

A Formalisation of Sturm's Theorem

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Abstract

Sturm sequences are a method for computing the number of real roots of a real polynomial inside a given interval efficiently. In this project, this fact and a number of methods to construct Sturm sequences efficiently have been formalised with the interactive theorem prover Isabelle/HOL. Building upon this, an Isabelle/HOL proof method was then implemented to prove statements about the number of roots of a real polynomial and related properties.

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1 Miscellaneous

```
theory Misc-Polynomial
imports HOL-Computational-Algebra.Polynomial HOL-Computational-Algebra.Polynomial-Factorial
Pure-ex.Guess
begin
```

1.1 Analysis

lemma *fun-eq-in-ivl*:

```
assumes  $a \leq b \ \forall x::\text{real}. a \leq x \wedge x \leq b \longrightarrow \text{eventually } (\lambda\xi. f \xi = f x) \text{ (at } x)$ 
shows  $f a = f b$ 
```

proof (*rule-connected-local-const*)

```
show connected { $a..b$ }  $a \in \{a..b\} \ b \in \{a..b\}$  using  $\langle a \leq b \rangle$  by (auto intro: connected-Icc)
```

```
show  $\forall aa \in \{a..b\}. \text{eventually } (\lambda b. f aa = f b) \text{ (at } aa \text{ within } \{a..b\})$ 
```

proof

```
fix  $x$  assume  $x \in \{a..b\}$ 
```

```
with assms(2)[rule-format, of x]
```

```
show eventually  $(\lambda b. f x = f b) \text{ (at } x \text{ within } \{a..b\})$ 
```

```
by (auto simp: eventually-at-filter elim: eventually-mono)
```

qed

qed

1.2 Polynomials

1.2.1 General simplification lemmas

lemma *pderiv-div*:

```
assumes [simp]:  $q \ \text{dvd} \ p \ q \neq 0$ 
```

```
shows  $\text{pderiv } (p \ \text{div} \ q) = (q * \text{pderiv } p - p * \text{pderiv } q) \ \text{div} \ (q * q)$ 
 $q * q \ \text{dvd} \ (q * \text{pderiv } p - p * \text{pderiv } q)$ 
```

proof –

```
from assms obtain  $r$  where  $p = q * r$  unfolding dvd-def by blast
```

```
hence  $q * \text{pderiv } p - p * \text{pderiv } q = (q * q) * \text{pderiv } r$ 
```

```
by (simp add: algebra-simps pderiv-mult)
```

```
thus  $q * q \ \text{dvd} \ (q * \text{pderiv } p - p * \text{pderiv } q)$  by simp
```

```
note  $0 = \text{pderiv-mult}[of \ q \ p \ \text{div} \ q]$ 
```

```
have  $1: q * (p \ \text{div} \ q) = p$ 
```

```
by (metis assms(1) assms(2) dvd-def nonzero-mult-div-cancel-left)
```

```
have  $f1: \text{pderiv } (p \ \text{div} \ q) * (q * q) \ \text{div} \ (q * q) = \text{pderiv } (p \ \text{div} \ q)$ 
```

```
by simp
```

```
have  $f2: \text{pderiv } p = q * \text{pderiv } (p \ \text{div} \ q) + p \ \text{div} \ q * \text{pderiv } q$ 
```

```
by (metis 0 1)
```

```
have  $p * \text{pderiv } q = \text{pderiv } q * (q * (p \ \text{div} \ q))$ 
```

```
by (metis 1 mult commute)
```

```
then have  $p * \text{pderiv } q = q * (p \ \text{div} \ q * \text{pderiv } q)$ 
```

```
by fastforce
```

```
then have  $q * \text{pderiv } p - p * \text{pderiv } q = q * (q * \text{pderiv } (p \ \text{div} \ q))$ 
```

using *f2* **by** (*metis add-diff-cancel-right' distrib-left*)
then show $pderiv (p \text{ div } q) = (q * pderiv p - p * pderiv q) \text{ div } (q * q)$
using *f1* **by** (*metis mult.commute mult.left-commute*)
qed

1.2.2 Divisibility of polynomials

Two polynomials that are coprime have no common roots.

lemma *coprime-imp-no-common-roots*:

$\neg (poly p x = 0 \wedge poly q x = 0)$ **if** *coprime p q*
for $x :: 'a :: field$

proof *clarify*

assume $poly p x = 0 \wedge poly q x = 0$
then have $[-x, 1:] \text{ dvd } p \ [-x, 1:] \text{ dvd } q$
by (*simp-all add: poly-eq-0-iff-dvd*)
with that have *is-unit* $[-x, 1:]$
by (*rule coprime-common-divisor*)
then show *False*
by (*auto simp add: is-unit-pCons-iff*)

qed

lemma *poly-div*:

assumes $poly q x \neq 0$ **and** $(q :: 'a :: field \text{ poly}) \text{ dvd } p$
shows $poly (p \text{ div } q) x = poly p x / poly q x$

proof –

from *assms* **have** $[simp]: q \neq 0$ **by** *force*
have $poly q x * poly (p \text{ div } q) x = poly (q * (p \text{ div } q)) x$ **by** *simp*
also have $q * (p \text{ div } q) = p$
using *assms* **by** (*simp add: div-mult-swap*)
finally show $poly (p \text{ div } q) x = poly p x / poly q x$
using *assms* **by** (*simp add: field-simps*)

