *assms* **by** (*urule mono-wrt-relD*)

lemma *antimono-Dep-Fun-Rel-rel-left*: *antimono* $(\lambda(R :: 'a \Rightarrow 'b \Rightarrow bool).\ ((x\ y ::$
$R) \Rrightarrow S\ x\ y))$
by (*intro antimonoI*) *auto*

lemma *antimono-Dep-Fun-Rel-pred-left*: *antimono* $(\lambda(P :: 'a \Rightarrow bool).\ ((x : P) \Rrightarrow$
$Q\ x))$
by (*intro antimonoI*) *auto*

lemma *antimono-dep-mono-wrt-rel-left*: *antimono* (λ(R :: $'a \Rightarrow 'a \Rightarrow$ *bool*). ((x y :: R) $\Rightarrow$ S x y))
by (*intro antimonoI*) *blast*

lemma *antimono-dep-mono-wrt-pred-left*: *antimono* (λ(P :: $'a \Rightarrow$ *bool*). ((x : P) $\Rightarrow$ Q x))
by (*intro antimonoI*) *blast*

lemma *Dep-Fun-Rel-rel-if-le-left-if-Dep-Fun-Rel-rel*:
fixes R R' :: $'a \Rightarrow 'b \Rightarrow$ *bool*
assumes ((x y :: R) $\Rrightarrow$ S x y) f g
and $R' \leq R$
shows ((x y :: R) $\Rrightarrow$ S x y) f g
using *assms* **by** *blast*

lemma *Dep-Fun-Rel-pred-if-le-left-if-Dep-Fun-Rel-pred*:
fixes P P' :: $'a \Rightarrow$ *bool*
assumes ((x : P) $\Rrightarrow$ Q x) f g
and $P' \leq P$
shows ((x : P') $\Rrightarrow$ Q x) f g
using *assms* **by** *blast*

lemma *dep-mono-wrt-rel-if-le-left-if-dep-mono-wrt-rel*:
fixes R R' :: $'a \Rightarrow 'a \Rightarrow$ *bool*
assumes ((x y :: R) $\Rightarrow$ S x y) f
and $R' \leq R$
shows ((x y :: R') $\Rightarrow$ S x y) f
using *assms* **by** *blast*

lemma *dep-mono-wrt-pred-if-le-left-if-dep-mono-wrt-pred*:
fixes P P' :: $'a \Rightarrow$ *bool*
assumes ((x : P) $\Rightarrow$ Q x) f
and $P' \leq P$
shows ((x : P') $\Rightarrow$ Q x) f
using *assms* **by** *blast*

lemma *mono-Dep-Fun-Rel-rel-right*: *mono* (λ(S :: $'a \Rightarrow 'b \Rightarrow 'c \Rightarrow 'd \Rightarrow$ *bool*). ((x y :: R) $\Rrightarrow$ S x y))
by (*intro monoI*) *blast*

lemma *mono-Dep-Fun-Rel-pred-right*: *mono* (λ(Q :: $'a \Rightarrow 'b \Rightarrow 'c \Rightarrow$ *bool*). ((x : P) $\Rrightarrow$ Q x))
by (*intro monoI*) *blast*

lemma *mono-dep-mono-wrt-rel-right*: *mono* (λ(S :: $'a \Rightarrow 'a \Rightarrow 'b \Rightarrow 'b \Rightarrow$ *bool*). ((x y :: R) $\Rightarrow$ S x y))
by (*intro monoI*) *blast*

lemma *mono-dep-mono-wrt-pred-right*: *mono* ($\lambda(Q :: 'a \Rightarrow 'b \Rightarrow bool).\ ((x : P) \Rightarrow Q\ x)$)
by (*intro monoI*) *blast*

lemma *Dep-Fun-Rel-rel-if-le-right-if-Dep-Fun-Rel-rel*:
assumes $((x\ y :: R) \Rrightarrow S\ x\ y)\ f\ g$
and $\bigwedge x\ y.\ R\ x\ y \Longrightarrow S\ x\ y\ (f\ x)\ (g\ y) \Longrightarrow T\ x\ y\ (f\ x)\ (g\ y)$
shows $((x\ y :: R) \Rrightarrow T\ x\ y)\ f\ g$
using *assms* **by** (*intro Dep-Fun-Rel-relI*) *blast*

lemma *Dep-Fun-Rel-pred-if-le-right-if-Dep-Fun-Rel-pred*:
assumes $((x : P) \Rrightarrow Q\ x)\ f\ g$
and $\bigwedge x.\ P\ x \Longrightarrow Q\ x\ (f\ x)\ (g\ x) \Longrightarrow T\ x\ (f\ x)\ (g\ x)$
shows $((x : P) \Rrightarrow T\ x)\ f\ g$
using *assms* **by** *blast*

lemma *dep-mono-wrt-rel-if-le-right-if-dep-mono-wrt-rel*:
assumes $((x\ y :: R) \Rightarrow S\ x\ y)\ f$
and $\bigwedge x\ y.\ R\ x\ y \Longrightarrow S\ x\ y\ (f\ x)\ (f\ y) \Longrightarrow T\ x\ y\ (f\ x)\ (f\ y)$
shows $((x\ y :: R) \Rightarrow T\ x\ y)\ f$
using *assms* **by** (*intro dep-mono-wrt-relI*) *blast*

lemma *dep-mono-wrt-pred-if-le-right-if-dep-mono-wrt-pred*:
assumes $((x : P) \Rightarrow Q\ x)\ f$
and $\bigwedge x.\ P\ x \Longrightarrow Q\ x\ (f\ x) \Longrightarrow T\ x\ (f\ x)$
shows $((x : P) \Rightarrow T\ x)\ f$
using *assms* **by** *blast*

Composition **lemma** *dep-mono-wrt-rel-compI*:
assumes $((x\ y :: R) \Rightarrow S\ x\ y)\ f$
and $\bigwedge x\ y.\ R\ x\ y \Longrightarrow ((x'\ y' :: T\ x\ y) \Rightarrow U\ x\ y\ x'\ y')\ f'$
and $\bigwedge x\ y.\ R\ x\ y \Longrightarrow S\ x\ y\ (f\ x)\ (f\ y) \Longrightarrow T\ x\ y\ (f\ x)\ (f\ y)$
shows $((x\ y :: R) \Rightarrow U\ x\ y\ (f\ x)\ (f\ y))\ (f' \circ f)$
using *assms* **by** (*intro dep-mono-wrt-relI*) *force*

corollary *dep-mono-wrt-rel-compI'*:
assumes $((x\ y :: R) \Rightarrow S\ x\ y)\ f$
and $\bigwedge x\ y.\ R\ x\ y \Longrightarrow ((x'\ y' :: S\ x\ y) \Rightarrow T\ x\ y\ x'\ y')\ f'$
shows $((x\ y :: R) \Rightarrow T\ x\ y\ (f\ x)\ (f\ y))\ (f' \circ f)$
using *assms* **by** (*intro dep-mono-wrt-rel-compI*)

lemma *dep-mono-wrt-pred-comp-dep-mono-wrt-rel-compI*:
assumes $((x : P) \Rightarrow Q\ x)\ f$
and $\bigwedge x.\ P\ x \Longrightarrow ((x'\ y' :: R\ x) \Rightarrow S\ x\ x'\ y')\ f'$
and $\bigwedge x.\ P\ x \Longrightarrow Q\ x\ (f\ x) \Longrightarrow R\ x\ (f\ x)\ (f\ x)$
shows $((x : P) \Rightarrow (\lambda y.\ S\ x\ (f\ x)\ (f\ x)\ y\ y))\ (f' \circ f)$
using *assms* **by** (*intro dep-mono-wrt-predI*) *force*

lemma *dep-mono-wrt-pred-comp-dep-mono-wrt-pred-compI*:

assumes $((x : P) \Rightarrow Q\ x)\ f$
and $\bigwedge x.\ P\ x \Longrightarrow ((x' : R\ x) \Rightarrow S\ x\ x')\ f'$
and $\bigwedge x.\ P\ x \Longrightarrow Q\ x\ (f\ x) \Longrightarrow R\ x\ (f\ x)$
shows $((x : P) \Rightarrow S\ x\ (f\ x))\ (f' \circ f)$
using *assms* **by** (*intro dep-mono-wrt-predI*) *force*

corollary *dep-mono-wrt-pred-comp-dep-mono-wrt-pred-compI'*:
assumes $((x : P) \Rightarrow Q\ x)\ f$
and $\bigwedge x.\ P\ x \Longrightarrow ((x' : Q\ x) \Rightarrow S\ x\ x')\ f'$
shows $((x : P) \Rightarrow S\ x\ (f\ x))\ (f' \circ f)$
using *assms* **by** (*intro dep-mono-wrt-pred-comp-dep-mono-wrt-pred-compI*)

lemma *mono-wrt-rel-top* [*iff*]:
fixes $R :: 'a \Rightarrow 'a \Rightarrow bool$ **and** $f :: 'a \Rightarrow 'b$
shows $(R \Rightarrow (\top :: 'b \Rightarrow 'b \Rightarrow bool))\ f$
by *auto*

lemma *mono-wrt-pred-top* [*iff*]:
fixes $P :: 'a \Rightarrow bool$ **and** $f :: 'a \Rightarrow 'b$
shows $(P \Rightarrow (\top :: 'b \Rightarrow bool))\ f$
by *auto*

Instantiations **lemma** *mono-wrt-rel-self-id*:
fixes $R :: 'a \Rightarrow 'a \Rightarrow bool$
shows $(R \Rightarrow R)\ (id :: 'a \Rightarrow 'a)$
by *auto*

lemma *mono-wrt-pred-self-id*:
fixes $P :: 'a \Rightarrow bool$
shows $(P \Rightarrow P)\ (id :: 'a \Rightarrow 'a)$
by *auto*

lemma *mono-dep-mono-wrt-dep-mono-wrt-comp*:
$(((x : (B :: 'b \Rightarrow bool)) \Rightarrow C\ x) \Rightarrow (f : A \Rightarrow B) \Rightarrow (x : A) \Rightarrow C\ (f\ x))\ (\circ)$
by *force*

lemma *mono-wrt-dep-mono-wrt-dep-fun-map*:
fixes $A :: 'a \Rightarrow bool$
shows $((f : A \Rightarrow B) \Rightarrow ((x : A) \Rightarrow (y : B) \Rightarrow (z : C\ (f\ x)) \Rightarrow D\ x\ y\ z) \Rightarrow$
$(h : (x : B) \Rightarrow C\ x) \Rightarrow (x : A) \Rightarrow D\ x\ (f\ x)\ (h\ (f\ x)))$ *dep-fun-map*
by *fastforce*

lemma *mono-wrt-dep-mono-wrt-fun-map*:
fixes $A :: 'a \Rightarrow bool$
shows $((f : A \Rightarrow B) \Rightarrow ((x : C) \Rightarrow D\ x) \Rightarrow (h : B \Rightarrow C) \Rightarrow (x : A) \Rightarrow D\ (h\ (f\ x)))$ *fun-map*
by *fastforce*

lemma *mono-in-dom*: *mono in-dom* **by** (*intro monoI*) *fast*

lemma *mono-in-codom*: *mono in-codom* **by** (*intro monoI*) *fast*
lemma *mono-in-field*: *mono in-field* **by** (*intro monoI*) *fast*
lemma *mono-rel-comp*: $((\leq) \Rightarrow (\leq) \Rrightarrow (\leq))$ $(\circ\circ)$ **by** (*intro mono-wrt-relI Fun-Rel-relI le-relI*) *fast*
lemma *mono-rel-inv*: *mono rel-inv* **by** (*intro monoI*) *fast*

lemma *mono-rel-restrict-left*:
$((\leq) \Rightarrow (\leq) \Rrightarrow (\leq))$ (*rel-restrict-left* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow ('a \Rightarrow bool) \Rightarrow 'a \Rightarrow 'b \Rightarrow bool$)
by (*intro mono-wrt-relI Fun-Rel-relI le-relI*) *fastforce*

lemma *mono-rel-restrict-right*:
$((\leq) \Rightarrow (\leq) \Rrightarrow (\leq))$ (*rel-restrict-right* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow ('b \Rightarrow bool) \Rightarrow 'a \Rightarrow 'b \Rightarrow bool$)
by (*intro mono-wrt-relI Fun-Rel-relI le-relI*) *fastforce*

lemma *mono-ball*: $((\geq) \Rightarrow (\leq) \Rrightarrow (\leq))$ *ball* **by** *fast*
lemma *mono-bex*: $((\leq) \Rightarrow (\leq) \Rrightarrow (\leq))$ *bex* **by** *fast*

end

theory *Reverse-Implies*
imports *Binary-Relation-Functions*
begin

definition *rev-implies* $\equiv (\longrightarrow)^{-1}$

open-bundle *rev-implies-syntax*
begin
notation *rev-implies* (**infixr** $\langle\longleftarrow\rangle$ *25*)
end

lemma *rev-imp-eq-imp-inv* [*simp*]: $(\longleftarrow) = (\longrightarrow)^{-1}$
unfolding *rev-implies-def* **by** *simp*

lemma *rev-impI* [*intro*!]:
assumes $Q \Longrightarrow P$
shows $P \longleftarrow Q$
using *assms* **by** *auto*

lemma *rev-impD* [*dest*!]:
assumes $P \longleftarrow Q$
shows $Q \Longrightarrow P$
using *assms* **by** *auto*

lemma *rev-imp-iff-imp*: $(P \longleftarrow Q) \longleftrightarrow (Q \longrightarrow P)$ **by** *auto*

end

Injective

theory *Binary-Relations-Injective*
  imports
    Functions-Monotone
    Reverse-Implies
begin

consts *rel-injective-on* :: $'a \Rightarrow 'b \Rightarrow bool$

overloading
  rel-injective-on-pred $\equiv$ *rel-injective-on* :: $('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool$
begin
  definition *rel-injective-on-pred* $P\ R \equiv \forall x\ x' : P.\ \forall y.\ R\ x\ y \wedge R\ x'\ y \longrightarrow x = x'$
end

lemma *rel-injective-onI* [*intro*]:
  assumes $\bigwedge x\ x'\ y.\ P\ x \Longrightarrow P\ x' \Longrightarrow R\ x\ y \Longrightarrow R\ x'\ y \Longrightarrow x = x'$
  shows *rel-injective-on* $P\ R$
  unfolding *rel-injective-on-pred-def* **using** *assms* **by** *blast*

lemma *rel-injective-onD*:
  assumes *rel-injective-on* $P\ R$
  and $P\ x\ P\ x'$
  and $R\ x\ y\ R\ x'\ y$
  shows $x = x'$
  using *assms* **unfolding** *rel-injective-on-pred-def* **by** *blast*

lemma *antimono-rel-injective-on*:
  $((\leq) \Rightarrow (\leq) \Rrightarrow (\geq))$ (*rel-injective-on* :: $('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool$)
  by (*intro mono-wrt-relI*) (*auto dest*: *rel-injective-onD intro*!: *rel-injective-onI*)

consts *rel-injective-at* :: $'a \Rightarrow 'b \Rightarrow bool$

overloading
  rel-injective-at-pred $\equiv$ *rel-injective-at* :: $('a \Rightarrow bool) \Rightarrow ('b \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$
begin
  definition *rel-injective-at-pred* $P\ R \equiv \forall x\ x'\ y.\ P\ y \wedge R\ x\ y \wedge R\ x'\ y \longrightarrow x = x'$
end

lemma *rel-injective-atI* [*intro*]:
  assumes $\bigwedge x\ x'\ y.\ P\ y \Longrightarrow R\ x\ y \Longrightarrow R\ x'\ y \Longrightarrow x = x'$
  shows *rel-injective-at* $P\ R$
  unfolding *rel-injective-at-pred-def* **using** *assms* **by** *blast*

lemma *rel-injective-atD*:
  assumes *rel-injective-at* $P\ R$
  and $P\ y$
  and $R\ x\ y\ R\ x'\ y$

shows $x = x'$
using *assms* **unfolding** *rel-injective-at-pred-def* **by** *blast*

lemma *rel-injective-on-if-Fun-Rel-imp-if-rel-injective-at*:
assumes *rel-injective-at* $(Q :: 'b \Rightarrow bool)$ $(R :: 'a \Rightarrow 'b \Rightarrow bool)$
and $(R \Rrightarrow (\longrightarrow))$ P Q
shows *rel-injective-on* P R
using *assms* **by** (*intro rel-injective-onI*) (*auto dest*: *rel-injective-atD*)

lemma *rel-injective-at-if-Fun-Rel-rev-imp-if-rel-injective-on*:
assumes *rel-injective-on* $(P :: 'a \Rightarrow bool)$ $(R :: 'a \Rightarrow 'b \Rightarrow bool)$
and $(R \Rrightarrow (\longleftarrow))$ P Q
shows *rel-injective-at* Q R
using *assms* **by** (*intro rel-injective-atI*) (*auto dest*: *rel-injective-onD*)

consts *rel-injective* :: $'a \Rightarrow bool$

overloading
rel-injective ≡ *rel-injective* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool$
begin
definition (*rel-injective* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool$) ≡ *rel-injective-on* ($\top$ :: $'a \Rightarrow bool$)
end

lemma *rel-injective-eq-rel-injective-on*:
(*rel-injective* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow$ -) = *rel-injective-on* ($\top$ :: $'a \Rightarrow bool$)
unfolding *rel-injective-def* **..**

lemma *rel-injective-eq-rel-injective-on-uhint* [*uhint*]:
assumes $P \equiv (\top :: 'a \Rightarrow bool)$
shows *rel-injective* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow$ - ≡ *rel-injective-on* P
using *assms* **by** (*simp add*: *rel-injective-eq-rel-injective-on*)

lemma *rel-injectiveI* [*intro*]:
assumes $\bigwedge x\ x'\ y.\ R\ x\ y \Longrightarrow R\ x'\ y \Longrightarrow x = x'$
shows *rel-injective* R
using *assms* **by** (*urule rel-injective-onI*)

lemma *rel-injectiveD*:
assumes *rel-injective* R
and $R\ x\ y\ R\ x'\ y$
shows $x = x'$
using *assms* **by** (*urule* (*d*) *rel-injective-onD chained*: *insert*) *simp-all*

lemma *rel-injective-eq-rel-injective-at*:
(*rel-injective* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool$) = *rel-injective-at* ($\top$ :: $'b \Rightarrow bool$)
by (*intro ext iffI rel-injectiveI*) (*auto dest*: *rel-injective-atD rel-injectiveD*)

lemma *rel-injective-eq-rel-injective-at-uhint* [*uhint*]:
assumes $P \equiv (\top :: 'b \Rightarrow bool)$
shows *rel-injective* $:: ('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool \equiv$ *rel-injective-at* P
using *assms* **by** (*simp add*: *rel-injective-eq-rel-injective-at*)

lemma *rel-injective-on-if-rel-injective*:
fixes $P :: 'a \Rightarrow bool$ **and** $R :: 'a \Rightarrow 'b \Rightarrow bool$
assumes *rel-injective* R
shows *rel-injective-on* P R
using *assms* **by** (*blast dest*: *rel-injectiveD*)

lemma *rel-injective-at-if-rel-injective*:
fixes $P :: 'a \Rightarrow bool$ **and** $R :: 'b \Rightarrow 'a \Rightarrow bool$
assumes *rel-injective* R
shows *rel-injective-at* P R
using *assms* **by** (*blast dest*: *rel-injectiveD*)

lemma *rel-injective-if-rel-injective-on-in-dom*:
assumes *rel-injective-on* (*in-dom* R) R
shows *rel-injective* R
using *assms* **by** (*blast dest*: *rel-injective-onD*)

lemma *rel-injective-if-rel-injective-at-in-codom*:
assumes *rel-injective-at* (*in-codom* R) R
shows *rel-injective* R
using *assms* **by** (*blast dest*: *rel-injective-atD*)

corollary *rel-injective-on-in-dom-iff-rel-injective* [*simp*]:
rel-injective-on (*in-dom* R) $R \longleftrightarrow$ *rel-injective* R
using *rel-injective-if-rel-injective-on-in-dom rel-injective-on-if-rel-injective*
by *blast*

corollary *rel-injective-at-in-codom-iff-rel-injective* [*iff*]:
rel-injective-at (*in-codom* R) $R \longleftrightarrow$ *rel-injective* R
using *rel-injective-if-rel-injective-at-in-codom rel-injective-at-if-rel-injective*
by *blast*

lemma *rel-injective-on-compI*:
fixes $P :: 'a \Rightarrow bool$
assumes *rel-injective* $(R :: 'a \Rightarrow 'b \Rightarrow bool)$
and *rel-injective-on* (*in-codom* $R \sqcap$ *in-dom* S) $(S :: 'b \Rightarrow 'c \Rightarrow bool)$
shows *rel-injective-on* P $(R \circ\circ S)$
proof (*rule rel-injective-onI*)
fix x x' y
assume P x P x' $(R \circ\circ S)$ x y $(R \circ\circ S)$ x' y
then obtain z z' **where** R x z S z y R x' z' S z' y **by** *blast*
with *assms* **have** $z = z'$ **by** (*auto dest*: *rel-injective-onD*)
with ‹R x z› ‹R x' z'› *assms* **show** $x = x'$ **by** (*auto dest*: *rel-injectiveD*)
qed

end

1.2.5 Agreement

theory *Binary-Relations-Agree*
imports
Functions-Monotone
begin

consts *rel-agree-on* :: $'a \Rightarrow 'b \Rightarrow bool$

overloading
rel-agree-on-pred $\equiv$ *rel-agree-on* :: $('a \Rightarrow bool) \Rightarrow (('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool) \Rightarrow bool$
begin
definition *rel-agree-on-pred* $(P :: 'a \Rightarrow bool)$ $(\mathcal{R} :: ('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool) \equiv$
$\forall R\ R' : \mathcal{R}.\ R{\restriction_P} = R'{\restriction_P}$
end

lemma *rel-agree-onI* [*intro*]:
assumes $\bigwedge R\ R'\ x\ y.\ \mathcal{R}\ R \Longrightarrow \mathcal{R}\ R' \Longrightarrow P\ x \Longrightarrow R\ x\ y \Longrightarrow R'\ x\ y$
shows *rel-agree-on* $P\ \mathcal{R}$
using *assms* **unfolding** *rel-agree-on-pred-def* **by** *blast*

lemma *rel-agree-onE*:
assumes *rel-agree-on* $P\ \mathcal{R}$
obtains $\bigwedge R\ R'.\ \mathcal{R}\ R \Longrightarrow \mathcal{R}\ R' \Longrightarrow R{\restriction_P} = R'{\restriction_P}$
using *assms* **unfolding** *rel-agree-on-pred-def* **by** *blast*

lemma *rel-restrict-left-eq-if-rel-agree-onI*:
assumes *rel-agree-on* $P\ \mathcal{R}$
and $\mathcal{R}\ R\ \mathcal{R}\ R'$
shows $R{\restriction_P} = R'{\restriction_P}$
using *assms* **by** (*blast elim*: *rel-agree-onE*)

lemma *rel-agree-onD*:
assumes *rel-agree-on* $P\ \mathcal{R}$
and $\mathcal{R}\ R\ \mathcal{R}\ R'$
and $P\ x$
and $R\ x\ y$
shows $R'\ x\ y$
using *assms* **by** (*blast elim*: *rel-agree-onE dest*: *fun-cong*)

lemma *antimono-rel-agree-on*:
$((\leq) \Rightarrow (\leq) \Rrightarrow (\geq))$ (*rel-agree-on* :: $('a \Rightarrow bool) \Rightarrow (('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool) \Rightarrow bool$)
by (*intro mono-wrt-relI Fun-Rel-relI*) (*fastforce dest*: *rel-agree-onD*)

lemma *le-if-in-dom-le-if-rel-agree-onI*:
assumes *rel-agree-on A* $\mathcal{R}$
and $\mathcal{R}$ *R* $\mathcal{R}$ *R'*
and *in-dom R* $\leq$ *A*
shows *R* $\leq$ *R'*
using *assms* **by** (*auto dest*: *rel-agree-onD*[**where** *?R=R*])

lemma *eq-if-in-dom-le-if-rel-agree-onI*:
assumes *rel-agree-on A* $\mathcal{R}$
and $\mathcal{R}$ *R* $\mathcal{R}$ *R'*
and *in-dom R* $\leq$ *A in-dom R'* $\leq$ *A*
shows *R* = *R'*
using *assms le-if-in-dom-le-if-rel-agree-onI* **by** *blast*

end

1.2.6 Dependent Binary Relations

theory *Dependent-Binary-Relations*
imports
Binary-Relations-Agree
begin

consts *dep-bin-rel* :: *'a* $\Rightarrow$ (*'b* $\Rightarrow$ *'c*) $\Rightarrow$ *'d*
consts *bin-rel* :: *'a* $\Rightarrow$ *'b* $\Rightarrow$ *'c*

open-bundle *bin-rel-syntax*
begin
syntax *-dep-bin-rel* :: ‹*idt* $\Rightarrow$ *'a* $\Rightarrow$ *'b* $\Rightarrow$ *'c*› (‹{$\sum$}- : -./ -› [*51*, *50*, *50*] *51*)
notation *bin-rel* (**infixr** ‹{×}› *51*)
syntax-consts
-dep-bin-rel $\rightleftharpoons$ *dep-bin-rel*
translations
{$\sum$}*x* : *A*. *B* $\rightleftharpoons$ *CONST dep-bin-rel A* (λ*x*. *B*)
end

definition *dep-bin-rel-pred* (*A* :: *'a* $\Rightarrow$ *bool*) (*B* :: *'a* $\Rightarrow$ *'b* $\Rightarrow$ *bool*) (*R* :: *'a* $\Rightarrow$ *'b* $\Rightarrow$ *bool*) $\equiv$
$\forall$ *x y*. *R x y* $\longrightarrow$ *A x* $\wedge$ *B x y*
adhoc-overloading *dep-bin-rel* $\rightleftharpoons$ *dep-bin-rel-pred*

definition *bin-rel-pred* (*A* :: *'a* $\Rightarrow$ *bool*) (*B* :: *'b* $\Rightarrow$ *bool*) :: (*'a* $\Rightarrow$ *'b* $\Rightarrow$ *bool*) $\Rightarrow$ *bool* $\equiv$
{$\sum$}(- :: *'a*) : *A*. *B*
adhoc-overloading *bin-rel* $\rightleftharpoons$ *bin-rel-pred*

lemma *bin-rel-pred-eq-dep-bin-rel-pred*: *A* {×} *B* = {$\sum$}- : *A*. *B*
unfolding *bin-rel-pred-def* **by** *auto*

lemma *bin-rel-pred-eq-dep-bin-rel-pred-uhint* [*uhint*]:
assumes $A \equiv A'$
and $\bigwedge x.\ B \equiv B'\ x$
shows $A\ \{\times\}\ B \equiv \{\sum\}x : A'.\ B'\ x$
using *assms* **by** (*simp add*: *bin-rel-pred-eq-dep-bin-rel-pred*)

lemma *bin-rel-pred-iff-dep-bin-rel-pred*: $(A\ \{\times\}\ B)\ R \longleftrightarrow (\{\sum\}\text{-} : A.\ B)\ R$
unfolding *bin-rel-pred-eq-dep-bin-rel-pred* **by** *auto*

lemma *dep-bin-relI* [*intro*]:
assumes $\bigwedge x\ y.\ R\ x\ y \Longrightarrow A\ x$
and $\bigwedge x\ y.\ R\ x\ y \Longrightarrow A\ x \Longrightarrow B\ x\ y$
shows $(\{\sum\}x : A.\ B\ x)\ R$
using *assms* **unfolding** *dep-bin-rel-pred-def* **by** *auto*

lemma *dep-bin-rel-if-bin-rel-and*:
assumes $\bigwedge x\ y.\ R\ x\ y \Longrightarrow A\ x \wedge B\ x\ y$
shows $(\{\sum\}x : A.\ B\ x)\ R$
using *assms* **by** *auto*

lemma *dep-bin-relE* [*elim*]:
assumes $(\{\sum\}x : A.\ B\ x)\ R$
and $R\ x\ y$
obtains $A\ x\ B\ x\ y$
using *assms* **unfolding** *dep-bin-rel-pred-def* **by** *auto*

lemma *dep-bin-relE'*:
assumes $(\{\sum\}x : A.\ B\ x)\ R$
obtains $\bigwedge x\ y.\ R\ x\ y \Longrightarrow A\ x \wedge B\ x\ y$
using *assms* **by** *auto*

lemma *bin-relI* [*intro*]:
assumes $\bigwedge x\ y.\ R\ x\ y \Longrightarrow A\ x$
and $\bigwedge x\ y.\ R\ x\ y \Longrightarrow A\ x \Longrightarrow B\ y$
shows $(A\ \{\times\}\ B)\ R$
using *assms* **by** (*urule dep-bin-relI chained*: *fact*)

lemma *bin-rel-if-bin-rel-and*:
assumes $\bigwedge x\ y.\ R\ x\ y \Longrightarrow A\ x \wedge B\ y$
shows $(A\ \{\times\}\ B)\ R$
using *assms* **by** (*urule dep-bin-rel-if-bin-rel-and*)

lemma *bin-relE* [*elim*]:
assumes $(A\ \{\times\}\ B)\ R$
and $R\ x\ y$
obtains $A\ x\ B\ y$
using *assms* **by** (*urule* (*e*) *dep-bin-relE*)

lemma *bin-relE'*:
assumes $(A\ \{\times\}\ B)\ R$
obtains $\bigwedge x\ y.\ R\ x\ y \Longrightarrow A\ x \wedge B\ y$
using *assms* **by** (*urule* (*e*) *dep-bin-relE'*)

lemma *dep-bin-rel-cong* [*cong*]:
$[\![A = A';\ \bigwedge x.\ A'\ x \Longrightarrow B\ x = B'\ x]\!] \Longrightarrow (\{\sum\}x : A.\ B\ x) = \{\sum\}x : A'.\ B'\ x$
by (*intro ext iffI dep-bin-relI*) *fastforce+*

lemma *le-dep-bin-rel-if-le-dom*:
assumes $A \leq A'$
shows $(\{\sum\}x : A.\ B\ x) \leq (\{\sum\}x : A'.\ B\ x)$
using *assms* **by** (*intro le-predI dep-bin-relI*) *auto*

lemma *dep-bin-rel-covariant-codom*:
assumes $(\{\sum\}x : A.\ B\ x)\ R$
and $\bigwedge x\ y.\ R\ x\ y \Longrightarrow A\ x \Longrightarrow B\ x\ y \Longrightarrow B'\ x\ y$
shows $(\{\sum\}x : A.\ B'\ x)\ R$
using *assms* **by** (*intro dep-bin-relI*) *auto*

lemma *mono-dep-bin-rel*: $((\leq) \Rightarrow (\leq) \Rrightarrow (\geq) \Rrightarrow (\leq))$ *dep-bin-rel*
by (*intro mono-wrt-relI Fun-Rel-relI dep-bin-relI*) *force*

lemma *mono-bin-rel*: $((\leq) \Rightarrow (\leq) \Rrightarrow (\geq) \Rrightarrow (\leq))\ (\{\times\})$
by (*intro mono-wrt-relI Fun-Rel-relI*) *auto*

lemma *in-dom-le-if-dep-bin-rel*:
assumes $(\{\sum\}x : A.\ B\ x)\ R$
shows *in-dom* $R \leq A$
using *assms* **by** *auto*

lemma *in-codom-le-in-codom-on-if-dep-bin-rel*:
assumes $(\{\sum\}x : A.\ B\ x)\ R$
shows *in-codom* $R \leq$ *in-codom-on* $A\ B$
using *assms* **by** *fast*

lemma *rel-restrict-left-eq-self-if-dep-bin-rel* [*simp*]:
assumes $(\{\sum\}x : A.\ B\ x)\ R$
shows $R\restriction_A = R$
using *assms rel-restrict-left-eq-self-if-in-dom-le* **by** *auto*

lemma *dep-bin-rel-bottom-dom-iff-eq-bottom* [*iff*]: $(\{\sum\}x : \bot.\ B\ x)\ R \longleftrightarrow R = \bot$
by *fastforce*

lemma *dep-bin-rel-bottom-codom-iff-eq-bottom* [*iff*]: $(\{\sum\}x : A.\ \bot)\ R \longleftrightarrow R = \bot$
by *fastforce*

lemma *mono-bin-rel-dep-bin-rel-bin-rel-in-codom-on-rel-comp*:
$(A\ \{\times\}\ B \Rightarrow (\{\sum\}x : B.\ C\ x) \Rightarrow A\ \{\times\}$ *in-codom-on* $B\ C)\ (\circ\circ)$

by *fastforce*

lemma *mono-bin-rel-bin-rel-bin-rel-rel-comp*: $(A \ \{\times\} \ B \Rightarrow B \ \{\times\} \ C \Rightarrow A \ \{\times\} \ C) \ (\circ\circ)$
by *fast*

lemma *mono-dep-bin-rel-bin-rel-rel-inv*: $((\{\sum\} x : A.\ B\ x) \Rightarrow \textit{in-codom-on}\ A\ B \ \{\times\} \ A)\ \textit{rel-inv}$
by *force*

lemma *mono-bin-rel-rel-inv*: $(A \ \{\times\} \ B \Rightarrow B \ \{\times\} \ A)\ \textit{rel-inv}$
by *auto*

lemma *mono-dep-bin-rel-top-dep-bin-rel-inf-rel-restrict-left*:
$((\{\sum\} x : A.\ B\ x) \Rightarrow (P : \top) \Rightarrow (\{\sum\} x : A \sqcap P.\ B\ x))\ \textit{rel-restrict-left}$
by *fast*

lemma *mono-dep-bin-rel-top-dep-bin-rel-inf-rel-restrict-right*:
$((\{\sum\} x : A.\ B\ x) \Rightarrow (P : \top) \Rightarrow (\{\sum\} x : A.\ B\ x \sqcap P))\ \textit{rel-restrict-right}$
by *fast*

lemma *mono-dep-bin-rel-top-dep-bin-rel-inf-rel-restrict*:
$((\{\sum\} x : A.\ B\ x) \Rightarrow (P : \top) \Rightarrow (\{\sum\} x : A \sqcap P.\ B\ x \sqcap P))\ \textit{rel-restrict}$
by *fast*

lemma *le-if-rel-agree-on-if-dep-bin-relI*:
assumes $(\{\sum\} x : A.\ B\ x)\ R$
and *rel-agree-on* $A\ \mathcal{R}$
and $\mathcal{R}\ R\ \mathcal{R}\ R'$
shows $R \leq R'$
using *assms* **by** (*intro le-if-in-dom-le-if-rel-agree-onI in-dom-le-if-dep-bin-rel*)

lemma *eq-if-rel-agree-on-if-dep-bin-relI*:
assumes $(\{\sum\} x : A.\ B\ x)\ R\ (\{\sum\} x : A.\ B'\ x)\ R'$
and *rel-agree-on* $A\ \mathcal{R}$
and $\mathcal{R}\ R\ \mathcal{R}\ R'$
shows $R = R'$
using *assms* **by** (*intro eq-if-in-dom-le-if-rel-agree-onI in-dom-le-if-dep-bin-rel*)

end

1.2.7 Extensions

theory *Binary-Relations-Extend*
imports
Dependent-Binary-Relations
begin

consts *extend* :: $'a \Rightarrow 'b \Rightarrow 'c \Rightarrow 'c$

definition *extend-rel* $x\ y\ R\ x'\ y' \equiv (x = x' \wedge y = y') \vee R\ x'\ y'$
adhoc-overloading *extend* $\rightleftharpoons$ *extend-rel*

lemma *extend-leftI* [*iff*]: (*extend* $x\ y\ R$) $x\ y$
unfolding *extend-rel-def* **by** *blast*

lemma *extend-rightI* [*intro*]:
assumes $R\ x'\ y'$
shows (*extend* $x\ y\ R$) $x'\ y'$
unfolding *extend-rel-def* **using** *assms* **by** *blast*

lemma *extendE* [*elim*]:
assumes (*extend* $x\ y\ R$) $x'\ y'$
obtains $x = x'\ y = y' \mid x \neq x' \vee y \neq y'\ R\ x'\ y'$
using *assms* **unfolding** *extend-rel-def* **by** *blast*

lemma *extend-eq-self-if-rel* [*simp*]: $R\ x\ y \Longrightarrow$ *extend* $x\ y\ R = R$
by *auto*

context
fixes $x :: 'a$ **and** $y :: 'b$ **and** $R :: 'a \Rightarrow 'b \Rightarrow$ *bool*
begin

lemma *in-dom-extend-eq*: *in-dom* (*extend* $x\ y\ R$) = *in-dom* $R \sqcup (=)\ x$
by (*intro ext*) *auto*

lemma *in-dom-extend-iff*: *in-dom* (*extend* $x\ y\ R$) $x' \longleftrightarrow$ *in-dom* $R\ x' \vee x = x'$
by (*auto simp*: *in-dom-extend-eq*)

lemma *codom-extend-eq*: *in-codom* (*extend* $x\ y\ R$) = *in-codom* $R \sqcup (=)\ y$
by (*intro ext*) *auto*

lemma *in-codom-extend-iff*: *in-codom* (*extend* $x\ y\ R$) $y' \longleftrightarrow$ *in-codom* $R\ y' \vee y = y'$
by (*auto simp*: *codom-extend-eq*)

end

lemma *in-field-extend-eq*: *in-field* (*extend* $x\ y\ R$) = *in-field* $R \sqcup (=)\ x \sqcup (=)\ y$
by (*intro ext*) *auto*

lemma *in-field-extend-iff*: *in-field* (*extend* $x\ y\ R$) $z \longleftrightarrow$ *in-field* $R\ z \vee z = x \vee z = y$
by (*auto simp*: *in-field-extend-eq*)

lemma *mono-extend*: *mono* (*extend* $x\ y :: ('a \Rightarrow 'b \Rightarrow bool) \Rightarrow 'a \Rightarrow 'b \Rightarrow bool$)
by (*intro monoI*) *force*

lemma *dep-mono-dep-bin-rel-extend*:
(($x : A$) $\Rightarrow$ $B\ x \Rightarrow$ ($\{\sum\}x : A'.\ B'\ x$) $\Rightarrow$ ($\{\sum\}x : A \sqcup A'.\ (B \sqcup B')\ x$)) *extend*
by *fastforce*

consts *glue* :: $'a \Rightarrow 'b$

definition *glue-rel* ($\mathcal{R}$ :: ($'a \Rightarrow 'b \Rightarrow$ *bool*) $\Rightarrow$ *bool*) $x \equiv$ *in-codom-on* $\mathcal{R}$ ($\lambda R.\ R\ x$)
adhoc-overloading *glue* $\rightleftharpoons$ *glue-rel*

lemma *glue-rel-eq-in-codom-on*: *glue* $\mathcal{R}\ x =$ *in-codom-on* $\mathcal{R}$ ($\lambda R.\ R\ x$)
unfolding *glue-rel-def* **by** *simp*

lemma *glueI* [*intro*]:
assumes $\mathcal{R}\ R$
and $R\ x\ y$
shows *glue* $\mathcal{R}\ x\ y$
using *assms* **unfolding** *glue-rel-def* **by** *blast*

lemma *glueE* [*elim*!]:
assumes *glue* $\mathcal{R}\ x\ y$
obtains R **where** $\mathcal{R}\ R\ R\ x\ y$
using *assms* **unfolding** *glue-rel-def* **by** *blast*

lemma *glue-bottom-eq* [*simp*]: *glue* ($\bot$ :: ($'a \Rightarrow 'b \Rightarrow$ *bool*) $\Rightarrow$ *bool*) $= \bot$
by (*intro ext*) *auto*

lemma *glue-eq-rel-eq-self* [*simp*]: *glue* ((=) R) = (R :: $'a \Rightarrow 'b \Rightarrow$ *bool*)
by (*intro ext*) *auto*

lemma *glue-sup-eq-glue-sup-glue* [*simp*]: *glue* ($A \sqcup B$) = *glue* $A \sqcup$ *glue* B
supply *glue-rel-eq-in-codom-on*[*simp*]
by (*rule ext*) (*urule in-codom-on-sup-pred-eq-in-codom-on-sup-in-codom-on*)

lemma *mono-glue*: *mono* (*glue* :: (($'a \Rightarrow 'b \Rightarrow$ *bool*) $\Rightarrow$ *bool*) $\Rightarrow$ ($'a \Rightarrow 'b \Rightarrow$ *bool*))
by *auto*

lemma *dep-bin-rel-glueI*:
fixes $\mathcal{R}$ **defines** [*simp*]: $D \equiv$ *in-codom-on* $\mathcal{R}$ *in-dom*
assumes $\bigwedge R.\ \mathcal{R}\ R \Longrightarrow \exists A.\ (\{\sum\}x : A.\ B\ x)\ R$
shows ($\{\sum\}x : D.\ B\ x$) (*glue* $\mathcal{R}$)
using *assms* **by** (*intro dep-bin-relI*) *auto*

end

Right Unique

theory *Binary-Relations-Right-Unique*

imports
Binary-Relations-Injective
Binary-Relations-Extend
begin

consts *right-unique-on* :: $'a \Rightarrow 'b \Rightarrow bool$

overloading
right-unique-on-pred $\equiv$ *right-unique-on* :: $('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool$
begin
definition *right-unique-on-pred* $P\ R \equiv \forall\, x : P.\ \forall\, y\ y'.\ R\ x\ y \wedge R\ x\ y' \longrightarrow y = y'$
end

lemma *right-unique-onI* [*intro*]:
assumes $\bigwedge x\ y\ y'.\ P\ x \Longrightarrow R\ x\ y \Longrightarrow R\ x\ y' \Longrightarrow y = y'$
shows *right-unique-on* $P\ R$
using *assms* **unfolding** *right-unique-on-pred-def* **by** *blast*

lemma *right-unique-onD*:
assumes *right-unique-on* $P\ R$
and $P\ x$
and $R\ x\ y\ R\ x\ y'$
shows $y = y'$
using *assms* **unfolding** *right-unique-on-pred-def* **by** *blast*

lemma *antimono-right-unique-on*:
$((\leq) \Rightarrow (\leq) \Rrightarrow (\geq))$ (*right-unique-on* :: $('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool$)
by (*fastforce dest*: *right-unique-onD*)

lemma *mono-right-unique-on-top-right-unique-on-inf-rel-restrict-left*:
$((R :$ *right-unique-on* $P) \Rightarrow (P' : \top) \Rightarrow$ *right-unique-on* $(P \sqcap P'))$ *rel-restrict-left*
by (*fast dest*: *right-unique-onD*)

lemma *mono-right-unique-on-comp*:
$((R :$ *right-unique-on* $P) \Rightarrow$ *right-unique-on* (*in-codom* $(R\restriction_P)) \Rightarrow$ *right-unique-on* $P)$ $(\circ\circ)$
by (*fast dest*: *right-unique-onD*)

context
fixes P :: $'a \Rightarrow bool$ **and** $\mathcal{R}$ :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool$
begin

lemma *right-unique-on-glue-if-right-unique-on-sup*:
assumes $\bigwedge R\ R'.\ \mathcal{R}\ R \Longrightarrow \mathcal{R}\ R' \Longrightarrow$ *right-unique-on* $P\ (R \sqcup R')$
shows *right-unique-on* P (*glue* $\mathcal{R}$)
using *assms* **by** (*fastforce dest*: *right-unique-onD*)

lemma *right-unique-on-if-right-unique-on-glue*:
assumes *right-unique-on* P (*glue* $\mathcal{R}$)

and $\mathcal{R}$ R
shows *right-unique-on* P R
using *assms* **by** (*intro right-unique-onI*) (*auto dest*: *right-unique-onD*)

end

consts *right-unique-at* :: $'a \Rightarrow 'b \Rightarrow$ *bool*

overloading
right-unique-at-pred $\equiv$ *right-unique-at* :: $('a \Rightarrow$ *bool*$) \Rightarrow ('b \Rightarrow 'a \Rightarrow$ *bool*$) \Rightarrow$ *bool*
begin
definition *right-unique-at-pred* P $R \equiv \forall x\ y\ y'.\ P\ y \wedge P\ y' \wedge R\ x\ y \wedge R\ x\ y' \longrightarrow y = y'$
end

lemma *right-unique-atI* [*intro*]:
assumes $\bigwedge x\ y\ y'.\ P\ y \Longrightarrow P\ y' \Longrightarrow R\ x\ y \Longrightarrow R\ x\ y' \Longrightarrow y = y'$
shows *right-unique-at* P R
using *assms* **unfolding** *right-unique-at-pred-def* **by** *blast*

lemma *right-unique-atD*:
assumes *right-unique-at* P R
and $P\ y$
and $P\ y'$
and $R\ x\ y$ $R\ x\ y'$
shows $y = y'$
using *assms* **unfolding** *right-unique-at-pred-def* **by** *blast*

lemma *right-unique-at-rel-inv-iff-rel-injective-on* [*iff*]:
right-unique-at $(P :: 'a \Rightarrow$ *bool*$)$ $(R^{-1} :: 'b \Rightarrow 'a \Rightarrow$ *bool*$) \longleftrightarrow$ *rel-injective-on* P R
by (*blast dest*: *right-unique-atD rel-injective-onD*)

lemma *rel-injective-on-rel-inv-iff-right-unique-at* [*iff*]:
rel-injective-on $(P :: 'a \Rightarrow$ *bool*$)$ $(R^{-1} :: 'a \Rightarrow 'b \Rightarrow$ *bool*$) \longleftrightarrow$ *right-unique-at* P R
by (*blast dest*: *right-unique-atD rel-injective-onD*)

lemma *right-unique-on-rel-inv-iff-rel-injective-at* [*iff*]:
right-unique-on $(P :: 'a \Rightarrow$ *bool*$)$ $(R^{-1} :: 'a \Rightarrow 'b \Rightarrow$ *bool*$) \longleftrightarrow$ *rel-injective-at* P R
by (*blast dest*: *right-unique-onD rel-injective-atD*)

lemma *rel-injective-at-rel-inv-iff-right-unique-on* [*iff*]:
rel-injective-at $(P :: 'b \Rightarrow$ *bool*$)$ $(R^{-1} :: 'a \Rightarrow 'b \Rightarrow$ *bool*$) \longleftrightarrow$ *right-unique-on* P R
by (*blast dest*: *right-unique-onD rel-injective-atD*)

lemma *right-unique-on-if-Fun-Rel-imp-if-right-unique-at*:

assumes *right-unique-at* (*Q* :: *'b* ⇒ *bool*) (*R* :: *'a* ⇒ *'b* ⇒ *bool*)
and (*R* ⇛ (⟶)) *P Q*
shows *right-unique-on P R*
using *assms* **by** (*intro right-unique-onI*) (*auto dest*: *right-unique-atD*)

lemma *right-unique-at-if-Fun-Rel-rev-imp-if-right-unique-on*:
assumes *right-unique-on* (*P* :: *'a* ⇒ *bool*) (*R* :: *'a* ⇒ *'b* ⇒ *bool*)
and (*R* ⇛ (⟵)) *P Q*
shows *right-unique-at Q R*
using *assms* **by** (*intro right-unique-atI*) (*auto dest*: *right-unique-onD*)

consts *right-unique* :: *'a* ⇒ *bool*

overloading
right-unique ≡ *right-unique* :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *bool*
begin
definition (*right-unique* :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *bool*) ≡ *right-unique-on* (⊤ :: *'a* ⇒ *bool*)
end

lemma *right-unique-eq-right-unique-on*:
(*right-unique* :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ -) = *right-unique-on* (⊤ :: *'a* ⇒ *bool*)
unfolding *right-unique-def* **..**

lemma *right-unique-eq-right-unique-on-uhint* [*uhint*]:
assumes *P* ≡ (⊤ :: *'a* ⇒ *bool*)
shows *right-unique* :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ - ≡ *right-unique-on P*
using *assms* **by** (*simp only*: *right-unique-eq-right-unique-on*)

lemma *right-uniqueI* [*intro*]:
assumes ⋀*x y y'*. *R x y* ⟹ *R x y'* ⟹ *y* = *y'*
shows *right-unique R*
using *assms* **by** (*urule right-unique-onI*)

lemma *right-uniqueD*:
assumes *right-unique R*
and *R x y R x y'*
shows *y* = *y'*
using *assms* **by** (*urule* (*d*) *right-unique-onD chained*: *insert*) *simp-all*

lemma *right-unique-eq-right-unique-at*:
(*right-unique* :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *bool*) = *right-unique-at* (⊤ :: *'b* ⇒ *bool*)
by (*intro ext iffI right-uniqueI*) (*auto dest*: *right-unique-atD right-uniqueD*)

lemma *right-unique-eq-right-unique-at-uhint* [*uhint*]:
assumes *P* ≡ (⊤ :: *'b* ⇒ *bool*)
shows *right-unique* :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ - ≡ *right-unique-at P*
using *assms* **by** (*simp only*: *right-unique-eq-right-unique-at*)

lemma *right-unique-on-if-right-unique*:
fixes $P :: 'a \Rightarrow bool$ **and** $R :: 'a \Rightarrow 'b \Rightarrow bool$
assumes *right-unique R*
shows *right-unique-on P R*
using *assms* **by** (*blast dest*: *right-uniqueD*)

lemma *right-unique-at-if-right-unique*:
fixes $P :: 'a \Rightarrow bool$ **and** $R :: 'b \Rightarrow 'a \Rightarrow bool$
assumes *right-unique R*
shows *right-unique-at P R*
using *assms* **by** (*blast dest*: *right-uniqueD*)

lemma *right-unique-if-right-unique-on-in-dom*:
assumes *right-unique-on* (*in-dom R*) *R*
shows *right-unique R*
using *assms* **by** (*blast dest*: *right-unique-onD*)

lemma *right-unique-if-right-unique-at-in-codom*:
assumes *right-unique-at* (*in-codom R*) *R*
shows *right-unique R*
using *assms* **by** (*blast dest*: *right-unique-atD*)

corollary *right-unique-on-in-dom-iff-right-unique* [*iff*]:
right-unique-on (*in-dom R*) $R \longleftrightarrow$ *right-unique R*
using *right-unique-if-right-unique-on-in-dom right-unique-on-if-right-unique*
by *blast*

corollary *right-unique-at-in-codom-iff-right-unique* [*iff*]:
right-unique-at (*in-codom R*) $R \longleftrightarrow$ *right-unique R*
using *right-unique-if-right-unique-at-in-codom right-unique-at-if-right-unique*
by *blast*

lemma *right-unique-rel-inv-iff-rel-injective* [*iff*]:
right-unique $R^{-1} \longleftrightarrow$ *rel-injective R*
by (*blast dest*: *right-uniqueD rel-injectiveD*)

lemma *rel-injective-rel-inv-iff-right-unique* [*iff*]:
rel-injective $R^{-1} \longleftrightarrow$ *right-unique R*
by (*blast dest*: *right-uniqueD rel-injectiveD*)

Instantiations **lemma** *right-unique-eq*: *right-unique* (=)
by (*rule right-uniqueI*) *blast*

end

1.2.8 Evaluation of Functions as Binary Relations

theory *Binary-Relations-Function-Evaluation*
imports
Binary-Relations-Right-Unique
Binary-Relations-Extend
begin

consts *eval* :: $'a \Rightarrow 'b \Rightarrow 'c$

definition *eval-rel* $R\ x \equiv THE\ y.\ R\ x\ y$
adhoc-overloading *eval* $\rightleftharpoons$ *eval-rel*

open-bundle *eval-syntax*
begin
notation *eval* ($\langle$'(‘')$\rangle$)
notation *eval* ($\langle$(-‘-)$\rangle$ [*999*, *1000*] *999*)
end

lemma *eval-eq-if-right-unique-onI*:
assumes *right-unique-on P R*
and *P x*
and *R x y*
shows *R‘x* = *y*
using *assms* **unfolding** *eval-rel-def* **by** (*auto dest*: *right-unique-onD*)

lemma *eval-eq-if-right-unique-on-eqI*:
assumes *right-unique-on* ((=) *x*) *R*
and *R x y*
shows *R‘x* = *y*
using *assms* **by** (*auto intro*: *eval-eq-if-right-unique-onI*)

lemma *rel-eval-if-ex1*:
assumes $\exists!y.\ R\ x\ y$
shows *R x* (*R‘x*)
using *assms* **unfolding** *eval-rel-def* **by** (*rule theI'*)

lemma *rel-if-eval-eq-if-in-dom-if-right-unique-on-eq*:
assumes *right-unique-on* ((=) *x*) *R*
and *in-dom R x*
and *R‘x* = *y*
shows *R x y*
using *assms* **by** (*blast intro*: *rel-eval-if-ex1*[*of R x*] *dest*: *right-unique-onD*)

corollary *rel-eval-if-in-dom-if-right-unique-on-eq*:
assumes *right-unique-on* ((=) *x*) *R*
and *in-dom R x*
shows *R x* (*R‘x*)
using *assms* **by** (*rule rel-if-eval-eq-if-in-dom-if-right-unique-on-eq*) *simp*

lemma *eq-app-eval-eq-eq* [*simp*]: $(\lambda x.\ (=)\ (f\ x))\ `x = f\ x$
by (*auto intro*: *eval-eq-if-right-unique-onI*)

lemma *extend-eval-eq-if-not-in-dom* [*simp*]:
assumes $\neg$(*in-dom R x*)
shows (*extend x y R*)`$x = y$
using *assms* **by** (*force intro*: *eval-eq-if-right-unique-on-eqI*)

corollary *extend-bottom-eval-eq* [*simp*]:
fixes $x :: 'a$ **and** $y :: 'b$
shows (*extend x y* ($\bot :: 'a \Rightarrow 'b \Rightarrow$ *bool*))`$x = y$
by (*intro extend-eval-eq-if-not-in-dom*) *auto*

lemma *glue-eval-eqI*:
assumes *right-unique-on P* (*glue* $\mathcal{R}$)
and $\mathcal{R}$ *R*
and *P x*
and *R x y*
shows (*glue* $\mathcal{R}$)`$x = y$
using *assms* **by** (*auto intro*: *eval-eq-if-right-unique-onI*)

lemma *glue-eval-eq-evalI*:
assumes *right-unique-on P* (*glue* $\mathcal{R}$)
and $\mathcal{R}$ *R*
and *P x*
and *in-dom R x*
shows (*glue* $\mathcal{R}$)`$x = R`x$
using *assms* **by** (*intro glue-eval-eqI*[*of P* $\mathcal{R}$ *R*])
(*auto intro*: *rel-if-eval-eq-if-in-dom-if-right-unique-on-eq*[*of x*] *dest*: *right-unique-onD*)

Note: the following rest on the definition of extend and eval:

lemma *extend-eval-eq-if-neq* [*simp*]:
fixes $R :: 'a \Rightarrow 'b \Rightarrow$ *bool*
shows $x \neq y \Longrightarrow$ (*extend y z R*)`$x = R`x$
unfolding *extend-rel-def eval-rel-def* **by** *auto*

lemma *sup-eval-eq-left-eval-if-not-in-dom* [*simp*]:
fixes *R S* $:: 'a \Rightarrow 'b \Rightarrow$ *bool*
shows $\neg$(*in-dom S x*) $\Longrightarrow$ ($R \sqcup S$)`$x = R`x$
unfolding *eval-rel-def* **by** (*cases* $\exists\, y.$ *S x y*) *auto*

lemma *sup-eval-eq-right-eval-if-not-in-dom* [*simp*]:
fixes *R S* $:: 'a \Rightarrow 'b \Rightarrow$ *bool*
shows $\neg$(*in-dom R x*) $\Longrightarrow$ ($R \sqcup S$)`$x = S`x$
unfolding *eval-rel-def* **by** (*cases* $\exists\, y.$ *R x y*) *auto*

lemma *rel-restrict-left-eval-eq-if-pred* [*simp*]:
fixes $R :: 'a \Rightarrow 'b \Rightarrow$ *bool*
assumes *P x*

shows $(R\restriction_P)`x = R`x$

using *assms* **unfolding** *eval-rel-def rel-restrict-left-pred-def* **by** *auto*

end

Left Total

theory *Binary-Relations-Left-Total*
imports
Functions-Monotone
begin

consts *left-total-on* :: $'a \Rightarrow 'b \Rightarrow bool$

overloading
left-total-on-pred $\equiv$ *left-total-on* :: $('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool$
begin
definition *left-total-on-pred* $P\ R \equiv \forall x : P.$ *in-dom* $R\ x$
end

lemma *left-total-onI* [*intro*]:
assumes $\bigwedge x.\ P\ x \Longrightarrow$ *in-dom* $R\ x$
shows *left-total-on* $P\ R$
unfolding *left-total-on-pred-def* **using** *assms* **by** *blast*

lemma *left-total-onE* [*elim*]:
assumes *left-total-on* $P\ R$
and $P\ x$
obtains y **where** $R\ x\ y$
using *assms* **unfolding** *left-total-on-pred-def* **by** *blast*

lemma *le-in-dom-if-left-total-on*:
assumes *left-total-on* $P\ R$
shows $P \leq$ *in-dom* R
using *assms* **by** *force*

lemma *left-total-on-if-le-in-dom*:
assumes $P \leq$ *in-dom* R
shows *left-total-on* $P\ R$
using *assms* **by** *fastforce*

lemma *mono-left-total-on*:
$((\geq) \Rightarrow (\leq) \Rrightarrow (\leq))$ (*left-total-on* :: $('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool$)
by *fastforce*

lemma *le-in-dom-iff-left-total-on*: $P \leq$ *in-dom* $R \longleftrightarrow$ *left-total-on* $P\ R$
using *le-in-dom-if-left-total-on left-total-on-if-le-in-dom* **by** *auto*

lemma *mono-left-total-on-top-left-total-on-inf-rel-restrict-left*:
((R : *left-total-on* P) ⇒ (P' : ⊤) ⇒ *left-total-on* (P ⊓ P')) *rel-restrict-left*
by *fast*

lemma *mono-left-total-on-comp*:
((R : *left-total-on* P) ⇒ *left-total-on* (*in-codom* ($R{\restriction}_P$)) ⇒ *left-total-on* P) (∘∘)
by *fast*

consts *left-total* :: $'a$ ⇒ *bool*

overloading
left-total ≡ *left-total* :: ($'a$ ⇒ $'b$ ⇒ *bool*) ⇒ *bool*
begin
definition (*left-total* :: ($'a$ ⇒ $'b$ ⇒ *bool*) ⇒ *bool*) ≡ *left-total-on* (⊤ :: $'a$ ⇒ *bool*)
end

lemma *left-total-eq-left-total-on*:
(*left-total* :: ($'a$ ⇒ $'b$ ⇒ *bool*) ⇒ -) = *left-total-on* (⊤ :: $'a$ ⇒ *bool*)
unfolding *left-total-def* **..**

lemma *left-total-eq-left-total-on-uhint* [*uhint*]:
assumes P ≡ ⊤ :: $'a$ ⇒ *bool*
shows *left-total* :: ($'a$ ⇒ $'b$ ⇒ *bool*) ⇒ - ≡ *left-total-on* P
using *assms* **by** (*simp add*: *left-total-eq-left-total-on*)

lemma *left-totalI* [*intro*]:
assumes ⋀x. *in-dom* R x
shows *left-total* R
using *assms* **by** (*urule left-total-onI*)

lemma *left-totalE*:
assumes *left-total* R
obtains y **where** R x y
using *assms* **by** (*urule* (*e*) *left-total-onE chained*: *insert*) *simp*

lemma *in-dom-if-left-total*:
assumes *left-total* R
shows *in-dom* R x
using *assms* **by** (*blast elim*: *left-totalE*)

lemma *left-total-on-if-left-total*:
fixes P :: $'a$ ⇒ *bool* **and** R :: $'a$ ⇒ $'b$ ⇒ *bool*
assumes *left-total* R
shows *left-total-on* P R
using *assms* **by** (*intro left-total-onI*) (*blast dest*: *in-dom-if-left-total*)

end

1.3 Functions as Binary Relations

theory *Binary-Relations-Function-Base*
imports
Binary-Relations-Function-Evaluation
Binary-Relations-Left-Total
begin

Relational functions may contain further elements outside their specification.

consts *rel-dep-mono-wrt* :: $'a \Rightarrow 'b \Rightarrow 'c$
consts *rel-mono-wrt* :: $'a \Rightarrow 'b \Rightarrow 'c$

open-bundle *rel-mono-wrt-syntax*
begin
notation *rel-mono-wrt* (**infixr** $\langle\rightarrow\rangle$ *40*)
syntax
-rel-dep-mono-wrt-pred :: *idt* $\Rightarrow 'a \Rightarrow 'b \Rightarrow 'c \Rightarrow$ *bool* ($\langle '(-/ :/ -') \rightarrow (-)\rangle$ [*41*, *41*, *40*] *40*)
syntax-consts *-rel-dep-mono-wrt-pred* $\rightleftharpoons$ *rel-dep-mono-wrt*
translations
$(x : A) \rightarrow B \rightleftharpoons$ *CONST rel-dep-mono-wrt A* $(\lambda x.\ B)$
end

definition *rel-dep-mono-wrt-pred* $(A :: 'a \Rightarrow$ *bool*$)$ $(B :: 'a \Rightarrow 'b \Rightarrow$ *bool*$)$ $(R :: 'a \Rightarrow 'b \Rightarrow$ *bool*$) \equiv$
left-total-on A R $\wedge$ *right-unique-on A R* $\wedge$ $((x : A) \Rightarrow B\ x)$ *(eval R)*
adhoc-overloading *rel-dep-mono-wrt* $\rightleftharpoons$ *rel-dep-mono-wrt-pred*

definition *rel-mono-wrt-pred* $(A :: 'a \Rightarrow$ *bool*$)$ $(B :: 'b \Rightarrow$ *bool*$)$ :: $('a \Rightarrow 'b \Rightarrow$ *bool*$) \Rightarrow$ *bool* $\equiv$
rel-dep-mono-wrt-pred A $(\lambda(- :: 'a).\ B)$
adhoc-overloading *rel-mono-wrt* $\rightleftharpoons$ *rel-mono-wrt-pred*

lemma *rel-mono-wrt-pred-eq-rel-dep-mono-wrt-pred*:
$(((A :: 'a \Rightarrow$ *bool*$) \rightarrow (B :: 'b \Rightarrow$ *bool*$)) :: ('a \Rightarrow 'b \Rightarrow$ *bool*$) \Rightarrow$ *bool*$) = (((- :: 'a) : A) \rightarrow B)$
by (*simp add*: *rel-mono-wrt-pred-def*)

lemma *rel-mono-wrt-pred-eq-rel-dep-mono-wrt-pred-uhint* [*uhint*]:
assumes $(A :: 'a \Rightarrow$ *bool*$) \equiv A'$
and $\bigwedge x.\ B :: 'b \Rightarrow$ *bool* $\equiv B'\ x$
shows $(A \rightarrow B) \equiv ((x : A') \rightarrow B'\ x)$
using *assms* **by** (*simp add*: *rel-mono-wrt-pred-eq-rel-dep-mono-wrt-pred*)

lemma *rel-mono-wrt-pred-iff-rel-dep-mono-wrt-pred*:
$((A :: 'a \Rightarrow$ *bool*$) \rightarrow (B :: 'b \Rightarrow$ *bool*$))$ $(R :: 'a \Rightarrow 'b \Rightarrow$ *bool*$) \longleftrightarrow (((- :: 'a) : A) \rightarrow B)\ R$
by (*simp add*: *rel-mono-wrt-pred-def*)

lemma *rel-dep-mono-wrt-predI* [*intro*]:
assumes *left-total-on A R*
and *right-unique-on A R*
and ((*x* : *A*) ⇒ *B x*) (*eval R*)
shows ((*x* : *A*) → *B x*) *R*
using *assms* **unfolding** *rel-dep-mono-wrt-pred-def* **by** *auto*

lemma *rel-dep-mono-wrt-predE* [*elim*]:
assumes ((*x* : *A*) → *B x*) *R*
obtains *left-total-on A R right-unique-on A R* ((*x* : *A*) ⇒ *B x*) (*eval R*)
using *assms* **unfolding** *rel-dep-mono-wrt-pred-def* **by** *auto*

lemma *rel-dep-mono-wrt-pred-cong* [*cong*]:
assumes *A* = *A*′
and ⋀*x y*. *A*′ *x* ⟹ *B x* = *B*′ *x*
shows ((*x* : *A*) → *B x*) = ((*x* : *A*′) → *B*′ *x*)
using *assms* **by** *fastforce*

lemma *rel-mono-wrt-predI* [*intro*]:
assumes *left-total-on A R*
and *right-unique-on A R*
and (*A* ⇒ *B*) (*eval R*)
shows (*A* → *B*) *R*
using *assms* **by** (*urule rel-dep-mono-wrt-predI*)

lemma *rel-mono-wrt-predE* [*elim*]:
assumes (*A* → *B*) *R*
obtains *left-total-on A R right-unique-on A R* (*A* ⇒ *B*) (*eval R*)
using *assms* **by** (*urule* (*e*) *rel-dep-mono-wrt-predE*)

lemma *mono-rel-dep-mono-wrt-pred-dep-mono-wrt-pred-eval*: (((*x* : *A*) → *B x*) ⇒ (*x* : *A*) ⇒ *B x*) *eval*
by *auto*

lemma *ex1-rel-right-if-rel-dep-mono-wrt-predI*:
assumes ((*x* : *A*) → *B x*) *R*
and *A x*
shows ∃!*y*. *R x y*
using *assms* **by** (*blast dest*: *right-unique-onD*)

lemma *eq-if-rel-if-rel-if-rel-dep-mono-wrt-predI*:
assumes ((*x* : *A*) → *B x*) *R*
and *A x*
and *R x y R x y*′
shows *y* = *y*′
using *assms* **by** (*blast dest*: *right-unique-onD*)

lemma *eval-eq-if-rel-if-rel-dep-mono-wrt-predI* [*simp*]:

assumes $((x : A) \to B\ x)\ R$
and $A\ x$
and $R\ x\ y$
shows $R`x = y$
using *assms* **by** (*intro eq-if-rel-if-rel-if-rel-dep-mono-wrt-predI*[*OF assms*, *sym-metric*])
(*blast intro*!: *rel-eval-if-ex1*[**where** *?R=R*] *ex1-rel-right-if-rel-dep-mono-wrt-predI*)

lemma *rel-if-eval-eq-if-rel-dep-mono-wrt-predI*:
assumes $((x : A) \to B\ x)\ R$
and $A\ x$
and $R`x = y$
shows $R\ x\ y$
by (*rule rel-if-eval-eq-if-in-dom-if-right-unique-on-eq*[**where** *?R=R*])
(*use assms* **in** ‹*blast dest*: *right-unique-onD*›)+

corollary *rel-eval-if-rel-dep-mono-wrt-predI*:
assumes $((x : A) \to B\ x)\ R$
and $A\ x$
shows $R\ x\ (R`x)$
using *assms* **by** (*rule rel-if-eval-eq-if-rel-dep-mono-wrt-predI*) *simp*

corollary *rel-iff-eval-eq-if-rel-dep-mono-wrt-predI*:
assumes $((x : A) \to B\ x)\ R$
and $A\ x$
shows $R\ x\ y \longleftrightarrow R`x = y$
using *assms* **by** (*intro iffI*; *rule eval-eq-if-rel-if-rel-dep-mono-wrt-predI rel-if-eval-eq-if-rel-dep-mono-wrt-predI*

lemma *rel-dep-mono-wrt-pred-relE*:
assumes $((x : A) \to B\ x)\ R$
and $A\ x$
and $R\ x\ y$
obtains $B\ x\ y\ R`x = y$
proof
from *assms* **show** $R`x = y$ **by** *simp*
with *assms* **show** $B\ x\ y$ **by** *blast*
qed

lemma *rel-dep-mono-wrt-pred-relE'*:
assumes $((x : A) \to B\ x)\ R$
obtains $\bigwedge x\ y.\ A\ x \Longrightarrow R\ x\ y \Longrightarrow B\ x\ y \wedge R`x = y$
using *assms* **by** (*auto elim*: *rel-dep-mono-wrt-pred-relE*)

lemma *rel-codom-if-rel-if-pred-if-rel-dep-mono-wrt-pred*:
assumes $((x : A) \to B\ x)\ R$
and $A\ x$
and $R\ x\ y$
shows $B\ x\ y$
using *assms* **by** (*auto elim*: *rel-dep-mono-wrt-pred-relE*)

lemma *rel-dep-mono-wrt-pred-contravariant-dom*:
assumes $((x : A) \to B\ x)\ R$
and [*dest*]: $\bigwedge x.\ A'\ x \Longrightarrow A\ x$
shows $((x : A') \to B\ x)\ R$
using *assms* **by** (*fast intro*!: *rel-dep-mono-wrt-predI dest*: *right-unique-onD*)

lemma *rel-dep-mono-wrt-pred-covariant-codom*:
assumes $((x : A) \to B\ x)\ R$
and $\bigwedge x.\ A\ x \Longrightarrow B\ x\ (R`x) \Longrightarrow B'\ x\ (R`x)$
shows $((x : A) \to B'\ x)\ R$
using *assms* **by** (*fast dest*: *right-unique-onD*)

lemma *rel-mono-wrt-pred-in-codom-on-if-rel-dep-mono-wrt-pred*:
assumes $((x : A) \to B\ x)\ R$
shows $(A \to \textit{in-codom-on}\ A\ B)\ R$
using *assms* **by** *fastforce*

lemma *eq-comp-eval-restrict-left-le-if-rel-dep-mono-wrt-pred*:
assumes $((x : A) \to B\ x)\ R$
shows $((=) \circ \textit{eval}\ R)\restriction_A \leq R\restriction_A$
using *assms* **by** (*intro le-relI*) (*force intro*: *rel-eval-if-rel-dep-mono-wrt-predI*)

lemma *restrict-left-le-eq-comp-eval-restrict-left-if-rel-dep-mono-wrt-pred*:
assumes $((x : A) \to B\ x)\ R$
shows $R\restriction_A \leq ((=) \circ \textit{eval}\ R)\restriction_A$
using *assms* **by** (*intro le-relI*) *force*

corollary *restrict-left-eq-eq-comp-eval-if-rel-dep-mono-wrt-pred*:
assumes $((x : A) \to B\ x)\ R$
shows $R\restriction_A = ((=) \circ \textit{eval}\ R)\restriction_A$
using *assms eq-comp-eval-restrict-left-le-if-rel-dep-mono-wrt-pred*
restrict-left-le-eq-comp-eval-restrict-left-if-rel-dep-mono-wrt-pred
by (*intro antisym*) *auto*

lemma *eval-eq-if-rel-dep-mono-wrt-predI*:
fixes $R :: 'a \Rightarrow 'b \Rightarrow \textit{bool}$
assumes $((x : A) \to B\ x)\ R\ ((x : A') \to B'\ x)\ R'$
and $R \leq R'$
and $(A \sqcap A')\ x$
shows $R`x = R'`x$
proof −
from *assms* **have** $R'\ x\ (R`x)\ R'\ x\ (R'`x)$ **by** (*auto intro*: *rel-eval-if-rel-dep-mono-wrt-predI*)
with *assms* **show** *?thesis* **by** (*intro eval-eq-if-rel-if-rel-dep-mono-wrt-predI*[*symmetric*])
force+
qed

lemma *rel-agree-on-if-eval-eq-if-rel-dep-mono-wrt-pred*:
assumes *rel-dep-mono-wrt-pred*: $\bigwedge R.\ \mathcal{R}\ R \Longrightarrow \exists B.\ ((x : A) \to B\ x)\ R$

and $\bigwedge R\ R'\ x.\ \mathcal{R}\ R \Longrightarrow \mathcal{R}\ R' \Longrightarrow A\ x \Longrightarrow R`x = R'`x$
shows *rel-agree-on* $A\ \mathcal{R}$
proof (*rule rel-agree-onI*)
fix $x\ y\ R\ R'$ **assume** *hyps*: $\mathcal{R}\ R\ \mathcal{R}\ R'\ A\ x\ R\ x\ y$
with *rel-dep-mono-wrt-pred* **have** $y = R`x$ **by** *fastforce*
also from *assms hyps* **have** $... = R'`x$ **by** *blast*
finally have $y = R'`x$ **.**
moreover from *rel-dep-mono-wrt-pred*[*OF* ‹$\mathcal{R}\ R'$›] **obtain** B **where** $((x : A) \to B\ x)\ R'$ **by** *blast*
ultimately show $R'\ x\ y$ **using** ‹$A\ x$› **by** (*auto intro*: *rel-eval-if-rel-dep-mono-wrt-predI*)
qed

lemma *mono-rel-dep-mono-wrt-pred-top-rel-dep-mono-wrt-pred-inf-rel-restrict-left*:
$(((x : A) \to B\ x) \Rightarrow (A' : \top) \Rightarrow ((x : A \sqcap A') \to B\ x))$ *rel-restrict-left*
by (*intro mono-wrt-predI dep-mono-wrt-predI rel-dep-mono-wrt-predI*

mono-right-unique-on-top-right-unique-on-inf-rel-restrict-left
[*THEN dep-mono-wrt-predD*, *THEN dep-mono-wrt-predD*]
mono-left-total-on-top-left-total-on-inf-rel-restrict-left
[*THEN dep-mono-wrt-predD*, *THEN dep-mono-wrt-predD*])
auto

end

1.3.1 Clean Functions

theory *Binary-Relations-Clean-Functions*
imports
Binary-Relations-Function-Base
begin

Clean relational functions may not contain further elements outside their specification.

consts *crel-dep-mono-wrt* :: $'a \Rightarrow 'b \Rightarrow 'c$
consts *crel-mono-wrt* :: $'a \Rightarrow 'b \Rightarrow 'c$

open-bundle *crel-mono-wrt-syntax*
begin
notation *crel-mono-wrt* (**infixr** ‹$\to_c$› *50*)
syntax
-crel-dep-mono-wrt :: *idt* $\Rightarrow 'a \Rightarrow 'b \Rightarrow 'c \Rightarrow$ *bool* (‹$'(-/\ :/\ -') \to_c (-)$›) [*51*, *50*, *50*] *50*)
syntax-consts
-crel-dep-mono-wrt $\rightleftharpoons$ *crel-dep-mono-wrt*
translations
$(x : A) \to_c B \rightleftharpoons$ *CONST crel-dep-mono-wrt* $A\ (\lambda x.\ B)$
end

definition *crel-dep-mono-wrt-pred* $(A :: 'a \Rightarrow bool)\ B\ R \equiv ((x : A) \to B\ x)\ R \wedge$ *in-dom* $R = A$

adhoc-overloading *crel-dep-mono-wrt* ⇌ *crel-dep-mono-wrt-pred*

definition *crel-mono-wrt-pred* (*A* :: *'a* ⇒ *bool*) *B* ≡ (((- :: *'a*) : *A*) →$_c$ *B*)
adhoc-overloading *crel-mono-wrt* ⇌ *crel-mono-wrt-pred*

lemma *crel-mono-wrt-pred-eq-crel-dep-mono-wrt-pred*:
(((*A* :: *'a* ⇒ *bool*) →$_c$ (*B* :: *'b* ⇒ *bool*)) :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *bool*) = (((- :: *'a*) : *A*) →$_c$ *B*)
by (*simp add*: *crel-mono-wrt-pred-def*)

lemma *crel-mono-wrt-pred-eq-crel-dep-mono-wrt-pred-uhint* [*uhint*]:
assumes (*A* :: *'a* ⇒ *bool*) ≡ *A'*
and ⋀*x*. *B* ≡ *B'* *x*
shows (*A* →$_c$ *B*) ≡ ((*x* : *A'*) →$_c$ *B'* *x*)
using *assms* **by** (*simp add*: *crel-mono-wrt-pred-eq-crel-dep-mono-wrt-pred*)

lemma *crel-mono-wrt-pred-iff-crel-dep-mono-wrt-pred*:
((*A* :: *'a* ⇒ *bool*) →$_c$ (*B* :: *'b* ⇒ *bool*)) (*R* :: *'a* ⇒ *'b* ⇒ *bool*) ⟷ (((- :: *'a*) : *A*) →$_c$ *B*) *R*
by (*simp add*: *crel-mono-wrt-pred-def*)

lemma *crel-dep-mono-wrt-predI* [*intro*]:
assumes ((*x* : *A*) → *B x*) *R*
and *in-dom R* ≤ *A*
shows ((*x* : *A*) →$_c$ *B x*) *R*
unfolding *crel-dep-mono-wrt-pred-def* **using** *assms*
by (*intro conjI antisym le-in-dom-if-left-total-on*) *auto*

lemma *crel-dep-mono-wrt-predI'*:
assumes *left-total-on A R*
and *right-unique-on A R*
and ({∑}*x* : *A*. *B x*) *R*
shows ((*x* : *A*) →$_c$ *B x*) *R*
proof (*intro crel-dep-mono-wrt-predI rel-dep-mono-wrt-predI dep-mono-wrt-predI*)
fix *x* **assume** *A x*
with *assms* **obtain** *y* **where** *B x y R x y* **by** *auto*
moreover with *assms* **have** *R*‘*x* = *y* **by** (*auto intro*: *eval-eq-if-right-unique-onI*)
ultimately show *B x* (*R*‘*x*) **by** *simp*
qed (*use assms* **in** *auto*)

lemma *crel-dep-mono-wrt-predE*:
assumes ((*x* : *A*) →$_c$ *B x*) *R*
obtains ((*x* : *A*) → *B x*) *R in-dom R* = *A*
using *assms* **unfolding** *crel-dep-mono-wrt-pred-def* **by** *auto*

lemma *crel-dep-mono-wrt-predE'* [*elim*]:
notes *crel-dep-mono-wrt-predE*[*elim*]
assumes ((*x* : *A*) →$_c$ *B x*) *R*
obtains ((*x* : *A*) → *B x*) *R* ({∑}*x* : *A*. *B x*) *R*

proof
  show $(\{\sum\}x : A.\ B\ x)\ R$
  proof (*rule dep-bin-relI*)
    fix $x\ y$ **assume** $R\ x\ y\ A\ x$
    with *assms* **have** $R`x = y\ B\ x\ (R`x)$ **by** *auto*
    then show $B\ x\ y$ **by** *simp*
  qed (*use assms* **in** *auto*)
qed (*use assms* **in** *auto*)

lemma *crel-dep-mono-wrt-pred-cong* [*cong*]:
  assumes $A = A'$
  and $\bigwedge x\ y.\ A'\ x \Longrightarrow B\ x = B'\ x$
  shows $((x : A) \to_c B\ x) = ((x : A') \to_c B'\ x)$
  using *assms* **by** (*intro ext*) (*auto elim*!: *crel-dep-mono-wrt-predE*)

lemma *in-dom-eq-if-crel-dep-mono-wrt-pred* [*simp*]:
  assumes $((x : A) \to_c B\ x)\ R$
  shows *in-dom* $R = A$
  using *assms* **by** (*auto elim*: *crel-dep-mono-wrt-predE*)

lemma *in-codom-le-in-codom-on-if-crel-dep-mono-wrt-pred*:
  assumes $((x : A) \to_c B\ x)\ R$
  shows *in-codom* $R \leq$ *in-codom-on* $A\ B$
  using *assms* **by** *fast*

lemma *crel-mono-wrt-predI* [*intro*]:
  assumes $(A \to B)\ R$
  and *in-dom* $R \leq A$
  shows $(A \to_c B)\ R$
  using *assms* **by** (*urule crel-dep-mono-wrt-predI*)

lemma *crel-mono-wrt-predI'*:
  assumes *left-total-on* $A\ R$
  and *right-unique-on* $A\ R$
  and $(A\ \{\times\}\ B)\ R$
  shows $(A \to_c B)\ R$
  using *assms* **by** (*urule crel-dep-mono-wrt-predI'*)

lemma *crel-mono-wrt-predE*:
  assumes $(A \to_c B)\ R$
  obtains $(A \to B)\ R$ *in-dom* $R = A$
  using *assms* **by** (*urule* (*e*) *crel-dep-mono-wrt-predE*)

lemma *crel-mono-wrt-predE'* [*elim*]:
  assumes $(A \to_c B)\ R$
  obtains $(A \to B)\ R\ (A\ \{\times\}\ B)\ R$
  using *assms* **by** (*urule* (*e*) *crel-dep-mono-wrt-predE'*)

lemma *in-dom-eq-if-crel-mono-wrt-pred* [*simp*]:

assumes $(A \rightarrow_c B)\ R$
shows *in-dom* $R = A$
using *assms* **by** (*urule in-dom-eq-if-crel-dep-mono-wrt-pred*)

lemma *eq-if-rel-if-rel-if-crel-dep-mono-wrt-predI*:
assumes $((x : A) \rightarrow_c B\ x)\ R$
and $R\ x\ y\ R\ x\ y'$
shows $y = y'$
using *assms* **by** (*auto intro*: *eq-if-rel-if-rel-if-rel-dep-mono-wrt-predI*)

lemma *eval-eq-if-rel-if-crel-dep-mono-wrt-predI* [*simp*]:
assumes $((x : A) \rightarrow_c B\ x)\ R$
and $R\ x\ y$
shows $R`x = y$
using *assms* **by** (*auto intro*: *eval-eq-if-rel-if-rel-dep-mono-wrt-predI*)

lemma *crel-dep-mono-wrt-pred-relE*:
assumes $((x : A) \rightarrow_c B\ x)\ R$
and $R\ x\ y$
obtains $A\ x\ B\ x\ y\ R`x = y$
using *assms* **by** (*auto elim*: *rel-dep-mono-wrt-pred-relE*)

lemma *crel-dep-mono-wrt-pred-relE'*:
assumes $((x : A) \rightarrow_c B\ x)\ R$
obtains $\bigwedge x\ y.\ R\ x\ y \Longrightarrow A\ x \wedge B\ x\ y \wedge R`x = y$
using *assms* **by** (*auto elim*: *crel-dep-mono-wrt-pred-relE*)

lemma *rel-restrict-left-eq-self-if-crel-dep-mono-wrt-pred* [*simp*]:
assumes $((x : A) \rightarrow_c B\ x)\ R$
shows $R\restriction_A = R$
using *assms* **by** *auto*

Note: clean function relations are not contravariant on their domain.

lemma *crel-dep-mono-wrt-pred-covariant-codom*:
assumes $((x : A) \rightarrow_c B\ x)\ R$
and $\bigwedge x.\ A\ x \Longrightarrow B\ x\ (R`x) \Longrightarrow B'\ x\ (R`x)$
shows $((x : A) \rightarrow_c B'\ x)\ R$
using *assms* **by** (*force intro*: *rel-dep-mono-wrt-pred-covariant-codom*)

lemma *eq-comp-eval-restrict-left-le-if-crel-dep-mono-wrt-pred*:
assumes [*uhint*]: $((x : A) \rightarrow_c B\ x)\ R$
shows $((=) \circ \mathit{eval}\ R)\restriction_A \leq R$
supply *rel-restrict-left-eq-self-if-crel-dep-mono-wrt-pred*[*uhint*]
by (*urule eq-comp-eval-restrict-left-le-if-rel-dep-mono-wrt-pred*) (*use assms* **in** *auto*)

lemma *le-eq-comp-eval-restrict-left-if-rel-dep-mono-wrt-pred*:
assumes [*uhint*]: $((x : A) \rightarrow_c B\ x)\ R$
shows $R \leq ((=) \circ \mathit{eval}\ R)\restriction_A$

supply *rel-restrict-left-eq-self-if-crel-dep-mono-wrt-pred*[*uhint*]
by (*urule restrict-left-le-eq-comp-eval-restrict-left-if-rel-dep-mono-wrt-pred*) (*use assms* **in** *auto*)

corollary *restrict-left-eq-eq-comp-eval-if-crel-dep-mono-wrt-pred*:
assumes $((x : A) \rightarrow_c B\ x)\ R$
shows $R = ((=) \circ \mathit{eval}\ R)\restriction_A$
using *assms eq-comp-eval-restrict-left-le-if-crel-dep-mono-wrt-pred*
le-eq-comp-eval-restrict-left-if-rel-dep-mono-wrt-pred
by (*intro antisym*) *auto*

lemma *eval-eq-if-crel-dep-mono-wrt-pred-if-rel-dep-mono-wrt-predI*:
fixes $R :: 'a \Rightarrow 'b \Rightarrow \mathit{bool}$
assumes $((x : A) \rightarrow B\ x)\ R\ ((x : A') \rightarrow_c B'\ x)\ R'$
and $R \leq R'$
and $A\ x$
shows $R`x = R'`x$
proof −
from *assms* **have** $A'\ x$ **by** (*blast elim*: *crel-dep-mono-wrt-pred-relE*)
with *assms* **show** *?thesis* **by** (*blast intro*: *eval-eq-if-rel-dep-mono-wrt-predI*)
qed

lemma *crel-dep-mono-wrt-pred-ext*:
assumes $((x : A) \rightarrow_c B\ x)\ R\ ((x : A) \rightarrow_c B'\ x)\ R'$
and $\bigwedge x.\ A\ x \Longrightarrow R`x = R'`x$
shows $R = R'$
using *assms*
by (*intro eq-if-rel-agree-on-if-dep-bin-relI*[**where** *?A*=*A* **and** *?B*=*B* **and** $?\mathcal{R}$=(=) $R \sqcup (=)\ R'$]
rel-agree-on-if-eval-eq-if-rel-dep-mono-wrt-pred)
auto

lemma *eq-if-le-if-crel-dep-mono-wrt-pred-if-rel-dep-mono-wrt-pred*:
assumes $((x : A) \rightarrow B\ x)\ R\ ((x : A) \rightarrow_c B'\ x)\ R'$
and $R \leq R'$
shows $R = R'$
proof (*intro ext iffI*)
fix *x y* **assume** $R'\ x\ y$
with *assms* **have** $R'`x = y\ A\ x$ **by** *auto*
moreover with *assms* **have** $R`x = R'`x$ **by** (*blast intro*: *eval-eq-if-crel-dep-mono-wrt-pred-if-rel-dep-mono-wr*
ultimately show $R\ x\ y$ **using** *assms* **by** (*auto intro*: *rel-if-eval-eq-if-rel-dep-mono-wrt-predI*)
qed (*use assms* **in** *auto*)

lemma *ex-dom-crel-dep-mono-wrt-pred-iff-crel-dep-mono-wrt-pred-in-dom*:
$(\exists (A :: 'a \Rightarrow \mathit{bool}).\ ((x : A) \rightarrow_c B\ x)\ R) \longleftrightarrow (((x : \mathit{in\text{-}dom}\ R) \rightarrow_c B\ x)\ R)$
by *auto*

lemma *crel-mono-wrt-pred-bottom-bottom*: $((\bot :: 'a \Rightarrow \mathit{bool}) \rightarrow_c A)\ (\bot :: 'a \Rightarrow 'b \Rightarrow \mathit{bool})$

by *fastforce*

lemma *crel-dep-mono-wrt-pred-bottom-iff-eq-bottom* [*iff*]: $((x : (\bot :: 'a \Rightarrow bool)) \rightarrow_c B\ x)\ R \longleftrightarrow R = \bot$
by *fastforce*

lemma *mono-crel-dep-mono-wrt-pred-top-crel-dep-mono-wrt-pred-inf-rel-restrict-left*:
$(((x : A) \rightarrow_c B\ x) \Rightarrow (A' : \top) \Rightarrow (x : A \sqcap A') \rightarrow_c B\ x)$ *rel-restrict-left*
by (*intro mono-wrt-predI dep-mono-wrt-predI crel-dep-mono-wrt-predI'*

mono-right-unique-on-top-right-unique-on-inf-rel-restrict-left
[*THEN dep-mono-wrt-predD*, *THEN dep-mono-wrt-predD*]
mono-left-total-on-top-left-total-on-inf-rel-restrict-left
[*THEN dep-mono-wrt-predD*, *THEN dep-mono-wrt-predD*]
mono-dep-bin-rel-top-dep-bin-rel-inf-rel-restrict-left
[*THEN mono-wrt-predD*, *THEN dep-mono-wrt-predD*])
auto

lemma *mono-rel-dep-mono-wrt-pred-ge-crel-dep-mono-wrt-pred-rel-restrict-left*:
$(((x : A) \rightarrow B\ x) \Rightarrow (A' : (\geq)\ A) \Rightarrow (x : A') \rightarrow_c B\ x)$ *rel-restrict-left*
proof (*intro mono-wrt-predI dep-mono-wrt-predI crel-dep-mono-wrt-predI*)
fix $A\ A' :: 'a \Rightarrow bool$ **and** B **and** $R :: 'a \Rightarrow 'b \Rightarrow bool$ **assume** $((x : A) \rightarrow B\ x)\ R$
with *mono-rel-dep-mono-wrt-pred-top-rel-dep-mono-wrt-pred-inf-rel-restrict-left*
have $((x : A \sqcap A') \rightarrow B\ x)\ R{\restriction_{A'}}$ **by** *force*
moreover assume $A' \leq A$
ultimately show $((x : A') \rightarrow B\ x)\ R{\restriction_{A'}}$ **by** (*simp only*: *inf-absorb2*)
qed *auto*

lemma *crel-dep-mono-wrt-pred-eq-restrict*: $((x : (A :: 'a \Rightarrow bool)) \rightarrow_c (=)\ x)\ (=){\restriction_A}$
by *fastforce*

end

Symmetric

theory *Binary-Relations-Symmetric*
imports
Functions-Monotone
begin

consts *symmetric-on* :: $'a \Rightarrow 'b \Rightarrow bool$

overloading
symmetric-on-pred $\equiv$ *symmetric-on* :: $('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$
begin
definition *symmetric-on-pred* $P\ R \equiv \forall\, x\ y : P.\ R\ x\ y \longrightarrow R\ y\ x$
end

lemma *symmetric-onI* [*intro*]:
assumes $\bigwedge x\ y.\ P\ x \Longrightarrow P\ y \Longrightarrow R\ x\ y \Longrightarrow R\ y\ x$
shows *symmetric-on P R*
unfolding *symmetric-on-pred-def* **using** *assms* **by** *blast*

lemma *symmetric-onD*:
assumes *symmetric-on P R*
and *P x P y*
and *R x y*
shows *R y x*
using *assms* **unfolding** *symmetric-on-pred-def* **by** *blast*

lemma *symmetric-on-rel-inv-iff-symmetric-on* [*iff*]:
symmetric-on P $R^{-1} \longleftrightarrow$ *symmetric-on* (*P* :: $'a \Rightarrow$ *bool*) (*R* :: $'a \Rightarrow 'a \Rightarrow$ *bool*)
by (*blast dest*: *symmetric-onD*)

lemma *antimono-symmetric-on*: *antimono* (*symmetric-on* :: ($'a \Rightarrow$ *bool*) $\Rightarrow$ ($'a \Rightarrow$ $'a \Rightarrow$ *bool*) $\Rightarrow$ *bool*)
by (*intro antimonoI*) (*auto dest*: *symmetric-onD*)

lemma *symmetric-on-if-le-pred-if-symmetric-on*:
fixes P' :: $'a \Rightarrow$ *bool* **and** *R* :: $'a \Rightarrow 'a \Rightarrow$ *bool*
assumes *symmetric-on P R*
and $P' \leq P$
shows *symmetric-on* P' *R*
using *assms antimono-symmetric-on* **by** *blast*

consts *symmetric* :: $'a \Rightarrow$ *bool*

overloading
symmetric $\equiv$ *symmetric* :: ($'a \Rightarrow 'a \Rightarrow$ *bool*) $\Rightarrow$ *bool*
begin
definition (*symmetric* :: ($'a \Rightarrow 'a \Rightarrow$ *bool*) $\Rightarrow$ -) $\equiv$ *symmetric-on* ($\top$ :: $'a \Rightarrow$ *bool*)
end

lemma *symmetric-eq-symmetric-on*:
(*symmetric* :: ($'a \Rightarrow 'a \Rightarrow$ *bool*) $\Rightarrow$ -) = *symmetric-on* ($\top$ :: $'a \Rightarrow$ *bool*)
unfolding *symmetric-def* **..**

lemma *symmetric-eq-symmetric-on-uhint* [*uhint*]:
P $\equiv$ ($\top$ :: $'a \Rightarrow$ *bool*) $\Longrightarrow$ (*symmetric* :: ($'a \Rightarrow 'a \Rightarrow$ *bool*) $\Rightarrow$ -) $\equiv$ *symmetric-on P*
by (*simp add*: *symmetric-eq-symmetric-on*)

lemma *symmetricI* [*intro*]:
assumes $\bigwedge x\ y.\ R\ x\ y \Longrightarrow R\ y\ x$
shows *symmetric R*
using *assms* **by** (*urule symmetric-onI*)

lemma *symmetricD*:
assumes *symmetric R*
and *R x y*
shows *R y x*
using *assms* **by** (*urule* (*d*) *symmetric-onD chained*: *insert*) *simp-all*

lemma *symmetric-on-if-symmetric*:
fixes *P* :: *'a* ⇒ *bool* **and** *R* :: *'a* ⇒ *'a* ⇒ *bool*
assumes *symmetric R*
shows *symmetric-on P R*
using *assms* **by** (*intro symmetric-onI*) (*blast dest*: *symmetricD*)

lemma *symmetric-rel-inv-iff-symmetric* [*iff*]: *symmetric* R^{-1} ⟷ *symmetric* (*R* :: *'a* ⇒ *'a* ⇒ *bool*)
by (*blast dest*: *symmetricD*)

lemma *rel-inv-eq-self-if-symmetric* [*simp*]:
assumes *symmetric R*
shows $R^{-1} = R$
using *assms* **by** (*blast dest*: *symmetricD*)

lemma *rel-iff-rel-if-symmetric*:
assumes *symmetric R*
shows *R x y* ⟷ *R y x*
using *assms* **by** (*blast dest*: *symmetricD*)

lemma *symmetric-if-rel-inv-eq-self*:
assumes $R^{-1} = R$
shows *symmetric R*
by (*intro symmetricI*, *subst assms*[*symmetric*]) *simp*

lemma *symmetric-iff-rel-inv-eq-self*: *symmetric R* ⟷ $R^{-1} = R$
using *rel-inv-eq-self-if-symmetric symmetric-if-rel-inv-eq-self* **by** *blast*

lemma *symmetric-if-symmetric-on-in-field*:
assumes *symmetric-on* (*in-field R*) *R*
shows *symmetric R*
using *assms* **by** (*intro symmetricI*) (*blast dest*: *symmetric-onD*)

corollary *symmetric-on-in-field-iff-symmetric* [*iff*]:
symmetric-on (*in-field R*) *R* ⟷ *symmetric R*
using *symmetric-if-symmetric-on-in-field symmetric-on-if-symmetric*
by *blast*

Instantiations **lemma** *symmetric-eq* [*iff*]: *symmetric* (=)
by (*rule symmetricI*) (*rule sym*)

lemma *symmetric-top*: *symmetric* (⊤ :: *'a* ⇒ *'a* ⇒ *bool*)
by (*rule symmetricI*) *auto*

end

Reflexive

theory *Binary-Relations-Reflexive*
  imports
    Functions-Monotone
    ML-Unification.ML-Unification-HOL-Setup
    ML-Unification.Unify-Resolve-Tactics
begin

consts *reflexive-on* :: $'a \Rightarrow 'b \Rightarrow bool$

overloading
  reflexive-on-pred $\equiv$ *reflexive-on* :: $('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$
begin
  definition *reflexive-on-pred* $P\ R \equiv \forall x.\ P\ x \longrightarrow R\ x\ x$
end

lemma *reflexive-onI* [*intro*]:
  assumes $\bigwedge x.\ P\ x \Longrightarrow R\ x\ x$
  shows *reflexive-on* $P\ R$
  using *assms* **unfolding** *reflexive-on-pred-def* **by** *blast*

lemma *reflexive-onD* [*dest*]:
  assumes *reflexive-on* $P\ R$
  and $P\ x$
  shows $R\ x\ x$
  using *assms* **unfolding** *reflexive-on-pred-def* **by** *blast*

context
  fixes $R :: 'a \Rightarrow 'a \Rightarrow bool$ **and** $P :: 'a \Rightarrow bool$
begin

lemma *le-in-dom-if-reflexive-on*:
  assumes *reflexive-on* $P\ R$
  shows $P \leq$ *in-dom* R
  using *assms* **by** *blast*

lemma *le-in-codom-if-reflexive-on*:
  assumes *reflexive-on* $P\ R$
  shows $P \leq$ *in-codom* R
  using *assms* **by** *blast*

lemma *in-codom-eq-in-dom-if-reflexive-on-in-field*:
  assumes *reflexive-on* (*in-field* R) R
  shows *in-codom* $R =$ *in-dom* R
  using *assms* **by** *blast*

lemma *reflexive-on-rel-inv-iff-reflexive-on* [*iff*]:
reflexive-on P R^{-1} $\longleftrightarrow$ *reflexive-on P R*
by *blast*

lemma *mono-reflexive-on*:
$((\geq) \Rightarrow (\leq) \Rrightarrow (\leq))$ (*reflexive-on* :: $('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$)
by *fastforce*

lemma *reflexive-on-if-le-pred-if-reflexive-on*:
fixes P' :: $'a \Rightarrow bool$
assumes *reflexive-on P R*
and $P' \leq P$
shows *reflexive-on* P' *R*
using *assms* **by** *blast*

end

lemma *reflexive-on-sup-eq* [*simp*]:
(*reflexive-on* :: $('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$) $(P \sqcup Q)$
$=$ *reflexive-on P* $\sqcap$ *reflexive-on Q*
by (*intro ext iffI reflexive-onI*)
(*auto intro*: *reflexive-on-if-le-pred-if-reflexive-on*)

lemma *reflexive-on-iff-eq-restrict-le*:
reflexive-on (P :: $'a \Rightarrow bool$) (R :: $'a \Rightarrow$ -) $\longleftrightarrow$ $((=)\restriction_P \leq R)$
by *blast*

consts *reflexive* :: $'a \Rightarrow bool$

overloading
reflexive $\equiv$ *reflexive* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$
begin
definition *reflexive* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool \equiv$ *reflexive-on* ($\top$ :: $'a \Rightarrow bool$)
end

lemma *reflexive-eq-reflexive-on*:
(*reflexive* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow$ -) $=$ *reflexive-on* ($\top$ :: $'a \Rightarrow bool$)
unfolding *reflexive-def* **..**

lemma *reflexive-eq-reflexive-on-uhint* [*uhint*]:
$P \equiv (\top$ :: $'a \Rightarrow bool) \Longrightarrow$ (*reflexive* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow$ -) $\equiv$ *reflexive-on P*
by (*simp add*: *reflexive-eq-reflexive-on*)

lemma *reflexiveI* [*intro*]:
assumes $\bigwedge x.$ *R x x*
shows *reflexive R*
using *assms* **by** (*urule reflexive-onI*)

lemma *reflexiveD*:
assumes *reflexive R*
shows *R x x*
using *assms* **by** (*urule* (*d*) *reflexive-onD chained*: *insert*) *simp*

lemma *reflexive-on-if-reflexive*:
fixes $P :: 'a \Rightarrow bool$ **and** $R :: 'a \Rightarrow 'a \Rightarrow bool$
assumes *reflexive R*
shows *reflexive-on P R*
using *assms* **by** (*intro reflexive-onI*) (*blast dest*: *reflexiveD*)

lemma *reflexive-rel-inv-iff-reflexive* [*iff*]:
reflexive $(R :: 'a \Rightarrow 'a \Rightarrow bool)^{-1} \longleftrightarrow$ *reflexive R*
by (*blast dest*: *reflexiveD*)

lemma *reflexive-iff-eq-le*: *reflexive R* $\longleftrightarrow ((=) \leq R)$
unfolding *reflexive-eq-reflexive-on reflexive-on-iff-eq-restrict-le*
by *auto*

Instantiations **lemma** *reflexive-eq*: *reflexive* $(=)$
by (*rule reflexiveI*) (*rule refl*)

lemma *reflexive-top*: *reflexive* $(\top :: 'a \Rightarrow 'a \Rightarrow bool)$
by (*rule reflexiveI*) *auto*

end

Transitive

theory *Binary-Relations-Transitive*
imports
Functions-Monotone
begin

consts *transitive-on* $:: 'a \Rightarrow 'b \Rightarrow bool$

overloading
transitive-on-pred $\equiv$ *transitive-on* $:: ('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$
begin
definition *transitive-on-pred P R* $\equiv \forall x\ y\ z : P.\ R\ x\ y \wedge R\ y\ z \longrightarrow R\ x\ z$
end

lemma *transitive-onI* [*intro*]:
assumes $\bigwedge x\ y\ z.\ P\ x \Longrightarrow P\ y \Longrightarrow P\ z \Longrightarrow R\ x\ y \Longrightarrow R\ y\ z \Longrightarrow R\ x\ z$
shows *transitive-on P R*
unfolding *transitive-on-pred-def* **using** *assms* **by** *blast*

lemma *transitive-onD*:
assumes *transitive-on P R*
and *P x P y P z*
and *R x y R y z*
shows *R x z*
using *assms* **unfolding** *transitive-on-pred-def* **by** *blast*

lemma *transitive-on-if-rel-comp-self-imp*:
assumes $\bigwedge x\ y.\ P\ x \Longrightarrow P\ y \Longrightarrow (R \circ\circ R)\ x\ y \Longrightarrow R\ x\ y$
shows *transitive-on P R*
proof (*rule transitive-onI*)
fix *x y z* **assume** *R x y R y z*
then have (*R* $\circ\circ$ *R*) *x z* **by** (*intro rel-compI*)
moreover assume *P x P y P z*
ultimately show *R x z* **by** (*simp only*: *assms*)
qed

lemma *transitive-on-rel-inv-iff-transitive-on* [*iff*]:
transitive-on P $R^{-1} \longleftrightarrow$ *transitive-on* (*P* :: $'a \Rightarrow$ *bool*) (*R* :: $'a \Rightarrow 'a \Rightarrow$ *bool*)
by (*auto intro*!: *transitive-onI dest*: *transitive-onD*)

lemma *antimono-transitive-on*: *antimono* (*transitive-on* :: ($'a \Rightarrow$ *bool*) $\Rightarrow$ ($'a \Rightarrow$ $'a \Rightarrow$ *bool*) $\Rightarrow$ *bool*)
by (*intro antimonoI*) (*auto dest*: *transitive-onD*)

lemma *transitive-on-rel-map-if-mono-wrt-pred-if-transitive-on*:
fixes *P* :: $'a \Rightarrow$ *bool* **and** *R* :: $'a \Rightarrow 'a \Rightarrow$ *bool*
assumes *transitive-on P R*
and (*Q* $\Rightarrow$ *P*) *f*
shows *transitive-on Q* (*rel-map f R*)
using *assms* **by** (*fastforce dest*: *transitive-onD*)

consts *transitive* :: $'a \Rightarrow$ *bool*

overloading
transitive $\equiv$ *transitive* :: ($'a \Rightarrow 'a \Rightarrow$ *bool*) $\Rightarrow$ *bool*
begin
definition (*transitive* :: ($'a \Rightarrow 'a \Rightarrow$ *bool*) $\Rightarrow$ *bool*) $\equiv$ *transitive-on* ($\top$ :: $'a \Rightarrow$ *bool*)
end

lemma *transitive-eq-transitive-on*:
(*transitive* :: ($'a \Rightarrow 'a \Rightarrow$ *bool*) $\Rightarrow$ -) = *transitive-on* ($\top$ :: $'a \Rightarrow$ *bool*)
unfolding *transitive-def* **..**

lemma *transitive-eq-transitive-on-uhint* [*uhint*]:
P $\equiv$ ($\top$ :: $'a \Rightarrow$ *bool*) $\Longrightarrow$ (*transitive* :: ($'a \Rightarrow 'a \Rightarrow$ *bool*) $\Rightarrow$ -) $\equiv$ *transitive-on P*
by (*simp add*: *transitive-eq-transitive-on*)

lemma *transitiveI* [*intro*]:
assumes $\bigwedge x\ y\ z.\ R\ x\ y \Longrightarrow R\ y\ z \Longrightarrow R\ x\ z$
shows *transitive R*
using *assms* **by** (*urule transitive-onI*)

lemma *transitiveD* [*dest*]:
assumes *transitive R*
and *R x y R y z*
shows *R x z*
using *assms* **by** (*urule* (*d*) *transitive-onD chained*: *insert*) *simp-all*

lemma *transitive-on-if-transitive*:
fixes $P :: 'a \Rightarrow bool$ **and** $R :: 'a \Rightarrow 'a \Rightarrow bool$
assumes *transitive R*
shows *transitive-on P R*
using *assms* **by** (*intro transitive-onI*) *blast*

lemma *transitive-if-rel-comp-le-self*:
assumes $R \circ\circ R \leq R$
shows *transitive R*
by (*urule transitive-on-if-rel-comp-self-imp*) (*use assms* **in** *auto*)

lemma *rel-comp-le-self-if-transitive*:
assumes *transitive R*
shows $R \circ\circ R \leq R$
using *assms* **by** *blast*

corollary *transitive-iff-rel-comp-le-self*: *transitive* $R \longleftrightarrow R \circ\circ R \leq R$
using *transitive-if-rel-comp-le-self rel-comp-le-self-if-transitive* **by** *blast*

lemma *transitive-if-transitive-on-in-field*:
assumes *transitive-on* (*in-field R*) *R*
shows *transitive R*
using *assms* **by** (*intro transitiveI*) (*blast dest*: *transitive-onD*)

corollary *transitive-on-in-field-iff-transitive* [*iff*]:
transitive-on (*in-field R*) $R \longleftrightarrow$ *transitive R*
using *transitive-if-transitive-on-in-field transitive-on-if-transitive*
by *blast*

lemma *transitive-rel-inv-iff-transitive* [*iff*]: *transitive* $R^{-1} \longleftrightarrow$ *transitive* ($R :: 'a \Rightarrow 'a \Rightarrow bool$)
by *fast*

Instantiations **lemma** *transitive-eq*: *transitive* (=)
by (*rule transitiveI*) (*rule trans*)

lemma *transitive-top*: *transitive* ($\top :: 'a \Rightarrow 'a \Rightarrow bool$)
by (*rule transitiveI*) *auto*

end

1.3.2 Preorders

theory *Preorders*
imports
Binary-Relations-Reflexive
Binary-Relations-Transitive
begin

definition *preorder-on* ≡ *reflexive-on* ⊓ *transitive-on*

lemma *preorder-onI* [*intro*]:
assumes *reflexive-on P R*
and *transitive-on P R*
shows *preorder-on P R*
unfolding *preorder-on-def* **using** *assms* **by** *auto*

lemma *preorder-onE* [*elim*]:
assumes *preorder-on P R*
obtains *reflexive-on P R transitive-on P R*
using *assms* **unfolding** *preorder-on-def* **by** *auto*

lemma *reflexive-on-if-preorder-on*:
assumes *preorder-on P R*
shows *reflexive-on P R*
using *assms* **by** (*elim preorder-onE*)

lemma *transitive-on-if-preorder-on*:
assumes *preorder-on P R*
shows *transitive-on P R*
using *assms* **by** (*elim preorder-onE*)

lemma *transitive-if-preorder-on-in-field*:
assumes *preorder-on* (*in-field R*) *R*
shows *transitive R*
using *assms* **by** (*elim preorder-onE*) (*rule transitive-if-transitive-on-in-field*)

corollary *preorder-on-in-fieldE* [*elim*]:
assumes *preorder-on* (*in-field R*) *R*
obtains *reflexive-on* (*in-field R*) *R transitive R*
using *assms*
by (*blast dest*: *reflexive-on-if-preorder-on transitive-if-preorder-on-in-field*)

lemma *preorder-on-rel-inv-if-preorder-on* [*iff*]:
preorder-on P R^{-1} ⟷ *preorder-on* (*P* :: *'a* ⇒ *bool*) (*R* :: *'a* ⇒ *'a* ⇒ *bool*)
by *auto*

lemma *rel-if-all-rel-if-rel-if-reflexive-on*:
assumes *reflexive-on P R*
and $\bigwedge z.\ P\ z \Longrightarrow R\ x\ z \Longrightarrow R\ y\ z$
and $P\ x$
shows $R\ y\ x$
using *assms* **by** *blast*

lemma *rel-if-all-rel-if-rel-if-reflexive-on'*:
assumes *reflexive-on P R*
and $\bigwedge z.\ P\ z \Longrightarrow R\ z\ x \Longrightarrow R\ z\ y$
and $P\ x$
shows $R\ x\ y$
using *assms* **by** *blast*

consts *preorder* :: $'a \Rightarrow bool$

overloading
preorder $\equiv$ *preorder* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$
begin
definition (*preorder* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool) \equiv$ *preorder-on* ($\top$ :: $'a \Rightarrow bool$)
end

lemma *preorder-eq-preorder-on*:
(*preorder* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool) =$ *preorder-on* ($\top$ :: $'a \Rightarrow bool$)
unfolding *preorder-def* **..**

lemma *preorder-eq-preorder-on-uhint* [*uhint*]:
assumes $P \equiv \top$ :: $'a \Rightarrow bool$
shows (*preorder* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool) \equiv$ *preorder-on P*
using *assms* **by** (*simp add*: *preorder-eq-preorder-on*)

context
fixes R :: $'a \Rightarrow 'a \Rightarrow bool$
begin

lemma *preorderI* [*intro*]:
assumes *reflexive R*
and *transitive R*
shows *preorder R*
using *assms* **by** (*urule preorder-onI*)

lemma *preorderE* [*elim*]:
assumes *preorder R*
obtains *reflexive R transitive R*
using *assms* **by** (*urule* (*e*) *preorder-onE*)

lemma *preorder-on-if-preorder*:
fixes P :: $'a \Rightarrow bool$

assumes *preorder R*
shows *preorder-on P R*
using *assms* **by** (*elim preorderE*)
(*intro preorder-onI reflexive-on-if-reflexive transitive-on-if-transitive*)

end

Instantiations **lemma** *preorder-eq*: *preorder* (=)
using *reflexive-eq transitive-eq* **by** (*rule preorderI*)

end

1.3.3 Partial Equivalence Relations

theory *Partial-Equivalence-Relations*
imports
Binary-Relations-Symmetric
Preorders
begin

definition *partial-equivalence-rel-on* ≡ *transitive-on* ⊓ *symmetric-on*

lemma *partial-equivalence-rel-onI* [*intro*]:
assumes *transitive-on P R*
and *symmetric-on P R*
shows *partial-equivalence-rel-on P R*
unfolding *partial-equivalence-rel-on-def* **using** *assms* **by** *auto*

lemma *partial-equivalence-rel-onE* [*elim*]:
assumes *partial-equivalence-rel-on P R*
obtains *transitive-on P R symmetric-on P R*
using *assms* **unfolding** *partial-equivalence-rel-on-def* **by** *auto*

lemma *partial-equivalence-rel-on-rel-self-if-rel-dom*:
assumes *partial-equivalence-rel-on* (*P* :: *'a* ⇒ *bool*) (*R* :: *'a* ⇒ *'a* ⇒ *bool*)
and *P x P y*
and *R x y*
shows *R x x*
using *assms* **by** (*blast dest*: *symmetric-onD transitive-onD*)

lemma *partial-equivalence-rel-on-rel-self-if-rel-codom*:
assumes *partial-equivalence-rel-on* (*P* :: *'a* ⇒ *bool*) (*R* :: *'a* ⇒ *'a* ⇒ *bool*)
and *P x P y*
and *R x y*
shows *R y y*
using *assms* **by** (*blast dest*: *symmetric-onD transitive-onD*)

lemma *partial-equivalence-rel-on-rel-inv-iff-partial-equivalence-rel-on* [*iff*]:

partial-equivalence-rel-on P R^{-1} $\longleftrightarrow$ *partial-equivalence-rel-on* (*P* :: $'a \Rightarrow$ *bool*) (*R* :: $'a \Rightarrow 'a \Rightarrow$ *bool*)
by *blast*

consts *partial-equivalence-rel* :: $'a \Rightarrow$ *bool*

overloading
partial-equivalence-rel ≡ *partial-equivalence-rel* :: ($'a \Rightarrow 'a \Rightarrow$ *bool*) ⇒ *bool*
begin
definition (*partial-equivalence-rel* :: ($'a \Rightarrow 'a \Rightarrow$ *bool*) ⇒ *bool*) ≡ *partial-equivalence-rel-on* (⊤ :: $'a \Rightarrow$ *bool*)
end

lemma *partial-equivalence-rel-eq-partial-equivalence-rel-on*:
(*partial-equivalence-rel* :: ($'a \Rightarrow 'a \Rightarrow$ *bool*) ⇒ *bool*) = *partial-equivalence-rel-on* (⊤ :: $'a \Rightarrow$ *bool*)
unfolding *partial-equivalence-rel-def* **..**

lemma *partial-equivalence-rel-eq-partial-equivalence-rel-on-uhint* [*uhint*]:
assumes *P* ≡ ⊤ :: $'a \Rightarrow$ *bool*
shows (*partial-equivalence-rel* :: ($'a \Rightarrow 'a \Rightarrow$ *bool*) ⇒ *bool*) ≡ *partial-equivalence-rel-on P*
using *assms* **by** (*simp add*: *partial-equivalence-rel-eq-partial-equivalence-rel-on*)

context
fixes *R* :: $'a \Rightarrow 'a \Rightarrow$ *bool*
begin

lemma *partial-equivalence-relI* [*intro*]:
assumes *transitive R*
and *symmetric R*
shows *partial-equivalence-rel R*
using *assms* **by** (*urule partial-equivalence-rel-onI*)

lemma *reflexive-on-in-field-if-partial-equivalence-rel*:
assumes *partial-equivalence-rel R*
shows *reflexive-on* (*in-field R*) *R*
using *assms* **unfolding** *partial-equivalence-rel-eq-partial-equivalence-rel-on*
by (*intro reflexive-onI*) (*blast*
intro: *top1I partial-equivalence-rel-on-rel-self-if-rel-dom*
partial-equivalence-rel-on-rel-self-if-rel-codom)

lemma *partial-equivalence-relE* [*elim*]:
assumes *partial-equivalence-rel R*
obtains *preorder-on* (*in-field R*) *R symmetric R*
using *assms* **by** (*urule* (*e*) *partial-equivalence-rel-onE chained*: *insert*)
(*auto intro*: *reflexive-on-in-field-if-partial-equivalence-rel*
simp flip: *transitive-eq-transitive-on symmetric-eq-symmetric-on*)

lemma *partial-equivalence-rel-on-if-partial-equivalence-rel*:
fixes $P :: 'a \Rightarrow bool$
assumes *partial-equivalence-rel R*
shows *partial-equivalence-rel-on P R*
using *assms* **by** (*elim partial-equivalence-relE preorder-on-in-fieldE*)
(*intro partial-equivalence-rel-onI transitive-on-if-transitive*
symmetric-on-if-symmetric)

lemma *partial-equivalence-rel-rel-inv-iff-partial-equivalence-rel* [*iff*]:
partial-equivalence-rel $R^{-1} \longleftrightarrow$ *partial-equivalence-rel R*
unfolding *partial-equivalence-rel-eq-partial-equivalence-rel-on* **by** *blast*

corollary *in-codom-eq-in-dom-if-partial-equivalence-rel*:
assumes *partial-equivalence-rel R*
shows *in-codom R* = *in-dom R*
using *assms reflexive-on-in-field-if-partial-equivalence-rel*
in-codom-eq-in-dom-if-reflexive-on-in-field
by *auto*

lemma *partial-equivalence-rel-rel-comp-self-eq-self*:
assumes *partial-equivalence-rel R*
shows $(R \circ\circ R) = R$
using *assms* **by** (*intro ext*) (*blast dest*: *symmetricD*)

lemma *partial-equivalence-rel-if-partial-equivalence-rel-on-in-field*:
assumes *partial-equivalence-rel-on* (*in-field R*) *R*
shows *partial-equivalence-rel R*
using *assms* **by** (*intro partial-equivalence-relI*)
(*auto intro*: *transitive-if-transitive-on-in-field symmetric-if-symmetric-on-in-field*)

corollary *partial-equivalence-rel-on-in-field-iff-partial-equivalence-rel* [*iff*]:
partial-equivalence-rel-on (*in-field R*) $R \longleftrightarrow$ *partial-equivalence-rel R*
using *partial-equivalence-rel-if-partial-equivalence-rel-on-in-field*
partial-equivalence-rel-on-if-partial-equivalence-rel
by *blast*

end

Instantiations **lemma** *partial-equivalence-rel-eq*: *partial-equivalence-rel* (=)
using *transitive-eq symmetric-eq* **by** (*rule partial-equivalence-relI*)

lemma *partial-equivalence-rel-top*: *partial-equivalence-rel* $(\top :: 'a \Rightarrow 'a \Rightarrow bool)$
using *transitive-top symmetric-top* **by** (*rule partial-equivalence-relI*)

end

1.3.4 Equivalences

theory *Equivalence-Relations*
imports
Partial-Equivalence-Relations
begin

definition *equivalence-rel-on* ≡ *partial-equivalence-rel-on* ⊓ *reflexive-on*

lemma *equivalence-rel-onI* [*intro*]:
assumes *partial-equivalence-rel-on P R*
and *reflexive-on P R*
shows *equivalence-rel-on P R*
unfolding *equivalence-rel-on-def* **using** *assms* **by** *auto*

lemma *equivalence-rel-onE* [*elim*]:
assumes *equivalence-rel-on P R*
obtains *partial-equivalence-rel-on P R reflexive-on P R*
using *assms* **unfolding** *equivalence-rel-on-def* **by** *auto*

lemma *equivalence-rel-on-in-field-if-partial-equivalence-rel*:
assumes *partial-equivalence-rel R*
shows *equivalence-rel-on* (*in-field R*) *R*
using *assms*
by (*intro equivalence-rel-onI reflexive-on-in-field-if-partial-equivalence-rel*) *auto*

corollary *partial-equivalence-rel-iff-equivalence-rel-on-in-field*:
partial-equivalence-rel R ⟷ *equivalence-rel-on* (*in-field R*) *R*
using *equivalence-rel-on-in-field-if-partial-equivalence-rel* **by** *auto*

consts *equivalence-rel* :: *'a* ⇒ *bool*

overloading
equivalence-rel ≡ *equivalence-rel* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*
begin
definition (*equivalence-rel* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*) ≡ *equivalence-rel-on* (⊤ :: *'a* ⇒ *bool*)
end

lemma *equivalence-rel-eq-equivalence-rel-on*:
(*equivalence-rel* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*) = *equivalence-rel-on* (⊤ :: *'a* ⇒ *bool*)
unfolding *equivalence-rel-def* **..**

lemma *equivalence-rel-eq-equivalence-rel-on-uhint* [*uhint*]:
assumes *P* ≡ ⊤ :: *'a* ⇒ *bool*
shows *equivalence-rel* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool* ≡ *equivalence-rel-on P*
using *assms* **by** (*simp add*: *equivalence-rel-eq-equivalence-rel-on*)

context
fixes *R* :: *'a* ⇒ *'a* ⇒ *bool*

begin

lemma *equivalence-relI* [*intro*]:
assumes *partial-equivalence-rel R*
and *reflexive R*
shows *equivalence-rel R*
using *assms* **by** (*urule equivalence-rel-onI*)

lemma *equivalence-relE* [*elim*]:
assumes *equivalence-rel R*
obtains *partial-equivalence-rel R reflexive R*
using *assms* **by** (*urule* (*e*) *equivalence-rel-onE*)

lemma *equivalence-rel-on-if-equivalence*:
fixes $P :: 'a \Rightarrow bool$
assumes *equivalence-rel R*
shows *equivalence-rel-on P R*
using *assms* **by** (*elim equivalence-relE*)
(*intro equivalence-rel-onI partial-equivalence-rel-on-if-partial-equivalence-rel*
reflexive-on-if-reflexive)

end

Instantiations **lemma** *equivalence-eq*: *equivalence-rel* (=)
using *partial-equivalence-rel-eq reflexive-eq* **by** (*rule equivalence-relI*)

lemma *equivalence-top*: *equivalence-rel* $(\top :: 'a \Rightarrow 'a \Rightarrow bool)$
using *partial-equivalence-rel-top reflexive-top* **by** (*rule equivalence-relI*)

end

Antisymmetric

theory *Binary-Relations-Antisymmetric*
imports
Binary-Relation-Functions
Functions-Monotone
begin

consts *antisymmetric-on* $:: 'a \Rightarrow 'b \Rightarrow bool$

overloading
antisymmetric-on-pred $\equiv$ *antisymmetric-on* $:: ('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$
begin
definition *antisymmetric-on-pred P R* $\equiv \forall x\ y : P.\ R\ x\ y \wedge R\ y\ x \longrightarrow x = y$
end

lemma *antisymmetric-onI* [*intro*]:

assumes $\bigwedge x\ y.\ P\ x \Longrightarrow P\ y \Longrightarrow R\ x\ y \Longrightarrow R\ y\ x \Longrightarrow x = y$
shows *antisymmetric-on P R*
unfolding *antisymmetric-on-pred-def* **using** *assms* **by** *blast*

lemma *antisymmetric-onD*:
assumes *antisymmetric-on P R*
and *P x P y*
and *R x y R y x*
shows $x = y$
using *assms* **unfolding** *antisymmetric-on-pred-def* **by** *blast*

lemma *antisymmetric-onE*:
assumes *antisymmetric-on P R*
obtains $\bigwedge x\ y.\ P\ x \Longrightarrow P\ y \Longrightarrow R\ x\ y \Longrightarrow R\ y\ x \Longrightarrow x = y$
using *assms* **unfolding** *antisymmetric-on-pred-def* **by** *blast*

lemma *antimono-antisymmetric-on*:
$((\leq) \Rightarrow (\leq) \Rrightarrow (\geq))$ (*antisymmetric-on* :: $('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$)
by (*blast intro*!: *antisymmetric-onI dest*: *antisymmetric-onD*)

consts *antisymmetric* :: $'a \Rightarrow bool$

overloading
antisymmetric $\equiv$ *antisymmetric* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$
begin
definition *antisymmetric* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow$ - $\equiv$ *antisymmetric-on* ($\top$ :: $'a \Rightarrow bool$)
end

lemma *antisymmetric-eq-antisymmetric-on*:
(*antisymmetric* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow$ -) = *antisymmetric-on* ($\top$ :: $'a \Rightarrow bool$)
unfolding *antisymmetric-def* **..**

lemma *antisymmetric-eq-antisymmetric-on-uhint* [*uhint*]:
$P \equiv (\top :: 'a \Rightarrow bool) \Longrightarrow$ (*antisymmetric* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow$ -) $\equiv$ *antisymmetric-on P*
by (*simp add*: *antisymmetric-eq-antisymmetric-on*)

lemma *antisymmetricI* [*intro*]:
assumes $\bigwedge x\ y.\ R\ x\ y \Longrightarrow R\ y\ x \Longrightarrow x = y$
shows *antisymmetric R*
using *assms* **by** (*urule antisymmetric-onI*)

lemma *antisymmetricD*:
assumes *antisymmetric R*
and *R x y R y x*
shows $x = y$
using *assms* **by** (*urule* (*d*) *antisymmetric-onD chained*: *insert*) *simp-all*

lemma *antisymmetricE*:
assumes *antisymmetric R*
obtains $\bigwedge x\ y.\ R\ x\ y \Longrightarrow R\ y\ x \Longrightarrow x = y$
using *assms* **by** (*urule* (*e*) *antisymmetric-onE chained*: *insert*) *simp-all*

lemma *antisymmetric-on-if-antisymmetric*:
fixes $P :: 'a \Rightarrow bool$ **and** $R :: 'a \Rightarrow 'a \Rightarrow bool$
assumes *antisymmetric R*
shows *antisymmetric-on P R*
using *assms* **by** (*intro antisymmetric-onI*) (*blast dest*: *antisymmetricD*)

lemma *antisymmetric-if-antisymmetric-on-in-field*:
assumes *antisymmetric-on* (*in-field R*) *R*
shows *antisymmetric R*
using *assms* **by** (*intro antisymmetricI*) (*blast dest*: *antisymmetric-onD*)

corollary *antisymmetric-on-in-field-iff-antisymmetric* [*iff*]:
antisymmetric-on (*in-field R*) *R* $\longleftrightarrow$ *antisymmetric R*
using *antisymmetric-if-antisymmetric-on-in-field antisymmetric-on-if-antisymmetric*
by *blast*

end

1.3.5 Partial Orders

theory *Partial-Orders*
imports
Binary-Relations-Antisymmetric
Preorders
begin

definition *partial-order-on* $\equiv$ *preorder-on* $\sqcap$ *antisymmetric-on*

lemma *partial-order-onI* [*intro*]:
assumes *preorder-on P R*
and *antisymmetric-on P R*
shows *partial-order-on P R*
unfolding *partial-order-on-def* **using** *assms* **by** *auto*

lemma *partial-order-onE* [*elim*]:
assumes *partial-order-on P R*
obtains *preorder-on P R antisymmetric-on P R*
using *assms* **unfolding** *partial-order-on-def* **by** *auto*

lemma *transitive-if-partial-order-on-in-field*:
assumes *partial-order-on* (*in-field R*) *R*
shows *transitive R*

using *assms* **by** (*elim partial-order-onE*) (*rule transitive-if-preorder-on-in-field*)

lemma *antisymmetric-if-partial-order-on-in-field*:
assumes *partial-order-on* (*in-field R*) *R*
shows *antisymmetric R*
using *assms* **by** (*elim partial-order-onE*)
(*rule antisymmetric-if-antisymmetric-on-in-field*)

consts *partial-order* :: *'a* ⇒ *bool*

overloading
partial-order ≡ *partial-order* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*
begin
definition (*partial-order* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*) ≡ *partial-order-on* (⊤ :: *'a* ⇒ *bool*)
end

lemma *partial-order-eq-partial-order-on*:
(*partial-order* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*) = *partial-order-on* (⊤ :: *'a* ⇒ *bool*)
unfolding *partial-order-def* **..**

lemma *partial-order-eq-partial-order-on-uhint* [*uhint*]:
assumes *P* ≡ ⊤ :: *'a* ⇒ *bool*
shows (*partial-order* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*) ≡ *partial-order-on P*
using *assms* **by** (*simp add*: *partial-order-eq-partial-order-on*)

context
fixes *R* :: *'a* ⇒ *'a* ⇒ *bool*
begin

lemma *partial-orderI* [*intro*]:
assumes *preorder R*
and *antisymmetric R*
shows *partial-order R*
using *assms* **by** (*urule partial-order-onI*)

lemma *partial-orderE* [*elim*]:
assumes *partial-order R*
obtains *preorder R antisymmetric R*
using *assms* **by** (*urule* (*e*) *partial-order-onE*)

lemma *partial-order-on-if-partial-order*:
fixes *P* :: *'a* ⇒ *bool*
assumes *partial-order R*
shows *partial-order-on P R*
using *assms* **by** (*elim partial-orderE*)
(*intro partial-order-onI preorder-on-if-preorder antisymmetric-on-if-antisymmetric*)

end

end

1.3.6 Restricted Equality

theory *Restricted-Equality*
imports
Binary-Relations-Order-Base
Binary-Relation-Functions
Equivalence-Relations
Partial-Orders
begin

Summary Introduces notations and theorems for restricted equalities. An equality (=) can be restricted to only apply to a subset of its elements. The restriction can be formulated, for example, by a predicate or a set.

open-bundle *eq-rel-restrict-syntax*
begin
syntax
-eq-restrict-infix :: $'a \Rightarrow 'b \Rightarrow 'a \Rightarrow bool$ ($\langle$(-) =(_) (-)$\rangle$ [*51*,*51*,*51*] *50*)
-eq-restrict :: $'b \Rightarrow 'a \Rightarrow bool$ ($\langle$'(=(_)')$\rangle$)
syntax-consts
-eq-restrict-infix $\rightleftharpoons$ *rel-restrict-left* **and**
-eq-restrict $\rightleftharpoons$ *rel-restrict-left*
translations
$(=_P) \rightleftharpoons$ *CONST rel-restrict-left* (=) *P*
$x =_P y \rightleftharpoons$ *CONST rel-restrict-left* (=) *P x y*
end

lemma *in-dom-eq-restrict-eq* [*simp*]: *in-dom* $(=_P) = P$ **by** *auto*
lemma *in-codom-eq-restrict-eq* [*simp*]: *in-codom* $(=_P) = P$ **by** *auto*
lemma *in-field-eq-restrict-eq* [*simp*]: *in-field* $(=_P) = P$ **by** *auto*

Order Properties **context**
fixes *P* :: $'a \Rightarrow bool$
begin

context
begin
lemma *reflexive-on-eq-restrict*: *reflexive-on P* $((=_P) :: 'a \Rightarrow$ -) **by** *auto*
lemma *transitive-eq-restrict*: *transitive* $((=_P) :: 'a \Rightarrow$ -) **by** *auto*
lemma *symmetric-eq-restrict*: *symmetric* $((=_P) :: 'a \Rightarrow$ -) **by** *auto*
lemma *antisymmetric-eq-restrict*: *antisymmetric* $((=_P) :: 'a \Rightarrow$ -) **by** *auto*
end

context
begin
lemma *preorder-on-eq-restrict*: *preorder-on P* $((=_P) :: 'a \Rightarrow$ -)

using *reflexive-on-eq-restrict transitive-eq-restrict* **by** *auto*
lemma *partial-equivalence-rel-eq-restrict*: *partial-equivalence-rel* $((=_P) :: 'a \Rightarrow -)$
using *symmetric-eq-restrict transitive-eq-restrict* **by** *auto*
end

lemma *partial-order-on-eq-restrict*: *partial-order-on* P $((=_P) :: 'a \Rightarrow -)$
using *preorder-on-eq-restrict antisymmetric-eq-restrict* **by** *auto*
lemma *equivalence-rel-on-eq-restrict*: *equivalence-rel-on* P $((=_P) :: 'a \Rightarrow -)$
using *partial-equivalence-rel-eq-restrict reflexive-on-eq-restrict* **by** *blast*
end

end

1.3.7 Composing Functions

theory *Binary-Relations-Function-Composition*
imports
Binary-Relations-Clean-Functions
Restricted-Equality
begin

lemma *dep-bin-rel-eval-rel-comp-if-dep-bin-rel-if-crel-dep-mono-wrt-pred*:
assumes $((x : A) \rightarrow_c B\ x)\ R$
and $\bigwedge x.\ A\ x \Longrightarrow (\{\sum\}y : B\ x.\ C\ x\ y)\ R'$
shows $(\{\sum\}x : A.\ C\ x\ (R`x))\ (R \circ\circ R')$
using *assms* **by** *force*

lemma *crel-dep-mono-wrt-pred-eval-rel-comp-if-rel-dep-mono-wrt-pred-if-crel-dep-mono-wrt-pred*:
assumes $((x : A) \rightarrow_c B\ x)\ R$
and $\bigwedge x.\ A\ x \Longrightarrow ((y : B\ x) \rightarrow C\ x\ y)\ R'$
shows $((x : A) \rightarrow_c C\ x\ (R`x))\ (R \circ\circ R')$
proof (*intro crel-dep-mono-wrt-predI' dep-bin-relI*

mono-left-total-on-comp[*THEN dep-mono-wrt-predD*, *THEN mono-wrt-predD*]
mono-right-unique-on-comp[*THEN dep-mono-wrt-predD*, *THEN mono-wrt-predD*])
from *assms* **show** *left-total-on* (*in-codom* $R{\restriction}_A$) R' **by** *force*
from *assms* **show** *right-unique-on* (*in-codom* $R{\restriction}_A$) R' **by** (*fast dest*: *right-unique-onD*)
qed (*use assms* **in** ‹*auto elim*: *rel-dep-mono-wrt-pred-relE crel-dep-mono-wrt-pred-relE*›)

corollary *mono-crel-mono-rel-mono-crel-mono-rel-comp*: $((A \rightarrow_c B) \Rightarrow (B \rightarrow C)$
$\Rightarrow A \rightarrow_c C)\ (\circ\circ)$
using *crel-dep-mono-wrt-pred-eval-rel-comp-if-rel-dep-mono-wrt-pred-if-crel-dep-mono-wrt-pred*
by (*intro mono-wrt-predI*) (*auto simp*: *rel-mono-wrt-pred-eq-rel-dep-mono-wrt-pred*
crel-mono-wrt-pred-eq-crel-dep-mono-wrt-pred)

lemma *rel-comp-eval-eq-if-rel-dep-mono-wrt-pred-if-crel-dep-mono-wrt-predI* [*simp*]:
assumes $((x : A) \rightarrow_c B\ x)\ R$
and $\bigwedge x.\ A\ x \Longrightarrow ((y : B\ x) \rightarrow C\ x\ y)\ R'$

and A x
shows $(R \circ\circ R')`x = R'`(R`x)$
proof (*rule eval-eq-if-right-unique-onI*)
from *assms* **have** $((x : A) \rightarrow_c C\ x\ (R`x))\ (R \circ\circ R')$
by (*intro crel-dep-mono-wrt-pred-eval-rel-comp-if-rel-dep-mono-wrt-pred-if-crel-dep-mono-wrt-pred*)
auto
then show *right-unique-on* A $(R \circ\circ R')$ **by** *blast*
from *assms* **show** $(R \circ\circ R')\ x\ (R'`(R`x))$ **by** (*blast intro*: *rel-eval-if-rel-dep-mono-wrt-predI*)
qed (*fact assms*)

lemma *eq-restrict-comp-eq-if-crel-dep-mono-wrt-pred* [*simp*]:
assumes $((x : A) \rightarrow_c B\ x)\ R$
shows $(=_A) \circ\circ R = R$
using *assms* **by** *force*

lemma *comp-eq-restrict-if-crel-dep-mono-wrt-pred* [*simp*]:
assumes $(A \rightarrow_c B)\ R$
shows $R \circ\circ (=_B) = R$
using *assms* **by** *force*

end

1.3.8 Extending Functions

theory *Binary-Relations-Function-Extend*
imports
Binary-Relations-Clean-Functions
begin

lemma *left-total-on-eq-sup-extend-if-left-total-on*:
assumes *left-total-on* A R
shows *left-total-on* $((=)\ x \sqcup A)$ $(extend\ x\ y\ R)$
using *assms* **by** *fastforce*

lemma *right-unique-on-eq-sup-extend-if-not-in-dom-if-right-unique-on*:
assumes *right-unique-on* A R
and $\neg(in\text{-}dom\ R\ x)$
shows *right-unique-on* $((=)\ x \sqcup A)$ $(extend\ x\ y\ R)$
using *assms* **by** (*intro right-unique-onI*) (*auto dest*: *right-unique-onD elim*!: *extendE*)

lemma *rel-dep-mono-wrt-eq-sup-if-extend-if-rel-dep-mono-wrt-predI*:
assumes $((x : A) \rightarrow B\ x)\ R$
and $\neg(in\text{-}dom\ R\ x)$
shows $((x' : (=)\ x \sqcup A) \rightarrow (if\ x' = x\ then\ (=)\ y\ else\ B\ x'))\ (extend\ x\ y\ R)$
using *assms* **by** (*intro rel-dep-mono-wrt-predI left-total-on-eq-sup-extend-if-left-total-on right-unique-on-eq-sup-extend-if-not-in-dom-if-right-unique-on dep-mono-wrt-predI*)
auto

lemma *rel-dep-mono-wrt-eq-sup-extend-if-rel-dep-mono-wrt-predI*:
assumes $((x : A) \to B\ x)\ R$
and $\neg(\textit{in-dom}\ R\ x)$
and $B\ x\ y$
shows $((x' : (=)\ x \sqcup A) \to B\ x')\ (\textit{extend}\ x\ y\ R)$
by (*urule rel-dep-mono-wrt-pred-covariant-codom*,
urule rel-dep-mono-wrt-eq-sup-if-extend-if-rel-dep-mono-wrt-predI)
(*use assms* **in** ‹*force split*: *if-split-asm*›)+

lemma *crel-dep-mono-wrt-eq-sup-if-extend-if-crel-dep-mono-wrt-predI*:
assumes $((x : A) \to_c B\ x)\ R$
and $\neg(\textit{in-dom}\ R\ x)$
shows $((x' : (=)\ x \sqcup A) \to_c (\textit{if}\ x' = x\ \textit{then}\ (=)\ y\ \textit{else}\ B\ x'))\ (\textit{extend}\ x\ y\ R)$
using *assms* **by** (*intro crel-dep-mono-wrt-predI*
rel-dep-mono-wrt-eq-sup-if-extend-if-rel-dep-mono-wrt-predI)
fastforce+

lemma *crel-dep-mono-wrt-eq-sup-extend-if-rel-dep-mono-wrt-predI*:
assumes $((x : A) \to_c B\ x)\ R$
and $\neg(\textit{in-dom}\ R\ x)$
and $B\ x\ y$
shows $((x' : (=)\ x \sqcup A) \to_c B\ x')\ (\textit{extend}\ x\ y\ R)$
using *assms*
by (*intro crel-dep-mono-wrt-predI rel-dep-mono-wrt-eq-sup-extend-if-rel-dep-mono-wrt-predI*)
fastforce+

context
fixes $\mathcal{R} :: ('a \Rightarrow 'b \Rightarrow \textit{bool}) \Rightarrow \textit{bool}$ **and** $A :: ('a \Rightarrow 'b \Rightarrow \textit{bool}) \Rightarrow 'a \Rightarrow \textit{bool}$ **and**
D
defines [*simp*]: $D \equiv \textit{in-codom-on}\ \mathcal{R}\ A$
begin

lemma *rel-dep-mono-wrt-pred-glue-if-right-unique-if-rel-dep-mono-wrt-pred*:
assumes *funs*: $\bigwedge R.\ \mathcal{R}\ R \Longrightarrow ((x : A\ R) \to B\ x)\ R$
and *runique*: *right-unique-on D* (*glue* $\mathcal{R}$)
shows $((x : D) \to B\ x)$ (*glue* $\mathcal{R}$)
proof (*intro rel-dep-mono-wrt-predI dep-mono-wrt-predI left-total-onI*)
fix x **assume** $D\ x$
with *funs* **obtain** R **where** *hyps*: $\mathcal{R}\ R\ A\ R\ x\ ((x : A\ R) \to B\ x)\ R$ **by** *auto*
then have $B\ x\ (R`x)$ **by** (*auto elim*: *rel-dep-mono-wrt-pred-relE*)
moreover have $(\textit{glue}\ \mathcal{R})`x = R`x$
proof (*intro glue-eval-eqI*)
show $\mathcal{R}\ R$ **by** (*fact hyps*)
then have $A\ R \leq D$ **by** *fastforce*
with *runique* **show** *right-unique-on* $(A\ R)$ (*glue* $\mathcal{R}$) **using** *antimono-right-unique-on*
by *blast*
qed (*use hyps* **in** ‹*auto elim*: *rel-dep-mono-wrt-pred-relE*›)
ultimately show $B\ x$ (*glue* $\mathcal{R}`x$) **by** *simp*
qed (*use assms* **in** ‹*fastforce+*›)

lemma *crel-dep-mono-wrt-pred-glue-if-right-unique-if-crel-dep-mono-wrt-pred*:
assumes $\bigwedge R.\ \mathcal{R}\ R \Longrightarrow ((x : A\ R) \to_c B\ x)\ R$
and *right-unique-on D* (*glue* $\mathcal{R}$)
shows $((x : D) \to_c B\ x)$ (*glue* $\mathcal{R}$)
using *assms*
by (*intro crel-dep-mono-wrt-predI rel-dep-mono-wrt-pred-glue-if-right-unique-if-rel-dep-mono-wrt-pred*)
fastforce+

end

lemma *right-unique-on-sup-if-rel-agree-on-sup-if-right-unique-on*:
assumes *right-unique-on P R right-unique-on P R′*
and *rel-agree-on* (*P* ⊓ *in-dom R* ⊓ *in-dom R′*) ((=) *R* ⊔ (=) *R′*)
shows *right-unique-on P* (*R* ⊔ *R′*)
proof (*intro right-unique-onI*)
fix *x y y′* **assume** *P x* (*R* ⊔ *R′*) *x y* (*R* ⊔ *R′*) *x y′*
then obtain *R1 R2* **where** *rels*: *R1 x y R2 x y′* **and** *ors*: *R1* = *R* ∨ *R1* = *R′* *R2* = *R* ∨ *R2* = *R′* **by** *auto*
then consider (*neq*) *R1* ≠ *R2* | *R1* = *R2* **by** *auto*
then show *y* = *y′*
proof *cases*
case *neq*
with *rels ors* **obtain** *z z′* **where** *R x z R′ x z′* **and** *yors*: *y* = *z* ∨ *y* = *z′ y′* = *z* ∨ *y′* = *z′*
by *auto*
with ‹*P x*› **have** *R′ x z* **by** (*intro rel-agree-onD*[**where** *?R*=*R* **and** *?R′*=*R′* **and** *?P*=*P* ⊓ *in-dom R* ⊓ *in-dom R′* **and** *?*$\mathcal{R}$=(=) *R* ⊔ (=) *R′*] *assms*)
auto
with ‹*P x*› ‹*R′ x z′*› *assms* **have** *z* = *z′* **by** (*blast dest*: *right-unique-onD*)
with *yors* **show** *y* = *y′* **by** *auto*
qed (*use assms* ‹*P x*› *rels ors* **in** ‹*auto dest*: *right-unique-onD*›)
qed

lemma *crel-dep-mono-wrt-pred-sup-if-eval-eq-if-crel-dep-mono-wrt-pred*:
assumes *dep-funs*: ((*x* : *A*) →$_c$ *B x*) *R* ((*x* : *A′*) →$_c$ *B x*) *R′*
and $\bigwedge x.\ A\ x \Longrightarrow A'\ x \Longrightarrow R`x = R'`x$
shows ((*x* : *A* ⊔ *A′*) →$_c$ *B x*) (*R* ⊔ *R′*)
proof −
let *?A* = λ*S*. *if S* = *R then A else A′*
from *dep-funs* **have** *A* = *A′* **if** *R* = *R′* **using** *that* **by** (*simp only*: *flip*: *in-dom-eq-if-crel-dep-mono-wrt-pred*[*OF dep-funs*(*1*)] *in-dom-eq-if-crel-dep-mono-wrt-pred*[*OF dep-funs*(*2*)])
then have [*uhint*]: *in-codom-on* ((=) *R* ⊔ (=) *R′*) *?A* = *A* ⊔ *A′*
by (*intro ext iffI*) (*fastforce split*: *if-splits*)+
show *?thesis* **by** (*urule* (*rr*) *crel-dep-mono-wrt-pred-glue-if-right-unique-if-crel-dep-mono-wrt-pred right-unique-on-sup-if-rel-agree-on-sup-if-right-unique-on rel-agree-on-if-eval-eq-if-rel-dep-mono-wrt-pred*)

(*use assms* **in** ‹*auto 7 0 intro*: *rel-dep-mono-wrt-pred-contravariant-dom dest*: *right-unique-onD*›)
qed

end

1.3.9 Lambda Abstractions

theory *Binary-Relations-Function-Lambda*
imports *Binary-Relations-Clean-Functions*
begin

consts *rel-lambda* :: $'a \Rightarrow ('b \Rightarrow 'c) \Rightarrow 'd$

definition *rel-lambda-pred* $A\ f\ x\ y \equiv A\ x \wedge f\ x = y$
adhoc-overloading *rel-lambda* $\rightleftharpoons$ *rel-lambda-pred*

open-bundle *rel-lambda-syntax*
begin
syntax
-rel-lambda :: *pttrns* $\Rightarrow 'a \Rightarrow 'b \Rightarrow 'c$ (‹(2λ- : -./ -)› 60)
syntax-consts
-rel-lambda $\rightleftharpoons$ *rel-lambda*
translations
$\lambda x\ xs : A.\ f \rightharpoonup CONST\ rel\text{-}lambda\ A\ (\lambda x.\ (\lambda xs : A.\ f))$
$\lambda x : A.\ f \rightleftharpoons CONST\ rel\text{-}lambda\ A\ (\lambda x.\ f)$
end

lemma *rel-lambdaI* [*intro*]:
assumes $A\ x$
and $f\ x = y$
shows $(\lambda x : A.\ f\ x)\ x\ y$
using *assms* **unfolding** *rel-lambda-pred-def* **by** *auto*

lemma *rel-lambda-appI*:
assumes $A\ x$
shows $(\lambda x : A.\ f\ x)\ x\ (f\ x)$
using *assms* **by** *auto*

lemma *rel-lambdaE* [*elim*!]:
assumes $(\lambda x : A.\ f\ x)\ x\ y$
obtains $A\ x\ y = f\ x$
using *assms* **unfolding** *rel-lambda-pred-def* **by** *auto*

lemma *rel-lambda-cong* [*cong*]:
$[\![\bigwedge x.\ A\ x \longleftrightarrow A'\ x;\ \bigwedge x.\ A'\ x \Longrightarrow f\ x = f'\ x]\!] \Longrightarrow (\lambda x : A.\ f\ x) = \lambda x : A'.\ f'\ x$
by (*intro ext*) *auto*

lemma *in-dom-rel-lambda-eq* [*simp*]: *in-dom* $(\lambda x : A.\ f\ x) = A$

by *auto*

lemma *in-codom-rel-lambda-eq-has-inverse-on* [*simp*]: *in-codom* $(\lambda x : A.\ f\ x) =$ *has-inverse-on* $A\ f$
by *fastforce*

lemma *left-total-on-rel-lambda*: *left-total-on* $A\ (\lambda x : A.\ f\ x)$
by *auto*

lemma *right-unique-on-rel-lambda*: *right-unique-on* $A\ (\lambda x : A.\ f\ x)$
by *auto*

lemma *crel-dep-mono-wrt-pred-rel-lambda*: $((x : A) \to_c ((=)\ (f\ x)))\ (\lambda x : A.\ f\ x)$
by (*intro crel-dep-mono-wrt-predI'*) *auto*

Compare the following with (*rel-dep-mono-wrt ?A ?B* $\Rightarrow$ *dep-mono-wrt ?A ?B*) (‘).

lemma *mono-dep-mono-wrt-pred-crel-dep-mono-wrt-pred-rel-lambda*:
$((A : \top) \Rightarrow ((x : A) \Rightarrow B\ x) \Rightarrow (x : A) \to_c B\ x)$ *rel-lambda*
by (*urule* (*rr*) *dep-mono-wrt-predI crel-dep-mono-wrt-predI'*) *auto*

lemma *rel-lambda-eval-eq* [*simp*]:
assumes $A\ x$
shows $(\lambda x : A.\ f\ x)`x = f\ x$
using *assms* **by** (*intro eval-eq-if-right-unique-onI*) *auto*

lemma *app-eq-if-rel-lambda-eqI*:
assumes $(\lambda x : A.\ f\ x) = (\lambda x : A.\ g\ x)$
and $A\ x$
shows $f\ x = g\ x$
using *assms* **by** (*auto dest*: *fun-cong*)

lemma *crel-dep-mono-wrt-pred-inf-rel-lambda-inf-if-rel-dep-mono-wrt-pred*:
assumes $((x : A) \to B\ x)\ R$
shows $((x : A \sqcap A') \to_c B\ x)\ (\lambda x : A \sqcap A'.\ R`x)$
using *assms* **by** *force*

corollary *crel-dep-mono-wrt-pred-rel-lambda-if-le-if-rel-dep-mono-wrt-pred*:
assumes $((x : A) \to B\ x)\ R$
and [*uhint*]: $A' \leq A$
shows $((x : A') \to_c B\ x)\ (\lambda x : A'.\ R`x)$
supply *inf-absorb2*[*uhint*]
by (*urule crel-dep-mono-wrt-pred-inf-rel-lambda-inf-if-rel-dep-mono-wrt-pred*) (*fact assms*)

lemma *rel-lambda-ext*:
assumes $((x : A) \to_c B\ x)\ R$
and $\bigwedge x.\ A\ x \Longrightarrow f\ x = R`x$
shows $(\lambda x : A.\ f\ x) = R$

using *assms* **by** (*intro ext iffI*) (*auto intro*!: *rel-lambdaI intro*: *rel-eval-if-rel-dep-mono-wrt-predI*)

lemma *rel-lambda-eval-eq-if-crel-dep-mono-wrt-pred* [*simp*]: $((x : A) \rightarrow_c B\ x)\ R \Longrightarrow (\lambda x : A.\ R`x) = R$
by (*rule rel-lambda-ext*) *auto*

Every element of *crel-dep-mono-wrt A B* may be expressed as a lambda abstraction.

lemma *eq-rel-lambda-if-crel-dep-mono-wrt-predE*:
assumes $((x : A) \rightarrow_c B\ x)\ R$
obtains f **where** $R = (\lambda x : A.\ f\ x)$
proof
let *?f*=$(\lambda x.\ R`x)$
from *assms* **show** $R = (\lambda x : A.\ (\lambda x.\ R`x)\ x)$ **by** *simp*
qed

lemma *rel-restrict-left-eq-rel-lambda-if-le-if-rel-dep-mono-wrt-pred*:
assumes $((x : A) \rightarrow B\ x)\ R$
and $A' \leq A$
shows $R\restriction_{A'} = (\lambda x : A'.\ R`x)$
proof −
from *assms mono-rel-dep-mono-wrt-pred-ge-crel-dep-mono-wrt-pred-rel-restrict-left*
have $((x : A') \rightarrow_c B\ x)\ R\restriction_{A'}$ **by** *force*
then show *?thesis*
supply $\langle A' \leq A\rangle$[*uhint*] *inf-absorb2*[*uhint*] **by** (*urule rel-lambda-ext*[*symmetric*])
auto
qed

lemma *mono-rel-lambda*: *mono* $(\lambda A.\ \lambda x : A.\ f\ x)$
by *auto*

end

theory *Binary-Relations-Functions*
imports
Binary-Relations-Clean-Functions
Binary-Relations-Function-Base
Binary-Relations-Function-Composition
Binary-Relations-Function-Evaluation
Binary-Relations-Function-Extend
Binary-Relations-Function-Lambda
begin

end

theory *Binary-Relations-Order*
imports

Binary-Relations-Order-Base
Binary-Relations-Reflexive
Binary-Relations-Symmetric
Binary-Relations-Transitive
begin

Summary Basic results about the order on binary relations.

