

Topological Groups

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Abstract

Topological groups are blends of groups and topological spaces with the property that the multiplication and inversion operations are continuous functions. They frequently occur in mathematics and physics, e.g. in the form of Lie groups. We formalize the theory of topological groups on top of HOL-Algebra and HOL-Analysis. Topological groups are defined via a locale. We also introduce a set-based notion of uniform spaces in order to define the uniform structures of topological groups. The most notable formalized result is the Birkhoff-Kakutani theorem which characterizes metrizable topological groups. Our formalization also defines the important matrix groups $GL_n(\mathbb{R})$, $SL_n(\mathbb{R})$, O_n , SO_n and proves them to be topological groups.

The formalized results and proofs have been taken from the textbooks of Arhangel'skii and Tkachenko [1], Bump [2] and James [4]. These lecture notes [5] have also been helpful.

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1 Uniform spaces

```

theory Uniform-Structure
  imports HOL-Analysis.Abstract-Topology HOL-Analysis.Abstract-Metric-Spaces
begin

```

Summary This section introduces a set-based notion of uniformities and connects it to the *uniform-space* type class.

1.1 Definitions and basic results

```

definition uniformity-on :: 'a set  $\Rightarrow$  (('a  $\times$  'a) set  $\Rightarrow$  bool)  $\Rightarrow$  bool where
  uniformity-on X  $\mathcal{E} \iff$ 
    ( $\exists E. \mathcal{E} E$ )  $\wedge$ 
    ( $\forall E. \mathcal{E} E \longrightarrow E \subseteq X \times X \wedge Id\text{-on } X \subseteq E \wedge \mathcal{E} (E^{-1}) \wedge (\exists F. \mathcal{E} F \wedge F \circ F \subseteq E) \wedge$ 
    ( $\forall F. E \subseteq F \wedge F \subseteq X \times X \longrightarrow \mathcal{E} F$ ))  $\wedge$ 
    ( $\forall E F. \mathcal{E} E \longrightarrow \mathcal{E} F \longrightarrow \mathcal{E} (E \cap F)$ )

```

```

typedef 'a uniformity = {(X :: 'a set,  $\mathcal{E}$ ). uniformity-on X  $\mathcal{E}$ }
morphisms uniformity-rep uniformity
proof -
  have uniformity-on UNIV ( $\lambda E. E = UNIV \times UNIV$ )
    unfolding uniformity-on-def Id-on-def relcomp-def by auto
  then show ?thesis by fast
qed

```

```

definition uspace :: 'a uniformity  $\Rightarrow$  'a set where
  uspace  $\Phi = (let (X,  $\mathcal{E}$ ) = uniformity-rep  $\Phi$  in X)$ 

```

```

definition entourage-in :: 'a uniformity  $\Rightarrow$  ('a  $\times$  'a) set  $\Rightarrow$  bool where
  entourage-in  $\Phi = (let (X,  $\mathcal{E}$ ) = uniformity-rep  $\Phi$  in  $\mathcal{E}$ )$ 

```

```

lemma uniformity-inverse'[simp]:
  assumes uniformity-on X  $\mathcal{E}$ 
  shows uspace (uniformity (X,  $\mathcal{E}$ )) = X  $\wedge$  entourage-in (uniformity (X,  $\mathcal{E}$ )) =  $\mathcal{E}$ 
proof -

```

from *assms* **have** *uniformity-rep* (*uniformity* $(X, \mathcal{E}) = (X, \mathcal{E})$)
using *uniformity-inverse* **by** *blast*
then show *?thesis* **by** (*auto simp: prod.splits uspace-def entourage-in-def*)
qed

lemma *uniformity-entourages*:
shows *uniformity-on* (*uspace* Φ) (*entourage-in* Φ)
by (*metis Product-Type.Collect-case-prodD entourage-in-def split-beta uspace-def uniformity-rep*)

lemma *entourages-exist*: $\exists E. \text{entourage-in } \Phi E$
using *uniformity-entourages unfolding uniformity-on-def* **by** *blast*

lemma *entourage-in-space[elim]*: $\text{entourage-in } \Phi E \implies E \subseteq \text{uspace } \Phi \times \text{uspace } \Phi$
using *uniformity-entourages unfolding uniformity-on-def* **by** *metis*

lemma *entourage-superset[intro]*:
 $\text{entourage-in } \Phi E \implies E \subseteq F \implies F \subseteq \text{uspace } \Phi \times \text{uspace } \Phi \implies \text{entourage-in } \Phi F$
using *uniformity-entourages unfolding uniformity-on-def* **by** *blast*

lemma *entourage-intersection[intro]*: $\text{entourage-in } \Phi E \implies \text{entourage-in } \Phi F \implies \text{entourage-in } \Phi (E \cap F)$
using *uniformity-entourages unfolding uniformity-on-def* **by** *metis*

lemma *entourage-converse[intro]*: $\text{entourage-in } \Phi E \implies \text{entourage-in } \Phi (E^{-1})$
using *uniformity-entourages unfolding uniformity-on-def* **by** *fast*

lemma *entourage-diagonal[dest]*:
assumes *entourage*: $\text{entourage-in } \Phi E$ **and** *in-space*: $x \in \text{uspace } \Phi$
shows $(x, x) \in E$
proof –
have *Id-on* (*uspace* Φ) $\subseteq E$
using *uniformity-entourages entourage unfolding uniformity-on-def* **by** *fast*
then show *?thesis* **using** *Id-onI[OF in-space]* **by** *blast*
qed

lemma *smaller-entourage*:
assumes *entourage*: $\text{entourage-in } \Phi E$
shows $\exists F. \text{entourage-in } \Phi F \wedge (\forall x y z. (x, y) \in F \wedge (y, z) \in F \longrightarrow (x, z) \in E)$
proof –
from *entourage* **obtain** F **where** $\text{entourage-in } \Phi F \wedge F \circ F \subseteq E$
using *uniformity-entourages entourage unfolding uniformity-on-def* **by** *meson*
moreover from this **have** $(x, z) \in E$ **if** $(x, y) \in F \wedge (y, z) \in F$ **for** $x y z$ **using** *that* **by** *blast*
ultimately show *?thesis* **by** *blast*
qed

lemma *entire-space-entourage*: $\text{entourage-in } \Phi (\text{uspace } \Phi \times \text{uspace } \Phi)$

by (metis entourages-exist entourage-in-space entourage-superset subset-refl)

definition *utopology* :: 'a uniformity \Rightarrow 'a topology **where**

utopology $\Phi = \text{topology } (\lambda U. U \subseteq \text{uspace } \Phi \wedge (\forall x \in U. \exists E. \text{entourage-in } \Phi E \wedge E''\{x\} \subseteq U))$

lemma *openin-utopology* [iff]:

fixes $\Phi ::$ 'a uniformity

defines *uopen* $U \equiv U \subseteq \text{uspace } \Phi \wedge (\forall x \in U. \exists E. \text{entourage-in } \Phi E \wedge E''\{x\} \subseteq U)$

shows *openin* (*utopology* Φ) = *uopen*

proof –

have *uopen* ($U \cap V$) if *hUV*: *uopen* $U \wedge$ *uopen* V **for** $U V$

proof –

have $\exists E. \text{entourage-in } \Phi E \wedge E''\{x\} \subseteq U \cap V$ if *hx*: $x \in U \cap V$ **for** x

proof –

from *hUV hx* **obtain** $E_1 E_2$ **where**

entourage-in $\Phi E_1 \wedge$ *entourage-in* $\Phi E_2 \wedge E_1''\{x\} \subseteq U \wedge E_2''\{x\} \subseteq V$

unfolding *uopen-def* **by** *blast*

then have *entourage-in* $\Phi (E_1 \cap E_2) \wedge (E_1 \cap E_2)''\{x\} \subseteq U \cap V$ **by** *blast*

then show *?thesis* **by** *fast*

qed

then show *?thesis* **using** *le-infI1 hUV* **unfolding** *uopen-def* **by** *auto*

qed

moreover have *uopen* ($\bigcup \mathcal{U}$) if *hU*: $\forall U \in \mathcal{U}. \text{uopen } U$ **for** \mathcal{U}

proof –

have $\exists E. \text{entourage-in } \Phi E \wedge E''\{x\} \subseteq \bigcup \mathcal{U}$ if *hx*: $x \in \bigcup \mathcal{U}$ **for** x

proof –

from *hx* **obtain** U **where** *hU*: $U \in \mathcal{U} \wedge x \in U$ **by** *blast*

from *this hU* **obtain** E **where** *entourage-in* $\Phi E \wedge E''\{x\} \subseteq U$ **unfolding**

uopen-def **by** *fast*

moreover from *this hU* have $E''\{x\} \subseteq \bigcup \mathcal{U}$ **by** *fast*

ultimately show *?thesis* **by** *blast*

qed

then show *?thesis* **using** *Union-least hU* **unfolding** *uopen-def* **by** *auto*

qed

ultimately have *istopology uopen* **unfolding** *istopology-def* **by** *presburger*

from *topology-inverse* [OF *this*] **show** *?thesis* **unfolding** *utopology-def uopen-def*

by *blast*

qed

lemma *topspace-utopology* [*simp*]:

shows *topspace* (*utopology* Φ) = *uspace* Φ

proof –

let *?T* = *utopology* Φ

have *topspace* *?T* \subseteq *uspace* Φ

using *openin-topspace openin-utopology* **by** *meson*

moreover have *openin* *?T* (*uspace* Φ)

unfolding *openin-utopology* **by** (*auto intro: entire-space-entourage*)

ultimately show *?thesis* **using** *topspace-def* **by** *fast*
qed

definition *ucontinuous* :: 'a *uniformity* \Rightarrow 'b *uniformity* \Rightarrow ('a \Rightarrow 'b) \Rightarrow *bool*
where

ucontinuous Φ Ψ $f \longleftrightarrow$
 $f \in \text{uspace } \Phi \rightarrow \text{uspace } \Psi \wedge$
 $(\forall E. \text{entourage-in } \Psi E \longrightarrow \text{entourage-in } \Phi \{(x, y) \in \text{uspace } \Phi \times \text{uspace } \Phi. (f x,$
 $f y) \in E\})$

lemma *ucontinuous-image-subset* [*dest*]: *ucontinuous* Φ Ψ $f \Longrightarrow f'(\text{uspace } \Phi) \subseteq$
 $\text{uspace } \Psi$

unfolding *ucontinuous-def* **by** *blast*

lemma *entourage-preimage-ucontinuous* [*dest*]:

assumes *ucontinuous* Φ Ψ f **and** *entourage-in* ΨE
shows *entourage-in* $\Phi \{(x, y) \in \text{uspace } \Phi \times \text{uspace } \Phi. (f x, f y) \in E\}$
using *assms* **unfolding** *ucontinuous-def* **by** *blast*

lemma *ucontinuous-imp-continuous*:

assumes *ucontinuous* Φ Ψ f
shows *continuous-map* (*utopology* Φ) (*utopology* Ψ) f
proof (*unfold continuous-map-def*, *intro conjI allI impI*)
show $f \in \text{topspace } (\text{utopology } \Phi) \rightarrow \text{topspace } (\text{utopology } \Psi)$
using *assms* **unfolding** *ucontinuous-def* **by** *auto*

next

fix U **assume** $hU: \text{openin } (\text{utopology } \Psi) U$
let $?V = \{x \in \text{topspace } (\text{utopology } \Phi). f x \in U\}$
have $\exists F. \text{entourage-in } \Phi F \wedge F''\{x\} \subseteq ?V$ **if** $hx: x \in \text{uspace } \Phi \wedge f x \in U$ **for** x
proof –
from *that* hU **obtain** E **where** $hE: \text{entourage-in } \Psi E \wedge E''\{f x\} \subseteq U$
unfolding *openin-utopology* **by** *blast*
let $?F = \{(x, y) \in \text{uspace } \Phi \times \text{uspace } \Phi. (f x, f y) \in E\}$
have $?F''\{x\} = \{y \in \text{uspace } \Phi. f y \in E''\{f x\}\}$ **unfolding** *Image-def* **using**
 hx **by** *auto*
then **have** $?F''\{x\} \subseteq ?V$ **using** hE **by** *auto*
moreover **have** *entourage-in* $\Phi ?F$
using *assms* *entourage-preimage-ucontinuous* hE **unfolding** *topspace-utopology*
by *blast*
ultimately show *?thesis* **by** *blast*
qed
then **show** *openin* (*utopology* Φ) $?V$ **unfolding** *openin-utopology* **by** *force*
qed

1.2 Metric spaces as uniform spaces

context *Metric-space*

begin

abbreviation *mentourage* :: *real* \Rightarrow (*'a* \times *'a*) **set where**
mentourage $\varepsilon \equiv \{(x,y) \in M \times M. d\ x\ y < \varepsilon\}$

definition *muniformity* :: *'a* **uniformity where**
muniformity = *uniformity* (*M*, $\lambda E. E \subseteq M \times M \wedge (\exists \varepsilon > 0. \text{mentourage } \varepsilon \subseteq E)$)

lemma

uspace-muniformity[simp]: *uspace muniformity* = *M* **and**
entourage-muniformity: *entourage-in muniformity* = ($\lambda E. E \subseteq M \times M \wedge (\exists \varepsilon > 0. \text{mentourage } \varepsilon \subseteq E)$)

proof –

have *uniformity-on* *M* ($\lambda E. E \subseteq M \times M \wedge (\exists \varepsilon > 0. \text{mentourage } \varepsilon \subseteq E)$)

unfolding *uniformity-on-def* *Id-on-def* *converse-def*

proof (*intro conjI allI impI, goal-cases*)

case 1

then show *?case* **by** (*rule exI[of - mentourage 1]*) *force*

next

case (5 *E*)

then obtain ε **where** $h\varepsilon: \varepsilon > 0 \wedge \text{mentourage } \varepsilon \subseteq E$ **by** *blast*

then have $\{(y, x). (x, y) \in \text{mentourage } \varepsilon\} \subseteq E$ **using** *commute* **by** *auto*

then have $\text{mentourage } \varepsilon \subseteq E^{-1}$ **by** *blast*

then show *?case* **using** $h\varepsilon$ **by** *auto*

next

case (6 *E*)

then obtain ε **where** $h\varepsilon: \varepsilon > 0 \wedge \text{mentourage } \varepsilon \subseteq E$ **by** *blast*

let $?F = \text{mentourage } (\varepsilon/2)$

have $(x,z) \in E$ **if** $(x,y) \in ?F \wedge (y,z) \in ?F$ **for** $x\ y\ z$

proof –

have $d\ x\ z < \varepsilon$ **using** *that triangle* **by** *fastforce*

then show *?thesis* **using** $h\varepsilon$ **by** *blast*

qed

then have $?F \subseteq M \times M \wedge ?F\ O\ ?F \subseteq E$ **by** *blast*

then show *?case* **by** (*meson* $h\varepsilon$ *order-refl* *zero-less-divide-iff* *zero-less-numeral*)

next

case (8 *E F*)

then show *?case* **by** *fast*

next

case (10 *E F*)

then obtain $\varepsilon\ \delta$ **where**

$\varepsilon > 0 \wedge \text{mentourage } \varepsilon \subseteq E$ **and**

$\delta > 0 \wedge \text{mentourage } \delta \subseteq F$ **by** *presburger*

then have $\min\ \varepsilon\ \delta > 0 \wedge \text{mentourage } (\min\ \varepsilon\ \delta) \subseteq E \cap F$ **by** *auto*

then show *?case* **by** *blast*

qed (*auto*)

then show

uspace muniformity = *M* **and**

entourage-in muniformity = ($\lambda E. E \subseteq M \times M \wedge (\exists \varepsilon > 0. \text{mentourage } \varepsilon \subseteq$

E)

unfolding *muniformity-def* **using** *uniformity-inverse'* **by** *auto*

qed

lemma *uniformity-induces-mtopology* [simp]: *utopology muniformity = mtopology*

proof –

have *mentourage-image*: $\text{mball } x \ \varepsilon = (\text{mentourage } \varepsilon) \{x\}$ **for** $x \ \varepsilon$ **unfolding**
mball-def **by** *auto*

have *openin (utopology muniformity)* $U \longleftrightarrow \text{openin mtopology } U$ **for** U

proof

assume hU : *openin (utopology muniformity)* U

have $\exists \varepsilon > 0. \text{mball } x \ \varepsilon \subseteq U$ **if** $x \in U$ **for** x

proof –

from hU **that obtain** E **where** hE : *entourage-in muniformity* $E \wedge E \{x\} \subseteq U$ **unfolding** *openin-utopology* **by** *blast*

then obtain ε **where** $h\varepsilon$: $\varepsilon > 0 \wedge \text{mentourage } \varepsilon \subseteq E$ **unfolding** *entourage-muniformity* **by** *presburger*

then have $(\text{mentourage } \varepsilon) \{x\} \subseteq U$ **using** hE **by** *fast*

then show *?thesis* **using** *mentourage-image* $h\varepsilon$ **by** *auto*

qed

then show *openin mtopology* U **unfolding** *openin-mtopology* **using** hU *openin-subset*
by *fastforce*

next

assume hU : *openin mtopology* U

have $\exists E. \text{entourage-in muniformity } E \wedge E \{x\} \subseteq U$ **if** $x \in U$ **for** x

proof –

from hU **that obtain** ε **where** $\varepsilon > 0 \wedge \text{mball } x \ \varepsilon \subseteq U$ **unfolding** *openin-mtopology*
by *blast*

then show *?thesis* **unfolding** *mentourage-image* *entourage-muniformity* **by**
auto

qed

then show *openin (utopology muniformity)* U **unfolding** *openin-utopology*
using hU *openin-subset* **by** *fastforce*

qed

then show *?thesis* **using** *topology-eq* **by** *blast*

qed

1.3 Connection to type class

end

The following connects the *uniform-space* class to the set based notion *Uniform-Structure.uniformity-on*.

Given a type $'a$ which is an instance of the class *uniform-space*, it is possible to introduce an *'a uniformity* on the entire universe: *UNIV*:

definition *uniformity-of-space* :: $('a :: \text{uniform-space}) \text{uniformity}$ **where**

uniformity-of-space = *uniformity* (*UNIV* :: *'a set*, $(\lambda S. \forall_F x \text{ in } \text{uniformity-class}.\text{uniformity}.\ x \in S)$)

The induced uniformity fulfills the required conditions, i.e., the class based notion implies the set-based notion.

lemma *uniformity-on-uniformity-of-space-aux*:
uniformity-on (*UNIV* :: ('a :: uniform-space) set) ($\lambda S. \forall_F x$ in *uniformity-class.uniformity*.
 $x \in S$)

proof –
let ?u = *uniformity-class.uniformity* :: ('a × 'a) filter

have $\exists S. (\forall_F x$ in ?u. $x \in S$) **by** (*intro exI*[**where** $x = UNIV \times UNIV$]) *simp*
moreover have $(\forall_F x$ in ?u. $x \in E \cap F$) **if** $(\forall_F x$ in ?u. $x \in E$) $(\forall_F x$ in ?u. $x \in F$) **for** $E F$
using *that eventually-conj* **by** *auto*
moreover have Id -on *UNIV* $\subseteq E$ **if** $\forall_F x$ in ?u. $x \in E$ **for** E
proof –
have $(x, x) \in E$ **for** x **using** *uniformity-refl*[*OF that*] **by** *auto*
thus ?thesis **unfolding** *Id-on-def* **by** *auto*
qed
moreover have $(\forall_F x$ in ?u. $x \in E^{-1}$) **if** $\forall_F x$ in ?u. $x \in E$ **for** E
using *uniformity-sym*[*OF that*] **by** (*simp add: converse-unfold*)
moreover have $\exists F. (\forall_F x$ in ?u. $x \in F$) $\wedge F \circ F \subseteq E$ **if** $\forall_F x$ in ?u. $x \in E$
for E
proof –
from *uniformity-trans*[*OF that*]
obtain D **where** *eventually D* ?u $(\forall x y z. D(x, y) \longrightarrow D(y, z) \longrightarrow (x, z) \in E)$ **by** *auto*
thus ?thesis **by** (*intro exI*[**where** $x = Collect D$]) *auto*
qed
moreover have $\forall_F x$ in ?u. $x \in F$ **if** $\forall_F x$ in ?u. $x \in E$ $E \subseteq F$ **for** $E F$
using *that(2)* **by** (*intro eventually-mono*[*OF that(1)*]) *auto*
ultimately show ?thesis
unfolding *uniformity-on-def* **by** *auto*
qed

lemma *uniformity-rep-uniformity-of-space*:
uniformity-rep uniformity-of-space = (*UNIV*, ($\lambda S. \forall_F x$ in *uniformity-class.uniformity*.
 $x \in S$))
unfolding *uniformity-of-space-def* **using** *uniformity-on-uniformity-of-space-aux*
by (*intro uniformity-inverse*) *auto*

lemma *uspace-uniformity-space* [*simp, iff*]:
uspace uniformity-of-space = *UNIV*
unfolding *uspace-def uniformity-rep-uniformity-of-space* **by** *simp*

lemma *entourage-in-uniformity-space*:
entourage-in uniformity-of-space $S = (\forall_F x$ in *uniformity-class.uniformity*. $x \in S$)
unfolding *entourage-in-def uniformity-rep-uniformity-of-space* **by** *simp*

Compatibility of the *Metric-space.muniformity* with the uniformity based on the class based hierarchy.

lemma (*uniformity-of-space* :: ('a :: metric-space) *uniformity*) = *Met-TC.muniformity*


```

proof –
  have  $(\forall x y. \text{dist } x y < \varepsilon \longrightarrow (x, y) \in E) = (\{(x, y). \text{dist } x y < \varepsilon\} \subseteq E)$ 
    for  $\varepsilon$  and  $E :: ('a \times 'a)$  set
    by auto
  thus ?thesis
  unfolding Met-TC.muniformality-def uniformity-of-space-def eventually-uniformity-metric
by simp
qed

end

```

2 General theory of Topological Groups

```

theory Topological-Group
  imports
    HOL-Algebra.Group
    HOL-Algebra.Coset
    HOL-Analysis.Abstract-Topology
    HOL-Analysis.Product-Topology
    HOL-Analysis.T1-Spaces
    HOL-Analysis.Abstract-Metric-Spaces
    Uniform-Structure
begin

```

Summary In this section we define topological groups and prove basic results about them. We also introduce the left and right uniform structures of topological groups and prove the Birkhoff-Kakutani theorem.

