Three squares theorem

Anna Danilkin, Loïc Chevalier

February 6, 2026

Abstract

We formalize the Legendre's three squares theorem and its consequences, in particular the following results:

1. A natural number can be represented as the sum of three squares of natural numbers if and only if it is not of the form $4^a(8k+7)$, where a and k are natural numbers.
2. If n is a natural number such that $n \equiv 3 \pmod 8$, then n can be be represented as the sum of three squares of odd natural numbers.

Consequences include the following:

1. An integer n can be written as $n = x^2 + y^2 + z^2 + z$, where x, y, z are integers, if and only if $n \geq 0$.
2. The Legendre's four squares theorem: any natural number can be represented as the sum of four squares of natural numbers.

We follow the book of Melvyn B. Nathanson 'Additive Number Theory: The Classical Bases' [1].

We plan to make use of the first consequence mentioned above in an upcoming AFP entry on Diophantine equations. More concretely, we intend to formalize universal pairs over the integers which requires expressing a natural number as a polynomial in integers while only using few variables.

Contents

1 Properties of residues, congruences, quadratic residues and the Legendre symbol **2**
1.1 Properties of residues and congruences 2
1.2 Properties of quadratic residues 4
1.3 Properties of the Legendre symbol 4

2 Vectors and matrices, determinants and their properties in dimensions 2 and 3 **9**

3 Properties of quadratic forms and their equivalences **22**

4 Legendre's three squares theorem and its consequences 42
4.1 Legendre's three squares theorem 42
4.2 Consequences 67

1 Properties of residues, congruences, quadratic residues and the Legendre symbol

theory *Residues-Properties*
imports *HOL−Number-Theory.Quadratic-Reciprocity*
begin

1.1 Properties of residues and congruences

lemma *mod-diff-eq-nat*:
fixes *a b m* :: *nat*
assumes $a \geq b$
shows $(a - b) \bmod m = (m + (a \bmod m) - (b \bmod m)) \bmod m$
proof *cases*
assume $m = 0$
thus *?thesis* **by** *auto*
next
assume *0*: $m \neq 0$
have $(a - b) \bmod m = nat\ (int\ (a - b) \bmod int\ m)$
unfolding *nat-mod-as-int* **by** *blast*
also have ... = $nat\ ((int\ a - int\ b) \bmod int\ m)$
using *assms* **by** (*simp add*: *of-nat-diff*)
also have ... = $nat\ ((int\ a \bmod int\ m - int\ b \bmod int\ m) \bmod int\ m)$
using *mod-diff-eq* **by** *metis*
also have ... = $nat\ ((int\ a \bmod int\ m + (int\ m - int\ b \bmod int\ m)) \bmod int\ m)$
by (*metis add.left-commute add-uminus-conv-diff mod-add-self1*)
also have ... = $nat\ ((int\ (nat\ (int\ a \bmod int\ m)) +$
$(int\ m - int\ b \bmod int\ m)) \bmod int\ m)$
by (*metis nat-int of-nat-mod*)
also have ... = $nat\ ((int\ (nat\ (int\ a \bmod int\ m)) +$
$int\ (m - nat\ (int\ b \bmod int\ m))) \bmod int\ m)$
by (*metis 0 less-eq-nat.simps*(*1*) *mod-less-divisor*
nat-int nat-less-le of-nat-diff zmod-int)
also have ... = $nat\ (int\ (m + nat\ (int\ a \bmod int\ m) -$
$nat\ (int\ b \bmod int\ m)) \bmod int\ m)$
by (*metis 0 Nat.add-diff-assoc add.commute bot-nat-0.not-eq-extremum*
mod-less-divisor nat-less-le nat-mod-as-int of-nat-add)
also have ... = $(m + (a \bmod m) - (b \bmod m)) \bmod m$
unfolding *nat-mod-as-int* **by** *blast*
finally show *?thesis* **.**
qed

lemma *prime-invertible-int*:
fixes *a p* :: *int*

assumes *prime p*
assumes ¬ *p dvd a*
shows ∃ *b*. [*a* ∗ *b* = *1*] (*mod p*)
using *assms coprime-commute coprime-iff-invertible-int prime-imp-coprime* **by** *blast*

lemma *power-cong*:
fixes *x y a m* :: *nat*
assumes *coprime a m*
assumes [*x* = *y*] (*mod totient m*)
shows [*a* ^ *x* = *a* ^ *y*] (*mod m*)
proof −
obtain *k* :: *int* **where** *0*: *x* + *totient m* ∗ *k* = *y*
using *assms* **by** (*metis cong-iff-lin cong-int-iff*)
show *?thesis*
proof *cases*
assume *k* ≥ *0*
hence *x* + *totient m* ∗ *nat k* = *y*
using *0* **by** (*metis int-eq-iff nat-int-add of-nat-mult*)
hence [*a* ^ *x* ∗ (*a* ^ *totient m*) ^ *nat k* = *a* ^ *y*] (*mod m*)
unfolding *cong-def* **by** (*metis power-add power-mult*)
hence [*a* ^ *x* ∗ (*a* ^ *totient m mod m*) ^ *nat k* = *a* ^ *y*] (*mod m*)
unfolding *cong-def* **by** (*metis mod-mult-right-eq power-mod*)
hence [*a* ^ *x* ∗ (*1 mod m*) ^ *nat k* = *a* ^ *y*] (*mod m*)
using *euler-theorem*[*OF assms*(*1*)] **unfolding** *cong-def* **by** *argo*
hence [*a* ^ *x* ∗ *1* ^ *nat k* = *a* ^ *y*] (*mod m*)
unfolding *cong-def* **by** (*metis mod-mult-right-eq power-mod*)
thus [*a* ^ *x* = *a* ^ *y*] (*mod m*) **by** *auto*
next
assume ¬ *k* ≥ *0*
hence *x* = *y* + *totient m* ∗ *nat* (− *k*)
using *0*
by (*smt* (*verit*) *int-nat-eq nat-int nat-minus-as-int of-nat-add of-nat-mult right-diff-distrib'*)
hence [*a* ^ *x* = *a* ^ *y* ∗ (*a* ^ *totient m*) ^ *nat* (− *k*)] (*mod m*)
unfolding *cong-def* **by** (*metis power-add power-mult*)
hence [*a* ^ *x* = *a* ^ *y* ∗ (*a* ^ *totient m mod m*) ^ *nat* (− *k*)] (*mod m*)
unfolding *cong-def* **by** (*metis mod-mult-right-eq power-mod*)
hence [*a* ^ *x* = *a* ^ *y* ∗ (*1 mod m*) ^ *nat* (− *k*)] (*mod m*)
using *euler-theorem*[*OF assms*(*1*)] **unfolding** *cong-def* **by** *argo*
hence [*a* ^ *x* = *a* ^ *y* ∗ *1* ^ *nat* (− *k*)] (*mod m*)
unfolding *cong-def* **by** (*metis mod-mult-right-eq power-mod*)
thus [*a* ^ *x* = *a* ^ *y*] (*mod m*) **by** *auto*
qed
qed

lemma *power-cong-alt*:
fixes *x a m* :: *nat*
assumes *coprime a m*

shows $a \mathbin{\hat{}} x \bmod m = a \mathbin{\hat{}} (x \bmod totient\ m) \bmod m$
using *power-cong*[*OF assms*] *cong-def cong-mod-left* **by** *blast*

1.2 Properties of quadratic residues

lemma *QuadRes-cong*:
fixes $a\ b\ p :: int$
assumes $[a = b]\ (mod\ p)$
assumes *QuadRes p a*
shows *QuadRes p b*
using *assms cong-trans* **unfolding** *QuadRes-def* **by** *blast*

lemma *QuadRes-mult*:
fixes $a\ b\ p :: int$
assumes *QuadRes p a*
assumes *QuadRes p b*
shows *QuadRes p* $(a * b)$
using *assms*
unfolding *QuadRes-def*
by (*metis cong-mult mult.assoc mult.commute power2-eq-square*)

lemma *QuadRes-inv*:
fixes $a\ b\ p :: int$
assumes *prime p*
assumes $[a * b = 1]\ (mod\ p)$
assumes *QuadRes p a*
shows *QuadRes p b*
proof −
have *0*: ¬ *p dvd a*
using *assms*
by (*metis cong-dvd-iff dvd-mult2 not-prime-unit*)
obtain *x* **where** *1*: $[x^2 = a]\ (mod\ p)$ **using** *assms* **unfolding** *QuadRes-def* **by** *blast*
have ¬ *p dvd x* **using** *0 1 assms cong-dvd-iff pos2 prime-dvd-power-iff* **by** *blast*
then obtain *y* **where** $[x * y = 1]\ (mod\ p)$
using *assms prime-invertible-int* **by** *blast*
hence *2*: $[(x * y)^2 = 1]\ (mod\ p)$ **using** *cong-pow* **by** *fastforce*
have $[x^2 * b = 1]\ (mod\ p)$ **using** *1 assms cong-scalar-right cong-trans* **by** *blast*
hence $[y^2 * (x^2 * b) = y^2 * 1]\ (mod\ p)$ **using** *cong-scalar-left* **by** *blast*
hence $[(x * y)^2 * b = y^2]\ (mod\ p)$ **by** (*simp add*: *algebra-simps*)
hence $[b = y^2]\ (mod\ p)$
using *2*
by (*metis cong-def cong-scalar-left mult.commute mult.right-neutral*)
hence $[y^2 = b]\ (mod\ p)$ **by** (*rule cong-sym*)
thus *?thesis* **unfolding** *QuadRes-def* **by** *blast*
qed

1.3 Properties of the Legendre symbol

lemma *Legendre-cong*:

fixes *a b p* :: *int*
assumes [*a* = *b*] (*mod p*)
shows *Legendre a p* = *Legendre b p*
using *assms QuadRes-cong*[*of a b p*] *QuadRes-cong*[*of b a p*]
unfolding *Legendre-def cong-def*
by *auto*

lemma *Legendre-one*:
fixes *p* :: *int*
assumes *p* > *2*
shows *Legendre 1 p* = *1*
using *assms*
by (*smt* (*verit*) *Legendre-def QuadRes-def cong-def*
cong-less-imp-eq-int one-power2)

lemma *Legendre-minus-one*:
fixes *p* :: *int*
assumes *prime p*
assumes *p* > *2*
shows *Legendre* (− *1*) *p* = *1* ⟷ [*p* = *1*] (*mod 4*)
proof −
have *Legendre* (− *1*) *p* = *1* ⟷ [*Legendre* (− *1*) *p* = *1*] (*mod p*)
using *assms*
by (*metis Legendre-def QuadRes-def cong-0-iff cong-def not-prime-unit*
one-power2)
also have ... ⟷ [(− *1*) ^ ((*nat p* − *1*) *div 2*) = *1*] (*mod p*)
using *assms euler-criterion*[*of nat p* − *1*]
by (*smt* (*verit*) *cong-def nat-0-le nat-one-as-int of-nat-add one-add-one*
prime-int-nat-transfer zless-nat-eq-int-zless)
also have ... ⟷ ((− *1* :: *int*) ^ ((*nat p* − *1*) *div 2*)) = *1*
using *assms*
by (*simp add*: *cong-iff-dvd-diff minus-one-power-iff zdvd-not-zless*)
also have ... ⟷ *even* ((*nat p* − *1*) *div 2*) **by** (*simp add*: *minus-one-power-iff*)
also have ... ⟷ *4 dvd* (*nat p* − *1*)
using *assms*
by (*metis One-nat-def add-Suc-right div-dvd-div even-Suc even-diff-nat*
even-numeral even-of-nat group-cancel.rule0 nat-0-le
numeral-Bit0-div-2 prime-int-nat-transfer prime-odd-int)
also have ... ⟷ [*p* = *1*] (*mod 4*)
using *assms*
unfolding *cong-iff-dvd-diff int-dvd-int-iff*[*symmetric*]
by (*simp add*: *int-ops*)
finally show *?thesis* **.**
qed

lemma *Legendre-minus-one-alt*:
fixes *p* :: *int*
assumes *prime p*
assumes *p* > *2*

shows *Legendre* (-1) $p = (\mathit{if}\ [p = 1]\ (\mathit{mod}\ 4)\ \mathit{then}\ 1\ \mathit{else}\ -1)$
using *assms Legendre-minus-one*[*OF assms*]
unfolding *Legendre-def cong-def*
by (*auto simp add*: *zmod-minus1*)

lemma *Legendre-two*:
fixes p :: *int*
assumes *prime p*
assumes $p > 2$
shows *Legendre* $2\ p = 1 \longleftrightarrow [p = 1]\ (\mathit{mod}\ 8) \vee [p = 7]\ (\mathit{mod}\ 8)$
proof −
let *?n* $= (p - 1)\ \mathit{div}\ 2 - (p - 1)\ \mathit{div}\ 4$
have *odd p* **using** *assms prime-odd-int* **by** *blast*
hence *0*: $(\exists k.\ p = 8 * k + 1) \vee (\exists k.\ p = 8 * k + 3) \vee$
$(\exists k.\ p = 8 * k + 5) \vee (\exists k.\ p = 8 * k + 7)$
by *presburger*
have *1*: $(j + 8 * k)\ \mathit{div}\ 4 = j\ \mathit{div}\ 4 + 2 * k$ **for** $k\ j$ :: *int* **by** *linarith*
have *2*: *GAUSS* $(\mathit{nat}\ p)\ 2$
using *assms* **unfolding** *GAUSS-def cong-def* **by** *auto*
have *Legendre* $2\ p = (-1)\ \hat{}\ \mathit{card}\ (\mathit{GAUSS.E}\ (\mathit{nat}\ p)\ 2)$
using *assms GAUSS.gauss-lemma*[*OF 2*] **by** *auto*
also have ... $= (-1)\ \hat{}\ \mathit{card}$
$((\lambda k.\ k\ \mathit{mod}\ p)\ `\ (*)\ 2\ `\ \{0<..(p - 1)\ \mathit{div}\ 2\} \cap \{(p - 1)\ \mathit{div}\ 2<..\})$
unfolding *GAUSS.E-def*[*OF 2*] *GAUSS.C-def*[*OF 2*]
GAUSS.B-def[*OF 2*] *GAUSS.A-def*[*OF 2*]
by (*simp add*: *algebra-simps*)
also have ... $= (-1)\ \hat{}\ \mathit{card}$
$((*)\ 2\ `\ \{0<..(p - 1)\ \mathit{div}\ 2\} \cap \{(p - 1)\ \mathit{div}\ 2<..\})$
by (*rule arg-cong*[*of - - λA. (−1) ^ card (A ∩ {(p − 1) div 2<..})*]; *force*)
also have ... $= (-1)\ \hat{}\ \mathit{card}$
$\{k \in (*)\ 2\ `\ \{0<..(p - 1)\ \mathit{div}\ 2\}.\ k > (p - 1)\ \mathit{div}\ 2\}$
by (*rule arg-cong*[*of - - λA. (−1) ^ card A*]; *blast*)
also have ... $= (-1)\ \hat{}\ \mathit{card}\ \{k \in \{0<..(p - 1)\ \mathit{div}\ 2\}.\ 2 * k > (p - 1)\ \mathit{div}\ 2\}$
apply (*rule arg-cong*[*of - - λn. (−1) ^ n*])
apply (*rule card-bij-eq*[**where** *?f* $= \lambda k.\ k\ \mathit{div}\ 2$ **and** *?g* $= (*)\ 2$])
subgoal unfolding *inj-on-def* **by** *auto*
subgoal by *auto*
subgoal by (*simp add*: *inj-on-mult*)
subgoal by *auto*
subgoal by (*rule finite-Collect-conjI*; *auto*)
subgoal by (*rule finite-Collect-conjI*; *auto*)
done
also have ... $= (-1)\ \hat{}\ \mathit{card}\ \{k \in \{0<..(p - 1)\ \mathit{div}\ 2\}.\ k > (p - 1)\ \mathit{div}\ 4\}$
by (*rule arg-cong*[*of - - λf. (−1) ^ card {k ∈ {0<..(p − 1) div 2}. f k}*];
auto)
also have ... $= (-1)\ \hat{}\ \mathit{card}\ \{(p - 1)\ \mathit{div}\ 4<..(p - 1)\ \mathit{div}\ 2\}$
by (*rule arg-cong*[*of - - λA. (−1) ^ card A*]; *fastforce*)
also have ... $= (-1)\ \hat{}\ (\mathit{nat}\ ?n)$ **by** *auto*
finally have *Legendre* $2\ p = 1 \longleftrightarrow \mathit{even}\ ?n$

```
    unfolding minus-one-power-iff
    by (simp add: assms even-nat-iff prime-gt-0-int zdiv-mono2)
  also have ... ⟷ [p = 1] (mod 8) ∨ [p = 7] (mod 8) unfolding cong-def
    using 0 by (auto simp add: 1)
  finally show ?thesis .
qed

lemma Legendre-two-alt:
  fixes p :: int
  assumes prime p
  assumes p > 2
  shows Legendre 2 p = (if [p = 1] (mod 8) ∨ [p = 7] (mod 8) then 1 else − 1)
  using assms Legendre-two[OF assms]
  unfolding Legendre-def cong-def
  by (auto simp add: zmod-minus1)

lemma Legendre-mult:
  fixes a b p :: int
  assumes prime p
  shows Legendre (a * b) p = Legendre a p * Legendre b p
proof cases
  assume 0: p = 2
  have 1: QuadRes p = (λx. True) using 0
    by (metis QuadRes-def add-0 cong-iff-dvd-diff even-add odd-one power-one
          power-zero-numeral uminus-add-conv-diff)
  thus ?thesis using 0 unfolding 1 Legendre-def cong-0-iff by auto
next
  assume p ≠ 2
  hence 2: p > 2 using assms by (simp add: order-less-le prime-ge-2-int)
  have [Legendre a p = a ^ ((nat p − 1) div 2)] (mod p)
      [Legendre b p = b ^ ((nat p − 1) div 2)] (mod p)
      [Legendre (a * b) p = (a * b) ^ ((nat p − 1) div 2)] (mod p)
    using 2 assms euler-criterion[of nat p a]
                 euler-criterion[of nat p b]
                 euler-criterion[of nat p a * b]
    by auto
  hence 3: [Legendre (a * b) p = Legendre a p * Legendre b p] (mod p)
    by (smt (verit, best) cong-mult cong-sym cong-trans power-mult-distrib)
  have 4: {Legendre a p, Legendre b p, Legendre (a * b) p} ⊆ {−1, 0, 1}
    unfolding Legendre-def by auto
  show ?thesis using 2 3 4 by (auto simp add: cong-iff-dvd-diff zdvd-not-zless)
qed

lemma Legendre-power:
  fixes a :: int
  fixes n :: nat
  fixes p :: int
  assumes prime p
  assumes p > 2
```

shows *Legendre* $(a \mathbin{\hat{}} n)$ p = $(Legendre\ a\ p) \mathbin{\hat{}} n$
proof (*induct n*)
case *0*
thus *?case* **using** *assms Legendre-one* **by** *auto*
next
case (*Suc n*)
thus *?case* **using** *assms Legendre-mult* **by** *auto*
qed

lemma *Legendre-prod*:
fixes $A :: 'a\ set$
fixes $f :: 'a \Rightarrow int$
fixes $p :: int$
assumes *prime p*
assumes $p > 2$
shows *Legendre* $(prod\ f\ A)\ p = (\prod x{\in}A.\ Legendre\ (f\ x)\ p)$
proof (*induct A rule*: *infinite-finite-induct*)
case (*infinite A*)
thus *?case* **using** *assms Legendre-one* **by** *auto*
next
case *empty*
thus *?case* **using** *assms Legendre-one* **by** *auto*
next
case (*insert x F*)
thus *?case* **using** *assms Legendre-mult* **by** *auto*
qed

lemma *Legendre-equal*:
fixes *p q* :: *int*
assumes *prime p prime q*
assumes $p > 2$ $q > 2$
assumes $p \neq q$
assumes $[p = 1]\ (mod\ 4) \vee [q = 1]\ (mod\ 4)$
shows *Legendre p q* = *Legendre q p*
proof −
have *0*: *even* $(p - 1)$ *even* $(q - 1)$ **using** *assms prime-odd-int* **by** *auto*
have *1*: $((p - 1)\ div\ 2) * ((q - 1)\ div\ 2) = (p - 1) * (q - 1)\ div\ 4$
using *0* **by** *fastforce*
have *2*: $\{Legendre\ p\ q, Legendre\ q\ p\} \subseteq \{-1, 0, 1\}$
unfolding *Legendre-def* **by** *auto*
have *Legendre p q * Legendre q p* =
$(-1) \mathbin{\hat{}} nat\ (((p - 1)\ div\ 2) * ((q - 1)\ div\ 2))$
using *assms Quadratic-Reciprocity-int*[*of p q*]
by *fastforce*
also have ... = $(-1) \mathbin{\hat{}} nat\ ((p - 1) * (q - 1)\ div\ 4)$ **unfolding** *1* **by** *rule*
also have ... = *1*
using *0 assms even-nat-iff*
unfolding *minus-one-power-iff cong-iff-dvd-diff*
by *auto*

finally show *?thesis* **using** *2* **by** *auto*
qed

lemma *Legendre-opposite*:
fixes *p q* :: *int*
assumes *prime p prime q*
assumes *p* > *2 q* > *2*
assumes *p* ≠ *q*
assumes [*p* = *3*] (*mod 4*) ∧ [*q* = *3*] (*mod 4*)
shows *Legendre p q* = − *Legendre q p*
proof −
have ‹[*p* − *1* = *2*] (*mod 4*)› ‹[*q* − *1* = *2*] (*mod 4*)›
using ‹[*p* = *3*] (*mod 4*) ∧ [*q* = *3*] (*mod 4*)›
by (*simp-all add*: *cong-iff-dvd-diff*)
from *assms* **have** ‹*even* (*p* − *1*)› ‹*even* (*q* − *1*)›
using *assms* **by** (*auto dest*: *prime-odd-int*)
then have ∗: (*p* − *1*) ∗ (*q* − *1*) *div 4* = ((*p* − *1*) *div 2*) ∗ ((*q* − *1*) *div 2*)
by *fastforce*
have *odd* ((*p* − *1*) ∗ (*q* − *1*) *div 4*)
using ‹*even* (*p* − *1*)› ‹*even* (*q* − *1*)› ‹[*p* − *1* = *2*] (*mod 4*)› ‹[*q* − *1* = *2*] (*mod 4*)›
by (*auto elim*!: *oddE evenE dest*: *cong-dvd-iff simp add*: ∗)
hence *2*: *odd* (*nat* ((*p* − *1*) ∗ (*q* − *1*) *div 4*))
using *assms even-nat-iff pos-imp-zdiv-nonneg-iff* **by** *fastforce*
have *3*: {*Legendre p q*, *Legendre q p*} ⊆ {−*1*, *0*, *1*}
unfolding *Legendre-def* **by** *auto*
have *Legendre p q* ∗ *Legendre q p* =
(− *1*) ^ *nat* (((*p* − *1*) *div 2*) ∗ ((*q* − *1*) *div 2*))
using *assms Quadratic-Reciprocity-int*[*of p q*]
by *fastforce*
also have ... = (− *1*) ^ *nat* ((*p* − *1*) ∗ (*q* − *1*) *div 4*) **unfolding** ∗ **by** *rule*
also have ... = − *1*
using *2 3*
unfolding *minus-one-power-iff*
by *auto*
finally show *?thesis* **using** *3* **by** *auto*
qed

end

2 Vectors and matrices, determinants and their properties in dimensions 2 and 3

theory *Low-Dimensional-Linear-Algebra*
imports *Main*
begin

datatype *vec2* =

vec2
(*vec2*$_1$: *int*)
(*vec2*$_2$: *int*)

datatype *vec3* =
vec3
(*vec3*$_1$: *int*)
(*vec3*$_2$: *int*)
(*vec3*$_3$: *int*)

datatype *mat2* =
mat2
(*mat2*$_{11}$: *int*) (*mat2*$_{12}$: *int*)
(*mat2*$_{21}$: *int*) (*mat2*$_{22}$: *int*)

datatype *mat3* =
mat3
(*mat3*$_{11}$: *int*) (*mat3*$_{12}$: *int*) (*mat3*$_{13}$: *int*)
(*mat3*$_{21}$: *int*) (*mat3*$_{22}$: *int*) (*mat3*$_{23}$: *int*)
(*mat3*$_{31}$: *int*) (*mat3*$_{32}$: *int*) (*mat3*$_{33}$: *int*)