qed

lemma *poly-div-gcd-squarefree-aux*:

assumes $pderiv (p :: ('a :: \{field-char-0, field-gcd\}) \text{ poly}) \neq 0$
defines $d \equiv gcd p (pderiv p)$
shows *coprime* $(p \text{ div } d) (pderiv (p \text{ div } d))$ **and**
 $\bigwedge x. poly (p \text{ div } d) x = 0 \longleftrightarrow poly p x = 0$

proof –

obtain $r \ s$ **where** *bezout-coefficients* $p (pderiv p) = (r, s)$
by (*auto simp add: prod-eq-iff*)
then have $r * p + s * pderiv p = gcd p (pderiv p)$
by (*rule bezout-coefficients*)
then have $rs: d = r * p + s * pderiv p$
by (*simp add: d-def*)
define t **where** $t = p \text{ div } d$
define p' **where** $[simp]: p' = pderiv p$
define d' **where** $[simp]: d' = pderiv d$

define u **where** $u = p' \text{ div } d$
have $A: p = t * d$ **and** $B: p' = u * d$
by (*simp-all add: t-def u-def d-def algebra-simps*)
from *poly-squarefree-decomp*[*OF assms(1) A B[unfolded p'-def] rs*]
show $\bigwedge x. \text{poly}(p \text{ div } d) x = 0 \iff \text{poly } p x = 0$ **by** (*auto simp: t-def*)

from rs **have** $C: s*t*d' = d * (1 - r*t - s*pderiv t)$
by (*simp add: A B algebra-simps pderiv-mult*)
from *assms* **have** [*simp*]: $p \neq 0 \ d \neq 0 \ t \neq 0$
by (*force, force, subst (asm) A, force*)

have $\bigwedge x. [x \text{ dvd } t; x \text{ dvd } (pderiv t)] \implies x \text{ dvd } 1$
proof –

fix x **assume** $x \text{ dvd } t \ x \text{ dvd } (pderiv t)$
then obtain $v \ w$ **where** vw :
 $t = x*v \ pderiv t = x*w$ **unfolding** *dvd-def* **by** *blast*
define $x' \ v'$ **where** [*simp*]: $x' = pderiv x$ **and** [*simp*]: $v' = pderiv v$
from vw **have** $x*v' + v*x' = x*w$ **by** (*simp add: pderiv-mult*)
hence $v*x' = x*(w - v')$ **by** (*simp add: algebra-simps*)
hence $x \text{ dvd } v*pderiv x$ **by** *simp*
then obtain y **where** $y: v*x' = x*y$ **unfolding** *dvd-def* **by** *force*
from $\langle t \neq 0 \rangle$ **and** vw **have** $x \neq 0$ **by** *simp*

have *x-pow-n-dvd-d*: $\bigwedge n. x^n \text{ dvd } d$
proof–

fix n **show** $x^n \text{ dvd } d$
proof (*induction n, simp, rename-tac n, case-tac n*)
fix n **assume** $n = (0::nat)$
from vw **and** C **have** $d = x*(d*r*v + d*s*w + s*v*d')$
by (*simp add: algebra-simps*)
with $\langle n = 0 \rangle$ **show** $x^{Suc\ n} \text{ dvd } d$ **by** (*force intro: dvdI*)
next
fix $n \ n'$ **assume** *IH*: $x^n \text{ dvd } d$ **and** $n = Suc\ n'$
hence [*simp*]: $Suc\ n' = n \ x * x^{n'} = x^n$ **by** *simp-all*
define $c :: 'a \text{ poly}$ **where** $c = [:of-nat\ n:]$
from *pderiv-power-Suc*[*of x n'*]
have [*simp*]: $pderiv (x^n) = c*x^{n'} * x'$ **unfolding** *c-def*
by *simp*

from *IH* **obtain** z **where** $d = x^n * z$ **unfolding** *dvd-def* **by** *blast*

define z' **where** [*simp*]: $z' = pderiv z$
from $d \ \langle d \neq 0 \rangle$ **have** $x^n \neq 0 \ z \neq 0$ **by** *force+*
from $C \ d$ **have** $x^n * z = z*r*v*x^{Suc\ n} + z*s*c*x^n*(v*x') +$
 $s*v*z'*x^{Suc\ n} + s*z*(v*x')*x^n + s*z*v'*x^{Suc\ n}$
by (*simp add: algebra-simps vw pderiv-mult*)
also have $\dots = x^n * x * (z*r*v + z*s*c*y + s*v*z' + s*z*y + s*z*v')$
by (*simp only: y, simp add: algebra-simps*)
finally have $z = x*(z*r*v + z*s*c*y + s*v*z' + s*z*y + s*z*v')$
using $\langle x^n \neq 0 \rangle$ **by** *force*

hence $x \text{ dvd } z$ **by** (*metis dvd-triv-left*)
with d **show** $x \wedge \text{Suc } n \text{ dvd } d$ **by** *simp*
qed
qed