2.1 Auxiliary definitions and results

2.1.1 Miscellaneous

```

lemma connected-components-homeo:
  assumes homeo: homeomorphic-map T1 T2 φ and in-space: x ∈ topspace T1
  shows  $\varphi(\text{connected-component-of-set } T_1 x) = \text{connected-component-of-set } T_2 (\varphi x)$ 
proof
  let  $?Z = \text{connected-component-of-set}$ 
  show  $\varphi(?Z T_1 x) \subseteq ?Z T_2 (\varphi x)$ 
    by (metis connected-component-of-eq connected-component-of-maximal connected-in-connected-component-of homeo homeomorphic-map-connectedness-eq imageI in-space mem-Collect-eq)
  next
  let  $?Z = \text{connected-component-of-set}$ 
  from homeo obtain  $\psi$  where ψ-homeo: homeomorphic-map T2 T1 ψ
    and ψ-inv: (∀ y ∈ topspace T1. ψ (φ y) = y) ∧ (∀ y ∈ topspace T2. φ (ψ y) = y)
  by (smt (verit) homeomorphic-map-maps homeomorphic-maps-map)

```

from *homeo in-space* **have** $\varphi x \in \text{topspace } T_2$
using *homeomorphic-imp-surjective-map* **by** *blast*
then have $\psi'(?Z T_2 (\varphi x)) \subseteq ?Z T_1 (\psi (\varphi x))$
by (*metis connected-component-of-eq connected-component-of-maximal connecte-*
din-connected-component-of ψ -homeo homeomorphic-map-connectedness-eq imageI
mem-Collect-eq)
then show $?Z T_2 (\varphi x) \subseteq \varphi'(?Z T_1 x)$
by (*smt (verit, del-insts) ψ -inv connected-component-of-subset-topspace im-*
age-subset-iff in-space subsetD subsetI)
qed

lemma *open-map-prod-top:*

assumes *open-map* $T_1 T_3 f$ **and** *open-map* $T_2 T_4 g$
shows *open-map (prod-topology $T_1 T_2$) (prod-topology $T_3 T_4$) ($\lambda(x, y). (f x, g y)$)*
proof (*unfold open-map-def, standard, standard*)
let $?p = \lambda(x, y). (f x, g y)$
fix U **assume** *openin (prod-topology $T_1 T_2$) U*
then obtain \mathcal{U} **where** $h\mathcal{U}: \mathcal{U} \subseteq \{V \times W \mid V W. \text{openin } T_1 V \wedge \text{openin } T_2 W\} \wedge \bigcup \mathcal{U} = U$
unfolding *openin-prod-topology union-of-def* **using** *arbitrary-def* **by** *auto*
then have $?p'U = \bigcup \{?p'VW \mid VW. VW \in \mathcal{U}\}$ **by** *blast*
then have $?p'U = \bigcup \{?p'(V \times W) \mid V W. V \times W \in \mathcal{U} \wedge \text{openin } T_1 V \wedge \text{openin } T_2 W\}$
using $h\mathcal{U}$ **by** *blast*
moreover have $?p'(V \times W) = (f'V) \times (g'W)$ **for** $V W$ **by** *fast*
ultimately have $?p'U = \bigcup \{(f'V) \times (g'W) \mid V W. V \times W \in \mathcal{U} \wedge \text{openin } T_1 V \wedge \text{openin } T_2 W\}$ **by** *presburger*
moreover have *openin (prod-topology $T_3 T_4$) (($f'V$) \times ($g'W$))* **if** *openin $T_1 V \wedge \text{openin } T_2 W$* **for** $V W$
using *openin-prod-Times-iff assms that open-map-def* **by** *metis*
ultimately show *openin (prod-topology $T_3 T_4$) ($?p'U$)* **by** *fastforce*
qed

lemma *injective-quotient-map-homeo:*

assumes *quotient-map* $T1 T2 q$ **and** *inj: inj-on q (topspace $T1$)*
shows *homeomorphic-map $T1 T2 q$* **using** *assms*
unfolding *homeomorphic-eq-everything-map injective-quotient-map[OF inj]* **by** *fast*

lemma (*in group*) *subgroupI-alt:*

assumes *subset: $H \subseteq \text{carrier } G$* **and** *nonempty: $H \neq \{\}$* **and**
closed: $\bigwedge \sigma \tau. \sigma \in H \wedge \tau \in H \implies \sigma \otimes \text{inv } \tau \in H$
shows *subgroup $H G$*
proof –
from *nonempty* **obtain** η **where** $\eta \in H$ **by** *blast*
then have $1 \in H$ **using** *closed[of $\eta \eta$] subset r-inv* **by** *fastforce*
then have *closed-inv: inv $\sigma \in H$ if $\sigma \in H$* **for** σ
using *closed[of 1 σ] r-inv r-one subset that* **by** *force*

then have $\sigma \otimes \tau \in H$ **if** $\sigma \in H \wedge \tau \in H$ **for** $\sigma \tau$
using *closed*[of σ *inv* τ] *inv-inv subset subset-iff* **that by auto**
then show *?thesis using assms closed-inv* **by** (*auto intro: subgroupI*)
qed

lemma *subgroup-intersection*:
assumes *subgroup* $H G$ **and** *subgroup* $H' G$
shows *subgroup* $(H \cap H') G$
using *assms unfolding subgroup-def* **by force**

2.1.2 Quotient topology

definition *quot-topology* :: $'a$ *topology* \Rightarrow $('a \Rightarrow 'b) \Rightarrow 'b$ *topology* **where**
quot-topology $T q = \text{topology } (\lambda U. U \subseteq q(\text{topspace } T) \wedge \text{openin } T \{x \in \text{topspace } T. q x \in U\})$

lemma *quot-topology-open*:
fixes $T :: 'a$ *topology* **and** $q :: 'a \Rightarrow 'b$
defines *openin-quot* $U \equiv U \subseteq q(\text{topspace } T) \wedge \text{openin } T \{x \in \text{topspace } T. q x \in U\}$
shows *openin* (*quot-topology* $T q$) = *openin-quot*
proof –
have *istopology openin-quot*
proof –
have *openin-quot* $(U_1 \cap U_2)$ **if** *openin-quot* $U_1 \wedge$ *openin-quot* U_2 **for** $U_1 U_2$
proof –
have $\{x \in \text{topspace } T. q x \in U_1 \cap U_2\} = \{x \in \text{topspace } T. q x \in U_1\} \cap \{x \in \text{topspace } T. q x \in U_2\}$ **by blast**
then show *?thesis using that unfolding openin-quot-def* **by auto**
qed
moreover have *openin-quot* $(\bigcup \mathcal{U})$ **if** $\forall U \in \mathcal{U}. \text{openin-quot } U$ **for** \mathcal{U}
proof –
have $\{x \in \text{topspace } T. q x \in \bigcup \mathcal{U}\} = \bigcup \{\{x \in \text{topspace } T. q x \in U\} \mid U \in \mathcal{U}\}$ **by blast**
then show *?thesis using that unfolding openin-quot-def* **by auto**
qed
ultimately show *?thesis using istopology-def*
by (*smt (verit) Collect-cong Sup-set-def UnionI Union-iff image-eqI mem-Collect-eq mem-Collect-eq openin-topspace subsetI subset-antisym topspace-def*)
qed
from *topology-inverse'*[*OF this*] **show** *?thesis using quot-topology-def unfolding openin-quot-def* **bymetis**
qed

lemma *projection-quotient-map*: *quotient-map* T (*quot-topology* $T q$) q
proof (*unfold quotient-map-def, intro conjI*)
have *openin* (*quot-topology* $T q$) $(q ` \text{topspace } T)$ **using** *quot-topology-open*
by (*smt (verit) image-subset-iff mem-Collect-eq openin-subtopology-refl subsetI subtopology-superset*)

then show $q \text{ ' } \textit{topspace } T = \textit{topspace } (\textit{quot-topology } T \textit{ } q)$ **using** *quot-topology-open*
by (*metis (no-types, opaque-lifting) openin-subset openin-topospace subset-antisym*)
next
show $\forall U \subseteq \textit{topspace } (\textit{quot-topology } T \textit{ } q).$
 $\textit{openin } T \{x \in \textit{topspace } T. q \ x \in U\} = \textit{openin } (\textit{quot-topology } T \textit{ } q) \ U$
using *quot-topology-open* **by** (*metis (mono-tags, lifting) openin-topospace or-der-trans*)
qed

corollary *topspace-quot-topology [simp]: topspace (quot-topology T q) = q'(topspace T)*
using *projection-quotient-map quotient-imp-surjective-map* **by** *metis*

corollary *projection-continuous: continuous-map T (quot-topology T q) q*
using *projection-quotient-map quotient-imp-continuous-map* **by** *fast*

2.2 Definition and basic results

locale *topological-group = group +*
fixes $T :: 'g \textit{ topology}$
assumes *group-is-space [simp]: topspace T = carrier G*
assumes *inv-continuous: continuous-map T T ($\lambda \sigma. \textit{inv } \sigma$)*
assumes *mul-continuous: continuous-map (prod-topology T T) T ($\lambda(\sigma, \tau). \sigma \otimes \tau$)*
begin

lemma *in-space-iff-in-group [iff]: $\sigma \in \textit{topspace } T \longleftrightarrow \sigma \in \textit{carrier } G$*
by *auto*

lemma *translations-continuous [intro]:*
assumes *in-group: $\sigma \in \textit{carrier } G$*
shows *continuous-map T T ($\lambda \tau. \sigma \otimes \tau$) and continuous-map T T ($\lambda \tau. \tau \otimes \sigma$)*

proof –

have *continuous-map T (prod-topology T T) ($\lambda \tau. (\sigma, \tau)$)*
by (*auto intro: continuous-map-pairedI simp: in-group*)
moreover have $(\lambda \tau. \sigma \otimes \tau) = (\lambda(\sigma, \tau). \sigma \otimes \tau) \circ (\lambda \tau. (\sigma, \tau))$ **by** *auto*
ultimately show *continuous-map T T ($\lambda \tau. \sigma \otimes \tau$)*
using *mul-continuous continuous-map-compose* **by** *metis*

next

have *continuous-map T (prod-topology T T) ($\lambda \tau. (\tau, \sigma)$)*
by (*auto intro: continuous-map-pairedI simp: in-group*)
moreover have $(\lambda \tau. \tau \otimes \sigma) = (\lambda(\sigma, \tau). \sigma \otimes \tau) \circ (\lambda \tau. (\tau, \sigma))$ **by** *auto*
ultimately show *continuous-map T T ($\lambda \tau. \tau \otimes \sigma$)*
using *mul-continuous continuous-map-compose* **by** *metis*

qed

lemma *translations-homeos:*
assumes *in-group: $\sigma \in \textit{carrier } G$*
shows *homeomorphic-map T T ($\lambda \tau. \sigma \otimes \tau$) and homeomorphic-map T T ($\lambda \tau. \tau \otimes \sigma$)*

proof –
have $\forall \tau \in \text{topspace } T. \text{inv } \sigma \otimes (\sigma \otimes \tau) = \tau$ **by** (*simp add: group.inv-solve-left' in-group*)
moreover have $\forall \tau \in \text{topspace } T. \sigma \otimes (\text{inv } \sigma \otimes \tau) = \tau$
by (*metis group-is-space in-group inv-closed l-one m-assoc r-inv*)
ultimately have *homeomorphic-maps* $T T (\lambda \tau. \sigma \otimes \tau) (\lambda \tau. (\text{inv } \sigma) \otimes \tau)$
using *homeomorphic-maps-def in-group* **by blast**
then show *homeomorphic-map* $T T (\lambda \tau. \sigma \otimes \tau)$ **using** *homeomorphic-maps-map*
by blast
next
have $\forall \tau \in \text{topspace } T. \tau \otimes \sigma \otimes \text{inv } \sigma = \tau$
by (*simp add: group.inv-solve-right' in-group*)
moreover have $\forall \tau \in \text{topspace } T. \tau \otimes \text{inv } \sigma \otimes \sigma = \tau$ **by** (*simp add: in-group m-assoc*)
ultimately have *homeomorphic-maps* $T T (\lambda \tau. \tau \otimes \sigma) (\lambda \tau. \tau \otimes (\text{inv } \sigma))$
using *homeomorphic-maps-def in-group* **by blast**
then show *homeomorphic-map* $T T (\lambda \tau. \tau \otimes \sigma)$ **using** *homeomorphic-maps-map*
by blast
qed

abbreviation *conjugation* :: $'g \Rightarrow 'g \Rightarrow 'g$ **where**
conjugation $\sigma \tau \equiv \sigma \otimes \tau \otimes \text{inv } \sigma$

corollary *conjugation-homeo*:

assumes *in-group*: $\sigma \in \text{carrier } G$
shows *homeomorphic-map* $T T (\text{conjugation } \sigma)$

proof –

have *conjugation* $\sigma = (\lambda \tau. \tau \otimes \text{inv } \sigma) \circ (\lambda \tau. \sigma \otimes \tau)$ **by auto**
then show *?thesis* **using** *translations-homeos homeomorphic-map-compose*
by (*metis in-group inv-closed*)
qed

corollary *open-set-translations*:

assumes *open-set*: *openin* $T U$ **and** *in-group*: $\sigma \in \text{carrier } G$
shows *openin* $T (\sigma <\# U)$ **and** *openin* $T (U \#> \sigma)$

proof –

let $? \varphi = \lambda \tau. \sigma \otimes \tau$
have $\sigma <\# U = ? \varphi 'U$ **unfolding** *l-coset-def* **by blast**
then show *openin* $T (\sigma <\# U)$ **using** *translations-homeos[OF in-group]*
by (*metis homeomorphic-map-openness-eq open-set*)

next

let $? \psi = \lambda \tau. \tau \otimes \sigma$
have $U \#> \sigma = ? \psi 'U$ **unfolding** *r-coset-def* **by fast**
then show *openin* $T (U \#> \sigma)$ **using** *translations-homeos[OF in-group]*
by (*metis homeomorphic-map-openness-eq open-set*)

qed

corollary *closed-set-translations*:

assumes *closed-set*: *closedin* $T U$ **and** *in-group*: $\sigma \in \text{carrier } G$

shows $\text{closedin } T (\sigma <\# U)$ **and** $\text{closedin } T (U \#> \sigma)$
proof –
let $?\varphi = \lambda\tau. \sigma \otimes \tau$
have $\sigma <\# U = ?\varphi'U$ **unfolding** $l\text{-coset-def}$ **by** *fast*
then show $\text{closedin } T (\sigma <\# U)$ **using** $\text{translations-homeos}[OF \text{ in-group}]$
by $(\text{metis } \text{homeomorphic-map-closedness-eq } \text{closed-set})$
next
let $?\psi = \lambda\tau. \tau \otimes \sigma$
have $U \#> \sigma = ?\psi'U$ **unfolding** $r\text{-coset-def}$ **by** *fast*
then show $\text{closedin } T (U \#> \sigma)$ **using** $\text{translations-homeos}[OF \text{ in-group}]$
by $(\text{metis } \text{homeomorphic-map-closedness-eq } \text{closed-set})$
qed

lemma $\text{inverse-homeo: homeomorphic-map } T T (\lambda\sigma. \text{inv } \sigma)$
using $\text{homeomorphic-map-involution}[OF \text{ inv-continuous}]$ **by** *auto*

2.3 Subspaces and quotient spaces

abbreviation $\text{connected-component-1} :: 'g \text{ set where}$
 $\text{connected-component-1} \equiv \text{connected-component-of-set } T \mathbf{1}$

lemma $\text{connected-component-1-props:}$

shows $\text{connected-component-1} \triangleleft G$ **and** $\text{closedin } T \text{ connected-component-1}$

proof –

let $?Z = \text{connected-component-of-set } T$

have $\text{in-space: } (?Z \mathbf{1}) \subseteq \text{topspace } T$

using $\text{connected-component-of-subset-topspace}$ **by** *fastforce*

have $\text{subgroup } (?Z \mathbf{1}) G$

proof $(\text{rule } \text{subgroupI})$

show $(?Z \mathbf{1}) \subseteq \text{carrier } G$ **using** in-space **by** *auto*

next

show $(?Z \mathbf{1}) \neq \{\}$

by $(\text{metis } \text{connected-component-of-eq-empty } \text{group-is-space } \text{one-closed})$

next

fix σ **assume** $h\sigma: \sigma \in (?Z \mathbf{1})$

let $?\varphi = \lambda\eta. \text{inv } \eta$

have $?\varphi'(?Z \mathbf{1}) = ?Z \mathbf{1}$ **using** $\text{connected-components-homeo}$

by $(\text{metis } \text{group-is-space } \text{inv-one } \text{inverse-homeo } \text{one-closed})$

then show $\text{inv } \sigma \in (?Z \mathbf{1})$ **using** $h\sigma$ **by** *blast*

next

fix $\sigma \tau$ **assume** $h\sigma: \sigma \in (?Z \mathbf{1})$ **and** $h\tau: \tau \in (?Z \mathbf{1})$

let $?\varphi = \lambda\eta. \sigma \otimes \eta$

have $?\varphi'(?Z \mathbf{1}) = ?Z \sigma$ **using** $\text{connected-components-homeo}$

by $(\text{metis } \text{group-is-space } h\sigma \text{ in-space } \text{one-closed } r\text{-one } \text{subset-eq } \text{translations-homeos}(1))$

moreover have $?Z \sigma = ?Z \mathbf{1}$ **using** $h\sigma$ **by** $(\text{simp add: } \text{connected-component-of-equiv})$

ultimately show $\sigma \otimes \tau \in ?Z \mathbf{1}$ **using** $h\tau$ **by** *blast*

qed

moreover have $\text{conjugation } \sigma \tau \in ?Z \mathbf{1}$ **if** $h\sigma\tau: \sigma \in \text{carrier } G \wedge \tau \in ?Z \mathbf{1}$ **for**

$\sigma \tau$
proof –
 let $? \varphi = \text{conjugation } \sigma$
 have $? \varphi'(?Z \mathbf{1}) = ?Z (? \varphi \mathbf{1})$ **using** *connected-components-homeo*
 by (*metis conjugation-homeo group-is-space one-closed hστ*)
 then show *?thesis* **using** *r-inv r-one hστ* **by** *auto*
qed
 ultimately show *connected-component-1* $\triangleleft G$ **using** *normal-inv-iff* **by** *blast*
next
 show *closedin T connected-component-1* **by** (*simp add: closedin-connected-component-of*)
qed

lemma *group-prod-space [simp]: topspace (prod-topology T T) = (carrier G) × (carrier G)*
by *auto*

no-notation *eq-closure-of (closure'-of1)*

lemma *subgroup-closure:*

assumes *H-subgroup: subgroup H G*

shows *subgroup (T closure-of H) G*

proof –

have *subset: T closure-of H ⊆ carrier G*

by (*metis closedin-closure-of closedin-subset group-is-space*)

have *nonempty: T closure-of H ≠ {}*

by (*simp add: assms closure-of-eq-empty group.subgroupE(1) subgroupE(2)*)

let $? \varphi = \lambda(\sigma, \tau). \sigma \otimes \text{inv } \tau$

have φ -*continuous: continuous-map (prod-topology T T) T ?φ*

proof –

have *continuous-map (prod-topology T T) (prod-topology T T) (λ(σ, τ). (σ, inv τ))*

using *continuous-map-prod-top inv-continuous* **by** *fastforce*

moreover have $? \varphi = (\lambda(\sigma, \tau). \sigma \otimes \tau) \circ (\lambda(\sigma, \tau). (\sigma, \text{inv } \tau))$ **by** *fastforce*

ultimately show *?thesis* **using** *mul-continuous continuous-map-compose* **by**

force

qed

have $\sigma \otimes \text{inv } \tau \in T \text{ closure-of } H$

if $h\sigma\tau: \sigma \in T \text{ closure-of } H \wedge \tau \in T \text{ closure-of } H$ **for** $\sigma \tau$

proof –

have *in-space: σ ⊗ inv τ ∈ topspace T* **using** *subset hστ* **by** *fast*

have $\exists \eta \in H. \eta \in U$ **if** $hU: \text{openin } T U \wedge \sigma \otimes \text{inv } \tau \in U$ **for** U

proof –

let $?V = \{x \in \text{topspace (prod-topology T T)}. ?\varphi x \in U\}$

have *openin (prod-topology T T) ?V*

using φ -*continuous hU openin-continuous-map-preimage* **by** *blast*

moreover have $(\sigma, \tau) \in ?V$

using *hU group-prod-space hστ subset* **by** *force*

ultimately obtain $V_1 V_2$ **where**
 $hV_1 V_2$: $\text{openin } T V_1 \wedge \text{openin } T V_2 \wedge \sigma \in V_1 \wedge \tau \in V_2 \wedge V_1 \times V_2 \subseteq ?V$
by (*smt* (*verit*) *openin-prod-topology-alt*)
then obtain $\sigma' \tau'$ **where** $h\sigma'\tau'$: $\sigma' \in V_1 \cap H \wedge \tau' \in V_2 \cap H$ **using** $h\sigma\tau$
by (*meson* *all-not-in-conv disjoint-iff openin-Int-closure-of-eq-empty*)
then have $? \varphi (\sigma', \tau') \in U$ **using** $hV_1 V_2$ **by** *blast*
moreover have $? \varphi (\sigma', \tau') \in H$ **using** $h\sigma'\tau'$ *H-subgroup subgroupE(3,4)* **by**
simp
ultimately show *?thesis* **by** *blast*
qed
then show *?thesis* **using** *closure-of-def in-space* **by** *force*
qed
then show *?thesis* **using** *subgroupI-alt subset nonempty* **by** *blast*
qed

lemma *normal-subgroup-closure*:
assumes *normal-subgroup*: $N \triangleleft G$
shows $(T \text{ closure-of } N) \triangleleft G$
proof –
have $(\text{conjugation } \sigma)'(T \text{ closure-of } N) \subseteq T \text{ closure-of } N$ **if** $h\sigma$: $\sigma \in \text{carrier } G$
for σ
proof –
have $(\text{conjugation } \sigma)'N \subseteq N$ **using** *normal-subgroup normal-invE(2)* $h\sigma$ **by**
auto
then have $T \text{ closure-of } (\text{conjugation } \sigma)'N \subseteq T \text{ closure-of } N$
using *closure-of-mono* **by** *meson*
moreover have $(\text{conjugation } \sigma)'(T \text{ closure-of } N) \subseteq T \text{ closure-of } (\text{conjugation } \sigma)'N$
using $h\sigma$ *conjugation-homeo*
by (*meson* *continuous-map-eq-image-closure-subset homeomorphic-imp-continuous-map*)
ultimately show *?thesis* **by** *blast*
qed
moreover have *subgroup* $(T \text{ closure-of } N) G$ **using** *subgroup-closure*
by (*simp* *add: normal-invE(1) normal-subgroup*)
ultimately show *?thesis* **using** *normal-inv-iff* **by** *auto*
qed

lemma *topological-subgroup*:
assumes *subgroup* $H G$
shows *topological-group* $(G (\text{carrier} := H)) (\text{subtopology } T H)$
proof –
interpret *subgroup* $H G$ **by** *fact*
let $?H = (G (\text{carrier} := H))$ **and** $?T' = \text{subtopology } T H$
have *H-subspace: topspace* $?T' = H$ **using** *topspace-subtopology-subset* **by** *force*
have *continuous-map* $?T' T (\lambda\sigma. \text{inv } \sigma)$ **using** *continuous-map-from-subtopology*
inv-continuous **by** *blast*
moreover have $(\lambda\sigma. \text{inv } \sigma) \in \text{topspace } ?T' \rightarrow H$ **unfolding** *Pi-def H-subspace*
by *blast*
ultimately have *continuous-map* $?T' ?T' (\lambda\sigma. \text{inv } \sigma)$ **using** *continuous-map-into-subtopology*

by *blast*
then have *sub-inv-continuous: continuous-map* $?T' ?T' (\lambda\sigma. \text{inv } ?\mathcal{H} \sigma)$
using *continuous-map-eq H-subspace m-inv-consistent assms* **by** *fastforce*
have *continuous-map (prod-topology ?T' ?T') T* $(\lambda(\sigma,\tau). \sigma \otimes \tau)$
unfolding *subtopology-Times[symmetric]* **using** *continuous-map-from-subtopology[OF mul-continuous]* **by** *fast*
moreover have $(\lambda(\sigma,\tau). \sigma \otimes \tau) \in \text{topspace (prod-topology ?T' ?T')} \rightarrow H$
unfolding *Pi-def topspace-prod-topology H-subspace* **by** *fast*
ultimately have *continuous-map (prod-topology ?T' ?T') ?T'* $(\lambda(\sigma,\tau). \sigma \otimes \tau)$
using *continuous-map-into-subtopology* **by** *blast*
then have *continuous-map (prod-topology ?T' ?T') ?T'* $(\lambda(\sigma,\tau). \sigma \otimes ?\mathcal{H} \tau)$ **by**
fastforce
then show *?thesis unfolding topological-group-def topological-group-axioms-def*
using *H-subspace sub-inv-continuous* **by** *auto*
qed