instantiation *vec2* :: *ab-group-add*
begin

definition *zero-vec2* **where**
zero-vec2 =
vec2
0
0

definition *uminus-vec2* **where**
uminus-vec2 v =
vec2
(− *vec2*$_1$ *v*)
(− *vec2*$_2$ *v*)

definition *plus-vec2* **where**
plus-vec2 v1 v2 =
vec2
(*vec2*$_1$ *v1* + *vec2*$_1$ *v2*)
(*vec2*$_2$ *v1* + *vec2*$_2$ *v2*)

definition *minus-vec2* **where**
minus-vec2 v1 v2 =
vec2
(*vec2*$_1$ *v1* − *vec2*$_1$ *v2*)
(*vec2*$_2$ *v1* − *vec2*$_2$ *v2*)

instance

apply *intro-classes*
unfolding *zero-vec2-def uminus-vec2-def plus-vec2-def minus-vec2-def*
apply *simp-all*
done

end

instantiation *vec3* :: *ab-group-add*
begin

definition *zero-vec3* **where**
zero-vec3 =
vec3
0
0
0

definition *uminus-vec3* **where**
uminus-vec3 v =
vec3
(− $vec3_1$ *v*)
(− $vec3_2$ *v*)
(− $vec3_3$ *v*)

definition *plus-vec3* **where**
plus-vec3 v1 v2 =
vec3
($vec3_1$ *v1* + $vec3_1$ *v2*)
($vec3_2$ *v1* + $vec3_2$ *v2*)
($vec3_3$ *v1* + $vec3_3$ *v2*)

definition *minus-vec3* **where**
minus-vec3 v1 v2 =
vec3
($vec3_1$ *v1* − $vec3_1$ *v2*)
($vec3_2$ *v1* − $vec3_2$ *v2*)
($vec3_3$ *v1* − $vec3_3$ *v2*)

instance
apply *intro-classes*
unfolding *zero-vec3-def uminus-vec3-def plus-vec3-def minus-vec3-def*
apply *simp-all*
done

end

instantiation *mat2* :: *ring-1*
begin

definition *zero-mat2* **where**
zero-mat2 =
mat2
0 0
0 0

definition *one-mat2* **where**
one-mat2 =
mat2
1 0
0 1

definition *uminus-mat2* **where**
uminus-mat2 m =
mat2
(− $mat2_{11}$ *m*) (− $mat2_{12}$ *m*)
(− $mat2_{21}$ *m*) (− $mat2_{22}$ *m*)

definition *plus-mat2* **where**
plus-mat2 m1 m2 =
mat2
($mat2_{11}$ *m1* + $mat2_{11}$ *m2*) ($mat2_{12}$ *m1* + $mat2_{12}$ *m2*)
($mat2_{21}$ *m1* + $mat2_{21}$ *m2*) ($mat2_{22}$ *m1* + $mat2_{22}$ *m2*)

definition *minus-mat2* **where**
minus-mat2 m1 m2 =
mat2
($mat2_{11}$ *m1* − $mat2_{11}$ *m2*) ($mat2_{12}$ *m1* − $mat2_{12}$ *m2*)
($mat2_{21}$ *m1* − $mat2_{21}$ *m2*) ($mat2_{22}$ *m1* − $mat2_{22}$ *m2*)

definition *times-mat2* **where**
times-mat2 m1 m2 =
mat2
($mat2_{11}$ *m1* ∗ $mat2_{11}$ *m2* + $mat2_{12}$ *m1* ∗ $mat2_{21}$ *m2*) ($mat2_{11}$ *m1* ∗ $mat2_{12}$ *m2* + $mat2_{12}$ *m1* ∗ $mat2_{22}$ *m2*)
($mat2_{21}$ *m1* ∗ $mat2_{11}$ *m2* + $mat2_{22}$ *m1* ∗ $mat2_{21}$ *m2*) ($mat2_{21}$ *m1* ∗ $mat2_{12}$ *m2* + $mat2_{22}$ *m1* ∗ $mat2_{22}$ *m2*)

instance
apply *intro-classes*
unfolding *zero-mat2-def one-mat2-def uminus-mat2-def plus-mat2-def minus-mat2-def times-mat2-def*
apply (*simp-all add*: *algebra-simps*)
done

end

instantiation *mat3* :: *ring-1*
begin

definition *zero-mat3* **where**
zero-mat3 =
mat3
0 0 0
0 0 0
0 0 0

definition *one-mat3* **where**
one-mat3 =
mat3
1 0 0
0 1 0
0 0 1

definition *uminus-mat3* **where**
uminus-mat3 m =
mat3
(− $mat3_{11}$ *m*) (− $mat3_{12}$ *m*) (− $mat3_{13}$ *m*)
(− $mat3_{21}$ *m*) (− $mat3_{22}$ *m*) (− $mat3_{23}$ *m*)
(− $mat3_{31}$ *m*) (− $mat3_{32}$ *m*) (− $mat3_{33}$ *m*)

definition *plus-mat3* **where**
plus-mat3 m1 m2 =
mat3
($mat3_{11}$ *m1* + $mat3_{11}$ *m2*) ($mat3_{12}$ *m1* + $mat3_{12}$ *m2*) ($mat3_{13}$ *m1* + $mat3_{13}$ *m2*)
($mat3_{21}$ *m1* + $mat3_{21}$ *m2*) ($mat3_{22}$ *m1* + $mat3_{22}$ *m2*) ($mat3_{23}$ *m1* + $mat3_{23}$ *m2*)
($mat3_{31}$ *m1* + $mat3_{31}$ *m2*) ($mat3_{32}$ *m1* + $mat3_{32}$ *m2*) ($mat3_{33}$ *m1* + $mat3_{33}$ *m2*)

definition *minus-mat3* **where**
minus-mat3 m1 m2 =
mat3
($mat3_{11}$ *m1* − $mat3_{11}$ *m2*) ($mat3_{12}$ *m1* − $mat3_{12}$ *m2*) ($mat3_{13}$ *m1* − $mat3_{13}$ *m2*)
($mat3_{21}$ *m1* − $mat3_{21}$ *m2*) ($mat3_{22}$ *m1* − $mat3_{22}$ *m2*) ($mat3_{23}$ *m1* − $mat3_{23}$ *m2*)
($mat3_{31}$ *m1* − $mat3_{31}$ *m2*) ($mat3_{32}$ *m1* − $mat3_{32}$ *m2*) ($mat3_{33}$ *m1* − $mat3_{33}$ *m2*)

definition *times-mat3* **where**
times-mat3 m1 m2 =
mat3
($mat3_{11}$ *m1* ∗ $mat3_{11}$ *m2* + $mat3_{12}$ *m1* ∗ $mat3_{21}$ *m2* + $mat3_{13}$ *m1* ∗ $mat3_{31}$ *m2*) ($mat3_{11}$ *m1* ∗ $mat3_{12}$ *m2* + $mat3_{12}$ *m1* ∗ $mat3_{22}$ *m2* + $mat3_{13}$ *m1* ∗ $mat3_{32}$ *m2*) ($mat3_{11}$ *m1* ∗ $mat3_{13}$ *m2* + $mat3_{12}$ *m1* ∗ $mat3_{23}$ *m2* + $mat3_{13}$ *m1* ∗ $mat3_{33}$ *m2*)

(*mat3*$_{21}$ *m1* * *mat3*$_{11}$ *m2* + *mat3*$_{22}$ *m1* * *mat3*$_{21}$ *m2* + *mat3*$_{23}$ *m1* * *mat3*$_{31}$ *m2*) (*mat3*$_{21}$ *m1* * *mat3*$_{12}$ *m2* + *mat3*$_{22}$ *m1* * *mat3*$_{22}$ *m2* + *mat3*$_{23}$ *m1* * *mat3*$_{32}$ *m2*) (*mat3*$_{21}$ *m1* * *mat3*$_{13}$ *m2* + *mat3*$_{22}$ *m1* * *mat3*$_{23}$ *m2* + *mat3*$_{23}$ *m1* * *mat3*$_{33}$ *m2*)
(*mat3*$_{31}$ *m1* * *mat3*$_{11}$ *m2* + *mat3*$_{32}$ *m1* * *mat3*$_{21}$ *m2* + *mat3*$_{33}$ *m1* * *mat3*$_{31}$ *m2*) (*mat3*$_{31}$ *m1* * *mat3*$_{12}$ *m2* + *mat3*$_{32}$ *m1* * *mat3*$_{22}$ *m2* + *mat3*$_{33}$ *m1* * *mat3*$_{32}$ *m2*) (*mat3*$_{31}$ *m1* * *mat3*$_{13}$ *m2* + *mat3*$_{32}$ *m1* * *mat3*$_{23}$ *m2* + *mat3*$_{33}$ *m1* * *mat3*$_{33}$ *m2*)

instance
apply *intro-classes*
unfolding *zero-mat3-def one-mat3-def uminus-mat3-def plus-mat3-def minus-mat3-def times-mat3-def*
apply (*simp-all add*: *algebra-simps*)
done

end

consts *vec-dot* :: *'a* ⇒ *'a* ⇒ *int* (‹<- | ->› *65*)

definition *vec2-dot* :: *vec2* ⇒ *vec2* ⇒ *int* **where**
vec2-dot v1 v2 = *vec2*$_1$ *v1* * *vec2*$_1$ *v2* + *vec2*$_2$ *v1* * *vec2*$_2$ *v2*

adhoc-overloading *vec-dot* ⇌ *vec2-dot*

definition *vec3-dot* :: *vec3* ⇒ *vec3* ⇒ *int* **where**
vec3-dot v1 v2 = *vec3*$_1$ *v1* * *vec3*$_1$ *v2* + *vec3*$_2$ *v1* * *vec3*$_2$ *v2* + *vec3*$_3$ *v1* * *vec3*$_3$ *v2*

adhoc-overloading *vec-dot* ⇌ *vec3-dot*

lemma *vec2-dot-zero-left* [*simp*]:
fixes *v* :: *vec2*
shows <*0* | *v*> = *0*
unfolding *vec2-dot-def zero-vec2-def* **by** *auto*

lemma *vec2-dot-zero-right* [*simp*]:
fixes *v* :: *vec2*
shows <*v* | *0*> = *0*
unfolding *vec2-dot-def zero-vec2-def* **by** *auto*

lemma *vec3-dot-zero-left* [*simp*]:
fixes *v* :: *vec3*
shows <*0* | *v*> = *0*
unfolding *vec3-dot-def zero-vec3-def* **by** *auto*

lemma *vec3-dot-zero-right* [*simp*]:
fixes *v* :: *vec3*
shows <*v* | *0*> = *0*

unfolding *vec3-dot-def zero-vec3-def* **by** *auto*

consts *mat-app* :: *'a* ⇒ *'b* ⇒ *'b* (**infixr** ‹$› *65*)

definition *mat2-app* :: *mat2* ⇒ *vec2* ⇒ *vec2* **where**
mat2-app m v =
vec2
(*mat2*$_{11}$ *m* ∗ *vec2*$_1$ *v* + *mat2*$_{12}$ *m* ∗ *vec2*$_2$ *v*)
(*mat2*$_{21}$ *m* ∗ *vec2*$_1$ *v* + *mat2*$_{22}$ *m* ∗ *vec2*$_2$ *v*)

adhoc-overloading *mat-app* ⇌ *mat2-app*

definition *mat3-app* :: *mat3* ⇒ *vec3* ⇒ *vec3* **where**
mat3-app m v =
vec3
(*mat3*$_{11}$ *m* ∗ *vec3*$_1$ *v* + *mat3*$_{12}$ *m* ∗ *vec3*$_2$ *v* + *mat3*$_{13}$ *m* ∗ *vec3*$_3$ *v*)
(*mat3*$_{21}$ *m* ∗ *vec3*$_1$ *v* + *mat3*$_{22}$ *m* ∗ *vec3*$_2$ *v* + *mat3*$_{23}$ *m* ∗ *vec3*$_3$ *v*)
(*mat3*$_{31}$ *m* ∗ *vec3*$_1$ *v* + *mat3*$_{32}$ *m* ∗ *vec3*$_2$ *v* + *mat3*$_{33}$ *m* ∗ *vec3*$_3$ *v*)

adhoc-overloading *mat-app* ⇌ *mat3-app*

lemma *mat2-app-zero* [*simp*]:
fixes *m* :: *mat2*
shows *m* $ *0* = *0*
unfolding *mat2-app-def zero-vec2-def* **by** *auto*

lemma *mat3-app-zero* [*simp*]:
fixes *m* :: *mat3*
shows *m* $ *0* = *0*
unfolding *mat3-app-def zero-vec3-def* **by** *auto*

lemma *mat2-app-one* [*simp*]:
fixes *v* :: *vec2*
shows *1* $ *v* = *v*
unfolding *mat2-app-def one-mat2-def* **by** *auto*

lemma *mat3-app-one* [*simp*]:
fixes *v* :: *vec3*
shows *1* $ *v* = *v*
unfolding *mat3-app-def one-mat3-def* **by** *auto*

lemma *mat2-app-mul* [*simp*]:
fixes *m1 m2* :: *mat2*
fixes *v* :: *vec2*
shows *m1* ∗ *m2* $ *v* = *m1* $ *m2* $ *v*
unfolding *times-mat2-def mat2-app-def* **by** (*simp add*: *algebra-simps*)

lemma *mat3-app-mul* [*simp*]:
fixes *m1 m2* :: *mat3*

fixes v :: *vec3*
shows *m1* $*$ *m2* $ $ v = *m1* $ *m2* $ $ v
unfolding *times-mat3-def mat3-app-def* **by** (*simp add*: *algebra-simps*)

consts *mat-det* :: $'a \Rightarrow$ *int*

definition *mat2-det* **where**
mat2-det m = $mat2_{11}$ *m* $*$ $mat2_{22}$ *m* $-$ $mat2_{12}$ *m* $*$ $mat2_{21}$ *m*

adhoc-overloading *mat-det* $\rightleftharpoons$ *mat2-det*

definition *mat3-det* **where**
mat3-det m =
$mat3_{11}$ *m* $*$ $mat3_{22}$ *m* $*$ $mat3_{33}$ *m*
$+$ $mat3_{12}$ *m* $*$ $mat3_{23}$ *m* $*$ $mat3_{31}$ *m*
$+$ $mat3_{13}$ *m* $*$ $mat3_{21}$ *m* $*$ $mat3_{32}$ *m*
$-$ $mat3_{11}$ *m* $*$ $mat3_{23}$ *m* $*$ $mat3_{32}$ *m*
$-$ $mat3_{12}$ *m* $*$ $mat3_{21}$ *m* $*$ $mat3_{33}$ *m*
$-$ $mat3_{13}$ *m* $*$ $mat3_{22}$ *m* $*$ $mat3_{31}$ *m*

adhoc-overloading *mat-det* $\rightleftharpoons$ *mat3-det*

lemma *mat2-mul-det* [*simp*]:
fixes *m1 m2* :: *mat2*
shows *mat-det* (*m1* $*$ *m2*) = *mat-det m1* $*$ *mat-det m2*
unfolding *times-mat2-def mat2-det-def* **by** (*simp*; *algebra*)

lemma *mat3-mul-det* [*simp*]:
fixes *m1 m2* :: *mat3*
shows *mat-det* (*m1* $*$ *m2*) = *mat-det m1* $*$ *mat-det m2*
unfolding *times-mat3-def mat3-det-def* **by** (*simp*; *algebra*)

consts *mat-sym* :: $'a \Rightarrow$ *bool*

definition *mat2-sym* :: *mat2* $\Rightarrow$ *bool* **where**
mat2-sym m = ($mat2_{12}$ *m* = $mat2_{21}$ *m*)

adhoc-overloading *mat-sym* $\rightleftharpoons$ *mat2-sym*

definition *mat3-sym* :: *mat3* $\Rightarrow$ *bool* **where**
mat3-sym m = ($mat3_{12}$ *m* = $mat3_{21}$ *m* $\wedge$ $mat3_{13}$ *m* = $mat3_{31}$ *m* $\wedge$ $mat3_{23}$ *m* = $mat3_{32}$ *m*)

adhoc-overloading *mat-sym* $\rightleftharpoons$ *mat3-sym*

consts *mat-transpose* :: $'a \Rightarrow$ $'a$ ($\langle$-$^T\rangle$ [*91*] *90*)

definition *mat2-transpose* :: *mat2* $\Rightarrow$ *mat2* **where**
mat2-transpose m =

mat2
(*mat2*$_{11}$ *m*) (*mat2*$_{21}$ *m*)
(*mat2*$_{12}$ *m*) (*mat2*$_{22}$ *m*)

adhoc-overloading *mat-transpose* ⇌ *mat2-transpose*

definition *mat3-transpose* :: *mat3* ⇒ *mat3* **where**
mat3-transpose m =
mat3
(*mat3*$_{11}$ *m*) (*mat3*$_{21}$ *m*) (*mat3*$_{31}$ *m*)
(*mat3*$_{12}$ *m*) (*mat3*$_{22}$ *m*) (*mat3*$_{32}$ *m*)
(*mat3*$_{13}$ *m*) (*mat3*$_{23}$ *m*) (*mat3*$_{33}$ *m*)

adhoc-overloading *mat-transpose* ⇌ *mat3-transpose*

lemma *mat2-transpose-involution* [*simp*]:
fixes *m* :: *mat2*
shows $(m^T)^T = m$
unfolding *mat2-transpose-def*
by *auto*

lemma *mat3-transpose-involution* [*simp*]:
fixes *m* :: *mat3*
shows $(m^T)^T = m$
unfolding *mat3-transpose-def*
by *auto*

lemma *mat2-sym-criterion*:
fixes *m* :: *mat2*
shows *mat-sym m* ⟷ $m^T = m$
unfolding *mat2-sym-def mat2-transpose-def*
by (*cases m*; *auto*)

lemma *mat3-sym-criterion*:
fixes *m* :: *mat3*
shows *mat-sym m* ⟷ $m^T = m$
unfolding *mat3-sym-def mat3-transpose-def*
by (*cases m*; *auto*)

lemma *mat2-transpose-one* [*simp*]: $(1 :: mat2)^T = 1$
unfolding *mat2-transpose-def one-mat2-def* **by** *auto*

lemma *mat3-transpose-one* [*simp*]: $(1 :: mat3)^T = 1$
unfolding *mat3-transpose-def one-mat3-def* **by** *auto*

lemma *mat2-transpose-mul* [*simp*]:
fixes *a b* :: *mat2*
shows $(a * b)^T = b^T * a^T$
unfolding *mat2-transpose-def times-mat2-def* **by** *auto*

lemma *mat3-transpose-mul* [*simp*]:
fixes *a b* :: *mat3*
shows $(a * b)^T = b^T * a^T$
unfolding *mat3-transpose-def times-mat3-def* **by** *auto*

lemma *vec2-dot-transpose-left*:
fixes *m* :: *mat2*
fixes *u v* :: *vec2*
shows $<m^T \$ u \mid v> = <u \mid m \$ v>$
unfolding *vec2-dot-def mat2-app-def mat2-transpose-def*
by (*simp add*: *algebra-simps*)

lemma *vec2-dot-transpose-right*:
fixes *m* :: *mat2*
fixes *u v* :: *vec2*
shows $<u \mid m^T \$ v> = <m \$ u \mid v>$
unfolding *vec2-dot-def mat2-app-def mat2-transpose-def*
by (*simp add*: *algebra-simps*)

lemma *vec3-dot-transpose-left*:
fixes *m* :: *mat3*
fixes *u v* :: *vec3*
shows $<m^T \$ u \mid v> = <u \mid m \$ v>$
unfolding *vec3-dot-def mat3-app-def mat3-transpose-def*
by (*simp add*: *algebra-simps*)

lemma *vec3-dot-transpose-right*:
fixes *m* :: *mat3*
fixes *u v* :: *vec3*
shows $<u \mid m^T \$ v> = <m \$ u \mid v>$
unfolding *vec3-dot-def mat3-app-def mat3-transpose-def*
by (*simp add*: *algebra-simps*)

lemma *mat2-det-tranpose* [*simp*]:
fixes *m* :: *mat2*
shows *mat-det* (m^T) = *mat-det m*
unfolding *mat2-det-def mat2-transpose-def* **by** *auto*

lemma *mat3-det-tranpose* [*simp*]:
fixes *m* :: *mat3*
shows *mat-det* (m^T) = *mat-det m*
unfolding *mat3-det-def mat3-transpose-def* **by** *auto*

consts *mat-inverse* :: $'a \Rightarrow 'a$ ($\langle$-$^{-1}\rangle$ [*91*] *90*)

definition *mat2-inverse* :: *mat2* ⇒ *mat2* **where**
mat2-inverse m =
mat2

$(mat2_{22}\ m)\ (-\ mat2_{12}\ m)$
$(-\ mat2_{21}\ m)\ (mat2_{11}\ m)$

adhoc-overloading *mat-inverse* $\rightleftharpoons$ *mat2-inverse*

definition *mat3-inverse* :: *mat3* $\Rightarrow$ *mat3* **where**
mat3-inverse m =
mat3
$(mat3_{22}\ m * mat3_{33}\ m - mat3_{23}\ m * mat3_{32}\ m)\ (mat3_{13}\ m * mat3_{32}\ m - mat3_{12}\ m * mat3_{33}\ m)\ (mat3_{12}\ m * mat3_{23}\ m - mat3_{13}\ m * mat3_{22}\ m)$
$(mat3_{23}\ m * mat3_{31}\ m - mat3_{21}\ m * mat3_{33}\ m)\ (mat3_{11}\ m * mat3_{33}\ m - mat3_{13}\ m * mat3_{31}\ m)\ (mat3_{13}\ m * mat3_{21}\ m - mat3_{11}\ m * mat3_{23}\ m)$
$(mat3_{21}\ m * mat3_{32}\ m - mat3_{22}\ m * mat3_{31}\ m)\ (mat3_{12}\ m * mat3_{31}\ m - mat3_{11}\ m * mat3_{32}\ m)\ (mat3_{11}\ m * mat3_{22}\ m - mat3_{12}\ m * mat3_{21}\ m)$

adhoc-overloading *mat-inverse* $\rightleftharpoons$ *mat3-inverse*

lemma *mat2-inverse-cancel*:
fixes *m* :: *mat2*
assumes *mat-det m* = *1*
shows $m * m^{-1} = 1\ m^{-1} * m = 1$
using *assms* **unfolding** *mat2-det-def mat2-inverse-def times-mat2-def one-mat2-def*
by (*auto simp add*: *algebra-simps*)

lemma *mat3-inverse-cancel*:
fixes *m* :: *mat3*
assumes *mat-det m* = *1*
shows $m * m^{-1} = 1\ m^{-1} * m = 1$
using *assms* **unfolding** *mat3-det-def mat3-inverse-def times-mat3-def one-mat3-def*
by (*auto simp add*: *algebra-simps*)

lemma *mat2-inverse-cancel-left*:
fixes *m a* :: *mat2*
assumes *mat-det m* = *1*
shows $m * (m^{-1} * a) = a\ m^{-1} * (m * a) = a$
unfolding *mult.assoc*[*symmetric*]
using *assms mat2-inverse-cancel*
by *auto*

lemma *mat3-inverse-cancel-left*:
fixes *m a* :: *mat3*
assumes *mat-det m* = *1*
shows $m * (m^{-1} * a) = a\ m^{-1} * (m * a) = a$
unfolding *mult.assoc*[*symmetric*]
using *assms mat3-inverse-cancel*
by *auto*

lemma *mat2-inverse-cancel-right*:
fixes m a :: *mat2*
assumes *mat-det* $m = 1$
shows $a * (m * m^{-1}) = a$ $a * (m^{-1} * m) = a$
using *assms mat2-inverse-cancel*
by *auto*

lemma *mat3-inverse-cancel-right*:
fixes m a :: *mat3*
assumes *mat-det* $m = 1$
shows $a * (m * m^{-1}) = a$ $a * (m^{-1} * m) = a$
using *assms mat3-inverse-cancel*
by *auto*