have $\text{degree } x = 0$
proof (*cases degree x, simp*)
case (*Suc n*)
hence $x \neq 0$ **by** *auto*
with *Suc* **have** $\text{degree } (x \wedge (\text{Suc } (\text{degree } d))) > \text{degree } d$
by (*subst degree-power-eq, simp-all*)
moreover from $x\text{-pow-}n\text{-dvd-}d$ [*of Suc (degree d)*] **and** $\langle d \neq 0 \rangle$
have $\text{degree } (x \wedge \text{Suc } (\text{degree } d)) \leq \text{degree } d$
by (*simp add: dvd-imp-degree-le*)
ultimately show *?thesis* **by** *simp*
qed
then obtain c **where** [*simp*]: $x = [:c:]$ **by** (*cases x, simp split: if-split-asm*)
moreover from $\langle x \neq 0 \rangle$ **have** $c \neq 0$ **by** *simp*
ultimately show $x \text{ dvd } 1$ **using** *dvdI* [*of 1 x [:inverse c:]*]
by *simp*
qed

then show *coprime t* (*pderiv t*)
by (*rule coprimeI*)
qed

lemma *normalize-field*:
 $\text{normalize } (x :: 'a :: \{\text{field}, \text{normalization-semidom}\}) = (\text{if } x = 0 \text{ then } 0 \text{ else } 1)$
by (*auto simp: is-unit-normalize dvd-field-iff*)

lemma *normalize-field-eq-1* [*simp*]:
 $x \neq 0 \implies \text{normalize } (x :: 'a :: \{\text{field}, \text{normalization-semidom}\}) = 1$
by (*simp add: normalize-field*)

lemma *unit-factor-field* [*simp*]:
 $\text{unit-factor } (x :: 'a :: \{\text{field}, \text{normalization-semidom}\}) = x$
by (*cases x = 0*) (*auto simp: is-unit-unit-factor dvd-field-iff*)

Dividing a polynomial by its gcd with its derivative yields a squarefree polynomial with the same roots.

lemma *poly-div-gcd-squarefree*:
assumes $(p :: ('a :: \{\text{field-char-0}, \text{field-gcd}\}) \text{ poly}) \neq 0$
defines $d \equiv \text{gcd } p \text{ (pderiv } p)$
shows *coprime* $(p \text{ div } d)$ $(\text{pderiv } (p \text{ div } d))$ (**is** *?A*) **and**
 $\bigwedge x. \text{poly } (p \text{ div } d) \ x = 0 \iff \text{poly } p \ x = 0$ (**is** $\bigwedge x. ?B \ x$)
proof–
have $?A \wedge (\forall x. ?B \ x)$
proof (*cases pderiv p = 0*)
case *False*

```

    from poly-div-gcd-squarefree-aux[OF this] show ?thesis
      unfolding d-def by auto
  next
  case True
  then obtain c where [simp]: p = [:c:] using pderiv-iszero by blast
  from assms(1) have c ≠ 0 by simp
  from True have d = smult (inverse c) p
    by (simp add: d-def normalize-poly-def map-poly-pCons field-simps)
  with ⟨p ≠ 0⟩ ⟨c ≠ 0⟩ have p div d = [:c:]
    by (simp add: pCons-one)
  with ⟨c ≠ 0⟩ show ?thesis
    by (simp add: normalize-const-poly is-unit-triv)
qed
thus ?A and  $\bigwedge x. ?B x$  by simp-all
qed

```

1.2.3 Sign changes of a polynomial

If a polynomial has different signs at two points, it has a root inbetween.

lemma *poly-different-sign-imp-root*:

assumes $a < b$ **and** $\text{sgn}(\text{poly } p \ a) \neq \text{sgn}(\text{poly } p \ (b::\text{real}))$

shows $\exists x. a \leq x \wedge x \leq b \wedge \text{poly } p \ x = 0$

proof (cases $\text{poly } p \ a = 0 \vee \text{poly } p \ b = 0$)

case True

thus ?thesis using assms(1)

by (elim disjE, rule-tac exI[of - a], simp,
rule-tac exI[of - b], simp)

next

case False

hence [simp]: $\text{poly } p \ a \neq 0 \ \text{poly } p \ b \neq 0$ by simp-all

show ?thesis

proof (cases $\text{poly } p \ a < 0$)

case True

hence $\text{sgn}(\text{poly } p \ a) = -1$ by simp

with assms True have $\text{poly } p \ b > 0$

by (auto simp: sgn-real-def split: if-split-asm)

from poly-IVT-pos[OF ⟨a < b⟩ True this] guess x ..

thus ?thesis by (intro exI[of - x], simp)

next

case False

hence $\text{poly } p \ a > 0$ by (simp add: not-less less-eq-real-def)

hence $\text{sgn}(\text{poly } p \ a) = 1$ by simp

with assms False have $\text{poly } p \ b < 0$

by (auto simp: sgn-real-def not-less
less-eq-real-def split: if-split-asm)

from poly-IVT-neg[OF ⟨a < b⟩ ⟨poly p a > 0⟩ this] guess x ..

