The Teichmüller–Tukey Lemma

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September 23, 2026

Abstract

This entry formalizes the Teichmüller–Tukey lemma in Isabelle/HOL: every nonempty family of sets of finite character has a member that is maximal under inclusion. Instead of deriving the result from Zorn's lemma, which is already available in Isabelle/HOL, the development follows the direct choice-function construction of Sun and Yu, originally formulated in Morse–Kelley set theory and checked in Coq. The Isabelle proof makes explicit the choice argument that turns a fixed point of the construction into a maximal member.

Contents

1 The Teichmüller–Tukey Lemma

theory *Teichmuller-Tukey-Lemma*
imports *Main*
begin

The Teichmüller–Tukey lemma states that every nonempty family of sets of finite character has a member that is maximal under inclusion. Although the result follows from Zorn's lemma, the development below formalizes the direct choice-function construction presented by Sun and Yu [2]. Their proof is formulated in Morse–Kelley set theory and checked in Coq; here it is adapted to polymorphic HOL sets.

This result was developed as infrastructure for a separate formalization of first-order logic following Shoenfield [1].

1.1 Families of finite character

definition *Maximal* :: *'a set* ⇒ *'a set set* ⇒ *bool* **where**
  $\textit{Maximal}\ A\ J \equiv A \in J \land (\forall B{\in}J.\ \neg\ A \subset B)$

definition *Nest* :: *'a set set* ⇒ *bool* **where**
  $\textit{Nest}\ A \equiv \forall x{\in}A.\ \forall y{\in}A.\ x \subseteq y \lor y \subseteq x$

definition *Finite-character* :: *'a set set* ⇒ *bool* **where**
  $\textit{Finite-character}\ J \equiv \forall \Delta.\ \Delta \in J \longleftrightarrow (\forall \delta{\subseteq}\Delta.\ \textit{finite}\ \delta \longrightarrow \delta \in J)$

A family of finite character is downward closed. It is also closed under the union of a nest. Nonemptiness is needed for the empty nest: it ensures that the empty set belongs to the family.

proposition *property-fin-char*:
  assumes $\textit{Finite-character}\ J$ **and** $J \neq \{\}$
  shows $\forall A{\in}J.\ \forall B{\subseteq}A.\ B \in J$
    and $\forall G.\ \textit{Nest}\ G \land G \subseteq J \longrightarrow \bigcup G \in J$
⟨*proof*⟩

1.2 The choice extension

fun *enlarge* :: *'a set* ⇒ *'a set set* ⇒ *'a set* **where**
  $\textit{enlarge}\ F\ J = \{x.\ x \in \bigcup J \land F{\cup}\{x\} \in J\}$

proposition *enlarge-subset*: $F \in J \Longrightarrow F \subseteq \textit{enlarge}\ F\ J$
  ⟨*proof*⟩

fun $\mathcal{X}$*-choice* :: *'a set* ⇒ *'a set set* ⇒ (*'a set* ⇒ *'a*) ⇒ *'a set* **where**
  $\mathcal{X}\textit{-choice}\ F\ J\ c =$
    $(\textit{if enlarge}\ F\ J - F = \{\}\ \textit{then}\ F\ \textit{else}\ F{\cup}\{c\ (\textit{enlarge}\ F\ J - F)\})$

For a member F of J, *enlarge* F J contains the elements that can be adjoined to F while staying in J. The operation $\mathcal{X}$*-choice* F J c adjoins an element chosen outside F, unless no such element exists.

The fixed-point argument below makes explicit a detail that is only implicit in Proof 4.7 of the source article. From $\mathcal{X}\textit{-choice}\ F\ J\ c = F$ one may conclude $\textit{enlarge}\ F\ J = F$ only by using the choice property of c: if *enlarge* F $J - F$ were nonempty, its chosen element would lie both outside and inside F. This is the step that turns a fixed point of the choice extension into a maximal member.

lemma $\mathcal{X}$*-implies-TTL*:

assumes *Finite-character J*
and $\exists F{\in}J.\ \exists c.\ ((\forall Y.\ Y \neq \{\} \longrightarrow c\ Y \in Y) \wedge \mathcal{X}\text{-}choice\ F\ J\ c = F)$
shows $\exists \Delta{\in}J.\ Maximal\ \Delta\ J$
⟨*proof*⟩

1.3 The least closed subclass

Following the notation of the source proof, a *t-subclass* contains the empty set and is closed under both choice extensions and unions of nests. Intersecting all t-subclasses gives the least such subclass.