lemma *mat2-inversable-cancel-left*:
fixes m *a1 a2* :: *mat2*
assumes *mat-det* $m = 1$
assumes $m * a1 = m * a2$
shows *a1* $=$ *a2*
by (*metis assms mat2-inverse-cancel-left*(*2*))

lemma *mat3-inversable-cancel-left*:
fixes m *a1 a2* :: *mat3*
assumes *mat-det* $m = 1$
assumes $m * a1 = m * a2$
shows *a1* $=$ *a2*
by (*metis assms mat3-inverse-cancel-left*(*2*))

lemma *mat2-inversable-cancel-right*:
fixes m *a1 a2* :: *mat2*
assumes *mat-det* $m = 1$
assumes $a1 * m = a2 * m$
shows *a1* $=$ *a2*
by (*metis assms mat2-inverse-cancel*(*1*) *mult.assoc mult.right-neutral*)

lemma *mat3-inversable-cancel-right*:
fixes m *a1 a2* :: *mat3*
assumes *mat-det* $m = 1$
assumes $a1 * m = a2 * m$
shows *a1* $=$ *a2*
by (*metis assms mat3-inverse-cancel*(*1*) *mult.assoc mult.right-neutral*)

lemma *mat2-inverse-det* [*simp*]:
fixes m :: *mat2*
shows *mat-det* (m^{-1}) $=$ *mat-det* m
unfolding *mat2-inverse-def mat2-det-def*
by *auto*

lemma *mat3-inverse-det* [*simp*]:

fixes m :: *mat3*
shows *mat-det* $(m^{-1}) = (\textit{mat-det}\ m)^2$
unfolding *mat3-inverse-def mat3-det-def power2-eq-square*
by (*simp add*: *algebra-simps*)

lemma *mat2-inverse-transpose*:
fixes m :: *mat2*
shows $(m^T)^{-1} = (m^{-1})^T$
unfolding *mat2-inverse-def mat2-transpose-def*
by *auto*

lemma *mat3-inverse-transpose*:
fixes m :: *mat3*
shows $(m^T)^{-1} = (m^{-1})^T$
unfolding *mat3-inverse-def mat3-transpose-def*
by *auto*

lemma *mat2-special-preserves-zero*:
fixes u :: *mat2*
fixes v :: *vec2*
assumes *mat-det* $u = 1$
shows $u\ \$\ v = 0 \longleftrightarrow v = 0$
proof
assume $u\ \$\ v = 0$
hence $u^{-1}\ \$\ u\ \$\ v = 0$ **by** *auto*
hence $(u^{-1} * u)\ \$\ v = 0$ **by** *auto*
thus $v = 0$ **using** *assms mat2-inverse-cancel* **by** *auto*
next
assume $v = 0$
thus $u\ \$\ v = 0$ **by** *auto*
qed

lemma *mat3-special-preserves-zero*:
fixes u :: *mat3*
fixes v :: *vec3*
assumes *mat-det* $u = 1$
shows $u\ \$\ v = 0 \longleftrightarrow v = 0$
proof
assume $u\ \$\ v = 0$
hence $u^{-1}\ \$\ u\ \$\ v = 0$ **by** *auto*
hence $(u^{-1} * u)\ \$\ v = 0$ **by** *auto*
thus $v = 0$ **using** *assms mat3-inverse-cancel* **by** *auto*
next
assume $v = 0$
thus $u\ \$\ v = 0$ **by** *auto*
qed

end

3 Properties of quadratic forms and their equivalences

theory *Quadratic-Forms*
  imports *Complex-Main Low-Dimensional-Linear-Algebra*
begin

consts *qf-app* :: *'a* ⇒ *'b* ⇒ *int* (**infixl** ‹$$› *65*)

definition *qf2-app* :: *mat2* ⇒ *vec2* ⇒ *int* **where**
qf2-app m v = *<v | m $ v>*

adhoc-overloading *qf-app* ⇌ *qf2-app*

definition *qf3-app* :: *mat3* ⇒ *vec3* ⇒ *int* **where**
qf3-app m v = *<v | m $ v>*

adhoc-overloading *qf-app* ⇌ *qf3-app*

lemma *qf2-app-zero* [*simp*]:
  fixes *m* :: *mat2*
  shows *m $$ 0* = *0*
  unfolding *qf2-app-def* **by** *auto*

lemma *qf3-app-zero* [*simp*]:
  fixes *m* :: *mat3*
  shows *m $$ 0* = *0*
  unfolding *qf3-app-def* **by** *auto*

consts *qf-positive-definite* :: *'a* ⇒ *bool*

definition *qf2-positive-definite* :: *mat2* ⇒ *bool* **where**
qf2-positive-definite m = (∀ *v*. *v* ≠ *0* ⟶ *m $$ v* > *0*)

adhoc-overloading *qf-positive-definite* ⇌ *qf2-positive-definite*

definition *qf3-positive-definite* :: *mat3* ⇒ *bool* **where**
qf3-positive-definite m = (∀ *v*. *v* ≠ *0* ⟶ *m $$ v* > *0*)

adhoc-overloading *qf-positive-definite* ⇌ *qf3-positive-definite*

lemma *qf2-positive-definite-positive*:
  fixes *m* :: *mat2*
  assumes *qf-positive-definite m*
  shows ∀ *v*. *m $$ v* ≥ *0*
  using *assms* **unfolding** *qf2-positive-definite-def*
  by (*metis order-less-le order-refl qf2-app-zero*)

lemma *qf3-positive-definite-positive*:

fixes $m :: mat3$
assumes *qf-positive-definite* m
shows $\forall v.\ m\ \$\$\ v \geq 0$
using *assms* **unfolding** *qf3-positive-definite-def*
by (*metis order-less-le order-refl qf3-app-zero*)

consts *qf-action* $:: 'a \Rightarrow 'a \Rightarrow 'a$ (**infixl** $\langle\cdot\rangle$ *55*)

definition *qf2-action* $:: mat2 \Rightarrow mat2 \Rightarrow mat2$ **where**
qf2-action $a\ u = u^T * a * u$

adhoc-overloading *qf-action* $\rightleftharpoons$ *qf2-action*

definition *qf3-action* $:: mat3 \Rightarrow mat3 \Rightarrow mat3$ **where**
qf3-action $a\ u = u^T * a * u$

adhoc-overloading *qf-action* $\rightleftharpoons$ *qf3-action*

lemma *qf2-action-id*:
fixes $a :: mat2$
shows $a \cdot 1 = a$
unfolding *qf2-action-def*
by *simp*

lemma *qf3-action-id*:
fixes $a :: mat3$
shows $a \cdot 1 = a$
unfolding *qf3-action-def*
by *simp*

lemma *qf2-action-mul* [*simp*]:
fixes $a\ u\ v :: mat2$
shows $a \cdot (u * v) = (a \cdot u) \cdot v$
unfolding *qf2-action-def*
by (*simp add*: *algebra-simps*)

lemma *qf3-action-mul* [*simp*]:
fixes $a\ u\ v :: mat3$
shows $a \cdot (u * v) = (a \cdot u) \cdot v$
unfolding *qf3-action-def*
by (*simp add*: *algebra-simps*)

consts *qf-equiv* $:: 'a \Rightarrow 'a \Rightarrow bool$ (**infix** $\langle\sim\rangle$ *65*)

definition *qf2-equiv* $:: mat2 \Rightarrow mat2 \Rightarrow bool$ **where**
qf2-equiv $a\ b = (\exists u.\ mat\text{-}det\ u = 1 \wedge a \cdot u = b)$

adhoc-overloading *qf-equiv* $\rightleftharpoons$ *qf2-equiv*

definition *qf3-equiv* :: *mat3* ⇒ *mat3* ⇒ *bool* **where**
qf3-equiv a b = (∃ *u*. *mat-det u* = *1* ∧ *a* · *u* = *b*)

adhoc-overloading *qf-equiv* ⇌ *qf3-equiv*

lemma *qf2-equiv-sym-impl*:
fixes *a b* :: *mat2*
shows *a* ~ *b* ⟹ *b* ~ *a*
unfolding *qf2-equiv-def qf2-action-def*
proof −
assume ∃ *u*. *mat-det u* = *1* ∧ u^T ∗ *a* ∗ *u* = *b*
then obtain *u* **where** *mat-det u* = *1* ∧ u^T ∗ *a* ∗ *u* = *b* **by** *blast*
hence *mat-det* (u^{-1}) = *1* ∧ ((u^{-1})^T) ∗ *b* ∗ (u^{-1}) = *a*
unfolding *mat2-inverse-transpose*[*symmetric*]
using *mat2-inverse-cancel-left*[*of* u^T] *mat2-inverse-cancel-right*
by (*auto simp add*: *algebra-simps*)
thus ∃ *u*. *mat-det u* = *1* ∧ u^T ∗ *b* ∗ *u* = *a* **by** *blast*
qed

lemma *qf3-equiv-sym-impl*:
fixes *a b* :: *mat3*
shows *a* ~ *b* ⟹ *b* ~ *a*
unfolding *qf3-equiv-def qf3-action-def*
proof −
assume ∃ *u*. *mat-det u* = *1* ∧ u^T ∗ *a* ∗ *u* = *b*
then obtain *u* **where** *mat-det u* = *1* ∧ u^T ∗ *a* ∗ *u* = *b* **by** *blast*
hence *mat-det* (u^{-1}) = *1* ∧ ((u^{-1})^T) ∗ *b* ∗ (u^{-1}) = *a*
unfolding *mat3-inverse-transpose*[*symmetric*]
using *mat3-inverse-cancel-left*[*of* u^T] *mat3-inverse-cancel-right*
by (*auto simp add*: *algebra-simps*)
thus ∃ *u*. *mat-det u* = *1* ∧ u^T ∗ *b* ∗ *u* = *a* **by** *blast*
qed

lemma *qf2-equiv-sym*:
fixes *a b* :: *mat2*
shows *a* ~ *b* ⟷ *b* ~ *a*
using *qf2-equiv-sym-impl* **by** *blast*

lemma *qf3-equiv-sym*:
fixes *a b* :: *mat3*
shows *a* ~ *b* ⟷ *b* ~ *a*
using *qf3-equiv-sym-impl* **by** *blast*

lemma *qf2-equiv-trans*:
fixes *a b c* :: *mat2*
assumes *a* ~ *b*
assumes *b* ~ *c*
shows *a* ~ *c*
using *assms* **by** (*metis mult-1 mat2-mul-det qf2-action-mul qf2-equiv-def*)

lemma *qf3-equiv-trans*:
fixes $a\ b\ c$ :: *mat3*
assumes $a \sim b$
assumes $b \sim c$
shows $a \sim c$
using *assms* **by** (*metis mult-1 mat3-mul-det qf3-action-mul qf3-equiv-def*)

lemma *qf2-action-app* [*simp*]:
fixes $a\ u$ :: *mat2*
fixes v :: *vec2*
shows $(a \cdot u)$ \$\$ $v = a$ \$\$ $(u$ \$ $v)$
unfolding *qf2-action-def qf2-app-def*
using *vec2-dot-transpose-right* **by** *auto*

lemma *qf3-action-app* [*simp*]:
fixes $a\ u$ :: *mat3*
fixes v :: *vec3*
shows $(a \cdot u)$ \$\$ $v = a$ \$\$ $(u$ \$ $v)$
unfolding *qf3-action-def qf3-app-def*
using *vec3-dot-transpose-right* **by** *auto*

lemma *qf2-equiv-preserves-positive-definite*:
fixes $a\ b$ :: *mat2*
assumes $a \sim b$
shows *qf-positive-definite a* $\longleftrightarrow$ *qf-positive-definite b*
unfolding *qf2-positive-definite-def*
by (*metis assms mat2-special-preserves-zero qf2-action-app*
qf2-equiv-def qf2-equiv-sym)

lemma *qf3-equiv-preserves-positive-definite*:
fixes $a\ b$ :: *mat3*
assumes $a \sim b$
shows *qf-positive-definite a* $\longleftrightarrow$ *qf-positive-definite b*
unfolding *qf3-positive-definite-def*
by (*metis assms mat3-special-preserves-zero qf3-action-app*
qf3-equiv-def qf3-equiv-sym)

lemma *qf2-equiv-preserves-sym*:
fixes $a\ b$ :: *mat2*
assumes $a \sim b$
shows *mat2-sym a* $\longleftrightarrow$ *mat2-sym b*
proof −
obtain u **where** *mat-det* $u = 1$ $u^T * a * u = b$
using *assms* **unfolding** *qf2-action-def qf2-equiv-def* **by** *auto*
thus *?thesis*
unfolding *mat2-sym-criterion*
using *mat2-inversable-cancel-left*[*of* u^T $a^T * u$ $a * u$]
mat2-inversable-cancel-right[*of* u a^T a]

by (*auto simp add*: *algebra-simps*)
qed

lemma *qf3-equiv-preserves-sym*:
fixes *a b* :: *mat3*
assumes $a \sim b$
shows *mat3-sym a* $\longleftrightarrow$ *mat3-sym b*
proof −
obtain *u* **where** *mat-det u* = *1* $u^T * a * u = b$
using *assms* **unfolding** *qf3-action-def qf3-equiv-def* **by** *auto*
thus *?thesis*
unfolding *mat3-sym-criterion*
using *mat3-inversable-cancel-left*[*of* u^T $a^T * u$ $a * u$]
mat3-inversable-cancel-right[*of u* a^T *a*]
by (*auto simp add*: *algebra-simps*)
qed

lemma *qf2-equiv-preserves-det*:
fixes *a b* :: *mat2*
assumes $a \sim b$
shows *mat-det a* = *mat-det b*
using *assms* **unfolding** *qf2-action-def qf2-equiv-def* **by** *auto*

lemma *qf3-equiv-preserves-det*:
fixes *a b* :: *mat3*
assumes $a \sim b$
shows *mat-det a* = *mat-det b*
using *assms* **unfolding** *qf3-action-def qf3-equiv-def* **by** *auto*

lemma *qf2-equiv-preserves-range-subset*:
fixes *a b* :: *mat2*
assumes $a \sim b$
shows *range* (($$) *b*) $\subseteq$ *range* (($$) *a*)
proof −
obtain *u* **where** *0*: *mat-det u* = *1* $a \cdot u = b$
using *assms* **unfolding** *qf2-equiv-def* **by** *auto*
show *?thesis* **unfolding** *0*[*symmetric*] *image-def* **by** *auto*
qed

lemma *qf3-equiv-preserves-range-subset*:
fixes *a b* :: *mat3*
assumes $a \sim b$
shows *range* (($$) *b*) $\subseteq$ *range* (($$) *a*)
proof −
obtain *u* **where** *0*: *mat-det u* = *1* $a \cdot u = b$
using *assms* **unfolding** *qf3-equiv-def* **by** *auto*
show *?thesis* **unfolding** *0*[*symmetric*] *image-def* **by** *auto*
qed

lemma *qf2-equiv-preserves-range*:
fixes *a b* :: *mat2*
assumes $a \sim b$
shows *range* ((\$\$) *a*) = *range* ((\$\$) *b*)
using *assms qf2-equiv-sym qf2-equiv-preserves-range-subset* **by** *blast*

lemma *qf3-equiv-preserves-range*:
fixes *a b* :: *mat3*
assumes $a \sim b$
shows *range* ((\$\$) *a*) = *range* ((\$\$) *b*)
using *assms qf3-equiv-sym qf3-equiv-preserves-range-subset* **by** *blast*

Lemma 1.1 from [1].

lemma *qf2-positive-definite-criterion*:
fixes *a*
assumes *mat-sym a*
shows *qf-positive-definite a* $\longleftrightarrow$ $mat2_{11}\ a > 0 \wedge$ *mat-det a* > 0
proof
assume *0*: *qf-positive-definite a*
have *vec2 1 0* $\neq 0 \longrightarrow$ *a* \$\$ *vec2 1 0* > 0 **using** *0*
unfolding *qf2-positive-definite-def* **by** *blast*
hence *1*: $mat2_{11}\ a > 0$
unfolding *zero-vec2-def vec2-dot-def times-mat2-def mat2-app-def qf2-app-def*
by *auto*
let *?v* = *vec2* $(-\ mat2_{12}\ a)\ (mat2_{11}\ a)$
have *?v* $\neq 0 \longrightarrow$ *a* \$\$ *?v* > 0 **using** *0*
unfolding *qf2-positive-definite-def* **by** *blast*
hence $mat2_{11}\ a *$ *mat-det a* > 0 **using** *assms 1*
unfolding *zero-vec2-def vec2-dot-def times-mat2-def mat2-app-def*
mat2-det-def mat2-sym-def qf2-app-def
by (*simp add*: *mult.commute*)
hence *2*: *mat-det a* > 0 **using** *1 zero-less-mult-pos* **by** *blast*
show $mat2_{11}\ a > 0 \wedge$ *mat-det a* > 0 **using** *1 2* **by** *blast*
next
assume *3*: $mat2_{11}\ a > 0 \wedge$ *mat-det a* > 0
show *qf-positive-definite a* **unfolding** *qf2-positive-definite-def*
proof (*rule allI*; *rule impI*)
fix *v* :: *vec2*
assume *v* $\neq 0$
hence *4*: $vec2_1\ v \neq 0 \vee vec2_2\ v \neq 0$ **unfolding** *zero-vec2-def*
by (*metis vec2.collapse*)
let *?n* = $(mat2_{11}\ a * vec2_1\ v + mat2_{12}\ a * vec2_2\ v)^2 + (mat\text{-}det\ a) * (vec2_2\ v)^2$
have *5*: $mat2_{11}\ a *$ (*a* \$\$ *v*) = *?n*
using *assms*
unfolding *vec2-dot-def times-mat2-def mat2-app-def mat2-det-def*
mat2-sym-def qf2-app-def power2-eq-square
by (*simp add*: *algebra-simps*)
have *?n* > 0 **using** *3 4*

```
    by (metis add.commute add-0 add-pos-nonneg mult-eq-0-iff
        mult-pos-pos power-zero-numeral zero-le-power2 zero-less-power2)
   thus a $$ v > 0 using 3 5 zero-less-mult-pos by metis
  qed
qed

lemma congruence-class-close:
  fixes k m :: int
  assumes m > 0
  shows ∃ t. 2 * |k + m * t| ≤ m (is ∃ t. ?P t)
proof −
  let ?s = k div m
  have 0: k − m * ?s ≥ 0 ∧ k − m * ?s < m using assms
    by (metis pos-mod-sign pos-mod-bound add.commute add-diff-cancel-right'
          div-mod-decomp-int mult.commute)
  show ?thesis proof cases
    assume 2 * (k − m * ?s) ≤ m
    hence ?P (− ?s) using 0 by auto
    thus ?thesis by blast
  next
    assume ¬ (2 * (k − m * ?s) ≤ m)
    hence 2 * (k − m * ?s) > m by auto
    hence ?P (− (?s + 1)) using 0 by (simp add: algebra-simps)
    thus ?thesis by blast
  qed
qed
```

Lemma 1.2 from [1].

```
lemma lemma-1-2:
  fixes b :: mat2
  assumes mat-sym b
  assumes qf-positive-definite b
  shows ∃ a. a ~ b ∧
            2 * |mat2_12 a| ≤ mat2_11 a ∧
            mat2_11 a ≤ (2 / sqrt 3) * sqrt (mat-det a) (is ∃ a. ?P a)
proof −
  define a_11 where a_11 ≡ LEAST y. y > 0 ∧ (∃ x. int y = b $$ x)
  have 0: ∃ y. y > 0 ∧ (∃ x. int y = b $$ x)
    apply (rule exI[of - nat (b $$ (vec2 1 1))])
    using assms(2) unfolding qf2-positive-definite-def zero-vec2-def
    apply (metis nat-0-le order-less-le vec2.inject zero-less-nat-eq zero-neq-one)
    done
  obtain r where 1: a_11 > 0 int a_11 = b $$ r
    using a_11-def LeastI-ex[OF 0] by auto
  let ?h = gcd (vec2_1 r) (vec2_2 r)
  have r ≠ 0 using assms(2) 1 by fastforce
  hence 2: ?h > 0 by (simp; metis vec2.collapse zero-vec2-def)
  let ?r' = vec2 (vec2_1 r div ?h) (vec2_2 r div ?h)
  have ?r' ≠ 0 using 2 unfolding zero-vec2-def
```

by (*simp add*: *algebra-simps dvd-div-eq-0-iff*)
hence *nat* (*b* $$ *?r'*) > *0* ∧ (∃*x*. *int* (*nat* (*b* $$ *?r'*)) = *b* $$ *x*)
using *assms*(*2*) **unfolding** *qf2-positive-definite-def* **by** *auto*
hence ∗: $a_{11} \leq$ *nat* (*b* $$ *?r'*) **unfolding** a_{11}*-def* **by** (*rule Least-le*)
hence $a_{11} \leq$ *b* $$ *?r'* **using** *1* **by** *auto*
also have ∗∗: ... = (*b* $$ *r*) *div* $?h^2$ **proof** −
have (*b* $$ *?r'*) ∗ $?h^2$ = *b* $$ *r* $?h^2$ *dvd b* $$ *r*
unfolding *qf2-app-def vec2-dot-def mat2-app-def power2-eq-square*
using *1*
by (*auto simp add*: *algebra-simps mult-dvd-mono*)
thus *b* $$ *?r'* = (*b* $$ *r*) *div* $?h^2$ **using** *2* **by** *auto*
qed
also have ... = a_{11} *div* $?h^2$ **using** *1* **by** *auto*
finally have $a_{11} \leq a_{11}$ *div* $?h^2$ **.**
also have ... ≤ a_{11} **using** *1 2*
by (*metis* ∗ ∗∗ *assms*(*2*) *div-le-dividend int-eq-iff nat-div-distrib order-refl*
qf2-positive-definite-positive verit-la-disequality)
finally have *?h* = *1* **using** *1 2*
by (*smt* (*verit*) *int-div-less-self of-nat-0-less-iff power2-eq-square*
zero-less-power zmult-eq-1-iff)
then obtain s_1 s_2 **where** *3*: *1* = (*vec2*$_1$ *r*) ∗ s_2 − (*vec2*$_2$ *r*) ∗ s_1
by (*metis bezout-int diff-minus-eq-add mult.commute mult-minus-right*)
define a'_{12} **where**
$a'_{12} \equiv$
(*mat2*$_{11}$ *b*) ∗ (*vec2*$_1$ *r*) ∗ s_1
+ (*mat2*$_{12}$ *b*) ∗ ((*vec2*$_1$ *r*) ∗ s_2 + (*vec2*$_2$ *r*) ∗ s_1)
+ (*mat2*$_{22}$ *b*) ∗ (*vec2*$_2$ *r*) ∗ s_2

obtain *t* **where** *4*: *2* ∗ $|a'_{12} + a_{11} * (t :: int)| \leq a_{11}$
using *1 congruence-class-close* **by** *fastforce*
define a_{12} **where** $a_{12} \equiv a'_{12} + a_{11} * t$
define a_{22} **where** $a_{22} \equiv$ *b* $$ (*vec2* (s_1 + (*vec2*$_1$ *r*) ∗ *t*) (s_2 + (*vec2*$_2$ *r*) ∗ *t*))
let *?u* =
mat2
(*vec2*$_1$ *r*) (s_1 + (*vec2*$_1$ *r*) ∗ *t*)
(*vec2*$_2$ *r*) (s_2 + (*vec2*$_2$ *r*) ∗ *t*)

let *?a* =
mat2
a_{11} a_{12}
a_{12} a_{22}

have *5*: *mat-det ?u* = *1* **unfolding** *mat2-det-def*
using *3* **by** (*simp add*: *algebra-simps*)
have *6*: *?a* = *b* · *?u* **using** *assms*(*1*) *1*
unfolding *qf2-action-def mat2-transpose-def qf2-app-def vec2-dot-def*
mat2-app-def times-mat2-def mat2-sym-def
a_{12}*-def* a_{12}*-def* a_{22}*-def* a'_{12}*-def*
by (*simp add*: *algebra-simps*)

```
  have b ~ ?a unfolding qf2-equiv-def using 5 6 by auto
  hence 7: ?a ~ b using qf2-equiv-sym by blast
  have 8: 2 * |mat2_12 ?a| ≤ mat2_11 ?a using 4 unfolding a_12-def by auto
  have a_11 ≤ int (nat a_22)
    unfolding a_11-def a_22-def
    apply (rule rev-iffD1[OF - nat-int-comparison(3)])
    apply (rule wellorder-Least-lemma(2))
   using assms(2) 5 unfolding qf2-positive-definite-def zero-vec2-def mat2-det-def
    apply (metis add.right-neutral diff-add-cancel mat2.sel
                 mult-zero-left mult-zero-right nat-0-le order-less-le vec2.inject
                 zero-less-nat-eq zero-neq-one)
    done
  hence a_11 ≤ a_22 using 1 by linarith
  hence 4 * a_11^2 ≤ 4 * a_11 * a_22 unfolding power2-eq-square using 1 by auto
  also have ... = 4 * (mat-det ?a + a_12^2)
    unfolding mat2-det-def power2-eq-square by auto
  also have ... = 4 * mat-det ?a + (2 * |a_12|)^2
    unfolding power2-eq-square by auto
  also have ... ≤ 4 * mat-det ?a + (int a_11)^2
    using 4 power2-le-iff-abs-le unfolding a_12-def by (smt (verit))
  finally have 3 * a_11^2 ≤ 4 * (mat-det ?a) by auto
  hence a_11^2 ≤ (4 / 3) * mat-det ?a by linarith
  hence sqrt (a_11^2) ≤ sqrt ((4 / 3) * mat-det ?a)
    using real-sqrt-le-mono by blast
  hence a_11 ≤ sqrt ((4 / 3) * mat-det ?a) by auto
  hence a_11 ≤ (sqrt 4) / (sqrt 3) * sqrt (mat-det ?a)
    unfolding real-sqrt-mult real-sqrt-divide by blast
  hence 9: mat2_11 ?a ≤ (2 / sqrt 3) * sqrt (mat-det ?a) by auto
  have ?P ?a using 7 8 9 by blast
  thus ?thesis by blast
qed
```