Topology on the set of cosets of some subgroup

abbreviation *coset-topology* $:: 'g \text{ set} \Rightarrow 'g \text{ set topology}$ **where**
coset-topology $H \equiv \text{quot-topology } T \text{ (r-coset } G \ H)$

lemma *coset-topology-topospace[simp]*:
shows *topspace (coset-topology H) = (r-coset G H) (carrier G)*
using *projection-quotient-map quotient-imp-surjective-map group-is-space* **by** *metis*

lemma *projection-open-map*:
assumes *subgroup: subgroup H G*
shows *open-map T (coset-topology H) (r-coset G H)*
proof (*unfold open-map-def, standard, standard*)
fix U **assume** $hU: \text{openin } T \ U$
let $? \pi = \text{r-coset } G \ H$
let $?V = \{\sigma \in \text{topspace } T. ? \pi \sigma \in ? \pi 'U\}$
have *subsets: H ⊆ carrier G ∧ U ⊆ carrier G*
using *subgroup hU openin-subset* **by** (*force elim!: subgroupE*)
have $?V = \{\sigma \in \text{carrier } G. \exists \tau \in U. H \#> \sigma = H \#> \tau\}$ **using** *image-def* **by**
blast
then have $?V = \{\sigma \in \text{carrier } G. \exists \tau \in U. \sigma \in H \#> \tau\}$ **using** *subsets*
by (*smt (verit) Collect-cong rcos-self repr-independence subgroup subset-eq*)
also have $\dots = (\bigcup \eta \in H. \eta <\# U)$ **unfolding** *r-coset-def l-coset-def* **using**
subsets **by** *auto*
moreover have *openin T (η <# U) if η ∈ H for η*
using *open-set-translations(1)[OF hU] subsets that* **by** *blast*
ultimately have *openin T ?V* **by** *fastforce*
then show *openin (coset-topology H) (?π 'U)* **using** *quot-topology-open hU*
by (*metis (mono-tags, lifting) Collect-cong image-mono openin-subset*)
qed

lemma *topological-quotient-group*:
assumes *normal-subgroup: N ◁ G*
shows *topological-group (G Mod N) (coset-topology N)*

proof –

interpret *normal* N G **by** *fact*

let $?\pi = r\text{-coset } G \ N$

let $?T' = \text{coset-topology } N$

have *quot-space: topspace* $?T' = ?\pi \text{'(carrier } G)$ **using** *coset-topology-topospace*

by *presburger*

then have *quot-group-quot-space: topspace* $?T' = \text{carrier } (G \ \text{Mod } N)$ **using** *carrier-FactGroup* **by** *metis*

let $?quot\text{-mul} = \lambda(N\sigma, N\tau). N\sigma \otimes_G \text{Mod } N \ N\tau$

have $\pi\text{-prod-space: topspace } (\text{prod-topology } ?T' \ ?T') = ?\pi \text{'(carrier } G) \times ?\pi \text{'(carrier } G)$

using *quot-space topspace-prod-topology* **by** *simp*

have *quot-mul-continuous: continuous-map* $(\text{prod-topology } ?T' \ ?T') \ ?T' \ ?quot\text{-mul}$

proof (*unfold continuous-map-def, intro conjI ballI allI impI*)

show $?quot\text{-mul} \in \text{topspace } (\text{prod-topology } ?T' \ ?T') \rightarrow \text{topspace } ?T'$

using *rcos-sum unfolding quot-space π -prod-space* **by** *auto*

next

fix U **assume** $hU: \text{openin } ?T' \ U$

let $?V = \{p \in \text{topspace } (\text{prod-topology } ?T' \ ?T'). \ ?quot\text{-mul } p \in U\}$

let $?W = \{(\sigma, \tau) \in \text{topspace } (\text{prod-topology } T \ T). \ N \ \#\!> \ (\sigma \otimes \tau) \in U\}$

let $?pi_2 = \lambda(\sigma, \tau). (N \ \#\!> \ \sigma, N \ \#\!> \ \tau)$

have $(\lambda(\sigma, \tau). N \ \#\!> \ (\sigma \otimes \tau)) = ?\pi \circ (\lambda(\sigma, \tau). \sigma \otimes \tau)$ **by** *fastforce*

then have *continuous-map* $(\text{prod-topology } T \ T) \ ?T' \ (\lambda(\sigma, \tau). N \ \#\!> \ (\sigma \otimes \tau))$

using *continuous-map-compose mul-continuous projection-continuous* **by** *fast-force*

then have *openin* $(\text{prod-topology } T \ T) \ ?W$

using hU *openin-continuous-map-preimage*

by (*smt* *verit*) *Collect-cong case-prodE case-prodI2 case-prod-conv*)

moreover have *open-map* $(\text{prod-topology } T \ T) \ (\text{prod-topology } ?T' \ ?T') \ ?pi_2$

using *projection-open-map open-map-prod-top* **by** (*metis subgroup-axioms*)

ultimately have *openin* $(\text{prod-topology } ?T' \ ?T') \ (?pi_2 \text{'?}W)$ **using** *open-map-def*

by *blast*

moreover have $?V = ?pi_2 \text{'?}W$

using *rcos-sum unfolding π -prod-space group-prod-space* **by** *auto*

ultimately show *openin* $(\text{prod-topology } ?T' \ ?T') \ ?V$ **by** *presburger*

qed

let $?quot\text{-inv} = \lambda N\sigma. \text{inv}_G \ \text{Mod } N \ N\sigma$

have $\pi\text{-inv: } ?quot\text{-inv} (N \ \#\!> \ \sigma) = ?\pi (\text{inv } \sigma)$ **if** $\sigma \in \text{carrier } G$ **for** σ

using *inv-FactGroup rcos-inv carrier-FactGroup* **that** **by** *blast*

have *continuous-map* $?T' \ ?T' \ ?quot\text{-inv}$

proof (*unfold continuous-map-def, intro conjI ballI allI impI*)

show $?quot\text{-inv} \in \text{topspace } ?T' \rightarrow \text{topspace } ?T'$ **using** $\pi\text{-inv quot-space}$ **by** *auto*

next

fix U **assume** $hU: \text{openin } ?T' \ U$

let $?V = \{N\sigma \in \text{topspace } ?T'. \ ?quot\text{-inv } N\sigma \in U\}$

let $?W = \{\sigma \in \text{topspace } T. \ N \ \#\!> \ (\text{inv } \sigma) \in U\}$

have $(\lambda\sigma. N \ \#\!> \ (\text{inv } \sigma)) = ?\pi \circ (\lambda\sigma. \text{inv } \sigma)$ **by** *fastforce*

then have *continuous-map* $T \ ?T' (\lambda\sigma. N \ \#\> (\text{inv } \sigma))$
using *continuous-map-compose projection-continuous inv-continuous*
by (*metis (no-types, lifting)*)
then have *openin* $T \ ?W$ **using** *hU openin-continuous-map-preimage* **by** *blast*
then have *openin* $?T' (\ ?\pi \ ' ?W)$
using *projection-open-map* **by** (*simp add: open-map-def subgroup-axioms*)
moreover have $?V = \ ?\pi \ ' ?W$ **using** *π -inv quot-space* **by** *force*
ultimately show *openin* $?T' \ ?V$ **by** *presburger*
qed

then show *?thesis unfolding topological-group-def topological-group-axioms-def*
using *quot-group-quot-space quot-mul-continuous factorgroup-is-group* **by** *blast*
qed

See [3] for our approach to proving that quotient groups of topological groups are topological.

abbreviation *neighborhood* $:: 'g \Rightarrow 'g \text{ set} \Rightarrow \text{bool}$ **where**
neighborhood $\sigma \ U \equiv \text{openin } T \ U \wedge \sigma \in U$

abbreviation *symmetric* $:: 'g \text{ set} \Rightarrow \text{bool}$ **where**
symmetric $S \equiv \{\text{inv } \sigma \mid \sigma. \sigma \in S\} \subseteq S$

Note that this implies the other inclusion, so symmetric subsets are equal to their image under inversion.

lemma *neighborhoods-of-1*:

assumes *neighborhood* $\mathbf{1} \ U$

shows $\exists V. \ \text{neighborhood } \mathbf{1} \ V \wedge \text{symmetric } V \wedge V \ \<\#\> \ V \subseteq U$

proof –

have $a: \exists V \subseteq U'. \ \text{neighborhood } \mathbf{1} \ V \wedge \text{symmetric } V$ **if** $hU': \text{neighborhood } \mathbf{1} \ U'$
for U'

proof –

let $?W = \{\sigma \in \text{carrier } G. \ \text{inv } \sigma \in U'\}$

let $?V = ?W \cap ((\lambda\sigma. \ \text{inv } \sigma) \ ' ?W)$

have *neighborhood* $\mathbf{1} \ ?W$ **using** *openin-continuous-map-preimage[OF inv-continuous]*
 $hU' \ \text{inv-one}$ **by** *fastforce*

moreover from *this* **have** *neighborhood* $\mathbf{1} \ ((\lambda\sigma. \ \text{inv } \sigma) \ ' ?W)$ **using** *inverse-homeo*

homeomorphic-imp-open-map inv-one image-eqI open-map-def **by** (*metis (mono-tags, lifting)*)

ultimately have *neighborhood*: *neighborhood* $\mathbf{1} \ ?V$ **by** *blast*

have $\text{inv } \sigma \in ?V$ **if** $\sigma \in ?V$ **for** σ **using** *that* **by** *auto*

then have *symmetric* $?V$ **by** *fast*

moreover have $\sigma \in U'$ **if** $\sigma \in ?V$ **for** σ **using** *that* **by** *blast*

ultimately show *?thesis* **using** *neighborhood* **by** *blast*

qed

have $b: \exists V. \ \text{neighborhood } \mathbf{1} \ V \wedge V \ \<\#\> \ V \subseteq U'$ **if** $hU': \text{neighborhood } \mathbf{1} \ U'$
for U'

proof –

let $?W = \{(\sigma, \tau) \in \text{carrier } G \times \text{carrier } G. \ \sigma \otimes \tau \in U'\}$

have *preimage-mul*: $?W = \{x \in \text{topspace } (\text{prod-topology } T \ T). (\lambda(\sigma, \tau). \sigma \otimes \tau) x \in U\}$
using *topspace-prod-topology* **by** *fastforce*
then have *openin* $(\text{prod-topology } T \ T) \ ?W \wedge (\mathbf{1}, \mathbf{1}) \in ?W$
using *openin-continuous-map-preimage*[*OF mul-continuous*] hU' *r-one* **by** *fastforce*
then obtain $W_1 \ W_2$ **where** $hW_1 W_2$: *neighborhood* $\mathbf{1} \ W_1 \wedge$ *neighborhood* $\mathbf{1} \ W_2 \wedge W_1 \times W_2 \subseteq ?W$
using *openin-prod-topology-alt*[*where* $S = ?W$] **by** *meson*
let $?V = W_1 \cap W_2$
from $hW_1 W_2$ **have** *neighborhood* $\mathbf{1} \ ?V$ **by** *fast*
moreover have $\sigma \otimes \tau \in U'$ **if** $\sigma \in ?V \wedge \tau \in ?V$ **for** $\sigma \ \tau$ **using** *preimage-mul* $hW_1 W_2$ **that by** *blast*
ultimately show *?thesis unfolding set-mult-def* **by** *blast*
qed
from $b[\text{OF assms}]$ **obtain** W **where** hW : *neighborhood* $\mathbf{1} \ W \wedge W \langle \# \rangle W \subseteq U$ **by** *presburger*
from this a obtain V **where** $V \subseteq W \wedge$ *neighborhood* $\mathbf{1} \ V \wedge$ *symmetric* V **by** *presburger*
moreover from this have $V \langle \# \rangle V \subseteq U$ **using** *hW mono-set-mult* **by** *blast*
ultimately show *?thesis unfolding set-mult-def* **by** *blast*
qed

lemma *Hausdorff-coset-space*:

assumes *subgroup*: *subgroup* $H \ G$ **and** *H-closed*: *closedin* $T \ H$
shows *Hausdorff-space* $(\text{coset-topology } H)$
proof $(\text{unfold Hausdorff-space-def, intro allI impI})$
interpret *subgroup* $H \ G$ **by** *fact*
let $? \pi = \text{r-coset } G \ H$
let $?T' = \text{coset-topology } H$
fix $H \sigma \ H \tau$ **assume** *cosets*: $H \sigma \in \text{topspace } ?T' \wedge H \tau \in \text{topspace } ?T' \wedge H \sigma \neq H \tau$
then obtain $\sigma \ \tau$ **where** $h \sigma \tau$: $\sigma \in \text{carrier } G \wedge \tau \in \text{carrier } G \wedge H \sigma = H \# \rangle \sigma \wedge H \tau = H \# \rangle \tau$ **by** *auto*
then have $\sigma \notin H \# \rangle \tau$ **using** *cosets subgroup repr-independence* **by** *blast*
have $\mathbf{1} \notin (\text{inv } \sigma) \langle \# \rangle (H \# \rangle \tau)$
proof
assume $\mathbf{1} \in \text{inv } \sigma \langle \# \rangle (H \# \rangle \tau)$
then obtain η **where** $h \eta$: $\eta \in H \wedge \mathbf{1} = (\text{inv } \sigma) \otimes (\eta \otimes \tau)$ **unfolding** *r-coset-def* *l-coset-def* **by** *auto*
then have $\sigma = \eta \otimes \tau$
by $(\text{metis } (\text{no-types, lifting}) \text{Units-eq Units-m-closed group.inv-comm group-l-invI } h \sigma \tau \text{ inv-closed inv-inv inv-unique' l-inv-ex mem-carrier})$
then show *False* **using** $\langle \sigma \notin H \# \rangle \tau \rangle h \eta$ *r-coset-def* **by** *fast*
qed
let $?U = \text{topspace } T - ((\text{inv } \sigma) \langle \# \rangle (H \# \rangle \tau))$
have *closedin* $T \ ((\text{inv } \sigma) \langle \# \rangle (H \# \rangle \tau))$
using *closed-set-translations closed-set-translations*[*OF H-closed*] $h \sigma \tau$ **by** *simp*
then have *neighborhood* $\mathbf{1} \ ?U$ **using** $\langle \mathbf{1} \notin (\text{inv } \sigma) \langle \# \rangle (H \# \rangle \tau) \rangle$ **by** *blast*

then obtain V **where** hV : *neighborhood* $\mathbf{1} \ V \wedge$ *symmetric* $V \wedge V <\#\> V \subseteq ?U$
using *neighborhoods-of-1* **by** *presburger*
let $?V_1 = \sigma <\#\ V$ **and** $?V_2 = \tau <\#\ V$
have *disjoint*: $? \pi ' ?V_1 \cap ? \pi ' ?V_2 = \{\}$
proof (*rule ccontr*)
assume $? \pi ' ?V_1 \cap ? \pi ' ?V_2 \neq \{\}$
then obtain $v_1 \ v_2$ **where** $h v_1 v_2$: $v_1 \in V \wedge v_2 \in V \wedge ? \pi (\sigma \otimes v_1) = ? \pi (\tau \otimes v_2)$

unfolding *l-coset-def* **by** *auto*
moreover then have $v_1 v_2$ -*in-group*: $v_1 \in \text{carrier } G \wedge v_2 \in \text{carrier } G$
using hV *openin-subset* **by** *force*
ultimately have *in-H*: $(\sigma \otimes v_1) \otimes \text{inv } (\tau \otimes v_2) \in H$
using *subgroup repr-independenceD rcos-module-imp h\sigma\tau m-closed*
by (*metis group.rcos-self is-group subgroup.m-closed subgroup-self*)
let $? \eta = (\sigma \otimes v_1) \otimes \text{inv } (\tau \otimes v_2)$
have $? \eta = \sigma \otimes (v_1 \otimes \text{inv } v_2) \otimes \text{inv } \tau$ **using** $h\sigma\tau$ $v_1 v_2$ -*in-group m-assoc*
by (*simp add: inv-mult-group subgroupE(4) subgroup-self*)
then have $\text{inv } \sigma \otimes (? \eta \otimes \tau) = v_1 \otimes \text{inv } v_2$
using $h\sigma\tau$ $v_1 v_2$ -*in-group m-assoc inv-solve-left'* **by** *auto*
then have $v_1 \otimes \text{inv } v_2 \in (\text{inv } \sigma) <\#\ (H \#> \tau)$
unfolding *l-coset-def r-coset-def* **using** $h\sigma\tau$ *inv-closed in-H* **by** *force*
moreover have $v_1 \otimes \text{inv } v_2 \in ?U$ **using** $h v_1 v_2 \ hV$ **unfolding** *set-mult-def*
by *blast*
ultimately show *False* **by** *force*
qed
have *neighborhood* $\sigma \ ?V_1 \wedge$ *neighborhood* $\tau \ ?V_2$
using *open-set-translations[of V] l-coset-def hV h\sigma\tau r-one* **by** *force*
then have *openin* $?T' (? \pi ' ?V_1) \wedge$ *openin* $?T' (? \pi ' ?V_2) \wedge H\sigma \in ? \pi ' ?V_1 \wedge H\tau \in ? \pi ' ?V_2$
using *projection-open-map open-map-def subgroup h\sigma\tau* **by** *fast*
then show $\exists W_1 \ W_2$. *openin* $?T' W_1 \wedge$ *openin* $?T' W_2 \wedge H\sigma \in W_1 \wedge H\tau \in W_2 \wedge \text{disjnt } W_1 \ W_2$
using *disjoint disjnt-def* **by** *meson*
qed

lemma *Hausdorff-coset-space-converse*:
assumes *subgroup*: *subgroup* $H \ G$
assumes *Hausdorff*: *Hausdorff-space* (*coset-topology* H)
shows *closedin* $T \ H$
proof –
interpret *subgroup* $H \ G$ **by** *fact*
let $?T' = \text{coset-topology } H$
have $H \in \text{topspace } ?T'$ **using** *coset-topology-topspace coset-join2[of 1 H]* *subgroup* **by** *auto*
then have *closedin* $?T' \ \{H\}$
using *t1-space-closedin-singleton Hausdorff-imp-t1-space[OF Hausdorff]* **by** *fast*
then have *preimage-closed*: *closedin* $T \ \{\sigma \in \text{carrier } G. H \#> \sigma = H\}$
using *projection-continuous closedin-continuous-map-preimage* **by** *fastforce*

have $\sigma \in H \iff H \#> \sigma = H$ **if** $\sigma \in \text{carrier } G$ **for** σ
using *coset-join1 coset-join2 subgroup that by metis*
then have $H = \{\sigma \in \text{carrier } G. H \#> \sigma = H\}$ **using** *subset by auto*
then show *?thesis using preimage-closed by presburger*
qed

corollary *Hausdorff-coset-space-iff:*
assumes *subgroup: subgroup H G*
shows *Hausdorff-space (coset-topology H) \iff closedin T H*
using *Hausdorff-coset-space Hausdorff-coset-space-converse subgroup by blast*

corollary *topological-group-hausdorff-iff-one-closed:*

shows *Hausdorff-space T \iff closedin T {1}*
proof –
let $? \pi = r\text{-coset } G \{1\}$
have *inj-on ? π (carrier G) unfolding inj-on-def r-coset-def by simp*
then have *homeomorphic-map T (coset-topology {1}) ? π*
using *projection-quotient-map injective-quotient-map-homeo group-is-space by metis*
then have *Hausdorff-space T \iff Hausdorff-space (coset-topology {1})*
using *homeomorphic-Hausdorff-space homeomorphic-map-imp-homeomorphic-space*
by *blast*
then show *?thesis using Hausdorff-coset-space-iff triv-subgroup by blast*
qed

lemma *set-mult-one-subset:*
assumes $A \subseteq \text{carrier } G \wedge B \subseteq \text{carrier } G$ **and** $1 \in B$
shows $A \subseteq A \#> B$
unfolding *set-mult-def using assms r-one by force*

lemma *open-set-mult-open:*
assumes *openin T U \wedge S \subseteq carrier G*
shows *openin T (S $\#>$ U)*
proof –
have $S \#> U = (\bigcup \sigma \in S. \sigma \#> U)$ **unfolding** *set-mult-def l-coset-def by blast*
moreover have *openin T ($\sigma \#> U$) if $\sigma \in S$ for σ using open-set-translations(1)*
assms that by auto
ultimately show *?thesis by auto*
qed

lemma *open-set-inv-open:*
assumes *openin T U*
shows *openin T (set-inv U)*
proof –
have $\text{set-inv } U = (\lambda \sigma. \text{inv } \sigma)'U$ **unfolding** *image-def SET-INV-def by blast*
then show *?thesis using inverse-homeo homeomorphic-imp-open-map open-map-def*
assms by metis
qed

lemma *open-set-in-carrier*[elim]:
assumes *openin* T U
shows $U \subseteq \text{carrier } G$
using *openin-subset* *assms* **by force**

2.4 Uniform structures

abbreviation *left-entourage* $:: 'g \text{ set} \Rightarrow ('g \times 'g) \text{ set}$ **where**
left-entourage $U \equiv \{(\sigma, \tau) \in \text{carrier } G \times \text{carrier } G. \text{inv } \sigma \otimes \tau \in U\}$

abbreviation *right-entourage* $:: 'g \text{ set} \Rightarrow ('g \times 'g) \text{ set}$ **where**
right-entourage $U \equiv \{(\sigma, \tau) \in \text{carrier } G \times \text{carrier } G. \sigma \otimes \text{inv } \tau \in U\}$

definition *left-uniformity* $:: 'g \text{ uniformity}$ **where** *left-uniformity* =
uniformity (*carrier* G , $\lambda E. E \subseteq \text{carrier } G \times \text{carrier } G \wedge (\exists U. \text{neighborhood } \mathbf{1} U \wedge \text{left-entourage } U \subseteq E)$)

definition *right-uniformity* $:: 'g \text{ uniformity}$ **where** *right-uniformity* =
uniformity (*carrier* G , $\lambda E. E \subseteq \text{carrier } G \times \text{carrier } G \wedge (\exists U. \text{neighborhood } \mathbf{1} U \wedge \text{right-entourage } U \subseteq E)$)

lemma

uspace-left-uniformity[simp]: *uspace* *left-uniformity* = *carrier* G (**is** *?space-def*)

and

entourage-left-uniformity: *entourage-in* *left-uniformity* =

($\lambda E. E \subseteq \text{carrier } G \times \text{carrier } G \wedge (\exists U. \text{neighborhood } \mathbf{1} U \wedge \text{left-entourage } U \subseteq E)$) (**is** *?entourage-def*)

proof –

let $? \Phi = \lambda E. E \subseteq \text{carrier } G \times \text{carrier } G \wedge (\exists U. \text{neighborhood } \mathbf{1} U \wedge \text{left-entourage } U \subseteq E)$

have $? \Phi$ (*carrier* $G \times \text{carrier } G$)

using *exI*[**where** $x = \text{carrier } G$] *openin-topspace* **by force**

moreover **have** *Id-on* (*carrier* G) $\subseteq E \wedge ? \Phi (E^{-1}) \wedge (\exists F. ? \Phi F \wedge F \circ F \subseteq E) \wedge$