definition *tSubclass* :: $'a\ set\ set \Rightarrow\ 'a\ set\ set \Rightarrow (\ 'a\ set \Rightarrow\ 'a) \Rightarrow bool$ **where**
$tSubclass\ G\ J\ c \equiv$
$G \subseteq J \wedge \{\} \in G \wedge (\forall F{\in}G.\ \mathcal{X}\text{-}choice\ F\ J\ c \in G) \wedge$
$(\forall L{\subseteq}G.\ Nest\ L \longrightarrow \bigcup L \in G)$

definition *inter-subclass* :: $'a\ set\ set \Rightarrow (\ 'a\ set \Rightarrow\ 'a) \Rightarrow\ 'a\ set\ set$ **where**
$inter\text{-}subclass\ J\ c \equiv \bigcap\{G.\ tSubclass\ G\ J\ c\}$

definition *μ-function* :: $'a\ set \Rightarrow\ 'a\ set\ set \Rightarrow (\ 'a\ set \Rightarrow\ 'a) \Rightarrow\ 'a\ set\ set$ **where**
$\mu\text{-}function\ C\ J\ c \equiv$
$\{A{\in}inter\text{-}subclass\ J\ c.\ A \subseteq C \vee C \subseteq A\}$

definition *total-subclass* :: $'a\ set\ set \Rightarrow (\ 'a\ set \Rightarrow\ 'a) \Rightarrow\ 'a\ set\ set$ **where**
$total\text{-}subclass\ J\ c \equiv$
$\{C{\in}inter\text{-}subclass\ J\ c.\ \mu\text{-}function\ C\ J\ c = inter\text{-}subclass\ J\ c\}$

definition *ν-function* :: $'a\ set \Rightarrow\ 'a\ set\ set \Rightarrow (\ 'a\ set \Rightarrow\ 'a) \Rightarrow\ 'a\ set\ set$ **where**
$\nu\text{-}function\ D\ J\ c \equiv$
$\{A{\in}inter\text{-}subclass\ J\ c.\ A \subseteq D \vee \mathcal{X}\text{-}choice\ D\ J\ c \subseteq A\}$

proposition *property-$\mathcal{X}$*: $F \in J \Longrightarrow F \subseteq \mathcal{X}\text{-}choice\ F\ J\ c$
⟨*proof*⟩

proposition *property-inter-subclass*:
assumes *Finite-character J* **and** $J \neq \{\}$
and $\forall Y.\ Y \neq \{\} \longrightarrow c\ Y \in Y$
shows $tSubclass\ (inter\text{-}subclass\ J\ c)\ J\ c$
and $\forall G.\ tSubclass\ G\ J\ c \longrightarrow inter\text{-}subclass\ J\ c \subseteq G$
⟨*proof*⟩

1.4 The least t-subclass is a Nest

The next three lemmas correspond to Steps I–III of Section 4.1 in the source. The auxiliary subclasses *μ-function C J c* and *ν-function D J c* isolate the members comparable with a given set and allow comparability to be propagated through the choice extension.

lemma *step1*:
assumes *Finite-character J* **and** $J \neq \{\}$

and $\forall\, Y.\ Y \neq \{\} \longrightarrow c\ Y \in Y$
and $D \in$ *total-subclass J c*
shows *tSubclass* (ν*-function D J c*) *J c*
⟨*proof*⟩

lemma *step2*:
assumes *Finite-character J* **and** $J \neq \{\}$
and $\forall\, Y.\ Y \neq \{\} \longrightarrow c\ Y \in Y$
and $D \in$ *total-subclass J c*
shows $\mathcal{X}$*-choice D J c* $\in$ *total-subclass J c*
⟨*proof*⟩

lemma *step3*:
assumes *Finite-character J* **and** $J \neq \{\}$
and $\forall\, Y.\ Y \neq \{\} \longrightarrow c\ Y \in Y$
shows *Nest* (*inter-subclass J c*)
⟨*proof*⟩

1.5 The fixed point and the main theorem

Since the least t-subclass is a nest, its union belongs to it. Closure under the choice extension then places the extension in the same nest, while the definition of union bounds it from above. Hence the union is a fixed point.

lemma *step4*:
assumes *Finite-character J* **and** $J \neq \{\}$
and $\forall\, Y.\ Y \neq \{\} \longrightarrow c\ Y \in Y$
shows $\mathcal{X}$*-choice* ($\bigcup$(*inter-subclass J c*)) *J c* $= \bigcup$(*inter-subclass J c*)
⟨*proof*⟩

theorem *teichmueller-tuckey-lemma*:
fixes *J* :: *'a set set*
assumes *Finite-character J* **and** $J \neq \{\}$
shows $\exists\,\Delta{\in}J.$ *Maximal* Δ *J*
⟨*proof*⟩

end

References

[1] J. R. Shoenfield. *Mathematical Logic*. Addison-Wesley, 1967.

[2] T. Sun and W. Yu. Formalization of the axiom of choice and its equivalent theorems. arXiv preprint arXiv:1906.03930v1, 2019. https://arxiv.org/abs/1906.03930v1.