Theorem 1.2 from [1].

```
theorem qf2-det-one-equiv-canonical:
  fixes f :: mat2
  assumes mat-sym f
  assumes qf-positive-definite f
  assumes mat-det f = 1
  shows f ~ 1
proof −
  obtain a where
    0: f ~ a
       2 * |mat2_12 a| ≤ mat2_11 a
       mat2_11 a ≤ (2 / sqrt 3) * sqrt (mat-det a)
    using assms lemma-1-2[of f] qf2-equiv-sym by auto
  have 1: mat-sym a
    using assms 0 qf2-equiv-preserves-sym by auto
  have 2: qf-positive-definite a
    using assms 0 qf2-equiv-preserves-positive-definite by auto
```

have *3*: *mat-det a = 1* **using** *assms 0 qf2-equiv-preserves-det* **by** *auto*
have *4*: $mat2_{11}\ a \geq 1$
apply (*rule allE*[*OF 2*[*unfolded qf2-positive-definite-def*], *of vec2 1 0*])
unfolding *zero-vec2-def vec2-dot-def mat2-app-def qf2-app-def*
qf2-positive-definite-def
apply *auto*
done
have *5*: $mat2_{11}\ a < 2$ **using** *0* **unfolding** *3*
by (*smt* (*verit*, *best*) *divide-le-eq int-less-real-le mult-cancel-left1*
of-int-1 real-sqrt-four real-sqrt-le-iff
real-sqrt-mult-self real-sqrt-one)
have *6*: $mat2_{11}\ a = 1$ **using** *4 5* **by** *auto*
have *7*: $mat2_{12}\ a = 0\ mat2_{21}\ a = 0$ **using** *0 1 6* **unfolding** *mat2-sym-def* **by** *auto*
have *8*: $mat2_{22}\ a = 1$ **using** *3 6 7* **unfolding** *mat2-det-def* **by** *auto*
have *a = 1* **using** *6 7 8* **unfolding** *one-mat2-def* **using** *mat2.collapse* **by** *metis*
thus *?thesis* **using** *0* **by** *metis*
qed

Lemma 1.3 from [1].

lemma *lemma-1-3*:
fixes *a* :: *mat3*
assumes *mat-sym a*
defines $a' \equiv$
mat2
$(mat3_{11}\ a * mat3_{22}\ a - (mat3_{12}\ a)^2)\ (mat3_{11}\ a * mat3_{23}\ a - mat3_{12}\ a * mat3_{13}\ a)$
$(mat3_{11}\ a * mat3_{23}\ a - mat3_{12}\ a * mat3_{13}\ a)\ (mat3_{11}\ a * mat3_{33}\ a - (mat3_{13}\ a)^2)$

defines $d' \equiv$
mat-det (
mat2
$(mat3_{11}\ a)\ (mat3_{12}\ a)$
$(mat3_{12}\ a)\ (mat3_{22}\ a)$
)

shows
mat-det a' = mat3$_{11}$ *a * mat-det a* (**is** *?P*)
$\bigwedge x.\ mat3_{11}\ a * (a\ \$\$\ x) =$
$(mat3_{11}\ a * vec3_1\ x + mat3_{12}\ a * vec3_2\ x + mat3_{13}\ a * vec3_3\ x)^2 +$
$(a'\ \$\$\ (vec2\ (vec3_2\ x)\ (vec3_3\ x)))$ (**is** $\bigwedge x.\ ?Q\ x$)
qf-positive-definite a $\Longrightarrow$ *qf-positive-definite a'*
qf-positive-definite a $\longleftrightarrow$ $mat3_{11}\ a > 0 \wedge d' > 0 \wedge$ *mat-det a* > 0
proof −
show *0*: *?P* **using** *assms*
unfolding *a'-def mat2-det-def mat3-det-def mat3-sym-def power2-eq-square*
by (*simp add*: *algebra-simps*)
show *1*: $\bigwedge x.\ ?Q\ x$ **using** *assms*

```
    unfolding a'-def vec2-dot-def vec3-dot-def mat2-app-def mat3-app-def
            mat3-sym-def qf2-app-def qf3-app-def power2-eq-square
    by (simp add: algebra-simps)
  have 2: qf-positive-definite a ⟹ mat3₁₁ a > 0
  proof −
    assume 3: qf-positive-definite a
    show mat3₁₁ a > 0
      using allE[OF iffD1[OF qf3-positive-definite-def 3], of vec3 1 0 0]
      unfolding zero-vec3-def vec3-dot-def mat3-app-def qf3-app-def
      by auto
  qed
  show 4: qf-positive-definite a ⟹ qf-positive-definite a'
  unfolding qf2-positive-definite-def
  proof
    fix v :: vec2
    assume 5: qf-positive-definite a
    hence 6: mat3₁₁ a > 0 using 2 by blast
    have a' $$ v ≤ 0 ⟹ v = 0 proof −
      assume 7: a' $$ v ≤ 0
      obtain x₂ x₃ where 8: v = vec2 x₂ x₃ by (rule vec2.exhaust)
      define x₁ where x₁ ≡ − (mat3₁₂ a * x₂ + mat3₁₃ a * x₃)
      have mat3₁₁ a * (a $$ (vec3 x₁ (mat3₁₁ a * x₂) (mat3₁₁ a * x₃))) =
       (mat3₁₁ a * x₁ + mat3₁₂ a * mat3₁₁ a * x₂ + mat3₁₃ a * mat3₁₁ a * x₃)²
+
       (a' $$ (vec2 (mat3₁₁ a * x₂) (mat3₁₁ a * x₃)))
        unfolding 1 by (simp add: algebra-simps)
      also have ... = a' $$ (vec2 (mat3₁₁ a * x₂) (mat3₁₁ a * x₃))
        unfolding x₁-def by (simp add: algebra-simps)
      also have ... = (mat3₁₁ a)² * (a' $$ v)
        unfolding 8 vec2-dot-def mat2-app-def qf2-app-def power2-eq-square
        by (simp add: algebra-simps)
      also have ... ≤ 0
        using 7 unfolding power2-eq-square by (simp add: mult-nonneg-nonpos)
      finally have mat3₁₁ a * (a $$ (vec3 x₁ (mat3₁₁ a * x₂) (mat3₁₁ a * x₃)))
≤ 0
        .
      hence a $$ (vec3 x₁ (mat3₁₁ a * x₂) (mat3₁₁ a * x₃)) ≤ 0
        using 6 by (simp add: mult-le-0-iff)
      hence vec3 x₁ (mat3₁₁ a * x₂) (mat3₁₁ a * x₃) = 0
        using 5 unfolding qf3-positive-definite-def by fastforce
      hence x₂ = 0 x₃ = 0 using 6 unfolding zero-vec3-def by fastforce+
      thus v = 0 unfolding 8 zero-vec2-def by blast
    qed
    thus v ≠ 0 ⟶ 0 < a' $$ v by fastforce
  qed
  have 9: qf-positive-definite a ⟹ d' > 0 ∧ mat-det a > 0
  proof −
    assume 10: qf-positive-definite a
    have 11: mat3₁₁ a > 0 using 2 10 by blast
```

```
  have qf-positive-definite a' using 4 10 by blast
  hence 12: mat2_11 a' > 0 ∧ mat-det a' > 0
    using qf2-positive-definite-criterion
    unfolding a'-def mat2-sym-def by fastforce
  have 13: d' > 0
    using 12 unfolding a'-def d'-def mat2-det-def power2-eq-square by fastforce
  have 14: mat-det a > 0
    using 11 12 unfolding 0 by (simp add: zero-less-mult-iff)
  show d' > 0 ∧ (mat-det a) > 0 using 13 14 by blast
 qed
 have 15: mat3_11 a > 0 ∧ d' > 0 ∧ mat-det a > 0 ⟹ qf-positive-definite a
 proof −
  assume 16: mat3_11 a > 0 ∧ d' > 0 ∧ mat-det a > 0
  have 17: mat2_11 a' > 0
    using 16 unfolding a'-def d'-def mat2-det-def power2-eq-square
    by (simp add: algebra-simps)
  have 18: mat-det a' > 0 using 16 unfolding 0 by fastforce
  have 19: qf-positive-definite a'
    using qf2-positive-definite-criterion 17 18
    unfolding a'-def mat2-sym-def by fastforce
  show qf-positive-definite a
  unfolding qf3-positive-definite-def
  proof
    fix x :: vec3
    have a $$ x ≤ 0 ⟹ x = 0 proof −
     assume a $$ x ≤ 0
     hence 20: mat3_11 a * (a $$ x) ≤ 0
       using 16 mult-le-0-iff order-less-le by auto
     have a' $$ (vec2 (vec3_2 x) (vec3_3 x)) ≤ 0
       using 20 unfolding 1 power2-eq-square by (smt (verit) zero-le-square)
     hence 21: a' $$ (vec2 (vec3_2 x) (vec3_3 x)) = 0
       using 19 qf2-positive-definite-positive
       using nle-le by blast
     have 22: vec3_2 x = 0 vec3_3 x = 0
       using 19 21 unfolding zero-vec2-def qf2-positive-definite-def
       by (smt (verit) vec2.inject)+
     have mat3_11 a * vec3_1 x + mat3_12 a * vec3_2 x + mat3_13 a * vec3_3 x = 0
       using 20 21 unfolding 1 by fastforce
     hence vec3_1 x = 0 using 16 22 by fastforce
     thus x = 0 using 22 unfolding zero-vec3-def by (metis vec3.collapse)
    qed
    thus x ≠ 0 ⟶ 0 < a $$ x by fastforce
  qed
 qed
 show qf-positive-definite a ⟷ mat3_11 a > 0 ∧ d' > 0 ∧ mat-det a > 0
  using 2 9 15 by blast
qed
```

Lemma 1.4 from [1].

lemma *lemma-1-4*:
fixes $b :: mat3$
fixes $v' :: mat2$
fixes $r\ s :: int$
assumes *mat-sym b*
assumes *qf-positive-definite b*
assumes *mat-det* $v' = 1$
defines $b' \equiv$
mat2
$(mat3_{11}\ b * mat3_{22}\ b - (mat3_{12}\ b)^2)\ (mat3_{11}\ b * mat3_{23}\ b - mat3_{12}\ b * mat3_{13}\ b)$
$(mat3_{11}\ b * mat3_{23}\ b - mat3_{12}\ b * mat3_{13}\ b)\ (mat3_{11}\ b * mat3_{33}\ b - (mat3_{13}\ b)^2)$

defines $a' \equiv b' \cdot v'$
defines $v \equiv$
mat3
1 r s
0 $(mat2_{11}\ v')\ (mat2_{12}\ v')$
0 $(mat2_{21}\ v')\ (mat2_{22}\ v')$

defines $a \equiv b \cdot v$
shows
$\bigwedge y.\ mat3_{11}\ b * (b\ \$\$\ y) =$
$(mat3_{11}\ b * vec3_1\ y + mat3_{12}\ b * vec3_2\ y + mat3_{13}\ b * vec3_3\ y)^2 +$
$(b'\ \$\$\ (vec2\ (vec3_2\ y)\ (vec3_3\ y)))$ (**is** $\bigwedge y.\ ?P\ y$)
$mat3_{11}\ a = mat3_{11}\ b$
$\bigwedge x.\ mat3_{11}\ a * (a\ \$\$\ x) =$
$(mat3_{11}\ a * vec3_1\ x + mat3_{12}\ a * vec3_2\ x + mat3_{13}\ a * vec3_3\ x)^2 +$
$(a'\ \$\$\ (vec2\ (vec3_2\ x)\ (vec3_3\ x)))$ (**is** $\bigwedge x.\ ?Q\ x$)
proof −
show $\bigwedge y.\ ?P\ y$ **using** *assms*
unfolding *b′-def vec2-dot-def vec3-dot-def mat2-app-def mat3-app-def mat3-sym-def qf2-app-def qf3-app-def power2-eq-square*
by (*simp add*: *algebra-simps*)
show $mat3_{11}\ a = mat3_{11}\ b$
unfolding *a-def v-def times-mat3-def mat3-transpose-def qf3-action-def*
by *force*
show $\bigwedge y.\ ?Q\ y$ **using** *assms*
by (*simp add*: *algebra-simps power2-eq-square*
a-def v-def a′-def b′-def vec2-dot-def vec3-dot-def
times-mat2-def times-mat3-def mat2-app-def mat3-app-def
mat2-transpose-def mat3-transpose-def mat3-sym-def
qf2-app-def qf3-app-def qf2-action-def qf3-action-def)
qed

Lemma 1.5 from [1].

lemma *lemma-1-5*:
fixes $u_{11}\ u_{21}\ u_{31}$

assumes *Gcd* $\{u_{11}, u_{21}, u_{31}\} = 1$
shows $\exists u.\ mat3_{11}\ u = u_{11} \wedge mat3_{21}\ u = u_{21} \wedge mat3_{31}\ u = u_{31} \wedge mat\text{-}det\ u = 1$
proof −
let *?a* = *gcd* $u_{11}\ u_{21}$
show *?thesis* **proof** *cases*
assume *?a* = *0*
hence *0*: $u_{11} = 0\ u_{21} = 0\ u_{31} = 1 \vee u_{31} = -1$ **using** *assms* **by** *auto*
let *?u* =
mat3
0 0 (− *1*)
0 u_{31} *0*
u_{31} *0 0*
show *?thesis*
apply (*rule exI*[*of* - *?u*])
unfolding *mat3-det-def* **using** *0* **apply** *auto*
done
next
assume *1*: *?a* ≠ *0*
obtain $u_{12}\ u_{22}$ **where** *2*: $u_{11} * u_{22} - u_{21} * u_{12} = ?a$ **using** *bezout-int*
by (*metis diff-minus-eq-add mult.commute mult-minus-right*)
have *gcd ?a* $u_{31} = 1$ **using** *assms* **by** (*simp add*: *gcd.assoc*)
then obtain $u_{33}\ b$ **where** *3*: $?a * u_{33} - b * u_{31} = 1$ **using** *bezout-int*
by (*metis diff-minus-eq-add mult.commute mult-minus-right*)
let *?u* =
mat3
$u_{11}\ u_{12}\ ((u_{11}\ div\ ?a) * b)$
$u_{21}\ u_{22}\ ((u_{21}\ div\ ?a) * b)$
u_{31} *0* u_{33}
have *mat-det ?u* =
$u_{11} * u_{22} * u_{33}$
$+ u_{12} * ((u_{21}\ div\ ?a) * b) * u_{31}$
$- ((u_{11}\ div\ ?a) * b) * u_{22} * u_{31}$
$- u_{12} * u_{21} * u_{33}$
unfolding *mat3-det-def* **by** *auto*
also have ... =
$u_{11} * u_{22} * u_{33}$
$+ u_{12} * (u_{21}\ div\ ?a) * (b * u_{31})$
$- u_{22} * (u_{11}\ div\ ?a) * (b * u_{31})$
$- u_{12} * u_{21} * u_{33}$
by *auto*
also have ... =
$u_{11} * u_{22} * u_{33}$
$+ u_{12} * (u_{21}\ div\ ?a) * (?a * u_{33} - 1)$
$- u_{22} * (u_{11}\ div\ ?a) * (?a * u_{33} - 1)$
$- u_{12} * u_{21} * u_{33}$
using *3* **by** (*simp add*: *algebra-simps*)
also have ... =
$u_{11} * u_{22} * u_{33}$

```
      + u12 * ((u21 div ?a) * ?a) * u33
      - u12 * (u21 div ?a)
      - u22 * ((u11 div ?a) * ?a) * u33
      + u22 * (u11 div ?a)
      - u12 * u21 * u33
    by (simp add: algebra-simps)
  also have ... =
      - u12 * (u21 div ?a)
      + u22 * (u11 div ?a)
    by (simp add: algebra-simps)
  also have ... =
      - u12 * (u21 div ?a)
      + u22 * u11 div ?a
    by (metis dvd-div-mult gcd-dvd1 mult.commute)
  also have ... =
      - u12 * (u21 div ?a)
      + (?a + u21 * u12) div ?a
    using 2 by (simp add: algebra-simps)
  also have ... =
      - u12 * (u21 div ?a)
      + 1 + (u21 * u12) div ?a
    using 1 by auto
  also have ... = 1 by (simp add: algebra-simps div-mult1-eq)
  finally have 4: mat-det ?u = 1 .
  show ?thesis
    apply (rule exI[of - ?u])
    using 4 apply auto
    done
 qed
qed
```

Lemma 1.6 from [1].

```
lemma lemma-1-6:
 fixes c :: mat3
 assumes mat-sym c
 assumes qf-positive-definite c
 shows ∃ a. a ~ c ∧
          2 * (max |mat3 1 2 a| |mat3 1 3 a|) ≤ mat3 1 1 a ∧
          mat3 1 1 a ≤ (4 / 3) * root 3 (mat-det a)
proof −
 define a11 where a11 ≡ LEAST y. y > 0 ∧ (∃ x. int y = c $$ x)
 have 0: ∃ y. y > 0 ∧ (∃ x. int y = c $$ x)
   apply (rule exI[of - nat (c $$ (vec3 1 1 1))])
   using assms(2) unfolding qf3-positive-definite-def zero-vec3-def
   apply (metis nat-0-le order-less-le vec3.inject zero-less-nat-eq zero-neq-one)
   done
 obtain t where 1: a11 > 0 int a11 = c $$ t
   using a11-def LeastI-ex[OF 0] by auto
 let ?h = Gcd {vec3 1 t, vec3 2 t, vec3 3 t}
```

have $t \neq 0$ **using** *assms*(2) *1* **by** *fastforce*
hence *2*: $?h > 0$ **by** (*simp*; *metis vec3.collapse zero-vec3-def*)
let $?t' = vec3\ (vec3_1\ t\ div\ ?h)\ (vec3_2\ t\ div\ ?h)\ (vec3_3\ t\ div\ ?h)$
have $?t' \neq 0$ **using** *2* **unfolding** *zero-vec3-def*
  by (*auto simp add*: *algebra-simps dvd-div-eq-0-iff*)
hence $nat\ (c\ \$\$\ ?t') > 0 \wedge (\exists x.\ int\ (nat\ (c\ \$\$\ ?t')) = c\ \$\$\ x)$
  using *assms*(2) **unfolding** *qf3-positive-definite-def* **by** *auto*
hence $a_{11} \leq nat\ (c\ \$\$\ ?t')$ **unfolding** a_{11}*-def* **by** (*rule Least-le*)
hence $a_{11} \leq c\ \$\$\ ?t'$ **using** *1* **by** *auto*
also have $... = (c\ \$\$\ t)\ div\ ?h^2$ **proof** −
  have *?h dvd vec3*$_1$ *t ?h dvd vec3*$_2$ *t ?h dvd vec3*$_3$ *t*
    by (*meson Gcd-dvd insertCI*)+
  then have $(c\ \$\$\ ?t') * ?h^2 = c\ \$\$\ t\ ?h^2\ dvd\ c\ \$\$\ t$
    unfolding *qf3-app-def vec3-dot-def mat3-app-def power2-eq-square*
    using *1*
    by (*auto simp add*: *algebra-simps mult-dvd-mono*)
  thus $c\ \$\$\ ?t' = (c\ \$\$\ t)\ div\ ?h^2$ **using** *2* **by** *auto*
qed
also have $... = a_{11}\ div\ ?h^2$ **using** *1* **by** *auto*
finally have $a_{11} \leq a_{11}\ div\ ?h^2$ **.**
also have $... \leq a_{11}$ **using** *1 2*
  by (*metis div-le-dividend nat-div-distrib nat-int nat-le-iff of-nat-0-le-iff*)
finally have *?h = 1* **using** *1 2*
  by (*smt* (*verit*) *int-div-less-self of-nat-0-less-iff power2-eq-square*
      zero-less-power zmult-eq-1-iff)
then obtain *u* **where** *3*: $mat3_{11}\ u = vec3_1\ t\ mat3_{21}\ u = vec3_2\ t$
    $mat3_{31}\ u = vec3_3\ t\ mat\text{-}det\ u = 1$
  using *lemma-1-5* **by** *blast*
define *b* **where** $b \equiv c \cdot u$
have *4*: *mat-sym b*
  using *3 assms*(1) *qf3-equiv-preserves-sym*
  unfolding *b-def qf3-equiv-def*
  by *auto*
have *5*: *qf-positive-definite b*
  using *3 assms*(2) *qf3-equiv-preserves-positive-definite*
  unfolding *b-def qf3-equiv-def*
  by *auto*
have *6*: $a_{11} = (LEAST\ y.\ y > 0 \wedge (\exists x.\ int\ y = b\ \$\$\ x))$
  unfolding a_{11}*-def* **apply** (*rule arg-cong*[*of - - Least*])
  using *3 qf3-equiv-preserves-range*[*of c b*]
  unfolding *b-def image-def qf3-equiv-def*
  apply *fast*
  done
have *7*: $a_{11} = mat3_{11}\ b$
  using *1 3*
  by (*simp add*: *algebra-simps*
      b-def times-mat3-def vec3-dot-def mat3-app-def
      mat3-transpose-def qf3-app-def qf3-action-def)
define b' **where** $b' \equiv$

mat2
(*mat3*$_{11}$ *b* * *mat3*$_{22}$ *b* − (*mat3*$_{12}$ *b*)^2) (*mat3*$_{11}$ *b* * *mat3*$_{23}$ *b* − *mat3*$_{12}$ *b* * *mat3*$_{13}$ *b*)
(*mat3*$_{11}$ *b* * *mat3*$_{23}$ *b* − *mat3*$_{12}$ *b* * *mat3*$_{13}$ *b*) (*mat3*$_{11}$ *b* * *mat3*$_{33}$ *b* − (*mat3*$_{13}$ *b*)^2)