lemma *in-dom-if-rel-if-rel-comp-le*:
assumes $(R \circ\circ S) \leq (S \circ\circ R)$
and $R\ x\ y\ S\ y\ z$
shows *in-dom* $S\ x$
using *assms* **by** (*blast intro*: *in-dom-if-in-dom-rel-comp*)

lemma *in-codom-if-rel-if-rel-comp-le*:
assumes $(R \circ\circ S) \leq (S \circ\circ R)$
and $R\ x\ y\ S\ y\ z$
shows *in-codom* $R\ z$
using *assms* **by** (*blast intro*: *in-codom-if-in-codom-rel-comp*)

lemma *rel-comp-le-rel-inv-if-rel-comp-le-if-symmetric*:
assumes *symms*: *symmetric R1 symmetric R2*
and *le*: $(R1 \circ\circ R2) \leq R3$
shows $(R2 \circ\circ R1) \leq R3^{-1}$
proof −
from *le* **have** $(R1 \circ\circ R2)^{-1} \leq R3^{-1}$ **by** *blast*
with *symms* **show** *?thesis* **by** *simp*
qed

lemma *rel-inv-le-rel-comp-if-le-rel-comp-if-symmetric*:
assumes *symms*: *symmetric R1 symmetric R2*
and *le*: $R3 \leq (R1 \circ\circ R2)$
shows $R3^{-1} \leq (R2 \circ\circ R1)$
proof −
from *le* **have** $R3^{-1} \leq (R1 \circ\circ R2)^{-1}$ **by** *blast*
with *symms* **show** *?thesis* **by** *simp*
qed

corollary *rel-comp-le-rel-comp-if-rel-comp-le-rel-comp-if-symmetric*:
assumes *symmetric* $(R1 :: 'a \Rightarrow 'a \Rightarrow bool)$ *symmetric R2 symmetric R3 symmetric R4*
and $(R1 \circ\circ R2) \leq (R3 \circ\circ R4)$
shows $(R2 \circ\circ R1) \leq (R4 \circ\circ R3)$
proof −
from *assms* **have** $(R2 \circ\circ R1) \leq (R3 \circ\circ R4)^{-1}$
by (*intro rel-comp-le-rel-inv-if-rel-comp-le-if-symmetric*)
with *assms* **show** *?thesis* **by** *simp*
qed

corollary *rel-comp-le-rel-comp-iff-if-symmetric*:

assumes *symmetric* (*R1* :: *'a* ⇒ *'a* ⇒ *bool*) *symmetric R2 symmetric R3 symmetric R4*
shows (*R1* ∘∘ *R2*) ≤ (*R3* ∘∘ *R4*) ⟷ (*R2* ∘∘ *R1*) ≤ (*R4* ∘∘ *R3*)
using *assms*
by (*blast intro*: *rel-comp-le-rel-comp-if-rel-comp-le-rel-comp-if-symmetric*)

corollary *eq-if-le-rel-if-symmetric*:
assumes *symmetric R symmetric S*
and (*R* ∘∘ *S*) ≤ (*S* ∘∘ *R*)
shows (*R* ∘∘ *S*) = (*S* ∘∘ *R*)
using *assms rel-comp-le-rel-comp-iff-if-symmetric*[*of R S*]
by (*intro antisym*) *auto*

lemma *rel-comp-le-rel-comp-if-le-rel-if-reflexive-on-in-codom-if-transitive*:
assumes *trans*: *transitive S*
and *refl-on*: *reflexive-on* (*in-codom S*) *R*
and *le-rel*: *R* ≤ *S*
shows *R* ∘∘ *S* ≤ *S* ∘∘ *R*
proof (*rule le-relI*)
fix *x1 x2* **assume**(*R* ∘∘ *S*) *x1 x2*
then obtain *x3* **where** *R x1 x3 S x3 x2* **by** *blast*
then have *S x1 x3* **using** *le-rel* **by** *blast*
with ‹*S x3 x2*› **have** *S x1 x2* **using** *trans* **by** *blast*
with *refl-on* **have** *R x2 x2* **by** *blast*
then show (*S* ∘∘ *R*) *x1 x2* **using** ‹*S x1 x2*› **by** *blast*
qed

end

Irreflexive

theory *Binary-Relations-Irreflexive*
imports
Functions-Monotone
begin

consts *irreflexive-on* :: *'a* ⇒ *'b* ⇒ *bool*

overloading
irreflexive-on-pred ≡ *irreflexive-on* :: (*'a* ⇒ *bool*) ⇒ (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*
begin
definition *irreflexive-on-pred P R* ≡ ∀ *x* : *P*. ¬(*R x x*)
end

lemma *irreflexive-onI* [*intro*]:
assumes ⋀*x*. *P x* ⟹ ¬(*R x x*)
shows *irreflexive-on P R*
using *assms* **unfolding** *irreflexive-on-pred-def* **by** *blast*

lemma *irreflexive-onD* [*dest*]:
assumes *irreflexive-on P R*
and *P x*
shows ¬(*R x x*)
using *assms* **unfolding** *irreflexive-on-pred-def* **by** *blast*

lemma *antimono-irreflexive-on*:
((≤) ⇒ (≤) ⇛ (≥)) (*irreflexive-on* :: ('*a* ⇒ *bool*) ⇒ ('*a* ⇒ '*a* ⇒ *bool*) ⇒ *bool*)
by *blast*

consts *irreflexive* :: '*a* ⇒ *bool*

overloading
irreflexive ≡ *irreflexive* :: ('*a* ⇒ '*a* ⇒ *bool*) ⇒ *bool*
begin
definition (*irreflexive* :: ('*a* ⇒ '*a* ⇒ *bool*) ⇒ *bool*) ≡ *irreflexive-on* (⊤ :: '*a* ⇒ *bool*)
end

lemma *irreflexive-eq-irreflexive-on*:
(*irreflexive* :: ('*a* ⇒ '*a* ⇒ *bool*) ⇒ -) = *irreflexive-on* (⊤ :: '*a* ⇒ *bool*)
unfolding *irreflexive-def* **..**

lemma *irreflexive-eq-irreflexive-on-uhint* [*uhint*]:
assumes *P* ≡ (⊤ :: '*a* ⇒ *bool*)
shows *irreflexive* :: ('*a* ⇒ '*a* ⇒ *bool*) ⇒ - ≡ *irreflexive-on P*
using *assms* **by** (*simp add*: *irreflexive-eq-irreflexive-on*)

lemma *irreflexiveI* [*intro*]:
assumes ⋀*x*. ¬(*R x x*)
shows *irreflexive R*
using *assms* **by** (*urule irreflexive-onI*)

lemma *irreflexiveD*:
assumes *irreflexive R*
shows ¬(*R x x*)
using *assms* **by** (*urule* (*d*) *irreflexive-onD chained*: *insert*) *simp*

lemma *irreflexive-on-if-irreflexive*:
fixes *P* :: '*a* ⇒ *bool* **and** *R* :: '*a* ⇒ '*a* ⇒ *bool*
assumes *irreflexive R*
shows *irreflexive-on P R*
using *assms* **by** (*intro irreflexive-onI*) (*blast dest*: *irreflexiveD*)

end

theory *Binary-Relations-Asymmetric*
imports
Binary-Relations-Irreflexive
Binary-Relations-Antisymmetric
Functions-Monotone
begin

consts *asymmetric-on* :: $'a \Rightarrow 'b \Rightarrow bool$

overloading
asymmetric-on-pred $\equiv$ *asymmetric-on* :: $('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$
begin
definition *asymmetric-on-pred* $(P :: 'a \Rightarrow bool)$ $(R :: 'a \Rightarrow 'a \Rightarrow bool)$
$\equiv \forall x\ y : P.\ R\ x\ y \longrightarrow \neg(R\ y\ x)$
end

lemma *asymmetric-onI* [*intro*]:
assumes $\bigwedge x\ y.\ P\ x \Longrightarrow P\ y \Longrightarrow R\ x\ y \Longrightarrow \neg(R\ y\ x)$
shows *asymmetric-on P R*
using *assms* **unfolding** *asymmetric-on-pred-def* **by** *blast*

lemma *asymmetric-onD* [*dest*]:
assumes *asymmetric-on P R*
and *P x P y*
and *R x y*
shows $\neg(R\ y\ x)$
using *assms* **unfolding** *asymmetric-on-pred-def* **by** *blast*

lemma *asymmetric-onE*:
assumes *asymmetric-on P R*
obtains $\bigwedge x\ y.\ P\ x \Longrightarrow P\ y \Longrightarrow R\ x\ y \Longrightarrow \neg(R\ y\ x)$
using *assms* **unfolding** *asymmetric-on-pred-def* **by** *blast*

lemma *asymmetric-on-le-irreflexive-on*:
(*asymmetric-on* :: $('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a \Rightarrow bool) \Rightarrow$ -) $\leq$ *irreflexive-on*
by *blast*

lemma *asymmetric-on-le-antisymmetric-on*:
(*asymmetric-on* :: $('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a \Rightarrow bool) \Rightarrow$ -) $\leq$ *antisymmetric-on*
by *blast*

lemma *antimono-asymmetric-on*:
$((\leq) \Rightarrow (\leq) \Rrightarrow (\geq))$ (*asymmetric-on* :: $('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$)
by *blast*

lemma *asymmetric-on-rel-map-if-mono-wrt-pred-if-asymmetric-on*:
fixes $P :: 'a \Rightarrow bool$ **and** $R :: 'a \Rightarrow 'a \Rightarrow bool$
assumes *asymmetric-on P R*

and $(Q \Rightarrow P)$ f
shows *asymmetric-on* Q (*rel-map* f R)
using *assms* **by** *fastforce*

consts *asymmetric* :: $'a \Rightarrow$ *bool*

overloading
asymmetric $\equiv$ *asymmetric* :: $('a \Rightarrow 'a \Rightarrow$ *bool*$) \Rightarrow$ *bool*
begin
definition *asymmetric* :: $('a \Rightarrow 'a \Rightarrow$ *bool*$) \Rightarrow$ - $\equiv$ *asymmetric-on* $(\top :: 'a \Rightarrow$ *bool*$)$
end

lemma *asymmetric-eq-asymmetric-on*:
(*asymmetric* :: $('a \Rightarrow 'a \Rightarrow$ *bool*$) \Rightarrow$ -$) =$ *asymmetric-on* $(\top :: 'a \Rightarrow$ *bool*$)$
unfolding *asymmetric-def* **..**

lemma *asymmetric-eq-asymmetric-on-uhint* [*uhint*]:
$P \equiv (\top :: 'a \Rightarrow$ *bool*$) \Longrightarrow$ (*asymmetric* :: $('a \Rightarrow 'a \Rightarrow$ *bool*$) \Rightarrow$ -$) \equiv$ *asymmetric-on* P
by (*simp add*: *asymmetric-eq-asymmetric-on*)

lemma *asymmetricI* [*intro*]:
assumes $\bigwedge x\ y.\ R\ x\ y \Longrightarrow \neg(R\ y\ x)$
shows *asymmetric* R
using *assms* **by** (*urule asymmetric-onI*)

lemma *asymmetricD* [*dest*]:
assumes *asymmetric* R
and $R\ x\ y$
shows $\neg(R\ y\ x)$
using *assms* **by** (*urule* (*d*) *asymmetric-onD chained*: *insert*) *simp-all*

lemma *asymmetricE*:
assumes *asymmetric* R
obtains $\bigwedge x\ y.\ R\ x\ y \Longrightarrow \neg(R\ y\ x)$
using *assms* **by** (*urule* (*e*) *asymmetric-onE chained*: *insert*) *simp-all*

lemma *asymmetric-on-if-asymmetric*:
fixes $P :: 'a \Rightarrow$ *bool* **and** $R :: 'a \Rightarrow 'a \Rightarrow$ *bool*
assumes *asymmetric* R
shows *asymmetric-on* $P\ R$
using *assms* **by** (*intro asymmetric-onI*) *blast*

lemma *asymmetric-if-asymmetric-on-in-field*:
assumes *asymmetric-on* (*in-field* R) R
shows *asymmetric* R
using *assms* **by** (*intro asymmetricI*) *blast*

corollary *asymmetric-on-in-field-iff-asymmetric* [*iff*]:
asymmetric-on (*in-field R*) *R* ⟷ *asymmetric R*
using *asymmetric-if-asymmetric-on-in-field asymmetric-on-if-asymmetric* **by** *blast*

lemma *asymmetric-on-if-asymmetric-restrict*:
assumes *asymmetric* $R{\restriction}_P$
shows *asymmetric-on P R*
using *assms* **by** *blast*

lemma *asymmetric-restrict-if-asymmetric-on*:
assumes *asymmetric-on P R*
shows *asymmetric* $R{\restriction}_P$
using *assms* **by** *blast*

corollary *asymmetric-restrict-iff-asymmetric-on* [*iff*]: *asymmetric* $R{\restriction}_P$ ⟷ *asymmetric-on P R*
using *asymmetric-on-if-asymmetric-restrict asymmetric-restrict-if-asymmetric-on*
by *blast*

lemma *asymmetric-le-irreflexive*:
(*asymmetric* :: ('*a* ⇒ '*a* ⇒ *bool*) ⇒ -) ≤ *irreflexive*
by *blast*

lemma *asymmetric-le-antisymmetric*:
(*asymmetric* :: ('*a* ⇒ '*a* ⇒ *bool*) ⇒ -) ≤ *antisymmetric*
by *blast*

lemma *antimono-asymmetric*: *antimono* (*asymmetric* :: ('*a* ⇒ '*a* ⇒ *bool*) ⇒ -)
by *blast*

end

Surjective

theory *Binary-Relations-Surjective*
imports
Binary-Relations-Left-Total
HOL-Syntax-Bundles-Lattices
begin

consts *rel-surjective-at* :: '*a* ⇒ '*b* ⇒ *bool*

overloading
rel-surjective-at-pred ≡ *rel-surjective-at* :: ('*a* ⇒ *bool*) ⇒ ('*b* ⇒ '*a* ⇒ *bool*) ⇒ *bool*
begin
definition *rel-surjective-at-pred P R* ≡ ∀ *y* : *P*. *in-codom R y*
end

lemma *rel-surjective-atI* [*intro*]:
assumes $\bigwedge y.\ P\ y \Longrightarrow$ *in-codom* $R\ y$
shows *rel-surjective-at* $P\ R$
unfolding *rel-surjective-at-pred-def* **using** *assms* **by** *blast*

lemma *rel-surjective-atE* [*elim*]:
assumes *rel-surjective-at* $P\ R$
and $P\ y$
obtains x **where** $R\ x\ y$
using *assms* **unfolding** *rel-surjective-at-pred-def* **by** *blast*

lemma *in-codom-if-rel-surjective-at*:
assumes *rel-surjective-at* $P\ R$
and $P\ y$
shows *in-codom* $R\ y$
using *assms* **by** *blast*

lemma *rel-surjective-at-rel-inv-iff-left-total-on* [*iff*]:
rel-surjective-at $(P :: 'a \Rightarrow bool)\ (R^{-1} :: 'b \Rightarrow 'a \Rightarrow bool) \longleftrightarrow$ *left-total-on* $P\ R$
by *fast*

lemma *left-total-on-rel-inv-iff-rel-surjective-at* [*iff*]:
left-total-on $(P :: 'a \Rightarrow bool)\ (R^{-1} :: 'a \Rightarrow 'b \Rightarrow bool) \longleftrightarrow$ *rel-surjective-at* $P\ R$
by *fast*

lemma *mono-rel-surjective-at*:
$((\geq) \Rightarrow (\leq) \Rrightarrow (\leq))$ $(\textit{rel-surjective-at} :: ('b \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool)$
by *fastforce*

lemma *rel-surjective-at-iff-le-codom*:
rel-surjective-at $(P :: 'b \Rightarrow bool)\ (R :: 'a \Rightarrow 'b \Rightarrow bool) \longleftrightarrow P \leq$ *in-codom* R
by *force*

lemma *rel-surjective-at-compI*:
fixes $P :: 'c \Rightarrow bool$ **and** $R :: 'a \Rightarrow 'b \Rightarrow bool$ **and** $S :: 'b \Rightarrow 'c \Rightarrow bool$
assumes *surj-R*: *rel-surjective-at* (*in-dom* S) R
and *surj-S*: *rel-surjective-at* $P\ S$
shows *rel-surjective-at* P $(R \circ\circ S)$
proof (*rule rel-surjective-atI*)
fix y **assume** $P\ y$
then obtain x **where** $S\ x\ y$ **using** *surj-S* **by** *auto*
moreover then have *in-dom* $S\ x$ **by** *auto*
moreover then obtain z **where** $R\ z\ x$ **using** *surj-R* **by** *auto*
ultimately show *in-codom* $(R \circ\circ S)\ y$ **by** *blast*
qed

consts *rel-surjective* :: $'a \Rightarrow bool$

overloading
rel-surjective ≡ *rel-surjective* :: (*'b* ⇒ *'a* ⇒ *bool*) ⇒ *bool*
begin
definition (*rel-surjective* :: (*'b* ⇒ *'a* ⇒ *bool*) ⇒ -) ≡ *rel-surjective-at* (⊤ :: *'a* ⇒ *bool*)
end

lemma *rel-surjective-eq-rel-surjective-at*:
(*rel-surjective* :: (*'b* ⇒ *'a* ⇒ *bool*) ⇒ -) = *rel-surjective-at* (⊤ :: *'a* ⇒ *bool*)
unfolding *rel-surjective-def* **..**

lemma *rel-surjective-eq-rel-surjective-at-uhint* [*uhint*]:
assumes *P* ≡ (⊤ :: *'a* ⇒ *bool*)
shows (*rel-surjective* :: (*'b* ⇒ *'a* ⇒ *bool*) ⇒ -) ≡ *rel-surjective-at P*
using *assms* **by** (*simp add*: *rel-surjective-eq-rel-surjective-at*)

lemma *rel-surjectiveI*:
assumes ⋀*y*. *in-codom R y*
shows *rel-surjective R*
using *assms* **by** (*urule rel-surjective-atI*)

lemma *rel-surjectiveE*:
assumes *rel-surjective R*
obtains *x* **where** *R x y*
using *assms* **by** (*urule* (*e*) *rel-surjective-atE chained*: *insert*) *simp*

lemma *in-codom-if-rel-surjective*:
assumes *rel-surjective R*
shows *in-codom R y*
using *assms* **by** (*blast elim*: *rel-surjectiveE*)

lemma *rel-surjective-rel-inv-iff-left-total* [*iff*]: *rel-surjective* R^{-1} ⟷ *left-total R*
by (*urule rel-surjective-at-rel-inv-iff-left-total-on*)

lemma *left-total-rel-inv-iff-rel-surjective* [*iff*]: *left-total* R^{-1} ⟷ *rel-surjective R*
by (*urule left-total-on-rel-inv-iff-rel-surjective-at*)

lemma *rel-surjective-at-if-surjective*:
fixes *P* :: *'a* ⇒ *bool* **and** *R* :: *'b* ⇒ *'a* ⇒ *bool*
assumes *rel-surjective R*
shows *rel-surjective-at P R*
using *assms* **by** (*intro rel-surjective-atI*) (*blast dest*: *in-codom-if-rel-surjective*)

end

Bi-Total

theory *Binary-Relations-Bi-Total*

imports
Binary-Relations-Left-Total
Binary-Relations-Surjective
begin

definition *bi-total-on P Q* ≡ *left-total-on P* ⊓ *rel-surjective-at Q*

lemma *bi-total-onI* [*intro*]:
assumes *left-total-on P R*
and *rel-surjective-at Q R*
shows *bi-total-on P Q R*
unfolding *bi-total-on-def* **using** *assms* **by** *auto*

lemma *bi-total-onE* [*elim*]:
assumes *bi-total-on P Q R*
obtains *left-total-on P R rel-surjective-at Q R*
using *assms* **unfolding** *bi-total-on-def* **by** *auto*

lemma *bi-total-on-restricts-if-Fun-Rel-iff-if-bi-total-on*:
assumes *bi-total-on P Q R*
and (*R* ⇛ (⟷)) *P Q*
shows *bi-total-on P Q R*↾$_P$↿$_Q$
using *assms* **by** *force*

definition *bi-total* ≡ *bi-total-on* (⊤ :: *'a* ⇒ *bool*) (⊤ :: *'b* ⇒ *bool*) :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *bool*

lemma *bi-total-eq-bi-total-on*:
(*bi-total* :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *bool*) = *bi-total-on* (⊤ :: *'a* ⇒ *bool*) (⊤ :: *'b* ⇒ *bool*)
unfolding *bi-total-def* **..**

lemma *bi-total-eq-bi-total-on-uhint* [*uhint*]:
assumes *P* ≡ (⊤ :: *'a* ⇒ *bool*)
and *Q* ≡ (⊤ :: *'b* ⇒ *bool*)
shows (*bi-total* :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *bool*) ≡ *bi-total-on P Q*
using *assms* **by** (*simp add*: *bi-total-eq-bi-total-on*)

lemma *bi-totalI* [*intro*]:
assumes *left-total R*
and *rel-surjective R*
shows *bi-total R*
using *assms* **by** (*urule bi-total-onI*)

lemma *bi-totalE* [*elim*]:
assumes *bi-total R*
obtains *left-total R rel-surjective R*
using *assms* **by** (*urule* (*e*) *bi-total-onE*)

end

Bi-Unique

theory *Binary-Relations-Bi-Unique*
imports
Binary-Relations-Injective
Binary-Relations-Right-Unique
begin

definition *bi-unique-on* ≡ *right-unique-on* ⊓ *rel-injective-on*

lemma *bi-unique-onI* [*intro*]:
assumes *right-unique-on P R*
and *rel-injective-on P R*
shows *bi-unique-on P R*
unfolding *bi-unique-on-def* **using** *assms* **by** *auto*

lemma *bi-unique-onE* [*elim*]:
assumes *bi-unique-on P R*
obtains *right-unique-on P R rel-injective-on P R*
using *assms* **unfolding** *bi-unique-on-def* **by** *auto*

lemma *bi-unique-on-rel-inv-if-Fun-Rel-iff-if-bi-unique-on*:
assumes *bi-unique-on P R*
and (*R* ⇛ (⟷)) *P Q*
shows *bi-unique-on Q* R^{-1}
using *assms* **by** (*intro bi-unique-onI*
rel-injective-on-if-Fun-Rel-imp-if-rel-injective-at
right-unique-on-if-Fun-Rel-imp-if-right-unique-at)
(*auto 0 3*)

definition *bi-unique* ≡ *bi-unique-on* (⊤ :: *'a* ⇒ *bool*) :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *bool*

lemma *bi-unique-eq-bi-unique-on*:
bi-unique = (*bi-unique-on* (⊤ :: *'a* ⇒ *bool*) :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *bool*)
unfolding *bi-unique-def* **..**

lemma *bi-unique-eq-bi-unique-on-uhint* [*uhint*]:
assumes *P* ≡ (⊤ :: *'a* ⇒ *bool*)
shows *bi-unique* ≡ (*bi-unique-on P* :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *bool*)
using *assms* **by** (*simp add*: *bi-unique-eq-bi-unique-on*)

lemma *bi-uniqueI* [*intro*]:
assumes *right-unique R*
and *rel-injective R*
shows *bi-unique R*
using *assms* **by** (*urule bi-unique-onI*)

lemma *bi-uniqueE* [*elim*]:
assumes *bi-unique R*
obtains *right-unique R rel-injective R*
using *assms* **by** (*urule* (*e*) *bi-unique-onE*)

end

Injective

theory *Functions-Injective*
imports
Bounded-Quantifiers
Functions-Monotone
HOL-Syntax-Bundles-Lattices
begin

consts *injective-on* :: $'a \Rightarrow 'b \Rightarrow bool$

overloading
injective-on-pred $\equiv$ *injective-on* :: $('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b) \Rightarrow bool$
begin
definition *injective-on-pred P f* $\equiv \forall\, x\ x' : P.\ f\ x = f\ x' \longrightarrow x = x'$
end

lemma *injective-onI* [*intro*]:
assumes $\bigwedge x\ x'.\ P\ x \Longrightarrow P\ x' \Longrightarrow f\ x = f\ x' \Longrightarrow x = x'$
shows *injective-on P f*
unfolding *injective-on-pred-def* **using** *assms* **by** *blast*

lemma *injective-onD*:
assumes *injective-on P f*
and *P x P x'*
and $f\ x = f\ x'$
shows $x = x'$
using *assms* **unfolding** *injective-on-pred-def* **by** *blast*

lemma *injective-on-comp-if-injective-onI*:
assumes *injective-on* (P :: $'a \Rightarrow bool$) *f injective-on Q g*
and $(P \Rightarrow Q)\ f$
shows *injective-on P* $(g \circ f)$
by (*urule injective-onI*) (*use assms* **in** ‹*auto dest*: *injective-onD*›)

consts *injective* :: $'a \Rightarrow bool$

overloading
injective $\equiv$ *injective* :: $('a \Rightarrow 'b) \Rightarrow bool$
begin
definition (*injective* :: $('a \Rightarrow 'b) \Rightarrow bool$) $\equiv$ *injective-on* ($\top$ :: $'a \Rightarrow bool$)

end

lemma *injective-eq-injective-on*:
(*injective* :: ('*a* ⇒ '*b*) ⇒ *bool*) = *injective-on* (⊤ :: '*a* ⇒ *bool*)
unfolding *injective-def* **..**

lemma *injective-eq-injective-on-uhint* [*uhint*]:
assumes *P* ≡ (⊤ :: '*a* ⇒ *bool*)
shows *injective* :: ('*a* ⇒ '*b*) ⇒ *bool* ≡ *injective-on P*
using *assms* **by** (*simp add*: *injective-eq-injective-on*)

lemma *injectiveI* [*intro*]:
assumes ⋀*x x'*. *f x* = *f x'* ⟹ *x* = *x'*
shows *injective f*
using *assms* **by** (*urule injective-onI*)

lemma *injectiveD*:
assumes *injective f*
and *f x* = *f x'*
shows *x* = *x'*
using *assms* **by** (*urule* (*d*) *injective-onD chained*: *insert*) *simp-all*

lemma *injective-on-if-injective*:
fixes *P* :: '*a* ⇒ *bool* **and** *f* :: '*a* ⇒ -
assumes *injective f*
shows *injective-on P f*
using *assms* **by** (*intro injective-onI*) (*blast dest*: *injectiveD*)

Instantiations **lemma** *injective-id*: *injective id* **by** *auto*

end

Connected

theory *Binary-Relations-Connected*
imports
Binary-Relation-Functions
Functions-Injective
begin

consts *connected-on* :: '*a* ⇒ '*b* ⇒ *bool*

overloading
connected-on-pred ≡ *connected-on* :: ('*a* ⇒ *bool*) ⇒ ('*a* ⇒ '*a* ⇒ *bool*) ⇒ *bool*
begin
definition *connected-on-pred P R* ≡ ∀ *x y* : *P*. *x* ≠ *y* ⟶ *R x y* ∨ *R y x*
end

lemma *connected-onI* [*intro*]:
  assumes $\bigwedge x\ y.\ P\ x \Longrightarrow P\ y \Longrightarrow x \neq y \Longrightarrow R\ x\ y \vee R\ y\ x$
  shows *connected-on P R*
  using *assms* **unfolding** *connected-on-pred-def* **by** *blast*

lemma *connected-onE* [*elim*]:
  assumes *connected-on P R*
  and *P x P y*
  obtains $x = y \mid R\ x\ y \mid R\ y\ x$
  using *assms* **unfolding** *connected-on-pred-def* **by** *auto*

lemma *connected-on-rel-inv-iff-connected-on* [*iff*]:
  connected-on $(P :: 'a \Rightarrow bool)\ (R :: 'a \Rightarrow 'a \Rightarrow bool)^{-1} \longleftrightarrow$ *connected-on P R*
  by *blast*

lemma *connected-on-rel-map-if-injective-on-if-mono-wrt-pred-if-connected-on*:
  fixes $P :: 'a \Rightarrow bool$ **and** $R :: 'a \Rightarrow 'a \Rightarrow bool$
  assumes *connected-on P R*
  and $(Q \Rightarrow P)\ f$
  and *injective-on Q f*
  shows *connected-on Q* (*rel-map f R*)
  using *assms* **by** (*fastforce dest*: *injective-onD*)

lemma *antimono-connected-on*: *antimono* (*connected-on* :: $('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$)
  by (*intro antimonoI*) *auto*

lemma *mono-connected-on*:
  $((\leq) \Rightarrow (\geq) \Rrightarrow (\geq))$ (*connected-on* :: $('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$)
  by *blast*

lemma *in-field-restrict-eq-if-ex-ne-if-connected-on*:
  assumes *conn*: *connected-on P R*
  and *eq-ne*: $\exists\, x\ y : P.\ x \neq y$
  shows *in-field* $R{\restriction_P} = P$
proof (*intro ext iffI*)
  fix *x* **assume** *P x*
  moreover with *eq-ne* **obtain** *y* **where** *P y x* $\neq$ *y* **by** *blast*
  ultimately have *R x y* $\vee$ *R y x* **using** *conn* **by** *blast*
  with ‹*P x*› ‹*P y*› **show** *in-field* $R{\restriction_P}\ x$ **by** *fastforce*
qed *auto*

consts *connected* :: $'a \Rightarrow bool$

overloading
  connected $\equiv$ *connected* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$
begin
  definition *connected* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool \equiv$ *connected-on* $(\top :: 'a \Rightarrow bool)$
end

lemma *connected-eq-connected-on*:
(*connected* :: ('*a* ⇒ '*a* ⇒ *bool*) ⇒ -) = *connected-on* (⊤ :: '*a* ⇒ *bool*)
unfolding *connected-def* **..**

lemma *connected-eq-connected-on-uhint* [*uhint*]:
P ≡ (⊤ :: '*a* ⇒ *bool*) ⟹ (*connected* :: ('*a* ⇒ '*a* ⇒ *bool*) ⇒ -) ≡ *connected-on P*
by (*simp add*: *connected-eq-connected-on*)

lemma *connectedI* [*intro*]:
assumes ⋀*x y*. *x* ≠ *y* ⟹ *R x y* ∨ *R y x*
shows *connected R*
using *assms* **by** (*urule connected-onI*)

lemma *connectedE* [*elim*]:
assumes *connected R*
and *x* ≠ *y*
obtains *R x y* | *R y x*
using *assms* **by** (*urule* (*e*) *connected-onE chained*: *insert*) *auto*

lemma *connected-on-if-connected*:
fixes *P* :: '*a* ⇒ *bool* **and** *R* :: '*a* ⇒ '*a* ⇒ *bool*
assumes *connected R*
shows *connected-on P R*
using *assms* **by** (*intro connected-onI*) *blast*

end

Well-Founded Relations

theory *Binary-Relations-Wellfounded*
imports
Functions-Monotone
begin

consts *wellfounded-on* :: '*a* ⇒ '*b* ⇒ *bool*

overloading
wellfounded-on-pred ≡ *wellfounded-on* :: ('*a* ⇒ *bool*) ⇒ ('*a* ⇒ '*a* ⇒ *bool*) ⇒ *bool*
begin
definition *wellfounded-on-pred* (*P* :: '*a* ⇒ *bool*) (*R* :: '*a* ⇒ '*a* ⇒ *bool*)
≡ ∀ *Q*. (∃ *x* : *P*. *Q x*) ⟶ (∃ *m* : *P*. *Q m* ∧ (∀ *y* : *P*. *R y m* ⟶ ¬(*Q y*)))
end

lemma *wellfounded-onI*:
assumes ⋀*Q x*. *P x* ⟹ *Q x* ⟹ ∃ *m* : *P*. *Q m* ∧ (∀ *y* : *P*. *R y m* ⟶ ¬(*Q y*))

shows *wellfounded-on P R*
using *assms* **unfolding** *wellfounded-on-pred-def* **by** *blast*

lemma *wellfounded-onE*:
assumes *wellfounded-on P R P x Q x*
obtains *m* **where** *P m Q m* $\bigwedge y.$ *P y* $\Longrightarrow$ *R y m* $\Longrightarrow$ $\neg$(*Q y*)
proof −
from *assms* **obtain** *m* **where** *P m Q m* $\forall\, y : P.$ *R y m* $\longrightarrow$ $\neg$(*Q y*)
unfolding *wellfounded-on-pred-def* **by** *auto*
then show *?thesis* **using** *that* **by** *blast*
qed

lemma *wellfounded-on-induct* [*consumes 1*, *case-names step*]:
assumes *wellfounded-on P R P z*
assumes *step*: $\bigwedge x.$ *P x* $\Longrightarrow$ ($\bigwedge y.$ *P y* $\Longrightarrow$ *R y x* $\Longrightarrow$ *Q y*) $\Longrightarrow$ *Q x*
shows *Q z*
proof (*rule ccontr*)
assume $\neg$(*Q z*)
then obtain *m* **where** *P m* $\neg$(*Q m*) $\bigwedge y.$ *P y* $\Longrightarrow$ *R y m* $\Longrightarrow$ *Q y*
using *assms* **by** (*blast elim*!: *wellfounded-onE*[**where** *Q* = $\lambda x.$ $\neg$(*Q x*)])
then show *False* **using** *step* **by** *auto*
qed

lemma *wellfounded-on-rel-map-if-mono-wrt-pred-if-wellfounded-on*:
fixes *A* :: *'a* $\Rightarrow$ *bool* **and** *B* :: *'b* $\Rightarrow$ *bool*
assumes *wf*: *wellfounded-on A R*
and (*B* $\Rightarrow$ *A*) *f*
shows *wellfounded-on B* (*rel-map f R*)
proof (*intro wellfounded-onI*)
fix *Q* **and** *x* :: *'b* **assume** *B x Q x*
moreover define *U* **where** *U* $\equiv$ *has-inverse-on* (*B* $\sqcap$ *Q*) *f*
ultimately have *A* (*f x*) *U* (*f x*) **using** ‹(*B* $\Rightarrow$ *A*) *f*› **by** *auto*
then obtain m_U **where** *A* m_U *U* m_U **and** *min*: $\bigwedge a.$ *A a* $\Longrightarrow$ *R a* m_U $\Longrightarrow$ $\neg$(*U a*)
using *wf* **by** (*blast elim*!: *wellfounded-onE*)
from ‹*U* m_U› **obtain** m_Q **where** *B* m_Q *f* m_Q = m_U *Q* m_Q **unfolding** *U-def* **by** *blast*
have $\neg$(*Q b*) **if** *B b rel-map f R b* m_Q **for** *b*
proof −
from *that* **have** *R* (*f b*) m_U **using** ‹*f* m_Q = m_U› **by** *auto*
then have $\neg$(*U* (*f b*)) **using** ‹*B b*› ‹(*B* $\Rightarrow$ *A*) *f*› *min* **by** *blast*
then show *?thesis* **unfolding** *U-def* **using** ‹*B b*› **by** *blast*
qed
then show $\exists\, m : B.$ *Q m* $\wedge$ ($\forall\, b : B.$ *rel-map f R b m* $\longrightarrow$ $\neg$(*Q b*)) **using** ‹*B* m_Q› ‹*Q* m_Q› **by** *blast*
qed

corollary *wellfounded-on-if-le-pred-if-wellfounded-on*:
fixes *P* :: *'a* $\Rightarrow$ *bool* **and** *R* :: *'a* $\Rightarrow$ *'a* $\Rightarrow$ *bool*

assumes *wellfounded-on P R Q* $\leq$ *P*
shows *wellfounded-on Q R*
proof −
have (*Q* $\Rightarrow$ *P*) *id* **using** ‹*Q* $\leq$ *P*› **by** *force*
then have *wellfounded-on Q* (*rel-map id R*)
using *assms wellfounded-on-rel-map-if-mono-wrt-pred-if-wellfounded-on* **by** *blast*
then show *?thesis* **by** *simp*
qed

lemma *wellfounded-on-if-le-rel-if-wellfounded-on*:
fixes *P* :: *'a* $\Rightarrow$ *bool* **and** *R* :: *'a* $\Rightarrow$ *'a* $\Rightarrow$ *bool*
assumes *wellfounded-on P R S* $\leq$ *R*
shows *wellfounded-on P S*
proof (*intro wellfounded-onI*)
fix *Q* **and** *x* :: *'a* **assume** *P x Q x*
then obtain *m* **where** *P m Q m* $\bigwedge$*y. P y* $\Longrightarrow$ *R y m* $\Longrightarrow$ ¬ *Q y*
using *assms* **by** (*force elim*!: *wellfounded-onE*)
then show ∃ *m* : *P*. *Q m* ∧ (∀ *y* : *P*. *S y m* ⟶ ¬ *Q y*) **using** ‹*S* $\leq$ *R*› **by** *blast*
qed

corollary *antimono-wellfounded-on*:
((≤) $\Rightarrow$ (≤) ⇛ (≥)) (*wellfounded-on* :: (*'a* $\Rightarrow$ *bool*) $\Rightarrow$ (*'a* $\Rightarrow$ *'a* $\Rightarrow$ *bool*) $\Rightarrow$ *bool*)
using *wellfounded-on-if-le-pred-if-wellfounded-on wellfounded-on-if-le-rel-if-wellfounded-on*
by (*intro mono-wrt-relI Fun-Rel-relI*) *blast*

consts *wellfounded* :: *'a* $\Rightarrow$ *bool*

overloading
wellfounded ≡ *wellfounded* :: (*'a* $\Rightarrow$ *'a* $\Rightarrow$ *bool*) $\Rightarrow$ *bool*
begin
definition (*wellfounded* :: (*'a* $\Rightarrow$ *'a* $\Rightarrow$ *bool*) $\Rightarrow$ *bool*) ≡ *wellfounded-on* (⊤ :: *'a* $\Rightarrow$ *bool*)
end

lemma *wellfounded-eq-wellfounded-on*:
(*wellfounded* :: (*'a* $\Rightarrow$ *'a* $\Rightarrow$ *bool*) $\Rightarrow$ -) = *wellfounded-on* (⊤ :: *'a* $\Rightarrow$ *bool*)
unfolding *wellfounded-def* **..**

lemma *wellfounded-eq-wellfounded-on-uhint* [*uhint*]:
P ≡ (⊤ :: *'a* $\Rightarrow$ *bool*) $\Longrightarrow$ (*wellfounded* :: (*'a* $\Rightarrow$ *'a* $\Rightarrow$ *bool*) $\Rightarrow$ -) ≡ *wellfounded-on P*
by (*simp add*: *wellfounded-eq-wellfounded-on*)

lemma *wellfoundedI* [*intro*]:
assumes $\bigwedge$*Q x*. *Q x* $\Longrightarrow$ (∃ *m* : *Q*. ∀ *y*. *R y m* ⟶ ¬(*Q y*))
shows *wellfounded R*
by (*urule wellfounded-onI*) (*use assms* **in** *fastforce*)

lemma *wellfoundedE*:
assumes *wellfounded R Q x*
obtains *m* **where** *Q m* $\bigwedge$*y*. *R y m* $\Longrightarrow$ $\neg$(*Q y*)
by (*urule wellfounded-onE*) (*urule assms* | *simp*)+

lemma *wellfounded-on-if-wellfounded*:
fixes *P* :: *'a* $\Rightarrow$ *bool* **and** *R* :: *'a* $\Rightarrow$ *'a* $\Rightarrow$ *bool*
assumes *wellfounded R*
shows *wellfounded-on P R*
using *assms* **by** (*urule wellfounded-on-if-le-pred-if-wellfounded-on*) *auto*

lemma *wellfounded-induct* [*consumes 1*, *case-names step*]:
assumes *wellfounded R*
assumes $\bigwedge$*x*. ($\bigwedge$*y*. *R y x* $\Longrightarrow$ *Q y*) $\Longrightarrow$ *Q x*
shows *Q x*
using *assms*(*1*) **by** (*urule wellfounded-on-induct*) (*use assms*(*2*$-$) **in** *auto*)

lemma *wellfounded-rel-restrict-if-wellfounded-on*:
assumes *wellfounded-on P R*
shows *wellfounded R*$\upharpoonright_P$
proof (*intro wellfoundedI*)
fix *Q* **and** *x* :: *'a* **assume** *Q x*
show $\exists$ *m* : *Q*. $\forall$ *y*. *R*$\upharpoonright_P$ *y m* $\longrightarrow$ $\neg$(*Q y*)
proof (*cases* $\exists$ *s* : *P*. *Q s*)
case *True*
then show *?thesis* **using** *assms* **by** (*fast elim*!: *wellfounded-onE*)
next
case *False*
then show *?thesis* **using** ‹*Q x*› **by** *blast*
qed
qed

end

1.3.10 Basic Properties

theory *Binary-Relation-Properties*
imports
Binary-Relations-Antisymmetric
Binary-Relations-Asymmetric
Binary-Relations-Bi-Total
Binary-Relations-Bi-Unique
Binary-Relations-Connected
Binary-Relations-Injective
Binary-Relations-Irreflexive
Binary-Relations-Left-Total
Binary-Relations-Reflexive
Binary-Relations-Right-Unique

Binary-Relations-Surjective
Binary-Relations-Symmetric
Binary-Relations-Transitive
Binary-Relations-Wellfounded
begin

end

1.3.11 Bounded Definite Description

theory *Bounded-Definite-Description*
imports
Bounded-Quantifiers
begin

consts *bthe* :: $'a \Rightarrow ('b \Rightarrow bool) \Rightarrow 'b$

open-bundle *bounded-the-syntax*
begin
syntax *-bthe* :: $[idt, 'a, bool] \Rightarrow 'b$ ($\langle(3THE$ - : -./ -$)\rangle$ $[0, 0, 10]$ 10)
syntax-consts *-bthe* $\rightleftharpoons$ *bthe*
translations *THE* x : P. $Q \rightleftharpoons$ *CONST bthe* P $(\lambda x.\ Q)$
end

definition *bthe-pred* P $Q \equiv$ *The* $(P \sqcap Q)$
adhoc-overloading *bthe* $\rightleftharpoons$ *bthe-pred*

lemma *bthe-eqI* [*intro*]:
assumes Q a
and P a
and $\bigwedge x.\ [\![P\ x;\ Q\ x]\!] \Longrightarrow x = a$
shows (*THE* x : P. Q x) $= a$
unfolding *bthe-pred-def* **by** (*auto intro*: *assms*)

lemma *pred-bthe-if-ex1E*:
assumes $\exists!x$: P. Q x
obtains P (*THE* x : P. Q x) Q (*THE* x : P. Q x)
unfolding *bthe-pred-def inf-fun-def* **using** *theI'*[*OF assms*[*unfolded bex1-pred-def*]]
by *auto*

lemma *pred-btheI*:
assumes $\exists!x$: P. Q x
shows P (*THE* x : P. Q x)
using *assms* **by** (*elim pred-bthe-if-ex1E*)

lemma *pred-btheI'*:
assumes $\exists!x$: P. Q x
shows Q (*THE* x : P. Q x)

using *assms* **by** (*elim pred-bthe-if-ex1E*)

end

1.3.12 Least Element With Respect To Some Relation

theory *Binary-Relations-Least*
imports
Binary-Relations-Antisymmetric
Bounded-Definite-Description
begin

definition *is-minimal-wrt-rel R* (*P* :: *'a* ⇒ *bool*) *x* ≡ *P x* ∧ (∀ *y* : *P*. *R x y*)

lemma *is-minimal-wrt-relI* [*intro*]:
assumes *P x*
and ⋀*y*. *P y* ⟹ *R x y*
shows *is-minimal-wrt-rel R P x*
unfolding *is-minimal-wrt-rel-def* **using** *assms* **by** *blast*

lemma *is-minimal-wrt-relE* [*elim*]:
assumes *is-minimal-wrt-rel R P x*
obtains *P x* ⋀*y*. *P y* ⟹ *R x y*
using *assms* **unfolding** *is-minimal-wrt-rel-def* **by** *blast*

definition *is-least-wrt-rel R* (*P* :: *'a* ⇒ *bool*) *x* ≡
is-minimal-wrt-rel R P x ∧ (∀ *x'*. *is-minimal-wrt-rel R P x'* ⟶ *x'* = *x*)

lemma *is-least-wrt-relI* [*intro*]:
assumes *is-minimal-wrt-rel R P x*
and ⋀*x'*. *is-minimal-wrt-rel R P x'* ⟹ *x'* = *x*
shows *is-least-wrt-rel R P x*
unfolding *is-least-wrt-rel-def* **using** *assms* **by** *blast*

lemma *is-least-wrt-relE* [*elim*]:
assumes *is-least-wrt-rel R P x*
obtains *is-minimal-wrt-rel R P x* ⋀*x'*. *is-minimal-wrt-rel R P x'* ⟹ *x'* = *x*
using *assms* **unfolding** *is-least-wrt-rel-def* **by** *blast*

lemma *is-least-wrt-rel-if-antisymmetric-onI*:
assumes *antisymmetric-on P R*
and *is-minimal-wrt-rel R P x*
shows *is-least-wrt-rel R P x*
using *assms* **by** (*intro is-least-wrt-relI*) (*blast dest*: *antisymmetric-onD*)+

corollary *is-least-wrt-rel-eq-is-minimal-wrt-rel-if-antisymmetric-on* [*simp*]:
assumes *antisymmetric-on P R*

shows *is-least-wrt-rel R P = is-minimal-wrt-rel R P*
using *assms is-least-wrt-rel-if-antisymmetric-onI* **by** (*intro ext iffI*) *auto*

definition *least-wrt-rel R* (*P* :: *'a* ⇒ *bool*) ≡ *THE x. is-least-wrt-rel R P x*

lemma *least-wrt-rel-eqI*:
assumes *is-least-wrt-rel R P x*
shows *least-wrt-rel R P = x*
unfolding *least-wrt-rel-def* **using** *assms* **by** (*intro the1-equality*) *blast*

lemma *least-wrt-rel-eq-if-antisymmetric-onI*:
assumes *antisymmetric-on P R*
and *is-minimal-wrt-rel R P x*
shows *least-wrt-rel R P = x*
using *assms* **by** (*intro least-wrt-rel-eqI*) *auto*

lemma *pred-least-wrt-relI*:
assumes *is-least-wrt-rel R P x*
shows *P* (*least-wrt-rel R P*)
unfolding *least-wrt-rel-def* **by** (*rule the1I2*) (*use assms* **in** *auto*)

lemma *pred-least-wrt-rel-if-antisymmetric-onI*:
assumes *antisymmetric-on P R*
and *is-minimal-wrt-rel R P x*
shows *P* (*least-wrt-rel R P*)
by (*rule pred-least-wrt-relI*) (*use assms* **in** *auto*)

end

1.3.13 Transitive Closure

theory *Binary-Relations-Transitive-Closure*
imports
Binary-Relations-Least
Binary-Relations-Transitive
begin

consts *trans-closure-on* :: *'a* ⇒ *'b* ⇒ *'b*

definition *trans-closure-on-pred* (*P* :: *'a* ⇒ *bool*) *R x y* ≡
∀(*R'* :: *'a* ⇒ *'a* ⇒ *bool*) : *transitive-on P. R* ≤ *R'* ⟶ *R' x y*
adhoc-overloading *trans-closure-on* ⇌ *trans-closure-on-pred*

lemma *trans-closure-on-if-all-trans-relI*:
assumes ⋀*R'. transitive-on P R'* ⟹ *R* ≤ *R'* ⟹ *R' x y*

shows *trans-closure-on P R x y*
using *assms* **unfolding** *trans-closure-on-pred-def* **by** *auto*

lemma *all-rel-if-trans-closure-onE*:
assumes *trans-closure-on P R x y*
obtains $\bigwedge R'$. *transitive-on P R'* $\Longrightarrow$ $R \leq R' \Longrightarrow$ *R' x y*
using *assms* **unfolding** *trans-closure-on-pred-def* **by** *auto*

lemma *transitive-on-trans-closure-on*: *transitive-on P* (*trans-closure-on P R*)
by (*intro transitive-onI trans-closure-on-if-all-trans-relI*, *elim all-rel-if-trans-closure-onE*)
(*blast dest*: *transitive-onD*)

lemma *trans-closure-on-le-if-le-if-transitive-on*:
assumes *transitive-on P S*
and $R \leq S$
shows *trans-closure-on P R* $\leq S$
using *assms* **by** (*intro le-relI trans-closure-on-if-all-trans-relI*) (*elim all-rel-if-trans-closure-onE*)

lemma *trans-closure-on-if-rel*: *R x y* $\Longrightarrow$ *trans-closure-on P R x y*
by (*intro trans-closure-on-if-all-trans-relI*) *auto*

corollary *le-trans-closure-on*: $R \leq$ *trans-closure-on P R*
using *trans-closure-on-if-rel* **by** *fast*

corollary *is-least-wrt-rel-trans-closure-on*:
is-least-wrt-rel ($\leq$) (($\leq$) *R* $\sqcap$ *transitive-on P*) (*trans-closure-on P R*)
by (*intro is-least-wrt-rel-if-antisymmetric-onI is-minimal-wrt-relI inf1I*
trans-closure-on-le-if-le-if-transitive-on le-trans-closure-on transitive-on-trans-closure-on)
auto

corollary *trans-closure-on-eq-least-wrt-rel*:
trans-closure-on P R = *least-wrt-rel* ($\leq$) (($\leq$) *R* $\sqcap$ *transitive-on P*)
by (*intro least-wrt-rel-eqI*[*symmetric*] *is-least-wrt-rel-trans-closure-on*)

lemma *trans-closure-on-le-sup*:
fixes *P R* **defines** $S \equiv R \sqcup (\lambda x\ y.\ P\ x \wedge P\ y \wedge (\exists z : P.$ *trans-closure-on P R x*
z $\wedge$ *R z y*))
shows *trans-closure-on P R* $\leq S$
proof (*rule trans-closure-on-le-if-le-if-transitive-on*)
show *transitive-on P S*
proof (*intro transitive-onI*)
fix *x y z* **assume** *Ps*: *P x P y P z* **and** *S x y*
then have *txy*: *trans-closure-on P R x y* **unfolding** *S-def*
using *transitive-on-trans-closure-on*[*of P R*] **by** (*elim sup2E conjE bexE*)
(*blast dest*: *trans-closure-on-if-rel*[*of R*] *transitive-onD*
intro: *trans-closure-on-if-all-trans-relI*)+
assume *S y z*
then consider *R y z* | *z'* **where** *P z' trans-closure-on P R y z' R z' z*
unfolding *S-def* **by** *auto*

then show $S\ x\ z$
proof *cases*
case *1* **with** *Ps txy* **show** *?thesis* **unfolding** *S-def* **by** *fastforce*
next
case *2* **with** *Ps txy* **show** *?thesis* **unfolding** *S-def*
using *transitive-on-trans-closure-on*[*of P R*] **by** (*fastforce dest*: *transitive-onD*)
qed
qed
show $R \leq S$ **unfolding** *S-def* **by** *fastforce*
qed

lemma *trans-closure-on-cases*:
assumes *trans-closure-on P R x y*
obtains (*rel*) $R\ x\ y$ | (*step*) z **where** $P\ x\ P\ z\ P\ y$ *trans-closure-on P R x z R z y*
using *le-relD*[*OF trans-closure-on-le-sup assms*] **by** *auto*

lemma *trans-closure-on-eq-rel-sup-trans-closure-on*:
trans-closure-on P R = $R \sqcup$ *trans-closure-on P* $R\restriction_P\restriction_P$
proof −
have $R'\ x\ y$ **if** $\bigwedge R'$. *transitive-on P R'* $\Longrightarrow$ $R \leq R' \Longrightarrow R'\ x\ y$ $\neg(R\ x\ y)$ *transitive-on P R'*
$R\restriction_P\restriction_P \leq R'$ **for** $R'\ x\ y$
proof −
let *?R'plus* = $R \sqcup R'$
from *that*(*3*−*4*) **have** *transitive-on P ?R'plus*
by (*intro transitive-onI sup2CI*) (*auto dest*: *transitive-onD*)+
with *that* **show** *?thesis* **by** *fastforce*
qed
moreover have $R'\ x\ y$ **if** $\bigwedge R'$. *transitive-on P R'* $\Longrightarrow$ $R\restriction_P\restriction_P \leq R' \Longrightarrow R'\ x\ y$ *transitive-on P R'*
$R \leq R'$ **for** $R'\ x\ y$
using *that* **by** *blast*
ultimately show *?thesis*
by (*intro ext iffI sup2CI*; *elim sup2E all-rel-if-trans-closure-onE*)
(*blast intro*!: *trans-closure-on-if-all-trans-relI dest*: *transitive-onD*)+
qed

consts *trans-closure* :: $'a \Rightarrow 'a$

definition *trans-closure-rel* ≡ *trans-closure-on* ⊤
adhoc-overloading *trans-closure* ⇌ *trans-closure-rel*

lemma *trans-closure-eq-trans-closure-on*: *trans-closure* = *trans-closure-on* ⊤
unfolding *trans-closure-rel-def* **..**

lemma *trans-closure-eq-trans-closure-on-uhint* [*uhint*]:
assumes $P \equiv \top$
shows *trans-closure* = *trans-closure-on P*

using *assms trans-closure-eq-trans-closure-on* **by** *simp*

lemma *trans-closure-iff-trans-closure-on*: *trans-closure R x y = trans-closure-on* ⊤ *R x y*
unfolding *trans-closure-eq-trans-closure-on* **by** *simp*

lemma *all-rel-if-trans-closureE*:
assumes *trans-closure R x y*
obtains ⋀*R'*. *transitive R'* ⟹ *R* ≤ *R'* ⟹ *R' x y*
using *assms* **by** (*urule* (*e*) *all-rel-if-trans-closure-onE*)

lemma *transitive-trans-closure*: *transitive* (*trans-closure R*)
by (*urule transitive-on-trans-closure-on*)

lemma *trans-closure-le-if-le-if-transitive*:
assumes *transitive S*
and *R* ≤ *S*
shows *trans-closure R* ≤ *S*
using *assms* **by** (*urule trans-closure-on-le-if-le-if-transitive-on*)

lemma *trans-closure-if-rel*: *R x y* ⟹ *trans-closure R x y*
by (*urule trans-closure-on-if-rel*)

corollary *trans-closure-eq-least-wrt-rel*: *trans-closure R = least-wrt-rel* (≤) ((≤) *R* ⊓ *transitive*)
by (*urule trans-closure-on-eq-least-wrt-rel*)

lemma *trans-closure-cases*:
assumes *trans-closure R x y*
obtains (*rel*) *R x y* | (*step*) *z* **where** *trans-closure R x z R z y*
using *assms* **unfolding** *trans-closure-eq-trans-closure-on* **by** (*urule* (*e*) *trans-closure-on-cases*)

end

theory *Functions-Restrict*
imports *HOL-Basics-Base*
begin

consts *fun-restrict* :: *'a* ⇒ *'b* ⇒ *'a*

bundle *fun-restrict-syntax*
begin
notation *fun-restrict* (‹(-)↾(_)› [*1000*])
end

definition *fun-restrict-pred f P x* ≡ *if P x then f x else undefined*
adhoc-overloading *fun-restrict* ⇌ *fun-restrict-pred*

context

includes *fun-restrict-syntax*
begin

lemma *fun-restrict-eq* [*simp*]:
assumes $P\ x$
shows $f\restriction_P\ x = f\ x$
using *assms* **unfolding** *fun-restrict-pred-def* **by** *auto*

lemma *fun-restrict-eq-if-not* [*simp*]:
assumes $\neg(P\ x)$
shows $f\restriction_P\ x = \mathit{undefined}$
using *assms* **unfolding** *fun-restrict-pred-def* **by** *auto*

lemma *fun-restrict-eq-if*: $f\restriction_P\ x = (\mathit{if}\ P\ x\ \mathit{then}\ f\ x\ \mathit{else}\ \mathit{undefined})$
by *auto*

lemma *fun-restrict-cong* [*cong*]:
assumes $P = P'$
and $\bigwedge x.\ P'\ x \Longrightarrow f\ x = g\ x$
shows $f\restriction_P = g\restriction_{P'}$
using *assms* **by** (*intro ext*) (*auto simp*: *fun-restrict-eq-if*)

end

end

1.3.14 Well-Founded Transitive Recursion

theory *Wellfounded-Transitive-Recursion*
imports
Binary-Relations-Transitive
Binary-Relations-Wellfounded
Binary-Relation-Functions
Functions-Restrict
begin

Summary Well-founded recursion on transitive, well-founded relations. One can extend this to well-founded recursion on non-transitive, well-founded relations since the transitive closure of a well-founded relation is well-founded.

context
includes *fun-restrict-syntax* **and** *no rel-restrict-syntax*
begin

definition *fun-rel-restrict f R y* $\equiv f\restriction_{R^{-1}}\ y :: 'a \Rightarrow \mathit{bool}$

lemma *fun-rel-restrict-eq* [*simp*]:

assumes $R\ x\ y$
shows *fun-rel-restrict f R y x* = $f\ x$
using *assms* **unfolding** *fun-rel-restrict-def* **by** *auto*

lemma *fun-rel-restrict-if-not* [*simp*]:
assumes $\neg(R\ x\ y)$
shows *fun-rel-restrict f R y x* = *undefined*
using *assms* **unfolding** *fun-rel-restrict-def* **by** *auto*

lemma *fun-rel-restrict-eq-fun-restrict*: *fun-rel-restrict f R y* = $f\restriction_{R^{-1}}\ y$
unfolding *fun-rel-restrict-def* **by** *auto*

lemma *fun-rel-restrict-cong* [*cong*]:
assumes $y = y'$
and $\bigwedge x.\ R\ x\ y' \longleftrightarrow R'\ x\ y'$
and $\bigwedge x.\ R'\ x\ y' \Longrightarrow f\ x = g\ x$
shows *fun-rel-restrict f R y* = *fun-rel-restrict g R' y*
using *assms* **by** (*intro ext*) (*auto simp*: *fun-rel-restrict-eq-fun-restrict fun-restrict-eq-if*)

lemma *fun-rel-restrict-rel-restrict-eq-fun-restrict-fun-rel-restrictI* [*simp*]:
assumes $P\ x$
shows *fun-rel-restrict f* (*rel-restrict R P*) x = (*fun-rel-restrict f R x*)$\restriction_P$
using *assms* **unfolding** *fun-rel-restrict-eq-fun-restrict fun-restrict-eq-if* **by** *fast-force*

context
fixes $R :: 'a \Rightarrow 'a \Rightarrow bool$ **and** $step :: ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b$
begin

definition *wf-rec-step f x* = *step* (*fun-rel-restrict* (*f x*) *R x*) *x*

lemma *wf-rec-step-eq*: *wf-rec-step f x* = *step* (*fun-rel-restrict* (*f x*) *R x*) *x*
unfolding *wf-rec-step-def* **by** *simp*

definition *is-recfun X f* ≡ *f* = *fun-rel-restrict* (*wf-rec-step* (λ-. *f*)) *R X*

definition *the-recfun X* = (*THE f*. *is-recfun X f*)

lemma *is-recfunI* [*intro*]:
assumes $\bigwedge x.\ R\ x\ X \Longrightarrow f\ x = wf\text{-}rec\text{-}step\ (\lambda\text{-}.\ f)\ x$
and $\bigwedge x.\ \neg(R\ x\ X) \Longrightarrow f\ x = undefined$
shows *is-recfun X f*
using *assms* **unfolding** *is-recfun-def wf-rec-step-eq fun-rel-restrict-eq-fun-restrict fun-restrict-eq-if*
by (*intro ext*) *simp*

lemma *is-recfunE* [*elim*]:
assumes *is-recfun X f*

obtains $\bigwedge x.\ R\ x\ X \Longrightarrow f\ x = \mathit{wf\text{-}rec\text{-}step}\ (\lambda\text{-}.\ f)\ x$
$\bigwedge x.\ \neg(R\ x\ X) \Longrightarrow f\ x = \mathit{undefined}$
using *assms*[*unfolded is-recfun-def*, *THEN fun-cong*]
unfolding *fun-rel-restrict-eq-fun-restrict fun-restrict-eq-if*
by *simp*

lemma *eq-if-rel-if-is-recfun*:
assumes *is-recfun X f*
and *R x X*
shows $f\ x = \mathit{wf\text{-}rec\text{-}step}\ (\lambda\text{-}.\ f)\ x$
using *assms* **by** *auto*

lemma *eq-if-not-rel-if-is-recfun*:
assumes *is-recfun X f*
and $\neg(R\ x\ X)$
shows $f\ x = \mathit{undefined}$
using *assms* **by** *auto*

context
assumes *wf*: *wellfounded R*
and *trans*: *transitive R*
begin

lemma *app-eq-app-if-is-recfunI*:
assumes *is-recfun X f is-recfun Y g*
and *R Z X R Z Y*
shows $f\ Z = g\ Z$
using *wf* ‹*R Z X*› ‹*R Z Y*›
proof (*induction Z rule*: *wellfounded-induct*)
case (*step Z*)
then have $f\ z = g\ z$ **if** *R z Z* **for** *z* **using** *that trans* **by** *auto*
moreover have $f\ Z = \mathit{wf\text{-}rec\text{-}step}\ (\lambda\text{-}.\ f)\ Z\ g\ Z = \mathit{wf\text{-}rec\text{-}step}\ (\lambda\text{-}.\ g)\ Z$ **using** *assms step.prems* **by** *auto*
ultimately show *?case* **by** (*auto simp*: *wf-rec-step-eq*)
qed

corollary *eq-if-is-recfunI*:
assumes *is-recfun X f is-recfun X g*
shows $f = g$
proof (*intro ext*)
fix *x* **from** *assms app-eq-app-if-is-recfunI* **show** $f\ x = g\ x$ **by** (*cases R x X*) *force*+
qed

corollary *is-recfun-the-recfun-if-is-recfunI*:
assumes *is-recfun X f*
shows *is-recfun X* (*the-recfun X*)
proof −
from *assms eq-if-is-recfunI* **have** $\exists!g.$ *is-recfun X g* **by** *auto*

```
  from theI'[OF this] show ?thesis unfolding the-recfun-def by auto
qed

corollary fun-rel-restrict-eq-fun-rel-restrict-if-is-recfunI:
  assumes recfuns: is-recfun x f is-recfun X g
  and R x X
  shows fun-rel-restrict f R x = fun-rel-restrict g R x
proof −
  have f y = g y if R y x for y
    using app-eq-app-if-is-recfunI[OF recfuns] ‹R y x› ‹R x X› trans by blast
  then show ?thesis by auto
qed

definition wft-rec ≡ wf-rec-step the-recfun

lemma ex-is-recfunI: ∃ f. is-recfun X f
using wf
proof (induction X rule: wellfounded-induct)
  case (step X)
  then have is-recfun: is-recfun x (the-recfun x) if R x X for x
    using is-recfun-the-recfun-if-is-recfunI that by blast
  define f where f ≡ fun-rel-restrict wft-rec R X
  have fun-rel-restrict f R x = fun-rel-restrict (the-recfun x) R x if R x X for x
  proof −
    have f y = (the-recfun x) y if R y x for y
    proof −
      have is-recfun y (the-recfun y) is-recfun x (the-recfun x)
        using is-recfun ‹R x X› ‹R y x› trans by auto
      with fun-rel-restrict-eq-fun-rel-restrict-if-is-recfunI have
        fun-rel-restrict (the-recfun y) R y = fun-rel-restrict (the-recfun x) R y
        using ‹R y x› by auto
      moreover have f y = wft-rec y
        unfolding f-def using ‹R y x› ‹R x X› trans by auto
      moreover have wf-rec-step (λ-. the-recfun x) y = (the-recfun x) y
        using ‹R y x› ‹R x X› is-recfun by fastforce
      ultimately show ?thesis by (auto simp: wft-rec-def wf-rec-step-eq)
    qed
    then show ?thesis by auto
  qed
 then have is-recfun X f unfolding f-def by (auto simp: wft-rec-def wf-rec-step-eq)
  then show ?case by auto
qed

corollary is-recfun-the-recfunI: is-recfun X (the-recfun X)
  using is-recfun-the-recfun-if-is-recfunI ex-is-recfunI by blast

theorem wft-rec-eq-wf-rec-stepI: wft-rec X = wf-rec-step (λ-. wft-rec) X
proof −
  have (the-recfun X) x = wf-rec-step the-recfun x if R x X for x
```

proof −
have (*the-recfun X*) *x* = *wf-rec-step* (λ-. *the-recfun X*) *x*
using *is-recfun-the-recfunI* ‹*R x X*› **by** *auto*
moreover have *fun-rel-restrict* (*the-recfun x*) *R x* = *fun-rel-restrict* (*the-recfun X*) *R x*
using *fun-rel-restrict-eq-fun-rel-restrict-if-is-recfunI is-recfun-the-recfunI* ‹*R x X*›
by *blast*
ultimately show *?thesis* **by** (*auto simp*: *wf-rec-step-eq*)
qed
then have *fun-rel-restrict* (*the-recfun X*) *R X* = *fun-rel-restrict* (*wf-rec-step the-recfun*) *R X* **by** *simp*
then show *?thesis* **by** (*simp add*: *wft-rec-def wf-rec-step-eq*)
qed

end
end
end

end

1.3.15 Well-Founded Recursion

theory *Wellfounded-Recursion*
imports
Binary-Relations-Transitive-Closure
Wellfounded-Transitive-Recursion
begin

Summary We use *wft-rec* to define well-founded recursion on non-transitive, well-founded relations. The fact that the transitive closure of a well-founded relation is itself well-founded can be used to remove the transitivity assumption of ⟦*wellfounded ?R*; *transitive ?R*⟧ ⟹ *wft-rec ?R ?step ?X* = *wf-rec-step ?R ?step* (λ-. *wft-rec ?R ?step*) *?X*.

lemma *wellfounded-trans-closure-if-wellfounded*:
assumes *wellfounded R*
shows *wellfounded* (*trans-closure R*)
proof (*rule ccontr*)
assume ¬(*wellfounded* (*trans-closure R*))
then obtain *P X* **where** *P X* **and** *no-minimal*: ⋀*Y*. *P Y* ⟹ (∃ *y*. *trans-closure R y Y* ∧ *P y*)
unfolding *wellfounded-eq-wellfounded-on wellfounded-on-pred-def* **by** *auto*
from *assms* **have** ¬(*P Y*) ∧ (∀ *y*. *trans-closure R y Y* ⟶ ¬(*P y*)) **for** *Y*
proof (*induction Y rule*: *wellfounded-induct*)
case (*step Y*)
show *?case*

```
    proof (rule ccontr)
      assume ¬(¬(P Y) ∧ (∀ y. trans-closure R y Y ⟶ ¬(P y)))
      then consider P Y | y where trans-closure R y Y P y by blast
      then show False
      proof cases
        case 1
        then obtain y where trans-closure R y Y P y using no-minimal by blast
        then show False using step.IH trans-closure-cases by force
      next
        case 2
        then show False using step.IH trans-closure-cases by force
      qed
    qed
  qed
  then show False using ‹P X› by blast
qed

consts wf-rec :: 'a ⇒ 'b ⇒ 'c

definition wf-rec-rel R step = wft-rec (trans-closure R) (λf. wf-rec-step R step (λ-.
f))
adhoc-overloading wf-rec ⇌ wf-rec-rel

theorem wf-rec-eq-wf-rec-stepI:
  assumes wellfounded R
  shows wf-rec R step X = wf-rec-step R step (λ-. wf-rec R step) X
proof −
  have fun-rel-restrict-eq:
    fun-rel-restrict (fun-rel-restrict f (trans-closure R) X) R X x = fun-rel-restrict
f R X x
    for f x by (cases R x X) (auto dest: trans-closure-if-rel)
   have wf-rec R step X = wf-rec-step R step (fun-rel-restrict (wf-rec R step)
(trans-closure R)) X
   using wellfounded-trans-closure-if-wellfounded[OF assms] transitive-trans-closure[of
R]
    wft-rec-eq-wf-rec-stepI[where ?step=λf. wf-rec-step R step (λ-. f)]
    by (simp add: wf-rec-step-eq wf-rec-rel-def)
  then show ?thesis unfolding wf-rec-step-eq fun-rel-restrict-eq by simp
qed

consts wf-rec-on :: 'a ⇒ 'b ⇒ 'c ⇒ 'd

definition wf-rec-on-pred :: ('a ⇒ bool) ⇒ ('a ⇒ 'a ⇒ bool) ⇒ (('a ⇒ 'b) ⇒ 'a
⇒ 'b) ⇒ 'a ⇒ 'b where
  wf-rec-on-pred P R ≡ wf-rec R↑P
adhoc-overloading wf-rec-on ⇌ wf-rec-on-pred

lemma wf-rec-on-eq-wf-rec-restrict: wf-rec-on P R = wf-rec R↑P
  unfolding wf-rec-on-pred-def by simp
```

theorem *wf-rec-on-eq-wf-rec-step-restrictI*:
assumes *wellfounded-on P R*
shows *wf-rec-on P R step X = wf-rec-step R↑P step (λ-. wf-rec-on P R step) X*
proof −
from *assms* **have** *wellfounded R↑P* **using** *wellfounded-rel-restrict-if-wellfounded-on*
by *blast*
with *wf-rec-eq-wf-rec-stepI* **show** *?thesis* **unfolding** *wf-rec-on-eq-wf-rec-restrict*
by *fastforce*
qed

lemma *wf-rec-on-eq-app-fun-restrict-fun-rel-restrictI*:
fixes *step*
assumes *wf*: *wellfounded-on P R*
and *P x*
defines *f ≡ wf-rec-on P R step*
shows *f x = step (fun-restrict (fun-rel-restrict f R x) P) x*
proof −
from *wf* **have** *f x = step (fun-rel-restrict f R↑P x) x*
unfolding *f-def* **by** (*simp only*: *wf-rec-step-eq wf-rec-on-eq-wf-rec-step-restrictI*)
moreover have *fun-rel-restrict f R↑P x = fun-restrict (fun-rel-restrict f R x) P*
using ‹*P x*› *fun-rel-restrict-rel-restrict-eq-fun-restrict-fun-rel-restrictI* **by** *simp*
ultimately show *?thesis* **by** *simp*
qed

end

1.3.16 Graph

theory *Binary-Relations-Graph*
imports
Functions-Monotone
Restricted-Equality
begin

Summary Basic functions on binary relations.

Graph

consts *Graph-on* :: *'a ⇒ 'b ⇒ 'c*

definition *Graph-on-pred P f ≡ λx y. P x ∧ y = f x*
adhoc-overloading *Graph-on ⇌ Graph-on-pred*

lemma *Graph-onI*:
assumes *P x*
shows *Graph-on P f x (f x)*
using *assms* **unfolding** *Graph-on-pred-def* **by** *auto*

lemma *Graph-on-if-eq-if-pred* [*intro*]:
assumes $P\ x$
and $y = f\ x$
shows *Graph-on* $P\ f\ x\ y$
using *assms* **by** (*auto intro*: *Graph-onI*)

lemma *Graph-onE* [*elim*]:
assumes *Graph-on* $P\ f\ x\ y$
obtains $P\ x\ y = f\ x$
using *assms* **unfolding** *Graph-on-pred-def* **by** *auto*

lemma *in-dom-Graph-on-eq* [*simp*]: *in-dom* (*Graph-on* $P\ f$) = P
by *auto*

lemma *in-codom-on-Graph-on-eq-has-inverse-on* [*simp*]:
in-codom-on P (*Graph-on* $Q\ f$) = *has-inverse-on* ($P \sqcap Q$) f
by *fastforce*

lemma *in-codom-Graph-on-eq-has-inverse-on* [*simp*]: *in-codom* (*Graph-on* $P\ f$) = *has-inverse-on* $P\ f$
by (*urule in-codom-on-Graph-on-eq-has-inverse-on*)

lemma *mono-Graph-on*: *mono Graph-on* **by** *fastforce*

lemma *Graph-on-eq-Graph-on-if-mono-eq*:
assumes ((P :: - $\Rightarrow$ *bool*) $\Rrightarrow$ (=)) $f\ g$
shows *Graph-on* $P\ f$ = *Graph-on* $P\ g$
using *assms* **by** *fastforce*

lemma *Graph-on-id-eq-eq-restrict* [*simp*]: *Graph-on* P *id* = $(=_P)$
by *fastforce*

consts *Graph* :: $'a \Rightarrow 'b$

definition *Graph-fun* $\equiv$ *Graph-on* ($\top$ :: $'a \Rightarrow$ *bool*)
adhoc-overloading *Graph* $\rightleftharpoons$ *Graph-fun*

lemma *Graph-eq-Graph-on*: *Graph* = *Graph-on* $\top$
unfolding *Graph-fun-def* **..**

lemma *Graph-eq-Graph-on-uhint* [*uhint*]:
assumes $P \equiv (\top :: 'a \Rightarrow bool)$
shows *Graph* :: ($'a \Rightarrow$ -) $\Rightarrow$ - $\equiv$ *Graph-on* P
using *assms* **by** (*simp add*: *Graph-eq-Graph-on*)

lemma *GraphI* [*intro*]:
assumes $y = f\ x$
shows *Graph* $f\ x\ y$

by (*urule Graph-on-if-eq-if-pred*) (*use assms* **in** *simp-all*)

lemma *GraphD* [*dest*]:
assumes *Graph f x y*
shows $y = f\ x$
using *assms* **by** (*urule* (*e*) *Graph-onE*)

lemma *in-dom-Graph-eq* [*simp*]: *in-dom* (*Graph f*) = $\top$
by (*urule in-dom-Graph-on-eq*)

lemma *in-codom-on-Graph-eq-has-inverse-on* [*simp*]: *in-codom-on P* (*Graph f*) = *has-inverse-on P f*
by (*urule in-codom-on-Graph-on-eq-has-inverse-on*)

lemma *in-codom-Graph-eq-has-inverse* [*simp*]: *in-codom* (*Graph f*) = *has-inverse f*
by (*urule in-codom-Graph-on-eq-has-inverse-on*)

lemma *Graph-eq-eq-comp*: *Graph f* = (=) $\circ$ *f*
by (*intro ext*) *auto*

lemma *Graph-id-eq-eq* [*simp*]: *Graph id* = (=)
by (*urule Graph-on-id-eq-eq-restrict*)

end

1.3.17 Lattice

theory *Binary-Relations-Lattice*
imports
Binary-Relations-Order-Base
HOL.Boolean-Algebras
begin

Summary Basic results about the lattice structure on binary relations.

lemma *rel-infI* [*intro*]:
assumes *R x y*
and *S x y*
shows ($R \sqcap S$) *x y*
using *assms* **by** (*rule inf2I*)

lemma *rel-infE* [*elim*]:
assumes ($R \sqcap S$) *x y*
obtains *R x y S x y*
using *assms* **by** (*rule inf2E*)

lemma *rel-infD*:
assumes ($R \sqcap S$) *x y*

shows $R\ x\ y$ **and** $S\ x\ y$
using *assms* **by** *auto*

lemma *in-dom-rel-infI* [*intro*]:
assumes $R\ x\ y$
and $S\ x\ y$
shows *in-dom* $(R \sqcap S)\ x$
using *assms* **by** *blast*

lemma *in-dom-rel-infE* [*elim*]:
assumes *in-dom* $(R \sqcap S)\ x$
obtains y **where** $R\ x\ y\ S\ x\ y$
using *assms* **by** *blast*

lemma *in-codom-rel-infI* [*intro*]:
assumes $R\ x\ y$
and $S\ x\ y$
shows *in-codom* $(R \sqcap S)\ y$
using *assms* **by** *blast*

lemma *in-codom-rel-infE* [*elim*]:
assumes *in-codom* $(R \sqcap S)\ y$
obtains x **where** $R\ x\ y\ S\ x\ y$
using *assms* **by** *blast*

lemma *in-field-eq-in-dom-sup-in-codom*: *in-field* $L = (\textit{in-dom}\ L \sqcup \textit{in-codom}\ L)$
by (*intro ext*) (*simp add*: *in-field-iff-in-dom-or-in-codom*)

lemma *in-dom-rel-restrict-left-eq* [*simp*]: *in-dom* $R\restriction_P = (\textit{in-dom}\ R \sqcap P)$
by (*intro ext*) *auto*

lemma *in-codom-rel-restrict-left-eq* [*simp*]: *in-codom* $R\restriction_P = (\textit{in-codom}\ R \sqcap P)$
by (*intro ext*) *auto*

lemma *rel-restrict-left-restrict-left-eq* [*simp*]:
fixes $R :: 'a \Rightarrow 'b \Rightarrow \textit{bool}$ **and** $P\ Q :: 'a \Rightarrow \textit{bool}$
shows $R\restriction_P\restriction_Q = R\restriction_P \sqcap R\restriction_Q$
by (*intro ext iffI rel-restrict-leftI*) *auto*

lemma *rel-restrict-left-restrict-right-eq* [*simp*]:
fixes $R :: 'a \Rightarrow 'b \Rightarrow \textit{bool}$ **and** $P :: 'a \Rightarrow \textit{bool}$ **and** $Q :: 'b \Rightarrow \textit{bool}$
shows $R\restriction_P\restriction Q = R\restriction_P \sqcap R\restriction Q$
by (*intro ext iffI rel-restrict-leftI rel-restrict-rightI*) *auto*

lemma *rel-restrict-right-restrict-left-eq* [*simp*]:
fixes $R :: 'a \Rightarrow 'b \Rightarrow \textit{bool}$ **and** $P :: 'b \Rightarrow \textit{bool}$ **and** $Q :: 'a \Rightarrow \textit{bool}$
shows $R\restriction P\restriction_Q = R\restriction P \sqcap R\restriction_Q$
by (*intro ext iffI rel-restrict-leftI rel-restrict-rightI*) *auto*

lemma *rel-restrict-right-restrict-right-eq* [*simp*]:
fixes $R :: 'a \Rightarrow 'b \Rightarrow bool$ **and** $P\ Q :: 'b \Rightarrow bool$
shows $R\restriction_P\restriction_Q = R\restriction_P \sqcap R\restriction_Q$
by (*intro ext iffI*) *auto*

lemma *rel-restrict-left-sup-eq* [*simp*]:
$(R :: 'a \Rightarrow 'b \Rightarrow bool)\restriction_{((P :: 'a \Rightarrow bool) \sqcup Q)} = R\restriction_P \sqcup R\restriction_Q$
by (*intro antisym le-relI*) (*auto elim*!: *rel-restrict-leftE*)

lemma *rel-restrict-left-inf-eq* [*simp*]:
$(R :: 'a \Rightarrow 'b \Rightarrow bool)\restriction_{((P :: 'a \Rightarrow bool) \sqcap Q)} = R\restriction_P \sqcap R\restriction_Q$
by (*intro antisym le-relI*) (*auto elim*!: *rel-restrict-leftE*)

lemma *inf-rel-bimap-and-eq-restrict-left-restrict-right*:
$R \sqcap (\textit{rel-bimap}\ P\ Q\ (\wedge)) = R\restriction_P\restriction_Q$
by (*intro ext*) *auto*

end

1.3.18 Reflexive Closure

theory *Binary-Relations-Reflexive-Closure*
imports
Binary-Relations-Reflexive
Restricted-Equality
begin

consts *refl-closure-on* :: $'a \Rightarrow 'b \Rightarrow 'b$

overloading
refl-closure-on-pred $\equiv$ *refl-closure-on* :: $('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a \Rightarrow bool)$
begin
definition *refl-closure-on-pred* $(P :: 'a \Rightarrow bool)\ (R :: 'a \Rightarrow 'a \Rightarrow bool) \equiv R \sqcup (=_P)$
end

lemma *refl-closure-on-if-pred* [*intro*]:
assumes $P\ x$
shows *refl-closure-on* $P\ R\ x\ x$
using *assms* **unfolding** *refl-closure-on-pred-def* **by** *auto*

lemma *refl-closure-on-if-rel* [*intro*]:
assumes $(R :: 'a \Rightarrow 'a \Rightarrow bool)\ x\ y$
shows *refl-closure-on* $(P :: 'a \Rightarrow bool)\ R\ x\ y$
using *assms* **unfolding** *refl-closure-on-pred-def* **by** *auto*

corollary *le-refl-closure-on-self*: $(R :: 'a \Rightarrow 'a \Rightarrow bool) \leq$ *refl-closure-on* $(P :: 'a$

$\Rightarrow$ *bool*) *R*
by *blast*

lemma *refl-closure-onE* [*elim*]:
assumes *refl-closure-on P R x y*
obtains *P x x* = *y* | *R x y*
using *assms* **unfolding** *refl-closure-on-pred-def* **by** *auto*

lemma *reflexive-on-refl-closure-on*:
reflexive-on (*P* :: *'a* $\Rightarrow$ *bool*) (*refl-closure-on P* (*R* :: *'a* $\Rightarrow$ *'a* $\Rightarrow$ *bool*))
by *fastforce*

definition *refl-closure-field R* $\equiv$ *refl-closure-on* (*in-field R*) *R*

open-bundle *refl-closure-field-syntax*
begin
notation *refl-closure-field* ($\langle$(-$^+$)$\rangle$ [*1000*])
end

lemma *refl-closure-fieldI* [*intro*]:
assumes *refl-closure-on* (*in-field R*) *R x y*
shows R^+ *x y*
using *assms* **unfolding** *refl-closure-field-def* **by** *auto*

lemma *refl-closure-fieldD* [*dest*]:
assumes R^+ *x y*
shows *refl-closure-on* (*in-field R*) *R x y*
using *assms* **unfolding** *refl-closure-field-def* **by** *auto*

lemma *refl-closure-field-eq-refl-closure-on-in-field*: R^+ = *refl-closure-on* (*in-field R*) *R*
by *auto*

end

theory *LBinary-Relations*
imports
Binary-Relations-Functions
Binary-Relations-Order
Binary-Relation-Properties
Wellfounded-Recursion
Wellfounded-Transitive-Recursion
Binary-Relation-Functions
Binary-Relations-Agree
Binary-Relations-Extend
Binary-Relations-Graph
Binary-Relations-Lattice

Binary-Relations-Least
Binary-Relations-Reflexive-Closure
Binary-Relations-Transitive-Closure
Dependent-Binary-Relations
Restricted-Equality
Reverse-Implies
begin

Summary Basic concepts on binary relations.

end

Inverse

theory *Functions-Inverse*
imports
Functions-Injective
Binary-Relations-Function-Evaluation
Bounded-Definite-Description
begin

consts *the-inverse-on* :: $'a \Rightarrow 'b \Rightarrow 'c$

definition *the-inverse-on-pred* $P\ f \equiv \lambda y.\ THE\ x : P.\ y = f\ x$
adhoc-overloading *the-inverse-on* $\rightleftharpoons$ *the-inverse-on-pred*

lemma *the-inverse-on-eq-if-injective-onI*:
assumes *injective-on* $P\ f$
and $y = f\ x$
and $P\ x$
shows *the-inverse-on* $P\ f\ y = x$
unfolding *the-inverse-on-pred-def* **using** *assms* **by** (*intro bthe-eqI*) (*auto dest*: *injective-onD*)

lemma *the-inverse-on-app-eq-if-injective-onI* [*simp*]:
assumes *injective-on* $P\ f$
and $P\ x$
shows *the-inverse-on* $P\ f\ (f\ x) = x$
using *assms* **by** (*intro the-inverse-on-eq-if-injective-onI*) *auto*

consts *the-inverse* :: $'a \Rightarrow 'b$

definition *the-inverse-fun* $\equiv$ *the-inverse-on* $(\top :: 'a \Rightarrow bool)$
adhoc-overloading *the-inverse* $\rightleftharpoons$ *the-inverse-fun*

lemma *the-inverse-eq-the-inverse-on*:
the-inverse = *the-inverse-on* $(\top :: 'a \Rightarrow bool)$
unfolding *the-inverse-fun-def* **..**

lemma *the-inverse-eq-the-inverse-on-uhint* [*uhint*]:

assumes $P \equiv \top :: 'a \Rightarrow bool$
shows *the-inverse* $:: ('a \Rightarrow 'b) \Rightarrow 'b \Rightarrow 'a \equiv$ *the-inverse-on* P
using *assms* **by** (*simp add*: *the-inverse-eq-the-inverse-on*)

lemma *the-inverse-eq-if-injectiveI*:
assumes *injective* f
and $y = f\ x$
shows *the-inverse* $f\ y = x$
using *assms* **by** (*urule the-inverse-on-eq-if-injective-onI*) *auto*

lemma *the-inverse-app-eq-if-injectiveI* [*simp*]:
assumes *injective* f
shows *the-inverse* $f\ (f\ x) = x$
using *assms* **by** (*urule the-inverse-on-app-eq-if-injective-onI*) *auto*

consts *inverse-on* $:: 'a \Rightarrow 'b \Rightarrow 'c \Rightarrow bool$

overloading
inverse-on-pred $\equiv$ *inverse-on* $:: ('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b) \Rightarrow ('b \Rightarrow 'a) \Rightarrow bool$
begin
definition *inverse-on-pred* $P\ f\ g \equiv \forall x : P.\ g\ (f\ x) = x$
end

lemma *inverse-onI* [*intro*]:
assumes $\bigwedge x.\ P\ x \Longrightarrow g\ (f\ x) = x$
shows *inverse-on* $P\ f\ g$
unfolding *inverse-on-pred-def* **using** *assms* **by** *blast*

lemma *inverse-onD*:
assumes *inverse-on* $P\ f\ g$
and $P\ x$
shows $g\ (f\ x) = x$
using *assms* **unfolding** *inverse-on-pred-def* **by** *blast*

lemma *inverse-onE*:
assumes *inverse-on* $P\ f\ g$
obtains $\bigwedge x.\ P\ x \Longrightarrow g\ (f\ x) = x$
using *assms inverse-onD* **by** *fastforce*

lemma *injective-on-if-inverse-on*:
assumes *inv*: *inverse-on* $(P :: 'a \Rightarrow bool)\ (f :: 'a \Rightarrow 'b)\ (g :: 'b \Rightarrow 'a)$
shows *injective-on* $P\ f$
proof (*rule injective-onI*)
fix $x\ x'$
assume *Px*: $P\ x$ **and** *Px'*: $P\ x'$ **and** *f-x-eq-f-x'*: $f\ x = f\ x'$
from *inv* **have** $x = g\ (f\ x)$ **using** *Px* **by** (*intro inverse-onD*[*symmetric*])
also have $... = g\ (f\ x')$ **by** (*simp only*: *f-x-eq-f-x'*)
also have $... = x'$ **using** *inv Px'* **by** (*intro inverse-onD*)
finally show $x = x'$ **.**

qed

lemma *inverse-on-the-inverse-on-if-injective-on*:
fixes $P :: 'a \Rightarrow bool$ **and** $f :: 'a \Rightarrow 'b$
assumes *injective-on P f*
shows *inverse-on P f* (*the-inverse-on P f*)
using *assms* **by** (*intro inverse-onI the-inverse-on-eq-if-injective-onI*) *auto*

lemma *inverse-on-has-inverse-on-the-inverse-on-if-injective-on*:
assumes *injective-on P f*
shows *inverse-on* (*has-inverse-on P f*) (*the-inverse-on P f*) *f*
using *assms* **by** (*intro inverse-onI*) *auto*

lemma *antimono-inverse-on*: *antimono* (*inverse-on* :: $('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b) \Rightarrow ('b \Rightarrow 'a) \Rightarrow bool$)
by (*fastforce dest*: *inverse-onD*)

lemma *inverse-on-compI*:
fixes $P :: 'a \Rightarrow bool$ **and** $P' :: 'b \Rightarrow bool$
and $f :: 'a \Rightarrow 'b$ **and** $g :: 'b \Rightarrow 'a$ **and** $f' :: 'b \Rightarrow 'c$ **and** $g' :: 'c \Rightarrow 'b$
assumes $(P \Rightarrow P')\ f$
and *inverse-on P f g*
and *inverse-on P' f' g'*
shows *inverse-on P* $(f' \circ f)$ $(g \circ g')$
using *assms* **by** (*intro inverse-onI*) (*force dest*: *inverse-onD*)

consts *inverse* :: $'a \Rightarrow 'b \Rightarrow bool$

overloading
inverse $\equiv$ *inverse* :: $('a \Rightarrow 'b) \Rightarrow ('b \Rightarrow 'a) \Rightarrow bool$
begin
definition (*inverse* :: $('a \Rightarrow 'b) \Rightarrow ('b \Rightarrow 'a) \Rightarrow bool$) $\equiv$ *inverse-on* ($\top$ :: $'a \Rightarrow bool$)
end

lemma *inverse-eq-inverse-on*:
(*inverse* :: $('a \Rightarrow 'b) \Rightarrow ('b \Rightarrow 'a) \Rightarrow bool$) = *inverse-on* ($\top$ :: $'a \Rightarrow bool$)
unfolding *inverse-def* **..**

lemma *inverse-eq-inverse-on-uhint* [*uhint*]:
assumes $P \equiv \top :: 'a \Rightarrow bool$
shows *inverse* :: $('a \Rightarrow 'b) \Rightarrow ('b \Rightarrow 'a) \Rightarrow bool \equiv$ *inverse-on P*
using *assms* **by** (*simp add*: *inverse-eq-inverse-on*)

lemma *inverseI* [*intro*]:
assumes $\bigwedge x.\ g\ (f\ x) = x$
shows *inverse f g*
using *assms* **by** (*urule inverse-onI*)

lemma *inverseD*:
assumes *inverse f g*
shows *g* (*f x*) = *x*
using *assms* **by** (*urule* (*d*) *inverse-onD chained*: *insert*) *simp-all*

lemma *inverseE*:
assumes *inverse f g*
obtains ⋀*x*. *g* (*f x*) = *x*
using *assms* **by** (*urule* (*e*) *inverse-onE chained*: *insert*) *simp-all*

lemma *inverse-on-if-inverse*:
fixes *P* :: *'a* ⇒ *bool* **and** *f* :: *'a* ⇒ *'b* **and** *g* :: *'b* ⇒ *'a*
assumes *inverse f g*
shows *inverse-on P f g*
using *assms* **by** (*intro inverse-onI*) (*blast dest*: *inverseD*)

lemma *injective-if-inverse*:
assumes *inverse* (*f* :: *'a* ⇒ *'b*) (*g* :: *'b* ⇒ *'a*)
shows *injective f*
using *assms* **by** (*urule injective-on-if-inverse-on*)

lemma *inverse-the-inverse-if-injective*:
assumes *injective f*
shows *inverse f* (*the-inverse f*)
using *assms* **by** (*urule inverse-on-the-inverse-on-if-injective-on*)

lemma *inverse-on-has-inverse-the-inverse-if-injective*:
assumes *injective f*
shows *inverse-on* (*has-inverse f*) (*the-inverse f*) *f*
using *assms* **by** (*urule inverse-on-has-inverse-on-the-inverse-on-if-injective-on*)

end

Bijections

theory *Functions-Bijection*
imports
Functions-Inverse
Functions-Monotone
begin

consts *bijection-on* :: *'a* ⇒ *'b* ⇒ *'c* ⇒ *'d* ⇒ *bool*

definition *bijection-on-pred* (*P* :: *'a* ⇒ *bool*) (*Q* :: *'b* ⇒ *bool*) *f g* ≡
(*P* ⇒ *Q*) *f* ∧
(*Q* ⇒ *P*) *g* ∧
inverse-on P f g ∧
inverse-on Q g f
adhoc-overloading *bijection-on* ⇌ *bijection-on-pred*

context
fixes $P :: 'a \Rightarrow bool$ **and** $Q :: 'b \Rightarrow bool$ **and** $f :: 'a \Rightarrow 'b$ **and** $g :: 'b \Rightarrow 'a$
begin

lemma *bijection-onI* [*intro*]:
assumes $(P \Rightarrow Q)\ f$
and $(Q \Rightarrow P)\ g$
and *inverse-on* $P\ f\ g$
and *inverse-on* $Q\ g\ f$
shows *bijection-on* $P\ Q\ f\ g$
using *assms* **unfolding** *bijection-on-pred-def* **by** *blast*

lemma *bijection-onE* [*elim*]:
assumes *bijection-on* $P\ Q\ f\ g$
obtains $(P \Rightarrow Q)\ f\ (Q \Rightarrow P)\ g$
inverse-on $P\ f\ g$ *inverse-on* $Q\ g\ f$
using *assms* **unfolding** *bijection-on-pred-def* **by** *blast*

lemma *mono-wrt-pred-if-bijection-on-left*:
assumes *bijection-on* $P\ Q\ f\ g$
shows $(P \Rightarrow Q)\ f$
using *assms* **by** (*elim bijection-onE*)

lemma *mono-wrt-pred-if-bijection-on-right*:
assumes *bijection-on* $P\ Q\ f\ g$
shows $(Q \Rightarrow P)\ g$
using *assms* **by** (*elim bijection-onE*)

lemma *bijection-on-pred-right*:
assumes *bijection-on* $P\ Q\ f\ g$
and $P\ x$
shows $Q\ (f\ x)$
using *assms* **by** *blast*

lemma *bijection-on-pred-left*:
assumes *bijection-on* $P\ Q\ f\ g$
and $Q\ y$
shows $P\ (g\ y)$
using *assms* **by** *blast*

lemma *inverse-on-if-bijection-on-left-right*:
assumes *bijection-on* $P\ Q\ f\ g$
shows *inverse-on* $P\ f\ g$
using *assms* **by** (*elim bijection-onE*)

lemma *inverse-on-if-bijection-on-right-left*:
assumes *bijection-on* $P\ Q\ f\ g$
shows *inverse-on* $Q\ g\ f$

using *assms* **by** (*elim bijection-onE*)

lemma *bijection-on-left-right-eq-self*:
assumes *bijection-on P Q f g*
and *P x*
shows *g* (*f x*) = *x*
using *assms inverse-on-if-bijection-on-left-right*
by (*intro inverse-onD*)

lemma *bijection-on-right-left-eq-self'*:
assumes *bijection-on P Q f g*
and *Q y*
shows *f* (*g y*) = *y*
using *assms inverse-on-if-bijection-on-right-left* **by** (*intro inverse-onD*)

end

lemma *bijection-on-has-inverse-on-the-inverse-on-if-injective-on*:
assumes *injective-on P f*
shows *bijection-on P* (*has-inverse-on P f*) *f* (*the-inverse-on P f*)
using *assms* **by** (*intro bijection-onI inverse-on-has-inverse-on-the-inverse-on-if-injective-on*
inverse-on-the-inverse-on-if-injective-on)
fastforce+

context
fixes *P* :: *'a* ⇒ *bool* **and** *Q* :: *'b* ⇒ *bool* **and** *f* :: *'a* ⇒ *'b* **and** *g* :: *'b* ⇒ *'a*
begin

lemma *bijection-on-right-left-if-bijection-on-left-right*:
assumes *bijection-on P Q f g*
shows *bijection-on Q P g f*
using *assms* **by** *auto*

lemma *injective-on-if-bijection-on-left*:
assumes *bijection-on P Q f g*
shows *injective-on P f*
using *assms*
by (*intro injective-on-if-inverse-on inverse-on-if-bijection-on-left-right*)

lemma *injective-on-if-bijection-on-right*:
assumes *bijection-on P Q f g*
shows *injective-on Q g*
by (*intro injective-on-if-inverse-on*)
(*fact inverse-on-if-bijection-on-right-left*[*OF assms*])

end

lemma *bijection-on-compI*:
fixes *P* :: *'a* ⇒ *bool* **and** *P'* :: *'b* ⇒ *bool* **and** *Q* :: *'c* ⇒ *bool*

assumes *bijection-on P P′ f g*
and *bijection-on P′ Q f′ g′*
shows *bijection-on P Q* (*f′* ∘ *f*) (*g* ∘ *g′*)
using *assms* **by** (*intro bijection-onI*)
(*auto intro*: *dep-mono-wrt-pred-comp-dep-mono-wrt-pred-compI′ inverse-on-compI*
elim!: *bijection-onE simp*: *mono-wrt-pred-eq-dep-mono-wrt-pred*)

consts *bijection* :: *'a* ⇒ *'b* ⇒ *bool*

definition (*bijection-rel* :: (*'a* ⇒ *'b*) ⇒ (*'b* ⇒ *'a*) ⇒ *bool*) ≡
bijection-on (⊤ :: *'a* ⇒ *bool*) (⊤ :: *'b* ⇒ *bool*)
adhoc-overloading *bijection* ⇌ *bijection-rel*

lemma *bijection-eq-bijection-on*:
(*bijection* :: (*'a* ⇒ *'b*) ⇒ (*'b* ⇒ *'a*) ⇒ *bool*) = *bijection-on* (⊤ :: *'a* ⇒ *bool*) (⊤ :: *'b* ⇒ *bool*)
unfolding *bijection-rel-def* **..**

lemma *bijection-eq-bijection-on-uhint* [*uhint*]:
assumes *P* ≡ (⊤ :: *'a* ⇒ *bool*)
and *Q* ≡ (⊤ :: *'b* ⇒ *bool*)
shows (*bijection* :: (*'a* ⇒ *'b*) ⇒ (*'b* ⇒ *'a*) ⇒ *bool*) = *bijection-on P Q*
using *assms* **by** (*simp add*: *bijection-eq-bijection-on*)

context
fixes *P* :: *'a* ⇒ *bool* **and** *Q* :: *'b* ⇒ *bool* **and** *f* :: *'a* ⇒ *'b* **and** *g* :: *'b* ⇒ *'a*
begin

lemma *bijectionI* [*intro*]:
assumes *inverse f g*
and *inverse g f*
shows *bijection f g*
by (*urule bijection-onI*) (*simp* | *urule assms*)+

lemma *bijectionE* [*elim*]:
assumes *bijection f g*
obtains *inverse f g inverse g f*
using *assms* **by** (*urule* (*e*) *bijection-onE*)

lemma *inverse-if-bijection-left-right*:
assumes *bijection f g*
shows *inverse f g*
using *assms* **by** (*elim bijectionE*)

lemma *inverse-if-bijection-right-left*:
assumes *bijection f g*
shows *inverse g f*
using *assms* **by** (*elim bijectionE*)

end

lemma *bijection-right-left-if-bijection-left-right*:
fixes $f :: 'a \Rightarrow 'b$ **and** $g :: 'b \Rightarrow 'a$
assumes *bijection f g*
shows *bijection g f*
using *assms* **by** *auto*

Instantiations **lemma** *bijection-on-self-id*: *bijection-on* $(P :: 'a \Rightarrow bool)$ *P id id*
by (*intro bijection-onI inverse-onI mono-wrt-predI*) *simp-all*

end

Surjective

theory *Functions-Surjective*
imports
Functions-Base
begin

consts *surjective-at* :: $'a \Rightarrow 'b \Rightarrow bool$

overloading
surjective-at-pred $\equiv$ *surjective-at* :: $('a \Rightarrow bool) \Rightarrow ('b \Rightarrow 'a) \Rightarrow bool$
begin
definition *surjective-at-pred* $(P :: 'a \Rightarrow bool)$ $(f :: 'b \Rightarrow 'a) \equiv \forall y : P.$ *has-inverse f y*
end

lemma *surjective-atI* [*intro*]:
assumes $\bigwedge y.$ *P y* $\Longrightarrow$ *has-inverse f y*
shows *surjective-at P f*
unfolding *surjective-at-pred-def* **using** *assms* **by** *blast*

lemma *surjective-atE* [*elim*]:
assumes *surjective-at P f*
and *P y*
obtains *x* **where** $y = f\ x$
using *assms* **unfolding** *surjective-at-pred-def* **by** *blast*

lemma *has-inverse-if-pred-if-surjective-at*:
assumes *surjective-at P f*
and *P y*
shows *has-inverse f y*
using *assms* **by** *blast*

consts *surjective* :: $'a \Rightarrow bool$

overloading
  surjective $\equiv$ *surjective* :: $('b \Rightarrow 'a) \Rightarrow bool$
begin
  definition (*surjective* :: $('b \Rightarrow 'a) \Rightarrow bool$) $\equiv$ *surjective-at* ($\top$ :: $'a \Rightarrow bool$)
end

lemma *surjective-eq-surjective-at*:
  (*surjective* :: $('b \Rightarrow 'a) \Rightarrow bool$) = *surjective-at* ($\top$ :: $'a \Rightarrow bool$)
  unfolding *surjective-def* **..**

lemma *surjective-eq-surjective-at-uhint* [*uhint*]:
  assumes $P \equiv \top$ :: $'a \Rightarrow bool$
  shows *surjective* :: $('b \Rightarrow 'a) \Rightarrow bool \equiv$ *surjective-at P*
  using *assms* **by** (*simp add*: *surjective-eq-surjective-at*)

lemma *surjectiveI* [*intro*]:
  assumes $\bigwedge y$. *has-inverse f y*
  shows *surjective f*
  using *assms* **by** (*urule surjective-atI*)

lemma *surjectiveE*:
  assumes *surjective f*
  obtains $\bigwedge y$. *has-inverse f y*
  using *assms* **unfolding** *surjective-eq-surjective-at* **by** (*force dest*: *has-inverseI*)

lemma *surjective-at-if-surjective*:
  fixes *P* :: $'a \Rightarrow bool$ **and** *f* :: $'b \Rightarrow 'a$
  assumes *surjective f*
  shows *surjective-at P f*
  using *assms* **by** (*intro surjective-atI*) (*blast elim*: *surjectiveE*)

end

1.3.19 Basic Properties

theory *Function-Properties*
  imports
    Functions-Bijection
    Functions-Injective
    Functions-Inverse
    Functions-Monotone
    Functions-Surjective
begin

Summary Basic properties on functions.

end

theory *LFunctions*
imports
Function-Properties
Function-Relators
Functions-Restrict
begin

Summary Basic concepts on functions.

end

1.3.20 Functions On Orders

Basics

theory *Order-Functions-Base*
imports
Functions-Monotone
Binary-Relations-Antisymmetric
Binary-Relations-Symmetric
Preorders
begin

Bi-Relation **consts** *bi-related* :: $'a \Rightarrow 'b \Rightarrow 'b \Rightarrow bool$

overloading
bi-related $\equiv$ *bi-related* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$
begin
definition *bi-related* $(R :: 'a \Rightarrow 'a \Rightarrow bool)\ x\ y \equiv R\ x\ y \wedge R\ y\ x$
end

open-bundle *bi-related-syntax*
begin
syntax
-bi-related :: $'a \Rightarrow 'b \Rightarrow 'a \Rightarrow bool$ $(\langle(\text{-})\ \equiv_{(\text{-})}\ (\text{-})\rangle\ [51,51,51]\ 50)$
notation *bi-related* $(\langle'(\equiv(_)')\rangle)$
syntax-consts
-bi-related $\rightleftharpoons$ *bi-related*
translations
$x \equiv_R y \rightleftharpoons$ *CONST bi-related* $R\ x\ y$
end

lemma *bi-relatedI* [*intro*]:
assumes $R\ x\ y$
and $R\ y\ x$
shows $x \equiv_R y$
unfolding *bi-related-def* **using** *assms* **by** *blast*

lemma *bi-relatedE* [*elim*]:

assumes $x \equiv_R y$
obtains $R\ x\ y\ R\ y\ x$
using *assms* **unfolding** *bi-related-def* **by** *blast*

context
fixes $P :: 'a \Rightarrow bool$ **and** $R\ S :: 'a \Rightarrow 'a \Rightarrow bool$ **and** $x\ y :: 'a$
begin

lemma *symmetric-bi-related* [*iff*]: *symmetric* $((\equiv_R) :: 'a \Rightarrow 'a \Rightarrow bool)$
by (*intro symmetricI*) *blast*

lemma *reflexive-bi-related-if-reflexive* [*intro*]:
assumes *reflexive R*
shows *reflexive* $((\equiv_R) :: 'a \Rightarrow 'a \Rightarrow bool)$
using *assms* **by** (*intro reflexiveI*) (*blast dest*: *reflexiveD*)

lemma *transitive-bi-related-if-transitive* [*intro*]:
assumes *transitive R*
shows *transitive* $((\equiv_R) :: 'a \Rightarrow 'a \Rightarrow bool)$
using *assms* **by** (*intro transitiveI bi-relatedI*) *auto*

lemma *mono-bi-related*: *mono* (*bi-related* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$)
by (*intro monoI*) *blast*

lemma *bi-related-if-le-rel-if-bi-related*:
assumes $x \equiv_R y$
and $R \leq S$
shows $x \equiv_S y$
using *assms* **by** *blast*

lemma *eq-if-bi-related-if-antisymmetric-on*:
assumes *antisymmetric-on P R*
and $x \equiv_R y$
and $P\ x\ P\ y$
shows $x = y$
using *assms* **by** (*blast dest*: *antisymmetric-onD*)

lemma *eq-if-bi-related-if-in-field-le-if-antisymmetric-on*:
assumes *antisymmetric-on P R*
and *in-field R* $\leq P$
and $x \equiv_R y$
shows $x = y$
using *assms* **by** (*intro eq-if-bi-related-if-antisymmetric-on*) *blast+*

lemma *bi-related-if-all-rel-iff-if-reflexive-on*:
assumes *reflexive-on P R*
and $\bigwedge z.\ P\ z \Longrightarrow R\ x\ z \longleftrightarrow R\ y\ z$
and $P\ x\ P\ y$
shows $x \equiv_R y$

using *assms* **by** *blast*

lemma *bi-related-if-all-rel-iff-if-reflexive-on'*:
assumes *reflexive-on P R*
and $\bigwedge z.\ P\ z \Longrightarrow R\ z\ x \longleftrightarrow R\ z\ y$
and *P x P y*
shows $x \equiv_R y$
using *assms* **by** *blast*

corollary *eq-if-all-rel-iff-if-antisymmetric-on-if-reflexive-on*:
assumes *reflexive-on P R* **and** *antisymmetric-on P R*
and $\bigwedge z.\ P\ z \Longrightarrow R\ x\ z \longleftrightarrow R\ y\ z$
and *P x P y*
shows $x = y$
using *assms* **by** (*blast intro*: *eq-if-bi-related-if-antisymmetric-on bi-related-if-all-rel-iff-if-reflexive-on*)

corollary *eq-if-all-rel-iff-if-antisymmetric-on-if-reflexive-on'*:
assumes *reflexive-on P R* **and** *antisymmetric-on P R*
and $\bigwedge z.\ P\ z \Longrightarrow R\ z\ x \longleftrightarrow R\ z\ y$
and *P x P y*
shows $x = y$
using *assms* **by** (*blast intro*: *eq-if-bi-related-if-antisymmetric-on bi-related-if-all-rel-iff-if-reflexive-on'*)

end

lemma *bi-related-le-eq-if-antisymmetric-on-in-field*:
assumes *antisymmetric-on* (*in-field R*) ($R :: 'a \Rightarrow 'a \Rightarrow bool$)
shows $((\equiv_R) :: 'a \Rightarrow 'a \Rightarrow bool) \leq (=)$
using *assms*
by (*intro le-relI eq-if-bi-related-if-in-field-le-if-antisymmetric-on*) *blast+*

Inflationary

consts *inflationary-on* :: $'a \Rightarrow 'b \Rightarrow 'c \Rightarrow bool$

overloading
inflationary-on-pred ≡ *inflationary-on* :: $('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b) \Rightarrow bool$
begin

Often also called "extensive".

definition *inflationary-on-pred P* ($R :: 'a \Rightarrow 'b \Rightarrow bool$) ($f :: 'a \Rightarrow 'b$) ≡ $\forall x : P.\ R\ x\ (f\ x)$
end

lemma *inflationary-onI* [*intro*]:
assumes $\bigwedge x.\ P\ x \Longrightarrow R\ x\ (f\ x)$
shows *inflationary-on P R f*
unfolding *inflationary-on-pred-def* **using** *assms* **by** *blast*

lemma *inflationary-onD* [*dest*]:
assumes *inflationary-on P R f*
and *P x*
shows *R x* (*f x*)
using *assms* **unfolding** *inflationary-on-pred-def* **by** *blast*

lemma *inflationary-on-eq-dep-mono-wrt-pred*: *inflationary-on* = *dep-mono-wrt-pred*
by *blast*

lemma *inflationary-on-if-le-rel-if-inflationary-on*:
assumes *inflationary-on P R f*
and $\bigwedge x.\ P\ x \Longrightarrow R\ x\ (f\ x) \Longrightarrow R'\ x\ (f\ x)$
shows *inflationary-on P R′ f*
using *assms* **by** *blast*

lemma *mono-inflationary-on-rel*:
((≥) ⇒ (≤) ⇛ (≤)) (*inflationary-on* :: (′*a* ⇒ *bool*) ⇒ (′*a* ⇒ ′*b* ⇒ *bool*) ⇒ (′*a* ⇒ ′*b*) ⇒ *bool*)
by (*intro mono-wrt-relI Fun-Rel-relI*) *auto*

context
fixes *P P′* :: ′*a* ⇒ *bool* **and** *R* :: ′*a* ⇒ ′*b* ⇒ *bool* **and** *f* :: ′*a* ⇒ ′*b*
begin

lemma *inflationary-on-if-le-pred-if-inflationary-on*:
assumes *inflationary-on P R f*
and *P′* ≤ *P*
shows *inflationary-on P′ R f*
using *assms* **by** *blast*

lemma *le-in-dom-if-inflationary-on*:
assumes *inflationary-on P R f*
shows *P* ≤ *in-dom R*
using *assms* **by** *blast*
end

lemma *inflationary-on-sup-eq* [*simp*]:
(*inflationary-on* :: (′*a* ⇒ *bool*) ⇒ (′*a* ⇒ ′*b* ⇒ *bool*) ⇒ (′*a* ⇒ ′*b*) ⇒ *bool*) (*P* ⊔ *Q*)
= *inflationary-on P* ⊓ *inflationary-on Q*
by (*intro ext iffI inflationary-onI*)
(*auto intro*: *inflationary-on-if-le-pred-if-inflationary-on*)

consts *inflationary* :: ′*a* ⇒ ′*b* ⇒ *bool*

overloading
inflationary ≡ *inflationary* :: (′*a* ⇒ ′*b* ⇒ *bool*) ⇒ (′*a* ⇒ ′*b*) ⇒ *bool*

begin
definition (*inflationary* :: ('*a* ⇒ '*b* ⇒ *bool*) ⇒ ('*a* ⇒ '*b*) ⇒ *bool*) ≡
inflationary-on (⊤ :: '*a* ⇒ *bool*)
end

lemma *inflationary-eq-inflationary-on*:
(*inflationary* :: ('*a* ⇒ '*b* ⇒ *bool*) ⇒ ('*a* ⇒ '*b*) ⇒ *bool*) = *inflationary-on* (⊤ :: '*a* ⇒ *bool*)
unfolding *inflationary-def* **..**

lemma *inflationary-eq-inflationary-on-uhint* [*uhint*]:
assumes *P* ≡ ⊤ :: '*a* ⇒ *bool*
shows *inflationary* :: ('*a* ⇒ '*b* ⇒ *bool*) ⇒ ('*a* ⇒ '*b*) ⇒ *bool* ≡ *inflationary-on P*
using *assms* **by** (*simp add*: *inflationary-eq-inflationary-on*)

lemma *inflationaryI* [*intro*]:
assumes ⋀*x*. *R x* (*f x*)
shows *inflationary R f*
using *assms* **by** (*urule inflationary-onI*)

lemma *inflationaryD*:
assumes *inflationary R f*
shows *R x* (*f x*)
using *assms* **by** (*urule* (*d*) *inflationary-onD chained*: *insert*) *simp*

lemma *inflationary-on-if-inflationary*:
fixes *P* :: '*a* ⇒ *bool* **and** *R* :: '*a* ⇒ '*b* ⇒ *bool* **and** *f* :: '*a* ⇒ '*b*
assumes *inflationary R f*
shows *inflationary-on P R f*
using *assms* **by** (*intro inflationary-onI*) (*blast dest*: *inflationaryD*)

lemma *inflationary-eq-dep-mono-wrt-pred*: *inflationary* = *dep-mono-wrt-pred* ⊤
by (*intro ext*) (*fastforce dest*: *inflationaryD*)

Deflationary

consts *deflationary-on* :: '*a* ⇒ '*b* ⇒ '*c* ⇒ *bool*

overloading
deflationary-on-pred ≡ *deflationary-on* :: ('*a* ⇒ *bool*) ⇒ ('*b* ⇒ '*a* ⇒ *bool*) ⇒ ('*a* ⇒ '*b*) ⇒ *bool*
begin
definition *deflationary-on-pred* (*P* :: '*a* ⇒ *bool*) (*R* :: '*b* ⇒ '*a* ⇒ *bool*) :: ('*a* ⇒ '*b*) ⇒ *bool* ≡
inflationary-on P R^{-1}
end

context
fixes *P* :: '*a* ⇒ *bool* **and** *R* :: '*b* ⇒ '*a* ⇒ *bool* **and** *f* :: '*a* ⇒ '*b*
begin

lemma *deflationary-on-eq-inflationary-on-rel-inv*:
(*deflationary-on P R* :: (*'a* ⇒ *'b*) ⇒ *bool*) = *inflationary-on P* R^{-1}
unfolding *deflationary-on-pred-def* **..**

declare *deflationary-on-eq-inflationary-on-rel-inv*[*symmetric*, *simp*]

lemma *deflationary-onI* [*intro*]:
assumes ⋀*x*. *P x* ⟹ *R* (*f x*) *x*
shows *deflationary-on P R f*
unfolding *deflationary-on-eq-inflationary-on-rel-inv* **using** *assms*
by (*intro inflationary-onI rel-invI*)

lemma *deflationary-onD* [*dest*]:
assumes *deflationary-on P R f*
and *P x*
shows *R* (*f x*) *x*
using *assms* **unfolding** *deflationary-on-eq-inflationary-on-rel-inv* **by** *blast*

end

context
fixes *P P'* :: *'a* ⇒ *bool* **and** *R* :: *'b* ⇒ *'a* ⇒ *bool* **and** *f* :: *'a* ⇒ *'b*
begin

corollary *deflationary-on-rel-inv-eq-inflationary-on* [*simp*]:
(*deflationary-on P* (*S* :: *'a* ⇒ *'b* ⇒ *bool*)$^{-1}$ :: (*'a* ⇒ *'b*) ⇒ *bool*) = *inflationary-on P S*
unfolding *deflationary-on-eq-inflationary-on-rel-inv* **by** *simp*

lemma *deflationary-on-eq-dep-mono-wrt-pred-rel-inv*:
(*deflationary-on P R* :: (*'a* ⇒ *'b*) ⇒ *bool*) = ((*x* : *P*) ⇒ R^{-1} *x*)
by *blast*

lemma *deflationary-on-if-le-rel-if-deflationary-on*:
assumes *deflationary-on P R f*
and ⋀*x*. *P x* ⟹ *R* (*f x*) *x* ⟹ *R'* (*f x*) *x*
shows *deflationary-on P R' f*
using *assms* **by** *auto*

lemma *mono-deflationary-on*:
((≥) ⇒ (≤) ⇛ (≤)) (*deflationary-on* :: (*'a* ⇒ *bool*) ⇒ (*'b* ⇒ *'a* ⇒ *bool*) ⇒ (*'a* ⇒ *'b*) ⇒ *bool*)
by *blast*

lemma *deflationary-on-if-le-pred-if-deflationary-on*:
assumes *deflationary-on P R f*
and *P'* ≤ *P*
shows *deflationary-on P' R f*

using *assms* **by** *blast*

lemma *le-in-dom-if-deflationary-on*:
assumes *deflationary-on P R f*
shows $P \leq$ *in-codom R*
using *assms* **by** *blast*

lemma *deflationary-on-sup-eq* [*simp*]:
(*deflationary-on* :: ('*a* ⇒ *bool*) ⇒ ('*b* ⇒ '*a* ⇒ *bool*) ⇒ ('*a* ⇒ '*b*) ⇒ *bool*) (*P* ⊔ *Q*)
= *deflationary-on P* ⊓ *deflationary-on Q*
unfolding *deflationary-on-eq-inflationary-on-rel-inv* **by** *auto*

end

consts *deflationary* :: '*a* ⇒ '*b* ⇒ *bool*

overloading
deflationary ≡ *deflationary* :: ('*b* ⇒ '*a* ⇒ *bool*) ⇒ ('*a* ⇒ '*b*) ⇒ *bool*
begin
definition (*deflationary* :: ('*b* ⇒ '*a* ⇒ *bool*) ⇒ ('*a* ⇒ '*b*) ⇒ *bool*) ≡
deflationary-on (⊤ :: '*a* ⇒ *bool*)
end

lemma *deflationary-eq-deflationary-on*:
(*deflationary* :: ('*b* ⇒ '*a* ⇒ *bool*) ⇒ ('*a* ⇒ '*b*) ⇒ *bool*) = *deflationary-on* (⊤ :: '*a* ⇒ *bool*)
unfolding *deflationary-def* **..**

lemma *deflationary-eq-deflationary-on-uhint* [*uhint*]:
assumes *P* ≡ ⊤ :: '*a* ⇒ *bool*
shows *deflationary* :: ('*b* ⇒ '*a* ⇒ *bool*) ⇒ ('*a* ⇒ '*b*) ⇒ *bool* ≡ *deflationary-on P*
using *assms* **by** (*simp add*: *deflationary-eq-deflationary-on*)

lemma *deflationaryI* [*intro*]:
assumes ⋀*x*. *R* (*f x*) *x*
shows *deflationary R f*
using *assms* **by** (*urule deflationary-onI*)

lemma *deflationaryD*:
assumes *deflationary R f*
shows *R* (*f x*) *x*
using *assms* **by** (*urule* (*d*) *deflationary-onD chained*: *insert*) *simp*

lemma *deflationary-on-if-deflationary*:
fixes *P* :: '*a* ⇒ *bool* **and** *R* :: '*b* ⇒ '*a* ⇒ *bool* **and** *f* :: '*a* ⇒ '*b*
assumes *deflationary R f*
shows *deflationary-on P R f*
using *assms* **by** (*intro deflationary-onI*) (*blast dest*: *deflationaryD*)

lemma *deflationary-eq-dep-mono-wrt-pred-rel-inv*:
deflationary R = *dep-mono-wrt-pred* $\top$ R^{-1}
by (*intro ext*) (*fastforce dest*: *deflationaryD*)

Relational Equivalence **definition** *rel-equivalence-on* $\equiv$ *inflationary-on* $\sqcap$ *deflationary-on*

lemma *rel-equivalence-on-eq*:
rel-equivalence-on = *inflationary-on* $\sqcap$ *deflationary-on*
unfolding *rel-equivalence-on-def* **..**

lemma *rel-equivalence-onI* [*intro*]:
assumes *inflationary-on* P R f
and *deflationary-on* P R f
shows *rel-equivalence-on* P R f
unfolding *rel-equivalence-on-eq* **using** *assms* **by** *auto*

lemma *rel-equivalence-onE* [*elim*]:
assumes *rel-equivalence-on* P R f
obtains *inflationary-on* P R f *deflationary-on* P R f
using *assms* **unfolding** *rel-equivalence-on-eq* **by** *auto*

lemma *rel-equivalence-on-eq-dep-mono-wrt-pred-inf*:
rel-equivalence-on P R = *dep-mono-wrt-pred* P $(R \sqcap R^{-1})$
by (*intro ext*) *fastforce*

context
fixes P P' :: $'a \Rightarrow$ *bool* **and** R :: $'a \Rightarrow$ $'a \Rightarrow$ *bool* **and** f :: $'a \Rightarrow$ $'a$
begin

lemma *bi-related-if-rel-equivalence-on*:
assumes *rel-equivalence-on* P R f
and P x
shows $x \equiv_R f\ x$
using *assms* **by** (*intro bi-relatedI*) *auto*

lemma *rel-equivalence-on-if-all-bi-related*:
assumes $\bigwedge x.\ P\ x \Longrightarrow x \equiv_R f\ x$
shows *rel-equivalence-on* P R f
using *assms* **by** *auto*

corollary *rel-equivalence-on-iff-all-bi-related*:
rel-equivalence-on P R f $\longleftrightarrow$ $(\forall x.\ P\ x \longrightarrow x \equiv_R f\ x)$
using *rel-equivalence-on-if-all-bi-related bi-related-if-rel-equivalence-on*
by *blast*

lemma *rel-equivalence-onD* [*dest*]:
assumes *rel-equivalence-on* P R f

and $P\ x$
shows $x \equiv_R f\ x$
using *assms* **by** (*auto dest*: *bi-related-if-rel-equivalence-on*)

lemma *rel-equivalence-on-rel-inv-eq-rel-equivalence-on* [*simp*]:
(*rel-equivalence-on* $P\ R^{-1}$ :: $('a \Rightarrow 'a) \Rightarrow bool$) = *rel-equivalence-on* $P\ R$
by (*intro ext*) *fastforce*

lemma *mono-rel-equivalence-on*:
$((\geq) \Rightarrow (\leq) \Rrightarrow (\leq))$ (*rel-equivalence-on* :: $('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a) \Rightarrow bool$)
by *blast*

lemma *rel-equivalence-on-if-le-pred-if-rel-equivalence-on*:
assumes *rel-equivalence-on* $P\ R\ f$
and $P' \leq P$
shows *rel-equivalence-on* $P'\ R\ f$
using *assms* **by** *blast*

lemma *rel-equivalence-on-sup-eq* [*simp*]:
(*rel-equivalence-on* :: $('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a) \Rightarrow bool$) ($P \sqcup Q$)
= *rel-equivalence-on* $P \sqcap$ *rel-equivalence-on* Q
unfolding *rel-equivalence-on-eq* **by** (*simp add*: *inf-aci*)

lemma *in-codom-eq-in-dom-if-rel-equivalence-on-in-field*:
assumes *rel-equivalence-on* (*in-field* R) $R\ f$
shows *in-codom* R = *in-dom* R
using *assms* **by** (*intro ext*) *blast*

lemma *reflexive-on-if-transitive-on-if-mon-wrt-pred-if-rel-equivalence-on*:
assumes *rel-equivalence-on* $P\ R\ f$
and $(P \Rightarrow P)\ f$
and *transitive-on* $P\ R$
shows *reflexive-on* $P\ R$
using *assms* **by** (*blast dest*: *transitive-onD*)

lemma *inflationary-on-eq-rel-equivalence-on-if-symmetric*:
assumes *symmetric* R
shows (*inflationary-on* $P\ R$ :: $('a \Rightarrow 'a) \Rightarrow bool$) = *rel-equivalence-on* $P\ R$
using *assms*
by (*simp add*: *rel-equivalence-on-eq deflationary-on-eq-inflationary-on-rel-inv*)

lemma *deflationary-on-eq-rel-equivalence-on-if-symmetric*:
assumes *symmetric* R
shows (*deflationary-on* $P\ R$ :: $('a \Rightarrow 'a) \Rightarrow bool$) = *rel-equivalence-on* $P\ R$
using *assms*
by (*simp add*: *deflationary-on-eq-inflationary-on-rel-inv rel-equivalence-on-eq*)

end

consts *rel-equivalence* :: $'a \Rightarrow 'b \Rightarrow bool$

overloading
  rel-equivalence ≡ *rel-equivalence* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a) \Rightarrow bool$
begin
  definition (*rel-equivalence* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a) \Rightarrow bool$) ≡
    rel-equivalence-on ($\top$ :: $'a \Rightarrow bool$)
end

lemma *rel-equivalence-eq-rel-equivalence-on*:
  (*rel-equivalence* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a) \Rightarrow bool$) = *rel-equivalence-on* ($\top$ :: $'a \Rightarrow bool$)
  unfolding *rel-equivalence-def* **..**

lemma *rel-equivalence-eq-rel-equivalence-on-uhint* [*uhint*]:
  assumes $P \equiv \top$ :: $'a \Rightarrow bool$
  shows *rel-equivalence* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a) \Rightarrow bool \equiv$ *rel-equivalence-on* *P*
  using *assms* **by** (*simp add*: *rel-equivalence-eq-rel-equivalence-on*)

context
  fixes *P* :: $'a \Rightarrow bool$ **and** *R* :: $'a \Rightarrow 'a \Rightarrow bool$ **and** *f* :: $'a \Rightarrow 'a$
begin

lemma *rel-equivalenceI* [*intro*]:
  assumes *inflationary R f*
  and *deflationary R f*
  shows *rel-equivalence R f*
  using *assms* **by** (*urule rel-equivalence-onI*)

lemma *rel-equivalenceE* [*elim*]:
  assumes *rel-equivalence R f*
  obtains *inflationary R f deflationary R f*
  using *assms* **by** (*urule* (*e*) *rel-equivalence-onE*)

lemma *inflationary-if-rel-equivalence*:
  assumes *rel-equivalence R f*
  shows *inflationary R f*
  using *assms* **by** (*rule rel-equivalenceE*)

lemma *deflationary-if-rel-equivalence*:
  assumes *rel-equivalence R f*
  shows *deflationary R f*
  using *assms* **by** (*rule rel-equivalenceE*)

lemma *rel-equivalence-on-if-rel-equivalence*:

assumes *rel-equivalence R f*
shows *rel-equivalence-on P R f*
using *assms* **by** (*intro rel-equivalence-onI*)
(*auto dest*: *inflationary-on-if-inflationary deflationary-on-if-deflationary*)

lemma *bi-related-if-rel-equivalence*:
assumes *rel-equivalence R f*
shows $x \equiv_R f\ x$
using *assms* **by** (*intro bi-relatedI*) (*auto dest*: *inflationaryD deflationaryD*)

lemma *rel-equivalence-if-all-bi-related*:
assumes $\bigwedge x.\ x \equiv_R f\ x$
shows *rel-equivalence R f*
using *assms* **by** *auto*

lemma *rel-equivalenceD*:
assumes *rel-equivalence R f*
shows *R x* (*f x*) *R* (*f x*) *x*
using *assms* **by** (*auto dest*: *bi-related-if-rel-equivalence*)

lemma *reflexive-on-in-field-if-transitive-if-rel-equivalence-on*:
assumes *rel-equivalence-on* (*in-field R*) *R f*
and *transitive R*
shows *reflexive-on* (*in-field R*) *R*
using *assms* **by** (*intro reflexive-onI*) *blast*

corollary *preorder-on-in-field-if-transitive-if-rel-equivalence-on*:
assumes *rel-equivalence-on* (*in-field R*) *R f*
and *transitive R*
shows *preorder-on* (*in-field R*) *R*
using *assms reflexive-on-in-field-if-transitive-if-rel-equivalence-on*
by *blast*

end

end

1.3.21 Order Functors

Basic Setup and Results

theory *Order-Functors-Base*
imports
Functions-Inverse
Order-Functions-Base
begin

In the following, we do not add any assumptions to our locales but rather add them as needed to the theorem statements. This allows consumers to state preciser results; particularly, the development of Transport depends on

this setup.

locale *orders* =
  fixes $L :: 'a \Rightarrow 'b \Rightarrow bool$
  and $R :: 'c \Rightarrow 'd \Rightarrow bool$
begin

notation L (**infix** $\langle \leq_L \rangle$ *50*)
notation R (**infix** $\langle \leq_R \rangle$ *50*)

We call $(\leq_L)$ the *left relation* and $(\leq_R)$ the *right relation*.

abbreviation (*input*) *ge-left* $\equiv (\leq_L)^{-1}$
notation *ge-left* (**infix** $\langle \geq_L \rangle$ *50*)

abbreviation (*input*) *ge-right* $\equiv (\leq_R)^{-1}$
notation *ge-right* (**infix** $\langle \geq_R \rangle$ *50*)

end

Homogeneous orders

locale *hom-orders* = *orders* L R
  for $L :: 'a \Rightarrow 'a \Rightarrow bool$
  and $R :: 'b \Rightarrow 'b \Rightarrow bool$

locale *order-functor* = *hom-orders* L R
  for $L :: 'a \Rightarrow 'a \Rightarrow bool$
  and $R :: 'b \Rightarrow 'b \Rightarrow bool$
  and $l :: 'a \Rightarrow 'b$
begin

lemma *left-right-rel-left-self-if-reflexive-on-left-if-mono-left*:
  assumes $((\leq_L) \Rightarrow (\leq_R))\ l$
  and *reflexive-on* P $(\leq_L)$
  and $P\ x$
  shows $l\ x \leq_R l\ x$
  using *assms* **by** *blast*

lemma *left-right-rel-left-self-if-reflexive-on-in-dom-right-if-mono-left*:
  assumes $((\leq_L) \Rightarrow (\leq_R))\ l$
  and *reflexive-on* (*in-dom* $(\leq_R)$) $(\leq_R)$
  and *in-dom* $(\leq_L)\ x$
  shows $l\ x \leq_R l\ x$
  using *assms* **by** *blast*

lemma *left-right-rel-left-self-if-reflexive-on-in-codom-right-if-mono-left*:
  assumes $((\leq_L) \Rightarrow (\leq_R))\ l$
  and *reflexive-on* (*in-codom* $(\leq_R)$) $(\leq_R)$
  and *in-codom* $(\leq_L)\ x$
  shows $l\ x \leq_R l\ x$
  using *assms* **by** *blast*

lemma *left-right-rel-left-self-if-reflexive-on-in-field-right-if-mono-left*:
assumes $((\leq_L) \Rightarrow (\leq_R))\ l$
and *reflexive-on* (*in-field* $(\leq_R)$) $(\leq_R)$
and *in-field* $(\leq_L)\ x$
shows $l\ x \leq_R l\ x$
using *assms* **by** *blast*

lemma *mono-wrt-rel-left-if-reflexive-on-if-le-eq-if-mono-wrt-in-field*:
assumes (*in-field* $(\leq_L) \Rightarrow P)\ l$
and $(\leq_L) \leq (=)$
and *reflexive-on* P $(\leq_R)$
shows $((\leq_L) \Rightarrow (\leq_R))\ l$
using *assms* **by** (*intro mono-wrt-relI*) *auto*

end

locale *order-functors* = *order-functor* $L\ R\ l$ + *flip-of* : *order-functor* $R\ L\ r$
for $L\ R\ l\ r$
begin

We call the composition $r \circ l$ the *unit* and the term $l \circ r$ the *counit* of the order functors pair. This is terminology is borrowed from category theory - the functors are an *adjoint*.

definition *unit* $\equiv r \circ l$

notation *unit* $(\langle\eta\rangle)$

lemma *unit-eq-comp*: $\eta = r \circ l$ **unfolding** *unit-def* **by** *simp*

lemma *unit-eq* [*simp*]: $\eta\ x = r\ (l\ x)$ **by** (*simp add*: *unit-eq-comp*)

context
begin

Note that by flipping the roles of the left and rights functors, we obtain a flipped interpretation of *order-functors*. In many cases, this allows us to obtain symmetric definitions and theorems for free. As such, in many cases, we do we do not explicitly state those free results but users can obtain them as needed by creating said flipped interpretation.

interpretation *flip* : *order-functors* $R\ L\ r\ l$ **.**

definition *counit* $\equiv$ *flip.unit*

notation *counit* $(\langle\varepsilon\rangle)$

lemma *counit-eq-comp*: $\varepsilon = l \circ r$ **unfolding** *counit-def flip.unit-def* **by** *simp*

lemma *counit-eq* [*simp*]: $\varepsilon\ x = l\ (r\ x)$ **by** (*simp add*: *counit-eq-comp*)

end

context
begin

interpretation *flip* : *order-functors R L r l* **.**

lemma *flip-counit-eq-unit*: *flip.counit* $= \eta$
by (*intro ext*) *simp*

lemma *flip-unit-eq-counit*: *flip.unit* $= \varepsilon$
by (*intro ext*) *simp*

lemma *inflationary-on-unit-if-left-rel-right-if-left-right-relI*:
assumes $((\leq_L) \Rightarrow (\leq_R))\ l$
and *reflexive-on P* $(\leq_L)$
and $\bigwedge x\ y.\ P\ x \Longrightarrow l\ x \leq_R y \Longrightarrow x \leq_L r\ y$
shows *inflationary-on P* $(\leq_L)\ \eta$
using *assms* **by** (*intro inflationary-onI*) *fastforce*

lemma *deflationary-on-unit-if-right-left-rel-if-right-rel-leftI*:
assumes $((\leq_L) \Rightarrow (\leq_R))\ l$
and *reflexive-on P* $(\leq_L)$
and $\bigwedge x\ y.\ P\ x \Longrightarrow y \leq_R l\ x \Longrightarrow r\ y \leq_L x$
shows *deflationary-on P* $(\leq_L)\ \eta$
using *assms* **by** (*intro deflationary-onI*) *fastforce*

context
fixes $P :: 'a \Rightarrow bool$
begin

lemma *rel-equivalence-on-unit-iff-inflationary-on-if-inverse-on*:
assumes *inverse-on P l r*
shows *rel-equivalence-on P* $(\leq_L)\ \eta \longleftrightarrow$ *inflationary-on P* $(\leq_L)\ \eta$
using *assms* **by** (*intro iffI rel-equivalence-onI inflationary-onI deflationary-onI*)
(*fastforce dest*: *inverse-onD*)+

lemma *reflexive-on-left-if-inflationary-on-unit-if-inverse-on*:
assumes *inverse-on P l r*
and *inflationary-on P* $(\leq_L)\ \eta$
shows *reflexive-on P* $(\leq_L)$
using *assms* **by** (*intro reflexive-onI*) (*force dest*: *inverse-onD*)

lemma *rel-equivalence-on-unit-if-reflexive-on-if-inverse-on*:
assumes *inverse-on P l r*
and *reflexive-on P* $(\leq_L)$
shows *rel-equivalence-on P* $(\leq_L)\ \eta$

using *assms* **by** (*intro rel-equivalence-onI inflationary-onI deflationary-onI*) (*force dest*: *inverse-onD*)+

end

corollary *rel-equivalence-on-unit-iff-reflexive-on-if-inverse-on*:
fixes $P :: 'a \Rightarrow bool$
assumes *inverse-on P l r*
shows *rel-equivalence-on P* $(\leq_L)$ η $\longleftrightarrow$ *reflexive-on P* $(\leq_L)$
using *assms reflexive-on-left-if-inflationary-on-unit-if-inverse-on rel-equivalence-on-unit-if-reflexive-on-if-inverse-on*
by (*intro iffI*) *auto*

end

Here is an example of a free theorem.

notepad
begin
interpret *flip* : *order-functors R L r l*
rewrites *flip.unit* $\equiv \varepsilon$ **by** (*simp only*: *flip-unit-eq-counit*)
have $[\![((\leq_R) \Rightarrow (\leq_L))\ r;$ *reflexive-on P* $(\leq_R);$ $\bigwedge x\ y.\ [\![P\ x;\ r\ x \leq_L y]\!] \Longrightarrow x \leq_R l\ y]\!]$
$\Longrightarrow$ *inflationary-on P* $(\leq_R)$ ε **for** *P*
by (*fact flip.inflationary-on-unit-if-left-rel-right-if-left-right-relI*)
end

end

end

1.4 Galois

1.4.1 Basic Abbreviations

theory *Galois-Base*
imports
Order-Functors-Base
begin

locale *galois* = *order-functors*
begin

The locale *galois* serves to define concepts that ultimately lead to the definition of Galois connections and Galois equivalences. Galois connections and equivalences are special cases of adjoints and adjoint equivalences, respectively, known from category theory. As such, in what follows, we sometimes borrow vocabulary from category theory to highlight this connection.

A *Galois connection* between two relations $(\leq_L)$ and $(\leq_R)$ consists of

two monotone functions (i.e. order functors) l and r such that $(x \leq_L r\ y) = (l\ x \leq_R y)$. We call this the *Galois property*. l is called the *left adjoint* and r the *right adjoint*. We call $(\leq_L)$ the *left relation* and $(\leq_R)$ the *right relation*. By composing the adjoints, we obtain the unit η and counit ε of the Galois connection.

end

end

1.4.2 Basics For Relator For Galois Connections

theory *Galois-Relator-Base*
  imports
    Galois-Base
begin

locale *galois-rel* = *orders L R*
  for $L :: 'a \Rightarrow 'b \Rightarrow bool$
  and $R :: 'c \Rightarrow 'd \Rightarrow bool$
  and $r :: 'd \Rightarrow 'b$
begin

Morally speaking, the Galois relator characterises when two terms x and y are "similar".

definition *Galois x y* $\equiv$ *in-codom* $(\leq_R)\ y \wedge x \leq_L r\ y$

abbreviation *left-Galois* $\equiv$ *Galois*
notation *left-Galois* (**infix** $\langle {}_L\lessapprox\rangle$ *50*)

abbreviation (*input*) *ge-Galois-left* $\equiv ({}_L\lessapprox)^{-1}$
notation *ge-Galois-left* (**infix** $\langle \gtrapprox_L\rangle$ *50*)

Here we only introduced the (left) Galois relator $({}_L\lessapprox)$. All other variants can be introduced by considering suitable flipped and inversed interpretations (see `Half_Galois_Property.thy`).

lemma *left-GaloisI* [*intro*]:
  assumes *in-codom* $(\leq_R)\ y$
  and $x \leq_L r\ y$
  shows $x\ {}_L\lessapprox y$
  unfolding *Galois-def* **using** *assms* **by** *blast*

lemma *left-GaloisE* [*elim*]:
  assumes $x\ {}_L\lessapprox y$
  obtains *in-codom* $(\leq_R)\ y\ x \leq_L r\ y$
  using *assms* **unfolding** *Galois-def* **by** *blast*

corollary *in-dom-left-if-left-Galois*:
  assumes $x\ {}_L\lessapprox y$

shows *in-dom* $(\leq_L)$ x
using *assms* **by** *blast*

corollary *left-Galois-iff-in-codom-and-left-rel-right*:
x $_L\overset{\leq}{\approx}$ $y \longleftrightarrow$ *in-codom* $(\leq_R)$ $y \wedge x \leq_L r\ y$
by *blast*

lemma *left-Galois-restrict-left-eq-left-Galois-left-restrict-left*:
$({}_L\overset{\leq}{\approx})\restriction_{P :: 'a \Rightarrow bool} =$ *galois-rel.Galois* $(\leq_L)\restriction_P (\leq_R)$ r
by (*intro ext iffI galois-rel.left-GaloisI rel-restrict-leftI*)
(*auto elim*: *galois-rel.left-GaloisE*)

lemma *left-Galois-restrict-right-eq-left-Galois-right-restrict-right*:
$({}_L\overset{\leq}{\approx})\restriction_{P :: 'd \Rightarrow bool} =$ *galois-rel.Galois* $(\leq_L)$ $(\leq_R)\restriction_P r$
by (*intro ext iffI galois-rel.left-GaloisI rel-restrict-rightI*)
(*auto elim*!: *galois-rel.left-GaloisE rel-restrict-rightE*)

end

end

Equivalences

theory *Order-Equivalences*
imports
Order-Functors-Base
Partial-Equivalence-Relations
Preorders
begin

context *order-functors*
begin

definition *order-equivalence* $\equiv$
$((\leq_L) \Rightarrow (\leq_R))$ $l \wedge$
$((\leq_R) \Rightarrow (\leq_L))$ $r \wedge$
rel-equivalence-on (*in-field* $(\leq_L)$) $(\leq_L)$ $\eta \wedge$
rel-equivalence-on (*in-field* $(\leq_R)$) $(\leq_R)$ ε

notation *order-functors.order-equivalence* (**infix** $\langle\equiv_o\rangle$ *50*)

lemma *order-equivalenceI* [*intro*]:
assumes $((\leq_L) \Rightarrow (\leq_R))$ l
and $((\leq_R) \Rightarrow (\leq_L))$ r
and *rel-equivalence-on* (*in-field* $(\leq_L)$) $(\leq_L)$ η
and *rel-equivalence-on* (*in-field* $(\leq_R)$) $(\leq_R)$ ε
shows $((\leq_L) \equiv_o (\leq_R))$ l r
unfolding *order-equivalence-def* **using** *assms* **by** *blast*

lemma *order-equivalenceE* [*elim*]:
assumes $((\leq_L) \equiv_o (\leq_R))\ l\ r$
obtains $((\leq_L) \Rightarrow (\leq_R))\ l\ ((\leq_R) \Rightarrow (\leq_L))\ r$
rel-equivalence-on (*in-field* $(\leq_L)$) $(\leq_L)\ \eta$
rel-equivalence-on (*in-field* $(\leq_R)$) $(\leq_R)\ \varepsilon$
using *assms* **unfolding** *order-equivalence-def* **by** *blast*

interpretation *of* : *order-functors* $S\ T\ f\ g$ **for** $S\ T\ f\ g$ **.**

lemma *rel-inv-order-equivalence-eq-order-equivalence* [*simp*]:
$((\leq_R) \equiv_o (\leq_L))^{-1} = ((\leq_L) \equiv_o (\leq_R))$
by (*intro ext*) (*auto intro*!: *of.order-equivalenceI simp*: *of.flip-unit-eq-counit*)

corollary *order-equivalence-right-left-iff-order-equivalence-left-right*:
$((\leq_R) \equiv_o (\leq_L))\ r\ l \longleftrightarrow ((\leq_L) \equiv_o (\leq_R))\ l\ r$
by (*simp flip*: *rel-inv-order-equivalence-eq-order-equivalence*)

Due to the symmetry given by $((\leq_R) \equiv_o (\leq_L))\ r\ l =$ *order-equivalence*, for any theorem on $(\leq_L)$, we obtain a corresponding theorem on $(\leq_R)$ by flipping the roles of the two functors. As such, in what follows, we do not explicitly state these free theorems but users can obtain them as needed by creating a flipped interpretation of *order-functors*.

lemma *order-equivalence-rel-inv-eq-order-equivalence* [*simp*]:
$((\geq_L) \equiv_o (\geq_R)) = ((\leq_L) \equiv_o (\leq_R))$
by (*intro ext*) (*auto 0 4 intro*!: *of.order-equivalenceI*)

lemma *in-codom-left-eq-in-dom-left-if-order-equivalence*:
assumes $((\leq_L) \equiv_o (\leq_R))\ l\ r$
shows *in-codom* $(\leq_L)$ = *in-dom* $(\leq_L)$
using *assms* **by** (*elim order-equivalenceE*)
(*rule in-codom-eq-in-dom-if-rel-equivalence-on-in-field*)

corollary *preorder-on-in-field-left-if-transitive-if-order-equivalence*:
assumes $((\leq_L) \equiv_o (\leq_R))\ l\ r$
and *transitive* $(\leq_L)$
shows *preorder-on* (*in-field* $(\leq_L)$) $(\leq_L)$
using *assms* **by** (*elim order-equivalenceE*)
(*rule preorder-on-in-field-if-transitive-if-rel-equivalence-on*)

lemma *order-equivalence-partial-equivalence-rel-not-reflexive-not-transitive*:
assumes $\exists (y :: 'b)\ y'.\ y \neq y'$
shows $\exists (L :: 'a \Rightarrow 'a \Rightarrow bool)\ (R :: 'b \Rightarrow 'b \Rightarrow bool)\ l\ r.$
$(L \equiv_o R)\ l\ r \wedge$ *partial-equivalence-rel* $L \wedge$
$\neg$(*reflexive-on* (*in-field* R) R) $\wedge\ \neg$(*transitive-on* (*in-field* R) R)
proof −
from *assms* **obtain** *cy cy'* **where** $(cy :: 'b) \neq cy'$ **by** *blast*
let *?cx* = *undefined* :: $'a$
let *?L* = $\lambda x\ x'.$ *?cx* = $x \wedge x = x'$

and *?R* = $\lambda y\ y'.\ (y = cy \vee y = cy') \wedge (y' = cy \vee y' = cy') \wedge (y \neq cy' \vee y' \neq cy')$
and *?l* = $\lambda(a :: 'a).\ cy$
and *?r* = $\lambda(b :: 'b).$ *?cx*
have (*?L* $\equiv_o$ *?R*) *?l ?r* **using** ‹$cy \neq cy'$›
by (*intro of.order-equivalenceI*) (*auto 0 4*)
moreover have *partial-equivalence-rel ?L* **by** *blast*
moreover have
¬(*transitive-on* (*in-field ?R*) *?R*) **and** ¬(*reflexive-on* (*in-field ?R*) *?R*)
using ‹$cy \neq cy'$› **by** *auto*
ultimately show *?thesis* **by** *blast*
qed

end

end

1.4.3 Half Galois Property

theory *Half-Galois-Property*
imports
Galois-Relator-Base
Order-Equivalences
begin

As the definition of the Galois property also works on heterogeneous relations, we define the concepts in a locale that generalises *galois*.

locale *galois-prop* = *orders L R*
for *L* :: $'a \Rightarrow 'b \Rightarrow$ *bool*
and *R* :: $'c \Rightarrow 'd \Rightarrow$ *bool*
and *l* :: $'a \Rightarrow 'c$
and *r* :: $'d \Rightarrow 'b$
begin

sublocale *galois-rel L R r* **.**

interpretation *gr-flip-inv* : *galois-rel* ($\geq_R$) ($\geq_L$) *l* **.**

abbreviation *right-ge-Galois* ≡ *gr-flip-inv.Galois*
notation *right-ge-Galois* (**infix** ‹$_R\gtrsim$› *50*)

abbreviation (*input*) *Galois-right* ≡ *gr-flip-inv.ge-Galois-left*
notation *Galois-right* (**infix** ‹$\lesssim_R$› *50*)

lemma *Galois-rightI* [*intro*]:
assumes *in-dom* ($\leq_L$) *x*
and *l x* $\leq_R$ *y*
shows *x* $\lesssim_R$ *y*

using *assms* **by** *blast*

lemma *Galois-rightE* [*elim*]:
assumes $x \lessapprox_R y$
obtains *in-dom* $(\leq_L)$ x l $x \leq_R y$
using *assms* **by** *blast*

corollary *Galois-right-iff-in-dom-and-left-right-rel*:
$x \lessapprox_R y \longleftrightarrow$ *in-dom* $(\leq_L)$ $x \wedge l\ x \leq_R y$
by *blast*

Unlike common literature, we split the definition of the Galois property into two halves. This has its merits in modularity of proofs and preciser statement of required assumptions.

definition *half-galois-prop-left* $\equiv \forall x\ y.\ x\ {}_L\lessapprox\ y \longrightarrow l\ x \leq_R y$

notation *galois-prop.half-galois-prop-left* (**infix** $\langle {}_h\trianglelefteq\rangle$ *50*)

lemma *half-galois-prop-leftI* [*intro*]:
assumes $\bigwedge x\ y.\ x\ {}_L\lessapprox\ y \Longrightarrow l\ x \leq_R y$
shows $((\leq_L)\ {}_h\trianglelefteq\ (\leq_R))\ l\ r$
unfolding *half-galois-prop-left-def* **using** *assms* **by** *blast*

lemma *half-galois-prop-leftD* [*dest*]:
assumes $((\leq_L)\ {}_h\trianglelefteq\ (\leq_R))\ l\ r$
and $x\ {}_L\lessapprox\ y$
shows $l\ x \leq_R y$
using *assms* **unfolding** *half-galois-prop-left-def* **by** *blast*

Observe that the second half can be obtained by creating an appropriately flipped and inverted interpretation of *galois-prop*. Indeed, many concepts in our formalisation are "closed" under inversion, i.e. taking their inversion yields a statement for a related concept. Many theorems can thus be derived for free by inverting (and flipping) the concepts at hand. In such cases, we only state those theorems that require some non-trivial setup. All other theorems can simply be obtained by creating a suitable locale interpretation.

interpretation *flip-inv* : *galois-prop* $(\geq_R)$ $(\geq_L)$ r l **.**

definition *half-galois-prop-right* $\equiv$ *flip-inv.half-galois-prop-left*

notation *galois-prop.half-galois-prop-right* (**infix** $\langle\trianglelefteq_h\rangle$ *50*)

lemma *half-galois-prop-rightI* [*intro*]:
assumes $\bigwedge x\ y.\ x \lessapprox_R y \Longrightarrow x \leq_L r\ y$
shows $((\leq_L)\ \trianglelefteq_h\ (\leq_R))\ l\ r$
unfolding *half-galois-prop-right-def* **using** *assms* **by** *blast*

lemma *half-galois-prop-rightD* [*dest*]:
assumes $((\leq_L) \trianglelefteq_h (\leq_R))\ l\ r$
and $x \lessapprox_R y$
shows $x \leq_L r\ y$
using *assms* **unfolding** *half-galois-prop-right-def* **by** *blast*

interpretation g : *galois-prop* $S\ T\ f\ g$ **for** $S\ T\ f\ g$ **.**

lemma *rel-inv-half-galois-prop-right-eq-half-galois-prop-left-rel-inv* [*simp*]:
$((\leq_R) \trianglelefteq_h (\leq_L))^{-1} = ((\geq_L)\ {}_h\trianglerighteq\ (\geq_R))$
by (*intro ext*) *blast*

corollary *half-galois-prop-left-rel-inv-iff-half-galois-prop-right* [*iff*]:
$((\geq_L)\ {}_h\trianglerighteq\ (\geq_R))\ f\ g \longleftrightarrow ((\leq_R) \trianglelefteq_h (\leq_L))\ g\ f$
by (*simp flip*: *rel-inv-half-galois-prop-right-eq-half-galois-prop-left-rel-inv*)

lemma *rel-inv-half-galois-prop-left-eq-half-galois-prop-right-rel-inv* [*simp*]:
$((\leq_R)\ {}_h\trianglerighteq\ (\leq_L))^{-1} = ((\geq_L) \trianglelefteq_h (\geq_R))$
by (*intro ext*) *blast*

corollary *half-galois-prop-right-rel-inv-iff-half-galois-prop-left* [*iff*]:
$((\geq_L) \trianglelefteq_h (\geq_R))\ f\ g \longleftrightarrow ((\leq_R)\ {}_h\trianglerighteq\ (\leq_L))\ g\ f$
by (*simp flip*: *rel-inv-half-galois-prop-left-eq-half-galois-prop-right-rel-inv*)

end

context *galois*
begin

sublocale *galois-prop* $L\ R\ l\ r$ **.**

interpretation *flip* : *galois* $R\ L\ r\ l$ **.**

abbreviation *right-Galois* ≡ *flip.Galois*
notation *right-Galois* (**infix** $\langle_R\lessapprox\rangle$ *50*)

abbreviation (*input*) *ge-Galois-right* ≡ *flip.ge-Galois-left*
notation *ge-Galois-right* (**infix** $\langle\gtrapprox_R\rangle$ *50*)

abbreviation *left-ge-Galois* ≡ *flip.right-ge-Galois*
notation *left-ge-Galois* (**infix** $\langle_L\gtrapprox\rangle$ *50*)

abbreviation (*input*) *Galois-left* ≡ *flip.Galois-right*
notation *Galois-left* (**infix** $\langle\lessapprox_L\rangle$ *50*)

context
begin

interpretation *flip-inv* : *galois* $(\geq_R)\ (\geq_L)\ r\ l$ **.**

lemma *rel-unit-if-left-rel-if-mono-wrt-relI*:
assumes $((\leq_L) \Rightarrow (\leq_R))\ l$
and $x \lessapprox_R l\ x' \Longrightarrow x \leq_L \eta\ x'$
and $x \leq_L x'$
shows $x \leq_L \eta\ x'$
using *assms* **by** *blast*

corollary *rel-unit-if-left-rel-if-half-galois-prop-right-if-mono-wrt-rel*:
assumes $((\leq_L) \Rightarrow (\leq_R))\ l$
and $((\leq_L) \trianglelefteq_h (\leq_R))\ l\ r$
and $x \leq_L x'$
shows $x \leq_L \eta\ x'$
using *assms* **by** (*fastforce intro*: *rel-unit-if-left-rel-if-mono-wrt-relI*)

corollary *rel-unit-if-reflexive-on-if-half-galois-prop-right-if-mono-wrt-rel*:
assumes $((\leq_L) \Rightarrow (\leq_R))\ l$
and $((\leq_L) \trianglelefteq_h (\leq_R))\ l\ r$
and *reflexive-on* $P\ (\leq_L)$
and $P\ x$
shows $x \leq_L \eta\ x$
using *assms* **by** (*blast intro*: *rel-unit-if-left-rel-if-half-galois-prop-right-if-mono-wrt-rel*)

corollary *inflationary-on-unit-if-reflexive-on-if-half-galois-prop-rightI*:
fixes $P :: 'a \Rightarrow bool$
assumes $((\leq_L) \Rightarrow (\leq_R))\ l$
and $((\leq_L) \trianglelefteq_h (\leq_R))\ l\ r$
and *reflexive-on* $P\ (\leq_L)$
shows *inflationary-on* $P\ (\leq_L)\ \eta$
using *assms* **by** (*intro inflationary-onI*)
(*fastforce intro*: *rel-unit-if-reflexive-on-if-half-galois-prop-right-if-mono-wrt-rel*)

interpretation *flip* : *galois-prop* $R\ L\ r\ l$ **.**

lemma *right-rel-if-Galois-left-right-if-deflationary-onI*:
assumes $((\leq_R) \Rightarrow (\leq_L))\ r$
and $((\leq_R) \trianglelefteq_h (\leq_L))\ r\ l$
and *deflationary-on* $P\ (\leq_R)\ \varepsilon$
and *transitive* $(\leq_R)$
and $y \lessapprox_L r\ y'$
and $P\ y'$
shows $y \leq_R y'$
using *assms* **by** *force*

lemma *half-galois-prop-left-left-right-if-transitive-if-deflationary-on-if-mono-wrt-rel*:
assumes $((\leq_L) \Rightarrow (\leq_R))\ l$
and *deflationary-on* $(\textit{in-codom}\ (\leq_R))\ (\leq_R)\ \varepsilon$
and *transitive* $(\leq_R)$
shows $((\leq_L)\ {}_h\trianglelefteq\ (\leq_R))\ l\ r$

using *assms* **by** (*intro half-galois-prop-leftI*) *fastforce*

end

interpretation *flip-inv* : *galois* $(\geq_R)$ $(\geq_L)$ *r l*
rewrites *flip-inv.unit* $\equiv \varepsilon$ **and** *flip-inv.counit* $\equiv \eta$
and $\bigwedge R\ S.\ (R^{-1} \Rightarrow S^{-1}) \equiv (R \Rightarrow S)$
and $\bigwedge R\ S\ f\ g.\ (R^{-1} \trianglelefteq_h S^{-1})\ f\ g \equiv (S\ {}_h\trianglelefteq\ R)\ g\ f$
and $((\geq_R)\ {}_h\trianglelefteq\ (\geq_L))\ r\ l \equiv ((\leq_L) \trianglelefteq_h (\leq_R))\ l\ r$
and $\bigwedge R.\ R^{-1-1} \equiv R$
and $\bigwedge(P :: 'c \Rightarrow bool)\ (R :: 'c \Rightarrow 'c \Rightarrow bool).$
(*inflationary-on P* $R^{-1} :: ('c \Rightarrow 'c) \Rightarrow bool) \equiv$ *deflationary-on P R*
and $\bigwedge(P :: 'c \Rightarrow bool)\ (R :: 'c \Rightarrow 'c \Rightarrow bool).$
(*deflationary-on P* R^{-1} $:: ('c \Rightarrow 'c) \Rightarrow bool) \equiv$ *inflationary-on P R*
and $\bigwedge(P :: 'b \Rightarrow bool).$ *reflexive-on P* $(\geq_R) \equiv$ *reflexive-on P* $(\leq_R)$
and $\bigwedge(R :: 'a \Rightarrow 'a \Rightarrow bool).$ *transitive* $R^{-1} \equiv$ *transitive R*
and $\bigwedge R.$ *in-codom* $R^{-1} \equiv$ *in-dom R*
by (*simp-all add*: *flip-unit-eq-counit flip-counit-eq-unit*
galois-prop.half-galois-prop-left-rel-inv-iff-half-galois-prop-right
galois-prop.half-galois-prop-right-rel-inv-iff-half-galois-prop-left
mono-wrt-rel-eq-dep-mono-wrt-rel)

corollary *counit-rel-if-right-rel-if-mono-wrt-relI*:
assumes $((\leq_R) \Rightarrow (\leq_L))\ r$
and $r\ y\ {}_L\lessapprox y' \Longrightarrow \varepsilon\ y \leq_R y'$
and $y \leq_R y'$
shows $\varepsilon\ y \leq_R y'$
using *assms*
by (*fact flip-inv.rel-unit-if-left-rel-if-mono-wrt-relI*
[*simplified rel-inv-iff-rel*])

corollary *counit-rel-if-right-rel-if-half-galois-prop-left-if-mono-wrt-rel*:
assumes $((\leq_R) \Rightarrow (\leq_L))\ r$
and $((\leq_L)\ {}_h\trianglelefteq\ (\leq_R))\ l\ r$
and $y \leq_R y'$
shows $\varepsilon\ y \leq_R y'$
using *assms*
by (*fact flip-inv.rel-unit-if-left-rel-if-half-galois-prop-right-if-mono-wrt-rel*
[*simplified rel-inv-iff-rel*])

corollary *counit-rel-if-reflexive-on-if-half-galois-prop-left-if-mono-wrt-rel*:
assumes $((\leq_R) \Rightarrow (\leq_L))\ r$
and $((\leq_L)\ {}_h\trianglelefteq\ (\leq_R))\ l\ r$
and *reflexive-on P* $(\leq_R)$
and *P y*
shows $\varepsilon\ y \leq_R y$
using *assms*
by (*fact flip-inv.rel-unit-if-reflexive-on-if-half-galois-prop-right-if-mono-wrt-rel*
[*simplified rel-inv-iff-rel*])

corollary *deflationary-on-counit-if-reflexive-on-if-half-galois-prop-leftI*:
fixes $P :: 'b \Rightarrow bool$
assumes $((\leq_R) \Rightarrow (\leq_L))\ r$
and $((\leq_L)\ {}_h\trianglelefteq\ (\leq_R))\ l\ r$
and *reflexive-on* $P\ (\leq_R)$
shows *deflationary-on* $P\ (\leq_R)\ \varepsilon$
using *assms*
by (*fact flip-inv.inflationary-on-unit-if-reflexive-on-if-half-galois-prop-rightI*)

corollary *left-rel-if-left-right-Galois-if-inflationary-onI*:
assumes $((\leq_L) \Rightarrow (\leq_R))\ l$
and $((\leq_R)\ {}_h\trianglelefteq\ (\leq_L))\ r\ l$
and *inflationary-on* $P\ (\leq_L)\ \eta$
and *transitive* $(\leq_L)$
and $l\ x\ {}_R\lessapprox\ x'$
and $P\ x$
shows $x \leq_L x'$
using *assms* **by** (*intro flip-inv.right-rel-if-Galois-left-right-if-deflationary-onI*
[*simplified rel-inv-iff-rel*])

corollary *half-galois-prop-right-left-right-if-transitive-if-inflationary-on-if-mono-wrt-rel*:
assumes $((\leq_R) \Rightarrow (\leq_L))\ r$
and *inflationary-on* $(\textit{in-dom}\ (\leq_L))\ (\leq_L)\ \eta$
and *transitive* $(\leq_L)$
shows $((\leq_L) \trianglelefteq_h\ (\leq_R))\ l\ r$
using *assms*
by (*fact flip-inv.half-galois-prop-left-left-right-if-transitive-if-deflationary-on-if-mono-wrt-rel*)

end

context *order-functors*
begin

interpretation *g* : *galois L R l r* **.**
interpretation *flip-g* : *galois R L r l*
rewrites *flip-g.unit* $\equiv \varepsilon$ **and** *flip-g.counit* $\equiv \eta$
by (*simp-all only*: *flip-unit-eq-counit flip-counit-eq-unit*)

lemma *left-rel-if-left-right-rel-left-if-order-equivalenceI*:
assumes $((\leq_L) \equiv_o (\leq_R))\ l\ r$
and *transitive* $(\leq_L)$
and $l\ x \leq_R l\ x'$
and *in-dom* $(\leq_L)\ x$
and *in-codom* $(\leq_L)\ x'$
shows $x \leq_L x'$
using *assms* **by** (*auto intro*!:
flip-g.right-rel-if-Galois-left-right-if-deflationary-onI
g.half-galois-prop-right-left-right-if-transitive-if-inflationary-on-if-mono-wrt-rel

elim!: *rel-equivalence-onE*
intro: *inflationary-on-if-le-pred-if-inflationary-on*
in-field-if-in-dom in-field-if-in-codom)

end

end

1.4.4 Galois Property

theory *Galois-Property*
imports
Half-Galois-Property
begin

context *galois-prop*
begin

definition *galois-prop* $\equiv ((\leq_L)\ {}_h\trianglelefteq\ (\leq_R)) \sqcap ((\leq_L)\ \trianglelefteq_h\ (\leq_R))$

notation *galois-prop.galois-prop* (**infix** $\langle\trianglelefteq\rangle$ *50*)

lemma *galois-propI* [*intro*]:
assumes $((\leq_L)\ {}_h\trianglelefteq\ (\leq_R))\ l\ r$
and $((\leq_L)\ \trianglelefteq_h\ (\leq_R))\ l\ r$
shows $((\leq_L)\ \trianglelefteq\ (\leq_R))\ l\ r$
unfolding *galois-prop-def* **using** *assms* **by** *auto*

lemma *galois-propI'*:
assumes $\bigwedge x\ y.$ *in-dom* $(\leq_L)\ x \Longrightarrow$ *in-codom* $(\leq_R)\ y \Longrightarrow x \leq_L r\ y \longleftrightarrow l\ x \leq_R y$
shows $((\leq_L)\ \trianglelefteq\ (\leq_R))\ l\ r$
using *assms* **by** (*blast elim*: *galois-rel.left-GaloisE*)

lemma *galois-propE* [*elim*]:
assumes $((\leq_L)\ \trianglelefteq\ (\leq_R))\ l\ r$
obtains $((\leq_L)\ {}_h\trianglelefteq\ (\leq_R))\ l\ r$ $((\leq_L)\ \trianglelefteq_h\ (\leq_R))\ l\ r$
using *assms* **unfolding** *galois-prop-def* **by** *auto*

interpretation *g* : *galois-prop S T f g* **for** *S T f g*.

lemma *galois-prop-eq-half-galois-prop-left-rel-inf-half-galois-prop-right*:
$((\leq_L)\ \trianglelefteq\ (\leq_R)) = ((\leq_L)\ {}_h\trianglelefteq\ (\leq_R)) \sqcap ((\leq_L)\ \trianglelefteq_h\ (\leq_R))$
by (*intro ext*) *auto*

lemma *galois-prop-left-rel-right-iff-left-right-rel*:
assumes $((\leq_L)\ \trianglelefteq\ (\leq_R))\ l\ r$
and *in-dom* $(\leq_L)\ x$ *in-codom* $(\leq_R)\ y$

shows $x \leq_L r\ y \longleftrightarrow l\ x \leq_R y$
using *assms* **by** *blast*

lemma *rel-inv-galois-prop-eq-galois-prop-rel-inv* [*simp*]:
$((\leq_R) \trianglelefteq (\leq_L))^{-1} = ((\geq_L) \trianglelefteq (\geq_R))$
by (*intro ext*) *blast*

corollary *galois-prop-rel-inv-iff-galois-prop* [*iff*]:
$((\geq_L) \trianglelefteq (\geq_R))\ f\ g \longleftrightarrow ((\leq_R) \trianglelefteq (\leq_L))\ g\ f$
by *auto*

end

context *galois*
begin

lemma *galois-prop-left-right-if-transitive-if-deflationary-on-if-inflationary-on-if-mono-wrt-rel*:
assumes $((\leq_L) \Rightarrow (\leq_R))\ l$ **and** $((\leq_R) \Rightarrow (\leq_L))\ r$
and *inflationary-on* (*in-dom* $(\leq_L)$) $(\leq_L)\ \eta$
and *deflationary-on* (*in-codom* $(\leq_R)$) $(\leq_R)\ \varepsilon$
and *transitive* $(\leq_L)$ *transitive* $(\leq_R)$
shows $((\leq_L) \trianglelefteq (\leq_R))\ l\ r$
using *assms*
by (*intro galois-propI*
half-galois-prop-left-left-right-if-transitive-if-deflationary-on-if-mono-wrt-rel
half-galois-prop-right-left-right-if-transitive-if-inflationary-on-if-mono-wrt-rel)

end

end

1.4.5 Galois Connections

theory *Galois-Connections*
imports
Galois-Property
begin

context *galois*
begin

definition *galois-connection* $\equiv ((\leq_L) \Rightarrow (\leq_R))\ l \wedge ((\leq_R) \Rightarrow (\leq_L))\ r \wedge ((\leq_L) \trianglelefteq (\leq_R))\ l\ r$

notation *galois.galois-connection* (**infix** $\langle\dashv\rangle$ *50*)

lemma *galois-connectionI* [*intro*]:
assumes $((\leq_L) \Rightarrow (\leq_R))\ l$ **and** $((\leq_R) \Rightarrow (\leq_L))\ r$

and $((\leq_L) \trianglelefteq (\leq_R))\ l\ r$
shows $((\leq_L) \dashv (\leq_R))\ l\ r$
unfolding *galois-connection-def* **using** *assms* **by** *blast*

lemma *galois-connectionE* [*elim*]:
assumes $((\leq_L) \dashv (\leq_R))\ l\ r$
obtains $((\leq_L) \Rightarrow (\leq_R))\ l\ ((\leq_R) \Rightarrow (\leq_L))\ r\ ((\leq_L) \trianglelefteq (\leq_R))\ l\ r$
using *assms* **unfolding** *galois-connection-def* **by** *blast*

context
begin

interpretation g : *galois* $S\ T\ f\ g$ **for** $S\ T\ f\ g$**.**

lemma *rel-inv-galois-connection-eq-galois-connection-rel-inv* [*simp*]:
$((\leq_R) \dashv (\leq_L))^{-1} = ((\geq_L) \dashv (\geq_R))$
by (*intro ext*) *blast*

corollary *galois-connection-rel-inv-iff-galois-connection* [*iff*]:
$((\geq_L) \dashv (\geq_R))\ l\ r \longleftrightarrow ((\leq_R) \dashv (\leq_L))\ r\ l$
by (*simp flip*: *rel-inv-galois-connection-eq-galois-connection-rel-inv*)

lemma *rel-unit-if-left-rel-if-galois-connection*:
assumes $((\leq_L) \dashv (\leq_R))\ l\ r$
and $x \leq_L x'$
shows $x \leq_L \eta\ x'$
using *assms*
by (*blast intro*: *rel-unit-if-left-rel-if-half-galois-prop-right-if-mono-wrt-rel*)

end

lemma *counit-rel-if-right-rel-if-galois-connection*:
assumes $((\leq_L) \dashv (\leq_R))\ l\ r$
and $y \leq_R y'$
shows $\varepsilon\ y \leq_R y'$
using *assms*
by (*blast intro*: *counit-rel-if-right-rel-if-half-galois-prop-left-if-mono-wrt-rel*)

lemma *rel-unit-if-reflexive-on-if-galois-connection*:
assumes $((\leq_L) \dashv (\leq_R))\ l\ r$
and *reflexive-on* $P\ (\leq_L)$
and $P\ x$
shows $x \leq_L \eta\ x$
using *assms*
by (*blast intro*: *rel-unit-if-reflexive-on-if-half-galois-prop-right-if-mono-wrt-rel*)

lemma *counit-rel-if-reflexive-on-if-galois-connection*:
assumes $((\leq_L) \dashv (\leq_R))\ l\ r$
and *reflexive-on* $P\ (\leq_R)$

and $P\ y$
shows $\varepsilon\ y \leq_R y$
using *assms*
by (*blast intro*: *counit-rel-if-reflexive-on-if-half-galois-prop-left-if-mono-wrt-rel*)

lemma *inflationary-on-unit-if-reflexive-on-if-galois-connection*:
fixes $P :: 'a \Rightarrow bool$
assumes $((\leq_L) \dashv (\leq_R))\ l\ r$
and *reflexive-on* $P\ (\leq_L)$
shows *inflationary-on* $P\ (\leq_L)\ \eta$
using *assms*
by (*blast intro*: *inflationary-on-unit-if-reflexive-on-if-half-galois-prop-rightI*)

lemma *deflationary-on-counit-if-reflexive-on-if-galois-connection*:
fixes $P :: 'b \Rightarrow bool$
assumes $((\leq_L) \dashv (\leq_R))\ l\ r$
and *reflexive-on* $P\ (\leq_R)$
shows *deflationary-on* $P\ (\leq_R)\ \varepsilon$
using *assms*
by (*blast intro*: *deflationary-on-counit-if-reflexive-on-if-half-galois-prop-leftI*)

end

end

1.4.6 Galois Equivalences

theory *Galois-Equivalences*
imports
Galois-Connections
Order-Equivalences
Partial-Equivalence-Relations
begin

context *galois*
begin

In the literature, an adjoint equivalence is an adjunction for which the unit and counit are natural isomorphisms. Translated to the category of orders, this means that a *Galois equivalence* between two relations $(\leq_L)$ and $(\leq_R)$ is a Galois connection for which the unit η is *deflationary* and the counit ε is *inflationary*.