($\forall F. E \subseteq F \wedge F \subseteq \text{carrier } G \times \text{carrier } G \longrightarrow ? \Phi F$) **if** $hE: ? \Phi E$ **for** E

proof –

from hE **obtain** U **where** $hU: \text{neighborhood } \mathbf{1} U \wedge \text{left-entourage } U \subseteq E$ **by** *presburger*

then **have** *U-subset*: $U \subseteq \text{carrier } G$ **using** *openin-subset* **by force**

from hU **have** *Id-on* (*carrier* G) $\subseteq E$ **by** *fastforce*

moreover **have** $? \Phi (E^{-1})$

proof –

have $(\tau, \sigma) \in E$ **if** $\sigma \in \text{carrier } G \wedge \tau \in \text{carrier } G \wedge \text{inv } \sigma \otimes \tau \in \text{set-inv } U$ **for** $\sigma \tau$

proof –

have $\text{inv } \tau \otimes \sigma = \text{inv } (\text{inv } \sigma \otimes \tau)$ **using** *that inv-mult-group* **by auto**

from this **have** $\text{inv } \tau \otimes \sigma \in U$ **using** *that inv-inv U-subset* **unfolding** *SET-INV-def* **by auto**

then show *?thesis using that hU by fast*
 qed
 then have *left-entourage (set-inv U) $\subseteq E^{-1}$ by blast*
 moreover have *neighborhood 1 (set-inv U) using inv-one hU open-set-inv-open SET-INV-def by fastforce*
 ultimately show *?thesis using hE by auto*
 qed
 moreover have $\exists F. ?\Phi F \wedge F \circ F \subseteq E$
 proof –
 obtain V where $hV: \text{neighborhood } 1 \ V \wedge V <\#\> V \subseteq U$
 using *neighborhoods-of-1 hU by meson*
 let $?F = \text{left-entourage } V$
 have $(\sigma, \rho) \in E$ if $(\sigma, \tau) \in ?F \wedge (\tau, \rho) \in ?F$ for $\sigma \ \tau \ \rho$
 proof –
 have $\sigma \in \text{carrier } G \wedge \tau \in \text{carrier } G \wedge \rho \in \text{carrier } G$ using *that by force*
 then have $\text{inv } \sigma \otimes \rho = (\text{inv } \sigma \otimes \tau) \otimes (\text{inv } \tau \otimes \rho)$
 using *m-assoc inv-closed m-closed r-inv r-one by metis*
 moreover have $(\text{inv } \sigma \otimes \tau) \otimes (\text{inv } \tau \otimes \rho) \in U$ using *that hV unfolding set-mult-def by fast*
 ultimately show *?thesis using hU that by force*
 qed
 moreover have $?\Phi ?F$ using *hV by blast*
 ultimately show *?thesis using hV by auto*
 qed
 moreover have $\forall F. E \subseteq F \wedge F \subseteq \text{carrier } G \times \text{carrier } G \longrightarrow ?\Phi F$ using *hE by auto*
 ultimately show *?thesis by blast*
 qed
 moreover have $?\Phi (E \cap F)$ if $hEF: ?\Phi E \wedge ?\Phi F$ for $E \ F$
 proof –
 from hEF obtain $U \ V$ where
 $hU: \text{neighborhood } 1 \ U \wedge \text{left-entourage } U \subseteq E$ and
 $hV: \text{neighborhood } 1 \ V \wedge \text{left-entourage } V \subseteq F$ by *presburger*
 then have $\text{neighborhood } 1 \ (U \cap V) \wedge \text{left-entourage } (U \cap V) \subseteq E \cap F$ by *fast*
 then show *?thesis using that by auto*
 qed
 ultimately have *uniformity-on (carrier G) ?\Phi*
 unfolding *uniformity-on-def by auto*
 from *uniformity-inverse'[OF this]* show *?space-def and ?entourage-def unfolding left-uniformity-def by auto*
 qed

lemma
uspace-right-uniformity[simp]: uspace right-uniformity = carrier G (is ?space-def)
and
entourage-right-uniformity: entourage-in right-uniformity =
 $(\lambda E. E \subseteq \text{carrier } G \times \text{carrier } G \wedge (\exists U. \text{neighborhood } 1 \ U \wedge \text{right-entourage } U \subseteq E))$ (is *?entourage-def*)

proof –
let $?\Phi = \lambda E. E \subseteq \text{carrier } G \times \text{carrier } G \wedge (\exists U. \text{neighborhood } \mathbf{1} \ U \wedge \text{right-entourage } U \subseteq E)$
have $?\Phi (\text{carrier } G \times \text{carrier } G)$
using $\text{exI}[\text{where } x = \text{carrier } G]$ **openin-topspace** **by** *force*
moreover **have** $\text{Id-on } (\text{carrier } G) \subseteq E \wedge ?\Phi (E^{-1}) \wedge (\exists F. ?\Phi F \wedge F \circ F \subseteq E) \wedge$
 $(\forall F. E \subseteq F \wedge F \subseteq \text{carrier } G \times \text{carrier } G \longrightarrow ?\Phi F)$ **if** $hE: ?\Phi E$ **for** E
proof –
from hE **obtain** U **where**
 $hU: \text{neighborhood } \mathbf{1} \ U \wedge \text{right-entourage } U \subseteq E$
by *presburger*
then **have** $U\text{-subset}: U \subseteq \text{carrier } G$ **using** *openin-subset* **by** *force*
from hU **have** $\text{Id-on } (\text{carrier } G) \subseteq E$ **by** *fastforce*
moreover **have** $?\Phi (E^{-1})$
proof –
have $(\tau, \sigma) \in E$ **if** $\sigma \in \text{carrier } G \wedge \tau \in \text{carrier } G \wedge \sigma \otimes \text{inv } \tau \in \text{set-inv } U$
for $\sigma \ \tau$
proof –
have $\tau \otimes \text{inv } \sigma = \text{inv } (\sigma \otimes \text{inv } \tau)$ **using** *that inv-mult-group* **by** *auto*
from *this* **have** $\tau \otimes \text{inv } \sigma \in U$ **using** *that inv-inv U-subset* **unfolding**
SET-INV-def **by** *auto*
then **show** *?thesis* **using** *that hU* **by** *fast*
qed
then **have** $\text{right-entourage } (\text{set-inv } U) \subseteq E^{-1}$ **by** *blast*
moreover **have** $\text{neighborhood } \mathbf{1} (\text{set-inv } U)$ **using** *inv-one hU open-set-inv-open*
SET-INV-def **by** *fastforce*
ultimately **show** *?thesis* **using** hE **by** *auto*
qed
moreover **have** $\exists F. ?\Phi F \wedge F \circ F \subseteq E$
proof –
obtain V **where** $hV: \text{neighborhood } \mathbf{1} \ V \wedge V <\#\> V \subseteq U$
using *neighborhoods-of-1 hU* **by** *meson*
let $?F = \text{right-entourage } V$
have $(\sigma, \rho) \in E$ **if** $(\sigma, \tau) \in ?F \wedge (\tau, \rho) \in ?F$ **for** $\sigma \ \tau \ \rho$
proof –
have $\sigma \in \text{carrier } G \wedge \tau \in \text{carrier } G \wedge \rho \in \text{carrier } G$ **using** *that* **by** *force*
then **have** $\sigma \otimes \text{inv } \rho = (\sigma \otimes \text{inv } \tau) \otimes (\tau \otimes \text{inv } \rho)$
using *m-assoc inv-closed m-closed l-inv r-one* **by** *metis*
moreover **have** $(\sigma \otimes \text{inv } \tau) \otimes (\tau \otimes \text{inv } \rho) \in U$ **using** *that hV* **unfolding**
set-mult-def **by** *fast*
ultimately **show** *?thesis* **using** hU **that** **by** *force*
qed
moreover **have** $?\Phi ?F$ **using** hV **by** *blast*
ultimately **show** *?thesis* **using** hV **by** *auto*
qed
moreover **have** $\forall F. E \subseteq F \wedge F \subseteq \text{carrier } G \times \text{carrier } G \longrightarrow ?\Phi F$ **using**
 hE **by** *auto*
ultimately **show** *?thesis* **by** *blast*

qed
moreover have $?Φ (E \cap F)$ **if** $hEF: ?Φ E \wedge ?Φ F$ **for** $E F$
proof –
from hEF **obtain** $U V$ **where**
 $hU: \text{neighborhood } \mathbf{1} U \wedge \text{right-entourage } U \subseteq E$ **and**
 $hV: \text{neighborhood } \mathbf{1} V \wedge \text{right-entourage } V \subseteq F$
by *presburger*
then have $\text{neighborhood } \mathbf{1} (U \cap V) \wedge \text{right-entourage } (U \cap V) \subseteq E \cap F$ **by**
fast
then show *?thesis using that by auto*
qed
ultimately have *uniformity-on (carrier G) ?Φ*
unfolding *uniformity-on-def by auto*
from *uniformity-inverse'[OF this]* **show** *?space-def and ?entourage-def unfolding right-uniformity-def by auto*
qed

lemma *left-uniformity-induces-group-topology [simp]:*
shows *utopology left-uniformity = T*
proof –
let $?Φ = \text{left-uniformity}$
let $?T' = \text{utopology } ?Φ$
have $\text{openin } T U \longleftrightarrow \text{openin } ?T' U$ **for** U
proof
assume $U\text{-open}: \text{openin } T U$
have $\exists E. \text{entourage-in } ?Φ E \wedge E''\{\sigma\} \subseteq U$ **if** $h\sigma: \sigma \in U$ **for** σ
proof –
let $?E = \text{left-entourage } (\text{inv } \sigma < \# U)$
have $\text{in-group}: \sigma \in \text{carrier } G$ **using** $h\sigma$ $U\text{-open}$ *open-set-in-carrier by blast*
then have $\text{openin } T (\text{inv } \sigma < \# U)$
using *inv-closed open-set-translations(1) U-open by presburger*
then have $\text{neighborhood } \mathbf{1} (\text{inv } \sigma < \# U)$
using $h\sigma$ *in-group r-inv unfolding l-coset-def SET-INV-def by force*
then have $\text{entourage-in } ?Φ ?E$ **unfolding** *entourage-left-uniformity by blast*
moreover have $\tau \in U$ **if** $\tau \in ?E''\{\sigma\}$ **for** τ
proof –
from that have $\text{inv } \sigma \otimes \tau \in \text{inv } \sigma < \# U$ **by force**
then obtain ρ **where** $h\rho: \rho \in U \wedge \text{inv } \sigma \otimes \tau = \text{inv } \sigma \otimes \rho$ **unfolding**
l-coset-def by fast
then have $\rho \in \text{carrier } G \wedge \tau \in \text{carrier } G$ **using** *that open-set-in-carrier U-open by fast*
then have $\tau = \rho$ **using** *in-group h\rho inv-closed by (metis Units-eq Units-l-cancel)*
then show *?thesis using h\rho by simp*
qed
ultimately show *?thesis by blast*
qed
moreover have $U \subseteq \text{uspace } ?Φ$ **using** *openin-subset U-open by force*
ultimately show $\text{openin } ?T' U$ **unfolding** *openin-utopology by force*
next

assume U -open: $\text{openin } ?T' U$
have $\exists W$. neighborhood $\sigma W \wedge W \subseteq U$ **if** $h\sigma: \sigma \in U$ **for** σ
proof –
have $\text{in-group: } \sigma \in \text{carrier } G$ **using** $h\sigma$ U -open $\text{openin-subset topspace-utopology}$
by force
from U -open $h\sigma$ **obtain** E **where** $hE: \text{entourage-in } ?\Phi E \wedge E''\{\sigma\} \subseteq U$
unfolding openin-utopology **by** blast
then obtain V **where** $hV: \text{neighborhood } \mathbf{1} V \wedge \text{left-entourage } V \subseteq E$
unfolding $\text{entourage-left-uniformity}$ **by** fastforce
let $?W = \{\tau \in \text{carrier } G. \text{inv } \sigma \otimes \tau \in V\}$
from hV **have** W -subset: $?W \subseteq E''\{\sigma\}$ **using** in-group **by** fast
have $\text{continuous-map } T T (\lambda\tau. \text{inv } \sigma \otimes \tau)$ **using** $\text{translations-continuous}$
 $\text{in-group inv-closed}$ **by** blast
then have $\text{openin } T ?W$ **using** $\text{openin-continuous-map-preimage } hV$ **by**
 fastforce
then have neighborhood $\sigma ?W$ **using** $\text{in-group r-inv } hV$ **by** simp
then show $?thesis$ **using** W -subset hE **by** fast
qed
then show $\text{openin } T U$ **using** openin-subopen **by** force
qed
then show $?thesis$ **using** topology-eq **by** blast
qed

lemma $\text{right-uniformity-induces-group-topology}$ [simp]:

shows $\text{utopology right-uniformity} = T$
proof –
let $?\Phi = \text{right-uniformity}$
let $?T' = \text{utopology } ?\Phi$
have $\text{openin } T U \longleftrightarrow \text{openin } ?T' U$ **for** U
proof
assume U -open: $\text{openin } T U$
have $\exists E$. $\text{entourage-in } ?\Phi E \wedge E''\{\sigma\} \subseteq U$ **if** $h\sigma: \sigma \in U$ **for** σ
proof –
let $?E = \text{right-entourage } (\sigma <\# \text{set-inv } U)$
have $\text{in-group: } \sigma \in \text{carrier } G$ **using** $h\sigma$ U -open $\text{open-set-in-carrier}$ **by** blast
then have $\text{openin } T (\sigma <\# \text{set-inv } U)$
using $\text{open-set-inv-open open-set-translations(1) } U$ -open **by** presburger
then have neighborhood $\mathbf{1} (\sigma <\# \text{set-inv } U)$
using $h\sigma$ in-group r-inv **unfolding** $\text{l-coset-def SET-INV-def}$ **by** force
then have $\text{entourage-in } ?\Phi ?E$ **unfolding** $\text{entourage-right-uniformity}$ **by** blast
moreover have $\tau \in U$ **if** $\tau \in ?E''\{\sigma\}$ **for** τ
proof –
from that have $\sigma \otimes \text{inv } \tau \in \sigma <\# \text{set-inv } U$ **by** force
then obtain ϱ **where** $h\varrho: \varrho \in U \wedge \sigma \otimes \text{inv } \tau = \sigma \otimes \text{inv } \varrho$
unfolding $\text{l-coset-def SET-INV-def}$ **by** fast
then have $\varrho \in \text{carrier } G \wedge \tau \in \text{carrier } G$ **using** $\text{that open-set-in-carrier}$
 U -open **by** fast
then have $\tau = \varrho$ **using** $\text{in-group } h\varrho$ inv-closed **by** ($\text{metis Units-eq Units-l-cancel}$
 inv-inv)

then show *?thesis* **using** *h ρ* **by** *simp*
qed
ultimately show *?thesis* **by** *blast*
qed
moreover have $U \subseteq \text{uspace } ?\Phi$ **using** *openin-subset* *U-open* **by** *force*
ultimately show *openin ?T' U* **unfolding** *openin-utopology* **by** *force*
next
assume *U-open: openin ?T' U*
have $\exists W. \text{neighborhood } \sigma \ W \wedge W \subseteq U$ **if** *h σ : $\sigma \in U$* **for** σ
proof –
have *in-group: $\sigma \in \text{carrier } G$* **using** *h σ U-open openin-subset topspace-utopology*
by *force*
from *U-open h σ* **obtain** E **where** *hE: entourage-in ? Φ $E \wedge E^{\{\sigma\}} \subseteq U$*
unfolding *openin-utopology* **by** *blast*
then obtain V **where** *hV: neighborhood $\mathbf{1} \ V \wedge \text{right-entourage } V \subseteq E$*
unfolding *entourage-right-uniformity* **by** *fastforce*
let $?W = \{\tau \in \text{carrier } G. \sigma \otimes \text{inv } \tau \in V\}$
from *hV* **have** *W-subset: ?W $\subseteq E^{\{\sigma\}}$* **using** *in-group* **by** *fast*
have $(\lambda\tau. \sigma \otimes \text{inv } \tau) = (\lambda\tau. \sigma \otimes \tau) \circ (\lambda\tau. \text{inv } \tau)$ **by** *fastforce*
then have *continuous-map T T* $(\lambda\tau. \sigma \otimes \text{inv } \tau)$ **using** *continuous-map-compose*
inv-continuous
translations-continuous[OF in-group] **by** *metis*
then have *openin T ?W* **using** *openin-continuous-map-preimage hV* **by**
fastforce
then have *neighborhood σ ?W* **using** *in-group r-inv hV* **by** *simp*
then show *?thesis* **using** *W-subset hE* **by** *fast*
qed
then show *openin T U* **using** *openin-subopen* **by** *force*
qed
then show *?thesis* **using** *topology-eq* **by** *blast*
qed

lemma *translations-ucontinuous:*
assumes *in-group: $\sigma \in \text{carrier } G$*
shows *ucontinuous left-uniformity left-uniformity $(\lambda\tau. \sigma \otimes \tau)$ and*
ucontinuous right-uniformity right-uniformity $(\lambda\tau. \tau \otimes \sigma)$
proof –
let $?\Phi = \text{left-uniformity}$
have *entourage-in ? Φ $\{(\tau_1, \tau_2) \in \text{uspace } ?\Phi \times \text{uspace } ?\Phi. (\sigma \otimes \tau_1, \sigma \otimes \tau_2) \in E\}$*
if *hE: entourage-in ? Φ E for E*
proof –
let $?F = \{(\tau_1, \tau_2) \in \text{uspace } ?\Phi \times \text{uspace } ?\Phi. (\sigma \otimes \tau_1, \sigma \otimes \tau_2) \in E\}$
from *hE* **obtain** U **where** *hU: neighborhood $\mathbf{1} \ U \wedge \text{left-entourage } U \subseteq E$*
unfolding *entourage-left-uniformity* **by** *auto*
have $(\tau_1, \tau_2) \in ?F$ **if** $\tau_1 \in \text{carrier } G \wedge \tau_2 \in \text{carrier } G \wedge \text{inv } \tau_1 \otimes \tau_2 \in U$ **for**
 $\tau_1 \ \tau_2$
proof –
have $\text{inv } (\sigma \otimes \tau_1) \otimes (\sigma \otimes \tau_2) = \text{inv } \tau_1 \otimes \tau_2$

using *that in-group m-closed inv-closed inv-mult-group m-assoc r-inv r-one*
by (*smt (verit, ccfv-threshold)*)
then have $(\sigma \otimes \tau_1, \sigma \otimes \tau_2) \in E$ **using** *that hU in-group m-closed by fastforce*
then show *?thesis using that by auto*
qed
then have *left-entourage* $U \subseteq ?F$ **by force**
then show *?thesis unfolding entourage-left-uniformity using hU by auto*
qed
moreover have $(\lambda\tau. \sigma \otimes \tau) \in \text{uspace } ?\Phi \rightarrow \text{uspace } ?\Phi$
unfolding *Pi-def using uspace-left-uniformity in-group m-closed by force*
ultimately show *ucontinuous ?\Phi ?\Phi* $(\lambda\tau. \sigma \otimes \tau)$
unfolding *ucontinuous-def by fast*
next
let $?\Phi = \text{right-uniformity}$
have *entourage-in ?\Phi* $\{(\tau_1, \tau_2) \in \text{uspace } ?\Phi \times \text{uspace } ?\Phi. (\tau_1 \otimes \sigma, \tau_2 \otimes \sigma) \in E\}$
if *hE: entourage-in ?\Phi E for E*
proof –
let $?F = \{(\tau_1, \tau_2) \in \text{uspace } ?\Phi \times \text{uspace } ?\Phi. (\tau_1 \otimes \sigma, \tau_2 \otimes \sigma) \in E\}$
from *hE obtain U where hU: neighborhood 1 U \wedge right-entourage U \subseteq E*
unfolding *entourage-right-uniformity by auto*
have $(\tau_1, \tau_2) \in ?F$ **if** $\tau_1 \in \text{carrier } G \wedge \tau_2 \in \text{carrier } G \wedge \tau_1 \otimes \text{inv } \tau_2 \in U$ **for**
 $\tau_1 \tau_2$
proof –
have $(\tau_1 \otimes \sigma) \otimes \text{inv } (\tau_2 \otimes \sigma) = \tau_1 \otimes \text{inv } \tau_2$
using *that in-group m-closed inv-closed inv-mult-group m-assoc r-inv r-one*
by (*smt (verit, ccfv-threshold)*)
then have $(\tau_1 \otimes \sigma, \tau_2 \otimes \sigma) \in E$ **using** *that hU in-group m-closed by fastforce*
then show *?thesis using that by simp*
qed
then have *right-entourage* $U \subseteq ?F$ **by force**
then show *?thesis unfolding entourage-right-uniformity using hU by auto*
qed
moreover have $(\lambda\tau. \tau \otimes \sigma) \in \text{uspace } ?\Phi \rightarrow \text{uspace } ?\Phi$
unfolding *Pi-def using entourage-right-uniformity in-group m-closed by force*
ultimately show *ucontinuous ?\Phi ?\Phi* $(\lambda\tau. \tau \otimes \sigma)$
unfolding *ucontinuous-def by fast*
qed

2.5 The Birkhoff-Kakutani theorem

2.5.1 Prenorms on groups

definition *group-prenorm* $:: ('g \Rightarrow \text{real}) \Rightarrow \text{bool}$ **where**

group-prenorm $N \longleftrightarrow$

$N \mathbf{1} = 0 \wedge$

$(\forall \sigma \tau. \sigma \in \text{carrier } G \wedge \tau \in \text{carrier } G \longrightarrow N (\sigma \otimes \tau) \leq N \sigma + N \tau) \wedge$

$(\forall \sigma \in \text{carrier } G. N (\text{inv } \sigma) = N \sigma)$

lemma *group-prenorm-clauses[elim]:*

assumes *group-prenorm* N

obtains

$N \mathbf{1} = 0$ **and**

$\bigwedge \sigma \tau. \sigma \in \text{carrier } G \implies \tau \in \text{carrier } G \implies N (\sigma \otimes \tau) \leq N \sigma + N \tau$ **and**