have *8*: *mat-sym b'* **unfolding** *b'-def mat2-sym-def* **by** *auto*
have *9*: *mat-det b'* = *mat3*$_{11}$ *b* * *mat-det b*
⋀*x*. *mat3*$_{11}$ *b* * (*b* $$ *x*) =
(*mat3*$_{11}$ *b* * *vec3*$_1$ *x* + *mat3*$_{12}$ *b* * *vec3*$_2$ *x* + *mat3*$_{13}$ *b* * *vec3*$_3$ *x*)^2 +
(*b'* $$ (*vec2* (*vec3*$_2$ *x*) (*vec3*$_3$ *x*)))
qf-positive-definite b'
using *4 5 b'-def lemma-1-3* **by** *blast+*
obtain *a' v'* **where** *10*: *a'* = *b'* · *v'*
mat-det v' = *1*
mat2$_{11}$ *a'* ≤ (*2* / *sqrt 3*) * *sqrt* (*mat3*$_{11}$ *b* * *mat-det b*)
using *8 9 qf2-equiv-sym qf2-equiv-preserves-det lemma-1-2*[*of b'*]
unfolding *qf2-equiv-def* **by** *metis*
obtain *r s* **where** *11*: *2* * |(*mat3*$_{12}$ *b*) * (*mat2*$_{11}$ *v'*) +
(*mat3*$_{13}$ *b*) * (*mat2*$_{21}$ *v'*) + *a*$_{11}$ * (*r* :: *int*)| ≤ *a*$_{11}$
2 * |(*mat3*$_{12}$ *b*) * (*mat2*$_{12}$ *v'*) +
(*mat3*$_{13}$ *b*) * (*mat2*$_{22}$ *v'*) + *a*$_{11}$ * (*s* :: *int*)| ≤ *a*$_{11}$
using *1 congruence-class-close* **by** *fastforce*
define *a*$_{12}$ **where** *a*$_{12}$ ≡ (*mat3*$_{12}$ *b*) * (*mat2*$_{11}$ *v'*) +
(*mat3*$_{13}$ *b*) * (*mat2*$_{21}$ *v'*) + *a*$_{11}$ * *r*
define *a*$_{13}$ **where** *a*$_{13}$ ≡ (*mat3*$_{12}$ *b*) * (*mat2*$_{12}$ *v'*) +
(*mat3*$_{13}$ *b*) * (*mat2*$_{22}$ *v'*) + *a*$_{11}$ * *s*
define *v* **where** *v* ≡
mat3
1 r s
0 (*mat2*$_{11}$ *v'*) (*mat2*$_{12}$ *v'*)
0 (*mat2*$_{21}$ *v'*) (*mat2*$_{22}$ *v'*)

have *12*: *mat-det v* = *1*
using *10* **unfolding** *v-def mat2-det-def mat3-det-def* **by** (*simp add*: *algebra-simps*)
define *a* **where** *a* ≡ *b* · *v*
have *13*: *mat-det a* = *mat-det b*
using *12 qf3-equiv-preserves-det*
unfolding *a-def qf3-equiv-def*
by *metis*
have *14*: *a*$_{11}$ = (*LEAST y*. *y* > *0* ∧ (∃ *x*. *int y* = *a* $$ *x*))
unfolding *6* **apply** (*rule arg-cong*[*of* - - *Least*])
using *12 qf3-equiv-preserves-range*[*of b a*]
unfolding *a-def image-def qf3-equiv-def*
apply *fast*
done
have *15*: *mat3*$_{11}$ *a* = *mat3*$_{11}$ *b*
⋀*x*. *mat3*$_{11}$ *a* * (*a* $$ *x*) =

```
        (mat3_11 a * vec3_1 x + mat3_12 a * vec3_2 x + mat3_13 a * vec3_3 x)^2 +
        (a' $$ (vec2 (vec3_2 x) (vec3_3 x)))
  using 4 5 10 lemma-1-4 unfolding b'-def v-def a-def by blast+
 have 16: mat2_11 a' = mat3_11 a * mat3_22 a - (mat3_12 a)^2
  using 15(2)[of vec3 0 1 0]
  unfolding vec3-dot-def vec2-dot-def mat3-app-def mat2-app-def
      qf3-app-def qf2-app-def
  by simp
 define a_22 where a_22 ≡ a $$ (vec3 0 1 0)
 have 17: a_11 = mat3_11 a unfolding 7 15 by auto
 have 18: a_12 = mat3_12 a a_13 = mat3_13 a a_22 = mat3_22 a a_22 = mat3_22 a
  using 13(1) 17
  unfolding a-def v-def a_12-def a_13-def a_22-def
  by (auto simp add: algebra-simps
            times-mat3-def vec3-dot-def mat3-app-def
            mat3-transpose-def qf3-app-def qf3-action-def)
 have 19: a ~ c
  unfolding qf3-equiv-sym[of a c]
  unfolding a-def b-def qf3-equiv-def qf3-action-mul[symmetric]
  using 3 12 by (metis mat3-mul-det mult-1)
 have 20: 2 * (max |mat3_12 a| |mat3_13 a|) ≤ mat3_11 a
  using 11 unfolding 17[symmetric] 18 a_12-def[symmetric] a_13-def[symmetric]
  by auto
 have 21: (2 / sqrt 3) ^ 6 = 64 / 27 unfolding power-def by auto
 have a_11 ≤ int (nat a_22)
  unfolding a_11-def a_22-def
  apply (rule rev-iffD1[OF - nat-int-comparison(3)])
  apply (rule wellorder-Least-lemma(2))
  using assms(2) 5 12
  apply (metis a-def b-def nat-0-le nat-int of-nat-0 qf3-action-app
        qf3-equiv-def qf3-equiv-preserves-positive-definite
        qf3-positive-definite-def qf3-positive-definite-positive
        vec3.sel(2) zero-neq-one zero-vec3-def zless-nat-conj)
  done
 hence a_11 ≤ a_22 using 1 by linarith
 hence (mat3_11 a)^2 ≤ a_11 * a_22 unfolding power2-eq-square using 1 17 by auto
 also have ... = mat2_11 a' + a_12^2 using 16 17 18 by auto
 also have ... ≤ (2 / sqrt 3) * sqrt (mat3_11 b * mat-det b) + a_12^2
  using 10 by auto
 also have ... ≤ (2 / sqrt 3) * sqrt (mat3_11 a * mat-det a) + (mat3_11 a)^2 / 4
  using 11 13 15 17 18 a_12-def
      power2-le-iff-abs-le[of real-of-int (mat3_11 a) (mat3_12 a) * 2]
  by auto
 finally have (3 / 4) * (mat3_11 a)^2 ≤ (2 / sqrt 3) * sqrt (mat3_11 a * mat-det
a)
  by (simp add: algebra-simps)
 hence (mat3_11 a)^2 ≤ (2 / sqrt 3) ^ 3 * sqrt (mat3_11 a * mat-det a)
  unfolding power2-eq-square power3-eq-cube by (simp add: algebra-simps)
 hence ((mat3_11 a)^2)^2 ≤ ((2 / sqrt 3) ^ 3 * sqrt (mat3_11 a * mat-det a))^2
```

```
    using 1(1) unfolding 17[symmetric] power2-eq-square
    by (metis of-int-power power2-eq-square power-mono zero-le-square)
  hence (mat3_11 a) ^ 4 ≤ (2 / sqrt 3) ^ 6 * (sqrt (mat3_11 a * mat-det a))^2
    by (simp add: power-mult-distrib)
  hence (mat3_11 a) ^ 4 ≤ (2 / sqrt 3) ^ 6 * mat3_11 a * mat-det a
    using 4 5 13 17 lemma-1-3 by auto
  hence (mat3_11 a) ^ 3 ≤ (2 / sqrt 3) ^ 6 * mat-det a
    using 1(1) unfolding 17[symmetric]
    unfolding power-def apply (simp add: algebra-simps)
    apply (metis mult-left-le-imp-le of-nat-0-less-iff times-divide-eq-right)
    done
  hence (mat3_11 a) ^ 3 ≤ (64 / 27) * mat-det a using 21 by auto
  hence root 3 ((mat3_11 a) ^ 3) ≤ root 3 ((64 / 27) * mat-det a) by auto
  hence root 3 ((mat3_11 a) ^ 3) ≤ root 3 (64 / 27) * root 3 (mat-det a)
    unfolding real-root-mult by auto
  hence mat3_11 a ≤ root 3 (64 / 27) * root 3 (mat-det a)
    using odd-real-root-power-cancel by auto
  hence 22: mat3_11 a ≤ (4 / 3) * root 3 (mat-det a)
    using real-root-divide by force
  show ?thesis using 19 20 22 by blast
qed
```

Theorem 1.3 from [1].

```
theorem qf3-det-one-equiv-canonical:
  fixes f :: mat3
  assumes mat-sym f
  assumes qf-positive-definite f
  assumes mat-det f = 1
  shows f ~ 1
proof −
  obtain a where
    0: f ~ a ∧
       2 * (max |mat3_12 a| |mat3_13 a|) ≤ mat3_11 a ∧
       mat3_11 a ≤ (4 / 3) * root 3 (mat-det a)
    using assms lemma-1-6[of f] qf3-equiv-sym by auto
  have 1: mat-sym a
    using assms 0 qf3-equiv-preserves-sym by auto
  have 2: qf-positive-definite a
    using assms 0 qf3-equiv-preserves-positive-definite by auto
  have 3: mat-det a = 1 using assms 0 qf3-equiv-preserves-det by auto
  have 4: mat3_12 a = 0 mat3_13 a = 0 using 0 3 by auto
  have mat3_11 a ≥ 1
    apply (rule allE[OF 2[unfolded qf3-positive-definite-def], of vec3 1 0 0])
    unfolding zero-vec3-def vec3-dot-def mat3-app-def qf3-app-def
              qf3-positive-definite-def
    apply auto
    done
  hence 6: mat3_11 a = 1 using 0 3 by auto
  define a' where a' ≡
```

```
    mat2
      (mat3_22 a) (mat3_23 a)
      (mat3_32 a) (mat3_33 a)

  have 7: mat-det a' = 1
    using 3 4 6 unfolding a'-def mat2-det-def mat3-det-def by auto
  have 8: mat-sym a' using 1 unfolding a'-def mat2-sym-def mat3-sym-def by
auto
  have 9: qf-positive-definite a'
  using 2 unfolding qf2-positive-definite-def qf3-positive-definite-def
  proof −
    assume 10: ∀ v. v ≠ 0 ⟶ a $$ v > 0
    show ∀ v. v ≠ 0 ⟶ a' $$ v > 0
    proof (rule; rule)
      fix v :: vec2
      assume v ≠ 0
      hence vec3 0 (vec2_1 v) (vec2_2 v) ≠ 0
        unfolding zero-vec2-def zero-vec3-def by (metis vec2.collapse vec3.inject)
      hence a $$ (vec3 0 (vec2_1 v) (vec2_2 v)) > 0 using 10 by auto
      thus a' $$ v > 0
        unfolding a'-def vec2-dot-def vec3-dot-def
                  mat2-app-def mat3-app-def qf2-app-def qf3-app-def
                  qf2-positive-definite-def qf3-positive-definite-def
        by auto
    qed
  qed
  obtain u' where 11: mat-det u' = 1 a' · u' = 1
    using 7 8 9 qf2-det-one-equiv-canonical[of a']
    unfolding qf2-equiv-def
    by auto
  define u where u ≡
    mat3
      1 0 0
      0 (mat2_11 u') (mat2_12 u')
      0 (mat2_21 u') (mat2_22 u')

  have 12: mat-det u = 1
    using 11 unfolding u-def mat2-det-def mat3-det-def by auto
  have 13: a · u = 1
    using 1 4 6 11
    by (simp add: algebra-simps
                  a'-def u-def one-mat2-def one-mat3-def
                  times-mat2-def times-mat3-def
                  mat2-transpose-def mat3-transpose-def
                  qf2-action-def qf3-action-def mat3-sym-def)
  have a ~ 1 using 12 13 unfolding qf3-equiv-def by blast
  thus ?thesis using 0 qf3-equiv-trans by blast
qed
```

end

4 Legendre's three squares theorem and its consequences

theory *Three-Squares*
imports *Dirichlet-L.Dirichlet-Theorem Residues-Properties Quadratic-Forms*
begin

4.1 Legendre's three squares theorem

definition *quadratic-residue-alt* :: *int* ⇒ *int* ⇒ *bool* **where**
quadratic-residue-alt m a = (∃ *x y*. $x^2 - a = y * m$)

lemma *quadratic-residue-alt-equiv*: *quadratic-residue-alt* = *QuadRes*
unfolding *quadratic-residue-alt-def QuadRes-def*
by (*metis cong-iff-dvd-diff cong-modulus-mult dvdE dvd-refl mult.commute*)

lemma *sq-nat-abs*: (*nat* |*v*|)^2 = *nat* (v^2)
by (*simp add*: *nat-power-eq*[*symmetric*])

Lemma 1.7 from [1].

lemma *three-squares-using-quadratic-residue*:
fixes *n d'* :: *nat*
assumes *n* ≥ *2*
assumes *d'* > *0*
assumes *QuadRes* (*d'* ∗ *n* − *1*) (− *d'*)
shows ∃ x_1 x_2 x_3. $n = {x_1}^2 + {x_2}^2 + {x_3}^2$
proof −
define *a* **where** *a* ≡ *d'* ∗ *n* − *1*
from *assms*(*3*) **obtain** *x y* **where** x^2 + *int d'* = *y* ∗ *int a*
unfolding *a-def quadratic-residue-alt-equiv*[*symmetric*]
quadratic-residue-alt-def
by *auto*
hence *Hxy*: $x^2 + d' = y * a$ **by** *auto*
have *y* ≥ *1*
using *assms Hxy*
unfolding *a-def*
by (*smt* (*verit*) *bot-nat-0.not-eq-extremum mult-le-0-iff of-nat-0-le-iff*
of-nat-le-0-iff power2-eq-square zero-le-square)
moreover from *Hxy* **have** *Hxy*$_2$: $d' = y * a - x^2$ **by** (*simp add*: *algebra-simps*)
define *M* **where** *M* ≡ *mat3 y x 1 x a 0 1 0 n*
moreover have *Msym*: *mat-sym M*
unfolding *mat3-sym-def M-def mat3.sel*
by *simp*
moreover have *Mdet-eq-1*: *mat-det M* = *1*
proof −
have *mat-det M* = ($y * a - x^2$) ∗ *n* − *a*

```
    unfolding mat3-det-def M-def mat3.sel power2-eq-square
    by (simp add: algebra-simps)
  also have ... = int d' * n - a unfolding Hxy₂ by simp
  also have ... = 1 unfolding a-def using assms int-ops by force
  finally show ?thesis .
qed
moreover have mat-det (mat2 y x x a) > 0
  using Hxy₂ assms
  unfolding mat2-det-def power2-eq-square
  by simp
ultimately have qf-positive-definite M
  by (auto simp add: lemma-1-3(4))
hence M ~ 1
  using Msym and Mdet-eq-1
  by (simp add: qf3-det-one-equiv-canonical)
moreover have M $$ vec3 0 0 1 = n
  unfolding M-def qf3-app-def mat3-app-def vec3-dot-def mat3.sel vec3.sel
  by (simp add: algebra-simps)
hence n ∈ range (($$) M) by (metis rangeI)
ultimately have n ∈ range (($$) (1 :: mat3))
  using qf3-equiv-preserves-range by simp
then obtain u :: vec3 where 1 $$ u = n
  by auto
hence <u | u> = n
  unfolding qf3-app-def mat3-app-def one-mat3-def
  by simp
hence ∃ x₁ x₂ x₃. int n = x₁² + x₂² + x₃²
  unfolding vec3-dot-def power2-eq-square by metis
hence ∃ x₁ x₂ x₃. n = (nat |x₁|)² + (nat |x₂|)² + (nat |x₃|)²
  unfolding sq-nat-abs
  apply (simp add: nat-add-distrib[symmetric])
  apply (metis nat-int)
  done
thus ∃ x₁ x₂ x₃. n = x₁² + x₂² + x₃² by blast
qed

lemma prime-linear-combination:
  fixes a m :: nat
  assumes m > 1
  assumes coprime a m
  obtains j :: nat where prime (a + m * j) ∧ j ≠ 0
proof −
  assume 0: ⋀j. prime (a + m * j) ∧ j ≠ 0 ⟹ thesis

  have 1: infinite {p. prime p ∧ [p = a] (mod m)}
    using assms
    by (rule Dirichlet-Theorem.residues-nat.Dirichlet[unfolded residues-nat-def])

  have 2: finite {j. prime (nat (int a + int m * j)) ∧ j ≤ 0}
```

```
  apply (rule finite-subset[of - {− (int a) div (int m)..0}])
  subgoal
    apply (rule subsetI)
    subgoal for j
    proof −
      assume 1: j ∈ {j. prime (nat (int a + int m * j)) ∧ j ≤ 0}
      have int a + int m * j ≥ 0 using 1 prime-ge-0-int by force
      hence int m * j ≥ − int a by auto
      hence j ≥ (− int a) div int m
        using assms 1
        by (smt (verit) unique-euclidean-semiring-class.cong-def
                        coprime-crossproduct-nat coprime-iff-invertible-int
                        coprime-int-iff int-distrib(3) int-ops(2) int-ops(7)
                        mem-Collect-eq mult-cancel-right1 zdiv-mono1
                        nonzero-mult-div-cancel-left of-nat-0-eq-iff
                        of-nat-le-0-iff prime-ge-2-int prime-nat-iff-prime)
      thus j ∈ {− (int a) div (int m)..0} using 1 by auto
    qed
    done
  subgoal by blast
  done

have {p. prime p ∧ [p = a] (mod m)} =
      {p. prime p ∧ (∃j. int p = int a + int m * j)}
  unfolding cong-sym-eq[of - a]
  unfolding cong-int-iff[symmetric] cong-iff-lin
  ..
also have ... = {p. prime p ∧ (∃j. p = nat (int a + int m * j))}
  by (metis (opaque-lifting) nat-int nat-eq-iff
                             prime-ge-0-int prime-nat-iff-prime)
also have ... = (λj. nat (int a + int m * j)) ‘
                {j. prime (nat (int a + int m * j))}
  by blast
finally have infinite ((λj. nat (int a + int m * j)) ‘
                       {j. prime (nat (int a + int m * j))})
  using 1 by metis
hence infinite {j. prime (nat (int a + int m * j))}
  using finite-imageI by blast
hence infinite ({j. prime (nat (int a + int m * j))} −
                {j. prime (nat (int a + int m * j)) ∧ j ≤ 0})
  using 2 Diff-infinite-finite by blast
hence infinite {j. prime (nat (int a + int m * j)) ∧ j > 0}
  by (rule back-subst[of infinite]; auto)
hence infinite (int ‘ {j. prime (nat (int a + int m * j)) ∧ j ≠ (0 :: nat)})
  apply (rule back-subst[of infinite])
  unfolding image-def using zero-less-imp-eq-int apply auto
  done
hence infinite {j. prime (nat (int a + int m * j)) ∧ (j :: nat) ≠ 0}
  using finite-imageI by blast
```

```
  hence infinite {j. prime (a + m * j) ∧ j ≠ 0}
    apply (rule back-subst[of infinite])
    apply (auto simp add: int-ops nat-plus-as-int)
    done
  thus thesis using 0 not-finite-existsD by blast
qed
```

Lemma 1.8 from [1].

```
lemma three-squares-using-mod-four:
  fixes n :: nat
  assumes n mod 4 = 2
  shows ∃ x1 x2 x3. n = x1^2 + x2^2 + x3^2
proof −
  have n > 1 using assms by auto
  have coprime (n − 1) (4 * n)
    by (smt (verit, del-insts) Suc-pred assms bot-nat-0.not-eq-extremum
                               coprime-commute coprime-diff-one-right-nat
                               coprime-mod-right-iff coprime-mult-left-iff
                               diff-Suc-1 mod-Suc mod-mult-self1-is-0 mult-0-right
                               numeral-2-eq-2 zero-neq-numeral)
  then obtain j where H-j:
     prime ((n − 1) + (4 * n) * j) ∧ j ≠ 0
    using prime-linear-combination[of 4 * n n − 1] ‹n > 1› by auto
  have j > 0 using H-j by blast

  define d' where d' ≡ 4 * j + 1
  define p where p ≡ d' * n − 1
  have prime p
    unfolding p-def d'-def
    using conjunct1[OF H-j] apply (rule back-subst[of prime])
   using ‹n > 1› apply (simp add: algebra-simps nat-minus-as-int nat-plus-as-int)
    done
  have p mod 4 = 1
    unfolding p-def
    apply (subst mod-diff-eq-nat)
    subgoal unfolding d'-def using ‹n > 1› ‹j > 0› by simp
    subgoal
      apply (subst mod-mult-eq[symmetric])
      unfolding assms d'-def apply simp
      done
    done
  have d' * n mod 4 = 2
    using assms p-def d'-def ‹p mod 4 = 1›
    by (metis mod-mult-cong mod-mult-self4 nat-mult-1)
  hence d' mod 4 = 1 using assms by (simp add: d'-def)

  have QuadRes p (− d')
  proof −
    have d'-expansion: d' = (∏ q∈prime-factors d'. q ^ multiplicity q d')
```

using *prime-factorization-nat* **unfolding** *d'-def* **by** *auto*

have *odd d'* **unfolding** *d'-def* **by** *simp*
hence *d'-prime-factors-odd*: $q \in$ *prime-factors d'* $\Longrightarrow$ *odd q* **for** *q*
by *fastforce*

have *d'-prime-factors-gt-2*: $q \in$ *prime-factors d'* $\Longrightarrow q > 2$ **for** *q*
using *d'-prime-factors-odd*
by (*metis even-numeral in-prime-factors-imp-prime*
order-less-le prime-ge-2-nat)

have $[p = -1]$ (*mod d'*)
unfolding *p-def cong-iff-dvd-diff* **apply** *simp*
using ‹$n > 1$› **apply** (*smt* (*verit*) *Suc-as-int Suc-pred add-gr-0 d'-def*
dvd-nat-abs-iff dvd-triv-left
less-numeral-extra(*1*) *mult-pos-pos*
of-nat-less-0-iff order-le-less-trans
zero-less-one-class.zero-le-one)
done
hence *d'-prime-factors-2-p-mod*:
$q \in$ *prime-factors d'* $\Longrightarrow [p = -1]$ (*mod q*) **for** *q*
by (*rule cong-dvd-modulus*; *auto*)