For reasons of symmetry, we give a different definition which next to *galois-connection* requires *galois-prop l r*. In other words, a Galois equivalence is a Galois connection for which the left and right adjoints are also right and left adjoints, respectively. As shown below, in the case of preorders, the definitions coincide.

definition *galois-equivalence* $\equiv ((\leq_L) \dashv (\leq_R))\ l\ r \wedge ((\leq_R) \trianglelefteq (\leq_L))\ r\ l$

notation *galois.galois-equivalence* (**infix** $\langle\equiv_G\rangle$ *50*)

lemma *galois-equivalenceI* [*intro*]:
assumes $((\leq_L) \dashv (\leq_R))\ l\ r$
and $((\leq_R) \trianglelefteq (\leq_L))\ r\ l$
shows $((\leq_L) \equiv_G (\leq_R))\ l\ r$
unfolding *galois-equivalence-def* **using** *assms* **by** *blast*

lemma *galois-equivalenceE* [*elim*]:
assumes $((\leq_L) \equiv_G (\leq_R))\ l\ r$
obtains $((\leq_L) \dashv (\leq_R))\ l\ r$ $((\leq_R) \dashv (\leq_L))\ r\ l$
using *assms* **unfolding** *galois-equivalence-def*
by (*blast intro*: *galois.galois-connectionI*)

context
begin

interpretation g : *galois* $S\ T\ f\ g$ **for** $S\ T\ f\ g$**.**

lemma *galois-equivalence-eq-galois-connection-rel-inf-galois-prop*:
$((\leq_L) \equiv_G (\leq_R)) = ((\leq_L) \dashv (\leq_R)) \sqcap ((\geq_L) \trianglelefteq (\geq_R))$
by (*intro ext*) *auto*

lemma *rel-inv-galois-equivalence-eq-galois-equivalence* [*simp*]:
$((\leq_R) \equiv_G (\leq_L))^{-1} = ((\leq_L) \equiv_G (\leq_R))$
by (*intro ext*) *auto*

corollary *galois-equivalence-right-left-iff-galois-equivalence-left-right*:
$((\leq_R) \equiv_G (\leq_L))\ r\ l \longleftrightarrow ((\leq_L) \equiv_G (\leq_R))\ l\ r$
by *auto*

lemma *galois-equivalence-rel-inv-eq-galois-equivalence* [*simp*]:
$((\geq_L) \equiv_G (\geq_R)) = ((\leq_L) \equiv_G (\leq_R))$
by (*intro ext*) *auto*

lemma *inflationary-on-unit-if-reflexive-on-if-galois-equivalence*:
fixes $P :: 'a \Rightarrow bool$
assumes $((\leq_L) \equiv_G (\leq_R))\ l\ r$
and *reflexive-on* $P\ (\leq_L)$
shows *inflationary-on* $P\ (\leq_L)\ \eta$
using *assms* **by** (*intro inflationary-on-unit-if-reflexive-on-if-galois-connection*)
(*elim galois-equivalenceE*)

end

lemma *deflationary-on-unit-if-reflexive-on-if-galois-equivalence*:
fixes $P :: 'a \Rightarrow bool$

assumes $((\leq_L) \equiv_G (\leq_R))$ *l r*
and *reflexive-on P* $(\leq_L)$
shows *deflationary-on P* $(\leq_L)$ η
proof −
interpret *flip* : *galois R L r l* **.**
show *?thesis* **using** *assms*
by (*auto intro*: *flip.deflationary-on-counit-if-reflexive-on-if-galois-connection*
simp only: *flip.flip-unit-eq-counit*)
qed

Every *galois-equivalence* on reflexive orders is a Galois equivalence in the sense of the common literature.

lemma *rel-equivalence-on-unit-if-reflexive-on-if-galois-equivalence*:
fixes $P :: 'a \Rightarrow bool$
assumes $((\leq_L) \equiv_G (\leq_R))$ *l r*
and *reflexive-on P* $(\leq_L)$
shows *rel-equivalence-on P* $(\leq_L)$ η
using *assms* **by** (*intro rel-equivalence-onI*
inflationary-on-unit-if-reflexive-on-if-galois-equivalence
deflationary-on-unit-if-reflexive-on-if-galois-equivalence)

lemma *galois-equivalence-partial-equivalence-rel-not-reflexive-not-transitive*:
assumes $\exists (y :: 'b)\ y'.\ y \neq y'$
shows $\exists (L :: 'a \Rightarrow 'a \Rightarrow bool)\ (R :: 'b \Rightarrow 'b \Rightarrow bool)\ l\ r.$
$(L \equiv_G R)\ l\ r \wedge$ *partial-equivalence-rel L* $\wedge$
$\neg$(*reflexive-on* (*in-field R*) *R*) $\wedge$ $\neg$(*transitive-on* (*in-field R*) *R*)
proof −
from *assms* **obtain** *cy cy'* **where** $(cy :: 'b) \neq cy'$ **by** *blast*
let *?cx* = *undefined* :: *'a*
let *?L* = $\lambda x\ x'.$ *?cx* = $x \wedge x = x'$
and *?R* = $\lambda y\ y'.\ (y = cy \vee y = cy') \wedge (y' = cy \vee y' = cy') \wedge (y \neq cy' \vee y' \neq cy')$
and *?l* = $\lambda(a :: 'a).$ *cy*
and *?r* = $\lambda(b :: 'b).$ *?cx*
interpret *g* : *galois ?L ?R ?l ?r* **.**
interpret *flip-g* : *galois ?R ?L ?r ?l* **.**
have (*?L* $\equiv_G$ *?R*) *?l ?r* **using** ‹$cy \neq cy'$› **by** *blast*
moreover have *partial-equivalence-rel ?L* **by** *blast*
moreover have
$\neg$(*transitive-on* (*in-field ?R*) *?R*) **and** $\neg$(*reflexive-on* (*in-field ?R*) *?R*)
using ‹$cy \neq cy'$› **by** *auto*
ultimately show *?thesis* **by** *blast*
qed

1.4.7 Equivalence of Order Equivalences and Galois Equivalences

In general categories, every adjoint equivalence is an equivalence but not vice versa. In the category of preorders, however, they are morally the same: the

adjoint zigzag equations are satisfied up to unique isomorphism rather than equality. In the category of partial orders, the concepts coincide.

lemma *half-galois-prop-left-left-right-if-transitive-if-order-equivalence*:
assumes $((\leq_L) \equiv_o (\leq_R))$ *l r*
and *transitive* $(\leq_L)$ *transitive* $(\leq_R)$
shows $((\leq_L)\ {}_h\trianglelefteq\ (\leq_R))$ *l r*
using *assms*
by (*intro half-galois-prop-left-left-right-if-transitive-if-deflationary-on-if-mono-wrt-rel*)
(*auto elim*!: *order-equivalenceE*
intro: *deflationary-on-if-le-pred-if-deflationary-on in-field-if-in-codom*
intro!: *le-predI*)

lemma *half-galois-prop-right-left-right-if-transitive-if-order-equivalence*:
assumes $((\leq_L) \equiv_o (\leq_R))$ *l r*
and *transitive* $(\leq_L)$ *transitive* $(\leq_R)$
shows $((\leq_L) \trianglelefteq_h (\leq_R))$ *l r*
using *assms*
by (*intro half-galois-prop-right-left-right-if-transitive-if-inflationary-on-if-mono-wrt-rel*)
(*auto elim*!: *order-equivalenceE*
intro: *inflationary-on-if-le-pred-if-inflationary-on in-field-if-in-dom*
intro!: *le-predI*
simp only: *flip-counit-eq-unit*)

lemma *galois-prop-left-right-if-transitive-if-order-equivalence*:
assumes $((\leq_L) \equiv_o (\leq_R))$ *l r*
and *transitive* $(\leq_L)$ *transitive* $(\leq_R)$
shows $((\leq_L) \trianglelefteq (\leq_R))$ *l r*
using *assms*
half-galois-prop-left-left-right-if-transitive-if-order-equivalence
half-galois-prop-right-left-right-if-transitive-if-order-equivalence
by *blast*

corollary *galois-connection-left-right-if-transitive-if-order-equivalence*:
assumes $((\leq_L) \equiv_o (\leq_R))$ *l r*
and *transitive* $(\leq_L)$ *transitive* $(\leq_R)$
shows $((\leq_L) \dashv (\leq_R))$ *l r*
using *assms galois-prop-left-right-if-transitive-if-order-equivalence*
by (*intro galois-connectionI*) *auto*

interpretation *flip* : *galois R L r l*
rewrites *flip.unit* $\equiv \varepsilon$
by (*simp only*: *flip-unit-eq-counit*)

corollary *galois-equivalence-left-right-if-transitive-if-order-equivalence*:
assumes $((\leq_L) \equiv_o (\leq_R))$ *l r*
and *transitive* $(\leq_L)$ *transitive* $(\leq_R)$
shows $((\leq_L) \equiv_G (\leq_R))$ *l r*
using *assms galois-connection-left-right-if-transitive-if-order-equivalence*
flip.galois-prop-left-right-if-transitive-if-order-equivalence

by (*intro galois-equivalenceI*)
(*auto simp only*: *order-equivalence-right-left-iff-order-equivalence-left-right*)

lemma *order-equivalence-if-reflexive-on-in-field-if-galois-equivalence*:
assumes $((\leq_L) \equiv_G (\leq_R))$ *l r*
and *reflexive-on* (*in-field* $(\leq_L)$) $(\leq_L)$ *reflexive-on* (*in-field* $(\leq_R)$) $(\leq_R)$
shows $((\leq_L) \equiv_o (\leq_R))$ *l r*
using *assms rel-equivalence-on-unit-if-reflexive-on-if-galois-equivalence*
flip.rel-equivalence-on-unit-if-reflexive-on-if-galois-equivalence
by (*intro order-equivalenceI*)
(*auto simp only*: *galois-equivalence-right-left-iff-galois-equivalence-left-right*)

corollary *galois-equivalence-eq-order-equivalence-if-preorder-on-in-field*:
assumes *preorder-on* (*in-field* $(\leq_L)$) $(\leq_L)$ *preorder-on* (*in-field* $(\leq_R)$) $(\leq_R)$
shows $((\leq_L) \equiv_G (\leq_R)) = ((\leq_L) \equiv_o (\leq_R))$
using *assms*
galois.order-equivalence-if-reflexive-on-in-field-if-galois-equivalence
galois.galois-equivalence-left-right-if-transitive-if-order-equivalence
by (*elim preorder-on-in-fieldE*, *intro ext*) *blast*

end

end

1.4.8 Relator For Galois Connections

theory *Galois-Relator*
imports
Galois-Relator-Base
Galois-Property
begin

context *galois-prop*
begin

interpretation *flip-inv* : *galois-rel* $(\geq_R)$ $(\geq_L)$ *l* **.**

lemma *left-Galois-if-Galois-right-if-half-galois-prop-right*:
assumes $((\leq_L) \trianglelefteq_h (\leq_R))$ *l r*
and $x \lessapprox_R y$
shows $x \ {}_L\lessapprox y$
using *assms* **by** (*intro left-GaloisI*) *auto*

lemma *Galois-right-if-left-Galois-if-half-galois-prop-left*:
assumes $((\leq_L) \ {}_h\trianglelefteq (\leq_R))$ *l r*
and $x \ {}_L\lessapprox y$
shows $x \lessapprox_R y$
using *assms* **by** *fast*

corollary *Galois-right-iff-left-Galois-if-galois-prop* [*iff*]:
assumes $((\leq_L) \trianglelefteq (\leq_R))$ l r
shows $x \lessapprox_R y \longleftrightarrow x \ {}_L\lessapprox y$
using *assms*
left-Galois-if-Galois-right-if-half-galois-prop-right
Galois-right-if-left-Galois-if-half-galois-prop-left
by *blast*

lemma *rel-inv-Galois-eq-flip-Galois-rel-inv-if-galois-prop* [*simp*]:
assumes $((\leq_L) \trianglelefteq (\leq_R))$ l r
shows $(\gtrsim_L) = ({}_R\gtrsim)$
using *assms* **by** *blast*

corollary *flip-Galois-rel-inv-iff-Galois-if-galois-prop* [*iff*]:
assumes $((\leq_L) \trianglelefteq (\leq_R))$ l r
shows $y \ {}_R\gtrsim x \longleftrightarrow x \ {}_L\lessapprox y$
using *assms* **by** *blast*

corollary *inv-flip-Galois-rel-inv-eq-Galois-if-galois-prop* [*simp*]:
assumes $((\leq_L) \trianglelefteq (\leq_R))$ l r
shows $(\lessapprox_R) = ({}_L\lessapprox)$ — Note that $(galois\text{-}rel.Galois\ (\leq_R)^{-1}\ (\leq_L)^{-1}\ l)^{-1} = (galois\text{-}rel.Galois\ (\leq_R)^{-1}\ (\leq_L)^{-1}\ l)^{-1}$
using *assms* **by** (*subst rel-inv-eq-iff-eq*[*symmetric*]) *simp*

end

context *galois*
begin

interpretation *flip-inv* : *galois* $(\geq_R)$ $(\geq_L)$ r l **.**

context
begin

interpretation *flip* : *galois* R L r l **.**

lemma *left-Galois-left-if-left-relI*:
assumes $((\leq_L) \Rightarrow (\leq_R))$ l
and $((\leq_L) \trianglelefteq_h (\leq_R))$ l r
and $x \leq_L x'$
shows $x \ {}_L\lessapprox l\ x'$
using *assms*
by (*intro left-Galois-if-Galois-right-if-half-galois-prop-right*) (*auto 5 0*)

corollary *left-Galois-left-if-reflexive-on-if-half-galois-prop-rightI*:
assumes $((\leq_L) \Rightarrow (\leq_R))$ l
and $((\leq_L) \trianglelefteq_h (\leq_R))$ l r
and *reflexive-on* P $(\leq_L)$

and $P\ x$
shows $x\ {}_L\lessapprox\ l\ x$
using *assms* **by** (*intro left-Galois-left-if-left-relI*) *auto*

lemma *left-Galois-left-if-in-codom-if-inflationary-onI*:
assumes $((\leq_L) \Rightarrow (\leq_R))\ l$
and *inflationary-on* $P\ (\leq_L)\ \eta$
and *in-codom* $(\leq_L)\ x$
and $P\ x$
shows $x\ {}_L\lessapprox\ l\ x$
using *assms* **by** (*intro left-GaloisI*) (*auto elim*!: *in-codomE*)

lemma *left-Galois-left-if-in-codom-if-inflationary-on-in-codomI*:
assumes $((\leq_L) \Rightarrow (\leq_R))\ l$
and *inflationary-on* (*in-codom* $(\leq_L)$) $(\leq_L)\ \eta$
and *in-codom* $(\leq_L)\ x$
shows $x\ {}_L\lessapprox\ l\ x$
using *assms* **by** (*auto intro*!: *left-Galois-left-if-in-codom-if-inflationary-onI*)

lemma *left-Galois-left-if-left-rel-if-inflationary-on-in-fieldI*:
assumes $((\leq_L) \Rightarrow (\leq_R))\ l$
and *inflationary-on* (*in-field* $(\leq_L)$) $(\leq_L)\ \eta$
and $x \leq_L x$
shows $x\ {}_L\lessapprox\ l\ x$
using *assms* **by** (*auto intro*!: *left-Galois-left-if-in-codom-if-inflationary-onI*)

lemma *right-left-Galois-if-right-relI*:
assumes $((\leq_R) \Rightarrow (\leq_L))\ r$
and $y \leq_R y'$
shows $r\ y\ {}_L\lessapprox\ y'$
using *assms* **by** (*intro left-GaloisI*) *auto*

corollary *right-left-Galois-if-reflexive-onI*:
assumes $((\leq_R) \Rightarrow (\leq_L))\ r$
and *reflexive-on* $P\ (\leq_R)$
and $P\ y$
shows $r\ y\ {}_L\lessapprox\ y$
using *assms* **by** (*intro right-left-Galois-if-right-relI*) *auto*

lemma *left-Galois-if-right-rel-if-left-GaloisI*:
assumes $((\leq_R) \Rightarrow (\leq_L))\ r$
and *transitive* $(\leq_L)$
and $x\ {}_L\lessapprox\ y$
and $y \leq_R z$
shows $x\ {}_L\lessapprox\ z$
using *assms* **by** (*intro left-GaloisI*) (*auto 5 0*)

lemma *left-Galois-if-left-Galois-if-left-relI*:
assumes *transitive* $(\leq_L)$

and $x \leq_L y$
and $y \ {}_L\lessapprox z$
shows $x \ {}_L\lessapprox z$
using *assms* **by** (*intro left-GaloisI*) *auto*

lemma *left-rel-if-right-Galois-if-left-GaloisI*:
assumes $((\leq_R) \ {}_h\trianglelefteq \ (\leq_L))$ *r l*
and *transitive* $(\leq_L)$
and $x \ {}_L\lessapprox y$
and $y \ {}_R\lessapprox z$
shows $x \leq_L z$
using *assms* **by** *auto*

lemma *Dep-Fun-Rel-left-Galois-right-Galois-if-mono-wrt-rel* [*intro*]:
assumes $((\leq_L) \Rightarrow (\leq_R))$ *l*
shows $(({}_L\lessapprox) \Rrightarrow ({}_R\lessapprox))$ *l r*
using *assms* **by** *blast*

lemma *left-ge-Galois-eq-left-Galois-if-in-codom-eq-in-dom-if-symmetric*:
assumes *symmetric* $(\leq_L)$
and *in-codom* $(\leq_R)$ = *in-dom* $(\leq_R)$
shows $({}_L\gtrapprox) = ({}_L\lessapprox)$ — Note that *galois-rel.Galois* $(\leq_L)^{-1}$ $(\leq_R)^{-1}$ *r* = *galois-rel.Galois* $(\leq_L)^{-1}$ $(\leq_R)^{-1}$ *r*
using *assms* **by** (*intro ext iffI*)
(*auto elim*!: *galois-rel.left-GaloisE intro*!: *galois-rel.left-GaloisI*)

end

interpretation *flip* : *galois R L r l* .

lemma *ge-Galois-right-eq-left-Galois-if-symmetric-if-in-codom-eq-in-dom-if-galois-prop*:
assumes $((\leq_L) \trianglelefteq (\leq_R))$ *l r*
and *in-codom* $(\leq_L)$ = *in-dom* $(\leq_L)$
and *symmetric* $(\leq_R)$
shows $(\gtrapprox_R) = ({}_L\lessapprox)$ — Note that *flip.left-Galois*$^{-1}$ = *flip.left-Galois*$^{-1}$
using *assms*
by (*simp only*: *inv-flip-Galois-rel-inv-eq-Galois-if-galois-prop*
flip: *flip.left-ge-Galois-eq-left-Galois-if-in-codom-eq-in-dom-if-symmetric*)

interpretation *gp* : *galois-prop* $({}_L\lessapprox)$ $({}_R\lessapprox)$ *l l* .

lemma *half-galois-prop-left-left-Galois-right-Galois-if-half-galois-prop-leftI* [*intro*]:
assumes $((\leq_L) \ {}_h\trianglelefteq \ (\leq_R))$ *l r*
shows $(({}_L\lessapprox) \ {}_h\trianglelefteq \ ({}_R\lessapprox))$ *l l*
using *assms* **by** *fast*

lemma *half-galois-prop-right-left-Galois-right-Galois-if-half-galois-prop-rightI* [*intro*]:
assumes $((\leq_L) \trianglelefteq_h (\leq_R))$ *l r*
shows $(({}_L\lessapprox) \trianglelefteq_h ({}_R\lessapprox))$ *l l*

using *assms* **by** *fast*

corollary *galois-prop-left-Galois-right-Galois-if-galois-prop* [*intro*]:
assumes $((\leq_L) \trianglelefteq (\leq_R))$ *l r*
shows $((_L\lesssim) \trianglelefteq (_R\lesssim))$ *l l*
using *assms* **by** *blast*

end

end

Chapter 2

Transport

2.1 Basic Setup

theory *Transport-Base*
  imports
    Galois-Equivalences
    Galois-Relator
begin

Summary Basic setup for commonly used concepts in Transport, including a suitable locale.

locale *transport* = *galois* L R l r
  for $L :: 'a \Rightarrow 'a \Rightarrow bool$
  and $R :: 'b \Rightarrow 'b \Rightarrow bool$
  and $l :: 'a \Rightarrow 'b$
  and $r :: 'b \Rightarrow 'a$
begin

2.1.1 Ordered Galois Connections

definition *preorder-galois-connection* $\equiv$
  $((\leq_L) \dashv (\leq_R))\ l\ r$
  $\wedge$ *preorder-on* (*in-field* $(\leq_L)$) $(\leq_L)$
  $\wedge$ *preorder-on* (*in-field* $(\leq_R)$) $(\leq_R)$

end

bundle *transport-preorder-galois-connection-syntax*
begin
notation *transport.preorder-galois-connection* (**infix** $\langle \dashv_{pre} \rangle$ *50*)
end

context *transport*
begin

unbundle *transport-preorder-galois-connection-syntax*

lemma *preorder-galois-connectionI* [*intro*]:
assumes $((\leq_L) \dashv (\leq_R))\ l\ r$
and *preorder-on* (*in-field* $(\leq_L)$) $(\leq_L)$
and *preorder-on* (*in-field* $(\leq_R)$) $(\leq_R)$
shows $((\leq_L) \dashv_{pre} (\leq_R))\ l\ r$
unfolding *preorder-galois-connection-def* **using** *assms* **by** *blast*

lemma *preorder-galois-connectionE* [*elim*]:
assumes $((\leq_L) \dashv_{pre} (\leq_R))\ l\ r$
obtains $((\leq_L) \dashv (\leq_R))\ l\ r$ *preorder-on* (*in-field* $(\leq_L)$) $(\leq_L)$
preorder-on (*in-field* $(\leq_R)$) $(\leq_R)$
using *assms* **unfolding** *preorder-galois-connection-def* **by** *blast*

context
begin

interpretation t : *transport* $S\ T\ f\ g$ **for** $S\ T\ f\ g$ **.**

lemma *rel-inv-preorder-galois-connection-eq-preorder-galois-connection-rel-inv* [*simp*]:
$((\leq_R) \dashv_{pre} (\leq_L))^{-1} = ((\geq_L) \dashv_{pre} (\geq_R))$
by (*intro ext*) (*auto intro*!: *t.preorder-galois-connectionI*)

end

corollary *preorder-galois-connection-rel-inv-iff-preorder-galois-connection* [*iff*]:
$((\geq_L) \dashv_{pre} (\geq_R))\ l\ r \longleftrightarrow ((\leq_R) \dashv_{pre} (\leq_L))\ r\ l$
by (*simp flip*:
rel-inv-preorder-galois-connection-eq-preorder-galois-connection-rel-inv)

definition *partial-equivalence-rel-galois-connection* $\equiv$
$((\leq_L) \dashv (\leq_R))\ l\ r$
$\wedge$ *partial-equivalence-rel* $(\leq_L)$
$\wedge$ *partial-equivalence-rel* $(\leq_R)$

end

bundle *transport-partial-equivalence-rel-galois-connection-syntax*
begin
notation *transport.partial-equivalence-rel-galois-connection* (**infix** $\langle \dashv_{PER} \rangle$ *50*)
end

context *transport*
begin

unbundle *transport-partial-equivalence-rel-galois-connection-syntax*

lemma *partial-equivalence-rel-galois-connectionI* [*intro*]:

assumes $((\leq_L) \dashv (\leq_R))$ *l r*
and *partial-equivalence-rel-on* (*in-field* $(\leq_L)$) $(\leq_L)$
and *partial-equivalence-rel-on* (*in-field* $(\leq_R)$) $(\leq_R)$
shows $((\leq_L) \dashv_{PER} (\leq_R))$ *l r*
unfolding *partial-equivalence-rel-galois-connection-def* **using** *assms* **by** *blast*

lemma *partial-equivalence-rel-galois-connectionE* [*elim*]:
assumes $((\leq_L) \dashv_{PER} (\leq_R))$ *l r*
obtains $((\leq_L) \dashv_{pre} (\leq_R))$ *l r symmetric* $(\leq_L)$ *symmetric* $(\leq_R)$
using *assms* **unfolding** *partial-equivalence-rel-galois-connection-def* **by** *blast*

context
begin

interpretation *t* : *transport S T f g* **for** *S T f g* **.**

lemma *rel-inv-partial-equivalence-rel-galois-connection-eq-partial-equivalence-rel-galois-connection-rel-inv*
[*simp*]: $((\leq_R) \dashv_{PER} (\leq_L))^{-1} = ((\geq_L) \dashv_{PER} (\geq_R))$
by (*intro ext*) *blast*

end

corollary *partial-equivalence-rel-galois-connection-rel-inv-iff-partial-equivalence-rel-galois-connection*
[*iff*]: $((\geq_L) \dashv_{PER} (\geq_R))$ *l r* $\longleftrightarrow$ $((\leq_R) \dashv_{PER} (\leq_L))$ *r l*
by (*simp flip*:
rel-inv-partial-equivalence-rel-galois-connection-eq-partial-equivalence-rel-galois-connection-rel-inv)

lemma *left-Galois-comp-ge-Galois-left-eq-left-if-partial-equivalence-rel-galois-connection*:
assumes $((\leq_L) \dashv_{PER} (\leq_R))$ *l r*
shows $(({}_L\lessapprox) \circ\circ (\gtrapprox_L)) = (\leq_L)$
proof (*intro ext iffI*)
fix *x x'* **assume** $(({}_L\lessapprox) \circ\circ (\gtrapprox_L))$ *x x'*
then obtain *y* **where** $x \leq_L r\ y\ r\ y \geq_L x'$ **by** *blast*
with *assms* **show** $x \leq_L x'$ **by** (*blast dest*: *symmetricD*)
next
fix *x x'* **assume** $x \leq_L x'$
with *assms* **have** $x\ {}_L\lessapprox\ l\ x'\ x'\ {}_L\lessapprox\ l\ x'$
by (*blast intro*: *left-Galois-left-if-left-relI*)+
with *assms* **show** $(({}_L\lessapprox) \circ\circ (\gtrapprox_L))$ *x x'* **by** *auto*
qed

2.1.2 Ordered Equivalences

definition *preorder-equivalence* $\equiv$
$((\leq_L) \equiv_G (\leq_R))$ *l r*
$\wedge$ *preorder-on* (*in-field* $(\leq_L)$) $(\leq_L)$
$\wedge$ *preorder-on* (*in-field* $(\leq_R)$) $(\leq_R)$

end

bundle *transport-preorder-equivalence-syntax*
begin
notation *transport.preorder-equivalence* (**infix** ‹$\equiv_{pre}$› *50*)
end

context *transport*
begin

unbundle *transport-preorder-equivalence-syntax*

lemma *preorder-equivalence-if-galois-equivalenceI* [*intro*]:
assumes $((\leq_L) \equiv_G (\leq_R))$ *l r*
and *preorder-on* (*in-field* $(\leq_L)$) $(\leq_L)$
and *preorder-on* (*in-field* $(\leq_R)$) $(\leq_R)$
shows $((\leq_L) \equiv_{pre} (\leq_R))$ *l r*
unfolding *preorder-equivalence-def* **using** *assms* **by** *blast*

lemma *preorder-equivalence-if-order-equivalenceI*:
assumes $((\leq_L) \equiv_o (\leq_R))$ *l r*
and *transitive* $(\leq_L)$
and *transitive* $(\leq_R)$
shows $((\leq_L) \equiv_{pre} (\leq_R))$ *l r*
unfolding *preorder-equivalence-def* **using** *assms*
by (*blast intro*: *reflexive-on-in-field-if-transitive-if-rel-equivalence-on*
dest: *galois-equivalence-left-right-if-transitive-if-order-equivalence*)

lemma *preorder-equivalence-galois-equivalenceE* [*elim*]:
assumes $((\leq_L) \equiv_{pre} (\leq_R))$ *l r*
obtains $((\leq_L) \equiv_G (\leq_R))$ *l r preorder-on* (*in-field* $(\leq_L)$) $(\leq_L)$
preorder-on (*in-field* $(\leq_R)$) $(\leq_R)$
using *assms* **unfolding** *preorder-equivalence-def* **by** *blast*

lemma *preorder-equivalence-order-equivalenceE*:
assumes $((\leq_L) \equiv_{pre} (\leq_R))$ *l r*
obtains $((\leq_L) \equiv_o (\leq_R))$ *l r preorder-on* (*in-field* $(\leq_L)$) $(\leq_L)$
preorder-on (*in-field* $(\leq_R)$) $(\leq_R)$
using *assms* **by** (*blast intro*:
order-equivalence-if-reflexive-on-in-field-if-galois-equivalence)

context
begin

interpretation *t* : *transport S T f g* **for** *S T f g* **.**

lemma *rel-inv-preorder-equivalence-eq-preorder-equivalence* [*simp*]:
$((\leq_R) \equiv_{pre} (\leq_L))^{-1} = ((\leq_L) \equiv_{pre} (\leq_R))$
by (*intro ext*) *blast*

end

corollary *preorder-equivalence-right-left-iff-preorder-equivalence-left-right*:
$((\leq_R) \equiv_{pre} (\leq_L))\ r\ l \longleftrightarrow ((\leq_L) \equiv_{pre} (\leq_R))\ l\ r$
by (*simp flip*: *rel-inv-preorder-equivalence-eq-preorder-equivalence*)

lemma *preorder-equivalence-rel-inv-eq-preorder-equivalence* [*simp*]:
$((\geq_L) \equiv_{pre} (\geq_R)) = ((\leq_L) \equiv_{pre} (\leq_R))$
by (*intro ext iffI*)
(*auto intro*!: *transport.preorder-equivalence-if-galois-equivalenceI*
elim!: *transport.preorder-equivalence-galois-equivalenceE*)

definition *partial-equivalence-rel-equivalence* $\equiv$
$((\leq_L) \equiv_G (\leq_R))\ l\ r$
$\wedge$ *partial-equivalence-rel* $(\leq_L)$
$\wedge$ *partial-equivalence-rel* $(\leq_R)$

end

bundle *transport-partial-equivalence-rel-equivalence-syntax*
begin
notation *transport.partial-equivalence-rel-equivalence* (**infix** $\langle\equiv_{PER}\rangle$ *50*)
end

context *transport*
begin

unbundle *transport-partial-equivalence-rel-equivalence-syntax*

lemma *partial-equivalence-rel-equivalence-if-galois-equivalenceI* [*intro*]:
assumes $((\leq_L) \equiv_G (\leq_R))\ l\ r$
and *partial-equivalence-rel* $(\leq_L)$
and *partial-equivalence-rel* $(\leq_R)$
shows $((\leq_L) \equiv_{PER} (\leq_R))\ l\ r$
unfolding *partial-equivalence-rel-equivalence-def* **using** *assms* **by** *blast*

lemma *partial-equivalence-rel-equivalence-if-order-equivalenceI*:
assumes $((\leq_L) \equiv_o (\leq_R))\ l\ r$
and *partial-equivalence-rel* $(\leq_L)$
and *partial-equivalence-rel* $(\leq_R)$
shows $((\leq_L) \equiv_{PER} (\leq_R))\ l\ r$
unfolding *partial-equivalence-rel-equivalence-def* **using** *assms*
by (*blast dest*: *galois-equivalence-left-right-if-transitive-if-order-equivalence*)

lemma *partial-equivalence-rel-equivalenceE* [*elim*]:
assumes $((\leq_L) \equiv_{PER} (\leq_R))\ l\ r$
obtains $((\leq_L) \equiv_{pre} (\leq_R))\ l\ r$ *symmetric* $(\leq_L)$ *symmetric* $(\leq_R)$
using *assms* **unfolding** *partial-equivalence-rel-equivalence-def* **by** *blast*

context
begin

interpretation t : *transport* S T f g **for** S T f g **.**

lemma *rel-inv-partial-equivalence-rel-equivalence-eq-partial-equivalence-rel-equivalence*
[*simp*]:
$((\leq_R) \equiv_{PER} (\leq_L))^{-1} = ((\leq_L) \equiv_{PER} (\leq_R))$
by (*intro ext*) *blast*

end

corollary *partial-equivalence-rel-equivalence-right-left-iff-partial-equivalence-rel-equivalence-left-right*:
$((\leq_R) \equiv_{PER} (\leq_L))\ r\ l \longleftrightarrow ((\leq_L) \equiv_{PER} (\leq_R))\ l\ r$
by (*simp flip*:
rel-inv-partial-equivalence-rel-equivalence-eq-partial-equivalence-rel-equivalence)

lemma *partial-equivalence-rel-equivalence-rel-inv-eq-partial-equivalence-rel-equivalence*
[*simp*]:
$((\geq_L) \equiv_{PER} (\geq_R)) = ((\leq_L) \equiv_{PER} (\leq_R))$
by (*intro ext iffI*)
(*auto intro*!: *transport.partial-equivalence-rel-equivalence-if-galois-equivalenceI*
elim!: *transport.partial-equivalence-rel-equivalenceE*
transport.preorder-equivalence-galois-equivalenceE
preorder-on-in-fieldE)

end

bundle *transport-equivalences-syntax*
begin
unbundle *transport-preorder-galois-connection-syntax*
and *transport-partial-equivalence-rel-galois-connection-syntax*
and *transport-partial-equivalence-rel-equivalence-syntax*
and *transport-preorder-equivalence-syntax*
end

end

2.2 Compositions With Agreeing Relations

2.2.1 Basic Setup

theory *Transport-Compositions-Agree-Base*
imports
Transport-Base
begin

locale *transport-comp-agree* =
g1 : *galois L1 R1 l1 r1* + *g2* : *galois L2 R2 l2 r2*

for *L1* :: $'a \Rightarrow 'a \Rightarrow bool$
and *R1* :: $'b \Rightarrow 'b \Rightarrow bool$
and *l1* :: $'a \Rightarrow 'b$
and *r1* :: $'b \Rightarrow 'a$
and *L2* :: $'b \Rightarrow 'b \Rightarrow bool$
and *R2* :: $'c \Rightarrow 'c \Rightarrow bool$
and *l2* :: $'b \Rightarrow 'c$
and *r2* :: $'c \Rightarrow 'b$
begin

This locale collects results about the composition of transportable components under the assumption that the relations *R1* and *L2* agree (in one sense or another) whenever required. Such an agreement may not necessarily hold in practice, and the resulting theorems are not particularly pretty. However, in the special case where $R1 = L2$, most side-conditions disappear and the results are very simple.

notation *L1* (**infix** $\langle \leq_{L1} \rangle$ *50*)
notation *R1* (**infix** $\langle \leq_{R1} \rangle$ *50*)
notation *L2* (**infix** $\langle \leq_{L2} \rangle$ *50*)
notation *R2* (**infix** $\langle \leq_{R2} \rangle$ *50*)

notation *g1.ge-left* (**infix** $\langle \geq_{L1} \rangle$ *50*)
notation *g1.ge-right* (**infix** $\langle \geq_{R1} \rangle$ *50*)
notation *g2.ge-left* (**infix** $\langle \geq_{L2} \rangle$ *50*)
notation *g2.ge-right* (**infix** $\langle \geq_{R2} \rangle$ *50*)

notation *g1.left-Galois* (**infix** $\langle {}_{L1}\lessapprox \rangle$ *50*)
notation *g1.right-Galois* (**infix** $\langle {}_{R1}\lessapprox \rangle$ *50*)
notation *g2.left-Galois* (**infix** $\langle {}_{L2}\lessapprox \rangle$ *50*)
notation *g2.right-Galois* (**infix** $\langle {}_{R2}\lessapprox \rangle$ *50*)

notation *g1.ge-Galois-left* (**infix** $\langle \gtrapprox_{L1} \rangle$ *50*)
notation *g1.ge-Galois-right* (**infix** $\langle \gtrapprox_{R1} \rangle$ *50*)
notation *g2.ge-Galois-left* (**infix** $\langle \gtrapprox_{L2} \rangle$ *50*)
notation *g2.ge-Galois-right* (**infix** $\langle \gtrapprox_{R2} \rangle$ *50*)

notation *g1.right-ge-Galois* (**infix** $\langle {}_{R1}\gtrapprox \rangle$ *50*)
notation *g1.Galois-right* (**infix** $\langle \lessapprox_{R1} \rangle$ *50*)
notation *g2.right-ge-Galois* (**infix** $\langle {}_{R2}\gtrapprox \rangle$ *50*)
notation *g2.Galois-right* (**infix** $\langle \lessapprox_{R2} \rangle$ *50*)

notation *g1.left-ge-Galois* (**infix** $\langle {}_{L1}\gtrapprox \rangle$ *50*)
notation *g1.Galois-left* (**infix** $\langle \lessapprox_{L1} \rangle$ *50*)
notation *g2.left-ge-Galois* (**infix** $\langle {}_{L2}\gtrapprox \rangle$ *50*)
notation *g2.Galois-left* (**infix** $\langle \lessapprox_{L2} \rangle$ *50*)

notation *g1.unit* ($\langle \eta_1 \rangle$)
notation *g1.counit* ($\langle \varepsilon_1 \rangle$)
notation *g2.unit* ($\langle \eta_2 \rangle$)

notation *g2.counit* (‹ε_2›)

abbreviation (*input*) $L \equiv L1$

definition $l \equiv l2 \circ l1$

lemma *left-eq-comp*: $l = l2 \circ l1$
  unfolding *l-def* **..**

lemma *left-eq* [*simp*]: $l\ x = l2\ (l1\ x)$
  unfolding *left-eq-comp* **by** *simp*

context
begin

interpretation *flip* : *transport-comp-agree R2 L2 r2 l2 R1 L1 r1 l1* **.**

abbreviation (*input*) $R \equiv$ *flip.L*
abbreviation $r \equiv$ *flip.l*

lemma *right-eq-comp*: $r = r1 \circ r2$
  unfolding *flip.l-def* **..**

lemma *right-eq* [*simp*]: $r\ z = r1\ (r2\ z)$
  unfolding *right-eq-comp* **by** *simp*

lemmas *transport-defs* = *left-eq-comp right-eq-comp*

end

sublocale *transport L R l r* **.**

notation L (**infix** ‹$\leq_L$› *50*)
notation R (**infix** ‹$\leq_R$› *50*)

end

locale *transport-comp-same* =
  transport-comp-agree L1 R1 l1 r1 R1 R2 l2 r2
  for *L1* :: $'a \Rightarrow 'a \Rightarrow$ *bool*
  and *R1* :: $'b \Rightarrow 'b \Rightarrow$ *bool*
  and *l1* :: $'a \Rightarrow 'b$
  and *r1* :: $'b \Rightarrow 'a$
  and *R2* :: $'c \Rightarrow 'c \Rightarrow$ *bool*
  and *l2* :: $'b \Rightarrow 'c$
  and *r2* :: $'c \Rightarrow 'b$
begin

This locale is a special case of *transport-comp-agree* where the left and

right components both use ($\leq_{R1}$) as their right and left relation, respectively. This is the special case that is most prominent in the literature. The resulting theorems are quite simple, but often not applicable in practice.

end

end

2.2.2 Monotonicity

theory *Transport-Compositions-Agree-Monotone*
imports
Transport-Compositions-Agree-Base
begin

context *transport-comp-agree*
begin

lemma *mono-wrt-rel-leftI*:
assumes $((\leq_{L1}) \Rightarrow (\leq_{R1}))\ l1\ ((\leq_{L2}) \Rightarrow (\leq_{R2}))\ l2$
and $\bigwedge x\ y.\ x \leq_{L1} y \Longrightarrow l1\ x \leq_{R1} l1\ y \Longrightarrow l1\ x \leq_{L2} l1\ y$
shows $((\leq_L) \Rightarrow (\leq_R))\ l$
unfolding *left-eq-comp* **using** *assms* **by** (*urule dep-mono-wrt-rel-compI*)

end

context *transport-comp-same*
begin

lemma *mono-wrt-rel-leftI*:
assumes $((\leq_{L1}) \Rightarrow (\leq_{R1}))\ l1\ ((\leq_{R1}) \Rightarrow (\leq_{R2}))\ l2$
shows $((\leq_L) \Rightarrow (\leq_R))\ l$
using *assms* **by** (*rule mono-wrt-rel-leftI*) *auto*

end

end

2.2.3 Galois Property

theory *Transport-Compositions-Agree-Galois-Property*
imports
Transport-Compositions-Agree-Base
begin

context *transport-comp-agree*
begin

lemma *galois-propI*:
assumes *galois1*: $((\leq_{L1}) \trianglelefteq (\leq_{R1}))$ *l1 r1*
and *galois2*: $((\leq_{L2}) \trianglelefteq (\leq_{R2}))$ *l2 r2*
and *mono-l1*: (*in-dom* $(\leq_{L1}) \Rightarrow$ *in-dom* $(\leq_{L2})$) *l1*
and *mono-r2*: (*in-codom* $(\leq_{R2}) \Rightarrow$ *in-codom* $(\leq_{R1})$) *r2*
and *agree*: (*in-dom* $(\leq_{L1}) \Rrightarrow$ *in-codom* $(\leq_{R2}) \Rrightarrow (\longleftrightarrow)$)
(*rel-bimap l1 r2* $(\leq_{R1})$) (*rel-bimap l1 r2* $(\leq_{L2})$)
shows $((\leq_L) \trianglelefteq (\leq_R))$ *l r*
proof (*rule galois-prop.galois-propI'*)
fix *x y* **assume** *in-dom* $(\leq_L)$ *x in-codom* $(\leq_R)$ *y*
with *mono-r2 mono-l1* **have** *in-dom* $(\leq_{L2})$ (*l1 x*) *in-codom* $(\leq_{R1})$ (*r2 y*) **by** *auto*
have $x \leq_L r\ y \longleftrightarrow x \leq_{L1} r1\ (r2\ y)$ **by** *simp*
also from *galois1* ‹*in-dom* $(\leq_{L1})$ *x*› ‹*in-codom* $(\leq_{R1})$ (*r2 y*)›
have ... $\longleftrightarrow$ *l1 x* $\leq_{R1}$ *r2 y*
by (*rule g1.galois-prop-left-rel-right-iff-left-right-rel*)
also from *agree* ‹*in-dom* $(\leq_{L1})$ *x*› ‹*in-codom* $(\leq_{R2})$ *y*›
have ... $\longleftrightarrow$ *l1 x* $\leq_{L2}$ *r2 y* **by** *fastforce*
also from *galois2* ‹*in-dom* $(\leq_{L2})$ (*l1 x*)› ‹*in-codom* $(\leq_{R2})$ *y*›
have ... $\longleftrightarrow$ *l x* $\leq_{R2}$ *y*
unfolding *l-def*
by (*simp add*: *g2.galois-prop-left-rel-right-iff-left-right-rel*)
finally show $x \leq_L r\ y \longleftrightarrow l\ x \leq_R y$ **.**
qed

end

context *transport-comp-same*
begin

corollary *galois-propI*:
assumes $((\leq_{L1}) \trianglelefteq (\leq_{R1}))$ *l1 r1*
and $((\leq_{R1}) \trianglelefteq (\leq_{R2}))$ *l2 r2*
and (*in-dom* $(\leq_{L1}) \Rightarrow$ *in-dom* $(\leq_{R1})$) *l1*
and (*in-codom* $(\leq_{R2}) \Rightarrow$ *in-codom* $(\leq_{R1})$) *r2*
shows $((\leq_L) \trianglelefteq (\leq_R))$ *l r*
using *assms* **by** (*rule galois-propI*) *auto*

end

end

2.2.4 Galois Connection

theory *Transport-Compositions-Agree-Galois-Connection*
imports
Transport-Compositions-Agree-Monotone
Transport-Compositions-Agree-Galois-Property

begin

context *transport-comp-agree*
begin

interpretation *flip* : *transport-comp-agree R2 L2 r2 l2 R1 L1 r1 l1* .

lemma *galois-connectionI*:
  assumes *galois*: $((\leq_{L1}) \dashv (\leq_{R1}))$ *l1 r1* $((\leq_{L2}) \dashv (\leq_{R2}))$ *l2 r2*
  and *mono-L1-L2-l1*: $\bigwedge x\ y.\ x \leq_{L1} y \Longrightarrow l1\ x \leq_{R1} l1\ y \Longrightarrow l1\ x \leq_{L2} l1\ y$
  and *mono-R2-R1-r2*: $\bigwedge x\ y.\ x \leq_{R2} y \Longrightarrow r2\ x \leq_{L2} r2\ y \Longrightarrow r2\ x \leq_{R1} r2\ y$
   and $(in\text{-}dom\ (\leq_{L1}) \Rrightarrow in\text{-}codom\ (\leq_{R2}) \Rrightarrow (\longleftrightarrow))$ $(rel\text{-}bimap\ l1\ r2\ (\leq_{R1}))$
$(rel\text{-}bimap\ l1\ r2\ (\leq_{L2}))$
  shows $((\leq_L) \dashv (\leq_R))$ *l r*
proof −
  from *galois mono-L1-L2-l1* **have** $(in\text{-}dom\ (\leq_{L1}) \Rightarrow in\text{-}dom\ (\leq_{L2}))$ *l1*
    by (*intro mono-wrt-predI*) (*blast elim*!: *in-domE g1.galois-connectionE*)
  moreover from *galois mono-R2-R1-r2*
    have $(in\text{-}codom\ (\leq_{R2}) \Rightarrow in\text{-}codom\ (\leq_{R1}))$ *r2*
    by (*intro mono-wrt-predI*) (*blast elim*!: *in-codomE g2.galois-connectionE*)
  ultimately show *?thesis* **using** *assms*
    by (*intro galois-connectionI galois-propI mono-wrt-rel-leftI*
      flip.mono-wrt-rel-leftI)
    auto
qed

lemma *galois-connectionI′*:
  assumes $((\leq_{L1}) \dashv (\leq_{R1}))$ *l1 r1* $((\leq_{L2}) \dashv (\leq_{R2}))$ *l2 r2*
  and $((\leq_{L1}) \Rightarrow (\leq_{L2}))$ *l1* $((\leq_{R2}) \Rightarrow (\leq_{R1}))$ *r2*
  and $(in\text{-}dom\ (\leq_{L1}) \Rrightarrow in\text{-}codom\ (\leq_{R2}) \Rrightarrow (\longleftrightarrow))$
    $(rel\text{-}bimap\ l1\ r2\ (\leq_{R1}))$ $(rel\text{-}bimap\ l1\ r2\ (\leq_{L2}))$
  shows $((\leq_L) \dashv (\leq_R))$ *l r*
  using *assms* **by** (*intro galois-connectionI*) *auto*

end

context *transport-comp-same*
begin

corollary *galois-connectionI*:
  assumes $((\leq_{L1}) \dashv (\leq_{R1}))$ *l1 r1* $((\leq_{R1}) \dashv (\leq_{R2}))$ *l2 r2*
  shows $((\leq_L) \dashv (\leq_R))$ *l r*
  using *assms* **by** (*rule galois-connectionI*) *auto*

end

end

2.2.5 Galois Equivalence

theory *Transport-Compositions-Agree-Galois-Equivalence*
imports
Transport-Compositions-Agree-Galois-Connection
begin

context *transport-comp-agree*
begin

interpretation *flip* : *transport-comp-agree R2 L2 r2 l2 R1 L1 r1 l1* **.**

lemma *galois-equivalenceI*:
assumes *galois*: $((\leq_{L1}) \equiv_G (\leq_{R1}))$ *l1 r1* $((\leq_{L2}) \equiv_G (\leq_{R2}))$ *l2 r2*
and *mono-L1-L2-l1*: $\bigwedge x\ y.\ x \leq_{L1} y \Longrightarrow l1\ x \leq_{R1} l1\ y \Longrightarrow l1\ x \leq_{L2} l1\ y$
and *mono-R2-R1-r2*: $\bigwedge x\ y.\ x \leq_{R2} y \Longrightarrow r2\ x \leq_{L2} r2\ y \Longrightarrow r2\ x \leq_{R1} r2\ y$
and $(\textit{in-dom}\ (\leq_{L1}) \Rrightarrow \textit{in-codom}\ (\leq_{R2}) \Rrightarrow (\longleftrightarrow))$
$(\textit{rel-bimap}\ l1\ r2\ (\leq_{R1}))\ (\textit{rel-bimap}\ l1\ r2\ (\leq_{L2}))$
and *mono-iff2*: $(\textit{in-dom}\ (\leq_{R2}) \Rrightarrow \textit{in-codom}\ (\leq_{L1}) \Rrightarrow (\longleftrightarrow))$
$(\textit{rel-bimap}\ r2\ l1\ (\leq_{R1}))\ (\textit{rel-bimap}\ r2\ l1\ (\leq_{L2}))$
shows $((\leq_L) \equiv_G (\leq_R))$ *l r*
proof −
from *galois mono-L1-L2-l1* **have** $(\textit{in-codom}\ (\leq_{L1}) \Rightarrow \textit{in-codom}\ (\leq_{L2}))$ *l1*
by (*intro mono-wrt-predI*) *blast*
moreover from *galois mono-R2-R1-r2* **have** $(\textit{in-dom}\ (\leq_{R2}) \Rightarrow \textit{in-dom}\ (\leq_{R1}))$
r2
by (*intro mono-wrt-predI*) *blast*
moreover from *mono-iff2* **have** $(\textit{in-dom}\ (\leq_{R2}) \Rrightarrow \textit{in-codom}\ (\leq_{L1}) \Rrightarrow (\longleftrightarrow))$
$(\textit{rel-bimap}\ r2\ l1\ (\leq_{L2}))\ (\textit{rel-bimap}\ r2\ l1\ (\leq_{R1}))$ **by** *blast*
ultimately show *?thesis* **using** *assms*
by (*intro galois-equivalenceI galois-connectionI flip.galois-propI*) *auto*
qed

lemma *galois-equivalenceI'*:
assumes $((\leq_{L1}) \equiv_G (\leq_{R1}))$ *l1 r1* $((\leq_{L2}) \equiv_G (\leq_{R2}))$ *l2 r2*
and $((\leq_{L1}) \Rightarrow (\leq_{L2}))$ *l1* $((\leq_{R2}) \Rightarrow (\leq_{R1}))$ *r2*
and $(\textit{in-dom}\ (\leq_{L1}) \Rrightarrow \textit{in-codom}\ (\leq_{R2}) \Rrightarrow (\longleftrightarrow))$
$(\textit{rel-bimap}\ l1\ r2\ (\leq_{R1}))\ (\textit{rel-bimap}\ l1\ r2\ (\leq_{L2}))$
and $(\textit{in-dom}\ (\leq_{R2}) \Rrightarrow \textit{in-codom}\ (\leq_{L1}) \Rrightarrow (\longleftrightarrow))$
$(\textit{rel-bimap}\ r2\ l1\ (\leq_{R1}))\ (\textit{rel-bimap}\ r2\ l1\ (\leq_{L2}))$
shows $((\leq_L) \equiv_G (\leq_R))$ *l r*
using *assms* **by** (*intro galois-equivalenceI*) *auto*

end

context *transport-comp-same*
begin

lemma *galois-equivalenceI*:
assumes $((\leq_{L1}) \equiv_G (\leq_{R1}))$ *l1 r1* $((\leq_{R1}) \equiv_G (\leq_{R2}))$ *l2 r2*

shows $((\leq_L) \equiv_G (\leq_R))\ l\ r$
using *assms* **by** (*rule galois-equivalenceI*) *auto*

end

end

theory *Galois-Equivalent*
imports
Transport-Compositions-Agree-Galois-Equivalence
begin

context
begin

interpretation *galois L R l r* **for** *L R l r* **.**

definition *galois-equivalent L R* $\equiv \exists\, l\ r.\ (L \equiv_G R)\ l\ r$

lemma *galois-equivalentI* [*intro*]:
assumes $(L \equiv_G R)\ l\ r$
shows *galois-equivalent L R*
using *assms* **unfolding** *galois-equivalent-def* **by** *auto*

lemma *galois-equivalentE* [*elim*]:
assumes *galois-equivalent L R*
obtains *l r* **where** $(L \equiv_G R)\ l\ r$
using *assms* **unfolding** *galois-equivalent-def* **by** *auto*

lemma *galois-equivalent-if-galois-equivalent*:
assumes *galois-equivalent L R*
shows *galois-equivalent R L*
using *assms* **by** (*elim galois-equivalentE*) *auto*

lemma *galois-equivalent-trans* [*trans*]:
assumes *galois-equivalent L M*
and *galois-equivalent M R*
shows *galois-equivalent L R*
using *assms* **by** (*elim galois-equivalentE*) (*auto intro*: *transport-comp-same.galois-equivalenceI*)

end

end

theory *Galois*
imports
Galois-Equivalences
Galois-Equivalent

Galois-Relator
begin

Summary We define the concept of (partial) Galois connections, Galois equivalences, and the Galois relator. For details refer to [2].

end

2.2.6 Linear Orders

theory *Linear-Orders*
imports
Binary-Relations-Connected
Partial-Orders
begin

definition *linear-order-on* ≡ *partial-order-on* ⊓ *connected-on*

lemma *linear-order-onI* [*intro*]:
assumes *partial-order-on P R*
and *connected-on P R*
shows *linear-order-on P R*
using *assms* **unfolding** *linear-order-on-def* **by** *auto*

lemma *linear-order-onE* [*elim*]:
assumes *linear-order-on P R*
obtains *partial-order-on P R connected-on P R*
using *assms* **unfolding** *linear-order-on-def* **by** *auto*

consts *linear-order* :: *'a* ⇒ *bool*

overloading
linear-order ≡ *linear-order* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*
begin
definition (*linear-order* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*) ≡ *linear-order-on* (⊤ :: *'a* ⇒ *bool*)
end

lemma *linear-order-eq-linear-order-on*:
(*linear-order* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*) = *linear-order-on* (⊤ :: *'a* ⇒ *bool*)
unfolding *linear-order-def* **..**

lemma *linear-order-eq-linear-order-on-uhint* [*uhint*]:
assumes *P* ≡ ⊤ :: *'a* ⇒ *bool*
shows (*linear-order* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*) ≡ *linear-order-on P*
using *assms* **by** (*simp add*: *linear-order-eq-linear-order-on*)

context
fixes *R* :: *'a* ⇒ *'a* ⇒ *bool*
begin

lemma *linear-orderI* [*intro*]:
assumes *partial-order R*
and *connected R*
shows *linear-order R*
using *assms* **by** (*urule linear-order-onI*)

lemma *linear-orderE* [*elim*]:
assumes *linear-order R*
obtains *partial-order R connected R*
using *assms* **by** (*urule* (*e*) *linear-order-onE*)

lemma *linear-order-on-if-linear-order*:
fixes *P* :: $'a \Rightarrow$ *bool*
assumes *linear-order R*
shows *linear-order-on P R*
using *assms* **by** (*elim linear-orderE*)
(*intro linear-order-onI partial-order-on-if-partial-order connected-on-if-connected*)

end

end

Closure Operators

theory *Closure-Operators*
imports
Order-Functions-Base
begin

consts *idempotent-on* :: $'a \Rightarrow 'b \Rightarrow 'c \Rightarrow$ *bool*

overloading
idempotent-on-pred $\equiv$ *idempotent-on* :: $('a \Rightarrow$ *bool*$) \Rightarrow ('a \Rightarrow 'a \Rightarrow$ *bool*$) \Rightarrow ('a \Rightarrow 'a) \Rightarrow$ *bool*
begin
definition *idempotent-on-pred* (*P* :: $'a \Rightarrow$ *bool*$)$ (*R* :: $'a \Rightarrow 'a \Rightarrow$ *bool*$)$ (*f* :: $'a \Rightarrow 'a$) $\equiv$
rel-equivalence-on P (*rel-map f R*) *f*
end

context
fixes *P* :: $'a \Rightarrow$ *bool* **and** *R* :: $'a \Rightarrow 'a \Rightarrow$ *bool* **and** *f* :: $'a \Rightarrow 'a$
begin

lemma *idempotent-onI* [*intro*]:
assumes $\bigwedge x.\ P\ x \Longrightarrow f\ x \equiv_R f\ (f\ x)$
shows *idempotent-on P R f*
unfolding *idempotent-on-pred-def* **using** *assms* **by** *fastforce*

lemma *idempotent-onE* [*elim*]:
assumes *idempotent-on P R f*
and *P x*
obtains $f\ x \equiv_R f\ (f\ x)$
using *assms* **unfolding** *idempotent-on-pred-def* **by** *fastforce*

lemma *rel-equivalence-on-rel-map-iff-idempotent-on* [*iff*]:
rel-equivalence-on P (*rel-map f R*) *f* $\longleftrightarrow$ *idempotent-on P R f*
unfolding *idempotent-on-pred-def* **by** *simp*

lemma *idempotent-onD*:
assumes *idempotent-on P R f*
and *P x*
shows $f\ x \equiv_R f\ (f\ x)$
using *assms* **by** *blast*

end

consts *idempotent* :: $'a \Rightarrow 'b \Rightarrow$ *bool*

overloading
idempotent $\equiv$ *idempotent* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a) \Rightarrow bool$
begin
definition (*idempotent* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a) \Rightarrow bool) \equiv$ *idempotent-on* ($\top$ :: $'a \Rightarrow bool$)
end

lemma *idempotent-eq-idempotent-on*:
(*idempotent* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a) \Rightarrow bool) =$ *idempotent-on* ($\top$ :: $'a \Rightarrow bool$)
unfolding *idempotent-def* **..**

lemma *idempotent-eq-idempotent-on-uhint* [*uhint*]:
assumes $P \equiv (\top :: 'a \Rightarrow bool)$
shows *idempotent* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a) \Rightarrow bool \equiv$ *idempotent-on P*
using *assms* **by** (*simp add*: *idempotent-eq-idempotent-on*)

context
fixes *P* :: $'a \Rightarrow bool$ **and** *R* :: $'a \Rightarrow 'a \Rightarrow bool$ **and** *f* :: $'a \Rightarrow 'a$
begin

lemma *idempotentI* [*intro*]:
assumes $\bigwedge x.\ f\ x \equiv_R f\ (f\ x)$
shows *idempotent R f*
using *assms* **by** (*urule idempotent-onI*)

lemma *idempotentD* [*dest*]:
assumes *idempotent R f*

shows $f\ x \equiv_R f\ (f\ x)$
using *assms* **by** (*urule* (*e*) *idempotent-onE chained*: *insert*) *simp*

lemma *idempotent-on-if-idempotent*:
assumes *idempotent R f*
shows *idempotent-on P R f*
using *assms* **by** (*intro idempotent-onI*) *auto*

end

consts *closure-operator* :: $'a \Rightarrow 'b \Rightarrow$ *bool*

overloading
closure-operator $\equiv$ *closure-operator* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a) \Rightarrow bool$
begin
definition *closure-operator* $(R :: 'a \Rightarrow 'a \Rightarrow bool) :: ('a \Rightarrow 'a) \Rightarrow bool \equiv$
$(R \Rightarrow R) \sqcap$ *inflationary-on* (*in-field R*) $R \sqcap$ *idempotent-on* (*in-field R*) R
end

lemma *closure-operatorI* [*intro*]:
assumes $(R \Rightarrow R)\ f$
and *inflationary-on* (*in-field R*) $R\ f$
and *idempotent-on* (*in-field R*) $R\ f$
shows *closure-operator R f*
unfolding *closure-operator-def* **using** *assms* **by** *blast*

lemma *closure-operatorE* [*elim*]:
assumes *closure-operator R f*
obtains $(R \Rightarrow R)\ f$ *inflationary-on* (*in-field R*) $R\ f$
idempotent-on (*in-field R*) $R\ f$
using *assms* **unfolding** *closure-operator-def* **by** *blast*

lemma *mono-wrt-rel-if-closure-operator*:
assumes *closure-operator* $(R :: 'a \Rightarrow 'a \Rightarrow bool)\ f$
shows $(R \Rightarrow R)\ f$
using *assms* **by** (*elim closure-operatorE*)

context
fixes $R :: 'a \Rightarrow 'a \Rightarrow bool$ **and** $f :: 'a \Rightarrow 'a$
begin

lemma *inflationary-on-in-field-if-closure-operator*:
assumes *closure-operator R f*
shows *inflationary-on* (*in-field R*) $R\ f$
using *assms* **by** (*elim closure-operatorE*)

lemma *idempotent-on-in-field-if-closure-operator*:
assumes *closure-operator R f*
shows *idempotent-on* (*in-field R*) $R\ f$

using *assms* **by** (*elim closure-operatorE*)

end

end

theory *Order-Functions*
imports
Order-Functions-Base
Closure-Operators
begin

Summary Basic functions on orders.

end

2.3 Transport using Bijections

theory *Transport-Bijections*
imports
Restricted-Equality
Functions-Bijection
Transport-Base
Binary-Relations-Asymmetric
Binary-Relations-Connected
begin

Summary Setup for Transport using bijective transport functions.

locale *transport-bijection* =
fixes $L :: 'a \Rightarrow 'a \Rightarrow bool$
and $R :: 'b \Rightarrow 'b \Rightarrow bool$
and $l :: 'a \Rightarrow 'b$
and $r :: 'b \Rightarrow 'a$
assumes *mono-wrt-rel-left*: $(L \Rightarrow R)\ l$
and *mono-wrt-rel-right*: $(R \Rightarrow L)\ r$
and *inverse-left-right*: *inverse-on* (*in-field L*) *l r*
and *inverse-right-left*: *inverse-on* (*in-field R*) *r l*
begin

interpretation *transport L R l r* .
interpretation *g-flip-inv* : *galois* $(\geq_R)$ $(\geq_L)$ *r l* .

lemma *bijection-on-in-field*: *bijection-on* (*in-field* $(\leq_L)$) (*in-field* $(\leq_R)$) *l r*
using *mono-wrt-rel-left mono-wrt-rel-right inverse-left-right inverse-right-left*
by (*intro bijection-onI in-field-if-in-field-if-mono-wrt-rel*)
auto

lemma *half-galois-prop-left*: $((\leq_L)\ {}_h\trianglelefteq\ (\leq_R))\ l\ r$

using *mono-wrt-rel-left inverse-right-left*
by (*intro half-galois-prop-leftI*) (*fastforce dest*: *inverse-onD*)

lemma *half-galois-prop-right*: $((\leq_L) \trianglelefteq_h (\leq_R))\ l\ r$
using *mono-wrt-rel-right inverse-left-right*
by (*intro half-galois-prop-rightI*)
(*force dest*: *in-field-if-in-dom inverse-onD*)

lemma *galois-prop*: $((\leq_L) \trianglelefteq (\leq_R))\ l\ r$
using *half-galois-prop-left half-galois-prop-right*
by (*intro galois-propI*)

lemma *galois-connection*: $((\leq_L) \dashv (\leq_R))\ l\ r$
using *mono-wrt-rel-left mono-wrt-rel-right galois-prop*
by (*intro galois-connectionI*)

lemma *rel-equivalence-on-unitI*:
assumes *reflexive-on* (*in-field* $(\leq_L)$) $(\leq_L)$
shows *rel-equivalence-on* (*in-field* $(\leq_L)$) $(\leq_L)\ \eta$
using *assms inverse-left-right*
by (*subst rel-equivalence-on-unit-iff-reflexive-on-if-inverse-on*)

interpretation *flip* : *transport-bijection R L r l*
rewrites *order-functors.unit r l* $\equiv \varepsilon$
using *mono-wrt-rel-left mono-wrt-rel-right inverse-left-right inverse-right-left*
by *unfold-locales* (*simp-all only*: *flip-unit-eq-counit*)

lemma *galois-equivalence*: $((\leq_L) \equiv_G (\leq_R))\ l\ r$
using *galois-connection flip.galois-prop* **by** (*intro galois-equivalenceI*)

lemmas *rel-equivalence-on-counitI* = *flip.rel-equivalence-on-unitI*

lemma *order-equivalenceI*:
assumes *reflexive-on* (*in-field* $(\leq_L)$) $(\leq_L)$
and *reflexive-on* (*in-field* $(\leq_R)$) $(\leq_R)$
shows $((\leq_L) \equiv_o (\leq_R))\ l\ r$
using *assms mono-wrt-rel-left mono-wrt-rel-right rel-equivalence-on-unitI*
rel-equivalence-on-counitI
by (*intro order-equivalenceI*)

lemma *preorder-equivalenceI*:
assumes *preorder-on* (*in-field* $(\leq_L)$) $(\leq_L)$
and *preorder-on* (*in-field* $(\leq_R)$) $(\leq_R)$
shows $((\leq_L) \equiv_{pre} (\leq_R))\ l\ r$
using *assms* **by** (*intro preorder-equivalence-if-galois-equivalenceI*
galois-equivalence)
simp-all

lemma *partial-equivalence-rel-equivalenceI*:

assumes *partial-equivalence-rel* $(\leq_L)$
and *partial-equivalence-rel* $(\leq_R)$
shows $((\leq_L) \equiv_{PER} (\leq_R))$ *l r*
using *assms* **by** (*intro partial-equivalence-rel-equivalence-if-galois-equivalenceI galois-equivalence*)
simp-all

end

locale *transport-reflexive-on-in-field-bijection* =
fixes $L :: 'a \Rightarrow 'a \Rightarrow bool$
and $R :: 'b \Rightarrow 'b \Rightarrow bool$
and $l :: 'a \Rightarrow 'b$
and $r :: 'b \Rightarrow 'a$
assumes *reflexive-on-in-field-left*: *reflexive-on* (*in-field L*) *L*
and *reflexive-on-in-field-right*: *reflexive-on* (*in-field R*) *R*
and *transport-bijection*: *transport-bijection L R l r*
begin

sublocale *tbij?* : *transport-bijection L R l r*
rewrites *reflexive-on* (*in-field L*) *L* ≡ *True*
and *reflexive-on* (*in-field R*) *R* ≡ *True*
and $\bigwedge P.$ (*True* ⟹ *P*) ≡ *Trueprop P*
using *transport-bijection reflexive-on-in-field-left reflexive-on-in-field-right*
by *auto*

lemmas *rel-equivalence-on-unit* = *rel-equivalence-on-unitI*
lemmas *rel-equivalence-on-counit* = *rel-equivalence-on-counitI*
lemmas *order-equivalence* = *order-equivalenceI*

end

locale *transport-preorder-on-in-field-bijection* =
fixes $L :: 'a \Rightarrow 'a \Rightarrow bool$
and $R :: 'b \Rightarrow 'b \Rightarrow bool$
and $l :: 'a \Rightarrow 'b$
and $r :: 'b \Rightarrow 'a$
assumes *preorder-on-in-field-left*: *preorder-on* (*in-field L*) *L*
and *preorder-on-in-field-right*: *preorder-on* (*in-field R*) *R*
and *transport-bijection*: *transport-bijection L R l r*
begin

sublocale *trefl-bij?* : *transport-reflexive-on-in-field-bijection L R l r*
rewrites *preorder-on* (*in-field L*) *L* ≡ *True*
and *preorder-on* (*in-field R*) *R* ≡ *True*
and $\bigwedge P.$ (*True* ⟹ *P*) ≡ *Trueprop P*
using *transport-bijection*
by (*intro transport-reflexive-on-in-field-bijection.intro*)
(*insert preorder-on-in-field-left preorder-on-in-field-right*, *auto*)

lemmas *preorder-equivalence* = *preorder-equivalenceI*

end

locale *transport-partial-equivalence-rel-bijection* =
  fixes $L :: 'a \Rightarrow 'a \Rightarrow bool$
  and $R :: 'b \Rightarrow 'b \Rightarrow bool$
  and $l :: 'a \Rightarrow 'b$
  and $r :: 'b \Rightarrow 'a$
  assumes *partial-equivalence-rel-left*: *partial-equivalence-rel L*
  and *partial-equivalence-rel-right*: *partial-equivalence-rel R*
  and *transport-bijection*: *transport-bijection L R l r*
begin

sublocale *tpre-bij?* : *transport-preorder-on-in-field-bijection L R l r*
  rewrites *partial-equivalence-rel L* ≡ *True*
  and *partial-equivalence-rel R* ≡ *True*
  and $\bigwedge P.$ (*True* ⟹ *P*) ≡ *Trueprop P*
  using *transport-bijection*
  by (*intro transport-preorder-on-in-field-bijection.intro*)
  (*insert partial-equivalence-rel-left partial-equivalence-rel-right*, *auto*)

lemmas *partial-equivalence-rel-equivalence* = *partial-equivalence-rel-equivalenceI*

end

locale *transport-eq-restrict-bijection* =
  fixes $P :: 'a \Rightarrow bool$
  and $Q :: 'b \Rightarrow bool$
  and $l :: 'a \Rightarrow 'b$
  and $r :: 'b \Rightarrow 'a$
  assumes *bijection-on-in-field*:
    bijection-on (*in-field* ((=$_P$) :: $'a \Rightarrow$ -)) (*in-field* ((=$_Q$) :: $'b \Rightarrow$ -)) *l r*
begin

interpretation *transport* (=$_P$) (=$_Q$) *l r* **.**

sublocale *tper-bij?* : *transport-partial-equivalence-rel-bijection* (=$_P$) (=$_Q$) *l r*
  using *bijection-on-in-field partial-equivalence-rel-eq-restrict*
  by *unfold-locales*
  (*auto elim*: *bijection-onE intro*!:
    mono-wrt-rel-left-if-reflexive-on-if-le-eq-if-mono-wrt-in-field[*of in-field* (=$_Q$)]
   flip-of.*mono-wrt-rel-left-if-reflexive-on-if-le-eq-if-mono-wrt-in-field*[*of in-field* (=$_P$)])

lemma *left-Galois-eq-Galois-eq-eq-restrict*: ($_L\lessapprox$) = (*galois-rel.Galois* (=) (=) *r*)↾$_P$⇂$_Q$
  by (*subst galois-rel.left-Galois-restrict-left-eq-left-Galois-left-restrict-left*
    galois-rel.left-Galois-restrict-right-eq-left-Galois-right-restrict-right
    rel-restrict-right-eq rel-inv-eq-self-if-symmetric)+

auto

end

locale *transport-eq-bijection* =
fixes $l :: 'a \Rightarrow 'b$
and $r :: 'b \Rightarrow 'a$
assumes *bijection-on-in-field*:
bijection-on (*in-field* ((=) :: $'a \Rightarrow$ -)) (*in-field* ((=) :: $'b \Rightarrow$ -)) *l r*
begin

sublocale *teq-restr-bij?* : *transport-eq-restrict-bijection* $\top$ $\top$ *l r*
rewrites $(=_{\top} :: 'a \Rightarrow bool) = ((=) :: 'a \Rightarrow \text{-})$
and $(=_{\top} :: 'b \Rightarrow bool) = ((=) :: 'b \Rightarrow \text{-})$
using *bijection-on-in-field* **by** *unfold-locales simp-all*

end

lemma *mono-wrt-rel-if-connected-on-if-asymmetric-if-mono-if-inverse-on*:
assumes *conn*: *connected-on* (*in-field L*) *L*
and *asym*: *asymmetric R*
and *monol*: (*L* ⇒ *R*) *l*
and *monor*: (*in-field R* ⇒ *in-field L*) *r*
and *inv*: *inverse-on* (*in-field R*) *r l*
shows (*R* ⇒ *L*) *r*
proof (*intro mono-wrt-relI*)
fix *x y* **assume** *R x y*
with *monor* **have** *in-field L* (*r x*) *in-field L* (*r y*) **by** *blast*+
with *conn* **consider** *r x* = *r y* | *L* (*r y*) (*r x*) | *L* (*r x*) (*r y*) **by** *blast*
then show *L* (*r x*) (*r y*)
proof *cases*
case *1*
moreover from *inv* **have** *injective-on* (*in-field R*) *r* **using** *injective-on-if-inverse-on*
by *blast*
ultimately have *x* = *y* **using** ‹*R x y*› **by** (*blast dest*: *injective-onD*)
with *asym* ‹*R x y*› **show** *?thesis* **by** *blast*
next
case *2*
with *monol* **have** *R* (*l* (*r y*)) (*l* (*r x*)) **by** *auto*
moreover from *inv* ‹*R x y*› **have** *l* (*r y*) = *y l* (*r x*) = *x* **by** (*blast dest*:
inverse-onD)+
ultimately have *R y x* **by** *simp*
with *asym* ‹*R x y*› **show** *?thesis* **by** *blast*
qed *auto*
qed

context *galois*
begin

lemma *transport-bijection-if-asymmetric-if-connected-on-if-mono-if-bijection-on*:
assumes *bijection-on* (*in-field* ($\leq_L$)) (*in-field* ($\leq_R$)) *l r*
and (($\leq_L$) $\Rightarrow$ ($\leq_R$)) *l*
and *connected-on* (*in-field* ($\leq_L$)) ($\leq_L$)
and *asymmetric* ($\leq_R$)
shows *transport-bijection* ($\leq_L$) ($\leq_R$) *l r*
proof (*intro transport-bijection.intro*)
from *assms* **show** (($\leq_R$) $\Rightarrow$ ($\leq_L$)) *r*
by (*auto intro*: *mono-wrt-rel-if-connected-on-if-asymmetric-if-mono-if-inverse-on*)
qed (*use assms* **in** *auto*)

lemma *inverse-on-if-connected-on-if-asymmetric-if-galois-connection*:
assumes *gconn*: (($\leq_L$) $\dashv$ ($\leq_R$)) *l r*
and *conn*: *connected-on* (*in-field* ($\leq_L$)) ($\leq_L$)
and *asym*: *asymmetric* ($\leq_R$)
shows *inverse-on* (*in-field* ($\leq_L$)) *l r*
proof (*intro inverse-onI*)
fix *x* **assume** *in-field* ($\leq_L$) *x*
moreover with *gconn* **have** *in-field* ($\leq_L$) (η *x*) **by** *force*
ultimately consider $\eta\ x \leq_L x \vee x \leq_L \eta\ x \mid \eta\ x = x$ **using** *conn* **by** *blast*
then show *r* (*l x*) = *x*
proof *cases*
case *1*
then show *?thesis* **using** *gconn asym* **by** (*cases rule*: *disjE*) *force*+
qed *auto*
qed

interpretation *flip* : *galois R L r l* **.**

corollary *bijection-on-if-connected-on-if-asymmetric-if-galois-equivalence*:
assumes (($\leq_L$) $\equiv_G$ ($\leq_R$)) *l r*
and *asymmetric* ($\leq_L$) *asymmetric* ($\leq_R$)
and *connected-on* (*in-field* ($\leq_L$)) ($\leq_L$) *connected-on* (*in-field* ($\leq_R$)) ($\leq_R$)
shows *bijection-on* (*in-field* ($\leq_L$)) (*in-field* ($\leq_R$)) *l r*
using *assms* **by** (*intro transport-bijection.bijection-on-in-field transport-bijection.intro*
inverse-on-if-connected-on-if-asymmetric-if-galois-connection
flip.inverse-on-if-connected-on-if-asymmetric-if-galois-connection)
auto

end

end

theory *Order-Isomorphisms*
imports
Transport-Bijections
begin

consts *order-isomorphism-on* :: $'a \Rightarrow 'b \Rightarrow 'c \Rightarrow 'd \Rightarrow 'e \Rightarrow 'f \Rightarrow bool$

definition *order-isomorphism-on-pred* :: $('a \Rightarrow bool) \Rightarrow ('b \Rightarrow bool) \Rightarrow$
$('a \Rightarrow 'a \Rightarrow bool) \Rightarrow ('b \Rightarrow 'b \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b) \Rightarrow ('b \Rightarrow 'a) \Rightarrow bool$ **where**
order-isomorphism-on-pred P Q L R l r $\equiv$ *bijection-on P Q l r* $\wedge$ $(\forall x\ y : P.\ L\ x\ y \longleftrightarrow R\ (l\ x)\ (l\ y))$
adhoc-overloading *order-isomorphism-on* $\rightleftharpoons$ *order-isomorphism-on-pred*

context *order-functors*
begin

lemma *order-isomorphism-onI* [*intro*]:
assumes *bij*: *bijection-on P Q l r*
and *monol*: $((\leq_L)\!\uparrow_P \Rightarrow (\leq_R))\ l$
and *monor*: $((\leq_R)\!\uparrow_Q \Rightarrow (\leq_L))\ r$
shows *order-isomorphism-on P Q* $(\leq_L)$ $(\leq_R)$ *l r*
proof −
have $x \leq_L x'$ **if** *P x P x' l x* $\leq_R$ *l x'* **for** *x x'*
proof −
have $\eta\ x \leq_L \eta\ x'$ **using** *that monor bij* **by** *fastforce*
then show *?thesis*
using ‹*P x*› ‹*P x'*› *bij inverse-on-if-bijection-on-left-right* **by** (*force dest*: *inverse-onD*)
qed
then have $x \leq_L x' \longleftrightarrow$ *l x* $\leq_R$ *l x'* **if** *P x P x'* **for** *x x'* **using** *that monol* **by** *fast*
then show *?thesis* **unfolding** *order-isomorphism-on-pred-def* **using** *bij* **by** *auto*
qed

lemma *order-isomorphism-onE*:
assumes *order-isomorphism-on P Q* $(\leq_L)$ $(\leq_R)$ *l r*
obtains *bijection-on P Q l r* $\bigwedge x\ y.\ P\ x \Longrightarrow P\ y \Longrightarrow L\ x\ y \longleftrightarrow R\ (l\ x)\ (l\ y)$
using *assms* **unfolding** *order-isomorphism-on-pred-def* **by** *auto*

context
notes *order-isomorphism-onE*[*elim*!]
begin

lemma *right-rel-right-if-right-rel-if-preds-if-order-isomorphism-on*:
assumes *order-isomorphism-on P Q* $(\leq_L)$ $(\leq_R)$ *l r*
and *Q y Q y'*
and $y \leq_R y'$
shows $r\ y \leq_L r\ y'$
proof −
from *assms* **have** $y = \varepsilon\ y\ y' = \varepsilon\ y'$
using *inverse-on-if-bijection-on-left-right inverse-on-if-bijection-on-right-left*
by (*force dest*!: *inverse-onD*)+
with *assms* **show** *?thesis* **by** *force*

qed

lemma *order-isomorphism-on-right-left-if-order-isomorphism-on-left-right*:
assumes *order-isomorphism-on P Q* $(\leq_L)$ $(\leq_R)$ *l r*
shows *order-isomorphism-on Q P* $(\leq_R)$ $(\leq_L)$ *r l*
using *assms bijection-on-right-left-if-bijection-on-left-right*[*of P Q l r*]
right-rel-right-if-right-rel-if-preds-if-order-isomorphism-on[*OF assms*]
by (*intro order-functors.order-isomorphism-onI mono-wrt-relI*)
(*auto elim*!: *order-isomorphism-onE rel-restrict-leftE rel-restrict-rightE*)

lemma *mono-wrt-restrict-restrict-left-if-order-isomorphism-on*:
assumes *order-isomorphism-on P Q* $(\leq_L)$ $(\leq_R)$ *l r*
shows $((\leq_L)\restriction_P \Rightarrow (\leq_R)\restriction_Q)$ *l*
using *assms* **unfolding** *order-isomorphism-on-pred-def* **by** *fastforce*

end
end

context *galois*
begin

interpretation *flip* : *galois R L r l* **.**

lemma *order-isomorphism-onE* [*elim*]:
assumes *order-isomorphism-on P Q* $(\leq_L)$ $(\leq_R)$ *l r*
obtains *bijection-on P Q l r* $((\leq_L)\restriction_P \equiv_G (\leq_R)\restriction_Q)$ *l r*
proof −
note *order-isomorphism-onE*[*elim*]
have $((\leq_L)\restriction_P \Rightarrow (\leq_R)\restriction_Q)$ *l*
using *assms* **by** (*intro mono-wrt-restrict-restrict-left-if-order-isomorphism-on*)
moreover have $((\leq_R)\restriction_Q \Rightarrow (\leq_L)\restriction_P)$ *r*
using *assms* **by** (*intro flip.mono-wrt-restrict-restrict-left-if-order-isomorphism-on*
order-isomorphism-on-right-left-if-order-isomorphism-on-left-right)
moreover have *inverse-on* (*in-field* $(\leq_L)\restriction_P$) *l r*
proof −
have *in-field* $(\leq_L)\restriction_P \leq P$ **by** *auto*
with *assms* **show** *?thesis* **using** *antimono-inverse-on* **by** *blast*
qed
moreover have *inverse-on* (*in-field* $(\leq_R)\restriction_Q$) *r l*
proof −
have *in-field* $(\leq_R)\restriction_Q \leq Q$ **by** *auto*
with *assms* **show** *?thesis* **using** *antimono-inverse-on* **by** *blast*
qed
ultimately interpret *transport-bijection* $(\leq_L)\restriction_P$ $(\leq_R)\restriction_Q$ *l r*
using *assms* **unfolding** *order-isomorphism-on-pred-def* **by** *unfold-locales auto*
from *assms galois-equivalence* **show** *?thesis* **using** *that* **by** *blast*
qed

lemma *order-isomorphism-on-in-field-if-connected-on-if-asymmetric-if-galois-equivalence*:

assumes $((\leq_L) \equiv_G (\leq_R))$ *l r*
and *asymmetric* $(\leq_L)$ *asymmetric* $(\leq_R)$
and *connected-on* (*in-field* $(\leq_L)$) $(\leq_L)$ *connected-on* (*in-field* $(\leq_R)$) $(\leq_R)$
shows *order-isomorphism-on* (*in-field* $(\leq_L)$) (*in-field* $(\leq_R)$) $(\leq_L)$ $(\leq_R)$ *l r*
using *assms* **by** (*intro order-isomorphism-onI*
bijection-on-if-connected-on-if-asymmetric-if-galois-equivalence)
blast+

end

context *order-functors*
begin

interpretation *of* : *order-functors A B a b* **for** *A B a b* **.**

lemma *order-isomorphism-on-compI*:
assumes *order-isomorphism-on P Q L M l1 r1*
and *order-isomorphism-on Q U M R l2 r2*
shows *order-isomorphism-on P U L R* (*l2* ∘ *l1*) (*r1* ∘ *r2*)
proof (*intro of.order-isomorphism-onI*)
from *assms* **show** *bijection-on P U* (*l2* ∘ *l1*) (*r1* ∘ *r2*)
by (*intro bijection-on-compI*) (*auto elim*: *of.order-isomorphism-onE*)
from *assms* **show** $((\leq_L)\uparrow_P \Rightarrow (\leq_R))$ (*l2* ∘ *l1*)
proof −
from *assms* **have** $((\leq_L)\uparrow_P \Rightarrow M\uparrow_Q)$ *l1* $(M\uparrow_Q \Rightarrow (\leq_R))$ *l2*
using *of.mono-wrt-restrict-restrict-left-if-order-isomorphism-on* **by** *blast+*
then show *?thesis* **by** (*urule dep-mono-wrt-rel-compI'*)
qed
from *assms* **show** $((\leq_R)\uparrow_U \Rightarrow (\leq_L))$ (*r1* ∘ *r2*)
proof −
from *assms* **have** *order-isomorphism-on Q P M L r1 l1 order-isomorphism-on*
U Q R M r2 l2
by (*auto intro*: *of.order-isomorphism-on-right-left-if-order-isomorphism-on-left-right*)
then have $((\leq_R)\uparrow_U \Rightarrow M\uparrow_Q)$ *r2* $(M\uparrow_Q \Rightarrow (\leq_L))$ *r1*
using *of.mono-wrt-restrict-restrict-left-if-order-isomorphism-on* **by** *blast+*
then show *?thesis* **by** (*urule dep-mono-wrt-rel-compI'*)
qed
qed

lemma *order-isomorphism-on-self-id*: *order-isomorphism-on P P R R id id*
by (*intro of.order-isomorphism-onI bijection-on-self-id*) *fastforce+*

end

consts *order-isomorphic-on* :: *'a* ⇒ *'b* ⇒ *'c* ⇒ *'d* ⇒ *bool*

definition *order-isomorphic-on-pred* ::
(*'a* ⇒ *bool*) ⇒ (*'b* ⇒ *bool*) ⇒ (*'a* ⇒ *'a* ⇒ *bool*) ⇒ (*'b* ⇒ *'b* ⇒ *bool*) ⇒ *bool*
where

order-isomorphic-on-pred P Q L R ≡ ∃ *l r*. *order-isomorphism-on P Q L R l r*
adhoc-overloading *order-isomorphic-on* ⇌ *order-isomorphic-on-pred*

lemma *order-isomorphic-onI* [*intro*]:
assumes *order-isomorphism-on P Q L R l r*
shows *order-isomorphic-on P Q L R*
using *assms* **unfolding** *order-isomorphic-on-pred-def* **by** *blast*

lemma *order-isomorphic-onE* [*elim*]:
assumes *order-isomorphic-on P Q L R*
obtains *l r* **where** *order-isomorphism-on P Q L R l r*
using *assms* **unfolding** *order-isomorphic-on-pred-def* **by** *blast*

lemma *order-isomorphic-on-if-order-isomorphic-on*:
assumes *order-isomorphic-on P Q L R*
shows *order-isomorphic-on Q P R L*
using *assms order-functors.order-isomorphism-on-right-left-if-order-isomorphism-on-left-right*
by *blast*

lemma *order-isomorphic-on-trans*:
assumes *order-isomorphic-on P Q L M*
and *order-isomorphic-on Q U M R*
shows *order-isomorphic-on P U L R*
using *assms* **by** (*elim order-isomorphic-onE*) (*blast intro*: *order-functors.order-isomorphism-on-compI*)

lemma *order-isomorphic-on-self*: *order-isomorphic-on P P R R*
using *order-functors.order-isomorphism-on-self-id* **by** *blast*

end

theory *Order-Functors*
imports
Order-Functors-Base
Order-Equivalences
Order-Isomorphisms
begin

Summary Functors between orders aka. order-homomorphisms aka. monotone functions.

end

2.3.1 Strict Partial Orders

theory *Strict-Partial-Orders*
imports
Binary-Relations-Asymmetric
Binary-Relations-Transitive
begin

definition *strict-partial-order-on* ≡ *transitive-on* ⊓ *asymmetric-on*

lemma *strict-partial-order-onI* [*intro*]:
assumes *transitive-on P R*
and *asymmetric-on P R*
shows *strict-partial-order-on P R*
unfolding *strict-partial-order-on-def* **using** *assms* **by** *auto*

lemma *strict-partial-order-onE* [*elim*]:
assumes *strict-partial-order-on P R*
obtains *transitive-on P R asymmetric-on P R*
using *assms* **unfolding** *strict-partial-order-on-def* **by** *auto*

lemma *transitive-if-strict-partial-order-on-in-field*:
assumes *strict-partial-order-on* (*in-field R*) *R*
shows *transitive R*
using *assms* **by** (*elim strict-partial-order-onE*) (*rule transitive-if-transitive-on-in-field*)

lemma *asymmetric-if-strict-partial-order-on-in-field*:
assumes *strict-partial-order-on* (*in-field R*) *R*
shows *asymmetric R*
using *assms* **by** (*elim strict-partial-order-onE*) (*rule asymmetric-if-asymmetric-on-in-field*)

lemma *antimono-strict-partial-order-on*:
antimono (*strict-partial-order-on* :: (′*a* ⇒ *bool*) ⇒ (′*a* ⇒ ′*a* ⇒ *bool*) ⇒ *bool*)
using *antimono-transitive-on antimono-asymmetric-on*
by (*intro antimonoI le-predI*; *elim strict-partial-order-onE*) *auto*

lemma *strict-partial-order-on-rel-map-if-mono-wrt-pred-if-strict-partial-order-on*:
fixes *P* :: ′*a* ⇒ *bool* **and** *R* :: ′*a* ⇒ ′*a* ⇒ *bool*
assumes *strict-partial-order-on P R*
and (*Q* ⇒ *P*) *f*
shows *strict-partial-order-on Q* (*rel-map f R*)
using *assms* **by** (*intro strict-partial-order-onI*
asymmetric-on-rel-map-if-mono-wrt-pred-if-asymmetric-on
transitive-on-rel-map-if-mono-wrt-pred-if-transitive-on)
auto

consts *strict-partial-order* :: ′*a* ⇒ *bool*

overloading
strict-partial-order ≡ *strict-partial-order* :: (′*a* ⇒ ′*a* ⇒ *bool*) ⇒ *bool*
begin
definition (*strict-partial-order* :: (′*a* ⇒ ′*a* ⇒ *bool*) ⇒ *bool*) ≡ *strict-partial-order-on*
(⊤ :: ′*a* ⇒ *bool*)
end

lemma *strict-partial-order-eq-strict-partial-order-on*:
(*strict-partial-order* :: (′*a* ⇒ ′*a* ⇒ *bool*) ⇒ *bool*) = *strict-partial-order-on* (⊤ ::

$'a \Rightarrow bool$)
unfolding *strict-partial-order-def* **..**

lemma *strict-partial-order-eq-strict-partial-order-on-uhint* [*uhint*]:
assumes $P \equiv \top :: 'a \Rightarrow bool$
shows (*strict-partial-order* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$) $\equiv$ *strict-partial-order-on* *P*
using *assms* **by** (*simp add*: *strict-partial-order-eq-strict-partial-order-on*)

context
fixes $R :: 'a \Rightarrow 'a \Rightarrow bool$
begin

lemma *strict-partial-orderI* [*intro*]:
assumes *transitive R*
and *asymmetric R*
shows *strict-partial-order R*
using *assms* **by** (*urule strict-partial-order-onI*)

lemma *strict-partial-orderE* [*elim*]:
assumes *strict-partial-order R*
obtains *transitive R asymmetric R*
using *assms* **by** (*urule* (*e*) *strict-partial-order-onE*)

lemma *strict-partial-order-on-if-strict-partial-order*:
fixes $P :: 'a \Rightarrow bool$
assumes *strict-partial-order R*
shows *strict-partial-order-on P R*
using *assms* **by** (*elim strict-partial-orderE*)
(*intro strict-partial-order-onI transitive-on-if-transitive asymmetric-on-if-asymmetric*)

end

end

2.3.2 Strict Linear Orders

theory *Strict-Linear-Orders*
imports
Binary-Relations-Connected
Strict-Partial-Orders
begin

definition *strict-linear-order-on* $\equiv$ *strict-partial-order-on* $\sqcap$ *connected-on*

lemma *strict-linear-order-onI* [*intro*]:
assumes *strict-partial-order-on P R connected-on P R*

shows *strict-linear-order-on P R*
using *assms* **unfolding** *strict-linear-order-on-def* **by** *auto*

lemma *strict-linear-order-onE* [*elim*]:
assumes *strict-linear-order-on P R*
obtains *strict-partial-order-on P R connected-on P R*
using *assms* **unfolding** *strict-linear-order-on-def* **by** *auto*

lemma *antimono-strict-linear-order-on*:
antimono (*strict-linear-order-on* :: (*'a* ⇒ *bool*) ⇒ (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*)
using *antimono-strict-partial-order-on antimono-connected-on*
by (*intro antimonoI le-predI*; *elim strict-linear-order-onE*) *auto*

lemma *strict-linear-order-on-rel-map-if-injective-on-if-mono-wrt-pred-if-strict-linear-order-on*:
fixes *P* :: *'a* ⇒ *bool* **and** *R* :: *'a* ⇒ *'a* ⇒ *bool*
assumes *strict-linear-order-on P R*
and (*Q* ⇒ *P*) *f*
and *injective-on Q f*
shows *strict-linear-order-on Q* (*rel-map f R*)
using *assms* **by** (*intro strict-linear-order-onI*
strict-partial-order-on-rel-map-if-mono-wrt-pred-if-strict-partial-order-on
connected-on-rel-map-if-injective-on-if-mono-wrt-pred-if-connected-on)
auto

consts *strict-linear-order* :: *'a* ⇒ *bool*

overloading
strict-linear-order ≡ *strict-linear-order* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*
begin
definition (*strict-linear-order* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*) ≡ *strict-linear-order-on*
(⊤ :: *'a* ⇒ *bool*)
end

lemma *strict-linear-order-eq-strict-linear-order-on*:
(*strict-linear-order* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*) = *strict-linear-order-on* (⊤ :: *'a*
⇒ *bool*)
unfolding *strict-linear-order-def* **..**

lemma *strict-linear-order-eq-strict-linear-order-on-uhint* [*uhint*]:
assumes *P* ≡ ⊤ :: *'a* ⇒ *bool*
shows (*strict-linear-order* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*) ≡ *strict-linear-order-on*
P
using *assms* **by** (*simp add*: *strict-linear-order-eq-strict-linear-order-on*)

context
fixes *R* :: *'a* ⇒ *'a* ⇒ *bool*
begin

lemma *strict-linear-orderI* [*intro*]:

assumes *strict-partial-order R*
and *connected R*
shows *strict-linear-order R*
using *assms* **by** (*urule strict-linear-order-onI*)

lemma *strict-linear-orderE* [*elim*]:
assumes *strict-linear-order R*
obtains *strict-partial-order R connected R*
using *assms* **by** (*urule* (*e*) *strict-linear-order-onE*)

lemma *strict-linear-order-on-if-strict-linear-order*:
fixes *P* :: *'a* ⇒ *bool*
assumes *strict-linear-order R*
shows *strict-linear-order-on P R*
using *assms* **by** (*elim strict-linear-orderE*)
(*intro strict-linear-order-onI strict-partial-order-on-if-strict-partial-order*
connected-on-if-connected)

end

end

Well-Orders

theory *Wellorders*
imports
Binary-Relations-Wellfounded
Strict-Linear-Orders
begin

definition *wellorder-on* ≡ *strict-linear-order-on* ⊓ *wellfounded-on*

lemma *wellorder-onI* [*intro*]:
assumes *strict-linear-order-on P R wellfounded-on P R*
shows *wellorder-on P R*
using *assms* **unfolding** *wellorder-on-def* **by** *auto*

lemma *wellorder-onE* [*elim*]:
assumes *wellorder-on P R*
obtains *strict-linear-order-on P R wellfounded-on P R*
using *assms* **unfolding** *wellorder-on-def* **by** *auto*

lemma *antimono-wellorder-on*:
antimono (*wellorder-on* :: (*'a* ⇒ *bool*) ⇒ (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*)
using *antimono-strict-linear-order-on wellfounded-on-if-le-pred-if-wellfounded-on*
by (*intro antimonoI le-predI*; *elim wellorder-onE*) (*auto 5 0*)

lemma *wellorder-on-rel-map-if-injective-on-if-mono-wrt-pred-if-wellorder-on*:
fixes $P :: 'a \Rightarrow bool$
assumes *wellorder-on* P R $(Q \Rightarrow P)$ f *injective-on* Q f
shows *wellorder-on* Q (*rel-map* f R)
using *assms*
strict-linear-order-on-rel-map-if-injective-on-if-mono-wrt-pred-if-strict-linear-order-on
wellfounded-on-rel-map-if-mono-wrt-pred-if-wellfounded-on
by *fastforce*

consts *wellorder* :: $'a \Rightarrow bool$

overloading
wellorder $\equiv$ *wellorder* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$
begin
definition (*wellorder* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$) $\equiv$ *wellorder-on* ($\top$ :: $'a \Rightarrow bool$)
end

lemma *wellorder-eq-wellorder-on*:
(*wellorder* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$) = *wellorder-on* ($\top$ :: $'a \Rightarrow bool$)
unfolding *wellorder-def* **..**

lemma *wellorder-eq-wellorder-on-uhint* [*uhint*]:
assumes $P \equiv \top :: 'a \Rightarrow bool$
shows (*wellorder* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$) $\equiv$ *wellorder-on* P
using *assms* **by** (*simp add*: *wellorder-eq-wellorder-on*)

context
fixes $R :: 'a \Rightarrow 'a \Rightarrow bool$
begin

lemma *wellorderI* [*intro*]:
assumes *strict-linear-order* R
and *wellfounded* R
shows *wellorder* R
using *assms* **by** (*urule wellorder-onI*)

lemma *wellorderE* [*elim*]:
assumes *wellorder* R
obtains *strict-linear-order* R *wellfounded* R
using *assms* **by** (*urule* (*e*) *wellorder-onE*)

lemma *wellorder-on-if-wellorder*:
fixes $P :: 'a \Rightarrow bool$
assumes *wellorder* R
shows *wellorder-on* P R
using *assms* **by** (*elim wellorderE*)
(*intro wellorder-onI strict-linear-order-on-if-strict-linear-order wellfounded-on-if-wellfounded*)

end

end

2.4 Orders

theory *Orders*
imports
Equivalence-Relations
Linear-Orders
Order-Functions
Order-Functors
Partial-Equivalence-Relations
Partial-Orders
Preorders
Strict-Linear-Orders
Strict-Partial-Orders
Wellorders
begin

Summary Basic order-theoretic concepts.

end

2.5 Predicates

theory *Predicates*
imports
Bounded-Definite-Description
Bounded-Quantifiers
Predicate-Functions
Predicates-Lattice
Predicates-Order
begin

Summary Basic concepts on predicates.

end

2.6 HOL-Basics

theory *HOL-Basics*
imports
LBinary-Relations
LFunctions
Galois

Orders
Predicates
begin

Summary Library on top of HOL axioms, as required for Transport [2]. Requires *only* the HOL axioms, nothing else. Includes:

1. Basic concepts on binary relations, relativised properties, and restricted equalities e.g. *left-total-on* and *eq-restrict*.
2. Basic concepts on functions, relativised properties, and relators, e.g. *injective-on* and *dep-mono-wrt*.
3. Basic concepts on orders and relativised order-theoretic properties, e.g. *partial-equivalence-rel-on*.
4. Galois connections, Galois equivalences, order equivalences, and other related concepts on order functors, e.g. *galois.galois-equivalence*.
5. Basic concepts on predicates.
6. Syntax bundles for HOL `HOL_Syntax_Bundles`.
7. Alignments for concepts that have counterparts in the HOL library - see `HOL_Alignments`.

end

theory *HOL-Mem-Of*
imports
HOL.Set
ML-Unification.ML-Unification-HOL-Setup
begin

definition *mem-of* $A\ x \equiv x \in A$
lemma *mem-of-eq*: *mem-of* $= (\lambda A\ x.\ x \in A)$ **unfolding** *mem-of-def* **by** *simp*
lemma *mem-of-iff* [*iff*]: *mem-of* $A\ x \longleftrightarrow x \in A$ **unfolding** *mem-of-def* **by** *simp*

lemma *mem-of-eq-mem-uhint* [*uhint*]:
assumes $x \equiv x'$
and $A \equiv A'$
shows *mem-of* $A\ x \equiv x' \in A'$
using *assms* **by** *simp*

lemma *mem-of-UNIV-eq-top* [*simp*]: *mem-of UNIV* $= \top$
by *auto*

lemma *mem-of-empty-eq-bot* [*simp*]: *mem-of* $\{\} = \bot$
by *auto*

end

2.6.1 Alignment With Binary Relation Definitions from HOL.Main

theory *HOL-Alignment-Binary-Relations*
imports
Main
HOL-Mem-Of
LBinary-Relations
begin

unbundle *no relcomp-syntax* **and** *no converse-syntax*

named-theorems *HOL-bin-rel-alignment*

Properties

Antisymmetric **overloading**
antisymmetric-on-set ≡ *antisymmetric-on* :: *'a set* ⇒ (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*
begin
definition *antisymmetric-on-set* (*S* :: *'a set*) :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ - ≡
antisymmetric-on (*mem-of S*)
end

lemma *antisymmetric-on-set-eq-antisymmetric-on-pred* [*simp*]:
(*antisymmetric-on* (*S* :: *'a set*) :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*) = *antisymmetric-on* (*mem-of S*)
unfolding *antisymmetric-on-set-def* **by** *simp*

lemma *antisymmetric-on-set-eq-antisymmetric-on-pred-uhint* [*uhint*]:
assumes *P* ≡ *mem-of S*
shows *antisymmetric-on* (*S* :: *'a set*) :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool* ≡ *antisymmetric-on P*
using *assms* **by** *simp*

lemma *antisymmetric-on-set-iff-antisymmetric-on-pred* [*iff*]:
antisymmetric-on (*S* :: *'a set*) (*R* :: *'a* ⇒ *'a* ⇒ *bool*) ⟷ *antisymmetric-on* (*mem-of S*) *R*
by *simp*

lemma *antisymp-on-eq-antisymmetric-on* [*HOL-bin-rel-alignment*]:
antisymp-on = *antisymmetric-on*
by (*intro ext*) (*auto intro*: *antisymp-onI dest*: *antisymmetric-onD antisymp-onD*)

lemma *antisymp-eq-antisymmetric* [*HOL-bin-rel-alignment*]:
antisymp = *antisymmetric*
by (*intro ext*) (*auto intro*: *antisympI dest*: *antisymmetricD antisympD*)

Asymmetric overloading
asymmetric-on-set ≡ *asymmetric-on* :: *'a set* ⇒ (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*
begin
definition *asymmetric-on-set* (*S* :: *'a set*) :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ - ≡ *asymmetric-on* (*mem-of S*)
end

lemma *asymmetric-on-set-eq-asymmetric-on-pred* [*simp*]:
(*asymmetric-on* (*S* :: *'a set*) :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*) = *asymmetric-on* (*mem-of S*)
unfolding *asymmetric-on-set-def* **by** *simp*

lemma *asymmetric-on-set-eq-asymmetric-on-pred-uhint* [*uhint*]:
assumes *P* ≡ *mem-of S*
shows *asymmetric-on* (*S* :: *'a set*) :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool* ≡ *asymmetric-on P*
using *assms* **by** *simp*

lemma *asymmetric-on-set-iff-asymmetric-on-pred* [*iff*]:
asymmetric-on (*S* :: *'a set*) (*R* :: *'a* ⇒ *'a* ⇒ *bool*) ⟷ *asymmetric-on* (*mem-of S*) *R*
by *simp*

lemma *asymp-on-eq-asymmetric-on* [*HOL-bin-rel-alignment*]: *asymp-on* = *asymmetric-on*
by (*intro ext*) (*auto dest*: *asymp-onD*)

lemma *asymp-eq-asymmetric* [*HOL-bin-rel-alignment*]: *asymp* = *asymmetric*
by (*intro ext*) (*auto dest*: *asympD*)

Injective overloading
rel-injective-on-set ≡ *rel-injective-on* :: *'a set* ⇒ (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *bool*
rel-injective-at-set ≡ *rel-injective-at* :: *'a set* ⇒ (*'b* ⇒ *'a* ⇒ *bool*) ⇒ *bool*
begin
definition *rel-injective-on-set* (*S* :: *'a set*) :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ - ≡
rel-injective-on (*mem-of S*)
definition *rel-injective-at-set* (*S* :: *'a set*) :: (*'b* ⇒ *'a* ⇒ *bool*) ⇒ - ≡
rel-injective-at (*mem-of S*)
end

lemma *rel-injective-on-set-eq-rel-injective-on-pred* [*simp*]:
(*rel-injective-on* (*S* :: *'a set*) :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *bool*) =
rel-injective-on (*mem-of S*)
unfolding *rel-injective-on-set-def* **by** *simp*

lemma *rel-injective-on-set-eq-rel-injective-on-pred-uhint* [*uhint*]:
assumes *P* ≡ *mem-of S*
shows *rel-injective-on* (*S* :: *'a set*) :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *bool* ≡ *rel-injective-on P*

using *assms* **by** *simp*

lemma *rel-injective-on-set-iff-rel-injective-on-pred* [*iff*]:
rel-injective-on (*S* :: *'a set*) (*R* :: *'a* ⇒ *'b* ⇒ *bool*) ⟷ *rel-injective-on* (*mem-of S*) *R*
by *simp*

lemma *rel-injective-at-set-eq-rel-injective-at-pred* [*simp*]:
(*rel-injective-at* (*S* :: *'a set*) :: (*'b* ⇒ *'a* ⇒ *bool*) ⇒ *bool*) =
rel-injective-at (*mem-of S*)
unfolding *rel-injective-at-set-def* **by** *simp*

lemma *rel-injective-at-set-eq-rel-injective-at-pred-uhint* [*uhint*]:
assumes *P* ≡ *mem-of S*
shows *rel-injective-at* (*S* :: *'a set*) :: (*'b* ⇒ *'a* ⇒ *bool*) ⇒ *bool* ≡ *rel-injective-at P*
using *assms* **by** *simp*

lemma *rel-injective-at-set-iff-rel-injective-at-pred* [*iff*]:
rel-injective-at (*S* :: *'a set*) (*R* :: *'b* ⇒ *'a* ⇒ *bool*) ⟷ *rel-injective-at* (*mem-of S*) *R*
by *simp*

lemma *left-unique-eq-rel-injective* [*HOL-bin-rel-alignment*]:
left-unique = *rel-injective*
by (*intro ext*) (*blast intro*: *left-uniqueI dest*: *rel-injectiveD left-uniqueD*)

Irreflexive **overloading**
irreflexive-on-set ≡ *irreflexive-on* :: *'a set* ⇒ (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*
begin
definition *irreflexive-on-set* (*S* :: *'a set*) :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool* ≡
irreflexive-on (*mem-of S*)
end

lemma *irreflexive-on-set-eq-irreflexive-on-pred* [*simp*]:
(*irreflexive-on* (*S* :: *'a set*) :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*) =
irreflexive-on (*mem-of S*)
unfolding *irreflexive-on-set-def* **by** *simp*

lemma *irreflexive-on-set-eq-irreflexive-on-pred-uhint* [*uhint*]:
assumes *P* ≡ *mem-of S*
shows *irreflexive-on* (*S* :: *'a set*) :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool* ≡ *irreflexive-on P*
using *assms* **by** *simp*

lemma *irreflexive-on-set-iff-irreflexive-on-pred* [*iff*]:
irreflexive-on (*S* :: *'a set*) (*R* :: *'a* ⇒ *'a* ⇒ *bool*) ⟷ *irreflexive-on* (*mem-of S*) *R*
by *simp*

lemma *irreflp-on-eq-irreflexive-on* [*HOL-bin-rel-alignment*]: *irreflp-on* = *irreflexive-on*
by (*intro ext*) (*blast intro*: *irreflp-onI dest*: *irreflp-onD*)

lemma *irreflp-eq-irreflexive* [*HOL-bin-rel-alignment*]: *irreflp* = *irreflexive*
by (*intro ext*) (*blast intro*: *irreflpI dest*: *irreflexiveD irreflpD*)

Left-Total **overloading**
left-total-on-set ≡ *left-total-on* :: *'a set* ⇒ (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *bool*
begin
definition *left-total-on-set* (*S* :: *'a set*) :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ - ≡
left-total-on (*mem-of S*)
end

lemma *left-total-on-set-eq-left-total-on-pred* [*simp*]:
(*left-total-on* (*S* :: *'a set*) :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *bool*) =
left-total-on (*mem-of S*)
unfolding *left-total-on-set-def* **by** *simp*

lemma *left-total-on-set-eq-left-total-on-pred-uhint* [*uhint*]:
assumes *P* ≡ *mem-of S*
shows *left-total-on* (*S* :: *'a set*) :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *bool* ≡ *left-total-on P*
using *assms* **by** *simp*

lemma *left-total-on-set-iff-left-total-on-pred* [*iff*]:
left-total-on (*S* :: *'a set*) (*R* :: *'a* ⇒ *'b* ⇒ *bool*) ⟷ *left-total-on* (*mem-of S*) *R*
by *simp*

lemma *Transfer-left-total-eq-left-total* [*HOL-bin-rel-alignment*]:
Transfer.left-total = *Binary-Relations-Left-Total.left-total*
by (*intro ext*) (*fast intro*: *Transfer.left-totalI*
elim: *Transfer.left-totalE Binary-Relations-Left-Total.left-totalE*)

Reflexive **overloading**
reflexive-on-set ≡ *reflexive-on* :: *'a set* ⇒ (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*
begin
definition *reflexive-on-set* (*S* :: *'a set*) :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ - ≡
reflexive-on (*mem-of S*)
end

lemma *reflexive-on-set-eq-reflexive-on-pred* [*simp*]:
(*reflexive-on* (*S* :: *'a set*) :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*) = *reflexive-on* (*mem-of S*)
unfolding *reflexive-on-set-def* **by** *simp*

lemma *reflexive-on-set-eq-reflexive-on-pred-uhint* [*uhint*]:
assumes *P* ≡ *mem-of S*
shows *reflexive-on* (*S* :: *'a set*) :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool* ≡ *reflexive-on P*
using *assms* **by** *simp*

lemma *reflexive-on-set-iff-reflexive-on-pred* [*iff*]:
reflexive-on (*S* :: *'a set*) (*R* :: *'a* ⇒ *'a* ⇒ *bool*) ⟷ *reflexive-on* (*mem-of S*) *R*
by *simp*

lemma *reflp-on-eq-reflexive-on* [*HOL-bin-rel-alignment*]:
reflp-on = *reflexive-on*
by (*intro ext*) (*blast intro*: *reflp-onI dest*: *reflp-onD*)

lemma *reflp-eq-reflexive* [*HOL-bin-rel-alignment*]: *reflp* = *reflexive*
by (*intro ext*) (*blast intro*: *reflpI dest*: *reflexiveD reflpD*)

Right-Unique overloading

right-unique-on-set ≡ *right-unique-on* :: *'a set* ⇒ (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *bool*
right-unique-at-set ≡ *right-unique-at* :: *'a set* ⇒ (*'b* ⇒ *'a* ⇒ *bool*) ⇒ *bool*
begin
definition *right-unique-on-set* (*S* :: *'a set*) :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ - ≡
right-unique-on (*mem-of S*)
definition *right-unique-at-set* (*S* :: *'a set*) :: (*'b* ⇒ *'a* ⇒ *bool*) ⇒ - ≡
right-unique-at (*mem-of S*)
end

lemma *right-unique-on-set-eq-right-unique-on-pred* [*simp*]:
(*right-unique-on* (*S* :: *'a set*) :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *bool*) = *right-unique-on* (*mem-of S*)
unfolding *right-unique-on-set-def* **by** *simp*

lemma *right-unique-on-set-eq-right-unique-on-pred-uhint* [*uhint*]:
assumes *P* ≡ *mem-of S*
shows *right-unique-on* (*S* :: *'a set*) :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *bool* ≡ *right-unique-on P*
using *assms* **by** *simp*

lemma *right-unique-on-set-iff-right-unique-on-pred* [*iff*]:
right-unique-on (*S* :: *'a set*) (*R* :: *'a* ⇒ *'b* ⇒ *bool*) ⟷ *right-unique-on* (*mem-of S*) *R*
by *simp*

lemma *right-unique-at-set-eq-right-unique-at-pred* [*simp*]:
(*right-unique-at* (*S* :: *'a set*) :: (*'b* ⇒ *'a* ⇒ *bool*) ⇒ *bool*) =
right-unique-at (*mem-of S*)
unfolding *right-unique-at-set-def* **by** *simp*

lemma *right-unique-at-set-iff-right-unique-at-pred* [*iff*]:
right-unique-at (*S* :: *'a set*) (*R* :: *'b* ⇒ *'a* ⇒ *bool*) ⟷ *right-unique-at* (*mem-of S*) *R*
by *simp*

lemma *Relation-right-unique-eq-right-unique* [*HOL-bin-rel-alignment*]:
Relation.right-unique = *Binary-Relations-Right-Unique.right-unique*

by (*intro ext*) (*blast intro*: *Relation.right-uniqueI*
dest: *Relation.right-uniqueD Binary-Relations-Right-Unique.right-uniqueD*)

Surjective **overloading**
rel-surjective-at-set ≡ *rel-surjective-at* :: *'a set* ⇒ (*'b* ⇒ *'a* ⇒ *bool*) ⇒ *bool*
begin
definition *rel-surjective-at-set* (*S* :: *'a set*) :: (*'b* ⇒ *'a* ⇒ *bool*) ⇒ - ≡
rel-surjective-at (*mem-of S*)
end

lemma *rel-surjective-at-set-eq-rel-surjective-at-pred* [*simp*]:
(*rel-surjective-at* (*S* :: *'a set*) :: (*'b* ⇒ *'a* ⇒ *bool*) ⇒ *bool*) = *rel-surjective-at* (*mem-of S*)
unfolding *rel-surjective-at-set-def* **by** *simp*

lemma *rel-surjective-at-set-eq-rel-surjective-at-pred-uhint* [*uhint*]:
assumes *P* ≡ *mem-of S*
shows *rel-surjective-at* (*S* :: *'a set*) :: (*'b* ⇒ *'a* ⇒ *bool*) ⇒ *bool* ≡ *rel-surjective-at P*
using *assms* **by** *simp*

lemma *rel-surjective-at-set-iff-rel-surjective-at-pred* [*iff*]:
rel-surjective-at (*S* :: *'a set*) (*R* :: *'b* ⇒ *'a* ⇒ *bool*) ⟷ *rel-surjective-at* (*mem-of S*) *R*
by *simp*

lemma *Transfer-right-total-eq-rel-surjective* [*HOL-bin-rel-alignment*]:
Transfer.right-total = *rel-surjective*
by (*intro ext*) (*fast intro*: *Transfer.right-totalI rel-surjectiveI*
elim: *Transfer.right-totalE rel-surjectiveE*)

Symmetric **overloading**
symmetric-on-set ≡ *symmetric-on* :: *'a set* ⇒ (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*
begin
definition *symmetric-on-set* (*S* :: *'a set*) :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ - ≡
symmetric-on (*mem-of S*)
end

lemma *symmetric-on-set-eq-symmetric-on-pred* [*simp*]:
(*symmetric-on* (*S* :: *'a set*) :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*) = *symmetric-on* (*mem-of S*)
unfolding *symmetric-on-set-def* **by** *simp*

lemma *symmetric-on-set-eq-symmetric-on-pred-uhint* [*uhint*]:
assumes *P* ≡ *mem-of S*
shows *symmetric-on* (*S* :: *'a set*) :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool* ≡ *symmetric-on P*
using *assms* **by** *simp*

lemma *symmetric-on-set-iff-symmetric-on-pred* [*iff*]:

symmetric-on (*S* :: *'a set*) (*R* :: *'a* ⇒ *'a* ⇒ *bool*) ⟷ *symmetric-on* (*mem-of S*) *R*
by *simp*

lemma *symp-on-eq-symmetric-on* [*HOL-bin-rel-alignment*]: *symp-on* = *symmetric-on*
by (*intro ext*) (*blast intro*: *symp-onI dest*: *symmetric-onD symp-onD*)

lemma *symp-eq-symmetric* [*HOL-bin-rel-alignment*]: *symp* = *symmetric*
by (*intro ext*) (*blast intro*: *sympI dest*: *symmetricD sympD*)

Transitive **overloading**
transitive-on-set ≡ *transitive-on* :: *'a set* ⇒ (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*
begin
definition *transitive-on-set* (*S* :: *'a set*) :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ - ≡ *transitive-on* (*mem-of S*)
end

lemma *transitive-on-set-eq-transitive-on-pred* [*simp*]:
(*transitive-on* (*S* :: *'a set*) :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*) = *transitive-on* (*mem-of S*)
unfolding *transitive-on-set-def* **by** *simp*

lemma *transitive-on-set-eq-transitive-on-pred-uhint* [*uhint*]:
assumes *P* ≡ *mem-of S*
shows *transitive-on* (*S* :: *'a set*) :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool* ≡ *transitive-on P*
using *assms* **by** *simp*

lemma *transitive-on-set-iff-transitive-on-pred* [*iff*]:
transitive-on (*S* :: *'a set*) (*R* :: *'a* ⇒ *'a* ⇒ *bool*) ⟷ *transitive-on* (*mem-of S*) *R*
by *simp*

lemma *transp-on-eq-transitive-on* [*HOL-bin-rel-alignment*]: *transp-on* = *transitive-on*
by (*intro ext*) (*blast intro*: *transp-onI dest*: *transp-onD transitive-onD*)

lemma *transp-eq-transitive* [*HOL-bin-rel-alignment*]: *transp* = *transitive*
by (*intro ext*) (*blast intro*: *transpI dest*: *transpD*)

Well-Founded **overloading**
wellfounded-on-set ≡ *wellfounded-on* :: *'a set* ⇒ (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*
begin
definition *wellfounded-on-set* (*S* :: *'a set*) :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ - ≡ *wellfounded-on* (*mem-of S*)
end

lemma *wellfounded-on-set-eq-wellfounded-on-pred* [*simp*]:
(*wellfounded-on* (*S* :: *'a set*) :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*) = *wellfounded-on* (*mem-of S*)
unfolding *wellfounded-on-set-def* **by** *simp*

lemma *wellfounded-on-set-eq-wellfounded-on-pred-uhint* [*uhint*]:
assumes $P \equiv$ *mem-of* S
shows *wellfounded-on* $(S :: 'a\ set) :: ('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool \equiv$ *wellfounded-on* P
using *assms* **by** *simp*

lemma *wellfounded-on-set-iff-wellfounded-on-pred* [*iff*]:
wellfounded-on $(S :: 'a\ set)\ (R :: 'a \Rightarrow 'a \Rightarrow bool) \longleftrightarrow$ *wellfounded-on* (*mem-of* S) R
by *simp*

lemma *wfp-on-eq-wellfounded-on* [*HOL-bin-rel-alignment*]: *wfp-on* = *wellfounded-on*
proof (*urule* (*rr*) *ext iffI wellfounded-onI*)
fix $A\ R\ Q\ x$ **assume** *wfp-on* $A\ R$ **and** *hyps*: *mem-of* $A\ x\ Q\ x$
then show $\exists\, m : mem\text{-}of\ A.\ Q\ m \wedge (\forall\, y : mem\text{-}of\ A.\ R\ y\ m \longrightarrow \neg\ Q\ y)$
by (*induction rule*: *wfp-on-induct*) (*use hyps* **in** *auto*)
next
fix $A :: 'a\ set$ **and** $R :: 'a \Rightarrow 'a \Rightarrow bool$ **assume** *wf*: *wellfounded-on* $A\ R$
{
fix $P\ x$ **assume** *hyps*: $x \in A\ \forall\, x \in A.\ (\forall\, y \in A.\ R\ y\ x \longrightarrow P\ y) \longrightarrow P\ x$
with *wf* **have** $P\ x$
unfolding *wellfounded-on-set-iff-wellfounded-on-pred*
by (*induction rule*: *wellfounded-on-induct*) (*use hyps* **in** *blast*)+
}
then show *wfp-on* $A\ R$ **unfolding** *wfp-on-def* **by** *blast*
qed

lemma *wfp-eq-wellfounded* [*HOL-bin-rel-alignment*]: *wfp* = *wellfounded*
by (*urule fun-cong*[*OF wfp-on-eq-wellfounded-on*])

Bi-Total **lemma** *bi-total-on-set-eq-bi-total-on-pred* [*simp*]:
(*bi-total-on* $(S :: 'a\ set)\ (T :: 'b\ set) :: ('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool =$
bi-total-on (*mem-of* S) (*mem-of* T)
by *auto*

lemma *bi-total-on-set-eq-bi-total-on-pred-uhint* [*uhint*]:
assumes $P \equiv$ *mem-of* S
and $Q \equiv$ *mem-of* T
shows *bi-total-on* $(S :: 'a\ set)\ (T :: 'b\ set) :: ('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool \equiv$ *bi-total-on* $P\ Q$
using *assms* **by** *simp*

lemma *bi-total-on-set-iff-bi-total-on-pred* [*iff*]:
bi-total-on $(S :: 'a\ set)\ (T :: 'b\ set)\ (R :: 'a \Rightarrow 'b \Rightarrow bool) \longleftrightarrow$
bi-total-on (*mem-of* S) (*mem-of* T) R
by *simp*

lemma *Transfer-bi-total-eq-bi-total* [*HOL-bin-rel-alignment*]:

Transfer.bi-total = Binary-Relations-Bi-Total.bi-total
unfolding *bi-total-alt-def* **by** (*auto simp add*: *HOL-bin-rel-alignment*)

Bi-Unique **lemma** *bi-unique-on-set-eq-bi-unique-on-pred* [*simp*]:
(*bi-unique-on* (*S* :: *'a set*) :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *bool*) = *bi-unique-on* (*mem-of S*)
by *auto*

lemma *bi-unique-on-set-eq-bi-unique-on-pred-uhint* [*uhint*]:
assumes *P* ≡ *mem-of S*
shows *bi-unique-on* (*S* :: *'a set*) :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *bool* ≡ *bi-unique-on P*
using *assms* **by** *simp*

lemma *bi-unique-on-set-iff-bi-unique-on-pred* [*iff*]:
bi-unique-on (*S* :: *'a set*) (*R* :: *'a* ⇒ *'b* ⇒ *bool*) ⟷ *bi-unique-on* (*mem-of S*) *R*
by *simp*

lemma *Transfer-bi-unique-eq-bi-unique* [*HOL-bin-rel-alignment*]:
Transfer.bi-unique = Binary-Relations-Bi-Unique.bi-unique
unfolding *bi-unique-alt-def* **by** (*auto simp add*: *HOL-bin-rel-alignment*)

Functions **lemma** *Domainp-eq-in-dom* [*HOL-bin-rel-alignment*]: *Domainp* = *in-dom*
by (*intro ext*) *auto*

lemma *Rangep-eq-in-codom* [*HOL-bin-rel-alignment*]: *Rangep* = *in-codom*
by (*intro ext*) *auto*

lemma *relcompp-eq-rel-comp* [*HOL-bin-rel-alignment*]: *relcompp* = *rel-comp*
by (*intro ext*) *auto*

lemma *conversep-eq-rel-inv* [*HOL-bin-rel-alignment*]: *conversep* = *rel-inv*
by (*intro ext*) *auto*

lemma *eq-onp-eq-eq-restrict* [*HOL-bin-rel-alignment*]: *eq-onp* = *rel-restrict-left* (=)
unfolding *eq-onp-def* **by** (*intro ext*) *auto*

definition *rel-restrict-left-set* (*R* :: *'a* ⇒ *'b* ⇒ *bool*) (*S* :: *'a set*) ≡ $R\restriction_{\textit{mem-of } S}$
adhoc-overloading *rel-restrict-left* ⇌ *rel-restrict-left-set*

definition *rel-restrict-right-set* (*R* :: *'a* ⇒ *'b* ⇒ *bool*) (*S* :: *'b set*) ≡ $R\restriction_{\textit{mem-of } S}$
adhoc-overloading *rel-restrict-right* ⇌ *rel-restrict-right-set*

definition *rel-restrict-set* (*R* :: *'a* ⇒ *'a* ⇒ *bool*) (*S* :: *'a set*) ≡ $R\uparrow_{\textit{mem-of } S}$
adhoc-overloading *rel-restrict* ⇌ *rel-restrict-set*

lemma *rel-restrict-left-set-eq-restrict-left-pred* [*simp*]:
$R\restriction_S = R\restriction_{\textit{mem-of } S}$
unfolding *rel-restrict-left-set-def* **by** *simp*

lemma *rel-restrict-left-set-eq-restrict-left-pred-uhint* [*uhint*]:
assumes $R \equiv R'$
and $P \equiv$ *mem-of* S
shows $R\restriction_S \equiv R'\restriction_P$
using *assms* **by** *simp*

lemma *rel-restrict-left-set-iff-restrict-left-pred* [*iff*]: $R\restriction_S\ x\ y \longleftrightarrow R\restriction_{\textit{mem-of}\ S}\ x\ y$
by *simp*

lemma *rel-restrict-right-set-eq-restrict-right-pred* [*simp*]:
$R\restriction_S = R\restriction_{\textit{mem-of}\ S}$
unfolding *rel-restrict-right-set-def* **by** *simp*

lemma *rel-restrict-right-set-eq-restrict-right-pred-uhint* [*uhint*]:
assumes $R \equiv R'$
and $P \equiv$ *mem-of* S
shows $R\restriction_S \equiv R'\restriction_P$
using *assms* **by** *simp*

lemma *rel-restrict-right-set-iff-restrict-right-pred* [*iff*]: $R\restriction_S\ x\ y \longleftrightarrow R\restriction_{\textit{mem-of}\ S}\ x\ y$
by *simp*

lemma *rel-restrict-set-eq-restrict-pred* [*simp*]:
$R\uparrow_S = R\uparrow_{\textit{mem-of}\ S}$
unfolding *rel-restrict-set-def* **by** *simp*

lemma *rel-restrict-set-eq-restrict-pred-uhint* [*uhint*]:
assumes $R \equiv R'$
and $P \equiv$ *mem-of* S
shows $R\uparrow_S \equiv R'\uparrow_P$
using *assms* **by** *simp*

lemma *rel-restrict-set-iff-restrict-pred* [*iff*]: $R\uparrow_S\ x\ y \longleftrightarrow R\uparrow_{\textit{mem-of}\ S}\ x\ y$
by *simp*

end

2.6.2 Function Syntax

theory *HOL-Syntax-Bundles-Functions*
imports *HOL.Fun*
begin

bundle *HOL-function-syntax*
begin
notation *comp* (**infixl** ‹∘› *55*)
end

end

2.6.3 Alignment With Function Definitions from HOL.Main

theory *HOL-Alignment-Functions*
imports
HOL-Alignment-Binary-Relations
HOL-Syntax-Bundles-Functions
LFunctions
begin

unbundle *no HOL-function-syntax*

named-theorems *HOL-fun-alignment*

Functions

Bijection **definition** *bijection-on-set* $(S :: 'a\ set)\ (S' :: 'b\ set) :: ('a \Rightarrow 'b) \Rightarrow ('b \Rightarrow 'a) \Rightarrow bool \equiv$
bijection-on (*mem-of S*) (*mem-of S'*)
adhoc-overloading *bijection-on* $\rightleftharpoons$ *bijection-on-set*

lemma *bijection-on-set-eq-bijection-on-pred* [*simp*]:
bijection-on $(S :: 'a\ set)\ (S' :: 'b\ set)$ = *bijection-on* (*mem-of S*) (*mem-of S'*)
unfolding *bijection-on-set-def* **by** *simp*

lemma *bijection-on-set-eq-bijection-on-pred-uhint* [*uhint*]:
assumes $P \equiv$ *mem-of S*
and $Q \equiv$ *mem-of S'*
shows *bijection-on S S'* $\equiv$ *bijection-on P Q*
using *assms* **by** *simp*

lemma *bijection-on-set-iff-bijection-on-pred* [*iff*]:
bijection-on $(S :: 'a\ set)\ (S' :: 'b\ set)\ (f :: 'a \Rightarrow 'b)\ (g :: 'b \Rightarrow 'a) \longleftrightarrow$
bijection-on (*mem-of S*) (*mem-of S'*) *f g*
by *simp*

lemma *bij-betw-bijection-onE*:
assumes *bij-betw* $(f :: 'a \Rightarrow 'b)$ *S S'*
obtains $g :: 'b \Rightarrow 'a$ **where** *bijection-on S S' f g*
proof
let *?g* = *the-inv-into S f*
from *assms bij-betw-the-inv-into* **have** *bij-betw ?g S' S* **by** *blast*
with *assms* **show** *bijection-on S S' f ?g*
by (*auto intro*!: *bijection-onI*
dest: *bij-betw-apply bij-betw-imp-inj-on the-inv-into-f-f*
simp: *f-the-inv-into-f-bij-betw*)
qed

lemma *bij-betw-if-bijection-on*:
assumes *bijection-on S S′* (*f* :: *′a* ⇒ *′b*) (*g* :: *′b* ⇒ *′a*)
shows *bij-betw f S S′*
using *assms* **by** (*intro bij-betw-byWitness*[**where** *?f′*=*g*])
(*auto elim*: *bijection-onE dest*: *inverse-onD*)

corollary *bij-betw-iff-ex-bijection-on* [*HOL-fun-alignment*]:
bij-betw (*f* :: *′a* ⇒ *′b*) *S S′* ⟷ (∃(*g* :: *′b* ⇒ *′a*). *bijection-on S S′ f g*)
by (*intro iffI*) (*auto elim*!: *bij-betw-bijection-onE intro*: *bij-betw-if-bijection-on*)

Injective overloading
injective-on-set ≡ *injective-on* :: *′a set* ⇒ (*′a* ⇒ *′b*) ⇒ *bool*
begin
definition *injective-on-set* (*S* :: *′a set*) :: (*′a* ⇒ *′b*) ⇒ *bool* ≡
injective-on (*mem-of S*)
end

lemma *injective-on-set-eq-injective-on-pred* [*simp*]:
(*injective-on* (*S* :: *′a set*) :: (*′a* ⇒ *′b*) ⇒ -) = *injective-on* (*mem-of S*)
unfolding *injective-on-set-def* **by** *simp*

lemma *injective-on-set-eq-injective-on-pred-uhint* [*uhint*]:
assumes *P* ≡ *mem-of S*
shows *injective-on* (*S* :: *′a set*) :: (*′a* ⇒ *′b*) ⇒ - ≡ *injective-on P*
using *assms* **by** *simp*

lemma *injective-on-set-iff-injective-on-pred* [*iff*]:
injective-on (*S* :: *′a set*) (*f* :: *′a* ⇒ *′b*) ⟷ *injective-on* (*mem-of S*) *f*
by *simp*

lemma *inj-on-iff-injective-on* [*HOL-fun-alignment*]: *inj-on f P* ⟷ *injective-on P f*
by (*auto intro*: *inj-onI dest*: *inj-onD injective-onD*)

lemma *inj-eq-injective* [*HOL-fun-alignment*]: *inj* = *injective*
by (*auto intro*: *injI dest*: *injD injectiveD*)

Inverse overloading
inverse-on-set ≡ *inverse-on* :: *′a set* ⇒ (*′a* ⇒ *′b*) ⇒ (*′b* ⇒ *′a*) ⇒ *bool*
begin
definition *inverse-on-set* (*S* :: *′a set*) :: (*′a* ⇒ *′b*) ⇒ (*′b* ⇒ *′a*) ⇒ *bool* ≡
inverse-on (*mem-of S*)
end

lemma *inverse-on-set-eq-inverse-on-pred* [*simp*]:
(*inverse-on* (*S* :: *′a set*) :: (*′a* ⇒ *′b*) ⇒ (*′b* ⇒ *′a*) ⇒ -) = *inverse-on* (*mem-of S*)
unfolding *inverse-on-set-def* **by** *simp*

lemma *inverse-on-set-eq-inverse-on-pred-uhint* [*uhint*]:
assumes $P \equiv$ *mem-of* S
shows *inverse-on* $(S :: 'a\ set) :: ('a \Rightarrow 'b) \Rightarrow ('b \Rightarrow 'a) \Rightarrow$ - $\equiv$ *inverse-on* P
using *assms* **by** *simp*

lemma *inverse-on-set-iff-inverse-on-pred* [*iff*]:
inverse-on $(S :: 'a\ set)$ $(f :: 'a \Rightarrow 'b)$ $(g :: 'b \Rightarrow 'a) \longleftrightarrow$ *inverse-on* $(mem\text{-}of\ S)$ $f\ g$
by *simp*

Monotone **lemma** *monotone-on-eq-mono-wrt-rel-restrict-left-right* [*HOL-fun-alignment*]:
monotone-on $S\ R =$ *mono-wrt-rel* $R\restriction_S\restriction_S$
by (*intro ext*) (*auto intro*!: *monotone-onI dest*: *monotone-onD*)

lemma *monotone-eq-mono-wrt-rel* [*HOL-fun-alignment*]: *monotone* = *mono-wrt-rel*
by (*intro ext*) (*auto intro*: *monotoneI dest*: *monotoneD*)

lemma *pred-fun-eq-mono-wrt-pred* [*HOL-fun-alignment*]: *pred-fun* = *mono-wrt-pred*
by (*intro ext*) *auto*

lemma *Fun-mono-eq-mono* [*HOL-fun-alignment*]: *Fun.mono* = *mono*
by (*intro ext*) (*auto intro*: *Fun.mono-onI dest*: *Fun.monoD*)

lemma *Fun-antimono-eq-antimono* [*HOL-fun-alignment*]: *Fun.antimono* = *antimono*
by (*intro ext*) (*auto intro*: *monotoneI dest*: *monotoneD*)

Surjective **overloading**
surjective-at-set $\equiv$ *surjective-at* $:: 'a\ set \Rightarrow ('b \Rightarrow 'a) \Rightarrow bool$
begin
definition *surjective-at-set* $(S :: 'a\ set) :: ('b \Rightarrow 'a) \Rightarrow bool \equiv$
surjective-at $(mem\text{-}of\ S)$
end

lemma *surjective-at-set-eq-surjective-at-pred* [*simp*]:
(*surjective-at* $(S :: 'a\ set) :: ('b \Rightarrow 'a) \Rightarrow$ -) = *surjective-at* $(mem\text{-}of\ S)$
unfolding *surjective-at-set-def* **by** *simp*

lemma *surjective-at-set-eq-surjective-at-pred-uhint* [*uhint*]:
assumes $P \equiv$ *mem-of* S
shows *surjective-at* $(S :: 'a\ set) :: ('b \Rightarrow 'a) \Rightarrow$ - $\equiv$ *surjective-at* P
using *assms* **by** *simp*

lemma *surjective-at-set-iff-surjective-at-pred* [*iff*]:
surjective-at $(S :: 'a\ set)$ $(f :: 'b \Rightarrow 'a) \longleftrightarrow$ *surjective-at* $(mem\text{-}of\ S)$ f
by *simp*

lemma *surj-eq-surjective* [*HOL-fun-alignment*]: *surj* = *surjective*
by (*intro ext*) (*fast intro*: *surjI dest*: *surjD elim*: *surjectiveE*)

Functions **lemma** *Fun-id-eq-id* [*HOL-fun-alignment*]: *Fun.id* = *Functions-Base.id*
by (*intro ext*) *simp*

lemma *Fun-comp-eq-comp* [*HOL-fun-alignment*]: *Fun.comp* = *Functions-Base.comp*
by (*intro ext*) *simp*

lemma *map-fun-eq-fun-map* [*HOL-fun-alignment*]: *map-fun* = *fun-map*
by (*intro ext*) *simp*

Relators **lemma** *rel-fun-eq-Fun-Rel-rel* [*HOL-fun-alignment*]: *BNF-Def.rel-fun* = *Fun-Rel*
by (*intro ext*) (*auto dest*: *rel-funD*)

end

2.7 Order Syntax

theory *HOL-Syntax-Bundles-Orders*
imports *HOL.Orderings*
begin

bundle *HOL-order-syntax*
begin
notation
less-eq (‹'(≤')›) **and**
less-eq (‹(‹notation=‹infix ≤››-/ ≤ -)› [51, 51] 50) **and**
less (‹'(<')›) **and**
less (‹(‹notation=‹infix <››-/ < -)› [51, 51] 50)
notation (*input*) *greater-eq* (**infix** ‹≥› 50)
notation (*input*) *greater* (**infix** ‹>› 50)
notation (*ASCII*)
less-eq (‹'(<=')›) **and**
less-eq (‹(‹notation=‹infix <=››-/ <= -)› [51, 51] 50)
notation (*input*) *greater-eq* (**infix** ‹>=› 50)
end

end

2.7.1 Alignment With Order Definitions from HOL

theory *HOL-Alignment-Orders*
imports
HOL−Library.Preorder
HOL-Alignment-Binary-Relations
HOL-Syntax-Bundles-Orders
Orders
begin

named-theorems *HOL-order-alignment*

Functions **definition** *rel R x y* $\equiv (x, y) \in R$
lemma *rel-of-eq* [*simp*]: *rel* $= (\lambda R\ x\ y.\ (x, y) \in R)$ **unfolding** *rel-def* **by** *simp*
lemma *rel-of-iff* [*iff*]: *rel R x y* $\longleftrightarrow (x, y) \in R$ **by** *simp*