thus ?thesis by (intro exI[of - x], simp)

qed

qed

lemma *poly-different-sign-imp-root'*:
assumes $\text{sgn}(\text{poly } p \ a) \neq \text{sgn}(\text{poly } p \ (b::\text{real}))$
shows $\exists x. \text{poly } p \ x = 0$
using *assms* **by** (*cases* $a < b$, *auto* *dest!*: *poly-different-sign-imp-root*
simp: *less-eq-real-def not-less*)

lemma *no-roots-inbetween-imp-same-sign*:
assumes $a < b \ \forall x. a \leq x \wedge x \leq b \longrightarrow \text{poly } p \ x \neq (0::\text{real})$
shows $\text{sgn}(\text{poly } p \ a) = \text{sgn}(\text{poly } p \ b)$
using *poly-different-sign-imp-root* *assms* **by** *auto*

1.2.4 Limits of polynomials

lemma *poly-neighbourhood-without-roots*:
assumes $(p :: \text{real poly}) \neq 0$
shows *eventually* $(\lambda x. \text{poly } p \ x \neq 0)$ (*at* x_0)
proof –
{
fix $\varepsilon :: \text{real}$ **assume** $\varepsilon > 0$
have *fin*: *finite* $\{x. |x - x_0| < \varepsilon \wedge x \neq x_0 \wedge \text{poly } p \ x = 0\}$
using *poly-roots-finite*[*OF* *assms*] **by** *simp*
with $\langle \varepsilon > 0 \rangle$ **have** $\exists \delta > 0. \delta \leq \varepsilon \wedge (\forall x. |x - x_0| < \delta \wedge x \neq x_0 \longrightarrow \text{poly } p \ x \neq 0)$
proof (*induction* *card* $\{x. |x - x_0| < \varepsilon \wedge x \neq x_0 \wedge \text{poly } p \ x = 0\}$
arbitrary: ε *rule*: *less-induct*)
case (*less* ε)
let $?A = \{x. |x - x_0| < \varepsilon \wedge x \neq x_0 \wedge \text{poly } p \ x = 0\}$
show *?case*
proof (*cases* *card* $?A$)
case 0
hence $?A = \{\}$ **using** *less* **by** *auto*
thus *?thesis* **using** *less(2)* **by** (*rule-tac* *exI*[*of* - ε], *auto*)
next
case (*Suc* -)
with *less(3)* **have** $\{x. |x - x_0| < \varepsilon \wedge x \neq x_0 \wedge \text{poly } p \ x = 0\} \neq \{\}$ **by** *force*
then **obtain** x **where** *x-props*: $|x - x_0| < \varepsilon \wedge x \neq x_0 \wedge \text{poly } p \ x = 0$ **by** *blast*
define ε' **where** $\varepsilon' = |x - x_0| / 2$
have $\varepsilon' > 0 \ \varepsilon' < \varepsilon$ **unfolding** ε' -*def* **using** *x-props* **by** *simp-all*
from *x-props*(1,2) **and** $\langle \varepsilon > 0 \rangle$
have $x \notin \{x'. |x' - x_0| < \varepsilon' \wedge x' \neq x_0 \wedge \text{poly } p \ x' = 0\}$ (*is* - \notin $?B$)
by (*auto* *simp*: ε' -*def*)
moreover **from** *x-props*
have $x \in \{x. |x - x_0| < \varepsilon \wedge x \neq x_0 \wedge \text{poly } p \ x = 0\}$ **by** *blast*
ultimately **have** $?B \subset ?A$ **by** *auto*
hence *card* $?B < \text{card } ?A$ *finite* $?B$
by (*rule* *psubset-card-mono*[*OF* *less(3)*],
blast *intro*: *finite-subset*[*OF* - *less(3)*])
from *less(1)*[*OF* *this(1)* $\langle \varepsilon' > 0 \rangle$ *this(2)*]

```

      show ?thesis using ‹ $\varepsilon' < \varepsilon$ › by force
    qed
  qed
}
from this[of 1]
  show ?thesis by (auto simp: eventually-at dist-real-def)
qed

```

lemma *poly-neighbourhood-same-sign*:

```

  assumes poly p (x0 :: real) ≠ 0
  shows eventually (λx. sgn (poly p x) = sgn (poly p x0)) (at x0)
proof -
  have cont: isCont (λx. sgn (poly p x)) x0
    by (rule isCont-sgn, rule poly-isCont, rule assms)
  then have eventually (λx. |sgn (poly p x) - sgn (poly p x0)| < 1) (at x0)
    by (auto simp: isCont-def tendsto-iff dist-real-def)
  then show ?thesis
    by (rule eventually-mono) (simp add: sgn-real-def split: if-split-asm)
qed

```

lemma *poly-lhopital*:

```

  assumes poly p (x::real) = 0 poly q x = 0 q ≠ 0
  assumes (λx. poly (pderiv p) x / poly (pderiv q) x) -x→ y
  shows (λx. poly p x / poly q x) -x→ y
using assms
proof (rule-tac lhopital)
  have isCont (poly p) x isCont (poly q) x by simp-all
  with assms(1,2) show poly p -x→ 0 poly q -x→ 0
    by (simp-all add: isCont-def)
  from ‹q ≠ 0› and ‹poly q x = 0› have pderiv q ≠ 0
    by (auto dest: pderiv-iszero)
  from poly-neighbourhood-without-roots[OF this]
  show eventually (λx. poly (pderiv q) x ≠ 0) (at x) .
qed (auto intro: poly-DERIV poly-neighbourhood-without-roots)