$\bigwedge \sigma. \sigma \in \text{carrier } G \implies N (\text{inv } \sigma) = N \sigma$

using *assms* **unfolding** *group-prenorm-def* **by** *auto*

proposition *group-prenorm-nonnegative*:

assumes *prenorm: group-prenorm* N

shows $\forall \sigma \in \text{carrier } G. N \sigma \geq 0$

proof

fix σ **assume** $\sigma \in \text{carrier } G$

from *r-inv this* **have** $0 \leq N \sigma + N \sigma$ **using** *assms inv-closed group-prenorm-clauses*
by *metis*

then show $N \sigma \geq 0$ **by** *fastforce*

qed

proposition *group-prenorm-reverse-triangle-ineq*:

assumes *prenorm: group-prenorm* N **and** *in-group: $\sigma \in \text{carrier } G \wedge \tau \in \text{carrier } G$*

shows $|N \sigma - N \tau| \leq N (\sigma \otimes \text{inv } \tau)$

proof –

have $\sigma = \sigma \otimes \text{inv } \tau \otimes \tau$ **using** *in-group inv-closed r-one l-inv m-assoc* **by** *metis*

then have $a: N \sigma \leq N (\sigma \otimes \text{inv } \tau) + N \tau$ **using** *in-group inv-closed m-closed*
prenorm group-prenorm-clauses **by** *metis*

have $\text{inv } \tau = \text{inv } \sigma \otimes (\sigma \otimes \text{inv } \tau)$ **using** *in-group inv-closed l-one l-inv m-assoc*
by *metis*

then have $b: N \tau \leq N \sigma + N (\sigma \otimes \text{inv } \tau)$ **using** *in-group inv-closed m-closed*
prenorm group-prenorm-clauses **by** *metis*

from a b **show** *?thesis* **by** *linarith*

qed

definition *induced-group-prenorm* $:: ('g \Rightarrow \text{real}) \Rightarrow 'g \Rightarrow \text{real}$ **where**

induced-group-prenorm $f \sigma = (\text{SUP } \tau \in \text{carrier } G. |f (\tau \otimes \sigma) - f \tau|)$

lemma *induced-group-prenorm-welldefined*:

fixes $f :: 'g \Rightarrow \text{real}$

assumes *f-bounded: $\exists c. \forall \tau \in \text{carrier } G. |f \tau| \leq c$* **and** *in-group: $\sigma \in \text{carrier } G$*

shows *bdd-above* $((\lambda \tau. |f (\tau \otimes \sigma) - f \tau|) (\text{carrier } G))$

proof –

from *f-bounded* **obtain** c **where** $hc: \forall \tau \in \text{carrier } G. |f \tau| \leq c$ **by** *blast*

have $|f (\tau \otimes \sigma) - f \tau| \leq 2 * c$ **if** $\tau \in \text{carrier } G$ **for** τ

proof –

have $|f (\tau \otimes \sigma) - f \tau| \leq |f (\tau \otimes \sigma)| + |f \tau|$ **using** *abs-triangle-ineq* **by** *simp*

then show *?thesis* **using** *in-group that m-closed f-bounded hc* **by** *(smt (verit, best))*

qed

then show *?thesis* **unfolding** *bdd-above-def image-def* **by** *blast*

qed

lemma *bounded-function-induces-group-prenorm*:

fixes $f :: 'g \Rightarrow \text{real}$

assumes $f\text{-bounded}$: $\exists c. \forall \sigma \in \text{carrier } G. |f \sigma| \leq c$

shows *group-prenorm (induced-group-prenorm f)*

proof –

let $?N = \lambda \sigma. \text{SUP } \tau \in \text{carrier } G. |f (\tau \otimes \sigma) - f \tau|$

have $?N \mathbf{1} = (\text{SUP } \tau \in \text{carrier } G. 0)$ **using** *r-one* **by** *simp*

then have $?N \mathbf{1} = 0$ **using** *carrier-not-empty* **by** *simp*

moreover have $?N (\sigma \otimes \tau) \leq ?N \sigma + ?N \tau$ **if** $h\sigma\tau$: $\sigma \in \text{carrier } G \wedge \tau \in \text{carrier } G$ **for** $\sigma \tau$

proof –

have $|f (\varrho \otimes (\sigma \otimes \tau)) - f \varrho| \leq ?N \sigma + ?N \tau$ **if** $\varrho \in \text{carrier } G$ **for** ϱ

proof –

have a : $|f (\varrho \otimes (\sigma \otimes \tau)) - f \varrho| \leq |f (\varrho \otimes (\sigma \otimes \tau)) - f (\varrho \otimes \sigma)| + |f (\varrho \otimes \sigma) - f \varrho|$

using *abs-triangle-ineq* **by** *linarith*

have $|f (\varrho \otimes \sigma \otimes \tau) - f (\varrho \otimes \sigma)| \leq ?N \tau$

using *induced-group-prenorm-welldefined[OF f-bounded]* **that** $h\sigma\tau$ *m-closed* *cSUP-upper* **by** *meson*

then have b : $|f (\varrho \otimes (\sigma \otimes \tau)) - f (\varrho \otimes \sigma)| \leq ?N \tau$ **using** *m-assoc* **that** $h\sigma\tau$ **by** *simp*

have c : $|f (\varrho \otimes \sigma) - f \varrho| \leq ?N \sigma$ **using** *induced-group-prenorm-welldefined[OF f-bounded]* $h\sigma\tau$ **that** *cSUP-upper* **by** *meson*

from $a \ b \ c$ **show** *?thesis* **by** *argo*

qed

then show *?thesis* **using** *cSUP-least* *carrier-not-empty* **by** *meson*

qed

moreover have $?N (\text{inv } \sigma) = ?N \sigma$ **if** $h\sigma$: $\sigma \in \text{carrier } G$ **for** σ

proof –

have $|f (\tau \otimes \text{inv } \sigma) - f \tau| \in \{|f (\varrho \otimes \sigma) - f \varrho| \mid \varrho. \varrho \in \text{carrier } G\}$ **if** $\tau \in \text{carrier } G$ **for** τ

proof –

have $|f (\tau \otimes \text{inv } \sigma) - f \tau| = |f (\tau \otimes \text{inv } \sigma) - f (\tau \otimes \text{inv } \sigma \otimes \sigma)|$

using $h\sigma$ **that** *m-assoc* *r-one* *l-inv* **by** *simp*

then have $|f (\tau \otimes \text{inv } \sigma) - f \tau| = |f (\tau \otimes \text{inv } \sigma \otimes \sigma) - f (\tau \otimes \text{inv } \sigma)|$ **by** *argo*

then show *?thesis* **using** $h\sigma$ **that** *m-closed* **by** *blast*

qed

moreover

have $|f (\varrho \otimes \sigma) - f \varrho| \in \{|f (\tau \otimes \text{inv } \sigma) - f \tau| \mid \tau. \tau \in \text{carrier } G\}$ **if** $\varrho \in \text{carrier } G$ **for** ϱ

proof –

have $|f (\varrho \otimes \sigma) - f \varrho| = |f (\varrho \otimes \sigma) - f (\varrho \otimes \sigma \otimes \text{inv } \sigma)|$

using $h\sigma$ **that** *m-assoc* *r-one* *r-inv* **by** *simp*

then have $|f (\varrho \otimes \sigma) - f \varrho| = |f (\varrho \otimes \sigma \otimes \text{inv } \sigma) - f (\varrho \otimes \sigma)|$ **by** *argo*

then show *?thesis* **using** $h\sigma$ **that** **by** *blast*

qed

ultimately have $\{|f (\tau \otimes \text{inv } \sigma) - f \tau| \mid \tau. \tau \in \text{carrier } G\} = \{|f (\varrho \otimes \sigma) - f \varrho| \mid \varrho. \varrho \in \text{carrier } G\}$

$f \varrho \mid \varrho. \varrho \in \text{carrier } G$ **by** *blast*
then show *?thesis* **by** (*simp add: setcompr-eq-image*)
qed
ultimately show *?thesis unfolding induced-group-prenorm-def group-prenorm-def*
by *fast*
qed

lemma *neighborhood-1-translation*:
assumes *neighborhood 1 U* **and** $\sigma \in \text{carrier } G \vee \sigma \in \text{topspace } T$
shows *neighborhood σ ($\sigma < \# U$)*
proof –
have *openin T ($\sigma < \# U$)* **using** *assms open-set-translations(1)* **by** *simp*
then show *?thesis unfolding l-coset-def* **using** *assms r-one* **by** *force*
qed

proposition *group-prenorm-continuous-if-continuous-at-1*:
assumes *prenorm: group-prenorm N* **and**
continuous-at-1: $\forall \varepsilon > 0. \exists U. \text{neighborhood } 1 U \wedge (\forall \sigma \in U. N \sigma < \varepsilon)$
shows *continuous-map T euclideanreal N*
proof –
have $\exists V. \text{neighborhood } \sigma V \wedge (\forall \tau \in V. N \tau \in \text{Met-TC.mball } (N \sigma) \varepsilon)$
if $h\sigma: \sigma \in \text{topspace } T$ **and** $h\varepsilon: \varepsilon > 0$ **for** $\sigma \varepsilon$
proof –
from *continuous-at-1* **obtain** U **where** $hU: \text{neighborhood } 1 U \wedge (\forall \tau \in U. N \tau < \varepsilon)$ **using** *h\varepsilon* **by** *presburger*
then have *neighborhood σ ($\sigma < \# U$)* **using** *h\sigma neighborhood-1-translation* **by** *blast*
moreover have $N (\sigma \otimes \tau) \in \text{Met-TC.mball } (N \sigma) \varepsilon$ **if** $\tau \in U$ **for** τ
proof –
have *in-group: $\sigma \in \text{carrier } G \wedge \tau \in \text{carrier } G$* **using** *h\sigma that openin-subset hU* **by** *blast*
then have $(\text{inv } \sigma) \otimes (\sigma \otimes \tau) = \tau$ **using** *l-inv l-one m-assoc inv-closed* **by** *metis*
then have $|N (\text{inv } \sigma) - N (\text{inv } (\sigma \otimes \tau))| \leq N \tau$ **using** *group-prenorm-reverse-triangle-ineq*

in-group inv-closed m-closed **by** (*metis inv-inv prenorm*)
then have $|N \sigma - N (\sigma \otimes \tau)| < \varepsilon$
using *prenorm in-group m-closed inv-closed hU that* **by** *fastforce*
then show *?thesis unfolding Met-TC.mball-def dist-real-def* **by** *fast*
qed
ultimately show *?thesis unfolding l-coset-def* **by** *blast*
qed
then show *?thesis using Metric-space.continuous-map-to-metric*
by (*metis Met-TC.Metric-space-axioms mtopology-is-euclidean*)
qed

2.5.2 A prenorm respecting the group topology

context


```

fixes  $U :: \text{nat} \Rightarrow 'g \text{ set}$ 
assumes  $U\text{-neighborhood}: \forall n. \text{neighborhood } \mathbf{1} (U\ n)$ 
assumes  $U\text{-props}: \forall n. \text{symmetric } (U\ n) \wedge (U\ (n + 1)) <\#\> (U\ (n + 1)) \subseteq$ 
 $(U\ n)$ 
begin

private fun  $V :: \text{nat} \Rightarrow \text{nat} \Rightarrow 'g \text{ set}$  where
 $V\ m\ n = ($ 
   $\text{if } m = 0 \text{ then } \{\} \text{ else}$ 
   $\text{if } m = 1 \text{ then } U\ n \text{ else}$ 
   $\text{if } m > 2^{\wedge}n \text{ then carrier } G \text{ else}$ 
   $\text{if even } m \text{ then } V\ (m \text{ div } 2)\ (n - 1) \text{ else}$ 
   $V\ ((m - 1) \text{ div } 2)\ (n - 1) <\#\> U\ n$ 
 $)$ 

private lemma  $U\text{-in-group}: U\ k \subseteq \text{carrier } G$  using  $U\text{-neighborhood open-set-in-carrier}$ 
by fast

private lemma  $V\text{-in-group}:$ 
  shows  $V\ m\ n \subseteq \text{carrier } G$ 
proof ( $\text{induction } n \text{ arbitrary}: m$ )
  case ( $\text{Suc } n$ )
  then have  $V\ ((m - 1) \text{ div } 2)\ n <\#\> U\ (\text{Suc } n) \subseteq \text{carrier } G$ 
  unfolding set-mult-def using  $U\text{-in-group}$  by fast
  then show  $?case$  using  $U\text{-in-group Suc}$  by simp
qed ( $\text{auto simp}: U\text{-in-group}$ )

private lemma  $V\text{-mult}:$ 
  shows  $m \geq 1 \implies V\ m\ n <\#\> U\ n \subseteq V\ (m + 1)\ n$ 
proof ( $\text{induction } n \text{ arbitrary}: m$ )
  case  $0$ 
  then have  $V\ (m + 1)\ 0 = \text{carrier } G$  by simp
  then show  $?case$  unfolding set-mult-def using  $V\text{-in-group } U\text{-in-group}$  by fast
next
  case ( $\text{Suc } n$ )
  then show  $?case$ 
  proof ( $\text{cases } m + 1 > 2^{\wedge}(\text{Suc } n)$ )
  case  $\text{True}$ 
  then have  $V\ (m + 1)\ (\text{Suc } n) = \text{carrier } G$  by force
  then show  $?thesis$  unfolding set-mult-def using  $V\text{-in-group } U\text{-in-group}$  by
 $\text{blast}$ 
  next
  case  $m\text{-in-bounds}: \text{False}$ 
  then show  $?thesis$ 
  proof ( $\text{cases } m = 1$ )
  case  $\text{True}$ 
  then show  $?thesis$  using  $U\text{-in-group } U\text{-props}$  by force
  next
  case  $m\text{-not-1}: \text{False}$ 

```

then show *?thesis*
proof (*cases even m*)
 case *True*
 then have $V\ m\ (Suc\ n) <\#\> U\ (Suc\ n) = V\ (m + 1)\ (Suc\ n)$ **using**
m-in-bounds m-not-1 Suc(2) **by** *auto*
 then show *?thesis* **by** *blast*
 next
 case *m-odd: False*
 have $U\text{-mult}: U\ (Suc\ n) <\#\> U\ (Suc\ n) \subseteq U\ n$ **using** *U-props* **by** *simp*
 have *not-zero: $(m - 1)\ div\ 2 \geq 1$* **using** *Suc(2) m-not-1 m-odd* **by** *presburger*
 have *arith: $(m - 1)\ div\ 2 + 1 = (m + 1)\ div\ 2$* **using** *Suc(2)* **by** *simp*
 have $V\ m\ (Suc\ n) <\#\> U\ (Suc\ n) = V\ ((m - 1)\ div\ 2)\ n <\#\> U\ (Suc\ n)$
 using *m-odd m-in-bounds m-not-1 Suc(2)* **by** *simp*
 also have $\dots = V\ ((m - 1)\ div\ 2)\ n <\#\> (U\ (Suc\ n) <\#\> U\ (Suc\ n))$
using *set-mult-assoc V-in-group U-in-group* **by** *simp*
 also have $\dots \subseteq V\ ((m - 1)\ div\ 2)\ n <\#\> U\ n$ **using** *mono-set-mult U-mult*
by *blast*
 also have $\dots \subseteq V\ ((m - 1)\ div\ 2 + 1)\ n$ **using** *Suc(1) not-zero* **by** *blast*
 also have $\dots = V\ ((m + 1)\ div\ 2)\ n$ **using** *arith* **by** *presburger*
 also have $\dots = V\ (m + 1)\ (Suc\ n)$ **using** *m-odd m-not-1 m-in-bounds*
 Suc(2) **by** *simp*
 finally show *?thesis* **by** *blast*
 qed
 qed
qed
qed

private lemma *V-mono:*
 assumes *smaller: $(real\ m_1)/2^{\wedge}n_1 \leq (real\ m_2)/2^{\wedge}n_2$* **and** *not-zero: $m_1 \geq 1 \wedge m_2 \geq 1$*
 shows $V\ m_1\ n_1 \subseteq V\ m_2\ n_2$
proof –
 have $V\ m\ n \subseteq V\ (m + 1)\ n$ **if** $m \geq 1$ **for** $m\ n$
 proof –
 have $V\ m\ n <\#\> U\ n \subseteq V\ (m + 1)\ n$ **using** *V-mult U-props that* **by** *presburger*
 moreover have $V\ m\ n \subseteq carrier\ G \wedge U\ n \subseteq carrier\ G$ **using** *U-in-group*
 V-in-group **by** *auto*
 ultimately show *?thesis* **using** *set-mult-one-subset U-neighborhood* **by** *blast*
 qed
 then have *subset-suc: $V\ m\ n \subseteq V\ (m + 1)\ n$* **for** $m\ n$ **by** *simp*
 have $V\ m\ n \subseteq V\ (m + k)\ n$ **for** $m\ n\ k$
 proof (*induction k*)
 case (*Suc k*)
 then show *?case unfolding Suc-eq-plus1* **using** *subset-suc Suc*
 by (*metis (no-types, opaque-lifting) add.assoc dual-order.trans*)
 qed (*simp*)
 then have *a: $V\ m\ n \subseteq V\ m'\ n$* **if** $m' \geq m$ **for** $m\ m'\ n$ **using** *that le-Suc-ex* **by**
 blast
 have *b: $V\ m\ n = V\ (m * 2^{\wedge}k)\ (n+k)$* **if** $m \geq 1$ **for** $m\ n\ k$

```

proof (induction k)
  case (Suc k)
    have  $V (m * 2^k * 2) (n + k + 1) = V (m * 2^k) (n + k)$  using that by
simp
    then show ?case unfolding Suc-eq-plus1 using Suc by simp
  qed (auto)
  show ?thesis
  proof (cases  $n_1 \leq n_2$ )
    case True
      have  $(\text{real } m_1) / 2^{n_1} = (\text{real } (m_1 * 2^{(n_2 - n_1)})) / (2^{n_1} * 2^{(n_2 - n_1)})$  by
fastforce
      also have  $\dots = (\text{real } (m_1 * 2^{(n_2 - n_1)})) / 2^{n_2}$  using True by (metis
le-add-diff-inverse power-add)
      finally have  $(\text{real } (m_1 * 2^{(n_2 - n_1)})) / 2^{n_2} \leq (\text{real } m_2) / 2^{n_2}$  using smaller
by fastforce
      then have ineq:  $m_1 * 2^{(n_2 - n_1)} \leq m_2$ 
      by (smt (verit) divide-cancel-right divide-right-mono linorder-le-cases of-nat-eq-iff
of-nat-mono order-antisym-conv power-not-zero zero-le-power)
      from b have  $V m_1 n_1 = V (m_1 * 2^{(n_2 - n_1)}) (n_1 + (n_2 - n_1))$  using
not-zero by blast
      also have  $\dots = V (m_1 * 2^{(n_2 - n_1)}) n_2$  using True by force
      finally show ?thesis using a[OF ineq] by blast
    next
      case False
        then have  $n_2\text{-leq-}n_1$ :  $n_2 \leq n_1$  by simp
        have  $(\text{real } m_2) / 2^{n_2} = (\text{real } (m_2 * 2^{(n_1 - n_2)})) / (2^{n_2} * 2^{(n_1 - n_2)})$  by
fastforce
        also have  $\dots = (\text{real } (m_2 * 2^{(n_1 - n_2)})) / 2^{n_1}$  using  $n_2\text{-leq-}n_1$  by (metis
le-add-diff-inverse power-add)
        finally have  $(\text{real } (m_2 * 2^{(n_1 - n_2)})) / 2^{n_1} \geq (\text{real } m_1) / 2^{n_1}$  using smaller
by fastforce
        then have ineq:  $m_2 * 2^{(n_1 - n_2)} \geq m_1$ 
        by (smt (verit) divide-cancel-right divide-right-mono linorder-le-cases of-nat-eq-iff
of-nat-mono order-antisym-conv power-not-zero zero-le-power)
        from b have  $V m_2 n_2 = V (m_2 * 2^{(n_1 - n_2)}) (n_2 + (n_1 - n_2))$  using
not-zero by blast
        also have  $\dots = V (m_2 * 2^{(n_1 - n_2)}) n_1$  using  $n_2\text{-leq-}n_1$  by force
        finally show ?thesis using a[OF ineq] by blast
      qed
    qed
  qed

```

```

private lemma approx-number-by-multiples:
  assumes hx:  $x \geq 0$  and hc:  $c > 0$ 
  shows  $\exists k :: \text{nat} \geq 1. (\text{real } (k-1)) / c \leq x \wedge x < (\text{real } k) / c$ 
proof -
  let ?k =  $\lfloor x * c \rfloor + 1$ 
  have ?k  $\geq 1$  using assms by simp
  moreover from this have  $\text{real } (\text{nat } ?k) = ?k$  by auto
  moreover have  $(?k-1) / c \leq x \wedge x < ?k / c$ 

```

using *assms* **by** (*simp add: mult-imp-div-pos-le pos-less-divide-eq*)
ultimately show *?thesis*
by (*smt (verit) nat-diff-distrib nat-le-eq-zle nat-one-as-int of-nat-nat*)
qed

lemma *construction-of-prenorm-respecting-topology:*

shows $\exists N. \text{group-prenorm } N \wedge$
 $(\forall n. \{\sigma \in \text{carrier } G. N \sigma < 1/2^{\wedge}n\} \subseteq U n) \wedge$
 $(\forall n. U n \subseteq \{\sigma \in \text{carrier } G. N \sigma \leq 2/2^{\wedge}n\})$

proof –

define $f :: 'g \Rightarrow \text{real}$ **where** $f \sigma = \text{Inf } \{(real\ m)/2^{\wedge}n \mid m\ n. \sigma \in V\ m\ n\}$ **for** σ
define $N :: 'g \Rightarrow \text{real}$ **where** $N = \text{induced-group-prenorm } f$

have $\sigma \in V\ 2\ 0$ **if** $\sigma \in \text{carrier } G$ **for** σ **using** *that by auto*

then have *contains-2*: $(real\ 2)/2^{\wedge}0 \in \{(real\ m)/2^{\wedge}n \mid m\ n. \sigma \in V\ m\ n\}$ **if** $\sigma \in \text{carrier } G$ **for** σ **using** *that by blast*

then have *nonempty*: $\{(real\ m)/2^{\wedge}n \mid m\ n. \sigma \in V\ m\ n\} \neq \{\}$ **if** $\sigma \in \text{carrier } G$ **for** σ **using** *that by fast*

have *positive*: $(real\ m)/2^{\wedge}n \geq 0$ **for** $m\ n$ **by** *simp*

then have *bdd-below*: *bdd-below* $\{(real\ m)/2^{\wedge}n \mid m\ n. \sigma \in V\ m\ n\}$ **for** σ **by** *fast*

have *f-bounds*: $0 \leq f\ \sigma \wedge f\ \sigma \leq 2$ **if** $h\sigma: \sigma \in \text{carrier } G$ **for** σ

proof –

from *bdd-below* **have** $f\ \sigma \leq (real\ 2)/2^{\wedge}0$ **unfolding** *f-def* **using** *cInf-lower contains-2[OF hσ]* **by** *meson*

moreover have $0 \leq f\ \sigma$ **using** *cInf-greatest contains-2[OF hσ]* **unfolding** *f-def* **using** *positive*

by (*smt (verit, del-insts) Collect-mem-eq empty-Collect-eq mem-Collect-eq*)

ultimately show *?thesis* **by** *fastforce*

qed

then have *N-welldefined*: *bdd-above* $((\lambda\tau. |f\ (\tau \otimes \sigma) - f\ \tau|) \text{ `carrier } G)$ **if** $\sigma \in \text{carrier } G$ **for** σ

using *induced-group-prenorm-welldefined* **that by** (*metis (full-types) abs-of-nonneg*)

have *in-V-if-f-smaller*: $\sigma \in V\ m\ n$ **if** $h\sigma: \sigma \in \text{carrier } G$ **and** *smaller*: $f\ \sigma < (real\ m)/2^{\wedge}n$ **for** $\sigma\ m\ n$

proof –

from *cInf-lessD* **obtain** q **where** $h\sigma: q \in \{(real\ m)/2^{\wedge}n \mid m\ n. \sigma \in V\ m\ n\} \wedge q < (real\ m)/2^{\wedge}n$

using *smaller nonempty[OF hσ]* **unfolding** *f-def* **by** (*metis (mono-tags, lifting)*)

then obtain $m'\ n'$ **where** $h\sigma: \sigma \in V\ m'\ n' \wedge q = (real\ m')/2^{\wedge}n'$ **by** *fast*

moreover have $m' \geq 1$

proof (*rule ccontr*)

assume $\neg m' \geq 1$

then have $V\ m'\ n' = \{\}$ **by** *force*

then show *False* **using** $h\sigma$ **by** *blast*

qed

moreover have $m \geq 1$ **using** *f-bounds* *smaller hσ*

by (*metis divide-eq-0-iff less-numeral-extra(3) less-one linorder-le-less-linear*)