have *d' mod 4* $= (\prod q \in$*prime-factors d'*. *q* ^ *multiplicity q d'*) *mod 4*
using *d'-expansion* **by** *argo*
also have ... $= (\prod q \in$*prime-factors d'*. (*q mod 4*) ^ *multiplicity q d'*) *mod 4*
apply (*subst mod-prod-eq*[*symmetric*])
apply (*subst power-mod*[*symmetric*])
apply (*subst mod-prod-eq*)
apply *blast*
done
also have ... $= (\prod q \in \{q \in$ *prime-factors d'*. $[q = 3]$ (*mod 4*)$\}$.
(*q mod 4*) ^ *multiplicity q d'*) *mod 4*
apply (*rule arg-cong*[*of - - λx. x mod 4*])
apply (*rule prod.mono-neutral-right*)
subgoal by *blast*
subgoal by *blast*
subgoal
unfolding *unique-euclidean-semiring-class.cong-def*
apply (*rule ballI*)
using *d'-prime-factors-odd* **apply** *simp*
apply (*metis One-nat-def dvd-0-right even-even-mod-4-iff*
even-numeral mod-exhaust-less-4)
done
done
also have ... $= (\prod q \in \{q \in$ *prime-factors d'*. $[q = 3]$ (*mod 4*)$\}$.
((*int q*) *mod 4*) ^ *multiplicity q d'*) *mod 4*
by (*simp add*: *int-ops*)
also have ... $= (\prod q \in \{q \in$ *prime-factors d'*. $[q = 3]$ (*mod 4*)$\}$.

```
                ((− 1) mod 4) ^ multiplicity q d') mod 4
    apply (rule arg-cong[of - - λx. x mod 4])
    apply (rule prod.cong[OF refl])
     unfolding unique-euclidean-semiring-class.cong-def nat-mod-as-int apply
simp
    apply (metis nat-int of-nat-mod of-nat-numeral)
    done
  also have ... = (∏ q∈{q ∈ prime-factors d'. [q = 3] (mod 4)}.
                (− 1) ^ multiplicity q d') mod 4
    apply (subst mod-prod-eq[symmetric])
    apply (subst power-mod)
    apply (subst mod-prod-eq)
    apply blast
    done
  finally have [∏ q∈{q ∈ prime-factors d'. [q = 3] (mod 4)}.
        (− 1) ^ multiplicity q d' = 1 :: int] (mod 4)
    using ‹d' mod 4 = 1›
    by (simp add: unique-euclidean-semiring-class.cong-def)
  hence prod-prime-factors-minus-one:
     (∏ q∈{q ∈ prime-factors d'. [q = 3] (mod 4)}.
       (− 1) ^ multiplicity q d') = (1 :: int)
    unfolding power-sum[symmetric]
    unfolding minus-one-power-iff unique-euclidean-semiring-class.cong-def
    by presburger

  have p > 2 using ‹prime p› ‹p mod 4 = 1› nat-less-le prime-ge-2-nat by force

  have d'-prime-factors-Legendre:
    q ∈ prime-factors d' ⟹
     Legendre p q = Legendre q p for q
  proof −
    assume q ∈ prime-factors d'
    have prime q using ‹q ∈ prime-factors d'› by blast
    have q > 2 using d'-prime-factors-gt-2 ‹q ∈ prime-factors d'› by blast
    show Legendre p q = Legendre q p
      using ‹prime p› ‹p > 2› ‹p mod 4 = 1›
          ‹prime q› ‹q > 2› Legendre-equal[of p q]
      unfolding unique-euclidean-semiring-class.cong-def
      using zmod-int[of p 4]
      by auto
  qed

  have Legendre (− d') p = Legendre (− 1) p * Legendre d' p
    using ‹prime p› Legendre-mult[of p − 1 d'] by auto
  also have ... = Legendre d' p
    using ‹prime p› ‹p > 2› ‹p mod 4 = 1› Legendre-minus-one[of p]
    unfolding unique-euclidean-semiring-class.cong-def nat-mod-as-int
    by auto
  also have ... = (∏ q∈prime-factors d'. Legendre (q ^ multiplicity q d') p)
```

```
    apply (subst d'-expansion)
    using ‹prime p› ‹p > 2› Legendre-prod[of p] apply auto
    done
  also have ... = (∏ q∈prime-factors d'. (Legendre q p) ^ multiplicity q d')
    using ‹prime p› ‹p > 2› Legendre-power by auto
  also have ... = (∏ q∈prime-factors d'. (Legendre p q) ^ multiplicity q d')
    using d'-prime-factors-Legendre by auto
  also have ... = (∏ q∈prime-factors d'.
                    (Legendre (- 1) q) ^ multiplicity q d')
    apply (rule prod.cong[OF refl])
    using d'-prime-factors-2-p-mod apply (metis Legendre-cong)
    done
  also have ... = (∏ q∈prime-factors d'.
                    (if [q = 1] (mod 4) then 1 else - 1) ^ multiplicity q d')
    apply (rule prod.cong[OF refl])
    subgoal for q
      using Legendre-minus-one-alt[of q] d'-prime-factors-gt-2[of q]
      by (smt (verit) cong-int-iff in-prime-factors-iff int-eq-iff-numeral
                      less-imp-of-nat-less numeral-Bit0 numeral-One of-nat-1
                      prime-nat-int-transfer)
    done
  also have ... = (∏ q∈{q ∈ prime-factors d'. [q = 3] (mod 4)}.
                    (if [q = 1] (mod 4) then 1 else - 1) ^ multiplicity q d')
    apply (rule prod.mono-neutral-right)
    subgoal by blast
    subgoal by blast
    subgoal
      unfolding unique-euclidean-semiring-class.cong-def
      apply (rule ballI)
      using d'-prime-factors-odd apply simp
      apply (metis One-nat-def dvd-0-right even-even-mod-4-iff
                   even-numeral mod-exhaust-less-4)
      done
    done
  also have ... = (∏ q∈{q ∈ prime-factors d'. [q = 3] (mod 4)}.
                    (- 1) ^ multiplicity q d')
    by (rule prod.cong[OF refl];
        simp add: unique-euclidean-semiring-class.cong-def)
  also have ... = 1 using prod-prime-factors-minus-one .
  finally show QuadRes p (- d')
    unfolding Legendre-def
    by (metis one-neq-neg-one one-neq-zero)
 qed
 thus ∃ x1 x2 x3. n = x1^2 + x2^2 + x3^2
   using ‹n > 1› three-squares-using-quadratic-residue[of n d']
   unfolding d'-def p-def
   by auto
qed
```

```
lemma three-mod-eight-power-iff:
  fixes n :: nat
  shows (3 :: int) ^ n mod 8 = (if even n then 1 else 3)
proof (induction n)
  case 0
  thus ?case by auto
next
  case (Suc n)
  thus ?case
    apply (cases even n)
    subgoal
      using mod-mult-left-eq[of 3 8 3 ^ n] apply simp
      apply presburger
      done
    subgoal
      using mod-mult-left-eq[of 3 8 3 * 3 ^ n] apply simp
      apply presburger
      done
    done
qed
```

Lemma 1.9 from [1].

```
lemma three-squares-using-mod-eight:
  fixes n :: nat
  assumes n mod 8 ∈ {1, 3, 5}
  shows ∃ x1 x2 x3. n = x1^2 + x2^2 + x3^2
proof cases
  assume n = 1
  hence n = 1^2 + 0^2 + 0^2 unfolding power2-eq-square by auto
  thus ∃ x1 x2 x3. n = x1^2 + x2^2 + x3^2 by blast
next
  assume n ≠ 1
  hence n > 1 using assms by auto

  have H-n:
    (n mod 8 = 1 ⟹ P) ⟹
     (n mod 8 = 3 ⟹ P) ⟹
     (n mod 8 = 5 ⟹ P) ⟹ P for P
    using assms by auto

  define c :: nat where c ≡ if n mod 8 = 3 then 1 else 3
  have c * n ≥ 1 unfolding c-def using ‹n > 1› by auto

  obtain k where H-k: 2 * k = c * n - 1
    using H-n
    unfolding c-def
    by (smt (verit, ccfv-threshold) dvd-mod even-mult-iff even-numeral
                                    odd-numeral odd-one odd-two-times-div-two-nat)
  have k-mod-4: k mod 4 = (if n mod 8 = 5 then 3 else 1) (is k mod 4 = ?v)
```

```
proof −
  have c * n mod 8 = (if n mod 8 = 5 then 7 else 3)
  using H-n
  proof cases
    assume n mod 8 = 1
    have 3 * n mod 8 = 3
      using ‹n mod 8 = 1› mod-mult-right-eq[of 3 n 8]
      by auto
    thus ?thesis unfolding c-def using ‹n mod 8 = 1› by auto
  next
    assume n mod 8 = 3
    thus ?thesis unfolding c-def by auto
  next
    assume n mod 8 = 5
    have 3 * n mod 8 = 7
      using ‹n mod 8 = 5› mod-mult-right-eq[of 3 n 8]
      by auto
    thus ?thesis unfolding c-def using ‹n mod 8 = 5› by auto
  qed
  hence 2 * k mod 8 = (if n mod 8 = 5 then 6 else 2)
    unfolding H-k using ‹c * n ≥ 1› mod-diff-eq-nat by simp
  hence 2 * (k mod 4) = 2 * ?v by (simp add: mult-mod-right)
  thus ?thesis by simp
qed

have coprime k (4 * n)
  using k-mod-4 H-k ‹c * n ≥ 1›
  by (metis One-nat-def coprime-Suc-left-nat coprime-commute
        coprime-diff-one-right-nat coprime-mod-left-iff
        coprime-mult-right-iff mult-2-right numeral-2-eq-2 numeral-3-eq-3
        numeral-Bit0 order-less-le-trans zero-less-one zero-neq-numeral)
then obtain j where H-j:
    prime (k + (4 * n) * j) ∧ j ≠ 0
  using prime-linear-combination[of k n − 1] ‹n > 1›
  by (metis One-nat-def Suc-leI bot-nat-0.not-eq-extremum mult-is-0
        nat-1-eq-mult-iff nat-less-le prime-linear-combination
        zero-neq-numeral)
have j > 0 using H-j by blast

define p where p ≡ k + (4 * n) * j
have prime p
  unfolding p-def
  using conjunct1[OF H-j] apply (rule back-subst[of prime])
  apply (simp add: int-ops nat-plus-as-int)
  done
have [p = k] (mod 4 * n)
  unfolding p-def unique-euclidean-semiring-class.cong-def by auto

have p > 2
```

```
  using ‹prime p› ‹[p = k] (mod 4 * n)› ‹coprime k (4 * n)›
  by (metis cong-dvd-iff cong-dvd-modulus-nat coprime-common-divisor-nat
        dvd-mult2 even-numeral le-neq-implies-less odd-one prime-ge-2-nat)

define d' where d' ≡ 8 * j + c
have d' > 1 unfolding d'-def using ‹j > 0› by simp
have H-2-p: 2 * p = d' * n - 1
  unfolding p-def d'-def
  using ‹c * n ≥ 1› H-k
  by (smt (verit, del-insts) Nat.add-diff-assoc add.commute
                        add-mult-distrib mult.commute mult-2 numeral-Bit0)

have QuadRes p (- d')
proof -
  have d'-expansion: d' = (∏ q∈prime-factors d'. q ^ multiplicity q d')
    using ‹j > 0› prime-factorization-nat unfolding d'-def by auto

  have odd d' unfolding c-def d'-def by simp
  hence d'-prime-factors-odd: q ∈ prime-factors d' ⟹ odd q for q
    by fastforce

  have d'-prime-factors-gt-2: q ∈ prime-factors d' ⟹ q > 2 for q
    using d'-prime-factors-odd
    by (metis even-numeral in-prime-factors-imp-prime
          order-less-le prime-ge-2-nat)

  have [2 * p = - 1] (mod d')
    using ‹n > 1› ‹d' > 1›
    unfolding H-2-p cong-iff-dvd-diff
    by (simp add: int-ops less-1-mult order-less-imp-not-less)
  hence d'-prime-factors-2-p-mod:
     q ∈ prime-factors d' ⟹ [2 * p = - 1] (mod q) for q
    by (rule cong-dvd-modulus; auto)

  have q ∈ prime-factors d' ⟹ coprime (2 * int p) q for q
    using d'-prime-factors-2-p-mod
    by (metis cong-imp-coprime cong-sym coprime-1-left
          coprime-minus-left-iff mult-2 of-nat-add)
  hence d'-prime-factors-coprime:
     q ∈ prime-factors d' ⟹ coprime (int p) q for q
    using d'-expansion by auto

  have Legendre-using-quadratic-reciprocity:
    Legendre (- d') p =
     (∏ q∈prime-factors d'. (Legendre p q) ^ multiplicity q d')
  proof cases
    assume n mod 8 ∈ {1, 3}

    have k mod 4 = 1 using ‹n mod 8 ∈ {1, 3}› k-mod-4 by auto
```

```
  hence p mod 4 = 1
    using ‹[p = k] (mod 4 * n)›
    by (metis unique-euclidean-semiring-class.cong-def cong-modulus-mult-nat)
  hence [int p = 1] (mod 4)
    by (metis cong-mod-left cong-refl int-ops(2) int-ops(3) of-nat-mod)

  have d'-prime-factors-Legendre:
    q ∈ prime-factors d' ⟹
     Legendre p q = Legendre q p for q
  proof −
    assume q ∈ prime-factors d'
    have prime q using ‹q ∈ prime-factors d'› by blast
    have q > 2 using d'-prime-factors-gt-2 ‹q ∈ prime-factors d'› by blast
    show Legendre p q = Legendre q p
      using ‹prime p› ‹p > 2› ‹[int p = 1] (mod 4)›
          ‹prime q› ‹q > 2› Legendre-equal[of p q]
      by auto
  qed

  have Legendre (− d') p = Legendre (− 1) p * Legendre d' p
    using ‹prime p› Legendre-mult[of p − 1 d'] by auto
  also have ... = Legendre d' p
    using ‹prime p› ‹p > 2› ‹[int p = 1] (mod 4)› Legendre-minus-one
    by auto
  also have ... = (∏ q∈prime-factors d'. Legendre (q ^ multiplicity q d') p)
    apply (subst d'-expansion)
    using ‹prime p› ‹p > 2› Legendre-prod[of p] apply auto
    done
  also have ... = (∏ q∈prime-factors d'. (Legendre q p) ^ multiplicity q d')
    using ‹prime p› ‹p > 2› Legendre-power by auto
  also have ... = (∏ q∈prime-factors d'. (Legendre p q) ^ multiplicity q d')
    using d'-prime-factors-Legendre by auto
  finally show ?thesis .
next
  assume n mod 8 ∉ {1, 3}
  hence n mod 8 = 5 using assms by auto

  have [p = 3] (mod 4)
    using ‹n mod 8 = 5› k-mod-4 ‹[p = k] (mod 4 * n)›
    by (metis cong-mod-right cong-modulus-mult-nat)
  have [d' = 3] (mod 8)
    using ‹n mod 8 = 5›
    unfolding d'-def c-def cong-iff-dvd-diff
    by (simp add: unique-euclidean-semiring-class.cong-def)
  have [d' = − 1] (mod 4)
    using ‹[d' = 3] (mod 8)› cong-modulus-mult[of d' 3 4 2]
    unfolding unique-euclidean-semiring-class.cong-def nat-mod-as-int
    by auto
```

have *d′-prime-factors-cases*:
  $q \in$ *prime-factors* $d' \Longrightarrow$
   multiplicity $q\ d' = 0 \vee [q = 1]\ (mod\ 4) \vee [q = 3]\ (mod\ 4)$ **for** q
proof −
  assume $q \in$ *prime-factors* d'
  consider $[q = 0]\ (mod\ 4)$
        | $[q = 1]\ (mod\ 4)$
        | $[q = 2]\ (mod\ 4)$
        | $[q = 3]\ (mod\ 4)$
    unfolding *unique-euclidean-semiring-class.cong-def* **by** (*simp*; *linarith*)
  thus *multiplicity* $q\ d' = 0 \vee [q = 1]\ (mod\ 4) \vee [q = 3]\ (mod\ 4)$
  proof *cases*
    assume $[q = 0]\ (mod\ 4)$
    hence *False*
      using *d′-prime-factors-odd* ‹$q \in$ *prime-factors* d'›
      by (*meson cong-0-iff dvd-trans even-numeral*)
    thus *multiplicity* $q\ d' = 0 \vee [q = 1]\ (mod\ 4) \vee [q = 3]\ (mod\ 4)$
      by *blast*
  next
    assume $[q = 1]\ (mod\ 4)$
    thus *multiplicity* $q\ d' = 0 \vee [q = 1]\ (mod\ 4) \vee [q = 3]\ (mod\ 4)$
      by *blast*
  next
    assume $[q = 2]\ (mod\ 4)$
    hence $q = 2$
      using *d′-prime-factors-odd* ‹$q \in$ *prime-factors* d'›
      by (*metis unique-euclidean-semiring-class.cong-def*
          dvd-mod-iff even-numeral)
    thus *multiplicity* $q\ d' = 0 \vee [q = 1]\ (mod\ 4) \vee [q = 3]\ (mod\ 4)$
      by (*simp add*: ‹*odd* d'› *not-dvd-imp-multiplicity-0*)
  next
    assume $[q = 3]\ (mod\ 4)$
    thus *multiplicity* $q\ d' = 0 \vee [q = 1]\ (mod\ 4) \vee [q = 3]\ (mod\ 4)$
      by *blast*
  qed
qed

have $d' = (\prod q{\in}\{q{\in}$*prime-factors* d'*. True*$\}$*.* $q \mathbin{\hat{}}$ *multiplicity* $q\ d')$
  using *d′-expansion* **by** *auto*
also have ... $= (\prod q{\in}\{q{\in}$*prime-factors* d'*.*
        multiplicity $q\ d' = 0 \vee [q = 1]\ (mod\ 4) \vee [q = 3]\ (mod\ 4)\}$*.*
        $q \mathbin{\hat{}}$ *multiplicity* $q\ d')$
  using *d′-prime-factors-cases* **by** *meson*
also have ... $= (\prod q{\in}\{q{\in}$*prime-factors* d'*.* *multiplicity* $q\ d' = 0\} \cup$
        $\{q{\in}$*prime-factors* d'*.* $[q = 1]\ (mod\ 4) \vee$
        $[q = 3]\ (mod\ 4)\}$*.* $q \mathbin{\hat{}}$ *multiplicity* $q\ d')$
  by (*rule prod.cong*; *blast*)
also have ... $= (\prod q{\in}\{q{\in}$*prime-factors* d'*.* $[q = 1]\ (mod\ 4) \vee$
        $[q = 3]\ (mod\ 4)\}$*.* $q \mathbin{\hat{}}$ *multiplicity* $q\ d')$

by (*rule prod.mono-neutral-left*[*symmetric*]; *auto*)
also have ... = ($\prod$ *q*∈{*q*∈*prime-factors d'*. [*q* = *1*] (*mod 4*)} ∪
{*q*∈*prime-factors d'*. [*q* = *3*] (*mod 4*)}.
q ^ *multiplicity q d'*)
by (*rule prod.cong*; *blast*)
also have ... = ($\prod$ *q*∈{*q*∈*prime-factors d'*. [*q* = *1*] (*mod 4*)}.
q ^ *multiplicity q d'*) ∗
($\prod$ *q*∈{*q*∈*prime-factors d'*. [*q* = *3*] (*mod 4*)}.
q ^ *multiplicity q d'*)
by (*rule prod.union-disjoint*;
auto simp add: *unique-euclidean-semiring-class.cong-def*)
finally have *d'-expansion-mod-4*:
d' = ($\prod$ *q*∈{*q*∈*prime-factors d'*. [*q* = *1*] (*mod 4*)}.
q ^ *multiplicity q d'*) ∗
($\prod$ *q*∈{*q*∈*prime-factors d'*. [*q* = *3*] (*mod 4*)}.
q ^ *multiplicity q d'*) **.**

have *int d' mod 4* = *int* (($\prod$ *q*∈{*q*∈*prime-factors d'*. [*q* = *1*] (*mod 4*)}.
q ^ *multiplicity q d'*) ∗
($\prod$ *q*∈{*q*∈*prime-factors d'*. [*q* = *3*] (*mod 4*)}.
q ^ *multiplicity q d'*) *mod 4*)
using *d'-expansion-mod-4*
by *presburger*
also have ... = (($\prod$ *q*∈{*q*∈*prime-factors d'*. [*q* = *1*] (*mod 4*)}.
(*q mod 4*) ^ *multiplicity q d'*) *mod 4*) *mod 4* ∗
(($\prod$ *q*∈{*q*∈*prime-factors d'*. [*q* = *3*] (*mod 4*)}.
(*q mod 4*) ^ *multiplicity q d'*) *mod 4*) *mod 4*) *mod 4*
unfolding *mod-mult-eq mod-prod-eq power-mod* **..**
also have ... = *int* ((($\prod$ *q*∈{*q*∈*prime-factors d'*. [*q* = *1*] (*mod 4*)}.
(*1* ^ *multiplicity q d'*) *mod 4*) *mod 4* ∗
(($\prod$ *q*∈{*q*∈*prime-factors d'*. [*q* = *3*] (*mod 4*)}.
(*3* ^ *multiplicity q d'*) *mod 4*) *mod 4*) *mod 4*)
unfolding *unique-euclidean-semiring-class.cong-def* **by** *auto*
also have ... = (($\prod$ *q*∈{*q*∈*prime-factors d'*. [*q* = *3*] (*mod 4*)}.
(((*int 3*) *mod 4*) ^ *multiplicity q d'*) *mod 4*) *mod 4*) *mod 4*
by (*simp add*: *int-ops*)
also have ... = (($\prod$ *q*∈{*q*∈*prime-factors d'*. [*q* = *3*] (*mod 4*)}.
(((− *1*) *mod 4*) ^ *multiplicity q d'*) *mod 4*) *mod 4*) *mod 4*
by *auto*
also have ... = ($\prod$ *q*∈{*q*∈*prime-factors d'*. [*q* = *3*] (*mod 4*)}.
((− *1*) ^ *multiplicity q d'*)) *mod 4*
unfolding *power-mod mod-prod-eq mod-mod-trivial* **..**
finally have [*d'* = $\prod$ *q*∈{*q*∈*prime-factors d'*. [*q* = *3*] (*mod 4*)}.
((− *1*) ^ *multiplicity q d'*)] (*mod 4*)
unfolding *unique-euclidean-semiring-class.cong-def* **.**
hence [$\prod$ *q*∈{*q*∈*prime-factors d'*. [*q* = *3*] (*mod 4*)}.
((− *1*) ^ *multiplicity q d'*) = − *1* :: *int*] (*mod 4*)
using ‹[*d'* = − *1*] (*mod 4*)›
unfolding *unique-euclidean-semiring-class.cong-def*

```
  by argo
hence prod-d'-prime-factors-q-3-mod-4-minus-one:
    (∏ q∈{q∈prime-factors d'. [q = 3] (mod 4)}.
      ((− 1) ^ multiplicity q d')) = (− 1 :: int)
  unfolding power-sum[symmetric]
  unfolding minus-one-power-iff unique-euclidean-semiring-class.cong-def
  by auto

have d'-prime-factors-q-1-mod-4-Legendre:
  q ∈ prime-factors d' ⟹
   [q = 1] (mod 4) ⟹
   Legendre p q = Legendre q p for q
proof −
  assume q ∈ prime-factors d'
  assume [q = 1] (mod 4)
  have prime q using ‹q ∈ prime-factors d'› by blast
  have q > 2 using d'-prime-factors-gt-2 ‹q ∈ prime-factors d'› by blast
  show Legendre p q = Legendre q p
    using ‹prime p› ‹p > 2› ‹[p = 3] (mod 4)› ‹[q = 1] (mod 4)›
        ‹prime q› ‹q > 2› Legendre-equal[of p q]
    unfolding unique-euclidean-semiring-class.cong-def
    using zmod-int[of q 4]
    by auto
qed

have d'-prime-factors-q-3-mod-4-Legendre:
  q ∈ prime-factors d' ⟹
   [q = 3] (mod 4) ⟹
   Legendre p q = − Legendre q p for q
proof −
  assume q ∈ prime-factors d'
  assume [q = 3] (mod 4)
  have prime q using ‹q ∈ prime-factors d'› by blast
  have q > 2 using d'-prime-factors-gt-2 ‹q ∈ prime-factors d'› by blast
  have p ≠ q
    using d'-prime-factors-coprime[of q] ‹q ∈ prime-factors d'› ‹prime p›
    by auto
  show Legendre p q = − Legendre q p
    using ‹prime p› ‹p > 2› ‹[p = 3] (mod 4)› ‹[q = 3] (mod 4)›
        ‹prime q› ‹q > 2› ‹p ≠ q› Legendre-opposite[of p q]
    unfolding unique-euclidean-semiring-class.cong-def
    using zmod-int[of p 4] zmod-int[of q 4]
    by fastforce
qed

have d'-prime-factors-q-0-2-mod-4:
    q ∈ prime-factors d' ⟹
     ([q = 0] (mod 4) ∨ [q = 2] (mod 4)) ⟹
     Legendre p q = 1 for q
```

unfolding *unique-euclidean-semiring-class.cong-def*
using *d′-prime-factors-odd mod-mod-cancel*[*of 2 4 q*]
by *fastforce*

have *Legendre* (− *d′*) *p* = *Legendre* (− *1*) *p* ∗ *Legendre d′ p*
using ‹*prime p*› *Legendre-mult*[*of p* − *1 d′*] **by** *auto*
also have ... = − *Legendre d′ p*
using ‹*prime p*› ‹*p* > *2*› ‹[*p* = *3*] (*mod 4*)› *Legendre-minus-one*[*of p*]
unfolding *unique-euclidean-semiring-class.cong-def nat-mod-as-int*
by (*auto simp add*: *cong-0-iff Legendre-def*)
also have ... = − (∏ *q*∈{*q*∈*prime-factors d′*. [*q* = *1*] (*mod 4*)}.
(*Legendre q p*) ^ *multiplicity q d′*) ∗
(∏ *q*∈{*q*∈*prime-factors d′*. [*q* = *3*] (*mod 4*)}.
(*Legendre q p*) ^ *multiplicity q d′*)
apply (*subst d′-expansion-mod-4*)
using ‹*prime p*› ‹*p* > *2*› *Legendre-mult*[*of p*]
Legendre-prod[*of p* λ*q*. *q* ^ *multiplicity q d′*] *Legendre-power*[*of p*]
apply *simp*
done
also have ... = − (∏ *q*∈{*q*∈*prime-factors d′*. [*q* = *1*] (*mod 4*)}.
(*Legendre p q*) ^ *multiplicity q d′*) ∗
(∏ *q*∈{*q*∈*prime-factors d′*. [*q* = *3*] (*mod 4*)}.
(− *Legendre p q*) ^ *multiplicity q d′*)
using *d′-prime-factors-q-1-mod-4-Legendre*
d′-prime-factors-q-3-mod-4-Legendre
by *auto*
also have ... = − (∏ *q*∈{*q*∈*prime-factors d′*. [*q* = *1*] (*mod 4*)}.
(*Legendre p q*) ^ *multiplicity q d′*) ∗
(∏ *q*∈{*q*∈*prime-factors d′*. [*q* = *3*] (*mod 4*)}.
(*Legendre p q* ∗ (− *1*)) ^ *multiplicity q d′*)
by *auto*
also have ... = − (∏ *q*∈{*q*∈*prime-factors d′*. [*q* = *1*] (*mod 4*)}.
(*Legendre p q*) ^ *multiplicity q d′*) ∗
(∏ *q*∈{*q*∈*prime-factors d′*. [*q* = *3*] (*mod 4*)}.
(*Legendre p q*) ^ *multiplicity q d′*) ∗
(∏ *q*∈{*q*∈*prime-factors d′*. [*q* = *3*] (*mod 4*)}.
(− *1*) ^ *multiplicity q d′*)
unfolding *power-mult-distrib prod.distrib* **by** *auto*
also have ... = (∏ *q*∈{*q*∈*prime-factors d′*. [*q* = *1*] (*mod 4*)}.
(*Legendre p q*) ^ *multiplicity q d′*) ∗
(∏ *q*∈{*q*∈*prime-factors d′*. [*q* = *3*] (*mod 4*)}.
(*Legendre p q*) ^ *multiplicity q d′*)
unfolding *prod-d′-prime-factors-q-3-mod-4-minus-one* **by** *auto*
also have ... = (∏ *q*∈{*q*∈*prime-factors d′*. [*q* = *0*] (*mod 4*)}.
(*Legendre p q*) ^ *multiplicity q d′*) ∗
(∏ *q*∈{*q*∈*prime-factors d′*. [*q* = *1*] (*mod 4*)}.
(*Legendre p q*) ^ *multiplicity q d′*) ∗
(∏ *q*∈{*q*∈*prime-factors d′*. [*q* = *2*] (*mod 4*)}.
(*Legendre p q*) ^ *multiplicity q d′*) ∗