```

lemma *poly-roots-bounds*:

```

  assumes p ≠ 0
  obtains l u
  where l ≤ (u :: real)
    and poly p l ≠ 0
    and poly p u ≠ 0
    and {x. x > l ∧ x ≤ u ∧ poly p x = 0} = {x. poly p x = 0}
    and ∧x. x ≤ l ⇒ sgn (poly p x) = sgn (poly p l)
    and ∧x. x ≥ u ⇒ sgn (poly p x) = sgn (poly p u)
proof
  from assms have finite {x. poly p x = 0} (is finite ?roots)
    using poly-roots-finite by fast

```

let $?roots' = insert\ 0\ ?roots$
define l **where** $l = Min\ ?roots' - 1$
define u **where** $u = Max\ ?roots' + 1$
from $\langle finite\ ?roots \rangle$ **have** $A: finite\ ?roots'$ **by** *auto*
from $Min-le[OF\ this,\ of\ 0]$ **and** $Max-ge[OF\ this,\ of\ 0]$
show $l \leq u$ **by** *(simp add: l-def u-def)*
from $Min-le[OF\ A]$ **have** $l-props: \bigwedge x. x \leq l \implies poly\ p\ x \neq 0$
by *(fastforce simp: l-def)*
from $Max-ge[OF\ A]$ **have** $u-props: \bigwedge x. x \geq u \implies poly\ p\ x \neq 0$
by *(fastforce simp: u-def)*
from $l-props\ u-props$ **show** $[simp]: poly\ p\ l \neq 0\ poly\ p\ u \neq 0$ **by** *auto*
from $l-props$ **have** $\bigwedge x. poly\ p\ x = 0 \implies x > l$ **by** *(metis not-le)*
moreover from $u-props$ **have** $\bigwedge x. poly\ p\ x = 0 \implies x \leq u$ **by** *(metis linear)*
ultimately show $\{x. x > l \wedge x \leq u \wedge poly\ p\ x = 0\} = ?roots$ **by** *auto*
{
 fix x **assume** $A: x < l\ sgn\ (poly\ p\ x) \neq sgn\ (poly\ p\ l)$
 with $poly-IVT-pos[OF\ A(1),\ of\ p]\ poly-IVT-neg[OF\ A(1),\ of\ p]\ A(2)$
 have *False* **by** *(auto split: if-split-asm)*
 simp: sgn-real-def l-props not-less less-eq-real-def
}
thus $\bigwedge x. x \leq l \implies sgn\ (poly\ p\ x) = sgn\ (poly\ p\ l)$
by *(case-tac x = l, auto simp: less-eq-real-def)*
{
 fix x **assume** $A: x > u\ sgn\ (poly\ p\ x) \neq sgn\ (poly\ p\ u)$
 with $u-props\ poly-IVT-neg[OF\ A(1),\ of\ p]\ poly-IVT-pos[OF\ A(1),\ of\ p]\ A(2)$
 have *False* **by** *(auto split: if-split-asm)*
 simp: sgn-real-def l-props not-less less-eq-real-def
}
thus $\bigwedge x. x \geq u \implies sgn\ (poly\ p\ x) = sgn\ (poly\ p\ u)$
by *(case-tac x = u, auto simp: less-eq-real-def)*
qed

definition $poly-inf :: ('a::real-normed-vector)\ poly \Rightarrow 'a$ **where**
 $poly-inf\ p \equiv sgn\ (coeff\ p\ (degree\ p))$

definition $poly-neg-inf :: ('a::real-normed-vector)\ poly \Rightarrow 'a$ **where**
 $poly-neg-inf\ p \equiv if\ even\ (degree\ p)\ then\ sgn\ (coeff\ p\ (degree\ p))$
 else $-sgn\ (coeff\ p\ (degree\ p))$

lemma $poly-inf-0-iff[simp]:$
 $poly-inf\ p = 0 \iff p = 0\ poly-neg-inf\ p = 0 \iff p = 0$
by *(auto simp: poly-inf-def poly-neg-inf-def sgn-zero-iff)*

lemma $poly-inf-mult[simp]:$

```

fixes p :: ('a::real-normed-field) poly
shows poly-inf (p*q) = poly-inf p * poly-inf q
        poly-neg-inf (p*q) = poly-neg-inf p * poly-neg-inf q
unfolding poly-inf-def poly-neg-inf-def
by ((cases p = 0 ∨ q = 0, auto simp: sgn-zero-iff
      degree-mult-eq[of p q] coeff-mult-degree-sum Real-Vector-Spaces.sgn-mult)[]) +

```

```

lemma poly-neg-0-at-infinity:
  assumes (p :: real poly) ≠ 0
  shows eventually (λx. poly p x ≠ 0) at-infinity
proof -
  from poly-roots-bounds[OF assms] guess l u .
  note lu-props = this
  define b where b = max (-l) u
  show ?thesis
  proof (subst eventually-at-infinity, rule exI[of - b], clarsimp)
    fix x assume A: |x| ≥ b and B: poly p x = 0
    show False
    proof (cases x ≥ 0)
      case True
        with A have x ≥ u unfolding b-def by simp
        with lu-props(3, 6) show False by (metis sgn-zero-iff B)
      next
      case False
        with A have x ≤ l unfolding b-def by simp
        with lu-props(2, 5) show False by (metis sgn-zero-iff B)
    qed
  qed
qed