nle-le of-nat-0 order-less-imp-le
ultimately have $V m' n' \subseteq V m n$ **using** *V-mono hq U-props open-set-in-carrier*
by *simp*
then show *?thesis using hm'n' by fast*
qed
have *f-1-vanishes: f 1 = 0*
proof (*rule ccontr*)
assume $f \mathbf{1} \neq 0$
then have $f \mathbf{1} > 0$ **using** *f-bounds by fastforce*
then obtain n **where** $hn: f \mathbf{1} > (\text{real } 1)/2^{\wedge}n$
by (*metis divide-less-eq-1 of-nat-1 one-less-numeral-iff power-one-over real-arch-pow-inv*
semiring-norm(76) zero-less-numeral)
have $\mathbf{1} \in V 1 n$ **using** *U-neighborhood by simp*
then have $(\text{real } 1)/2^{\wedge}n \in \{(\text{real } m)/2^{\wedge}n \mid m n. \mathbf{1} \in V m n\}$ **by** *fast*
then show *False using hn cInf-lower bdd-below[of 1] unfolding f-def by (smt*
(verit, ccfv-threshold))
qed
have *in-U-if-N-small: $\sigma \in U n$ if in-group: $\sigma \in \text{carrier } G$ and N-small: $N \sigma < 1/2^{\wedge}n$ for σn*
proof –
have $f \sigma = |f (\mathbf{1} \otimes \sigma) - f \mathbf{1}|$ **using** *in-group l-one f-1-vanishes f-bounds by*
force
moreover have $\dots \leq N \sigma$ **unfolding** *N-def induced-group-prenorm-def*
using *cSUP-upper N-welldefined[OF in-group] by (metis (mono-tags, lifting)*
one-closed)
ultimately have $\sigma \in V 1 n$ **using** *in-V-if-f-smaller[OF in-group] N-small by*
(smt (verit) of-nat-1)
then show *?thesis by fastforce*
qed
have *N-bounds: $N \sigma \leq 2/2^{\wedge}n$ if h σ : $\sigma \in U n$ for σn*
proof –
have *diff-bounded: $f (\tau \otimes \sigma) - f \tau \leq 2/2^{\wedge}n \wedge f (\tau \otimes \text{inv } \sigma) - f \tau \leq 2/2^{\wedge}n$*
if $h\tau: \tau \in \text{carrier } G$ **for** τ
proof –
obtain k **where** $hk: k \geq 1 \wedge (\text{real } (k-1))/2^{\wedge}n \leq f \tau \wedge f \tau < (\text{real } k)/2^{\wedge}n$
using *approx-number-by-multiples[of f τ $2^{\wedge}n$] f-bounds[OF h τ] by auto*
then have $\tau \in V k n$ **using** *in-V-if-f-smaller[OF h τ] by blast*
moreover have $\sigma \in V 1 n \wedge \text{inv } \sigma \in V 1 n$ **using** *h σ U-props by auto*
moreover have $V k n <\#\> V 1 n \subseteq V (k+1) n$
using *V-mult U-props open-set-in-carrier hk by auto*
ultimately have $\tau \otimes \sigma \in V (k+1) n \wedge \tau \otimes \text{inv } \sigma \in V (k+1) n$
unfolding *set-mult-def by fast*
then have $a: (\text{real } (k+1))/2^{\wedge}n \in \{(\text{real } m)/2^{\wedge}n \mid m n. \tau \otimes \sigma \in V m n\}$
 $\wedge (\text{real } (k+1))/2^{\wedge}n \in \{(\text{real } m)/2^{\wedge}n \mid m n. \tau \otimes \text{inv } \sigma \in V m n\}$ **by** *fast*
then have $f (\tau \otimes \sigma) \leq (\text{real } (k+1))/2^{\wedge}n$
unfolding *f-def using cInf-lower[of (real (k+1))/2^{\wedge}n] bdd-below by*
presburger
moreover from a **have** $f (\tau \otimes \text{inv } \sigma) \leq (\text{real } (k+1))/2^{\wedge}n$
unfolding *f-def using cInf-lower[of (real (k+1))/2^{\wedge}n] bdd-below by*

presburger
ultimately show *?thesis* **using** *hk*
by (*smt (verit, ccfv-SIG) diff-divide-distrib of-nat-1 of-nat-add of-nat-diff*)
qed
have $|f(\varrho \otimes \sigma) - f \varrho| \leq 2/2^n$ **if** *h ϱ : $\varrho \in \text{carrier } G$ for ϱ*
proof –
have *in-group: $\sigma \in \text{carrier } G$ using $h\sigma$ U-in-group* **by** *fast*
then have $f(\varrho \otimes \sigma \otimes \text{inv } \sigma) - f(\varrho \otimes \sigma) \leq 2/2^n$ **using** *diff-bounded[$of \varrho \otimes \sigma$] $h\varrho$ m-closed* **by** *fast*
moreover have $\varrho \otimes \sigma \otimes \text{inv } \sigma = \varrho$ **using** *m-assoc r-inv r-one in-group inv-closed $h\varrho$* **by** *presburger*
ultimately have $f \varrho - f(\varrho \otimes \sigma) \leq 2/2^n$ **by** *force*
moreover have $f(\varrho \otimes \sigma) - f \varrho \leq 2/2^n$ **using** *diff-bounded[OF $h\varrho$]* **by** *fast*
ultimately show *?thesis* **by** *force*
qed
then show *?thesis unfolding N-def induced-group-prenorm-def* **using** *cSUP-least carrier-not-empty* **by** *meson*
qed
then have $U n \subseteq \{\sigma \in \text{carrier } G. N \sigma \leq 2/2^n\}$ **for** *n* **using** *U-in-group* **by** *blast*
moreover have *group-prenorm N unfolding N-def*
using *bounded-function-induces-group-prenorm f-bounds* **by** (*metis abs-of-nonneg*)
ultimately show *?thesis* **using** *in-U-if-N-small* **by** *blast*
qed
end

2.5.3 Proof of Birkhoff-Kakutani

lemma *first-countable-neighborhoods-of-1-sequence:*

assumes *first-countable T*

shows $\exists U :: \text{nat} \Rightarrow 'g \text{ set}$.

$(\forall n. \text{neighborhood } \mathbf{1} (U n) \wedge \text{symmetric } (U n) \wedge U (n + 1) <\#\> U (n + 1) \subseteq U n) \wedge$

$(\forall W. \text{neighborhood } \mathbf{1} W \longrightarrow (\exists n. U n \subseteq W))$

proof –

from *assms* **obtain** \mathcal{B} **where** *h \mathcal{B} :*

countable $\mathcal{B} \wedge (\forall W \in \mathcal{B}. \text{openin } T W) \wedge (\forall U. \text{neighborhood } \mathbf{1} U \longrightarrow (\exists W \in \mathcal{B}. \mathbf{1} \in W \wedge W \subseteq U))$

unfolding *first-countable-def* **by** *fastforce*

define $\mathfrak{B} :: 'g \text{ set set}$ **where** $\mathfrak{B} = \text{insert } (\text{carrier } G) \{W \in \mathcal{B}. \mathbf{1} \in W\}$

define $B :: \text{nat} \Rightarrow 'g \text{ set}$ **where** $B = \text{from-nat-into } \mathfrak{B}$

have $\mathfrak{B} \neq \{\}$ $\wedge (\forall W \in \mathfrak{B}. \text{neighborhood } \mathbf{1} W)$ **unfolding** \mathfrak{B} -*def* **using** *h \mathcal{B}*

by (*metis group-is-space insert-iff insert-not-empty mem-Collect-eq one-closed openin-topspace*)

then have *B-neighborhood: $\forall n. \text{neighborhood } \mathbf{1} (B n)$* **unfolding** *B-def* **by** (*simp add: from-nat-into*)

define P **where** $P n V \longleftrightarrow V \subseteq B n \wedge \text{neighborhood } \mathbf{1} V \wedge \text{symmetric } V$ **for** $n V$

define Q **where** $Q (n :: \text{nat}) V W \longleftrightarrow W <\#\> W \subseteq V$ **for** $n V W$

have $\exists V. P \ 0 \ V$
proof –
 obtain W **where** *neighborhood* $\mathbf{1} \ W \wedge$ *symmetric* $W \wedge W <\#\> W \subseteq B \ 0$
 using *neighborhoods-of-1 B-neighborhood* **by** *fastforce*
 moreover from this have $W \subseteq B \ 0$ **using** *set-mult-one-subset open-set-in-carrier*
by *blast*
 ultimately show *?thesis unfolding P-def* **by** *auto*
qed
moreover have $\exists W. P \ (Suc \ n) \ W \wedge Q \ n \ V \ W$ **if** $P \ n \ V$ **for** $n \ V$
proof –
 have *neighborhood* $\mathbf{1} \ (V \cap B \ (Suc \ n))$ **using** *B-neighborhood that unfolding*
P-def **by** *auto*
 then obtain W **where** *neighborhood* $\mathbf{1} \ W \wedge$ *symmetric* $W \wedge W <\#\> W \subseteq$
 $V \cap B \ (Suc \ n)$
 using *neighborhoods-of-1* **by** *fastforce*
 moreover from this have $W \subseteq B \ (Suc \ n)$
 using *set-mult-one-subset[of W W] open-set-in-carrier[of W]* **by** *fast*
 ultimately show *?thesis unfolding P-def Q-def* **by** *auto*
qed
ultimately obtain U **where** $hU: \forall n. P \ n \ (U \ n) \wedge Q \ n \ (U \ n) \ (U \ (Suc \ n))$
 using *dependent-nat-choice* **by** *metis*
moreover have $\exists n. U \ n \subseteq W$ **if** *neighborhood* $\mathbf{1} \ W$ **for** W
proof –
 from that obtain W' **where** $hW': W' \in \mathcal{B} \wedge \mathbf{1} \in W' \wedge W' \subseteq W$ **using** *hB*
by *blast*
 then have $W' \in \mathfrak{B} \wedge$ *countable* \mathfrak{B} **unfolding** *\mathfrak{B}-def* **using** *hB* **by** *simp*
 then obtain n **where** $B \ n = W'$ **unfolding** *B-def* **using** *from-nat-into-to-nat-on*
by *fast*
 then show *?thesis using hW' hU unfolding P-def* **by** *blast*
qed
 ultimately show *?thesis unfolding P-def Q-def* **by** *auto*
qed

definition *left-invariant-metric* $\Delta \longleftrightarrow$ *Metric-space (carrier G) \Delta \wedge*
 $(\forall \sigma \ \tau \ \varrho. \sigma \in \text{carrier } G \wedge \tau \in \text{carrier } G \wedge \varrho \in \text{carrier } G \longrightarrow \Delta \ (\varrho \otimes \sigma) \ (\varrho \otimes \tau)$
 $= \Delta \ \sigma \ \tau)$

definition *right-invariant-metric* $\Delta \longleftrightarrow$ *Metric-space (carrier G) \Delta \wedge*
 $(\forall \sigma \ \tau \ \varrho. \sigma \in \text{carrier } G \wedge \tau \in \text{carrier } G \wedge \varrho \in \text{carrier } G \longrightarrow \Delta \ (\sigma \otimes \varrho) \ (\tau \otimes \varrho)$
 $= \Delta \ \sigma \ \tau)$

lemma *left-invariant-metricE*:

assumes *left-invariant-metric* $\Delta \ \sigma \in \text{carrier } G \ \tau \in \text{carrier } G \ \varrho \in \text{carrier } G$
shows $\Delta \ (\varrho \otimes \sigma) \ (\varrho \otimes \tau) = \Delta \ \sigma \ \tau$
using *assms unfolding left-invariant-metric-def* **by** *blast*

lemma *right-invariant-metricE*:

assumes *right-invariant-metric* $\Delta \ \sigma \in \text{carrier } G \ \tau \in \text{carrier } G \ \varrho \in \text{carrier } G$
shows $\Delta \ (\sigma \otimes \varrho) \ (\tau \otimes \varrho) = \Delta \ \sigma \ \tau$

using *assms unfolding right-invariant-metric-def* by *blast*

theorem *Birkhoff-Kakutani-left*:

assumes *Hausdorff*: *Hausdorff-space* T **and** *first-countable*: *first-countable* T

shows $\exists \Delta$. *left-invariant-metric* $\Delta \wedge$ *Metric-space.mtopology* (*carrier* G) $\Delta = T$

proof –

from *first-countable* **obtain** $U :: \text{nat} \Rightarrow 'g \text{ set}$ **where**

U-props: $\forall n$. *neighborhood* $\mathbf{1}$ $(U\ n) \wedge$ *symmetric* $(U\ n) \wedge U\ (n + 1) <\#\> U\ (n + 1) \subseteq U\ n$ **and**

neighborhood-base: $\forall W$. *neighborhood* $\mathbf{1}$ $W \longrightarrow (\exists n$. $U\ n \subseteq W)$

using *first-countable-neighborhoods-of-1-sequence* **by** *auto*

from *U-props* **obtain** N **where**

prenorm: *group-prenorm* N **and**

norm-ball-in-U: $\forall n$. $\{\sigma \in \text{carrier } G. N\ \sigma < 1/2^{\wedge}n\} \subseteq U\ n$ **and**

U-in-norm-ball: $\forall n$. $U\ n \subseteq \{\sigma \in \text{carrier } G. N\ \sigma \leq 2/2^{\wedge}n\}$

using *construction-of-prenorm-respecting-topology* **by** *meson*

have *continuous*: *continuous-map* T *euclideanreal* N **using** *prenorm*

proof (*rule group-prenorm-continuous-if-continuous-at-1*, *intro allI impI*)

fix $\varepsilon :: \text{real}$ **assume** $\varepsilon > 0$

then obtain n **where** *hn*: $1/2^{\wedge}n < \varepsilon$

by (*metis divide-less-eq-1-pos one-less-numeral-iff power-one-over real-arch-pow-inv semiring-norm(76) zero-less-numeral*)

then have $N\ \sigma < \varepsilon$ **if** $\sigma \in U\ (n + 1)$ **for** σ **using** *that U-in-norm-ball* **by** *fastforce*

then show $\exists U$. *neighborhood* $\mathbf{1}$ $U \wedge (\forall \sigma \in U$. $N\ \sigma < \varepsilon)$ **using** *U-props* **by** *meson*

qed

let $?B = \lambda \varepsilon$. $\{\sigma \in \text{carrier } G. N\ \sigma < \varepsilon\}$

let $? \Delta = \lambda \sigma \tau$. $N\ (\text{inv } \sigma \otimes \tau)$

let $? \delta = \lambda \sigma \tau$. *if* $\sigma \in \text{carrier } G \wedge \tau \in \text{carrier } G$ *then* $? \Delta\ \sigma\ \tau$ *else* $4/2$

have $? \Delta\ \sigma\ \tau \geq 0$ **if** $\sigma \in \text{carrier } G \wedge \tau \in \text{carrier } G$ **for** $\sigma\ \tau$

using *group-prenorm-nonnegative* *prenorm* **that** **by** *blast*

moreover have $? \Delta\ \sigma\ \tau = ? \Delta\ \tau\ \sigma$ **if** $\sigma \in \text{carrier } G \wedge \tau \in \text{carrier } G$ **for** $\sigma\ \tau$

proof –

have $\text{inv } \tau \otimes \sigma = \text{inv } (\text{inv } \sigma \otimes \tau)$ **using** *inv-mult-group inv-inv* **that** **by** *auto*

then show *?thesis* **using** *prenorm* **that** **by** *fastforce*

qed

moreover have $? \Delta\ \sigma\ \tau = 0 \longleftrightarrow \sigma = \tau$ **if** $\sigma \in \text{carrier } G \wedge \tau \in \text{carrier } G$ **for** $\sigma\ \tau$

proof

assume $? \Delta\ \sigma\ \tau = 0$

then have $\text{inv } \sigma \otimes \tau \in U\ n$ **for** n **using** *norm-ball-in-U* **that** **by** *fastforce*

then have $\text{inv } \sigma \otimes \tau \in W$ **if** *neighborhood* $\mathbf{1}$ W **for** W **using** *neighborhood-base* **that** **by** *auto*

then have $\text{inv } \sigma \otimes \tau = \mathbf{1}$ **using** *Hausdorff-space-sing-Inter-opens[of T 1]* *Hausdorff* **by** *blast*

then show $\sigma = \tau$ **using** *inv-comm inv-equality* **that** **by** *fastforce*

next

assume $\sigma = \tau$

then show $?\Delta \sigma \tau = 0$ **using** *that prenorm by force*
qed
moreover have $? \Delta \sigma \varrho \leq ? \Delta \sigma \tau + ? \Delta \tau \varrho$ **if** $\sigma \in \text{carrier } G \wedge \tau \in \text{carrier } G$
 $\wedge \varrho \in \text{carrier } G$ **for** $\sigma \tau \varrho$
proof –
have $\text{inv } \sigma \otimes \varrho = (\text{inv } \sigma \otimes \tau) \otimes (\text{inv } \tau \otimes \varrho)$ **using** *m-assoc[symmetric] that*
by *(simp add: inv-solve-right)*
then show *?thesis using prenorm that by auto*
qed
ultimately have *metric: Metric-space (carrier G) ?δ unfolding Metric-space-def*
by *auto*
then interpret *Metric-space carrier G ?δ by blast*
have $? \Delta (\varrho \otimes \sigma) (\varrho \otimes \tau) = ? \Delta \sigma \tau$ **if** $\sigma \in \text{carrier } G \wedge \tau \in \text{carrier } G \wedge \varrho \in$
 $\text{carrier } G$ **for** $\sigma \tau \varrho$
proof –
have $\text{inv } \sigma \otimes \tau = \text{inv } (\varrho \otimes \sigma) \otimes (\varrho \otimes \tau)$ **using** *that m-assoc[symmetric] by*
(simp add: inv-solve-left inv-solve-right)
then show *?thesis by simp*
qed
then have *left-invariant: left-invariant-metric ?δ*
unfolding *left-invariant-metric-def using metric by auto*
have *mball-coset-of-norm-ball: mball $\sigma \varepsilon = \sigma < \# ?B \varepsilon$ if $h\sigma: \sigma \in \text{carrier } G$ for*
 $\sigma \varepsilon$
proof –
have $\text{mball } \sigma \varepsilon = \{\tau \in \text{carrier } G. N (\text{inv } \sigma \otimes \tau) < \varepsilon\}$ **unfolding** *mball-def*
using *hσ by auto*
also have $\dots = \sigma < \# (?B \varepsilon)$
proof –
have $\tau \in \sigma < \# (?B \varepsilon)$ **if** $\tau \in \text{carrier } G \wedge N (\text{inv } \sigma \otimes \tau) < \varepsilon$ **for** τ
proof –
have $\sigma \otimes (\text{inv } \sigma \otimes \tau) = \tau$ **using** *hσ that by (metis inv-closed inv-solve-left*
m-closed)
moreover have $\text{inv } \sigma \otimes \tau \in ?B \varepsilon$ **using** *hσ that by fastforce*
ultimately show *?thesis unfolding l-coset-def by force*
qed
moreover have $\tau \in \text{carrier } G \wedge N (\text{inv } \sigma \otimes \tau) < \varepsilon$ **if** $\tau \in \sigma < \# (?B \varepsilon)$ **for**
 τ
proof –
from *that obtain ϱ where $\varrho \in ?B \varepsilon \wedge \tau = \sigma \otimes \varrho$ unfolding l-coset-def*
by *blast*
moreover from *this have $\text{inv } \sigma \otimes \tau = \varrho$ using hσ by (simp add:*
inv-solve-left')
ultimately show *?thesis using hσ by simp*
qed
ultimately show *?thesis by blast*
qed
finally show *?thesis by presburger*
qed
define *ball where ball $S \longleftrightarrow (\exists \sigma \varepsilon. \sigma \in \text{carrier } G \wedge S = \text{mball } \sigma \varepsilon)$ for S*

have *openin mtopology V if ball V for V* **using** *that unfolding ball-def by fast*
moreover have $\exists W. \text{ball } W \wedge \sigma \in W \wedge W \subseteq V$ **if** *openin mtopology V $\wedge \sigma \in V$ for σV*

unfolding *ball-def using openin-mtopology that by (smt (verit, best) centre-in-mball-iff subset-iff)*

ultimately have *openin-metric: openin mtopology = arbitrary union-of ball*
by *(simp add: openin-topology-base-unique)*

have *openin T V if ball V for V*

proof –

from *that obtain $\sigma \varepsilon$ where $\sigma \in \text{carrier } G \wedge V = \sigma <\# ?B \varepsilon$*
unfolding *ball-def using mball-coset-of-norm-ball by blast*
moreover have *openin T ($?B \varepsilon$) using continuous*
by *(simp add: continuous-map-upper-lower-semicontinuous-lt)*
ultimately show *?thesis using open-set-translations(1) by presburger*

qed

moreover have $\exists W. \text{ball } W \wedge \sigma \in W \wedge W \subseteq V$ **if** *neighborhood σV for σV*

proof –

from *that have in-group: $\sigma \in \text{carrier } G$ using open-set-in-carrier by fast*
then have *neighborhood 1 (inv $\sigma <\# V$)*
using *l-coset-def open-set-translations(1) that l-inv by fastforce*
then obtain *n where $U n \subseteq \text{inv } \sigma <\# V$ using neighborhood-base by presburger*

then have *$?B (1/2^{\wedge}n) \subseteq \text{inv } \sigma <\# V$ using norm-ball-in-U by blast*
then have *$\sigma <\# ?B (1/2^{\wedge}n) \subseteq \sigma <\# (\text{inv } \sigma <\# V)$ unfolding l-coset-def*
by fast

also have *... = V using in-group that open-set-in-carrier by (simp add: lcos-m-assoc lcos-mult-one)*

finally have *mball $\sigma (1/2^{\wedge}n) \subseteq V$ using mball-coset-of-norm-ball in-group by blast*

then show *?thesis unfolding ball-def*
by *(smt (verit) centre-in-mball-iff divide-pos-pos in-group one-add-one zero-less-power zero-less-two)*

qed

ultimately have *openin T = arbitrary union-of ball by (simp add: openin-topology-base-unique)*
then show *?thesis using left-invariant openin-metric topology-eq by fastforce*

qed

theorem *Birkhoff-Kakutani-right:*

assumes *Hausdorff: Hausdorff-space T and first-countable: first-countable T*
shows $\exists \Delta. \text{right-invariant-metric } \Delta \wedge \text{Metric-space.mtopology (carrier } G) \Delta = T$

proof –

from *first-countable obtain $U :: \text{nat} \Rightarrow 'g \text{ set}$ where*
U-props: $\forall n. \text{neighborhood } 1 (U n) \wedge \text{symmetric } (U n) \wedge U (n + 1) <\#> U (n + 1) \subseteq U n$ and
neighborhood-base: $\forall W. \text{neighborhood } 1 W \longrightarrow (\exists n. U n \subseteq W)$
using *first-countable-neighborhoods-of-1-sequence by auto*
from *U-props obtain N where*
prenorm: group-prenorm N and

norm-ball-in-U: $\forall n. \{\sigma \in \text{carrier } G. N \sigma < 1/2^{\wedge}n\} \subseteq U n$ **and**
U-in-norm-ball: $\forall n. U n \subseteq \{\sigma \in \text{carrier } G. N \sigma \leq 2/2^{\wedge}n\}$
using *construction-of-prenorm-respecting-topology* **by** *meson*
have *continuous: continuous-map T euclideanreal N* **using** *prenorm*
proof (*rule group-prenorm-continuous-if-continuous-at-1, intro allI impI*)
fix $\varepsilon :: \text{real}$ **assume** $\varepsilon > 0$
then obtain n **where** $hn: 1/2^{\wedge}n < \varepsilon$
by (*metis divide-less-eq-1-pos one-less-numeral-iff power-one-over real-arch-pow-inv*
semiring-norm(76) zero-less-numeral)
then have $N \sigma < \varepsilon$ **if** $\sigma \in U (n + 1)$ **for** σ **using** *that U-in-norm-ball* **by**
fastforce
then show $\exists U. \text{neighborhood } \mathbf{1} U \wedge (\forall \sigma \in U. N \sigma < \varepsilon)$ **using** *U-props* **by**
meson
qed
let $?B = \lambda \varepsilon. \{\sigma \in \text{carrier } G. N \sigma < \varepsilon\}$
let $? \Delta = \lambda \sigma \tau. N (\sigma \otimes \text{inv } \tau)$
let $? \delta = \lambda \sigma \tau. \text{if } \sigma \in \text{carrier } G \wedge \tau \in \text{carrier } G \text{ then } ? \Delta \sigma \tau \text{ else } 4/2$
have $? \Delta \sigma \tau \geq 0$ **if** $\sigma \in \text{carrier } G \wedge \tau \in \text{carrier } G$ **for** $\sigma \tau$
using *group-prenorm-nonnegative* *prenorm* **that** **by** *blast*
moreover have $? \Delta \sigma \tau = ? \Delta \tau \sigma$ **if** $\sigma \in \text{carrier } G \wedge \tau \in \text{carrier } G$ **for** $\sigma \tau$
proof –
have $\tau \otimes \text{inv } \sigma = \text{inv } (\sigma \otimes \text{inv } \tau)$ **using** *inv-mult-group inv-inv* **that** **by** *auto*
then show *?thesis* **using** *prenorm* **that** **by** *auto*
qed
moreover have $? \Delta \sigma \tau = 0 \iff \sigma = \tau$ **if** $\sigma \in \text{carrier } G \wedge \tau \in \text{carrier } G$ **for**
 $\sigma \tau$
proof
assume $? \Delta \sigma \tau = 0$
then have $\sigma \otimes \text{inv } \tau \in U n$ **for** n **using** *norm-ball-in-U* **that** **by** *fastforce*
then have $\sigma \otimes \text{inv } \tau \in W$ **if** *neighborhood* $\mathbf{1} W$ **for** W **using** *neighborhood-base*
that **by** *auto*
then have $\sigma \otimes \text{inv } \tau = \mathbf{1}$ **using** *Hausdorff-space-sing-Inter-opens[of T 1]*
Hausdorff **by** *blast*
then show $\sigma = \tau$ **using** *inv-equality* **that** **by** *fastforce*
next
assume $\sigma = \tau$
then show $? \Delta \sigma \tau = 0$ **using** *that* *prenorm* **by** *force*
qed
moreover have $? \Delta \sigma \rho \leq ? \Delta \sigma \tau + ? \Delta \tau \rho$ **if** $\sigma \in \text{carrier } G \wedge \tau \in \text{carrier } G$
 $\wedge \rho \in \text{carrier } G$ **for** $\sigma \tau \rho$
proof –
have $\sigma \otimes \text{inv } \rho = (\sigma \otimes \text{inv } \tau) \otimes (\tau \otimes \text{inv } \rho)$ **using** *m-assoc* **that** **by** (*simp*
add: inv-solve-left)
then show *?thesis* **using** *prenorm* **that** **by** *auto*
qed
ultimately have *metric: Metric-space (carrier G) ?δ unfolding Metric-space-def*
by auto
then interpret *Metric-space carrier G ?δ* **by** *blast*
have $? \Delta (\sigma \otimes \rho) (\tau \otimes \rho) = ? \Delta \sigma \tau$ **if** $\sigma \in \text{carrier } G \wedge \tau \in \text{carrier } G \wedge \rho \in$