Bi-Related **overloading**
bi-related-set $\equiv$ *bi-related* :: $'a$ *rel* $\Rightarrow$ $'a \Rightarrow$ $'a \Rightarrow$ *bool*
begin
definition *bi-related-set* $(S :: 'a\ rel) \equiv$ *bi-related* $(rel\ S) :: 'a \Rightarrow 'a \Rightarrow bool$
end

lemma *bi-related-set-eq-bi-related-pred* [*simp*]:
$((\equiv_{S\ ::\ 'a\ rel}) :: 'a \Rightarrow 'a \Rightarrow bool) = (\equiv_{rel\ S})$
unfolding *bi-related-set-def* **by** *simp*

lemma *bi-related-set-eq-bi-related-pred-uhint* [*uhint*]:
assumes $R \equiv rel\ S$
shows $(\equiv_{S\ ::\ 'a\ rel}) :: 'a \Rightarrow 'a \Rightarrow bool \equiv (\equiv_R)$
using *assms* **by** *simp*

lemma *bi-related-set-iff-bi-related-pred* [*iff*]: $(x :: 'a) \equiv_{(S\ ::\ 'a\ rel)} (y :: 'a) \longleftrightarrow x \equiv_{rel\ S} y$
by *simp*

lemma (**in** *preorder-equiv*) *equiv-eq-bi-related* [*HOL-order-alignment*]:
equiv $=$ *bi-related* $(\leq)$
by (*intro ext*) (*auto intro*: *equiv-antisym dest*: *equivD1 equivD2*)

Inflationary **overloading**
inflationary-on-set $\equiv$ *inflationary-on* :: $'a$ *set* $\Rightarrow$ $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b) \Rightarrow bool$
begin
definition *inflationary-on-set* $(S :: 'a\ set) :: ('a \Rightarrow 'b \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b) \Rightarrow bool \equiv$
inflationary-on (*mem-of S*)
end

lemma *inflationary-on-set-eq-inflationary-on-pred* [*simp*]:
$(inflationary\text{-}on\ (S :: 'a\ set) :: ('a \Rightarrow 'b \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b) \Rightarrow bool) =$
inflationary-on (*mem-of S*)
unfolding *inflationary-on-set-def* **by** *simp*

lemma *inflationary-on-set-eq-inflationary-on-pred-uhint* [*uhint*]:
assumes $P \equiv mem\text{-}of\ S$
shows $inflationary\text{-}on\ (S :: 'a\ set) :: ('a \Rightarrow 'b \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b) \Rightarrow bool \equiv inflationary\text{-}on\ P$
using *assms* **by** *simp*

lemma *inflationary-on-set-iff-inflationary-on-pred* [*iff*]:
inflationary-on (*S* :: *'a set*) (*R* :: *'a* ⇒ *'b* ⇒ *bool*) (*f* :: *'a* ⇒ *'b*) ⟷
inflationary-on (*mem-of S*) *R f*
by *simp*

Deflationary **overloading**
deflationary-on-set ≡ *deflationary-on* :: *'a set* ⇒ (*'b* ⇒ *'a* ⇒ *bool*) ⇒ (*'a* ⇒ *'b*) ⇒ *bool*
begin
definition *deflationary-on-set* (*S* :: *'a set*) :: (*'b* ⇒ *'a* ⇒ *bool*) ⇒ (*'a* ⇒ *'b*) ⇒ *bool* ≡
deflationary-on (*mem-of S*)
end

lemma *deflationary-on-set-eq-deflationary-on-pred* [*simp*]:
(*deflationary-on* (*S* :: *'a set*) :: (*'b* ⇒ *'a* ⇒ *bool*) ⇒ (*'a* ⇒ *'b*) ⇒ *bool*) = *deflationary-on* (*mem-of S*)
unfolding *deflationary-on-set-def* **by** *simp*

lemma *deflationary-on-set-eq-deflationary-on-pred-uhint* [*uhint*]:
assumes *P* ≡ *mem-of S*
shows *deflationary-on* (*S* :: *'a set*) :: (*'b* ⇒ *'a* ⇒ *bool*) ⇒ (*'a* ⇒ *'b*) ⇒ *bool* ≡ *deflationary-on P*
using *assms* **by** *simp*

lemma *deflationary-on-set-iff-deflationary-on-pred* [*iff*]:
deflationary-on (*S* :: *'a set*) (*R* :: *'b* ⇒ *'a* ⇒ *bool*) (*f* :: *'a* ⇒ *'b*) ⟷ *deflationary-on* (*mem-of S*) *R f*
by *simp*

Idempotent **overloading**
idempotent-on-set ≡ *idempotent-on* :: *'a set* ⇒ (*'a* ⇒ *'a* ⇒ *bool*) ⇒ (*'a* ⇒ *'a*) ⇒ *bool*
begin
definition *idempotent-on-set* (*S* :: *'a set*) :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ (*'a* ⇒ *'a*) ⇒ *bool* ≡
idempotent-on (*mem-of S*)
end

lemma *idempotent-on-set-eq-idempotent-on-pred* [*simp*]:
(*idempotent-on* (*S* :: *'a set*) :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ (*'a* ⇒ *'a*) ⇒ *bool*) = *idempotent-on* (*mem-of S*)
unfolding *idempotent-on-set-def* **by** *simp*

lemma *idempotent-on-set-iff-idempotent-on-pred* [*iff*]:
idempotent-on (*S* :: *'a set*) (*R* :: *'a* ⇒ *'a* ⇒ *bool*) (*f* :: *'a* ⇒ *'a*) ⟷ *idempotent-on* (*mem-of S*) *R f*
by *simp*

Properties

Equivalence Relations **lemma** *equiv-eq-equivalence-rel* [*HOL-order-alignment*]:
equivp = *equivalence-rel*
by (*intro ext*) (*fastforce intro*: *equivpI*
simp: *HOL-bin-rel-alignment equivalence-rel-eq-equivalence-rel-on*
elim: *equivpE*)

Partial Equivalence Relations **lemma** *part-equiv-eq-partial-equivalence-rel-if-rel*
[*HOL-order-alignment*]:
assumes *R x y*
shows *part-equivp R* = *partial-equivalence-rel R*
using *assms* **by** (*fastforce intro*!: *part-equivpI simp*: *HOL-bin-rel-alignment*
partial-equivalence-rel-eq-partial-equivalence-rel-on
elim!: *part-equivpE*)

Partial Orders **lemma** (**in** *order*) *partial-order* [*HOL-order-alignment*]: *partial-order* (≤)
using *order-refl order-trans order-antisym* **by** *blast*

Preorders **lemma** (**in** *partial-preordering*) *preorder* [*HOL-order-alignment*]: *preorder* (≤)
using *refl trans* **by** *blast*

lemma *partial-preordering-eq* [*HOL-order-alignment*]:
partial-preordering = *Preorders.preorder*
by (*intro ext*) (*auto intro*: *partial-preordering.intro*
dest: *partial-preordering.trans partial-preordering.refl reflexiveD*)

Linear Orders **lemma** (**in** *linorder*) *linear-order*: *linear-order* (≤)
using *linear partial-order* **by** *blast*

Strict Parital Orders **lemma** (**in** *preordering*) *linear-order*: *strict-partial-order* (<)
using *strict-iff-not trans* **by** *blast*

Strict Linear Orders **lemma** (**in** *linorder*) *strict-linear-order*: *strict-linear-order* (<)
using *preordering.linear-order local.order.linear-order*
by (*intro strict-linear-orderI*) *auto*

Well-Orders **lemma** (**in** *wellorder*) *wellorder*: *wellorder* (<)
by (*metis wellorderI local.strict-linear-order local.wfp-on-less wfp-eq-wellfounded*)

end

2.7.2 Alignment With Predicate Definitions from HOL

theory *HOL-Alignment-Predicates*
imports
Main
HOL-Mem-Of
Predicates
begin

named-theorems *HOL-predicate-alignment*

Quantifiers **adhoc-overloading** *ball* $\rightleftharpoons$ *Ball*

lemma *Ball-eq-ball-pred* [*HOL-predicate-alignment*]: $\forall_A = \forall_{\textit{mem-of } A}$ **by** *auto*

lemma *Ball-eq-ball-pred-uhint* [*uhint*]:
assumes $P \equiv \textit{mem-of } A$
shows $\forall_A = \forall_P$
using *assms* **by** (*simp add*: *Ball-eq-ball-pred*)

lemma *Ball-iff-ball-pred* [*HOL-predicate-alignment*]: $(\forall x : A.\ Q\ x) \longleftrightarrow (\forall x : \textit{mem-of } A.\ Q\ x)$
by (*simp add*: *Ball-eq-ball-pred*)

adhoc-overloading *bex* $\rightleftharpoons$ *Bex*

lemma *Bex-eq-bex-pred* [*HOL-predicate-alignment*]: $\exists_A = \exists_{\textit{mem-of } A}$ **by** *fast*

lemma *Bex-eq-bex-pred-uhint* [*uhint*]:
assumes $P \equiv \textit{mem-of } A$
shows $\exists_A = \exists_P$
using *assms* **by** (*simp add*: *Bex-eq-bex-pred*)

lemma *Bex-iff-bex-pred* [*HOL-predicate-alignment*]: $(\exists x : A.\ Q\ x) \longleftrightarrow (\exists x : \textit{mem-of } A.\ Q\ x)$
by (*simp add*: *Bex-eq-bex-pred*)

end

2.8 HOL Alignments

theory *HOL-Alignments*
imports
HOL-Alignment-Binary-Relations
HOL-Alignment-Functions
HOL-Alignment-Orders
HOL-Alignment-Predicates
begin

Summary Alignment of concepts with HOL counterparts

end

2.8.1 Alignment With Order Definitions from HOL-Algebra

theory *HOL-Algebra-Alignment-Orders*
imports
HOL−Algebra.Order
HOL-Alignment-Orders
begin

named-theorems *HOL-Algebra-order-alignment*

context *equivalence*
begin

lemma *reflexive-on-carrier* [*HOL-Algebra-order-alignment*]:
reflexive-on (*carrier S*) (.=)
by *blast*

lemma *transitive-on-carrier* [*HOL-Algebra-order-alignment*]:
transitive-on (*carrier S*) (.=)
using *trans* **by** *blast*

lemma *preorder-on-carrier* [*HOL-Algebra-order-alignment*]:
preorder-on (*carrier S*) (.=)
using *reflexive-on-carrier transitive-on-carrier* **by** *blast*

lemma *symmetric-on-carrier* [*HOL-Algebra-order-alignment*]:
symmetric-on (*carrier S*) (.=)
using *sym* **by** *blast*

lemma *partial-equivalence-rel-on-carrier* [*HOL-Algebra-order-alignment*]:
partial-equivalence-rel-on (*carrier S*) (.=)
using *transitive-on-carrier symmetric-on-carrier* **by** *blast*

lemma *equivalence-rel-on-carrier* [*HOL-Algebra-order-alignment*]:
equivalence-rel-on (*carrier S*) (.=)
using *reflexive-on-carrier partial-equivalence-rel-on-carrier* **by** *blast*

end

lemma *equivalence-iff-equivalence-rel-on-carrier* [*HOL-Algebra-order-alignment*]:
equivalence S $\longleftrightarrow$ *equivalence-rel-on* (*carrier S*) ($.=_S$)
using *equivalence.equivalence-rel-on-carrier*
by (*blast dest*: *intro*!: *equivalence.intro dest*: *symmetric-onD transitive-onD*)

context *partial-order*
begin

lemma *reflexive-on-carrier* [*HOL-Algebra-order-alignment*]:
reflexive-on (*carrier L*) (⊑)
by *blast*

lemma *transitive-on-carrier* [*HOL-Algebra-order-alignment*]:
transitive-on (*carrier L*) (⊑)
using *le-trans* **by** *blast*

lemma *preorder-on-carrier* [*HOL-Algebra-order-alignment*]:
preorder-on (*carrier L*) (⊑)
using *reflexive-on-carrier transitive-on-carrier* **by** *blast*

lemma *antisymmetric-on-carrier* [*HOL-Algebra-order-alignment*]:
antisymmetric-on (*carrier L*) (⊑)
by *blast*

lemma *partial-order-on-carrier* [*HOL-Algebra-order-alignment*]:
partial-order-on (*carrier L*) (⊑)
using *preorder-on-carrier antisymmetric-on-carrier* **by** *blast*

end

end

2.8.2 Alignment With Galois Definitions from HOL-Algebra

theory *HOL-Algebra-Alignment-Galois*
imports
HOL−Algebra.Galois-Connection
HOL-Algebra-Alignment-Orders
Galois
begin

named-theorems *HOL-Algebra-galois-alignment*

context *galois-connection*
begin

context
fixes *L R l r*
defines $L \equiv (\sqsubseteq_{\mathcal{X}})\restriction_{carrier\ \mathcal{X}}\restriction_{carrier\ \mathcal{X}}$ **and** $R \equiv (\sqsubseteq_{\mathcal{Y}})\restriction_{carrier\ \mathcal{Y}}\restriction_{carrier\ \mathcal{Y}}$
and $l \equiv \pi^*$ **and** $r \equiv \pi_*$
notes *defs*[*simp*] = *L-def R-def l-def r-def* **and** *rel-restrict-right-eq*[*simp*]
and *rel-restrict-leftI*[*intro*!] *rel-restrict-leftE*[*elim*!]
begin

interpretation *galois L R l r* **.**

lemma *mono-wrt-rel-lower* [*HOL-Algebra-galois-alignment*]: $(L \Rightarrow R)\ l$
using *lower-closed upper-closed* **by** (*fastforce intro*: *use-iso2*[*OF lower-iso*])

lemma *mono-wrt-rel-upper* [*HOL-Algebra-galois-alignment*]: $(R \Rightarrow L)\ r$
using *lower-closed upper-closed* **by** (*fastforce intro*: *use-iso2*[*OF upper-iso*])

lemma *half-galois-prop-left* [*HOL-Algebra-galois-alignment*]: $(L\ {}_h\trianglelefteq\ R)\ l\ r$
using *galois-property lower-closed* **by** (*intro half-galois-prop-leftI*) *fastforce*

lemma *half-galois-prop-right* [*HOL-Algebra-galois-alignment*]: $(L \trianglelefteq_h\ R)\ l\ r$
using *galois-property upper-closed* **by** (*intro half-galois-prop-rightI*) *fastforce*

lemma *galois-prop* [*HOL-Algebra-galois-alignment*]: $(L \trianglelefteq R)\ l\ r$
using *half-galois-prop-left half-galois-prop-right* **by** *blast*

lemma *galois-connection* [*HOL-Algebra-galois-alignment*]: $(L \dashv R)\ l\ r$
using *mono-wrt-rel-lower mono-wrt-rel-upper galois-prop* **by** *blast*

end
end

context *galois-bijection*
begin

context
fixes $L\ R\ l\ r$
defines $L \equiv (\sqsubseteq_{\mathcal{X}})\restriction_{carrier\ \mathcal{X}}\restriction_{carrier\ \mathcal{X}}$ **and** $R \equiv (\sqsubseteq_{\mathcal{Y}})\restriction_{carrier\ \mathcal{Y}}\restriction_{carrier\ \mathcal{Y}}$
and $l \equiv \pi^*$ **and** $r \equiv \pi_*$
notes *defs*[*simp*] = *L-def R-def l-def r-def* **and** *rel-restrict-right-eq*[*simp*]
and *rel-restrict-leftI*[*intro*!] *rel-restrict-leftE*[*elim*!] *in-codomE*[*elim*!]
begin

interpretation *galois R L r l* **.**

lemma *half-galois-prop-left-right-left* [*HOL-Algebra-galois-alignment*]: $(R\ {}_h\trianglelefteq\ L)\ r\ l$
using *gal-bij-conn.right lower-inv-eq upper-closed upper-inv-eq*
by (*intro half-galois-prop-leftI*; *elim left-GaloisE*) (*auto*; *metis*)

lemma *half-galois-prop-right-right-left* [*HOL-Algebra-galois-alignment*]: $(R \trianglelefteq_h\ L)\ r\ l$
using *gal-bij-conn.left lower-closed lower-inv-eq upper-inv-eq*
by (*intro half-galois-prop-rightI*; *elim Galois-rightE*) (*auto*; *metis*)

lemma *prop-right-right-left* [*HOL-Algebra-galois-alignment*]: $(R \trianglelefteq L)\ r\ l$
using *half-galois-prop-left-right-left half-galois-prop-right-right-left* **by** *blast*

lemma *galois-equivalence* [*HOL-Algebra-galois-alignment*]: $(L \equiv_G\ R)\ l\ r$

using *gal-bij-conn.galois-connection prop-right-right-left*
by (*intro galois.galois-equivalenceI*) *auto*

end
end

end

2.9 HOL-Algebra Alignments

theory *HOL-Algebra-Alignments*
imports
HOL-Algebra-Alignment-Galois
HOL-Algebra-Alignment-Orders
begin

Summary Alignment of concepts with HOL-Algebra counterparts

end

2.10 HOL Syntax Bundles

2.10.1 Basic Syntax

theory *HOL-Syntax-Bundles-Base*
imports *HOL-Basics-Base*
begin

bundle *HOL-ascii-syntax*
begin
notation (*ASCII*) *Not* (‹(‹*open-block notation*=‹*prefix* ~››~ -)› [*40*] *40*)
and *conj* (**infixr** ‹&› *35*)
and *disj* (**infixr** ‹|› *30*)
and *implies* (**infixr** ‹−−>› *25*)
and *not-equal* (**infixl** ‹~=› *50*)
end

end

2.10.2 Group Syntax

theory *HOL-Syntax-Bundles-Groups*
imports *HOL.Groups*
begin

bundle *HOL-groups-syntax*
begin
notation *Groups.zero* (‹*0*›)

notation *Groups.one* (‹*1*›)
notation *Groups.plus* (**infixl** ‹+› *65*)
notation *Groups.minus* (**infixl** ‹−› *65*)
unbundle *uminus-syntax*
notation *Groups.times* (**infixl** ‹∗› *70*)
unbundle *abs-syntax*
end

end

theory *HOL-Syntax-Bundles*
imports
HOL-Syntax-Bundles-Base
HOL-Syntax-Bundles-Functions
HOL-Syntax-Bundles-Groups
HOL-Syntax-Bundles-Lattices
HOL-Syntax-Bundles-Orders
begin

Summary Bundles to enable and disable syntax from HOL.

end

2.10.3 Galois Relator

theory *Transport-Compositions-Agree-Galois-Relator*
imports
Transport-Compositions-Agree-Base
begin

context *transport-comp-agree*
begin

lemma *left-Galois-le-comp-left-GaloisI*:
assumes *in-codom-mono-r2*: (*in-codom* ($\leq_{R2}$) ⇒ *in-codom* ($\leq_{R1}$)) *r2*
and *r2-L2-self-if-in-codom*: $\bigwedge z.$ *in-codom* ($\leq_{R2}$) *z* ⟹ *r2 z* $\leq_{L2}$ *r2 z*
shows ($_{L}\lessapprox$) ≤ (($_{L1}\lessapprox$) ∘∘ ($_{L2}\lessapprox$))
proof (*rule le-relI*)
fix *x z* **assume** *x* $_{L}\lessapprox$ *z*
then have *x* $\leq_{L1}$ *r z in-codom* ($\leq_{R}$) *z* **by** *auto*
with ‹*x* $\leq_{L1}$ *r z*› *in-codom-mono-r2* **have** *x* $_{L1}\lessapprox$ *r2 z* **by** *fastforce*
moreover from ‹*in-codom* ($\leq_{R2}$) *z*› *r2-L2-self-if-in-codom* **have** *r2 z* $_{L2}\lessapprox$ *z*
by (*intro g2.left-GaloisI*) *auto*
ultimately show (($_{L1}\lessapprox$) ∘∘ ($_{L2}\lessapprox$)) *x z* **by** *blast*
qed

lemma *comp-left-Galois-le-left-GaloisI*:
assumes *mono-r1*: (($\leq_{R1}$) ⇒ ($\leq_{L1}$)) *r1*
and *trans-L1*: *transitive* ($\leq_{L1}$)
and *R1-r2-if-in-codom*: $\bigwedge y\ z.$ *in-codom* ($\leq_{R2}$) *z* ⟹ *y* $\leq_{L2}$ *r2 z* ⟹ *y* $\leq_{R1}$ *r2 z*

shows $((_{L1}\lessapprox) \circ\circ (_{L2}\lessapprox)) \leq (_{L}\lessapprox)$
proof (*rule le-relI*)
fix *x z* **assume** $((_{L1}\lessapprox) \circ\circ (_{L2}\lessapprox))\ x\ z$
then obtain *y* **where** $x\ _{L1}\lessapprox\ y\ y\ _{L2}\lessapprox\ z$ **by** *blast*
then have $x \leq_{L1} r1\ y\ y \leq_{L2} r2\ z$ *in-codom* $(\leq_R)\ z$ **by** *auto*
with *R1-r2-if-in-codom* **have** $y \leq_{R1} r2\ z$ **by** *blast*
with *mono-r1* **have** $r1\ y \leq_{L1} r\ z$ **by** *auto*
with ‹$x \leq_{L1} r1\ y$› ‹*in-codom* $(\leq_R)\ z$› **show** $x\ _{L}\lessapprox\ z$ **using** *trans-L1* **by** *blast*
qed

corollary *left-Galois-eq-comp-left-GaloisI*:
assumes $((\leq_{R1}) \Rightarrow (\leq_{L1}))\ r1$
and *transitive* $(\leq_{L1})$
and $\bigwedge z.$ *in-codom* $(\leq_{R2})\ z \Longrightarrow r2\ z \leq_{L2} r2\ z$
and $\bigwedge y\ z.$ *in-codom* $(\leq_{R2})\ z \Longrightarrow y \leq_{L2} r2\ z \Longrightarrow y \leq_{R1} r2\ z$
shows $(_{L}\lessapprox) = ((_{L1}\lessapprox) \circ\circ (_{L2}\lessapprox))$
using *assms*
by (*intro antisym left-Galois-le-comp-left-GaloisI comp-left-Galois-le-left-GaloisI dep-mono-wrt-predI*)
fastforce

corollary *left-Galois-eq-comp-left-GaloisI'*:
assumes $((\leq_{R1}) \Rightarrow (\leq_{L1}))\ r1$
and *transitive* $(\leq_{L1})$
and $((\leq_{R2}) \Rightarrow (\leq_{L2}))\ r2$
and *reflexive-on* (*in-codom* $(\leq_{R2})$) $(\leq_{R2})$
and $\bigwedge y\ z.$ *in-codom* $(\leq_{R2})\ z \Longrightarrow y \leq_{L2} r2\ z \Longrightarrow y \leq_{R1} r2\ z$
shows $(_{L}\lessapprox) = ((_{L1}\lessapprox) \circ\circ (_{L2}\lessapprox))$
using *assms* **by** (*intro left-Galois-eq-comp-left-GaloisI*) (*auto 5 0*)

corollary *left-Galois-eq-comp-left-GaloisI''*:
assumes $((\leq_{R1}) \Rightarrow (\leq_{L1}))\ r1$
and *transitive* $(\leq_{L1})$
and $((\leq_{R2}) \Rightarrow (\leq_{L2}))\ r2$
and *reflexive-on* (*in-codom* $(\leq_{L2})$) $(\leq_{L2})$
and $\bigwedge y\ z.$ *in-codom* $(\leq_{R2})\ z \Longrightarrow y \leq_{L2} r2\ z \Longrightarrow y \leq_{R1} r2\ z$
shows $(_{L}\lessapprox) = ((_{L1}\lessapprox) \circ\circ (_{L2}\lessapprox))$
using *assms* **by** (*intro left-Galois-eq-comp-left-GaloisI*) (*auto 0 6*)

end

context *transport-comp-same*
begin

lemma *left-Galois-eq-comp-left-GaloisI*:
assumes $((\leq_{R1}) \Rightarrow (\leq_{L1}))\ r1$
and *transitive* $(\leq_{L1})$
and $((\leq_{R2}) \Rightarrow (\leq_{R1}))\ r2$
and *reflexive-on* (*in-codom* $(\leq_{R2})$) $(\leq_{R2})$

shows $({}_{L}\lessapprox) = (({}_{L1}\lessapprox) \circ\circ ({}_{L2}\lessapprox))$
using *assms* **by** (*intro left-Galois-eq-comp-left-GaloisI'*) *auto*

lemma *left-Galois-eq-comp-left-GaloisI'*:
assumes $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and *transitive* $(\leq_{L1})$
and *reflexive-on* (*in-codom* $(\leq_{R1})$) $(\leq_{R1})$
and $((\leq_{R2}) \Rightarrow (\leq_{R1}))$ *r2*
shows $({}_{L}\lessapprox) = (({}_{L1}\lessapprox) \circ\circ ({}_{L2}\lessapprox))$
using *assms* **by** (*intro left-Galois-eq-comp-left-GaloisI''*) *auto*

end

end

2.10.4 Order Equivalence

theory *Transport-Compositions-Agree-Order-Equivalence*
imports
Transport-Compositions-Agree-Monotone
begin

context *transport-comp-agree*
begin

Unit

Inflationary **lemma** *inflationary-on-unitI*:
assumes *mono-l1*: $(P \Rightarrow P')$ *l1*
and *mono-r1*: $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and *inflationary-unit1*: *inflationary-on* P $(\leq_{L1})$ η_1
and *trans-L1*: *transitive* $(\leq_{L1})$
and *inflationary-unit2*: *inflationary-on* P' $(\leq_{L2})$ η_2
and *L2-le-R1*: $\bigwedge x.\ P\ x \Longrightarrow l1\ x \leq_{L2} r2\ (l\ x) \Longrightarrow l1\ x \leq_{R1} r2\ (l\ x)$
shows *inflationary-on* P $(\leq_{L})$ η
proof (*rule inflationary-onI*)
fix x **assume** $P\ x$
with *mono-l1* **have** $P'\ (l1\ x)$ **by** *blast*
with *inflationary-unit2* **have** $l1\ x \leq_{L2} r2\ (l\ x)$ **by** *auto*
with *L2-le-R1* ‹$P\ x$› **have** $l1\ x \leq_{R1} r2\ (l\ x)$ **by** *blast*
with *mono-r1* **have** $\eta_1\ x \leq_{L1} \eta\ x$ **by** *auto*
moreover from *inflationary-unit1* ‹$P\ x$› **have** $x \leq_{L1} \eta_1\ x$ **by** *auto*
ultimately show $x \leq_{L} \eta\ x$ **using** *trans-L1* **by** *blast*
qed

corollary *inflationary-on-in-field-unitI*:
assumes $((\leq_{L1}) \Rightarrow (\leq_{L2}))$ *l1*
and $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and *inflationary-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$ η_1

and *transitive* $(\leq_{L1})$
and *inflationary-on* (*in-field* $(\leq_{L2})$) $(\leq_{L2})$ η_2
and $\bigwedge x$. *in-field* $(\leq_{L1})$ $x \Longrightarrow$ *l1* $x \leq_{L2}$ *r2* (*l* x) $\Longrightarrow$ *l1* $x \leq_{R1}$ *r2* (*l* x)
shows *inflationary-on* (*in-field* $(\leq_L)$) $(\leq_L)$ η
using *assms* **by** (*intro inflationary-on-unitI dep-mono-wrt-predI*) (*auto 5 0*)

Deflationary context
begin

interpretation *inv* :
transport-comp-agree $(\geq_{L1})$ $(\geq_{R1})$ *l1 r1* $(\geq_{L2})$ $(\geq_{R2})$ *l2 r2*
rewrites $\bigwedge R\ S.\ (R^{-1} \Rightarrow S^{-1}) \equiv (R \Rightarrow S)$
and $\bigwedge(P :: 'i \Rightarrow bool)\ (R :: 'j \Rightarrow 'i \Rightarrow bool).$
(*inflationary-on P* R^{-1} $:: ('i \Rightarrow 'j) \Rightarrow bool) \equiv$ *deflationary-on P R*
and $\bigwedge(R :: 'i \Rightarrow 'i \Rightarrow bool).$ *transitive* $R^{-1} \equiv$ *transitive R*
and $\bigwedge R.$ *in-field* $R^{-1} \equiv$ *in-field R*
by (*simp-all add*: *mono-wrt-rel-eq-dep-mono-wrt-rel*)

lemma *deflationary-on-in-field-unitI*:
assumes $((\leq_{L1}) \Rightarrow (\leq_{L2}))$ *l1*
and $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and *deflationary-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$ η_1
and *transitive* $(\leq_{L1})$
and *deflationary-on* (*in-field* $(\leq_{L2})$) $(\leq_{L2})$ η_2
and $\bigwedge x$. *in-field* $(\leq_{L1})$ $x \Longrightarrow$ *r2* (*l* x) $\leq_{L2}$ *l1* $x \Longrightarrow$ *r2* (*l* x) $\leq_{R1}$ *l1* x
shows *deflationary-on* (*in-field* $(\leq_L)$) $(\leq_L)$ η
using *assms* **by** (*intro inv.inflationary-on-in-field-unitI*[*simplified rel-inv-iff-rel*])
auto

end

Relational Equivalence

corollary *rel-equivalence-on-in-field-unitI*:
assumes $((\leq_{L1}) \Rightarrow (\leq_{L2}))$ *l1*
and $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and *rel-equivalence-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$ η_1
and *transitive* $(\leq_{L1})$
and *rel-equivalence-on* (*in-field* $(\leq_{L2})$) $(\leq_{L2})$ η_2
and $\bigwedge x$. *in-field* $(\leq_{L1})$ $x \Longrightarrow$ *l1* $x \leq_{L2}$ *r2* (*l* x) $\Longrightarrow$ *l1* $x \leq_{R1}$ *r2* (*l* x)
and $\bigwedge x$. *in-field* $(\leq_{L1})$ $x \Longrightarrow$ *r2* (*l* x) $\leq_{L2}$ *l1* $x \Longrightarrow$ *r2* (*l* x) $\leq_{R1}$ *l1* x
shows *rel-equivalence-on* (*in-field* $(\leq_L)$) $(\leq_L)$ η
using *assms* **by** (*intro rel-equivalence-onI*
inflationary-on-in-field-unitI deflationary-on-in-field-unitI)
auto

Counit

Corresponding lemmas for the counit can be obtained by flipping the interpretation of the locale.

Order Equivalence

interpretation *flip* : *transport-comp-agree R2 L2 r2 l2 R1 L1 r1 l1*
rewrites $flip.g1.unit \equiv \varepsilon_2$ **and** $flip.g2.unit \equiv \varepsilon_1$ **and** $flip.unit \equiv \varepsilon$
by (*simp-all only*: *g1.flip-unit-eq-counit g2.flip-unit-eq-counit flip-unit-eq-counit*)

lemma *order-equivalenceI*:
assumes $((\leq_{L1}) \equiv_o (\leq_{R1}))$ *l1 r1*
and *transitive* $(\leq_{L1})$
and $((\leq_{L2}) \equiv_o (\leq_{R2}))$ *l2 r2*
and *transitive* $(\leq_{R2})$
and $\bigwedge x\ y.\ x \leq_{L1} y \Longrightarrow l1\ x \leq_{R1} l1\ y \Longrightarrow l1\ x \leq_{L2} l1\ y$
and $\bigwedge x\ y.\ x \leq_{R2} y \Longrightarrow r2\ x \leq_{L2} r2\ y \Longrightarrow r2\ x \leq_{R1} r2\ y$
and $\bigwedge x.$ *in-field* $(\leq_{L1})\ x \Longrightarrow l1\ x \leq_{L2} r2\ (l\ x) \Longrightarrow l1\ x \leq_{R1} r2\ (l\ x)$
and $\bigwedge x.$ *in-field* $(\leq_{L1})\ x \Longrightarrow r2\ (l\ x) \leq_{L2} l1\ x \Longrightarrow r2\ (l\ x) \leq_{R1} l1\ x$
and $\bigwedge x.$ *in-field* $(\leq_{R2})\ x \Longrightarrow r2\ x \leq_{R1} l1\ (r\ x) \Longrightarrow r2\ x \leq_{L2} l1\ (r\ x)$
and $\bigwedge x.$ *in-field* $(\leq_{R2})\ x \Longrightarrow l1\ (r\ x) \leq_{R1} r2\ x \Longrightarrow l1\ (r\ x) \leq_{L2} r2\ x$
shows $((\leq_L) \equiv_o (\leq_R))$ *l r*
using *assms* **by** (*intro order-equivalenceI rel-equivalence-on-in-field-unitI*
flip.rel-equivalence-on-in-field-unitI
mono-wrt-rel-leftI flip.mono-wrt-rel-leftI mono-wrt-relI)
(*auto elim*!: *g1.order-equivalenceE g2.order-equivalenceE*)

end

context *transport-comp-same*
begin

lemma *order-equivalenceI*:
assumes $((\leq_{L1}) \equiv_o (\leq_{R1}))$ *l1 r1*
and *transitive* $(\leq_{L1})$
and $((\leq_{R1}) \equiv_o (\leq_{R2}))$ *l2 r2*
and *transitive* $(\leq_{R2})$
shows $((\leq_L) \equiv_o (\leq_R))$ *l r*
using *assms* **by** (*rule order-equivalenceI*) *auto*

end

end

theory *Transport-Compositions-Agree*
imports
Transport-Compositions-Agree-Galois-Equivalence
Transport-Compositions-Agree-Galois-Relator
Transport-Compositions-Agree-Order-Equivalence
begin

Summary The general - though probably not very useful - results for the composition of transportable components under the condition of agreeing middle relations can be found in *transport-comp-agree*. The special case of a coinciding middle relation can be found in *transport-comp-same*. The latter corresponds to the well-know result in the literature, generalised to partial Galois connections and equivalences.

end

2.11 Generic Compositions

2.11.1 Basic Setup

theory *Transport-Compositions-Generic-Base*
imports
Equivalence-Relations
Transport-Base
begin

locale *transport-comp* =
t1 : *transport L1 R1 l1 r1* + *t2* : *transport L2 R2 l2 r2*
for *L1* :: $'a \Rightarrow 'a \Rightarrow bool$
and *R1* :: $'b \Rightarrow 'b \Rightarrow bool$
and *l1* :: $'a \Rightarrow 'b$
and *r1* :: $'b \Rightarrow 'a$
and *L2* :: $'b \Rightarrow 'b \Rightarrow bool$
and *R2* :: $'c \Rightarrow 'c \Rightarrow bool$
and *l2* :: $'b \Rightarrow 'c$
and *r2* :: $'c \Rightarrow 'b$
begin

This locale collects results about the composition of transportable components under some generic compatibility conditions on *R1* and *L2* (cf. below). The composition is rather subtle, but in return can cover quite general cases.

Explanations and intuition about the construction can be found in [2].

notation *L1* (**infix** $\langle \leq_{L1} \rangle$ *50*)
notation *R1* (**infix** $\langle \leq_{R1} \rangle$ *50*)
notation *L2* (**infix** $\langle \leq_{L2} \rangle$ *50*)
notation *R2* (**infix** $\langle \leq_{R2} \rangle$ *50*)

notation *t1.ge-left* (**infix** $\langle \geq_{L1} \rangle$ *50*)
notation *t1.ge-right* (**infix** $\langle \geq_{R1} \rangle$ *50*)
notation *t2.ge-left* (**infix** $\langle \geq_{L2} \rangle$ *50*)
notation *t2.ge-right* (**infix** $\langle \geq_{R2} \rangle$ *50*)

notation *t1.left-Galois* (**infix** $\langle {}_{L1}\lessapprox \rangle$ *50*)
notation *t1.right-Galois* (**infix** $\langle {}_{R1}\lessapprox \rangle$ *50*)

notation *t2.left-Galois* (**infix** $\langle_{L2}\lessapprox\rangle$ *50*)
notation *t2.right-Galois* (**infix** $\langle_{R2}\lessapprox\rangle$ *50*)

notation *t1.ge-Galois-left* (**infix** $\langle\gtrapprox_{L1}\rangle$ *50*)
notation *t1.ge-Galois-right* (**infix** $\langle\gtrapprox_{R1}\rangle$ *50*)
notation *t2.ge-Galois-left* (**infix** $\langle\gtrapprox_{L2}\rangle$ *50*)
notation *t2.ge-Galois-right* (**infix** $\langle\gtrapprox_{R2}\rangle$ *50*)

notation *t1.right-ge-Galois* (**infix** $\langle_{R1}\gtrapprox\rangle$ *50*)
notation *t1.Galois-right* (**infix** $\langle\lessapprox_{R1}\rangle$ *50*)
notation *t2.right-ge-Galois* (**infix** $\langle_{R2}\gtrapprox\rangle$ *50*)
notation *t2.Galois-right* (**infix** $\langle\lessapprox_{R2}\rangle$ *50*)

notation *t1.left-ge-Galois* (**infix** $\langle_{L1}\gtrapprox\rangle$ *50*)
notation *t1.Galois-left* (**infix** $\langle\lessapprox_{L1}\rangle$ *50*)
notation *t2.left-ge-Galois* (**infix** $\langle_{L2}\gtrapprox\rangle$ *50*)
notation *t2.Galois-left* (**infix** $\langle\lessapprox_{L2}\rangle$ *50*)

notation *t1.unit* ($\langle\eta_1\rangle$)
notation *t1.counit* ($\langle\varepsilon_1\rangle$)
notation *t2.unit* ($\langle\eta_2\rangle$)
notation *t2.counit* ($\langle\varepsilon_2\rangle$)

definition $L \equiv ({}_{L1}\lessapprox) \circ\circ (\leq_{L2}) \circ\circ ({}_{R1}\lessapprox)$

lemma *left-rel-eq-comp*: $L = ({}_{L1}\lessapprox) \circ\circ (\leq_{L2}) \circ\circ ({}_{R1}\lessapprox)$
  unfolding *L-def* **..**

definition $l \equiv l2 \circ l1$

lemma *left-eq-comp*: $l = l2 \circ l1$
  unfolding *l-def* **..**

lemma *left-eq* [*simp*]: $l\ x = l2\ (l1\ x)$
  unfolding *left-eq-comp* **by** *simp*

context
begin

interpretation *flip* : *transport-comp R2 L2 r2 l2 R1 L1 r1 l1* **.**

abbreviation $R \equiv flip.L$
abbreviation $r \equiv flip.l$

lemma *right-rel-eq-comp*: $R = ({}_{R2}\lessapprox) \circ\circ (\leq_{R1}) \circ\circ ({}_{L2}\lessapprox)$
  unfolding *flip.L-def* **..**

lemma *right-eq-comp*: $r = r1 \circ r2$
  unfolding *flip.l-def* **..**

lemma *right-eq* [*simp*]: $r\ z = r1\ (r2\ z)$
unfolding *right-eq-comp* **by** *simp*

lemmas *transport-defs* = *left-rel-eq-comp left-eq-comp right-rel-eq-comp right-eq-comp*

end

sublocale *transport L R l r* .

notation L (**infix** $\langle\leq_L\rangle$ *50*)
notation R (**infix** $\langle\leq_R\rangle$ *50*)

lemma *left-relI* [*intro*]:
assumes $x\ _{L1}\lessapprox\ y$
and $y \leq_{L2} y'$
and $y'\ _{R1}\lessapprox\ x'$
shows $x \leq_L x'$
unfolding *left-rel-eq-comp* **using** *assms* **by** *blast*

lemma *left-relE* [*elim*]:
assumes $x \leq_L x'$
obtains $y\ y'$ **where** $x\ _{L1}\lessapprox\ y\ y \leq_{L2} y'\ y'\ _{R1}\lessapprox\ x'$
using *assms* **unfolding** *left-rel-eq-comp* **by** *blast*

context
begin

interpretation *flip* : *transport-comp R2 L2 r2 l2 R1 L1 r1 l1* .
interpretation *inv* : *transport-comp* $(\geq_{L1})$ $(\geq_{R1})$ *l1 r1* $(\geq_{L2})$ $(\geq_{R2})$ *l2 r2* .

lemma *ge-left-rel-eq-left-rel-inv-if-galois-prop* [*simp*]:
assumes $((\leq_{L1}) \trianglelefteq (\leq_{R1}))$ *l1 r1* $((\leq_{R1}) \trianglelefteq (\leq_{L1}))$ *r1 l1*
shows $(\geq_L)$ = *transport-comp.L* $(\geq_{L1})$ $(\geq_{R1})$ *l1 r1* $(\geq_{L2})$
using *assms* **unfolding** *left-rel-eq-comp inv.left-rel-eq-comp*
by (*simp add*: *rel-comp-assoc*)

corollary *left-rel-inv-iff-left-rel-if-galois-prop* [*iff*]:
assumes $((\leq_{L1}) \trianglelefteq (\leq_{R1}))$ *l1 r1* $((\leq_{R1}) \trianglelefteq (\leq_{L1}))$ *r1 l1*
shows (*transport-comp.L* $(\geq_{L1})$ $(\geq_{R1})$ *l1 r1* $(\geq_{L2})$) $x\ x' \longleftrightarrow x' \leq_L x$
using *assms* **by** (*simp flip*: *ge-left-rel-eq-left-rel-inv-if-galois-prop*)

Simplification of Relations

lemma *left-rel-le-left-rel1I*:
assumes $((\leq_{L1}) \trianglelefteq_h (\leq_{R1}))$ *l1 r1*
and $((\leq_{R1})\ {}_h\trianglelefteq (\leq_{L1}))$ *r1 l1*
and *trans-L1*: *transitive* $(\leq_{L1})$

and *mono-l1*: $((\leq_L) \Rightarrow ((\leq_{R1}) \circ\circ (\leq_{R1})))\ l1$
shows $(\leq_L) \leq (\leq_{L1})$
proof (*rule le-relI*)
fix $x\ x'$ **assume** $x \leq_L x'$
with *mono-l1* **obtain** y **where** $l1\ x \leq_{R1} y\ y \leq_{R1} l1\ x'$ **by** *blast*
with $\langle((\leq_{L1}) \trianglelefteq_h (\leq_{R1}))\ l1\ r1\rangle\ \langle x \leq_L x'\rangle$ **have** $x \leq_{L1} r1\ y$ **by** *blast*
moreover from $\langle((\leq_{R1})\ {}_h\trianglerighteq (\leq_{L1}))\ r1\ l1\rangle\ \langle y \leq_{R1} l1\ x'\rangle\ \langle x \leq_L x'\rangle$
have ... $\leq_{L1} x'$ **by** *blast*
ultimately show $x \leq_{L1} x'$ **using** *trans-L1* **by** *blast*
qed

lemma *left-rel1-le-left-relI*:
assumes $((\leq_{L1}) \trianglelefteq_h (\leq_{R1}))\ l1\ r1$
and *mono-l1*: $((\leq_{L1}) \Rightarrow ((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})))\ l1$
shows $(\leq_{L1}) \leq (\leq_L)$
proof (*rule le-relI*)
fix $x\ x'$ **assume** $x \leq_{L1} x'$
with *mono-l1* **obtain** $y\ y'$ **where**
$l1\ x \leq_{R1} y\ y \leq_{L2} y'\ y' \leq_{R1} l1\ x'$ **by** *blast*
with $\langle((\leq_{L1}) \trianglelefteq_h (\leq_{R1}))\ l1\ r1\rangle\ \langle x \leq_{L1} x'\rangle$ **have** $x\ {}_{L1}\lessapprox y$ **by** *blast*
moreover note $\langle y \leq_{L2} y'\rangle$
moreover from $\langle y' \leq_{R1} l1\ x'\rangle\ \langle x \leq_{L1} x'\rangle$ **have** $y'\ {}_{R1}\lessapprox x'$ **by** *blast*
ultimately show $x \leq_L x'$ **by** *blast*
qed

corollary *left-rel-eq-left-rel1I*:
assumes $((\leq_{L1}) \trianglelefteq_h (\leq_{R1}))\ l1\ r1$
and $((\leq_{R1})\ {}_h\trianglerighteq (\leq_{L1}))\ r1\ l1$
and *transitive* $(\leq_{L1})$
and $((\leq_L) \Rightarrow ((\leq_{R1}) \circ\circ (\leq_{R1})))\ l1$
and $((\leq_{L1}) \Rightarrow ((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})))\ l1$
shows $(\leq_L) = (\leq_{L1})$
using *assms* **by** (*intro antisym left-rel-le-left-rel1I left-rel1-le-left-relI*)

Note that we may not necessarily have $\mathit{flip}.R = (\leq_{L1})$, even in case of equivalence relations. Depending on the use case, one thus may wish to use an alternative composition operation.

lemma *ex-order-equiv-left-rel-neq-left-rel1*:
$\exists (L1 :: \mathit{bool} \Rightarrow \text{-})\ (R1 :: \mathit{bool} \Rightarrow \text{-})\ l1\ r1$
$(L2 :: \mathit{bool} \Rightarrow \text{-})\ (R2 :: \mathit{bool} \Rightarrow \text{-})\ l2\ r2.$
$(L1 \equiv_o R1)\ l1\ r1$
$\wedge$ *equivalence-rel L1* $\wedge$ *equivalence-rel R1*
$\wedge\ (L2 \equiv_o R2)\ l2\ r2$
$\wedge$ *equivalence-rel L2* $\wedge$ *equivalence-rel R2*
$\wedge$ *transport-comp.L L1 R1 l1 r1 L2* $\neq$ *L1*
proof (*intro exI conjI*)
let *?L1* $= (=) ::$ *bool* $\Rightarrow$ - **let** *?R1* $=$ *?L1* **let** *?l1* $=$ *id* **let** *?r1* $=$ *?l1*
let *?L2* $= \top ::$ *bool* $\Rightarrow$ *bool* $\Rightarrow$ *bool* **let** *?R2* $=$ *?L2* **let** *?l2* $=$ *id* **let** *?r2* $=$ *?l2*
interpret *tc* : *transport-comp ?L1 ?R1 ?l1 ?r1 ?L2 ?R2 ?l2 ?r2* **.**

show (*?L1* $\equiv_o$ *?R1*) *?l1 ?r1* **by** *fastforce*
show *equivalence-rel ?L1 equivalence-rel ?R1* **by** (*fact equivalence-eq*)+
show (*?L2* $\equiv_o$ *?R2*) *?l2 ?r2* **by** *fastforce*
show *equivalence-rel ?L2 equivalence-rel ?R2* **by** (*fact equivalence-top*)+
show *tc.L* $\neq$ *?L1*
proof −
have ¬(*?L1 False True*) **by** *blast*
moreover have *tc.L False True* **by** (*intro tc.left-relI*) *auto*
ultimately show *?thesis* **by** *auto*
qed
qed

end

Generic Left to Right Introduction Rules

The following lemmas generalise the proof outline used, for example, when proving monotonicity and the Galois property (cf. the paper [2]).

interpretation *flip* : *transport-comp R2 L2 r2 l2 R1 L1 r1 l1* **.**

lemma *right-rel-if-left-relI*:
assumes $x \leq_L x'$
and *l1-R1-if-L1-r1*: $\bigwedge y.$ *in-codom* $(\leq_{R1})$ $y \Longrightarrow x \leq_{L1} r1\ y \Longrightarrow l1\ x \leq_{R1} y$
and *t-R1-if-l1-R1*: $\bigwedge y.$ *l1* $x \leq_{R1} y \Longrightarrow t\ y \leq_{R1} y$
and *R2-l2-if-t-L2-if-l1-R1*:
$\bigwedge y\ y'.$ *l1* $x \leq_{R1} y \Longrightarrow t\ y \leq_{L2} y' \Longrightarrow z \leq_{R2} l2\ y'$
and *R1-b-if-R1-l1-if-R1-l1*:
$\bigwedge y\ y'.$ $y \leq_{R1} l1\ x' \Longrightarrow y' \leq_{R1} l1\ x' \Longrightarrow y' \leq_{R1} b\ y$
and *b-L2-r2-if-in-codom-L2-b-if-R1-l1*:
$\bigwedge y.$ $y \leq_{R1} l1\ x' \Longrightarrow$ *in-codom* $(\leq_{L2})$ $(b\ y) \Longrightarrow b\ y \leq_{L2} r2\ z'$
and *in-codom-R2-if-in-codom-L2-b-if-R1-l1*:
$\bigwedge y.$ $y \leq_{R1} l1\ x' \Longrightarrow$ *in-codom* $(\leq_{L2})$ $(b\ y) \Longrightarrow$ *in-codom* $(\leq_{R2})$ z'
and *rel-comp-le*: $(\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1}) \leq (\leq_{L2}) \circ\circ (\leq_{R1})$
and *in-codom-rel-comp-le*: *in-codom* $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq$ *in-codom* $(\leq_{L2})$
shows $z \leq_R z'$
proof −
from ‹$x \leq_L x'$› **obtain** *yl yl'* **where** *l1* $x \leq_{R1} yl\ yl \leq_{L2} yl'\ yl' \leq_{R1} l1\ x'$
using *l1-R1-if-L1-r1* **by** *blast*
moreover then have $t\ yl \leq_{R1} yl$ **by** (*intro t-R1-if-l1-R1*)
ultimately have $((\leq_{L2}) \circ\circ (\leq_{R1}))$ $(t\ yl)$ $(l1\ x')$ **using** *rel-comp-le* **by** *blast*
then obtain *y* **where** $t\ yl \leq_{L2} y\ y \leq_{R1} l1\ x'$ **by** *blast*
show $z \leq_R z'$
proof (*rule flip.left-relI*)
from ‹$t\ yl \leq_{L2} y$› ‹$l1\ x \leq_{R1} yl$› **show** $z\ {}_{R2}\!\lessapprox y$
by (*auto intro*: *R2-l2-if-t-L2-if-l1-R1*)
from ‹$yl' \leq_{R1} l1\ x'$› ‹$y \leq_{R1} l1\ x'$› **show** $y \leq_{R1} b\ yl'$
by (*rule R1-b-if-R1-l1-if-R1-l1*)
show $b\ yl'\ {}_{L2}\!\lessapprox z'$

```
    proof (rule t2.left-GaloisI)
      from ‹yl' ≤R1 l1 x'› have yl' ≤R1 b yl'
        by (intro R1-b-if-R1-l1-if-R1-l1)
      with ‹l1 x ≤R1 yl› ‹yl ≤L2 yl'› in-codom-rel-comp-le
        have in-codom (≤L2) (b yl') by blast
      with ‹yl' ≤R1 l1 x'› show b yl' ≤L2 r2 z' in-codom (≤R2) z'
        by (auto intro: b-L2-r2-if-in-codom-L2-b-if-R1-l1
          in-codom-R2-if-in-codom-L2-b-if-R1-l1)
    qed
  qed
qed

lemma right-rel-if-left-relI':
  assumes x ≤L x'
  and l1-R1-if-L1-r1: ⋀y. in-codom (≤R1) y ⟹ x ≤L1 r1 y ⟹ l1 x ≤R1 y
  and R1-b-if-R1-l1: ⋀y. y ≤R1 l1 x' ⟹ y ≤R1 b y
  and L2-r2-if-L2-b-if-R1-l1:
    ⋀y y'. y ≤R1 l1 x' ⟹ y' ≤L2 b y ⟹ y' ≤L2 r2 z'
  and in-codom-R2-if-L2-b-if-R1-l1:
    ⋀y y'. y ≤R1 l1 x' ⟹ y' ≤L2 b y ⟹ in-codom (≤R2) z'
  and t-R1-if-R1-l1-if-l1-R1:
    ⋀y y' y''. l1 x ≤R1 y ⟹ l1 x ≤R1 y' ⟹ t y ≤R1 y'
  and R2-l2-t-if-in-dom-L2-t-if-l1-R1:
    ⋀y y'. l1 x ≤R1 y ⟹ in-dom (≤L2) (t y) ⟹ z ≤R2 l2 (t y)
  and in-codom-L2-t-if-in-dom-L2-t-if-l1-R1:
    ⋀y y'. l1 x ≤R1 y ⟹ in-dom (≤L2) (t y) ⟹ in-codom (≤L2) (t y)
  and rel-comp-le: ((≤R1) ∘∘ (≤L2) ∘∘ (≤R1)) ≤ ((≤R1) ∘∘ (≤L2))
  and in-dom-rel-comp-le: in-dom ((≤R1) ∘∘ (≤L2) ∘∘ (≤R1)) ≤ in-dom (≤L2)
  shows z ≤R z'
proof −
  from ‹x ≤L x'› obtain yl yl' where l1 x ≤R1 yl yl ≤L2 yl' yl' ≤R1 l1 x'
    using l1-R1-if-L1-r1 by blast
  moreover then have yl' ≤R1 b yl' by (intro R1-b-if-R1-l1)
  ultimately have ((≤R1) ∘∘ (≤L2)) (l1 x) (b yl') using rel-comp-le by blast
  then obtain y where l1 x ≤R1 y y ≤L2 b yl' by blast
  show z ≤R z'
  proof (rule flip.left-relI)
    from ‹yl' ≤R1 l1 x'› ‹y ≤L2 b yl'›
      have in-codom (≤R2) z' y ≤L2 r2 z'
      by (auto intro: in-codom-R2-if-L2-b-if-R1-l1 L2-r2-if-L2-b-if-R1-l1)
    then show y L2≲≈ z' by blast
  from ‹l1 x ≤R1 yl› ‹l1 x ≤R1 y› show t yl ≤R1 y by (rule t-R1-if-R1-l1-if-l1-R1)
    show z R2≲≈ t yl
    proof (rule flip.t1.left-GaloisI)
      from ‹l1 x ≤R1 yl› have t yl ≤R1 yl by (intro t-R1-if-R1-l1-if-l1-R1)
      with ‹yl ≤L2 yl'› ‹yl' ≤R1 l1 x'› in-dom-rel-comp-le have in-dom (≤L2) (t
yl)
        by blast
      with ‹l1 x ≤R1 yl›
```

show $z \leq_{R2} l2\ (t\ yl)$ *in-codom* $(\leq_{L2})\ (t\ yl)$ **by** (*auto intro*: *R2-l2-t-if-in-dom-L2-t-if-l1-R1 in-codom-L2-t-if-in-dom-L2-t-if-l1-R1*)
qed
qed
qed

Simplification of Monotonicity Assumptions

Some sufficient conditions for monotonicity assumptions that repeatedly arise in various places.

lemma *mono-in-dom-left-rel-left1-if-in-dom-rel-comp-le*:
assumes $((\leq_{L1})\ {}_h\trianglelefteq\ (\leq_{R1}))$ *l1 r1*
and *in-dom* $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq$ *in-dom* $(\leq_{L2})$
shows (*in-dom* $(\leq_L) \Rightarrow$ *in-dom* $(\leq_{L2})$) *l1*
using *assms* **by** (*intro mono-wrt-predI*) *blast*

lemma *mono-in-codom-left-rel-left1-if-in-codom-rel-comp-le*:
assumes $((\leq_{L1})\ {}_h\trianglelefteq\ (\leq_{R1}))$ *l1 r1*
and *in-codom* $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq$ *in-codom* $(\leq_{L2})$
shows (*in-codom* $(\leq_L) \Rightarrow$ *in-codom* $(\leq_{L2})$) *l1*
using *assms* **by** (*intro mono-wrt-predI*) *blast*

Simplification of Compatibility Conditions

Most results will depend on certain compatibility conditions between $(\leq_{R1})$ and $(\leq_{L2})$. We next derive some sufficient assumptions for these conditions.

end

lemma *rel-comp-comp-le-rel-comp-if-rel-comp-comp-if-in-dom-leI*:
assumes *trans-R*: *transitive R*
and *refl-S*: *reflexive-on P S*
and *in-dom-le*: *in-dom* $(R \circ\circ S \circ\circ R) \leq P$
and *rel-comp-le*: $(S \circ\circ R \circ\circ S) \leq (S \circ\circ R)$
shows $(R \circ\circ S \circ\circ R) \leq (S \circ\circ R)$
proof (*intro le-relI*)
fix *x y* **assume** $(R \circ\circ S \circ\circ R)\ x\ y$
moreover with *in-dom-le refl-S* **have** $S\ x\ x$ **by** *blast*
ultimately have $((S \circ\circ R \circ\circ S) \circ\circ R)\ x\ y$ **by** *blast*
with *rel-comp-le* **have** $(S \circ\circ R \circ\circ R)\ x\ y$ **by** *blast*
with *trans-R* **show** $(S \circ\circ R)\ x\ y$ **by** *blast*
qed

lemma *rel-comp-comp-le-rel-comp-if-rel-comp-comp-if-in-codom-leI*:
assumes *trans-R*: *transitive R*
and *refl-S*: *reflexive-on P S*
and *in-codom-le*: *in-codom* $(R \circ\circ S \circ\circ R) \leq P$
and *rel-comp-le*: $(S \circ\circ R \circ\circ S) \leq (R \circ\circ S)$
shows $(R \circ\circ S \circ\circ R) \leq (R \circ\circ S)$

proof (*intro le-relI*)
fix *x y* **assume** $(R \circ\circ S \circ\circ R)\ x\ y$
moreover with *in-codom-le refl-S* **have** $S\ y\ y$ **by** *blast*
ultimately have $(R \circ\circ (S \circ\circ R \circ\circ S))\ x\ y$ **by** *blast*
with *rel-comp-le* **have** $(R \circ\circ R \circ\circ S)\ x\ y$ **by** *blast*
with *trans-R* **show** $(R \circ\circ S)\ x\ y$ **by** *blast*
qed
thm *mono-rel-comp*
lemma *rel-comp-comp-le-rel-comp-if-rel-comp-le-if-transitive*:
assumes *trans-R*: *transitive R*
and *R-S-le*: $(R \circ\circ S) \leq (S \circ\circ R)$
shows $(R \circ\circ S \circ\circ R) \leq (S \circ\circ R)$
proof −
from *trans-R* **have** *R-R-le*: $(R \circ\circ R) \leq R$ **by** (*intro rel-comp-le-self-if-transitive*)
have $(R \circ\circ S \circ\circ R) \leq (S \circ\circ R \circ\circ R)$
using *mono-rel-comp R-S-le* **by** *blast*
also have ... $= S \circ\circ (R \circ\circ R)$ **by** (*simp flip*: *rel-comp-assoc*)
also have ... $\leq (S \circ\circ R)$ **using** *mono-rel-comp R-R-le* **by** *blast*
finally show *?thesis* **.**
qed

lemma *rel-comp-comp-le-rel-comp-if-rel-comp-le-if-transitive'*:
assumes *trans-R*: *transitive R*
and *S-R-le*: $(S \circ\circ R) \leq (R \circ\circ S)$
shows $(R \circ\circ S \circ\circ R) \leq (R \circ\circ S)$
proof −
from *trans-R* **have** *R-R-le*: $(R \circ\circ R) \leq R$ **by** (*intro rel-comp-le-self-if-transitive*)
have $(R \circ\circ S \circ\circ R) \leq (R \circ\circ R \circ\circ S)$
using *mono-rel-comp S-R-le* **by** (*auto simp flip*: *rel-comp-assoc*)
also have ... $\leq (R \circ\circ S)$ **using** *mono-rel-comp R-R-le* **by** *blast*
finally show *?thesis* **.**
qed

lemma *rel-comp-eq-rel-comp-if-le-if-transitive-if-reflexive*:
assumes *refl-R*: *reflexive-on* (*in-field S*) *R*
and *trans-S*: *transitive S*
and *R-le*: $R \leq S \sqcup (=)$
shows $(R \circ\circ S) = (S \circ\circ R)$
proof (*intro ext iffI*)
fix *x y* **assume** $(R \circ\circ S)\ x\ y$
then obtain *z* **where** $R\ x\ z\ S\ z\ y$ **by** *blast*
with *R-le* **have** $(S \sqcup (=))\ x\ z$ **by** *blast*
with ‹$S\ z\ y$› *trans-S* **have** $S\ x\ y$ **by** *auto*
moreover from ‹$S\ z\ y$› *refl-R* **have** $R\ y\ y$ **by** *blast*
ultimately show $(S \circ\circ R)\ x\ y$ **by** *blast*
next
fix *x y* **assume** $(S \circ\circ R)\ x\ y$
then obtain *z* **where** $S\ x\ z\ R\ z\ y$ **by** *blast*
with *R-le* **have** $(S \sqcup (=))\ z\ y$ **by** *blast*

with ‹S x z› *trans-S* **have** S x y **by** *auto*
moreover from ‹S x y› *refl-R* **have** R x x **by** *blast*
ultimately show $(R \circ\circ S)$ x y **by** *blast*
qed

lemma *rel-comp-eq-rel-comp-if-in-field-le-if-le-eq*:
assumes *le-eq*: $R \leq (=)$
and *in-field-le*: *in-field* $S \leq$ *in-field* R
shows $(R \circ\circ S) = (S \circ\circ R)$
proof (*intro ext iffI*)
fix x y **assume** $(R \circ\circ S)$ x y
then obtain z **where** R x z S z y **by** *blast*
with *le-eq* **have** S x y **by** *blast*
with *assms* **show** $(S \circ\circ R)$ x y **by** *blast*
next
fix x y **assume** $(S \circ\circ R)$ x y
then obtain z **where** S x z R z y **by** *blast*
with *le-eq* **have** S x y **by** *blast*
with *assms* **show** $(R \circ\circ S)$ x y **by** *blast*
qed

context *transport-comp*
begin

lemma *left2-right1-left2-le-left2-right1-if-right1-left2-right1-le-left2-right1*:
assumes *reflexive-on* (*in-codom* $(\leq_{R1})$) $(\leq_{R1})$
and *transitive* $(\leq_{L2})$
and $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq ((\leq_{L2}) \circ\circ (\leq_{R1}))$
and *in-codom* $((\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2})) \leq$ *in-codom* $(\leq_{R1})$
shows $((\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2})) \leq ((\leq_{L2}) \circ\circ (\leq_{R1}))$
using *assms* **by** (*intro rel-comp-comp-le-rel-comp-if-rel-comp-comp-if-in-codom-leI*)
auto

lemma *left2-right1-left2-le-right1-left2-if-right1-left2-right1-le-right1-left2I*:
assumes *reflexive-on* (*in-dom* $(\leq_{R1})$) $(\leq_{R1})$
and *transitive* $(\leq_{L2})$
and $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq ((\leq_{R1}) \circ\circ (\leq_{L2}))$
and *in-dom* $((\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2})) \leq$ *in-dom* $(\leq_{R1})$
shows $((\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2})) \leq ((\leq_{R1}) \circ\circ (\leq_{L2}))$
using *assms* **by** (*intro rel-comp-comp-le-rel-comp-if-rel-comp-comp-if-in-dom-leI*)
auto

lemma *in-dom-right1-left2-right1-le-if-right1-left2-right1-le*:
assumes $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq ((\leq_{L2}) \circ\circ (\leq_{R1}))$
shows *in-dom* $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq$ *in-dom* $(\leq_{L2})$
using *monoD*[*OF mono-in-dom assms*] **by** (*auto intro*: *in-dom-if-in-dom-rel-comp*)

lemma *in-codom-right1-left2-right1-le-if-right1-left2-right1-le*:
assumes $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq ((\leq_{R1}) \circ\circ (\leq_{L2}))$

shows *in-codom* $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq$ *in-codom* $(\leq_{L2})$
using *monoD*[*OF mono-in-codom assms*]
by (*auto intro*: *in-codom-if-in-codom-rel-comp*)

Our main results will be derivable for two different sets of compatibility conditions. The next two lemmas show the equivalence between those two sets under certain assumptions. In cases where these assumptions are met, we will only state the result for one of the two compatibility conditions. The other one will then be derivable using one of the following lemmas.

definition *middle-compatible-dom* $\equiv$
$(\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1}) \leq (\leq_{R1}) \circ\circ (\leq_{L2})$
$\wedge$ *in-dom* $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq$ *in-dom* $(\leq_{L2})$
$\wedge\ ((\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2})) \leq ((\leq_{L2}) \circ\circ (\leq_{R1}))$
$\wedge$ *in-dom* $((\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2})) \leq$ *in-dom* $(\leq_{R1})$

lemma *middle-compatible-domI* [*intro*]:
assumes $(\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1}) \leq (\leq_{R1}) \circ\circ (\leq_{L2})$
and *in-dom* $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq$ *in-dom* $(\leq_{L2})$
and $((\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2})) \leq ((\leq_{L2}) \circ\circ (\leq_{R1}))$
and *in-dom* $((\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2})) \leq$ *in-dom* $(\leq_{R1})$
shows *middle-compatible-dom*
unfolding *middle-compatible-dom-def* **using** *assms* **by** *blast*

lemma *middle-compatible-domE* [*elim*]:
assumes *middle-compatible-dom*
obtains $(\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1}) \leq (\leq_{R1}) \circ\circ (\leq_{L2})$
and *in-dom* $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq$ *in-dom* $(\leq_{L2})$
and $((\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2})) \leq ((\leq_{L2}) \circ\circ (\leq_{R1}))$
and *in-dom* $((\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2})) \leq$ *in-dom* $(\leq_{R1})$
using *assms* **unfolding** *middle-compatible-dom-def* **by** *blast*

definition *middle-compatible-codom* $\equiv$
$((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq ((\leq_{L2}) \circ\circ (\leq_{R1}))$
$\wedge$ *in-codom* $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq$ *in-codom* $(\leq_{L2})$
$\wedge\ (\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2}) \leq (\leq_{R1}) \circ\circ (\leq_{L2})$
$\wedge$ *in-codom* $((\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2})) \leq$ *in-codom* $(\leq_{R1})$

lemma *middle-compatible-codomI* [*intro*]:
assumes $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq ((\leq_{L2}) \circ\circ (\leq_{R1}))$
and *in-codom* $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq$ *in-codom* $(\leq_{L2})$
and $(\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2}) \leq (\leq_{R1}) \circ\circ (\leq_{L2})$
and *in-codom* $((\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2})) \leq$ *in-codom* $(\leq_{R1})$
shows *middle-compatible-codom*
unfolding *middle-compatible-codom-def* **using** *assms* **by** *blast*

lemma *middle-compatible-codomE* [*elim*]:
assumes *middle-compatible-codom*
obtains $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq ((\leq_{L2}) \circ\circ (\leq_{R1}))$
and *in-codom* $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq$ *in-codom* $(\leq_{L2})$

and $(\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2}) \leq (\leq_{R1}) \circ\circ (\leq_{L2})$
and *in-codom* $((\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2})) \leq$ *in-codom* $(\leq_{R1})$
using *assms* **unfolding** *middle-compatible-codom-def* **by** *blast*

context
begin

interpretation *flip* : *transport-comp R2 L2 r2 l2 R1 L1 r1 l1* **.**

lemma *rel-comp-comp-le-assms-if-in-codom-rel-comp-comp-leI*:
assumes *preorder-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$
and *preorder-on* (*in-field* $(\leq_{L2})$) $(\leq_{L2})$
and *middle-compatible-codom*
shows *middle-compatible-dom*
using *assms* **by** (*intro middle-compatible-domI*)
(*auto intro*!:
left2-right1-left2-le-left2-right1-if-right1-left2-right1-le-left2-right1
flip.left2-right1-left2-le-left2-right1-if-right1-left2-right1-le-left2-right1
in-dom-right1-left2-right1-le-if-right1-left2-right1-le
flip.in-dom-right1-left2-right1-le-if-right1-left2-right1-le
intro: *reflexive-on-if-le-pred-if-reflexive-on in-field-if-in-codom*)

lemma *rel-comp-comp-le-assms-if-in-dom-rel-comp-comp-leI*:
assumes *preorder-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$
and *preorder-on* (*in-field* $(\leq_{L2})$) $(\leq_{L2})$
and *middle-compatible-dom*
shows *middle-compatible-codom*
using *assms* **by** (*intro middle-compatible-codomI*)
(*auto intro*!:
left2-right1-left2-le-right1-left2-if-right1-left2-right1-le-right1-left2I
flip.left2-right1-left2-le-right1-left2-if-right1-left2-right1-le-right1-left2I
in-codom-right1-left2-right1-le-if-right1-left2-right1-le
flip.in-codom-right1-left2-right1-le-if-right1-left2-right1-le
intro: *reflexive-on-if-le-pred-if-reflexive-on in-field-if-in-dom*)

lemma *middle-compatible-dom-iff-middle-compatible-codom-if-preorder-on*:
assumes *preorder-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$
and *preorder-on* (*in-field* $(\leq_{L2})$) $(\leq_{L2})$
shows *middle-compatible-dom* $\longleftrightarrow$ *middle-compatible-codom*
using *assms* **by** (*intro iffI rel-comp-comp-le-assms-if-in-codom-rel-comp-comp-leI*)
(*auto intro*!: *rel-comp-comp-le-assms-if-in-dom-rel-comp-comp-leI*)

end

Finally we derive some sufficient assumptions for the compatibility conditions.

lemma *right1-left2-right1-le-assms-if-right1-left2-eqI*:
assumes *transitive* $(\leq_{R1})$
and $((\leq_{R1}) \circ\circ (\leq_{L2})) = ((\leq_{L2}) \circ\circ (\leq_{R1}))$

shows $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq ((\leq_{L2}) \circ\circ (\leq_{R1}))$
and $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq ((\leq_{R1}) \circ\circ (\leq_{L2}))$
using *assms rel-comp-comp-le-rel-comp-if-rel-comp-le-if-transitive*[*of R1 L2*]
by *auto*

interpretation *flip* : *transport-comp R2 L2 r2 l2 R1 L1 r1 l1*
rewrites $((\leq_{L2}) \circ\circ (\leq_{R1})) = ((\leq_{R1}) \circ\circ (\leq_{L2})) \equiv ((\leq_{R1}) \circ\circ (\leq_{L2})) = ((\leq_{L2}) \circ\circ (\leq_{R1}))$
by (*simp only*: *eq-commute*)

lemma *middle-compatible-codom-if-rel-comp-eq-if-transitive*:
assumes *transitive* $(\leq_{R1})$ *transitive* $(\leq_{L2})$
and $((\leq_{R1}) \circ\circ (\leq_{L2})) = ((\leq_{L2}) \circ\circ (\leq_{R1}))$
shows *middle-compatible-codom*
using *assms* **by** (*intro middle-compatible-codomI*
in-codom-right1-left2-right1-le-if-right1-left2-right1-le
flip.in-codom-right1-left2-right1-le-if-right1-left2-right1-le
right1-left2-right1-le-assms-if-right1-left2-eqI
flip.right1-left2-right1-le-assms-if-right1-left2-eqI)
auto

lemma *middle-compatible-codom-if-right1-le-left2-eqI*:
assumes *preorder-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$ *transitive* $(\leq_{L2})$
and $(\leq_{R1}) \leq (\leq_{L2}) \sqcup (=)$
and *in-field* $(\leq_{L2}) \leq$ *in-field* $(\leq_{R1})$
shows *middle-compatible-codom*
using *assms* **by** (*intro middle-compatible-codomI*
in-codom-right1-left2-right1-le-if-right1-left2-right1-le
flip.in-codom-right1-left2-right1-le-if-right1-left2-right1-le
right1-left2-right1-le-assms-if-right1-left2-eqI
flip.right1-left2-right1-le-assms-if-right1-left2-eqI
rel-comp-eq-rel-comp-if-le-if-transitive-if-reflexive)
(*auto intro*: *reflexive-on-if-le-pred-if-reflexive-on*)

lemma *middle-compatible-codom-if-right1-le-eqI*:
assumes $(\leq_{R1}) \leq (=)$
and *transitive* $(\leq_{L2})$
and *in-field* $(\leq_{L2}) \leq$ *in-field* $(\leq_{R1})$
shows *middle-compatible-codom*
using *assms* **by** (*intro middle-compatible-codomI*
in-codom-right1-left2-right1-le-if-right1-left2-right1-le
flip.in-codom-right1-left2-right1-le-if-right1-left2-right1-le
right1-left2-right1-le-assms-if-right1-left2-eqI
flip.right1-left2-right1-le-assms-if-right1-left2-eqI
rel-comp-eq-rel-comp-if-in-field-le-if-le-eq)
auto

end

end

2.11.2 Galois Property

theory *Transport-Compositions-Generic-Galois-Property*
imports
Transport-Compositions-Generic-Base
begin

context *transport-comp*
begin

interpretation *flip* : *transport-comp R2 L2 r2 l2 R1 L1 r1 l1*
rewrites *flip.t2.unit* $= \varepsilon_1$ **and** *flip.t1.counit* $\equiv \eta_2$
by (*simp-all only*: *t1.flip-unit-eq-counit t2.flip-counit-eq-unit*)

lemma *half-galois-prop-left-left-rightI*:
assumes $((\leq_{L1}) \ {}_h\trianglelefteq \ (\leq_{R1}))$ *l1 r1*
and *deflationary-counit1*: *deflationary-on* (*in-codom* $(\leq_{R1})$) $(\leq_{R1})$ ε_1
and *trans-R1*: *transitive* $(\leq_{R1})$
and $((\leq_{L2}) \Rightarrow (\leq_{R2}))$ *l2*
and *reflexive-on* (*in-codom* $(\leq_{L2})$) $(\leq_{L2})$
and $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq ((\leq_{L2}) \circ\circ (\leq_{R1}))$
and *in-codom* $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq$ *in-codom* $(\leq_{L2})$
and *mono-in-codom-r2*: (*in-codom* $(\leq_R) \Rightarrow$ *in-codom* $(\leq_{R1})$) *r2*
shows $((\leq_L) \ {}_h\trianglelefteq \ (\leq_R))$ *l r*
proof (*rule half-galois-prop-leftI*)
fix *x z* **assume** *x* $\ {}_L\lessapprox$ *z*
then show *l x* $\leq_R$ *z*
proof (*intro right-rel-if-left-relI*)
from ‹*x* $\ {}_L\lessapprox$ *z*› **show** *in-codom* $(\leq_{R2})$ *z* **by** *blast*
fix *y* **assume** *y* $\leq_{R1}$ *l1* (*r z*)
moreover have *l1* (*r z*) $\leq_{R1}$ *r2 z*
proof −
from *mono-in-codom-r2* ‹*x* $\ {}_L\lessapprox$ *z*› **have** *in-codom* $(\leq_{R1})$ (*r2 z*) **by** *blast*
with *deflationary-counit1* **show** *l1* (*r z*) $\leq_{R1}$ *r2 z* **by** *auto*
qed
ultimately show *y* $\leq_{R1}$ *r2 z* **using** *trans-R1* **by** *blast*
next
fix *y* **assume** *l1 x* $\leq_{L2}$ *y*
with ‹$((\leq_{L2}) \Rightarrow (\leq_{R2}))$ *l2*› **show** *l x* $\leq_{R2}$ *l2 y* **by** *auto*
qed (*insert assms*, *auto*)
qed

lemma *half-galois-prop-left-left-rightI′*:
assumes $((\leq_{L1}) \ {}_h\trianglelefteq \ (\leq_{R1}))$ *l1 r1*
and *deflationary-counit1*: *deflationary-on* (*in-codom* $(\leq_{R1})$) $(\leq_{R1})$ ε_1
and *trans-R1*: *transitive* $(\leq_{R1})$

and $((\leq_{L2}) \Rightarrow (\leq_{R2}))$ *l2*
and *refl-L2*: *reflexive-on* (*in-dom* $(\leq_{L2})$) $(\leq_{L2})$
and $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq ((\leq_{R1}) \circ\circ (\leq_{L2}))$
and *in-dom* $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq$ *in-dom* $(\leq_{L2})$
and *mono-in-codom-r2*: (*in-codom* $(\leq_R) \Rightarrow$ *in-codom* $(\leq_{R1}))$ *r2*
shows $((\leq_L)\ {}_h\trianglelefteq\ (\leq_R))$ *l r*
proof (*rule half-galois-prop-leftI*)
fix *x z* **assume** *x* ${}_L\lessapprox$ *z*
then show *l x* $\leq_R$ *z*
proof (*intro right-rel-if-left-relI'*)
from ‹*x* ${}_L\lessapprox$ *z*› **show** *in-codom* $(\leq_{R2})$ *z* **by** *blast*
fix *y* **assume** *y* $\leq_{R1}$ *l1* (*r z*)
moreover have *l1* (*r z*) $\leq_{R1}$ *r2 z*
proof −
from *mono-in-codom-r2* ‹*x* ${}_L\lessapprox$ *z*› **have** *in-codom* $(\leq_{R1})$ (*r2 z*) **by** *blast*
with *deflationary-counit1* **show** *l1* (*r z*) $\leq_{R1}$ *r2 z* **by** *auto*
qed
ultimately show *y* $\leq_{R1}$ *r2 z* **using** *trans-R1* **by** *blast*
next
assume *in-dom* $(\leq_{L2})$ (*l1 x*)
with *refl-L2* **have** *l1 x* $\leq_{L2}$ *l1 x* **by** *blast*
with ‹$((\leq_{L2}) \Rightarrow (\leq_{R2}))$ *l2*› **show** *in-codom* $(\leq_{L2})$ (*l1 x*) *l x* $\leq_{R2}$ *l2* (*l1 x*)
by *auto*
qed (*insert assms*, *auto*)
qed

lemma *half-galois-prop-right-left-rightI*:
assumes $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and $((\leq_{L1}) \trianglelefteq_h (\leq_{R1}))$ *l1 r1*
and *inflationary-counit1*: *inflationary-on* (*in-codom* $(\leq_{R1})$) $(\leq_{R1})$ ε_1
and $((\leq_{R2})\ {}_h\trianglelefteq\ (\leq_{L2}))$ *r2 l2*
and *inflationary-unit2*: *inflationary-on* (*in-dom* $(\leq_{L2})$) $(\leq_{L2})$ η_2
and *trans-L2*: *transitive* $(\leq_{L2})$
and *mono-in-dom-l1*: (*in-dom* $(\leq_L) \Rightarrow$ *in-dom* $(\leq_{L2}))$ *l1*
and $((\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2})) \leq ((\leq_{R1}) \circ\circ (\leq_{L2}))$
and *in-codom* $((\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2})) \leq$ *in-codom* $(\leq_{R1})$
shows $((\leq_L) \trianglelefteq_h (\leq_R))$ *l r*
proof (*rule half-galois-prop-rightI*)
fix *x z* **assume** *x* $\gtrapprox_R$ *z*
then show *x* $\leq_L$ *r z*
proof (*intro flip.right-rel-if-left-relI*)
fix *y* **assume** *r2* (*l x*) $\leq_{L2}$ *y*
moreover have *l1 x* $\leq_{L2}$ *r2* (*l x*)
proof −
from *mono-in-dom-l1* ‹*x* $\gtrapprox_R$ *z*› **have** *in-dom* $(\leq_{L2})$ (*l1 x*) **by** *blast*
with *inflationary-unit2* **show** *l1 x* $\leq_{L2}$ *r2* (*l x*) **by** *auto*
qed
ultimately show *l1 x* $\leq_{L2}$ *y* **using** *trans-L2* **by** *blast*
fix *y* **assume** *l1 x* $\leq_{R1}$ *y*

with ‹$((\leq_{L1}) \trianglelefteq_h (\leq_{R1}))$ *l1 r1*› ‹$x \lessapprox_R z$› **show** $x \leq_{L1} r1\ y$ **by** *blast*
next
assume *in-codom* $(\leq_{R1})$ $(r2\ z)$
with *inflationary-counit1* **show** $r2\ z \leq_{R1} l1\ (r\ z)$ **by** *auto*
from ‹$((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*› ‹*in-codom* $(\leq_{R1})$ $(r2\ z)$› **show** *in-codom* $(\leq_{L1})$ $(r\ z)$
by (*auto intro*: *in-codom-if-rel-if-dep-mono-wrt-rel* *simp*: *mono-wrt-rel-eq-dep-mono-wrt-rel*)
qed (*insert assms*, *auto elim*: *galois-rel.left-GaloisE*)
qed

lemma *half-galois-prop-right-left-rightI'*:
assumes $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and *inflationary-unit1*: *inflationary-on* (*in-dom* $(\leq_{L1})$) $(\leq_{L1})$ η_1
and *inflationary-counit1*: $\bigwedge y\ z.\ y \leq_{R1} r2\ z \Longrightarrow y \leq_{R1} l1\ (r\ z)$
and *in-dom* $(\leq_{R1}) \leq$ *in-codom* $(\leq_{R1})$
and $((\leq_{R2})\ {}_h\trianglelefteq\ (\leq_{L2}))$ *r2 l2*
and *inflationary-unit2*: *inflationary-on* (*in-dom* $(\leq_{L2})$) $(\leq_{L2})$ η_2
and *trans-L2*: *transitive* $(\leq_{L2})$
and *mono-in-dom-l1*: (*in-dom* $(\leq_L) \Rightarrow$ *in-dom* $(\leq_{L2})$) *l1*
and $((\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2})) \leq ((\leq_{L2}) \circ\circ (\leq_{R1}))$
and *in-dom* $((\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2})) \leq$ *in-dom* $(\leq_{R1})$
shows $((\leq_L) \trianglelefteq_h (\leq_R))\ l\ r$
proof (*rule half-galois-prop-rightI*)
fix $x\ z$ **assume** $x \lessapprox_R z$
then show $x \leq_L r\ z$
proof (*intro flip.right-rel-if-left-relI'*)
from ‹$x \lessapprox_R z$› *inflationary-unit1* **show** $x \leq_{L1} r1\ (l1\ x)$
by (*fastforce elim*: *galois-rel.left-GaloisE*)
fix y **assume** $y \leq_{R1} r2\ z$
with *inflationary-counit1* **show** $y \leq_{R1} l1\ (r\ z)$ **by** *auto*
next
fix y
from *mono-in-dom-l1* ‹$x \lessapprox_R z$› **have** *in-dom* $(\leq_{L2})$ $(l1\ x)$ **by** *blast*
with *inflationary-unit2* **have** $l1\ x \leq_{L2} r2\ (l\ x)$ **by** *auto*
moreover assume $r2\ (l\ x) \leq_{L2} y$
ultimately show $l1\ x \leq_{L2} y$ **using** *trans-L2* **by** *blast*
qed (*insert assms*, *auto elim*: *galois-rel.left-GaloisE*)
qed

lemma *galois-prop-left-rightI*:
assumes $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and $((\leq_{L1}) \trianglelefteq (\leq_{R1}))$ *l1 r1*
and *rel-equivalence-on* (*in-codom* $(\leq_{R1})$) $(\leq_{R1})$ ε_1
and *transitive* $(\leq_{R1})$
and $((\leq_{L2}) \Rightarrow (\leq_{R2}))$ *l2*
and $((\leq_{R2})\ {}_h\trianglelefteq\ (\leq_{L2}))$ *r2 l2*
and *inflationary-on* (*in-dom* $(\leq_{L2})$) $(\leq_{L2})$ η_2
and *preorder-on* (*in-field* $(\leq_{L2})$) $(\leq_{L2})$

and *middle-compatible-codom*
shows $((\leq_L) \trianglelefteq (\leq_R))\ l\ r$
using *assms* **by** (*intro galois-propI*
half-galois-prop-left-left-rightI half-galois-prop-right-left-rightI
flip.mono-in-codom-left-rel-left1-if-in-codom-rel-comp-le
mono-in-dom-left-rel-left1-if-in-dom-rel-comp-le
in-dom-right1-left2-right1-le-if-right1-left2-right1-le)
(*auto elim*!: *preorder-on-in-fieldE*
intro: *reflexive-on-if-le-pred-if-reflexive-on in-field-if-in-codom*)

lemma *galois-prop-left-rightI'*:
assumes $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and $((\leq_{L1})\ {}_h\trianglelefteq\ (\leq_{R1}))$ *l1 r1*
and *inflationary-on* (*in-dom* $(\leq_{L1})$) $(\leq_{L1})$ η_1
and *rel-equiv-counit1*: *rel-equivalence-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$ ε_1
and *trans-R1*: *transitive* $(\leq_{R1})$
and $((\leq_{L2}) \Rightarrow (\leq_{R2}))$ *l2*
and $((\leq_{R2})\ {}_h\trianglelefteq\ (\leq_{L2}))$ *r2 l2*
and *inflationary-on* (*in-dom* $(\leq_{L2})$) $(\leq_{L2})$ η_2
and *preorder-on* (*in-field* $(\leq_{L2})$) $(\leq_{L2})$
and *middle-compatible-dom*
shows $((\leq_L) \trianglelefteq (\leq_R))\ l\ r$
proof (*rule galois-propI*)
show $((\leq_L)\ {}_h\trianglelefteq\ (\leq_R))\ l\ r$ **using** *assms*
by (*intro half-galois-prop-left-left-rightI'*
flip.mono-in-codom-left-rel-left1-if-in-codom-rel-comp-le
flip.in-codom-right1-left2-right1-le-if-right1-left2-right1-le)
(*auto elim*!: *rel-equivalence-onE preorder-on-in-fieldE*
intro: *deflationary-on-if-le-pred-if-deflationary-on*
reflexive-on-if-le-pred-if-reflexive-on
in-field-if-in-dom in-field-if-in-codom)
have $y \leq_{R1} l1\ (r1\ (r2\ z))$ **if** $y \leq_{R1} r2\ z$ **for** $y\ z$
proof −
note ‹$y \leq_{R1} r2\ z$›
moreover with *rel-equiv-counit1* **have** $r2\ z \leq_{R1} \varepsilon_1\ (r2\ z)$ **by** *blast*
ultimately show *?thesis* **using** *trans-R1* **by** *auto*
qed
moreover have *in-dom* $(\leq_{R1}) \leq$ *in-codom* $(\leq_{R1})$
proof −
from *rel-equiv-counit1 trans-R1* **have** *reflexive-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$
by (*intro reflexive-on-in-field-if-transitive-if-rel-equivalence-on*) *auto*
then show *?thesis* **by** (*simp only*: *in-codom-eq-in-dom-if-reflexive-on-in-field*)
qed
ultimately show $((\leq_L) \trianglelefteq_h (\leq_R))\ l\ r$ **using** *assms*
by (*intro half-galois-prop-right-left-rightI'*
mono-in-dom-left-rel-left1-if-in-dom-rel-comp-le)
auto
qed

end

end

2.11.3 Monotonicity

theory *Transport-Compositions-Generic-Monotone*
imports
Transport-Compositions-Generic-Base
begin

context *transport-comp*
begin

lemma *mono-wrt-rel-leftI*:
assumes $((\leq_{L1}) \ {}_h\trianglelefteq \ (\leq_{R1}))$ *l1 r1*
and $((\leq_{L2}) \Rightarrow (\leq_{R2}))$ *l2*
and *inflationary-unit2*: *inflationary-on* (*in-codom* $(\leq_{L2})$) $(\leq_{L2})$ η_2
and $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq ((\leq_{L2}) \circ\circ (\leq_{R1}))$
and *in-codom* $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq$ *in-codom* $(\leq_{L2})$
shows $((\leq_L) \Rightarrow (\leq_R))$ *l*
proof (*rule mono-wrt-relI*)
fix *x x'* **assume** $x \leq_L x'$
then show $l\ x \leq_R l\ x'$
proof (*rule right-rel-if-left-relI*)
fix y' **assume** *l1 x* $\leq_{L2}$ y'
with ‹$((\leq_{L2}) \Rightarrow (\leq_{R2}))$ *l2*› **show** *l x* $\leq_{R2}$ *l2 y'* **by** *auto*
next
assume *in-codom* $(\leq_{L2})$ (*l1 x'*)
with *inflationary-unit2* **show** *l1 x'* $\leq_{L2}$ *r2* (*l x'*) **by** *auto*
from ‹*in-codom* $(\leq_{L2})$ (*l1 x'*)› ‹$((\leq_{L2}) \Rightarrow (\leq_{R2}))$ *l2*›
show *in-codom* $(\leq_{R2})$ (*l x'*) **by** *fastforce*
qed (*insert assms*, *auto*)
qed

lemma *mono-wrt-rel-leftI'*:
assumes $((\leq_{L1}) \ {}_h\trianglelefteq \ (\leq_{R1}))$ *l1 r1*
and $((\leq_{L2}) \Rightarrow (\leq_{R2}))$ *l2*
and $((\leq_{L2}) \trianglelefteq_h (\leq_{R2}))$ *l2 r2*
and *refl-L2*: *reflexive-on* (*in-dom* $(\leq_{L2})$) $(\leq_{L2})$
and $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq ((\leq_{R1}) \circ\circ (\leq_{L2}))$
and *in-dom* $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq$ *in-dom* $(\leq_{L2})$
shows $((\leq_L) \Rightarrow (\leq_R))$ *l*
proof (*rule mono-wrt-relI*)
fix *x x'* **assume** $x \leq_L x'$
then show $l\ x \leq_R l\ x'$
proof (*rule right-rel-if-left-relI'*)
fix y' **assume** $y' \leq_{L2}$ *l1 x'*

moreover with ‹$((\leq_{L2}) \Rightarrow (\leq_{R2}))$ $l2$› **have** $l2\ y' \leq_{R2} l\ x'$ **by** *auto*
ultimately show *in-codom* $(\leq_{R2})$ $(l\ x')$ $y' \leq_{L2} r2\ (l\ x')$
using ‹$((\leq_{L2}) \trianglelefteq_h (\leq_{R2}))$ $l2$ $r2$› **by** *auto*
next
assume *in-dom* $(\leq_{L2})$ $(l1\ x)$
with *refl-L2* ‹$((\leq_{L2}) \Rightarrow (\leq_{R2}))$ $l2$› **show** $l\ x \leq_{R2} l2\ (l1\ x)$ **by** *auto*
qed (*insert assms*, *auto*)
qed

end

end

2.11.4 Galois Connection

theory *Transport-Compositions-Generic-Galois-Connection*
imports
Transport-Compositions-Generic-Galois-Property
Transport-Compositions-Generic-Monotone
begin

context *transport-comp*
begin

interpretation *flip* : *transport-comp R2 L2 r2 l2 R1 L1 r1 l1*
rewrites *flip.t2.unit* $= \varepsilon_1$ **and** *flip.t1.counit* $\equiv \eta_2$
by (*simp-all only*: *t1.flip-unit-eq-counit t2.flip-counit-eq-unit*)

lemma *galois-connection-left-rightI*:
assumes $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and $((\leq_{L1}) \trianglelefteq (\leq_{R1}))$ *l1 r1*
and *rel-equivalence-on* (*in-codom* $(\leq_{R1})$) $(\leq_{R1})$ ε_1
and *transitive* $(\leq_{R1})$
and $((\leq_{L2}) \Rightarrow (\leq_{R2}))$ *l2*
and $((\leq_{R2})\ {}_h\trianglelefteq\ (\leq_{L2}))$ *r2 l2*
and *inflationary-on* (*in-field* $(\leq_{L2})$) $(\leq_{L2})$ η_2
and *preorder-on* (*in-field* $(\leq_{L2})$) $(\leq_{L2})$
and *middle-compatible-codom*
shows $((\leq_L) \dashv (\leq_R))$ *l r*
using *assms* **by** (*intro galois-connectionI galois-prop-left-rightI*
mono-wrt-rel-leftI flip.mono-wrt-rel-leftI)
(*auto intro*: *inflationary-on-if-le-pred-if-inflationary-on*
in-field-if-in-dom in-field-if-in-codom)

lemma *galois-connection-left-rightI'*:
assumes $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and $((\leq_{L1})\ {}_h\trianglelefteq\ (\leq_{R1}))$ *l1 r1*
and $((\leq_{R1}) \trianglelefteq_h (\leq_{L1}))$ *r1 l1*

and *inflationary-on* (*in-dom* ($\leq_{L1}$)) ($\leq_{L1}$) η_1
and *rel-equivalence-on* (*in-field* ($\leq_{R1}$)) ($\leq_{R1}$) ε_1
and *transitive* ($\leq_{R1}$)
and (($\leq_{L2}$) $\Rightarrow$ ($\leq_{R2}$)) *l2*
and (($\leq_{L2}$) $\trianglelefteq_h$ ($\leq_{R2}$)) *l2 r2*
and (($\leq_{R2}$) ${}_h\trianglerighteq$ ($\leq_{L2}$)) *r2 l2*
and *inflationary-on* (*in-dom* ($\leq_{L2}$)) ($\leq_{L2}$) η_2
and *preorder-on* (*in-field* ($\leq_{L2}$)) ($\leq_{L2}$)
and *middle-compatible-dom*
shows (($\leq_L$) $\dashv$ ($\leq_R$)) *l r*
using *assms* **by** (*intro galois-connectionI galois-prop-left-rightI'*
mono-wrt-rel-leftI' flip.mono-wrt-rel-leftI')
(*auto elim*!: *preorder-on-in-fieldE*
intro!: *reflexive-on-in-field-if-transitive-if-rel-equivalence-on*
intro: *reflexive-on-if-le-pred-if-reflexive-on in-field-if-in-dom*)

corollary *galois-connection-left-right-if-galois-equivalenceI*:
assumes (($\leq_{L1}$) $\equiv_G$ ($\leq_{R1}$)) *l1 r1*
and *preorder-on* (*in-field* ($\leq_{R1}$)) ($\leq_{R1}$)
and (($\leq_{L2}$) $\equiv_G$ ($\leq_{R2}$)) *l2 r2*
and *preorder-on* (*in-field* ($\leq_{L2}$)) ($\leq_{L2}$)
and *middle-compatible-codom*
shows (($\leq_L$) $\dashv$ ($\leq_R$)) *l r*
using *assms* **by** (*intro galois-connection-left-rightI*)
(*auto elim*!: *galois.galois-connectionE*
intro!: *flip.t2.rel-equivalence-on-unit-if-reflexive-on-if-galois-equivalence*
t2.inflationary-on-unit-if-reflexive-on-if-galois-equivalence
intro: *in-field-if-in-codom*)

corollary *galois-connection-left-right-if-order-equivalenceI*:
assumes (($\leq_{L1}$) $\equiv_o$ ($\leq_{R1}$)) *l1 r1*
and *transitive* ($\leq_{R1}$)
and (($\leq_{L2}$) $\equiv_o$ ($\leq_{R2}$)) *l2 r2*
and *transitive* ($\leq_{L2}$)
and *middle-compatible-codom*
shows (($\leq_L$) $\dashv$ ($\leq_R$)) *l r*
using *assms* **by** (*intro galois-connection-left-rightI'*)
(*auto elim*!: *rel-equivalence-onE*
intro!: *t1.half-galois-prop-left-left-right-if-transitive-if-deflationary-on-if-mono-wrt-rel*
flip.t1.half-galois-prop-left-left-right-if-transitive-if-deflationary-on-if-mono-wrt-rel
t2.half-galois-prop-right-left-right-if-transitive-if-inflationary-on-if-mono-wrt-rel
flip.t2.half-galois-prop-right-left-right-if-transitive-if-inflationary-on-if-mono-wrt-rel
preorder-on-in-field-if-transitive-if-rel-equivalence-on
rel-comp-comp-le-assms-if-in-codom-rel-comp-comp-leI
intro: *inflationary-on-if-le-pred-if-inflationary-on*
deflationary-on-if-le-pred-if-deflationary-on
in-field-if-in-dom in-field-if-in-codom)

end

end

2.11.5 Galois Equivalence

theory *Transport-Compositions-Generic-Galois-Equivalence*
imports
Transport-Compositions-Generic-Galois-Connection
begin

context *transport-comp*
begin

interpretation *flip* : *transport-comp R2 L2 r2 l2 R1 L1 r1 l1*
rewrites *flip.t2.unit* $= \varepsilon_1$ **and** *flip.t1.counit* $\equiv \eta_2$ **and** *flip.t1.unit* $\equiv \varepsilon_2$
by (*simp-all only*: *order-functors.flip-unit-eq-counit*)

lemma *galois-equivalenceI*:
assumes $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and $((\leq_{L1}) \trianglelefteq (\leq_{R1}))$ *l1 r1*
and *rel-equivalence-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$ ε_1
and *transitive* $(\leq_{R1})$
and $((\leq_{L2}) \Rightarrow (\leq_{R2}))$ *l2*
and $((\leq_{R2}) \trianglelefteq (\leq_{L2}))$ *r2 l2*
and *rel-equivalence-on* (*in-field* $(\leq_{L2})$) $(\leq_{L2})$ η_2
and *transitive* $(\leq_{L2})$
and *middle-compatible-codom*
shows $((\leq_L) \equiv_G (\leq_R))$ *l r*
using *assms* **by** (*intro galois-equivalenceI galois-connection-left-rightI*
flip.galois-prop-left-rightI)
(*auto intro*!: *preorder-on-in-field-if-transitive-if-rel-equivalence-on*
intro: *rel-equivalence-on-if-le-pred-if-rel-equivalence-on*
inflationary-on-if-le-pred-if-inflationary-on
in-field-if-in-dom in-field-if-in-codom)

lemma *galois-equivalenceI'*:
assumes $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and $((\leq_{L1}) \ {}_h\trianglelefteq (\leq_{R1}))$ *l1 r1*
and $((\leq_{R1}) \trianglelefteq_h (\leq_{L1}))$ *r1 l1*
and *inflationary-on* (*in-dom* $(\leq_{L1})$) $(\leq_{L1})$ η_1
and *rel-equivalence-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$ ε_1
and *transitive* $(\leq_{R1})$
and $((\leq_{L2}) \Rightarrow (\leq_{R2}))$ *l2*
and $((\leq_{L2}) \trianglelefteq_h (\leq_{R2}))$ *l2 r2*
and $((\leq_{R2}) \ {}_h\trianglelefteq (\leq_{L2}))$ *r2 l2*
and *rel-equivalence-on* (*in-field* $(\leq_{L2})$) $(\leq_{L2})$ η_2
and *inflationary-on* (*in-dom* $(\leq_{R2})$) $(\leq_{R2})$ ε_2
and *transitive* $(\leq_{L2})$

and *middle-compatible-dom*
shows $((\leq_L) \equiv_G (\leq_R))$ *l r*
using *assms* **by** (*intro galois.galois-equivalenceI galois-connection-left-rightI'*
flip.galois-prop-left-rightI')
(*auto elim*!: *rel-equivalence-onE*
intro!: *preorder-on-in-field-if-transitive-if-rel-equivalence-on*
intro: *inflationary-on-if-le-pred-if-inflationary-on*
in-field-if-in-dom)

corollary *galois-equivalence-if-galois-equivalenceI*:
assumes $((\leq_{L1}) \equiv_G (\leq_{R1}))$ *l1 r1*
and *preorder-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$
and $((\leq_{L2}) \equiv_G (\leq_{R2}))$ *l2 r2*
and *preorder-on* (*in-field* $(\leq_{L2})$) $(\leq_{L2})$
and *middle-compatible-codom*
shows $((\leq_L) \equiv_G (\leq_R))$ *l r*
using *assms* **by** (*intro galois-equivalenceI*)
(*auto intro*!: *t2.rel-equivalence-on-unit-if-reflexive-on-if-galois-equivalence*
flip.t2.rel-equivalence-on-unit-if-reflexive-on-if-galois-equivalence
intro: *reflexive-on-if-le-pred-if-reflexive-on*
in-field-if-in-dom in-field-if-in-codom)

corollary *galois-equivalence-if-order-equivalenceI*:
assumes $((\leq_{L1}) \equiv_o (\leq_{R1}))$ *l1 r1*
and *transitive* $(\leq_{R1})$
and $((\leq_{L2}) \equiv_o (\leq_{R2}))$ *l2 r2*
and *transitive* $(\leq_{L2})$
and *middle-compatible-codom*
shows $((\leq_L) \equiv_G (\leq_R))$ *l r*
using *assms* **by** (*intro galois-equivalenceI'*)
(*auto elim*!: *rel-equivalence-onE*
intro!: *t1.half-galois-prop-left-left-right-if-transitive-if-deflationary-on-if-mono-wrt-rel*
flip.t1.half-galois-prop-left-left-right-if-transitive-if-deflationary-on-if-mono-wrt-rel
t2.half-galois-prop-right-left-right-if-transitive-if-inflationary-on-if-mono-wrt-rel
flip.t2.half-galois-prop-right-left-right-if-transitive-if-inflationary-on-if-mono-wrt-rel
rel-comp-comp-le-assms-if-in-codom-rel-comp-comp-leI
preorder-on-in-field-if-transitive-if-rel-equivalence-on
intro: *deflationary-on-if-le-pred-if-deflationary-on*
inflationary-on-if-le-pred-if-inflationary-on
in-field-if-in-dom in-field-if-in-codom)

end

end

2.11.6 Galois Relator

theory *Transport-Compositions-Generic-Galois-Relator*

imports
Transport-Compositions-Generic-Base
begin

context *transport-comp*
begin

interpretation *flip* : *transport-comp R2 L2 r2 l2 R1 L1 r1 l1*
rewrites *flip.t2.unit* $\equiv \varepsilon_1$
by (*simp only*: *t1.flip-unit-eq-counit*)

lemma *left-Galois-le-comp-left-GaloisI*:
assumes *mono-r1*: $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and *galois-prop1*: $((\leq_{L1}) \trianglelefteq (\leq_{R1}))$ *l1 r1*
and *preorder-R1*: *preorder-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$
and *rel-comp-le*: $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq ((\leq_{R1}) \circ\circ (\leq_{L2}))$
and *mono-in-codom-r2*: (*in-codom* $(\leq_R) \Rightarrow$ *in-codom* $(\leq_{R1})$) *r2*
shows $({}_L\lessapprox) \leq (({}_{L1}\lessapprox) \circ\circ ({}_{L2}\lessapprox))$
proof (*rule le-relI*)
fix *x z* **assume** *x* ${}_L\lessapprox$ *z*
then have *in-codom* $(\leq_R)$ *z x* $\leq_L$ *r z* **by** *auto*
with *galois-prop1* **obtain** *y y′* **where** *in-dom* $(\leq_{L1})$ *x l1 x* $\leq_{R1}$ *y y* $\leq_{L2}$ *y′ y′* $\leq_{R1}$ ε_1 *(r2 z)*
by (*auto elim*!: *left-relE*)
moreover have ε_1 *(r2 z)* $\leq_{R1}$ *r2 z*
proof −
from *mono-in-codom-r2* ‹*in-codom* $(\leq_R)$ *z*› **have** *in-codom* $(\leq_{R1})$ *(r2 z)* **by** *blast*
with *mono-r1 galois-prop1 preorder-R1* **show** *?thesis* **by** (*blast intro*!: *t1.counit-rel-if-reflexive-on-if-half-galois-prop-left-if-mono-wrt-rel*)
qed
ultimately have *y′* $\leq_{R1}$ *r2 z* **using** *preorder-R1* **by** *blast*
with ‹*l1 x* $\leq_{R1}$ *y*› ‹*y* $\leq_{L2}$ *y′*› **have** $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1}))$ *(l1 x) (r2 z)*
by *blast*
with *rel-comp-le* **obtain** *y″* **where** *l1 x* $\leq_{R1}$ *y″ y″* $\leq_{L2}$ *r2 z* **by** *blast*
with *galois-prop1* ‹*in-dom* $(\leq_{L1})$ *x*› **have** *x* ${}_{L1}\lessapprox$ *y″*
by (*intro t1.left-Galois-if-Galois-right-if-half-galois-prop-right t1.left-GaloisI*) *auto*
moreover from ‹*in-codom* $(\leq_R)$ *z*› ‹*y″* $\leq_{L2}$ *r2 z*› **have** *y″* ${}_{L2}\lessapprox$ *z*
by (*intro t2.left-GaloisI*) *auto*
ultimately show $(({}_{L1}\lessapprox) \circ\circ ({}_{L2}\lessapprox))$ *x z* **by** *blast*
qed

lemma *comp-left-Galois-le-left-GaloisI*:
assumes *mono-r1*: $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and *half-galois-prop-left1*: $((\leq_{L1}) \trianglelefteq_h (\leq_{R1}))$ *l1 r1*
and *half-galois-prop-right1*: $((\leq_{R1}) \trianglelefteq_h (\leq_{L1}))$ *r1 l1*
and *refl-R1*: *reflexive-on* (*in-codom* $(\leq_{R1})$) $(\leq_{R1})$
and *mono-l2*: $((\leq_{L2}) \Rightarrow (\leq_{R2}))$ *l2*

```
  and refl-L2: reflexive-on (in-dom (≤L2)) (≤L2)
  and in-codom-rel-comp-le: in-codom ((≤L2) ∘∘ (≤R1) ∘∘ (≤L2)) ≤ in-codom
(≤R1)
  shows ((L1≲≈) ∘∘ (L2≲≈)) ≤ (L≲≈)
proof (intro le-relI left-GaloisI)
  fix x z assume ((L1≲≈) ∘∘ (L2≲≈)) x z
  from ‹((L1≲≈) ∘∘ (L2≲≈)) x z› obtain y where x L1≲≈ y y L2≲≈ z by blast
  with half-galois-prop-left1 have l1 x ≤R1 y y ≤L2 r2 z by auto
  with refl-R1 refl-L2 have y ≤R1 y y ≤L2 y by auto
  show in-codom (≤R) z
  proof (intro in-codomI flip.left-relI)
    from mono-l2 ‹y ≤L2 y› show l2 y R2≲≈ y by blast
    show y ≤R1 y y L2≲≈ z by fact+
  qed
  show x ≤L r z
  proof (intro left-relI)
    show x L1≲≈ y y ≤L2 r2 z by fact+
    show r2 z R1≲≈ r z
    proof (intro flip.t2.left-GaloisI)
      from ‹y ≤L2 y› ‹y ≤R1 y› ‹y ≤L2 r2 z› have ((≤L2) ∘∘ (≤R1) ∘∘ (≤L2)) y
(r2 z)
        by blast
      with in-codom-rel-comp-le have in-codom (≤R1) (r2 z) by blast
      with refl-R1 have r2 z ≤R1 r2 z by blast
      with mono-r1 show in-codom (≤L1) (r z) by auto
      with ‹r2 z ≤R1 r2 z›  half-galois-prop-right1 mono-r1
        show r2 z ≤R1 l1 (r z) by (fastforce intro:
        flip.t2.rel-unit-if-left-rel-if-half-galois-prop-right-if-mono-wrt-rel)
    qed
  qed
qed

corollary left-Galois-eq-comp-left-GaloisI:
  assumes ((≤R1) ⇒ (≤L1)) r1
  and ((≤L1) ⊴ (≤R1)) l1 r1
  and ((≤R1) ⊴h (≤L1)) r1 l1
  and preorder-on (in-field (≤R1)) (≤R1)
  and ((≤L2) ⇒ (≤R2)) l2
  and reflexive-on (in-dom (≤L2)) (≤L2)
  and ((≤R1) ∘∘ (≤L2) ∘∘ (≤R1)) ≤ ((≤R1) ∘∘ (≤L2))
  and (in-codom (≤R) ⇒ in-codom (≤R1)) r2
  and in-codom ((≤L2) ∘∘ (≤R1) ∘∘ (≤L2)) ≤ in-codom (≤R1)
  shows (L≲≈) = ((L1≲≈) ∘∘ (L2≲≈))
  using assms
 by (intro antisym left-Galois-le-comp-left-GaloisI comp-left-Galois-le-left-GaloisI)
  (auto elim!: preorder-on-in-fieldE
    intro: reflexive-on-if-le-pred-if-reflexive-on in-field-if-in-codom)

corollary left-Galois-eq-comp-left-GaloisI':
```

assumes $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and $((\leq_{L1}) \trianglelefteq (\leq_{R1}))$ *l1 r1*
and $((\leq_{R1}) \trianglelefteq_h (\leq_{L1}))$ *r1 l1*
and *preorder-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$
and $((\leq_{L2}) \Rightarrow (\leq_{R2}))$ *l2*
and $((\leq_{R2}) \ {}_h\trianglelefteq (\leq_{L2}))$ *r2 l2*
and *reflexive-on* (*in-dom* $(\leq_{L2})$) $(\leq_{L2})$
and $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq ((\leq_{R1}) \circ\circ (\leq_{L2}))$
and *in-codom* $((\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2})) \leq$ *in-codom* $(\leq_{R1})$
shows $({}_L\lessapprox) = (({}_{L1}\lessapprox) \circ\circ ({}_{L2}\lessapprox))$
using *assms* **by** (*intro left-Galois-eq-comp-left-GaloisI*
flip.mono-in-codom-left-rel-left1-if-in-codom-rel-comp-le)
auto

theorem *left-Galois-eq-comp-left-Galois-if-galois-connection-if-galois-equivalenceI'*:
assumes $((\leq_{L1}) \equiv_G (\leq_{R1}))$ *l1 r1*
and *preorder-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$
and $((\leq_{R2}) \dashv (\leq_{L2}))$ *r2 l2*
and *reflexive-on* (*in-dom* $(\leq_{L2})$) $(\leq_{L2})$
and $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq ((\leq_{R1}) \circ\circ (\leq_{L2}))$
and *in-codom* $((\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2})) \leq$ *in-codom* $(\leq_{R1})$
shows $({}_L\lessapprox) = (({}_{L1}\lessapprox) \circ\circ ({}_{L2}\lessapprox))$
using *assms* **by** (*intro left-Galois-eq-comp-left-GaloisI'*)
(*auto elim*!: *t1.galois-equivalenceE*)

corollary *left-Galois-eq-comp-left-Galois-if-galois-connection-if-galois-equivalenceI*:
assumes $((\leq_{L1}) \equiv_G (\leq_{R1}))$ *l1 r1*
and *preorder-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$
and $((\leq_{R2}) \dashv (\leq_{L2}))$ *r2 l2*
and *reflexive-on* (*in-field* $(\leq_{L2})$) $(\leq_{L2})$
and *in-codom* $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq$ *in-codom* $(\leq_{L2})$
and $((\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2})) \leq ((\leq_{R1}) \circ\circ (\leq_{L2}))$
and *in-codom* $((\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2})) \leq$ *in-codom* $(\leq_{R1})$
shows $({}_L\lessapprox) = (({}_{L1}\lessapprox) \circ\circ ({}_{L2}\lessapprox))$
using *assms*
by (*intro left-Galois-eq-comp-left-Galois-if-galois-connection-if-galois-equivalenceI'*
flip.left2-right1-left2-le-left2-right1-if-right1-left2-right1-le-left2-right1)
auto

corollary *left-Galois-eq-comp-left-Galois-if-preorder-equivalenceI*:
assumes $((\leq_{L1}) \equiv_{pre} (\leq_{R1}))$ *l1 r1*
and $((\leq_{L2}) \equiv_{pre} (\leq_{R2}))$ *l2 r2*
and *in-codom* $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq$ *in-codom* $(\leq_{L2})$
and $(\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2}) \leq (\leq_{R1}) \circ\circ (\leq_{L2})$
and *in-codom* $((\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2})) \leq$ *in-codom* $(\leq_{R1})$
shows $({}_L\lessapprox) = (({}_{L1}\lessapprox) \circ\circ ({}_{L2}\lessapprox))$
using *assms* **by** (*intro*
left-Galois-eq-comp-left-Galois-if-galois-connection-if-galois-equivalenceI)
auto

end

end

2.11.7 Basic Order Properties

theory *Transport-Compositions-Generic-Order-Base*
imports
Transport-Compositions-Generic-Base
begin

context *transport-comp*
begin

interpretation *flip1* : *galois R1 L1 r1 l1* **.**

Reflexivity

lemma *reflexive-on-in-dom-leftI*:
assumes *galois-prop*: $((\leq_{L1}) \trianglelefteq (\leq_{R1}))$ *l1 r1*
and *in-dom-L1-le*: *in-dom* $(\leq_{L1}) \leq$ *in-codom* $(\leq_{L1})$
and *refl-R1*: *reflexive-on* (*in-dom* $(\leq_{R1})$) $(\leq_{R1})$
and *refl-L2*: *reflexive-on* (*in-dom* $(\leq_{L2})$) $(\leq_{L2})$
and *mono-in-dom-l1*: (*in-dom* $(\leq_L) \Rightarrow$ *in-dom* $(\leq_{L2})$) *l1*
shows *reflexive-on* (*in-dom* $(\leq_L)$) $(\leq_L)$
proof (*rule reflexive-onI*)
fix *x* **assume** *in-dom* $(\leq_L)$ *x*
then obtain x' **where** $x \leq_L x'$ *in-dom* $(\leq_{L1})$ *x* **by** *blast*
show $x \leq_L x$
proof (*rule left-relI*)
from *refl-R1* **have** *l1 x* $\leq_{R1}$ *l1 x*
proof (*rule reflexive-onD*)
from ‹$x \leq_L x'$› *galois-prop* **show** *in-dom* $(\leq_{R1})$ (*l1 x*) **by** *blast*
qed
then show *x* $_{L1}\lessapprox$ *l1 x*
proof (*intro t1.left-GaloisI*)
from *galois-prop* ‹*in-dom* $(\leq_{L1})$ *x*› ‹*l1 x* $\leq_{R1}$ *l1 x*› **show** *x* $\leq_{L1}$ *r1* (*l1 x*)
by *blast*
qed *blast*
from *refl-L2* **show** *l1 x* $\leq_{L2}$ *l1 x*
proof (*rule reflexive-onD*)
from *mono-in-dom-l1* ‹$x \leq_L x'$› **show** *in-dom* $(\leq_{L2})$ (*l1 x*) **by** *blast*
qed
from ‹*l1 x* $\leq_{R1}$ *l1 x*› **show** *l1 x* $_{R1}\gtrapprox$ *x*
proof (*intro flip1.left-GaloisI*)
from ‹*in-dom* $(\leq_{L1})$ *x*› *in-dom-L1-le* **show** *in-codom* $(\leq_{L1})$ *x* **by** *blast*
qed
qed

qed

lemma *reflexive-on-in-codom-leftI*:
assumes *L1-r1-l1I*: $\bigwedge x.\ \textit{in-dom}\ (\leq_{L1})\ x \Longrightarrow \textit{l1}\ x \leq_{R1} \textit{l1}\ x \Longrightarrow x \leq_{L1} \textit{r1}\ (\textit{l1}\ x)$
and *in-codom-L1-le*: $\textit{in-codom}\ (\leq_{L1}) \leq \textit{in-dom}\ (\leq_{L1})$
and *refl-R1*: $\textit{reflexive-on}\ (\textit{in-codom}\ (\leq_{R1}))\ (\leq_{R1})$
and *refl-L2*: $\textit{reflexive-on}\ (\textit{in-codom}\ (\leq_{L2}))\ (\leq_{L2})$
and *mono-in-codom-l1*: $(\textit{in-codom}\ (\leq_L) \Rightarrow \textit{in-codom}\ (\leq_{L2}))\ \textit{l1}$
shows $\textit{reflexive-on}\ (\textit{in-codom}\ (\leq_L))\ (\leq_L)$
proof (*rule reflexive-onI*)
fix x **assume** $\textit{in-codom}\ (\leq_L)\ x$
then obtain x' **where** $x' \leq_L x\ \textit{in-codom}\ (\leq_{L1})\ x\ \textit{in-codom}\ (\leq_{R1})\ (\textit{l1}\ x)$
by *blast*
show $x \leq_L x$
proof (*rule left-relI*)
from *refl-R1* ‹$\textit{in-codom}\ (\leq_{R1})\ (\textit{l1}\ x)$› **have** $\textit{l1}\ x \leq_{R1} \textit{l1}\ x$ **by** *blast*
show $x\ {}_{L1}\lessapprox \textit{l1}\ x$
proof (*rule t1.left-GaloisI*)
from *in-codom-L1-le* ‹$\textit{in-codom}\ (\leq_{L1})\ x$› **have** $\textit{in-dom}\ (\leq_{L1})\ x$ **by** *blast*
with ‹$\textit{l1}\ x \leq_{R1} \textit{l1}\ x$› **show** $x \leq_{L1} \textit{r1}\ (\textit{l1}\ x)$ **by** (*intro L1-r1-l1I*)
qed *fact*
from *refl-L2* **show** $\textit{l1}\ x \leq_{L2} \textit{l1}\ x$
proof (*rule reflexive-onD*)
from *mono-in-codom-l1* ‹$x' \leq_L x$› **show** $\textit{in-codom}\ (\leq_{L2})\ (\textit{l1}\ x)$ **by** *blast*
qed
show $\textit{l1}\ x\ {}_{R1}\gtrapprox x$ **by** (*rule flip1.left-GaloisI*) *fact+*
qed
qed

corollary *reflexive-on-in-field-leftI*:
assumes $((\leq_{L1}) \trianglelefteq (\leq_{R1}))\ \textit{l1}\ \textit{r1}$
and $\textit{in-codom}\ (\leq_{L1}) = \textit{in-dom}\ (\leq_{L1})$
and $\textit{reflexive-on}\ (\textit{in-field}\ (\leq_{R1}))\ (\leq_{R1})$
and $\textit{reflexive-on}\ (\textit{in-field}\ (\leq_{L2}))\ (\leq_{L2})$
and $(\textit{in-field}\ (\leq_L) \Rightarrow \textit{in-field}\ (\leq_{L2}))\ \textit{l1}$
shows $\textit{reflexive-on}\ (\textit{in-field}\ (\leq_L))\ (\leq_L)$
proof −
from *assms* **have** $\textit{reflexive-on}\ (\textit{in-dom}\ (\leq_L))\ (\leq_L)$
by (*intro reflexive-on-in-dom-leftI*)
(*auto 0 5 intro*: *reflexive-on-if-le-pred-if-reflexive-on in-field-if-in-dom*)
moreover from *assms* **have** $\textit{reflexive-on}\ (\textit{in-codom}\ (\leq_L))\ (\leq_L)$
by (*intro reflexive-on-in-codom-leftI*)
(*auto 0 5 intro*: *reflexive-on-if-le-pred-if-reflexive-on in-field-if-in-codom*)
ultimately show *?thesis* **by** (*auto iff*: *in-field-iff-in-dom-or-in-codom*)
qed

Transitivity

There are many similar proofs for transitivity. They slightly differ in their assumptions, particularly which of $(\leq_{R1})$ and $(\leq_{L2})$ has to be transitive and the order of commutativity for the relations.

In the following, we just give two of them that suffice for many purposes.

lemma *transitive-leftI*:
assumes $((\leq_{L1}) \ {}_h\trianglelefteq \ (\leq_{R1}))$ *l1 r1*
and *trans-L2*: *transitive* $(\leq_{L2})$
and *R1-L2-R1-le*: $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq ((\leq_{L2}) \circ\circ (\leq_{R1}))$
shows *transitive* $(\leq_L)$
proof (*rule transitiveI*)
fix *x1 x2 x3* **assume** $x1 \leq_L x2$ $x2 \leq_L x3$
from ‹$x1 \leq_L x2$› **obtain** *y1 y2* **where** $x1 \ {}_{L1}\lessapprox y1$ $y1 \leq_{L2} y2$ $y2 \leq_{R1} l1$ *x2*
by *blast*
from ‹$x2 \leq_L x3$› ‹$((\leq_{L1}) \ {}_h\trianglelefteq \ (\leq_{R1}))$ *l1 r1*› **obtain** *y3 y4* **where**
$l1$ $x2 \leq_{R1} y3$ $y3 \leq_{L2} y4$ $y4 \leq_{R1} l1$ *x3 in-codom* $(\leq_{L1})$ *x3* **by** *blast*
with *R1-L2-R1-le* **have** $((\leq_{L2}) \circ\circ (\leq_{R1}))$ $(l1\ x2)$ $(l1\ x3)$ **by** *blast*
then obtain *y* **where** $l1$ $x2 \leq_{L2} y$ $y \leq_{R1} l1$ *x3* **by** *blast*
with ‹$y2 \leq_{R1} l1$ *x2*› *R1-L2-R1-le* **have** $((\leq_{L2}) \circ\circ (\leq_{R1}))$ *y2* $(l1\ x3)$ **by** *blast*
then obtain y' **where** $y2 \leq_{L2} y'$ $y' \leq_{R1} l1$ *x3* **by** *blast*
with ‹$y1 \leq_{L2} y2$› **have** $y1 \leq_{L2} y'$ **using** *trans-L2* **by** *blast*
show $x1 \leq_L x3$
proof (*rule left-relI*)
show $x1 \ {}_{L1}\lessapprox y1$ $y1 \leq_{L2} y'$ **by** *fact+*
show $y' \ {}_{R1}\lessapprox x3$ **by** (*rule flip1.left-GaloisI*) *fact+*
qed
qed

lemma *transitive-leftI′*:
assumes *galois-prop*: $((\leq_{L1}) \trianglelefteq (\leq_{R1}))$ *l1 r1*
and *trans-L2*: *transitive* $(\leq_{L2})$
and *R1-L2-R1-le*: $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq ((\leq_{R1}) \circ\circ (\leq_{L2}))$
shows *transitive* $(\leq_L)$
proof (*rule transitiveI*)
fix *x1 x2 x3* **assume** $x1 \leq_L x2$ $x2 \leq_L x3$
from ‹$x1 \leq_L x2$› *galois-prop* **obtain** *y1 y2* **where**
in-dom $(\leq_{L1})$ *x1 l1 x1* $\leq_{R1} y1$ $y1 \leq_{L2} y2$ $y2 \leq_{R1} l1$ *x2* **by** *blast*
with *R1-L2-R1-le* **have** $((\leq_{R1}) \circ\circ (\leq_{L2}))$ $(l1\ x1)$ $(l1\ x2)$ **by** *blast*
then obtain *y* **where** $l1$ $x1 \leq_{R1} y$ $y \leq_{L2} l1$ *x2* **by** *blast*
moreover from ‹$x2 \leq_L x3$› *galois-prop* **obtain** *y3 y4* **where**
$l1$ $x2 \leq_{R1} y3$ $y3 \leq_{L2} y4$ $y4 \ {}_{R1}\gtrapprox x3$ **by** *blast*
moreover note *R1-L2-R1-le*
ultimately have $((\leq_{R1}) \circ\circ (\leq_{L2}))$ $(l1\ x1)$ *y3* **by** *blast*
then obtain y' **where** $l1$ $x1 \leq_{R1} y'$ $y' \leq_{L2} y3$ **by** *blast*
with ‹$y3 \leq_{L2} y4$› **have** $y' \leq_{L2} y4$ **using** *trans-L2* **by** *blast*
show $x1 \leq_L x3$
proof (*rule left-relI*)
from ‹*in-dom* $(\leq_{L1})$ *x1*› ‹$l1$ $x1 \leq_{R1} y'$› *galois-prop* **show** $x1 \ {}_{L1}\lessapprox y'$

by (*intro t1.left-Galois-if-Galois-right-if-half-galois-prop-right t1.left-GaloisI*)
auto
show $y' \leq_{L2} y4$ **by** *fact*
from ‹$y' \leq_{L2} y4$› ‹$y4\ {}_{R1}\lessapprox x3$› **show** $y4\ {}_{R1}\lessapprox x3$ **by** *blast*
qed
qed

Preorders

lemma *preorder-on-in-field-leftI*:
assumes $((\leq_{L1}) \trianglelefteq (\leq_{R1}))$ *l1 r1*
and *in-codom* $(\leq_{L1})$ = *in-dom* $(\leq_{L1})$
and *reflexive-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$
and *preorder-on* (*in-field* $(\leq_{L2})$) $(\leq_{L2})$
and *mono-in-codom-l1*: (*in-codom* $(\leq_L) \Rightarrow$ *in-codom* $(\leq_{L2})$) *l1*
and *R1-L2-R1-le*: $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq ((\leq_{L2}) \circ\circ (\leq_{R1}))$
shows *preorder-on* (*in-field* $(\leq_L)$) $(\leq_L)$
proof −
have (*in-field* $(\leq_L) \Rightarrow$ *in-field* $(\leq_{L2})$) *l1*
proof −
from ‹$((\leq_{L1}) \trianglelefteq (\leq_{R1}))$ *l1 r1*› *R1-L2-R1-le*
have (*in-dom* $(\leq_L) \Rightarrow$ *in-dom* $(\leq_{L2})$) *l1*
by (*intro mono-in-dom-left-rel-left1-if-in-dom-rel-comp-le*
in-dom-right1-left2-right1-le-if-right1-left2-right1-le)
auto
with *mono-in-codom-l1* **show** *?thesis* **by** (*intro mono-wrt-predI*) *blast*
qed
with *assms* **show** *?thesis* **by** (*intro preorder-onI*)
(*auto intro*: *reflexive-on-in-field-leftI transitive-leftI*)
qed

lemma *preorder-on-in-field-leftI'*:
assumes $((\leq_{L1}) \trianglelefteq (\leq_{R1}))$ *l1 r1*
and *in-codom* $(\leq_{L1})$ = *in-dom* $(\leq_{L1})$
and *reflexive-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$
and *preorder-on* (*in-field* $(\leq_{L2})$) $(\leq_{L2})$
and *mono-in-dom-l1*: (*in-dom* $(\leq_L) \Rightarrow$ *in-dom* $(\leq_{L2})$) *l1*
and *R1-L2-R1-le*: $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq ((\leq_{R1}) \circ\circ (\leq_{L2}))$
shows *preorder-on* (*in-field* $(\leq_L)$) $(\leq_L)$
proof −
have (*in-field* $(\leq_L) \Rightarrow$ *in-field* $(\leq_{L2})$) *l1*
proof −
from ‹$((\leq_{L1}) \trianglelefteq (\leq_{R1}))$ *l1 r1*› *R1-L2-R1-le*
have (*in-codom* $(\leq_L) \Rightarrow$ *in-codom* $(\leq_{L2})$) *l1*
by (*intro mono-in-codom-left-rel-left1-if-in-codom-rel-comp-le*
in-codom-right1-left2-right1-le-if-right1-left2-right1-le)
auto
with *mono-in-dom-l1* **show** *?thesis* **by** (*intro mono-wrt-predI*) *blast*
qed

with *assms* **show** *?thesis* **by** (*intro preorder-onI*)
(*auto intro*: *reflexive-on-in-field-leftI transitive-leftI'*)
qed

Symmetry

lemma *symmetric-leftI*:
assumes $((\leq_{L1}) \trianglelefteq (\leq_{R1}))$ *l1 r1*
and *in-codom* $(\leq_{L1})$ = *in-dom* $(\leq_{L1})$
and *symmetric* $(\leq_{R1})$
and *symmetric* $(\leq_{L2})$
shows *symmetric* $(\leq_L)$
proof −
from *assms* **have** $(\gtrsim_{R1}) = ({}_{L1}\lesssim)$ **by** (*intro*
t1.ge-Galois-right-eq-left-Galois-if-symmetric-if-in-codom-eq-in-dom-if-galois-prop)
moreover then have $({}_{R1}\lesssim) = (\gtrsim_{L1})$
by (*subst rel-inv-eq-iff-eq*[*symmetric*]) *simp*
ultimately show *?thesis* **using** *assms* **unfolding** *left-rel-eq-comp*
by (*subst symmetric-iff-rel-inv-eq-self*) (*simp add*: *rel-comp-assoc*)
qed

lemma *partial-equivalence-rel-leftI*:
assumes $((\leq_{L1}) \trianglelefteq (\leq_{R1}))$ *l1 r1*
and *in-codom* $(\leq_{L1})$ = *in-dom* $(\leq_{L1})$
and *symmetric* $(\leq_{R1})$
and *partial-equivalence-rel* $(\leq_{L2})$
and $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq ((\leq_{L2}) \circ\circ (\leq_{R1}))$
shows *partial-equivalence-rel* $(\leq_L)$
using *assms* **by** (*intro partial-equivalence-relI transitive-leftI symmetric-leftI*)
auto

lemma *partial-equivalence-rel-leftI'*:
assumes $((\leq_{L1}) \trianglelefteq (\leq_{R1}))$ *l1 r1*
and *in-codom* $(\leq_{L1})$ = *in-dom* $(\leq_{L1})$
and *symmetric* $(\leq_{R1})$
and *partial-equivalence-rel* $(\leq_{L2})$
and $((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq ((\leq_{R1}) \circ\circ (\leq_{L2}))$
shows *partial-equivalence-rel* $(\leq_L)$
using *assms* **by** (*intro partial-equivalence-relI transitive-leftI' symmetric-leftI*)
auto

end

end

2.11.8 Order Equivalence

theory *Transport-Compositions-Generic-Order-Equivalence*
imports

Transport-Compositions-Generic-Monotone
begin

context *transport-comp*
begin

context
begin

interpretation *flip* : *transport-comp R2 L2 r2 l2 R1 L1 r1 l1* **.**

Unit

Inflationary **lemma** *inflationary-on-in-dom-unitI*:
assumes $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and $((\leq_{L1})\ {}_h\trianglelefteq\ (\leq_{R1}))$ *l1 r1*
and *inflationary-unit1*: *inflationary-on* (*in-dom* $(\leq_{L1})$) $(\leq_{L1})$ η_1
and *inflationary-counit1*: *inflationary-on* (*in-codom* $(\leq_{R1})$) $(\leq_{R1})$ ε_1
and *refl-R1*: *reflexive-on* (*in-dom* $(\leq_{R1})$) $(\leq_{R1})$
and *inflationary-unit2*: *inflationary-on* (*in-dom* $(\leq_{L2})$) $(\leq_{L2})$ η_2
and *refl-L2*: *reflexive-on* (*in-dom* $(\leq_{L2})$) $(\leq_{L2})$
and *mono-in-dom-l1*: (*in-dom* $(\leq_L) \Rightarrow$ *in-dom* $(\leq_{L2})$) *l1*
and *in-codom-rel-comp-le*: *in-codom* $((\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2})) \leq$ *in-codom* $((\leq_{R1}))$
shows *inflationary-on* (*in-dom* $(\leq_L)$) $(\leq_L)$ η
proof (*rule inflationary-onI*)
fix x **assume** *in-dom* $(\leq_L)$ x
show $x \leq_L \eta\ x$
proof (*rule left-relI*)
from ‹*in-dom* $(\leq_L)$ x› ‹$((\leq_{L1})\ {}_h\trianglelefteq\ (\leq_{R1}))$ *l1 r1*› **have** *in-dom* $(\leq_{R1})$ $(l1\ x)$ **by** *blast*
with *refl-R1* **have** $l1\ x \leq_{R1} l1\ x$ **by** *blast*
moreover from ‹*in-dom* $(\leq_L)$ x› **have** *in-dom* $(\leq_{L1})$ x **by** *blast*
moreover note *inflationary-unit1*
ultimately show $x\ {}_{L1}\lessapprox l1\ x$ **by** (*intro t1.left-GaloisI*) *auto*
from ‹*in-dom* $(\leq_L)$ x› *mono-in-dom-l1* **have** *in-dom* $(\leq_{L2})$ $(l1\ x)$ **by** *blast*
with *inflationary-unit2* **show** $l1\ x \leq_{L2} r2\ (l\ x)$ **by** *auto*
show $r2\ (l\ x)\ {}_{R1}\gtrapprox \eta\ x$
proof (*rule flip.t2.left-GaloisI*)
from *refl-L2* ‹*in-dom* $(\leq_{L2})$ $(l1\ x)$› **have** $l1\ x \leq_{L2} l1\ x$ **by** *blast*
with *in-codom-rel-comp-le* ‹$l1\ x \leq_{R1} l1\ x$› ‹$l1\ x \leq_{L2} r2\ (l\ x)$›
have *in-codom* $(\leq_{R1})$ $(r2\ (l\ x))$ **by** *blast*
with ‹$((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*› **show** *in-codom* $(\leq_{L1})$ $(\eta\ x)$
by (*auto intro*: *in-codom-if-rel-if-dep-mono-wrt-rel simp*: *mono-wrt-rel-eq-dep-mono-wrt-rel*)
from ‹*in-codom* $(\leq_{R1})$ $(r2\ (l\ x))$› *inflationary-counit1*
show $r2\ (l\ x) \leq_{R1} l1\ (\eta\ x)$ **by** *auto*
qed
qed
qed

lemma *inflationary-on-in-codom-unitI*:
assumes $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and *inflationary-unit1*: *inflationary-on* (*in-codom* $(\leq_{L1})$) $(\leq_{L1})$ η_1
and *inflationary-counit1*: *inflationary-on* (*in-codom* $(\leq_{R1})$) $(\leq_{R1})$ ε_1
and *refl-R1*: *reflexive-on* (*in-codom* $(\leq_{R1})$) $(\leq_{R1})$
and *inflationary-unit2*: *inflationary-on* (*in-codom* $(\leq_{L2})$) $(\leq_{L2})$ η_2
and *refl-L2*: *reflexive-on* (*in-codom* $(\leq_{L2})$) $(\leq_{L2})$
and *mono-in-codom-l1*: (*in-codom* $(\leq_L)$ $\Rightarrow$ *in-codom* $(\leq_{L2})$) *l1*
and *in-codom-rel-comp-le*: *in-codom* $((\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2})) \leq$ *in-codom* $((\leq_{R1}))$
shows *inflationary-on* (*in-codom* $(\leq_L)$) $(\leq_L)$ η
proof (*rule inflationary-onI*)
fix *x* **assume** *in-codom* $(\leq_L)$ *x*
show $x \leq_L \eta\ x$
proof (*rule left-relI*)
from ‹*in-codom* $(\leq_L)$ *x*› **have** *in-codom* $(\leq_{L1})$ *x in-codom* $(\leq_{R1})$ (*l1 x*) **by** *blast+*
with *inflationary-unit1* **show** $x\ {}_{L1}\lessapprox$ *l1 x* **by** (*intro t1.left-GaloisI*) *auto*
from *mono-in-codom-l1* ‹*in-codom* $(\leq_L)$ *x*› **have** *in-codom* $(\leq_{L2})$ (*l1 x*) **by** *blast*
with *inflationary-unit2* **show** *l1 x* $\leq_{L2}$ *r2* (*l x*) **by** *auto*
show *r2* (*l x*) ${}_{R1}\gtrapprox$ $\eta\ x$
proof (*rule flip.t2.left-GaloisI*)
from *refl-L2* ‹*in-codom* $(\leq_{L2})$ (*l1 x*)› **have** *l1 x* $\leq_{L2}$ *l1 x* **by** *blast*
moreover from *refl-R1* ‹*in-codom* $(\leq_{R1})$ (*l1 x*)› **have** *l1 x* $\leq_{R1}$ *l1 x* **by** *blast*
moreover note *in-codom-rel-comp-le* ‹*l1 x* $\leq_{L2}$ *r2* (*l x*)›
ultimately have *in-codom* $(\leq_{R1})$ (*r2* (*l x*)) **by** *blast*
with ‹$((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*› **show** *in-codom* $(\leq_{L1})$ $(\eta\ x)$
by (*auto intro*: *in-codom-if-rel-if-dep-mono-wrt-rel simp*: *mono-wrt-rel-eq-dep-mono-wrt-rel*)
from ‹*in-codom* $(\leq_{R1})$ (*r2* (*l x*))› *inflationary-counit1*
show *r2* (*l x*) $\leq_{R1}$ *l1* ($\eta\ x$) **by** *auto*
qed
qed
qed

corollary *inflationary-on-in-field-unitI*:
assumes $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and $((\leq_{L1})\ {}_h\trianglelefteq\ (\leq_{R1}))$ *l1 r1*
and *inflationary-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$ η_1
and *inflationary-on* (*in-codom* $(\leq_{R1})$) $(\leq_{R1})$ ε_1
and *reflexive-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$
and *inflationary-on* (*in-field* $(\leq_{L2})$) $(\leq_{L2})$ η_2
and *reflexive-on* (*in-field* $(\leq_{L2})$) $(\leq_{L2})$
and (*in-dom* $(\leq_L)$ $\Rightarrow$ *in-dom* $(\leq_{L2})$) *l1*
and (*in-codom* $(\leq_L)$ $\Rightarrow$ *in-codom* $(\leq_{L2})$) *l1*
and *in-codom* $((\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2})) \leq$ *in-codom* $((\leq_{R1}))$
shows *inflationary-on* (*in-field* $(\leq_L)$) $(\leq_L)$ η
proof −

from *assms* **have** *inflationary-on* (*in-dom* ($\leq_L$)) ($\leq_L$) η
by (*intro inflationary-on-in-dom-unitI*)
(*auto intro*: *inflationary-on-if-le-pred-if-inflationary-on reflexive-on-if-le-pred-if-reflexive-on in-field-if-in-dom*)
moreover from *assms* **have** *inflationary-on* (*in-codom* ($\leq_L$)) ($\leq_L$) η
by (*intro inflationary-on-in-codom-unitI*)
(*auto intro*: *inflationary-on-if-le-pred-if-inflationary-on reflexive-on-if-le-pred-if-reflexive-on in-field-if-in-codom*)
ultimately show *?thesis* **by** (*auto iff*: *in-field-iff-in-dom-or-in-codom*)
qed

Deflationary

lemma *deflationary-on-in-dom-unitI*:
assumes (($\leq_{L1}$) $\Rightarrow$ ($\leq_{R1}$)) *l1* (($\leq_{R1}$) $\Rightarrow$ ($\leq_{L1}$)) *r1*
and *refl-L1*: *reflexive-on* (*in-dom* ($\leq_{L1}$)) ($\leq_{L1}$)
and *in-dom-R1-le-in-codom-R1*: *in-dom* ($\leq_{R1}$) $\leq$ *in-codom* ($\leq_{R1}$)
and *deflationary-L2*: *deflationary-on* (*in-dom* ($\leq_{L2}$)) ($\leq_{L2}$) η_2
and *refl-L2*: *reflexive-on* (*in-dom* ($\leq_{L2}$)) ($\leq_{L2}$)
and *mono-in-dom-l1*: (*in-dom* ($\leq_L$) $\Rightarrow$ *in-dom* ($\leq_{L2}$)) *l1*
and *in-dom-rel-comp-le*: *in-dom* (($\leq_{L2}$) $\circ\circ$ ($\leq_{R1}$) $\circ\circ$ ($\leq_{L2}$)) $\leq$ *in-dom* (($\leq_{R1}$))
shows *deflationary-on* (*in-dom* ($\leq_L$)) ($\leq_L$) η
proof (*rule deflationary-onI*)
fix *x* **assume** *in-dom* ($\leq_L$) *x*
show η *x* $\leq_L$ *x*
proof (*rule left-relI*)
from *refl-L1* ‹*in-dom* ($\leq_L$) *x*› **have** *x* $\leq_{L1}$ *x* **by** *blast*
moreover with ‹(($\leq_{L1}$) $\Rightarrow$ ($\leq_{R1}$)) *l1*› **have** *l1 x* $\leq_{R1}$ *l1 x* **by** *blast*
ultimately show *l1 x* $_{R1}\lessapprox$ *x* **by** *auto*
from *mono-in-dom-l1* ‹*in-dom* ($\leq_L$) *x*› **have** *in-dom* ($\leq_{L2}$) (*l1 x*) **by** *blast*
with *deflationary-L2* **show** *r2* (*l x*) $\leq_{L2}$ *l1 x* **by** *auto*
show η *x* $_{L1}\gtrapprox$ *r2* (*l x*)
proof (*rule t1.left-GaloisI*)
from *refl-L2* ‹*in-dom* ($\leq_{L2}$) (*l1 x*)› **have** *l1 x* $\leq_{L2}$ *l1 x* **by** *blast*
with *in-dom-rel-comp-le* ‹*r2* (*l x*) $\leq_{L2}$ *l1 x*› ‹*l1 x* $\leq_{R1}$ *l1 x*›
have *in-dom* ($\leq_{R1}$) (*r2* (*l x*)) **by** *blast*
with ‹(($\leq_{R1}$) $\Rightarrow$ ($\leq_{L1}$)) *r1*› **have** *in-dom* ($\leq_{L1}$) (η *x*)
by (*auto intro*: *in-dom-if-rel-if-dep-mono-wrt-rel simp*: *mono-wrt-rel-eq-dep-mono-wrt-rel*)
with *refl-L1* **show** η *x* $\leq_{L1}$ *r1* (*r2* (*l x*))
by (*auto intro*: *in-field-if-in-codom*)
from ‹*in-dom* ($\leq_{R1}$) (*r2* (*l x*))› *in-dom-R1-le-in-codom-R1*
show *in-codom* ($\leq_{R1}$) (*r2* (*l x*)) **by** *blast*
qed
qed
qed

lemma *deflationary-on-in-codom-unitI*:
assumes (($\leq_{L1}$) $\Rightarrow$ ($\leq_{R1}$)) *l1* (($\leq_{R1}$) $\Rightarrow$ ($\leq_{L1}$)) *r1*
and *refl-L1*: *reflexive-on* (*in-codom* ($\leq_{L1}$)) ($\leq_{L1}$)
and *in-dom-R1-le-in-codom-R1*: *in-dom* ($\leq_{R1}$) $\leq$ *in-codom* ($\leq_{R1}$)

```
  and deflationary-L2: deflationary-on (in-codom (≤L2)) (≤L2) η2
  and refl-L2: reflexive-on (in-codom (≤L2)) (≤L2)
  and mono-in-codom-l1: (in-codom (≤L) ⇒ in-codom (≤L2)) l1
  and in-dom-rel-comp-le: in-dom ((≤L2) ∘∘ (≤R1) ∘∘ (≤L2)) ≤ in-dom ((≤R1))
  shows deflationary-on (in-codom (≤L)) (≤L) η
proof (rule deflationary-onI)
  fix x assume in-codom (≤L) x
  show η x ≤L x
  proof (rule left-relI)
    from refl-L1 ‹in-codom (≤L) x› have x ≤L1 x by blast
    moreover with ‹((≤L1) ⇒ (≤R1)) l1› have l1 x ≤R1 l1 x by blast
    ultimately show l1 x R1≲≈ x by auto
     from mono-in-codom-l1 ‹in-codom (≤L) x› have in-codom (≤L2) (l1 x) by
blast
    with deflationary-L2 show r2 (l x) ≤L2 l1 x by auto
    show η x L1≲≈ r2 (l x)
    proof (rule t1.left-GaloisI)
      from refl-L2 ‹in-codom (≤L2) (l1 x)› have l1 x ≤L2 l1 x by blast
      with in-dom-rel-comp-le ‹r2 (l x) ≤L2 l1 x› ‹l1 x ≤R1 l1 x›
        have in-dom (≤R1) (r2 (l x)) by blast
      with in-dom-R1-le-in-codom-R1 show in-codom (≤R1) (r2 (l x)) by blast
      with ‹((≤R1) ⇒ (≤L1)) r1› have in-codom (≤L1) (η x)
      by (auto intro: in-codom-if-rel-if-dep-mono-wrt-rel simp: mono-wrt-rel-eq-dep-mono-wrt-rel)
      with refl-L1 show η x ≤L1 r1 (r2 (l x)) by auto
    qed
  qed
qed

corollary deflationary-on-in-field-unitI:
  assumes ((≤L1) ⇒ (≤R1)) l1 ((≤R1) ⇒ (≤L1)) r1
  and reflexive-on (in-field (≤L1)) (≤L1)
  and in-dom (≤R1) ≤ in-codom (≤R1)
  and deflationary-on (in-field (≤L2)) (≤L2) η2
  and reflexive-on (in-field (≤L2)) (≤L2)
  and (in-dom (≤L) ⇒ in-dom (≤L2)) l1
  and (in-codom (≤L) ⇒ in-codom (≤L2)) l1
  and in-dom ((≤L2) ∘∘ (≤R1) ∘∘ (≤L2)) ≤ in-dom ((≤R1))
  shows deflationary-on (in-field (≤L)) (≤L) η
proof −
  from assms have deflationary-on (in-dom (≤L)) (≤L) η
    by (intro deflationary-on-in-dom-unitI)
    (auto intro: deflationary-on-if-le-pred-if-deflationary-on
      reflexive-on-if-le-pred-if-reflexive-on in-field-if-in-dom)
  moreover from assms have deflationary-on (in-codom (≤L)) (≤L) η
    by (intro deflationary-on-in-codom-unitI)
    (auto intro: deflationary-on-if-le-pred-if-deflationary-on
      reflexive-on-if-le-pred-if-reflexive-on in-field-if-in-codom)
  ultimately show ?thesis by (auto iff: in-field-iff-in-dom-or-in-codom)
qed
```

corollary *rel-equivalence-on-in-field-unitI*:
assumes $((\leq_{L1}) \Rightarrow (\leq_{R1}))$ *l1* $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and $((\leq_{L1})\ _h\trianglelefteq\ (\leq_{R1}))$ *l1 r1*
and *inflationary-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$ η_1
and *inflationary-on* (*in-codom* $(\leq_{R1})$) $(\leq_{R1})$ ε_1
and *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
and *reflexive-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$
and *rel-equivalence-on* (*in-field* $(\leq_{L2})$) $(\leq_{L2})$ η_2
and *reflexive-on* (*in-field* $(\leq_{L2})$) $(\leq_{L2})$
and (*in-dom* $(\leq_L)$ $\Rightarrow$ *in-dom* $(\leq_{L2})$) *l1*
and (*in-codom* $(\leq_L)$ $\Rightarrow$ *in-codom* $(\leq_{L2})$) *l1*
and *in-dom* $((\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2})) \leq$ *in-dom* $((\leq_{R1}))$
and *in-codom* $((\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2})) \leq$ *in-codom* $((\leq_{R1}))$
shows *rel-equivalence-on* (*in-field* $(\leq_L)$) $(\leq_L)$ η
using *assms* **by** (*intro rel-equivalence-onI*
inflationary-on-in-field-unitI deflationary-on-in-field-unitI)
(*auto simp only*: *in-codom-eq-in-dom-if-reflexive-on-in-field*)

Counit

Corresponding lemmas for the counit can be obtained by flipping the interpretation of the locale, i.e. *interpretation flip* : *transport-comp R2 L2 r2 l2 R1 L1 r1 l1 rewrites flip.t2.unit* $\equiv \varepsilon_1$ *and flip.t2.counit* $\equiv \eta_1$ *and flip.t1.unit* $\equiv \varepsilon_2$ *and flip.t1.counit* $\equiv \eta_2$ *and flip.unit* $\equiv \varepsilon$ *and flip.counit* $\equiv \eta$ *unfolding transport-comp.transport-defs by* (*auto simp*: *order-functors.flip-counit-eq-unit*)

end

Order Equivalence

interpretation *flip* : *transport-comp R2 L2 r2 l2 R1 L1 r1 l1*
rewrites *flip.t2.unit* $\equiv \varepsilon_1$ **and** *flip.t2.counit* $\equiv \eta_1$
and *flip.t1.unit* $\equiv \varepsilon_2$ **and** *flip.t1.counit* $\equiv \eta_2$
and *flip.counit* $\equiv \eta$ **and** *flip.unit* $\equiv \varepsilon$
by (*simp-all only*: *order-functors.flip-counit-eq-unit*)

lemma *order-equivalenceI*:
assumes $((\leq_{L1}) \Rightarrow (\leq_{R1}))$ *l1* $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and $((\leq_{L1})\ _h\trianglelefteq\ (\leq_{R1}))$ *l1 r1*
and *inflationary-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$ η_1
and *rel-equiv-counit1*: *rel-equivalence-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$ ε_1
and *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
and *reflexive-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$
and $((\leq_{R2}) \Rightarrow (\leq_{L2}))$ *r2* $((\leq_{L2}) \Rightarrow (\leq_{R2}))$ *l2*
and $((\leq_{R2})\ _h\trianglelefteq\ (\leq_{L2}))$ *r2 l2*
and *rel-equiv-unit2*: *rel-equivalence-on* (*in-field* $(\leq_{L2})$) $(\leq_{L2})$ η_2
and *inflationary-on* (*in-field* $(\leq_{R2})$) $(\leq_{R2})$ ε_2
and *reflexive-on* (*in-field* $(\leq_{L2})$) $(\leq_{L2})$

and *reflexive-on* (*in-field* ($\leq_{R2}$)) ($\leq_{R2}$)
and *middle-compatible*: *middle-compatible-codom*
shows (($\leq_L$) $\equiv_o$ ($\leq_R$)) *l r*
proof (*rule order-equivalenceI*)
show (($\leq_L$) $\Rightarrow$ ($\leq_R$)) *l* **using** *rel-equiv-unit2* ‹(($\leq_{L1}$) $_h\trianglelefteq$ ($\leq_{R1}$)) *l1 r1*›
‹(($\leq_{L2}$) $\Rightarrow$ ($\leq_{R2}$)) *l2*› *middle-compatible*
by (*intro mono-wrt-rel-leftI*) *auto*
show (($\leq_R$) $\Rightarrow$ ($\leq_L$)) *r* **using** *rel-equiv-counit1* ‹(($\leq_{R2}$) $_h\trianglelefteq$ ($\leq_{L2}$)) *r2 l2*›
‹(($\leq_{R1}$) $\Rightarrow$ ($\leq_{L1}$)) *r1*› *middle-compatible*
by (*intro flip.mono-wrt-rel-leftI*)
(*auto intro*: *inflationary-on-if-le-pred-if-inflationary-on*
in-field-if-in-codom)
from *middle-compatible* **have** *in-dom-rel-comp-les*:
in-dom (($\leq_{R1}$) $\circ\circ$ ($\leq_{L2}$) $\circ\circ$ ($\leq_{R1}$)) $\leq$ *in-dom* ($\leq_{L2}$)
in-dom (($\leq_{L2}$) $\circ\circ$ ($\leq_{R1}$) $\circ\circ$ ($\leq_{L2}$)) $\leq$ *in-dom* (($\leq_{R1}$))
by (*auto intro*: *in-dom-right1-left2-right1-le-if-right1-left2-right1-le*
flip.in-dom-right1-left2-right1-le-if-right1-left2-right1-le)
moreover then have (*in-dom* ($\leq_L$) $\Rightarrow$ *in-dom* ($\leq_{L2}$)) *l1*
and (*in-codom* ($\leq_L$) $\Rightarrow$ *in-codom* ($\leq_{L2}$)) *l1*
using ‹(($\leq_{L1}$) $_h\trianglelefteq$ ($\leq_{R1}$)) *l1 r1*› *middle-compatible*
by (*auto intro*: *mono-in-dom-left-rel-left1-if-in-dom-rel-comp-le*
mono-in-codom-left-rel-left1-if-in-codom-rel-comp-le)
ultimately show *rel-equivalence-on* (*in-field* ($\leq_L$)) ($\leq_L$) η
using *assms* **by** (*intro rel-equivalence-on-in-field-unitI*)
(*auto intro*: *inflationary-on-if-le-pred-if-inflationary-on*
intro!: *in-field-if-in-codom*)
note *in-dom-rel-comp-les*
moreover then have (*in-dom* ($\leq_R$) $\Rightarrow$ *in-dom* ($\leq_{R1}$)) *r2*
and (*in-codom* ($\leq_R$) $\Rightarrow$ *in-codom* ($\leq_{R1}$)) *r2*
using ‹(($\leq_{R2}$) $_h\trianglelefteq$ ($\leq_{L2}$)) *r2 l2*› *middle-compatible*
by (*auto intro*!: *flip.mono-in-dom-left-rel-left1-if-in-dom-rel-comp-le*
flip.mono-in-codom-left-rel-left1-if-in-codom-rel-comp-le)
ultimately show *rel-equivalence-on* (*in-field* ($\leq_R$)) ($\leq_R$) ε
using *assms* **by** (*intro flip.rel-equivalence-on-in-field-unitI*)
(*auto intro*: *inflationary-on-if-le-pred-if-inflationary-on*
intro!: *in-field-if-in-codom*)
qed

corollary *order-equivalence-if-order-equivalenceI*:
assumes (($\leq_{L1}$) $\equiv_o$ ($\leq_{R1}$)) *l1 r1*
and *reflexive-on* (*in-field* ($\leq_{L1}$)) ($\leq_{L1}$)
and *transitive* ($\leq_{R1}$)
and (($\leq_{L2}$) $\equiv_o$ ($\leq_{R2}$)) *l2 r2*
and *transitive* ($\leq_{L2}$)
and *reflexive-on* (*in-field* ($\leq_{R2}$)) ($\leq_{R2}$)
and *middle-compatible-codom*
shows (($\leq_L$) $\equiv_o$ ($\leq_R$)) *l r*
using *assms* **by** (*intro order-equivalenceI*) (*auto*
elim!: *t1.order-equivalenceE t2.order-equivalenceE rel-equivalence-onE*

intro!: *reflexive-on-in-field-if-transitive-if-rel-equivalence-on*
t1.half-galois-prop-left-left-right-if-transitive-if-deflationary-on-if-mono-wrt-rel
flip.t1.half-galois-prop-left-left-right-if-transitive-if-deflationary-on-if-mono-wrt-rel
intro: *deflationary-on-if-le-pred-if-deflationary-on in-field-if-in-codom*)

corollary *order-equivalence-if-galois-equivalenceI*:
assumes $((\leq_{L1}) \equiv_G (\leq_{R1}))$ *l1 r1*
and *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
and *reflexive-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$
and $((\leq_{L2}) \equiv_G (\leq_{R2}))$ *l2 r2*
and *reflexive-on* (*in-field* $(\leq_{L2})$) $(\leq_{L2})$
and *reflexive-on* (*in-field* $(\leq_{R2})$) $(\leq_{R2})$
and *middle-compatible-codom*
shows $((\leq_L) \equiv_o (\leq_R))$ *l r*
using *assms* **by** (*intro order-equivalenceI*)
(*auto elim*!: *t1.galois-equivalenceE t2.galois-equivalenceE*
intro!: *t1.inflationary-on-unit-if-reflexive-on-if-galois-equivalence*
flip.t1.inflationary-on-unit-if-reflexive-on-if-galois-equivalence
t2.rel-equivalence-on-unit-if-reflexive-on-if-galois-equivalence
flip.t2.rel-equivalence-on-unit-if-reflexive-on-if-galois-equivalence)

end

end

theory *Transport-Compositions-Generic*
imports
Transport-Compositions-Generic-Galois-Equivalence
Transport-Compositions-Generic-Galois-Relator
Transport-Compositions-Generic-Order-Base
Transport-Compositions-Generic-Order-Equivalence
begin

Summary of Main Results

Closure of Order and Galois Concepts **context** *transport-comp* **begin**

interpretation *flip* : *transport-comp R2 L2 r2 l2 R1 L1 r1 l1* .

lemma *preorder-galois-connection-if-galois-equivalenceI*:
assumes $((\leq_{L1}) \equiv_G (\leq_{R1}))$ *l1 r1*
and *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
and *preorder-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$
and $((\leq_{L2}) \equiv_G (\leq_{R2}))$ *l2 r2*
and *preorder-on* (*in-field* $(\leq_{L2})$) $(\leq_{L2})$
and *reflexive-on* (*in-field* $(\leq_{R2})$) $(\leq_{R2})$
and *middle-compatible-codom*

shows $((\leq_L) \dashv_{pre} (\leq_R))$ *l r*
using *assms* **by** (*intro preorder-galois-connectionI*)
(*auto elim*!: *t1.galois-equivalenceE t2.galois-equivalenceE*
intro!: *galois-connection-left-right-if-galois-equivalenceI*
preorder-on-in-field-leftI flip.preorder-on-in-field-leftI
mono-in-codom-left-rel-left1-if-in-codom-rel-comp-le
flip.mono-in-codom-left-rel-left1-if-in-codom-rel-comp-le
in-codom-eq-in-dom-if-reflexive-on-in-field)

theorem *preorder-galois-connection-if-preorder-equivalenceI*:
assumes $((\leq_{L1}) \equiv_{pre} (\leq_{R1}))$ *l1 r1*
and $((\leq_{L2}) \equiv_{pre} (\leq_{R2}))$ *l2 r2*
and *middle-compatible-codom*
shows $((\leq_L) \dashv_{pre} (\leq_R))$ *l r*
using *assms* **by** (*intro preorder-galois-connection-if-galois-equivalenceI*)
auto

lemma *preorder-equivalence-if-galois-equivalenceI*:
assumes $((\leq_{L1}) \equiv_G (\leq_{R1}))$ *l1 r1*
and *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
and *preorder-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$
and $((\leq_{L2}) \equiv_G (\leq_{R2}))$ *l2 r2*
and *preorder-on* (*in-field* $(\leq_{L2})$) $(\leq_{L2})$
and *reflexive-on* (*in-field* $(\leq_{R2})$) $(\leq_{R2})$
and *middle-compatible-codom*
shows $((\leq_L) \equiv_{pre} (\leq_R))$ *l r*
proof −
from *assms* **have** $((\leq_L) \dashv_{pre} (\leq_R))$ *l r*
by (*intro preorder-galois-connection-if-galois-equivalenceI*) *auto*
with *assms* **show** *?thesis* **by** (*intro preorder-equivalence-if-galois-equivalenceI*)
(*auto intro*!: *galois-equivalence-if-galois-equivalenceI*
preorder-galois-connection-if-galois-equivalenceI)
qed

theorem *preorder-equivalenceI*:
assumes $((\leq_{L1}) \equiv_{pre} (\leq_{R1}))$ *l1 r1*
and $((\leq_{L2}) \equiv_{pre} (\leq_{R2}))$ *l2 r2*
and *middle-compatible-codom*
shows $((\leq_L) \equiv_{pre} (\leq_R))$ *l r*
using *assms* **by** (*intro preorder-equivalence-if-galois-equivalenceI*) *auto*

theorem *partial-equivalence-rel-equivalenceI*:
assumes $((\leq_{L1}) \equiv_{PER} (\leq_{R1}))$ *l1 r1*
and $((\leq_{L2}) \equiv_{PER} (\leq_{R2}))$ *l2 r2*
and *middle-compatible-codom*
shows $((\leq_L) \equiv_{PER} (\leq_R))$ *l r*
using *assms* **by** (*intro partial-equivalence-rel-equivalence-if-galois-equivalenceI*
galois-equivalence-if-galois-equivalenceI
partial-equivalence-rel-leftI flip.partial-equivalence-rel-leftI

in-codom-eq-in-dom-if-partial-equivalence-rel)
auto

Simplification of Galois relator **theorem** *left-Galois-eq-comp-left-GaloisI*:
assumes $((\leq_{L1}) \equiv_{pre} (\leq_{R1}))$ *l1 r1*
and $((\leq_{R2}) \dashv_{pre} (\leq_{L2}))$ *r2 l2*
and *middle-compatible-codom*
shows $({}_{L}\lessapprox) = (({}_{L1}\lessapprox) \circ\circ ({}_{L2}\lessapprox))$
using *assms* **by** (*intro left-Galois-eq-comp-left-Galois-if-galois-connection-if-galois-equivalenceI*)
auto

For theorems with weaker assumptions, see $[\![((\leq_{R1}) \Rightarrow (\leq_{L1}))\ r1;\ t1.galois\text{-}prop\ l1\ r1;\ flip.t2.half\text{-}galois\text{-}prop\text{-}right;\ preorder\text{-}on\ (in\text{-}field\ (\leq_{R1}))\ (\leq_{R1});\ ((\leq_{L2}) \Rightarrow (\leq_{R2}))\ l2;\ flip.t1.half\text{-}galois\text{-}prop\text{-}left;\ reflexive\text{-}on\ (in\text{-}dom\ (\leq_{L2}))\ (\leq_{L2});\ (\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1}) \leq (\leq_{R1}) \circ\circ (\leq_{L2});\ in\text{-}codom\ ((\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2})) \leq in\text{-}codom\ (\leq_{R1})]\!] \Longrightarrow flip.right\text{-}Galois = flip.t2.right\text{-}Galois \circ\circ flip.t1.right\text{-}Galois$

$[\![t1.galois\text{-}equivalence;\ preorder\text{-}on\ (in\text{-}field\ (\leq_{R1}))\ (\leq_{R1});\ flip.t1.galois\text{-}connection;\ reflexive\text{-}on\ (in\text{-}field\ (\leq_{L2}))\ (\leq_{L2});\ in\text{-}codom\ ((\leq_{R1}) \circ\circ (\leq_{L2}) \circ\circ (\leq_{R1})) \leq in\text{-}codom\ (\leq_{L2});\ (\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2}) \leq (\leq_{R1}) \circ\circ (\leq_{L2});\ in\text{-}codom\ ((\leq_{L2}) \circ\circ (\leq_{R1}) \circ\circ (\leq_{L2})) \leq in\text{-}codom\ (\leq_{R1})]\!] \Longrightarrow flip.right\text{-}Galois = flip.t2.right\text{-}Galois \circ\circ flip.t1.right\text{-}Galois$.