```

```

lemma poly-limit-aux:
  fixes p :: real poly
  defines n ≡ degree p
  shows ((λx. poly p x / x ^ n) → coeff p n) at-infinity
proof (subst filterlim-cong, rule refl, rule refl)
  show eventually (λx. poly p x / x ^ n = (∑ i ≤ n. coeff p i / x ^ (n - i)))
    at-infinity
  proof (rule eventually-mono)
    show eventually (λx::real. x ≠ 0) at-infinity
      by (simp add: eventually-at-infinity, rule exI[of - 1], auto)
    fix x :: real assume [simp]: x ≠ 0
    show poly p x / x ^ n = (∑ i ≤ n. coeff p i / x ^ (n - i))
      by (simp add: n-def sum-divide-distrib power-diff poly-altdef)
  qed

```

let $?a = \lambda i. \text{if } i = n \text{ then coeff } p \ n \text{ else } 0$
have $\forall i \in \{..n\}. ((\lambda x. \text{coeff } p \ i / x^{(n-i)}) \longrightarrow ?a \ i) \text{ at-infinity}$
proof
fix i **assume** $i \in \{..n\}$
hence $i \leq n$ **by** *simp*
show $((\lambda x. \text{coeff } p \ i / x^{(n-i)}) \longrightarrow ?a \ i) \text{ at-infinity}$
proof (*cases* $i = n$)
case *True*
thus $?thesis$ **by** (*intro* *tendstoI*, *subst* *eventually-at-infinity*,
intro *exI*[*of - 1*], *simp* *add: dist-real-def*)
next
case *False*
hence $n - i > 0$ **using** $\langle i \leq n \rangle$ **by** *simp*
from *tendsto-inverse-0* **and** *divide-real-def*[*of 1*]
have $((\lambda x. 1 / x :: \text{real}) \longrightarrow 0) \text{ at-infinity}$ **by** *simp*
from *tendsto-power*[*OF this*, *of* $n - i$]
have $((\lambda x::\text{real}. 1 / x^{(n-i)}) \longrightarrow 0) \text{ at-infinity}$
using $\langle n - i > 0 \rangle$ **by** (*simp* *add: power-0-left power-one-over*)
from *tendsto-mult-right-zero*[*OF this*, *of* *coeff* $p \ i$]
have $((\lambda x. \text{coeff } p \ i / x^{(n-i)}) \longrightarrow 0) \text{ at-infinity}$
by (*simp* *add: field-simps*)
thus $?thesis$ **using** *False* **by** *simp*
qed
qed
hence $((\lambda x. \sum_{i \leq n}. \text{coeff } p \ i / x^{(n-i)}) \longrightarrow (\sum_{i \leq n}. ?a \ i)) \text{ at-infinity}$
by (*force* *intro!*: *tendsto-sum*)
also **have** $(\sum_{i \leq n}. ?a \ i) = \text{coeff } p \ n$ **by** (*subst* *sum.delta*, *simp-all*)
finally **show** $((\lambda x. \sum_{i \leq n}. \text{coeff } p \ i / x^{(n-i)}) \longrightarrow \text{coeff } p \ n) \text{ at-infinity}$.
qed

lemma *poly-at-top-at-top*:

fixes $p :: \text{real}$ *poly*

assumes $\text{degree } p \geq 1$ *coeff* $p \ (\text{degree } p) > 0$

shows $LIM \ x \ \text{at-top}. \ \text{poly } p \ x \ :=> \ \text{at-top}$

proof –

let $?n = \text{degree } p$

define $f \ g$ **where** $f \ x = \text{poly } p \ x / x^{?n}$ **and** $g \ x = x^{?n}$ **for** $x :: \text{real}$

from *poly-limit-aux* **have** $(f \longrightarrow \text{coeff } p \ (\text{degree } p)) \text{ at-top}$

using *tendsto-mono at-top-le-at-infinity* **unfolding** *f-def* **by** *blast*

moreover **from** *assms*

have $LIM \ x \ \text{at-top}. \ g \ x \ :=> \ \text{at-top}$

by (*auto* *simp* *add: g-def* *intro!*: *filterlim-pow-at-top* *filterlim-ident*)

ultimately **have** $LIM \ x \ \text{at-top}. \ f \ x * g \ x \ :=> \ \text{at-top}$

using *filterlim-tendsto-pos-mult-at-top* *assms* **by** *simp*

also **have** *eventually* $(\lambda x. f \ x * g \ x = \text{poly } p \ x) \text{ at-top}$

unfolding *f-def* *g-def*

by (subst eventually-at-top-linorder, rule exI[of - 1],
 simp add: poly-altdef field-simps sum-distrib-left power-diff)
 note filterlim-cong[OF refl refl this]
 finally show ?thesis .
 qed

lemma poly-at-bot-at-top:
 fixes $p :: \text{real poly}$
 assumes $\text{degree } p \geq 1$ $\text{coeff } p (\text{degree } p) < 0$
 shows $\text{LIM } x \text{ at-top. } \text{poly } p \ x \ :> \text{ at-bot}$
proof –
 from poly-at-top-at-top[of -p] and assms
 have $\text{LIM } x \text{ at-top. } -\text{poly } p \ x \ :> \text{ at-top}$ by simp
 thus ?thesis by (simp add: filterlim-uminus-at-bot)
 qed