carrier G for $\sigma \tau \rho$
proof –
 have $\sigma \otimes \text{inv } \tau = (\sigma \otimes \rho) \otimes \text{inv } (\tau \otimes \rho)$ **using** *that m-assoc by (simp add: inv-solve-left inv-solve-right)*
 then show *?thesis by simp*
qed
 then have *right-invariant: right-invariant-metric ? δ*
 unfolding *right-invariant-metric-def using metric by auto*
 have *mball-coset-of-norm-ball: mball $\sigma \ \varepsilon = ?B \ \varepsilon \ \#\> \ \sigma$ if $h\sigma: \sigma \in \text{carrier } G$ for $\sigma \ \varepsilon$*
proof –
 have *mball $\sigma \ \varepsilon = \{\tau \in \text{carrier } G. N(\sigma \otimes \text{inv } \tau) < \varepsilon\}$ unfolding mball-def using $h\sigma$ by auto*
 also have *... = (?B ε) $\#\> \ \sigma$*
proof –
 have $\tau \in (?B \ \varepsilon) \ \#\> \ \sigma$ **if** $\tau \in \text{carrier } G \wedge N(\sigma \otimes \text{inv } \tau) < \varepsilon$ **for** τ
proof –
 have *inv $(\sigma \otimes \text{inv } \tau) \otimes \sigma = \tau$ using $h\sigma$ that by (simp add: inv-mult-group m-assoc)*
 moreover have *inv $(\sigma \otimes \text{inv } \tau) \in ?B \ \varepsilon$ using $h\sigma$ that prenorn by fastforce*
 ultimately show *?thesis unfolding r-coset-def by force*
qed
 moreover have $\tau \in \text{carrier } G \wedge N(\sigma \otimes \text{inv } \tau) < \varepsilon$ **if** $\tau \in (?B \ \varepsilon) \ \#\> \ \sigma$ **for** τ
proof –
 from that obtain ρ where $\rho \in ?B \ \varepsilon \wedge \tau = \rho \otimes \sigma$ **unfolding r-coset-def by blast**
 moreover from this have $\sigma \otimes \text{inv } \tau = \text{inv } \rho$ **using $h\sigma$**
 by (*metis (no-types, lifting) inv-closed inv-mult-group inv-solve-left m-closed mem-Collect-eq*)
 ultimately show *?thesis using $h\sigma$ prenorn by fastforce*
qed
 ultimately show *?thesis by blast*
qed
 finally show *?thesis by presburger*
qed
 define *ball where ball $S \iff (\exists \sigma \ \varepsilon. \sigma \in \text{carrier } G \wedge S = \text{mball } \sigma \ \varepsilon)$ for S*
 have *openin mtopology V if ball V for V using that unfolding ball-def by fast*
 moreover have $\exists W. \text{ball } W \wedge \sigma \in W \wedge W \subseteq V$ **if** *openin mtopology $V \wedge \sigma \in V$ for $\sigma \ V$*
 unfolding *ball-def using openin-mtopology that by (smt (verit, best) centre-in-mball-iff subset-iff)*
 ultimately have *openin-metric: openin mtopology = arbitrary union-of ball*
 by (*simp add: openin-topology-base-unique*)
 have *openin $T \ V$ if ball V for V*
proof –
 from that obtain $\sigma \ \varepsilon$ where $\sigma \in \text{carrier } G \wedge V = ?B \ \varepsilon \ \#\> \ \sigma$
 unfolding *ball-def using mball-coset-of-norm-ball by blast*
 moreover have *openin $T \ (?B \ \varepsilon)$ using continuous*

by (simp add: continuous-map-upper-lower-semicontinuous-lt)
 ultimately show ?thesis using open-set-translations(2) by presburger
 qed
 moreover have $\exists W. \text{ball } W \wedge \sigma \in W \wedge W \subseteq V$ if neighborhood σV for σV
 proof –
 from that have in-group: $\sigma \in \text{carrier } G$ using open-set-in-carrier by fast
 then have neighborhood 1 ($V \#> \text{inv } \sigma$)
 using r-coset-def open-set-translations(2) that r-inv by fastforce
 then obtain n where $U n \subseteq V \#> \text{inv } \sigma$ using neighborhood-base by presburger
 then have $?B (1/2^n) \subseteq V \#> \text{inv } \sigma$ using norm-ball-in-U by blast
 then have $?B (1/2^n) \#> \sigma \subseteq (V \#> \text{inv } \sigma) \#> \sigma$ unfolding r-coset-def
 by fast
 also have $\dots = V$ using in-group that open-set-in-carrier by (simp add: coset-mult-assoc)
 finally have $\text{mball } \sigma (1/2^n) \subseteq V$ using mball-coset-of-norm-ball in-group by blast
 then show ?thesis unfolding ball-def
 by (smt (verit) centre-in-mball-iff divide-pos-pos in-group one-add-one zero-less-power zero-less-two)
 qed
 ultimately have $\text{openin } T = \text{arbitrary union-of ball}$ by (simp add: openin-topology-base-unique)
 then show ?thesis using right-invariant openin-metric topology-eq by fastforce
 qed

corollary Birkhoff-Kakutani-iff:

shows metrizable-space $T \longleftrightarrow \text{Hausdorff-space } T \wedge \text{first-countable } T$

using Birkhoff-Kakutani-left Metric-space.metrizable-space-mtopology metrizable-imp-Hausdorff-space

metrizable-imp-first-countable unfolding left-invariant-metric-def by metis

end

end

3 Examples of Topological Groups

theory Topological-Group-Examples

imports Topological-Group

begin

Summary This section gives examples of topological groups.

lemma (in group) discrete-topological-group:

shows topological-group G (discrete-topology (carrier G))

proof –

let $?T = \text{discrete-topology (carrier } G)$

have topspace $?T = \text{carrier } G$ using topspace-discrete-topology by force

moreover have continuous-map (prod-topology $?T ?T$) $?T (\lambda(\sigma, \tau). \sigma \otimes \tau)$

unfolding *prod-topology-discrete-topology[symmetric]* **by** *auto*
ultimately show *?thesis unfolding topological-group-def topological-group-axioms-def*
by *fastforce*
qed

lemma *topological-group-real-power-space:*

defines $\mathfrak{R} :: (\text{real}^n) \text{ monoid} \equiv (\text{carrier} = \text{UNIV}, \text{mult} = (+), \text{one} = 0)$

defines $T :: (\text{real}^n) \text{ topology} \equiv \text{euclidean}$

shows *topological-group* $\mathfrak{R} T$

proof –

have $x \in \text{Units } \mathfrak{R}$ **for** x

proof –

have $x \otimes_{\mathfrak{R}} -x = \mathbf{1}_{\mathfrak{R}} -x \otimes_{\mathfrak{R}} x = \mathbf{1}_{\mathfrak{R}}$ **using** $\mathfrak{R}\text{-def}$ **by** *auto*

then show *?thesis unfolding Units-def* $\mathfrak{R}\text{-def}$ **by** *fastforce*

qed

then have *group: group* \mathfrak{R} **by** (*unfold-locales*) (*auto simp:* $\mathfrak{R}\text{-def}$)

then interpret *group* \mathfrak{R} **by** *auto*

have *group-is-space: topspace* $T = \text{carrier } \mathfrak{R}$

unfolding $\mathfrak{R}\text{-def}$ $T\text{-def}$ **by** *force*

have *mul-continuous: continuous-map* (*prod-topology* $T T$) $T (\lambda(x,y). x \otimes_{\mathfrak{R}} y)$

using *continuous-map-add[OF continuous-map-fst continuous-map-snd]*

unfolding $T\text{-def}$ $\mathfrak{R}\text{-def}$ **by** (*simp add: case-prod-beta'*)

have $(-x) \otimes_{\mathfrak{R}} x = \mathbf{1}_{\mathfrak{R}}$ **for** x **unfolding** $\mathfrak{R}\text{-def}$ **by** *auto*

then have *inv* $_{\mathfrak{R}}$ $x = -x$ **for** x **using** *inv-equality* $\mathfrak{R}\text{-def}$ **by** *simp*

moreover have *continuous-map* $T T$ *uminus* **unfolding** $T\text{-def}$ **by** *force*

ultimately have *continuous-map* $T T (\lambda x. \text{inv}_{\mathfrak{R}} x)$ **by** *simp*

then show *?thesis using group-is-space mul-continuous group*

unfolding *topological-group-def topological-group-axioms-def* **by** *blast*

qed

definition *unit-group* $:: ('a :: \text{field}) \text{ monoid where}$

unit-group $= (\text{carrier} = \text{UNIV} - \{0\}, \text{mult} = (*), \text{one} = 1)$

lemma

group-unit-group: group *unit-group* **and**

inv-unit-group: x $\in \text{carrier } \text{unit-group} \implies \text{inv}_{\text{unit-group}} x = \text{inverse } x$

proof –

have $x \in \text{Units } \text{unit-group}$ **if** $x \neq 0$ **for** x

proof –

have $x \otimes_{\text{unit-group}} 1/x = \mathbf{1}_{\text{unit-group}} 1/x \otimes_{\text{unit-group}} x = \mathbf{1}_{\text{unit-group}}$

using *that* **unfolding** *unit-group-def* **by** *auto*

then show *?thesis unfolding Units-def unit-group-def* **using** *that* **by** *fastforce*

qed

then show *group* *unit-group* **by** (*unfold-locales*) (*auto simp: unit-group-def*)

then interpret *group* *unit-group* **by** *blast*

show *inv* $_{\text{unit-group}}$ $x = \text{inverse } x$ **if** $x \in \text{carrier } \text{unit-group}$

using *that inv-equality[of inverse x]* **unfolding** *unit-group-def* **by** *simp*

qed

```

lemma topological-group-real-unit-group:
  defines  $T :: \text{real topology} \equiv \text{subtopology euclidean } (UNIV - \{0\})$ 
  shows topological-group unit-group T
proof –
  let  $\mathfrak{R} = \text{unit-group} :: \text{real monoid}$ 
  have group-is-space: topspace T = carrier  $\mathfrak{R}$  unfolding unit-group-def T-def
by force
  have continuous-map (prod-topology euclidean euclidean) euclidean  $(\lambda(x,y). x$ 
 $\otimes_{\mathfrak{R}} y)$ 
  using continuous-map-real-mult[OF continuous-map-fst continuous-map-snd]
  unfolding T-def unit-group-def by (simp add: case-prod-beta')
  then have continuous-map (prod-topology T T) euclideanreal  $(\lambda(x,y). x \otimes_{\mathfrak{R}} y)$ 
  unfolding T-def subtopology-Times[symmetric] using continuous-map-from-subtopology
by blast
  moreover have  $(\lambda(x,y). x \otimes_{\mathfrak{R}} y) \in \text{topspace } (\text{prod-topology } T \ T) \rightarrow UNIV - \{0\}$ 
  unfolding T-def unit-group-def by fastforce
  ultimately have mul-continuous: continuous-map (prod-topology T T) T  $(\lambda(x,y).$ 
 $x \otimes_{\mathfrak{R}} y)$ 
  unfolding T-def using continuous-map-into-subtopology by blast
  have continuous-map T euclideanreal inverse
  using continuous-map-real-inverse[of T id] unfolding T-def by auto
  moreover have inverse  $\in \text{topspace } T \rightarrow \text{topspace } T$  unfolding T-def by fast-
force
  ultimately have continuous-map T T inverse
  unfolding T-def using continuous-map-into-subtopology by auto
  then have continuous-map T T  $(\lambda x. \text{inv}_{\mathfrak{R}} x)$ 
  using group-is-space continuous-map-eq inv-unit-group by metis
  then show ?thesis using group-is-space mul-continuous group-unit-group
  unfolding topological-group-def topological-group-axioms-def by blast
qed

end

```

4 Matrix groups

```

theory Matrix-Group
  imports
    Topological-Group
    Topological-Group-Examples
    HOL-Analysis.Determinants
begin

```

Summary In this section we define the general linear group and some of its subgroups. We also introduce topologies on vector types and use them to prove the aforementioned groups to be topological groups.

4.1 Topologies on vector types

definition *vec-topology* :: 'a topology \Rightarrow ('a[~]n) topology **where**
vec-topology T = quot-topology (product-topology ($\lambda i. T$) UNIV) *vec-lambda*

lemma *producttop-vec-top-homeo*:

shows *homeomorphic-map* (product-topology ($\lambda i. T$) UNIV) (*vec-topology* T) *vec-lambda*

proof –

have *inj-on vec-lambda* (topspace (product-topology ($\lambda i. T$) UNIV)) **unfolding** *inj-on-def* **by** *force*

then show *?thesis* **unfolding** *vec-topology-def*

using *injective-quotient-map-homeo*[OF *projection-quotient-map*] **by** *blast*

qed

lemma *homeo-inverse-homeo*:

assumes *homeo*: *homeomorphic-map* X Y *f* **and** *fg-id*: $\forall y \in \text{topspace } Y. f (g y) = y$ **and**

g-image: $\forall y \in \text{topspace } Y. g y \in \text{topspace } X$

shows *homeomorphic-map* Y X *g*

proof –

from *homeo* **obtain** *h* **where**

h-homeo: *homeomorphic-map* Y X *h* **and** *hf-id*: $(\forall x \in \text{topspace } X. h (f x) = x)$

by (*smt* (*verit*) *homeomorphic-map-maps* *homeomorphic-maps-map*)

have $g y = h y$ **if** $y \in \text{topspace } Y$ **for** *y*

proof –

have $g y = h (f (g y))$ **using** *hf-id* **that** *g-image* **by** *fastforce*

then show *?thesis* **using** *fg-id* **that** **by** *simp*

qed

then show *?thesis* **using** *homeomorphic-map-eq*[OF *h-homeo*] **by** *presburger*

qed

lemma *vec-top-producttop-homeo*:

shows *homeomorphic-map* (*vec-topology* T) (product-topology ($\lambda i. T$) UNIV) *vec-nth*

proof –

let $?T' = \text{product-topology } (\lambda i. T) \text{ UNIV}$

have *vec-lambda* (*vec-nth* v) = v **for** $v :: 'a^{\sim}n$ **by** *simp*

moreover have *vec-nth* v $\in \text{topspace } ?T'$ **if** $v \in \text{topspace } (\text{vec-topology } T)$ **for** $v :: 'a^{\sim}n$

proof –

have $\exists f \in \text{topspace } ?T'. v = \text{vec-lambda } f$ **using** *that*

unfolding *vec-topology-def* *topspace-quot-topology* *image-def* **by** *fast*

then show *?thesis* **by** *fastforce*

qed

ultimately show *?thesis* **using** *homeo-inverse-homeo*[OF *producttop-vec-top-homeo*]

by *blast*

qed

lemma *vec-topology-euclidean* [*simp*]:

defines $T :: ('a :: \text{topological-space}) \text{ topology} \equiv \text{euclidean}$
defines $T_{vec} :: ('a \wedge 'n) \text{ topology} \equiv \text{euclidean}$
shows $\text{vec-topology } T = T_{vec}$
proof –
have $\text{openin } (\text{vec-topology } T) U \text{ if } \text{openin } T_{vec} U \text{ for } U$
proof –
have $hU: \text{open } U$ **using** open-openin **that** **unfolding** $T_{vec}\text{-def}$ **by** blast
have $\exists U'. \text{openin } (\text{vec-topology } T) U' \wedge x \in U' \wedge U' \subseteq U \text{ if } x \in U \text{ for } x$
proof –
from $that$ **hU** **obtain** $V :: 'n \Rightarrow 'a \text{ set}$ **where**
 $hV: (\forall i. \text{open } (V i) \wedge x \$ i \in V i) \wedge (\forall y. (\forall i. y \$ i \in V i) \longrightarrow y \in U)$
unfolding open-vec-def **by** force
let $?W = \Pi_E i \in UNIV. V i$
from hV **have** $\text{openin } T (V i) \text{ for } i$ **using** open-openin **unfolding** $T\text{-def}$ **by**
 blast
then **have** $\text{openin } (\text{product-topology } (\lambda i. T) UNIV) ?W$ **by** $(\text{simp add:}$
 $\text{openin-PiE})$
then **have** $\text{is-open: } \text{openin } (\text{vec-topology } T) (\text{vec-lambda}'?W)$
using $\text{producttop-vectop-homeo homeomorphic-map-openness openin-subset}$
by metis
have $\text{vec-nth } x \in ?W$ **using** hV **by** fast
then **have** $\text{contains-x: } x \in (\text{vec-lambda}'?W)$ **unfolding** image-def **by** force
have $y \in U \text{ if } \text{vec-nth } y \in ?W \text{ for } y$
proof –
from $that$ **have** $y \$ i \in V i \text{ for } i$ **by** fast
then **show** $?thesis$ **using** hV **by** blast
qed
then **have** $(\text{vec-lambda}'?W) \subseteq U$ **by** force
then **show** $?thesis$ **using** $\text{contains-x is-open}$ **by** meson
qed
then **show** $?thesis$ **by** $(\text{meson openin-subopen})$
qed
moreover **have** $\text{openin } T_{vec} U \text{ if } \text{openin } (\text{vec-topology } T) U \text{ for } U$
proof –
from $that$ **have** $hU: \text{openin } (\text{product-topology } (\lambda i. T) UNIV) (\text{vec-nth}'U)$
using $\text{vectop-producttop-homeo homeomorphic-map-openness openin-subset}$ **by**
 metis
have $\exists V. (\forall i. \text{open } (V i) \wedge x \$ i \in V i) \wedge (\forall y. (\forall i. y \$ i \in V i) \longrightarrow y \in U)$
if $x \in U \text{ for } x$
proof –
from $that$ **have** $\text{vec-nth } x \in (\text{vec-nth}'U)$ **unfolding** image-def **by** blast
then **obtain** $V :: 'n \Rightarrow 'a \text{ set}$
where $hV: (\forall i. \text{openin } T (V i)) \wedge \text{vec-nth } x \in (\Pi_E i \in UNIV. V i) \wedge (\Pi_E$
 $i \in UNIV. V i) \subseteq (\text{vec-nth}'U)$
using hU $\text{product-topology-open-contains-basis}$ **by** $(\text{metis } (\text{no-types, lifting}))$
then **have** $\text{open } (V i) \wedge x \$ i \in V i \text{ for } i$ **unfolding** $T\text{-def}$ **using** open-openin
by fast
moreover **have** $y \in U \text{ if } \forall i. y \$ i \in V i \text{ for } y$
proof –

have $vec\text{-}nth\ y \in (\prod_{E \in UNIV}. V\ i)$ **using** *that* **by** *blast*
then show *?thesis* **using** hV **by** (*metis image-iff in-mono vec-nth-inject*)
qed
ultimately show *?thesis* **by** *blast*
qed
then have *open U* **unfolding** *open-vec-def* **by** *blast*
then show *?thesis* **unfolding** $T_{vec\text{-}def}$ **using** *open-openin* **by** *blast*
qed
ultimately show *?thesis* **using** *topology-eq* **by** *meson*
qed

lemma *vec-projection-continuous*:
shows *continuous-map (vec-topology T) T* $(\lambda v. v\$i)$
using *homeomorphic-imp-continuous-map[OF vectop-producttop-homeo]* **by** *fast*

lemma *vec-components-continuous-imp-continuous*:
fixes $f :: 'x \Rightarrow 'a^{n^m}$
assumes $\forall i. \text{continuous-map } X\ T\ (\lambda x. (f\ x)\ \$\ i)$
shows *continuous-map X (vec-topology T) f*
proof –
have *continuous-map X (product-topology (λi. T) UNIV) (vec-nth ∘ f)* **using** *assms* **by** *auto*
moreover have $f = \text{vec-lambda} \circ (\text{vec-nth} \circ f)$ **by** *fastforce*
ultimately show *?thesis* **using** *continuous-map-compose*
homeomorphic-imp-continuous-map[OF producttop-vectop-homeo] **by** *fastforce*
qed

definition *matrix-topology* $:: 'a\ topology \Rightarrow ('a^{n^m})\ topology$ **where**
matrix-topology T = vec-topology (vec-topology T)

lemma *matrix-topology-euclidean[simp]*:
shows *matrix-topology euclidean = euclidean*
unfolding *matrix-topology-def* **by** *simp*

lemma *matrix-projection-continuous*:
shows *continuous-map (matrix-topology T) T* $(\lambda A. A\$i\$j)$
proof –
have $(\lambda A. A\$i\$j) = (\lambda x. x\$j) \circ (\lambda A. A\$i)$ **by** *fastforce*
then show *?thesis* **unfolding** *matrix-topology-def*
using *vec-projection-continuous continuous-map-compose* **by** *metis*
qed

lemma *matrix-components-continuous-imp-continuous*:
fixes $f :: 'x \Rightarrow 'a^{n^m}$
assumes $\bigwedge i\ j. \text{continuous-map } X\ T\ (\lambda x. (f\ x)\ \$\ i\ \$\ j)$
shows *continuous-map X (matrix-topology T) f*
unfolding *matrix-topology-def* **using** *vec-components-continuous-imp-continuous*
assms **by** *metis*

4.2 The general linear group as a topological group

definition $GL :: ((\text{'a} :: \text{field}) \text{~}^n \text{~}^n)$ monoid **where**
 $GL = \langle \text{carrier} = \{A. \text{invertible } A\}, \text{monoid.mult} = (**), \text{one} = \text{mat } 1 \rangle$

definition $GL\text{-topology} :: (\text{real} \text{~}^n \text{~}^n)$ topology **where**
 $GL\text{-topology} = \text{subtopology euclidean } (\text{carrier } GL)$

lemma topspace-GL : $\text{topspace } GL\text{-topology} = \{A. \text{invertible } A\}$
unfolding $GL\text{-topology-def}$ $\text{topspace-subtopology } GL\text{-def}$ **by** simp