Simplification of Compatibility Assumption See *Transport.Transport-Compositions-Gener*

end

end

2.12 Transport For Compositions

theory *Transport-Compositions*
imports
Transport-Compositions-Agree
Transport-Compositions-Generic
begin

Summary We provide two ways to compose transportable components: a slightly intricate, generic one in *transport-comp* and another straightforward but less general one in *transport-comp-agree*. As a special case from the latter, we obtain *transport-comp-same*, which includes the cases most prominently covered in the literature.

Refer to [2] for more details.

end

2.13 Reflexive Relator

theory *Reflexive-Relator*
imports
Galois-Equivalences
Galois-Relator
begin

definition *Refl-Rel* $R\ x\ y \equiv R\ x\ x \wedge R\ y\ y \wedge R\ x\ y$

open-bundle *Refl-Rel-syntax*
begin
notation *Refl-Rel* ($\langle$(-$^{\oplus}$)$\rangle$ [*1000*])
end

lemma *Refl-RelI* [*intro*]:
assumes $R\ x\ x$
and $R\ y\ y$
and $R\ x\ y$
shows $R^{\oplus}\ x\ y$
using *assms* **unfolding** *Refl-Rel-def* **by** *blast*

lemma *Refl-Rel-selfI* [*intro*]:
assumes $R\ x\ x$
shows $R^{\oplus}\ x\ x$
using *assms* **by** *blast*

lemma *Refl-RelE* [*elim*]:
assumes $R^{\oplus}\ x\ y$
obtains $R\ x\ x\ R\ y\ y\ R\ x\ y$
using *assms* **unfolding** *Refl-Rel-def* **by** *blast*

lemma *Refl-Rel-reflexive-on-in-field* [*iff*]:
reflexive-on (*in-field* $R^{\oplus}$) $R^{\oplus}$
by (*rule reflexive-onI*) *auto*

lemma *Refl-Rel-le-self* [*iff*]: $R^{\oplus} \leq R$ **by** *blast*

lemma *Refl-Rel-eq-self-if-reflexive-on* [*simp*]:
assumes *reflexive-on* (*in-field* R) R
shows $R^{\oplus} = R$
using *assms* **by** *blast*

lemma *reflexive-on-in-field-if-Refl-Rel-eq-self*:
assumes $R^{\oplus} = R$
shows *reflexive-on* (*in-field* R) R
by (*fact Refl-Rel-reflexive-on-in-field*[*of* R, *simplified assms*])

corollary *Refl-Rel-eq-self-iff-reflexive-on*:

$R^{\oplus} = R \longleftrightarrow$ *reflexive-on* (*in-field R*) *R*
using *Refl-Rel-eq-self-if-reflexive-on reflexive-on-in-field-if-Refl-Rel-eq-self*
by *blast*

lemma *Refl-Rel-Refl-Rel-eq* [*simp*]: $(R^{\oplus})^{\oplus} = R^{\oplus}$
by (*intro ext*) *auto*

lemma *rel-inv-Refl-Rel-eq* [*simp*]: $(R^{\oplus})^{-1} = (R^{-1})^{\oplus}$
by (*intro ext iffI Refl-RelI rel-invI*) *auto*

lemma *Refl-Rel-transitive-onI* [*intro*]:
assumes *transitive-on* (*P* :: *'a* ⇒ *bool*) (*R* :: *'a* ⇒ -)
shows *transitive-on P* $R^{\oplus}$
using *assms* **by** (*intro transitive-onI*) (*blast dest*: *transitive-onD*)

corollary *Refl-Rel-transitiveI* [*intro*]:
assumes *transitive R*
shows *transitive* $R^{\oplus}$
using *assms* **by** *blast*

corollary *Refl-Rel-preorder-onI*:
assumes *transitive-on P R*
and $P \leq$ *in-field* $R^{\oplus}$
shows *preorder-on P* $R^{\oplus}$
using *assms* **by** (*intro preorder-onI*
reflexive-on-if-le-pred-if-reflexive-on[**where** *?P*=*in-field* $R^{\oplus}$ **and** *?P'*=*P*])
auto

corollary *Refl-Rel-preorder-on-in-fieldI* [*intro*]:
assumes *transitive R*
shows *preorder-on* (*in-field* $R^{\oplus}$) $R^{\oplus}$
using *assms* **by** (*intro Refl-Rel-preorder-onI*) *auto*

lemma *Refl-Rel-symmetric-onI* [*intro*]:
assumes *symmetric-on* (*P* :: *'a* ⇒ *bool*) (*R* :: *'a* ⇒ -)
shows *symmetric-on P* $R^{\oplus}$
using *assms* **by** (*intro symmetric-onI*) (*auto dest*: *symmetric-onD*)

lemma *Refl-Rel-symmetricI* [*intro*]:
assumes *symmetric R*
shows *symmetric* $R^{\oplus}$
by (*urule symmetric-on-if-le-pred-if-symmetric-on*)
(*use assms* **in** ‹*urule Refl-Rel-symmetric-onI*›, *simp*)

lemma *Refl-Rel-partial-equivalence-rel-onI* [*intro*]:
assumes *partial-equivalence-rel-on* (*P* :: *'a* ⇒ *bool*) (*R* :: *'a* ⇒ -)
shows *partial-equivalence-rel-on P* $R^{\oplus}$
using *assms* **by** (*intro partial-equivalence-rel-onI Refl-Rel-transitive-onI*
Refl-Rel-symmetric-onI) *auto*

lemma *Refl-Rel-partial-equivalence-relI* [*intro*]:
assumes *partial-equivalence-rel R*
shows *partial-equivalence-rel* $R^{\oplus}$
using *assms*
by (*intro partial-equivalence-relI Refl-Rel-transitiveI Refl-Rel-symmetricI*) *auto*

lemma *Refl-Rel-app-leftI*:
assumes $R\ (f\ x)\ y$
and *in-field* $S^{\oplus}\ x$
and *in-field* $R^{\oplus}\ y$
and $(S \Rightarrow R)\ f$
shows $R^{\oplus}\ (f\ x)\ y$
proof (*rule Refl-RelI*)
from ‹*in-field* $R^{\oplus}\ y$› **show** $R\ y\ y$ **by** *blast*
from ‹*in-field* $S^{\oplus}\ x$› **have** $S\ x\ x$ **by** *blast*
with ‹$(S \Rightarrow R)\ f$› **show** $R\ (f\ x)\ (f\ x)$ **by** *blast*
qed *fact*

corollary *Refl-Rel-app-rightI*:
assumes $R\ x\ (f\ y)$
and *in-field* $S^{\oplus}\ y$
and *in-field* $R^{\oplus}\ x$
and $(S \Rightarrow R)\ f$
shows $R^{\oplus}\ x\ (f\ y)$
proof −
from *assms* **have** $(R^{-1})^{\oplus}\ (f\ y)\ x$ **by** (*intro Refl-Rel-app-leftI*[**where** *?S*=S^{-1}])
(*auto 5 0 simp flip*: *rel-inv-Refl-Rel-eq*)
then show *?thesis* **by** *blast*
qed

lemma *mono-wrt-rel-Refl-Rel-Refl-Rel-if-mono-wrt-rel* [*intro*]:
assumes $(R \Rightarrow S)\ f$
shows $(R^{\oplus} \Rightarrow S^{\oplus})\ f$
using *assms* **by** (*intro mono-wrt-relI*) *blast*

context *galois*
begin

interpretation *gR* : *galois* $(\leq_L)^{\oplus}\ (\leq_R)^{\oplus}\ l\ r$ **.**

lemma *Galois-Refl-RelI*:
assumes $((\leq_R) \Rightarrow (\leq_L))\ r$
and *in-field* $(\leq_L)^{\oplus}\ x$
and *in-field* $(\leq_R)^{\oplus}\ y$
and *in-codom* $(\leq_R)\ y \Longrightarrow x\ {}_L\lessapprox\ y$
shows *gR.Galois x y*
using *assms* **by** (*intro gR.left-GaloisI in-codomI Refl-Rel-app-rightI*[**where** *?f*=*r*])
auto

lemma *half-galois-prop-left-Refl-Rel-left-rightI*:
assumes $((\leq_L) \Rightarrow (\leq_R))\ l$
and $((\leq_L)\ {}_h\trianglelefteq\ (\leq_R))\ l\ r$
shows $((\leq_L)^{\oplus}\ {}_h\trianglelefteq\ (\leq_R)^{\oplus})\ l\ r$
using *assms* **by** (*intro gR.half-galois-prop-leftI Refl-RelI*)
(*auto elim*!: *in-codomE gR.left-GaloisE Refl-RelE*)

interpretation *flip-inv* : *galois* $(\geq_R)\ (\geq_L)\ r\ l$
rewrites $((\geq_R) \Rightarrow (\geq_L)) \equiv ((\leq_R) \Rightarrow (\leq_L))$
and $\bigwedge R.\ (R^{-1})^{\oplus} \equiv (R^{\oplus})^{-1}$
and $\bigwedge R\ S\ f\ g.\ (R^{-1}\ {}_h\trianglelefteq\ S^{-1})\ f\ g \equiv (S\ \trianglelefteq_h\ R)\ g\ f$
by (*simp-all add*: *galois-prop.half-galois-prop-left-rel-inv-iff-half-galois-prop-right*
mono-wrt-rel-eq-dep-mono-wrt-rel)

lemma *half-galois-prop-right-Refl-Rel-right-leftI*:
assumes $((\leq_R) \Rightarrow (\leq_L))\ r$
and $((\leq_L) \trianglelefteq_h (\leq_R))\ l\ r$
shows $((\leq_L)^{\oplus} \trianglelefteq_h (\leq_R)^{\oplus})\ l\ r$
using *assms* **by** (*fact flip-inv.half-galois-prop-left-Refl-Rel-left-rightI*)

corollary *galois-prop-Refl-Rel-left-rightI*:
assumes $((\leq_L) \dashv (\leq_R))\ l\ r$
shows $((\leq_L)^{\oplus} \trianglelefteq (\leq_R)^{\oplus})\ l\ r$
using *assms*
by (*intro gR.galois-propI half-galois-prop-left-Refl-Rel-left-rightI*
half-galois-prop-right-Refl-Rel-right-leftI) *auto*

lemma *galois-connection-Refl-Rel-left-rightI*:
assumes $((\leq_L) \dashv (\leq_R))\ l\ r$
shows $((\leq_L)^{\oplus} \dashv (\leq_R)^{\oplus})\ l\ r$
using *assms*
by (*intro gR.galois-connectionI galois-prop-Refl-Rel-left-rightI*) *auto*

lemma *galois-equivalence-Refl-RelI*:
assumes $((\leq_L) \equiv_G (\leq_R))\ l\ r$
shows $((\leq_L)^{\oplus} \equiv_G (\leq_R)^{\oplus})\ l\ r$
proof −
interpret *flip* : *galois R L r l* **.**
show *?thesis* **using** *assms* **by** (*intro gR.galois-equivalenceI*
galois-connection-Refl-Rel-left-rightI flip.galois-prop-Refl-Rel-left-rightI)
auto
qed

end

context *order-functors*
begin

lemma *inflationary-on-in-field-Refl-Rel-left*:
assumes $((\leq_L) \Rightarrow (\leq_R))\ l$
and $((\leq_R) \Rightarrow (\leq_L))\ r$
and *inflationary-on* (*in-dom* $(\leq_L)$) $(\leq_L)\ \eta$
shows *inflationary-on* (*in-field* $(\leq_L)^{\oplus}$) $(\leq_L)^{\oplus}\ \eta$
using *assms*
by (*intro inflationary-onI Refl-RelI*) (*auto 0 3 elim*!: *in-fieldE Refl-RelE*)

lemma *inflationary-on-in-field-Refl-Rel-left'*:
assumes $((\leq_L) \Rightarrow (\leq_R))\ l$
and $((\leq_R) \Rightarrow (\leq_L))\ r$
and *inflationary-on* (*in-codom* $(\leq_L)$) $(\leq_L)\ \eta$
shows *inflationary-on* (*in-field* $(\leq_L)^{\oplus}$) $(\leq_L)^{\oplus}\ \eta$
using *assms*
by (*intro inflationary-onI Refl-RelI*) (*auto 0 3 elim*!: *in-fieldE Refl-RelE*)

interpretation *inv* : *galois* $(\geq_L)$ $(\geq_R)$ *l r*
rewrites $((\geq_L) \Rightarrow (\geq_R)) \equiv ((\leq_L) \Rightarrow (\leq_R))$
and $((\geq_R) \Rightarrow (\geq_L)) \equiv ((\leq_R) \Rightarrow (\leq_L))$
and $\bigwedge R.\ (R^{-1})^{\oplus} \equiv (R^{\oplus})^{-1}$
and $\bigwedge R.$ *in-dom* $R^{-1} \equiv$ *in-codom* R
and $\bigwedge R.$ *in-codom* $R^{-1} \equiv$ *in-dom* R
and $\bigwedge R.$ *in-field* $R^{-1} \equiv$ *in-field* R
and $\bigwedge(P :: 'c \Rightarrow bool)\ (R :: 'd \Rightarrow 'c \Rightarrow bool).$
(*inflationary-on* $P\ R^{-1} :: ('c \Rightarrow 'd) \Rightarrow bool) \equiv$ *deflationary-on* $P\ R$
by (*simp-all add*: *mono-wrt-rel-eq-dep-mono-wrt-rel*)

lemma *deflationary-on-in-field-Refl-Rel-leftI*:
assumes $((\leq_L) \Rightarrow (\leq_R))\ l$
and $((\leq_R) \Rightarrow (\leq_L))\ r$
and *deflationary-on* (*in-dom* $(\leq_L)$) $(\leq_L)\ \eta$
shows *deflationary-on* (*in-field* $(\leq_L)^{\oplus}$) $(\leq_L)^{\oplus}\ \eta$
using *assms* **by** (*fact inv.inflationary-on-in-field-Refl-Rel-left'*)

lemma *deflationary-on-in-field-Refl-RelI-left'*:
assumes $((\leq_L) \Rightarrow (\leq_R))\ l$
and $((\leq_R) \Rightarrow (\leq_L))\ r$
and *deflationary-on* (*in-codom* $(\leq_L)$) $(\leq_L)\ \eta$
shows *deflationary-on* (*in-field* $(\leq_L)^{\oplus}$) $(\leq_L)^{\oplus}\ \eta$
using *assms* **by** (*fact inv.inflationary-on-in-field-Refl-Rel-left*)

lemma *rel-equivalence-on-in-field-Refl-Rel-leftI*:
assumes $((\leq_L) \Rightarrow (\leq_R))\ l$
and $((\leq_R) \Rightarrow (\leq_L))\ r$
and *rel-equivalence-on* (*in-dom* $(\leq_L)$) $(\leq_L)\ \eta$
shows *rel-equivalence-on* (*in-field* $(\leq_L)^{\oplus}$) $(\leq_L)^{\oplus}\ \eta$
using *assms* **by** (*intro rel-equivalence-onI*
inflationary-on-in-field-Refl-Rel-left
deflationary-on-in-field-Refl-Rel-leftI)

auto

lemma *rel-equivalence-on-in-field-Refl-Rel-leftI'*:
assumes $((\leq_L) \Rightarrow (\leq_R))$ *l*
and $((\leq_R) \Rightarrow (\leq_L))$ *r*
and *rel-equivalence-on* (*in-codom* $(\leq_L)$) $(\leq_L)$ η
shows *rel-equivalence-on* (*in-field* $(\leq_L)^{\oplus}$) $(\leq_L)^{\oplus}$ η
using *assms* **by** (*intro rel-equivalence-onI*
inflationary-on-in-field-Refl-Rel-left'
deflationary-on-in-field-Refl-RelI-left')
auto

interpretation *oR* : *order-functors* $(\leq_L)^{\oplus}$ $(\leq_R)^{\oplus}$ *l r* **.**

lemma *order-equivalence-Refl-RelI*:
assumes $((\leq_L) \equiv_o (\leq_R))$ *l r*
shows $((\leq_L)^{\oplus} \equiv_o (\leq_R)^{\oplus})$ *l r*
proof −
interpret *flip* : *galois R L r l*
rewrites *flip.unit* $\equiv \varepsilon$
by (*simp only*: *flip-unit-eq-counit*)
show *?thesis* **using** *assms* **by** (*intro oR.order-equivalenceI*
mono-wrt-rel-Refl-Rel-Refl-Rel-if-mono-wrt-rel
rel-equivalence-on-in-field-Refl-Rel-leftI
flip.rel-equivalence-on-in-field-Refl-Rel-leftI)
(*auto intro*: *rel-equivalence-on-if-le-pred-if-rel-equivalence-on*
in-field-if-in-dom)
qed

end

end

2.14 Monotone Function Relator

theory *Monotone-Function-Relator*
imports
Reflexive-Relator
begin

abbreviation *Mono-Dep-Fun-Rel* $(R :: 'a \Rightarrow 'a \Rightarrow bool)$ $(S :: 'a \Rightarrow 'a \Rightarrow 'b \Rightarrow 'b \Rightarrow bool) \equiv$
$((x\ y :: R) \Rrightarrow S\ x\ y)^{\oplus}$
abbreviation *Mono-Fun-Rel R S* ≡ *Mono-Dep-Fun-Rel R* (λ- -. *S*)

open-bundle *Mono-Dep-Fun-Rel-syntax*
begin
notation *Mono-Fun-Rel* (**infixr** ‹⇛⊕› *40*)

syntax
-Mono-Dep-Fun-Rel-rel :: *idt* ⇒ *idt* ⇒ (*'a* ⇒ *'b* ⇒ *bool*) ⇒ (*'c* ⇒ *'d* ⇒ *bool*) ⇒ (*'a* ⇒ *'c*) ⇒ (*'b* ⇒ *'d*) ⇒ *bool* (‹*'*(*-/ -/ ::/ -*)*'*) ⇛⊕ (*-*)› [*41*, *41*, *41*, *40*] *40*)
-Mono-Dep-Fun-Rel-rel-if :: *idt* ⇒ *idt* ⇒ (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *bool* ⇒ (*'c* ⇒ *'d* ⇒ *bool*) ⇒ (*'a* ⇒ *'c*) ⇒ (*'b* ⇒ *'d*) ⇒ *bool* (‹*'*(*-/ -/ ::/ -/ |/ -*)*'*) ⇛⊕ (*-*)› [*41*, *41*, *41*, *41*, *40*] *40*)
syntax-consts
-Mono-Dep-Fun-Rel-rel -Mono-Dep-Fun-Rel-rel-if ⇌ *Mono-Dep-Fun-Rel*
translations
(*x y* :: *R*) ⇛⊕ *S* ⇌ *CONST Mono-Dep-Fun-Rel R* (λ*x y*. *S*)
(*x y* :: *R* | *B*) ⇛⊕ *S* ⇌ *CONST Mono-Dep-Fun-Rel R* (λ*x y*. *CONST rel-if B S*)
end

The monotone function relator is the function relator restricted to monotone functions:

lemma *Mono-Dep-Fun-Rel-eq-Dep-Fun-Rel-restrict-dep-monos*:
fixes *R S* **defines** *dep-mono* ≡ ((*x y* :: *R*) ⇒ *S x y*)
shows ((*x y* :: *R*) ⇛⊕ *S x y*) = ((*x y* :: *R*) ⇛ *S x y*)↾*dep-mono*⌉*dep-mono*
unfolding *dep-mono-def* **by** *force*

locale *Dep-Fun-Rel-orders* =
fixes *L* :: *'a* ⇒ *'b* ⇒ *bool*
and *R* :: *'a* ⇒ *'b* ⇒ *'c* ⇒ *'d* ⇒ *bool*
begin

sublocale *o* : *orders L R a b* **for** *a b* **.**

notation *L* (**infix** ‹≤$_L$› *50*)
notation *o.ge-left* (**infix** ‹≥$_L$› *50*)

notation *R* (‹(≤$_R$ (*-*) (*-*))› *50*)
abbreviation *right-infix c a b d* ≡ (≤$_R$ *a b*) *c d*
notation *right-infix* (‹(*-*) ≤$_R$ (*-*) (*-*) (*-*)› [*51*,*51*,*51*,*51*] *50*)

notation *o.ge-right* (‹(≥$_R$ (*-*) (*-*))› *50*)

abbreviation (*input*) *ge-right-infix d a b c* ≡ (≥$_R$ *a b*) *d c*
notation *ge-right-infix* (‹(*-*) ≥$_R$ (*-*) (*-*) (*-*)› [*51*,*51*,*51*,*51*] *50*)

abbreviation (*input*) *DFR* ≡ ((*a b* :: *L*) ⇛ *R a b*)

end

locale *hom-Dep-Fun-Rel-orders* = *Dep-Fun-Rel-orders L R*
for *L* :: *'a* ⇒ *'a* ⇒ *bool*
and *R* :: *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* ⇒ *bool*
begin

sublocale *ho* : *hom-orders L R a b* **for** *a b* **.**

lemma *Mono-Dep-Fun-Refl-Rel-right-eq-Mono-Dep-Fun-if-le-if-reflexive-onI*:
assumes *refl-L*: *reflexive-on* (*in-field* ($\leq_L$)) ($\leq_L$)
and $\bigwedge$*x1 x2*. *x1* $\leq_L$ *x2* $\Longrightarrow$ ($\leq_R$ *x2 x2*) $\leq$ ($\leq_R$ *x1 x2*)
and $\bigwedge$*x1 x2*. *x1* $\leq_L$ *x2* $\Longrightarrow$ ($\leq_R$ *x1 x1*) $\leq$ ($\leq_R$ *x1 x2*)
shows (($x\ y$ :: ($\leq_L$)) $\Rightarrow\!\!\oplus$ ($\leq_R$ *x y*)$^{\oplus}$) = (($x\ y$ :: ($\leq_L$)) $\Rightarrow\!\!\oplus$ ($\leq_R$ *x y*))
proof −
have *f x1* $\leq_R$ *x1 x2 g x1 f x2* $\leq_R$ *x1 x2 g x2*
if (($x\ y$ :: ($\leq_L$)) $\Rrightarrow$ ($\leq_R$ *x y*)) *f g x1* $\leq_L$ *x1 x1* $\leq_L$ *x2* **for** *f g x1 x2*
using *that assms* **by** *blast+*
with *refl-L* **show** *?thesis*
by (*intro ext iffI Refl-RelI Dep-Fun-Rel-relI*) (*auto elim*!: *Refl-RelE*)
qed

lemma *Mono-Dep-Fun-Refl-Rel-right-eq-Mono-Dep-Fun-if-mono-if-reflexive-onI*:
assumes *reflexive-on* (*in-field* ($\leq_L$)) ($\leq_L$)
and (($x1\ x2$ :: ($\geq_L$)) $\Rightarrow$ ($x3\ x4$ :: ($\leq_L$) | *x1* $\leq_L$ *x3*) $\Rrightarrow$ ($\leq$)) *R*
shows (($x\ y$ :: ($\leq_L$)) $\Rightarrow\!\!\oplus$ ($\leq_R$ *x y*)$^{\oplus}$) = (($x\ y$ :: ($\leq_L$)) $\Rightarrow\!\!\oplus$ ($\leq_R$ *x y*))
using *assms*
by (*intro Mono-Dep-Fun-Refl-Rel-right-eq-Mono-Dep-Fun-if-le-if-reflexive-onI*)
blast+

end

context *hom-orders*
begin

sublocale *fro* : *hom-Dep-Fun-Rel-orders L* λ*- -. R* **.**

corollary *Mono-Fun-Rel-Refl-Rel-right-eq-Mono-Fun-RelI*:
assumes *reflexive-on* (*in-field* ($\leq_L$)) ($\leq_L$)
shows (($\leq_L$) $\Rightarrow\!\!\oplus$ ($\leq_R$)$^{\oplus}$) = (($\leq_L$) $\Rightarrow\!\!\oplus$ ($\leq_R$))
using *assms* **by** (*intro fro.Mono-Dep-Fun-Refl-Rel-right-eq-Mono-Dep-Fun-if-le-if-reflexive-onI*)
simp-all

end

end

2.15 Transport For Functions

2.15.1 Basic Setup

theory *Transport-Functions-Base*
imports
Monotone-Function-Relator
Transport-Base
begin

Summary Basic setup for closure proofs. We introduce locales for the syntax, the dependent relator, the non-dependent relator, the monotone dependent relator, and the monotone non-dependent relator.

definition *flip2 f x1 x2 x3 x4* ≡ *f x2 x1 x4 x3*

lemma *flip2-eq*: *flip2 f x1 x2 x3 x4* = *f x2 x1 x4 x3*
unfolding *flip2-def* **by** *simp*

lemma *flip2-eq-rel-inv* [*simp*]: *flip2 R x y* = $(R\ y\ x)^{-1}$
by (*intro ext*) (*simp only*: *flip2-eq rel-inv-iff-rel*)

lemma *flip2-flip2-eq-self* [*simp*]: *flip2* (*flip2 f*) = *f*
by (*intro ext*) (*simp add*: *flip2-eq*)

lemma *flip2-eq-flip2-iff-eq* [*iff*]: *flip2 f* = *flip2 g* ⟷ *f* = *g*
unfolding *flip2-def* **by** (*intro iffI ext*) (*auto dest*: *fun-cong*)

Dependent Function Relator **locale** *transport-Dep-Fun-Rel-syntax* =
t1 : *transport L1 R1 l1 r1* +
dfro1 : *hom-Dep-Fun-Rel-orders L1 L2* +
dfro2 : *hom-Dep-Fun-Rel-orders R1 R2*
for *L1* :: *'a1* ⇒ *'a1* ⇒ *bool*
and *R1* :: *'a2* ⇒ *'a2* ⇒ *bool*
and *l1* :: *'a1* ⇒ *'a2*
and *r1* :: *'a2* ⇒ *'a1*
and *L2* :: *'a1* ⇒ *'a1* ⇒ *'b1* ⇒ *'b1* ⇒ *bool*
and *R2* :: *'a2* ⇒ *'a2* ⇒ *'b2* ⇒ *'b2* ⇒ *bool*
and *l2* :: *'a2* ⇒ *'a1* ⇒ *'b1* ⇒ *'b2*
and *r2* :: *'a1* ⇒ *'a2* ⇒ *'b2* ⇒ *'b1*
begin

notation *L1* (**infix** ‹$\leq_{L1}$› *50*)
notation *R1* (**infix** ‹$\leq_{R1}$› *50*)

notation *t1.ge-left* (**infix** ‹$\geq_{L1}$› *50*)
notation *t1.ge-right* (**infix** ‹$\geq_{R1}$› *50*)

notation *t1.left-Galois* (**infix** ‹$_{L1}\lessapprox$› *50*)
notation *t1.ge-Galois-left* (**infix** ‹$\gtrapprox_{L1}$› *50*)
notation *t1.right-Galois* (**infix** ‹$_{R1}\lessapprox$› *50*)
notation *t1.ge-Galois-right* (**infix** ‹$\gtrapprox_{R1}$› *50*)
notation *t1.right-ge-Galois* (**infix** ‹$_{R1}\gtrapprox$› *50*)
notation *t1.Galois-right* (**infix** ‹$\lessapprox_{R1}$› *50*)
notation *t1.left-ge-Galois* (**infix** ‹$_{L1}\gtrapprox$› *50*)
notation *t1.Galois-left* (**infix** ‹$\lessapprox_{L1}$› *50*)

notation *t1.unit* (‹η_1›)
notation *t1.counit* (‹ε_1›)

notation *L2* (‹(≤$_{L2}$ (-) (-))› *50*)
notation *R2* (‹(≤$_{R2}$ (-) (-))› *50*)

notation *dfro1.right-infix* (‹(-) ≤$_{L2}$ (-) (-) (-)› [*51*,*51*,*51*,*51*] *50*)
notation *dfro2.right-infix* (‹(-) ≤$_{R2}$ (-) (-) (-)› [*51*,*51*,*51*,*51*] *50*)

notation *dfro1.o.ge-right* (‹(≥$_{L2}$ (-) (-))› *50*)
notation *dfro2.o.ge-right* (‹(≥$_{R2}$ (-) (-))› *50*)

notation *dfro1.ge-right-infix* (‹(-) ≥$_{L2}$ (-) (-) (-)› [*51*,*51*,*51*,*51*] *50*)
notation *dfro2.ge-right-infix* (‹(-) ≥$_{R2}$ (-) (-) (-)› [*51*,*51*,*51*,*51*] *50*)

notation *l2* (‹*l2*$_{(-)\ (-)}$›)
notation *r2* (‹*r2*$_{(-)\ (-)}$›)

sublocale *t2* : *transport* (≤$_{L2\ x\ (r1\ x')}$) (≤$_{R2\ (l1\ x)\ x'}$) *l2*$_{x'\ x}$ *r2*$_{x\ x'}$ **for** *x x'* **.**

notation *t2.left-Galois* (‹($_{L2\ (-)\ (-)}\lessapprox$)› *50*)
notation *t2.right-Galois* (‹($_{R2\ (-)\ (-)}\lessapprox$)› *50*)

abbreviation *left2-Galois-infix y x x' y'* ≡ ($_{L2\ x\ x'}\lessapprox$) *y y'*
notation *left2-Galois-infix* (‹(-) $_{L2\ (-)\ (-)}\lessapprox$ (-)› [*51*,*51*,*51*,*51*] *50*)
abbreviation *right2-Galois-infix y' x x' y* ≡ ($_{R2\ x\ x'}\lessapprox$) *y' y*
notation *right2-Galois-infix* (‹(-) $_{R2\ (-)\ (-)}\lessapprox$ (-)› [*51*,*51*,*51*,*51*] *50*)

notation *t2.ge-Galois-left* (‹(≳$_{L2\ (-)\ (-)}$)› *50*)
notation *t2.ge-Galois-right* (‹(≳$_{R2\ (-)\ (-)}$)› *50*)

abbreviation (*input*) *ge-Galois-left-left2-infix y' x x' y* ≡ (≳$_{L2\ x\ x'}$) *y' y*
notation *ge-Galois-left-left2-infix* (‹(-) ≳$_{L2\ (-)\ (-)}$ (-)› [*51*,*51*,*51*,*51*] *50*)
abbreviation (*input*) *ge-Galois-left-right2-infix y x x' y'* ≡ (≳$_{R2\ x\ x'}$) *y y'*
notation *ge-Galois-left-right2-infix* (‹(-) ≳$_{R2\ (-)\ (-)}$ (-)› [*51*,*51*,*51*,*51*] *50*)

notation *t2.right-ge-Galois* (‹($_{R2\ (-)\ (-)}\gtrapprox$)› *50*)
notation *t2.left-ge-Galois* (‹($_{L2\ (-)\ (-)}\gtrapprox$)› *50*)

abbreviation *left2-ge-Galois-left-infix y x x' y'* ≡ ($_{L2\ x\ x'}\gtrapprox$) *y y'*
notation *left2-ge-Galois-left-infix* (‹(-) $_{L2\ (-)\ (-)}\gtrapprox$ (-)› [*51*,*51*,*51*,*51*] *50*)
abbreviation *right2-ge-Galois-left-infix y' x x' y* ≡ ($_{R2\ x\ x'}\gtrapprox$) *y' y*
notation *right2-ge-Galois-left-infix* (‹(-) $_{R2\ (-)\ (-)}\gtrapprox$ (-)› [*51*,*51*,*51*,*51*] *50*)

notation *t2.Galois-right* (‹(≲$_{R2\ (-)\ (-)}$)› *50*)
notation *t2.Galois-left* (‹(≲$_{L2\ (-)\ (-)}$)› *50*)

abbreviation (*input*) *Galois-left2-infix y' x x' y* ≡ (≲$_{L2\ x\ x'}$) *y' y*
notation *Galois-left2-infix* (‹(-) ≲$_{L2\ (-)\ (-)}$ (-)› [*51*,*51*,*51*,*51*] *50*)
abbreviation (*input*) *Galois-right2-infix y x x' y'* ≡ (≲$_{R2\ x\ x'}$) *y y'*

notation *Galois-right2-infix* ($\langle$(-) $\lessapprox_{R2}$ (-) (-) (-)$\rangle$ [*51*,*51*,*51*,*51*] *50*)

abbreviation *t2-unit x x'* ≡ *t2.unit x' x*
notation *t2-unit* ($\langle\eta_{2}$ (-) (-)$\rangle$)
abbreviation *t2-counit x x'* ≡ *t2.counit x' x*
notation *t2-counit* ($\langle\varepsilon_{2}$ (-) (-)$\rangle$)

end

locale *transport-Dep-Fun-Rel* =
  transport-Dep-Fun-Rel-syntax L1 R1 l1 r1 L2 R2 l2 r2
  for *L1* :: *'a1* ⇒ *'a1* ⇒ *bool*
  and *R1* :: *'a2* ⇒ *'a2* ⇒ *bool*
  and *l1* :: *'a1* ⇒ *'a2*
  and *r1* :: *'a2* ⇒ *'a1*
  and *L2* :: *'a1* ⇒ *'a1* ⇒ *'b1* ⇒ *'b1* ⇒ *bool*
  and *R2* :: *'a2* ⇒ *'a2* ⇒ *'b2* ⇒ *'b2* ⇒ *bool*
  and *l2* :: *'a2* ⇒ *'a1* ⇒ *'b1* ⇒ *'b2*
  and *r2* :: *'a1* ⇒ *'a2* ⇒ *'b2* ⇒ *'b1*
begin

definition *L* ≡ (*x1 x2* :: (≤$_{L1}$)) ⇛ (≤$_{L2\ x1\ x2}$)

lemma *left-rel-eq-Dep-Fun-Rel*: *L* = ((*x1 x2* :: (≤$_{L1}$)) ⇛ (≤$_{L2\ x1\ x2}$))
  unfolding *L-def* **..**

definition *l* ≡ ((*x'* : *r1*) ⇝ *l2 x'*)

lemma *left-eq-dep-fun-map*: *l* = ((*x'* : *r1*) ⇝ *l2 x'*)
  unfolding *l-def* **..**

lemma *left-eq* [*simp*]: *l f x'* = *l2*$_{x'\ (r1\ x')}$ (*f* (*r1 x'*))
  unfolding *left-eq-dep-fun-map* **by** *simp*

context
begin

interpretation *flip* : *transport-Dep-Fun-Rel R1 L1 r1 l1 R2 L2 r2 l2* **.**

abbreviation *R* ≡ *flip.L*
abbreviation *r* ≡ *flip.l*

lemma *right-rel-eq-Dep-Fun-Rel*: *R* = ((*x1' x2'* :: (≤$_{R1}$)) ⇛ (≤$_{R2\ x1'\ x2'}$))
  unfolding *flip.L-def* **..**

lemma *right-eq-dep-fun-map*: *r* = ((*x* : *l1*) ⇝ *r2 x*)
  unfolding *flip.l-def* **..**

end

lemma *right-eq* [*simp*]: $r\ g\ x = r2_{x\ (l1\ x)}\ (g\ (l1\ x))$
unfolding *right-eq-dep-fun-map* **by** *simp*

lemmas *transport-defs* = *left-rel-eq-Dep-Fun-Rel left-eq-dep-fun-map*
right-rel-eq-Dep-Fun-Rel right-eq-dep-fun-map

sublocale *transport L R l r* .

notation L (**infix** $\langle\leq_L\rangle$ *50*)
notation R (**infix** $\langle\leq_R\rangle$ *50*)

lemma *left-relI* [*intro*]:
assumes $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow f\ x1 \leq_{L2\ x1\ x2} f'\ x2$
shows $f \leq_L f'$
unfolding *left-rel-eq-Dep-Fun-Rel* **using** *assms* **by** *blast*

lemma *left-relE* [*elim*]:
assumes $f \leq_L f'$
and $x1 \leq_{L1} x2$
obtains $f\ x1 \leq_{L2\ x1\ x2} f'\ x2$
using *assms* **unfolding** *left-rel-eq-Dep-Fun-Rel* **by** *blast*

interpretation *flip-inv* :
transport-Dep-Fun-Rel $(\geq_{R1})$ $(\geq_{L1})$ *r1 l1 flip2 R2 flip2 L2 r2 l2* .

lemma *flip-inv-right-eq-ge-left*: *flip-inv.R* $= (\geq_L)$
unfolding *left-rel-eq-Dep-Fun-Rel flip-inv.right-rel-eq-Dep-Fun-Rel*
by (*simp only*: *rel-inv-Dep-Fun-Rel-rel-eq flip2-eq-rel-inv*[*symmetric*, *of L2*])

interpretation *flip* : *transport-Dep-Fun-Rel R1 L1 r1 l1 R2 L2 r2 l2* .

lemma *flip-inv-left-eq-ge-right*: *flip-inv.L* $\equiv (\geq_R)$
unfolding *flip.flip-inv-right-eq-ge-left* .

Useful Rewritings for Dependent Relation **lemma** *left-rel2-unit-eqs-left-rel2I*:
assumes $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x2\ x2}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow (\leq_{L2\ (\eta_1\ x)\ x}) \leq (\leq_{L2\ x\ x})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ x1}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow (\leq_{L2\ x\ (\eta_1\ x)}) \leq (\leq_{L2\ x\ x})$
and $x \leq_{L1} x$
and $x \equiv_{L1} \eta_1\ x$
shows $(\leq_{L2\ (\eta_1\ x)\ x}) = (\leq_{L2\ x\ x})$
and $(\leq_{L2\ x\ (\eta_1\ x)}) = (\leq_{L2\ x\ x})$
using *assms* **by** (*auto intro*!: *antisym*)

lemma *left2-eq-if-bi-related-if-monoI*:

assumes *mono-L2*: $((x1\ x2 :: (\geq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid x1 \leq_{L1} x3) \Rrightarrow (\leq))$ *L2*
and $x1 \leq_{L1} x2$
and $x1 \equiv_{L1} x3$
and $x2 \equiv_{L1} x4$
and *trans-L1*: *transitive* $(\leq_{L1})$
shows $(\leq_{L2\ x1\ x2}) = (\leq_{L2\ x3\ x4})$
proof (*intro antisym*)
from $\langle x1 \equiv_{L1} x3\rangle$ $\langle x2 \equiv_{L1} x4\rangle$ **have** $x3 \leq_{L1} x1\ x2 \leq_{L1} x4$ **by** *auto*
with $\langle x1 \leq_{L1} x2\rangle$ *mono-L2* **show** $(\leq_{L2\ x1\ x2}) \leq (\leq_{L2\ x3\ x4})$ **by** *blast*
from $\langle x1 \equiv_{L1} x3\rangle$ $\langle x2 \equiv_{L1} x4\rangle$ **have** $x1 \leq_{L1} x3\ x4 \leq_{L1} x2$ **by** *auto*
moreover from $\langle x3 \leq_{L1} x1\rangle$ $\langle x1 \leq_{L1} x2\rangle$ $\langle x2 \leq_{L1} x4\rangle$ **have** $x3 \leq_{L1} x4$
using *trans-L1* **by** *blast*
ultimately show $(\leq_{L2\ x3\ x4}) \leq (\leq_{L2\ x1\ x2})$ **using** *mono-L2* **by** *blast*
qed

end

Function Relator **locale** *transport-Fun-Rel-syntax* =
tdfrs : *transport-Dep-Fun-Rel-syntax L1 R1 l1 r1* λ- -. *L2* λ- -. *R2* λ- -. *l2* λ- -. *r2*
for *L1* :: *'a1* ⇒ *'a1* ⇒ *bool*
and *R1* :: *'a2* ⇒ *'a2* ⇒ *bool*
and *l1* :: *'a1* ⇒ *'a2*
and *r1* :: *'a2* ⇒ *'a1*
and *L2* :: *'b1* ⇒ *'b1* ⇒ *bool*
and *R2* :: *'b2* ⇒ *'b2* ⇒ *bool*
and *l2* :: *'b1* ⇒ *'b2*
and *r2* :: *'b2* ⇒ *'b1*
begin

notation *L1* (**infix** $\langle\leq_{L1}\rangle$ *50*)
notation *R1* (**infix** $\langle\leq_{R1}\rangle$ *50*)

notation *tdfrs.t1.ge-left* (**infix** $\langle\geq_{L1}\rangle$ *50*)
notation *tdfrs.t1.ge-right* (**infix** $\langle\geq_{R1}\rangle$ *50*)

notation *tdfrs.t1.left-Galois* (**infix** $\langle{}_{L1}\overset{\leq}{\approx}\rangle$ *50*)
notation *tdfrs.t1.ge-Galois-left* (**infix** $\langle\overset{\geq}{\approx}_{L1}\rangle$ *50*)
notation *tdfrs.t1.right-Galois* (**infix** $\langle{}_{R1}\overset{\leq}{\approx}\rangle$ *50*)
notation *tdfrs.t1.ge-Galois-right* (**infix** $\langle\overset{\geq}{\approx}_{R1}\rangle$ *50*)
notation *tdfrs.t1.right-ge-Galois* (**infix** $\langle{}_{R1}\overset{\geq}{\approx}\rangle$ *50*)
notation *tdfrs.t1.Galois-right* (**infix** $\langle\overset{\leq}{\approx}_{R1}\rangle$ *50*)
notation *tdfrs.t1.left-ge-Galois* (**infix** $\langle{}_{L1}\overset{\geq}{\approx}\rangle$ *50*)
notation *tdfrs.t1.Galois-left* (**infix** $\langle\overset{\leq}{\approx}_{L1}\rangle$ *50*)

notation *tdfrs.t1.unit* ($\langle\eta_1\rangle$)
notation *tdfrs.t1.counit* ($\langle\varepsilon_1\rangle$)

notation *L2* (**infix** ‹$\leq_{L2}$› *50*)
notation *R2* (**infix** ‹$\leq_{R2}$› *50*)

notation *tdfrs.t2.ge-left* (**infix** ‹$\geq_{L2}$› *50*)
notation *tdfrs.t2.ge-right* (**infix** ‹$\geq_{R2}$› *50*)

notation *tdfrs.t2.left-Galois* (**infix** ‹$_{L2}\lessapprox$› *50*)
notation *tdfrs.t2.ge-Galois-left* (**infix** ‹$\gtrapprox_{L2}$› *50*)
notation *tdfrs.t2.right-Galois* (**infix** ‹$_{R2}\lessapprox$› *50*)
notation *tdfrs.t2.ge-Galois-right* (**infix** ‹$\gtrapprox_{R2}$› *50*)
notation *tdfrs.t2.right-ge-Galois* (**infix** ‹$_{R2}\gtrapprox$› *50*)
notation *tdfrs.t2.Galois-right* (**infix** ‹$\lessapprox_{R2}$› *50*)
notation *tdfrs.t2.left-ge-Galois* (**infix** ‹$_{L2}\gtrapprox$› *50*)
notation *tdfrs.t2.Galois-left* (**infix** ‹$\lessapprox_{L2}$› *50*)

notation *tdfrs.t2.unit* (‹η_2›)
notation *tdfrs.t2.counit* (‹ε_2›)

end

locale *transport-Fun-Rel* =
  transport-Fun-Rel-syntax L1 R1 l1 r1 L2 R2 l2 r2 +
  tdfr : *transport-Dep-Fun-Rel L1 R1 l1 r1* λ- -. *L2* λ- -. *R2*
    λ- -. *l2* λ- -. *r2*
  for *L1* :: *'a1* ⇒ *'a1* ⇒ *bool*
  and *R1* :: *'a2* ⇒ *'a2* ⇒ *bool*
  and *l1* :: *'a1* ⇒ *'a2*
  and *r1* :: *'a2* ⇒ *'a1*
  and *L2* :: *'b1* ⇒ *'b1* ⇒ *bool*
  and *R2* :: *'b2* ⇒ *'b2* ⇒ *bool*
  and *l2* :: *'b1* ⇒ *'b2*
  and *r2* :: *'b2* ⇒ *'b1*
begin

notation *tdfr.L* (‹*L*›)
notation *tdfr.R* (‹*R*›)

abbreviation *l* ≡ *tdfr.l*
abbreviation *r* ≡ *tdfr.r*

notation *tdfr.L* (**infix** ‹$\leq_L$› *50*)
notation *tdfr.R* (**infix** ‹$\leq_R$› *50*)

notation *tdfr.ge-left* (**infix** ‹$\geq_L$› *50*)
notation *tdfr.ge-right* (**infix** ‹$\geq_R$› *50*)

notation *tdfr.left-Galois* (**infix** ‹$_L\lessapprox$› *50*)
notation *tdfr.ge-Galois-left* (**infix** ‹$\gtrapprox_L$› *50*)

notation *tdfr.right-Galois* (**infix** $\langle_R \lessapprox\rangle$ *50*)
notation *tdfr.ge-Galois-right* (**infix** $\langle \gtrapprox_R\rangle$ *50*)
notation *tdfr.right-ge-Galois* (**infix** $\langle_R \gtrapprox\rangle$ *50*)
notation *tdfr.Galois-right* (**infix** $\langle \lessapprox_R\rangle$ *50*)
notation *tdfr.left-ge-Galois* (**infix** $\langle_L \gtrapprox\rangle$ *50*)
notation *tdfr.Galois-left* (**infix** $\langle \lessapprox_L\rangle$ *50*)

notation *tdfr.unit* ($\langle\eta\rangle$)
notation *tdfr.counit* ($\langle\varepsilon\rangle$)

lemma *left-rel-eq-Fun-Rel*: $(\leq_L) = ((\leq_{L1}) \Rrightarrow (\leq_{L2}))$
by (*urule tdfr.left-rel-eq-Dep-Fun-Rel*)

lemma *left-eq-fun-map*: $l = (r1 \rightsquigarrow l2)$
by (*intro ext*) *simp*

interpretation *flip* : *transport-Fun-Rel R1 L1 r1 l1 R2 L2 r2 l2* **.**

lemma *right-rel-eq-Fun-Rel*: $(\leq_R) = ((\leq_{R1}) \Rrightarrow (\leq_{R2}))$
unfolding *flip.left-rel-eq-Fun-Rel* **..**

lemma *right-eq-fun-map*: $r = (l1 \rightsquigarrow r2)$
unfolding *flip.left-eq-fun-map* **..**

lemmas *transport-defs* = *left-rel-eq-Fun-Rel right-rel-eq-Fun-Rel*
left-eq-fun-map right-eq-fun-map

end

Monotone Dependent Function Relator

locale *transport-Mono-Dep-Fun-Rel* =
transport-Dep-Fun-Rel-syntax L1 R1 l1 r1 L2 R2 l2 r2
+ *tdfr* : *transport-Dep-Fun-Rel L1 R1 l1 r1 L2 R2 l2 r2*
for *L1* :: *'a1* ⇒ *'a1* ⇒ *bool*
and *R1* :: *'a2* ⇒ *'a2* ⇒ *bool*
and *l1* :: *'a1* ⇒ *'a2*
and *r1* :: *'a2* ⇒ *'a1*
and *L2* :: *'a1* ⇒ *'a1* ⇒ *'b1* ⇒ *'b1* ⇒ *bool*
and *R2* :: *'a2* ⇒ *'a2* ⇒ *'b2* ⇒ *'b2* ⇒ *bool*
and *l2* :: *'a2* ⇒ *'a1* ⇒ *'b1* ⇒ *'b2*
and *r2* :: *'a1* ⇒ *'a2* ⇒ *'b2* ⇒ *'b1*
begin

definition $L \equiv tdfr.L^{\oplus}$

lemma *left-rel-eq-tdfr-left-Refl-Rel*: $L = tdfr.L^{\oplus}$
unfolding *L-def* **..**

lemma *left-rel-eq-Mono-Dep-Fun-Rel*: $L = ((x1\ x2 :: (\leq_{L1})) \Rrightarrow\oplus (\leq_{L2}\ x1\ x2))$

unfolding *left-rel-eq-tdfr-left-Refl-Rel tdfr.left-rel-eq-Dep-Fun-Rel* **by** *simp*

lemma *left-rel-eq-tdfr-left-rel-if-reflexive-on*:
assumes *reflexive-on* (*in-field tdfr.L*) *tdfr.L*
shows $L = tdfr.L$
unfolding *left-rel-eq-tdfr-left-Refl-Rel* **using** *assms*
by (*rule Refl-Rel-eq-self-if-reflexive-on*)

abbreviation $l \equiv tdfr.l$

lemma *left-eq-tdfr-left*: $l = tdfr.l$ **..**

interpretation *flip* : *transport-Mono-Dep-Fun-Rel R1 L1 r1 l1 R2 L2 r2 l2* **.**

abbreviation $R \equiv flip.L$

lemma *right-rel-eq-tdfr-right-Refl-Rel*: $R = tdfr.R^{\oplus}$
unfolding *flip.left-rel-eq-tdfr-left-Refl-Rel* **..**

lemma *right-rel-eq-Mono-Dep-Fun-Rel*: $R = ((y1\ y2 :: (\leq_{R1})) \Rrightarrow\oplus (\leq_{R2\ y1\ y2}))$
unfolding *flip.left-rel-eq-Mono-Dep-Fun-Rel* **..**

lemma *right-rel-eq-tdfr-right-rel-if-reflexive-on*:
assumes *reflexive-on* (*in-field tdfr.R*) *tdfr.R*
shows $R = tdfr.R$
using *assms* **by** (*rule flip.left-rel-eq-tdfr-left-rel-if-reflexive-on*)

abbreviation $r \equiv tdfr.r$

lemma *right-eq-tdfr-right*: $r = tdfr.r$ **..**

lemmas *transport-defs* = *left-rel-eq-tdfr-left-Refl-Rel*
right-rel-eq-tdfr-right-Refl-Rel

sublocale *transport L R l r* **.**

notation L (**infix** $\langle\leq_L\rangle$ *50*)
notation R (**infix** $\langle\leq_R\rangle$ *50*)

end

Monotone Function Relator **locale** *transport-Mono-Fun-Rel* =
transport-Fun-Rel-syntax L1 R1 l1 r1 L2 R2 l2 r2 +
tfr : *transport-Fun-Rel L1 R1 l1 r1 L2 R2 l2 r2* +
tmdfr : *transport-Mono-Dep-Fun-Rel L1 R1 l1 r1* λ- -. *L2* λ- -. *R2*
λ- -. *l2* λ- -. *r2*
for *L1* :: *'a1* ⇒ *'a1* ⇒ *bool*
and *R1* :: *'a2* ⇒ *'a2* ⇒ *bool*

and $l1 :: 'a1 \Rightarrow 'a2$
and $r1 :: 'a2 \Rightarrow 'a1$
and $L2 :: 'b1 \Rightarrow 'b1 \Rightarrow bool$
and $R2 :: 'b2 \Rightarrow 'b2 \Rightarrow bool$
and $l2 :: 'b1 \Rightarrow 'b2$
and $r2 :: 'b2 \Rightarrow 'b1$
begin

notation *tmdfr.L* ($\langle L \rangle$)
notation *tmdfr.R* ($\langle R \rangle$)

abbreviation $l \equiv tmdfr.l$
abbreviation $r \equiv tmdfr.r$

notation *tmdfr.L* (**infix** $\langle \leq_L \rangle$ *50*)
notation *tmdfr.R* (**infix** $\langle \leq_R \rangle$ *50*)

notation *tmdfr.ge-left* (**infix** $\langle \geq_L \rangle$ *50*)
notation *tmdfr.ge-right* (**infix** $\langle \geq_R \rangle$ *50*)

notation *tmdfr.left-Galois* (**infix** $\langle {}_L\lessapprox \rangle$ *50*)
notation *tmdfr.ge-Galois-left* (**infix** $\langle \gtrapprox_L \rangle$ *50*)
notation *tmdfr.right-Galois* (**infix** $\langle {}_R\lessapprox \rangle$ *50*)
notation *tmdfr.ge-Galois-right* (**infix** $\langle \gtrapprox_R \rangle$ *50*)
notation *tmdfr.right-ge-Galois* (**infix** $\langle {}_R\gtrapprox \rangle$ *50*)
notation *tmdfr.Galois-right* (**infix** $\langle \lessapprox_R \rangle$ *50*)
notation *tmdfr.left-ge-Galois* (**infix** $\langle {}_L\gtrapprox \rangle$ *50*)
notation *tmdfr.Galois-left* (**infix** $\langle \lessapprox_L \rangle$ *50*)

notation *tmdfr.unit* ($\langle \eta \rangle$)
notation *tmdfr.counit* ($\langle \varepsilon \rangle$)

lemma *left-rel-eq-Mono-Fun-Rel*: $(\leq_L) = ((\leq_{L1}) \Rrightarrow\oplus (\leq_{L2}))$
unfolding *tmdfr.left-rel-eq-Mono-Dep-Fun-Rel* **by** *simp*

lemma *left-eq-fun-map*: $l = (r1 \rightsquigarrow l2)$
unfolding *tfr.left-eq-fun-map* **..**

interpretation *flip* : *transport-Mono-Fun-Rel R1 L1 r1 l1 R2 L2 r2 l2* **.**

lemma *right-rel-eq-Mono-Fun-Rel*: $(\leq_R) = ((\leq_{R1}) \Rrightarrow\oplus (\leq_{R2}))$
unfolding *flip.left-rel-eq-Mono-Fun-Rel* **..**

lemma *right-eq-fun-map*: $r = (l1 \rightsquigarrow r2)$
unfolding *flip.left-eq-fun-map* **..**

lemmas *transport-defs* = *tmdfr.transport-defs*

end

end

2.15.2 Monotonicity

theory *Transport-Functions-Monotone*
imports
Transport-Functions-Base
begin

Dependent Function Relator **context** *transport-Dep-Fun-Rel*
begin

interpretation *flip* : *transport-Dep-Fun-Rel R1 L1 r1 l1 R2 L2 r2 l2* .

lemma *mono-wrt-rel-leftI*:
assumes *mono-r1*: $((\leq_{R1}) \Rightarrow (\leq_{L1}))\ r1$
and *mono-l2*: $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow$
$((\leq_{L2}\ (r1\ x1')\ (r1\ x2')) \Rightarrow (\leq_{R2}\ (\varepsilon_1\ x1')\ x2'))\ (l2_{x2'}\ (r1\ x1'))$
and *R2-le1*: $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2}\ (\varepsilon_1\ x1')\ x2') \leq (\leq_{R2\ x1'\ x2'})$
and *R2-l2-le1*: $\bigwedge x1'\ x2'\ y.\ x1' \leq_{R1} x2' \Longrightarrow$ *in-dom* $(\leq_{L2}\ (r1\ x1')\ (r1\ x2'))\ y$
$\Longrightarrow$
$(\leq_{R2\ x1'\ x2'})\ (l2_{x2'}\ (r1\ x1')\ y) \leq (\leq_{R2\ x1'\ x2'})\ (l2_{x1'}\ (r1\ x1')\ y)$
and *ge-R2-l2-le2*: $\bigwedge x1'\ x2'\ y.\ x1' \leq_{R1} x2' \Longrightarrow$ *in-codom* $(\leq_{L2}\ (r1\ x1')\ (r1\ x2'))$
$y \Longrightarrow$
$(\geq_{R2\ x1'\ x2'})\ (l2_{x2'}\ (r1\ x1')\ y) \leq (\geq_{R2\ x1'\ x2'})\ (l2_{x2'}\ (r1\ x2')\ y)$
shows $((\leq_L) \Rightarrow (\leq_R))\ l$
proof (*intro mono-wrt-relI flip.left-relI*)
fix *f1 f2 x1' x2'* **assume** [*iff*]: $x1' \leq_{R1} x2'$
with *mono-r1* **have** $r1\ x1' \leq_{L1} r1\ x2'$ (**is** $?x1 \leq_{L1} ?x2$) **by** *blast*
moreover assume $f1 \leq_L f2$
ultimately have $f1\ ?x1 \leq_{L2}\ ?x1\ ?x2\ f2\ ?x2$ (**is** $?y1 \leq_{L2}\ ?x1\ ?x2\ ?y2$) **by** *blast*
with *mono-l2* **have** $l2_{x2'}\ ?x1\ ?y1 \leq_{R2}\ (\varepsilon_1\ x1')\ x2'\ l2_{x2'}\ ?x1\ ?y2$ **by** *blast*
with *R2-le1* **have** $l2_{x2'}\ ?x1\ ?y1 \leq_{R2\ x1'\ x2'} l2_{x2'}\ ?x1\ ?y2$ **by** *blast*
with *R2-l2-le1* **have** $l2_{x1'}\ ?x1\ ?y1 \leq_{R2\ x1'\ x2'} l2_{x2'}\ ?x1\ ?y2$
using ‹$?y1 \leq_{L2}\ ?x1\ ?x2$ -› **by** *blast*
with *ge-R2-l2-le2* **have** $l2_{x1'}\ ?x1\ ?y1 \leq_{R2\ x1'\ x2'} l2_{x2'}\ ?x2\ ?y2$
using ‹- $\leq_{L2}\ ?x1\ ?x2\ ?y2$› **by** *blast*
then show $l\ f1\ x1' \leq_{R2\ x1'\ x2'} l\ f2\ x2'$ **by** *simp*
qed

lemma *mono-wrt-rel-left-in-dom-mono-left-assm*:
assumes (*in-dom* $(\leq_{L2}\ (r1\ x1')\ (r1\ x2'))) \Rightarrow (\leq_{R2\ x1'\ x2'}))\ (l2_{x1'}\ (r1\ x1'))\ (l2_{x2'}\ (r1\ x1'))$
and *transitive* $(\leq_{R2\ x1'\ x2'})$
and $x1' \leq_{R1} x2'$
and *in-dom* $(\leq_{L2}\ (r1\ x1')\ (r1\ x2'))\ y$

shows $(\leq_{R2\ x1'\ x2'})\ (l2_{x2'\ (r1\ x1')}\ y) \leq (\leq_{R2\ x1'\ x2'})\ (l2_{x1'\ (r1\ x1')}\ y)$
using *assms* **by** *blast*

lemma *mono-wrt-rel-left-in-codom-mono-left-assm*:
assumes $(\textit{in-codom}\ (\leq_{L2\ (r1\ x1')\ (r1\ x2')}) \Rrightarrow (\leq_{R2\ x1'\ x2'}))\ (l2_{x2'\ (r1\ x1')})$
$(l2_{x2'\ (r1\ x2')})$
and *transitive* $(\leq_{R2\ x1'\ x2'})$
and $x1' \leq_{R1\ x2'}$
and $\textit{in-codom}\ (\leq_{L2\ (r1\ x1')\ (r1\ x2')})\ y$
shows $(\geq_{R2\ x1'\ x2'})\ (l2_{x2'\ (r1\ x1')}\ y) \leq (\geq_{R2\ x1'\ x2'})\ (l2_{x2'\ (r1\ x2')}\ y)$
using *assms* **by** *blast*

lemma *mono-wrt-rel-left-if-transitiveI*:
assumes $((\leq_{R1}) \Rightarrow (\leq_{L1}))\ r1$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1\ x2'} \Longrightarrow$
$((\leq_{L2\ (r1\ x1')\ (r1\ x2')}) \Rightarrow (\leq_{R2\ (\varepsilon_1\ x1')\ x2'}))\ (l2_{x2'\ (r1\ x1')})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1\ x2'} \Longrightarrow (\leq_{R2\ (\varepsilon_1\ x1')\ x2'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1\ x2'} \Longrightarrow$
$(\textit{in-dom}\ (\leq_{L2\ (r1\ x1')\ (r1\ x2')}) \Rrightarrow (\leq_{R2\ x1'\ x2'}))\ (l2_{x1'\ (r1\ x1')})\ (l2_{x2'\ (r1\ x1')})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1\ x2'} \Longrightarrow$
$(\textit{in-codom}\ (\leq_{L2\ (r1\ x1')\ (r1\ x2')}) \Rrightarrow (\leq_{R2\ x1'\ x2'}))\ (l2_{x2'\ (r1\ x1')})\ (l2_{x2'\ (r1\ x2')})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1\ x2'} \Longrightarrow \textit{transitive}\ (\leq_{R2\ x1'\ x2'})$
shows $((\leq_L) \Rightarrow (\leq_R))\ l$
using *assms* **by** (*intro mono-wrt-rel-leftI*
mono-wrt-rel-left-in-dom-mono-left-assm
mono-wrt-rel-left-in-codom-mono-left-assm)
auto

lemma *mono-wrt-rel-left2-if-mono-wrt-rel-left2-if-left-GaloisI*:
assumes $((\leq_{R1}) \Rightarrow (\leq_{L1}))\ r1$
and $\bigwedge x\ x'.\ x\ {}_{L1}\!\lessapprox x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \Rightarrow (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})$
shows $\bigwedge x1'\ x2'.\ x1' \leq_{R1\ x2'} \Longrightarrow$
$((\leq_{L2\ (r1\ x1')\ (r1\ x2')}) \Rightarrow (\leq_{R2\ (\varepsilon_1\ x1')\ x2'}))\ (l2_{x2'\ (r1\ x1')})$
using *assms* **by** (*intro mono-wrt-relI*) *fastforce*

interpretation *flip-inv* :
transport-Dep-Fun-Rel $(\geq_{R1})\ (\geq_{L1})\ r1\ l1\ flip2\ R2\ flip2\ L2\ r2\ l2$
rewrites *flip-inv.R* $\equiv (\geq_L)$ **and** *flip-inv.L* $\equiv (\geq_R)$
and *flip-inv.t1.counit* $\equiv \eta_1$
and $\bigwedge R\ x\ y.\ (flip2\ R\ x\ y)^{-1} \equiv R\ y\ x$
and $\bigwedge R\ x1\ x2.\ in\text{-}dom\ (flip2\ R\ x1\ x2) \equiv in\text{-}codom\ (R\ x2\ x1)$
and $\bigwedge R\ x1\ x2.\ in\text{-}codom\ (flip2\ R\ x1\ x2) \equiv in\text{-}dom\ (R\ x2\ x1)$
and $\bigwedge R\ S.\ (R^{-1} \Rightarrow S^{-1}) \equiv (R \Rightarrow S)$
and $\bigwedge x1\ x2\ x1'\ x2'.\ (flip2\ R2\ x1'\ x2' \Rightarrow flip2\ L2\ x1\ x2) \equiv$
$((\leq_{R2\ x2'\ x1'}) \Rightarrow (\leq_{L2\ x2\ x1}))$
and $\bigwedge x1\ x2\ x3\ x4.\ flip2\ L2\ x1\ x2 \leq flip2\ L2\ x3\ x4 \equiv (\leq_{L2\ x2\ x1}) \leq (\leq_{L2\ x4\ x3})$
and $\bigwedge x1'\ x2'\ y1\ y2.$

flip-inv.dfro2.right-infix y1 x1' x2' ≤ *flip-inv.dfro2.right-infix y2 x1' x2'* ≡
(≥$_{L2\ x2'\ x1'}$) *y1* ≤ (≥$_{L2\ x2'\ x1'}$) *y2*
and ⋀(*P* :: *'f* ⇒ *bool*) *x1 x2*. (*P* ⇛ *flip2 L2 x1 x2*) ≡ (*P* ⇛ (≥$_{L2\ x2\ x1}$))
and ⋀(*P* :: *'f* ⇒ *bool*) (*R* :: *'g* ⇒ *'g* ⇒ *bool*). (*P* ⇛ *R*$^{-1}$) ≡ (*P* ⇛ *R*)$^{-1}$
and ⋀*x1 x2*. *transitive* (*flip2 L2 x1 x2*) ≡ *transitive* (≤$_{L2\ x2\ x1}$)
by (*simp-all add*: *flip-inv-left-eq-ge-right flip-inv-right-eq-ge-left*
t1.flip-counit-eq-unit mono-wrt-rel-eq-dep-mono-wrt-rel Fun-Rel-pred-eq-Dep-Fun-Rel-pred
del: *rel-inv-iff-rel*)

lemma *mono-wrt-rel-rightI*:
assumes ((≤$_{L1}$) ⇒ (≤$_{R1}$)) *l1*
and ⋀*x1 x2*. *x1* ≤$_{L1}$ *x2* ⟹ ((≤$_{R2\ (l1\ x1)\ (l1\ x2)}$) ⇒ (≤$_{L2\ x1\ (\eta_1\ x2)}$)) ($r2_{x1\ (l1\ x2)}$)
and ⋀*x1 x2*. *x1* ≤$_{L1}$ *x2* ⟹ (≤$_{L2\ x1\ (\eta_1\ x2)}$) ≤ (≤$_{L2\ x1\ x2}$)
and ⋀*x1 x2 y'*. *x1* ≤$_{L1}$ *x2* ⟹ *in-codom* (≤$_{R2\ (l1\ x1)\ (l1\ x2)}$) *y'* ⟹
(≥$_{L2\ x1\ x2}$) ($r2_{x1\ (l1\ x2)}$ *y'*) ≤ (≥$_{L2\ x1\ x2}$) ($r2_{x2\ (l1\ x2)}$ *y'*)
and ⋀*x1 x2 y'*. *x1* ≤$_{L1}$ *x2* ⟹ *in-dom* (≤$_{R2\ (l1\ x1)\ (l1\ x2)}$) *y'* ⟹
(≤$_{L2\ x1\ x2}$) ($r2_{x1\ (l1\ x2)}$ *y'*) ≤ (≤$_{L2\ x1\ x2}$) ($r2_{x1\ (l1\ x1)}$ *y'*)
shows ((≤$_R$) ⇒ (≤$_L$)) *r*
using *assms* **by** (*intro flip-inv.mono-wrt-rel-leftI*[*simplified rel-inv-iff-rel*])

lemma *mono-wrt-rel-right-if-transitiveI*:
assumes ((≤$_{L1}$) ⇒ (≤$_{R1}$)) *l1*
and ⋀*x1 x2*. *x1* ≤$_{L1}$ *x2* ⟹ ((≤$_{R2\ (l1\ x1)\ (l1\ x2)}$) ⇒ (≤$_{L2\ x1\ (\eta_1\ x2)}$)) ($r2_{x1\ (l1\ x2)}$)
and ⋀*x1 x2*. *x1* ≤$_{L1}$ *x2* ⟹ (≤$_{L2\ x1\ (\eta_1\ x2)}$) ≤ (≤$_{L2\ x1\ x2}$)
and ⋀*x1 x2*. *x1* ≤$_{L1}$ *x2* ⟹ (*in-codom* (≤$_{R2\ (l1\ x1)\ (l1\ x2)}$) ⇛ (≤$_{L2\ x1\ x2}$))
($r2_{x1\ (l1\ x2)}$) ($r2_{x2\ (l1\ x2)}$)
and ⋀*x1 x2*. *x1* ≤$_{L1}$ *x2* ⟹ (*in-dom* (≤$_{R2\ (l1\ x1)\ (l1\ x2)}$) ⇛ (≤$_{L2\ x1\ x2}$))
($r2_{x1\ (l1\ x1)}$) ($r2_{x1\ (l1\ x2)}$)
and ⋀*x1 x2*. *x1* ≤$_{L1}$ *x2* ⟹ *transitive* (≤$_{L2\ x1\ x2}$)
shows ((≤$_R$) ⇒ (≤$_L$)) *r*
using *assms* **by** (*intro flip-inv.mono-wrt-rel-left-if-transitiveI*
[*simplified rel-inv-iff-rel*])

lemma *mono-wrt-rel-right2-if-mono-wrt-rel-right2-if-left-GaloisI*:
assumes *assms1*: ((≤$_{L1}$) ⇒ (≤$_{R1}$)) *l1* ((≤$_{L1}$) ⊴$_h$ (≤$_{R1}$)) *l1 r1*
and *mono-r2*: ⋀*x x'*. *x* $_{L1}\lessapprox$ *x'* ⟹ ((≤$_{R2\ (l1\ x)\ x'}$) ⇒ (≤$_{L2\ x\ (r1\ x')}$)) ($r2_{x\ x'}$)
shows ⋀*x1 x2*. *x1* ≤$_{L1}$ *x2* ⟹ ((≤$_{R2\ (l1\ x1)\ (l1\ x2)}$) ⇒ (≤$_{L2\ x1\ (\eta_1\ x2)}$))
($r2_{x1\ (l1\ x2)}$)
proof −
show ((≤$_{R2\ (l1\ x1)\ (l1\ x2)}$) ⇒ (≤$_{L2\ x1\ (\eta_1\ x2)}$)) ($r2_{x1\ (l1\ x2)}$) **if** *x1* ≤$_{L1}$ *x2* **for**
x1 x2
proof −
from ‹*x1* ≤$_{L1}$ *x2*› **have** *x1* $_{L1}\lessapprox$ *l1 x2*
using *assms1* **by** (*intro t1.left-Galois-left-if-left-relI*) *blast*
with *mono-r2* **show** *?thesis* **by** *auto*
qed
qed

end

Function Relator **context** *transport-Fun-Rel*
begin

lemma *mono-wrt-rel-leftI*:
assumes $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and $((\leq_{L2}) \Rightarrow (\leq_{R2}))$ *l2*
shows $((\leq_L) \Rightarrow (\leq_R))$ *l*
using *assms* **by** (*intro tdfr.mono-wrt-rel-leftI*) *simp-all*

end

Monotone Dependent Function Relator **context** *transport-Mono-Dep-Fun-Rel*
begin

lemmas *mono-wrt-rel-leftI* = *mono-wrt-rel-Refl-Rel-Refl-Rel-if-mono-wrt-rel*
[*of tdfr.L tdfr.R l*, *folded transport-defs*]

end

Monotone Function Relator **context** *transport-Mono-Fun-Rel*
begin

lemmas *mono-wrt-rel-leftI* = *tmdfr.mono-wrt-rel-leftI*[*OF tfr.mono-wrt-rel-leftI*]

end

end

2.15.3 Galois Property

theory *Transport-Functions-Galois-Property*
imports
Transport-Functions-Monotone
begin

Dependent Function Relator **context** *transport-Dep-Fun-Rel*
begin

context
begin

interpretation *flip* : *transport-Dep-Fun-Rel R1 L1 r1 l1 R2 L2 r2 l2* .

lemma *left-right-rel-if-left-rel-rightI*:
assumes *mono-r1*: $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*

and *half-galois-prop-left1*: $((\leq_{L1}) \ {}_h\trianglelefteq \ (\leq_{R1}))\ l1\ r1$
and *refl-R1*: *reflexive-on* (*in-dom* $(\leq_{R1})$) $(\leq_{R1})$
and *half-galois-prop-left2*: $\bigwedge x'.\ x' \leq_{R1} x' \Longrightarrow$
$((\leq_{L2\ (r1\ x')\ (r1\ x')})\ {}_h\trianglelefteq\ (\leq_{R2\ (\varepsilon_1\ x')\ x'}))\ (l2_{\ x'\ (r1\ x')})\ (r2_{(r1\ x')\ x'})$
and *R2-le1*: $\bigwedge x'.\ x' \leq_{R1} x' \Longrightarrow (\leq_{R2\ (\varepsilon_1\ x')\ x'}) \leq (\leq_{R2\ x'\ x'})$
and *R2-le2*: $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ x1'\ x1'}) \leq (\leq_{R2\ x1'\ x2'})$
and *ge-L2-r2-le2*: $\bigwedge x'\ y'.\ x' \leq_{R1} x' \Longrightarrow$ *in-codom* $(\leq_{R2\ (\varepsilon_1\ x')\ x'})\ y' \Longrightarrow$
$(\geq_{L2\ (r1\ x')\ (r1\ x')})\ (r2_{(r1\ x')\ (\varepsilon_1\ x')\ y'}) \leq (\geq_{L2\ (r1\ x')\ (r1\ x')})\ (r2_{(r1\ x')\ x'\ y'})$
and *trans-R2*: $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow$ *transitive* $(\leq_{R2\ x1'\ x2'})$
and $g \leq_R g$
and $f \leq_L r\ g$
shows $l\ f \leq_R g$
proof (*rule flip.left-relI*)
fix *x1' x2'*
assume [*iff*]: $x1' \leq_{R1} x2'$
with *refl-R1* **have** [*iff*]: $x1' \leq_{R1} x1'$ **by** *auto*
with *mono-r1 half-galois-prop-left1* **have** [*iff*]: $\varepsilon_1\ x1' \leq_{R1} x1'$
by (*intro t1.counit-rel-if-right-rel-if-half-galois-prop-left-if-mono-wrt-rel*)
with *refl-R1* **have** $\varepsilon_1\ x1' \leq_{R1} \varepsilon_1\ x1'$ **by** *blast*
with $\langle g \leq_R g\rangle$ **have** $g\ (\varepsilon_1\ x1') \leq_{R2\ (\varepsilon_1\ x1')\ (\varepsilon_1\ x1')}\ g\ (\varepsilon_1\ x1')$ **by** *blast*
with *R2-le2* **have** $g\ (\varepsilon_1\ x1') \leq_{R2\ (\varepsilon_1\ x1')\ x1'}\ g\ (\varepsilon_1\ x1')$ **by** *blast*

let *?x1* = *r1 x1'*
from $\langle f \leq_L r\ g\rangle$ $\langle x1' \leq_{R1} x1'\rangle$ **have** *f ?x1* $\leq_{L2\ ?x1\ ?x1}$ *r g ?x1* **using** *mono-r1*
by *blast*
then have *f ?x1* $\leq_{L2\ ?x1\ ?x1}$ $r2_{\ ?x1\ (\varepsilon_1\ x1')}\ (g\ (\varepsilon_1\ x1'))$ **by** *simp*
with *ge-L2-r2-le2* **have** *f ?x1* $\leq_{L2\ ?x1\ ?x1}$ $r2_{\ ?x1\ x1'}\ (g\ (\varepsilon_1\ x1'))$
using $\langle$*-* $\leq_{R2\ (\varepsilon_1\ x1')\ x1'}\ g\ (\varepsilon_1\ x1')\rangle$ **by** *blast*
with *half-galois-prop-left2* **have** $l2_{\ x1'\ ?x1}\ (f\ ?x1) \leq_{R2\ (\varepsilon_1\ x1')\ x1'}\ g\ (\varepsilon_1\ x1')$
using $\langle$*-* $\leq_{R2\ (\varepsilon_1\ x1')\ x1'}\ g\ (\varepsilon_1\ x1')\rangle$ **by** *auto*
moreover from $\langle g \leq_R g\rangle$ $\langle \varepsilon_1\ x1' \leq_{R1} x1'\rangle$ **have** ... $\leq_{R2\ (\varepsilon_1\ x1')\ x1'}\ g\ x1'$ **by**
blast
ultimately have $l2_{\ x1'\ ?x1}\ (f\ ?x1) \leq_{R2\ (\varepsilon_1\ x1')\ x1'}\ g\ x1'$ **using** *trans-R2* **by**
blast
with *R2-le1 R2-le2* **have** $l2_{\ x1'\ ?x1}\ (f\ ?x1) \leq_{R2\ x1'\ x2'}\ g\ x1'$ **by** *blast*
moreover from $\langle g \leq_R g\rangle$ $\langle x1' \leq_{R1} x2'\rangle$ **have** ... $\leq_{R2\ x1'\ x2'}\ g\ x2'$ **by** *blast*
ultimately have $l2_{\ x1'\ ?x1}\ (f\ ?x1) \leq_{R2\ x1'\ x2'}\ g\ x2'$ **using** *trans-R2* **by** *blast*
then show $l\ f\ x1' \leq_{R2\ x1'\ x2'}\ g\ x2'$ **by** *simp*
qed

lemma *left-right-rel-if-left-rel-right-ge-left2-assmI*:
assumes *mono-r1*: $((\leq_{R1}) \Rightarrow (\leq_{L1}))\ r1$
and $((\leq_{L1})\ {}_h\trianglelefteq\ (\leq_{R1}))\ l1\ r1$
and $((\textit{in-codom}\ (\leq_{R2\ (\varepsilon_1\ x')\ x'})) \Rrightarrow (\leq_{L2\ (r1\ x')\ (r1\ x')}))$
$(r2_{(r1\ x')\ (\varepsilon_1\ x')})\ (r2_{(r1\ x')\ x'})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *transitive* $(\leq_{L2\ x1\ x2})$

and $x' \leq_{R1} x'$
and *in-codom* $(\leq_{R2\ (\varepsilon_1\ x')\ x'})\ y'$
shows $(\geq_{L2\ (r1\ x')\ (r1\ x')})\ (r2_{(r1\ x')\ (\varepsilon_1\ x')}\ y') \leq (\geq_{L2\ (r1\ x')\ (r1\ x')})\ (r2_{(r1\ x')\ x'}\ y')$
using *mono-wrt-relD*[*OF mono-r1*] *assms*(2−) **by** *blast*

interpretation *flip-inv* :
transport-Dep-Fun-Rel $(\geq_{R1})\ (\geq_{L1})$ *r1 l1 flip2 R2 flip2 L2 r2 l2*
rewrites *flip-inv.L* $\equiv (\geq_R)$ **and** *flip-inv.R* $\equiv (\geq_L)$
and *flip-inv.t1.counit* $\equiv \eta_1$
and $\bigwedge R\ x\ y.$ *(flip2 R x y)*$^{-1} \equiv R\ y\ x$
and $\bigwedge R.$ *in-dom* $R^{-1} \equiv$ *in-codom R*
and $\bigwedge R\ x1\ x2.$ *in-codom (flip2 R x1 x2)* $\equiv$ *in-dom (R x2 x1)*
and $\bigwedge R\ S.\ (R^{-1} \Rightarrow S^{-1}) \equiv (R \Rightarrow S)$
and $\bigwedge R\ S\ x1\ x2\ x1'\ x2'.$ *(flip2 R x1 x2* $_h\trianglelefteq$ *flip2 S x1' x2')* $\equiv$ *(S x2' x1'* $\trianglelefteq_h$ *R x2 x1)*$^{-1}$
and $\bigwedge R\ S.\ (R^{-1}\ {}_h\trianglelefteq\ S^{-1}) \equiv (S \trianglelefteq_h R)^{-1}$
and $\bigwedge x1\ x2\ x3\ x4.$ *flip2 L2 x1 x2* $\leq$ *flip2 L2 x3 x4* $\equiv (\leq_{L2\ x2\ x1}) \leq (\leq_{L2\ x4\ x3})$
and $\bigwedge (R :: 'z \Rightarrow 'z \Rightarrow bool)\ (P :: 'z \Rightarrow bool).$ *reflexive-on P* $R^{-1} \equiv$ *reflexive-on P R*
and $\bigwedge R\ x1\ x2.$ *transitive (flip2 R x1 x2* $:: 'z \Rightarrow 'z \Rightarrow bool) \equiv$ *transitive (R x2 x1)*
and $\bigwedge x\ x.$ $((\textit{in-dom}\ (\leq_{L2\ x'\ (\eta_1\ x')}))) \Rrightarrow \textit{flip2 R2}\ (l1\ x')\ (l1\ x')$
$\equiv ((\textit{in-dom}\ (\leq_{L2\ x'\ (\eta_1\ x')}))) \Rrightarrow (\leq_{R2\ (l1\ x')\ (l1\ x')})^{-1}$
by (*simp-all add*: *flip-inv-left-eq-ge-right flip-inv-right-eq-ge-left*
t1.flip-counit-eq-unit
galois-prop.rel-inv-half-galois-prop-right-eq-half-galois-prop-left-rel-inv
mono-wrt-rel-eq-dep-mono-wrt-rel Fun-Rel-pred-eq-Dep-Fun-Rel-pred)

lemma *left-rel-right-if-left-right-relI*:
assumes $((\leq_{L1}) \Rightarrow (\leq_{R1}))$ *l1*
and $((\leq_{L1}) \trianglelefteq_h (\leq_{R1}))$ *l1 r1*
and *reflexive-on* $(\textit{in-codom}\ (\leq_{L1}))\ (\leq_{L1})$
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow ((\leq_{L2\ x\ (\eta_1\ x)}) \trianglelefteq_h (\leq_{R2\ (l1\ x)\ (l1\ x)}))\ (l2_{(l1\ x)\ x})\ (r2_{x\ (l1\ x)})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x2\ x2}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow (\leq_{L2\ x\ (\eta_1\ x)}) \leq (\leq_{L2\ x\ x})$
and $\bigwedge x\ y.\ x \leq_{L1} x \Longrightarrow \textit{in-dom}\ (\leq_{L2\ x\ (\eta_1\ x)})\ y \Longrightarrow$
$(\leq_{R2\ (l1\ x)\ (l1\ x)})\ (l2_{(l1\ x)\ (\eta_1\ x)}\ y) \leq (\leq_{R2\ (l1\ x)\ (l1\ x)})\ (l2_{(l1\ x)\ x}\ y)$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow \textit{transitive}\ (\leq_{L2\ x1\ x2})$
and $f \leq_L f$
and $l\ f \leq_R g$
shows $f \leq_L r\ g$
using *assms*
by (*intro flip-inv.left-right-rel-if-left-rel-rightI*[*simplified rel-inv-iff-rel*])

lemma *left-rel-right-if-left-right-rel-le-right2-assmI*:
assumes $((\leq_{L1}) \Rightarrow (\leq_{R1}))$ *l1*
and $((\leq_{L1}) \trianglelefteq_h (\leq_{R1}))^{-1}$ *r1 l1*

and $((\textit{in-dom}\ (\leq_{L2}\ x\ (\eta_1\ x)))) \Rrightarrow (\leq_{R2}\ (l1\ x)\ (l1\ x)))\ (l2_{(l1\ x)}\ x)\ (l2_{(l1\ x)}\ (\eta_1\ x))$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1}\ x2' \Longrightarrow \textit{transitive}\ (\leq_{R2\ x1'\ x2'})$
and $x \leq_{L1}\ x$
and $\textit{in-dom}\ (\leq_{L2\ x\ (\eta_1\ x)})\ y$
shows $(\leq_{R2\ (l1\ x)\ (l1\ x)})\ (l2_{(l1\ x)\ (\eta_1\ x)}\ y) \leq (\leq_{R2\ (l1\ x)\ (l1\ x)})\ (l2_{(l1\ x)\ x}\ y)$
using *assms* **by** (*intro flip-inv.left-right-rel-if-left-rel-right-ge-left2-assmI*
[*simplified rel-inv-iff-rel*])
auto

end

lemma *left-rel-right-iff-left-right-relI*:
assumes $((\leq_{L1}) \dashv (\leq_{R1}))\ l1\ r1$
and $\textit{reflexive-on}\ (\textit{in-codom}\ (\leq_{L1}))\ (\leq_{L1})$
and $\textit{reflexive-on}\ (\textit{in-dom}\ (\leq_{R1}))\ (\leq_{R1})$
and $\bigwedge x'.\ x' \leq_{R1}\ x' \Longrightarrow$
$((\leq_{L2\ (r1\ x')\ (r1\ x')})\ {}_h\trianglelefteq\ (\leq_{R2\ (\varepsilon_1\ x')\ x'}))\ (l2_{\ x'\ (r1\ x')})\ (r2_{(r1\ x')\ x'})$
and $\bigwedge x.\ x \leq_{L1}\ x \Longrightarrow ((\leq_{L2\ x\ (\eta_1\ x)}) \trianglelefteq_h (\leq_{R2\ (l1\ x)\ (l1\ x)}))\ (l2_{(l1\ x)\ x})\ (r2_{x\ (l1\ x)})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1}\ x2 \Longrightarrow (\leq_{L2\ x2\ x2}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x.\ x \leq_{L1}\ x \Longrightarrow (\leq_{L2\ x\ (\eta_1\ x)}) \leq (\leq_{L2\ x\ x})$
and $\bigwedge x'.\ x' \leq_{R1}\ x' \Longrightarrow (\leq_{R2\ (\varepsilon_1\ x')\ x'}) \leq (\leq_{R2\ x'\ x'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1}\ x2' \Longrightarrow (\leq_{R2\ x1'\ x1'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x\ y.\ x \leq_{L1}\ x \Longrightarrow \textit{in-dom}\ (\leq_{L2\ x\ (\eta_1\ x)})\ y \Longrightarrow$
$(\leq_{R2\ (l1\ x)\ (l1\ x)})\ (l2_{(l1\ x)\ (\eta_1\ x)}\ y) \leq (\leq_{R2\ (l1\ x)\ (l1\ x)})\ (l2_{(l1\ x)\ x}\ y)$
and $\bigwedge x'\ y'.\ x' \leq_{R1}\ x' \Longrightarrow \textit{in-codom}\ (\leq_{R2\ (\varepsilon_1\ x')\ x'})\ y' \Longrightarrow$
$(\geq_{L2\ (r1\ x')\ (r1\ x')})\ (r2_{(r1\ x')\ (\varepsilon_1\ x')}\ y') \leq (\geq_{L2\ (r1\ x')\ (r1\ x')})\ (r2_{(r1\ x')\ x'}$
$y')$
and $\bigwedge x1\ x2.\ x1 \leq_{L1}\ x2 \Longrightarrow \textit{transitive}\ (\leq_{L2\ x1\ x2})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1}\ x2' \Longrightarrow \textit{transitive}\ (\leq_{R2\ x1'\ x2'})$
and $f \leq_L f$
and $g \leq_R g$
shows $f \leq_L r\ g \longleftrightarrow l\ f \leq_R g$
using *assms* **by** (*intro iffI left-right-rel-if-left-rel-rightI*)
(*auto intro*!: *left-rel-right-if-left-right-relI*)

lemma *half-galois-prop-left2-if-half-galois-prop-left2-if-left-GaloisI*:
assumes $((\leq_{R1}) \Rightarrow (\leq_{L1}))\ r1$
and $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox\ x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')})\ {}_h\trianglelefteq\ (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $x' \leq_{R1}\ x'$
shows $((\leq_{L2\ (r1\ x')\ (r1\ x')})\ {}_h\trianglelefteq\ (\leq_{R2\ (\varepsilon_1\ x')\ x'}))\ (l2_{\ x'\ (r1\ x')})\ (r2_{(r1\ x')\ x'})$
using *assms* **by** (*auto intro*: *t1.right-left-Galois-if-right-relI*)

lemma *half-galois-prop-right2-if-half-galois-prop-right2-if-left-GaloisI*:
assumes $((\leq_{L1}) \Rightarrow (\leq_{R1}))\ l1$
and $((\leq_{L1}) \trianglelefteq_h (\leq_{R1}))\ l1\ r1$
and $\bigwedge x\ x'.\ x\ {}_{L1}\gtrapprox\ x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \trianglelefteq_h (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $x \leq_{L1}\ x$

shows $((\leq_{L2\ x\ (\eta_1\ x)}) \trianglelefteq_h (\leq_{R2\ (l1\ x)\ (l1\ x)}))\ (l2_{(l1\ x)\ x})\ (r2_{x\ (l1\ x)})$
by (*auto intro*!: *assms t1.left-Galois-left-if-left-relI*)

lemma *left-rel-right-iff-left-right-relI'*:
assumes $((\leq_{L1}) \dashv (\leq_{R1}))$ *l1 r1*
and *reflexive-on* (*in-codom* $(\leq_{L1})$) $(\leq_{L1})$
and *reflexive-on* (*in-dom* $(\leq_{R1})$) $(\leq_{R1})$
and *galois-prop2*: $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox x' \Longrightarrow$
$((\leq_{L2\ x\ (r1\ x')}) \trianglelefteq (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x2\ x2}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow (\leq_{L2\ x\ (\eta_1\ x)}) \leq (\leq_{L2\ x\ x})$
and $\bigwedge x'.\ x' \leq_{R1} x' \Longrightarrow (\leq_{R2\ (\varepsilon_1\ x')\ x'}) \leq (\leq_{R2\ x'\ x'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ x1'\ x1'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow$
$((in\text{-}dom\ (\leq_{L2\ x\ (\eta_1\ x)})) \Rrightarrow (\leq_{R2\ (l1\ x)\ (l1\ x)}))\ (l2_{(l1\ x)\ x})\ (l2_{(l1\ x)\ (\eta_1\ x)})$
and $\bigwedge x'.\ x' \leq_{R1} x' \Longrightarrow$
$((in\text{-}codom\ (\leq_{R2\ (\varepsilon_1\ x')\ x'})) \Rrightarrow (\leq_{L2\ (r1\ x')\ (r1\ x')}))\ (r2_{(r1\ x')\ (\varepsilon_1\ x')})\ (r2_{(r1\ x')\ x'})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow transitive\ (\leq_{L2\ x1\ x2})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow transitive\ (\leq_{R2\ x1'\ x2'})$
and $f \leq_L f$
and $g \leq_R g$
shows $f \leq_L r\ g \longleftrightarrow l\ f \leq_R g$
proof −
from *galois-prop2* **have**
$((\leq_{L2\ x\ (r1\ x')})\ {}_h\trianglelefteq (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
$((\leq_{L2\ x\ (r1\ x')}) \trianglelefteq_h (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
if $x\ {}_{L1}\lessapprox x'$ **for** $x\ x'$
using $\langle x\ {}_{L1}\lessapprox x' \rangle$ **by** *blast+*
with *assms* **show** *?thesis*
by (*intro left-rel-right-iff-left-right-relI*
left-right-rel-if-left-rel-right-ge-left2-assmI
left-rel-right-if-left-right-rel-le-right2-assmI
half-galois-prop-left2-if-half-galois-prop-left2-if-left-GaloisI
half-galois-prop-right2-if-half-galois-prop-right2-if-left-GaloisI)
auto
qed

lemma *left-rel-right-iff-left-right-rel-if-galois-prop-le-assms-leftI*:
assumes *galois-conn1*: $((\leq_{L1}) \dashv (\leq_{R1}))$ *l1 r1*
and *refl-L1*: *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
and *antimono-L2*:
$((x1\ x2 :: (\leq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid (x2 \leq_{L1} x3 \wedge x4 \leq_{L1} \eta_1\ x3)) \Rrightarrow (\geq))$
L2
shows $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x2\ x2}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ (\eta_1\ x2)}) \leq (\leq_{L2\ x1\ x2})$
proof −
fix *x1 x2* **assume** $x1 \leq_{L1} x2$
with *galois-conn1 refl-L1* **have** $x1 \leq_{L1} x1\ x2 \leq_{L1} \eta_1\ x2$

by (*blast intro*:
t1.rel-unit-if-left-rel-if-half-galois-prop-right-if-mono-wrt-rel)+
moreover with *refl-L1* **have** $x2 \leq_{L1} x2\ \eta_1\ x2 \leq_{L1} \eta_1\ x2$ **by** *auto*
moreover note *dep-mono-wrt-relD*[*OF antimono-L2* ‹$x1 \leq_{L1} x2$›]
and *dep-mono-wrt-relD*[*OF antimono-L2* ‹$x1 \leq_{L1} x1$›]
ultimately show $(\leq_{L2\ x2\ x2}) \leq (\leq_{L2\ x1\ x2})\ (\leq_{L2\ x1\ (\eta_1\ x2)}) \leq (\leq_{L2\ x1\ x2})$
using ‹$x1 \leq_{L1} x2$› **by** *auto*
qed

lemma *left-rel-right-iff-left-right-rel-if-galois-prop-le-assms-rightI*:
assumes *galois-conn1*: $((\leq_{L1}) \dashv (\leq_{R1}))$ *l1 r1*
and *refl-R1*: *reflexive-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$
and *mono-R2*:
$((x1'\ x2' :: (\leq_{R1}) \mid \varepsilon_1\ x2' \leq_{R1} x1') \Rightarrow (x3'\ x4' :: (\leq_{R1}) \mid x2' \leq_{R1} x3') \Rrightarrow (\leq))\ R2$
shows $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ (\varepsilon_1\ x1')\ x2'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ x1'\ x1'}) \leq (\leq_{R2\ x1'\ x2'})$
proof −
fix $x1'\ x2'$ **assume** $x1' \leq_{R1} x2'$
with *galois-conn1 refl-R1* **have** $x2' \leq_{R1} x2'\ \varepsilon_1\ x1' \leq_{R1} x1'$
by (*blast intro*:
t1.counit-rel-if-right-rel-if-half-galois-prop-left-if-mono-wrt-rel)+
moreover with *refl-R1* **have** $x1' \leq_{R1} x1'\ \varepsilon_1\ x1' \leq_{R1} \varepsilon_1\ x1'$ **by** *auto*
moreover note *dep-mono-wrt-relD*[*OF mono-R2* ‹$\varepsilon_1\ x1' \leq_{R1} x1'$›]
and *dep-mono-wrt-relD*[*OF mono-R2* ‹$x1' \leq_{R1} x1'$›]
ultimately show $(\leq_{R2\ (\varepsilon_1\ x1')\ x2'}) \leq (\leq_{R2\ x1'\ x2'})\ (\leq_{R2\ x1'\ x1'}) \leq (\leq_{R2\ x1'\ x2'})$
using ‹$x1' \leq_{R1} x2'$› **by** *auto*
qed

corollary *left-rel-right-iff-left-right-rel-if-monoI*:
assumes $((\leq_{L1}) \dashv (\leq_{R1}))$ *l1 r1*
and *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
and *reflexive-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$
and $\bigwedge x\ x'.\ x\ {}_{L1}\!\lessapprox x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \trianglelefteq (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid (x2 \leq_{L1} x3 \wedge x4 \leq_{L1} \eta_1\ x3)) \Rrightarrow (\geq))\ L2$
and $((x1'\ x2' :: (\leq_{R1}) \mid \varepsilon_1\ x2' \leq_{R1} x1') \Rightarrow (x3'\ x4' :: (\leq_{R1}) \mid x2' \leq_{R1} x3') \Rrightarrow (\leq))\ R2$
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow$
$((\textit{in-dom}\ (\leq_{L2\ x\ (\eta_1\ x)})) \Rrightarrow (\leq_{R2\ (l1\ x)\ (l1\ x)}))\ (l2_{(l1\ x)\ x})\ (l2_{(l1\ x)\ (\eta_1\ x)})$
and $\bigwedge x'.\ x' \leq_{R1} x' \Longrightarrow$
$((\textit{in-codom}\ (\leq_{R2\ (\varepsilon_1\ x')\ x'})) \Rrightarrow (\leq_{L2\ (r1\ x')\ (r1\ x')}))\ (r2_{(r1\ x')\ (\varepsilon_1\ x')})\ (r2_{(r1\ x')\ x'})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow \textit{transitive}\ (\leq_{L2\ x1\ x2})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow \textit{transitive}\ (\leq_{R2\ x1'\ x2'})$
and $f \leq_L f$
and $g \leq_R g$
shows $f \leq_L r\ g \longleftrightarrow l\ f \leq_R g$
using *assms* **by** (*intro left-rel-right-iff-left-right-relI'*

left-rel-right-iff-left-right-rel-if-galois-prop-le-assms-leftI
left-rel-right-iff-left-right-rel-if-galois-prop-le-assms-rightI)
(*auto intro*: *reflexive-on-if-le-pred-if-reflexive-on*
in-field-if-in-dom in-field-if-in-codom)

end

Function Relator **context** *transport-Fun-Rel*
begin

corollary *left-right-rel-if-left-rel-rightI*:
assumes $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and $((\leq_{L1}) \ {}_h\trianglelefteq\ (\leq_{R1}))$ *l1 r1*
and *reflexive-on* (*in-dom* $(\leq_{R1})$) $(\leq_{R1})$
and $((\leq_{L2}) \ {}_h\trianglelefteq\ (\leq_{R2}))$ *l2 r2*
and *transitive* $(\leq_{R2})$
and $g \leq_R g$
and $f \leq_L r\ g$
shows $l\ f \leq_R g$
using *assms* **by** (*intro tdfr.left-right-rel-if-left-rel-rightI*) *simp-all*

corollary *left-rel-right-if-left-right-relI*:
assumes $((\leq_{L1}) \Rightarrow (\leq_{R1}))$ *l1*
and $((\leq_{L1}) \trianglelefteq_h (\leq_{R1}))$ *l1 r1*
and *reflexive-on* (*in-codom* $(\leq_{L1})$) $(\leq_{L1})$
and $((\leq_{L2}) \trianglelefteq_h (\leq_{R2}))$ *l2 r2*
and *transitive* $(\leq_{L2})$
and $f \leq_L f$
and $l\ f \leq_R g$
shows $f \leq_L r\ g$
using *assms* **by** (*intro tdfr.left-rel-right-if-left-right-relI*) *simp-all*

corollary *left-rel-right-iff-left-right-relI*:
assumes $((\leq_{L1}) \dashv (\leq_{R1}))$ *l1 r1*
and *reflexive-on* (*in-codom* $(\leq_{L1})$) $(\leq_{L1})$
and *reflexive-on* (*in-dom* $(\leq_{R1})$) $(\leq_{R1})$
and $((\leq_{L2}) \trianglelefteq (\leq_{R2}))$ *l2 r2*
and *transitive* $(\leq_{L2})$
and *transitive* $(\leq_{R2})$
and $f \leq_L f$
and $g \leq_R g$
shows $f \leq_L r\ g \longleftrightarrow l\ f \leq_R g$
using *assms* **by** (*intro tdfr.left-rel-right-iff-left-right-relI*) *auto*

end

Monotone Dependent Function Relator **context** *transport-Mono-Dep-Fun-Rel*
begin

lemma *half-galois-prop-left-left-rightI*:
assumes (*tdfr.L* ⇒ *tdfr.R*) *l*
and ((≤$_{R1}$) ⇒ (≤$_{L1}$)) *r1*
and ((≤$_{L1}$) $_h\trianglelefteq$ (≤$_{R1}$)) *l1 r1*
and *reflexive-on* (*in-dom* (≤$_{R1}$)) (≤$_{R1}$)
and ⋀*x′*. *x′* ≤$_{R1}$ *x′* ⟹
((≤$_{L2}$ (*r1 x′*) (*r1 x′*)) $_h\trianglelefteq$ (≤$_{R2}$ (*ε*$_1$ *x′*) *x′*)) (*l2* *x′* (*r1 x′*)) (*r2*(*r1 x′*) *x′*)
and ⋀*x′*. *x′* ≤$_{R1}$ *x′* ⟹ (≤$_{R2}$ (*ε*$_1$ *x′*) *x′*) ≤ (≤$_{R2}$ *x′ x′*)
and ⋀*x1′ x2′*. *x1′* ≤$_{R1}$ *x2′* ⟹ (≤$_{R2}$ *x1′ x1′*) ≤ (≤$_{R2}$ *x1′ x2′*)
and ⋀*x′ y′*. *x′* ≤$_{R1}$ *x′* ⟹ *in-codom* (≤$_{R2}$ (*ε*$_1$ *x′*) *x′*) *y′* ⟹
(≥$_{L2}$ (*r1 x′*) (*r1 x′*)) (*r2*(*r1 x′*) (*ε*$_1$ *x′*) *y′*) ≤ (≥$_{L2}$ (*r1 x′*) (*r1 x′*)) (*r2*(*r1 x′*) *x′*
y′)
and ⋀*x1′ x2′*. *x1′* ≤$_{R1}$ *x2′* ⟹ *transitive* (≤$_{R2}$ *x1′ x2′*)
shows ((≤$_L$) $_h\trianglelefteq$ (≤$_R$)) *l r*
unfolding *left-rel-eq-tdfr-left-Refl-Rel right-rel-eq-tdfr-right-Refl-Rel* **using** *assms*
by (*intro*
half-galois-prop-leftI[*unfolded left-rel-eq-tdfr-left-Refl-Rel right-rel-eq-tdfr-right-Refl-Rel*]
Refl-Rel-app-leftI[**where** *?f*=*l*]
tdfr.left-right-rel-if-left-rel-rightI)
(*auto elim*!: *galois-rel.left-GaloisE*)

lemma *half-galois-prop-right-left-rightI*:
assumes (*tdfr.R* ⇒ *tdfr.L*) *r*
and ((≤$_{L1}$) ⇒ (≤$_{R1}$)) *l1*
and ((≤$_{L1}$) ⊴$_h$ (≤$_{R1}$)) *l1 r1*
and *reflexive-on* (*in-codom* (≤$_{L1}$)) (≤$_{L1}$)
and ⋀*x*. *x* ≤$_{L1}$ *x* ⟹ ((≤$_{L2}$ *x* (*η*$_1$ *x*)) ⊴$_h$ (≤$_{R2}$ (*l1 x*) (*l1 x*))) (*l2*(*l1 x*) *x*) (*r2*$_x$ (*l1 x*))
and ⋀*x1 x2*. *x1* ≤$_{L1}$ *x2* ⟹ (≤$_{L2}$ *x2 x2*) ≤ (≤$_{L2}$ *x1 x2*)
and ⋀*x*. *x* ≤$_{L1}$ *x* ⟹ (≤$_{L2}$ *x* (*η*$_1$ *x*)) ≤ (≤$_{L2}$ *x x*)
and ⋀*x y*. *x* ≤$_{L1}$ *x* ⟹ *in-dom* (≤$_{L2}$ *x* (*η*$_1$ *x*)) *y* ⟹
(≤$_{R2}$ (*l1 x*) (*l1 x*)) (*l2*(*l1 x*) (*η*$_1$ *x*) *y*) ≤ (≤$_{R2}$ (*l1 x*) (*l1 x*)) (*l2*(*l1 x*) *x y*)
and ⋀*x1 x2*. *x1* ≤$_{L1}$ *x2* ⟹ *transitive* (≤$_{L2}$ *x1 x2*)
shows ((≤$_L$) ⊴$_h$ (≤$_R$)) *l r*
unfolding *left-rel-eq-tdfr-left-Refl-Rel right-rel-eq-tdfr-right-Refl-Rel* **using** *assms*
by (*intro*
half-galois-prop-rightI[*unfolded left-rel-eq-tdfr-left-Refl-Rel right-rel-eq-tdfr-right-Refl-Rel*]
Refl-Rel-app-rightI[**where** *?f*=*r*]
tdfr.left-rel-right-if-left-right-relI)
(*auto elim*!: *galois-rel.left-GaloisE in-codomE Refl-RelE intro*!: *in-fieldI*)

corollary *galois-prop-left-rightI*:
assumes (*tdfr.L* ⇒ *tdfr.R*) *l* **and** (*tdfr.R* ⇒ *tdfr.L*) *r*
and ((≤$_{L1}$) ⊣ (≤$_{R1}$)) *l1 r1*
and *reflexive-on* (*in-codom* (≤$_{L1}$)) (≤$_{L1}$)
and *reflexive-on* (*in-dom* (≤$_{R1}$)) (≤$_{R1}$)
and ⋀*x′*. *x′* ≤$_{R1}$ *x′* ⟹
((≤$_{L2}$ (*r1 x′*) (*r1 x′*)) $_h\trianglelefteq$ (≤$_{R2}$ (*ε*$_1$ *x′*) *x′*)) (*l2* *x′* (*r1 x′*)) (*r2*(*r1 x′*) *x′*)
and ⋀*x*. *x* ≤$_{L1}$ *x* ⟹ ((≤$_{L2}$ *x* (*η*$_1$ *x*)) ⊴$_h$ (≤$_{R2}$ (*l1 x*) (*l1 x*))) (*l2*(*l1 x*) *x*) (*r2*$_x$ (*l1 x*))

and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x2\ x2}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow (\leq_{L2\ x\ (\eta_1\ x)}) \leq (\leq_{L2\ x\ x})$
and $\bigwedge x'.\ x' \leq_{R1} x' \Longrightarrow (\leq_{R2\ (\varepsilon_1\ x')\ x'}) \leq (\leq_{R2\ x'\ x'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ x1'\ x1'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x\ y.\ x \leq_{L1} x \Longrightarrow$ *in-dom* $(\leq_{L2\ x\ (\eta_1\ x)})\ y \Longrightarrow$
$(\leq_{R2\ (l1\ x)\ (l1\ x)})\ (l2_{(l1\ x)\ (\eta_1\ x)}\ y) \leq (\leq_{R2\ (l1\ x)\ (l1\ x)})\ (l2_{(l1\ x)\ x}\ y)$
and $\bigwedge x'\ y'.\ x' \leq_{R1} x' \Longrightarrow$ *in-codom* $(\leq_{R2\ (\varepsilon_1\ x')\ x'})\ y' \Longrightarrow$
$(\geq_{L2\ (r1\ x')\ (r1\ x')})\ (r2_{(r1\ x')\ (\varepsilon_1\ x')}\ y') \leq (\geq_{L2\ (r1\ x')\ (r1\ x')})\ (r2_{(r1\ x')\ x'}\ y')$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *transitive* $(\leq_{L2\ x1\ x2})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow$ *transitive* $(\leq_{R2\ x1'\ x2'})$
shows $((\leq_L) \trianglelefteq (\leq_R))\ l\ r$
using *assms* **by** (*intro galois-propI half-galois-prop-left-left-rightI half-galois-prop-right-left-rightI*)
auto

corollary *galois-prop-left-rightI'*:
assumes $(tdfr.L \Rightarrow tdfr.R)\ l$ **and** $(tdfr.R \Rightarrow tdfr.L)\ r$
and $((\leq_{L1}) \dashv (\leq_{R1}))\ l1\ r1$
and *reflexive-on* $(in\text{-}codom\ (\leq_{L1}))\ (\leq_{L1})$
and *reflexive-on* $(in\text{-}dom\ (\leq_{R1}))\ (\leq_{R1})$
and *galois-prop2*: $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox x' \Longrightarrow$
$((\leq_{L2\ x\ (r1\ x')}) \trianglelefteq (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x2\ x2}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow (\leq_{L2\ x\ (\eta_1\ x)}) \leq (\leq_{L2\ x\ x})$
and $\bigwedge x'.\ x' \leq_{R1} x' \Longrightarrow (\leq_{R2\ (\varepsilon_1\ x')\ x'}) \leq (\leq_{R2\ x'\ x'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ x1'\ x1'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow$
$((in\text{-}dom\ (\leq_{L2\ x\ (\eta_1\ x)})) \Rrightarrow (\leq_{R2\ (l1\ x)\ (l1\ x)}))\ (l2_{(l1\ x)\ x})\ (l2_{(l1\ x)\ (\eta_1\ x)})$
and $\bigwedge x'.\ x' \leq_{R1} x' \Longrightarrow$
$((in\text{-}codom\ (\leq_{R2\ (\varepsilon_1\ x')\ x'})) \Rrightarrow (\leq_{L2\ (r1\ x')\ (r1\ x')}))\ (r2_{(r1\ x')\ (\varepsilon_1\ x')})\ (r2_{(r1\ x')\ x'})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *transitive* $(\leq_{L2\ x1\ x2})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow$ *transitive* $(\leq_{R2\ x1'\ x2'})$
shows $((\leq_L) \trianglelefteq (\leq_R))\ l\ r$
proof −
from *galois-prop2* **have**
$((\leq_{L2\ x\ (r1\ x')}) \ {}_h\trianglelefteq (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
$((\leq_{L2\ x\ (r1\ x')}) \trianglelefteq_h (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
if $x\ {}_{L1}\lessapprox x'$ **for** $x\ x'$
using $\langle x\ {}_{L1}\lessapprox x' \rangle$ **by** *blast+*
with *assms* **show** *?thesis* **by** (*intro galois-prop-left-rightI*
tdfr.left-right-rel-if-left-rel-right-ge-left2-assmI
tdfr.left-rel-right-if-left-right-rel-le-right2-assmI
tdfr.half-galois-prop-left2-if-half-galois-prop-left2-if-left-GaloisI
tdfr.half-galois-prop-right2-if-half-galois-prop-right2-if-left-GaloisI)
auto
qed

corollary *galois-prop-left-right-if-mono-if-galois-propI*:
assumes $(tdfr.L \Rightarrow tdfr.R)\ l$ **and** $(tdfr.R \Rightarrow tdfr.L)\ r$
and $((\leq_{L1}) \dashv (\leq_{R1}))\ l1\ r1$
and *reflexive-on* $(in\text{-}field\ (\leq_{L1}))\ (\leq_{L1})$
and *reflexive-on* $(in\text{-}field\ (\leq_{R1}))\ (\leq_{R1})$
and $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox\ x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \trianglelefteq (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid (x2 \leq_{L1} x3 \wedge x4 \leq_{L1} \eta_1\ x3)) \Rrightarrow (\geq))\ L2$
and $((x1'\ x2' :: (\leq_{R1}) \mid \varepsilon_1\ x2' \leq_{R1} x1') \Rightarrow (x3'\ x4' :: (\leq_{R1}) \mid x2' \leq_{R1} x3') \Rrightarrow (\leq))\ R2$
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow$
$((in\text{-}dom\ (\leq_{L2\ x\ (\eta_1\ x)})) \Rrightarrow (\leq_{R2\ (l1\ x)\ (l1\ x)}))\ (l2_{(l1\ x)\ x})\ (l2_{(l1\ x)\ (\eta_1\ x)})$
and $\bigwedge x'.\ x' \leq_{R1} x' \Longrightarrow$
$((in\text{-}codom\ (\leq_{R2\ (\varepsilon_1\ x')\ x'})) \Rrightarrow (\leq_{L2\ (r1\ x')\ (r1\ x')}))\ (r2_{(r1\ x')\ (\varepsilon_1\ x')})\ (r2_{(r1\ x')\ x'})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow transitive\ (\leq_{L2\ x1\ x2})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow transitive\ (\leq_{R2\ x1'\ x2'})$
shows $((\leq_L) \trianglelefteq (\leq_R))\ l\ r$
using *assms* **by** (*intro galois-prop-left-rightI'*
tdfr.left-rel-right-iff-left-right-rel-if-galois-prop-le-assms-leftI
tdfr.left-rel-right-iff-left-right-rel-if-galois-prop-le-assms-rightI)
(*auto intro*: *reflexive-on-if-le-pred-if-reflexive-on*
in-field-if-in-dom in-field-if-in-codom)

Note that we could further rewrite $[\![(tdfr.L \Rightarrow tdfr.R)\ l;\ (tdfr.R \Rightarrow tdfr.L)\ r;\ t1.galois\text{-}connection;\ reflexive\text{-}on\ (in\text{-}field\ (\leq_{L1}))\ (\leq_{L1});\ reflexive\text{-}on\ (in\text{-}field\ (\leq_{R1}))\ (\leq_{R1});\ \bigwedge x\ x'.\ x\ {}_{L1}\lessapprox\ x' \Longrightarrow t2.galois\text{-}prop\ x\ x'\ l2_{x'\ x}\ r2_{x\ x'};\ ((x1\ x2 :: (\leq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1})) \Rrightarrow (x2 \leq_{L1} x3 \wedge x4 \leq_{L1} \eta_1\ x3) \longrightarrow (\lambda x\ y.\ y \leq x))\ L2;\ ((x1'\ x2' :: (\leq_{R1})) \Rightarrow \varepsilon_1\ x2' \leq_{R1} x1' \longrightarrow ((x3'\ x4' :: (\leq_{R1})) \Rrightarrow x2' \leq_{R1} x3' \longrightarrow (\leq)))\ R2;\ \bigwedge x.\ x \leq_{L1} x \Longrightarrow (in\text{-}dom\ (\leq_{L2\ x\ \eta_1\ x}) \Rrightarrow \leq_{R2\ l1\ x\ l1\ x})\ l2_{l1\ x\ x}\ l2_{l1\ x\ \eta_1\ x};\ \bigwedge x'.\ x' \leq_{R1} x' \Longrightarrow (in\text{-}codom\ (\leq_{R2\ \varepsilon_1\ x'\ x'}) \Rrightarrow \leq_{L2\ r1\ x'\ r1\ x'})\ r2_{r1\ x'\ \varepsilon_1\ x'}\ r2_{r1\ x'\ x'};\ \bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow transitive\ (\leq_{L2\ x1\ x2});\ \bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow transitive\ (\leq_{R2\ x1'\ x2'})]\!] \Longrightarrow galois\text{-}prop\ l\ r$, as we will do later for Galois connections, by applying $[\![((\leq_{R1}) \Rightarrow (\leq_{L1}))\ r1;\ \bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow ((\leq_{L2\ r1\ x1'\ r1\ x2'}) \Rrightarrow \leq_{R2\ \varepsilon_1\ x1'\ x2'})\ l2_{x2'\ r1\ x1'};\ \bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ \varepsilon_1\ x1'\ x2'}) \leq (\leq_{R2\ x1'\ x2'});\ \bigwedge x1'\ x2'\ y.\ [\![x1' \leq_{R1} x2';\ in\text{-}dom\ (\leq_{L2\ r1\ x1'\ r1\ x2'})\ y]\!] \Longrightarrow dfro2.right\text{-}infix\ (l2_{x2'\ r1\ x1'}\ y)\ x1'\ x2' \leq dfro2.right\text{-}infix\ (l2_{x1'\ r1\ x1'}\ y)\ x1'\ x2';\ \bigwedge x1'\ x2'\ y.\ [\![x1' \leq_{R1} x2';\ in\text{-}codom\ (\leq_{L2\ r1\ x1'\ r1\ x2'})\ y]\!] \Longrightarrow (\leq_{R2\ x1'\ x2'})^{-1}\ (l2_{x2'\ r1\ x1'}\ y) \leq (\leq_{R2\ x1'\ x2'})^{-1}\ (l2_{x2'\ r1\ x2'}\ y)]\!] \Longrightarrow (tdfr.L \Rightarrow tdfr.R)\ l$ and $[\![((\leq_{L1}) \Rightarrow (\leq_{R1}))\ l1;\ \bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow ((\leq_{R2\ l1\ x1\ l1\ x2}) \Rightarrow \leq_{L2\ x1\ \eta_1\ x2})\ r2_{x1\ l1\ x2};\ \bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ \eta_1\ x2}) \leq (\leq_{L2\ x1\ x2});\ \bigwedge x1\ x2\ y'.\ [\![x1 \leq_{L1} x2;\ in\text{-}codom\ (\leq_{R2\ l1\ x1\ l1\ x2})\ y']\!] \Longrightarrow (\leq_{L2\ x1\ x2})^{-1}\ (r2_{x1\ l1\ x2}\ y') \leq (\leq_{L2\ x1\ x2})^{-1}\ (r2_{x2\ l1\ x2}\ y');\ \bigwedge x1\ x2\ y'.\ [\![x1 \leq_{L1} x2;\ in\text{-}dom\ (\leq_{R2\ l1\ x1\ l1\ x2})\ y']\!] \Longrightarrow dfro1.right\text{-}infix\ (r2_{x1\ l1\ x2}\ y')\ x1\ x2 \leq dfro1.right\text{-}infix\ (r2_{x1\ l1\ x1}\ y')\ x1\ x2]\!] \Longrightarrow (tdfr.R \Rightarrow$

tdfr.L) *r* to the first premises. However, this is not really helpful here. Moreover, the resulting theorem will not result in a useful lemma for the flipped instance of *transport-Dep-Fun-Rel* since $[\![((\leq_{R1}) \Rightarrow (\leq_{L1}))\ r1;\ \bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow ((\leq_{L2\ r1\ x1'\ r1\ x2'}) \Rightarrow \leq_{R2\ \varepsilon_1\ x1'\ x2'})\ l2_{x2'\ r1\ x1'};\ \bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ \varepsilon_1\ x1'\ x2'}) \leq (\leq_{R2\ x1'\ x2'});\ \bigwedge x1'\ x2'\ y.\ [\![x1' \leq_{R1} x2';\ in\text{-}dom\ (\leq_{L2\ r1\ x1'\ r1\ x2'})\ y]\!] \Longrightarrow dfro2.right\text{-}infix\ (l2_{x2'\ r1\ x1'}\ y)\ x1'\ x2' \leq dfro2.right\text{-}infix\ (l2_{x1'\ r1\ x1'}\ y)\ x1'\ x2';\ \bigwedge x1'\ x2'\ y.\ [\![x1' \leq_{R1} x2';\ in\text{-}codom\ (\leq_{L2\ r1\ x1'\ r1\ x2'})\ y]\!] \Longrightarrow (\leq_{R2\ x1'\ x2'})^{-1}\ (l2_{x2'\ r1\ x1'}\ y) \leq (\leq_{R2\ x1'\ x2'})^{-1}\ (l2_{x2'\ r1\ x2'}\ y)]\!] \Longrightarrow (tdfr.L \Rightarrow tdfr.R)\ l$ and $[\![((\leq_{L1}) \Rightarrow (\leq_{R1}))\ l1;\ \bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow ((\leq_{R2\ l1\ x1\ l1\ x2}) \Rightarrow \leq_{L2\ x1\ \eta_1\ x2})\ r2_{x1\ l1\ x2};\ \bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ \eta_1\ x2}) \leq (\leq_{L2\ x1\ x2});\ \bigwedge x1\ x2\ y'.\ [\![x1 \leq_{L1} x2;\ in\text{-}codom\ (\leq_{R2\ l1\ x1\ l1\ x2})\ y']\!] \Longrightarrow (\leq_{L2\ x1\ x2})^{-1}\ (r2_{x1\ l1\ x2}\ y') \leq (\leq_{L2\ x1\ x2})^{-1}\ (r2_{x2\ l1\ x2}\ y');\ \bigwedge x1\ x2\ y'.\ [\![x1 \leq_{L1} x2;\ in\text{-}dom\ (\leq_{R2\ l1\ x1\ l1\ x2})\ y']\!] \Longrightarrow dfro1.right\text{-}infix\ (r2_{x1\ l1\ x2}\ y')\ x1\ x2 \leq dfro1.right\text{-}infix\ (r2_{x1\ l1\ x1}\ y')\ x1\ x2]\!] \Longrightarrow (tdfr.R \Rightarrow tdfr.L)\ r$ are not flipped dual but only flipped-inversed dual.

end

Monotone Function Relator **context** *transport-Mono-Fun-Rel*
begin

lemma *half-galois-prop-left-left-rightI*:
assumes $((\leq_{R1}) \Rightarrow (\leq_{L1}))\ r1$
and $((\leq_{L1})\ {}_h\trianglelefteq\ (\leq_{R1}))\ l1\ r1$
and *reflexive-on* $(in\text{-}dom\ (\leq_{R1}))\ (\leq_{R1})$
and $((\leq_{L2}) \Rightarrow (\leq_{R2}))\ l2$
and $((\leq_{L2})\ {}_h\trianglelefteq\ (\leq_{R2}))\ l2\ r2$
and *transitive* $(\leq_{R2})$
shows $((\leq_L)\ {}_h\trianglelefteq\ (\leq_R))\ l\ r$
using *assms* **by** (*intro tmdfr.half-galois-prop-left-left-rightI tfr.mono-wrt-rel-leftI*)
simp-all

interpretation *flip* : *transport-Mono-Fun-Rel R1 L1 r1 l1 R2 L2 r2 l2* **.**

lemma *half-galois-prop-right-left-rightI*:
assumes $((\leq_{L1}) \Rightarrow (\leq_{R1}))\ l1$
and $((\leq_{L1})\ \trianglelefteq_h\ (\leq_{R1}))\ l1\ r1$
and *reflexive-on* $(in\text{-}codom\ (\leq_{L1}))\ (\leq_{L1})$
and $((\leq_{R2}) \Rightarrow (\leq_{L2}))\ r2$
and $((\leq_{L2})\ \trianglelefteq_h\ (\leq_{R2}))\ l2\ r2$
and *transitive* $(\leq_{L2})$
shows $((\leq_L)\ \trianglelefteq_h\ (\leq_R))\ l\ r$
using *assms* **by** (*intro tmdfr.half-galois-prop-right-left-rightI flip.tfr.mono-wrt-rel-leftI*)
simp-all

corollary *galois-prop-left-rightI*:
assumes $((\leq_{L1}) \dashv (\leq_{R1}))\ l1\ r1$
and *reflexive-on* $(in\text{-}codom\ (\leq_{L1}))\ (\leq_{L1})$

and *reflexive-on* (*in-dom* ($\leq_{R1}$)) ($\leq_{R1}$)
and (($\leq_{L2}$) $\dashv$ ($\leq_{R2}$)) *l2 r2*
and *transitive* ($\leq_{L2}$)
and *transitive* ($\leq_{R2}$)
shows (($\leq_L$) $\trianglelefteq$ ($\leq_R$)) *l r*
using *assms* **by** (*intro tmdfr.galois-propI*
half-galois-prop-left-left-rightI half-galois-prop-right-left-rightI)
auto

end

end

2.15.4 Galois Connection

theory *Transport-Functions-Galois-Connection*
imports
Transport-Functions-Galois-Property
Transport-Functions-Monotone
begin

Dependent Function Relator **context** *transport-Dep-Fun-Rel*
begin

Lemmas for Monotone Function Relator **lemma** *galois-connection-left-right-if-galois-connection-m*
assumes *galois-conn1*: (($\leq_{L1}$) $\dashv$ ($\leq_{R1}$)) *l1 r1*
and *refl-R1*: *reflexive-on* (*in-codom* ($\leq_{R1}$)) ($\leq_{R1}$)
and *R2-le1*: $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ (\varepsilon_1\ x1')\ x2'}) \leq (\leq_{R2\ x1'\ x2'})$
and *mono-l2-2*: $((x' : \textit{in-codom}\ (\leq_{R1})) \Rightarrow (x1\ x2 :: (\leq_{L1}) \mid x2\ {}_{L1}\lessapprox x') \Rightarrow$
(*in-field* ($\leq_{L2\ x1\ (r1\ x')}$)) $\Rrightarrow$ ($\leq_{R2\ (l1\ x1)\ x'}$)) *l2*
shows $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow$
((*in-codom* ($\leq_{L2\ (r1\ x1')\ (r1\ x2')}$)) $\Rrightarrow$ ($\leq_{R2\ x1'\ x2'}$)) ($l2_{x2'\ (r1\ x1')}$) ($l2_{x2'\ (r1\ x2')}$)
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow$
((*in-dom* ($\leq_{L2\ x\ (\eta_1\ x)}$)) $\Rrightarrow$ ($\leq_{R2\ (l1\ x)\ (l1\ x)}$)) ($l2_{(l1\ x)\ x}$) ($l2_{(l1\ x)\ (\eta_1\ x)}$)
proof −
show ((*in-codom* ($\leq_{L2\ (r1\ x1')\ (r1\ x2')}$)) $\Rrightarrow$ ($\leq_{R2\ x1'\ x2'}$)) ($l2_{x2'\ (r1\ x1')}$) ($l2_{x2'\ (r1\ x2')}$)
if $x1' \leq_{R1} x2'$ **for** $x1'\ x2'$
proof −
from *galois-conn1* ‹$x1' \leq_{R1} x2'$› **have** $r1\ x1' \leq_{L1} r1\ x2'\ r1\ x2'\ {}_{L1}\lessapprox x2'$
using *refl-R1* **by** (*auto intro*: *t1.right-left-Galois-if-reflexive-onI*)
with *mono-l2-2* **show** *?thesis* **using** *R2-le1* ‹$x1' \leq_{R1} x2'$› **by** *fastforce*
qed
show ((*in-dom* ($\leq_{L2\ x\ (\eta_1\ x)}$)) $\Rrightarrow$ ($\leq_{R2\ (l1\ x)\ (l1\ x)}$)) ($l2_{(l1\ x)\ x}$) ($l2_{(l1\ x)\ (\eta_1\ x)}$)
if $x \leq_{L1} x$ **for** x
proof −
from *galois-conn1* ‹$x \leq_{L1} x$› **have** $x \leq_{L1} \eta_1\ x\ \eta_1\ x\ {}_{L1}\lessapprox l1\ x$
by (*auto intro*!: *t1.right-left-Galois-if-right-relI*

```
    t1.rel-unit-if-left-rel-if-half-galois-prop-right-if-mono-wrt-rel
      [unfolded t1.unit-eq])
  with mono-l2-2 show ?thesis by fastforce
 qed
qed

lemma galois-connection-left-right-if-galois-connection-mono-assms-leftI:
 assumes galois-conn1: ((≤L1) ⊣ (≤R1)) l1 r1
 and refl-R1: reflexive-on (in-field (≤R1)) (≤R1)
 and R2-le1: ⋀x1' x2'. x1' ≤R1 x2' ⟹ (≤R2 (ε1 x1') x2') ≤ (≤R2 x1' x2')
 and mono-l2: ((x1' x2' :: (≤R1)) ⇛ (x1 x2 :: (≤L1) | x2 L1≲≈ x1') ⇛
   in-field (≤L2 x1 (r1 x2')) ⇛ (≤R2 (l1 x1) x2')) l2
 shows ⋀x1' x2'. x1' ≤R1 x2' ⟹
  ((in-dom (≤L2 (r1 x1') (r1 x2'))) ⇛ (≤R2 x1' x2')) (l2x1' (r1 x1')) (l2x2' (r1 x1'))
 and ((x' : in-codom (≤R1)) ⇛ (x1 x2 :: (≤L1) | x2 L1≲≈ x') ⇛
   (in-field (≤L2 x1 (r1 x'))) ⇛ (≤R2 (l1 x1) x')) l2
proof −
 show ((in-dom (≤L2 (r1 x1') (r1 x2'))) ⇛ (≤R2 x1' x2')) (l2x1' (r1 x1')) (l2x2' (r1 x1'))
   if x1' ≤R1 x2' for x1' x2'
 proof −
   from galois-conn1 ‹x1' ≤R1 x2'› have r1 x1' ≤L1 r1 x1' r1 x1' L1≲≈ x1'
     using refl-R1 by force+
   with mono-l2 show ?thesis using ‹x1' ≤R1 x2'› R2-le1 by (auto 11 0)
 qed
 from mono-l2 show ((x' : in-codom (≤R1)) ⇛ (x1 x2 :: (≤L1) | x2 L1≲≈ x') ⇛
   (in-field (≤L2 x1 (r1 x'))) ⇛ (≤R2 (l1 x1) x')) l2 using refl-R1 by blast
qed
```

In theory, the following lemmas can be obtained by taking the flipped, inverse interpretation of the locale; however, rewriting the assumptions is more involved than simply copying and adapting above proofs.

```
lemma galois-connection-left-right-if-galois-connection-mono-2-assms-rightI:
 assumes galois-conn1: ((≤L1) ⊣ (≤R1)) l1 r1
 and refl-L1: reflexive-on (in-dom (≤L1)) (≤L1)
 and L2-le2: ⋀x1 x2. x1 ≤L1 x2 ⟹ (≤L2 x1 (η1 x2)) ≤ (≤L2 x1 x2)
 and mono-r2-2: ((x : in-dom (≤L1)) ⇛ (x1' x2' :: (≤R1) | x L1≲≈ x1') ⇛
   (in-field (≤R2 (l1 x) x2')) ⇛ (≤L2 x (r1 x2'))) r2
 shows ⋀x1 x2. x1 ≤L1 x2 ⟹
   ((in-dom (≤R2 (l1 x1) (l1 x2))) ⇛ (≤L2 x1 x2)) (r2x1 (l1 x1)) (r2x1 (l1 x2))
 and ⋀x'. x' ≤R1 x' ⟹
  ((in-codom (≤R2 (ε1 x') x')) ⇛ (≤L2 (r1 x') (r1 x'))) (r2(r1 x') (ε1 x')) (r2(r1 x') x')
proof −
 show ((in-dom (≤R2 (l1 x1) (l1 x2))) ⇛ (≤L2 x1 x2)) (r2x1 (l1 x1)) (r2x1 (l1 x2))
   if x1 ≤L1 x2 for x1 x2
 proof −
   from galois-conn1 ‹x1 ≤L1 x2› have x1 L1≲≈ l1 x1 l1 x1 ≤R1 l1 x2
    using refl-L1 by (auto intro!: t1.left-Galois-left-if-reflexive-on-if-half-galois-prop-rightI)
```

with *mono-r2-2* **show** *?thesis* **using** *L2-le2* ‹*x1* $\leq_{L1}$ *x2*› **by** (*auto 12 0*)
qed
show ((*in-codom* ($\leq_{R2}$ (ε_1 *x'*) *x'*)) $\Rrightarrow$ ($\leq_{L2}$ (*r1 x'*) (*r1 x'*))) ($r2_{(r1\ x')\ (\varepsilon_1\ x')}$) ($r2_{(r1\ x')\ x'}$)
if $x' \leq_{R1} x'$ **for** *x'*
proof −
from *galois-conn1* ‹$x' \leq_{R1} x'$› **have** *r1 x'* $_{L1}\lessapprox$ ε_1 *x'* ε_1 *x'* $\leq_{R1}$ *x'*
by (*auto intro*!: *t1.left-Galois-left-if-left-relI*
t1.counit-rel-if-right-rel-if-half-galois-prop-left-if-mono-wrt-rel
[*unfolded t1.counit-eq*])
with *mono-r2-2* **show** *?thesis* **by** *fastforce*
qed
qed

lemma *galois-connection-left-right-if-galois-connection-mono-assms-rightI*:
assumes *galois-conn1*: (($\leq_{L1}$) ⊣ ($\leq_{R1}$)) *l1 r1*
and *refl-L1*: *reflexive-on* (*in-field* ($\leq_{L1}$)) ($\leq_{L1}$)
and *L2-le2*: $\bigwedge$*x1 x2*. *x1* $\leq_{L1}$ *x2* $\Longrightarrow$ ($\leq_{L2\ x1\ (\eta_1\ x2)}$) $\leq$ ($\leq_{L2\ x1\ x2}$)
and *mono-r2*: ((*x1 x2* :: ($\leq_{L1}$)) ⇒ (*x1'* *x2'* :: ($\leq_{R1}$) | *x2* $_{L1}\lessapprox$ *x1'*) ⇛
in-field ($\leq_{R2\ (l1\ x1)\ x2'}$) ⇛ ($\leq_{L2\ x1\ (r1\ x2')}$)) *r2*
shows $\bigwedge$*x1 x2*. *x1* $\leq_{L1}$ *x2* $\Longrightarrow$
((*in-codom* ($\leq_{R2\ (l1\ x1)\ (l1\ x2)}$)) ⇛ ($\leq_{L2\ x1\ x2}$)) ($r2_{x1\ (l1\ x2)}$) ($r2_{x2\ (l1\ x2)}$)
and ((*x* : *in-dom* ($\leq_{L1}$)) ⇒ (*x1'* *x2'* :: ($\leq_{R1}$) | *x* $_{L1}\lessapprox$ *x1'*) ⇒
(*in-field* ($\leq_{R2\ (l1\ x)\ x2'}$)) ⇛ ($\leq_{L2\ x\ (r1\ x2')}$)) *r2*
proof −
show ((*in-codom* ($\leq_{R2\ (l1\ x1)\ (l1\ x2)}$)) ⇛ ($\leq_{L2\ x1\ x2}$)) ($r2_{x1\ (l1\ x2)}$) ($r2_{x2\ (l1\ x2)}$)
if *x1* $\leq_{L1}$ *x2* **for** *x1 x2*
proof −
from *galois-conn1* ‹*x1* $\leq_{L1}$ *x2*› **have** *x2* $_{L1}\lessapprox$ *l1 x2 l1 x2* $\leq_{R1}$ *l1 x2*
using *refl-L1* **by** (*blast intro*: *t1.left-Galois-left-if-reflexive-on-if-half-galois-prop-rightI*)+
with *mono-r2* **show** *?thesis* **using** ‹*x1* $\leq_{L1}$ *x2*› *L2-le2* **by** *fastforce*
qed
from *mono-r2* **show** ((*x* : *in-dom* ($\leq_{L1}$)) ⇒ (*x1'* *x2'* :: ($\leq_{R1}$) | *x* $_{L1}\lessapprox$ *x1'*) ⇒
(*in-field* ($\leq_{R2\ (l1\ x)\ x2'}$)) ⇛ ($\leq_{L2\ x\ (r1\ x2')}$)) *r2* **using** *refl-L1* **by** *blast*
qed

end

Monotone Dependent Function Relator **context** *transport-Mono-Dep-Fun-Rel* **begin**

interpretation *flip* : *transport-Mono-Dep-Fun-Rel R1 L1 r1 l1 R2 L2 r2 l2* **.**

lemma *galois-connection-left-rightI*:
assumes (*tdfr.L* ⇒ *tdfr.R*) *l* **and** (*tdfr.R* ⇒ *tdfr.L*) *r*
and (($\leq_{L1}$) ⊣ ($\leq_{R1}$)) *l1 r1*
and *reflexive-on* (*in-codom* ($\leq_{L1}$)) ($\leq_{L1}$)
and *reflexive-on* (*in-dom* ($\leq_{R1}$)) ($\leq_{R1}$)

and $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox\ x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \trianglelefteq (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x2\ x2}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow (\leq_{L2\ x\ (\eta_1\ x)}) \leq (\leq_{L2\ x\ x})$
and $\bigwedge x'.\ x' \leq_{R1} x' \Longrightarrow (\leq_{R2\ (\varepsilon_1\ x')\ x'}) \leq (\leq_{R2\ x'\ x'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ x1'\ x1'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow$
$((in\text{-}dom\ (\leq_{L2\ x\ (\eta_1\ x)})) \Rrightarrow (\leq_{R2\ (l1\ x)\ (l1\ x)}))\ (l2_{(l1\ x)\ x})\ (l2_{(l1\ x)\ (\eta_1\ x)})$
and $\bigwedge x'.\ x' \leq_{R1} x' \Longrightarrow$
$((in\text{-}codom\ (\leq_{R2\ (\varepsilon_1\ x')\ x'})) \Rrightarrow (\leq_{L2\ (r1\ x')\ (r1\ x')}))\ (r2_{(r1\ x')\ (\varepsilon_1\ x')})\ (r2_{(r1\ x')\ x'})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow transitive\ (\leq_{L2\ x1\ x2})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow transitive\ (\leq_{R2\ x1'\ x2'})$
shows $((\leq_L) \dashv (\leq_R))\ l\ r$
using *assms*
by (*intro galois-connectionI galois-prop-left-rightI' mono-wrt-rel-leftI*
flip.mono-wrt-rel-leftI)
auto

lemma *galois-connection-left-rightI'*:
assumes $((\leq_{L1}) \dashv (\leq_{R1}))\ l1\ r1$
and *reflexive-on* $(in\text{-}codom\ (\leq_{L1}))\ (\leq_{L1})$
and *reflexive-on* $(in\text{-}dom\ (\leq_{R1}))\ (\leq_{R1})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow$
$(\leq_{L2\ (r1\ x1')\ (r1\ x2')}) \Rightarrow (\leq_{R2\ (\varepsilon_1\ x1')\ x2'}))\ (l2_{x2'\ (r1\ x1')})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow ((\leq_{R2\ (l1\ x1)\ (l1\ x2)}) \Rightarrow (\leq_{L2\ x1\ (\eta_1\ x2)}))\ (r2_{x1\ (l1\ x2)})$
and $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox\ x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \trianglelefteq (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x2\ x2}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ (\eta_1\ x2)}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ (\varepsilon_1\ x1')\ x2'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ x1'\ x1'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow$
$((in\text{-}dom\ (\leq_{L2\ (r1\ x1')\ (r1\ x2')})) \Rrightarrow (\leq_{R2\ x1'\ x2'}))\ (l2_{x1'\ (r1\ x1')})\ (l2_{x2'\ (r1\ x1')})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow$
$((in\text{-}codom\ (\leq_{L2\ (r1\ x1')\ (r1\ x2')})) \Rrightarrow (\leq_{R2\ x1'\ x2'}))\ (l2_{x2'\ (r1\ x1')})\ (l2_{x2'\ (r1\ x2')})$
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow$
$((in\text{-}dom\ (\leq_{L2\ x\ (\eta_1\ x)})) \Rrightarrow (\leq_{R2\ (l1\ x)\ (l1\ x)}))\ (l2_{(l1\ x)\ x})\ (l2_{(l1\ x)\ (\eta_1\ x)})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$
$((in\text{-}codom\ (\leq_{R2\ (l1\ x1)\ (l1\ x2)})) \Rrightarrow (\leq_{L2\ x1\ x2}))\ (r2_{x1\ (l1\ x2)})\ (r2_{x2\ (l1\ x2)})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$
$((in\text{-}dom\ (\leq_{R2\ (l1\ x1)\ (l1\ x2)})) \Rrightarrow (\leq_{L2\ x1\ x2}))\ (r2_{x1\ (l1\ x1)})\ (r2_{x1\ (l1\ x2)})$
and $\bigwedge x'.\ x' \leq_{R1} x' \Longrightarrow$
$((in\text{-}codom\ (\leq_{R2\ (\varepsilon_1\ x')\ x'})) \Rrightarrow (\leq_{L2\ (r1\ x')\ (r1\ x')}))\ (r2_{(r1\ x')\ (\varepsilon_1\ x')})\ (r2_{(r1\ x')\ x'})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow transitive\ (\leq_{L2\ x1\ x2})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow transitive\ (\leq_{R2\ x1'\ x2'})$
shows $((\leq_L) \dashv (\leq_R))\ l\ r$
using *assms*
by (*intro galois-connection-left-rightI tdfr.mono-wrt-rel-left-if-transitiveI*

tdfr.mono-wrt-rel-right-if-transitiveI)
auto

lemma *galois-connection-left-right-if-galois-connectionI*:
assumes $((\leq_{L1}) \dashv (\leq_{R1}))$ *l1 r1*
and *reflexive-on* (*in-codom* $(\leq_{L1})$) $(\leq_{L1})$
and *reflexive-on* (*in-dom* $(\leq_{R1})$) $(\leq_{R1})$
and $\bigwedge x\ x'.\ x\ _{L1}\lessapprox x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \dashv (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x2\ x2}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ (\eta_1\ x2)}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ (\varepsilon_1\ x1')\ x2'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ x1'\ x1'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow$
$((in\text{-}dom\ (\leq_{L2\ (r1\ x1')\ (r1\ x2')}))) \Rrightarrow (\leq_{R2\ x1'\ x2'}))\ (l2_{x1'\ (r1\ x1')})\ (l2_{x2'\ (r1\ x1')})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow$
$((in\text{-}codom\ (\leq_{L2\ (r1\ x1')\ (r1\ x2')}))) \Rrightarrow (\leq_{R2\ x1'\ x2'}))\ (l2_{x2'\ (r1\ x1')})\ (l2_{x2'\ (r1\ x2')})$
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow$
$((in\text{-}dom\ (\leq_{L2\ x\ (\eta_1\ x)}))) \Rrightarrow (\leq_{R2\ (l1\ x)\ (l1\ x)}))\ (l2_{(l1\ x)\ x})\ (l2_{(l1\ x)\ (\eta_1\ x)})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$
$((in\text{-}codom\ (\leq_{R2\ (l1\ x1)\ (l1\ x2)}))) \Rrightarrow (\leq_{L2\ x1\ x2}))\ (r2_{x1\ (l1\ x2)})\ (r2_{x2\ (l1\ x2)})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$
$((in\text{-}dom\ (\leq_{R2\ (l1\ x1)\ (l1\ x2)}))) \Rrightarrow (\leq_{L2\ x1\ x2}))\ (r2_{x1\ (l1\ x1)})\ (r2_{x1\ (l1\ x2)})$
and $\bigwedge x'.\ x' \leq_{R1} x' \Longrightarrow$
$((in\text{-}codom\ (\leq_{R2\ (\varepsilon_1\ x')\ x'}))) \Rrightarrow (\leq_{L2\ (r1\ x')\ (r1\ x')}))\ (r2_{(r1\ x')\ (\varepsilon_1\ x')})\ (r2_{(r1\ x')\ x'})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *transitive* $(\leq_{L2\ x1\ x2})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow$ *transitive* $(\leq_{R2\ x1'\ x2'})$
shows $((\leq_L) \dashv (\leq_R))\ l\ r$
using *assms*
by (*intro galois-connection-left-rightI'*
tdfr.mono-wrt-rel-left2-if-mono-wrt-rel-left2-if-left-GaloisI
tdfr.mono-wrt-rel-right2-if-mono-wrt-rel-right2-if-left-GaloisI)
(*auto 7 0*)

corollary *galois-connection-left-right-if-galois-connectionI'*:
assumes $((\leq_{L1}) \dashv (\leq_{R1}))$ *l1 r1*
and *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
and *reflexive-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$
and $\bigwedge x\ x'.\ x\ _{L1}\lessapprox x' \Longrightarrow$
$((\leq_{L2\ x\ (r1\ x')}) \dashv (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x2\ x2}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ (\eta_1\ x2)}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ (\varepsilon_1\ x1')\ x2'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ x1'\ x1'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow$
$((in\text{-}dom\ (\leq_{L2\ (r1\ x1')\ (r1\ x2')}))) \Rrightarrow (\leq_{R2\ x1'\ x2'}))\ (l2_{x1'\ (r1\ x1')})\ (l2_{x2'\ (r1\ x1')})$
and $((x' : in\text{-}codom\ (\leq_{R1})) \Rightarrow (x1\ x2 :: (\leq_{L1}) \mid x2\ _{L1}\lessapprox x') \Rightarrow$
$(in\text{-}field\ (\leq_{L2\ x1\ (r1\ x')}))) \Rrightarrow (\leq_{R2\ (l1\ x1)\ x'}))\ l2$

and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$
$((\mathit{in\text{-}codom}\ (\leq_{R2\ (l1\ x1)\ (l1\ x2)})) \Rrightarrow (\leq_{L2\ x1\ x2})) \ (r2_{x1\ (l1\ x2)})\ (r2_{x2\ (l1\ x2)})$
and $((x : \mathit{in\text{-}dom}\ (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x\ _{L1}\lessapprox x1') \Rightarrow$
$(\mathit{in\text{-}field}\ (\leq_{R2\ (l1\ x)\ x2'})) \Rrightarrow (\leq_{L2\ x\ (r1\ x2')}))\ r2$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow \mathit{transitive}\ (\leq_{L2\ x1\ x2})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow \mathit{transitive}\ (\leq_{R2\ x1'\ x2'})$
shows $((\leq_L) \dashv (\leq_R))\ l\ r$
using *assms* **by** (*intro galois-connection-left-right-if-galois-connectionI*
tdfr.galois-connection-left-right-if-galois-connection-mono-2-assms-leftI
tdfr.galois-connection-left-right-if-galois-connection-mono-2-assms-rightI)
(*auto intro*: *reflexive-on-if-le-pred-if-reflexive-on in-field-if-in-dom in-field-if-in-codom*)

corollary *galois-connection-left-right-if-mono-if-galois-connectionI*:
assumes $((\leq_{L1}) \dashv (\leq_{R1}))\ l1\ r1$
and *reflexive-on* $(\mathit{in\text{-}field}\ (\leq_{L1}))\ (\leq_{L1})$
and *reflexive-on* $(\mathit{in\text{-}field}\ (\leq_{R1}))\ (\leq_{R1})$
and $\bigwedge x\ x'.\ x\ _{L1}\lessapprox x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \dashv (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x2\ x2}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ (\eta_1\ x2)}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ (\varepsilon_1\ x1')\ x2'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ x1'\ x1'}) \leq (\leq_{R2\ x1'\ x2'})$
and $((x1'\ x2' :: (\leq_{R1})) \Rightarrow (x1\ x2 :: (\leq_{L1}) \mid x2\ _{L1}\lessapprox x1') \Rrightarrow$
$\mathit{in\text{-}field}\ (\leq_{L2\ x1\ (r1\ x2')}) \Rrightarrow (\leq_{R2\ (l1\ x1)\ x2'}))\ l2$
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ _{L1}\lessapprox x1') \Rrightarrow$
$\mathit{in\text{-}field}\ (\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))\ r2$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow \mathit{transitive}\ (\leq_{L2\ x1\ x2})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow \mathit{transitive}\ (\leq_{R2\ x1'\ x2'})$
shows $((\leq_L) \dashv (\leq_R))\ l\ r$
using *assms* **by** (*intro galois-connection-left-right-if-galois-connectionI'*
tdfr.galois-connection-left-right-if-galois-connection-mono-assms-leftI
tdfr.galois-connection-left-right-if-galois-connection-mono-assms-rightI)
auto

corollary *galois-connection-left-right-if-mono-if-galois-connectionI'*:
assumes $((\leq_{L1}) \dashv (\leq_{R1}))\ l1\ r1$
and *reflexive-on* $(\mathit{in\text{-}field}\ (\leq_{L1}))\ (\leq_{L1})$
and *reflexive-on* $(\mathit{in\text{-}field}\ (\leq_{R1}))\ (\leq_{R1})$
and $\bigwedge x\ x'.\ x\ _{L1}\lessapprox x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \dashv (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $((\text{-}\ x2 :: (\leq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid (x2 \leq_{L1} x3 \wedge x4 \leq_{L1} \eta_1\ x3)) \Rrightarrow (\geq))$
L2
and $((x1'\ x2' :: (\leq_{R1}) \mid \varepsilon_1\ x2' \leq_{R1} x1') \Rightarrow (x3'\ \text{-} :: (\leq_{R1}) \mid x2' \leq_{R1} x3') \Rrightarrow$
$(\leq))\ R2$
and $((x1'\ x2' :: (\leq_{R1})) \Rightarrow (x1\ x2 :: (\leq_{L1}) \mid x2\ _{L1}\lessapprox x1') \Rrightarrow$
$\mathit{in\text{-}field}\ (\leq_{L2\ x1\ (r1\ x2')}) \Rrightarrow (\leq_{R2\ (l1\ x1)\ x2'}))\ l2$
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ _{L1}\lessapprox x1') \Rrightarrow$
$\mathit{in\text{-}field}\ (\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))\ r2$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow \mathit{transitive}\ (\leq_{L2\ x1\ x2})$

and $\bigwedge x1'\ x2'.\ x1' \leq_{R1}\ x2' \Longrightarrow$ *transitive* $(\leq_{R2\ x1'\ x2'})$
shows $((\leq_L) \dashv (\leq_R))\ l\ r$
using *assms* **by** (*intro galois-connection-left-right-if-mono-if-galois-connectionI*
tdfr.left-rel-right-iff-left-right-rel-if-galois-prop-le-assms-leftI
tdfr.left-rel-right-iff-left-right-rel-if-galois-prop-le-assms-rightI)
auto

end

Monotone Function Relator **context** *transport-Mono-Fun-Rel*
begin

interpretation *flip* : *transport-Mono-Fun-Rel R1 L1 r1 l1 R2 L2 r2 l2* **.**

lemma *galois-connection-left-rightI*:
assumes $((\leq_{L1}) \dashv (\leq_{R1}))$ *l1 r1*
and *reflexive-on* (*in-codom* $(\leq_{L1})$) $(\leq_{L1})$
and *reflexive-on* (*in-dom* $(\leq_{R1})$) $(\leq_{R1})$
and $((\leq_{L2}) \dashv (\leq_{R2}))$ *l2 r2*
and *transitive* $(\leq_{L2})$
and *transitive* $(\leq_{R2})$
shows $((\leq_L) \dashv (\leq_R))\ l\ r$
using *assms* **by** (*intro tmdfr.galois-connectionI galois-prop-left-rightI*
mono-wrt-rel-leftI flip.mono-wrt-rel-leftI)
auto

end

end

2.15.5 Basic Order Properties

theory *Transport-Functions-Order-Base*
imports
Transport-Functions-Base
begin

Dependent Function Relator **context** *hom-Dep-Fun-Rel-orders*
begin

lemma *reflexive-on-in-domI*:
assumes *refl-L*: *reflexive-on* (*in-codom* $(\leq_L)$) $(\leq_L)$
and *R-le-R-if-L*: $\bigwedge x1\ x2.\ x1 \leq_L x2 \Longrightarrow (\leq_{R\ x2\ x2}) \leq (\leq_{R\ x1\ x2})$
and *pequiv-R*: $\bigwedge x1\ x2.\ x1 \leq_L x2 \Longrightarrow$ *partial-equivalence-rel* $(\leq_{R\ x1\ x2})$
shows *reflexive-on* (*in-dom DFR*) *DFR*
proof (*intro reflexive-onI Dep-Fun-Rel-relI*)
fix *f x1 x2*
assume *in-dom DFR f*

then obtain *g* **where** *DFR f g* **by** *auto*
moreover assume *x1* $\leq_L$ *x2*
moreover with *refl-L* **have** *x2* $\leq_L$ *x2* **by** *blast*
ultimately have *f x1* $\leq_{R\ x1\ x2}$ *g x2 f x2* $\leq_{R\ x1\ x2}$ *g x2*
using *R-le-R-if-L* **by** *auto*
moreover with *pequiv-R* ‹*x1* $\leq_L$ *x2*› **have** *g x2* $\leq_{R\ x1\ x2}$ *f x2*
by (*blast dest*: *symmetricD*)
ultimately show *f x1* $\leq_{R\ x1\ x2}$ *f x2* **using** *pequiv-R* ‹*x1* $\leq_L$ *x2*› **by** *blast*
qed

lemma *reflexive-on-in-codomI*:
assumes *refl-L*: *reflexive-on* (*in-dom* ($\leq_L$)) ($\leq_L$)
and *R-le-R-if-L*: $\bigwedge$*x1 x2*. *x1* $\leq_L$ *x2* $\Longrightarrow$ ($\leq_{R\ x1\ x1}$) $\leq$ ($\leq_{R\ x1\ x2}$)
and *pequiv-R*: $\bigwedge$*x1 x2*. *x1* $\leq_L$ *x2* $\Longrightarrow$ *partial-equivalence-rel* ($\leq_{R\ x1\ x2}$)
shows *reflexive-on* (*in-codom DFR*) *DFR*
proof (*intro reflexive-onI Dep-Fun-Rel-relI*)
fix *f x1 x2*
assume *in-codom DFR f*
then obtain *g* **where** *DFR g f* **by** *auto*
moreover assume *x1* $\leq_L$ *x2*
moreover with *refl-L* **have** *x1* $\leq_L$ *x1* **by** *blast*
ultimately have *g x1* $\leq_{R\ x1\ x2}$ *f x2 g x1* $\leq_{R\ x1\ x2}$ *f x1*
using *R-le-R-if-L* **by** *auto*
moreover with *pequiv-R* ‹*x1* $\leq_L$ *x2*› **have** *f x1* $\leq_{R\ x1\ x2}$ *g x1*
by (*blast dest*: *symmetricD*)
ultimately show *f x1* $\leq_{R\ x1\ x2}$ *f x2* **using** *pequiv-R* ‹*x1* $\leq_L$ *x2*› **by** *blast*
qed

corollary *reflexive-on-in-fieldI*:
assumes *reflexive-on* (*in-field* ($\leq_L$)) ($\leq_L$)
and $\bigwedge$*x1 x2*. *x1* $\leq_L$ *x2* $\Longrightarrow$ ($\leq_{R\ x2\ x2}$) $\leq$ ($\leq_{R\ x1\ x2}$)
and $\bigwedge$*x1 x2*. *x1* $\leq_L$ *x2* $\Longrightarrow$ ($\leq_{R\ x1\ x1}$) $\leq$ ($\leq_{R\ x1\ x2}$)
and $\bigwedge$*x1 x2*. *x1* $\leq_L$ *x2* $\Longrightarrow$ *partial-equivalence-rel* ($\leq_{R\ x1\ x2}$)
shows *reflexive-on* (*in-field DFR*) *DFR*
proof −
from *assms* **have** *reflexive-on* (*in-dom DFR*) *DFR*
by (*intro reflexive-on-in-domI*)
(*auto intro*: *reflexive-on-if-le-pred-if-reflexive-on in-field-if-in-codom*)
moreover from *assms* **have** *reflexive-on* (*in-codom DFR*) *DFR*
by (*intro reflexive-on-in-codomI*)
(*auto intro*: *reflexive-on-if-le-pred-if-reflexive-on in-field-if-in-dom*)
ultimately show *?thesis* **by** (*auto iff*: *in-field-iff-in-dom-or-in-codom*)
qed

lemma *transitiveI*:
assumes *refl-L*: *reflexive-on* (*in-dom* ($\leq_L$)) ($\leq_L$)
and *R-le-R-if-L*: $\bigwedge$*x1 x2*. *x1* $\leq_L$ *x2* $\Longrightarrow$ ($\leq_{R\ x1\ x1}$) $\leq$ ($\leq_{R\ x1\ x2}$)
and *trans*: $\bigwedge$*x1 x2*. *x1* $\leq_L$ *x2* $\Longrightarrow$ *transitive* ($\leq_{R\ x1\ x2}$)
shows *transitive DFR*

proof (*intro transitiveI Dep-Fun-Rel-relI*)
fix *f1 f2 f3 x1 x2* **assume** *x1* $\leq_L$ *x2*
with *refl-L* **have** *x1* $\leq_L$ *x1* **by** *blast*
moreover assume *DFR f1 f2*
ultimately have *f1 x1* $\leq_{R\ x1\ x1}$ *f2 x1* **by** *blast*
with *R-le-R-if-L* **have** *f1 x1* $\leq_{R\ x1\ x2}$ *f2 x1* **using** ‹*x1* $\leq_L$ *x2*› **by** *blast*
assume *DFR f2 f3*
with ‹*x1* $\leq_L$ *x2*› **have** *f2 x1* $\leq_{R\ x1\ x2}$ *f3 x2* **by** *blast*
with ‹*f1 x1* $\leq_{R\ x1\ x2}$ *f2 x1*› **show** *f1 x1* $\leq_{R\ x1\ x2}$ *f3 x2*
using *trans* ‹*x1* $\leq_L$ *x2*› **by** *blast*
qed

lemma *transitiveI'*:
assumes *refl-L*: *reflexive-on* (*in-codom* ($\leq_L$)) ($\leq_L$)
and *R-le-R-if-L*: $\bigwedge$*x1 x2*. *x1* $\leq_L$ *x2* $\Longrightarrow$ ($\leq_{R\ x2\ x2}$) $\leq$ ($\leq_{R\ x1\ x2}$)
and *trans*: $\bigwedge$*x1 x2*. *x1* $\leq_L$ *x2* $\Longrightarrow$ *transitive* ($\leq_{R\ x1\ x2}$)
shows *transitive DFR*
proof (*intro Binary-Relations-Transitive.transitiveI Dep-Fun-Rel-relI*)
fix *f1 f2 f3 x1 x2* **assume** *DFR f1 f2 x1* $\leq_L$ *x2*
then have *f1 x1* $\leq_{R\ x1\ x2}$ *f2 x2* **by** *blast*
from ‹*x1* $\leq_L$ *x2*› *refl-L* **have** *x2* $\leq_L$ *x2* **by** *blast*
moreover assume *DFR f2 f3*
ultimately have *f2 x2* $\leq_{R\ x2\ x2}$ *f3 x2* **by** *blast*
with *R-le-R-if-L* **have** *f2 x2* $\leq_{R\ x1\ x2}$ *f3 x2* **using** ‹*x1* $\leq_L$ *x2*› **by** *blast*
with ‹*f1 x1* $\leq_{R\ x1\ x2}$ *f2 x2*› **show** *f1 x1* $\leq_{R\ x1\ x2}$ *f3 x2*
using *trans* ‹*x1* $\leq_L$ *x2*› **by** *blast*
qed

lemma *preorder-on-in-fieldI*:
assumes *reflexive-on* (*in-field* ($\leq_L$)) ($\leq_L$)
and $\bigwedge$*x1 x2*. *x1* $\leq_L$ *x2* $\Longrightarrow$ ($\leq_{R\ x2\ x2}$) $\leq$ ($\leq_{R\ x1\ x2}$)
and $\bigwedge$*x1 x2*. *x1* $\leq_L$ *x2* $\Longrightarrow$ ($\leq_{R\ x1\ x1}$) $\leq$ ($\leq_{R\ x1\ x2}$)
and *pequiv-R*: $\bigwedge$*x1 x2*. *x1* $\leq_L$ *x2* $\Longrightarrow$ *partial-equivalence-rel* ($\leq_{R\ x1\ x2}$)
shows *preorder-on* (*in-field DFR*) *DFR*
using *assms* **by** (*intro preorder-onI reflexive-on-in-fieldI*)
(*auto intro*!: *transitiveI dest*: *pequiv-R elim*!: *partial-equivalence-relE*)

lemma *symmetricI*:
assumes *sym-L*: *symmetric* ($\leq_L$)
and *R-le-R-if-L*: $\bigwedge$*x1 x2*. *x1* $\leq_L$ *x2* $\Longrightarrow$ ($\leq_{R\ x1\ x2}$) $\leq$ ($\leq_{R\ x2\ x1}$)
and *sym-R*: $\bigwedge$*x1 x2*. *x1* $\leq_L$ *x2* $\Longrightarrow$ *symmetric* ($\leq_{R\ x1\ x2}$)
shows *symmetric DFR*
proof (*intro symmetricI Dep-Fun-Rel-relI*)
fix *f g x y* **assume** *x* $\leq_L$ *y*
with *sym-L* **have** *y* $\leq_L$ *x* **by** (*rule symmetricD*)
moreover assume *DFR f g*
ultimately have *f y* $\leq_{R\ y\ x}$ *g x* **by** *blast*
with *sym-R* ‹*y* $\leq_L$ *x*› **have** *g x* $\leq_{R\ y\ x}$ *f y* **by** (*blast dest*: *symmetricD*)
with *R-le-R-if-L* ‹*y* $\leq_L$ *x*› **show** *g x* $\leq_{R\ x\ y}$ *f y* **by** *blast*

qed

corollary *partial-equivalence-relI*:
assumes *reflexive-on* (*in-field* ($\leq_L$)) ($\leq_L$)
and *sym-L*: *symmetric* ($\leq_L$)
and *mono-R*: ((*x1 x2* :: ($\leq_L$)) $\Rightarrow$ (*x3 x4* :: ($\leq_L$) | *x1* $\leq_L$ *x3*) $\Rrightarrow$ ($\leq$)) *R*
and $\bigwedge$*x1 x2*. *x1* $\leq_L$ *x2* $\Longrightarrow$ *partial-equivalence-rel* ($\leq_R$ *x1 x2*)
shows *partial-equivalence-rel DFR*
proof −
have ($\leq_R$ *x1 x2*) $\leq$ ($\leq_R$ *x2 x1*) **if** *x1* $\leq_L$ *x2* **for** *x1 x2*
proof −
from *sym-L* ‹*x1* $\leq_L$ *x2*› **have** *x2* $\leq_L$ *x1* **by** (*rule symmetricD*)
with *mono-R* ‹*x1* $\leq_L$ *x2*› **show** *?thesis* **by** *blast*
qed
with *assms* **show** *?thesis*
by (*intro partial-equivalence-relI transitiveI symmetricI*)
(*blast elim*: *partial-equivalence-relE*[*OF assms*(*4*)])+
qed

end

context *transport-Dep-Fun-Rel*
begin

lemmas *reflexive-on-in-field-leftI* = *dfro1.reflexive-on-in-fieldI*
[*folded left-rel-eq-Dep-Fun-Rel*]
lemmas *transitive-leftI* = *dfro1.transitiveI*[*folded left-rel-eq-Dep-Fun-Rel*]
lemmas *transitive-leftI′* = *dfro1.transitiveI′*[*folded left-rel-eq-Dep-Fun-Rel*]
lemmas *preorder-on-in-field-leftI* = *dfro1.preorder-on-in-fieldI*
[*folded left-rel-eq-Dep-Fun-Rel*]
lemmas *symmetric-leftI* = *dfro1.symmetricI*[*folded left-rel-eq-Dep-Fun-Rel*]
lemmas *partial-equivalence-rel-leftI* = *dfro1.partial-equivalence-relI*
[*folded left-rel-eq-Dep-Fun-Rel*]