$(\prod q{\in}\{q{\in}\textit{prime-factors } d'.\ [q = 3]\ (\textit{mod } 4)\}.$
$(\textit{Legendre } p\ q)\ \widehat{}\ \textit{multiplicity } q\ d')$
using *d'-prime-factors-q-0-2-mod-4* **by** *auto*
also have ... $= (\prod q{\in}\{q{\in}\textit{prime-factors } d'.\ [q = 0]\ (\textit{mod } 4)\} \cup$
$\{q{\in}\textit{prime-factors } d'.\ [q = 1]\ (\textit{mod } 4)\}.$
$(\textit{Legendre } p\ q)\ \widehat{}\ \textit{multiplicity } q\ d') *$
$(\prod q{\in}\{q{\in}\textit{prime-factors } d'.\ [q = 2]\ (\textit{mod } 4)\} \cup$
$\{q{\in}\textit{prime-factors } d'.\ [q = 3]\ (\textit{mod } 4)\}.$
$(\textit{Legendre } p\ q)\ \widehat{}\ \textit{multiplicity } q\ d')$
using *prod.union-disjoint*[*of* $\{q{\in}\textit{prime-factors } d'.\ [q = 0]\ (\textit{mod } 4)\}$
$\{q{\in}\textit{prime-factors } d'.\ [q = 1]\ (\textit{mod } 4)\}$
$\lambda q.\ (\textit{Legendre } p\ (\textit{int } q))\ \widehat{}$
$\textit{multiplicity } q\ d'$]
prod.union-disjoint[*of* $\{q{\in}\textit{prime-factors } d'.\ [q = 2]\ (\textit{mod } 4)\}$
$\{q{\in}\textit{prime-factors } d'.\ [q = 3]\ (\textit{mod } 4)\}$
$\lambda q.\ (\textit{Legendre } p\ (\textit{int } q))\ \widehat{}$
$\textit{multiplicity } q\ d'$]
by (*fastforce simp add*: *unique-euclidean-semiring-class.cong-def*)
also have ... $= (\prod q{\in}(\{q{\in}\textit{prime-factors } d'.\ [q = 0]\ (\textit{mod } 4)\} \cup$
$\{q{\in}\textit{prime-factors } d'.\ [q = 1]\ (\textit{mod } 4)\}) \cup$
$(\{q{\in}\textit{prime-factors } d'.\ [q = 2]\ (\textit{mod } 4)\} \cup$
$\{q{\in}\textit{prime-factors } d'.\ [q = 3]\ (\textit{mod } 4)\}).$
$(\textit{Legendre } p\ q)\ \widehat{}\ \textit{multiplicity } q\ d')$
by (*rule prod.union-disjoint*[*symmetric*];
auto simp add: *unique-euclidean-semiring-class.cong-def*)
also have ... $= (\prod q{\in}\{q{\in}\textit{prime-factors } d'.$
$[q = 0]\ (\textit{mod } 4) \vee [q = 1]\ (\textit{mod } 4) \vee$
$[q = 2]\ (\textit{mod } 4) \vee [q = 3]\ (\textit{mod } 4)\}.$
$(\textit{Legendre } p\ q)\ \widehat{}\ \textit{multiplicity } q\ d')$
by (*rule prod.cong*; *auto*)
also have ... $= (\prod q{\in}\textit{prime-factors } d'.\ (\textit{Legendre } p\ q)\ \widehat{}\ \textit{multiplicity } q\ d')$
by (*rule prod.cong*;
auto simp add: *unique-euclidean-semiring-class.cong-def*)
finally show *?thesis* **.**
qed

have $q \in \textit{prime-factors } d' \Longrightarrow \textit{Legendre } 4\ q = 1$ **for** q
proof −
assume $q \in \textit{prime-factors } d'$
have $q\ \textit{dvd } 4 \Longrightarrow q \leq 4$ **by** (*simp add*: *dvd-imp-le*)
hence $q\ \textit{dvd } 4 \Longrightarrow q \in \{0, 1, 2, 3, 4\}$ **by** *auto*
hence $q\ \textit{dvd } 4 \Longrightarrow q \in \{1, 2, 4\}$ **by** *auto*
hence $\neg\ q\ \textit{dvd } 4$ **using** ‹$q \in \textit{prime-factors } d'$› *d'-prime-factors-odd*[*of q*]
by (*metis empty-iff even-numeral in-prime-factors-imp-prime*
insert-iff not-prime-1)
hence $\neg\ \textit{int } q\ \textit{dvd } 4$ **by** *presburger*
thus *Legendre 4 q = 1*
unfolding *Legendre-def QuadRes-def cong-0-iff power2-eq-square*
by (*metis cong-refl mult-2 numeral-Bit0*)

qed
hence *Legendre* (− *d′*) *p* = (∏ *q*∈*prime-factors d′*.
(*Legendre* (*2* ∗ *2*) *q* ∗ *Legendre p q*) ^ *multiplicity q d′*)
using *Legendre-using-quadratic-reciprocity* **by** *auto*
also have ... = (∏ *q*∈*prime-factors d′*.
(*Legendre 2 q* ∗ *Legendre* (*2* ∗ *p*) *q*) ^ *multiplicity q d′*)
apply (*rule prod.cong*[*OF refl*])
using *d′-prime-factors-gt-2 Legendre-mult in-prime-factors-imp-prime*
by (*metis int-ops*(*7*) *of-nat-numeral prime-nat-int-transfer mult.assoc*)
also have ... = (∏ *q*∈*prime-factors d′*.
(*Legendre 2 q* ∗ *Legendre* (− *1*) *q*) ^ *multiplicity q d′*)
apply (*rule prod.cong*[*OF refl*])
using *d′-prime-factors-2-p-mod Legendre-cong*
unfolding *unique-euclidean-semiring-class.cong-def*
apply *metis*
done
also have ... = (∏ *q*∈*prime-factors d′*.
((*if* [*q* = *1*] (*mod 8*) ∨ [*q* = *7*] (*mod 8*) *then 1 else* − *1*) ∗
(*if* [*q* = *1*] (*mod 4*) *then 1 else* − *1*)) ^ *multiplicity q d′*)
apply (*rule prod.cong*[*OF refl*])
subgoal for *q*
apply (*rule arg-cong2*[*of - - - - λa b*. (*a* ∗ *b*) ^ *multiplicity q d′*])
subgoal
using *Legendre-two-alt*[*of q*] *d′-prime-factors-gt-2*[*of q*]
unfolding *unique-euclidean-semiring-class.cong-def nat-mod-as-int*
by *force*
subgoal
using *Legendre-minus-one-alt*[*of q*] *d′-prime-factors-gt-2*[*of q*]
unfolding *unique-euclidean-semiring-class.cong-def nat-mod-as-int*
by *force*
done
done
also have ... = (∏ *q*∈*prime-factors d′*.
((*if* [*q* = *5*] (*mod 8*) ∨ [*q* = *7*] (*mod 8*) *then* − *1 else 1*)) ^
multiplicity q d′)
apply (*rule prod.cong*)
subgoal by *blast*
subgoal for *q*
apply (*rule arg-cong*[*of - - λa*. *a* ^ *multiplicity q d′*])
unfolding *unique-euclidean-semiring-class.cong-def* **apply** (*simp*; *presburger*)
done
done
also have ... = (∏ *q*∈*prime-factors d′*.
(*if* [*q* = *5*] (*mod 8*) ∨ [*q* = *7*] (*mod 8*)
then (− *1*) ^ *multiplicity q d′ else 1*))
by (*rule prod.cong*; *auto*)
also have ... = (∏ *q*∈{*q*∈*prime-factors d′*. [*q* = *5*] (*mod 8*) ∨ [*q* = *7*] (*mod 8*)}.

```
                (− 1) ^ multiplicity q d′)
  using prod.inter-filter[symmetric] by fast
also have ... = (− 1) ^ (∑ q∈{q∈prime-factors d′.
                              [q = 5] (mod 8) ∨ [q = 7] (mod 8)}.
                          multiplicity q d′)
  by (simp add: power-sum)
finally have Legendre-using-sum:
  Legendre (− d′) p =
    (− 1) ^ (∑ q∈{q∈prime-factors d′. [q = 5] (mod 8) ∨ [q = 7] (mod 8)}.
          multiplicity q d′) .

have [∑ q∈{q∈prime-factors d′. [q = 5] (mod 8) ∨ [q = 7] (mod 8)}.
      multiplicity q d′ = 0] (mod 2)
proof −
  have d′ = (∏ q∈{q ∈ prime-factors d′.
                [q = 1] (mod 8) ∨ [q = 3] (mod 8) ∨
                [q = 5] (mod 8) ∨ [q = 7] (mod 8)}. q ^ multiplicity q d′)
    apply (subst d′-expansion)
    apply (rule prod.cong)
    subgoal
      apply (rule Set.set-eqI)
      subgoal for q
        apply (rule iffI)
        subgoal
          using d′-prime-factors-odd[of q]
          unfolding unique-euclidean-semiring-class.cong-def
          apply simp
          apply presburger
          done
        subgoal by blast
        done
      done
    subgoal by blast
    done
  also have ... = (∏ q∈({q ∈ prime-factors d′. [q = 1] (mod 8)} ∪
                        {q ∈ prime-factors d′. [q = 3] (mod 8)}) ∪
                       ({q ∈ prime-factors d′. [q = 5] (mod 8)} ∪
                        {q ∈ prime-factors d′. [q = 7] (mod 8)}).
                   q ^ multiplicity q d′)
    by (rule prod.cong; auto)
  also have ... = (∏ q∈({q ∈ prime-factors d′. [q = 1] (mod 8)} ∪
                        {q ∈ prime-factors d′. [q = 3] (mod 8)}).
                   q ^ multiplicity q d′) *
                  (∏ q∈({q ∈ prime-factors d′. [q = 5] (mod 8)} ∪
                        {q ∈ prime-factors d′. [q = 7] (mod 8)}).
                   q ^ multiplicity q d′)
    by (rule prod.union-disjoint;
        force simp add: unique-euclidean-semiring-class.cong-def)
  also have ... = (∏ q∈{q ∈ prime-factors d′. [q = 1] (mod 8)}.
```

$q \mathbin{\hat{}} \mathit{multiplicity}\ q\ d') *$
$(\prod q{\in}\{q \in \mathit{prime\text{-}factors}\ d'.\ [q = 3]\ (\mathit{mod}\ 8)\}.$
$q \mathbin{\hat{}} \mathit{multiplicity}\ q\ d') *$
$(\prod q{\in}\{q \in \mathit{prime\text{-}factors}\ d'.\ [q = 5]\ (\mathit{mod}\ 8)\}.$
$q \mathbin{\hat{}} \mathit{multiplicity}\ q\ d') *$
$(\prod q{\in}\{q \in \mathit{prime\text{-}factors}\ d'.\ [q = 7]\ (\mathit{mod}\ 8)\}.$
$q \mathbin{\hat{}} \mathit{multiplicity}\ q\ d')$
using *prod.union-disjoint*[*of* $\{q{\in}\mathit{prime\text{-}factors}\ d'.\ [q = 1]\ (\mathit{mod}\ 8)\}$
$\{q{\in}\mathit{prime\text{-}factors}\ d'.\ [q = 3]\ (\mathit{mod}\ 8)\}$
$\lambda q.\ q \mathbin{\hat{}} \mathit{multiplicity}\ q\ d'$]
prod.union-disjoint[*of* $\{q{\in}\mathit{prime\text{-}factors}\ d'.\ [q = 5]\ (\mathit{mod}\ 8)\}$
$\{q{\in}\mathit{prime\text{-}factors}\ d'.\ [q = 7]\ (\mathit{mod}\ 8)\}$
$\lambda q.\ q \mathbin{\hat{}} \mathit{multiplicity}\ q\ d'$]
by (*force simp add*: *unique-euclidean-semiring-class.cong-def*)
finally have *int* $(d'\ \mathit{mod}\ 8) = (\prod q{\in}\{q \in \mathit{prime\text{-}factors}\ d'.\ [q = 1]\ (\mathit{mod}\ 8)\}.$
$q \mathbin{\hat{}} \mathit{multiplicity}\ q\ d') *$
$(\prod q{\in}\{q \in \mathit{prime\text{-}factors}\ d'.\ [q = 3]\ (\mathit{mod}\ 8)\}.$
$q \mathbin{\hat{}} \mathit{multiplicity}\ q\ d') *$
$(\prod q{\in}\{q \in \mathit{prime\text{-}factors}\ d'.\ [q = 5]\ (\mathit{mod}\ 8)\}.$
$q \mathbin{\hat{}} \mathit{multiplicity}\ q\ d') *$
$(\prod q{\in}\{q \in \mathit{prime\text{-}factors}\ d'.\ [q = 7]\ (\mathit{mod}\ 8)\}.$
$q \mathbin{\hat{}} \mathit{multiplicity}\ q\ d')\ \mathit{mod}\ 8$
by *auto*
also have $\ldots = ((\prod q{\in}\{q \in \mathit{prime\text{-}factors}\ d'.\ [q = 1]\ (\mathit{mod}\ 8)\}.$
$q \mathbin{\hat{}} \mathit{multiplicity}\ q\ d')\ \mathit{mod}\ 8) *$
$((\prod q{\in}\{q \in \mathit{prime\text{-}factors}\ d'.\ [q = 3]\ (\mathit{mod}\ 8)\}.$
$q \mathbin{\hat{}} \mathit{multiplicity}\ q\ d')\ \mathit{mod}\ 8) *$
$((\prod q{\in}\{q \in \mathit{prime\text{-}factors}\ d'.\ [q = 5]\ (\mathit{mod}\ 8)\}.$
$q \mathbin{\hat{}} \mathit{multiplicity}\ q\ d')\ \mathit{mod}\ 8) *$
$((\prod q{\in}\{q \in \mathit{prime\text{-}factors}\ d'.\ [q = 7]\ (\mathit{mod}\ 8)\}.$
$q \mathbin{\hat{}} \mathit{multiplicity}\ q\ d')\ \mathit{mod}\ 8)\ \mathit{mod}\ 8$
by (*metis* (*no-types*, *lifting*) *mod-mult-eq mod-mod-trivial*)
also have $\ldots = ((\prod q{\in}\{q \in \mathit{prime\text{-}factors}\ d'.\ [q = 1]\ (\mathit{mod}\ 8)\}.$
$(q \mathbin{\hat{}} \mathit{multiplicity}\ q\ d')\ \mathit{mod}\ 8)\ \mathit{mod}\ 8) *$
$((\prod q{\in}\{q \in \mathit{prime\text{-}factors}\ d'.\ [q = 3]\ (\mathit{mod}\ 8)\}.$
$(q \mathbin{\hat{}} \mathit{multiplicity}\ q\ d')\ \mathit{mod}\ 8)\ \mathit{mod}\ 8) *$
$((\prod q{\in}\{q \in \mathit{prime\text{-}factors}\ d'.\ [q = 5]\ (\mathit{mod}\ 8)\}.$
$(q \mathbin{\hat{}} \mathit{multiplicity}\ q\ d')\ \mathit{mod}\ 8)\ \mathit{mod}\ 8) *$
$((\prod q{\in}\{q \in \mathit{prime\text{-}factors}\ d'.\ [q = 7]\ (\mathit{mod}\ 8)\}.$
$(q \mathbin{\hat{}} \mathit{multiplicity}\ q\ d')\ \mathit{mod}\ 8)\ \mathit{mod}\ 8)\ \mathit{mod}\ 8$
unfolding *mod-prod-eq* **..**
also have $\ldots = ((\prod q{\in}\{q \in \mathit{prime\text{-}factors}\ d'.\ [q = 1]\ (\mathit{mod}\ 8)\}.$
$((q\ \mathit{mod}\ 8) \mathbin{\hat{}} \mathit{multiplicity}\ q\ d')\ \mathit{mod}\ 8)\ \mathit{mod}\ 8) *$
$((\prod q{\in}\{q \in \mathit{prime\text{-}factors}\ d'.\ [q = 3]\ (\mathit{mod}\ 8)\}.$
$((q\ \mathit{mod}\ 8) \mathbin{\hat{}} \mathit{multiplicity}\ q\ d')\ \mathit{mod}\ 8)\ \mathit{mod}\ 8) *$
$((\prod q{\in}\{q \in \mathit{prime\text{-}factors}\ d'.\ [q = 5]\ (\mathit{mod}\ 8)\}.$
$((q\ \mathit{mod}\ 8) \mathbin{\hat{}} \mathit{multiplicity}\ q\ d')\ \mathit{mod}\ 8)\ \mathit{mod}\ 8) *$
$((\prod q{\in}\{q \in \mathit{prime\text{-}factors}\ d'.\ [q = 7]\ (\mathit{mod}\ 8)\}.$
$((q\ \mathit{mod}\ 8) \mathbin{\hat{}} \mathit{multiplicity}\ q\ d')\ \mathit{mod}\ 8)\ \mathit{mod}\ 8)\ \mathit{mod}\ 8$

unfolding *power-mod* **..**
also have ... = ((∏ q∈{q ∈ *prime-factors d′*. [q = *1*] (*mod 8*)}.
(((*int q*) *mod 8*) ^ *multiplicity q d′*) *mod 8*) *mod 8*) ∗
((∏ q∈{q ∈ *prime-factors d′*. [q = *3*] (*mod 8*)}.
(((*int q*) *mod 8*) ^ *multiplicity q d′*) *mod 8*) *mod 8*) ∗
((∏ q∈{q ∈ *prime-factors d′*. [q = *5*] (*mod 8*)}.
(((*int q*) *mod 8*) ^ *multiplicity q d′*) *mod 8*) *mod 8*) ∗
((∏ q∈{q ∈ *prime-factors d′*. [q = *7*] (*mod 8*)}.
(((*int q*) *mod 8*) ^ *multiplicity q d′*) *mod 8*) *mod 8*) *mod 8*
by (*simp add*: *int-ops*)
also have ... = ((∏ q∈{q ∈ *prime-factors d′*. [q = *1*] (*mod 8*)}.
((*1 mod 8*) ^ *multiplicity q d′*) *mod 8*) *mod 8*) ∗
((∏ q∈{q ∈ *prime-factors d′*. [q = *3*] (*mod 8*)}.
((*3 mod 8*) ^ *multiplicity q d′*) *mod 8*) *mod 8*) ∗
((∏ q∈{q ∈ *prime-factors d′*. [q = *5*] (*mod 8*)}.
(((− *3*) *mod 8*) ^ *multiplicity q d′*) *mod 8*) *mod 8*) ∗
((∏ q∈{q ∈ *prime-factors d′*. [q = *7*] (*mod 8*)}.
(((− *1*) *mod 8*) ^ *multiplicity q d′*) *mod 8*) *mod 8*) *mod 8*
unfolding *cong-int-iff*[*symmetric*] *unique-euclidean-semiring-class.cong-def*
by *auto*
also have ... = ((∏ q∈{q ∈ *prime-factors d′*. [q = *1*] (*mod 8*)}.
(*1* ^ *multiplicity q d′*) *mod 8*) *mod 8*) ∗
((∏ q∈{q ∈ *prime-factors d′*. [q = *3*] (*mod 8*)}.
(*3* ^ *multiplicity q d′*) *mod 8*) *mod 8*) ∗
((∏ q∈{q ∈ *prime-factors d′*. [q = *5*] (*mod 8*)}.
((− *3*) ^ *multiplicity q d′*) *mod 8*) *mod 8*) ∗
((∏ q∈{q ∈ *prime-factors d′*. [q = *7*] (*mod 8*)}.
((− *1*) ^ *multiplicity q d′*) *mod 8*) *mod 8*) *mod 8*
unfolding *power-mod* **..**
also have ... = ((∏ q∈{q ∈ *prime-factors d′*. [q = *1*] (*mod 8*)}.
1 ^ *multiplicity q d′*) *mod 8*) ∗
((∏ q∈{q ∈ *prime-factors d′*. [q = *3*] (*mod 8*)}.
3 ^ *multiplicity q d′*) *mod 8*) ∗
((∏ q∈{q ∈ *prime-factors d′*. [q = *5*] (*mod 8*)}.
(− *3*) ^ *multiplicity q d′*) *mod 8*) ∗
((∏ q∈{q ∈ *prime-factors d′*. [q = *7*] (*mod 8*)}.
(− *1*) ^ *multiplicity q d′*) *mod 8*) *mod 8*
unfolding *mod-prod-eq* **..**
also have ... = (∏ q∈{q ∈ *prime-factors d′*. [q = *1*] (*mod 8*)}.
1 ^ *multiplicity q d′*) ∗
(∏ q∈{q ∈ *prime-factors d′*. [q = *3*] (*mod 8*)}.
3 ^ *multiplicity q d′*) ∗
(∏ q∈{q ∈ *prime-factors d′*. [q = *5*] (*mod 8*)}.
(− *3*) ^ *multiplicity q d′*) ∗
(∏ q∈{q ∈ *prime-factors d′*. [q = *7*] (*mod 8*)}.
(− *1*) ^ *multiplicity q d′*) *mod 8*
by (*metis* (*no-types*, *lifting*) *mod-mult-eq mod-mod-trivial*)
also have ... = (∏ q∈{q ∈ *prime-factors d′*. [q = *3*] (*mod 8*)}.
3 ^ *multiplicity q d′*) ∗