4.2.1 Continuity of matrix operations

lemma det-continuous :

defines $T :: (\text{real} \text{~}^n \text{~}^n)$ topology \equiv euclidean
shows $\text{continuous-map } T$ euclideanreal det

proof –

let $?T' = \text{matrix-topology euclideanreal}$

let $?S = \{\pi. \pi \text{ permutes } (\text{UNIV} :: 'n \text{ set})\}$

have $S\text{-finite}$: $\text{finite } ?S$ **by** simp

have $\text{finite } (\text{UNIV} :: 'n \text{ set})$ **by** simp

then have $\text{continuous-map } ?T'$ $\text{euclideanreal } (\lambda A. \prod i \in (\text{UNIV} :: 'n \text{ set}). (A \ \$ \ i \ \$ \ \pi \ i))$

for $\pi :: 'n \Rightarrow 'n$ **using** $\text{continuous-map-prod}[OF \ \text{matrix-projection-continuous}]$
by fast

then have $\text{continuous-map } ?T'$ $\text{euclideanreal } (\lambda A. \text{of-int } (\text{sign } \pi) * (\prod i \in (\text{UNIV} :: 'n \text{ set}). (A \ \$ \ i \ \$ \ \pi \ i)))$

for $\pi :: 'n \Rightarrow 'n$ **using** $\text{continuous-map-real-mult-left}$ **by** fast

from $\text{continuous-map-sum}[OF \ S\text{-finite this}]$ **have** $\text{continuous-map } ?T'$ euclideanreal

$(\lambda A. \sum \pi \in ?S. \text{of-int } (\text{sign } \pi) * (\prod i \in (\text{UNIV} :: 'n \text{ set}). A \ \$ \ i \ \$ \ \pi \ i))$ **by** fast

then show $?thesis$ **unfolding** $T\text{-def}$ $\text{matrix-topology-euclidean det-def}$ **by** force qed

lemma $\text{matrix-mul-continuous}$:

defines $T1 :: (\text{real} \text{~}^n \text{~}^m)$ topology \equiv euclidean

defines $T2 :: (\text{real} \text{~}^r \text{~}^n)$ topology \equiv euclidean

defines $T3 :: (\text{real} \text{~}^r \text{~}^m)$ topology \equiv euclidean

shows $\text{continuous-map } (\text{prod-topology } T1 \ T2) \ T3$ $(\lambda(A,B). A ** B)$

proof –

let $?T = \text{prod-topology } T1 \ T2$

have $\text{continuous-map } ?T$ $\text{euclideanreal } (\lambda AB. (\text{fst } AB ** \text{snd } AB) \ \$ \ i \ \$ \ j)$ **for** $i :: 'm$ **and** $j :: 'r$

proof –

have eq : $(\lambda AB. (\text{fst } AB ** \text{snd } AB) \ \$ \ i \ \$ \ j) = (\lambda AB. (\sum (k :: 'n) \in \text{UNIV}. \text{fst } AB \ \$ \ i \ \$ \ k * \text{snd } AB \ \$ \ k \ \$ \ j))$

unfolding $\text{matrix-matrix-mult-def}$ **by** auto

have

comp1 : $(\lambda AB. \text{fst } AB \ \$ \ i \ \$ \ k) = (\lambda A. A \ \$ \ i \ \$ \ k) \circ \text{fst}$ **and**

comp2 : $(\lambda AB. \text{snd } AB \ \$ \ k \ \$ \ j) = (\lambda B. B \ \$ \ k \ \$ \ j) \circ \text{snd}$

```

    for k :: 'n by auto
  from comp1 have continuous-map ?T euclideanreal ( $\lambda AB. \text{fst } AB \ \$ \ i \ \$ \ k$ ) for
k :: 'n
    unfolding T1-def matrix-topology-euclidean[symmetric]
  using continuous-map-compose[OF continuous-map-fst matrix-projection-continuous]
by metis
  moreover from comp2 have continuous-map ?T euclideanreal ( $\lambda AB. \text{snd } AB \ \$ \ k \ \$ \ j$ ) for k :: 'n
    unfolding T2-def matrix-topology-euclidean[symmetric]
  using continuous-map-compose[OF continuous-map-snd matrix-projection-continuous]
by metis
  ultimately have summand-continuous:
    continuous-map ?T euclideanreal ( $\lambda AB. \text{fst } AB \ \$ \ i \ \$ \ k * \ \text{snd } AB \ \$ \ k \ \$ \ j$ ) for
k :: 'n
    using continuous-map-real-mult by blast
  have finite: finite (UNIV :: 'n set) by simp
  have continuous-map ?T euclideanreal ( $\lambda AB. (\sum (k::'n) \in UNIV. \text{fst } AB \ \$ \ i \ \$ \ k * \ \text{snd } AB \ \$ \ k \ \$ \ j)$ )
    using continuous-map-sum[OF finite summand-continuous] by fast
  then show ?thesis unfolding eq by blast
qed
from matrix-components-continuous-imp-continuous[OF this] show ?thesis
  unfolding T3-def matrix-topology-euclidean[symmetric] by (simp add: case-prod-beta')
qed

```

lemma *transpose-continuous*:

shows continuous-map (euclidean :: (('a :: topological-space)ⁿ^m) topology) euclidean transpose

proof –

have continuous-map euclidean euclidean ($\lambda A. (\text{transpose } A) \ \$ \ i \ \$ \ j$) for $i :: 'n$ and $j :: 'm$

unfolding transpose-def matrix-topology-euclidean[symmetric]

using matrix-projection-continuous[of euclidean j i] by fastforce

from matrix-components-continuous-imp-continuous[OF this] show ?thesis

unfolding matrix-topology-euclidean by blast

qed

4.2.2 Continuity of matrix inversion

lemma *matrix-mul-columns*:

fixes $A :: ('a :: semiring-1)ⁿ^m$ and $B :: 'a^kⁿ$

shows column j (A ** B) = A *v (column j B)

unfolding column-def matrix-matrix-mult-def matrix-vector-mult-def by force

lemma *matrix-columns-unique*:

assumes $\forall j. \text{column } j \ A = \text{column } j \ B$

shows $A = B$

using assms unfolding column-def by (simp add: vec-eq-iff)

lemma *matrix-inv-is-inv*:
assumes *invertible A*
shows $A ** (\text{matrix-inv } A) = \text{mat } 1$ **and** $(\text{matrix-inv } A) ** A = \text{mat } 1$
proof –
show $A ** \text{matrix-inv } A = \text{mat } 1$
using *assms unfolding invertible-def matrix-inv-def* **by** (*simp add: verit-sko-ex'*)
show $(\text{matrix-inv } A) ** A = \text{mat } 1$
using *assms unfolding invertible-def matrix-inv-def* **by** (*simp add: verit-sko-ex'*)
qed

lemma *invertible-imp-right-inverse-is-inverse*:
assumes *invertible: invertible A and $A ** B = \text{mat } 1$*
shows $\text{matrix-inv } A = B$
using *matrix-inv-is-inv[OF invertible] assms* **by** (*metis matrix-mul-assoc matrix-mul-lid*)

lemma *matrix-inv-invertible*:
assumes *invertible A*
shows *invertible (matrix-inv A)*
using *assms matrix-inv-is-inv invertible-def* **by** *fast*

lemma *det-inv*:
fixes $A :: ('a :: \text{field})^{n \times n}$
assumes $\det A \neq 0$
shows $\det (\text{matrix-inv } A) = 1 / \det A$
proof –
have $A ** (\text{matrix-inv } A) = \text{mat } 1$ **using** *assms invertible-det-nz matrix-inv-is-inv(1)*
by *fast*
then have $\det A * \det (\text{matrix-inv } A) = 1$ **using** *det-mul[of A matrix-inv A]* **by**
auto
then show *?thesis* **using** *assms* **by** (*metis nonzero-mult-div-cancel-left*)
qed

See proposition "cramer" from HOL-Analysis.Determinants

definition *cramer-inv* :: $('a :: \text{field})^{n \times n} \Rightarrow 'a^{n \times n}$ **where**
 $\text{cramer-inv } A = (\chi \ i \ j. \det(\chi \ k \ l. \text{if } l = i \text{ then } (\text{axis } j \ 1) \ \$ \ k \ \text{else } A\$k\$l) / \det A)$

lemma *cramer-inv-is-inverse*:
assumes *invertible: invertible (A :: ('a :: field)^{n × n})*
shows $\text{matrix-inv } A = \text{cramer-inv } A$
proof –
have $A ** (\text{cramer-inv } A) = \text{mat } 1$
proof –
have $\text{column } j (\text{cramer-inv } A) = (\chi \ i. \det(\chi \ k \ l. \text{if } l = i \text{ then } (\text{axis } j \ 1) \ \$ \ k \ \text{else } A\$k\$l) / \det A)$ **for** j
unfolding *cramer-inv-def column-def* **by** *simp*
moreover have $\det A \neq 0$ **using** *invertible* **unfolding** *invertible-det-nz* **by**
force
ultimately have $A * v (\text{column } j (\text{cramer-inv } A)) = \text{axis } j \ 1$ **for** j **using** *cramer*

by *auto*
then have $\text{column } j (A ** (\text{cramer-inv } A)) = \text{axis } j \ 1$ **for** j **unfolding** *matrix-mul-columns* **by** *auto*
moreover have $\text{column } j (\text{mat } 1) = \text{axis } j \ 1$ **for** $j :: 'n$ **unfolding** *column-def mat-def axis-def* **by** *simp*
ultimately show *?thesis* **using** *matrix-columns-unique* **by** *metis*
qed
then show *?thesis* **using** *invertible invertible-imp-right-inverse-is-inverse* **unfolding** *GL-def* **by** *fastforce*
qed

lemma *matrix-inv-continuous*:
shows *continuous-map (GL-topology :: (realⁿ) topology) GL-topology matrix-inv*
proof –
define $B :: \text{real}^n \Rightarrow 'n \Rightarrow 'n \Rightarrow 'n \Rightarrow 'n \Rightarrow \text{real}$ **where**
 $B = (\lambda A \ i \ j \ k \ l. \text{if } l = i \text{ then } (\text{axis } j \ 1) \ \$ \ k \ \text{else } A \$ k \$ l)$
define $C :: \text{real}^n \Rightarrow 'n \Rightarrow 'n \Rightarrow \text{real}^n$ **where**
 $C \ A \ i \ j = (\chi \ k \ l. B \ A \ i \ j \ k \ l)$ **for** $A \ i \ j$
have *det-GL-continuous: continuous-map GL-topology euclideanreal det*
unfolding *GL-topology-def* **using** *continuous-map-from-subtopology[OF det-continuous]*
by *fast*
have *continuous-map euclidean euclideanreal* $(\lambda A. B \ A \ i \ j \ k \ l)$ **for** $i \ j \ k \ l$
proof (*cases* $l = i$)
case *True*
then have $(\lambda A. B \ A \ i \ j \ k \ l) = (\lambda A. (\text{axis } j \ 1) \ \$ \ k)$ **unfolding** *B-def* **by** *force*
moreover have *continuous-map euclidean euclideanreal* $(\lambda A. (\text{axis } j \ 1) \ \$ \ k)$
by *simp*
ultimately show *?thesis* **by** (*smt* (*verit*) *continuous-map-eq*)
next
case *False*
then have $(\lambda A. B \ A \ i \ j \ k \ l) = (\lambda A. A \$ k \$ l)$ **unfolding** *B-def* **by** *simp*
then show *?thesis* **unfolding** *matrix-topology-euclidean[symmetric]*
using *matrix-projection-continuous[of euclideanreal k l]* **by** *force*
qed
then have *continuous-map euclidean euclideanreal* $(\lambda A. (C \ A \ i \ j) \ \$ \ k \ \$ \ l)$
for $i \ j \ k \ l$ **unfolding** *C-def* **by** *simp*
from *matrix-components-continuous-imp-continuous[OF this]*
have *continuous-map euclidean euclidean* $(\lambda A. C \ A \ i \ j)$ **for** $i \ j$
unfolding *matrix-topology-euclidean[symmetric]* **by** *blast*
from *continuous-map-compose[OF this det-continuous]*
have *continuous-map euclidean euclideanreal* $(\lambda A. \text{det } (C \ A \ i \ j))$ **for** $i \ j$ **by** *force*
then have *continuous-map GL-topology euclideanreal* $(\lambda A. \text{det } (C \ A \ i \ j))$ **for** $i \ j$
unfolding *GL-topology-def* **using** *continuous-map-from-subtopology* **by** *fast*
from *continuous-map-real-divide[OF this det-GL-continuous]*
have *continuous-map GL-topology euclideanreal* $(\lambda A. \text{det } (C \ A \ i \ j) / \text{det } A)$ **for**
 $i \ j$
unfolding *topspace-GL invertible-det-nz* **by** *simp*
then have *continuous-map GL-topology euclideanreal* $(\lambda A. (\chi \ i \ j. \text{det } (C \ A \ i \ j))$

```

/ det A) $ i $ j) for i j by simp
from matrix-components-continuous-imp-continuous[OF this]
have continuous-map (GL-topology :: (real^n^n) topology) euclidean cramer-inv

  unfolding cramer-inv-def C-def B-def matrix-topology-euclidean[symmetric] by
blast
from continuous-map-eq[OF this] have continuous-map (GL-topology :: (real^n^n)
topology) euclidean matrix-inv
  unfolding topspace-GL using cramer-inv-is-inverse by (metis mem-Collect-eq)
  moreover have matrix-inv A ∈ topspace GL-topology if A ∈ topspace GL-topology
for A :: real^n^n
  using that unfolding topspace-GL
  by (metis invertible-imp-right-inverse-is-inverse invertible-left-inverse invert-
ible-right-inverse mem-Collect-eq)
  ultimately show ?thesis unfolding GL-topology-def Pi-def image-def using
continuous-map-into-subtopology by auto
qed

```

4.2.3 The general linear group is topological

lemma

```

GL-group: group GL and
GL-carrier [simp]: carrier GL = {A. invertible A} and
GL-inv [simp]: A ∈ carrier GL ⇒ inv_GL A = matrix-inv A
proof -
show carrier GL = {A. invertible A} unfolding GL-def by simp
show group GL
proof (unfold-locale, goal-cases)
case 3
then show ?case unfolding GL-def by (simp add: invertible-def)
case 6
then show ?case using GL-def unfolding Units-def invertible-def
  by (smt (verit, ccfv-threshold) Collect-mono invertible-def mem-Collect-eq
monoid.select-convs(1) monoid.select-convs(2) partial-object.select-convs(1))
qed (unfold GL-def, auto simp: matrix-mul-assoc invertible-mult)
interpret group GL by fact
show A ∈ carrier GL ⇒ inv_GL A = matrix-inv A
  using matrix-inv-is-inv matrix-inv-invertible inv-equality unfolding GL-def by
fastforce
qed

```

lemma

```

GL-topological-group: topological-group GL GL-topology and
GL-open: openin (euclidean :: (real^n^n) topology) (carrier GL)
proof -
have group-is-space: topspace GL-topology = carrier GL unfolding topspace-GL
GL-def by simp
have continuous-map (prod-topology GL-topology GL-topology) euclidean (λ(A,B).
A ** B)

```

unfolding *GL-topology-def subtopology-Times*[*symmetric*] **using** *matrix-mul-continuous continuous-map-from-subtopology* **by** *fast*
from *continuous-map-into-subtopology*[*OF this*]
have *continuous-map (prod-topology GL-topology GL-topology) GL-topology* $(\lambda(A,B). A \otimes_{GL} B)$
unfolding *GL-topology-def Pi-def topspace-prod-topology topspace-subtopology GL-def* **using** *invertible-mult* **by** *auto*
moreover from *continuous-map-eq*[*OF matrix-inv-continuous*]
have *continuous-map GL-topology GL-topology* $(\lambda A. inv_{GL} A)$ **unfolding** *group-is-space*
using *GL-inv* **by** *metis*
ultimately show *topological-group GL GL-topology* **using** *GL-group group-is-space*
unfolding *topological-group-def topological-group-axioms-def* **by** *blast*
have *openin euclideanreal* $((topspace euclideanreal) - \{0\})$ **by** *auto*
from *openin-continuous-map-preimage*[*OF det-continuous this*]
have *openin euclidean* $\{(A :: real^{^n}n) \in topspace euclidean. det A \in ((topspace euclideanreal) - \{0\})\}$ **by** *blast*
moreover have *carrier GL* $= \{A :: real^{^n}n. det A \neq 0\}$
using *group-is-space*[*symmetric*] *invertible-det-nz* **unfolding** *topspace-GL* **by** *blast*
ultimately show *openin (euclidean :: (real^{^n}n) topology)* $(carrier GL)$ **by** *fastforce*
qed

4.3 Subgroups of the general linear group

definition *SL* $:: ((a :: field)^{^n}n) monoid$ **where**
 $SL = GL \ (carrier := \{A. det A = 1\})$

lemma *det-homomorphism: group-hom GL unit-group det*

proof –

have *det* $\in carrier GL \rightarrow carrier unit-group$
unfolding *GL-carrier unit-group-def* **using** *invertible-det-nz* **by** *fastforce*
moreover have *det* $(A \otimes_{GL} B) = det A \otimes_{unit-group} det B$ **for** *A B*
unfolding *GL-def unit-group-def* **using** *det-mul* **by** *auto*
ultimately have *det* $\in hom GL unit-group$ **unfolding** *hom-def* **by** *blast*
then show *?thesis* **using** *GL-group group-unit-group*
unfolding *group-hom-def group-hom-axioms-def* **by** *blast*
qed

lemma

SL-kernel-det: carrier (SL :: ((a :: field)^{^n}n) monoid) = kernel GL unit-group det **and**

SL-subgroup: subgroup (carrier SL) (GL :: (a^{^n}n) monoid) **and**

SL-carrier [simp]: carrier SL = \{A. det A = 1\}

proof –

interpret *group-hom GL :: (a^{^n}n) monoid unit-group det* **using** *det-homomorphism*
by *blast*

show *carrier SL* $= \{A. det A = 1\}$ **unfolding** *SL-def* **by** *simp*

then show *carrier (SL :: (a^{^n}n) monoid) = kernel GL unit-group det*

unfolding *kernel-def GL-carrier unit-group-def using invertible-det-nz by force*
then show *subgroup (carrier SL) (GL :: ('a ^n ^n) monoid) using subgroup-kernel*
by *presburger*
qed

lemma

SL-topological-group: topological-group SL (subtopology GL-topology (carrier SL))

and

SL-closed: closedin GL-topology (carrier SL)

proof –

interpret *topological-group GL GL-topology using GL-topological-group by blast*

show *topological-group SL (subtopology GL-topology (carrier SL))*

unfolding *SL-def using topological-subgroup[OF SL-subgroup] by force*

have *closedin euclideanreal {1} by simp*

then have *closedin GL-topology {A ∈ topspace GL-topology. det A = 1} un-*

folding *GL-topology-def*

using *continuous-map-from-subtopology[OF det-continuous] closedin-continuous-map-preimage*

by *(smt (verit, ccfv-SIG) Collect-cong singleton-iff)*

moreover have *{A ∈ topspace GL-topology. det A = 1} = {A. det A = 1}*

using *topspace-GL using invertible-det-nz by fastforce*

ultimately show *closedin GL-topology (carrier SL) unfolding SL-carrier by*

(smt (verit))

qed

definition *GO :: (real ^n ^n) monoid where*

GO = GL (|carrier := {A. orthogonal-matrix A})

lemma

GO-subgroup: subgroup {A :: real ^n ^n. orthogonal-matrix A} GL and

GO-carrier [simp]: carrier GO = {A. orthogonal-matrix A}

proof –

show *carrier GO = {A. orthogonal-matrix A} unfolding GO-def by force*

show *subgroup {A :: real ^n ^n. orthogonal-matrix A} GL*

proof *(unfold-locales, goal-cases)*

case 1

then show *?case unfolding GL-carrier orthogonal-matrix-def invertible-def*

by *blast*

next

case (2 A B)

then show *?case unfolding GL-def using orthogonal-matrix-mul[of A B] by*

force

next

case 3

then show *?case unfolding GL-def using orthogonal-matrix-id by simp*

next

case (4 A)

then have *A ∈ carrier GL unfolding GL-carrier orthogonal-matrix-def invert-*
ible-def by blast

moreover from $_4$ **have** *orthogonal-matrix* (*matrix-inv* A)
by (*metis invertible-imp-right-inverse-is-inverse invertible-right-inverse mem-Collect-eq*
orthogonal-matrix-def orthogonal-matrix-transpose)
ultimately show *?case* **using** *GL-inv* **by** *fastforce*
qed
qed

lemma

GO-topological-group: topological-group GO (*subtopology* *GL-topology* (*carrier* GO))

and

GO-closed: closedin (*GL-topology* :: (*realⁿ* *topology*) (*carrier* GO))

proof –

interpret *topological-group* GL *GL-topology* **using** *GL-topological-group* **by** *blast*

show *topological-group* GO (*subtopology* *GL-topology* (*carrier* GO))

unfolding *GO-def* **using** *topological-subgroup*[*OF GO-subgroup*] **by** *simp*

have *one-closed: closedin euclidean* $\{(\text{mat } 1) :: \text{real}^n\}$ **by** *fastforce*

have *continuous-map euclidean* (*prod-topology euclidean euclidean*) $(\lambda A :: \text{real}^n.$
(transpose $A, A))$

using *continuous-map-pairedI*[*OF transpose-continuous continuous-map-id*] **by**
force

from *continuous-map-compose*[*OF this matrix-mul-continuous*]

have *continuous-map euclidean euclidean* $(\lambda A :: \text{real}^n. (\text{transpose } A) ** A)$

by *force*

from *closedin-continuous-map-preimage*[*OF this one-closed*]

have *closedin euclidean* $\{A :: \text{real}^n. (\text{transpose } A) ** A = \text{mat } 1\}$ **by** *force*

moreover have *carrier* $GO = \{A :: \text{real}^n. (\text{transpose } A) ** A = \text{mat } 1\}$

using *orthogonal-matrix* **unfolding** *GO-carrier* **by** *blast*

ultimately have *closedin* (*euclidean* :: (*realⁿ* *topology*) (*carrier* GO)) **by**
(smt (verit, del-insts))

moreover have *carrier* $GO \subseteq \text{carrier } GL$

unfolding *GO-carrier* *GL-carrier* *orthogonal-matrix-def invertible-def* **by** *blast*

ultimately show *closedin* (*GL-topology* :: (*realⁿ* *topology*) (*carrier* GO))

unfolding *GL-topology-def* **using** *closedin-subset-topospace* **by** *blast*

qed

definition $SO :: (\text{real}^n)$ *monoid* **where**

$SO = GL$ (*carrier* := $\{A. \text{orthogonal-matrix } A \wedge \det A = 1\}$)

lemma

SO-carrier [*simp*]: *carrier* $SO = \{A. \text{orthogonal-matrix } A \wedge \det A = 1\}$ **and**

SO-subgroup: subgroup $\{A :: \text{real}^n. \text{orthogonal-matrix } A \wedge \det A = 1\}$ *GL*

proof –

show *carrier* $SO = \{A. \text{orthogonal-matrix } A \wedge \det A = 1\}$ **unfolding** *SO-def*

by *auto*

have *eq*: $\{A :: \text{real}^n. \text{orthogonal-matrix } A \wedge \det A = 1\} = \{A. \text{ortho-}$
*nal-matrix } A\} \cap \{A. \det A = 1\} **by** *fastforce**

show *subgroup* $\{A :: \text{real}^n. \text{orthogonal-matrix } A \wedge \det A = 1\}$ *GL*

unfolding *eq* **using** *subgroup-intersection*[*OF GO-subgroup SL-subgroup*] **by**

simp

qed

lemma

SO-topological-group: topological-group SO (subtopology GL-topology (carrier SO))

and

SO-closed: closedin GL-topology (carrier SO)

proof –

interpret *topological-group GL GL-topology* **using** *GL-topological-group by blast*

show *topological-group SO (subtopology GL-topology (carrier SO))*

unfolding *SO-def* **using** *topological-subgroup[OF SO-subgroup]* **by** *simp*

have *carrier SO = carrier SL \cap carrier GO* **unfolding** *SO-carrier SL-carrier GO-carrier* **by** *blast*

then show *closedin GL-topology (carrier SO)* **using** *closedin-Int[OF SL-closed GO-closed]* **by** *metis*

qed

end

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