Introduction Rules for Assumptions **lemma** *transitive-left2-if-transitive-left2-if-left-GaloisI*:
assumes (($\leq_{L1}$) $\Rightarrow$ ($\leq_{R1}$)) *l1*
and (($\leq_{L1}$) $\trianglelefteq_h$ ($\leq_{R1}$)) *l1 r1*
and *L2-eq*: ($\leq_{L2\ x1\ x2}$) = ($\leq_{L2\ x1\ (\eta_1\ x2)}$)
and $\bigwedge$*x x′*. *x* $_{L1}\lessapprox$ *x′* $\Longrightarrow$ *transitive* ($\leq_{L2\ x\ (r1\ x')}$)
and *x1* $\leq_{L1}$ *x2*
shows *transitive* ($\leq_{L2\ x1\ x2}$)
by (*subst L2-eq*) (*auto intro*!: *assms t1.left-Galois-left-if-left-relI*)

lemma *symmetric-left2-if-symmetric-left2-if-left-GaloisI*:
assumes (($\leq_{L1}$) $\Rightarrow$ ($\leq_{R1}$)) *l1*
and (($\leq_{L1}$) $\trianglelefteq_h$ ($\leq_{R1}$)) *l1 r1*
and *L2-eq*: ($\leq_{L2\ x1\ x2}$) = ($\leq_{L2\ x1\ (\eta_1\ x2)}$)
and $\bigwedge$*x x′*. *x* $_{L1}\lessapprox$ *x′* $\Longrightarrow$ *symmetric* ($\leq_{L2\ x\ (r1\ x')}$)

and *x1* $\leq_{L1}$ *x2*
shows *symmetric* $(\leq_{L2\ x1\ x2})$
by (*subst L2-eq*) (*auto intro*!: *assms t1.left-Galois-left-if-left-relI*)

lemma *partial-equivalence-rel-left2-if-partial-equivalence-rel-left2-if-left-GaloisI*:
assumes $((\leq_{L1}) \Rightarrow (\leq_{R1}))$ *l1*
and $((\leq_{L1}) \trianglelefteq_h (\leq_{R1}))$ *l1 r1*
and *L2-eq*: $(\leq_{L2\ x1\ x2}) = (\leq_{L2\ x1\ (\eta_1\ x2)})$
and $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox\ x' \Longrightarrow$ *partial-equivalence-rel* $(\leq_{L2\ x\ (r1\ x')})$
and *x1* $\leq_{L1}$ *x2*
shows *partial-equivalence-rel* $(\leq_{L2\ x1\ x2})$
by (*subst L2-eq*) (*auto intro*!: *assms t1.left-Galois-left-if-left-relI*)

context
assumes *galois-prop*: $((\leq_{L1}) \trianglelefteq (\leq_{R1}))$ *l1 r1*
begin

interpretation *flip-inv* :
transport-Dep-Fun-Rel $(\geq_{R1})$ $(\geq_{L1})$ *r1 l1 flip2 R2 flip2 L2 r2 l2*
rewrites *flip-inv.t1.unit* $\equiv \varepsilon_1$
and $\bigwedge R\ x\ y.\ (flip2\ R\ x\ y) \equiv (R\ y\ x)^{-1}$
and $\bigwedge R\ S.\ R^{-1} = S^{-1} \equiv R = S$
and $\bigwedge R\ S.\ (R^{-1} \Rightarrow S^{-1}) \equiv (R \Rightarrow S)$
and $\bigwedge x\ x'.\ x'\ {}_{R1}\gtrsim\ x \equiv x\ {}_{L1}\lessapprox\ x'$
and $((\geq_{R1}) \trianglelefteq_h (\geq_{L1}))$ *r1 l1* $\equiv$ *True*
and $\bigwedge(R :: 'z \Rightarrow 'z \Rightarrow bool).$ *transitive* $R^{-1} \equiv$ *transitive R*
and $\bigwedge(R :: 'z \Rightarrow 'z \Rightarrow bool).$ *symmetric* $R^{-1} \equiv$ *symmetric R*
and $\bigwedge(R :: 'z \Rightarrow 'z \Rightarrow bool).$ *partial-equivalence-rel* $R^{-1} \equiv$ *partial-equivalence-rel R*
and $\bigwedge P.\ (True \Longrightarrow P) \equiv Trueprop\ P$
and $\bigwedge P\ Q.\ (True \Longrightarrow PROP\ P \Longrightarrow PROP\ Q) \equiv (PROP\ P \Longrightarrow True \Longrightarrow PROP\ Q)$
using *galois-prop*
by (*auto intro*!: *Eq-TrueI simp*: *t1.flip-unit-eq-counit*
galois-prop.half-galois-prop-right-rel-inv-iff-half-galois-prop-left
mono-wrt-rel-eq-dep-mono-wrt-rel
simp del: *rel-inv-iff-rel*)

lemma *transitive-right2-if-transitive-right2-if-left-GaloisI*:
assumes $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and $(\leq_{R2\ x1\ x2}) = (\leq_{R2\ (\varepsilon_1\ x1)\ x2})$
and $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox\ x' \Longrightarrow$ *transitive* $(\leq_{R2\ (l1\ x)\ x'})$
and *x1* $\leq_{R1}$ *x2*
shows *transitive* $(\leq_{R2\ x1\ x2})$
using *galois-prop assms*
by (*intro flip-inv.transitive-left2-if-transitive-left2-if-left-GaloisI*
[*simplified rel-inv-iff-rel*, *of x1*])
auto

lemma *symmetric-right2-if-symmetric-right2-if-left-GaloisI*:
assumes $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and $(\leq_{R2\ x1\ x2}) = (\leq_{R2\ (\varepsilon_1\ x1)\ x2})$
and $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox x' \Longrightarrow$ *symmetric* $(\leq_{R2\ (l1\ x)\ x'})$
and $x1 \leq_{R1} x2$
shows *symmetric* $(\leq_{R2\ x1\ x2})$
using *galois-prop assms*
by (*intro flip-inv.symmetric-left2-if-symmetric-left2-if-left-GaloisI*
[*simplified rel-inv-iff-rel*, *of x1*])
auto

lemma *partial-equivalence-rel-right2-if-partial-equivalence-rel-right2-if-left-GaloisI*:
assumes $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and $(\leq_{R2\ x1\ x2}) = (\leq_{R2\ (\varepsilon_1\ x1)\ x2})$
and $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox x' \Longrightarrow$ *partial-equivalence-rel* $(\leq_{R2\ (l1\ x)\ x'})$
and $x1 \leq_{R1} x2$
shows *partial-equivalence-rel* $(\leq_{R2\ x1\ x2})$
using *galois-prop assms*
by (*intro flip-inv.partial-equivalence-rel-left2-if-partial-equivalence-rel-left2-if-left-GaloisI*
[*simplified rel-inv-iff-rel*, *of x1*])
auto

end

lemma *transitive-left2-if-preorder-equivalenceI*:
assumes *pre-equiv1*: $((\leq_{L1}) \equiv_{pre} (\leq_{R1}))$ *l1 r1*
and $((x1\ x2 :: (\geq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid x1 \leq_{L1} x3) \Rrightarrow (\leq))$ *L2*
and $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \equiv_{pre} (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $x1 \leq_{L1} x2$
shows *transitive* $(\leq_{L2\ x1\ x2})$
proof −
from ‹$x1 \leq_{L1} x2$› *pre-equiv1* **have** $x2 \equiv_{L1} \eta_1\ x2$
by (*blast elim*: *t1.preorder-equivalence-order-equivalenceE*
intro: *bi-related-if-rel-equivalence-on*)
with *assms* **have** $(\leq_{L2\ x1\ x2}) = (\leq_{L2\ x1\ (\eta_1\ x2)})$
by (*intro left2-eq-if-bi-related-if-monoI*) *blast+*
with *assms* **show** *?thesis*
by (*intro transitive-left2-if-transitive-left2-if-left-GaloisI*[*of x1*]) *blast+*
qed

lemma *symmetric-left2-if-partial-equivalence-rel-equivalenceI*:
assumes *PER-equiv1*: $((\leq_{L1}) \equiv_{PER} (\leq_{R1}))$ *l1 r1*
and $((x1\ x2 :: (\geq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid x1 \leq_{L1} x3) \Rrightarrow (\leq))$ *L2*
and $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \equiv_{PER} (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $x1 \leq_{L1} x2$
shows *symmetric* $(\leq_{L2\ x1\ x2})$
proof −
from ‹$x1 \leq_{L1} x2$› *PER-equiv1* **have** $x2 \equiv_{L1} \eta_1\ x2$

by (*blast elim*: *t1.preorder-equivalence-order-equivalenceE*
intro: *bi-related-if-rel-equivalence-on*)
with *assms* **have** $(\leq_{L2\ x1\ x2}) = (\leq_{L2\ x1\ (\eta_1\ x2)})$
by (*intro left2-eq-if-bi-related-if-monoI*) *blast+*
with *assms* **show** *?thesis*
by (*intro symmetric-left2-if-symmetric-left2-if-left-GaloisI*[*of x1*]) *blast+*
qed

lemma *partial-equivalence-rel-left2-if-partial-equivalence-rel-equivalenceI*:
assumes *PER-equiv1*: $((\leq_{L1}) \equiv_{PER} (\leq_{R1}))$ *l1 r1*
and $((x1\ x2 :: (\geq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid x1 \leq_{L1} x3) \Rrightarrow (\leq))$ *L2*
and $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \equiv_{PER} (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $x1 \leq_{L1} x2$
shows *partial-equivalence-rel* $(\leq_{L2\ x1\ x2})$
proof −
from $\langle x1 \leq_{L1} x2 \rangle$ *PER-equiv1* **have** $x2 \equiv_{L1} \eta_1\ x2$
by (*blast elim*: *t1.preorder-equivalence-order-equivalenceE*
intro: *bi-related-if-rel-equivalence-on*)
with *assms* **have** $(\leq_{L2\ x1\ x2}) = (\leq_{L2\ x1\ (\eta_1\ x2)})$
by (*intro left2-eq-if-bi-related-if-monoI*) *blast+*
with *assms* **show** *?thesis*
by (*intro partial-equivalence-rel-left2-if-partial-equivalence-rel-left2-if-left-GaloisI*[*of x1*])
blast+
qed

interpretation *flip* : *transport-Dep-Fun-Rel R1 L1 r1 l1 R2 L2 r2 l2*
rewrites $flip.t1.counit \equiv \eta_1$ **and** $flip.t1.unit \equiv \varepsilon_1$
by (*simp-all only*: *t1.flip-counit-eq-unit t1.flip-unit-eq-counit*)

lemma *transitive-right2-if-preorder-equivalenceI*:
assumes *pre-equiv1*: $((\leq_{L1}) \equiv_{pre} (\leq_{R1}))$ *l1 r1*
and $((x1'\ x2' :: (\geq_{R1})) \Rightarrow (x3'\ x4' :: (\leq_{R1}) \mid x1' \leq_{R1} x3') \Rrightarrow (\leq))$ *R2*
and $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \equiv_{pre} (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $x1' \leq_{R1} x2'$
shows *transitive* $(\leq_{R2\ x1'\ x2'})$
proof −
from $\langle x1' \leq_{R1} x2' \rangle$ *pre-equiv1* **have** $x1' \equiv_{R1} \varepsilon_1\ x1'$
by (*blast elim*: *t1.preorder-equivalence-order-equivalenceE*
intro: *bi-related-if-rel-equivalence-on*)
with *assms* **have** $(\leq_{R2\ x1'\ x2'}) = (\leq_{R2\ (\varepsilon_1\ x1')\ x2'})$
by (*intro flip.left2-eq-if-bi-related-if-monoI*) *blast+*
with *assms* **show** *?thesis*
by (*intro transitive-right2-if-transitive-right2-if-left-GaloisI*[*of x1'*]) *blast+*
qed

lemma *symmetric-right2-if-partial-equivalence-rel-equivalenceI*:
assumes *PER-equiv1*: $((\leq_{L1}) \equiv_{PER} (\leq_{R1}))$ *l1 r1*

and $((x1'\ x2' :: (\geq_{R1})) \Rightarrow (x3'\ x4' :: (\leq_{R1}) \mid x1' \leq_{R1}\ x3') \Rrightarrow (\leq))\ R2$
and $\bigwedge x\ x'.\ x\ _{L1}\lessapprox x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \equiv_{PER} (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $x1' \leq_{R1} x2'$
shows *symmetric* $(\leq_{R2\ x1'\ x2'})$
proof −
from $\langle x1' \leq_{R1} x2' \rangle$ *PER-equiv1* **have** $x1' \equiv_{R1} \varepsilon_1\ x1'$
by (*blast elim*: *t1.preorder-equivalence-order-equivalenceE*
intro: *bi-related-if-rel-equivalence-on*)
with *assms* **have** $(\leq_{R2\ x1'\ x2'}) = (\leq_{R2\ (\varepsilon_1\ x1')\ x2'})$
by (*intro flip.left2-eq-if-bi-related-if-monoI*) *blast*+
with *assms* **show** *?thesis*
by (*intro symmetric-right2-if-symmetric-right2-if-left-GaloisI*[*of x1'*]) *blast*+
qed

lemma *partial-equivalence-rel-right2-if-partial-equivalence-rel-equivalenceI*:
assumes *PER-equiv1*: $((\leq_{L1}) \equiv_{PER} (\leq_{R1}))$ *l1 r1*
and $((x1'\ x2' :: (\geq_{R1})) \Rightarrow (x3'\ x4' :: (\leq_{R1}) \mid x1' \leq_{R1}\ x3') \Rrightarrow (\leq))\ R2$
and $\bigwedge x\ x'.\ x\ _{L1}\lessapprox x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \equiv_{PER} (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $x1' \leq_{R1} x2'$
shows *partial-equivalence-rel* $(\leq_{R2\ x1'\ x2'})$
proof −
from $\langle x1' \leq_{R1} x2' \rangle$ *PER-equiv1* **have** $x1' \equiv_{R1} \varepsilon_1\ x1'$
by (*blast elim*: *t1.preorder-equivalence-order-equivalenceE*
intro: *bi-related-if-rel-equivalence-on*)
with *assms* **have** $(\leq_{R2\ x1'\ x2'}) = (\leq_{R2\ (\varepsilon_1\ x1')\ x2'})$
by (*intro flip.left2-eq-if-bi-related-if-monoI*) *blast*+
with *assms* **show** *?thesis*
by (*intro partial-equivalence-rel-right2-if-partial-equivalence-rel-right2-if-left-GaloisI*[*of x1'*])
blast+
qed

end

Function Relator **context** *transport-Fun-Rel*
begin

lemma *reflexive-on-in-field-leftI*:
assumes *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
and *partial-equivalence-rel* $(\leq_{L2})$
shows *reflexive-on* (*in-field* $(\leq_{L})$) $(\leq_{L})$
using *assms* **by** (*intro tdfr.reflexive-on-in-field-leftI*) *simp-all*

lemma *transitive-leftI*:
assumes *reflexive-on* (*in-dom* $(\leq_{L1})$) $(\leq_{L1})$
and *transitive* $(\leq_{L2})$
shows *transitive* $(\leq_{L})$
using *assms* **by** (*intro tdfr.transitive-leftI*) *simp-all*

lemma *transitive-leftI'*:
assumes *reflexive-on* (*in-codom* ($\leq_{L1}$)) ($\leq_{L1}$)
and *transitive* ($\leq_{L2}$)
shows *transitive* ($\leq_{L}$)
using *assms* **by** (*intro tdfr.transitive-leftI'*) *simp-all*

lemma *preorder-on-in-field-leftI*:
assumes *reflexive-on* (*in-field* ($\leq_{L1}$)) ($\leq_{L1}$)
and *partial-equivalence-rel* ($\leq_{L2}$)
shows *preorder-on* (*in-field* ($\leq_{L}$)) ($\leq_{L}$)
using *assms* **by** (*intro tdfr.preorder-on-in-field-leftI*) *simp-all*

lemma *symmetric-leftI*:
assumes *symmetric* ($\leq_{L1}$)
and *symmetric* ($\leq_{L2}$)
shows *symmetric* ($\leq_{L}$)
using *assms* **by** (*intro tdfr.symmetric-leftI*) *simp-all*

corollary *partial-equivalence-rel-leftI*:
assumes *reflexive-on* (*in-field* ($\leq_{L1}$)) ($\leq_{L1}$)
and *symmetric* ($\leq_{L1}$)
and *partial-equivalence-rel* ($\leq_{L2}$)
shows *partial-equivalence-rel* ($\leq_{L}$)
using *assms* **by** (*intro tdfr.partial-equivalence-rel-leftI*) *auto*

end

Monotone Dependent Function Relator **context** *transport-Mono-Dep-Fun-Rel*
begin

lemmas *reflexive-on-in-field-leftI* = *Refl-Rel-reflexive-on-in-field*[*of tdfr.L*,
folded left-rel-eq-tdfr-left-Refl-Rel]
lemmas *transitive-leftI* = *Refl-Rel-transitiveI*
[*of tdfr.L*, *folded left-rel-eq-tdfr-left-Refl-Rel*]
lemmas *preorder-on-in-field-leftI* = *Refl-Rel-preorder-on-in-fieldI*[*of tdfr.L*,
folded left-rel-eq-tdfr-left-Refl-Rel]
lemmas *symmetric-leftI* = *Refl-Rel-symmetricI*[*of tdfr.L*,
OF tdfr.symmetric-leftI, *folded left-rel-eq-tdfr-left-Refl-Rel*]
lemmas *partial-equivalence-rel-leftI* = *Refl-Rel-partial-equivalence-relI*[*of tdfr.L*,
OF tdfr.partial-equivalence-rel-leftI, *folded left-rel-eq-tdfr-left-Refl-Rel*]

end

Monotone Function Relator **context** *transport-Mono-Fun-Rel*
begin

lemma *symmetric-leftI*:
assumes *symmetric* ($\leq_{L1}$)

and *symmetric* $(\leq_{L2})$
shows *symmetric* $(\leq_{L})$
using *assms* **by** (*intro tmdfr.symmetric-leftI*) *auto*

lemma *partial-equivalence-rel-leftI*:
assumes *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
and *symmetric* $(\leq_{L1})$
and *partial-equivalence-rel* $(\leq_{L2})$
shows *partial-equivalence-rel* $(\leq_{L})$
using *assms* **by** (*intro tmdfr.partial-equivalence-rel-leftI*) *auto*

end

end

2.15.6 Galois Equivalence

theory *Transport-Functions-Galois-Equivalence*
imports
Transport-Functions-Galois-Connection
Transport-Functions-Order-Base
begin

Dependent Function Relator **context** *transport-Dep-Fun-Rel*
begin

Lemmas for Monotone Function Relator **lemma** *flip-half-galois-prop-left2-if-half-galois-prop-left2-i*
assumes $((\leq_{L1}) \Rightarrow (\leq_{R1}))$ *l1*
and $((\leq_{L1}) \trianglelefteq_h (\leq_{R1}))$ *l1 r1*
and *half-galois-prop-left2*: $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox x' \Longrightarrow$
$((\leq_{R2}\ (l1\ x)\ x') \ {}_h\trianglelefteq\ (\leq_{L2}\ x\ (r1\ x'))) \ (r2_{x\ x'}) \ (l2_{x'\ x})$
and $(\leq_{L2}\ (\eta_1\ x)\ x) = (\leq_{L2}\ x\ x)$
and $(\leq_{L2}\ x\ (\eta_1\ x)) = (\leq_{L2}\ x\ x)$
and $x \leq_{L1} x$
shows $((\leq_{R2}\ (l1\ x)\ (l1\ x)) \ {}_h\trianglelefteq\ (\leq_{L2}\ (\eta_1\ x)\ x)) \ (r2_{x\ (l1\ x)}) \ (l2_{(l1\ x)\ x})$
proof −
from *assms* **have** $x\ {}_{L1}\lessapprox l1\ x$ **by** (*intro t1.left-Galois-left-if-left-relI*) *auto*
with *half-galois-prop-left2*
have $((\leq_{R2}\ (l1\ x)\ (l1\ x)) \ {}_h\trianglelefteq\ (\leq_{L2}\ x\ (\eta_1\ x))) \ (r2_{x\ (l1\ x)}) \ (l2_{(l1\ x)\ x})$ **by** *auto*
with *assms* **show** *?thesis* **by** *simp*
qed

lemma *flip-half-galois-prop-right2-if-half-galois-prop-right2-if-GaloisI*:
assumes $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and *half-galois-prop-right2*: $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox x' \Longrightarrow$
$((\leq_{R2}\ (l1\ x)\ x') \trianglelefteq_h\ (\leq_{L2}\ x\ (r1\ x'))) \ (r2_{x\ x'}) \ (l2_{x'\ x})$
and $(\leq_{R2}\ (\varepsilon_1\ x')\ x') = (\leq_{R2}\ x'\ x')$

and $(\leq_{R2\ x'\ (\varepsilon_1\ x')}) = (\leq_{R2\ x'\ x'})$
and $x' \leq_{R1} x'$
shows $((\leq_{R2\ x'\ (\varepsilon_1\ x')}) \trianglelefteq_h (\leq_{L2\ (r1\ x')\ (r1\ x')})) (r2_{(r1\ x')\ x'}) (l2_{x'\ (r1\ x')})$
proof −
from *assms* **have** $r1\ x'\ _{L1}\lessapprox x'$ **by** (*intro t1.right-left-Galois-if-right-relI*) *auto*
with *half-galois-prop-right2*
have $((\leq_{R2\ (\varepsilon_1\ x')\ x'}) \trianglelefteq_h (\leq_{L2\ (r1\ x')\ (r1\ x')})) (r2_{(r1\ x')\ x'}) (l2_{x'\ (r1\ x')})$ **by** *auto*
with *assms* **show** *?thesis* **by** *simp*
qed

interpretation *flip* : *transport-Dep-Fun-Rel R1 L1 r1 l1 R2 L2 r2 l2*
rewrites *flip.t1.counit* $\equiv \eta_1$ **and** *flip.t1.unit* $\equiv \varepsilon_1$
by (*simp-all only*: *t1.flip-counit-eq-unit t1.flip-unit-eq-counit*)

lemma *galois-equivalence-if-mono-if-galois-equivalence-mono-assms-leftI*:
assumes *galois-equiv1*: $((\leq_{L1}) \equiv_G (\leq_{R1}))$ *l1 r1*
and *preorder-L1*: *preorder-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
and *mono-L2*: $((x1\ x2 :: (\geq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid x1 \leq_{L1} x3) \Rrightarrow (\leq))\ L2$
shows $((x1\ x2 :: (\leq_{L1}) \mid \eta_1\ x2 \leq_{L1} x1) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid x2 \leq_{L1} x3) \Rrightarrow (\leq))\ L2$ (**is** *?goal1*)
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid (x2 \leq_{L1} x3 \wedge x4 \leq_{L1} \eta_1\ x3)) \Rrightarrow (\geq))\ L2$ (**is** *?goal2*)
proof −
show *?goal1*
proof (*intro dep-mono-wrt-relI rel-if-if-impI Dep-Fun-Rel-relI*)
fix *x1 x2 x3 x4* **assume** $x1 \leq_{L1} x2$
moreover with *galois-equiv1 preorder-L1* **have** $x2 \leq_{L1} \eta_1\ x2$
by (*blast intro*: *t1.rel-unit-if-reflexive-on-if-galois-connection*)
moreover assume $\eta_1\ x2 \leq_{L1} x1$
ultimately have $x2 \equiv_{L1} x1$ **using** *preorder-L1* **by** *blast*
moreover assume $x3 \leq_{L1} x4\ x2 \leq_{L1} x3$
ultimately show $(\leq_{L2\ x1\ x3}) \leq (\leq_{L2\ x2\ x4})$ **using** *preorder-L1 mono-L2*
by (*intro le-relI*) (*blast dest*!: *rel-ifD elim*!: *dep-mono-wrt-relE*)
qed
show *?goal2*
proof (*intro dep-mono-wrt-relI rel-if-if-impI Dep-Fun-Rel-relI*)
fix *x1 x2 x3 x4* **presume** $x3 \leq_{L1} x4\ x4 \leq_{L1} \eta_1\ x3$
moreover with *galois-equiv1 preorder-L1* **have** $\eta_1\ x3 \leq_{L1} x3$
by (*blast intro*: *flip.t1.counit-rel-if-reflexive-on-if-galois-connection*)
ultimately have $x3 \equiv_{L1} x4$ **using** *preorder-L1* **by** *blast*
moreover presume $x1 \leq_{L1} x2\ x2 \leq_{L1} x3$
ultimately show $(\leq_{L2\ x2\ x4}) \leq (\leq_{L2\ x1\ x3})$ **using** *preorder-L1 mono-L2* **by** *fast*
qed *auto*
qed

lemma *galois-equivalence-if-mono-if-galois-equivalence-Dep-Fun-Rel-pred-assm-leftI*:
assumes *galois-equiv1*: $((\leq_{L1}) \equiv_G (\leq_{R1}))$ *l1 r1*

and *refl-L1*: *reflexive-on* (*in-field* ($\leq_{L1}$)) ($\leq_{L1}$)
and *refl-R1*: *reflexive-on* (*in-field* ($\leq_{R1}$)) ($\leq_{R1}$)
and *mono-L2*: ((*x1 x2* :: ($\geq_{L1}$)) $\Rightarrow$ (*x3 x4* :: ($\leq_{L1}$) | *x1* $\leq_{L1}$ *x3*) $\Rrightarrow$ ($\leq$)) *L2*
and *mono-R2*: ((*x1'* *x2'* :: ($\geq_{R1}$)) $\Rightarrow$ (*x3'* *x4'* :: ($\leq_{R1}$) | *x1'* $\leq_{R1}$ *x3'*) $\Rrightarrow$ ($\leq$)) *R2*
and *mono-l2*: ((*x1'* *x2'* :: ($\leq_{R1}$)) $\Rightarrow$ (*x1 x2* :: ($\leq_{L1}$) | *x2* $_{L1}\lessapprox$ *x1'*) $\Rrightarrow$ *in-field* ($\leq_{L2\ x1\ (r1\ x2')}$) $\Rrightarrow$ ($\leq_{R2\ (l1\ x1)\ x2'}$)) *l2*
and $x \leq_{L1} x$
shows (*in-codom* ($\leq_{L2\ (\eta_1\ x)\ x}$) $\Rrightarrow$ ($\leq_{R2\ (l1\ x)\ (l1\ x)}$)) ($l2_{(l1\ x)\ (\eta_1\ x)}$) ($l2_{(l1\ x)\ x}$)
proof (*intro Fun-Rel-predI*)
fix *y* **assume** *in-codom* ($\leq_{L2\ (\eta_1\ x)\ x}$) *y*
moreover from ‹$x \leq_{L1} x$› *galois-equiv1 refl-L1* **have** $x \equiv_{L1} \eta_1\ x$
by (*blast intro*: *bi-related-if-rel-equivalence-on t1.rel-equivalence-on-unit-if-reflexive-on-if-galois-equivalence*)
moreover with *refl-L1* **have** $\eta_1\ x \leq_{L1} \eta_1\ x$ **by** *blast*
ultimately have *in-codom* ($\leq_{L2\ (\eta_1\ x)\ (\eta_1\ x)}$) *y* **using** *mono-L2* **by** *blast*
moreover from ‹$x \leq_{L1} x$› *galois-equiv1*
have *l1 x* $\leq_{R1}$ *l1 x* η_1 *x* $\leq_{L1}$ *x x* $_{L1}\lessapprox$ *l1 x*
by (*blast intro*: *t1.left-Galois-left-if-left-relI flip.t1.counit-rel-if-right-rel-if-galois-connection*)+
moreover note
Dep-Fun-Rel-relD[*OF dep-mono-wrt-relD*[*OF mono-l2* ‹*l1 x* $\leq_{R1}$ *l1 x*›] ‹η_1 *x* $\leq_{L1}$ *x*›]
ultimately have $l2_{(l1\ x)\ (\eta_1\ x)}$ *y* $\leq_{R2\ (\varepsilon_1\ (l1\ x))\ (l1\ x)}$ $l2_{(l1\ x)\ x}$ *y* **by** *auto*
moreover note ‹*l1 x* $\leq_{R1}$ *l1 x*›
moreover with *galois-equiv1 refl-R1* **have** *l1 x* $\equiv_{R1} \varepsilon_1\ (l1\ x)$
by (*blast intro*: *bi-related-if-rel-equivalence-on flip.t1.rel-equivalence-on-unit-if-reflexive-on-if-galois-equivalence*)
ultimately show $l2_{(l1\ x)\ (\eta_1\ x)}$ *y* $\leq_{R2\ (l1\ x)\ (l1\ x)}$ $l2_{(l1\ x)\ x}$ *y*
using *mono-R2* **by** *blast*
qed

lemma *galois-equivalence-if-mono-if-galois-equivalence-Dep-Fun-Rel-pred-assm-right*:
assumes *galois-equiv1*: (($\leq_{L1}$) $\equiv_G$ ($\leq_{R1}$)) *l1 r1*
and *refl-R1*: *reflexive-on* (*in-field* ($\leq_{R1}$)) ($\leq_{R1}$)
and *mono-L2*: ((*x1 x2* :: ($\geq_{L1}$)) $\Rightarrow$ (*x3 x4* :: ($\leq_{L1}$) | *x1* $\leq_{L1}$ *x3*) $\Rrightarrow$ ($\leq$)) *L2*
and *mono-R2*: ((*x1'* *x2'* :: ($\geq_{R1}$)) $\Rightarrow$ (*x3'* *x4'* :: ($\leq_{R1}$) | *x1'* $\leq_{R1}$ *x3'*) $\Rrightarrow$ ($\leq$)) *R2*
and *mono-r2*: ((*x1 x2* :: ($\leq_{L1}$)) $\Rightarrow$ (*x1'* *x2'* :: ($\leq_{R1}$) | *x2* $_{L1}\lessapprox$ *x1'*) $\Rrightarrow$ *in-field* ($\leq_{R2\ (l1\ x1)\ x2'}$) $\Rrightarrow$ ($\leq_{L2\ x1\ (r1\ x2')}$)) *r2*
and $x' \leq_{R1} x'$
shows (*in-dom* ($\leq_{R2\ x'\ (\varepsilon_1\ x')}$) $\Rrightarrow$ ($\leq_{L2\ (r1\ x')\ (r1\ x')}$)) ($r2_{(r1\ x')\ x'}$) ($r2_{(r1\ x')\ (\varepsilon_1\ x')}$)
proof (*intro Fun-Rel-predI*)
fix *y* **assume** *in-dom* ($\leq_{R2\ x'\ (\varepsilon_1\ x')}$) *y*
moreover from ‹$x' \leq_{R1} x'$› *galois-equiv1 refl-R1* **have** $x' \equiv_{R1} \varepsilon_1\ x'$
by (*blast intro*: *bi-related-if-rel-equivalence-on flip.t1.rel-equivalence-on-unit-if-reflexive-on-if-galois-equivalence*)
moreover with *refl-R1* **have** $\varepsilon_1\ x' \leq_{R1} \varepsilon_1\ x'$ **by** *blast*

ultimately have *in-dom* $(\leq_{R2\ (\varepsilon_1\ x')\ (\varepsilon_1\ x'')})\ y$ **using** *mono-R2* **by** *blast*
moreover from $\langle x' \leq_{R1} x''\rangle$ *galois-equiv1*
have $r1\ x' \leq_{L1} r1\ x'\ x'' \leq_{R1} \varepsilon_1\ x'\ r1\ x''\ {}_{L1}\lessapprox x'$
by (*blast intro*: *t1.right-left-Galois-if-right-relI*
flip.t1.rel-unit-if-left-rel-if-galois-connection)+
moreover note
Dep-Fun-Rel-relD[*OF dep-mono-wrt-relD*[*OF mono-r2* $\langle r1\ x' \leq_{L1} r1\ x''\rangle$] $\langle x' \leq_{R1} \varepsilon_1\ x''\rangle$]
ultimately have ${}^{r2}{}_{(r1\ x')\ x'}\ y \leq_{L2\ (r1\ x')\ (\eta_1\ (r1\ x'))}\ {}^{r2}{}_{(r1\ x')\ (\varepsilon_1\ x')}\ y$ **by** *auto*
moreover note $\langle r1\ x' \leq_{L1} r1\ x''\rangle$
moreover with *galois-equiv1 refl-R1* **have** $r1\ x' \equiv_{L1} \eta_1\ (r1\ x')$
by (*blast intro*: *bi-related-if-rel-equivalence-on*
t1.rel-equivalence-on-unit-if-reflexive-on-if-galois-equivalence)
ultimately show ${}^{r2}{}_{(r1\ x')\ x'}\ y \leq_{L2\ (r1\ x')\ (r1\ x')}\ {}^{r2}{}_{(r1\ x')\ (\varepsilon_1\ x')}\ y$
using *mono-L2* **by** *blast*
qed

end

Monotone Dependent Function Relator **context** *transport-Mono-Dep-Fun-Rel*
begin

context
begin

interpretation *flip* : *transport-Mono-Dep-Fun-Rel R1 L1 r1 l1 R2 L2 r2 l2*
rewrites $flip.t1.counit \equiv \eta_1$ **and** $flip.t1.unit \equiv \varepsilon_1$
by (*simp-all only*: *t1.flip-counit-eq-unit t1.flip-unit-eq-counit*)

lemma *galois-equivalence-if-galois-equivalenceI*:
assumes *galois-equiv1*: $((\leq_{L1}) \equiv_G (\leq_{R1}))\ l1\ r1$
and *refl-L1*: *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
and *reflexive-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$
and *galois-equiv2*: $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox x' \Longrightarrow$
$((\leq_{L2\ x\ (r1\ x')}) \equiv_G (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x2\ x2}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow (\leq_{L2\ (\eta_1\ x)\ x}) \leq (\leq_{L2\ x\ x})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ x1}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ (\eta_1\ x2)}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ x2'\ x2'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ (\varepsilon_1\ x1')\ x2'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ x1'\ x1'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x'.\ x' \leq_{R1} x' \Longrightarrow (\leq_{R2\ x'\ (\varepsilon_1\ x')}) \leq (\leq_{R2\ x'\ x'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow$
$(in\text{-}dom\ (\leq_{L2\ (r1\ x1')\ (r1\ x2')}) \Rrightarrow (\leq_{R2\ x1'\ x2'}))\ (l2_{x1'\ (r1\ x1')})\ (l2_{x2'\ (r1\ x1')})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow$
$(in\text{-}codom\ (\leq_{L2\ (r1\ x1')\ (r1\ x2')}) \Rrightarrow (\leq_{R2\ x1'\ x2'}))\ (l2_{x2'\ (r1\ x1')})\ (l2_{x2'\ (r1\ x2')})$

and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow$
$(\mathit{in\text{-}dom}\ (\leq_{L2\ x\ (\eta_1\ x)}) \Rrightarrow (\leq_{R2\ (l1\ x)\ (l1\ x)}))\ (l2_{(l1\ x)\ x})\ (l2_{(l1\ x)\ (\eta_1\ x)})$
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow$
$(\mathit{in\text{-}codom}\ (\leq_{L2\ (\eta_1\ x)\ x}) \Rrightarrow (\leq_{R2\ (l1\ x)\ (l1\ x)}))\ (l2_{(l1\ x)\ (\eta_1\ x)})\ (l2_{(l1\ x)\ x})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$
$(\mathit{in\text{-}codom}\ (\leq_{R2\ (l1\ x1)\ (l1\ x2)}) \Rrightarrow (\leq_{L2\ x1\ x2}))\ (r2_{x1\ (l1\ x2)})\ (r2_{x2\ (l1\ x2)})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$
$(\mathit{in\text{-}dom}\ (\leq_{R2\ (l1\ x1)\ (l1\ x2)}) \Rrightarrow (\leq_{L2\ x1\ x2}))\ (r2_{x1\ (l1\ x1)})\ (r2_{x1\ (l1\ x2)})$
and $\bigwedge x'.\ x' \leq_{R1} x' \Longrightarrow$
$(\mathit{in\text{-}codom}\ (\leq_{R2\ (\varepsilon_1\ x')\ x'}) \Rrightarrow (\leq_{L2\ (r1\ x')\ (r1\ x')}))\ (r2_{(r1\ x')\ (\varepsilon_1\ x')})\ (r2_{(r1\ x')\ x'})$
and $\bigwedge x'.\ x' \leq_{R1} x' \Longrightarrow$
$(\mathit{in\text{-}dom}\ (\leq_{R2\ x'\ (\varepsilon_1\ x')}) \Rrightarrow (\leq_{L2\ (r1\ x')\ (r1\ x')}))\ (r2_{(r1\ x')\ x'})\ (r2_{(r1\ x')\ (\varepsilon_1\ x')})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow \mathit{transitive}\ (\leq_{L2\ x1\ x2})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow \mathit{transitive}\ (\leq_{R2\ x1'\ x2'})$
shows $((\leq_L) \equiv_G (\leq_R))\ l\ r$
proof −
from *galois-equiv2* **have**
$((\leq_{L2\ x\ (r1\ x')}) \dashv (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
$((\leq_{R2\ (l1\ x)\ x'})\ {}_h\trianglelefteq\ (\leq_{L2\ x\ (r1\ x')}))\ (r2_{x\ x'})\ (l2_{x'\ x})$
$((\leq_{R2\ (l1\ x)\ x'}) \trianglelefteq_h (\leq_{L2\ x\ (r1\ x')}))\ (r2_{x\ x'})\ (l2_{x'\ x})$
if $x\ {}_{L1}\lessapprox\ x'$ **for** $x\ x'$ **using** $\langle x\ {}_{L1}\lessapprox\ x'\rangle$
by (*blast elim*: *galois.galois-connectionE galois-prop.galois-propE*)+
moreover from *galois-equiv1 galois-equiv2* **have**
$\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow$
$((\leq_{L2\ (r1\ x1')\ (r1\ x2')}) \Rightarrow (\leq_{R2\ (\varepsilon_1\ x1')\ x2'}))\ (l2_{x2'\ (r1\ x1')})$
by (*intro tdfr.mono-wrt-rel-left2-if-mono-wrt-rel-left2-if-left-GaloisI*) *auto*
moreover from *galois-equiv1 galois-equiv2* **have**
$\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow ((\leq_{R2\ (l1\ x1)\ (l1\ x2)}) \Rightarrow (\leq_{L2\ x1\ (\eta_1\ x2)}))\ (r2_{x1\ (l1\ x2)})$
by (*intro tdfr.mono-wrt-rel-right2-if-mono-wrt-rel-right2-if-left-GaloisI*)
(*auto elim*!: *t1.galois-equivalenceE*)
moreover from *galois-equiv1 refl-L1* **have**
$\bigwedge x.\ x \leq_{L1} x \Longrightarrow x \equiv_{L1} \eta_1\ x$
$\bigwedge x'.\ x' \leq_{R1} x' \Longrightarrow x' \equiv_{R1} \varepsilon_1\ x'$
by (*blast intro*!: *bi-related-if-rel-equivalence-on*
t1.rel-equivalence-on-unit-if-reflexive-on-if-galois-equivalence
flip.t1.rel-equivalence-on-unit-if-reflexive-on-if-galois-equivalence)+
ultimately show *?thesis* **using** *assms*
by (*intro galois-equivalenceI*
galois-connection-left-right-if-galois-connectionI flip.galois-prop-left-rightI
tdfr.flip-half-galois-prop-left2-if-half-galois-prop-left2-if-left-GaloisI
tdfr.flip-half-galois-prop-right2-if-half-galois-prop-right2-if-GaloisI
tdfr.mono-wrt-rel-left-if-transitiveI tdfr.mono-wrt-rel-right-if-transitiveI
flip.tdfr.left-rel-right-if-left-right-rel-le-right2-assmI
flip.tdfr.left-right-rel-if-left-rel-right-ge-left2-assmI
tdfr.left-rel2-unit-eqs-left-rel2I
flip.tdfr.left-rel2-unit-eqs-left-rel2I)
(*auto elim*!: *t1.galois-equivalenceE*

intro: *reflexive-on-if-le-pred-if-reflexive-on in-field-if-in-dom in-field-if-in-codom*)
qed

corollary *galois-equivalence-if-galois-equivalenceI'*:
assumes $((\leq_{L1}) \equiv_G (\leq_{R1}))$ *l1 r1*
and *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
and *reflexive-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$
and $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox\ x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \equiv_G (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x2\ x2}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow (\leq_{L2\ (\eta_1\ x)\ x}) \leq (\leq_{L2\ x\ x})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ x1}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ (\eta_1\ x2)}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ x2'\ x2'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ (\varepsilon_1\ x1')\ x2'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ x1'\ x1'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x'.\ x' \leq_{R1} x' \Longrightarrow (\leq_{R2\ x'\ (\varepsilon_1\ x')}) \leq (\leq_{R2\ x'\ x'})$
and $((x1'\ x2' :: (\leq_{R1})) \Rightarrow (x1\ x2 :: (\leq_{L1}) \mid x2\ {}_{L1}\lessapprox\ x1') \Rrightarrow$
in-field $(\leq_{L2\ x1\ (r1\ x2')}) \Rrightarrow (\leq_{R2\ (l1\ x1)\ x2'}))$ *l2*
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow$
$(\mathit{in\text{-}codom}\ (\leq_{L2\ (\eta_1\ x)\ x}) \Rrightarrow (\leq_{R2\ (l1\ x)\ (l1\ x)}))\ (l2_{(l1\ x)\ (\eta_1\ x)})\ (l2_{(l1\ x)\ x})$
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ {}_{L1}\lessapprox\ x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))$ *r2*
and $\bigwedge x'.\ x' \leq_{R1} x' \Longrightarrow$
$(\mathit{in\text{-}dom}\ (\leq_{R2\ x'\ (\varepsilon_1\ x')}) \Rrightarrow (\leq_{L2\ (r1\ x')\ (r1\ x')}))\ (r2_{(r1\ x')\ x'})\ (r2_{(r1\ x')\ (\varepsilon_1\ x')})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *transitive* $(\leq_{L2\ x1\ x2})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow$ *transitive* $(\leq_{R2\ x1'\ x2'})$
shows $((\leq_L) \equiv_G (\leq_R))$ *l r*
using *assms* **by** (*intro galois-equivalence-if-galois-equivalenceI*
tdfr.galois-connection-left-right-if-galois-connection-mono-assms-leftI
tdfr.galois-connection-left-right-if-galois-connection-mono-assms-rightI
tdfr.galois-connection-left-right-if-galois-connection-mono-2-assms-leftI
tdfr.galois-connection-left-right-if-galois-connection-mono-2-assms-rightI)
(*auto intro*: *reflexive-on-if-le-pred-if-reflexive-on in-field-if-in-dom*
in-field-if-in-codom)

corollary *galois-equivalence-if-mono-if-galois-equivalenceI*:
assumes $((\leq_{L1}) \equiv_G (\leq_{R1}))$ *l1 r1*
and *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
and *reflexive-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$
and $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox\ x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \equiv_G (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $((x1\ x2 :: (\leq_{L1}) \mid \eta_1\ x2 \leq_{L1} x1) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid x2 \leq_{L1} x3) \Rrightarrow (\leq))$ *L2*
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid (x2 \leq_{L1} x3 \wedge x4 \leq_{L1} \eta_1\ x3)) \Rrightarrow (\geq))$ *L2*
and $((x1'\ x2' :: (\leq_{R1}) \mid \varepsilon_1\ x2' \leq_{R1} x1') \Rightarrow (x3'\ x4' :: (\leq_{R1}) \mid x2' \leq_{R1} x3') \Rrightarrow (\leq))$ *R2*

and $((x1\,'\ x2\,' :: (\leq_{R1})) \Rightarrow (x3\,'\ x4\,' :: (\leq_{R1}) \mid (x2\,' \leq_{R1} x3\,' \wedge x4\,' \leq_{R1} \varepsilon_1\ x3\,')) \Rrightarrow (\geq))\ R2$
and $((x1\,'\ x2\,' :: (\leq_{R1})) \Rightarrow (x1\ x2 :: (\leq_{L1}) \mid x2\ _{L1}\lessapprox x1\,') \Rrightarrow$
in-field $(\leq_{L2\ x1\ (r1\ x2\,')}) \Rrightarrow (\leq_{R2\ (l1\ x1)\ x2\,'}))\ l2$
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow$
$(\textit{in-codom}\ (\leq_{L2\ (\eta_1\ x)\ x}) \Rrightarrow (\leq_{R2\ (l1\ x)\ (l1\ x)}))\ (l2_{(l1\ x)\ (\eta_1\ x)})\ (l2_{(l1\ x)\ x})$
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1\,'\ x2\,' :: (\leq_{R1}) \mid x2\ _{L1}\lessapprox x1\,') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2\,'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2\,')}))\ r2$
and $\bigwedge x'.\ x' \leq_{R1} x' \Longrightarrow$
$(\textit{in-dom}\ (\leq_{R2\ x'\ (\varepsilon_1\ x')}) \Rrightarrow (\leq_{L2\ (r1\ x')\ (r1\ x')}))\ (r2_{(r1\ x')\ x'})\ (r2_{(r1\ x')\ (\varepsilon_1\ x')})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow \textit{transitive}\ (\leq_{L2\ x1\ x2})$
and $\bigwedge x1\,'\ x2\,'.\ x1\,' \leq_{R1} x2\,' \Longrightarrow \textit{transitive}\ (\leq_{R2\ x1\,'\ x2\,'})$
shows $((\leq_L) \equiv_G (\leq_R))\ l\ r$
using *assms* **by** (*intro galois-equivalence-if-galois-equivalenceI'*
tdfr.left-rel-right-iff-left-right-rel-if-galois-prop-le-assms-leftI
flip.tdfr.left-rel-right-iff-left-right-rel-if-galois-prop-le-assms-leftI
tdfr.left-rel-right-iff-left-right-rel-if-galois-prop-le-assms-rightI
flip.tdfr.left-rel-right-iff-left-right-rel-if-galois-prop-le-assms-rightI)
auto

end

interpretation *flip* : *transport-Mono-Dep-Fun-Rel R1 L1 r1 l1 R2 L2 r2 l2*
rewrites *flip.t1.counit* $\equiv \eta_1$ **and** *flip.t1.unit* $\equiv \varepsilon_1$
by (*simp-all only*: *t1.flip-counit-eq-unit t1.flip-unit-eq-counit*)

lemma *galois-equivalence-if-mono-if-preorder-equivalenceI*:
assumes $((\leq_{L1}) \equiv_{pre} (\leq_{R1}))\ l1\ r1$
and $\bigwedge x\ x'.\ x\ _{L1}\lessapprox x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \equiv_G (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $((x1\ x2 :: (\geq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid x1 \leq_{L1} x3) \Rrightarrow (\leq))\ L2$
and $((x1\,'\ x2\,' :: (\geq_{R1})) \Rightarrow (x3\,'\ x4\,' :: (\leq_{R1}) \mid x1\,' \leq_{R1} x3\,') \Rrightarrow (\leq))\ R2$
and $((x1\,'\ x2\,' :: (\leq_{R1})) \Rightarrow (x1\ x2 :: (\leq_{L1}) \mid x2\ _{L1}\lessapprox x1\,') \Rrightarrow$
in-field $(\leq_{L2\ x1\ (r1\ x2\,')}) \Rrightarrow (\leq_{R2\ (l1\ x1)\ x2\,'}))\ l2$
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1\,'\ x2\,' :: (\leq_{R1}) \mid x2\ _{L1}\lessapprox x1\,') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2\,'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2\,')}))\ r2$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow \textit{transitive}\ (\leq_{L2\ x1\ x2})$
and $\bigwedge x1\,'\ x2\,'.\ x1\,' \leq_{R1} x2\,' \Longrightarrow \textit{transitive}\ (\leq_{R2\ x1\,'\ x2\,'})$
shows $((\leq_L) \equiv_G (\leq_R))\ l\ r$
using *assms* **by** (*intro galois-equivalence-if-mono-if-galois-equivalenceI*
tdfr.galois-equivalence-if-mono-if-galois-equivalence-mono-assms-leftI
flip.tdfr.galois-equivalence-if-mono-if-galois-equivalence-mono-assms-leftI
tdfr.galois-equivalence-if-mono-if-galois-equivalence-Dep-Fun-Rel-pred-assm-leftI
tdfr.galois-equivalence-if-mono-if-galois-equivalence-Dep-Fun-Rel-pred-assm-right)
auto

theorem *galois-equivalence-if-mono-if-preorder-equivalenceI'*:
assumes $((\leq_{L1}) \equiv_{pre} (\leq_{R1}))\ l1\ r1$
and $\bigwedge x\ x'.\ x\ _{L1}\lessapprox x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \equiv_{pre} (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$

and $((x1\ x2 :: (\geq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid x1 \leq_{L1} x3) \Rrightarrow (\leq))\ L2$
and $((x1'\ x2' :: (\geq_{R1})) \Rightarrow (x3'\ x4' :: (\leq_{R1}) \mid x1' \leq_{R1} x3') \Rrightarrow (\leq))\ R2$
and $((x1'\ x2' :: (\leq_{R1})) \Rightarrow (x1\ x2 :: (\leq_{L1}) \mid x2\ {}_{L1}\overset{\leq}{\approx}\ x1') \Rrightarrow$
in-field $(\leq_{L2\ x1\ (r1\ x2')}) \Rrightarrow (\leq_{R2\ (l1\ x1)\ x2'}))\ l2$
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ {}_{L1}\overset{\leq}{\approx}\ x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))\ r2$
shows $((\leq_L) \equiv_G (\leq_R))\ l\ r$
using *assms* **by** (*intro galois-equivalence-if-mono-if-preorder-equivalenceI*
tdfr.transitive-left2-if-preorder-equivalenceI
tdfr.transitive-right2-if-preorder-equivalenceI)
auto

end

Monotone Function Relator **context** *transport-Mono-Fun-Rel*
begin

interpretation *flip* : *transport-Mono-Fun-Rel R1 L1 r1 l1 R2 L2 r2 l2* **.**

lemma *galois-equivalenceI*:
assumes $((\leq_{L1}) \equiv_G (\leq_{R1}))\ l1\ r1$
and *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
and *reflexive-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$
and $((\leq_{L2}) \equiv_G (\leq_{R2}))\ l2\ r2$
and *transitive* $(\leq_{L2})$
and *transitive* $(\leq_{R2})$
shows $((\leq_L) \equiv_G (\leq_R))\ l\ r$
using *assms* **by** (*intro tmdfr.galois-equivalenceI*
galois-connection-left-rightI flip.galois-prop-left-rightI)
(*auto intro*: *reflexive-on-if-le-pred-if-reflexive-on*
in-field-if-in-dom in-field-if-in-codom)

end

end

2.15.7 Simplification of Left and Right Relations

theory *Transport-Functions-Relation-Simplifications*
imports
Transport-Functions-Order-Base
Transport-Functions-Galois-Equivalence
begin

Dependent Function Relator **context** *transport-Dep-Fun-Rel*
begin

Due to *reflexive-on* (*in-field* (*transport-Dep-Fun-Rel.L ?L1.0 ?L2.0*))

(*transport-Dep-Fun-Rel.L ?L1.0 ?L2.0*) $\Longrightarrow$ *transport-Mono-Dep-Fun-Rel.L ?L1.0 ?L2.0 = transport-Dep-Fun-Rel.L ?L1.0 ?L2.0*, we can apply all results from *transport-Mono-Dep-Fun-Rel* to *transport-Dep-Fun-Rel* whenever $(\leq_L)$ and $(\leq_R)$ are reflexive.

lemma *reflexive-on-in-field-left-rel2-le-assmI*:
  assumes *refl-L1*: *reflexive-on* (*in-dom* $(\leq_{L1})$) $(\leq_{L1})$
  and *mono-L2*: $((x1 : \top) \Rightarrow (x2\ x3 :: (\leq_{L1}) \mid x1 \leq_{L1} x2) \Rightarrow (\leq))$ *L2*
  and $x1 \leq_{L1} x2$
  shows $(\leq_{L2\ x1\ x1}) \leq (\leq_{L2\ x1\ x2})$
proof −
  from *refl-L1* ‹$x1 \leq_{L1} x2$› **have** $x1 \leq_{L1} x1$ **by** *blast*
  with *dep-mono-wrt-relD*[*OF dep-mono-wrt-predD*[*OF mono-L2*] ‹$x1 \leq_{L1} x2$›]
    show $(\leq_{L2\ x1\ x1}) \leq (\leq_{L2\ x1\ x2})$ **by** *auto*
qed

lemma *reflexive-on-in-field-mono-assm-left2I*:
  assumes *mono-L2*: $((x1\ x2 :: (\geq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid x1 \leq_{L1} x3) \Rrightarrow (\leq))$ *L2*
  and *refl-L1*: *reflexive-on* (*in-dom* $(\leq_{L1})$) $(\leq_{L1})$
  shows $((x1 : \top) \Rightarrow (x2\ x3 :: (\leq_{L1}) \mid x1 \leq_{L1} x2) \Rightarrow (\leq))$ *L2*
proof (*intro dep-mono-wrt-predI dep-mono-wrt-relI rel-if-if-impI*)
  fix *x1 x2 x3* **assume** $x1 \leq_{L1} x2$ $x2 \leq_{L1} x3$
  with *refl-L1* **have** $x1 \geq_{L1} x1$ **by** *blast*
  from *Dep-Fun-Rel-relD*[*OF dep-mono-wrt-relD*[*OF mono-L2* ‹$x1 \geq_{L1} x1$›]
    ‹$x2 \leq_{L1} x3$›] ‹$x1 \leq_{L1} x2$›
    show $(\leq_{L2\ x1\ x2}) \leq (\leq_{L2\ x1\ x3})$ **by** *blast*
qed

lemma *reflexive-on-in-field-left-if-equivalencesI*:
  assumes $((\leq_{L1}) \equiv_G (\leq_{R1}))$ *l1 r1*
  and *preorder-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
  and $((x1\ x2 :: (\geq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid x1 \leq_{L1} x3) \Rrightarrow (\leq))$ *L2*
  and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *partial-equivalence-rel* $(\leq_{L2\ x1\ x2})$
  shows *reflexive-on* (*in-field* $(\leq_L)$) $(\leq_L)$
  using *assms*
  by (*intro reflexive-on-in-field-leftI*
    left-rel-right-iff-left-right-rel-if-galois-prop-le-assms-leftI
    galois-equivalence-if-mono-if-galois-equivalence-mono-assms-leftI
    reflexive-on-in-field-left-rel2-le-assmI
    reflexive-on-in-field-mono-assm-left2I)
  (*auto intro*: *reflexive-on-if-le-pred-if-reflexive-on in-field-if-in-dom*)

end

Monotone Dependent Function Relator **context** *transport-Mono-Dep-Fun-Rel* **begin**

lemma *left-rel-eq-tdfr-leftI*:

assumes *reflexive-on* (*in-field* ($\leq_{L1}$)) ($\leq_{L1}$)
and $\bigwedge$*x1 x2*. *x1* $\leq_{L1}$ *x2* $\Longrightarrow$ ($\leq_{L2\ x2\ x2}$) $\leq$ ($\leq_{L2\ x1\ x2}$)
and $\bigwedge$*x1 x2*. *x1* $\leq_{L1}$ *x2* $\Longrightarrow$ ($\leq_{L2\ x1\ x1}$) $\leq$ ($\leq_{L2\ x1\ x2}$)
and $\bigwedge$*x1 x2*. *x1* $\leq_{L1}$ *x2* $\Longrightarrow$ *partial-equivalence-rel* ($\leq_{L2\ x1\ x2}$)
shows ($\leq_L$) = *tdfr.L*
using *assms* **by** (*intro left-rel-eq-tdfr-left-rel-if-reflexive-on tdfr.reflexive-on-in-field-leftI*)
auto

lemma *left-rel-eq-tdfr-leftI-if-equivalencesI*:
assumes (($\leq_{L1}$) $\equiv_G$ ($\leq_{R1}$)) *l1 r1*
and *preorder-on* (*in-field* ($\leq_{L1}$)) ($\leq_{L1}$)
and ((*x1 x2* :: ($\geq_{L1}$)) $\Rightarrow$ (*x3 x4* :: ($\leq_{L1}$) | *x1* $\leq_{L1}$ *x3*) $\Rrightarrow$ ($\leq$)) *L2*
and $\bigwedge$*x1 x2*. *x1* $\leq_{L1}$ *x2* $\Longrightarrow$ *partial-equivalence-rel* ($\leq_{L2\ x1\ x2}$)
shows ($\leq_L$) = *tdfr.L*
using *assms* **by** (*intro left-rel-eq-tdfr-left-rel-if-reflexive-on tdfr.reflexive-on-in-field-left-if-equivalencesI*)
auto

end

Monotone Function Relator **context** *transport-Mono-Fun-Rel*
begin

lemma *left-rel-eq-tfr-leftI*:
assumes *reflexive-on* (*in-field* ($\leq_{L1}$)) ($\leq_{L1}$)
and *partial-equivalence-rel* ($\leq_{L2}$)
shows ($\leq_L$) = *tfr.tdfr.L*
using *assms* **by** (*intro tmdfr.left-rel-eq-tdfr-leftI*) *auto*

end

end

2.15.8 Galois Relator

theory *Transport-Functions-Galois-Relator*
imports
Transport-Functions-Relation-Simplifications
begin

Dependent Function Relator **context** *transport-Dep-Fun-Rel*
begin

interpretation *flip* : *transport-Dep-Fun-Rel R1 L1 r1 l1 R2 L2 r2 l2*
rewrites *flip.t1.counit* $\equiv \eta_1$ **by** (*simp only*: *t1.flip-counit-eq-unit*)

lemma *Dep-Fun-Rel-left-Galois-if-left-GaloisI*:

assumes $((\leq_{L1})\ {}_h\trianglelefteq\ (\leq_{R1}))$ *l1 r1*
and *refl-L1*: *reflexive-on* (*in-dom* $(\leq_{L1})$) $(\leq_{L1})$
and *mono-r2*: $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox\ x' \Longrightarrow ((\leq_{R2\ (l1\ x)\ x'}) \Rightarrow (\leq_{L2\ x\ (r1\ x')}))\ ({r2}_{x\ x'})$
and *L2-le2*: $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ x1}) \leq (\leq_{L2\ x1\ x2})$
and *ge-L2-r2-le2*: $\bigwedge x\ x'\ y'.\ x\ {}_{L1}\lessapprox\ x' \Longrightarrow$ *in-dom* $(\leq_{R2\ (l1\ x)\ x'})\ y' \Longrightarrow$
$(\geq_{L2\ x\ (r1\ x')})\ ({r2}_{x\ (l1\ x)}\ y') \leq (\geq_{L2\ x\ (r1\ x')})\ ({r2}_{x\ x'}\ y')$
and *trans-L2*: $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *transitive* $(\leq_{L2\ x1\ x2})$
and $g \leq_R g$
and $f\ {}_{L}\lessapprox\ g$
shows $((x\ x' :: ({}_{L1}\lessapprox)) \Rrightarrow ({}_{L2\ x\ x'}\lessapprox))\ f\ g$
proof (*intro Dep-Fun-Rel-relI*)
fix $x\ x'$ **assume** $x\ {}_{L1}\lessapprox\ x'$
show $f\ x\ {}_{L2\ x\ x'}\lessapprox\ g\ x'$
proof (*intro t2.left-GaloisI*)
from $\langle x\ {}_{L1}\lessapprox\ x'\rangle$ $\langle((\leq_{L1})\ {}_h\trianglelefteq\ (\leq_{R1}))\ l1\ r1\rangle$ **have** $x \leq_{L1} r1\ x'\ l1\ x \leq_{R1} x'$ **by** *auto*
with $\langle g \leq_R g\rangle$ **have** $g\ (l1\ x) \leq_{R2\ (l1\ x)\ x'} g\ x'$ **by** *blast*
then show *in-codom* $(\leq_{R2\ (l1\ x)\ x'})\ (g\ x')$ **by** *blast*

from $\langle f\ {}_{L}\lessapprox\ g\rangle$ **have** $f \leq_L r\ g$ **by** *blast*
moreover from *refl-L1* $\langle x\ {}_{L1}\lessapprox\ x'\rangle$ **have** $x \leq_{L1} x$ **by** *blast*
ultimately have $f\ x \leq_{L2\ x\ x} (r\ g)\ x$ **by** *blast*
with *L2-le2* $\langle x \leq_{L1} r1\ x'\rangle$ **have** $f\ x \leq_{L2\ x\ (r1\ x')} (r\ g)\ x$ **by** *blast*
then have $f\ x \leq_{L2\ x\ (r1\ x')}\ {r2}_{x\ (l1\ x)}\ (g\ (l1\ x))$ **by** *simp*
with *ge-L2-r2-le2* **have** $f\ x \leq_{L2\ x\ (r1\ x')}\ {r2}_{x\ x'}\ (g\ (l1\ x))$
using $\langle x\ {}_{L1}\lessapprox\ x'\rangle$ $\langle g\ (l1\ x) \leq_{R2\ (l1\ x)\ x'} \text{-}\rangle$ **by** *blast*
moreover have $\ldots \leq_{L2\ x\ (r1\ x')}\ {r2}_{x\ x'}\ (g\ x')$
using *mono-r2* $\langle x\ {}_{L1}\lessapprox\ x'\rangle$ $\langle g\ (l1\ x) \leq_{R2\ (l1\ x)\ x'} g\ x'\rangle$ **by** *blast*
ultimately show $f\ x \leq_{L2\ x\ (r1\ x')}\ {r2}_{x\ x'}\ (g\ x')$
using *trans-L2* $\langle x\ {}_{L1}\lessapprox\ x'\rangle$ **by** *blast*
qed
qed

lemma *left-rel-right-if-Dep-Fun-Rel-left-GaloisI*:
assumes *mono-l1*: $((\leq_{L1}) \Rightarrow (\leq_{R1}))\ l1$
and *half-galois-prop-right1*: $((\leq_{L1})\ \trianglelefteq_h\ (\leq_{R1}))\ l1\ r1$
and *L2-unit-le2*: $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ (\eta_1\ x2)}) \leq (\leq_{L2\ x1\ x2})$
and *ge-L2-r2-le1*: $\bigwedge x1\ x2\ y'.\ x1 \leq_{L1} x2 \Longrightarrow$ *in-codom* $(\leq_{R2\ (l1\ x1)\ (l1\ x2)})\ y' \Longrightarrow$
$(\geq_{L2\ x1\ x2})\ ({r2}_{x1\ (l1\ x2)}\ y') \leq (\geq_{L2\ x1\ x2})\ ({r2}_{x2\ (l1\ x2)}\ y')$
and *rel-f-g*: $((x\ x' :: ({}_{L1}\lessapprox)) \Rrightarrow ({}_{L2\ x\ x'}\lessapprox))\ f\ g$
shows $f \leq_L r\ g$
proof (*intro left-relI*)
fix *x1 x2* **assume** $x1 \leq_{L1} x2$
with *mono-l1 half-galois-prop-right1* **have** $x1\ {}_{L1}\lessapprox\ l1\ x2$
by (*intro t1.left-Galois-left-if-left-relI*) *auto*
with *rel-f-g* **have** $f\ x1\ {}_{L2\ x1\ (l1\ x2)}\lessapprox\ g\ (l1\ x2)$ **by** *blast*

then have *in-codom* $(\leq_{R2\ (l1\ x1)\ (l1\ x2)})\ (g\ (l1\ x2))$
$f\ x1 \leq_{L2\ x1\ (\eta_1\ x2)}\ r2_{x1\ (l1\ x2)}\ (g\ (l1\ x2))$ **by** *auto*
with *L2-unit-le2* ‹*x1* $\leq_{L1}$ *x2*› **have** $f\ x1 \leq_{L2\ x1\ x2}\ r2_{x1\ (l1\ x2)}\ (g\ (l1\ x2))$ **by** *blast*
with *ge-L2-r2-le1* ‹*x1* $\leq_{L1}$ *x2*› ‹*in-codom* $(\leq_{R2\ (l1\ x1)\ (l1\ x2)})\ (g\ (l1\ x2))$›
have $f\ x1 \leq_{L2\ x1\ x2}\ r2_{x2\ (l1\ x2)}\ (g\ (l1\ x2))$ **by** *blast*
then show $f\ x1 \leq_{L2\ x1\ x2}\ r\ g\ x2$ **by** *simp*
qed

lemma *left-Galois-if-Dep-Fun-Rel-left-GaloisI*:
assumes $((\leq_{L1}) \Rightarrow (\leq_{R1}))\ l1$
and $((\leq_{L1}) \trianglelefteq_h (\leq_{R1}))\ l1\ r1$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ (\eta_1\ x2)}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1\ x2\ y'.\ x1 \leq_{L1} x2 \Longrightarrow$ *in-codom* $(\leq_{R2\ (l1\ x1)\ (l1\ x2)})\ y' \Longrightarrow$
$(\geq_{L2\ x1\ x2})\ (r2_{x1\ (l1\ x2)}\ y') \leq (\geq_{L2\ x1\ x2})\ (r2_{x2\ (l1\ x2)}\ y')$
and *in-codom* $(\leq_R)\ g$
and $((x\ x' :: (_{L1}\overset{\leqslant}{\approx})) \Rrightarrow (_{L2\ x\ x'}\overset{\leqslant}{\approx}))\ f\ g$
shows $f\ _L\overset{\leqslant}{\approx}\ g$
using *assms* **by** (*intro left-GaloisI left-rel-right-if-Dep-Fun-Rel-left-GaloisI*) *auto*

lemma *left-right-rel-if-Dep-Fun-Rel-left-GaloisI*:
assumes *mono-r1*: $((\leq_{R1}) \Rightarrow (\leq_{L1}))\ r1$
and *half-galois-prop-left2*: $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow$
$((\leq_{L2\ (r1\ x1')\ (r1\ x2')})\ {}_h\trianglelefteq\ (\leq_{R2\ (\varepsilon_1\ x1')\ x2'}))\ (l2_{x2'\ (r1\ x1')})\ (r2_{(r1\ x1')\ x2'})$
and *R2-le1*: $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ (\varepsilon_1\ x1')\ x2'}) \leq (\leq_{R2\ x1'\ x2'})$
and *R2-l2-le1*: $\bigwedge x1'\ x2'\ y.\ x1' \leq_{R1} x2' \Longrightarrow$ *in-dom* $(\leq_{L2\ (r1\ x1')\ (r1\ x2')})\ y$
$\Longrightarrow$
$(\leq_{R2\ x1'\ x2'})\ (l2_{x2'\ (r1\ x1')}\ y) \leq (\leq_{R2\ x1'\ x2'})\ (l2_{x1'\ (r1\ x1')}\ y)$
and *rel-f-g*: $((x\ x' :: (_{L1}\overset{\leqslant}{\approx})) \Rrightarrow (_{L2\ x\ x'}\overset{\leqslant}{\approx}))\ f\ g$
shows $l\ f \leq_R g$
proof (*rule flip.left-relI*)
fix $x1'\ x2'$ **assume** $x1' \leq_{R1} x2'$
with *mono-r1* **have** $r1\ x1'\ _{L1}\overset{\leqslant}{\approx}\ x2'$ **by** *blast*
with *rel-f-g* **have** $f\ (r1\ x1')\ _{L2\ (r1\ x1')\ x2'}\overset{\leqslant}{\approx}\ g\ x2'$ **by** *blast*
with *half-galois-prop-left2*[*OF* ‹$x1' \leq_{R1} x2'$›]
have $l2_{x2'\ (r1\ x1')}\ (f\ (r1\ x1')) \leq_{R2\ (\varepsilon_1\ x1')\ x2'}\ g\ x2'$ **by** *auto*
with *R2-le1* ‹$x1' \leq_{R1} x2'$› **have** $l2_{x2'\ (r1\ x1')}\ (f\ (r1\ x1')) \leq_{R2\ x1'\ x2'}\ g\ x2'$
by *blast*
with *R2-l2-le1* ‹$x1' \leq_{R1} x2'$› ‹$f\ (r1\ x1')\ _{L2\ (r1\ x1')\ x2'}\overset{\leqslant}{\approx}$ -›
have $l2_{x1'\ (r1\ x1')}\ (f\ (r1\ x1')) \leq_{R2\ x1'\ x2'}\ g\ x2'$ **by** *blast*
then show $l\ f\ x1' \leq_{R2\ x1'\ x2'}\ g\ x2'$ **by** *simp*
qed

lemma *left-Galois-if-Dep-Fun-Rel-left-GaloisI'*:
assumes $((\leq_{L1}) \Rightarrow (\leq_{R1}))\ l1$ **and** $((\leq_{R1}) \Rightarrow (\leq_{L1}))\ r1$
and $((\leq_{L1}) \trianglelefteq_h (\leq_{R1}))\ l1\ r1$

and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow$
$((\leq_{L2\ (r1\ x1')\ (r1\ x2')})\ {}_h\trianglelefteq\ (\leq_{R2\ (\varepsilon_1\ x1')\ x2'}))\ (l2_{x2'\ (r1\ x1')})\ (r2_{(r1\ x1')\ x2'})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ (\eta_1\ x2)}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ (\varepsilon_1\ x1')\ x2'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x1\ x2\ y'.\ x1 \leq_{L1} x2 \Longrightarrow$ *in-codom* $(\leq_{R2\ (l1\ x1)\ (l1\ x2)})\ y' \Longrightarrow$
$(\geq_{L2\ x1\ x2})\ (r2_{x1\ (l1\ x2)}\ y') \leq (\geq_{L2\ x1\ x2})\ (r2_{x2\ (l1\ x2)}\ y')$
and $\bigwedge x1'\ x2'\ y.\ x1' \leq_{R1} x2' \Longrightarrow$ *in-dom* $(\leq_{L2\ (r1\ x1')\ (r1\ x2')})\ y \Longrightarrow$
$(\leq_{R2\ x1'\ x2'})\ (l2_{x2'\ (r1\ x1')}\ y) \leq (\leq_{R2\ x1'\ x2'})\ (l2_{x1'\ (r1\ x1')}\ y)$
and $((x\ x' :: (_{L1}\lessapprox)) \Rrightarrow (_{L2\ x\ x'}\lessapprox))\ f\ g$
shows $f\ _{L}\lessapprox\ g$
using *assms* **by** (*intro left-Galois-if-Dep-Fun-Rel-left-GaloisI in-codomI*[**where** *?x=l f*])
(*auto intro*!: *left-right-rel-if-Dep-Fun-Rel-left-GaloisI*)

lemma *left-Galois-iff-Dep-Fun-Rel-left-GaloisI*:
assumes $((\leq_{L1}) \Rightarrow (\leq_{R1}))\ l1$
and $((\leq_{L1}) \trianglelefteq (\leq_{R1}))\ l1\ r1$
and *reflexive-on* (*in-dom* $(\leq_{L1})$) $(\leq_{L1})$
and $\bigwedge x\ x'.\ x\ _{L1}\lessapprox\ x' \Longrightarrow ((\leq_{R2\ (l1\ x)\ x'}) \Rightarrow (\leq_{L2\ x\ (r1\ x')}))\ (r2_{x\ x'})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ x1}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ (\eta_1\ x2)}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1\ x2\ y'.\ x1 \leq_{L1} x2 \Longrightarrow$ *in-codom* $(\leq_{R2\ (l1\ x1)\ (l1\ x2)})\ y' \Longrightarrow$
$(\geq_{L2\ x1\ x2})\ (r2_{x1\ (l1\ x2)}\ y') \leq (\geq_{L2\ x1\ x2})\ (r2_{x2\ (l1\ x2)}\ y')$
and $\bigwedge x\ x'\ y'.\ x\ _{L1}\lessapprox\ x' \Longrightarrow$ *in-dom* $(\leq_{R2\ (l1\ x)\ x'})\ y' \Longrightarrow$
$(\geq_{L2\ x\ (r1\ x')})\ (r2_{x\ (l1\ x)}\ y') \leq (\geq_{L2\ x\ (r1\ x')})\ (r2_{x\ x'}\ y')$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *transitive* $(\leq_{L2\ x1\ x2})$
and $g \leq_R g$
shows $f\ _{L}\lessapprox\ g \longleftrightarrow ((x\ x' :: (_{L1}\lessapprox)) \Rrightarrow (_{L2\ x\ x'}\lessapprox))\ f\ g$
using *assms* **by** (*intro iffI*)
(*auto intro*!: *Dep-Fun-Rel-left-Galois-if-left-GaloisI left-Galois-if-Dep-Fun-Rel-left-GaloisI*)

lemma *left-Galois-iff-Dep-Fun-Rel-left-Galois-ge-left-rel2-assmI*:
assumes $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$
$((\textit{in-codom}\ (\leq_{R2\ (l1\ x1)\ (l1\ x2)})) \Rrightarrow (\leq_{L2\ x1\ x2}))\ (r2_{x1\ (l1\ x2)})\ (r2_{x2\ (l1\ x2)})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *transitive* $(\leq_{L2\ x1\ x2})$
shows $\bigwedge x1\ x2\ y'.\ x1 \leq_{L1} x2 \Longrightarrow$ *in-codom* $(\leq_{R2\ (l1\ x1)\ (l1\ x2)})\ y' \Longrightarrow$
$(\geq_{L2\ x1\ x2})\ (r2_{x1\ (l1\ x2)}\ y') \leq (\geq_{L2\ x1\ x2})\ (r2_{x2\ (l1\ x2)}\ y')$
using *assms* **by** *blast*

lemma *left-Galois-iff-Dep-Fun-Rel-left-Galois-ge-left-rel2-assmI'*:
assumes $\bigwedge x\ x'.\ x\ _{L1}\lessapprox\ x' \Longrightarrow$
$((\textit{in-dom}\ (\leq_{R2\ (l1\ x)\ x'})) \Rrightarrow (\leq_{L2\ x\ (r1\ x')}))\ (r2_{x\ (l1\ x)})\ (r2_{x\ x'})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *transitive* $(\leq_{L2\ x1\ x2})$
shows $\bigwedge x\ x'\ y'.\ x\ _{L1}\lessapprox\ x' \Longrightarrow$ *in-dom* $(\leq_{R2\ (l1\ x)\ x'})\ y' \Longrightarrow$
$(\geq_{L2\ x\ (r1\ x')})\ (r2_{x\ (l1\ x)}\ y') \leq (\geq_{L2\ x\ (r1\ x')})\ (r2_{x\ x'}\ y')$
using *assms* **by** *blast*

lemma *left-Galois-iff-Dep-Fun-Rel-left-Galois-mono-assm-in-codom-rightI*:
assumes *mono-l1*: $((\leq_{L1}) \Rightarrow (\leq_{R1}))\ l1$
and *half-galois-prop-right1*: $((\leq_{L1}) \trianglelefteq_h (\leq_{R1}))\ l1\ r1$
and *refl-L1*: *reflexive-on* (*in-codom* $(\leq_{L1})$) $(\leq_{L1})$
and *L2-le-unit2*: $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ (\eta_1\ x2)}) \leq (\leq_{L2\ x1\ x2})$
and *mono-r2*: $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ {}_{L1}\lessapprox x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')})\ r2$
and $x1 \leq_{L1} x2$
shows $((\textit{in-codom}\ (\leq_{R2\ (l1\ x1)\ (l1\ x2)})) \Rrightarrow (\leq_{L2\ x1\ x2}))\ (r2_{x1\ (l1\ x2)})\ (r2_{x2\ (l1\ x2)})$
proof (*intro Fun-Rel-predI*)
from *mono-l1 half-galois-prop-right1 refl-L1* ‹$x1 \leq_{L1} x2$›
have $l1\ x2 \leq_{R1} l1\ x2\ x2\ {}_{L1}\lessapprox l1\ x2$
by (*blast intro*: *t1.left-Galois-left-if-left-relI*)+
fix y' **assume** *in-codom* $(\leq_{R2\ (l1\ x1)\ (l1\ x2)})\ y'$
with *Dep-Fun-Rel-relD*[*OF*
dep-mono-wrt-relD[*OF mono-r2* ‹$x1 \leq_{L1} x2$›] ‹$l1\ x2 \leq_{R1} l1\ x2$›]
have $r2_{x1\ (l1\ x2)}\ y' \leq_{L2\ x1\ (\eta_1\ x2)}\ r2_{x2\ (l1\ x2)}\ y'$
using ‹$x2\ {}_{L1}\lessapprox l1\ x2$› **by** (*auto dest*: *in-field-if-in-codom*)
with *L2-le-unit2* ‹$x1 \leq_{L1} x2$› **show** $r2_{x1\ (l1\ x2)}\ y' \leq_{L2\ x1\ x2}\ r2_{x2\ (l1\ x2)}\ y'$
by *blast*
qed

lemma *left-Galois-iff-Dep-Fun-Rel-left-Galois-mono-assm-in-dom-rightI*:
assumes *mono-l1*: $((\leq_{L1}) \Rightarrow (\leq_{R1}))\ l1$
and *half-galois-prop-right1*: $((\leq_{L1}) \trianglelefteq (\leq_{R1}))\ l1\ r1$
and *refl-L1*: *reflexive-on* (*in-dom* $(\leq_{L1})$) $(\leq_{L1})$
and *mono-r2*: $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ {}_{L1}\lessapprox x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')})\ r2$
and $x\ {}_{L1}\lessapprox x'$
shows $((\textit{in-dom}\ (\leq_{R2\ (l1\ x)\ x'})) \Rrightarrow (\leq_{L2\ x\ (r1\ x')}))\ (r2_{x\ (l1\ x)})\ (r2_{x\ x'})$
proof −
from *mono-l1 half-galois-prop-right1 refl-L1* ‹$x\ {}_{L1}\lessapprox x'$›
have $x \leq_{L1} x\ l1\ x \leq_{R1} x'\ x\ {}_{L1}\lessapprox l1\ x$
by (*auto intro*!: *t1.half-galois-prop-leftD t1.left-Galois-left-if-left-relI*)
with *Dep-Fun-Rel-relD*[*OF dep-mono-wrt-relD*[*OF mono-r2* ‹$x \leq_{L1} x$›] ‹$l1\ x \leq_{R1} x'$›]
show *?thesis* **by** *blast*
qed

lemma *left-Galois-iff-Dep-Fun-Rel-left-Galois-if-monoI*:
assumes $((\leq_{L1}) \Rightarrow (\leq_{R1}))\ l1$
and $((\leq_{L1}) \trianglelefteq (\leq_{R1}))\ l1\ r1$
and *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
and $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox x' \Longrightarrow ((\leq_{R2\ (l1\ x)\ x'}) \Rightarrow (\leq_{L2\ x\ (r1\ x')}))\ (r2_{x\ x'})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ x1}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ (\eta_1\ x2)}) \leq (\leq_{L2\ x1\ x2})$

and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ {}_{L1}\lessapprox\ x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))\ r2$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *transitive* $(\leq_{L2\ x1\ x2})$
and $g \leq_R g$
shows $f\ {}_L\lessapprox\ g \longleftrightarrow ((x\ x' :: ({}_{L1}\lessapprox)) \Rrightarrow ({}_{L2\ x\ x'}\lessapprox))\ f\ g$
using *assms* **by** (*intro left-Galois-iff-Dep-Fun-Rel-left-GaloisI*
left-Galois-iff-Dep-Fun-Rel-left-Galois-ge-left-rel2-assmI
left-Galois-iff-Dep-Fun-Rel-left-Galois-ge-left-rel2-assmI'
left-Galois-iff-Dep-Fun-Rel-left-Galois-mono-assm-in-dom-rightI
left-Galois-iff-Dep-Fun-Rel-left-Galois-mono-assm-in-codom-rightI)
(*auto intro*: *reflexive-on-if-le-pred-if-reflexive-on in-field-if-in-dom*
in-field-if-in-codom)

lemma *left-Galois-iff-Dep-Fun-Rel-left-Galois-left-rel2-le-assmI*:
assumes *refl-L1*: *reflexive-on* (*in-dom* $(\leq_{L1})$) $(\leq_{L1})$
and *mono-L2*: $((x1 : \top) \Rightarrow (x2\ \text{-} :: (\leq_{L1}) \mid x1 \leq_{L1} x2) \Rightarrow (\leq))\ L2$
and $x1 \leq_{L1} x2$
shows $(\leq_{L2\ x1\ x1}) \leq (\leq_{L2\ x1\ x2})$
proof −
from *refl-L1* ‹$x1 \leq_{L1} x2$› **have** $x1 \leq_{L1} x1$ **by** *blast*
with *dep-mono-wrt-relD*[*OF dep-mono-wrt-predD*[*OF mono-L2*] ‹$x1 \leq_{L1} x2$›]
show $(\leq_{L2\ x1\ x1}) \leq (\leq_{L2\ x1\ x2})$ **by** *auto*
qed

lemma *left-Galois-iff-Dep-Fun-Rel-left-Galois-left-rel2-unit1-le-assmI*:
assumes *mono-l1*: $((\leq_{L1}) \Rightarrow (\leq_{R1}))\ l1$
and *half-galois-prop-right1*: $((\leq_{L1}) \trianglelefteq_h (\leq_{R1}))\ l1\ r1$
and *refl-L1*: *reflexive-on* (*in-codom* $(\leq_{L1})$) $(\leq_{L1})$
and *antimono-L2*:
$((x1 : \top) \Rightarrow (x2\ x3 :: (\leq_{L1}) \mid (x1 \leq_{L1} x2 \wedge x3 \leq_{L1} \eta_1\ x2)) \Rightarrow (\geq))\ L2$
and $x1 \leq_{L1} x2$
shows $(\leq_{L2\ x1\ (\eta_1\ x2)}) \leq (\leq_{L2\ x1\ x2})$
proof −
from *mono-l1 half-galois-prop-right1 refl-L1* ‹$x1 \leq_{L1} x2$› **have** $x2 \leq_{L1} \eta_1\ x2$
by (*blast intro*: *t1.rel-unit-if-reflexive-on-if-half-galois-prop-right-if-mono-wrt-rel*)
with *refl-L1* **have** $\eta_1\ x2 \leq_{L1} \eta_1\ x2$ **by** *blast*
with *dep-mono-wrt-relD*[*OF dep-mono-wrt-predD*[*OF antimono-L2*] ‹$x2 \leq_{L1} \eta_1\ x2$›]
show $(\leq_{L2\ x1\ (\eta_1\ x2)}) \leq (\leq_{L2\ x1\ x2})$ **using** ‹$x1 \leq_{L1} x2$› **by** *auto*
qed

lemma *left-Galois-iff-Dep-Fun-Rel-left-Galois-if-monoI'*:
assumes $((\leq_{L1}) \Rightarrow (\leq_{R1}))\ l1$
and $((\leq_{L1}) \trianglelefteq (\leq_{R1}))\ l1\ r1$
and *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
and $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox\ x' \Longrightarrow ((\leq_{R2\ (l1\ x)\ x'}) \Rightarrow (\leq_{L2\ x\ (r1\ x')}))\ (r2_{x\ x'})$
and $((x1 : \top) \Rightarrow (x2\ \text{-} :: (\leq_{L1}) \mid x1 \leq_{L1} x2) \Rightarrow (\leq))\ L2$
and $((x1 : \top) \Rightarrow (x2\ x3 :: (\leq_{L1}) \mid (x1 \leq_{L1} x2 \wedge x3 \leq_{L1} \eta_1\ x2)) \Rightarrow (\geq))\ L2$
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ {}_{L1}\lessapprox\ x1') \Rrightarrow$

in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')})$) *r2*
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *transitive* $(\leq_{L2\ x1\ x2})$
and $g \leq_R g$
shows $f\ {}_{L}\lessapprox\ g \longleftrightarrow ((x\ x' :: ({}_{L1}\lessapprox)) \Rrightarrow ({}_{L2\ x\ x'}\lessapprox))\ f\ g$
using *assms* **by** (*intro left-Galois-iff-Dep-Fun-Rel-left-Galois-if-monoI*
left-Galois-iff-Dep-Fun-Rel-left-Galois-left-rel2-unit1-le-assmI
left-Galois-iff-Dep-Fun-Rel-left-Galois-left-rel2-le-assmI)
(*auto intro*: *reflexive-on-if-le-pred-if-reflexive-on in-field-if-in-codom*
in-field-if-in-dom)

corollary *left-Galois-iff-Dep-Fun-Rel-left-Galois-if-mono-if-galois-connectionI*:
assumes $((\leq_{L1}) \dashv (\leq_{R1}))$ *l1 r1*
and *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
and $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox\ x' \Longrightarrow ((\leq_{R2\ (l1\ x)\ x'}) \Rightarrow (\leq_{L2\ x\ (r1\ x')}))\ ({r2}_{x\ x'})$
and $((x1 : \top) \Rightarrow (x2\ \text{-} :: (\leq_{L1}) \mid x1 \leq_{L1} x2) \Rightarrow (\leq))\ L2$
and $((x1 : \top) \Rightarrow (x2\ x3 :: (\leq_{L1}) \mid (x1 \leq_{L1} x2 \wedge x3 \leq_{L1} \eta_1\ x2)) \Rightarrow (\geq))\ L2$
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ {}_{L1}\lessapprox\ x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')})$) *r2*
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *transitive* $(\leq_{L2\ x1\ x2})$
and $g \leq_R g$
shows $f\ {}_{L}\lessapprox\ g \longleftrightarrow ((x\ x' :: ({}_{L1}\lessapprox)) \Rrightarrow ({}_{L2\ x\ x'}\lessapprox))\ f\ g$
using *assms* **by** (*intro left-Galois-iff-Dep-Fun-Rel-left-Galois-if-monoI'*) *auto*

interpretation *flip-inv* : *galois* $(\geq_{R1})$ $(\geq_{L1})$ *r1 l1* **.**

lemma *left-Galois-iff-Dep-Fun-Rel-left-Galois-left-rel2-unit1-le-assm-if-galois-equivI*:
assumes *galois-equiv1*: $((\leq_{L1}) \equiv_G (\leq_{R1}))$ *l1 r1*
and *refl-L1*: *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
and *mono-L2*: $((x1 : \top) \Rightarrow (x2\ \text{-} :: (\leq_{L1}) \mid x1 \leq_{L1} x2) \Rightarrow (\leq))\ L2$
and $x1 \leq_{L1} x2$
shows $(\leq_{L2\ x1\ (\eta_1\ x2)}) \leq (\leq_{L2\ x1\ x2})$
proof −
from *refl-L1* ‹$x1 \leq_{L1} x2$› **have** $x1 \leq_{L1} x1$ **by** *blast*
from *galois-equiv1 refl-L1* ‹$x1 \leq_{L1} x2$› **have** $\eta_1\ x2 \leq_{L1} x2$ **by** (*intro*
flip.t1.counit-rel-if-reflexive-on-if-half-galois-prop-left-if-mono-wrt-rel)
blast+
have $x1 \leq_{L1} \eta_1\ x2$ **by** (*rule t1.rel-unit-if-left-rel-if-mono-wrt-relI*)
(*insert galois-equiv1 refl-L1* ‹$x1 \leq_{L1} x2$›, *auto*)
with *dep-mono-wrt-relD*[*OF dep-mono-wrt-predD*[*OF mono-L2*] ‹$\eta_1\ x2 \leq_{L1} x2$›]
show $(\leq_{L2\ x1\ (\eta_1\ x2)}) \leq (\leq_{L2\ x1\ x2})$ **by** *auto*
qed

lemma *left-Galois-iff-Dep-Fun-Rel-left-Galois-if-galois-equivalenceI*:
assumes $((\leq_{L1}) \equiv_G (\leq_{R1}))$ *l1 r1*
and *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
and $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox\ x' \Longrightarrow ((\leq_{R2\ (l1\ x)\ x'}) \Rightarrow (\leq_{L2\ x\ (r1\ x')}))\ ({r2}_{x\ x'})$
and $((x1 : \top) \Rightarrow (x2\ \text{-} :: (\leq_{L1}) \mid x1 \leq_{L1} x2) \Rightarrow (\leq))\ L2$
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ {}_{L1}\lessapprox\ x1') \Rrightarrow$

in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')})$ *r2*
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *transitive* $(\leq_{L2\ x1\ x2})$
and $g \leq_R g$
shows $f\ {}_L\lessapprox\ g \longleftrightarrow ((x\ x' :: (_{L1}\lessapprox)) \Rrightarrow (_{L2\ x\ x'}\lessapprox))\ f\ g$
using *assms* **by** (*intro*
left-Galois-iff-Dep-Fun-Rel-left-Galois-if-monoI
left-Galois-iff-Dep-Fun-Rel-left-Galois-left-rel2-le-assmI
left-Galois-iff-Dep-Fun-Rel-left-Galois-left-rel2-unit1-le-assm-if-galois-equivI)
auto

corollary *left-Galois-iff-Dep-Fun-Rel-left-Galois-if-galois-equivalenceI'*:
assumes $((\leq_{L1}) \equiv_G (\leq_{R1}))$ *l1 r1*
and *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
and $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox\ x' \Longrightarrow ((\leq_{R2\ (l1\ x)\ x'}) \Rightarrow (\leq_{L2\ x\ (r1\ x')}))\ (r2_{x\ x'})$
and $((x1\ x2 :: (\geq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid x1 \leq_{L1} x3) \Rrightarrow (\leq))\ L2$
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ {}_{L1}\lessapprox\ x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))\ r2$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *transitive* $(\leq_{L2\ x1\ x2})$
and $g \leq_R g$
shows $f\ {}_L\lessapprox\ g \longleftrightarrow ((x\ x' :: (_{L1}\lessapprox)) \Rrightarrow (_{L2\ x\ x'}\lessapprox))\ f\ g$
using *assms* **by** (*intro left-Galois-iff-Dep-Fun-Rel-left-Galois-if-galois-equivalenceI*
reflexive-on-in-field-mono-assm-left2I)
auto

corollary *left-Galois-iff-Dep-Fun-Rel-left-Galois-if-preorder-equivalenceI*:
assumes $((\leq_{L1}) \equiv_{pre} (\leq_{R1}))$ *l1 r1*
and $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox\ x' \Longrightarrow ((\leq_{R2\ (l1\ x)\ x'}) \Rightarrow (\leq_{L2\ x\ (r1\ x')}))\ (r2_{x\ x'})$
and $((x1\ x2 :: (\geq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid x1 \leq_{L1} x3) \Rrightarrow (\leq))\ L2$
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ {}_{L1}\lessapprox\ x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))\ r2$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *transitive* $(\leq_{L2\ x1\ x2})$
and $g \leq_R g$
shows $f\ {}_L\lessapprox\ g \longleftrightarrow ((x\ x' :: (_{L1}\lessapprox)) \Rrightarrow (_{L2\ x\ x'}\lessapprox))\ f\ g$
using *assms* **by** (*intro left-Galois-iff-Dep-Fun-Rel-left-Galois-if-galois-equivalenceI'*)
auto

corollary *left-Galois-iff-Dep-Fun-Rel-left-Galois-if-preorder-equivalenceI'*:
assumes $((\leq_{L1}) \equiv_{pre} (\leq_{R1}))$ *l1 r1*
and $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox\ x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \equiv_{pre} (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $((x1\ x2 :: (\geq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid x1 \leq_{L1} x3) \Rrightarrow (\leq))\ L2$
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ {}_{L1}\lessapprox\ x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))\ r2$
and $g \leq_R g$
shows $f\ {}_L\lessapprox\ g \longleftrightarrow ((x\ x' :: (_{L1}\lessapprox)) \Rrightarrow (_{L2\ x\ x'}\lessapprox))\ f\ g$
using *assms* **by** (*intro left-Galois-iff-Dep-Fun-Rel-left-Galois-if-preorder-equivalenceI*
transitive-left2-if-preorder-equivalenceI)
(*auto 5 0*)

Simplification of Restricted Function Relator **lemma** *Dep-Fun-Rel-left-Galois-restrict-left-right-eq*
assumes $((\leq_{L1}) \Rightarrow (\leq_{R1}))$ *l1* **and** $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and $((\leq_{L1}) \trianglelefteq_h (\leq_{R1}))$ *l1 r1*
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow$
$((\leq_{L2\ (r1\ x1')\ (r1\ x2')})\ {}_h\trianglelefteq\ (\leq_{R2\ (\varepsilon_1\ x1')\ x2'}))\ (l2_{x2'\ (r1\ x1')})\ (r2_{(r1\ x1')\ x2'})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ (\eta_1\ x2)}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ (\varepsilon_1\ x1')\ x2'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x1'\ x2'\ y.\ x1' \leq_{R1} x2' \Longrightarrow$ *in-dom* $(\leq_{L2\ (r1\ x1')\ (r1\ x2')})\ y \Longrightarrow$
$(\leq_{R2\ x1'\ x2'})\ (l2_{x2'\ (r1\ x1')}\ y) \leq (\leq_{R2\ x1'\ x2'})\ (l2_{x1'\ (r1\ x1')}\ y)$
and $\bigwedge x1\ x2\ y'.\ x1 \leq_{L1} x2 \Longrightarrow$ *in-codom* $(\leq_{R2\ (l1\ x1)\ (l1\ x2)})\ y' \Longrightarrow$
$(\geq_{L2\ x1\ x2})\ (r2_{x1\ (l1\ x2)}\ y') \leq (\geq_{L2\ x1\ x2})\ (r2_{x2\ (l1\ x2)}\ y')$
shows $((x\ x' :: ({}_{L1}\lessapprox)) \Rrightarrow ({}_{L2\ x\ x'}\lessapprox))\restriction_{\textit{in-dom}\ (\leq_L)}{}^{\uparrow}{}_{\textit{in-codom}\ (\leq_R)}$
$= ((x\ x' :: ({}_{L1}\lessapprox)) \Rrightarrow ({}_{L2\ x\ x'}\lessapprox))$
using *assms* **by** (*intro ext iffI rel-restrict-leftI rel-restrict-rightI*
in-domI[**where** *?y=r* -] *left-rel-right-if-Dep-Fun-Rel-left-GaloisI*
in-codomI[**where** *?x=l* -] *left-right-rel-if-Dep-Fun-Rel-left-GaloisI*)
auto

lemma *Dep-Fun-Rel-left-Galois-restrict-left-right-eq-Dep-Fun-Rel-left-GaloisI'*:
assumes $((\leq_{L1}) \dashv (\leq_{R1}))$ *l1 r1*
and *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
and *reflexive-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow$
$((\leq_{L2\ (r1\ x1')\ (r1\ x2')})\ {}_h\trianglelefteq\ (\leq_{R2\ (\varepsilon_1\ x1')\ x2'}))\ (l2_{x2'\ (r1\ x1')})\ (r2_{(r1\ x1')\ x2'})$
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid (x2 \leq_{L1} x3 \wedge x4 \leq_{L1} \eta_1\ x3)) \Rrightarrow (\geq))\ L2$
and $((x1'\ x2' :: (\leq_{R1}) \mid \varepsilon_1\ x2' \leq_{R1} x1') \Rightarrow (x3'\ x4' :: (\leq_{R1}) \mid x2' \leq_{R1} x3') \Rrightarrow (\leq))\ R2$
and $((x1'\ x2' :: (\leq_{R1})) \Rightarrow (x1\ x2 :: (\leq_{L1}) \mid x2\ {}_{L1}\lessapprox\ x1') \Rrightarrow$
in-field $(\leq_{L2\ x1\ (r1\ x2')}) \Rrightarrow (\leq_{R2\ (l1\ x1)\ x2'}))\ l2$
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ {}_{L1}\lessapprox\ x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))\ r2$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *transitive* $(\leq_{L2\ x1\ x2})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow$ *transitive* $(\leq_{R2\ x1'\ x2'})$
shows $((x\ x' :: ({}_{L1}\lessapprox)) \Rrightarrow ({}_{L2\ x\ x'}\lessapprox))\restriction_{\textit{in-dom}\ (\leq_L)}{}^{\uparrow}{}_{\textit{in-codom}\ (\leq_R)}$
$= ((x\ x' :: ({}_{L1}\lessapprox)) \Rrightarrow ({}_{L2\ x\ x'}\lessapprox))$
using *assms* **by** (*intro*
Dep-Fun-Rel-left-Galois-restrict-left-right-eq-Dep-Fun-Rel-left-GaloisI
left-rel-right-iff-left-right-rel-if-galois-prop-le-assms-leftI
reflexive-on-in-field-mono-assm-left2I
left-rel-right-iff-left-right-rel-if-galois-prop-le-assms-rightI
mono-wrt-rel-left-in-dom-mono-left-assm
galois-connection-left-right-if-galois-connection-mono-assms-leftI
galois-connection-left-right-if-galois-connection-mono-assms-rightI
left-Galois-iff-Dep-Fun-Rel-left-Galois-ge-left-rel2-assmI)
auto

lemma *Dep-Fun-Rel-left-Galois-restrict-left-right-restrict-left-right-eq*:
fixes $S :: 'a1 \Rightarrow 'a2 \Rightarrow 'b1 \Rightarrow 'b2 \Rightarrow bool$
assumes $((\leq_{L1}) \ {}_h\trianglelefteq \ (\leq_{R1}))$ *l1 r1*
shows $((x\ x' :: (_{L1}\overset{<}{\approx})) \Rrightarrow (S\ x\ x')\restriction_{in\text{-}dom\ (\leq_{L2}\ x\ (r1\ x'))}\rceil_{in\text{-}codom\ (\leq_{R2}\ (l1\ x)\ x')})$
$\restriction_{in\text{-}dom\ (\leq_L)}\rceil_{in\text{-}codom\ (\leq_R)} =$
$((x\ x' :: (_{L1}\overset{<}{\approx})) \Rrightarrow S\ x\ x')\restriction_{in\text{-}dom\ (\leq_L)}\rceil_{in\text{-}codom\ (\leq_R)}$ (**is** *?lhs = ?rhs*)
proof −
have $?lhs = ((x\ x' :: (_{L1}\overset{<}{\approx})) \Rrightarrow (S\ x\ x')\rceil_{in\text{-}codom\ (\leq_{R2}\ (l1\ x)\ x')})\restriction_{in\text{-}dom\ (\leq_L)}\rceil_{in\text{-}codom\ (\leq_R)}$
by (*subst rel-restrict-left-right-eq-restrict-right-left*,
subst Dep-Fun-Rel-restrict-left-restrict-left-eq)
auto
also have *... = ?rhs*
using *assms* **by** (*subst rel-restrict-left-right-eq-restrict-right-left*,
subst Dep-Fun-Rel-restrict-right-restrict-right-eq)
(*auto elim*!: *in-codomE t1.left-GaloisE*
simp only: *rel-restrict-left-right-eq-restrict-right-left*)
finally show *?thesis* **.**
qed

end

Function Relator **context** *transport-Fun-Rel*
begin

corollary *Fun-Rel-left-Galois-if-left-GaloisI*:
assumes $((\leq_{L1}) \ {}_h\trianglelefteq \ (\leq_{R1}))$ *l1 r1*
and *reflexive-on* (*in-dom* $(\leq_{L1})$) $(\leq_{L1})$
and $((\leq_{R2}) \Rightarrow (\leq_{L2}))$ *r2*
and *transitive* $(\leq_{L2})$
and $g \leq_R g$
and $f\ {}_L\overset{<}{\approx}\ g$
shows $((_{L1}\overset{<}{\approx}) \Rrightarrow (_{L2}\overset{<}{\approx}))\ f\ g$
by (*urule tdfr.Dep-Fun-Rel-left-Galois-if-left-GaloisI assms* | *simp*)+

corollary *left-Galois-if-Fun-Rel-left-GaloisI*:
assumes $((\leq_{L1}) \Rightarrow (\leq_{R1}))$ *l1*
and $((\leq_{L1}) \trianglelefteq_h (\leq_{R1}))$ *l1 r1*
and *in-codom* $(\leq_R)$ *g*
and $((_{L1}\overset{<}{\approx}) \Rrightarrow (_{L2}\overset{<}{\approx}))\ f\ g$
shows $f\ {}_L\overset{<}{\approx}\ g$
by (*urule tdfr.left-Galois-if-Dep-Fun-Rel-left-GaloisI assms* | *simp*)+

lemma *left-Galois-if-Fun-Rel-left-GaloisI'*:
assumes $((\leq_{L1}) \Rightarrow (\leq_{R1}))$ *l1* **and** $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and $((\leq_{L1}) \trianglelefteq_h (\leq_{R1}))$ *l1 r1*
and $((\leq_{L2}) \ {}_h\trianglelefteq \ (\leq_{R2}))$ *l2 r2*

and $((_{L1}\lessapprox) \Rrightarrow (_{L2}\lessapprox))\ f\ g$
shows $f\ _{L}\lessapprox\ g$
by (*urule tdfr.left-Galois-if-Dep-Fun-Rel-left-GaloisI′ assms* | *simp*)+

corollary *left-Galois-iff-Fun-Rel-left-GaloisI*:
assumes $((\leq_{L1}) \Rightarrow (\leq_{R1}))\ l1$
and $((\leq_{L1}) \trianglelefteq (\leq_{R1}))\ l1\ r1$
and *reflexive-on* (*in-dom* $(\leq_{L1})$) $(\leq_{L1})$
and $((\leq_{R2}) \Rightarrow (\leq_{L2}))\ r2$
and *transitive* $(\leq_{L2})$
and $g \leq_{R} g$
shows $f\ _{L}\lessapprox\ g \longleftrightarrow ((_{L1}\lessapprox) \Rrightarrow (_{L2}\lessapprox))\ f\ g$
by (*urule tdfr.left-Galois-iff-Dep-Fun-Rel-left-GaloisI assms* | *simp*)+

Simplification of Restricted Function Relator **lemma** *Fun-Rel-left-Galois-restrict-left-right-eq-Fun*
assumes $((\leq_{L1}) \Rightarrow (\leq_{R1}))\ l1$ **and** $((\leq_{R1}) \Rightarrow (\leq_{L1}))\ r1$
and $((\leq_{L1}) \trianglelefteq_{h} (\leq_{R1}))\ l1\ r1$
and $((\leq_{L2})\ {}_{h}\trianglelefteq (\leq_{R2}))\ l2\ r2$
shows $((_{L1}\lessapprox) \Rrightarrow (_{L2}\lessapprox))\restriction_{in\text{-}dom\ (\leq_{L})}\rceil_{in\text{-}codom\ (\leq_{R})} = ((_{L1}\lessapprox) \Rrightarrow (_{L2}\lessapprox))$
by (*urule tdfr.Dep-Fun-Rel-left-Galois-restrict-left-right-eq-Dep-Fun-Rel-left-GaloisI assms* | *simp*)+

Simplification of Restricted Function Relator for Nested Transports

lemma *Fun-Rel-left-Galois-restrict-left-right-restrict-left-right-eq*:
fixes $S :: 'b1 \Rightarrow 'b2 \Rightarrow bool$
assumes $((\leq_{L1})\ {}_{h}\trianglelefteq (\leq_{R1}))\ l1\ r1$
shows $((_{L1}\lessapprox) \Rrightarrow S\restriction_{in\text{-}dom\ (\leq_{L2})}\rceil_{in\text{-}codom\ (\leq_{R2})})\restriction_{in\text{-}dom\ (\leq_{L})}\rceil_{in\text{-}codom\ (\leq_{R})}$
$=$
$((_{L1}\lessapprox) \Rrightarrow S)\restriction_{in\text{-}dom\ (\leq_{L})}\rceil_{in\text{-}codom\ (\leq_{R})}$
by (*urule tdfr.Dep-Fun-Rel-left-Galois-restrict-left-right-restrict-left-right-eq assms* | *simp*)+

end

Monotone Dependent Function Relator **context** *transport-Mono-Dep-Fun-Rel* **begin**

lemma *Dep-Fun-Rel-left-Galois-if-left-GaloisI*:
assumes $((\leq_{L1})\ {}_{h}\trianglelefteq (\leq_{R1}))\ l1\ r1$
and *reflexive-on* (*in-dom* $(\leq_{L1})$) $(\leq_{L1})$
and $\bigwedge x\ x'.\ x\ _{L1}\lessapprox\ x' \Longrightarrow ((\leq_{R2}\ (l1\ x)\ x') \Rightarrow (\leq_{L2}\ x\ (r1\ x')))\ (r2_{x\ x'})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2}\ x1\ x1) \leq (\leq_{L2}\ x1\ x2)$
and $\bigwedge x\ x'\ y'.\ x\ _{L1}\lessapprox\ x' \Longrightarrow in\text{-}dom\ (\leq_{R2}\ (l1\ x)\ x')\ y' \Longrightarrow$
$(\geq_{L2}\ x\ (r1\ x'))\ (r2_{x}\ (l1\ x)\ y') \leq (\geq_{L2}\ x\ (r1\ x'))\ (r2_{x\ x'}\ y')$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow transitive\ (\leq_{L2}\ x1\ x2)$
and $f\ _{L}\lessapprox\ g$
shows $((x\ x' :: (_{L1}\lessapprox)) \Rrightarrow (_{L2\ x\ x'}\lessapprox))\ f\ g$

using *assms* **unfolding** *left-rel-eq-tdfr-left-Refl-Rel right-rel-eq-tdfr-right-Refl-Rel*
by (*intro tdfr.Dep-Fun-Rel-left-Galois-if-left-GaloisI tdfr.left-GaloisI*)
(*auto elim*!: *galois-rel.left-GaloisE in-codomE*)

lemma *left-Galois-if-Dep-Fun-Rel-left-GaloisI*:
assumes (*tdfr.R* ⇒ *tdfr.L*) *r*
and ((≤$_{L1}$) ⇒ (≤$_{R1}$)) *l1*
and ((≤$_{L1}$) ⊴$_h$ (≤$_{R1}$)) *l1 r1*
and ⋀*x1 x2*. *x1* ≤$_{L1}$ *x2* ⟹ (≤$_{L2\ x1\ (\eta_1\ x2)}$) ≤ (≤$_{L2\ x1\ x2}$)
and ⋀*x1 x2 y'*. *x1* ≤$_{L1}$ *x2* ⟹ *in-codom* (≤$_{R2\ (l1\ x1)\ (l1\ x2)}$) *y'* ⟹
(≥$_{L2\ x1\ x2}$) ($r2_{x1\ (l1\ x2)}$ *y'*) ≤ (≥$_{L2\ x1\ x2}$) ($r2_{x2\ (l1\ x2)}$ *y'*)
and *in-dom* (≤$_L$) *f*
and *in-codom* (≤$_R$) *g*
and ((*x x'* :: ($_{L1}\lessapprox$)) ⇛ ($_{L2\ x\ x'}\lessapprox$)) *f g*
shows *f* $_L\lessapprox$ *g*
using *assms* **unfolding** *left-rel-eq-tdfr-left-Refl-Rel right-rel-eq-tdfr-right-Refl-Rel*
by (*intro tdfr.Galois-Refl-RelI tdfr.left-Galois-if-Dep-Fun-Rel-left-GaloisI*)
(*auto simp*: *in-codom-eq-in-dom-if-reflexive-on-in-field*)

lemma *left-Galois-iff-Dep-Fun-Rel-left-GaloisI*:
assumes (*tdfr.R* ⇒ *tdfr.L*) *r*
and ((≤$_{L1}$) ⇒ (≤$_{R1}$)) *l1*
and ((≤$_{L1}$) ⊴ (≤$_{R1}$)) *l1 r1*
and *reflexive-on* (*in-field* (≤$_{L1}$)) (≤$_{L1}$)
and ⋀*x x'*. *x* $_{L1}\lessapprox$ *x'* ⟹ ((≤$_{R2\ (l1\ x)\ x'}$) ⇒ (≤$_{L2\ x\ (r1\ x')}$)) ($r2_{x\ x'}$)
and ⋀*x1 x2*. *x1* ≤$_{L1}$ *x2* ⟹ (≤$_{L2\ x1\ x1}$) ≤ (≤$_{L2\ x1\ x2}$)
and ⋀*x1 x2*. *x1* ≤$_{L1}$ *x2* ⟹ (≤$_{L2\ x1\ (\eta_1\ x2)}$) ≤ (≤$_{L2\ x1\ x2}$)
and ((*x1 x2* :: (≤$_{L1}$)) ⇒ (*x1' x2'* :: (≤$_{R1}$) | *x2* $_{L1}\lessapprox$ *x1'*) ⇛
in-field (≤$_{R2\ (l1\ x1)\ x2'}$) ⇛ (≤$_{L2\ x1\ (r1\ x2')}$)) *r2*
and ⋀*x1 x2*. *x1* ≤$_{L1}$ *x2* ⟹ *transitive* (≤$_{L2\ x1\ x2}$)
and *in-dom* (≤$_L$) *f*
and *in-codom* (≤$_R$) *g*
shows *f* $_L\lessapprox$ *g* ⟷ ((*x x'* :: ($_{L1}\lessapprox$)) ⇛ ($_{L2\ x\ x'}\lessapprox$)) *f g*
using *assms* **by** (*intro iffI Dep-Fun-Rel-left-Galois-if-left-GaloisI*
tdfr.left-Galois-iff-Dep-Fun-Rel-left-Galois-ge-left-rel2-assmI'
tdfr.left-Galois-iff-Dep-Fun-Rel-left-Galois-mono-assm-in-dom-rightI)
(*auto intro*!: *left-Galois-if-Dep-Fun-Rel-left-GaloisI*
tdfr.left-Galois-iff-Dep-Fun-Rel-left-Galois-ge-left-rel2-assmI
tdfr.left-Galois-iff-Dep-Fun-Rel-left-Galois-mono-assm-in-codom-rightI
intro: *reflexive-on-if-le-pred-if-reflexive-on*
in-field-if-in-dom in-field-if-in-codom)

lemma *left-Galois-iff-Dep-Fun-Rel-left-Galois-if-mono-if-galois-connectionI*:
assumes *galois-conn1*: ((≤$_{L1}$) ⊣ (≤$_{R1}$)) *l1 r1*
and *refl-L1*: *reflexive-on* (*in-field* (≤$_{L1}$)) (≤$_{L1}$)
and ⋀*x x'*. *x* $_{L1}\lessapprox$ *x'* ⟹ ((≤$_{R2\ (l1\ x)\ x'}$) ⇒ (≤$_{L2\ x\ (r1\ x')}$)) ($r2_{x\ x'}$)
and ⋀*x1 x2*. *x1* ≤$_{L1}$ *x2* ⟹ (≤$_{L2\ x1\ x1}$) ≤ (≤$_{L2\ x1\ x2}$)
and *L2-le-unit2*: ⋀*x1 x2*. *x1* ≤$_{L1}$ *x2* ⟹ (≤$_{L2\ x1\ (\eta_1\ x2)}$) ≤ (≤$_{L2\ x1\ x2}$)

and *mono-r2*: $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ {}_{L1}\lesssim\!\approx\ x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))\ r2$
and *trans-L2*: $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *transitive* $(\leq_{L2\ x1\ x2})$
and *in-dom* $(\leq_L)\ f$
and *in-codom* $(\leq_R)\ g$
shows $f\ {}_L\lesssim\!\approx\ g \longleftrightarrow ((x\ x' :: ({}_{L1}\lesssim\!\approx)) \Rrightarrow ({}_{L2\ x\ x'}\lesssim\!\approx))\ f\ g$ (**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof −
have $(\leq_{L2\ x1\ x2})\ ({r2}_{x1\ (l1\ x2)}\ y') \leq (\leq_{L2\ x1\ x2})\ ({r2}_{x1\ (l1\ x1)}\ y')$
if *hyps*: $x1 \leq_{L1} x2$ *in-dom* $(\leq_{R2\ (l1\ x1)\ (l1\ x2)})\ y'$ **for** $x1\ x2\ y'$
proof −
have $((\textit{in-dom}\ (\leq_{R2\ (l1\ x1)\ (l1\ x2)})) \Rrightarrow (\leq_{L2\ x1\ x2}))\ ({r2}_{x1\ (l1\ x1)})\ ({r2}_{x1\ (l1\ x2)})$
proof (*intro Fun-Rel-predI*)
from *galois-conn1 refl-L1* ‹$x1 \leq_{L1} x2$›
have $x1 \leq_{L1} x1\ l1\ x1 \leq_{R1} l1\ x2\ x1\ {}_{L1}\lesssim\!\approx\ l1\ x1$
by (*blast intro*: *t1.left-Galois-left-if-left-relI*)+
fix y' **assume** *in-dom* $(\leq_{R2\ (l1\ x1)\ (l1\ x2)})\ y'$
with *Dep-Fun-Rel-relD*[*OF dep-mono-wrt-relD*[*OF mono-r2* ‹$x1 \leq_{L1} x1$›]
‹$l1\ x1 \leq_{R1} l1\ x2$›]
have ${r2}_{x1\ (l1\ x1)}\ y' \leq_{L2\ x1\ (\eta_1\ x2)}\ {r2}_{x1\ (l1\ x2)}\ y'$
using ‹$x1\ {}_{L1}\lesssim\!\approx\ l1\ x1$› **by** (*auto dest*: *in-field-if-in-dom*)
with *L2-le-unit2* ‹$x1 \leq_{L1} x2$› **show** ${r2}_{x1\ (l1\ x1)}\ y' \leq_{L2\ x1\ x2}\ {r2}_{x1\ (l1\ x2)}\ y'$
by *blast*
qed
with *hyps* **show** *?thesis* **using** *trans-L2* **by** *blast*
qed
then show *?thesis* **using** *assms*
using *assms* **by** (*intro left-Galois-iff-Dep-Fun-Rel-left-GaloisI*
tdfr.mono-wrt-rel-rightI
tdfr.mono-wrt-rel-right2-if-mono-wrt-rel-right2-if-left-GaloisI
tdfr.left-Galois-iff-Dep-Fun-Rel-left-Galois-ge-left-rel2-assmI
tdfr.left-Galois-iff-Dep-Fun-Rel-left-Galois-mono-assm-in-codom-rightI)
(*auto intro*: *reflexive-on-if-le-pred-if-reflexive-on in-field-if-in-codom*)
qed

corollary *left-Galois-iff-Dep-Fun-Rel-left-Galois-if-mono-if-galois-connectionI'*:
assumes $((\leq_{L1}) \dashv (\leq_{R1}))\ l1\ r1$
and *reflexive-on* $(\textit{in-field}\ (\leq_{L1}))\ (\leq_{L1})$
and $\bigwedge x\ x'.\ x\ {}_{L1}\lesssim\!\approx\ x' \Longrightarrow ((\leq_{R2\ (l1\ x)\ x'}) \Rightarrow (\leq_{L2\ x\ (r1\ x')}))\ ({r2}_{x\ x'})$
and $((x1 : \top) \Rightarrow (x2\ \text{-} :: (\leq_{L1}) \mid x1 \leq_{L1} x2) \Rightarrow (\leq))\ L2$
and $((x1 : \top) \Rightarrow (x2\ x3 :: (\leq_{L1}) \mid (x1 \leq_{L1} x2 \wedge x3 \leq_{L1} \eta_1\ x2)) \Rightarrow (\geq))\ L2$
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ {}_{L1}\lesssim\!\approx\ x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))\ r2$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *transitive* $(\leq_{L2\ x1\ x2})$
and *in-dom* $(\leq_L)\ f$
and *in-codom* $(\leq_R)\ g$
shows $f\ {}_L\lesssim\!\approx\ g \longleftrightarrow ((x\ x' :: ({}_{L1}\lesssim\!\approx)) \Rrightarrow ({}_{L2\ x\ x'}\lesssim\!\approx))\ f\ g$ (**is** *?lhs* $\longleftrightarrow$ *?rhs*)
using *assms* **by** (*intro*

left-Galois-iff-Dep-Fun-Rel-left-Galois-if-mono-if-galois-connectionI
tdfr.left-Galois-iff-Dep-Fun-Rel-left-Galois-left-rel2-unit1-le-assmI
tdfr.left-Galois-iff-Dep-Fun-Rel-left-Galois-left-rel2-le-assmI)
auto

corollary *left-Galois-eq-Dep-Fun-Rel-left-Galois-restrict-if-mono-if-galois-connectionI*:
assumes $((\leq_{L1}) \dashv (\leq_{R1}))$ *l1 r1*
and *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
and $\bigwedge x\ x'.\ x\ _{L1}\lessapprox x' \Longrightarrow ((\leq_{R2\ (l1\ x)\ x'}) \Rightarrow (\leq_{L2\ x\ (r1\ x')}))\ (r2_{x\ x'})$
and $((x1 : \top) \Rightarrow (x2\ \text{-} :: (\leq_{L1}) \mid x1 \leq_{L1} x2) \Rightarrow (\leq))\ L2$
and $((x1 : \top) \Rightarrow (x2\ x3 :: (\leq_{L1}) \mid (x1 \leq_{L1} x2 \wedge x3 \leq_{L1} \eta_1\ x2)) \Rightarrow (\geq))\ L2$
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ _{L1}\lessapprox x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))\ r2$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow transitive\ (\leq_{L2\ x1\ x2})$
shows $(_L\lessapprox) = ((x\ x' :: (_{L1}\lessapprox)) \Rightarrow (_{L2\ x\ x'}\lessapprox))\restriction_{in\text{-}dom\ (\leq_L)}\restriction_{in\text{-}codom\ (\leq_R)}$
using *assms* **by** (*intro ext iffI rel-restrict-leftI rel-restrict-rightI*
iffD1[*OF left-Galois-iff-Dep-Fun-Rel-left-Galois-if-mono-if-galois-connectionI'*])
(*auto intro*!:
iffD2[*OF left-Galois-iff-Dep-Fun-Rel-left-Galois-if-mono-if-galois-connectionI'*])

lemma *left-Galois-iff-Dep-Fun-Rel-left-Galois-if-galois-equivalenceI*:
assumes $((\leq_{L1}) \equiv_G (\leq_{R1}))$ *l1 r1*
and *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
and $\bigwedge x\ x'.\ x\ _{L1}\lessapprox x' \Longrightarrow ((\leq_{R2\ (l1\ x)\ x'}) \Rightarrow (\leq_{L2\ x\ (r1\ x')}))\ (r2_{x\ x'})$
and $((x1\ x2 :: (\geq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid x1 \leq_{L1} x3) \Rrightarrow (\leq))\ L2$
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ _{L1}\lessapprox x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))\ r2$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow transitive\ (\leq_{L2\ x1\ x2})$
and *in-dom* $(\leq_L)\ f$
and *in-codom* $(\leq_R)\ g$
shows $f\ _L\lessapprox\ g \longleftrightarrow ((x\ x' :: (_{L1}\lessapprox)) \Rrightarrow (_{L2\ x\ x'}\lessapprox))\ f\ g$
using *assms* **by** (*intro left-Galois-iff-Dep-Fun-Rel-left-Galois-if-mono-if-galois-connectionI*
tdfr.left-Galois-iff-Dep-Fun-Rel-left-Galois-left-rel2-le-assmI
tdfr.reflexive-on-in-field-mono-assm-left2I
tdfr.left-Galois-iff-Dep-Fun-Rel-left-Galois-left-rel2-unit1-le-assm-if-galois-equivI)
auto

theorem *left-Galois-eq-Dep-Fun-Rel-left-Galois-restrict-if-galois-equivalenceI*:
assumes $((\leq_{L1}) \equiv_G (\leq_{R1}))$ *l1 r1*
and *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
and $\bigwedge x\ x'.\ x\ _{L1}\lessapprox x' \Longrightarrow ((\leq_{R2\ (l1\ x)\ x'}) \Rightarrow (\leq_{L2\ x\ (r1\ x')}))\ (r2_{x\ x'})$
and $((x1\ x2 :: (\geq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid x1 \leq_{L1} x3) \Rrightarrow (\leq))\ L2$
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ _{L1}\lessapprox x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))\ r2$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow transitive\ (\leq_{L2\ x1\ x2})$
shows $(_L\lessapprox) = ((x\ x' :: (_{L1}\lessapprox)) \Rightarrow (_{L2\ x\ x'}\lessapprox))\restriction_{in\text{-}dom\ (\leq_L)}\restriction_{in\text{-}codom\ (\leq_R)}$
using *assms* **by** (*intro ext iffI rel-restrict-leftI rel-restrict-rightI*
iffD1[*OF left-Galois-iff-Dep-Fun-Rel-left-Galois-if-galois-equivalenceI*])

(*auto intro*!: *iffD2*[*OF left-Galois-iff-Dep-Fun-Rel-left-Galois-if-galois-equivalenceI*])

corollary *left-Galois-iff-Dep-Fun-Rel-left-Galois-if-preorder-equivalenceI*:
assumes $((\leq_{L1}) \equiv_{pre} (\leq_{R1}))$ *l1 r1*
and $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox\ x' \Longrightarrow ((\leq_{R2\ (l1\ x)\ x'}) \Rightarrow (\leq_{L2\ x\ (r1\ x')}))\ (r2_{x\ x'})$
and $((x1\ x2 :: (\geq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid x1 \leq_{L1} x3) \Rrightarrow (\leq))$ *L2*
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ {}_{L1}\lessapprox\ x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))$ *r2*
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *transitive* $(\leq_{L2\ x1\ x2})$
and *in-dom* $(\leq_L)$ *f*
and *in-codom* $(\leq_R)$ *g*
shows $f\ {}_L\lessapprox\ g \longleftrightarrow ((x\ x' :: ({}_{L1}\lessapprox)) \Rrightarrow ({}_{L2\ x\ x'}\lessapprox))\ f\ g$
using *assms* **by** (*intro left-Galois-iff-Dep-Fun-Rel-left-Galois-if-galois-equivalenceI*)
auto

corollary *left-Galois-eq-Dep-Fun-Rel-left-Galois-restrict-if-preorder-equivalenceI*:
assumes $((\leq_{L1}) \equiv_{pre} (\leq_{R1}))$ *l1 r1*
and $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox\ x' \Longrightarrow ((\leq_{R2\ (l1\ x)\ x'}) \Rightarrow (\leq_{L2\ x\ (r1\ x')}))\ (r2_{x\ x'})$
and $((x1\ x2 :: (\geq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid x1 \leq_{L1} x3) \Rrightarrow (\leq))$ *L2*
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ {}_{L1}\lessapprox\ x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))$ *r2*
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *transitive* $(\leq_{L2\ x1\ x2})$
shows $({}_L\lessapprox) = ((x\ x' :: ({}_{L1}\lessapprox)) \Rrightarrow ({}_{L2\ x\ x'}\lessapprox))\restriction_{in\text{-}dom\ (\leq_L)}\restriction_{in\text{-}codom\ (\leq_R)}$
using *assms* **by** (*intro ext iffI rel-restrict-leftI rel-restrict-rightI*
iffD1[*OF left-Galois-iff-Dep-Fun-Rel-left-Galois-if-preorder-equivalenceI*])
(*auto intro*!: *iffD2*[*OF left-Galois-iff-Dep-Fun-Rel-left-Galois-if-preorder-equivalenceI*])

corollary *left-Galois-iff-Dep-Fun-Rel-left-Galois-if-preorder-equivalenceI'*:
assumes $((\leq_{L1}) \equiv_{pre} (\leq_{R1}))$ *l1 r1*
and $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox\ x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \equiv_{pre} (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $((x1\ x2 :: (\geq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid x1 \leq_{L1} x3) \Rrightarrow (\leq))$ *L2*
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ {}_{L1}\lessapprox\ x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))$ *r2*
and *in-dom* $(\leq_L)$ *f*
and *in-codom* $(\leq_R)$ *g*
shows $f\ {}_L\lessapprox\ g \longleftrightarrow ((x\ x' :: ({}_{L1}\lessapprox)) \Rrightarrow ({}_{L2\ x\ x'}\lessapprox))\ f\ g$
using *assms* **by** (*intro left-Galois-iff-Dep-Fun-Rel-left-Galois-if-preorder-equivalenceI*
tdfr.transitive-left2-if-preorder-equivalenceI)
(*auto 5 0*)

corollary *left-Galois-eq-Dep-Fun-Rel-left-Galois-restrict-if-preorder-equivalenceI'*:
assumes $((\leq_{L1}) \equiv_{pre} (\leq_{R1}))$ *l1 r1*
and $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox\ x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \equiv_{pre} (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $((x1\ x2 :: (\geq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid x1 \leq_{L1} x3) \Rrightarrow (\leq))$ *L2*
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ {}_{L1}\lessapprox\ x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))$ *r2*
shows $({}_L\lessapprox) = ((x\ x' :: ({}_{L1}\lessapprox)) \Rrightarrow ({}_{L2\ x\ x'}\lessapprox))\restriction_{in\text{-}dom\ (\leq_L)}\restriction_{in\text{-}codom\ (\leq_R)}$

using *assms* **by** (*intro ext iffI rel-restrict-leftI rel-restrict-rightI*
iffD1[*OF left-Galois-iff-Dep-Fun-Rel-left-Galois-if-preorder-equivalenceI'*])
(*auto intro*!: *iffD2*[*OF left-Galois-iff-Dep-Fun-Rel-left-Galois-if-preorder-equivalenceI'*])

Simplification of Restricted Function Relator **lemma** *Dep-Fun-Rel-left-Galois-restrict-left-right-eq*
assumes *reflexive-on* (*in-field tdfr.L*) *tdfr.L*
and *reflexive-on* (*in-field tdfr.R*) *tdfr.R*
and $((\leq_{L1}) \dashv (\leq_{R1}))$ *l1 r1*
and *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
and *reflexive-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow$
$((\leq_{L2\ (r1\ x1')\ (r1\ x2')})\ {}_h\trianglelefteq\ (\leq_{R2\ (\varepsilon_1\ x1')\ x2'}))\ (l2_{x2'\ (r1\ x1')})\ (r2_{(r1\ x1')\ x2'})$
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid (x2 \leq_{L1} x3 \wedge x4 \leq_{L1} \eta_1\ x3)) \Rrightarrow (\geq))$ *L2*
and $((x1'\ x2' :: (\leq_{R1}) \mid \varepsilon_1\ x2' \leq_{R1} x1') \Rightarrow (x3'\ x4' :: (\leq_{R1}) \mid x2' \leq_{R1} x3') \Rrightarrow (\leq))$ *R2*
and $((x1'\ x2' :: (\leq_{R1})) \Rightarrow (x1\ x2 :: (\leq_{L1}) \mid x2\ {}_{L1}\lessapprox\ x1') \Rrightarrow$
in-field $(\leq_{L2\ x1\ (r1\ x2')}) \Rrightarrow (\leq_{R2\ (l1\ x1)\ x2'}))$ *l2*
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ {}_{L1}\lessapprox\ x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))$ *r2*
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *transitive* $(\leq_{L2\ x1\ x2})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow$ *transitive* $(\leq_{R2\ x1'\ x2'})$
shows $((x\ x' :: ({}_{L1}\lessapprox)) \Rrightarrow ({}_{L2\ x\ x'}\lessapprox))\restriction_{in\text{-}dom\ (\leq_L)}\rceil_{in\text{-}codom\ (\leq_R)}$
$= ((x\ x' :: ({}_{L1}\lessapprox)) \Rrightarrow ({}_{L2\ x\ x'}\lessapprox))$
using *assms* **by** (*auto simp only*: *left-rel-eq-tdfr-left-rel-if-reflexive-on*
right-rel-eq-tdfr-right-rel-if-reflexive-on
intro!: *tdfr.Dep-Fun-Rel-left-Galois-restrict-left-right-eq-Dep-Fun-Rel-left-GaloisI'*)

interpretation *flip* : *transport-Dep-Fun-Rel R1 L1 r1 l1 R2 L2 r2 l2*
rewrites $flip.t1.unit \equiv \varepsilon_1$ **by** (*simp only*: *t1.flip-unit-eq-counit*)

lemma *Dep-Fun-Rel-left-Galois-restrict-left-right-eq-Dep-Fun-Rel-left-GaloisI*:
assumes $((\leq_{L1}) \equiv_{pre} (\leq_{R1}))$ *l1 r1*
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow$
$((\leq_{L2\ (r1\ x1')\ (r1\ x2')})\ {}_h\trianglelefteq\ (\leq_{R2\ (\varepsilon_1\ x1')\ x2'}))\ (l2_{x2'\ (r1\ x1')})\ (r2_{(r1\ x1')\ x2'})$
and $((x1\ x2 :: (\geq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid x1 \leq_{L1} x3) \Rrightarrow (\leq))$ *L2*
and $((x1'\ x2' :: (\geq_{R1})) \Rightarrow (x3'\ x4' :: (\leq_{R1}) \mid x1' \leq_{R1} x3') \Rrightarrow (\leq))$ *R2*
and $((x1'\ x2' :: (\leq_{R1})) \Rightarrow (x1\ x2 :: (\leq_{L1}) \mid x2\ {}_{L1}\lessapprox\ x1') \Rrightarrow$
in-field $(\leq_{L2\ x1\ (r1\ x2')}) \Rrightarrow (\leq_{R2\ (l1\ x1)\ x2'}))$ *l2*
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ {}_{L1}\lessapprox\ x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))$ *r2*
and *PERS*: $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *partial-equivalence-rel* $(\leq_{L2\ x1\ x2})$
$\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow$ *partial-equivalence-rel* $(\leq_{R2\ x1'\ x2'})$
shows $((x\ x' :: ({}_{L1}\lessapprox)) \Rrightarrow ({}_{L2\ x\ x'}\lessapprox))\restriction_{in\text{-}dom\ (\leq_L)}\rceil_{in\text{-}codom\ (\leq_R)}$
$= ((x\ x' :: ({}_{L1}\lessapprox)) \Rrightarrow ({}_{L2\ x\ x'}\lessapprox))$
using *assms* **by** (*intro*
Dep-Fun-Rel-left-Galois-restrict-left-right-eq-Dep-Fun-Rel-left-Galois-if-reflexive-onI

tdfr.reflexive-on-in-field-left-if-equivalencesI
flip.reflexive-on-in-field-left-if-equivalencesI
tdfr.galois-equivalence-if-mono-if-galois-equivalence-mono-assms-leftI
flip.galois-equivalence-if-mono-if-galois-equivalence-mono-assms-leftI)
(*auto dest*!: *PERS*)

Simplification of Restricted Function Relator for Nested Transports

lemma *Dep-Fun-Rel-left-Galois-restrict-left-right-restrict-left-right-eq*:
fixes $S :: 'a1 \Rightarrow 'a2 \Rightarrow 'b1 \Rightarrow 'b2 \Rightarrow bool$
assumes $((\leq_{L1}) \ {}_h\trianglelefteq \ (\leq_{R1}))$ *l1 r1*
shows $((x\ x' :: (_{L1}\lessapprox)) \Rrightarrow (S\ x\ x')\restriction_{in\text{-}dom\ (\leq_{L2}\ x\ (r1\ x'))}\rceil_{in\text{-}codom\ (\leq_{R2}\ (l1\ x)\ x')})$
$\restriction_{in\text{-}dom\ (\leq_L)}\rceil_{in\text{-}codom\ (\leq_R)} =$
$((x\ x' :: (_{L1}\lessapprox)) \Rrightarrow S\ x\ x')\restriction_{in\text{-}dom\ (\leq_L)}\rceil_{in\text{-}codom\ (\leq_R)}$
(**is** $?lhs\restriction_{?DL}\rceil_{?CR} = ?rhs\restriction_{?DL}\rceil_{?CR}$)
proof (*intro ext*)
fix *f g*
have $?lhs\restriction_{?DL}\rceil_{?CR}\ f\ g \longleftrightarrow ?lhs\ f\ g \wedge ?DL\ f \wedge ?CR\ g$ **by** *blast*
also have ... $\longleftrightarrow ?lhs\restriction_{in\text{-}dom\ tdfr.L}\rceil_{in\text{-}codom\ tdfr.R}\ f\ g \wedge ?DL\ f \wedge ?CR\ g$
unfolding *left-rel-eq-tdfr-left-Refl-Rel right-rel-eq-tdfr-right-Refl-Rel*
by *blast*
also with *assms* **have** ... $\longleftrightarrow ?rhs\restriction_{in\text{-}dom\ tdfr.L}\rceil_{in\text{-}codom\ tdfr.R}\ f\ g \wedge ?DL\ f \wedge ?CR\ g$
by (*simp only*:
tdfr.Dep-Fun-Rel-left-Galois-restrict-left-right-restrict-left-right-eq)
also have ... $\longleftrightarrow ?rhs\restriction_{?DL}\rceil_{?CR}\ f\ g$
unfolding *left-rel-eq-tdfr-left-Refl-Rel right-rel-eq-tdfr-right-Refl-Rel*
by *blast*
finally show $?lhs\restriction_{?DL}\rceil_{?CR}\ f\ g \longleftrightarrow ?rhs\restriction_{?DL}\rceil_{?CR}\ f\ g$.
qed

end

Monotone Function Relator **context** *transport-Mono-Fun-Rel*
begin

corollary *Fun-Rel-left-Galois-if-left-GaloisI*:
assumes $((\leq_{L1}) \ {}_h\trianglelefteq \ (\leq_{R1}))$ *l1 r1*
and *reflexive-on* (*in-dom* $(\leq_{L1})$) $(\leq_{L1})$
and $((\leq_{R2}) \Rightarrow (\leq_{L2}))$ $(r2)$
and *transitive* $(\leq_{L2})$
and $f\ {}_{L}\lessapprox\ g$
shows $((_{L1}\lessapprox) \Rrightarrow (_{L2}\lessapprox))\ f\ g$
by (*urule tmdfr.Dep-Fun-Rel-left-Galois-if-left-GaloisI assms* | *simp*)+

interpretation *flip* : *transport-Mono-Fun-Rel R1 L1 r1 l1 R2 L2 r2 l2* .