lemma poly-lim-inf:
 eventually ($\lambda x::\text{real. } \text{sgn } (\text{poly } p \ x) = \text{poly-inf } p$) at-top
proof (cases degree $p \geq 1$)
 case False
 hence $\text{degree } p = 0$ by simp
 then obtain c where $p = [c]$ by (cases p, auto split: if-split-asm)
 thus ?thesis
 by (simp add: eventually-at-top-linorder poly-inf-def)

next
 case True
 note $\text{deg} = \text{this}$
 let $?lc = \text{coeff } p (\text{degree } p)$
 from True have $?lc \neq 0$ by force
 show ?thesis
proof (cases $?lc > 0$)
 case True
 from poly-at-top-at-top[OF deg this]
 obtain x_0 where $\bigwedge x. x \geq x_0 \implies \text{poly } p \ x \geq 1$
 by (fastforce simp: filterlim-at-top
 eventually-at-top-linorder less-eq-real-def)
 hence $\bigwedge x. x \geq x_0 \implies \text{sgn } (\text{poly } p \ x) = 1$ by force
 thus ?thesis by (simp only: eventually-at-top-linorder poly-inf-def,
 intro exI[of - x_0], simp add: True)

next
 case False
 hence $?lc < 0$ using $\langle ?lc \neq 0 \rangle$ by linarith
 from poly-at-bot-at-top[OF deg this]
 obtain x_0 where $\bigwedge x. x \geq x_0 \implies \text{poly } p \ x \leq -1$
 by (fastforce simp: filterlim-at-bot
 eventually-at-top-linorder less-eq-real-def)
 hence $\bigwedge x. x \geq x_0 \implies \text{sgn } (\text{poly } p \ x) = -1$ by force
 thus ?thesis by (simp only: eventually-at-top-linorder poly-inf-def,
 intro exI[of - x_0], simp add: $\langle ?lc < 0 \rangle$)

qed
qed

lemma *poly-at-top-or-bot-at-bot*:

fixes $p :: \text{real poly}$

assumes $\text{degree } p \geq 1 \text{ coeff } p (\text{degree } p) > 0$

shows $\text{LIM } x \text{ at-bot. } \text{poly } p \ x :=> (\text{if even } (\text{degree } p) \text{ then at-top else at-bot})$

proof –

let $?n = \text{degree } p$

define $f \ g$ **where** $f \ x = \text{poly } p \ x / x \wedge ?n$ **and** $g \ x = x \wedge ?n$ **for** $x :: \text{real}$

from *poly-limit-aux* **have** $(f \longrightarrow \text{coeff } p (\text{degree } p)) \text{ at-bot}$

using *tendsto-mono at-bot-le-at-infinity* **by** (*force simp: f-def [abs-def]*)

moreover from *assms*

have $\text{LIM } x \text{ at-bot. } g \ x :=> (\text{if even } (\text{degree } p) \text{ then at-top else at-bot})$

by (*auto simp add: g-def split: if-split-asm intro: filterlim-pow-at-bot-even filterlim-pow-at-bot-odd filterlim-ident*)

ultimately have $\text{LIM } x \text{ at-bot. } f \ x * g \ x :=>$

$(\text{if even } ?n \text{ then at-top else at-bot})$

by (*auto simp: assms intro: filterlim-tendsto-pos-mult-at-top filterlim-tendsto-pos-mult-at-bot*)

also have *eventually* $(\lambda x. f \ x * g \ x = \text{poly } p \ x) \text{ at-bot}$

unfolding *f-def g-def*

by (*subst eventually-at-bot-linorder, rule exI[of - -1],*

simp add: poly-altdef field-simps sum-distrib-left power-diff)

note *filterlim-cong[OF refl refl this]*

finally show *?thesis* .

qed

lemma *poly-at-bot-or-top-at-bot*:

fixes $p :: \text{real poly}$

assumes $\text{degree } p \geq 1 \text{ coeff } p (\text{degree } p) < 0$

shows $\text{LIM } x \text{ at-bot. } \text{poly } p \ x :=> (\text{if even } (\text{degree } p) \text{ then at-bot else at-top})$

proof –

from *poly-at-top-or-bot-at-bot[of -p]* **and** *assms*

have $\text{LIM } x \text{ at-bot. } -\text{poly } p \ x :=>$

$(\text{if even } (\text{degree } p) \text{ then at-top else at-bot})$ **by** *simp*

thus *?thesis* **by** (*auto simp: filterlim-uminus-at-bot*)

qed

lemma *poly-lim-neg-inf*:

eventually $(\lambda x::\text{real. } \text{sgn } (\text{poly } p \ x) = \text{poly-neg-inf } p) \text{ at-bot}$

proof (*cases degree p ≥ 1*)

case *False*

hence $\text{degree } p = 0$ **by** *simp*

then obtain c **where** $p = [c:]$ **by** (*cases p, auto split: if-split-asm*)