$(\prod q{\in}\{q \in \textit{prime-factors}\ d'.\ [q = 5]\ (\textit{mod}\ 8)\}.$
$(-3) \mathbin{\hat{}} \textit{multiplicity}\ q\ d') *$
$(\prod q{\in}\{q \in \textit{prime-factors}\ d'.\ [q = 7]\ (\textit{mod}\ 8)\}.$
$(-1) \mathbin{\hat{}} \textit{multiplicity}\ q\ d')\ \textit{mod}\ 8$
by *auto*
also have ... = $(\prod q{\in}\{q \in \textit{prime-factors}\ d'.\ [q = 3]\ (\textit{mod}\ 8)\}.$
$3 \mathbin{\hat{}} \textit{multiplicity}\ q\ d') *$
$(\prod q{\in}\{q \in \textit{prime-factors}\ d'.\ [q = 5]\ (\textit{mod}\ 8)\}.$
$3 \mathbin{\hat{}} \textit{multiplicity}\ q\ d' * (-1) \mathbin{\hat{}} \textit{multiplicity}\ q\ d') *$
$(\prod q{\in}\{q \in \textit{prime-factors}\ d'.\ [q = 7]\ (\textit{mod}\ 8)\}.$
$(-1) \mathbin{\hat{}} \textit{multiplicity}\ q\ d')\ \textit{mod}\ 8$
unfolding *power-mult-distrib*[*symmetric*] **by** *auto*
also have ... = $((\prod q{\in}\{q \in \textit{prime-factors}\ d'.\ [q = 3]\ (\textit{mod}\ 8)\}.$
$3 \mathbin{\hat{}} \textit{multiplicity}\ q\ d') *$
$(\prod q{\in}\{q \in \textit{prime-factors}\ d'.\ [q = 5]\ (\textit{mod}\ 8)\}.$
$3 \mathbin{\hat{}} \textit{multiplicity}\ q\ d')) *$
$((\prod q{\in}\{q \in \textit{prime-factors}\ d'.\ [q = 5]\ (\textit{mod}\ 8)\}.$
$(-1) \mathbin{\hat{}} \textit{multiplicity}\ q\ d') *$
$(\prod q{\in}\{q \in \textit{prime-factors}\ d'.\ [q = 7]\ (\textit{mod}\ 8)\}.$
$(-1) \mathbin{\hat{}} \textit{multiplicity}\ q\ d'))\ \textit{mod}\ 8$
unfolding *prod.distrib* **by** (*simp add*: *algebra-simps*)
also have ... = $((\prod q{\in}\{q \in \textit{prime-factors}\ d'.\ [q = 3]\ (\textit{mod}\ 8)\} \cup$
$\{q \in \textit{prime-factors}\ d'.\ [q = 5]\ (\textit{mod}\ 8)\}.$
$3 \mathbin{\hat{}} \textit{multiplicity}\ q\ d') *$
$(\prod q{\in}\{q \in \textit{prime-factors}\ d'.\ [q = 5]\ (\textit{mod}\ 8)\} \cup$
$\{q \in \textit{prime-factors}\ d'.\ [q = 7]\ (\textit{mod}\ 8)\}.$
$(-1) \mathbin{\hat{}} \textit{multiplicity}\ q\ d'))\ \textit{mod}\ 8$
apply (*subst*
prod.union-disjoint[*of* $\{q{\in}\textit{prime-factors}\ d'.\ [q = 5]\ (\textit{mod}\ 8)\}$
$\{q{\in}\textit{prime-factors}\ d'.\ [q = 7]\ (\textit{mod}\ 8)\}$
$\lambda q.\ (-1) \mathbin{\hat{}} \textit{multiplicity}\ q\ d'$]
)
apply ((*force simp add*: *unique-euclidean-semiring-class.cong-def*)+)[3]
apply (*subst*
prod.union-disjoint[*of* $\{q{\in}\textit{prime-factors}\ d'.\ [q = 3]\ (\textit{mod}\ 8)\}$
$\{q{\in}\textit{prime-factors}\ d'.\ [q = 5]\ (\textit{mod}\ 8)\}$
$\lambda q.\ 3 \mathbin{\hat{}} \textit{multiplicity}\ q\ d'$]
)
apply ((*force simp add*: *unique-euclidean-semiring-class.cong-def*)+)[3]
apply *blast*
done
also have ... = $((\prod q{\in}\{q \in \textit{prime-factors}\ d'.$
$[q = 3]\ (\textit{mod}\ 8) \vee [q = 5]\ (\textit{mod}\ 8)\}.$
$3 \mathbin{\hat{}} \textit{multiplicity}\ q\ d') *$
$(\prod q{\in}\{q \in \textit{prime-factors}\ d'.$
$[q = 5]\ (\textit{mod}\ 8) \vee [q = 7]\ (\textit{mod}\ 8)\}.$
$(-1) \mathbin{\hat{}} \textit{multiplicity}\ q\ d'))\ \textit{mod}\ 8$
by (*rule arg-cong2*[*of* - - - - $\lambda A\ B.\ ((\prod q{\in}A.\ -\ q) * (\prod q{\in}B.\ -\ q))\ \textit{mod}\ 8$];
auto)

```
  also have ... = (((∏ q∈{q ∈ prime-factors d'.
                        [q = 3] (mod 8) ∨ [q = 5] (mod 8)}.
                    3 ^ multiplicity q d') mod 8) *
                  ((∏ q∈{q ∈ prime-factors d'.
                        [q = 5] (mod 8) ∨ [q = 7] (mod 8)}.
                    (− 1) ^ multiplicity q d') mod 8)) mod 8
    unfolding mod-mult-eq ..
  also have ... = ((3 ^ (∑ q∈{q ∈ prime-factors d'.
                              [q = 3] (mod 8) ∨ [q = 5] (mod 8)}.
                         multiplicity q d') mod 8) *
                  ((− 1) ^ (∑ q∈{q ∈ prime-factors d'.
                                 [q = 5] (mod 8) ∨ [q = 7] (mod 8)}.
                            multiplicity q d') mod 8)) mod 8
    unfolding power-sum ..
  also have ... =
      int (case ((∑ q∈{q ∈ prime-factors d'.
                       [q = 3] (mod 8) ∨ [q = 5] (mod 8)}.
                  multiplicity q d') mod 2,
                 (∑ q∈{q ∈ prime-factors d'.
                       [q = 5] (mod 8) ∨ [q = 7] (mod 8)}.
                  multiplicity q d') mod 2) of
            (0    , 0    ) ⇒ 1
          | (0    , Suc 0) ⇒ 7
          | (Suc 0, 0    ) ⇒ 3
          | (Suc 0, Suc 0) ⇒ 5) (is - = int ?d'-mod-8)
    unfolding three-mod-eight-power-iff minus-one-power-iff
    by (simp; simp add: odd-iff-mod-2-eq-one)
  finally have d'-mod-8: d' mod 8 = ?d'-mod-8 by linarith

  have [d' = 1] (mod 8) ∨ [d' = 3] (mod 8)
    unfolding d'-def c-def unique-euclidean-semiring-class.cong-def
    using assms
    by auto
  hence ?d'-mod-8 = 1 ∨ ?d'-mod-8 = 3
    unfolding unique-euclidean-semiring-class.cong-def d'-mod-8 by auto
  thus ?thesis
    unfolding unique-euclidean-semiring-class.cong-def
    by (smt (verit) Collect-cong One-nat-def cong-exp-iff-simps(11)
          even-mod-2-iff even-numeral nat.case(2) numeral-eq-iff
          numerals(1) old.nat.simps(4) parity-cases prod.simps(2)
          semiring-norm(84))
qed
hence Legendre (− d') p = 1
  using Legendre-using-sum
  unfolding minus-one-power-iff cong-0-iff
  by argo
thus QuadRes p (− d')
  unfolding Legendre-def
  by (metis one-neq-neg-one one-neq-zero)
```

```
  qed

  from ‹QuadRes p (− d')› obtain x₀ y where x₀² − (− d') = y * (int p)
    unfolding quadratic-residue-alt-equiv[symmetric]
          quadratic-residue-alt-def
    by auto
  hence (int p) dvd (x₀² − − d') by simp

  define x :: int where x ≡ if odd x₀ then x₀ else (x₀ + p)

  have even (4 * int n * j) by simp
  moreover have odd k using ‹coprime k (4 * n)› by auto
  ultimately have odd (int p) unfolding p-def by simp

  have odd x unfolding x-def using ‹odd (int p)› by auto

  have QuadRes (2 * p) (− d')
  unfolding quadratic-residue-alt-equiv[symmetric]
        quadratic-residue-alt-def
  proof −
    have 2 dvd (x² − − d') unfolding d'-def c-def using ‹odd x› by auto
    moreover from ‹(int p) dvd (x₀² − − d')›
    have (int p) dvd (x² − − d')
      unfolding x-def power2-eq-square
      apply (simp add: algebra-simps)
      unfolding add.assoc[symmetric, of d' x₀ * x₀]
      apply auto
      done
    moreover have coprime 2 (int p) using ‹odd (int p)› by auto
    ultimately have (int (2 * p)) dvd (x² − − d') by (simp add: divides-mult)
    hence (x² − − d') mod (int (2 * p)) = 0 by simp
    hence ∃ y. x² − − d' = int (2 * p) * y by auto
    hence ∃ y. x² − − d' = y * int (2 * p) by (simp add: algebra-simps)
    thus ∃ x y. x² − − d' = y * int (2 * p) by (rule exI[where ?x=x])
  qed

  have n ≥ 2 using ‹1 < n› by auto
  moreover have 0 < nat d' unfolding d'-def using ‹j > 0› by simp
  moreover have QuadRes (int (nat d' * n − 1)) (− d')
    using ‹d' > 1› H-2-p ‹QuadRes (2 * p) (− d')› by (simp add: int-ops)
  ultimately show ∃ x₁ x₂ x₃. n = x₁² + x₂² + x₃²
    using three-squares-using-quadratic-residue[of n nat d']
    by auto
qed

lemma power-two-mod-eight:
  fixes n :: nat
  shows n² mod 8 ∈ {0, 1, 4}
proof −
```

```
  have 0: n^2 mod 8 = (n mod 8)^2 mod 8
    unfolding power2-eq-square by (simp add: mod-mult-eq)
  have n mod 8 ∈ {0, 1, 2, 3, 4, 5, 6, 7} by auto
  hence (n mod 8)^2 mod 8 ∈ {0, 1, 4}
    unfolding power2-eq-square by auto
  thus n^2 mod 8 ∈ {0, 1, 4} unfolding 0 .
qed

lemma power-two-mod-four:
  fixes n :: nat
  shows n^2 mod 4 ∈ {0, 1}
  using power-two-mod-eight[of n] mod-mod-cancel[of 4 8 n^2] by auto
```

Theorem 1.4 from [1].

```
theorem three-squares-iff:
  fixes n :: nat
  shows (∃ x_1 x_2 x_3. n = x_1^2 + x_2^2 + x_3^2) ⟷ (∄ a k. n = 4 ^ a * (8 * k + 7))
proof
  assume ∃ x_1 x_2 x_3. n = x_1^2 + x_2^2 + x_3^2
  then obtain x_1 x_2 x_3 where 0: n = x_1^2 + x_2^2 + x_3^2 by blast
  show ∄ a k. n = 4 ^ a * (8 * k + 7)
  proof
    assume ∃ a k. n = 4 ^ a * (8 * k + 7)
    then obtain a k where 1: n = 4 ^ a * (8 * k + 7) by blast
    from 0 1 show False
    proof (induction a arbitrary: n x_1 x_2 x_3 rule: nat.induct)
      fix n x_1 x_2 x_3 :: nat
      assume 2: n = x_1^2 + x_2^2 + x_3^2
      assume n = 4 ^ 0 * (8 * k + 7)
      hence 3: n mod 8 = 7 unfolding 1 by auto
      have (x_1^2 mod 8 + x_2^2 mod 8 + x_3^2 mod 8) mod 8 = 7
        unfolding 2 3[symmetric]
        by (meson mod-add-cong mod-mod-trivial)
      thus False
        using power-two-mod-eight[of x_1]
              power-two-mod-eight[of x_2]
              power-two-mod-eight[of x_3]
        by auto
    next
      fix a' n x_1 x_2 x_3 :: nat
      assume 4: ⋀n' x_1' x_2' x_3' :: nat.
                  n' = x_1'^2 + x_2'^2 + x_3'^2 ⟹ n' = 4 ^ a' * (8 * k + 7) ⟹ False
      assume 5: n = x_1^2 + x_2^2 + x_3^2
      assume n = 4 ^ Suc a' * (8 * k + 7)
      hence n = 4 * (4 ^ a' * (8 * k + 7)) (is n = 4 * ?m) by auto
      hence 6: 4 * ?m = x_1^2 + x_2^2 + x_3^2 unfolding 5 by auto
      have (x_1^2 + x_2^2 + x_3^2) mod 4 = 0 using 6 by presburger
      hence ((x_1^2 mod 4) + (x_2^2 mod 4) + (x_3^2 mod 4)) mod 4 = 0
        by presburger
```

```
    hence x₁² mod 4 = 0 ∧ x₂² mod 4 = 0 ∧ x₃² mod 4 = 0
      using power-two-mod-four[of x₁]
            power-two-mod-four[of x₂]
            power-two-mod-four[of x₃]
      by (auto; presburger)
    hence even x₁ ∧ even x₂ ∧ even x₃
      by (metis dvd-0-right even-even-mod-4-iff even-power)
    hence 4 * ?m = 4 * ((x₁ div 2)² + (x₂ div 2)² + (x₃ div 2)²)
      unfolding 6 by fastforce
    hence ?m = (x₁ div 2)² + (x₂ div 2)² + (x₃ div 2)² by auto
    thus False using 4 by blast
   qed
  qed
next
  assume 7: ∄a k. n = 4 ^ a * (8 * k + 7)
  show ∃x₁ x₂ x₃. n = x₁² + x₂² + x₃²
  proof cases
    assume n = 0
    thus ∃x₁ x₂ x₃. n = x₁² + x₂² + x₃² by auto
  next
    assume 8: n ≠ 0
    have n > 0 ⟹ ∃a m. n = 4 ^ a * m ∧ ¬ 4 dvd m
    proof (induction n rule: less-induct)
      fix n :: nat
      assume 9: ⋀n'. n' < n ⟹ n' > 0 ⟹ ∃a' m'. n' = 4 ^ a' * m' ∧ ¬ 4 dvd
m'
      assume 10: n > 0
      show ∃a m. n = 4 ^ a * m ∧ ¬ 4 dvd m
      proof cases
        assume 11: 4 dvd n
        have n div 4 < n n div 4 > 0 using 10 11 by auto
        then obtain a' m' where 12: n div 4 = 4 ^ a' * m' ∧ ¬ 4 dvd m'
          using 9 by blast
        have n = 4 ^ (Suc a') * m' ∧ ¬ 4 dvd m'
          using 11 12 by auto
        thus ∃a m. n = 4 ^ a * m ∧ ¬ 4 dvd m by blast
      next
        assume ¬ 4 dvd n
        hence n = 4 ^ 0 * n ∧ ¬ 4 dvd n by auto
        thus ∃a m. n = 4 ^ a * m ∧ ¬ 4 dvd m by blast
      qed
    qed
    then obtain a m where 13: n = 4 ^ a * m ¬ 4 dvd m using 8 by auto
    have 14: m mod 8 ≠ 7
    proof
      assume m mod 8 = 7
    then obtain k where m = 8 * k + 7 by (metis div-mod-decomp mult.commute)
      hence n = 4 ^ a * (8 * k + 7) unfolding 13 by blast
      thus False using 7 by blast
```

```
    qed
    have m mod 4 = 2 ∨ m mod 8 ∈ {1, 3, 5, 7}
      using 13 by (simp; presburger)
    hence m mod 4 = 2 ∨ m mod 8 ∈ {1, 3, 5}
      using 14 by blast
    hence ∃ x1 x2 x3. m = x1^2 + x2^2 + x3^2
      using three-squares-using-mod-four three-squares-using-mod-eight
      by blast
    hence ∃ x1 x2 x3. n = (2 ^ a * x1)^2 + (2 ^ a * x2)^2 + (2 ^ a * x3)^2
      unfolding 13 power2-eq-square
      unfolding mult.assoc[of 2 ^ a]
      unfolding mult.commute[of 2 ^ a]
      unfolding mult.assoc
      unfolding power-add[symmetric]
      unfolding mult-2[symmetric]
      unfolding power-mult
      unfolding mult.assoc[symmetric]
      unfolding add-mult-distrib[symmetric]
      unfolding mult.commute[of 4 ^ a]
      by simp
    thus ∃ x1 x2 x3. n = x1^2 + x2^2 + x3^2 by blast
  qed
qed
```

Theorem 1.5 from [1].

```
theorem odd-three-squares-using-mod-eight:
  fixes n :: nat
  assumes n mod 8 = 3
  shows ∃ x1 x2 x3. odd x1 ∧ odd x2 ∧ odd x3 ∧ n = x1^2 + x2^2 + x3^2
proof −
  obtain x1 x2 x3 where 0: n = x1^2 + x2^2 + x3^2
    using assms three-squares-using-mod-eight by blast
  have (x1^2 mod 8 + x2^2 mod 8 + x3^2 mod 8) mod 8 = 3
    unfolding 0 assms[symmetric]
    by (meson mod-add-cong mod-mod-trivial)
  hence x1^2 mod 8 = 1 ∧ x2^2 mod 8 = 1 ∧ x3^2 mod 8 = 1
    using power-two-mod-eight[of x1]
          power-two-mod-eight[of x2]
          power-two-mod-eight[of x3]
    by auto
  hence odd x1 ∧ odd x2 ∧ odd x3
    by (metis dvd-mod even-numeral even-power odd-one pos2)
  hence odd x1 ∧ odd x2 ∧ odd x3 ∧ n = x1^2 + x2^2 + x3^2 using 0 by blast
  thus ∃ x1 x2 x3. odd x1 ∧ odd x2 ∧ odd x3 ∧ n = x1^2 + x2^2 + x3^2 by blast
qed
```

4.2 Consequences

```
lemma four-decomposition:
```

```
  fixes n :: nat
  shows ∃x y z. n = x^2 + y^2 + z^2 + z
proof −
  have (4 * n + 1) mod 8 ∈ {1, 3, 5} by (simp; presburger)
  then obtain x y z where 0: 4 * n + 1 = x^2 + y^2 + z^2
    using three-squares-using-mod-eight by blast
  hence 1: 1 = (x^2 + y^2 + z^2) mod 4
    by (metis Suc-0-mod-numeral(2) Suc-eq-plus1 mod-add-left-eq
          mod-mult-self1-is-0)
  show ?thesis
  proof cases
    assume even x
    then obtain x' where H-x: x = 2 * x' by blast
    show ?thesis
    proof cases
      assume even y
      then obtain y' where H-y: y = 2 * y' by blast
      show ?thesis
      proof cases
        assume even z
        then obtain z' where H-z: z = 2 * z' by blast
        show ?thesis using 1 unfolding H-x H-y H-z by auto
      next
        assume odd z
        then obtain z' where H-z: z = 2 * z' + 1 using oddE by blast
        have n = x'^2 + y'^2 + z'^2 + z'
          using 0 unfolding H-x H-y H-z power2-eq-square by auto
        thus ?thesis by blast
      qed
    next
      assume odd y
      then obtain y' where H-y: y = 2 * y' + 1 using oddE by blast
      show ?thesis
      proof cases
        assume even z
        then obtain z' where H-z: z = 2 * z' by blast
        have n = x'^2 + z'^2 + y'^2 + y'
          using 0 unfolding H-x H-y H-z power2-eq-square by auto
        thus ?thesis by blast
      next
        assume odd z
        then obtain z' where H-z: z = 2 * z' + 1 using oddE by blast
        show ?thesis
          using 1
          unfolding H-x H-y H-z power2-eq-square
          by (metis dvd-mod even-add even-mult-iff even-numeral odd-one)
      qed
    qed
  next
```

```
    assume odd x
    then obtain x' where H-x: x = 2 * x' + 1 using oddE by blast
    show ?thesis
    proof cases
      assume even y
      then obtain y' where H-y: y = 2 * y' by blast
      show ?thesis
      proof cases
        assume even z
        then obtain z' where H-z: z = 2 * z' by blast
        have n = y'^2 + z'^2 + x'^2 + x'
          using 0 unfolding H-x H-y H-z power2-eq-square by auto
        thus ?thesis by blast
      next
        assume odd z
        then obtain z' where H-z: z = 2 * z' + 1 using oddE by blast
        show ?thesis
          using 1
          unfolding H-x H-y H-z power2-eq-square
          by (metis dvd-mod even-add even-mult-iff even-numeral odd-one)
      qed
    next
      assume odd y
      then obtain y' where H-y: y = 2 * y' + 1 using oddE by blast
      show ?thesis
      proof cases
        assume even z
        then obtain z' where H-z: z = 2 * z' by blast
        show ?thesis
          using 1
          unfolding H-x H-y H-z power2-eq-square
          by (metis dvd-mod even-add even-mult-iff even-numeral odd-one)
      next
        assume odd z
        then obtain z' where H-z: z = 2 * z' + 1 using oddE by blast
        show ?thesis
          using 1
          unfolding H-x H-y H-z power2-eq-square
          by (simp add: mod-add-eq[symmetric])
      qed
    qed
  qed
qed

theorem four-decomposition-int:
  fixes n :: int
  shows (∃ x y z. n = x^2 + y^2 + z^2 + z) ⟷ n ≥ 0
proof
  assume ∃ x y z. n = x^2 + y^2 + z^2 + z
```

```
  then obtain x y z where 0: n = x^2 + y^2 + z^2 + z by blast
  show n ≥ 0
    unfolding 0 power2-eq-square
    by (smt (verit) div-pos-neg-trivial mult-le-0-iff
                    nonzero-mult-div-cancel-right sum-squares-ge-zero)
next
  assume n ≥ 0
  thus ∃ x y z. n = x^2 + y^2 + z^2 + z
    using four-decomposition[of nat n]
    by (smt (verit) int-eq-iff int-plus of-nat-power)
qed

theorem four-squares:
  fixes n :: nat
  shows ∃ x1 x2 x3 x4. n = x1^2 + x2^2 + x3^2 + x4^2
proof cases
  assume ∃ a k. n = 4 ^ a * (8 * k + 7)
  then obtain a k where n = 4 ^ a * (8 * k + 7) by blast
  hence 0: n = 4 ^ a * (8 * k + 6) + (2 ^ a)^2
    by (metis add-mult-distrib left-add-mult-distrib mult.commute mult-numeral-1
        numeral-Bit0 numeral-plus-numeral power2-eq-square power-mult-distrib
        semiring-norm(5))
  have ∄ a' k'. 4 ^ a * (8 * k + 6) = 4 ^ a' * (8 * k' + 7)
  proof
    assume ∃ a' k'. 4 ^ a * (8 * k + 6) = 4 ^ a' * (8 * k' + 7)
    then obtain a' k' where 1: 4 ^ a * (8 * k + 6) = 4 ^ a' * (8 * k' + 7)
      by blast
    show False
    proof (cases rule: linorder-cases[of a a'])
      assume a < a'
      hence 2: a' = a + (a' - a) a' - a > 0 by auto
      have 3: 4 ^ a * (8 * k + 6) = 4 ^ a * 4 ^ (a' - a) * (8 * k' + 7)
        using 1 2 by (metis power-add)
      have 2 = (8 * k + 6) mod 4 by presburger
      also have ... = (4 ^ (a' - a) * (8 * k' + 7)) mod 4 using 3 by auto
      also have ... = 0 using 2 by auto
      finally show False by auto
    next
      assume a = a'
      hence 8 * k + 6 = 8 * k' + 7 using 1 by auto
      thus False by presburger
    next
      assume a > a'
      hence 4: a = a' + (a - a') a - a' > 0 by auto
      have 5: 4 ^ a' * 4 ^ (a - a') * (8 * k + 6) = 4 ^ a' * (8 * k' + 7)
        using 1 4 by (metis power-add)
      have 0 = (4 ^ (a - a') * (8 * k + 6)) mod 4 using 4 by auto
      also have ... = (8 * k' + 7) mod 4 using 5 by auto
      also have ... = 3 by presburger
```

```
    finally show False by auto
   qed
  qed
  then obtain x1 x2 x3 where 4 ^ a * (8 * k + 6) = x1^2 + x2^2 + x3^2
   using three-squares-iff by blast
  thus ∃x1 x2 x3 x4. n = x1^2 + x2^2 + x3^2 + x4^2 unfolding 0 by auto
next
  assume ∄a k. n = 4 ^ a * (8 * k + 7)
  hence ∃x1 x2 x3. n = x1^2 + x2^2 + x3^2 using three-squares-iff by blast
  thus ∃x1 x2 x3 x4. n = x1^2 + x2^2 + x3^2 + x4^2 by (metis zero-power2 add-0)
qed

end
```

References

[1] M. B. Nathanson. *Additive Number Theory: The Classical Bases*, volume 164 of *Graduate Texts in Mathematics*. Springer, New York, 1996.