lemma *left-Galois-if-Fun-Rel-left-GaloisI*:
assumes $((\leq_{L1}) \Rightarrow (\leq_{R1}))$ *l1*

and $((\leq_{L1}) \trianglelefteq_h (\leq_{R1}))$ *l1 r1*
and $((\leq_{R2}) \Rightarrow (\leq_{L2}))$ *r2*
and *in-dom* $(\leq_L)$ *f*
and *in-codom* $(\leq_R)$ *g*
and $((_{L1}\lessapprox) \Rrightarrow (_{L2}\lessapprox))$ *f g*
shows *f* $_{L}\lessapprox$ *g*
by (*urule tmdfr.left-Galois-if-Dep-Fun-Rel-left-GaloisI flip.tfr.mono-wrt-rel-leftI*)+
(*urule assms* | *simp*)+

corollary *left-Galois-iff-Fun-Rel-left-GaloisI*:
assumes $((\leq_{L1}) \Rightarrow (\leq_{R1}))$ *l1*
and $((\leq_{L1}) \trianglelefteq (\leq_{R1}))$ *l1 r1*
and *reflexive-on* (*in-dom* $(\leq_{L1})$) $(\leq_{L1})$
and $((\leq_{R2}) \Rightarrow (\leq_{L2}))$ $(r2)$
and *transitive* $(\leq_{L2})$
and *in-dom* $(\leq_L)$ *f*
and *in-codom* $(\leq_R)$ *g*
shows *f* $_{L}\lessapprox$ *g* $\longleftrightarrow$ $((_{L1}\lessapprox) \Rrightarrow (_{L2}\lessapprox))$ *f g*
using *assms* **by** (*intro iffI Fun-Rel-left-Galois-if-left-GaloisI*)
(*auto intro*!: *left-Galois-if-Fun-Rel-left-GaloisI*)

theorem *left-Galois-eq-Fun-Rel-left-Galois-restrictI*:
assumes $((\leq_{L1}) \Rightarrow (\leq_{R1}))$ *l1*
and $((\leq_{L1}) \trianglelefteq (\leq_{R1}))$ *l1 r1*
and *reflexive-on* (*in-dom* $(\leq_{L1})$) $(\leq_{L1})$
and $((\leq_{R2}) \Rightarrow (\leq_{L2}))$ *r2*
and *transitive* $(\leq_{L2})$
shows $(_{L}\lessapprox) = ((_{L1}\lessapprox) \Rrightarrow (_{L2}\lessapprox))\restriction_{\textit{in-dom}\ (\leq_L)}\restriction_{\textit{in-codom}\ (\leq_R)}$
using *assms* **by** (*intro ext iffI rel-restrict-leftI rel-restrict-rightI*
iffD1[*OF left-Galois-iff-Fun-Rel-left-GaloisI*])
(*auto elim*!: *tmdfr.left-GaloisE intro*!: *iffD2*[*OF left-Galois-iff-Fun-Rel-left-GaloisI*])

Simplification of Restricted Function Relator **lemma** *Fun-Rel-left-Galois-restrict-left-right-eq-Fun*
assumes *reflexive-on* (*in-field tfr.tdfr.L*) *tfr.tdfr.L*
and *reflexive-on* (*in-field tfr.tdfr.R*) *tfr.tdfr.R*
and $((\leq_{L1}) \Rightarrow (\leq_{R1}))$ *l1* **and** $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and $((\leq_{L1}) \trianglelefteq_h (\leq_{R1}))$ *l1 r1*
and $((\leq_{L2})\ {}_h\trianglelefteq (\leq_{R2}))$ *l2 r2*
shows $((_{L1}\lessapprox) \Rrightarrow (_{L2}\lessapprox))\restriction_{\textit{in-dom}\ (\leq_L)}\restriction_{\textit{in-codom}\ (\leq_R)} = ((_{L1}\lessapprox) \Rrightarrow (_{L2}\lessapprox))$
using *assms* **by** (*auto simp only*: *tmdfr.left-rel-eq-tdfr-left-rel-if-reflexive-on*
tmdfr.right-rel-eq-tdfr-right-rel-if-reflexive-on
intro!: *tfr.Fun-Rel-left-Galois-restrict-left-right-eq-Fun-Rel-left-GaloisI*)

lemma *Fun-Rel-left-Galois-restrict-left-right-eq-Fun-Rel-left-GaloisI*:
assumes $((\leq_{L1}) \Rightarrow (\leq_{R1}))$ *l1* **and** $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and $((\leq_{L1}) \trianglelefteq_h (\leq_{R1}))$ *l1 r1*
and *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
and *reflexive-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$
and $((\leq_{L2})\ {}_h\trianglelefteq (\leq_{R2}))$ *l2 r2*

and *partial-equivalence-rel* $(\leq_{L2})$
and *partial-equivalence-rel* $(\leq_{R2})$
shows $((_{L1}\lessapprox) \Rrightarrow (_{L2}\lessapprox))\restriction_{\textit{in-dom}\ (\leq_L)}\rceil_{\textit{in-codom}\ (\leq_R)} = ((_{L1}\lessapprox) \Rrightarrow (_{L2}\lessapprox))$
using *assms* **by** (*intro*
Fun-Rel-left-Galois-restrict-left-right-eq-Fun-Rel-left-Galois-if-reflexive-onI
tfr.reflexive-on-in-field-leftI
flip.tfr.reflexive-on-in-field-leftI)
auto

Simplification of Restricted Function Relator for Nested Transports

lemma *Fun-Rel-left-Galois-restrict-left-right-restrict-left-right-eq*:
fixes $S :: 'b1 \Rightarrow 'b2 \Rightarrow bool$
assumes $((\leq_{L1})\ {}_h\trianglelefteq\ (\leq_{R1}))\ l1\ r1$
shows $((_{L1}\lessapprox) \Rrightarrow S\restriction_{\textit{in-dom}\ (\leq_{L2})}\rceil_{\textit{in-codom}\ (\leq_{R2})})\restriction_{\textit{in-dom}\ (\leq_L)}\rceil_{\textit{in-codom}\ (\leq_R)}$
=
$((_{L1}\lessapprox) \Rrightarrow S)\restriction_{\textit{in-dom}\ (\leq_L)}\rceil_{\textit{in-codom}\ (\leq_R)}$
by (*urule tmdfr.Dep-Fun-Rel-left-Galois-restrict-left-right-restrict-left-right-eq assms*
| *simp*)+

end

end

2.15.9 Order Equivalence

theory *Transport-Functions-Order-Equivalence*
imports
Transport-Functions-Monotone
Transport-Functions-Galois-Equivalence
begin

Dependent Function Relator **context** *transport-Dep-Fun-Rel*
begin

Inflationary **lemma** *rel-unit-self-if-rel-selfI*:
assumes *inflationary-unit1*: *inflationary-on* $(\textit{in-codom}\ (\leq_{L1}))\ (\leq_{L1})\ \eta_1$
and *refl-L1*: *reflexive-on* $(\textit{in-codom}\ (\leq_{L1}))\ (\leq_{L1})$
and *trans-L1*: *transitive* $(\leq_{L1})$
and *mono-l2*: $\bigwedge x.\ x \leq_{L1} x \Longrightarrow ((\leq_{L2\ x\ x}) \Rightarrow (\leq_{R2\ (l1\ x)\ (l1\ x)}))\ (l2_{(l1\ x)\ x})$
and *mono-r2*: $\bigwedge x.\ x \leq_{L1} x \Longrightarrow ((\leq_{R2\ (l1\ x)\ (l1\ x)}) \Rightarrow (\leq_{L2\ x\ (\eta_1\ x)}))\ (r2_{x\ (l1\ x)})$
and *inflationary-unit2*: $\bigwedge x.\ x \leq_{L1} x \Longrightarrow$
inflationary-on $(\textit{in-codom}\ (\leq_{L2\ x\ x}))\ (\leq_{L2\ x\ x})\ (\eta_{2\ x\ (l1\ x)})$
and *L2-le1*: $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x2\ x2}) \leq (\leq_{L2\ x1\ x2})$
and *L2-unit-le2*: $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ (\eta_1\ x2)}) \leq (\leq_{L2\ x1\ x2})$
and *ge-R2-l2-le2*: $\bigwedge x\ y.\ x \leq_{L1} x \Longrightarrow \textit{in-codom}\ (\leq_{L2\ x\ (\eta_1\ x)})\ y \Longrightarrow$
$(\geq_{R2\ (l1\ x)\ (l1\ x)})\ (l2_{(l1\ x)\ x\ y}) \leq (\geq_{R2\ (l1\ x)\ (l1\ x)})\ (l2_{(l1\ x)\ (\eta_1\ x)\ y})$
and *trans-L2*: $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *transitive* $(\leq_{L2\ x1\ x2})$

and $f \leq_L f$
shows $f \leq_L \eta\ f$
proof (*intro left-relI*)
fix *x1 x2* **assume** [*iff*]: $x1 \leq_{L1} x2$
moreover with *inflationary-unit1* **have** $x2 \leq_{L1} \eta_1\ x2$ **by** *blast*
ultimately have $x1 \leq_{L1} \eta_1\ x2$ **using** *trans-L1* **by** *blast*
with ‹$f \leq_L f$› **have** $f\ x1 \leq_{L2\ x1\ (\eta_1\ x2)} f\ (\eta_1\ x2)$ **by** *blast*
with *L2-unit-le2* **have** $f\ x1 \leq_{L2\ x1\ x2} f\ (\eta_1\ x2)$ **by** *blast*
moreover have ... $\leq_{L2\ x1\ x2} \eta\ f\ x2$
proof −
from *refl-L1* ‹$x2 \leq_{L1} \eta_1\ x2$› **have** $\eta_1\ x2 \leq_{L1} \eta_1\ x2$ **by** *blast*
with ‹$f \leq_L f$› **have** $f\ (\eta_1\ x2) \leq_{L2\ (\eta_1\ x2)\ (\eta_1\ x2)} f\ (\eta_1\ x2)$ **by** *blast*
with *L2-le1* **have** $f\ (\eta_1\ x2) \leq_{L2\ x2\ (\eta_1\ x2)} f\ (\eta_1\ x2)$
using ‹$x2 \leq_{L1} \eta_1\ x2$› **by** *blast*
moreover from *refl-L1* ‹$x1 \leq_{L1} x2$› **have** [*iff*]: $x2 \leq_{L1} x2$ **by** *blast*
ultimately have $f\ (\eta_1\ x2) \leq_{L2\ x2\ x2} f\ (\eta_1\ x2)$ **using** *L2-unit-le2* **by** *blast*
with *inflationary-unit2* **have** $f\ (\eta_1\ x2) \leq_{L2\ x2\ x2\ \eta_2\ x2\ (l1\ x2)} (f\ (\eta_1\ x2))$ **by** *blast*
moreover have ... $\leq_{L2\ x2\ x2} \eta\ f\ x2$
proof −
from ‹$f\ (\eta_1\ x2) \leq_{L2\ x2\ x2} f\ (\eta_1\ x2)$› *mono-l2*
have ${l2}_{(l1\ x2)\ x2}\ (f\ (\eta_1\ x2)) \leq_{R2\ (l1\ x2)\ (l1\ x2)}\ {l2}_{(l1\ x2)\ x2}\ (f\ (\eta_1\ x2))$
by *blast*
with *ge-R2-l2-le2*
have ${l2}_{(l1\ x2)\ x2}\ (f\ (\eta_1\ x2)) \leq_{R2\ (l1\ x2)\ (l1\ x2)}\ {l2}_{(l1\ x2)\ (\eta_1\ x2)}\ (f\ (\eta_1\ x2))$
using ‹$f\ (\eta_1\ x2) \leq_{L2\ x2\ (\eta_1\ x2)} f\ (\eta_1\ x2)$› **by** *blast*
with *mono-r2* **have** $\eta_{2\ x2\ (l1\ x2)}\ (f\ (\eta_1\ x2)) \leq_{L2\ x2\ (\eta_1\ x2)} \eta\ f\ x2$
by *auto*
with *L2-unit-le2* **show** *?thesis* **by** *blast*
qed
ultimately have $f\ (\eta_1\ x2) \leq_{L2\ x2\ x2} \eta\ f\ x2$ **using** *trans-L2* **by** *blast*
with *L2-le1* **show** *?thesis* **by** *blast*
qed
ultimately show $f\ x1 \leq_{L2\ x1\ x2} \eta\ f\ x2$ **using** *trans-L2* **by** *blast*
qed

Deflationary interpretation *flip-inv* :
transport-Dep-Fun-Rel $(\geq_{R1})$ $(\geq_{L1})$ *r1 l1 flip2 R2 flip2 L2 r2 l2*
rewrites *flip-inv.L* $\equiv (\geq_R)$ **and** *flip-inv.R* $\equiv (\geq_L)$
and *flip-inv.unit* $\equiv \varepsilon$
and *flip-inv.t1.unit* $\equiv \varepsilon_1$
and $\bigwedge x\ y.$ *flip-inv.t2-unit x y* $\equiv \varepsilon_{2\ y\ x}$
and $\bigwedge R\ x\ y.$ $(flip2\ R\ x\ y)^{-1} \equiv R\ y\ x$
and $\bigwedge R.$ *in-codom* $R^{-1} \equiv$ *in-dom R*
and $\bigwedge R\ x1\ x2.$ *in-codom* $(flip2\ R\ x1\ x2) \equiv$ *in-dom* $(R\ x2\ x1)$
and $\bigwedge x1\ x2\ x1'\ x2'.$ $(flip2\ R2\ x1'\ x2' \Rightarrow flip2\ L2\ x1\ x2) \equiv ((\leq_{R2\ x2'\ x1'}) \Rightarrow (\leq_{L2\ x2\ x1}))$
and $\bigwedge x1\ x2\ x1'\ x2'.$ $(flip2\ L2\ x1\ x2 \Rightarrow flip2\ R2\ x1'\ x2') \equiv ((\leq_{L2\ x2\ x1}) \Rightarrow$

$(\leq_{R2\ x2'\ x1'}))$
and $\bigwedge(R :: 'z \Rightarrow 'z \Rightarrow bool)\ (P :: 'z \Rightarrow bool).$
$(inflationary\text{-}on\ P\ R^{-1} :: ('z \Rightarrow 'z) \Rightarrow bool) \equiv deflationary\text{-}on\ P\ R$
and $\bigwedge(P :: 'b2 \Rightarrow bool)\ x.$
$(inflationary\text{-}on\ P\ (flip2\ R2\ x\ x) :: ('b2 \Rightarrow 'b2) \Rightarrow bool) \equiv deflationary\text{-}on\ P\ (\leq_{R2\ x\ x})$
and $\bigwedge x1\ x2\ x3\ x4.\ flip2\ R2\ x1\ x2 \leq flip2\ R2\ x3\ x4 \equiv (\leq_{R2\ x2\ x1}) \leq (\leq_{R2\ x4\ x3})$
and $\bigwedge(R :: 'z \Rightarrow 'z \Rightarrow bool)\ (P :: 'z \Rightarrow bool).\ reflexive\text{-}on\ P\ R^{-1} \equiv reflexive\text{-}on\ P\ R$
and $\bigwedge(R :: 'z \Rightarrow 'z \Rightarrow bool).\ transitive\ R^{-1} \equiv transitive\ R$
and $\bigwedge x1'\ x2'.\ transitive\ (flip2\ R2\ x1'\ x2') \equiv transitive\ (\leq_{R2\ x2'\ x1'})$
by (*simp-all add*: *flip-inv-left-eq-ge-right flip-inv-right-eq-ge-left*
flip-unit-eq-counit t1.flip-unit-eq-counit t2.flip-unit-eq-counit
galois-prop.rel-inv-half-galois-prop-right-eq-half-galois-prop-left-rel-inv
mono-wrt-rel-eq-dep-mono-wrt-rel)

lemma *counit-rel-self-if-rel-selfI*:
assumes $deflationary\text{-}on\ (in\text{-}dom\ (\leq_{R1}))\ (\leq_{R1})\ \varepsilon_1$
and $reflexive\text{-}on\ (in\text{-}dom\ (\leq_{R1}))\ (\leq_{R1})$
and $transitive\ (\leq_{R1})$
and $\bigwedge x'.\ x' \leq_{R1} x' \Longrightarrow ((\leq_{L2\ (r1\ x')\ (r1\ x')}) \Rightarrow (\leq_{R2\ (\varepsilon_1\ x')\ x'}))\ (l2\ x'\ (r1\ x'))$
and $\bigwedge x'\ x'.\ x' \leq_{R1} x' \Longrightarrow ((\leq_{R2\ x'\ x'}) \Rightarrow (\leq_{L2\ (r1\ x')\ (r1\ x')}))\ (r2_{(r1\ x')\ x'})$
and $\bigwedge x'.\ x' \leq_{R1} x' \Longrightarrow deflationary\text{-}on\ (in\text{-}dom\ (\leq_{R2\ x'\ x'}))\ (\leq_{R2\ x'\ x'})\ (\varepsilon_{2\ (r1\ x')\ x'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ (\varepsilon_1\ x1')\ x2'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ x1'\ x1'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x'\ y'.\ x' \leq_{R1} x' \Longrightarrow in\text{-}dom\ (\leq_{R2\ (\varepsilon_1\ x')\ x'})\ y' \Longrightarrow$
$(\leq_{L2\ (r1\ x')\ (r1\ x')})\ (r2_{(r1\ x')\ x'}\ y') \leq (\leq_{L2\ (r1\ x')\ (r1\ x')})\ (r2_{(r1\ x')\ (\varepsilon_1\ x')}\ y')$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow transitive\ (\leq_{R2\ x1'\ x2'})$
and $g \leq_R g$
shows $\varepsilon\ g \leq_R g$
using *assms* **by** (*intro flip-inv.rel-unit-self-if-rel-selfI*[*simplified rel-inv-iff-rel*])

Relational Equivalence **lemma** *bi-related-unit-self-if-rel-self-aux*:
assumes *rel-equiv-unit1*: $rel\text{-}equivalence\text{-}on\ (in\text{-}field\ (\leq_{L1}))\ (\leq_{L1})\ \eta_1$
and *mono-r2*: $\bigwedge x.\ x \leq_{L1} x \Longrightarrow ((\leq_{R2\ (l1\ x)\ (l1\ x)}) \Rightarrow (\leq_{L2\ x\ x}))\ (r2_{x\ (l1\ x)})$
and *rel-equiv-unit2*: $\bigwedge x.\ x \leq_{L1} x \Longrightarrow$
$rel\text{-}equivalence\text{-}on\ (in\text{-}field\ (\leq_{L2\ x\ x}))\ (\leq_{L2\ x\ x})\ (\eta_{2\ x\ (l1\ x)})$
and *L2-le1*: $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x2\ x2}) \leq (\leq_{L2\ x1\ x2})$
and *L2-le2*: $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ x1}) \leq (\leq_{L2\ x1\ x2})$
and [*iff*]: $x \leq_{L1} x$
shows $((\leq_{R2\ (l1\ x)\ (l1\ x)}) \Rightarrow (\leq_{L2\ x\ (\eta_1\ x)}))\ (r2_{x\ (l1\ x)})$
and $((\leq_{R2\ (l1\ x)\ (l1\ x)}) \Rightarrow (\leq_{L2\ (\eta_1\ x)\ x}))\ (r2_{x\ (l1\ x)})$
and $deflationary\text{-}on\ (in\text{-}dom\ (\leq_{L2\ x\ x}))\ (\leq_{L2\ x\ x})\ \eta_{2\ x\ (l1\ x)}$
and $inflationary\text{-}on\ (in\text{-}codom\ (\leq_{L2\ x\ x}))\ (\leq_{L2\ x\ x})\ \eta_{2\ x\ (l1\ x)}$
proof −
from *rel-equiv-unit1* **have** $x \equiv_{L1} \eta_1\ x$ **by** *blast*

with *mono-r2* **show** $((\leq_{R2\ (l1\ x)\ (l1\ x)}) \Rightarrow (\leq_{L2\ x\ (\eta_1\ x)}))\ (r2_{x\ (l1\ x)})$
and $((\leq_{R2\ (l1\ x)\ (l1\ x)}) \Rightarrow (\leq_{L2\ (\eta_1\ x)\ x}))\ (r2_{x\ (l1\ x)})$
using *L2-le1 L2-le2* **by** *blast+*
qed (*insert rel-equiv-unit2*, *blast+*)

interpretation *flip* : *transport-Dep-Fun-Rel R1 L1 r1 l1 R2 L2 r2 l2*
rewrites *flip.counit* $\equiv \eta$ **and** *flip.t1.counit* $\equiv \eta_1$
and $\bigwedge x\ y$. *flip.t2-counit x y* $\equiv \eta_{2\ y\ x}$
by (*simp-all add*: *order-functors.flip-counit-eq-unit*)

lemma *bi-related-unit-self-if-rel-selfI*:
assumes *rel-equiv-unit1*: *rel-equivalence-on* $(\textit{in-field}\ (\leq_{L1}))\ (\leq_{L1})\ \eta_1$
and *trans-L1*: *transitive* $(\leq_{L1})$
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow ((\leq_{L2\ x\ x}) \Rightarrow (\leq_{R2\ (l1\ x)\ (l1\ x)}))\ (l2_{(l1\ x)\ x})$
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow ((\leq_{R2\ (l1\ x)\ (l1\ x)}) \Rightarrow (\leq_{L2\ x\ x}))\ (r2_{x\ (l1\ x)})$
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow$
rel-equivalence-on $(\textit{in-field}\ (\leq_{L2\ x\ x}))\ (\leq_{L2\ x\ x})\ (\eta_{2\ x\ (l1\ x)})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x2\ x2}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ (\eta_1\ x1)\ x2}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ x1}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ (\eta_1\ x2)}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x\ y.\ x \leq_{L1} x \Longrightarrow \textit{in-dom}\ (\leq_{L2\ (\eta_1\ x)\ x})\ y \Longrightarrow$
$(\leq_{R2\ (l1\ x)\ (l1\ x)})\ (l2_{(l1\ x)\ x\ y}) \leq (\leq_{R2\ (l1\ x)\ (l1\ x)})\ (l2_{(l1\ x)\ (\eta_1\ x)\ y})$
and $\bigwedge x\ y.\ x \leq_{L1} x \Longrightarrow \textit{in-codom}\ (\leq_{L2\ x\ (\eta_1\ x)})\ y \Longrightarrow$
$(\geq_{R2\ (l1\ x)\ (l1\ x)})\ (l2_{(l1\ x)\ x\ y}) \leq (\geq_{R2\ (l1\ x)\ (l1\ x)})\ (l2_{(l1\ x)\ (\eta_1\ x)\ y})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow \textit{transitive}\ (\leq_{L2\ x1\ x2})$
and $f \leq_L f$
shows $f \equiv_L \eta\ f$
proof −
from *rel-equiv-unit1 trans-L1* **have** *reflexive-on* $(\textit{in-field}\ (\leq_{L1}))\ (\leq_{L1})$
by (*intro reflexive-on-in-field-if-transitive-if-rel-equivalence-on*)
with *assms* **show** *?thesis*
by (*intro bi-relatedI rel-unit-self-if-rel-selfI*
flip.counit-rel-self-if-rel-selfI
bi-related-unit-self-if-rel-self-aux)
(*auto intro*: *inflationary-on-if-le-pred-if-inflationary-on*
deflationary-on-if-le-pred-if-deflationary-on
reflexive-on-if-le-pred-if-reflexive-on
in-field-if-in-dom in-field-if-in-codom)
qed

Lemmas for Monotone Function Relator **lemma** *order-equivalence-if-order-equivalence-mono-assm*
assumes *order-equiv1*: $((\leq_{L1}) \equiv_o (\leq_{R1}))$ *l1 r1*
and *refl-R1*: *reflexive-on* $(\textit{in-field}\ (\leq_{R1}))\ (\leq_{R1})$
and *R2-counit-le1*: $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ (\varepsilon_1\ x1')\ x2'}) \leq (\leq_{R2\ x1'\ x2'})$
and *mono-l2*: $((x1'\ x2' :: (\leq_{R1})) \Rightarrow (x1\ x2 :: (\leq_{L1}) \mid x2\ {}_{L1}\!\lessapprox x1') \Rrightarrow$
in-field $(\leq_{L2\ x1\ (r1\ x2')}) \Rrightarrow (\leq_{R2\ (l1\ x1)\ x2'}))\ l2$

and [*iff*]: $x1' \leq_{R1} x2'$
shows $((\textit{in-dom}\ (\leq_{L2}\ (r1\ x1')\ (r1\ x2'))) \Rrightarrow (\leq_{R2\ x1'\ x2'}))\ (l2_{x1'}\ (r1\ x1'))\ (l2_{x2'}\ (r1\ x1'))$
and $((\textit{in-codom}\ (\leq_{L2}\ (r1\ x1')\ (r1\ x2'))) \Rrightarrow (\leq_{R2\ x1'\ x2'}))\ (l2_{x2'}\ (r1\ x1'))\ (l2_{x2'}\ (r1\ x2'))$
proof −
from *refl-R1* **have** $x1' \leq_{R1} x1'\ x2' \leq_{R1} x2'$ **by** *auto*
moreover with *order-equiv1*
have $r1\ x1' \leq_{L1} r1\ x2'\ r1\ x1' \leq_{L1} r1\ x1'\ r1\ x2' \leq_{L1} r1\ x2'$ **by** *auto*
ultimately have $r1\ x1'\ _{L1}\!\lessapprox x1'\ r1\ x2'\ _{L1}\!\lessapprox x2'$ **by** *blast+*
note *Dep-Fun-Rel-relD*[*OF dep-mono-wrt-relD*[*OF mono-l2* ‹$x1' \leq_{R1} x2'$›]
‹$r1\ x1' \leq_{L1} r1\ x1'$›]
with ‹$r1\ x1'\ _{L1}\!\lessapprox x1'$› *R2-counit-le1*
show $((\textit{in-dom}\ (\leq_{L2}\ (r1\ x1')\ (r1\ x2'))) \Rrightarrow (\leq_{R2\ x1'\ x2'}))\ (l2_{x1'}\ (r1\ x1'))\ (l2_{x2'}\ (r1\ x1'))$
by (*intro Fun-Rel-predI*) (*auto dest*!: *in-field-if-in-dom*)
note *Dep-Fun-Rel-relD*[*OF dep-mono-wrt-relD*[*OF mono-l2* ‹$x2' \leq_{R1} x2'$›]
‹$r1\ x1' \leq_{L1} r1\ x2'$›]
with ‹$r1\ x2'\ _{L1}\!\lessapprox x2'$› *R2-counit-le1*
show $((\textit{in-codom}\ (\leq_{L2}\ (r1\ x1')\ (r1\ x2'))) \Rrightarrow (\leq_{R2\ x1'\ x2'}))\ (l2_{x2'}\ (r1\ x1'))\ (l2_{x2'}\ (r1\ x2'))$
by (*intro Fun-Rel-predI*) (*auto dest*!: *in-field-if-in-codom*)
qed

lemma *order-equivalence-if-order-equivalence-mono-assms-rightI*:
assumes *order-equiv1*: $((\leq_{L1}) \equiv_o (\leq_{R1}))\ l1\ r1$
and *refl-L1*: *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
and *L2-unit-le2*: $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ (\eta_1\ x2)}) \leq (\leq_{L2\ x1\ x2})$
and *mono-r2*: $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ _{L1}\!\lessapprox x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))\ r2$
and [*iff*]: $x1 \leq_{L1} x2$
shows $((\textit{in-codom}\ (\leq_{R2\ (l1\ x1)\ (l1\ x2)})) \Rrightarrow (\leq_{L2\ x1\ x2}))\ (r2_{x1\ (l1\ x2)})\ (r2_{x2\ (l1\ x2)})$
and $((\textit{in-dom}\ (\leq_{R2\ (l1\ x1)\ (l1\ x2)})) \Rrightarrow (\leq_{L2\ x1\ x2}))\ (r2_{x1\ (l1\ x1)})\ (r2_{x1\ (l1\ x2)})$
proof −
from *refl-L1* **have** $x1 \leq_{L1} x1\ x2 \leq_{L1} x2$ **by** *auto*
moreover with *order-equiv1*
have $l1\ x1 \leq_{R1} l1\ x2\ l1\ x1 \leq_{R1} l1\ x1\ l1\ x2 \leq_{R1} l1\ x2$ **by** *auto*
ultimately have $x1\ _{L1}\!\lessapprox l1\ x1\ x2\ _{L1}\!\lessapprox l1\ x2$ **using** *order-equiv1*
by (*auto intro*!: *t1.left-Galois-left-if-in-codom-if-inflationary-onI*)
note *Dep-Fun-Rel-relD*[*OF dep-mono-wrt-relD*[*OF mono-r2* ‹$x1 \leq_{L1} x2$›]
‹$l1\ x2 \leq_{R1} l1\ x2$›]
with ‹$x2\ _{L1}\!\lessapprox l1\ x2$› *L2-unit-le2*
show $((\textit{in-codom}\ (\leq_{R2\ (l1\ x1)\ (l1\ x2)})) \Rrightarrow (\leq_{L2\ x1\ x2}))\ (r2_{x1\ (l1\ x2)})\ (r2_{x2\ (l1\ x2)})$
by (*intro Fun-Rel-predI*) (*auto dest*!: *in-field-if-in-codom*)
note *Dep-Fun-Rel-relD*[*OF dep-mono-wrt-relD*[*OF mono-r2* ‹$x1 \leq_{L1} x1$›]
‹$l1\ x1 \leq_{R1} l1\ x2$›]
with ‹$x1\ _{L1}\!\lessapprox l1\ x1$› *L2-unit-le2*
show $((\textit{in-dom}\ (\leq_{R2\ (l1\ x1)\ (l1\ x2)})) \Rrightarrow (\leq_{L2\ x1\ x2}))\ (r2_{x1\ (l1\ x1)})\ (r2_{x1\ (l1\ x2)})$
by (*intro Fun-Rel-predI*) (*auto dest*!: *in-field-if-in-dom*)
qed

lemma *l2-unit-bi-rel-selfI*:
assumes *pre-equiv1*: $((\leq_{L1}) \equiv_{pre} (\leq_{R1}))$ *l1 r1*
and *mono-L2*:
$((x1\ x2 :: (\leq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid (x2 \leq_{L1} x3 \wedge x4 \leq_{L1} \eta_1\ x3)) \Rrightarrow (\geq))$ *L2*
and *mono-R2*:
$((x1'\ x2' :: (\leq_{R1})) \Rightarrow (x3'\ x4' :: (\leq_{R1}) \mid (x2' \leq_{R1} x3' \wedge x4' \leq_{R1} \varepsilon_1\ x3')) \Rrightarrow (\geq))$ *R2*
and *mono-l2*: $((x1'\ x2' :: (\leq_{R1})) \Rightarrow (x1\ x2 :: (\leq_{L1}) \mid x2\ {}_{L1}\lessapprox x1') \Rrightarrow$
in-field $(\leq_{L2\ x1\ (r1\ x2')}) \Rrightarrow (\leq_{R2\ (l1\ x1)\ x2'}))$ *l2*
and $x \leq_{L1} x$
and *in-field* $(\leq_{L2\ x\ x})\ y$
shows ${}^{l2}{}_{(l1\ x)\ (\eta_1\ x)}\ y \equiv_{R2\ (l1\ x)\ (l1\ x)}\ {}^{l2}{}_{(l1\ x)\ x}\ y$
proof (*rule bi-relatedI*)
note *t1.preorder-equivalence-order-equivalenceE*[*elim*!]
from ‹$x \leq_{L1} x$› *pre-equiv1* **have** $l1\ x \leq_{R1} l1\ x\ x \leq_{L1} \eta_1\ x\ \eta_1\ x \leq_{L1} x$ **by** *blast*+
with *pre-equiv1* **have** $x\ {}_{L1}\lessapprox\ l1\ x\ \eta_1\ x\ {}_{L1}\lessapprox\ l1\ x$ **by** (*auto 4 3*)
from *pre-equiv1* ‹$x \leq_{L1} \eta_1\ x$› **have** $x \leq_{L1} \eta_1\ (\eta_1\ x)$ **by** *fastforce*
moreover note ‹*in-field* $(\leq_{L2\ x\ x})\ y$›
Dep-Fun-Rel-relD[*OF dep-mono-wrt-relD*[*OF mono-L2* ‹$\eta_1\ x \leq_{L1} x$›] ‹$\eta_1\ x \leq_{L1} x$›]
Dep-Fun-Rel-relD[*OF dep-mono-wrt-relD*[*OF mono-L2* ‹$x \leq_{L1} x$›] ‹$\eta_1\ x \leq_{L1} x$›]
ultimately have *in-field* $(\leq_{L2\ (\eta_1\ x)\ (\eta_1\ x)})\ y$ *in-field* $(\leq_{L2\ x\ (\eta_1\ x)})\ y$
using ‹$x \leq_{L1} \eta_1\ x$› **by** *blast*+
moreover note ‹$x\ {}_{L1}\lessapprox\ l1\ x$›
Dep-Fun-Rel-relD[*OF dep-mono-wrt-relD*[*OF mono-l2* ‹$l1\ x \leq_{R1} l1\ x$›] ‹$\eta_1\ x \leq_{L1} x$›]
ultimately have ${}^{l2}{}_{(l1\ x)\ (\eta_1\ x)}\ y \leq_{R2\ (\varepsilon_1\ (l1\ x))\ (l1\ x)}\ {}^{l2}{}_{(l1\ x)\ x}\ y$ **by** *auto*
moreover from *pre-equiv1* ‹$l1\ x \leq_{R1} l1\ x$›
have $\varepsilon_1\ (l1\ x) \leq_{R1} l1\ x\ l1\ x \leq_{R1} \varepsilon_1\ (l1\ x)$ **by** *fastforce*+
moreover note *Dep-Fun-Rel-relD*[*OF dep-mono-wrt-relD*
[*OF mono-R2* ‹$l1\ x \leq_{R1} \varepsilon_1\ (l1\ x)$›] ‹$l1\ x \leq_{R1} l1\ x$›]
ultimately show ${}^{l2}{}_{(l1\ x)\ (\eta_1\ x)}\ y \leq_{R2\ (l1\ x)\ (l1\ x)}\ {}^{l2}{}_{(l1\ x)\ x}\ y$ **by** *blast*
note ‹$\eta_1\ x\ {}_{L1}\lessapprox\ l1\ x$› ‹*in-field* $(\leq_{L2\ x\ (\eta_1\ x)})\ y$›
Dep-Fun-Rel-relD[*OF dep-mono-wrt-relD*[*OF mono-l2* ‹$l1\ x \leq_{R1} l1\ x$›] ‹$x \leq_{L1} \eta_1\ x$›]
then show ${}^{l2}{}_{(l1\ x)\ x}\ y \leq_{R2\ (l1\ x)\ (l1\ x)}\ {}^{l2}{}_{(l1\ x)\ (\eta_1\ x)}\ y$ **by** *auto*
qed

lemma *r2-counit-bi-rel-selfI*:
assumes *pre-equiv1*: $((\leq_{L1}) \equiv_{pre} (\leq_{R1}))$ *l1 r1*
and *mono-L2*:
$((x1\ x2 :: (\leq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid (x2 \leq_{L1} x3 \wedge x4 \leq_{L1} \eta_1\ x3)) \Rrightarrow (\geq))$ *L2*
and *mono-R2*:
$((x1'\ x2' :: (\leq_{R1})) \Rightarrow (x3'\ x4' :: (\leq_{R1}) \mid (x2' \leq_{R1} x3' \wedge x4' \leq_{R1} \varepsilon_1\ x3')) \Rrightarrow$

$(\geq))$ *R2*
and *mono-r2*: $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ {}_{L1}\lessapprox\ x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))$ *r2*
and $x' \leq_{R1} x'$
and *in-field* $(\leq_{R2\ x'\ x'})\ y'$
shows $r2_{(r1\ x')\ (\varepsilon_1\ x')}\ y' \equiv_{L2\ (r1\ x')\ (r1\ x')}\ r2_{(r1\ x')\ x'}\ y'$
proof (*rule bi-relatedI*)
note *t1.preorder-equivalence-order-equivalenceE*[*elim*!]
from $\langle x' \leq_{R1} x'\rangle$ *pre-equiv1* **have** $r1\ x' \leq_{L1} r1\ x'\ x' \leq_{R1} \varepsilon_1\ x'\ \varepsilon_1\ x' \leq_{R1} x'$
by *blast*+
with *pre-equiv1* **have** $r1\ x'\ {}_{L1}\lessapprox\ x'\ r1\ x'\ {}_{L1}\lessapprox\ \varepsilon_1\ x'$ **by** *fastforce*+
from *pre-equiv1* $\langle x' \leq_{R1} \varepsilon_1\ x'\rangle$ **have** $x' \leq_{R1} \varepsilon_1\ (\varepsilon_1\ x')$ **by** *fastforce*
moreover note $\langle$*in-field* $(\leq_{R2\ x'\ x'})\ y'\rangle$
Dep-Fun-Rel-relD[*OF dep-mono-wrt-relD*[*OF mono-R2* $\langle\varepsilon_1\ x' \leq_{R1} x'\rangle$] $\langle\varepsilon_1\ x' \leq_{R1} x'\rangle$]
Dep-Fun-Rel-relD[*OF dep-mono-wrt-relD*[*OF mono-R2* $\langle\varepsilon_1\ x' \leq_{R1} x'\rangle$] $\langle x' \leq_{R1} x'\rangle$]
ultimately have *in-field* $(\leq_{R2\ (\varepsilon_1\ x')\ (\varepsilon_1\ x')})\ y'$ *in-field* $(\leq_{R2\ (\varepsilon_1\ x')\ x'})\ y'$
using $\langle x' \leq_{R1} \varepsilon_1\ x'\rangle\ \langle x' \leq_{R1} x'\rangle$ **by** *blast*+
moreover note $\langle r1\ x'\ {}_{L1}\lessapprox\ \varepsilon_1\ x'\rangle$
Dep-Fun-Rel-relD[*OF dep-mono-wrt-relD*[*OF mono-r2* $\langle r1\ x' \leq_{L1} r1\ x'\rangle$] $\langle\varepsilon_1\ x' \leq_{R1} x'\rangle$]
ultimately show $r2_{(r1\ x')\ (\varepsilon_1\ x')}\ y' \leq_{L2\ (r1\ x')\ (r1\ x')}\ r2_{(r1\ x')\ x'}\ y'$ **by** *auto*
note $\langle r1\ x'\ {}_{L1}\lessapprox\ x'\rangle\ \langle$*in-field* $(\leq_{R2\ (\varepsilon_1\ x')\ (\varepsilon_1\ x')})\ y'\rangle$
Dep-Fun-Rel-relD[*OF dep-mono-wrt-relD*[*OF mono-r2* $\langle r1\ x' \leq_{L1} r1\ x'\rangle$] $\langle x' \leq_{R1} \varepsilon_1\ x'\rangle$]
then have $r2_{(r1\ x')\ x'}\ y' \leq_{L2\ (r1\ x')\ (\eta_1\ (r1\ x'))}\ r2_{(r1\ x')\ (\varepsilon_1\ x')}\ y'$ **by** *auto*
moreover from *pre-equiv1* $\langle r1\ x' \leq_{L1} r1\ x'\rangle$
have $\eta_1\ (r1\ x') \leq_{L1} r1\ x'\ r1\ x' \leq_{L1} \eta_1\ (r1\ x')$ **by** *fastforce*+
moreover note *Dep-Fun-Rel-relD*[*OF dep-mono-wrt-relD*
[*OF mono-L2* $\langle r1\ x' \leq_{L1} r1\ x'\rangle$] $\langle r1\ x' \leq_{L1} \eta_1\ (r1\ x')\rangle$]
ultimately show $r2_{(r1\ x')\ x'}\ y' \leq_{L2\ (r1\ x')\ (r1\ x')}\ r2_{(r1\ x')\ (\varepsilon_1\ x')}\ y'$
using *pre-equiv1* **by** *blast*
qed

end

Function Relator **context** *transport-Fun-Rel*
begin

corollary *rel-unit-self-if-rel-selfI*:
assumes *inflationary-on* (*in-codom* $(\leq_{L1})$) $(\leq_{L1})\ \eta_1$
and *reflexive-on* (*in-codom* $(\leq_{L1})$) $(\leq_{L1})$
and *transitive* $(\leq_{L1})$
and $((\leq_{L2}) \Rightarrow (\leq_{R2}))$ *l2*
and $((\leq_{R2}) \Rightarrow (\leq_{L2}))$ *r2*
and *inflationary-on* (*in-codom* $(\leq_{L2})$) $(\leq_{L2})\ \eta_2$
and *transitive* $(\leq_{L2})$

and $f \leq_L f$
shows $f \leq_L \eta\ f$
using *assms* **by** (*intro tdfr.rel-unit-self-if-rel-selfI*) *simp-all*

corollary *counit-rel-self-if-rel-selfI*:
assumes *deflationary-on* (*in-dom* ($\leq_{R1}$)) ($\leq_{R1}$) ε_1
and *reflexive-on* (*in-dom* ($\leq_{R1}$)) ($\leq_{R1}$)
and *transitive* ($\leq_{R1}$)
and (($\leq_{L2}$) $\Rightarrow$ ($\leq_{R2}$)) *l2*
and (($\leq_{R2}$) $\Rightarrow$ ($\leq_{L2}$)) *r2*
and *deflationary-on* (*in-dom* ($\leq_{R2}$)) ($\leq_{R2}$) ε_2
and *transitive* ($\leq_{R2}$)
and $g \leq_R g$
shows $\varepsilon\ g \leq_R g$
using *assms* **by** (*intro tdfr.counit-rel-self-if-rel-selfI*) *simp-all*

lemma *bi-related-unit-self-if-rel-selfI*:
assumes *rel-equivalence-on* (*in-field* ($\leq_{L1}$)) ($\leq_{L1}$) η_1
and *transitive* ($\leq_{L1}$)
and (($\leq_{L2}$) $\Rightarrow$ ($\leq_{R2}$)) *l2*
and (($\leq_{R2}$) $\Rightarrow$ ($\leq_{L2}$)) *r2*
and *rel-equivalence-on* (*in-field* ($\leq_{L2}$)) ($\leq_{L2}$) η_2
and *transitive* ($\leq_{L2}$)
and $f \leq_L f$
shows $f \equiv_L \eta\ f$
using *assms* **by** (*intro tdfr.bi-related-unit-self-if-rel-selfI*) *simp-all*

end

Monotone Dependent Function Relator **context** *transport-Mono-Dep-Fun-Rel* **begin**

Inflationary **lemma** *inflationary-on-unitI*:
assumes (*tdfr.L* $\Rightarrow$ *tdfr.R*) *l* **and** (*tdfr.R* $\Rightarrow$ *tdfr.L*) *r*
and *inflationary-on* (*in-codom* ($\leq_{L1}$)) ($\leq_{L1}$) η_1
and *reflexive-on* (*in-codom* ($\leq_{L1}$)) ($\leq_{L1}$)
and *transitive* ($\leq_{L1}$)
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow ((\leq_{L2\ x\ x}) \Rightarrow (\leq_{R2\ (l1\ x)\ (l1\ x)}))\ (l2_{(l1\ x)\ x})$
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow ((\leq_{R2\ (l1\ x)\ (l1\ x)}) \Rightarrow (\leq_{L2\ x\ (\eta_1\ x)}))\ (r2_{x\ (l1\ x)})$
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow$ *inflationary-on* (*in-codom* ($\leq_{L2\ x\ x}$)) ($\leq_{L2\ x\ x}$) ($\eta_{2\ x\ (l1\ x)}$)
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x2\ x2}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ (\eta_1\ x2)}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x\ y.\ x \leq_{L1} x \Longrightarrow$ *in-codom* ($\leq_{L2\ x\ (\eta_1\ x)}$) $y \Longrightarrow$
($\geq_{R2\ (l1\ x)\ (l1\ x)}$) ($l2_{(l1\ x)\ x\ y}$) $\leq$ ($\geq_{R2\ (l1\ x)\ (l1\ x)}$) ($l2_{(l1\ x)\ (\eta_1\ x)\ y}$)
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *transitive* ($\leq_{L2\ x1\ x2}$)
shows *inflationary-on* (*in-field* ($\leq_L$)) ($\leq_L$) η
unfolding *left-rel-eq-tdfr-left-Refl-Rel* **using** *assms*
by (*intro inflationary-onI Refl-RelI*)

(*auto 6 2 intro*: *tdfr.rel-unit-self-if-rel-selfI*[*simplified unit-eq*]
elim!: *Refl-RelE in-fieldE*)

Deflationary **lemma** *deflationary-on-counitI*:
assumes (*tdfr.L* ⇒ *tdfr.R*) *l* **and** (*tdfr.R* ⇒ *tdfr.L*) *r*
and *deflationary-on* (*in-dom* (≤$_{R1}$)) (≤$_{R1}$) ε_1
and *reflexive-on* (*in-dom* (≤$_{R1}$)) (≤$_{R1}$)
and *transitive* (≤$_{R1}$)
and $\bigwedge x'.\ x' \leq_{R1} x' \Longrightarrow ((\leq_{L2\ (r1\ x')\ (r1\ x')}) \Rightarrow (\leq_{R2\ (\varepsilon_1\ x')\ x'}))\ (l2_{\ x'\ (r1\ x')})$
and $\bigwedge x'.\ x' \leq_{R1} x' \Longrightarrow$
$((\leq_{R2\ x'\ x'}) \Rightarrow (\leq_{L2\ (r1\ x')\ (r1\ x')}))\ (r2_{(r1\ x')\ x'})$
and $\bigwedge x'.\ x' \leq_{R1} x' \Longrightarrow$ *deflationary-on* (*in-dom* (≤$_{R2\ x'\ x'}$)) (≤$_{R2\ x'\ x'}$) ($\varepsilon_{2\ (r1\ x')\ x'}$)
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ (\varepsilon_1\ x1')\ x2'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ x1'\ x1'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x'\ y'.\ x' \leq_{R1} x' \Longrightarrow$ *in-dom* $(\leq_{R2\ (\varepsilon_1\ x')\ x'})\ y' \Longrightarrow$
$(\leq_{L2\ (r1\ x')\ (r1\ x')})\ (r2_{(r1\ x')\ x'\ y'}) \leq (\leq_{L2\ (r1\ x')\ (r1\ x')})\ (r2_{(r1\ x')\ (\varepsilon_1\ x')\ y'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow$ *transitive* (≤$_{R2\ x1'\ x2'}$)
shows *deflationary-on* (*in-field* (≤$_R$)) (≤$_R$) ε
unfolding *right-rel-eq-tdfr-right-Refl-Rel* **using** *assms*
by (*intro deflationary-onI Refl-RelI*)
(*auto 6 2 intro*: *tdfr.counit-rel-self-if-rel-selfI*[*simplified counit-eq*]
elim!: *Refl-RelE in-fieldE*)

Relational Equivalence **context**
begin

interpretation *flip* : *transport-Mono-Dep-Fun-Rel R1 L1 r1 l1 R2 L2 r2 l2*
rewrites *flip.counit* ≡ η **and** *flip.t1.counit* ≡ η_1
and $\bigwedge x\ y.$ *flip.t2-counit x y* ≡ $\eta_{2\ y\ x}$
by (*simp-all add*: *order-functors.flip-counit-eq-unit*)

lemma *rel-equivalence-on-unitI*:
assumes (*tdfr.L* ⇒ *tdfr.R*) *l* **and** (*tdfr.R* ⇒ *tdfr.L*) *r*
and *rel-equiv-unit1*: *rel-equivalence-on* (*in-field* (≤$_{L1}$)) (≤$_{L1}$) η_1
and *trans-L1*: *transitive* (≤$_{L1}$)
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow ((\leq_{L2\ x\ x}) \Rightarrow (\leq_{R2\ (l1\ x)\ (l1\ x)}))\ (l2_{(l1\ x)\ x})$
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow ((\leq_{R2\ (l1\ x)\ (l1\ x)}) \Rightarrow (\leq_{L2\ x\ x}))\ (r2_{x\ (l1\ x)})$
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow$ *rel-equivalence-on* (*in-field* (≤$_{L2\ x\ x}$)) (≤$_{L2\ x\ x}$) ($\eta_{2\ x\ (l1\ x)}$)
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x2\ x2}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ (\eta_1\ x1)\ x2}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ x1}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ (\eta_1\ x2)}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x\ y.\ x \leq_{L1} x \Longrightarrow$ *in-dom* $(\leq_{L2\ (\eta_1\ x)\ x})\ y \Longrightarrow$
$(\leq_{R2\ (l1\ x)\ (l1\ x)})\ (l2_{(l1\ x)\ x\ y}) \leq (\leq_{R2\ (l1\ x)\ (l1\ x)})\ (l2_{(l1\ x)\ (\eta_1\ x)\ y})$
and $\bigwedge x\ y.\ x \leq_{L1} x \Longrightarrow$ *in-codom* $(\leq_{L2\ x\ (\eta_1\ x)})\ y \Longrightarrow$

$(\geq_{R2\ (l1\ x)\ (l1\ x)})\ (l2_{(l1\ x)\ x\ y}) \leq (\geq_{R2\ (l1\ x)\ (l1\ x)})\ (l2_{(l1\ x)\ (\eta_1\ x)\ y})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *transitive* $(\leq_{L2\ x1\ x2})$
shows *rel-equivalence-on* (*in-field* $(\leq_L)$) $(\leq_L)$ η
proof −
from *rel-equiv-unit1 trans-L1* **have** *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
by (*intro reflexive-on-in-field-if-transitive-if-rel-equivalence-on*)
with *assms* **show** *?thesis*
by (*intro rel-equivalence-onI inflationary-on-unitI*
flip.deflationary-on-counitI)
(*auto intro*!: *tdfr.bi-related-unit-self-if-rel-self-aux*
intro: *inflationary-on-if-le-pred-if-inflationary-on*
deflationary-on-if-le-pred-if-deflationary-on
reflexive-on-if-le-pred-if-reflexive-on
in-field-if-in-dom in-field-if-in-codom
elim!: *rel-equivalence-onE*
simp only:)
qed

end

Order Equivalence **interpretation** *flip* : *transport-Mono-Dep-Fun-Rel R1 L1 r1 l1 R2 L2 r2 l2*
rewrites *flip.unit* $\equiv \varepsilon$ **and** *flip.t1.unit* $\equiv \varepsilon_1$
and *flip.counit* $\equiv \eta$ **and** *flip.t1.counit* $\equiv \eta_1$
and $\bigwedge x\ y.$ *flip.t2-unit* $x\ y \equiv \varepsilon_{2\ y\ x}$
by (*simp-all add*: *order-functors.flip-counit-eq-unit*)

lemma *order-equivalenceI*:
assumes (*tdfr.L* $\Rightarrow$ *tdfr.R*) l **and** (*tdfr.R* $\Rightarrow$ *tdfr.L*) r
and *rel-equivalence-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$ η_1
and *rel-equivalence-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$ ε_1
and *transitive* $(\leq_{L1})$ **and** *transitive* $(\leq_{R1})$
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow ((\leq_{L2\ x\ x}) \Rightarrow (\leq_{R2\ (l1\ x)\ (l1\ x)}))\ (l2_{(l1\ x)\ x})$
and $\bigwedge x'.\ x' \leq_{R1} x' \Longrightarrow ((\leq_{L2\ (r1\ x')\ (r1\ x')}) \Rightarrow (\leq_{R2\ x'\ x'}))\ (l2_{x'\ (r1\ x')})$
and $\bigwedge x'.\ x' \leq_{R1} x' \Longrightarrow ((\leq_{R2\ x'\ x'}) \Rightarrow (\leq_{L2\ (r1\ x')\ (r1\ x')}))\ (r2_{(r1\ x')\ x'})$
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow ((\leq_{R2\ (l1\ x)\ (l1\ x)}) \Rightarrow (\leq_{L2\ x\ x}))\ (r2_{x\ (l1\ x)})$
and $\bigwedge x.\ x \leq_{L1} x \Longrightarrow$ *rel-equivalence-on* (*in-field* $(\leq_{L2\ x\ x})$) $(\leq_{L2\ x\ x})$ $(\eta_{2\ x\ (l1\ x)})$
and $\bigwedge x'.\ x' \leq_{R1} x' \Longrightarrow$
rel-equivalence-on (*in-field* $(\leq_{R2\ x'\ x'})$) $(\leq_{R2\ x'\ x'})$ $(\varepsilon_{2\ (r1\ x')\ x'})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x2\ x2}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ (\eta_1\ x1)\ x2}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ x1}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ (\eta_1\ x2)}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ x2'\ x2'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ (\varepsilon_1\ x1')\ x2'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ x1'\ x1'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ x1'\ (\varepsilon_1\ x2')}) \leq (\leq_{R2\ x1'\ x2'})$

and $\bigwedge x\ y.\ x \leq_{L1} x \Longrightarrow$ *in-dom* $(\leq_{L2\ (\eta_1\ x)\ x})\ y \Longrightarrow$
$(\leq_{R2\ (l1\ x)\ (l1\ x)})\ (l2_{(l1\ x)\ x\ y}) \leq (\leq_{R2\ (l1\ x)\ (l1\ x)})\ (l2_{(l1\ x)\ (\eta_1\ x)\ y})$
and $\bigwedge x\ y.\ x \leq_{L1} x \Longrightarrow$ *in-codom* $(\leq_{L2\ x\ (\eta_1\ x)})\ y \Longrightarrow$
$(\geq_{R2\ (l1\ x)\ (l1\ x)})\ (l2_{(l1\ x)\ x\ y}) \leq (\geq_{R2\ (l1\ x)\ (l1\ x)})\ (l2_{(l1\ x)\ (\eta_1\ x)\ y})$
and $\bigwedge x'\ y'.\ x' \leq_{R1} x' \Longrightarrow$ *in-dom* $(\leq_{R2\ (\varepsilon_1\ x')\ x'})\ y' \Longrightarrow$
$(\leq_{L2\ (r1\ x')\ (r1\ x')})\ (r2_{(r1\ x')\ x'\ y'}) \leq (\leq_{L2\ (r1\ x')\ (r1\ x')})\ (r2_{(r1\ x')\ (\varepsilon_1\ x')\ y'})$
and $\bigwedge x'\ y'.\ x' \leq_{R1} x' \Longrightarrow$ *in-codom* $(\leq_{R2\ x'\ (\varepsilon_1\ x')})\ y' \Longrightarrow$
$(\geq_{L2\ (r1\ x')\ (r1\ x')})\ (r2_{(r1\ x')\ x'\ y'}) \leq (\geq_{L2\ (r1\ x')\ (r1\ x')})\ (r2_{(r1\ x')\ (\varepsilon_1\ x')\ y'})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *transitive* $(\leq_{L2\ x1\ x2})$
and $\bigwedge x1\ x2.\ x1 \leq_{R1} x2 \Longrightarrow$ *transitive* $(\leq_{R2\ x1\ x2})$
shows $((\leq_L) \equiv_o (\leq_R))\ l\ r$
using *assms*
by (*intro order-equivalenceI rel-equivalence-on-unitI flip.rel-equivalence-on-unitI mono-wrt-rel-leftI flip.mono-wrt-rel-leftI*)
auto

lemma *order-equivalence-if-preorder-equivalenceI*:
assumes *pre-equiv1*: $((\leq_{L1}) \equiv_{pre} (\leq_{R1}))\ l1\ r1$
and *order-equiv2*: $\bigwedge x\ x'.\ x\ {}_{L1}\!\lessapprox x' \Longrightarrow$
$((\leq_{L2\ x\ (r1\ x')}) \equiv_o (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and *L2-les*: $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x2\ x2}) \leq (\leq_{L2\ x1\ x2})$
$\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ (\eta_1\ x1)\ x2}) \leq (\leq_{L2\ x1\ x2})$
$\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ x1}) \leq (\leq_{L2\ x1\ x2})$
$\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ (\eta_1\ x2)}) \leq (\leq_{L2\ x1\ x2})$
and *R2-les*: $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ x2'\ x2'}) \leq (\leq_{R2\ x1'\ x2'})$
$\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ (\varepsilon_1\ x1')\ x2'}) \leq (\leq_{R2\ x1'\ x2'})$
$\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ x1'\ x1'}) \leq (\leq_{R2\ x1'\ x2'})$
$\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ x1'\ (\varepsilon_1\ x2')}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow$
$((\mathit{in\text{-}dom}\ (\leq_{L2\ (r1\ x1')\ (r1\ x2')}))) \Rrightarrow (\leq_{R2\ x1'\ x2'}))\ (l2_{x1'\ (r1\ x1')})\ (l2_{x2'\ (r1\ x1')})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow$
$((\mathit{in\text{-}codom}\ (\leq_{L2\ (r1\ x1')\ (r1\ x2')}))) \Rrightarrow (\leq_{R2\ x1'\ x2'}))\ (l2_{x2'\ (r1\ x1')})\ (l2_{x2'\ (r1\ x2')})$
and *l2-bi-rel*: $\bigwedge x\ y.\ x \leq_{L1} x \Longrightarrow$ *in-field* $(\leq_{L2\ x\ x})\ y \Longrightarrow$
$l2_{(l1\ x)\ (\eta_1\ x)\ y} \equiv_{R2\ (l1\ x)\ (l1\ x)} l2_{(l1\ x)\ x\ y}$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$
$((\mathit{in\text{-}codom}\ (\leq_{R2\ (l1\ x1)\ (l1\ x2)}))) \Rrightarrow (\leq_{L2\ x1\ x2}))\ (r2_{x1\ (l1\ x2)})\ (r2_{x2\ (l1\ x2)})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$
$((\mathit{in\text{-}dom}\ (\leq_{R2\ (l1\ x1)\ (l1\ x2)}))) \Rrightarrow (\leq_{L2\ x1\ x2}))\ (r2_{x1\ (l1\ x1)})\ (r2_{x1\ (l1\ x2)})$
and *r2-bi-rel*: $\bigwedge x'\ y'.\ x' \leq_{R1} x' \Longrightarrow$ *in-field* $(\leq_{R2\ x'\ x'})\ y' \Longrightarrow$
$r2_{(r1\ x')\ (\varepsilon_1\ x')\ y'} \equiv_{L2\ (r1\ x')\ (r1\ x')} r2_{(r1\ x')\ x'\ y'}$
and *trans-L2*: $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *transitive* $(\leq_{L2\ x1\ x2})$
and *trans-R2*: $\bigwedge x1\ x2.\ x1 \leq_{R1} x2 \Longrightarrow$ *transitive* $(\leq_{R2\ x1\ x2})$
shows $((\leq_L) \equiv_o (\leq_R))\ l\ r$
proof −

from *pre-equiv1 L2-les* **have** *L2-unit-eq1*: $(\leq_{L2\ (\eta_1\ x)\ x}) = (\leq_{L2\ x\ x})$
and *L2-unit-eq2*: $(\leq_{L2\ x\ (\eta_1\ x)}) = (\leq_{L2\ x\ x})$
if $x \leq_{L1} x$ **for** x **using** $\langle x \leq_{L1} x\rangle$
by (*auto elim*!: *t1.preorder-equivalence-order-equivalenceE*
intro!: *tdfr.left-rel2-unit-eqs-left-rel2I bi-related-if-rel-equivalence-on*
simp del: *t1.unit-eq*)
from *pre-equiv1 R2-les* **have** *R2-counit-eq1*: $(\leq_{R2\ (\varepsilon_1\ x')\ x'}) = (\leq_{R2\ x'\ x'})$
and *R2-counit-eq2*: $(\leq_{R2\ x'\ (\varepsilon_1\ x')}) = (\leq_{R2\ x'\ x'})$ (**is** *?goal2*)
if $x' \leq_{R1} x'$ **for** x' **using** $\langle x' \leq_{R1} x'\rangle$
by (*auto elim*!: *t1.preorder-equivalence-order-equivalenceE*
intro!: *flip.tdfr.left-rel2-unit-eqs-left-rel2I bi-related-if-rel-equivalence-on*
simp del: *t1.counit-eq*)
from *order-equiv2* **have**
mono-l2: $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \Rightarrow (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})$
and *mono-r2*: $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox x' \Longrightarrow ((\leq_{R2\ (l1\ x)\ x'}) \Rightarrow (\leq_{L2\ x\ (r1\ x')}))\ (r2_{x\ x'})$
by *auto*
moreover have *rel-equivalence-on* (*in-field* $(\leq_{L2\ x\ x})$) $(\leq_{L2\ x\ x})$ $(\eta_{2\ x\ (l1\ x)})$ (**is** *?goal1*)
and $((\leq_{L2\ x\ x}) \Rightarrow (\leq_{R2\ (l1\ x)\ (l1\ x)}))\ (l2_{(l1\ x)\ x})$ (**is** *?goal2*)
if [*iff*]: $x \leq_{L1} x$ **for** x
proof −
from *pre-equiv1* **have** $x\ {}_{L1}\lessapprox l1\ x$ **by** (*intro t1.left-GaloisI*)
(*auto elim*!: *t1.preorder-equivalence-order-equivalenceE t1.order-equivalenceE bi-relatedE*)
with *order-equiv2* **have** $((\leq_{L2\ x\ x}) \equiv_o (\leq_{R2\ (l1\ x)\ (l1\ x)}))\ (l2_{(l1\ x)\ x})\ (r2_{x\ (l1\ x)})$
by (*auto simp flip*: *L2-unit-eq2*)
then show *?goal1 ?goal2* **by** (*auto elim*: *order-functors.order-equivalenceE*)
qed
moreover have
rel-equivalence-on (*in-field* $(\leq_{R2\ x'\ x'})$) $(\leq_{R2\ x'\ x'})$ $(\varepsilon_{2\ (r1\ x')\ x'})$ (**is** *?goal1*)
and $((\leq_{R2\ x'\ x'}) \Rightarrow (\leq_{L2\ (r1\ x')\ (r1\ x')}))\ (r2_{(r1\ x')\ x'})$ (**is** *?goal2*)
if [*iff*]: $x' \leq_{R1} x'$ **for** x'
proof −
from *pre-equiv1* **have** *r1 x'* ${}_{L1}\lessapprox x'$ **by** *blast*
with *order-equiv2* **have** $((\leq_{L2\ (r1\ x')\ (r1\ x')}) \equiv_o (\leq_{R2\ x'\ x'}))\ (l2_{x'\ (r1\ x')})\ (r2_{(r1\ x')\ x'})$
by (*auto simp flip*: *R2-counit-eq1*)
then show *?goal1 ?goal2* **by** (*auto elim*: *order-functors.order-equivalenceE*)
qed
moreover from *mono-l2 tdfr.mono-wrt-rel-left2-if-mono-wrt-rel-left2-if-left-GaloisI*
have $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow ((\leq_{L2\ (r1\ x1')\ (r1\ x2')}) \Rightarrow (\leq_{R2\ x1'\ x2'}))\ (l2_{x2'\ (r1\ x1')})$
using *pre-equiv1 R2-les*(*2*) **by** (*blast elim*!: *le-relE*)
moreover from *pre-equiv1* **have** $((\leq_{L1}) \trianglelefteq_h (\leq_{R1}))$ *l1 r1*
by (*intro t1.half-galois-prop-right-left-right-if-transitive-if-order-equivalence*)
(*auto elim*!: *t1.preorder-equivalence-order-equivalenceE*)

moreover with *mono-r2 tdfr.mono-wrt-rel-right2-if-mono-wrt-rel-right2-if-left-GaloisI*
have $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow ((\leq_{R2\ (l1\ x1)\ (l1\ x2)}) \Rightarrow (\leq_{L2\ x1\ (\eta_1\ x2)}))\ (r2_{x1\ (l1\ x2)})$
using *pre-equiv1* **by** *blast*
moreover with *L2-les*
have $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow ((\leq_{R2\ (l1\ x1)\ (l1\ x2)}) \Rightarrow (\leq_{L2\ x1\ x2}))\ (r2_{x1\ (l1\ x2)})$
by *blast*
moreover have *in-dom* $(\leq_{L2\ (\eta_1\ x)\ x})\ y \Longrightarrow$
$(\leq_{R2\ (l1\ x)\ (l1\ x)})\ (l2_{(l1\ x)\ x}\ y) \leq (\leq_{R2\ (l1\ x)\ (l1\ x)})\ (l2_{(l1\ x)\ (\eta_1\ x)}\ y)$
(**is** - $\Longrightarrow$ *?goal1*)
and *in-codom* $(\leq_{L2\ x\ (\eta_1\ x)})\ y \Longrightarrow$
$(\geq_{R2\ (l1\ x)\ (l1\ x)})\ (l2_{(l1\ x)\ x}\ y) \leq (\geq_{R2\ (l1\ x)\ (l1\ x)})\ (l2_{(l1\ x)\ (\eta_1\ x)}\ y)$
(**is** - $\Longrightarrow$ *?goal2*)
if [*iff*]: $x \leq_{L1} x$ **for** $x\ y$
proof −
presume *in-dom* $(\leq_{L2\ (\eta_1\ x)\ x})\ y \vee$ *in-codom* $(\leq_{L2\ x\ (\eta_1\ x)})\ y$
then have *in-field* $(\leq_{L2\ x\ x})\ y$ **using** *L2-unit-eq1 L2-unit-eq2* **by** *auto*
with *l2-bi-rel* **have** $l2_{(l1\ x)\ (\eta_1\ x)}\ y \equiv_{R2\ (l1\ x)\ (l1\ x)}\ l2_{(l1\ x)\ x}\ y$ **by** *blast*
moreover from *pre-equiv1* **have** $\langle l1\ x \leq_{R1} l1\ x\rangle$ **by** *blast*
ultimately show *?goal1 ?goal2* **using** *trans-R2* **by** *blast+*
qed *auto*
moreover have *in-dom* $(\leq_{R2\ (\varepsilon_1\ x')\ x'})\ y' \Longrightarrow$
$(\leq_{L2\ (r1\ x')\ (r1\ x')})\ (r2_{(r1\ x')\ x'}\ y') \leq (\leq_{L2\ (r1\ x')\ (r1\ x')})\ (r2_{(r1\ x')\ (\varepsilon_1\ x')}\ y')$
(**is** - $\Longrightarrow$ *?goal1*)
and *in-codom* $(\leq_{R2\ x'\ (\varepsilon_1\ x')})\ y' \Longrightarrow$
$(\geq_{L2\ (r1\ x')\ (r1\ x')})\ (r2_{(r1\ x')\ x'}\ y') \leq (\geq_{L2\ (r1\ x')\ (r1\ x')})\ (r2_{(r1\ x')\ (\varepsilon_1\ x')}\ y')$
(**is** - $\Longrightarrow$ *?goal2*)
if [*iff*]: $x' \leq_{R1} x'$ **for** $x'\ y'$
proof −
presume *in-dom* $(\leq_{R2\ (\varepsilon_1\ x')\ x'})\ y' \vee$ *in-codom* $(\leq_{R2\ x'\ (\varepsilon_1\ x')})\ y'$
then have *in-field* $(\leq_{R2\ x'\ x'})\ y'$ **using** *R2-counit-eq1 R2-counit-eq2* **by** *auto*
with *r2-bi-rel* **have** $r2_{(r1\ x')\ (\varepsilon_1\ x')}\ y' \equiv_{L2\ (r1\ x')\ (r1\ x')}\ r2_{(r1\ x')\ x'}\ y'$
by *blast*
moreover from *pre-equiv1* **have** $\langle r1\ x' \leq_{L1} r1\ x'\rangle$ **by** *blast*
ultimately show *?goal1 ?goal2* **using** *trans-L2* **by** *blast+*
qed *auto*
ultimately show *?thesis* **using** *assms*
by (*intro order-equivalenceI*
tdfr.mono-wrt-rel-left-if-transitiveI
tdfr.mono-wrt-rel-left2-if-mono-wrt-rel-left2-if-left-GaloisI
tdfr.mono-wrt-rel-right-if-transitiveI
tdfr.mono-wrt-rel-right2-if-mono-wrt-rel-right2-if-left-GaloisI)
(*auto elim*!: *t1.preorder-equivalence-order-equivalenceE*)
qed

lemma *order-equivalence-if-preorder-equivalenceI'*:
assumes $((\leq_{L1}) \equiv_{pre} (\leq_{R1}))$ *l1 r1*
and $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \equiv_o (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x2\ x2}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ (\eta_1\ x1)\ x2}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ x1}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow (\leq_{L2\ x1\ (\eta_1\ x2)}) \leq (\leq_{L2\ x1\ x2})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ x2'\ x2'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ (\varepsilon_1\ x1')\ x2'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ x1'\ x1'}) \leq (\leq_{R2\ x1'\ x2'})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow (\leq_{R2\ x1'\ (\varepsilon_1\ x2')}) \leq (\leq_{R2\ x1'\ x2'})$
and $((x1'\ x2' :: (\leq_{R1})) \Rightarrow (x1\ x2 :: (\leq_{L1}) \mid x2\ {}_{L1}\lessapprox x1') \Rrightarrow$
in-field $(\leq_{L2\ x1\ (r1\ x2')}) \Rrightarrow (\leq_{R2\ (l1\ x1)\ x2'}))$ *l2*
and $\bigwedge x\ y.\ x \leq_{L1} x \Longrightarrow$ *in-field* $(\leq_{L2\ x\ x})\ y \Longrightarrow$
${}^{l2}{}_{(l1\ x)\ (\eta_1\ x)}\ y \equiv_{R2\ (l1\ x)\ (l1\ x)}\ {}^{l2}{}_{(l1\ x)\ x}\ y$
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ {}_{L1}\lessapprox x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))$ *r2*
and $\bigwedge x'\ y'.\ x' \leq_{R1} x' \Longrightarrow$ *in-field* $(\leq_{R2\ x'\ x'})\ y' \Longrightarrow$
${}^{r2}{}_{(r1\ x')\ (\varepsilon_1\ x')}\ y' \equiv_{L2\ (r1\ x')\ (r1\ x')}\ {}^{r2}{}_{(r1\ x')\ x'}\ y'$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *transitive* $(\leq_{L2\ x1\ x2})$
and $\bigwedge x1\ x2.\ x1 \leq_{R1} x2 \Longrightarrow$ *transitive* $(\leq_{R2\ x1\ x2})$
shows $((\leq_L) \equiv_o (\leq_R))$ *l r*
using *assms* **by** (*intro order-equivalence-if-preorder-equivalenceI*
tdfr.order-equivalence-if-order-equivalence-mono-assms-leftI
tdfr.order-equivalence-if-order-equivalence-mono-assms-rightI
reflexive-on-in-field-if-transitive-if-rel-equivalence-on)
(*auto elim*!: *t1.preorder-equivalence-order-equivalenceE*)

lemma *order-equivalence-if-mono-if-preorder-equivalenceI*:
assumes $((\leq_{L1}) \equiv_{pre} (\leq_{R1}))$ *l1 r1*
and $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \equiv_o (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $((x1\ x2 :: (\leq_{L1}) \mid \eta_1\ x2 \leq_{L1} x1) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid x2 \leq_{L1} x3) \Rrightarrow (\leq))$ *L2*
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid (x2 \leq_{L1} x3 \wedge x4 \leq_{L1} \eta_1\ x3)) \Rrightarrow (\geq))$ *L2*
and $((x1'\ x2' :: (\leq_{R1}) \mid \varepsilon_1\ x2' \leq_{R1} x1') \Rightarrow (x3'\ x4' :: (\leq_{R1}) \mid x2' \leq_{R1} x3') \Rrightarrow (\leq))$ *R2*
and $((x1'\ x2' :: (\leq_{R1})) \Rightarrow (x3'\ x4' :: (\leq_{R1}) \mid (x2' \leq_{R1} x3' \wedge x4' \leq_{R1} \varepsilon_1\ x3')) \Rrightarrow (\geq))$ *R2*
and $((x1'\ x2' :: (\leq_{R1})) \Rightarrow (x1\ x2 :: (\leq_{L1}) \mid x2\ {}_{L1}\lessapprox x1') \Rrightarrow$
in-field $(\leq_{L2\ x1\ (r1\ x2')}) \Rrightarrow (\leq_{R2\ (l1\ x1)\ x2'}))$ *l2*
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ {}_{L1}\lessapprox x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))$ *r2*
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *transitive* $(\leq_{L2\ x1\ x2})$
and $\bigwedge x1\ x2.\ x1 \leq_{R1} x2 \Longrightarrow$ *transitive* $(\leq_{R2\ x1\ x2})$
shows $((\leq_L) \equiv_o (\leq_R))$ *l r*
using *assms* **by** (*intro order-equivalence-if-preorder-equivalenceI'*

tdfr.l2-unit-bi-rel-selfI tdfr.r2-counit-bi-rel-selfI
tdfr.left-rel-right-iff-left-right-rel-if-galois-prop-le-assms-leftI
flip.tdfr.left-rel-right-iff-left-right-rel-if-galois-prop-le-assms-leftI
tdfr.left-rel-right-iff-left-right-rel-if-galois-prop-le-assms-rightI
flip.tdfr.left-rel-right-iff-left-right-rel-if-galois-prop-le-assms-rightI
t1.galois-connection-left-right-if-transitive-if-order-equivalence
flip.t1.galois-connection-left-right-if-transitive-if-order-equivalence
reflexive-on-in-field-if-transitive-if-rel-equivalence-on)
(*auto elim*!: *t1.preorder-equivalence-order-equivalenceE*)

theorem *order-equivalence-if-mono-if-preorder-equivalenceI'*:
assumes $((\leq_{L1}) \equiv_{pre} (\leq_{R1}))$ *l1 r1*
and $\bigwedge x\ x'.\ x\ {}_{L1}\lessapprox x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \equiv_{pre} (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $((x1\ x2 :: (\geq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid x1 \leq_{L1} x3) \Rrightarrow (\leq))\ L2$
and $((x1'\ x2' :: (\geq_{R1})) \Rightarrow (x3'\ x4' :: (\leq_{R1}) \mid x1' \leq_{R1} x3') \Rrightarrow (\leq))\ R2$
and $((x1'\ x2' :: (\leq_{R1})) \Rightarrow (x1\ x2 :: (\leq_{L1}) \mid x2\ {}_{L1}\lessapprox x1') \Rrightarrow$
in-field $(\leq_{L2\ x1\ (r1\ x2')}) \Rrightarrow (\leq_{R2\ (l1\ x1)\ x2'}))\ l2$
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ {}_{L1}\lessapprox x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))\ r2$
shows $((\leq_L) \equiv_o (\leq_R))\ l\ r$
using *assms* **by** (*intro order-equivalence-if-mono-if-preorder-equivalenceI*
tdfr.galois-equivalence-if-mono-if-galois-equivalence-mono-assms-leftI
flip.tdfr.galois-equivalence-if-mono-if-galois-equivalence-mono-assms-leftI
tdfr.transitive-left2-if-preorder-equivalenceI
tdfr.transitive-right2-if-preorder-equivalenceI
t1.preorder-on-in-field-left-if-transitive-if-order-equivalence
flip.t1.preorder-on-in-field-left-if-transitive-if-order-equivalence
t1.galois-equivalence-left-right-if-transitive-if-order-equivalence
flip.t1.galois-equivalence-left-right-if-transitive-if-order-equivalence)
(*auto elim*!: *t1.preorder-equivalence-order-equivalenceE*
t2.preorder-equivalence-order-equivalenceE)

end

Monotone Function Relator **context** *transport-Mono-Fun-Rel* **begin**

interpretation *flip* : *transport-Mono-Fun-Rel R1 L1 r1 l1 R2 L2 r2 l2* **.**

lemma *inflationary-on-unitI*:
assumes $((\leq_{L1}) \Rightarrow (\leq_{R1}))$ *l1*
and $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*
and *inflationary-on* (*in-codom* $(\leq_{L1})$) $(\leq_{L1})$ η_1
and *reflexive-on* (*in-codom* $(\leq_{L1})$) $(\leq_{L1})$
and *transitive* $(\leq_{L1})$
and $((\leq_{L2}) \Rightarrow (\leq_{R2}))$ *l2*
and $((\leq_{R2}) \Rightarrow (\leq_{L2}))$ *r2*
and *inflationary-on* (*in-codom* $(\leq_{L2})$) $(\leq_{L2})$ η_2
and *transitive* $(\leq_{L2})$

shows *inflationary-on* (*in-field* ($\leq_L$)) ($\leq_L$) η
using *assms* **by** (*intro tmdfr.inflationary-on-unitI*
tfr.mono-wrt-rel-leftI flip.tfr.mono-wrt-rel-leftI)
simp-all

lemma *deflationary-on-counitI*:
assumes (($\leq_{L1}$) $\Rightarrow$ ($\leq_{R1}$)) *l1*
and (($\leq_{R1}$) $\Rightarrow$ ($\leq_{L1}$)) *r1*
and *deflationary-on* (*in-dom* ($\leq_{R1}$)) ($\leq_{R1}$) ε_1
and *reflexive-on* (*in-dom* ($\leq_{R1}$)) ($\leq_{R1}$)
and *transitive* ($\leq_{R1}$)
and (($\leq_{L2}$) $\Rightarrow$ ($\leq_{R2}$)) *l2*
and (($\leq_{R2}$) $\Rightarrow$ ($\leq_{L2}$)) *r2*
and *deflationary-on* (*in-dom* ($\leq_{R2}$)) ($\leq_{R2}$) ε_2
and *transitive* ($\leq_{R2}$)
shows *deflationary-on* (*in-field* ($\leq_R$)) ($\leq_R$) ε
using *assms* **by** (*intro tmdfr.deflationary-on-counitI*
tfr.mono-wrt-rel-leftI flip.tfr.mono-wrt-rel-leftI)
simp-all

lemma *rel-equivalence-on-unitI*:
assumes (($\leq_{L1}$) $\Rightarrow$ ($\leq_{R1}$)) *l1*
and (($\leq_{R1}$) $\Rightarrow$ ($\leq_{L1}$)) *r1*
and *rel-equivalence-on* (*in-field* ($\leq_{L1}$)) ($\leq_{L1}$) η_1
and *transitive* ($\leq_{L1}$)
and (($\leq_{L2}$) $\Rightarrow$ ($\leq_{R2}$)) *l2*
and (($\leq_{R2}$) $\Rightarrow$ ($\leq_{L2}$)) *r2*
and *rel-equivalence-on* (*in-field* ($\leq_{L2}$)) ($\leq_{L2}$) η_2
and *transitive* ($\leq_{L2}$)
shows *rel-equivalence-on* (*in-field* ($\leq_L$)) ($\leq_L$) η
using *assms* **by** (*intro tmdfr.rel-equivalence-on-unitI*
tfr.mono-wrt-rel-leftI flip.tfr.mono-wrt-rel-leftI)
simp-all

lemma *order-equivalenceI*:
assumes (($\leq_{L1}$) $\equiv_{pre}$ ($\leq_{R1}$)) *l1 r1*
and (($\leq_{L2}$) $\equiv_{pre}$ ($\leq_{R2}$)) *l2 r2*
shows (($\leq_L$) $\equiv_o$ ($\leq_R$)) *l r*
using *assms* **by** (*intro tmdfr.order-equivalenceI*
tfr.mono-wrt-rel-leftI flip.tfr.mono-wrt-rel-leftI)
(*auto elim*!: *tdfrs.t1.preorder-equivalence-order-equivalenceE*
tdfrs.t2.preorder-equivalence-order-equivalenceE)

end

end

theory *Transport-Functions*

imports
Transport-Functions-Galois-Equivalence
Transport-Functions-Galois-Relator
Transport-Functions-Order-Base
Transport-Functions-Order-Equivalence
Transport-Functions-Relation-Simplifications
begin

Summary Composition under (dependent) (monotone) function relators. Refer to [2] for more details.

2.15.10 Summary of Main Results

More precise results can be found in the corresponding subtheories.

Monotone Dependent Function Relator **context** *transport-Mono-Dep-Fun-Rel*
begin

interpretation *flip* : *transport-Mono-Dep-Fun-Rel R1 L1 r1 l1 R2 L2 r2 l2*
rewrites *flip.t1.counit* $\equiv \eta_1$ **and** *flip.t1.unit* $\equiv \varepsilon_1$
by (*simp-all only*: *t1.flip-counit-eq-unit t1.flip-unit-eq-counit*)

Closure of Order and Galois Concepts **theorem** *preorder-galois-connection-if-galois-connectionI*:
assumes $((\leq_{L1}) \dashv (\leq_{R1}))$ *l1 r1*
and *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$
and *reflexive-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$
and $\bigwedge x\ x'.\ x\ _{L1}\lessapprox x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \dashv (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $((\text{-}\ x2 :: (\leq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid (x2 \leq_{L1} x3 \wedge x4 \leq_{L1} \eta_1\ x3)) \Rrightarrow (\geq))$
L2
and $((x1'\ x2' :: (\leq_{R1}) \mid \varepsilon_1\ x2' \leq_{R1} x1') \Rightarrow (x3'\ \text{-} :: (\leq_{R1}) \mid x2' \leq_{R1} x3') \Rrightarrow$
$(\leq))$ *R2*
and $((x1'\ x2' :: (\leq_{R1})) \Rightarrow (x1\ x2 :: (\leq_{L1}) \mid x2\ _{L1}\gtrapprox x1') \Rrightarrow$
in-field $(\leq_{L2\ x1\ (r1\ x2')}) \Rrightarrow (\leq_{R2\ (l1\ x1)\ x2'}))\ l2$
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ _{L1}\gtrapprox x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))\ r2$
and $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow transitive\ (\leq_{L2\ x1\ x2})$
and $\bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow transitive\ (\leq_{R2\ x1'\ x2'})$
shows $((\leq_L) \dashv_{pre} (\leq_R))\ l\ r$
using *assms* **by** (*intro preorder-galois-connectionI*
galois-connection-left-right-if-mono-if-galois-connectionI'
preorder-on-in-field-leftI flip.preorder-on-in-field-leftI
tdfr.transitive-leftI' flip.tdfr.transitive-leftI
tdfr.left-rel-right-iff-left-right-rel-if-galois-prop-le-assms-leftI
tdfr.left-rel-right-iff-left-right-rel-if-galois-prop-le-assms-rightI)
(*auto intro*: *reflexive-on-if-le-pred-if-reflexive-on*
in-field-if-in-dom in-field-if-in-codom)

theorem *preorder-equivalenceI*:
assumes $((\leq_{L1}) \equiv_{pre} (\leq_{R1}))$ *l1 r1*
and $\bigwedge x\ x'.\ x\ _{L1}\lessapprox x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \equiv_{pre} (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $((x1\ \text{-} :: (\geq_{L1})) \Rightarrow (x3\ \text{-} :: (\leq_{L1}) \mid x1 \leq_{L1} x3) \Rrightarrow (\leq))\ L2$
and $((x1'\ \text{-} :: (\geq_{R1})) \Rightarrow (x3'\ \text{-} :: (\leq_{R1}) \mid x1' \leq_{R1} x3') \Rrightarrow (\leq))\ R2$
and $((x1'\ x2' :: (\leq_{R1})) \Rightarrow (x1\ x2 :: (\leq_{L1}) \mid x2\ _{L1}\lessapprox x1') \Rrightarrow$
in-field $(\leq_{L2\ x1\ (r1\ x2')}) \Rrightarrow (\leq_{R2\ (l1\ x1)\ x2'}))\ l2$
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ _{L1}\lessapprox x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))\ r2$
shows $((\leq_L) \equiv_{pre} (\leq_R))\ l\ r$
using *assms* **by** (*intro preorder-equivalence-if-galois-equivalenceI*
galois-equivalence-if-mono-if-preorder-equivalenceI'
preorder-on-in-field-leftI flip.preorder-on-in-field-leftI
tdfr.transitive-leftI' flip.tdfr.transitive-leftI
tdfr.transitive-left2-if-preorder-equivalenceI
tdfr.transitive-right2-if-preorder-equivalenceI
tdfr.left-rel-right-iff-left-right-rel-if-galois-prop-le-assms-leftI
tdfr.left-rel-right-iff-left-right-rel-if-galois-prop-le-assms-rightI
tdfr.galois-equivalence-if-mono-if-galois-equivalence-mono-assms-leftI
flip.tdfr.galois-equivalence-if-mono-if-galois-equivalence-mono-assms-leftI)
(*auto intro*: *reflexive-on-if-le-pred-if-reflexive-on*
in-field-if-in-dom in-field-if-in-codom)

theorem *partial-equivalence-rel-equivalenceI*:
assumes $((\leq_{L1}) \equiv_{PER} (\leq_{R1}))$ *l1 r1*
and $\bigwedge x\ x'.\ x\ _{L1}\lessapprox x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \equiv_{PER} (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $((x1\ \text{-} :: (\geq_{L1})) \Rightarrow (x3\ \text{-} :: (\leq_{L1}) \mid x1 \leq_{L1} x3) \Rrightarrow (\leq))\ L2$
and $((x1'\ \text{-} :: (\geq_{R1})) \Rightarrow (x3'\ \text{-} :: (\leq_{R1}) \mid x1' \leq_{R1} x3') \Rrightarrow (\leq))\ R2$
and $((x1'\ x2' :: (\leq_{R1})) \Rightarrow (x1\ x2 :: (\leq_{L1}) \mid x2\ _{L1}\lessapprox x1') \Rrightarrow$
in-field $(\leq_{L2\ x1\ (r1\ x2')}) \Rrightarrow (\leq_{R2\ (l1\ x1)\ x2'}))\ l2$
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ _{L1}\lessapprox x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))\ r2$
shows $((\leq_L) \equiv_{PER} (\leq_R))\ l\ r$
using *assms*
by (*intro partial-equivalence-rel-equivalence-if-galois-equivalenceI*
galois-equivalence-if-mono-if-preorder-equivalenceI'
tdfr.transitive-left2-if-preorder-equivalenceI
tdfr.transitive-right2-if-preorder-equivalenceI
partial-equivalence-rel-leftI flip.partial-equivalence-rel-leftI
tdfr.partial-equivalence-rel-left2-if-partial-equivalence-rel-equivalenceI
tdfr.partial-equivalence-rel-right2-if-partial-equivalence-rel-equivalenceI)
auto

Simplification of Left and Right Relations See $[\![$*t1.galois-equivalence*; *preorder-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$; $((x1\ x2 :: (\leq_{L1})^{-1}) \Rightarrow (x3\ x4 :: (\leq_{L1})) \Rrightarrow x1 \leq_{L1} x3 \longrightarrow (\leq))\ L2$; $\bigwedge x1\ x2.\ x1 \leq_{L1} x2 \Longrightarrow$ *partial-equivalence-rel* $(\leq_{L2\ x1\ x2})]\!] \Longrightarrow$ *flip.R* = *flip.tdfr.R*.

Simplification of Galois relator See $[\![t1.galois\text{-}connection;\ reflexive\text{-}on\ (in\text{-}field\ (\leq_{L1}))\ (\leq_{L1});\ \bigwedge x\ x'.\ flip.t1.right\text{-}Galois\ x\ x' \Longrightarrow ((\leq_{R2\ l1\ x\ x'}) \Rightarrow \leq_{L2\ x\ r1\ x'})\ r2_{x\ x'};\ ((x1\ :\ \top) \Rightarrow (x2\ \text{-}\ ::\ (\leq_{L1})) \Rightarrow x1\ \leq_{L1}\ x2 \longrightarrow (\leq))\ L2;\ ((x1\ :\ \top) \Rightarrow (x2\ x3\ ::\ (\leq_{L1})) \Rightarrow (x1\ \leq_{L1}\ x2 \wedge x3\ \leq_{L1}\ \eta_1\ x2) \longrightarrow (\lambda x\ y.\ y \leq x))\ L2;\ ((x1\ x2\ ::\ (\leq_{L1})) \Rightarrow (x1'\ x2'\ ::\ (\leq_{R1})) \Rrightarrow flip.t1.right\text{-}Galois\ x2\ x1' \longrightarrow (in\text{-}field\ (\leq_{R2\ l1\ x1\ x2'}) \Rrightarrow \leq_{L2\ x1\ r1\ x2'}))\ r2;\ \bigwedge x1\ x2.\ x1\ \leq_{L1}\ x2 \Longrightarrow transitive\ (\leq_{L2\ x1\ x2})]\!] \Longrightarrow flip.right\text{-}Galois = (Dep\text{-}Fun\text{-}Rel\ flip.t1.right\text{-}Galois\ t2.left\text{-}Galois)\restriction_{in\text{-}dom\ flip.R}\restriction_{in\text{-}codom\ flip.L}$

$[\![t1.preorder\text{-}equivalence;\ \bigwedge x\ x'.\ flip.t1.right\text{-}Galois\ x\ x' \Longrightarrow ((\leq_{R2\ l1\ x\ x'}) \Rightarrow \leq_{L2\ x\ r1\ x'})\ r2_{x\ x'};\ ((x1\ x2\ ::\ (\leq_{L1})^{-1}) \Rightarrow (x3\ x4\ ::\ (\leq_{L1})) \Rrightarrow x1\ \leq_{L1}\ x3 \longrightarrow (\leq))\ L2;\ ((x1\ x2\ ::\ (\leq_{L1})) \Rightarrow (x1'\ x2'\ ::\ (\leq_{R1})) \Rrightarrow flip.t1.right\text{-}Galois\ x2\ x1' \longrightarrow (in\text{-}field\ (\leq_{R2\ l1\ x1\ x2'}) \Rrightarrow \leq_{L2\ x1\ r1\ x2'}))\ r2;\ \bigwedge x1\ x2.\ x1\ \leq_{L1}\ x2 \Longrightarrow transitive\ (\leq_{L2\ x1\ x2})]\!] \Longrightarrow flip.right\text{-}Galois = (Dep\text{-}Fun\text{-}Rel\ flip.t1.right\text{-}Galois\ t2.left\text{-}Galois)\restriction_{in\text{-}dom\ flip.R}\restriction_{in\text{-}codom\ flip.L}$

$[\![t1.preorder\text{-}equivalence;\ \bigwedge x\ x'.\ flip.t1.right\text{-}Galois\ x\ x' \Longrightarrow t2.preorder\text{-}equivalence\ x\ x';\ ((x1\ x2\ ::\ (\leq_{L1})^{-1}) \Rightarrow (x3\ x4\ ::\ (\leq_{L1})) \Rrightarrow x1\ \leq_{L1}\ x3 \longrightarrow (\leq))\ L2;\ ((x1\ x2\ ::\ (\leq_{L1})) \Rightarrow (x1'\ x2'\ ::\ (\leq_{R1})) \Rrightarrow flip.t1.right\text{-}Galois\ x2\ x1' \longrightarrow (in\text{-}field\ (\leq_{R2\ l1\ x1\ x2'}) \Rrightarrow \leq_{L2\ x1\ r1\ x2'}))\ r2]\!] \Longrightarrow flip.right\text{-}Galois = (Dep\text{-}Fun\text{-}Rel\ flip.t1.right\text{-}Galois\ t2.left\text{-}Galois)\restriction_{in\text{-}dom\ flip.R}\restriction_{in\text{-}codom\ flip.L}$

$[\![t1.preorder\text{-}equivalence;\ \bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow ((\leq_{L2\ r1\ x1'\ r1\ x2'})\ {}_h\trianglelefteq\ (\leq_{R2\ \varepsilon_1\ x1'\ x2'}))\ l2_{x2'\ r1\ x1'}\ r2_{r1\ x1'\ x2'};\ ((x1\ x2\ ::\ (\leq_{L1})^{-1}) \Rightarrow (x3\ x4\ ::\ (\leq_{L1})) \Rrightarrow x1\ \leq_{L1}\ x3 \longrightarrow (\leq))\ L2;\ ((x1'\ x2'\ ::\ (\leq_{R1})^{-1}) \Rightarrow (x3'\ x4'\ ::\ (\leq_{R1})) \Rrightarrow x1' \leq_{R1} x3' \longrightarrow (\leq))\ R2;\ ((x1'\ x2'\ ::\ (\leq_{R1})) \Rightarrow (x1\ x2\ ::\ (\leq_{L1})) \Rrightarrow flip.t1.right\text{-}Galois\ x2\ x1' \longrightarrow (in\text{-}field\ (\leq_{L2\ x1\ r1\ x2'}) \Rrightarrow \leq_{R2\ l1\ x1\ x2'}))\ l2;\ ((x1\ x2\ ::\ (\leq_{L1})) \Rightarrow (x1'\ x2'\ ::\ (\leq_{R1})) \Rrightarrow flip.t1.right\text{-}Galois\ x2\ x1' \longrightarrow (in\text{-}field\ (\leq_{R2\ l1\ x1\ x2'}) \Rrightarrow \leq_{L2\ x1\ r1\ x2'}))\ r2;\ \bigwedge x1\ x2.\ x1\ \leq_{L1}\ x2 \Longrightarrow partial\text{-}equivalence\text{-}rel\ (\leq_{L2\ x1\ x2});\ \bigwedge x1'\ x2'.\ x1' \leq_{R1} x2' \Longrightarrow partial\text{-}equivalence\text{-}rel\ (\leq_{R2\ x1'\ x2'})]\!] \Longrightarrow (Dep\text{-}Fun\text{-}Rel\ flip.t1.right\text{-}Galois\ t2.left\text{-}Galois)\restriction_{in\text{-}dom\ flip.R}\restriction_{in\text{-}codom\ flip.L} = Dep\text{-}Fun\text{-}Rel\ flip.t1.right\text{-}Galois\ t2.left\text{-}Galois$

$t1.half\text{-}galois\text{-}prop\text{-}left \Longrightarrow ((x\ x' ::\ flip.t1.right\text{-}Galois) \Rrightarrow (?S\ x\ x')\restriction_{in\text{-}dom\ (\leq_{L2\ x\ r1\ x'})}\restriction_{in\text{-}codom\ (\leq$
$= (Dep\text{-}Fun\text{-}Rel\ flip.t1.right\text{-}Galois\ ?S)\restriction_{in\text{-}dom\ flip.R}\restriction_{in\text{-}codom\ flip.L}$

end

Monotone Function Relator **context** *transport-Mono-Fun-Rel*
begin

interpretation *flip* : *transport-Mono-Fun-Rel R1 L1 r1 l1 R2 L2 r2 l2* .

Closure of Order and Galois Concepts **lemma** *preorder-galois-connection-if-galois-connectionI*:
assumes $((\leq_{L1}) \dashv (\leq_{R1}))$ *l1 r1*
and *reflexive-on* (*in-codom* $(\leq_{L1})$) $(\leq_{L1})$ *reflexive-on* (*in-dom* $(\leq_{R1})$) $(\leq_{R1})$
and $((\leq_{L2}) \dashv (\leq_{R2}))$ *l2 r2*
and *transitive* $(\leq_{L2})$ *transitive* $(\leq_{R2})$
shows $((\leq_L) \dashv_{pre} (\leq_R))$ *l r*

using *assms* **by** (*intro tmdfr.preorder-galois-connectionI*
galois-connection-left-rightI
tmdfr.preorder-on-in-field-leftI flip.tmdfr.preorder-on-in-field-leftI
tfr.transitive-leftI' flip.tfr.transitive-leftI)
auto

theorem *preorder-galois-connectionI*:
assumes $((\leq_{L1}) \dashv_{pre} (\leq_{R1}))$ *l1 r1*
and $((\leq_{L2}) \dashv_{pre} (\leq_{R2}))$ *l2 r2*
shows $((\leq_{L}) \dashv_{pre} (\leq_{R}))$ *l r*
using *assms* **by** (*intro preorder-galois-connection-if-galois-connectionI*)
(*auto intro*: *reflexive-on-if-le-pred-if-reflexive-on*
in-field-if-in-dom in-field-if-in-codom)

theorem *preorder-equivalence-if-galois-equivalenceI*:
assumes $((\leq_{L1}) \equiv_{G} (\leq_{R1}))$ *l1 r1*
and *reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$ *reflexive-on* (*in-field* $(\leq_{R1})$) $(\leq_{R1})$
and $((\leq_{L2}) \equiv_{G} (\leq_{R2}))$ *l2 r2*
and *transitive* $(\leq_{L2})$ *transitive* $(\leq_{R2})$
shows $((\leq_{L}) \equiv_{pre} (\leq_{R}))$ *l r*
using *assms* **by** (*intro tmdfr.preorder-equivalence-if-galois-equivalenceI*
galois-equivalenceI
tmdfr.preorder-on-in-field-leftI flip.tmdfr.preorder-on-in-field-leftI
tfr.transitive-leftI flip.tfr.transitive-leftI)
(*auto intro*: *reflexive-on-if-le-pred-if-reflexive-on in-field-if-in-dom*)

theorem *preorder-equivalenceI*:
assumes $((\leq_{L1}) \equiv_{pre} (\leq_{R1}))$ *l1 r1*
and $((\leq_{L2}) \equiv_{pre} (\leq_{R2}))$ *l2 r2*
shows $((\leq_{L}) \equiv_{pre} (\leq_{R}))$ *l r*
using *assms* **by** (*intro preorder-equivalence-if-galois-equivalenceI*) *auto*

theorem *partial-equivalence-rel-equivalenceI*:
assumes $((\leq_{L1}) \equiv_{PER} (\leq_{R1}))$ *l1 r1*
and $((\leq_{L2}) \equiv_{PER} (\leq_{R2}))$ *l2 r2*
shows $((\leq_{L}) \equiv_{PER} (\leq_{R}))$ *l r*
using *assms* **by** (*intro tmdfr.partial-equivalence-rel-equivalence-if-galois-equivalenceI*
galois-equivalenceI
partial-equivalence-rel-leftI flip.partial-equivalence-rel-leftI)
auto

Simplification of Left and Right Relations See $[\![$*reflexive-on* (*in-field* $(\leq_{L1})$) $(\leq_{L1})$; *partial-equivalence-rel* $(\leq_{L2})]\!] \Longrightarrow$ *flip.tmdfr.R* = *flip.tfr.tdfr.R*.

Simplification of Galois relator See $[\![((\leq_{L1}) \Rightarrow (\leq_{R1}))$ *l1*; *tdfrs.t1.galois-prop l1 r1*; *reflexive-on* (*in-dom* $(\leq_{L1})$) $(\leq_{L1})$; $((\leq_{R2}) \Rightarrow (\leq_{L2}))$ *r2*; *transitive* $(\leq_{L2})]\!] \Longrightarrow$ *flip.tmdfr.right-Galois* = (*flip.tdfrs.t1.right-Galois* $\Rrightarrow$ *flip.tdfrs.t2.right-Galois*)$\restriction_{in\text{-}dom\ flip}$
$[\![((\leq_{L1}) \Rightarrow (\leq_{R1}))$ *l1*; $((\leq_{R1}) \Rightarrow (\leq_{L1}))$ *r1*; *tdfrs.t1.half-galois-prop-right*;

reflexive-on (*in-field* ($\leq_{L1}$)) ($\leq_{L1}$); *reflexive-on* (*in-field* ($\leq_{R1}$)) ($\leq_{R1}$); *tdfrs.t2.half-galois-prop-left*; *partial-equivalence-rel* ($\leq_{L2}$); *partial-equivalence-rel* ($\leq_{R2}$)$\rrbracket \Longrightarrow$ (*flip.tdfrs.t1.right-Galois* $\Rrightarrow$ *flip.tdfrs.t2.right-Galois*)$\restriction_{\textit{in-dom flip.tmdfr.R}}\restriction_{\textit{in-codom flip.tmdfr}}$
= (*flip.tdfrs.t1.right-Galois* $\Rrightarrow$ *flip.tdfrs.t2.right-Galois*)
tdfrs.t1.half-galois-prop-left $\Longrightarrow$ (*flip.tdfrs.t1.right-Galois* $\Rrightarrow$ *?S*$\restriction_{\textit{in-dom}\ (\leq_{L2})}\restriction_{\textit{in-codom}\ (\leq_{R2})}$)$\restriction_{\textit{in-d}}$
= (*flip.tdfrs.t1.right-Galois* $\Rrightarrow$ *?S*)$\restriction_{\textit{in-dom flip.tmdfr.R}}\restriction_{\textit{in-codom flip.tmdfr.L}}$.

end

Dependent Function Relator While a general transport of functions is only possible for the monotone function relator (see above), the locales *transport-Dep-Fun-Rel* and *transport-Fun-Rel* contain special cases to transport functions that are proven to be monotone using the standard function space.

Moreover, in the special case of equivalences on partial equivalence relations, the standard function space is monotone - see $\llbracket$*galois.galois-equivalence ?L1.0 ?R1.0 ?l1.0 ?r1.0*; *preorder-on* (*in-field ?L1.0*) *?L1.0*; ((*x1 x2* :: *?L1.0*$^{-1}$) $\Rightarrow$ (*x3 x4* :: *?L1.0*) $\Rrightarrow$ *?L1.0 x1 x3* $\longrightarrow$ ($\leq$)) *?L2.0*; $\bigwedge$*x1 x2*. *?L1.0 x1 x2* $\Longrightarrow$ *partial-equivalence-rel* (*?L2.0 x1 x2*)$\rrbracket \Longrightarrow$ *transport-Mono-Dep-Fun-Rel.L ?L1.0 ?L2.0* = *transport-Dep-Fun-Rel.L ?L1.0 ?L2.0* As such, we can derive general transport theorems from the monotone cases above.

context *transport-Dep-Fun-Rel*
begin

interpretation *tmdfr* : *transport-Mono-Dep-Fun-Rel L1 R1 l1 r1 L2 R2 l2 r2* .
interpretation *flip* : *transport-Mono-Dep-Fun-Rel R1 L1 r1 l1 R2 L2 r2 l2*
rewrites *flip.t1.counit* $\equiv \eta_1$ **and** *flip.t1.unit* $\equiv \varepsilon_1$
by (*simp-all only*: *t1.flip-counit-eq-unit t1.flip-unit-eq-counit*)

theorem *partial-equivalence-rel-equivalenceI*:
assumes (($\leq_{L1}$) $\equiv_{PER}$ ($\leq_{R1}$)) *l1 r1*
and $\bigwedge x\ x'.\ x\ _{L1}\lessapprox x' \Longrightarrow$ (($\leq_{L2\ x\ (r1\ x')}$) $\equiv_{PER}$ ($\leq_{R2\ (l1\ x)\ x'}$)) ($l2_{x'\ x}$) ($r2_{x\ x'}$)
and ((*x1 x2* :: ($\geq_{L1}$)) $\Rightarrow$ (*x3 x4* :: ($\leq_{L1}$) | *x1* $\leq_{L1}$ *x3*) $\Rrightarrow$ ($\leq$)) *L2*
and ((*x1' x2'* :: ($\geq_{R1}$)) $\Rightarrow$ (*x3' x4'* :: ($\leq_{R1}$) | *x1'* $\leq_{R1}$ *x3'*) $\Rrightarrow$ ($\leq$)) *R2*
and ((*x1' x2'* :: ($\leq_{R1}$)) $\Rightarrow$ (*x1 x2* :: ($\leq_{L1}$) | *x2* $_{L1}\lessapprox$ *x1'*) $\Rrightarrow$
in-field ($\leq_{L2\ x1\ (r1\ x2')}$) $\Rrightarrow$ ($\leq_{R2\ (l1\ x1)\ x2'}$)) *l2*
and ((*x1 x2* :: ($\leq_{L1}$)) $\Rightarrow$ (*x1' x2'* :: ($\leq_{R1}$) | *x2* $_{L1}\lessapprox$ *x1'*) $\Rrightarrow$
in-field ($\leq_{R2\ (l1\ x1)\ x2'}$) $\Rrightarrow$ ($\leq_{L2\ x1\ (r1\ x2')}$)) *r2*
shows (($\leq_L$) $\equiv_{PER}$ ($\leq_R$)) *l r*
proof −
from *assms* **have** (($\leq_L$) $\equiv_{PER}$ ($\leq_R$)) = (*tmdfr.L* $\equiv_{PER}$ *tmdfr.R*)
by (*subst tmdfr.left-rel-eq-tdfr-leftI-if-equivalencesI*
flip.left-rel-eq-tdfr-leftI-if-equivalencesI,
auto intro!: *partial-equivalence-rel-left2-if-partial-equivalence-rel-equivalenceI*
partial-equivalence-rel-right2-if-partial-equivalence-rel-equivalenceI
iff: *t1.galois-equivalence-right-left-iff-galois-equivalence-left-right*)+

with *assms* **show** *?thesis*
by (*auto intro*!: *tmdfr.partial-equivalence-rel-equivalenceI*)
qed

theorem *left-Galois-eq-Dep-Fun-Rel-left-Galois-if-partial-equivalence-rel-equivalenceI*:
assumes $((\leq_{L1}) \equiv_{PER} (\leq_{R1}))$ *l1 r1*
and $\bigwedge x\, x'.\ x\ {}_{L1}\lessapprox x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \equiv_{PER} (\leq_{R2\ (l1\ x)\ x'}))\ (l2_{x'\ x})\ (r2_{x\ x'})$
and $((x1\ x2 :: (\geq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid x1 \leq_{L1} x3) \Rrightarrow (\leq))$ *L2*
and $((x1'\ x2' :: (\geq_{R1})) \Rightarrow (x3'\ x4' :: (\leq_{R1}) \mid x1' \leq_{R1} x3') \Rrightarrow (\leq))$ *R2*
and $((x1'\ x2' :: (\leq_{R1})) \Rightarrow (x1\ x2 :: (\leq_{L1}) \mid x2\ {}_{L1}\lessapprox x1') \Rrightarrow$
in-field $(\leq_{L2\ x1\ (r1\ x2')}) \Rrightarrow (\leq_{R2\ (l1\ x1)\ x2'}))$ *l2*
and $((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ {}_{L1}\lessapprox x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))$ *r2*
shows $({}_{L}\lessapprox) = ((x\ x' :: ({}_{L1}\lessapprox)) \Rrightarrow ({}_{L2\ x\ x'}\lessapprox))$
proof −
from *assms* **have** *rel-eqs*: $(\leq_L) = tmdfr.L\ (\leq_R) = tmdfr.R$
by (*subst tmdfr.left-rel-eq-tdfr-leftI-if-equivalencesI flip.left-rel-eq-tdfr-leftI-if-equivalencesI*,
auto intro!: *partial-equivalence-rel-left2-if-partial-equivalence-rel-equivalenceI*
partial-equivalence-rel-right2-if-partial-equivalence-rel-equivalenceI
iff: *t1.galois-equivalence-right-left-iff-galois-equivalence-left-right*)+
then have $({}_{L}\lessapprox) = tmdfr.Galois$ **by** *simp*
also with *assms* **have** $... = ((x\ x' :: ({}_{L1}\lessapprox)) \Rrightarrow ({}_{L2\ x\ x'}\lessapprox))\restriction_{in\text{-}dom\ tmdfr.L}\restriction_{in\text{-}codom\ tmdfr.R}$
by (*intro tmdfr.left-Galois-eq-Dep-Fun-Rel-left-Galois-restrict-if-preorder-equivalenceI*
transitive-left2-if-preorder-equivalenceI)
(*auto 6 2 intro*!: *partial-equivalence-rel-left2-if-partial-equivalence-rel-equivalenceI*)
also from *rel-eqs* **have** $... = ((x\ x' :: ({}_{L1}\lessapprox)) \Rrightarrow ({}_{L2\ x\ x'}\lessapprox))\restriction_{in\text{-}dom\ (\leq_L)}\restriction_{in\text{-}codom\ (\leq_R)}$
by *simp*
also with *assms* **have** $... = ((x\ x' :: ({}_{L1}\lessapprox)) \Rrightarrow ({}_{L2\ x\ x'}\lessapprox))$
by (*intro Dep-Fun-Rel-left-Galois-restrict-left-right-eq-Dep-Fun-Rel-left-GaloisI'*
transitive-left2-if-preorder-equivalenceI transitive-right2-if-preorder-equivalenceI
galois-equivalence-if-mono-if-galois-equivalence-mono-assms-leftI
flip.tdfr.galois-equivalence-if-mono-if-galois-equivalence-mono-assms-leftI)
(*auto iff*: *t1.galois-equivalence-right-left-iff-galois-equivalence-left-right*, *fast?*)
finally show *?thesis* **.**
qed

end

Function Relator **context** *transport-Fun-Rel*
begin

interpretation *tmfr* : *transport-Mono-Fun-Rel L1 R1 l1 r1 L2 R2 l2 r2* **.**
interpretation *flip* : *transport-Mono-Fun-Rel R1 L1 r1 l1 R2 L2 r2 l2* **.**

theorem *partial-equivalence-rel-equivalenceI*:
assumes $((\leq_{L1}) \equiv_{PER} (\leq_{R1}))$ *l1 r1*
and $((\leq_{L2}) \equiv_{PER} (\leq_{R2}))$ *l2 r2*
shows $((\leq_L) \equiv_{PER} (\leq_R))$ *l r*

proof −
from *assms* **have** $((\leq_L) \equiv_{PER} (\leq_R)) = (tmfr.tmdfr.L \equiv_{PER} tmfr.tmdfr.R)$
by (*subst tmfr.left-rel-eq-tfr-leftI flip.left-rel-eq-tfr-leftI*; *auto*)+
with *assms* **show** *?thesis* **by** (*auto intro*!: *tmfr.partial-equivalence-rel-equivalenceI*)
qed

theorem *left-Galois-eq-Fun-Rel-left-Galois-if-partial-equivalence-rel-equivalenceI*:
assumes $((\leq_{L1}) \equiv_{PER} (\leq_{R1}))$ *l1 r1*
and $((\leq_{L2}) \equiv_{PER} (\leq_{R2}))$ *l2 r2*
shows $({}_L\lessapprox) = (({}_{L1}\lessapprox) \Rrightarrow ({}_{L2}\lessapprox))$
proof −
from *assms* **have** *rel-eqs*: $(\leq_L) = tmfr.tmdfr.L$ $(\leq_R) = tmfr.tmdfr.R$
by (*subst tmfr.left-rel-eq-tfr-leftI flip.left-rel-eq-tfr-leftI*; *auto*)+
then have $({}_L\lessapprox) = tmfr.tmdfr.Galois$ **by** *simp*
also with *assms* **have** ... $= (({}_{L1}\lessapprox) \Rrightarrow ({}_{L2}\lessapprox))\restriction_{in\text{-}dom\ tmfr.tmdfr.L}\restriction_{in\text{-}codom\ tmfr.tmdfr.R}$
by (*intro tmfr.left-Galois-eq-Fun-Rel-left-Galois-restrictI*)
(*auto intro*: *reflexive-on-if-le-pred-if-reflexive-on in-field-if-in-dom*)
also from *rel-eqs* **have** ... $= (({}_{L1}\lessapprox) \Rrightarrow ({}_{L2}\lessapprox))\restriction_{in\text{-}dom\ (\leq_L)}\restriction_{in\text{-}codom\ (\leq_R)}$ **by** *simp*
also with *assms* **have** ... $= (({}_{L1}\lessapprox) \Rrightarrow ({}_{L2}\lessapprox))$
by (*intro Fun-Rel-left-Galois-restrict-left-right-eq-Fun-Rel-left-GaloisI*) *fastforce*+
finally show *?thesis* **.**
qed

end

end

2.16 Transport using Identity

theory *Transport-Identity*
imports
Transport-Bijections
begin

Summary Setup for Transport using the identity transport function.

locale *transport-id* =
fixes $L :: 'a \Rightarrow 'a \Rightarrow bool$
begin

sublocale *tbij?* : *transport-bijection L L id id*
by (*intro transport-bijection.intro*) *auto*

interpretation *transport L L id id* **.**

lemma *left-Galois-eq-left*: $({}_L\lessapprox) = (\leq_L)$

by (*intro ext iffI*) *auto*

end

locale *transport-reflexive-on-in-field-id* =
fixes $L :: 'a \Rightarrow 'a \Rightarrow bool$
assumes *reflexive-on-in-field*: *reflexive-on* (*in-field L*) *L*
begin

sublocale *trefl-bij?* : *transport-reflexive-on-in-field-bijection L L id id*
using *reflexive-on-in-field* **by** *unfold-locales auto*

end

locale *transport-preorder-on-in-field-id* =
fixes $L :: 'a \Rightarrow 'a \Rightarrow bool$
assumes *preorder-on-in-field*: *preorder-on* (*in-field L*) *L*
begin

sublocale *tpre-bij?* : *transport-preorder-on-in-field-bijection L L id id*
using *preorder-on-in-field* **by** *unfold-locales auto*

end

locale *transport-partial-equivalence-rel-id* =
fixes $L :: 'a \Rightarrow 'a \Rightarrow bool$
assumes *partial-equivalence-rel*: *partial-equivalence-rel L*
begin

sublocale *tper-bij?* : *transport-partial-equivalence-rel-bijection L L id id*
using *partial-equivalence-rel* **by** *unfold-locales auto*

end

interpretation *transport-eq-restrict-id* :
transport-eq-restrict-bijection P P id id **for** $P :: 'a \Rightarrow bool$
using *bijection-on-self-id* **by** (*unfold-locales*) *auto*

interpretation *transport-eq-id* : *transport-eq-bijection id id*
using *bijection-on-self-id* **by** (*unfold-locales*) *auto*

end

theory *Transport-Black-Box*
imports
Transport-Bijections
Transport-Compositions
Transport-Functions

Transport-Identity
begin

Summary The theory for black-box transports. For details, refer to [2].

end

2.17 White-Box Transport of (Restricted) Equality

theory *Transport-Equality*
imports
Restricted-Equality
Binary-Relations-Bi-Unique
begin

Summary Theorems for white-box transports of (restricted) equalities.

context
fixes $R :: 'a \Rightarrow 'b \Rightarrow bool$ **and** $P :: 'a \Rightarrow bool$ **and** $Q :: 'b \Rightarrow bool$
begin

lemma *Fun-Rel-imp-eq-restrict-if-right-unique-onI*:
assumes *runique*: *right-unique-on* P R
and *rel*: $(R \Rrightarrow (\longrightarrow))$ P Q
shows $(R \Rrightarrow R \Rrightarrow (\longrightarrow))$ $(=_P)$ $(=_Q)$
proof (*intro Fun-Rel-relI impI*)
fix x x' y y'
assume R x y R x' y' $x =_P x'$
moreover with *rel* **have** R x y' Q y' **by** *auto*
ultimately show $y =_Q y'$ **using** *runique* **by** (*auto dest*: *right-unique-onD*)
qed

lemma *Fun-Rel-rev-imp-eq-restrict-if-rel-injective-atI*:
assumes *rinjective*: *rel-injective-at* Q R
and *rel*: $(R \Rrightarrow (\longleftarrow))$ P Q
shows $(R \Rrightarrow R \Rrightarrow (\longleftarrow))$ $(=_P)$ $(=_Q)$
proof (*intro Fun-Rel-relI rev-impI*)
fix x x' y y'
assume R x y R x' y' $y =_Q y'$
moreover with *rel* **have** R x y' P x **by** *auto*
ultimately show $x =_P x'$ **using** *rinjective* **by** (*auto dest*: *rel-injective-atD*)
qed

corollary *Fun-Rel-iff-eq-restrict-if-bi-unique-onI*:
assumes *bi-unique*: *bi-unique-on* P R
and $(R \Rrightarrow (\longleftrightarrow))$ P Q

shows $(R \Rightarrow R \Rightarrow (\longleftrightarrow))\ (=_P)\ (=_Q)$
proof −
from *assms* **have** $(R \Rightarrow (\longrightarrow))\ P\ Q\ (R \Rightarrow (\longleftarrow))\ P\ Q$ *bi-unique-on* $Q\ R^{-1}$
using *bi-unique-on-rel-inv-if-Fun-Rel-iff-if-bi-unique-on* **by** *auto*
with *bi-unique Fun-Rel-imp-eq-restrict-if-right-unique-onI*
Fun-Rel-rev-imp-eq-restrict-if-rel-injective-atI
have $(R \Rightarrow R \Rightarrow (\longrightarrow))\ (=_P)\ (=_Q)\ (R \Rightarrow R \Rightarrow (\longleftarrow))\ (=_P)\ (=_Q)$ **by** *auto*
then show *?thesis* **by** *blast*
qed

lemma *right-unique-on-if-Fun-Rel-imp-eq-restrict*:
assumes $(R \Rightarrow R \Rightarrow (\longrightarrow))\ (=_P)\ (=)$
shows *right-unique-on P R*
using *assms* **by** (*intro right-unique-onI*) *auto*

lemma *Fun-Rel-imp-if-Fun-Rel-imp-eq-restrict*:
assumes $(R \Rightarrow R \Rightarrow (\longrightarrow))\ (=_P)\ ((S :: 'b \Rightarrow 'b \Rightarrow bool)\restriction_Q)$
shows $(R \Rightarrow (\longrightarrow))\ P\ Q$
using *assms* **by** (*intro Fun-Rel-relI*) *blast*

corollary *Fun-Rel-imp-eq-restrict-iff-right-unique-on-and-Fun-Rel-imp*:
$(R \Rightarrow R \Rightarrow (\longrightarrow))\ (=_P)\ (=_Q) \longleftrightarrow$ (*right-unique-on P R* $\wedge$ $(R \Rightarrow (\longrightarrow))\ P\ Q$)
using *Fun-Rel-imp-eq-restrict-if-right-unique-onI*
right-unique-on-if-Fun-Rel-imp-eq-restrict Fun-Rel-imp-if-Fun-Rel-imp-eq-restrict
by *blast*

lemma *rel-injective-at-if-Fun-Rel-rev-imp-eq-restrict*:
assumes $(R \Rightarrow R \Rightarrow (\longleftarrow))\ (=)\ (=_Q)$
shows *rel-injective-at Q R*
using *assms* **by** (*intro rel-injective-atI*) *auto*

lemma *Fun-Rel-rev-imp-if-Fun-Rel-rev-imp-eq-restrict*:
assumes $(R \Rightarrow R \Rightarrow (\longleftarrow))\ ((S :: 'a \Rightarrow 'a \Rightarrow bool)\restriction_P)\ (=_Q)$
shows $(R \Rightarrow (\longleftarrow))\ P\ Q$
using *assms* **by** (*intro Fun-Rel-relI rev-impI*) *auto*

corollary *Fun-Rel-rev-imp-eq-restrict-iff-rel-injective-at-and-Fun-Rel-rev-imp*:
$(R \Rightarrow R \Rightarrow (\longleftarrow))\ (=_P)\ (=_Q) \longleftrightarrow$ (*rel-injective-at Q R* $\wedge$ $(R \Rightarrow (\longleftarrow))\ P\ Q$)
using *Fun-Rel-rev-imp-eq-restrict-if-rel-injective-atI*
rel-injective-at-if-Fun-Rel-rev-imp-eq-restrict Fun-Rel-rev-imp-if-Fun-Rel-rev-imp-eq-restrict
by *blast*

lemma *bi-unique-on-if-Fun-Rel-iff-eq-restrict*:
assumes $(R \Rightarrow R \Rightarrow (\longleftrightarrow))\ (=_P)\ (=_Q)$
shows *bi-unique-on P R*
using *assms* **by** (*intro bi-unique-onI*) *blast*+

lemma *Fun-Rel-iff-if-Fun-Rel-iff-eq-restrict*:
assumes $(R \Rightarrow R \Rightarrow (\longleftrightarrow))\ (=_P)\ (=_Q)$

shows $(R \Rrightarrow (\longleftrightarrow))\ P\ Q$
using *assms* **by** (*intro Fun-Rel-relI*) *blast*

corollary *Fun-Rel-iff-eq-restrict-iff-bi-unique-on-and-Fun-Rel-iff*:
$(R \Rrightarrow R \Rrightarrow (\longleftrightarrow))\ (=_P)\ (=_Q) \longleftrightarrow$ (*bi-unique-on P R* $\wedge$ $(R \Rrightarrow (\longleftrightarrow))\ P\ Q$)
using *Fun-Rel-iff-eq-restrict-if-bi-unique-onI*
bi-unique-on-if-Fun-Rel-iff-eq-restrict Fun-Rel-iff-if-Fun-Rel-iff-eq-restrict
by *blast*

end

context
fixes $R :: 'a \Rightarrow 'b \Rightarrow bool$
begin

corollary *Fun-Rel-imp-eq-if-right-unique*:
assumes *right-unique R*
shows $(R \Rrightarrow R \Rrightarrow (\longrightarrow))\ (=)\ (=)$
using *assms* **by** (*urule Fun-Rel-imp-eq-restrict-if-right-unique-onI*) *auto*

corollary *Fun-Rel-rev-imp-eq-if-rel-injective*:
assumes *rel-injective R*
shows $(R \Rrightarrow R \Rrightarrow (\longleftarrow))\ (=)\ (=)$
using *assms* **by** (*urule Fun-Rel-rev-imp-eq-restrict-if-rel-injective-atI*) *auto*

corollary *Fun-Rel-iff-eq-if-bi-unique*:
assumes *bi-unique R*
shows $(R \Rrightarrow R \Rrightarrow (\longleftrightarrow))\ (=)\ (=)$
using *assms* **by** (*urule Fun-Rel-iff-eq-restrict-if-bi-unique-onI*) *auto*

corollary *right-unique-if-Fun-Rel-imp-eq*:
assumes $(R \Rrightarrow R \Rrightarrow (\longrightarrow))\ (=)\ (=)$
shows *right-unique R*
using *assms* **by** (*urule right-unique-on-if-Fun-Rel-imp-eq-restrict*)

corollary *Fun-Rel-imp-eq-iff-right-unique*: $(R \Rrightarrow R \Rrightarrow (\longrightarrow))\ (=)\ (=) \longleftrightarrow$ *right-unique R*
using *right-unique-if-Fun-Rel-imp-eq Fun-Rel-imp-eq-if-right-unique* **by** *blast*

corollary *rel-injective-if-Fun-Rel-rev-imp-eq*:
assumes $(R \Rrightarrow R \Rrightarrow (\longleftarrow))\ (=)\ (=)$
shows *rel-injective R*
using *assms* **by** (*urule rel-injective-at-if-Fun-Rel-rev-imp-eq-restrict*)

corollary *Fun-Rel-rev-imp-eq-iff-rel-injective*: $(R \Rrightarrow R \Rrightarrow (\longleftarrow))\ (=)\ (=) \longleftrightarrow$ *rel-injective R*
using *rel-injective-if-Fun-Rel-rev-imp-eq Fun-Rel-rev-imp-eq-if-rel-injective* **by** *blast*

corollary *bi-unique-if-Fun-Rel-iff-eq*:

assumes $(R \Rrightarrow R \Rrightarrow (\longleftrightarrow))$ $(=)$ $(=)$
shows *bi-unique R*
using *assms* **by** (*urule bi-unique-on-if-Fun-Rel-iff-eq-restrict*)

corollary *Fun-Rel-iff-eq-iff-bi-unique*: $(R \Rrightarrow R \Rrightarrow (\longleftrightarrow))$ $(=)$ $(=) \longleftrightarrow$ *bi-unique R*
using *bi-unique-if-Fun-Rel-iff-eq Fun-Rel-iff-eq-if-bi-unique* **by** *blast*

end

end

2.18 White-Box Transport of (Bounded) Universal Quantifier

theory *Transport-Universal-Quantifier*
imports
Bounded-Quantifiers
Binary-Relations-Bi-Total
Reverse-Implies
begin

Summary Theorems for white-box transports of (bounded) universal quantifiers.

context
fixes $R :: 'a \Rightarrow 'b \Rightarrow bool$
and $P :: 'a \Rightarrow bool$ **and** $Q :: 'b \Rightarrow bool$
begin

lemma *Fun-Rel-restricts-imp-ball-if-rel-surjective-atI*:
assumes *rel-surjective-at Q* $R{\restriction}_P$
shows $((R{\restriction}_P){\restriction}_Q \Rrightarrow (\longrightarrow)) \Rrightarrow (\longrightarrow)) \forall_P \forall_Q$
using *assms* **by** (*intro Fun-Rel-relI*) *blast*

lemma *Fun-Rel-restricts-rev-imp-ball-if-left-total-onI*:
assumes *left-total-on P* $R{\restriction}_Q$
shows $((R{\restriction}_P){\restriction}_Q \Rrightarrow (\longleftarrow)) \Rrightarrow (\longleftarrow)) \forall_P \forall_Q$
using *assms* **by** (*intro Fun-Rel-relI*) *blast*

lemma *Fun-Rel-restricts-iff-ball-if-left-total-on-if-rel-surjective-at*:
assumes *rel-surjective-at Q* $R{\restriction}_P$
and *left-total-on P* $R{\restriction}_Q$
shows $((R{\restriction}_P){\restriction}_Q \Rrightarrow (\longleftrightarrow)) \Rrightarrow (\longleftrightarrow)) \forall_P \forall_Q$
using *assms* **by** (*intro Fun-Rel-relI*) *blast*

corollary *Fun-Rel-restricts-iff-ball-if-bi-total-on*:
assumes *bi-total-on P Q* $R\restriction_P\restriction_Q$
shows $((R\restriction_P\restriction_Q \Rrightarrow (\longleftrightarrow)) \Rrightarrow (\longleftrightarrow))\ \forall_P\ \forall_Q$
using *assms* **by** (*intro Fun-Rel-restricts-iff-ball-if-left-total-on-if-rel-surjective-at*)
force+

lemma *rel-surjective-at-restrict-left-if-Fun-Rel-imp-ball*:
assumes $((R \Rrightarrow (\longrightarrow)) \Rrightarrow (\longrightarrow))\ \forall_P\ \forall_Q$
shows *rel-surjective-at Q* $R\restriction_P$
proof −
let *?P2* = *in-codom* $R\restriction_P$
have $(R \Rrightarrow (\longrightarrow))$ *P ?P2* **by** *blast*
with *assms* **have** $(\forall x : P.\ P\ x) \longrightarrow (\forall y : Q.\ ?P2\ y)$ **by** *blast*
then show *rel-surjective-at Q* $R\restriction_P$ **by** *fast*
qed

lemma *Fun-Rel-Fun-Rel-if-le-left-if-Fun-Rel-Fun-Rel*:
assumes $(((O :: 'a \Rightarrow 'b \Rightarrow bool) \Rrightarrow S) \Rrightarrow T)\ U\ V$
and $O \leq O'$
shows $((O' \Rrightarrow S) \Rrightarrow T)\ U\ V$
proof −
from *assms* **have** $(O' \Rrightarrow S) \leq (O \Rrightarrow S)$ **by** *blast*
with *assms antimonoD*[*OF antimono-Dep-Fun-Rel-rel-left*] **show** *?thesis*
unfolding *Fun-Rel-rel-iff-Dep-Fun-Rel-rel* **by** *blast*
qed

corollary *Fun-Rel-imp-ball-iff-rel-surjective-at-restrict-left*:
$((R \Rrightarrow (\longrightarrow)) \Rrightarrow (\longrightarrow))\ \forall_P\ \forall_Q \longleftrightarrow$ *rel-surjective-at Q* $R\restriction_P$
by (*blast intro*: *Fun-Rel-restricts-imp-ball-if-rel-surjective-atI*
Fun-Rel-Fun-Rel-if-le-left-if-Fun-Rel-Fun-Rel
rel-surjective-at-restrict-left-if-Fun-Rel-imp-ball)

lemma *left-total-on-restrict-right-if-Fun-Rel-rev-imp-ball*:
assumes $((R \Rrightarrow (\longleftarrow)) \Rrightarrow (\longleftarrow))\ \forall_P\ \forall_Q$
shows *left-total-on P* $R\restriction_Q$
proof −
let *?P1* = *in-dom* $R\restriction_Q$
have $(R \Rrightarrow (\longleftarrow))$ *?P1 Q* **by** *blast*
with *assms* **have** $(\forall x : P.\ ?P1\ x) \longleftarrow (\forall y : Q.\ Q\ y)$ **by** *blast*
then show *left-total-on P* $R\restriction_Q$ **by** *fast*
qed

corollary *Fun-Rel-rev-imp-ball-iff-left-total-on-restrict-right*:
$((R \Rrightarrow (\longleftarrow)) \Rrightarrow (\longleftarrow))\ \forall_P\ \forall_Q \longleftrightarrow$ *left-total-on P* $R\restriction_Q$
by (*blast intro*: *Fun-Rel-restricts-rev-imp-ball-if-left-total-onI*
Fun-Rel-Fun-Rel-if-le-left-if-Fun-Rel-Fun-Rel
left-total-on-restrict-right-if-Fun-Rel-rev-imp-ball)

lemma *bi-total-on-if-Fun-Rel-iff-ball*:

assumes $((R \Rrightarrow (\longleftrightarrow)) \Rrightarrow (\longleftrightarrow)) \forall_P \forall_Q$
shows *bi-total-on P Q R*
proof (*rule bi-total-onI*)
let *?P1* = *in-dom R* **and** *?P2* = λ-. *True*
have $(R \Rrightarrow (\longleftrightarrow))$ *?P1 ?P2* **by** *blast*
with *assms* **have** $(\forall x : P.\ ?P1\ x) \longleftrightarrow (\forall y : Q.\ ?P2\ y)$ **by** *blast*
then show *left-total-on P R* **by** *fast*
next
let *?P1* = λ-. *True* **and** *?P2* = *in-codom R*
have $(R \Rrightarrow (\longleftrightarrow))$ *?P1 ?P2* **by** *blast*
with *assms* **have** $(\forall x : P.\ ?P1\ x) \longleftrightarrow (\forall y : Q.\ ?P2\ y)$ **by** *blast*
then show *rel-surjective-at Q R* **by** *fast*
qed

corollary *Fun-Rel-iff-ball-iff-bi-total-on-if-Fun-Rel-iff*:
assumes $(R \Rrightarrow (\longleftrightarrow))$ *P Q*
shows $((R \Rrightarrow (\longleftrightarrow)) \Rrightarrow (\longleftrightarrow)) \forall_P \forall_Q \longleftrightarrow$ *bi-total-on P Q R*
using *assms* **by** (*blast intro*: *Fun-Rel-restricts-iff-ball-if-bi-total-on*
Fun-Rel-Fun-Rel-if-le-left-if-Fun-Rel-Fun-Rel
bi-total-on-restricts-if-Fun-Rel-iff-if-bi-total-on
bi-total-on-if-Fun-Rel-iff-ball)

end

context
fixes $R :: 'a \Rightarrow 'b \Rightarrow bool$
begin

corollary *Fun-Rel-imp-all-if-rel-surjective*:
assumes *rel-surjective R*
shows $((R \Rrightarrow (\longrightarrow)) \Rrightarrow (\longrightarrow))$ *All All*
using *assms* **by** (*urule Fun-Rel-restricts-imp-ball-if-rel-surjective-atI*)

corollary *Fun-Rel-rev-imp-all-if-left-total*:
assumes *left-total R*
shows $((R \Rrightarrow (\longleftarrow)) \Rrightarrow (\longleftarrow))$ *All All*
using *assms* **by** (*urule Fun-Rel-restricts-rev-imp-ball-if-left-total-onI*)

corollary *Fun-Rel-iff-all-if-bi-total*:
assumes *bi-total R*
shows $((R \Rrightarrow (\longleftrightarrow)) \Rrightarrow (\longleftrightarrow))$ *All All*
using *assms* **by** (*urule Fun-Rel-restricts-iff-ball-if-bi-total-on*)

corollary *rel-surjective-if-Fun-Rel-imp-all*:
assumes $((R \Rrightarrow (\longrightarrow)) \Rrightarrow (\longrightarrow))$ *All All*
shows *rel-surjective R*
using *assms* **by** (*urule rel-surjective-at-restrict-left-if-Fun-Rel-imp-ball*)

corollary *Fun-Rel-imp-all-iff-rel-surjective*:

((R ⇛ (⟶)) ⇛ (⟶)) All All ⟷ rel-surjective R
using *rel-surjective-if-Fun-Rel-imp-all Fun-Rel-imp-all-if-rel-surjective* **by** *blast*

corollary *left-total-if-Fun-Rel-rev-imp-all*:
assumes ((R ⇛ (⟵)) ⇛ (⟵)) All All
shows *left-total R*
using *assms* **by** (*urule left-total-on-restrict-right-if-Fun-Rel-rev-imp-ball*)

corollary *Fun-Rel-rev-imp-all-iff-left-total*:
((R ⇛ (⟵)) ⇛ (⟵)) All All ⟷ left-total R
using *left-total-if-Fun-Rel-rev-imp-all Fun-Rel-rev-imp-all-if-left-total* **by** *blast*

corollary *bi-total-if-Fun-Rel-iff-all*:
assumes ((R ⇛ (⟷)) ⇛ (⟷)) All All
shows *bi-total R*
using *assms* **by** (*urule bi-total-on-if-Fun-Rel-iff-ball*)

corollary *Fun-Rel-iff-all-iff-bi-total*:
((R ⇛ (⟷)) ⇛ (⟷)) All All ⟷ bi-total R
using *bi-total-if-Fun-Rel-iff-all Fun-Rel-iff-all-if-bi-total* **by** *blast*

end

end

2.19 White-Box Transport of (Bounded) Existential Quantifier

theory *Transport-Existential-Quantifier*
imports
Transport-Universal-Quantifier
begin

Summary Theorems for white-box transports of (bounded) existential quantifiers.

context
fixes R :: 'a ⇒ 'b ⇒ *bool* **and** P :: 'a ⇒ *bool* **and** Q :: 'b ⇒ *bool*
begin

lemma *Fun-Rel-restricts-imp-bex-if-left-total-onI*:
assumes *left-total-on P R*⌉ Q
shows ((R⌈P⌉ Q ⇛ (⟶)) ⇛ (⟶)) ∃ P ∃ Q
using *assms* **by** (*intro Fun-Rel-relI*) *fast*

lemma *Fun-Rel-restricts-rev-imp-bex-if-rel-surjective-at*:
assumes *rel-surjective-at Q* $R\restriction_P$
shows $((R\restriction_P)\restriction_Q \Rrightarrow (\longleftarrow)) \Rrightarrow (\longleftarrow)) \ni_P \ni_Q$
using *assms* **by** (*intro Fun-Rel-relI*) *fast*

lemma *Fun-Rel-restricts-iff-bex-if-left-total-on-if-rel-surjective-at*:
assumes *rel-surjective-at Q* $R\restriction_P$
and *left-total-on P* $R\restriction_Q$
shows $((R\restriction_P)\restriction_Q \Rrightarrow (\longleftrightarrow)) \Rrightarrow (\longleftrightarrow)) \ni_P \ni_Q$
using *assms* **by** (*intro Fun-Rel-relI*) *fast*

corollary *Fun-Rel-restricts-iff-bex-if-bi-total-on*:
assumes *bi-total-on P Q* $R\restriction_P\restriction_Q$
shows $((R\restriction_P)\restriction_Q \Rrightarrow (\longleftrightarrow)) \Rrightarrow (\longleftrightarrow)) \ni_P \ni_Q$
using *assms* **by** (*intro Fun-Rel-restricts-iff-bex-if-left-total-on-if-rel-surjective-at*)
fast+

lemma *left-total-on-restrict-right-if-Fun-Rel-imp-bex*:
assumes $((R \Rrightarrow (\longrightarrow)) \Rrightarrow (\longrightarrow)) \ni_P \ni_Q$
shows *left-total-on P* $R\restriction_Q$
proof
fix *x* **assume** *P x*
let *?P1* $= (=)$ *x* **and** *?P2* $= R$ *x*
have $(R \Rrightarrow (\longrightarrow))$ *?P1 ?P2* **by** *auto*
with *assms* **have** $\ni_P$ *?P1* $\longrightarrow \ni_Q$ *?P2* **by** *blast*
with ‹*P x*› **show** *in-dom* $R\restriction_Q$ *x* **by** *blast*
qed

corollary *Fun-Rel-imp-all-on-iff-left-total-on-restrict-right*:
$((R \Rrightarrow (\longrightarrow)) \Rrightarrow (\longrightarrow)) \ni_P \ni_Q \longleftrightarrow$ *left-total-on P* $R\restriction_Q$
by (*blast intro*: *Fun-Rel-restricts-imp-bex-if-left-total-onI*
Fun-Rel-Fun-Rel-if-le-left-if-Fun-Rel-Fun-Rel
left-total-on-restrict-right-if-Fun-Rel-imp-bex)

lemma *rel-surjective-at-restrict-left-if-Fun-Rel-rev-imp-bex*:
assumes $((R \Rrightarrow (\longleftarrow)) \Rrightarrow (\longleftarrow)) \ni_P \ni_Q$
shows *rel-surjective-at Q* $R\restriction_P$
proof
fix *y* **assume** *Q y*
let *?P1* $= R^{-1}$ *y* **and** *?P2* $= (=)$ *y*
have $(R \Rrightarrow (\longleftarrow))$ *?P1 ?P2* **by** *auto*
with *assms* **have** $\ni_P$ *?P1* $\longleftarrow \ni_Q$ *?P2* **by** *blast*
with ‹*Q y*› **show** *in-codom* $R\restriction_P$ *y* **by** *blast*
qed

corollary *Fun-Rel-rev-imp-bex-iff-rel-surjective-at-restrict-left*:
$((R \Rrightarrow (\longleftarrow)) \Rrightarrow (\longleftarrow)) \ni_P \ni_Q \longleftrightarrow$ *rel-surjective-at Q* $R\restriction_P$
by (*blast intro*: *Fun-Rel-restricts-rev-imp-bex-if-rel-surjective-at*
Fun-Rel-Fun-Rel-if-le-left-if-Fun-Rel-Fun-Rel

rel-surjective-at-restrict-left-if-Fun-Rel-rev-imp-bex)

lemma *bi-total-on-if-Fun-Rel-iff-bex*:
assumes $((R \Rrightarrow (\longleftarrow)) \Rrightarrow (\longleftarrow))\ \exists_P\ \exists_Q$
shows *bi-total-on P Q R*
proof (*intro bi-total-onI left-total-onI rel-surjective-atI*; *rule ccontr*)
fix *x* **assume** *P x* **and** *not-dom*: $\neg$(*in-dom R x*)
let *?P1* = (=) *x* **and** *?P2* = *R x*
from *not-dom* **have** $(R \Rrightarrow (\longleftarrow))$ *?P1 ?P2* **by** *blast*
with *assms* **have** $(\exists\, x : P.\ ?P1\ x) \longleftrightarrow (\exists\, y : Q.\ ?P2\ y)$ **by** *blast*
with ‹*P x*› *not-dom* **show** *False* **by** *blast*
next
fix *y* **assume** *Q y* **and** *not-codom*: $\neg$(*in-codom R y*)
let *?P1* = $\lambda x.$ *R x y* **and** *?P2* = (=) *y*
from *not-codom* **have** $(R \Rrightarrow (\longleftarrow))$ *?P1 ?P2* **by** *blast*
with *assms* **have** $(\exists\, x : P.\ ?P1\ x) \longleftrightarrow (\exists\, y : Q.\ ?P2\ y)$ **by** *blast*
with ‹*Q y*› *not-codom* **show** *False* **by** *blast*
qed

corollary *Fun-Rel-iff-bex-iff-bi-total-on-if-Fun-Rel-iff*:
assumes $(R \Rrightarrow (\longleftarrow))$ *P Q*
shows $((R \Rrightarrow (\longleftarrow)) \Rrightarrow (\longleftarrow))\ \exists_P\ \exists_Q \longleftrightarrow$ *bi-total-on P Q R*
using *assms* **by** (*blast intro*: *Fun-Rel-restricts-iff-bex-if-bi-total-on*
Fun-Rel-Fun-Rel-if-le-left-if-Fun-Rel-Fun-Rel
bi-total-on-restricts-if-Fun-Rel-iff-if-bi-total-on
bi-total-on-if-Fun-Rel-iff-bex)

end

context
fixes *R* :: $'a \Rightarrow 'b \Rightarrow$ *bool*
begin

corollary *Fun-Rel-imp-ex-if-left-total*:
assumes *left-total R*
shows $((R \Rrightarrow (\longrightarrow)) \Rrightarrow (\longrightarrow))$ *Ex Ex*
using *assms* **by** (*urule Fun-Rel-restricts-imp-bex-if-left-total-onI*)

corollary *Fun-Rel-rev-imp-ex-if-rel-surjective*:
assumes *rel-surjective R*
shows $((R \Rrightarrow (\longleftarrow)) \Rrightarrow (\longleftarrow))$ *Ex Ex*
using *assms* **by** (*urule Fun-Rel-restricts-rev-imp-bex-if-rel-surjective-at*)

corollary *Fun-Rel-iff-ex-if-bi-total*:
assumes *bi-total R*
shows $((R \Rrightarrow (\longleftrightarrow)) \Rrightarrow (\longleftrightarrow))$ *Ex Ex*
using *assms* **by** (*urule Fun-Rel-restricts-iff-bex-if-bi-total-on*)

corollary *left-total-if-Fun-Rel-imp-ex*:

assumes $((R \Rightarrow (\longrightarrow)) \Rightarrow (\longrightarrow))$ *Ex Ex*
shows *left-total R*
using *assms* **by** (*urule left-total-on-restrict-right-if-Fun-Rel-imp-bex*)

corollary *Fun-Rel-imp-ex-iff-left-total*:
$((R \Rightarrow (\longrightarrow)) \Rightarrow (\longrightarrow))$ *Ex Ex* $\longleftrightarrow$ *left-total R*
using *left-total-if-Fun-Rel-imp-ex Fun-Rel-imp-ex-if-left-total* **by** *blast*

corollary *rel-surjective-if-Fun-Rel-rev-imp-ex*:
assumes $((R \Rightarrow (\longleftarrow)) \Rightarrow (\longleftarrow))$ *Ex Ex*
shows *rel-surjective R*
using *assms* **by** (*urule rel-surjective-at-restrict-left-if-Fun-Rel-rev-imp-bex*)

corollary *Fun-Rel-rev-imp-ex-iff-rel-surjective*:
$((R \Rightarrow (\longleftarrow)) \Rightarrow (\longleftarrow))$ *Ex Ex* $\longleftrightarrow$ *rel-surjective R*
using *rel-surjective-if-Fun-Rel-rev-imp-ex Fun-Rel-rev-imp-ex-if-rel-surjective* **by** *blast*

corollary *bi-total-if-Fun-Rel-iff-ex*:
assumes $((R \Rightarrow (\longleftrightarrow)) \Rightarrow (\longleftrightarrow))$ *Ex Ex*
shows *bi-total R*
using *assms* **by** (*urule bi-total-on-if-Fun-Rel-iff-bex*)

corollary *Fun-Rel-iff-ex-iff-bi-total*:
$((R \Rightarrow (\longleftrightarrow)) \Rightarrow (\longleftrightarrow))$ *Ex Ex* $\longleftrightarrow$ *bi-total R*
using *bi-total-if-Fun-Rel-iff-ex Fun-Rel-iff-ex-if-bi-total* **by** *blast*

end

end

2.20 Properties of Galois Relator for White-Box Transport Side Conditions

theory *Transport-Galois-Relator-Properties*
imports
Binary-Relations-Bi-Total
Binary-Relations-Bi-Unique
Galois-Connections
Galois-Relator
begin

Summary Properties of Galois relator arising as side conditions for white-box transport.

context *galois*
begin

Right-Uniqueness context
fixes $P :: 'a \Rightarrow bool$ **and** $Q :: 'b \Rightarrow bool$
begin

lemma *right-unique-at-left-Galois-if-right-unique-at-rightI*:
assumes *right-unique-at* Q $(\leq_R)$
and $((\leq_L)$ $_h\trianglelefteq$ $(\leq_R))$ l r
shows *right-unique-at* Q $(_L\lessapprox)$
using *assms* **by** (*auto dest*: *right-unique-atD*)

lemma *right-unique-at-right-if-right-unique-at-left-GaloisI*:
assumes *right-unique-at* Q $(_L\lessapprox)$
and $((\leq_R) \Rightarrow (\leq_L))$ r
shows *right-unique-at* Q $(\leq_R)$
using *assms* **by** (*blast dest*: *right-unique-atD*)

corollary *right-unique-at-left-Galois-iff-right-unique-at-rightI*:
assumes $((\leq_L)$ $_h\trianglelefteq$ $(\leq_R))$ l r
and $((\leq_R) \Rightarrow (\leq_L))$ r
shows *right-unique-at* Q $(_L\lessapprox) \longleftrightarrow$ *right-unique-at* Q $(\leq_R)$
using *assms right-unique-at-left-Galois-if-right-unique-at-rightI*
right-unique-at-right-if-right-unique-at-left-GaloisI
by *blast*

corollary *right-unique-on-left-Galois-if-right-unique-at-rightI*:
assumes *right-unique-at* Q $(\leq_R)$
and $((_L\lessapprox) \Rrightarrow (\longrightarrow))$ P Q
and $((\leq_L)$ $_h\trianglelefteq$ $(\leq_R))$ l r
shows *right-unique-on* P $(_L\lessapprox)$
using *assms* **by** (*intro right-unique-on-if-Fun-Rel-imp-if-right-unique-at*)
(*blast intro*: *right-unique-at-left-Galois-if-right-unique-at-rightI*)

corollary *right-unique-at-right-if-right-unique-on-left-GaloisI*:
assumes *right-unique-on* P $(_L\lessapprox)$
and $((_L\lessapprox) \Rrightarrow (\longleftarrow))$ P Q
and $((\leq_R) \Rightarrow (\leq_L))$ r
shows *right-unique-at* Q $(\leq_R)$
using *assms* **by** (*intro right-unique-at-right-if-right-unique-at-left-GaloisI*
right-unique-at-if-Fun-Rel-rev-imp-if-right-unique-on)

corollary *right-unique-on-left-Galois-iff-right-unique-at-rightI*:
assumes $((_L\lessapprox) \Rrightarrow (\longleftrightarrow))$ P Q
and $((\leq_L)$ $_h\trianglelefteq$ $(\leq_R))$ l r
and $((\leq_R) \Rightarrow (\leq_L))$ r
shows *right-unique-on* P $(_L\lessapprox) \longleftrightarrow$ *right-unique-at* Q $(\leq_R)$
using *assms right-unique-on-left-Galois-if-right-unique-at-rightI*
right-unique-at-right-if-right-unique-on-left-GaloisI
by *blast*

end

corollary *right-unique-left-Galois-if-right-unique-rightI*:
assumes *right-unique* $(\leq_R)$
and $((\leq_L)\ {}_h\trianglelefteq\ (\leq_R))\ l\ r$
shows *right-unique* $({}_L\lessapprox)$
using *assms* **by** (*urule right-unique-at-left-Galois-if-right-unique-at-rightI*)

corollary *right-unique-right-if-right-unique-left-GaloisI*:
assumes *right-unique* $({}_L\lessapprox)$
and $((\leq_R) \Rightarrow (\leq_L))\ r$
shows *right-unique* $(\leq_R)$
using *assms* **by** (*urule right-unique-at-right-if-right-unique-at-left-GaloisI*)

corollary *right-unique-left-Galois-iff-right-unique-rightI*:
assumes $((\leq_L)\ {}_h\trianglelefteq\ (\leq_R))\ l\ r$
and $((\leq_R) \Rightarrow (\leq_L))\ r$
shows *right-unique* $({}_L\lessapprox) \longleftrightarrow$ *right-unique* $(\leq_R)$
using *assms right-unique-left-Galois-if-right-unique-rightI*
right-unique-right-if-right-unique-left-GaloisI
by *blast*

Injectivity **context**
fixes $P :: 'a \Rightarrow bool$ **and** $Q :: 'b \Rightarrow bool$
begin

lemma *injective-on-left-Galois-if-rel-injective-on-left*:
assumes *rel-injective-on* $P\ (\leq_L)$
shows *rel-injective-on* $P\ ({}_L\lessapprox)$
using *assms* **by** (*auto dest*: *rel-injective-onD*)

lemma *rel-injective-on-left-if-injective-on-left-GaloisI*:
assumes *rel-injective-on* $P\ ({}_L\lessapprox)$
and $((\leq_L) \Rightarrow (\leq_R))\ l$
and $((\leq_L)\ \trianglelefteq_h\ (\leq_R))\ l\ r$
shows *rel-injective-on* $P\ (\leq_L)$
using *assms* **by** (*intro rel-injective-onI*) (*blast dest*!: *rel-injective-onD*)

corollary *injective-on-left-Galois-iff-rel-injective-on-leftI*:
assumes $((\leq_L) \Rightarrow (\leq_R))\ l$
and $((\leq_L)\ \trianglelefteq_h\ (\leq_R))\ l\ r$
shows *rel-injective-on* $P\ ({}_L\lessapprox) \longleftrightarrow$ *rel-injective-on* $P\ (\leq_L)$
using *assms injective-on-left-Galois-if-rel-injective-on-left*
rel-injective-on-left-if-injective-on-left-GaloisI
by *blast*

corollary *injective-at-left-Galois-if-rel-injective-on-leftI*:
assumes *rel-injective-on* $P\ (\leq_L)$
and $({}_L\lessapprox) \Rrightarrow (\longleftarrow)\ P\ Q$

shows *rel-injective-at* Q $({}_L\lessapprox)$
using *assms* **by** (*intro rel-injective-at-if-Fun-Rel-rev-imp-if-rel-injective-on*)
(*blast intro*: *injective-on-left-Galois-if-rel-injective-on-left*)

corollary *rel-injective-on-left-if-injective-at-left-GaloisI*:
assumes *rel-injective-at* Q $({}_L\lessapprox)$
and $(({}_L\lessapprox) \Rrightarrow (\longrightarrow))$ P Q
and $((\leq_L) \Rightarrow (\leq_R))$ l
and $((\leq_L) \trianglelefteq_h (\leq_R))$ l r
shows *rel-injective-on* P $(\leq_L)$
using *assms* **by** (*intro rel-injective-on-left-if-injective-on-left-GaloisI*
rel-injective-on-if-Fun-Rel-imp-if-rel-injective-at)

corollary *injective-at-left-Galois-iff-rel-injective-on-leftI*:
assumes $(({}_L\lessapprox) \Rrightarrow (\longleftrightarrow))$ P Q
and $((\leq_L) \Rightarrow (\leq_R))$ l
and $((\leq_L) \trianglelefteq_h (\leq_R))$ l r
shows *rel-injective-at* Q $({}_L\lessapprox) \longleftrightarrow$ *rel-injective-on* P $(\leq_L)$
using *assms injective-at-left-Galois-if-rel-injective-on-leftI*
rel-injective-on-left-if-injective-at-left-GaloisI
by *blast*

end

corollary *injective-left-Galois-if-rel-injective-left*:
assumes *rel-injective* $(\leq_L)$
shows *rel-injective* $({}_L\lessapprox)$
using *assms* **by** (*urule injective-on-left-Galois-if-rel-injective-on-left*)

corollary *rel-injective-left-if-injective-left-GaloisI*:
assumes *rel-injective* $({}_L\lessapprox)$
and $((\leq_L) \Rightarrow (\leq_R))$ l
and $((\leq_L) \trianglelefteq_h (\leq_R))$ l r
shows *rel-injective* $(\leq_L)$
using *assms* **by** (*urule rel-injective-on-left-if-injective-on-left-GaloisI*)

corollary *rel-injective-left-Galois-iff-rel-injective-leftI*:
assumes $((\leq_L) \Rightarrow (\leq_R))$ l
and $((\leq_L) \trianglelefteq_h (\leq_R))$ l r
shows *rel-injective* $({}_L\lessapprox) \longleftrightarrow$ *rel-injective* $(\leq_L)$
using *assms injective-left-Galois-if-rel-injective-left*
rel-injective-left-if-injective-left-GaloisI
by *blast*

Bi-Uniqueness **context**
fixes $P :: 'a \Rightarrow bool$ **and** $Q :: 'b \Rightarrow bool$
begin

corollary *bi-unique-on-left-Galois-if-right-unique-at-right-if-rel-injective-on-leftI*:

assumes *rel-injective-on* P $(\leq_L)$
and *right-unique-at* Q $(\leq_R)$
and $((_L\lessapprox) \Rrightarrow (\longrightarrow))$ P Q
and $((\leq_L)$ $_h\trianglelefteq$ $(\leq_R))$ l r
shows *bi-unique-on* P $(_L\lessapprox)$
using *assms*
by (*intro bi-unique-onI right-unique-on-left-Galois-if-right-unique-at-rightI injective-on-left-Galois-if-rel-injective-on-left*)

corollary *rel-injective-on-left-and-right-unique-at-right-if-bi-unique-on-left-GaloisI*:
assumes *bi-unique-on* P $(_L\lessapprox)$
and $((_L\lessapprox) \Rrightarrow (\longleftarrow))$ P Q
and $((\leq_L) \Rightarrow (\leq_R))$ l
and $((\leq_R) \Rightarrow (\leq_L))$ r
and $((\leq_L) \trianglelefteq_h (\leq_R))$ l r
shows *rel-injective-on* P $(\leq_L)$ $\wedge$ *right-unique-at* Q $(\leq_R)$
using *assms* **by** (*intro conjI rel-injective-on-left-if-injective-on-left-GaloisI right-unique-at-right-if-right-unique-on-left-GaloisI*)
auto

corollary *bi-unique-on-left-Galois-iff-rel-injective-on-left-and-right-unique-at-rightI*:
assumes $((_L\lessapprox) \Rrightarrow (\longleftrightarrow))$ P Q
and $((\leq_L) \dashv (\leq_R))$ l r
shows *bi-unique-on* P $(_L\lessapprox)$ $\longleftrightarrow$ *rel-injective-on* P $(\leq_L)$ $\wedge$ *right-unique-at* Q $(\leq_R)$
using *assms bi-unique-on-left-Galois-if-right-unique-at-right-if-rel-injective-on-leftI rel-injective-on-left-and-right-unique-at-right-if-bi-unique-on-left-GaloisI*
by *blast*

end

corollary *bi-unique-left-Galois-if-right-unique-right-if-rel-injective-leftI*:
assumes *rel-injective* $(\leq_L)$
and *right-unique* $(\leq_R)$
and $((\leq_L)$ $_h\trianglelefteq$ $(\leq_R))$ l r
shows *bi-unique* $(_L\lessapprox)$
by (*urule* (*rr*) *bi-unique-on-left-Galois-if-right-unique-at-right-if-rel-injective-on-leftI assms*)
auto

corollary *rel-injective-left-and-right-unique-right-if-bi-unique-left-GaloisI*:
assumes *bi-unique* $(_L\lessapprox)$
and $((\leq_L) \Rightarrow (\leq_R))$ l
and $((\leq_R) \Rightarrow (\leq_L))$ r
and $((\leq_L) \trianglelefteq_h (\leq_R))$ l r
shows *rel-injective* $(\leq_L)$ $\wedge$ *right-unique* $(\leq_R)$
by (*urule* (*rr*) *rel-injective-on-left-and-right-unique-at-right-if-bi-unique-on-left-GaloisI assms*)
auto

corollary *bi-unique-left-Galois-iff-rel-injective-left-and-right-unique-rightI*:
assumes $((\leq_L) \dashv (\leq_R))$ *l r*
shows *bi-unique* $({}_L\lessapprox)$ $\longleftrightarrow$ *rel-injective* $(\leq_L) \wedge$ *right-unique* $(\leq_R)$
using *assms bi-unique-left-Galois-if-right-unique-right-if-rel-injective-leftI*
rel-injective-left-and-right-unique-right-if-bi-unique-left-GaloisI
by *blast*

Surjectivity **context**
fixes $Q :: 'b \Rightarrow bool$
begin

lemma *surjective-at-left-Galois-if-rel-surjective-at-rightI*:
assumes *rel-surjective-at* Q $(\leq_R)$
and $((\leq_R) \Rightarrow (\leq_L))$ *r*
shows *rel-surjective-at* Q $({}_L\lessapprox)$
using *assms* **by** (*intro rel-surjective-atI*) *fast*

lemma *rel-surjective-at-right-if-surjective-at-left-Galois*:
assumes *rel-surjective-at* Q $({}_L\lessapprox)$
shows *rel-surjective-at* Q $(\leq_R)$
using *assms* **by** (*intro rel-surjective-atI*) (*auto*)

corollary *rel-surjective-at-right-iff-surjective-at-left-GaloisI*:
assumes $((\leq_R) \Rightarrow (\leq_L))$ *r*
shows *rel-surjective-at* Q $({}_L\lessapprox)$ $\longleftrightarrow$ *rel-surjective-at* Q $(\leq_R)$
using *assms surjective-at-left-Galois-if-rel-surjective-at-rightI*
rel-surjective-at-right-if-surjective-at-left-Galois
by *blast*

end

corollary *surjective-left-Galois-if-rel-surjective-rightI*:
assumes *rel-surjective* $(\leq_R)$
and $((\leq_R) \Rightarrow (\leq_L))$ *r*
shows *rel-surjective* $({}_L\lessapprox)$
using *assms* **by** (*urule surjective-at-left-Galois-if-rel-surjective-at-rightI*)

corollary *rel-surjective-right-if-surjective-left-Galois*:
assumes *rel-surjective* $({}_L\lessapprox)$
shows *rel-surjective* $(\leq_R)$
using *assms* **by** (*urule rel-surjective-at-right-if-surjective-at-left-Galois*)

corollary *rel-surjective-right-iff-surjective-left-GaloisI*:
assumes $((\leq_R) \Rightarrow (\leq_L))$ *r*
shows *rel-surjective* $({}_L\lessapprox)$ $\longleftrightarrow$ *rel-surjective* $(\leq_R)$
using *assms surjective-left-Galois-if-rel-surjective-rightI*
rel-surjective-right-if-surjective-left-Galois
by *blast*

Left-Totality **context**
fixes $P :: 'a \Rightarrow bool$
begin

lemma *left-total-on-left-Galois-if-left-total-on-leftI*:
assumes *left-total-on* P $(\leq_L)$
and $((\leq_L) \Rightarrow (\leq_R))$ l
and $((\leq_L) \trianglelefteq_h (\leq_R))$ l r
shows *left-total-on* P $({}_L\lessapprox)$
using *assms* **by** (*intro left-total-onI*) *force*

lemma *left-total-on-left-if-left-total-on-left-GaloisI*:
assumes *left-total-on* P $({}_L\gtrapprox)$
shows *left-total-on* P $(\leq_L)$
using *assms* **by** (*intro left-total-onI*) *auto*

corollary *left-total-on-left-Galois-iff-left-total-on-leftI*:
assumes $((\leq_L) \Rightarrow (\leq_R))$ l
and $((\leq_L) \trianglelefteq_h (\leq_R))$ l r
shows *left-total-on* P $({}_L\lessapprox) \longleftrightarrow$ *left-total-on* P $(\leq_L)$
using *assms left-total-on-left-Galois-if-left-total-on-leftI*
left-total-on-left-if-left-total-on-left-GaloisI
by *blast*

end

corollary *left-total-left-Galois-if-left-total-leftI*:
assumes *left-total* $(\leq_L)$
and $((\leq_L) \Rightarrow (\leq_R))$ l
and $((\leq_L) \trianglelefteq_h (\leq_R))$ l r
shows *left-total* $({}_L\lessapprox)$
using *assms* **by** (*urule left-total-on-left-Galois-if-left-total-on-leftI*)

corollary *left-total-left-if-left-total-left-Galois*:
assumes *left-total* $({}_L\gtrapprox)$
shows *left-total* $(\leq_L)$
using *assms* **by** (*urule left-total-on-left-if-left-total-on-left-GaloisI*)

corollary *left-total-left-Galois-iff-left-total-leftI*:
assumes $((\leq_L) \Rightarrow (\leq_R))$ l
and $((\leq_L) \trianglelefteq_h (\leq_R))$ l r
shows *left-total* $({}_L\lessapprox) \longleftrightarrow$ *left-total* $(\leq_L)$
using *assms left-total-left-Galois-if-left-total-leftI*
left-total-left-if-left-total-left-Galois
by *blast*

Bi-Totality **context**
fixes $P :: 'a \Rightarrow bool$ **and** $Q :: 'b \Rightarrow bool$
begin

lemma *bi-total-on-left-GaloisI*:
assumes *left-total-on* P $(\leq_L)$
and *rel-surjective-at* Q $(\leq_R)$
and $((\leq_L) \Rightarrow (\leq_R))$ l
and $((\leq_R) \Rightarrow (\leq_L))$ r
and $((\leq_L) \trianglelefteq_h (\leq_R))$ l r
shows *bi-total-on* P Q $({}_L\lessapprox)$
using *assms surjective-at-left-Galois-if-rel-surjective-at-rightI*
left-total-on-left-Galois-if-left-total-on-leftI
by *blast*

lemma *left-total-on-left-rel-surjective-at-right-if-bi-total-on-left-GaloisE*:
assumes *bi-total-on* P Q $({}_L\lessapprox)$
obtains *left-total-on* P $(\leq_L)$ *rel-surjective-at* Q $(\leq_R)$
using *assms rel-surjective-at-right-if-surjective-at-left-Galois*
left-total-on-left-if-left-total-on-left-GaloisI
by *auto*

corollary *bi-total-on-left-Galois-iff-left-total-on-left-and-rel-surjective-on-rightI*:
assumes $((\leq_L) \Rightarrow (\leq_R))$ l
and $((\leq_R) \Rightarrow (\leq_L))$ r
and $((\leq_L) \trianglelefteq_h (\leq_R))$ l r
shows *bi-total-on* P Q $({}_L\lessapprox) \longleftrightarrow$ *left-total-on* P $(\leq_L) \wedge$ *rel-surjective-at* Q $(\leq_R)$
using *assms bi-total-on-left-GaloisI*
left-total-on-left-rel-surjective-at-right-if-bi-total-on-left-GaloisE
by *blast*

end

corollary *bi-total-left-GaloisI*:
assumes *left-total* $(\leq_L)$
and *rel-surjective* $(\leq_R)$
and $((\leq_L) \Rightarrow (\leq_R))$ l
and $((\leq_R) \Rightarrow (\leq_L))$ r
and $((\leq_L) \trianglelefteq_h (\leq_R))$ l r
shows *bi-total* $({}_L\lessapprox)$
using *assms* **by** (*urule bi-total-on-left-GaloisI*)

corollary *left-total-left-rel-surjective-right-if-bi-total-left-GaloisE*:
assumes *bi-total* $({}_L\lessapprox)$
obtains *left-total* $(\leq_L)$ *rel-surjective* $(\leq_R)$
using *assms* **by** (*urule* (*e*) *left-total-on-left-rel-surjective-at-right-if-bi-total-on-left-GaloisE*)

corollary *bi-total-left-Galois-iff-left-total-left-and-rel-surjective-rightI*:
assumes $((\leq_L) \Rightarrow (\leq_R))$ l
and $((\leq_R) \Rightarrow (\leq_L))$ r
and $((\leq_L) \trianglelefteq_h (\leq_R))$ l r
shows *bi-total* $({}_L\lessapprox) \longleftrightarrow$ *left-total* $(\leq_L) \wedge$ *rel-surjective* $(\leq_R)$

using *assms bi-total-left-GaloisI*
left-total-left-rel-surjective-right-if-bi-total-left-GaloisE
by *blast*

Function Relator **lemma** *Fun-Rel-left-Galois-left-Galois-imp-left-rightI*:
assumes *monol*: $((\leq_L) \Rightarrow (\leq_R))\ l$
and *half-gal*: $((\leq_L)\ {}_h\trianglelefteq\ (\leq_R))\ l\ r$
and *perR*: *partial-equivalence-rel* $(\leq_R)$
shows $((_L\lessapprox) \Rrightarrow (_L\lessapprox) \Rrightarrow (\longrightarrow))\ (\leq_L)\ (\leq_R)$
proof (*intro Fun-Rel-relI impI*)
fix $x\ y\ x'\ y'$
assume $x\ {}_L\lessapprox\ y\ x'\ {}_L\lessapprox\ y'\ x \leq_L x'$
with *half-gal* **have** $l\ x' \leq_R y'$ **by** *auto*
with $\langle x \leq_L x'\rangle$ *monol perR* **have** $l\ x \leq_R y'$ **by** *blast*
with $\langle x\ {}_L\lessapprox\ y\rangle$ *half-gal perR* **have** $y \leq_R l\ x$ **by** (*blast dest*: *symmetricD*)
with *perR* $\langle l\ x \leq_R y'\rangle$ **show** $y \leq_R y'$ **by** *blast*
qed

lemma *Fun-Rel-left-Galois-left-Galois-rev-imp-left-rightI*:
assumes *monor*: $((\leq_R) \Rightarrow (\leq_L))\ r$
and *perL*: *partial-equivalence-rel* $(\leq_L)$
shows $((_L\lessapprox) \Rrightarrow (_L\lessapprox) \Rrightarrow (\longleftarrow))\ (\leq_L)\ (\leq_R)$
proof (*intro Fun-Rel-relI rev-impI*)
fix $x\ y\ x'\ y'$
assume $x\ {}_L\lessapprox\ y$ **and** $x'\ {}_L\lessapprox\ y'$ **and** $y \leq_R y'$
with *monor perL* **have** $x \leq_L r\ y'$ **by** *fastforce*
with $\langle x'\ {}_L\lessapprox\ y'\rangle$ **have** $x' \leq_L r\ y'$ **by** *auto*
with *perL* $\langle x \leq_L r\ y'\rangle$ **show** $x \leq_L x'$ **by** (*blast dest*: *symmetricD*)
qed

corollary *Fun-Rel-left-Galois-left-Galois-iff-left-rightI*:
assumes $((\leq_L) \Rightarrow (\leq_R))\ l$
and $((\leq_R) \Rightarrow (\leq_L))\ r$
and $((\leq_L)\ {}_h\trianglelefteq\ (\leq_R))\ l\ r$
and *partial-equivalence-rel* $(\leq_L)$
and *partial-equivalence-rel* $(\leq_R)$
shows $((_L\lessapprox) \Rrightarrow (_L\lessapprox) \Rrightarrow (\longleftrightarrow))\ (\leq_L)\ (\leq_R)$
using *assms Fun-Rel-left-Galois-left-Galois-imp-left-rightI*
Fun-Rel-left-Galois-left-Galois-rev-imp-left-rightI
by (*intro Fun-Rel-relI*) *blast*

end

end

theory *Transport-Implies*
imports
Function-Relators

Reverse-Implies
begin

lemma *Fun-Rel-iff-imp-imp*: $((\longleftrightarrow) \Rrightarrow (\longleftrightarrow) \Rrightarrow (\longleftrightarrow))\ (\longrightarrow)\ (\longrightarrow)$
by *auto*

lemma *Fun-Rel-imp-imp-imp*: $((\longleftarrow) \Rrightarrow (\longrightarrow) \Rrightarrow (\longrightarrow))\ (\longrightarrow)\ (\longrightarrow)$
by *auto*

lemma *Fun-Rel-rev-imp-imp-imp*: $((\longrightarrow) \Rrightarrow (\longleftarrow) \Rrightarrow (\longleftarrow))\ (\longrightarrow)\ (\longrightarrow)$
by *auto*

lemma *Fun-Rel-iff-rev-imp-rev-imp*: $((\longleftrightarrow) \Rrightarrow (\longleftrightarrow) \Rrightarrow (\longleftrightarrow))\ (\longleftarrow)\ (\longleftarrow)$
by *auto*

lemma *Fun-Rel-imp-rev-imp-rev-imp*: $((\longrightarrow) \Rrightarrow (\longleftarrow) \Rrightarrow (\longrightarrow))\ (\longleftarrow)\ (\longleftarrow)$
by *force*

lemma *Fun-Rel-rev-imp-rev-imp-rev-imp*: $((\longleftarrow) \Rrightarrow (\longrightarrow) \Rrightarrow (\longleftarrow))\ (\longleftarrow)\ (\longleftarrow)$
by *force*

end

theory *Transport-White-Box*
imports
Transport-Equality
Transport-Existential-Quantifier
Transport-Galois-Relator-Properties
Transport-Implies
Transport-Universal-Quantifier
begin

Summary Theorems for white-box transports.

context *galois*
begin

notepad
begin
print-statement *Fun-Rel-imp-eq-restrict-if-right-unique-onI*
[*of in-field* $(\leq_L)$ $({}_L\lessapprox)$ *in-field* $(\leq_R)$]

have $(({}_L\lessapprox) \Rrightarrow (\longrightarrow))\ (\textit{in-field}\ (\leq_L))\ (\textit{in-field}\ (\leq_R))$ **by** (*intro Fun-Rel-relI*) *auto*
end

end

end

theory *Transport*
imports
Transport-Black-Box
Transport-White-Box
begin

Summary We formalise the theory for the Transport framework. The Transport framework allows us to transport terms along (partial) Galois connections (*galois.galois-connection*) and equivalences (*galois.galois-equivalence*). For details, refer to [2].

end

2.21 Transport for Dependent Function Relator with Non-Dependent Functions

theory *Transport-Rel-If*
imports
Transport-Black-Box
begin

Summary We introduce a special case of *transport-Dep-Fun-Rel*. The derived theorem is easier to apply and supported by the current prototype.

context
fixes $P :: 'a \Rightarrow bool$ **and** $R :: 'a \Rightarrow 'a \Rightarrow bool$
begin

lemma *reflexive-on-rel-if-if-reflexive-onI* [*intro*]:
assumes $B \Longrightarrow$ *reflexive-on P R*
shows *reflexive-on P* (*rel-if B R*)
using *assms* **by** (*intro reflexive-onI*) *blast*

lemma *transitive-on-rel-if-if-transitive-onI* [*intro*]:
assumes $B \Longrightarrow$ *transitive-on P R*
shows *transitive-on P* (*rel-if B R*)
using *assms* **by** (*intro transitive-onI*) (*blast dest*: *transitive-onD*)

lemma *preorder-on-rel-if-if-preorder-onI* [*intro*]:
assumes $B \Longrightarrow$ *preorder-on P R*
shows *preorder-on P* (*rel-if B R*)
using *assms* **by** (*intro preorder-onI*) *auto*

lemma *symmetric-on-rel-if-if-symmetric-onI* [*intro*]:
assumes $B \Longrightarrow$ *symmetric-on P R*
shows *symmetric-on P* (*rel-if B R*)
using *assms* **by** (*intro symmetric-onI*) (*blast dest*: *symmetric-onD*)

lemma *partial-equivalence-rel-on-rel-if-if-partial-equivalence-rel-onI* [*intro*]:
assumes $B \Longrightarrow$ *partial-equivalence-rel-on P R*
shows *partial-equivalence-rel-on P* (*rel-if B R*)
using *assms* **by** (*intro partial-equivalence-rel-onI*)
auto

lemma *rel-if-dep-mono-wrt-rel-if-iff-if-dep-mono-wrt-relI*:
assumes $B \Longrightarrow B' \Longrightarrow ((x\ y :: R) \Rightarrow S\ x\ y)\ f$
and $B \longleftrightarrow B'$
shows $((x\ y ::$ *rel-if B R*$) \Rightarrow ($*rel-if* $B'\ (S\ x\ y))) f$
using *assms* **by** (*intro dep-mono-wrt-relI*) *auto*

corollary *reflexive-rel-if-if-reflexiveI* [*intro*]:
assumes $B \Longrightarrow$ *reflexive R*
shows *reflexive* (*rel-if B R*)
using *assms* **unfolding** *reflexive-eq-reflexive-on* **by** *blast*

corollary *transitive-rel-if-if-transitiveI* [*intro*]:
assumes $B \Longrightarrow$ *transitive R*
shows *transitive* (*rel-if B R*)
using *assms* **unfolding** *transitive-eq-transitive-on*
by (*intro transitive-onI*) (*force dest*: *transitive-onD*)

end

context
fixes $P :: 'a \Rightarrow$ *bool* **and** $R :: 'a \Rightarrow 'a \Rightarrow$ *bool*
begin

corollary *preorder-rel-if-if-preorderI* [*intro*]:
assumes $B \Longrightarrow$ *preorder R*
shows *preorder* (*rel-if B R*)
using *assms* **unfolding** *preorder-eq-preorder-on* **by** *blast*

corollary *symmetric-rel-if-if-symmetricI* [*intro*]:
assumes $B \Longrightarrow$ *symmetric R*
shows *symmetric* (*rel-if B R*)
using *assms* **unfolding** *symmetric-eq-symmetric-on* **by** *blast*

corollary *partial-equivalence-rel-rel-if-if-partial-equivalence-relI* [*intro*]:
assumes $B \Longrightarrow$ *partial-equivalence-rel R*
shows *partial-equivalence-rel* (*rel-if B R*)
using *assms* **unfolding** *partial-equivalence-rel-eq-partial-equivalence-rel-on*
by *blast*

end

context *galois-prop*
begin

interpretation *rel-if* : *galois-prop rel-if B* $(\leq_L)$ *rel-if B'* $(\leq_R)$ *l r* **.**
interpretation *flip-inv* : *galois-prop* $(\geq_R)$ $(\geq_L)$ *r l* **.**

lemma *rel-if-half-galois-prop-left-if-iff-if-half-galois-prop-leftI*:
assumes $B \Longrightarrow B' \Longrightarrow ((\leq_L)\ {}_h\trianglelefteq\ (\leq_R))\ l\ r$
and $B \longleftrightarrow B'$
shows $((\textit{rel-if}\ B\ (\leq_L))\ {}_h\trianglelefteq\ (\textit{rel-if}\ B'\ (\leq_R)))\ l\ r$
using *assms* **by** (*intro rel-if.half-galois-prop-leftI*) *auto*

lemma *rel-if-half-galois-prop-right-if-iff-if-half-galois-prop-rightI*:
assumes $B \Longrightarrow B' \Longrightarrow ((\leq_L)\ \trianglelefteq_h\ (\leq_R))\ l\ r$
and $B \longleftrightarrow B'$
shows $((\textit{rel-if}\ B\ (\leq_L))\ \trianglelefteq_h\ (\textit{rel-if}\ B'\ (\leq_R)))\ l\ r$
using *assms* **by** (*intro rel-if.half-galois-prop-rightI*) *fastforce*

lemma *rel-if-galois-prop-if-iff-if-galois-propI*:
assumes $B \Longrightarrow B' \Longrightarrow ((\leq_L)\ \trianglelefteq\ (\leq_R))\ l\ r$
and $B \longleftrightarrow B'$
shows $((\textit{rel-if}\ B\ (\leq_L))\ \trianglelefteq\ (\textit{rel-if}\ B'\ (\leq_R)))\ l\ r$
using *assms* **by** (*intro rel-if.galois-propI*
rel-if-half-galois-prop-left-if-iff-if-half-galois-prop-leftI
rel-if-half-galois-prop-right-if-iff-if-half-galois-prop-rightI)
auto

end

context *galois*
begin

interpretation *rel-if* : *galois rel-if B* $(\leq_L)$ *rel-if B'* $(\leq_R)$ *l r* **.**

lemma *rel-if-galois-connection-if-iff-if-galois-connectionI*:
assumes $B \Longrightarrow B' \Longrightarrow ((\leq_L)\ \dashv\ (\leq_R))\ l\ r$
and $B \longleftrightarrow B'$
shows $((\textit{rel-if}\ B\ (\leq_L))\ \dashv\ (\textit{rel-if}\ B'\ (\leq_R)))\ l\ r$
using *assms* **by** (*intro rel-if.galois-connectionI*
rel-if-dep-mono-wrt-rel-if-iff-if-dep-mono-wrt-relI
rel-if-galois-prop-if-iff-if-galois-propI)
auto

lemma *rel-if-galois-equivalence-if-iff-if-galois-equivalenceI*:
assumes $B \Longrightarrow B' \Longrightarrow ((\leq_L)\ \equiv_G\ (\leq_R))\ l\ r$
and $B \longleftrightarrow B'$
shows $((\textit{rel-if}\ B\ (\leq_L))\ \equiv_G\ (\textit{rel-if}\ B'\ (\leq_R)))\ l\ r$
using *assms* **by** (*intro rel-if.galois-equivalenceI*
rel-if-galois-connection-if-iff-if-galois-connectionI
galois-prop.rel-if-galois-prop-if-iff-if-galois-propI)
(*auto elim*: *galois.galois-connectionE*)

end

context *transport*
begin

interpretation *rel-if* : *transport rel-if B* $(\leq_L)$ *rel-if B'* $(\leq_R)$ *l r* **.**

lemma *rel-if-preorder-equivalence-if-iff-if-preorder-equivalenceI*:
assumes $B \Longrightarrow B' \Longrightarrow ((\leq_L) \equiv_{pre} (\leq_R))$ *l r*
and $B \longleftrightarrow B'$
shows $((rel\text{-}if\ B\ (\leq_L)) \equiv_{pre} (rel\text{-}if\ B'\ (\leq_R)))$ *l r*
using *assms* **by** (*intro rel-if.preorder-equivalence-if-galois-equivalenceI*
rel-if-galois-equivalence-if-iff-if-galois-equivalenceI
preorder-on-rel-if-if-preorder-onI)
blast+

lemma *rel-if-partial-equivalence-rel-equivalence-if-iff-if-partial-equivalence-rel-equivalenceI*:
assumes $B \Longrightarrow B' \Longrightarrow ((\leq_L) \equiv_{PER} (\leq_R))$ *l r*
and $B \longleftrightarrow B'$
shows $((rel\text{-}if\ B\ (\leq_L)) \equiv_{PER} (rel\text{-}if\ B'\ (\leq_R)))$ *l r*
using *assms* **by** (*intro*
rel-if.partial-equivalence-rel-equivalence-if-galois-equivalenceI
rel-if-galois-equivalence-if-iff-if-galois-equivalenceI)
blast+

end

locale *transport-Dep-Fun-Rel-no-dep-fun* =
transport-Dep-Fun-Rel-syntax L1 R1 l1 r1 L2 R2 λ- -. *l2* λ- -. *r2* +
tdfr : *transport-Dep-Fun-Rel L1 R1 l1 r1 L2 R2* λ- -. *l2* λ- -. *r2*
for *L1* :: $'a1 \Rightarrow 'a1 \Rightarrow bool$
and *R1* :: $'a2 \Rightarrow 'a2 \Rightarrow bool$
and *l1* :: $'a1 \Rightarrow 'a2$
and *r1* :: $'a2 \Rightarrow 'a1$
and *L2* :: $'a1 \Rightarrow 'a1 \Rightarrow 'b1 \Rightarrow 'b1 \Rightarrow bool$
and *R2* :: $'a2 \Rightarrow 'a2 \Rightarrow 'b2 \Rightarrow 'b2 \Rightarrow bool$
and *l2* :: $'b1 \Rightarrow 'b2$
and *r2* :: $'b2 \Rightarrow 'b1$
begin

notation *t2.unit* ($\langle\eta_2\rangle$)
notation *t2.counit* ($\langle\varepsilon_2\rangle$)

abbreviation *L* $\equiv$ *tdfr.L*
abbreviation *R* $\equiv$ *tdfr.R*

abbreviation *l* $\equiv$ *tdfr.l*
abbreviation *r* $\equiv$ *tdfr.r*

notation *tdfr.L* (**infix** $\langle\leq_L\rangle$ *50*)
notation *tdfr.R* (**infix** $\langle\leq_R\rangle$ *50*)

notation *tdfr.ge-left* (**infix** $\langle\geq_L\rangle$ *50*)
notation *tdfr.ge-right* (**infix** $\langle\geq_R\rangle$ *50*)

notation *tdfr.unit* ($\langle\eta\rangle$)
notation *tdfr.counit* ($\langle\varepsilon\rangle$)

theorem *partial-equivalence-rel-equivalenceI*:
assumes *per-equiv1*: $((\leq_{L1}) \equiv_{PER} (\leq_{R1}))$ *l1 r1*
and *per-equiv2*: $\bigwedge x\ x'.\ x\ _{L1}\lessapprox x' \Longrightarrow ((\leq_{L2\ x\ (r1\ x')}) \equiv_{PER} (\leq_{R2\ (l1\ x)\ x'}))$ *l2 r2*
and $((x1\ x2 :: (\geq_{L1})) \Rightarrow (x3\ x4 :: (\leq_{L1}) \mid x1 \leq_{L1} x3) \Rrightarrow (\leq))$ *L2*
and $((x1'\ x2' :: (\geq_{R1})) \Rightarrow (x3'\ x4' :: (\leq_{R1}) \mid x1' \leq_{R1} x3') \Rrightarrow (\leq))$ *R2*
shows $((\leq_L) \equiv_{PER} (\leq_R))$ *l r*
proof −
have *per2I*: $((\leq_{L2\ x1\ (r1\ x2')}) \equiv_{PER} (\leq_{R2\ (l1\ x1)\ x2'}))$ *l2 r2*
if *hyps*: *x1* $\leq_{L1}$ *x2 x2* $_{L1}\lessapprox$ *x1' x1'* $\leq_{R1}$ *x2'* **for** *x1 x2 x1' x2'*
proof −
from *hyps* **have** *x1* $_{L1}\lessapprox$ *x2'*
using *per-equiv1 t1.left-Galois-if-left-Galois-if-left-relI t1.left-Galois-if-right-rel-if-left-GaloisI*
by *fast*
with *per-equiv2* **show** *?thesis* **by** *blast*
qed
have $((x1'\ x2' :: (\leq_{R1})) \Rightarrow (x1\ x2 :: (\leq_{L1}) \mid x2\ _{L1}\lessapprox x1') \Rrightarrow$
in-field $(\leq_{L2\ x1\ (r1\ x2')}) \Rrightarrow (\leq_{R2\ (l1\ x1)\ x2'}))$ $(\lambda\text{-}\ \text{-}.\ l2)$
by (*intro dep-mono-wrt-relI Dep-Fun-Rel-relI Dep-Fun-Rel-predI rel-if-if-impI*)
(*auto 10 0 dest*!: *per2I*)
moreover have
$((x1\ x2 :: (\leq_{L1})) \Rightarrow (x1'\ x2' :: (\leq_{R1}) \mid x2\ _{L1}\lessapprox x1') \Rrightarrow$
in-field $(\leq_{R2\ (l1\ x1)\ x2'}) \Rrightarrow (\leq_{L2\ x1\ (r1\ x2')}))$ $(\lambda\text{-}\ \text{-}.\ r2)$
by (*intro dep-mono-wrt-relI Dep-Fun-Rel-relI Dep-Fun-Rel-predI rel-if-if-impI*)
(*auto 10 0 dest*!: *per2I*)
ultimately show *?thesis*
using *assms* **by** (*intro tdfr.partial-equivalence-rel-equivalenceI*) *auto*
qed

end

end

2.22 Transport via Equivalences on PERs (Prototype)

theory *Transport-Prototype*
imports
Transport-Rel-If
ML-Unification.ML-Unification-HOL-Setup
ML-Unification.Unify-Resolve-Tactics
keywords *trp-term* :: *thy-goal-defn*
begin

Summary We implement a simple Transport prototype. The prototype is restricted to work with equivalences on partial equivalence relations. It is also not forming the compositions of equivalences so far. The support for dependent function relators is restricted to the form described in ⟦*transport.partial-equivalence-rel-equivalence ?L1.0 ?R1.0 ?l1.0 ?r1.0*; ⋀*x x'*. *galois-rel.Galois ?L1.0 ?R1.0 ?r1.0 x x'* ⟹ *transport.partial-equivalence-rel-equivalence* (*?L2.0 x* (*?r1.0 x'*)) (*?R2.0* (*?l1.0 x*) *x'*) *?l2.0 ?r2.0*; ((*x1 x2* :: *?L1.0*$^{-1}$) ⇒ (*x3 x4* :: *?L1.0*) ⇛ *?L1.0 x1 x3* ⟶ (≤)) *?L2.0*; ((*x1' x2'* :: *?R1.0*$^{-1}$) ⇒ (*x3' x4'* :: *?R1.0*) ⇛ *?R1.0 x1' x3'* ⟶ (≤)) *?R2.0*⟧ ⟹ *transport.partial-equivalence-rel-equivalence* (*transport-Dep-Fun-Rel.L ?L1.0 ?L2.0*) (*transport-Dep-Fun-Rel.L ?R1.0 ?R2.0*) (*transport-Dep-Fun-Rel.l ?r1.0* (λ- -. *?l2.0*)) (*transport-Dep-Fun-Rel.l ?l1.0* (λ- -. *?r2.0*)): The relations can be dependent, but the functions must be simple. This is not production ready, but a proof of concept.

The package provides a command **trp-term**, which sets up the required goals to prove a given term. See the examples in this directory for some use cases and refer to [2] for more details.

Theorem Setups **context** *transport*
begin

lemma *left-Galois-left-if-left-rel-if-partial-equivalence-rel-equivalence*:
assumes ((≤$_L$) ≡$_{PER}$ (≤$_R$)) *l r*
and *x* ≤$_L$ *x'*
shows *x* $_L\lesssim$ *l x*
using *assms* **by** (*intro left-Galois-left-if-left-rel-if-inflationary-on-in-fieldI*)
(*blast elim*: *preorder-equivalence-order-equivalenceE*)+

definition *transport-per x y* ≡ ((≤$_L$) ≡$_{PER}$ (≤$_R$)) *l r* ∧ *x* $_L\lesssim$ *y*

The choice of x' is arbitrary. All we need is *in-dom* (≤$_L$) *x*.

lemma *transport-per-start*:
assumes ((≤$_L$) ≡$_{PER}$ (≤$_R$)) *l r*
and *x* ≤$_L$ *x'*
shows *transport-per x* (*l x*)
using *assms* **unfolding** *transport-per-def*

by (*blast intro*: *left-Galois-left-if-left-rel-if-partial-equivalence-rel-equivalence*)

lemma *left-Galois-if-transport-per*:
assumes *transport-per x y*
shows $x\ {}_{L}\lessapprox\ y$
using *assms* **unfolding** *transport-per-def* **by** *blast*

end

context *transport-Fun-Rel*
begin

Simplification of Galois relator for simple function relator.

corollary *left-Galois-eq-Fun-Rel-left-Galois*:
assumes $((\leq_{L1}) \equiv_{PER} (\leq_{R1}))$ *l1 r1*
and $((\leq_{L2}) \equiv_{PER} (\leq_{R2}))$ *l2 r2*
shows $({}_{L}\lessapprox) = (({}_{L1}\lessapprox) \Rrightarrow ({}_{L2}\lessapprox))$
proof (*intro ext*)
fix *f g*
show $f\ {}_{L}\lessapprox\ g \longleftrightarrow (({}_{L1}\lessapprox) \Rrightarrow ({}_{L2}\lessapprox))\ f\ g$
proof
assume $f\ {}_{L}\lessapprox\ g$
moreover have $g \leq_{R} g$
proof −
from *assms* **have** *per*: $((\leq_{L}) \equiv_{PER} (\leq_{R}))$ *l r*
by (*intro partial-equivalence-rel-equivalenceI*) *auto*
with ‹$f\ {}_{L}\lessapprox\ g$› **show** *?thesis* **by** *blast*
qed
ultimately show $(({}_{L1}\lessapprox) \Rrightarrow ({}_{L2}\lessapprox))\ f\ g$ **using** *assms*
by (*intro Fun-Rel-left-Galois-if-left-GaloisI*)
(*auto elim*!: *tdfrs.t1.partial-equivalence-rel-equivalenceE*
tdfrs.t1.preorder-equivalence-galois-equivalenceE
tdfrs.t1.galois-equivalenceE
intro: *reflexive-on-if-le-pred-if-reflexive-on in-field-if-in-dom*)
next
assume $(({}_{L1}\lessapprox) \Rrightarrow ({}_{L2}\lessapprox))\ f\ g$
with *assms* **have** $(({}_{L1}\lessapprox) \Rrightarrow ({}_{L2}\lessapprox))\restriction_{in\text{-}dom\ (\leq_{L})}\uparrow_{in\text{-}codom\ (\leq_{R})}\ f\ g$
by (*subst Fun-Rel-left-Galois-restrict-left-right-eq-Fun-Rel-left-GaloisI*) *blast*+
with *assms* **show** $f\ {}_{L}\lessapprox\ g$
by (*intro left-Galois-if-Fun-Rel-left-GaloisI*) *blast*+
qed
qed

end

lemmas *related-Fun-Rel-combI* = *Fun-Rel-relD*[*rotated*]

lemma *related-Fun-Rel-lambdaI*:
assumes $\bigwedge x\ y.\ R\ x\ y \Longrightarrow S\ (f\ x)\ (g\ y)$

and $T = (R \Rrightarrow S)$
shows $T\ f\ g$
using *assms* **by** *blast*

General ML setups **ML-file**‹*transport-util.ML*›

Unification Setup **ML**‹
functor-instance ‹*struct-name*: *Transport-Unification-Combine*
functor-name: *Unification-Combine*
id: *Transport-Util.transport-id*›
›
local-setup ‹*Transport-Unification-Combine.setup-attribute NONE*›

ML‹
functor-instance ‹*struct-name*: *Transport-Mixed-Comb-Unification*
functor-name: *Mixed-Comb-Unification*
id: *Transport-Util.transport-id*
more-args: ‹*structure UC = Transport-Unification-Combine*››
›
declare [[*trp-ucombine* ‹*Transport-Unification-Combine.eunif-data*
(*Transport-Unification-Combine.metadata* ***binding*** ‹*var-hop-unif*› *Prio.HIGH1*,
Var-Higher-Order-Pattern-Unification.e-unify
#> *Unification-Combinator.norm-unifier* (*Unification-Util.inst-norm-term'*
Transport-Mixed-Comb-Unification.norms-first-higherp-comb-unify))›]]
ML‹
functor-instance ‹*struct-name*: *Transport-Unification-Hints*
functor-name: *Term-Index-Unification-Hints*
id: *Transport-Util.transport-id*
more-args: ‹
structure TI = Discrimination-Tree
structure Args = Term-Index-Unification-Hints-Args
val init-args = {
concl-unifier = SOME (*Higher-Order-Pattern-Unification.unify*
|> *Type-Unification.e-unify Unification-Util.unify-types*),
prems-unifier = SOME (*Transport-Mixed-Comb-Unification.first-higherp-comb-unify*
|> *Unification-Combinator.norm-unifier Envir-Normalisation.beta-norm-term-unif*),
normalisers = SOME Transport-Mixed-Comb-Unification.norms-first-higherp-comb-unify,
retrieval = SOME (*Args.mk-retrieval-sym-pair* (*K TI.unifiables* |> *Args.retrieve-transfer*
TI.norm-term),
hint-preprocessor = SOME (*K I*)
}››
›
local-setup ‹*Transport-Unification-Hints.setup-attribute NONE*›
declare [[*trp-uhint config hint-preprocessor*: ‹*Unification-Hints-Base.obj-logic-hint-preprocessor*
@{*thm atomize-eq*[*symmetric*]} (*Conv.rewr-conv* @{*thm eq-eq-True*})›]]
declare [[*trp-ucombine* ‹*Transport-Unification-Combine.eunif-data*
(*Transport-Unification-Combine.default-metadata Transport-Unification-Hints.binding*,
Transport-Unification-Hints.try-hints
|> *Unification-Combinator.norm-unifier* (*Unification-Util.inst-norm-term'*

Transport-Mixed-Comb-Unification.norms-first-higherp-comb-unify)
|> *K*)›]]

Prototype **ML-file**‹*transport.ML*›

declare
transport-Dep-Fun-Rel.transport-defs[*trp-def*]
transport-Fun-Rel.transport-defs[*trp-def*]

declare

transport-Fun-Rel.partial-equivalence-rel-equivalenceI[*rotated*, *per-intro*]
transport-eq-id.partial-equivalence-rel-equivalenceI[*per-intro*]
transport-eq-restrict-id.partial-equivalence-rel-equivalence[*per-intro*]

declare
transport-id.left-Galois-eq-left[*trp-relator-rewrite*]
transport-Fun-Rel.left-Galois-eq-Fun-Rel-left-Galois[*trp-relator-rewrite*]

end

2.23 Syntax Bundles for Galois Relator

theory *Galois-Relator-Syntax*
imports
Galois-Relator-Base
begin

abbreviation *Galois-infix x L R r y* ≡ *galois-rel.Galois L R r x y*
abbreviation (*input*) *ge-Galois R r L* ≡ *galois-rel.ge-Galois-left L R r*
abbreviation (*input*) *ge-Galois-infix y R r L x* ≡ *ge-Galois R r L y x*

bundle *galois-rel-Galois-syntax*
begin
notation *galois-rel.Galois* (‹'((-)≲≈(-) (-))'›)
notation *Galois-infix* (‹(-) (-)≲≈(-) (-)) (-)› [*51*,*51*,*51*,*51*,*51*] *50*)
notation *ge-Galois* (‹'((-) (-))≳≈(-)'›)
notation *ge-Galois-infix* (‹(-) ((-) (-))≳≈(-) (-)› [*51*,*51*,*51*,*51*,*51*] *50*)
end

end

2.24 Example Transports for Dependent Function Relator

theory *Transport-Dep-Fun-Rel-Examples*
imports
Transport-Prototype
Galois-Relator-Syntax
HOL-Alignment-Binary-Relations
HOL−Library.IArray
begin

Summary Dependent function relator examples from [2]. Refer to the paper for more details.

context
includes *galois-rel-Galois-syntax* **and** *transport-equivalences-syntax*
notes
transport.rel-if-partial-equivalence-rel-equivalence-if-iff-if-partial-equivalence-rel-equivalenceI
[*rotated*, *per-intro*]
transport-Dep-Fun-Rel-no-dep-fun.partial-equivalence-rel-equivalenceI
[*ML-Krattr* ‹*Drule.rearrange-prems* [*1*] #> *Drule.rearrange-prems* [*2*,*3*]›, *per-intro*]
begin

interpretation *transport L R l r* **for** *L R l r* **.**

abbreviation *Zpos* ≡ $((=_{(\leq)(0 :: int)}) :: int \Rightarrow$ -)

lemma *Zpos-per* [*per-intro*]: (*Zpos* $\equiv_{PER}$ (=)) *nat int*
by *fastforce*

lemma *sub-parametric* [*trp-in-dom*]:
((*i* - :: *Zpos*) ⇛ (*j* - :: *Zpos* | *j* ≤ *i*) ⇛ *Zpos*) (−) (−)
by *fastforce*

trp-term *nat-sub* :: *nat* ⇒ *nat* ⇒ *nat* **where** *x* = (−) :: *int* ⇒ -
and *L* = (*i* - :: *Zpos*) ⇛ (*j* - :: *Zpos* | *j* ≤ *i*) ⇛ *Zpos*
and *R* = (*n* - :: (=)) ⇛ (*m* - :: (=)| *m* ≤ *n*) ⇛ (=)

by *trp-prover fastforce+*

thm *nat-sub-app-eq*

Note: as of now, **trp-term** does not rewrite the Galois relator of dependent function relators.

thm *nat-sub-related′*

abbreviation *LRel* ≡ *list-all2*
abbreviation *IARel* ≡ *rel-iarray*

lemma [*per-intro*]:
assumes *partial-equivalence-rel R*
shows (*LRel R* $\equiv_{PER}$ *IARel R*) *IArray.IArray IArray.list-of*
using *assms* **by** (*fastforce simp flip*: *transp-eq-transitive symp-eq-symmetric intro*: *list.rel-transp list.rel-symp iarray.rel-transp iarray.rel-symp elim*: *iarray.rel-cases*)+

lemma [*trp-in-dom*]:
((*xs* - :: *LRel R*) ⇛ (*i* - :: (=) | *i* < *length xs*) ⇛ *R*) (!) (!)
by (*fastforce simp*: *list-all2-lengthD list-all2-nthD2*)

context
fixes *R* :: *'a* ⇒ *'a* ⇒ *bool* **assumes** [*per-intro*]: *partial-equivalence-rel R*
begin

interpretation *Rper* : *transport-partial-equivalence-rel-id R*
by *unfold-locales per-prover*

declare *Rper.partial-equivalence-rel-equivalence* [*per-intro*]

trp-term *iarray-index* **where** *x* = (!) :: *'a list* ⇒ -
and *L* = ((*xs* - :: *LRel R*) ⇛ (*i* - :: (=) | *i* < *length xs*) ⇛ *R*)
and *R* = ((*xs* - :: *IARel R*) ⇛ (*i* - :: (=) | *i* < *IArray.length xs*) ⇛ *R*)
by *trp-prover*

(*fastforce simp*: *list-all2-lengthD elim*: *iarray.rel-cases*)+

end
end

end

2.25 Example Transports Between Lists and Sets

theory *Transport-Lists-Sets-Examples*
imports
Galois-Relator-Syntax
Transport-Prototype
HOL−Library.FSet
begin

Summary Introductory examples from [2]. Transports between lists and (finite) sets. Refer to the paper for more details.

context includes *galois-rel-Galois-syntax* **and** *transport-equivalences-syntax*
begin

Introductory examples from paper Left and right relations.

definition *LFSL xs xs′* ≡ *fset-of-list xs* = *fset-of-list xs′*
abbreviation (*input*) (*LFSR* :: *′a fset* ⇒ -) ≡ (=)
definition *LSL xs xs′* ≡ *set xs* = *set xs′*
abbreviation (*input*) (*LSR* :: *′a set* ⇒ -) ≡ ($=_{finite\ ::\ 'a\ set\ \Rightarrow\ bool}$)

interpretation *t* : *transport LSL R l r* **for** *LSL R l r* **.**

Proofs of equivalences.

lemma *list-fset-PER* [*per-intro*]: (*LFSL* $\equiv_{PER}$ *LFSR*) *fset-of-list sorted-list-of-fset*
unfolding *LFSL-def* **by** *fastforce*

lemma *list-set-PER* [*per-intro*]: (*LSL* $\equiv_{PER}$ *LSR*) *set sorted-list-of-set*
unfolding *LSL-def* **by** *fastforce*

We can rewrite the Galois relators in the following theorems to the relator of the paper.

definition *LFS xs s* ≡ *fset-of-list xs* = *s*
definition *LS xs s* ≡ *set xs* = *s*

lemma *LFSL-Galois-eq-LFS*: ($_{LFSL}\overset{<}{\approx}_{LFSR\ sorted\text{-}list\text{-}of\text{-}fset}$) ≡ *LFS*
unfolding *LFS-def LFSL-def* **by** (*intro eq-reflection ext*) (*auto*)
lemma *LFSR-Galois-eq-inv-LFS*: ($_{LFSR}\overset{<}{\approx}_{LFSL\ fset\text{-}of\text{-}list}$) ≡ LFS^{-1}
unfolding *LFS-def LFSL-def* **by** (*intro eq-reflection ext*) (*auto*)
lemma *LSL-Galois-eq-LS*: ($_{LSL}\overset{<}{\approx}_{LSR\ sorted\text{-}list\text{-}of\text{-}set}$) ≡ *LS*
unfolding *LS-def LSL-def* **by** (*intro eq-reflection ext*) (*auto*)

declare *LFSL-Galois-eq-LFS*[*trp-relator-rewrite*, *trp-uhint*]
LFSR-Galois-eq-inv-LFS[*trp-relator-rewrite*, *trp-uhint*]
LSL-Galois-eq-LS[*trp-relator-rewrite*, *trp-uhint*]

definition *max-list xs* ≡ *foldr max xs* (*0* :: *nat*)

Proof of parametricity for *max-list*.

lemma *max-max-list-removeAll-eq-maxlist*:
assumes *x* ∈ *set xs*
shows *max x* (*max-list* (*removeAll x xs*)) = *max-list xs*
unfolding *max-list-def* **using** *assms* **by** (*induction xs*)
(*simp-all*, (*metis max.left-idem removeAll-id max.left-commute*)+)

lemma *max-list-parametric* [*trp-in-dom*]: (*LSL* ⇛ (=)) *max-list max-list*
proof (*intro Fun-Rel-relI*)
fix *xs xs′* :: *nat list* **assume** *LSL xs xs′*
then have *finite* (*set xs*) *set xs* = *set xs′* **unfolding** *LSL-def* **by** *auto*
then show *max-list xs* = *max-list xs′*
proof (*induction set xs arbitrary*: *xs xs′ rule*: *finite-induct*)
case (*insert x F*)
then have *F* = *set* (*removeAll x xs*) **by** *auto*
moreover from *insert* **have** *...* = *set* (*removeAll x xs′*) **by** *auto*
ultimately have *max-list* (*removeAll x xs*) = *max-list* (*removeAll x xs′*)

```
    (is ?lhs = ?rhs) using insert by blast
  then have max x ?lhs = max x ?rhs by simp
  then show ?case
    using insert max-max-list-removeAll-eq-maxlist insertI1 by metis
 qed auto
qed

lemma LFSL-eq-LSL: LFSL ≡ LSL
 unfolding LFSL-def LSL-def by (intro eq-reflection ext) (auto simp: fset-of-list-elem)

lemma max-list-parametricfin [trp-in-dom]: (LFSL ⇛ (=)) max-list max-list
 using max-list-parametric by (simp only: LFSL-eq-LSL)
```

Transport from lists to finite sets.

```
trp-term max-fset :: nat fset ⇒ nat where x = max-list
 and L = (LFSL ⇛ (=))
 by trp-prover
```

Use **print-theorems** to show all theorems. Here's the correctness theorem:

```
lemma (LFS ⇛ (=)) max-list max-fset by (trp-hints-resolve max-fset-related′)

lemma [trp-in-dom]: (LFSR ⇛ (=)) max-fset max-fset by simp
```

Transport from lists to sets.

```
trp-term max-set :: nat set ⇒ nat where x = max-list
 by trp-prover

lemma (LS ⇛ (=)) max-list max-set by (trp-hints-resolve max-set-related′)
```

The registration of symmetric equivalence rules is not done by default as of now, but that would not be a problem in principle.

```
lemma list-fset-PER-sym [per-intro]:
 (LFSR ≡_PER LFSL) sorted-list-of-fset fset-of-list
 by (subst transport.partial-equivalence-rel-equivalence-right-left-iff-partial-equivalence-rel-equivalence-left-righ
 (fact list-fset-PER)
```

Transport from finite sets to lists.

```
trp-term max-list′ :: nat list ⇒ nat where x = max-fset
 by trp-prover

lemma (LFS⁻¹ ⇛ (=)) max-fset max-list′ by (trp-hints-resolve max-list′-related′)
```

Transporting higher-order functions.

```
lemma map-parametric [trp-in-dom]:
 (((=) ⇛ (=)) ⇛ LSL ⇛ LSL) map map
 unfolding LSL-def by (intro Fun-Rel-relI) simp

lemma [trp-uhint]: P ≡ (=) ⟹ P ≡ (=) ⇛ (=) by simp
```

lemma [*trp-uhint*]: $P \equiv \top \Longrightarrow (=_{P\ ::\ 'a \Rightarrow bool}) \equiv ((=)\ ::\ 'a \Rightarrow \text{-})$ **by** *simp*

trp-term *map-set* :: $('a ::$ *linorder* $\Rightarrow\ 'b) \Rightarrow\ 'a$ *set* $\Rightarrow ('b ::$ *linorder*$)$ *set*
  where $x =$ *map* :: $('a ::$ *linorder* $\Rightarrow\ 'b) \Rightarrow\ 'a$ *list* $\Rightarrow ('b ::$ *linorder*$)$ *list*
  by *trp-prover*

lemma $(((=) \Rrightarrow (=)) \Rrightarrow LS \Rrightarrow LS)$ *map map-set* **by** (*trp-hints-resolve map-set-related'*)

lemma *filter-parametric* [*trp-in-dom*]:
  $(((=) \Rrightarrow (\longleftrightarrow)) \Rrightarrow LSL \Rrightarrow LSL)$ *filter filter*
  unfolding *LSL-def* **by** (*intro Fun-Rel-relI*) *simp*

trp-term *filter-set* :: $('a ::$ *linorder* $\Rightarrow$ *bool*$) \Rightarrow\ 'a$ *set* $\Rightarrow\ 'a$ *set*
  where $x =$ *filter* :: $('a ::$ *linorder* $\Rightarrow$ *bool*$) \Rightarrow\ 'a$ *list* $\Rightarrow\ 'a$ *list*
  by *trp-prover*

lemma $(((=) \Rrightarrow (=)) \Rrightarrow LS \Rrightarrow LS)$ *filter filter-set* **by** (*trp-hints-resolve filter-set-related'*)

lemma *append-parametric* [*trp-in-dom*]:
  $(LSL \Rrightarrow LSL \Rrightarrow LSL)$ *append append*
  unfolding *LSL-def* **by** (*intro Fun-Rel-relI*) *simp*

trp-term *append-set* :: $('a ::$ *linorder*$)$ *set* $\Rightarrow\ 'a$ *set* $\Rightarrow\ 'a$ *set*
  where $x =$ *append* :: $('a ::$ *linorder*$)$ *list* $\Rightarrow\ 'a$ *list* $\Rightarrow\ 'a$ *list*
  by *trp-prover*

lemma $(LS \Rrightarrow LS \Rrightarrow LS)$ *append append-set* **by** (*trp-hints-resolve append-set-related'*)

The prototype also provides a simplified definition.

lemma *append-set s s'* $\equiv$ *set* (*sorted-list-of-set s*) $\cup$ *set* (*sorted-list-of-set s'*)
  by (*fact append-set-app-eq*)

lemma *finite s* $\Longrightarrow$ *finite s'* $\Longrightarrow$ *append-set s s'* $=$ *s* $\cup$ *s'*
  by (*auto simp*: *append-set-app-eq*)

end

end

2.26 Transport for Partial Quotient Types

theory *Transport-Partial-Quotient-Types*
  imports
    HOL.Lifting
    Transport
begin

Summary Every partial quotient type *Quotient*, as used by the Lifting package, is transportable.

context *transport*
begin

interpretation t : *transport* L $(=)$ l r **.**

lemma *Quotient-T-eq-Galois*:
  assumes *Quotient* $(\leq_L)$ l r T
  shows T = *t.Galois*
proof (*intro ext iffI*)
  fix x y **assume** T x y
  with *assms* **have** $x \leq_L x$ l $x = y$ **using** *Quotient-cr-rel* **by** *auto*
  with *assms* **have** r $(l$ $x) \leq_L x$ r $(l$ $x) \leq_L r$ y
    using *Quotient-rep-abs Quotient-rep-reflp* **by** *auto*
  with *assms* **have** $x \leq_L r$ y **using** *Quotient-part-equivp*
    by (*blast elim*: *part-equivpE dest*: *transpD sympD*)
  then show *t.Galois* x y **by** *blast*
next
  fix x y **assume** *t.Galois* x y
  with *assms* **show** T x y **using** *Quotient-cr-rel Quotient-refl1 Quotient-symp*
    by (*fastforce intro*: *Quotient-rel-abs2*[*symmetric*] *dest*: *sympD*)
qed

lemma *Quotient-if-preorder-equivalence*:
  assumes $((\leq_L) \equiv_{pre} (=))$ l r
  shows *Quotient* $(\leq_L)$ l r *t.Galois*
proof (*rule QuotientI*)
  from *assms* **show** *g2*: l $(r$ $y) = y$ **for** y **by** *fastforce*
  from *assms* **show** r $y \leq_L r$ y **for** y **by** *blast*
  show *g1*: $x \leq_L x' \longleftrightarrow x \leq_L x \wedge x' \leq_L x' \wedge l$ $x = l$ x'
    (**is** *?lhs* $\longleftrightarrow$ *?rhs*) **for** x x'
  proof (*rule iffI*)
    assume *?rhs*
    with *assms* **have** η $x \leq_L \eta$ x' **by** *fastforce*
    moreover from ‹*?rhs*› *assms* **have** $x \leq_L \eta$ x η $x' \leq_L x'$
      by (*auto 0 4 elim*!: *t.preorder-equivalence-order-equivalenceE*)
    moreover from *assms* **have** *transitive* $(\leq_L)$ **by** *blast*
    ultimately show $x \leq_L x'$ **by** *blast*
  next
    assume *?lhs*
    with *assms* **show** *?rhs* **by** *blast*
  qed
  from *assms* **show** *t.Galois* = $(\lambda x$ $y.$ $x \leq_L x \wedge l$ $x = y)$
    by (*intro ext iffI*)
    (*metis g1 g2 t.left-GaloisE*,
      auto intro!: *t.left-Galois-left-if-left-rel-if-inflationary-on-in-fieldI*
      elim!: *t.preorder-equivalence-order-equivalenceE*)
qed

lemma *partial-equivalence-rel-equivalence-if-Quotient*:
assumes *Quotient* $(\leq_L)$ *l r T*
shows $((\leq_L) \equiv_{PER} (=))$ *l r*
proof (*rule t.partial-equivalence-rel-equivalence-if-order-equivalenceI*)
from *Quotient-part-equivp*[*OF assms*] **show** *partial-equivalence-rel* $(\leq_L)$
by (*blast elim*: *part-equivpE dest*: *transpD sympD*)
have $x \equiv_L r\ (l\ x)$ **if** *in-field* $(\leq_L)$ *x* **for** *x*
proof −
from *assms* ‹*in-field* $(\leq_L)$ *x*› **have** $x \leq_L x$
using *Quotient-refl1 Quotient-refl2* **by** *fastforce*
with *assms Quotient-rep-abs Quotient-symp* **show** *?thesis*
by (*fastforce dest*: *sympD*)
qed
with *assms* **show** $((\leq_L) \equiv_o (=))$ *l r*
using *Quotient-abs-rep Quotient-rel-abs Quotient-rep-reflp*
Quotient-abs-rep[*symmetric*]
by (*intro t.order-equivalenceI mono-wrt-relI rel-equivalence-onI*
inflationary-onI deflationary-onI)
auto
qed *auto*

corollary *Quotient-iff-partial-equivalence-rel-equivalence*:
Quotient $(\leq_L)$ *l r t.Galois* $\longleftrightarrow$ $((\leq_L) \equiv_{PER} (=))$ *l r*
using *Quotient-if-preorder-equivalence partial-equivalence-rel-equivalence-if-Quotient*
by *blast*

corollary *Quotient-T-eq-ge-Galois-right*:
assumes *Quotient* $(\leq_L)$ *l r T*
shows *T* = *t.ge-Galois-right*
using *assms*
by (*subst t.ge-Galois-right-eq-left-Galois-if-symmetric-if-in-codom-eq-in-dom-if-galois-prop*)
(*blast dest*: *partial-equivalence-rel-equivalence-if-Quotient*
intro: *in-codom-eq-in-dom-if-reflexive-on-in-field Quotient-T-eq-Galois*)+

end

end

2.27 Transport for HOL Type Definitions

theory *Transport-Typedef-Base*
imports
HOL-Alignment-Binary-Relations
Transport-Bijections
HOL.Typedef
begin

context *type-definition*
begin

abbreviation (*input*) $L :: 'a \Rightarrow 'a \Rightarrow bool \equiv (=_A)$
abbreviation (*input*) $R :: 'b \Rightarrow 'b \Rightarrow bool \equiv (=)$

sublocale *transport?* :
transport-eq-restrict-bijection mem-of A $\top :: 'b \Rightarrow bool$ *Abs Rep*
rewrites $(=_{mem\text{-}of\ A}) \equiv L$
and $(=_{\top :: 'b \Rightarrow bool}) \equiv R$
and $(galois\text{-}rel.Galois\ (=)\ (=)\ Rep)\restriction_{mem\text{-}of\ A}\restriction_{\top :: 'b \Rightarrow bool} \equiv (galois\text{-}rel.Galois\ (=)\ (=)\ Rep)$
using *Abs-inverse Rep-inverse*
by (*intro transport-eq-restrict-bijection.intro bijection-onI*)
(*auto intro*!: *eq-reflection galois-rel.left-GaloisI Rep elim*: *galois-rel.left-GaloisE*)

interpretation *galois L R Abs Rep* **.**

lemma *Rep-left-Galois-self*: $Rep\ y\ {}_L\lessapprox y$
using *Rep* **by** (*intro left-GaloisI*) *auto*

definition $AR\ x\ y \equiv x = Rep\ y$

lemma *left-Galois-eq-AR*: *left-Galois* $= AR$
unfolding *AR-def*
by (*auto intro*!: *galois-rel.left-GaloisI Rep elim*: *galois-rel.left-GaloisE*)

end

end

theory *Transport-Typedef*
imports
HOL−Library.FSet
Galois-Relator-Syntax
Transport-Typedef-Base
Transport-Prototype
HOL-Alignment-Functions
begin

context includes *galois-rel-Galois-syntax* **and** *transport-equivalences-syntax*
begin

typedef $pint = \{i :: int.\ 0 \leq i\}$ **by** *auto*

interpretation *typedef-pint* : *type-definition Rep-pint Abs-pint* $\{i :: int.\ 0 \leq i\}$
by (*fact type-definition-pint*)

lemma [*trp-relator-rewrite*, *trp-uhint*]:
$(({=}_{\mathit{Collect}\ ((\leq)\ (0 :: \mathit{int}))})^{\leq}\approx(=)\ \mathit{Rep\text{-}pint}) \equiv \mathit{typedef\text{-}pint.AR}$
using *typedef-pint.left-Galois-eq-AR* **by** (*intro eq-reflection*) *simp*

typedef *'a fset* = $\{s :: 'a\ \mathit{set}.\ \mathit{finite}\ s\}$ **by** *auto*

interpretation *typedef-fset* :
type-definition Rep-fset Abs-fset $\{s :: 'a\ \mathit{set}.\ \mathit{finite}\ s\}$
by (*fact type-definition-fset*)

lemma [*trp-relator-rewrite*, *trp-uhint*]:
$(({=}_{\{s :: 'a\ \mathit{set}.\ \mathit{finite}\ s\}}) :: 'a\ \mathit{set} \Rightarrow \text{-}^{\leq}\approx(=)\ \mathit{Rep\text{-}fset}) \equiv \mathit{typedef\text{-}fset.AR}$
using *typedef-fset.left-Galois-eq-AR* **by** (*intro eq-reflection*) *simp*

lemma *eq-restrict-set-eq-eq-uhint* [*trp-uhint*]:
$(\bigwedge x.\ P\ x \equiv x \in A) \Longrightarrow (({=}_{A :: 'a\ \mathit{set}}) :: 'a \Rightarrow \text{-}) \equiv ({=}_P)$
by (*intro eq-reflection*) (*auto simp*: *fun-eq-iff*)

declare
typedef-pint.partial-equivalence-rel-equivalence[*per-intro*]
typedef-fset.partial-equivalence-rel-equivalence[*per-intro*]

lemma *one-parametric* [*trp-in-dom*]: *typedef-pint.L 1 1* **by** *auto*

trp-term *pint-one* :: *pint* **where** $x = 1 :: \mathit{int}$
by *trp-prover*

lemma *add-parametric* [*trp-in-dom*]:
(*typedef-pint.L* $\Rrightarrow$ *typedef-pint.L* $\Rrightarrow$ *typedef-pint.L*) (+) (+)
by (*intro Fun-Rel-relI*) *fastforce*

trp-term *pint-add* :: *pint* $\Rightarrow$ *pint* $\Rightarrow$ *pint*
where $x = (+) :: \mathit{int} \Rightarrow \text{-}$
by *trp-prover*

lemma *empty-parametric* [*trp-in-dom*]: *typedef-fset.L* {} {}
by *auto*

trp-term *fempty* :: *'a fset* **where** $x = \{\} :: 'a\ \mathit{set}$
by *trp-prover*

lemma *insert-parametric* [*trp-in-dom*]:
((=) $\Rrightarrow$ *typedef-fset.L* $\Rrightarrow$ *typedef-fset.L*) *insert insert*
by *auto*

trp-term *finsert* :: *'a* ⇒ *'a fset* ⇒ *'a fset* **where** *x* = *insert*
and *L* = ((=) ⇛ *typedef-fset.L* ⇛ *typedef-fset.L*)
and *R* = ((=) ⇛ *typedef-fset.R* ⇛ *typedef-fset.R*)
by *trp-prover*

context
notes *refl*[*trp-related-intro*]
begin

trp-term *insert-add-int-fset-whitebox* :: *int fset*
where *x* = *insert* (*1* + (*1* :: *int*)) {} !
by *trp-whitebox-prover*

lemma *empty-parametric'* [*trp-related-intro*]: (*rel-set R*) {} {}
by (*intro Dep-Fun-Rel-relI rel-setI*) (*auto dest*: *rel-setD1 rel-setD2*)

lemma *insert-parametric'* [*trp-related-intro*]:
(*R* ⇛ *rel-set R* ⇛ *rel-set R*) *insert insert*
by (*intro Fun-Rel-relI rel-setI*) (*auto dest*: *rel-setD1 rel-setD2*)

context
assumes [*trp-uhint*]:

L ≡ *rel-set* (*L1* :: *int* ⇒ *int* ⇒ *bool*) ⟹ *R* ≡ *rel-set* (*R1* :: *pint* ⇒ *pint* ⇒ *bool*)
⟹ *r* ≡ *Set.image r1* ⟹ *S* ≡ ($_{L1}\lessapprox_{R1\ r1}$) ⟹ ($_{L}\lessapprox_{R\ r}$) ≡ *rel-set S*
begin

trp-term *insert-add-pint-set-whitebox* :: *pint set*
where *x* = *insert* (*1* + (*1* :: *int*)) {} !
by *trp-whitebox-prover*

print-statement *insert-add-int-fset-whitebox-def insert-add-pint-set-whitebox-def*

end
end

lemma *image-parametric* [*trp-in-dom*]:
(((=) ⇛ (=)) ⇛ *typedef-fset.L* ⇛ *typedef-fset.L*) *Set.image Set.image*
by (*intro Fun-Rel-relI*) *auto*

trp-term *fimage* :: (*'a* ⇒ *'b*) ⇒ *'a fset* ⇒ *'b fset* **where** *x* = *Set.image*
by *trp-prover*

lemma *image-parametric'* [*trp-related-intro*]:

$((R \Rightarrow S) \Rightarrow \textit{rel-set}\ R \Rightarrow \textit{rel-set}\ S)\ \textit{Set.image}\ \textit{Set.image}$
using *transfer-raw*[*simplified HOL-fun-alignment Transfer.Rel-def*]
by *simp*

lemma *Galois-id-hint* [*trp-uhint*]:
$(L :: 'a \Rightarrow 'a \Rightarrow \textit{bool}) \equiv R \Longrightarrow r \equiv \textit{id} \Longrightarrow E \equiv L \Longrightarrow ({}_{L}\lessapprox_{R\ r}) \equiv E$
by (*simp add*: *eq-reflection*[*OF transport-id.left-Galois-eq-left*])

lemma *Freq* [*trp-uhint*]: $L \equiv (=) \Rightarrow (=) \Longrightarrow L \equiv (=)$
by *auto*

context
fixes *L1 R1 l1 r1 L R l r*
assumes *per1*: $(L1 \equiv_{PER} R1)\ l1\ r1$
defines $L \equiv \textit{rel-set}\ L1$ **and** $R \equiv \textit{rel-set}\ R1$
and $l \equiv \textit{Set.image}\ l1$ **and** $r \equiv \textit{Set.image}\ r1$
begin

interpretation *transport L R l r* **.**

context

assumes *transport-per-set*: $((\leq_L) \equiv_{PER} (\leq_R))\ l\ r$
and *compat*: *transport-comp.middle-compatible-codom R typedef-fset.L*
begin

trp-term *fempty-param* :: *'b fset*
where $x = \{\} :: 'a\ \textit{set}$
and $L = \textit{transport-comp.L}\ ?L1\ ?R1\ (?l1 :: 'a\ \textit{set} \Rightarrow 'b\ \textit{set})\ ?r1\ \textit{typedef-fset.L}$
and $R = \textit{transport-comp.L}\ \textit{typedef-fset.R}\ \textit{typedef-fset.L}\ ?r2\ ?l2\ ?R1$
apply (*rule transport-comp.partial-equivalence-rel-equivalenceI*)
apply (*rule transport-per-set*)
apply *per-prover*
apply (*fact compat*)
apply (*rule transport-comp.left-relI*[**where** $?y=\{\}$ **and** $?y'=\{\}$])
apply (*unfold L-def R-def l-def r-def*)
apply (*auto intro*!: *galois-rel.left-GaloisI in-codomI empty-transfer*)
done

definition *set-succ* $\equiv$ *Set.image* $((+)\ (1 :: \textit{int}))$

lemma *set-succ-parametric* [*trp-in-dom*]:
$(\textit{typedef-fset.L} \Rightarrow \textit{typedef-fset.L})\ \textit{set-succ}\ \textit{set-succ}$
unfolding *set-succ-def* **by** *auto*

trp-term *fset-succ* :: $\textit{int fset} \Rightarrow \textit{int fset}$
where $x = \textit{set-succ}$
and $L = \textit{typedef-fset.L} \Rightarrow \textit{typedef-fset.L}$
and $R = \textit{typedef-fset.R} \Rightarrow \textit{typedef-fset.R}$

by *trp-prover*

trp-term *fset-succ'* :: *int fset* ⇒ *int fset*
where *x* = *set-succ*
and *L* = *typedef-fset.L* ⇛ *typedef-fset.L*
and *R* = *typedef-fset.R* ⇛ *typedef-fset.R*
unfold set-succ-def !

using *refl*[*trp-related-intro*]
apply (*tactic* ‹*Transport.instantiate-skeleton-tac* @{*context*} *1*›)
apply (*tactic* ‹*Transport.transport-related-step-tac* @{*context*} *1*›)
apply (*tactic* ‹*Transport.transport-related-step-tac* @{*context*} *1*›)
apply (*tactic* ‹*Transport.transport-related-step-tac* @{*context*} *1*›)
apply (*tactic* ‹*Transport.transport-related-step-tac* @{*context*} *1*›)
apply (*tactic* ‹*Transport.transport-related-step-tac* @{*context*} *1*›)
apply (*tactic* ‹*Transport.transport-related-step-tac* @{*context*} *1*›)
apply *assumption*
apply *assumption*
prefer *3*
apply (*tactic* ‹*Transport.transport-related-step-tac* @{*context*} *1*›)
apply (*tactic* ‹*Transport.transport-related-step-tac* @{*context*} *1*›)
apply (*fold trp-def*)
apply (*trp-relator-rewrite*)+
apply (*unfold trp-def*)
apply (*trp-hints-resolve refl*)
done

lemma *pint-middle-compat*:
transport-comp.middle-compatible-codom (*rel-set* ((=) :: *pint* ⇒ -))
(=*Collect* (*finite* :: *pint set* ⇒ -))
by (*intro transport-comp.middle-compatible-codom-if-right1-le-eqI*)
(*auto simp*: *rel-set-eq intro*!: *transitiveI*)

trp-term *pint-fset-succ* :: *pint fset* ⇒ *pint fset*
where *x* = *set-succ* :: *int set* ⇒ *int set*

oops

end
end
end

end

2.28 Transport for Natural Functors

2.28.1 Basic Setup

theory *Transport-Natural-Functors-Base*

imports
HOL.BNF-Def
HOL-Alignment-Functions
Transport-Base
begin

Summary Basic setup for closure proofs and simple lemmas.

In the following, we willingly use granular apply-style proofs since, in practice, these theorems have to be automatically generated whenever we declare a new natural functor.

Note that "HOL-Library" provides a command *bnf-axiomatization* which allows one to axiomatically declare a bounded natural functor. However, we only need a subset of these axioms - the boundedness of the functor is irrelevant for our purposes. For this reason - and the sake of completeness - we state all the required axioms explicitly below.

lemma *Grp-UNIV-eq-eq-comp*: *BNF-Def.Grp UNIV f* = (=) ∘ *f*
by (*intro ext*) (*auto elim*: *GrpE intro*: *GrpI*)

lemma *eq-comp-rel-comp-eq-comp*: (=) ∘ *f* ∘∘ *R* = *R* ∘ *f*
by (*intro ext*) *auto*

lemma *Domain-Collect-case-prod-eq-Collect-in-dom*:
Domain {(*x*, *y*). *R x y*} = {*x*. *in-dom R x*}
by *blast*

lemma *ball-in-dom-iff-ball-ex*:
(∀ *x* ∈ *S*. *in-dom R x*) ⟷ (∀ *x* ∈ *S*. ∃ *y*. *R x y*)
by *blast*

lemma *pair-mem-Collect-case-prod-iff*: (*x*, *y*) ∈ {(*x*, *y*). *R x y*} ⟷ *R x y*
by *blast*

Natural Functor Axiomatisation **typedecl** ('*d*, '*a*, '*b*, '*c*) *F*

consts *Fmap* :: ('*a1* ⇒ '*a2*) ⇒ ('*b1* ⇒ '*b2*) ⇒ ('*c1* ⇒ '*c2*) ⇒
('*d*, '*a1*, '*b1*, '*c1*) *F* ⇒ ('*d*, '*a2*, '*b2*, '*c2*) *F*
Fset1 :: ('*d*, '*a*, '*b*, '*c*) *F* ⇒ '*a set*
Fset2 :: ('*d*, '*a*, '*b*, '*c*) *F* ⇒ '*b set*
Fset3 :: ('*d*, '*a*, '*b*, '*c*) *F* ⇒ '*c set*

axiomatization
where *Fmap-id*: *Fmap id id id* = *id*
and *Fmap-comp*: ⋀*f1 f2 f3 g1 g2 g3*.
Fmap (*g1* ∘ *f1*) (*g2* ∘ *f2*) (*g3* ∘ *f3*) = *Fmap g1 g2 g3* ∘ *Fmap f1 f2 f3*
and *Fmap-cong*: ⋀*f1 f2 f3 g1 g2 g3 x*.
(⋀*x1*. *x1* ∈ *Fset1 x* ⟹ *f1 x1* = *g1 x1*) ⟹
(⋀*x2*. *x2* ∈ *Fset2 x* ⟹ *f2 x2* = *g2 x2*) ⟹

$(\bigwedge x3.\ x3 \in Fset3\ x \Longrightarrow f3\ x3 = g3\ x3) \Longrightarrow$
$Fmap\ f1\ f2\ f3\ x = Fmap\ g1\ g2\ g3\ x$
and *Fset1-natural*: $\bigwedge f1\ f2\ f3.\ Fset1 \circ Fmap\ f1\ f2\ f3 = Set.image\ f1 \circ Fset1$
and *Fset2-natural*: $\bigwedge f1\ f2\ f3.\ Fset2 \circ Fmap\ f1\ f2\ f3 = Set.image\ f2 \circ Fset2$
and *Fset3-natural*: $\bigwedge f1\ f2\ f3.\ Fset3 \circ Fmap\ f1\ f2\ f3 = Set.image\ f3 \circ Fset3$

lemma *Fmap-id-eq-self*: $Fmap\ id\ id\ id\ x = x$
by (*subst Fmap-id*, *subst id-eq-self*, *rule refl*)

lemma *Fmap-comp-eq-Fmap-Fmap*:
$Fmap\ (g1 \circ f1)\ (g2 \circ f2)\ (g3 \circ f3)\ x = Fmap\ g1\ g2\ g3\ (Fmap\ f1\ f2\ f3\ x)$
by (*fact fun-cong*[*OF Fmap-comp*, *simplified comp-eq*])

lemma *Fset1-Fmap-eq-image-Fset1*: $Fset1\ (Fmap\ f1\ f2\ f3\ x) = f1\ `\ Fset1\ x$
by (*fact fun-cong*[*OF Fset1-natural*, *simplified comp-eq*])

lemma *Fset2-Fmap-eq-image-Fset2*: $Fset2\ (Fmap\ f1\ f2\ f3\ x) = f2\ `\ Fset2\ x$
by (*fact fun-cong*[*OF Fset2-natural*, *simplified comp-eq*])

lemma *Fset3-Fmap-eq-image-Fset3*: $Fset3\ (Fmap\ f1\ f2\ f3\ x) = f3\ `\ Fset3\ x$
by (*fact fun-cong*[*OF Fset3-natural*, *simplified comp-eq*])

lemmas *Fset-Fmap-eqs* = *Fset1-Fmap-eq-image-Fset1 Fset2-Fmap-eq-image-Fset2*
Fset3-Fmap-eq-image-Fset3

Relator **definition** $Frel :: ('a1 \Rightarrow 'a2 \Rightarrow bool) \Rightarrow ('b1 \Rightarrow 'b2 \Rightarrow bool) \Rightarrow ('c1 \Rightarrow 'c2 \Rightarrow bool) \Rightarrow$
$('d, 'a1, 'b1, 'c1)\ F \Rightarrow ('d, 'a2, 'b2, 'c2)\ F \Rightarrow bool$
where $Frel\ R1\ R2\ R3\ x\ y \equiv (\exists z.$
$z \in \{x.\ Fset1\ x \subseteq \{(x, y).\ R1\ x\ y\} \wedge Fset2\ x \subseteq \{(x, y).\ R2\ x\ y\}$
$\wedge\ Fset3\ x \subseteq \{(x, y).\ R3\ x\ y\}\}$
$\wedge\ Fmap\ fst\ fst\ fst\ z = x$
$\wedge\ Fmap\ snd\ snd\ snd\ z = y)$

lemma *FrelI*:
assumes $Fset1\ z \subseteq \{(x, y).\ R1\ x\ y\}$
and $Fset2\ z \subseteq \{(x, y).\ R2\ x\ y\}$
and $Fset3\ z \subseteq \{(x, y).\ R3\ x\ y\}$
and $Fmap\ fst\ fst\ fst\ z = x$
and $Fmap\ snd\ snd\ snd\ z = y$
shows $Frel\ R1\ R2\ R3\ x\ y$
apply (*subst Frel-def*)
apply (*intro exI conjI CollectI*)
apply (*fact assms*)+
done

lemma *FrelE*:
assumes $Frel\ R1\ R2\ R3\ x\ y$
obtains z **where** $Fset1\ z \subseteq \{(x, y).\ R1\ x\ y\}\ Fset2\ z \subseteq \{(x, y).\ R2\ x\ y\}$

Fset3 z ⊆ {(x, y). R3 x y} Fmap fst fst fst z = x Fmap snd snd snd z = y
apply (*insert assms*)
apply (*subst* (*asm*) *Frel-def*)
apply (*elim exE CollectE conjE*)
apply *assumption*
done

lemma *Grp-UNIV-Fmap-eq-Frel-Grp*: *BNF-Def.Grp UNIV* (*Fmap f1 f2 f3*) =
Frel (*BNF-Def.Grp UNIV f1*) (*BNF-Def.Grp UNIV f2*) (*BNF-Def.Grp UNIV f3*)
apply (*intro ext iffI*)
apply (*rule FrelI*[**where**
?z=Fmap (*BNF-Def.convol id f1*) (*BNF-Def.convol id f2*) (*BNF-Def.convol id f3*) -])
apply (*subst Fset-Fmap-eqs*,
rule image-subsetI,
rule convol-mem-GrpI[*simplified Fun-id-eq-id*],
rule UNIV-I)+
apply (*unfold Fmap-comp-eq-Fmap-Fmap*[*symmetric*]
fst-convol[*simplified Fun-comp-eq-comp*]
snd-convol[*simplified Fun-comp-eq-comp*]
Fmap-id id-eq-self)
apply (*rule refl*)
apply (*subst* (*asm*) *Grp-UNIV-eq-eq-comp*)
apply (*subst* (*asm*) *comp-eq*)
apply *assumption*
apply (*erule FrelE*)
apply *hypsubst*
apply (*subst Grp-UNIV-eq-eq-comp*)
apply (*subst comp-eq*)
apply (*subst Fmap-comp-eq-Fmap-Fmap*[*symmetric*])
apply (*rule Fmap-cong*;
rule Collect-case-prod-Grp-eqD[*simplified Fun-comp-eq-comp*],
drule rev-subsetD,
assumption+)
done

lemma *Frel-Grp-UNIV-Fmap*:
Frel (*BNF-Def.Grp UNIV f1*) (*BNF-Def.Grp UNIV f2*) (*BNF-Def.Grp UNIV f3*)
x (*Fmap f1 f2 f3 x*)
apply (*subst Grp-UNIV-Fmap-eq-Frel-Grp*[*symmetric*])
apply (*subst Grp-UNIV-eq-eq-comp*)
apply (*subst comp-eq*)
apply (*rule refl*)
done

lemma *Frel-Grp-UNIV-iff-eq-Fmap*:
Frel (*BNF-Def.Grp UNIV f1*) (*BNF-Def.Grp UNIV f2*) (*BNF-Def.Grp UNIV*

f3) *x y* ⟷
(*y* = *Fmap f1 f2 f3 x*)
by (*subst eq-commute*[*of y*])
(*fact fun-cong*[*OF fun-cong*[*OF Grp-UNIV-Fmap-eq-Frel-Grp*],
simplified Grp-UNIV-eq-eq-comp comp-eq, *folded Grp-UNIV-eq-eq-comp*, *symmetric*])

lemma *Frel-eq*: *Frel* (=) (=) (=) = (=)
apply (*unfold BNF-Def.eq-alt*[*simplified Fun-id-eq-id*])
apply (*subst Grp-UNIV-Fmap-eq-Frel-Grp*[*symmetric*])
apply (*subst Fmap-id*)
apply (*fold BNF-Def.eq-alt*[*simplified Fun-id-eq-id*])
apply (*rule refl*)
done

corollary *Frel-eq-self*: *Frel* (=) (=) (=) *x x*
by (*fact iffD2*[*OF fun-cong*[*OF fun-cong*[*OF Frel-eq*]] *refl*])

lemma *Frel-mono-strong*:
assumes *Frel R1 R2 R3 x y*
and ⋀*x1 y1*. *x1* ∈ *Fset1 x* ⟹ *y1* ∈ *Fset1 y* ⟹ *R1 x1 y1* ⟹ *S1 x1 y1*
and ⋀*x2 y2*. *x2* ∈ *Fset2 x* ⟹ *y2* ∈ *Fset2 y* ⟹ *R2 x2 y2* ⟹ *S2 x2 y2*
and ⋀*x3 y3*. *x3* ∈ *Fset3 x* ⟹ *y3* ∈ *Fset3 y* ⟹ *R3 x3 y3* ⟹ *S3 x3 y3*
shows *Frel S1 S2 S3 x y*
apply (*insert assms*(*1*))
apply (*erule FrelE*)
apply (*rule FrelI*)
apply (*rule subsetI*,
frule rev-subsetD,
assumption,
frule imageI[*of - Fset1 - fst*]
imageI[*of - Fset2 - fst*]
imageI[*of - Fset3 - fst*],
drule imageI[*of - Fset1 - snd*]
imageI[*of - Fset2 - snd*]
imageI[*of - Fset3 - snd*],
(*subst* (*asm*) *Fset-Fmap-eqs*[*symmetric*])+,
intro CollectI case-prodI2,
rule assms;
hypsubst,
unfold fst-conv snd-conv,
(*elim CollectE case-prodE Pair-inject*, *hypsubst*)*?*,
assumption)+
apply *assumption*+
done

corollary *Frel-mono*:
assumes *R1* ≤ *S1 R2* ≤ *S2 R3* ≤ *S3*
shows *Frel R1 R2 R3* ≤ *Frel S1 S2 S3*

```
  apply (intro le-relI)
  apply (rule Frel-mono-strong)
   apply assumption
   apply (insert assms)
   apply (drule le-relD[OF assms(1)] le-relD[OF assms(2)] le-relD[OF assms(3)],
     assumption)+
  done

lemma Frel-refl-strong:
  assumes ⋀x1. x1 ∈ Fset1 x ⟹ R1 x1 x1
  and ⋀x2. x2 ∈ Fset2 x ⟹ R2 x2 x2
  and ⋀x3. x3 ∈ Fset3 x ⟹ R3 x3 x3
  shows Frel R1 R2 R3 x x
  by (rule Frel-mono-strong[OF Frel-eq-self[of x]];
    drule assms, hypsubst, assumption)

lemma Frel-cong:
  assumes ⋀x1 y1. x1 ∈ Fset1 x ⟹ y1 ∈ Fset1 y ⟹ R1 x1 y1 ⟷ R1' x1 y1
  and ⋀x2 y2. x2 ∈ Fset2 x ⟹ y2 ∈ Fset2 y ⟹ R2 x2 y2 ⟷ R2' x2 y2
  and ⋀x3 y3. x3 ∈ Fset3 x ⟹ y3 ∈ Fset3 y ⟹ R3 x3 y3 ⟷ R3' x3 y3
  shows Frel R1 R2 R3 x y = Frel R1' R2' R3' x y
  by (rule iffI;
    rule Frel-mono-strong,
    assumption;
    rule iffD1[OF assms(1)] iffD1[OF assms(2)] iffD1[OF assms(3)]
      iffD2[OF assms(1)] iffD2[OF assms(2)] iffD2[OF assms(3)];
    assumption)

lemma Frel-rel-inv-eq-rel-inv-Frel: Frel R1^-1 R2^-1 R3^-1 = (Frel R1 R2 R3)^-1
  by (intro ext iffI;
    unfold rel-inv-iff-rel,
    erule FrelE,
    hypsubst,
    rule FrelI[where ?z=Fmap prod.swap prod.swap prod.swap -];
      ((subst Fset-Fmap-eqs,
      rule image-subsetI,
      drule rev-subsetD,
      assumption,
      elim CollectE case-prodE,
      hypsubst,
      subst swap-simp,
      subst pair-mem-Collect-case-prod-iff,
      assumption) |
      (subst Fmap-comp-eq-Fmap-Fmap[symmetric],
        rule Fmap-cong;
        unfold comp-eq fst-swap snd-swap,
        rule refl)))
```

Given the former axioms, the following axiom - subdistributivity of the relator - is equivalent to the (F, Fmap) functor preserving weak pullbacks.

axiomatization
where *Frel-comp-le-Frel-rel-comp*: $\bigwedge R1\ R2\ R3\ S1\ S2\ S3$.
Frel R1 R2 R3 $\circ\circ$ *Frel S1 S2 S3* $\leq$ *Frel* (*R1* $\circ\circ$ *S1*) (*R2* $\circ\circ$ *S2*) (*R3* $\circ\circ$ *S3*)

lemma *fst-sndOp-eq-snd-fstOp*: *fst* $\circ$ *BNF-Def.sndOp P Q* = *snd* $\circ$ *BNF-Def.fstOp P Q*
unfolding *fstOp-def sndOp-def* **by** (*intro ext*) *simp*

lemma *Frel-rel-comp-le-Frel-comp*:
Frel (*R1* $\circ\circ$ *S1*) (*R2* $\circ\circ$ *S2*) (*R3* $\circ\circ$ *S3*) $\leq$ (*Frel R1 R2 R3* $\circ\circ$ *Frel S1 S2 S3*)
apply (*rule le-relI*)
apply (*erule FrelE*)
apply (*rule rel-compI*[**where** *?y=Fmap* (*snd* $\circ$ *BNF-Def.fstOp R1 S1*)
(*snd* $\circ$ *BNF-Def.fstOp R2 S2*) (*snd* $\circ$ *BNF-Def.fstOp R3 S3*) -])
apply (*rule FrelI*[**where** *?z=Fmap* (*BNF-Def.fstOp R1 S1*)
(*BNF-Def.fstOp R2 S2*) (*BNF-Def.fstOp R3 S3*) -])
apply (*subst Fset-Fmap-eqs*,
intro image-subsetI,
rule fstOp-in[*unfolded relcompp-eq-rel-comp*],
drule rev-subsetD,
assumption+)+
apply (*subst Fmap-comp-eq-Fmap-Fmap*[*symmetric*])
apply (*fold ext*[*of fst*, *OF fst-fstOp*[*unfolded Fun-comp-eq-comp*]])
apply *hypsubst*
apply (*rule refl*)
apply (*subst Fmap-comp-eq-Fmap-Fmap*[*symmetric*])
apply (*rule refl*)
apply (*rule FrelI*[**where** *?z=Fmap* (*BNF-Def.sndOp R1 S1*)
(*BNF-Def.sndOp R2 S2*) (*BNF-Def.sndOp R3 S3*) -])
apply (*subst Fset-Fmap-eqs*,
intro image-subsetI,
rule sndOp-in[*unfolded relcompp-eq-rel-comp*],
drule rev-subsetD,
assumption+)+
apply (*subst Fmap-comp-eq-Fmap-Fmap*[*symmetric*])
apply (*unfold fst-sndOp-eq-snd-fstOp*)
apply (*rule refl*)
apply (*subst Fmap-comp-eq-Fmap-Fmap*[*symmetric*])
apply (*fold ext*[*of snd*, *OF snd-sndOp*[*unfolded Fun-comp-eq-comp*]])
apply *hypsubst*
apply (*rule refl*)
done

corollary *Frel-comp-eq-Frel-rel-comp*:
Frel R1 R2 R3 $\circ\circ$ *Frel S1 S2 S3* = *Frel* (*R1* $\circ\circ$ *S1*) (*R2* $\circ\circ$ *S2*) (*R3* $\circ\circ$ *S3*)
by (*rule antisym*; *rule Frel-comp-le-Frel-rel-comp Frel-rel-comp-le-Frel-comp*)

lemma *Frel-Fmap-eq1*: *Frel R1 R2 R3* (*Fmap f1 f2 f3 x*) *y* =
Frel ($\lambda x.$ *R1* (*f1 x*)) ($\lambda x.$ *R2* (*f2 x*)) ($\lambda x.$ *R3* (*f3 x*)) *x y*

apply (*rule iffI*)
apply (*fold comp-eq*[*of R1*] *comp-eq*[*of R2*] *comp-eq*[*of R3*])
apply (*drule rel-compI*[**where** *?R=Frel - - -* **and** *?S=Frel - - -*,
OF Frel-Grp-UNIV-Fmap])
apply (*unfold Grp-UNIV-eq-eq-comp*)
apply (*drule le-relD*[*OF Frel-comp-le-Frel-rel-comp*])
apply (*unfold eq-comp-rel-comp-eq-comp*)
apply *assumption*
apply (*fold eq-comp-rel-comp-eq-comp*[**where** *?R=R1*]
eq-comp-rel-comp-eq-comp[**where** *?R=R2*]
eq-comp-rel-comp-eq-comp[**where** *?R=R3*]
Grp-UNIV-eq-eq-comp)
apply (*drule le-relD*[*OF Frel-rel-comp-le-Frel-comp*])
apply (*erule rel-compE*)
apply (*subst* (*asm*) *Frel-Grp-UNIV-iff-eq-Fmap*)
apply *hypsubst*
apply *assumption*
done

lemma *Frel-Fmap-eq2*: *Frel R1 R2 R3 x* (*Fmap g1 g2 g3 y*) =
Frel (λ*x y*. *R1 x* (*g1 y*)) (λ*x y*. *R2 x* (*g2 y*)) (λ*x y*. *R3 x* (*g3 y*)) *x y*
apply (*subst rel-inv-iff-rel*[*of Frel - - -*, *symmetric*])
apply (*subst Frel-rel-inv-eq-rel-inv-Frel*[*symmetric*])
apply (*subst Frel-Fmap-eq1*)
apply (*rule sym*)
apply (*subst rel-inv-iff-rel*[*of Frel - - -*, *symmetric*])
apply (*subst Frel-rel-inv-eq-rel-inv-Frel*[*symmetric*])
apply (*unfold rel-inv-iff-rel*)
apply (*rule refl*)
done

lemmas *Frel-Fmap-eqs* = *Frel-Fmap-eq1 Frel-Fmap-eq2*

Predicator **definition** *Fpred* :: ('*a* ⇒ *bool*) ⇒ ('*b* ⇒ *bool*) ⇒ ('*c* ⇒ *bool*) ⇒
('*d*, '*a*, '*b*, '*c*) *F* ⇒ *bool*
where *Fpred P1 P2 P3 x* ≡ *Frel* ((=)↾$_{P1}$) ((=)↾$_{P2}$) ((=)↾$_{P3}$) *x x*

lemma *Fpred-mono-strong*:
assumes *Fpred P1 P2 P3 x*
and ⋀*x1*. *x1* ∈ *Fset1 x* ⟹ *P1 x1* ⟹ *Q1 x1*
and ⋀*x2*. *x2* ∈ *Fset2 x* ⟹ *P2 x2* ⟹ *Q2 x2*
and ⋀*x3*. *x3* ∈ *Fset3 x* ⟹ *P3 x3* ⟹ *Q3 x3*
shows *Fpred Q1 Q2 Q3 x*
apply (*insert assms*(*1*))
apply (*unfold Fpred-def*)
apply (*rule Frel-mono-strong*,
assumption;
erule rel-restrict-leftE,
rule rel-restrict-leftI,

assumption,
rule assms,
assumption+)
done

lemma *Fpred-top*: *Fpred* ⊤ ⊤ ⊤ *x*
apply (*subst Fpred-def*)
apply (*rule Frel-refl-strong*;
subst rel-restrict-left-top-eq,
rule refl)
done

lemma *FpredI*:
assumes ⋀*x1*. *x1* ∈ *Fset1 x* ⟹ *P1 x1*
and ⋀*x2*. *x2* ∈ *Fset2 x* ⟹ *P2 x2*
and ⋀*x3*. *x3* ∈ *Fset3 x* ⟹ *P3 x3*
shows *Fpred P1 P2 P3 x*
using *assms* **by** (*rule Fpred-mono-strong*[*OF Fpred-top*])

lemma *FpredE*:
assumes *Fpred P1 P2 P3 x*
obtains ⋀*x1*. *x1* ∈ *Fset1 x* ⟹ *P1 x1*
⋀*x2*. *x2* ∈ *Fset2 x* ⟹ *P2 x2*
⋀*x3*. *x3* ∈ *Fset3 x* ⟹ *P3 x3*
by (*elim meta-impE*; (*assumption* |
insert assms,
subst (*asm*) *Fpred-def*,
erule FrelE,
hypsubst,
subst (*asm*) *Fset-Fmap-eqs*,
subst (*asm*) *Domain-fst*[*symmetric*],
drule rev-subsetD,
rule Domain-mono,
assumption,
unfold Domain-Collect-case-prod-eq-Collect-in-dom in-dom-rel-restrict-left-eq,
elim CollectE inf1E,
assumption))

lemma *Fpred-eq-ball*: *Fpred P1 P2 P3* =
(λ*x*. *Ball* (*Fset1 x*) *P1* ∧ *Ball* (*Fset2 x*) *P2* ∧ *Ball* (*Fset3 x*) *P3*)
by (*intro ext iffI conjI Set.ballI FpredI*; *elim FpredE conjE bspec*)

lemma *Fpred-Fmap-eq*:
Fpred P1 P2 P3 (*Fmap f1 f2 f3 x*) = *Fpred* (*P1* ∘ *f1*) (*P2* ∘ *f2*) (*P3* ∘ *f3*) *x*
by (*unfold Fpred-def Frel-Fmap-eqs*)
(*rule iffI*;
erule FrelE,
hypsubst,
unfold Frel-Fmap-eqs,

rule Frel-refl-strong;
rule rel-restrict-leftI,
rule refl,
drule rev-subsetD,
assumption,
elim CollectE case-prodE rel-restrict-leftE,
hypsubst,
unfold comp-eq fst-conv,
assumption)

lemma *Fpred-in-dom-if-in-dom-Frel*:
assumes *in-dom* (*Frel R1 R2 R3*) *x*
shows *Fpred* (*in-dom R1*) (*in-dom R2*) (*in-dom R3*) *x*
apply (*insert assms*)
apply (*elim in-domE FrelE*)
apply *hypsubst*
apply (*subst Fpred-Fmap-eq*)
apply (*rule FpredI*;
drule rev-subsetD,
assumption,
elim CollectE case-prodE,
hypsubst,
unfold comp-eq fst-conv,
rule in-domI,
assumption)
done

lemma *in-dom-Frel-if-Fpred-in-dom*:
assumes *Fpred* (*in-dom R1*) (*in-dom R2*) (*in-dom R3*) *x*
shows *in-dom* (*Frel R1 R2 R3*) *x*
apply (*insert assms*)
apply (*subst* (*asm*) *Fpred-eq-ball*)
apply (*elim conjE*)
apply (*subst* (*asm*) *ball-in-dom-iff-ball-ex*,
drule bchoice, — requires the axiom of choice.
erule exE)+
apply (*rule in-domI*[**where** *?x=x* **and** *?y=Fmap - - - x* **for** *x*])
apply (*subst Frel-Fmap-eq2*)
apply (*rule Frel-refl-strong*)
apply (*drule bspec*[*of Fset1* -])
apply *assumption*+
apply (*drule bspec*[*of Fset2* -])
apply *assumption*+
apply (*drule bspec*[*of Fset3* -])
apply *assumption*+
done

lemma *in-dom-Frel-eq-Fpred-in-dom*:
in-dom (*Frel R1 R2 R3*) = *Fpred* (*in-dom R1*) (*in-dom R2*) (*in-dom R3*)

by (*intro ext iffI*; *rule Fpred-in-dom-if-in-dom-Frel in-dom-Frel-if-Fpred-in-dom*)

lemma *in-codom-Frel-eq-Fpred-in-codom*:
in-codom (*Frel R1 R2 R3*) = *Fpred* (*in-codom R1*) (*in-codom R2*) (*in-codom R3*)
apply (*subst in-dom-rel-inv-eq-in-codom*[*symmetric*])
apply (*subst Frel-rel-inv-eq-rel-inv-Frel*[*symmetric*])
apply (*subst in-dom-Frel-eq-Fpred-in-dom*)
apply (*subst in-dom-rel-inv-eq-in-codom*)+
apply (*rule refl*)
done

lemma *in-field-Frel-eq-Fpred-in-in-field*:
in-field (*Frel R1 R2 R3*) =
Fpred (*in-dom R1*) (*in-dom R2*) (*in-dom R3*) ⊔
Fpred (*in-codom R1*) (*in-codom R2*) (*in-codom R3*)
apply (*subst in-field-eq-in-dom-sup-in-codom*)
apply (*subst in-dom-Frel-eq-Fpred-in-dom*)
apply (*subst in-codom-Frel-eq-Fpred-in-codom*)
apply (*rule refl*)
done

lemma *Frel-restrict-left-Fpred-eq-Frel-restrict-left*:
fixes *R1* :: *'a1* ⇒ *'a2* ⇒ *bool*
and *R2* :: *'b1* ⇒ *'b2* ⇒ *bool*
and *R3* :: *'c1* ⇒ *'c2* ⇒ *bool*
and *P1* :: *'a1* ⇒ *bool*
and *P2* :: *'b1* ⇒ *bool*
and *P3* :: *'c1* ⇒ *bool*
shows (*Frel R1 R2 R3* :: (*'d*, *'a1*, *'b1*, *'c1*) *F* ⇒ -)↾$_{Fpred\ P1\ P2\ P3}$ :: (*'d*, *'a1*, *'b1*, *'c1*) *F* ⇒ -
=
Frel (*R1*↾$_{P1}$) (*R2*↾$_{P2}$) (*R3*↾$_{P3}$)
apply (*intro ext*)
apply (*rule iffI*)
apply (*erule rel-restrict-leftE*)
apply (*elim FpredE*)
apply (*rule Frel-mono-strong*,
assumption;
rule rel-restrict-leftI,
assumption+)
apply (*rule rel-restrict-leftI*)
apply (*rule Frel-mono-strong*,
assumption;
erule rel-restrict-leftE,
assumption)
apply (*drule in-domI*[*of Frel* (*R1*↾$_{P1}$) (*R2*↾$_{P2}$) (*R3*↾$_{P3}$)])
apply (*drule Fpred-in-dom-if-in-dom-Frel*)
apply (*rule Fpred-mono-strong*,
assumption;
unfold in-dom-rel-restrict-left-eq inf-apply inf-bool-def;

rule conjunct2,
assumption)
done

lemma *Frel-restrict-right-Fpred-eq-Frel-restrict-right*:
fixes *R1* :: *'a1* ⇒ *'a2* ⇒ *bool*
and *R2* :: *'b1* ⇒ *'b2* ⇒ *bool*
and *R3* :: *'c1* ⇒ *'c2* ⇒ *bool*
and *P1* :: *'a2* ⇒ *bool*
and *P2* :: *'b2* ⇒ *bool*
and *P3* :: *'c2* ⇒ *bool*
shows (*Frel R1 R2 R3* :: - ⇒ (*'d*, *'a2*, *'b2*, *'c2*) *F* ⇒ -)⌈$_{Fpred\ P1\ P2\ P3}$ :: (*'d*, *'a2*, *'b2*, *'c2*) *F* ⇒ -
=
Frel (*R1*⌈$_{P1}$) (*R2*⌈$_{P2}$) (*R3*⌈$_{P3}$)
apply (*subst rel-restrict-right-eq*)
apply (*subst Frel-rel-inv-eq-rel-inv-Frel*[*symmetric*])
apply (*subst Frel-restrict-left-Fpred-eq-Frel-restrict-left*)
apply (*subst Frel-rel-inv-eq-rel-inv-Frel*[*symmetric*])
apply (*fold rel-restrict-right-eq*)
apply (*rule refl*)
done

locale *transport-natural-functor* =
t1 : *transport L1 R1 l1 r1* + *t2* : *transport L2 R2 l2 r2* +
t3 : *transport L3 R3 l3 r3*
for *L1* :: *'a1* ⇒ *'a1* ⇒ *bool*
and *R1* :: *'b1* ⇒ *'b1* ⇒ *bool*
and *l1* :: *'a1* ⇒ *'b1*
and *r1* :: *'b1* ⇒ *'a1*
and *L2* :: *'a2* ⇒ *'a2* ⇒ *bool*
and *R2* :: *'b2* ⇒ *'b2* ⇒ *bool*
and *l2* :: *'a2* ⇒ *'b2*
and *r2* :: *'b2* ⇒ *'a2*
and *L3* :: *'a3* ⇒ *'a3* ⇒ *bool*
and *R3* :: *'b3* ⇒ *'b3* ⇒ *bool*
and *l3* :: *'a3* ⇒ *'b3*
and *r3* :: *'b3* ⇒ *'a3*
begin

notation *L1* (**infix** ‹≤$_{L1}$› *50*)
notation *R1* (**infix** ‹≤$_{R1}$› *50*)
notation *L2* (**infix** ‹≤$_{L2}$› *50*)
notation *R2* (**infix** ‹≤$_{R2}$› *50*)
notation *L3* (**infix** ‹≤$_{L3}$› *50*)
notation *R3* (**infix** ‹≤$_{R3}$› *50*)

notation *t1.ge-left* (**infix** ‹≥$_{L1}$› *50*)
notation *t1.ge-right* (**infix** ‹≥$_{R1}$› *50*)
notation *t2.ge-left* (**infix** ‹≥$_{L2}$› *50*)

notation *t2.ge-right* (**infix** ‹$\geq_{R2}$› *50*)
notation *t3.ge-left* (**infix** ‹$\geq_{L3}$› *50*)
notation *t3.ge-right* (**infix** ‹$\geq_{R3}$› *50*)

notation *t1.left-Galois* (**infix** ‹$_{L1}\lessapprox$› *50*)
notation *t1.right-Galois* (**infix** ‹$_{R1}\lessapprox$› *50*)
notation *t2.left-Galois* (**infix** ‹$_{L2}\lessapprox$› *50*)
notation *t2.right-Galois* (**infix** ‹$_{R2}\lessapprox$› *50*)
notation *t3.left-Galois* (**infix** ‹$_{L3}\lessapprox$› *50*)
notation *t3.right-Galois* (**infix** ‹$_{R3}\lessapprox$› *50*)

notation *t1.ge-Galois-left* (**infix** ‹$\gtrapprox_{L1}$› *50*)
notation *t1.ge-Galois-right* (**infix** ‹$\gtrapprox_{R1}$› *50*)
notation *t2.ge-Galois-left* (**infix** ‹$\gtrapprox_{L2}$› *50*)
notation *t2.ge-Galois-right* (**infix** ‹$\gtrapprox_{R2}$› *50*)
notation *t3.ge-Galois-left* (**infix** ‹$\gtrapprox_{L3}$› *50*)
notation *t3.ge-Galois-right* (**infix** ‹$\gtrapprox_{R3}$› *50*)

notation *t1.right-ge-Galois* (**infix** ‹$_{R1}\gtrapprox$› *50*)
notation *t1.Galois-right* (**infix** ‹$\lessapprox_{R1}$› *50*)
notation *t2.right-ge-Galois* (**infix** ‹$_{R2}\gtrapprox$› *50*)
notation *t2.Galois-right* (**infix** ‹$\lessapprox_{R2}$› *50*)
notation *t3.right-ge-Galois* (**infix** ‹$_{R3}\gtrapprox$› *50*)
notation *t3.Galois-right* (**infix** ‹$\lessapprox_{R3}$› *50*)

notation *t1.left-ge-Galois* (**infix** ‹$_{L1}\gtrapprox$› *50*)
notation *t1.Galois-left* (**infix** ‹$\lessapprox_{L1}$› *50*)
notation *t2.left-ge-Galois* (**infix** ‹$_{L2}\gtrapprox$› *50*)
notation *t2.Galois-left* (**infix** ‹$\lessapprox_{L2}$› *50*)
notation *t3.left-ge-Galois* (**infix** ‹$_{L3}\gtrapprox$› *50*)
notation *t3.Galois-left* (**infix** ‹$\lessapprox_{L3}$› *50*)

notation *t1.unit* (‹η_1›)
notation *t1.counit* (‹ε_1›)
notation *t2.unit* (‹η_2›)
notation *t2.counit* (‹ε_2›)
notation *t3.unit* (‹η_3›)
notation *t3.counit* (‹ε_3›)

definition $L \equiv$ *Frel* ($\leq_{L1}$) ($\leq_{L2}$) ($\leq_{L3}$)

lemma *left-rel-eq-Frel*: $L =$ *Frel* ($\leq_{L1}$) ($\leq_{L2}$) ($\leq_{L3}$)
  unfolding *L-def* **..**

definition $l \equiv$ *Fmap l1 l2 l3*

lemma *left-eq-Fmap*: $l =$ *Fmap l1 l2 l3*
  unfolding *l-def* **..**

context
begin

interpretation *flip* :
transport-natural-functor R1 L1 r1 l1 R2 L2 r2 l2 R3 L3 r3 l3 **.**

abbreviation $R \equiv$ *flip.L*
abbreviation $r \equiv$ *flip.l*

lemma *right-rel-eq-Frel*: $R =$ *Frel* $(\leq_{R1})$ $(\leq_{R2})$ $(\leq_{R3})$
unfolding *flip.left-rel-eq-Frel* **..**

lemma *right-eq-Fmap*: $r =$ *Fmap r1 r2 r3*
unfolding *flip.left-eq-Fmap* **..**

lemmas *transport-defs* = *left-rel-eq-Frel left-eq-Fmap*
right-rel-eq-Frel right-eq-Fmap

end

sublocale *transport L R l r* **.**

notation L (**infix** $\langle\leq_L\rangle$ *50*)
notation R (**infix** $\langle\leq_R\rangle$ *50*)

lemma *unit-eq-Fmap*: $\eta =$ *Fmap* η_1 η_2 η_3
unfolding *unit-eq-comp* **by** (*simp only*: *right-eq-Fmap left-eq-Fmap*
flip: *Fmap-comp t1.unit-eq-comp t2.unit-eq-comp t3.unit-eq-comp*)

interpretation *flip-inv* : *transport-natural-functor* $(\geq_{R1})$ $(\geq_{L1})$ *r1 l1*
$(\geq_{R2})$ $(\geq_{L2})$ *r2 l2* $(\geq_{R3})$ $(\geq_{L3})$ *r3 l3*
rewrites *flip-inv.unit* $\equiv \varepsilon$ **and** *flip-inv.t1.unit* $\equiv \varepsilon_1$
and *flip-inv.t2.unit* $\equiv \varepsilon_2$ **and** *flip-inv.t3.unit* $\equiv \varepsilon_3$
by (*simp-all only*: *order-functors.flip-counit-eq-unit*)

lemma *counit-eq-Fmap*: $\varepsilon =$ *Fmap* ε_1 ε_2 ε_3
by (*fact flip-inv.unit-eq-Fmap*)

lemma *flip-inv-right-eq-ge-left*: *flip-inv.R* $= (\geq_L)$
unfolding *left-rel-eq-Frel flip-inv.right-rel-eq-Frel*
by (*fact Frel-rel-inv-eq-rel-inv-Frel*)

interpretation *flip* :
transport-natural-functor R1 L1 r1 l1 R2 L2 r2 l2 R3 L3 r3 l3 **.**

lemma *flip-inv-left-eq-ge-right*: *flip-inv.L* $\equiv (\geq_R)$
unfolding *flip.flip-inv-right-eq-ge-left* **.**

lemma *mono-wrt-rel-leftI*:
assumes $((\leq_{L1}) \Rightarrow (\leq_{R1}))$ *l1*
and $((\leq_{L2}) \Rightarrow (\leq_{R2}))$ *l2*
and $((\leq_{L3}) \Rightarrow (\leq_{R3}))$ *l3*
shows $((\leq_L) \Rightarrow (\leq_R))$ *l*
apply (*unfold left-rel-eq-Frel right-rel-eq-Frel left-eq-Fmap*)
apply (*rule mono-wrt-relI*)
apply (*unfold Frel-Fmap-eqs*)
apply (*fold rel-map-eq*)
apply (*rule le-relD*[*OF Frel-mono*])
apply (*subst mono-wrt-rel-iff-le-rel-map*[*symmetric*], *rule assms*)+
apply *assumption*
done

end

end

2.28.2 Galois Concepts

theory *Transport-Natural-Functors-Galois*
imports
Transport-Natural-Functors-Base
begin

context *transport-natural-functor*
begin

lemma *half-galois-prop-leftI*:
assumes $((\leq_{L1})\ {}_h\trianglelefteq\ (\leq_{R1}))$ *l1 r1*
and $((\leq_{L2})\ {}_h\trianglelefteq\ (\leq_{R2}))$ *l2 r2*
and $((\leq_{L3})\ {}_h\trianglelefteq\ (\leq_{R3}))$ *l3 r3*
shows $((\leq_L)\ {}_h\trianglelefteq\ (\leq_R))$ *l r*
apply (*rule half-galois-prop-leftI*)
apply (*erule left-GaloisE*)
apply (*unfold left-rel-eq-Frel right-rel-eq-Frel left-eq-Fmap right-eq-Fmap*)
apply (*subst* (*asm*) *in-codom-Frel-eq-Fpred-in-codom*)
apply (*erule FpredE*)
apply (*unfold Frel-Fmap-eqs*)
apply (*rule Frel-mono-strong*,
assumption;
rule t1.half-galois-prop-leftD t2.half-galois-prop-leftD t3.half-galois-prop-leftD,
rule assms,
rule t1.left-GaloisI t2.left-GaloisI t3.left-GaloisI;
assumption)
done

interpretation *flip-inv* : *transport-natural-functor* $(\geq_{R1})$ $(\geq_{L1})$ *r1 l1*

$(\geq_{R2})$ $(\geq_{L2})$ *r2 l2* $(\geq_{R3})$ $(\geq_{L3})$ *r3 l3*
rewrites *flip-inv.R* $\equiv (\geq_L)$
and *flip-inv.L* $\equiv (\geq_R)$
and $\bigwedge R\ S\ f\ g.\ (R^{-1}\ {}_h\trianglelefteq\ S^{-1})\ f\ g \equiv (S\ \trianglelefteq_h\ R)\ g\ f$
by (*simp-all only*: *flip-inv-left-eq-ge-right flip-inv-right-eq-ge-left*
galois-prop.half-galois-prop-left-rel-inv-iff-half-galois-prop-right)

lemma *half-galois-prop-rightI*:
assumes $((\leq_{L1}) \trianglelefteq_h (\leq_{R1}))$ *l1 r1*
and $((\leq_{L2}) \trianglelefteq_h (\leq_{R2}))$ *l2 r2*
and $((\leq_{L3}) \trianglelefteq_h (\leq_{R3}))$ *l3 r3*
shows $((\leq_L) \trianglelefteq_h (\leq_R))$ *l r*
using *assms* **by** (*intro flip-inv.half-galois-prop-leftI*)

corollary *galois-propI*:
assumes $((\leq_{L1}) \trianglelefteq (\leq_{R1}))$ *l1 r1*
and $((\leq_{L2}) \trianglelefteq (\leq_{R2}))$ *l2 r2*
and $((\leq_{L3}) \trianglelefteq (\leq_{R3}))$ *l3 r3*
shows $((\leq_L) \trianglelefteq (\leq_R))$ *l r*
using *assms* **by** (*elim galois-prop.galois-propE*)
(*intro galois-propI half-galois-prop-leftI half-galois-prop-rightI*)

interpretation *flip* :
transport-natural-functor R1 L1 r1 l1 R2 L2 r2 l2 R3 L3 r3 l3 **.**

corollary *galois-connectionI*:
assumes $((\leq_{L1}) \dashv (\leq_{R1}))$ *l1 r1*
and $((\leq_{L2}) \dashv (\leq_{R2}))$ *l2 r2*
and $((\leq_{L3}) \dashv (\leq_{R3}))$ *l3 r3*
shows $((\leq_L) \dashv (\leq_R))$ *l r*
using *assms* **by** (*elim galois.galois-connectionE*) (*intro*
galois-connectionI galois-propI mono-wrt-rel-leftI flip.mono-wrt-rel-leftI)

corollary *galois-equivalenceI*:
assumes $((\leq_{L1}) \equiv_G (\leq_{R1}))$ *l1 r1*
and $((\leq_{L2}) \equiv_G (\leq_{R2}))$ *l2 r2*
and $((\leq_{L3}) \equiv_G (\leq_{R3}))$ *l3 r3*
shows $((\leq_L) \equiv_G (\leq_R))$ *l r*
using *assms* **by** (*elim galois.galois-equivalenceE flip.t1.galois-connectionE*
flip.t2.galois-connectionE flip.t3.galois-connectionE)
(*intro galois-equivalenceI galois-connectionI flip.galois-propI*)

end

end

2.28.3 Galois Relator

theory *Transport-Natural-Functors-Galois-Relator*
imports
Transport-Natural-Functors-Base
begin

context *transport-natural-functor*
begin

lemma *left-Galois-Frel-left-Galois*: $({}_{L}\lessapprox) \leq$ *Frel* $({}_{L1}\lessapprox)$ $({}_{L2}\lessapprox)$ $({}_{L3}\lessapprox)$
apply (*rule le-relI*)
apply (*erule left-GaloisE*)
apply (*unfold left-rel-eq-Frel right-rel-eq-Frel right-eq-Fmap*)
apply (*subst* (*asm*) *in-codom-Frel-eq-Fpred-in-codom*)
apply (*erule FpredE*)
apply (*subst* (*asm*) *Frel-Fmap-eq2*)
apply (*rule Frel-mono-strong*,
assumption;
rule t1.left-GaloisI t2.left-GaloisI t3.left-GaloisI;
assumption)
done

lemma *Frel-left-Galois-le-left-Galois*:
Frel $({}_{L1}\lessapprox)$ $({}_{L2}\lessapprox)$ $({}_{L3}\lessapprox) \leq ({}_{L}\lessapprox)$
apply (*rule le-relI*)
apply (*unfold t1.left-Galois-iff-in-codom-and-left-rel-right*
t2.left-Galois-iff-in-codom-and-left-rel-right
t3.left-Galois-iff-in-codom-and-left-rel-right)
apply (*fold*
rel-restrict-right-eq[*of* $\lambda x\ y.\ x \leq_{L1} r1\ y$ *in-codom* $(\leq_{R1})$,
unfolded rel-restrict-left-pred-def rel-inv-iff-rel]
rel-restrict-right-eq[*of* $\lambda x\ y.\ x \leq_{L2} r2\ y$ *in-codom* $(\leq_{R2})$,
unfolded rel-restrict-left-pred-def rel-inv-iff-rel]
rel-restrict-right-eq[*of* $\lambda x\ y.\ x \leq_{L3} r3\ y$ *in-codom* $(\leq_{R3})$,
unfolded rel-restrict-left-pred-def rel-inv-iff-rel])
apply (*subst* (*asm*) *Frel-restrict-right-Fpred-eq-Frel-restrict-right*[*symmetric*])
apply (*erule rel-restrict-rightE*)
apply (*subst* (*asm*) *in-codom-Frel-eq-Fpred-in-codom*[*symmetric*])
apply (*erule in-codomE*)
apply (*rule left-GaloisI*)
apply (*rule in-codomI*)
apply (*subst right-rel-eq-Frel*)
apply *assumption*
apply (*unfold left-rel-eq-Frel right-eq-Fmap Frel-Fmap-eq2*)
apply *assumption*
done

corollary *left-Galois-eq-Frel-left-Galois*: $({}_{L}\lessapprox)$ = *Frel* $({}_{L1}\lessapprox)$ $({}_{L2}\lessapprox)$ $({}_{L3}\lessapprox)$
by (*intro antisym left-Galois-Frel-left-Galois Frel-left-Galois-le-left-Galois*)

end

end

2.28.4 Basic Order Properties

theory *Transport-Natural-Functors-Order-Base*
imports
Transport-Natural-Functors-Base
begin

context
fixes *R1* :: $'a \Rightarrow 'a \Rightarrow bool$ **and** *R2* :: $'b \Rightarrow 'b \Rightarrow bool$ **and** *R3* :: $'c \Rightarrow 'c \Rightarrow bool$
begin

lemma *reflexive-on-in-field-FrelI*:
assumes *reflexive-on* (*in-field R1*) *R1*
and *reflexive-on* (*in-field R2*) *R2*
and *reflexive-on* (*in-field R3*) *R3*
defines $R \equiv$ *Frel R1 R2 R3*
shows *reflexive-on* (*in-field R*) *R*
apply (*subst reflexive-on-iff-eq-restrict-le*)
apply (*subst Frel-eq*[*symmetric*])
apply (*unfold R-def*)
apply (*subst in-field-Frel-eq-Fpred-in-in-field*)
apply (*subst rel-restrict-left-sup-eq*)
apply (*subst Frel-restrict-left-Fpred-eq-Frel-restrict-left*)+
apply (*rule le-supI*;
rule Frel-mono;
subst reflexive-on-iff-eq-restrict-le[*symmetric*],
rule reflexive-on-if-le-pred-if-reflexive-on,
rule assms,
rule le-predI[*OF in-field-if-in-dom*]
le-predI[*OF in-field-if-in-codom*],
assumption)
done

lemma *transitive-FrelI*:
assumes *transitive R1*
and *transitive R2*
and *transitive R3*
shows *transitive* (*Frel R1 R2 R3*)
apply (*subst transitive-iff-rel-comp-le-self*)
apply (*subst Frel-comp-eq-Frel-rel-comp*)
apply (*rule Frel-mono*;
subst transitive-iff-rel-comp-le-self[*symmetric*],
rule assms)

done

lemma *preorder-on-in-field-FrelI*:
assumes *preorder-on* (*in-field R1*) *R1*
and *preorder-on* (*in-field R2*) *R2*
and *preorder-on* (*in-field R3*) *R3*
defines $R \equiv$ *Frel R1 R2 R3*
shows *preorder-on* (*in-field R*) *R*
apply (*unfold R-def*)
apply (*insert assms*)
apply (*elim preorder-on-in-fieldE*)
apply (*rule preorder-onI*)
apply (*rule reflexive-on-in-field-FrelI*; *assumption*)
apply (*subst transitive-on-in-field-iff-transitive*)
apply (*rule transitive-FrelI*; *assumption*)
done

lemma *symmetric-FrelI*:
assumes *symmetric R1*
and *symmetric R2*
and *symmetric R3*
shows *symmetric* (*Frel R1 R2 R3*)
apply (*subst symmetric-iff-rel-inv-eq-self*)
apply (*subst Frel-rel-inv-eq-rel-inv-Frel*[*symmetric*])
apply (*subst rel-inv-eq-self-if-symmetric*, *fact*)+
apply (*rule refl*)
done

lemma *partial-equivalence-rel-FrelI*:
assumes *partial-equivalence-rel R1*
and *partial-equivalence-rel R2*
and *partial-equivalence-rel R3*
shows *partial-equivalence-rel* (*Frel R1 R2 R3*)
apply (*insert assms*)
apply (*elim partial-equivalence-relE preorder-on-in-fieldE*)
apply (*rule partial-equivalence-relI*;
rule transitive-FrelI symmetric-FrelI;
assumption)
done

end

context *transport-natural-functor*
begin

lemmas *reflexive-on-in-field-leftI* = *reflexive-on-in-field-FrelI*
[*of L1 L2 L3*, *folded transport-defs*]

lemmas *transitive-leftI* = *transitive-FrelI*[*of L1 L2 L3*, *folded transport-defs*]

lemmas *preorder-on-in-field-leftI* = *preorder-on-in-field-FrelI*
[*of L1 L2 L3*, *folded transport-defs*]

lemmas *symmetricI* = *symmetric-FrelI*[*of L1 L2 L3*, *folded transport-defs*]

lemmas *partial-equivalence-rel-leftI* = *partial-equivalence-rel-FrelI*
[*of L1 L2 L3*, *folded transport-defs*]

end

end

2.28.5 Order Equivalence

theory *Transport-Natural-Functors-Order-Equivalence*
imports
Transport-Natural-Functors-Base
begin

context
fixes *R1* :: *'a* ⇒ *'a* ⇒ *bool* **and** *R2* :: *'b* ⇒ *'b* ⇒ *bool* **and** *R3* :: *'c* ⇒ *'c* ⇒ *bool*
and *f1* :: *'a* ⇒ *'a* **and** *f2* :: *'b* ⇒ *'b* **and** *f3* :: *'c* ⇒ *'c*
and *R* :: (*'d*, *'a*, *'b*, *'c*) *F* ⇒ (*'d*, *'a*, *'b*, *'c*) *F* ⇒ *bool*
and *f* :: (*'d*, *'a*, *'b*, *'c*) *F* ⇒ (*'d*, *'a*, *'b*, *'c*) *F*
defines *R* ≡ *Frel R1 R2 R3* **and** *f* ≡ *Fmap f1 f2 f3*
begin

lemma *inflationary-on-in-dom-FrelI*:
assumes *inflationary-on* (*in-dom R1*) *R1 f1*
and *inflationary-on* (*in-dom R2*) *R2 f2*
and *inflationary-on* (*in-dom R3*) *R3 f3*
shows *inflationary-on* (*in-dom R*) *R f*
apply (*unfold R-def f-def*)
apply (*rule inflationary-onI*)
apply (*subst* (*asm*) *in-dom-Frel-eq-Fpred-in-dom*)
apply (*erule FpredE*)
apply (*subst Frel-Fmap-eq2*)
apply (*rule Frel-refl-strong*)
apply (*rule inflationary-onD*[**where** *?R*=*R1*] *inflationary-onD*[**where** *?R*=*R2*]
inflationary-onD[**where** *?R*=*R3*],
rule assms,
assumption+)+
done

lemma *inflationary-on-in-codom-FrelI*:
assumes *inflationary-on* (*in-codom R1*) *R1 f1*
and *inflationary-on* (*in-codom R2*) *R2 f2*

and *inflationary-on* (*in-codom R3*) *R3 f3*
shows *inflationary-on* (*in-codom R*) *R f*
apply (*unfold R-def f-def*)
apply (*rule inflationary-onI*)
apply (*subst* (*asm*) *in-codom-Frel-eq-Fpred-in-codom*)
apply (*erule FpredE*)
apply (*subst Frel-Fmap-eq2*)
apply (*rule Frel-refl-strong*)
apply (*rule inflationary-onD*[**where** *?R=R1*] *inflationary-onD*[**where** *?R=R2*]
inflationary-onD[**where** *?R=R3*],
rule assms,
assumption+)+
done

end

context
fixes *R1* :: $'a \Rightarrow 'a \Rightarrow$ *bool* **and** *R2* :: $'b \Rightarrow 'b \Rightarrow$ *bool* **and** *R3* :: $'c \Rightarrow 'c \Rightarrow$ *bool*
and *f1* :: $'a \Rightarrow 'a$ **and** *f2* :: $'b \Rightarrow 'b$ **and** *f3* :: $'c \Rightarrow 'c$
and *R* :: $('d, 'a, 'b, 'c)\ F \Rightarrow ('d, 'a, 'b, 'c)\ F \Rightarrow$ *bool*
and *f* :: $('d, 'a, 'b, 'c)\ F \Rightarrow ('d, 'a, 'b, 'c)\ F$
defines *R* ≡ *Frel R1 R2 R3* **and** *f* ≡ *Fmap f1 f2 f3*
begin

lemma *inflationary-on-in-field-FrelI*:
assumes *inflationary-on* (*in-field R1*) *R1 f1*
and *inflationary-on* (*in-field R2*) *R2 f2*
and *inflationary-on* (*in-field R3*) *R3 f3*
shows *inflationary-on* (*in-field R*) *R f*
apply (*unfold R-def f-def*)
apply (*subst in-field-eq-in-dom-sup-in-codom*)
apply (*subst inflationary-on-sup-eq*)
apply (*unfold inf-apply*)
apply (*subst inf-bool-def*)
apply (*rule conjI*;
rule inflationary-on-in-dom-FrelI inflationary-on-in-codom-FrelI;
rule inflationary-on-if-le-pred-if-inflationary-on,
rule assms,
rule le-predI,
rule in-field-if-in-dom in-field-if-in-codom,
assumption)
done

lemma *deflationary-on-in-dom-FrelI*:
assumes *deflationary-on* (*in-dom R1*) *R1 f1*
and *deflationary-on* (*in-dom R2*) *R2 f2*
and *deflationary-on* (*in-dom R3*) *R3 f3*
shows *deflationary-on* (*in-dom R*) *R f*
apply (*unfold R-def f-def*)

apply (*subst deflationary-on-eq-inflationary-on-rel-inv*)
apply (*subst in-codom-rel-inv-eq-in-dom*[*symmetric*])
apply (*unfold Frel-rel-inv-eq-rel-inv-Frel*[*symmetric*])
apply (*rule inflationary-on-in-codom-FrelI*;
subst deflationary-on-eq-inflationary-on-rel-inv[*symmetric*],
subst in-codom-rel-inv-eq-in-dom,
rule assms)
done

lemma *deflationary-on-in-codom-FrelI*:
assumes *deflationary-on* (*in-codom R1*) *R1 f1*
and *deflationary-on* (*in-codom R2*) *R2 f2*
and *deflationary-on* (*in-codom R3*) *R3 f3*
shows *deflationary-on* (*in-codom R*) *R f*
apply (*unfold R-def f-def*)
apply (*subst deflationary-on-eq-inflationary-on-rel-inv*)
apply (*subst in-dom-rel-inv-eq-in-codom*[*symmetric*])
apply (*unfold Frel-rel-inv-eq-rel-inv-Frel*[*symmetric*])
apply (*rule inflationary-on-in-dom-FrelI*;
subst deflationary-on-eq-inflationary-on-rel-inv[*symmetric*],
subst in-dom-rel-inv-eq-in-codom,
rule assms)
done

end

context
fixes *R1* :: *'a* ⇒ *'a* ⇒ *bool* **and** *R2* :: *'b* ⇒ *'b* ⇒ *bool* **and** *R3* :: *'c* ⇒ *'c* ⇒ *bool*
and *f1* :: *'a* ⇒ *'a* **and** *f2* :: *'b* ⇒ *'b* **and** *f3* :: *'c* ⇒ *'c*
and *R* :: (*'d*, *'a*, *'b*, *'c*) *F* ⇒ (*'d*, *'a*, *'b*, *'c*) *F* ⇒ *bool*
and *f* :: (*'d*, *'a*, *'b*, *'c*) *F* ⇒ (*'d*, *'a*, *'b*, *'c*) *F*
defines *R* ≡ *Frel R1 R2 R3* **and** *f* ≡ *Fmap f1 f2 f3*
begin

lemma *deflationary-on-in-field-FrelI*:
assumes *deflationary-on* (*in-field R1*) *R1 f1*
and *deflationary-on* (*in-field R2*) *R2 f2*
and *deflationary-on* (*in-field R3*) *R3 f3*
shows *deflationary-on* (*in-field R*) *R f*
apply (*unfold R-def f-def*)
apply (*subst deflationary-on-eq-inflationary-on-rel-inv*)
apply (*subst in-field-rel-inv-eq-in-field*[*symmetric*])
apply (*unfold Frel-rel-inv-eq-rel-inv-Frel*[*symmetric*])
apply (*rule inflationary-on-in-field-FrelI*;
subst deflationary-on-eq-inflationary-on-rel-inv[*symmetric*],
subst in-field-rel-inv-eq-in-field,
rule assms)
done

lemma *rel-equivalence-on-in-field-FrelI*:
assumes *rel-equivalence-on* (*in-field R1*) *R1 f1*
and *rel-equivalence-on* (*in-field R2*) *R2 f2*
and *rel-equivalence-on* (*in-field R3*) *R3 f3*
shows *rel-equivalence-on* (*in-field R*) *R f*
apply (*unfold R-def f-def*)
apply (*subst rel-equivalence-on-eq*)
apply (*unfold inf-apply*)
apply (*subst inf-bool-def*)
apply (*insert assms*)
apply (*elim rel-equivalence-onE*)
apply (*rule conjI*)
apply (*rule inflationary-on-in-field-FrelI*; *assumption*)
apply (*fold R-def f-def*)
apply (*rule deflationary-on-in-field-FrelI*; *assumption*)
done

end

context *transport-natural-functor*
begin

lemmas *inflationary-on-in-field-unitI* = *inflationary-on-in-field-FrelI*
[*of L1* η_1 *L2* η_2 *L3* η_3, *folded transport-defs unit-eq-Fmap*]

lemmas *deflationary-on-in-field-unitI* = *deflationary-on-in-field-FrelI*
[*of L1* η_1 *L2* η_2 *L3* η_3, *folded transport-defs unit-eq-Fmap*]

lemmas *rel-equivalence-on-in-field-unitI* = *rel-equivalence-on-in-field-FrelI*
[*of L1* η_1 *L2* η_2 *L3* η_3, *folded transport-defs unit-eq-Fmap*]

interpretation *flip* :
transport-natural-functor R1 L1 r1 l1 R2 L2 r2 l2 R3 L3 r3 l3
rewrites *flip.unit* $\equiv \varepsilon$ **and** *flip.t1.unit* $\equiv \varepsilon_1$
and *flip.t2.unit* $\equiv \varepsilon_2$ **and** *flip.t3.unit* $\equiv \varepsilon_3$
by (*simp-all only*: *order-functors.flip-counit-eq-unit*)

lemma *order-equivalenceI*:
assumes $((\leq_{L1}) \equiv_o (\leq_{R1}))$ *l1 r1*
and $((\leq_{L2}) \equiv_o (\leq_{R2}))$ *l2 r2*
and $((\leq_{L3}) \equiv_o (\leq_{R3}))$ *l3 r3*
shows $((\leq_L) \equiv_o (\leq_R))$ *l r*
apply (*insert assms*)
apply (*elim order-functors.order-equivalenceE*)
apply (*rule order-equivalenceI*;
rule mono-wrt-rel-leftI
flip.mono-wrt-rel-leftI
rel-equivalence-on-in-field-unitI
flip.rel-equivalence-on-in-field-unitI;

assumption)
done

end

end

theory *Transport-Natural-Functors*
imports
Transport-Natural-Functors-Galois
Transport-Natural-Functors-Galois-Relator
Transport-Natural-Functors-Order-Base
Transport-Natural-Functors-Order-Equivalence
begin

Summary Summary of results for a fixed natural functor with 3 parameters. All apply-style proofs are written such that they also apply to functors with other arities. An automatic derivation of these results for all natural functors needs to be implemented in the BNF package. This is future work.

context *transport-natural-functor*
begin

interpretation *flip* :
transport-natural-functor R1 L1 r1 l1 R2 L2 r2 l2 R3 L3 r3 l3 **.**

theorem *preorder-equivalenceI*:
assumes $((\leq_{L1}) \equiv_{pre} (\leq_{R1}))$ *l1 r1*
and $((\leq_{L2}) \equiv_{pre} (\leq_{R2}))$ *l2 r2*
and $((\leq_{L3}) \equiv_{pre} (\leq_{R3}))$ *l3 r3*
shows $((\leq_{L}) \equiv_{pre} (\leq_{R}))$ *l r*
apply (*insert assms*)
apply (*elim transport.preorder-equivalence-galois-equivalenceE*)
apply (*intro preorder-equivalence-if-galois-equivalenceI*
galois-equivalenceI
preorder-on-in-field-leftI flip.preorder-on-in-field-leftI)
apply *assumption*+
done

theorem *partial-equivalence-rel-equivalenceI*:
assumes $((\leq_{L1}) \equiv_{PER} (\leq_{R1}))$ *l1 r1*
and $((\leq_{L2}) \equiv_{PER} (\leq_{R2}))$ *l2 r2*
and $((\leq_{L3}) \equiv_{PER} (\leq_{R3}))$ *l3 r3*
shows $((\leq_{L}) \equiv_{PER} (\leq_{R}))$ *l r*
apply (*insert assms*)
apply (*elim transport.partial-equivalence-rel-equivalenceE*
transport.preorder-equivalence-galois-equivalenceE
preorder-on-in-fieldE)

apply (*intro partial-equivalence-rel-equivalence-if-galois-equivalenceI*
galois-equivalenceI
partial-equivalence-rel-leftI flip.partial-equivalence-rel-leftI
partial-equivalence-relI)
apply *assumption+*
done

For the simplification of the Galois relator see *flip.right-Galois = Frel flip.t1.right-Galois flip.t2.right-Galois flip.t3.right-Galois.*

end

end

2.29 Transport Paper Guide

theory *Transport-Via-Partial-Galois-Connections-Equivalences-Paper*
imports
Transport-Black-Box
Transport-Natural-Functors
Transport-Partial-Quotient-Types
Transport-Prototype
Transport-Lists-Sets-Examples
Transport-Dep-Fun-Rel-Examples
Transport-Typedef-Base
begin

- Section 3.1: Order basics can be found in *Transport.Binary-Relation-Properties*, *Transport.Preorders*, *Transport.Partial-Equivalence-Relations*, *Transport.Equivalence-Relations*, and *Transport.Order-Functions-Base.* Theorem
- Section 3.2: Function relators and monotonicity can be found in *Transport.Function-Relators* and *Transport.Functions-Monotone*
- Section 3.3: Galois relator can be found in *Transport.Galois-Relator-Base.*
  - Lemma 1: *Transport.Transport-Partial-Quotient-Types* (*results from Appendix*)
  - Lemma 3: *galois-prop.galois-prop ?L ?R ?l ?r* $\Longrightarrow$ (*galois-rel.Galois* $?R^{-1}$ $?L^{-1}$ $?l)^{-1}$ *?x ?y = Galois-infix ?x ?L ?R ?r ?y*
- Section 3.4: Partial Galois Connections and Equivalences can be found in *Transport.Half-Galois-Property*, *Transport.Galois-Property*, *Transport.Galois-Connections*, *Transport.Galois-Equivalences*, and *Transport.Order-Equivalences.*

- Lemma 2: *Transport.Transport-Partial-Quotient-Types* (*results from Appendix*)
- Lemma 4: ⟦*order-functors.order-equivalence ?L ?R ?l ?r*; *transitive ?L*; *transitive ?R*⟧ ⟹ *galois.galois-equivalence ?L ?R ?l ?r*
- Lemma 5: ⟦*galois.galois-equivalence ?L ?R ?l ?r*; *reflexive-on* (*in-field ?L*) *?L*; *reflexive-on* (*in-field ?R*) *?R*⟧ ⟹ *order-functors.order-equivalence ?L ?R ?l ?r*

- Section 4.1: Closure (Dependent) Function Relator can be found in `Black_Box/Functions`.
  - Monotone function relator *Transport.Monotone-Function-Relator*.
  - Setup of construction *Transport.Transport-Functions-Base*.
  - Theorem 1: see *Transport.Transport-Functions*
  - Theorem 2: see ⟦*transport.preorder-equivalence ?L1.0 ?R1.0 ?l1.0 ?r1.0*; ⋀*x x'*. *Galois-infix x ?L1.0 ?R1.0 ?r1.0 x'* ⟹ *transport.preorder-equivalence* (*?L2.0 x* (*?r1.0 x'*)) (*?R2.0* (*?l1.0 x*) *x'*) (*?l2.0 x' x*) (*?r2.0 x x'*); ((*x1 x2* :: *?L1.0*$^{-1}$) ⇒ (*x3 x4* :: *?L1.0*) ⇛ *?L1.0 x1 x3* ⟶ (≤)) *?L2.0*; ((*x1 x2* :: *?L1.0*) ⇒ (*x1' x2'* :: *?R1.0*) ⇛ *Galois-infix x2 ?L1.0 ?R1.0 ?r1.0 x1'* ⟶ (*in-field* (*?R2.0* (*?l1.0 x1*) *x2'*) ⇛ *?L2.0 x1* (*?r1.0 x2'*))) *?r2.0*; *in-dom* (*transport-Mono-Dep-Fun-Rel.L ?L1.0 ?L2.0*) *?f*; *in-codom* (*transport-Mono-Dep-Fun-Rel.L ?R1.0 ?R2.0*) *?g*⟧ ⟹ *Galois-infix ?f* (*transport-Mono-Dep-Fun-Rel.L ?L1.0 ?L2.0*) (*transport-Mono-Dep-Fun-R ?R1.0 ?R2.0*) (*transport-Dep-Fun-Rel.l ?l1.0 ?r2.0*) *?g* = ((*x x'* :: *galois-rel.Galois ?L1.0 ?R1.0 ?r1.0*) ⇛ *galois-rel.Galois* (*?L2.0 x* (*?r1.0 x'*)) (*?R2.0* (*?l1.0 x*) *x'*) (*?r2.0 x x'*)) *?f ?g* (*results from Appendix*)
  - Lemma 6: ⟦*galois.galois-connection ?L1.0 ?R1.0 ?l1.0 ?r1.0*; *reflexive-on* (*in-codom ?L1.0*) *?L1.0*; *reflexive-on* (*in-dom ?R1.0*) *?R1.0*; *galois.galois-connection ?L2.0 ?R2.0 ?l2.0 ?r2.0*; *transitive ?L2.0*; *transitive ?R2.0*⟧ ⟹ *galois.galois-connection* (*transport-Mono-Dep-Fun-Rel.L ?L1.0* (λ- -. *?L2.0*)) (*transport-Mono-Dep-Fun-Rel.L ?R1.0* (λ- -. *?R2.0*)) (*transport-Dep-Fun-Rel.l ?r1.0* (λ- -. *?l2.0*)) (*transport-Dep-Fun-Rel.l ?l1.0* (λ- -. *?r2.0*))
  - Lemma 7: ⟦(*?L1.0* ⇒ *?R1.0*) *?l1.0*; *galois-prop.galois-prop ?L1.0 ?R1.0 ?l1.0 ?r1.0*; *reflexive-on* (*in-dom ?L1.0*) *?L1.0*; (*?R2.0* ⇒ *?L2.0*) *?r2.0*; *transitive ?L2.0*; *in-dom* (*transport-Mono-Dep-Fun-Rel.L ?L1.0* (λ- -. *?L2.0*)) *?f*; *in-codom* (*transport-Mono-Dep-Fun-Rel.L ?R1.0* (λ- -. *?R2.0*)) *?g*⟧ ⟹ *Galois-infix ?f* (*transport-Mono-Dep-Fun-Rel.L ?L1.0* (λ- -. *?L2.0*)) (*transport-Mono-Dep-Fun-Rel.L ?R1.0* (λ-

-. ?R2.0)) (*transport-Dep-Fun-Rel.l ?l1.0* (λ- *-. ?r2.0*)) *?g* = (*galois-rel.Galois ?L1.0 ?R1.0 ?r1.0* ⇛ *galois-rel.Galois ?L2.0 ?R2.0 ?r2.0*) *?f ?g*

– Theorem 7: ⟦*galois.galois-connection ?L1.0 ?R1.0 ?l1.0 ?r1.0*; *reflexive-on* (*in-field ?L1.0*) *?L1.0*; *reflexive-on* (*in-field ?R1.0*) *?R1.0*; ⋀*x x′. Galois-infix x ?L1.0 ?R1.0 ?r1.0 x′* ⟹ *galois.galois-connection* (*?L2.0 x* (*?r1.0 x′*)) (*?R2.0* (*?l1.0 x*) *x′*) (*?l2.0 x′ x*) (*?r2.0 x x′*); ((*- x2* :: *?L1.0*) ⇒ (*x3 x4* :: *?L1.0*) ⇛ (*?L1.0 x2 x3* ∧ *?L1.0 x4* (*order-functors.unit ?l1.0 ?r1.0 x3*)) ⟶ (λ*x y. y* ≤ *x*)) *?L2.0*; ((*x1′ x2′* :: *?R1.0*) ⇒ *?R1.0* (*order-functors.counit ?l1.0 ?r1.0 x2′*) *x1′* ⟶ ((*x3′ -* :: *?R1.0*) ⇛ *?R1.0 x2′ x3′* ⟶ (≤))) *?R2.0*; ((*x1′ x2′* :: *?R1.0*) ⇒ (*x1 x2* :: *?L1.0*) ⇛ *Galois-infix x2 ?L1.0 ?R1.0 ?r1.0 x1′* ⟶ (*in-field* (*?L2.0 x1* (*?r1.0 x2′*)) ⇛ *?R2.0* (*?l1.0 x1*) *x2′*)) *?l2.0*; ((*x1 x2* :: *?L1.0*) ⇒ (*x1′ x2′* :: *?R1.0*) ⇛ *Galois-infix x2 ?L1.0 ?R1.0 ?r1.0 x1′* ⟶ (*in-field* (*?R2.0* (*?l1.0 x1*) *x2′*) ⇛ *?L2.0 x1* (*?r1.0 x2′*))) *?r2.0*; ⋀*x1 x2. ?L1.0 x1 x2* ⟹ *transitive* (*?L2.0 x1 x2*); ⋀*x1′ x2′. ?R1.0 x1′ x2′* ⟹ *transitive* (*?R2.0 x1′ x2′*)⟧ ⟹ *galois.galois-connection* (*transport-Mono-Dep-Fun-Rel.L ?L1.0 ?L2.0*) (*transport-Mono-Dep-Fun-Rel.L ?R1.0 ?R2.0*) (*transport-Dep-Fun-Rel.l ?r1.0 ?l2.0*) (*transport-Dep-Fun-Rel.l ?l1.0 ?r2.0*)

– Theorem 8: ⟦*galois.galois-connection ?L1.0 ?R1.0 ?l1.0 ?r1.0*; *reflexive-on* (*in-field ?L1.0*) *?L1.0*; ⋀*x x′. Galois-infix x ?L1.0 ?R1.0 ?r1.0 x′* ⟹ (*?R2.0* (*?l1.0 x*) *x′* ⇒ *?L2.0 x* (*?r1.0 x′*)) (*?r2.0 x x′*); ((*x1* : ⊤) ⇒ (*x2 -* :: *?L1.0*) ⇒ *?L1.0 x1 x2* ⟶ (≤)) *?L2.0*; ((*x1* : ⊤) ⇒ (*x2 x3* :: *?L1.0*) ⇒ (*?L1.0 x1 x2* ∧ *?L1.0 x3* (*order-functors.unit ?l1.0 ?r1.0 x2*)) ⟶ (λ*x y. y* ≤ *x*)) *?L2.0*; ((*x1 x2* :: *?L1.0*) ⇒ (*x1′ x2′* :: *?R1.0*) ⇛ *Galois-infix x2 ?L1.0 ?R1.0 ?r1.0 x1′* ⟶ (*in-field* (*?R2.0* (*?l1.0 x1*) *x2′*) ⇛ *?L2.0 x1* (*?r1.0 x2′*))) *?r2.0*; ⋀*x1 x2. ?L1.0 x1 x2* ⟹ *transitive* (*?L2.0 x1 x2*); *in-dom* (*transport-Mono-Dep-Fun-Rel.L ?L1.0 ?L2.0*) *?f*; *in-codom* (*transport-Mono-Dep-Fun-Rel.L ?R1.0 ?R2.0*) *?g*⟧ ⟹ *Galois-infix ?f* (*transport-Mono-Dep-Fun-Rel.L ?L1.0 ?L2.0*) (*transport-Mono-Dep-Fun-Rel.L ?R1.0 ?R2.0*) (*transport-Dep-Fun-Rel.l ?l1.0 ?r2.0*) *?g* = ((*x x′* :: *galois-rel.Galois ?L1.0 ?R1.0 ?r1.0*) ⇛ *galois-rel.Galois* (*?L2.0 x* (*?r1.0 x′*)) (*?R2.0* (*?l1.0 x*) *x′*) (*?r2.0 x x′*)) *?f ?g*

– Lemma 8 ⟦*galois.galois-equivalence ?L1.0 ?R1.0 ?l1.0 ?r1.0*; *Pre-orders.preorder-on* (*in-field ?L1.0*) *?L1.0*; ((*x1 x2* :: *?L1.0*$^{-1}$) ⇒ (*x3 x4* :: *?L1.0*) ⇛ *?L1.0 x1 x3* ⟶ (≤)) *?L2.0*; ⋀*x1 x2. ?L1.0 x1 x2* ⟹ *partial-equivalence-rel* (*?L2.0 x1 x2*)⟧ ⟹ *transport-Mono-Dep-Fun-Rel.L ?L1.0 ?L2.0* = *transport-Dep-Fun-Rel.L ?L1.0 ?L2.0*

- Lemma 9: ⟦*reflexive-on* (*in-field ?L1.0*) *?L1.0*; *partial-equivalence-rel ?L2.0*⟧ ⟹ *transport-Mono-Dep-Fun-Rel.L ?L1.0* (λ- -. *?L2.0*) = *transport-Dep-Fun-Rel.L ?L1.0* (λ- -. *?L2.0*)

- Section 4.2: Closure Natural Functors can be found in `Black_Box/Natural_Functors`.
  - Theorem 3: see *Transport.Transport-Natural-Functors*
  - Theorem 4: *galois-rel.Galois* (*transport-natural-functor.L ?L1.0 ?L2.0 ?L3.0*) (*transport-natural-functor.L ?R1.0 ?R2.0 ?R3.0*) (*transport-natural-functor.l ?r1.0 ?r2.0 ?r3.0*) = *Frel* (*galois-rel.Galois ?L1.0 ?R1.0 ?r1.0*) (*galois-rel.Galois ?L2.0 ?R2.0 ?r2.0*) (*galois-rel.Galois ?L3.0 ?R3.0 ?r3.0*)

- Section 4.3: Closure Compositions can be found in `Black_Box/Compositions`.
  - Setup for simple case in *Transport.Transport-Compositions-Agree-Base*
  - Setup for generic case in *Transport.Transport-Compositions-Generic-Base*
  - Theorem 5: ⟦*transport.preorder-equivalence ?L1.0 ?R1.0 ?l1.0 ?r1.0*; *transport.preorder-equivalence ?L2.0 ?R2.0 ?l2.0 ?r2.0*; *transport-comp.middle-compatible-codom ?R1.0 ?L2.0*⟧ ⟹ *transport.preorder-equivalence* (*transport-comp.L ?L1.0 ?R1.0 ?l1.0 ?r1.0 ?L2.0*) (*transport-comp.L ?R2.0 ?L2.0 ?r2.0 ?l2.0 ?R1.0*) (*transport-comp.l ?l1.0 ?l2.0*) (*transport-comp.l ?r2.0 ?r1.0*) and ⟦*transport.partial-equivalence-rel-equivalence ?L1.0 ?R1.0 ?l1.0 ?r1.0*; *transport.partial-equivalence-rel-equivalence ?L2.0 ?R2.0 ?l2.0 ?r2.0*; *transport-comp.middle-compatible-codom ?R1.0 ?L2.0*⟧ ⟹ *transport.partial-equivalence-rel-equivalence* (*transport-comp.L ?L1.0 ?R1.0 ?l1.0 ?r1.0 ?L2.0*) (*transport-comp.L ?R2.0 ?L2.0 ?r2.0 ?l2.0 ?R1.0*) (*transport-comp.l ?l1.0 ?l2.0*) (*transport-comp.l ?r2.0 ?r1.0*)
  - Theorem 6: ⟦*transport.preorder-equivalence ?L1.0 ?R1.0 ?l1.0 ?r1.0*; *transport.preorder-galois-connection ?R2.0 ?L2.0 ?r2.0 ?l2.0*; *transport-comp.middle-compatible-codom ?R1.0 ?L2.0*⟧ ⟹ *galois-rel.Galois* (*transport-comp.L ?L1.0 ?R1.0 ?l1.0 ?r1.0 ?L2.0*) (*transport-comp.L ?R2.0 ?L2.0 ?r2.0 ?l2.0 ?R1.0*) (*transport-comp.l ?r2.0 ?r1.0*) = *galois-rel.Galois ?L1.0 ?R1.0 ?r1.0* ∘∘ *galois-rel.Galois ?L2.0 ?R2.0 ?r2.0*
    (*results from Appendix*)
  - Theorem 9: see `Black_Box/Compositions/Agree`, results in *transport-comp-same*.

- Theorem 10: ⟦*galois.galois-equivalence ?L1.0 ?R1.0 ?l1.0 ?r1.0*; *Preorders.preorder-on* (*in-field ?R1.0*) *?R1.0*; *galois.galois-equivalence ?L2.0 ?R2.0 ?l2.0 ?r2.0*; *Preorders.preorder-on* (*in-field ?L2.0*) *?L2.0*; *transport-comp.middle-compatible-codom ?R1.0 ?L2.0*⟧ ⟹ *galois.galois-connection* (*transport-comp.L ?L1.0 ?R1.0 ?l1.0 ?r1.0 ?L2.0*) (*transport-comp.L ?R2.0 ?L2.0 ?r2.0 ?l2.0 ?R1.0*) (*transport-comp.l ?l1.0 ?l2.0*) (*transport-comp.l ?r2.0 ?r1.0*)
- Theorem 11: ⟦(*?R1.0* ⇒ *?L1.0*) *?r1.0*; *galois-prop.galois-prop ?L1.0 ?R1.0 ?l1.0 ?r1.0*; *galois-prop.half-galois-prop-right ?R1.0 ?L1.0 ?r1.0 ?l1.0*; *Preorders.preorder-on* (*in-field ?R1.0*) *?R1.0*; (*?L2.0* ⇒ *?R2.0*) *?l2.0*; *galois-prop.half-galois-prop-left ?R2.0 ?L2.0 ?r2.0 ?l2.0*; *reflexive-on* (*in-dom ?L2.0*) *?L2.0*; *?R1.0* ∘∘ *?L2.0* ∘∘ *?R1.0* ≤ *?R1.0* ∘∘ *?L2.0*; *in-codom* (*?L2.0* ∘∘ *?R1.0* ∘∘ *?L2.0*) ≤ *in-codom ?R1.0*⟧ ⟹ *galois-rel.Galois* (*transport-comp.L ?L1.0 ?R1.0 ?l1.0 ?r1.0 ?L2.0*) (*transport-comp.L ?R2.0 ?L2.0 ?r2.0 ?l2.0 ?R1.0*) (*transport-comp.l ?r2.0 ?r1.0*) = *galois-rel.Galois ?L1.0 ?R1.0 ?r1.0* ∘∘ *galois-rel.Galois ?L2.0 ?R2.0 ?r2.0*

- Section 5:
  - Implementation *Transport.Transport-Prototype*: Note: the command "trp" from the paper is called **trp-term** and the method "trprover" is called "trp_term_prover".
  - Example 1: *Transport.Transport-Lists-Sets-Examples*
  - Example 2: *Transport.Transport-Dep-Fun-Rel-Examples*
  - Example 3: https://github.com/kappelmann/Isabelle-Set/blob/fdf59444d9a53b5279080fb4d24893c9efa31160/Isabelle_Set/Integers/Integers_Transport.thy
- Proof: Partial Quotient Types are a special case: *Transport.Transport-Partial-Quotient-Types*
- Proof: Typedefs are a special case: *Transport.Transport-Typedef-Base*
- Proof: Set-Extensions are a special case: https://github.com/kappelmann/Isabelle-Set/blob/fdf59444d9a53b5279080fb4d24893c9efa31160/Isabelle_Set/Set_Extensions/Set_Extensions_Transport.thy
- Proof: Bijections as special case: *Transport.Transport-Bijections*

end

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