thus *?thesis*

by (*simp add: eventually-at-bot-linorder poly-neg-inf-def*)

```

next
case True
  note deg = this
  let ?lc = coeff p (degree p)
  from True have ?lc ≠ 0 by force
  show ?thesis
  proof (cases ?lc > 0)
    case True
      note lc-pos = this
      show ?thesis
      proof (cases even (degree p))
        case True
          from poly-at-top-or-bot-at-bot[OF deg lc-pos] and True
          obtain x0 where  $\bigwedge x. x \leq x_0 \implies \text{poly } p \ x \geq 1$ 
            by (fastforce simp add: filterlim-at-top filterlim-at-bot
              eventually-at-bot-linorder less-eq-real-def)
          hence  $\bigwedge x. x \leq x_0 \implies \text{sgn } (\text{poly } p \ x) = 1$  by force
          thus ?thesis
            by (simp add: True eventually-at-bot-linorder poly-neg-inf-def,
              intro exI[of - x0], simp add: lc-pos)
        case False
          from poly-at-top-or-bot-at-bot[OF deg lc-pos] and False
          obtain x0 where  $\bigwedge x. x \leq x_0 \implies \text{poly } p \ x \leq -1$ 
            by (fastforce simp add: filterlim-at-bot
              eventually-at-bot-linorder less-eq-real-def)
          hence  $\bigwedge x. x \leq x_0 \implies \text{sgn } (\text{poly } p \ x) = -1$  by force
          thus ?thesis
            by (simp add: False eventually-at-bot-linorder poly-neg-inf-def,
              intro exI[of - x0], simp add: lc-pos)
      qed
    next
      case False
        hence lc-neg: ?lc < 0 using ‹?lc ≠ 0› by linarith
        show ?thesis
        proof (cases even (degree p))
          case True
            with poly-at-bot-or-top-at-bot[OF deg lc-neg]
            obtain x0 where  $\bigwedge x. x \leq x_0 \implies \text{poly } p \ x \leq -1$ 
              by (fastforce simp: filterlim-at-bot
                eventually-at-bot-linorder less-eq-real-def)
            hence  $\bigwedge x. x \leq x_0 \implies \text{sgn } (\text{poly } p \ x) = -1$  by force
            thus ?thesis
              by (simp only: True eventually-at-bot-linorder poly-neg-inf-def,
                intro exI[of - x0], simp add: lc-neg)
          case False
            with poly-at-bot-or-top-at-bot[OF deg lc-neg]
            obtain x0 where  $\bigwedge x. x \leq x_0 \implies \text{poly } p \ x \geq 1$ 

```

by (fastforce simp: filterlim-at-top
 eventually-at-bot-linorder less-eq-real-def)
 hence $\bigwedge x. x \leq x_0 \implies \text{sgn}(\text{poly } p \ x) = 1$ by force
 thus ?thesis
 by (simp only: False eventually-at-bot-linorder poly-neg-inf-def,
 intro exI[of - x₀], simp add: lc-neg)
 qed
 qed
 qed

1.2.5 Signs of polynomials for sufficiently large values

lemma polys-inf-sign-thresholds:

assumes finite (ps :: real poly set)

obtains l u

where $l \leq u$

and $\bigwedge p. \llbracket p \in ps; p \neq 0 \rrbracket \implies$

$\{x. l < x \wedge x \leq u \wedge \text{poly } p \ x = 0\} = \{x. \text{poly } p \ x = 0\}$

and $\bigwedge p \ x. \llbracket p \in ps; x \geq u \rrbracket \implies \text{sgn}(\text{poly } p \ x) = \text{poly-inf } p$

and $\bigwedge p \ x. \llbracket p \in ps; x \leq l \rrbracket \implies \text{sgn}(\text{poly } p \ x) = \text{poly-neg-inf } p$

proof goal-cases

case prems: 1

have $\exists l \ u. l \leq u \wedge (\forall p \ x. p \in ps \wedge x \geq u \longrightarrow \text{sgn}(\text{poly } p \ x) = \text{poly-inf } p) \wedge$
 $(\forall p \ x. p \in ps \wedge x \leq l \longrightarrow \text{sgn}(\text{poly } p \ x) = \text{poly-neg-inf } p)$

(is $\exists l \ u. ?P \ ps \ l \ u$)

proof (induction rule: finite-subset-induct[OF assms(1), where A = UNIV])

case 1

show ?case by simp

next

case 2

show ?case by (intro exI[of - 42], simp)

next

case prems: (3 p ps)

from prems(4) obtain l u where lu-props: ?P ps l u by blast

from poly-lim-inf obtain u'

where u'-props: $\forall x \geq u'. \text{sgn}(\text{poly } p \ x) = \text{poly-inf } p$

by (force simp add: eventually-at-top-linorder)

from poly-lim-neg-inf obtain l'

where l'-props: $\forall x \leq l'. \text{sgn}(\text{poly } p \ x) = \text{poly-neg-inf } p$

by (force simp add: eventually-at-bot-linorder)

show ?case

by (rule exI[of - min l l'], rule exI[of - max u u'],
 insert lu-props l'-props u'-props, auto)

qed

then obtain l u where lu-props: $l \leq u$

$\bigwedge p \ x. p \in ps \implies u \leq x \implies \text{sgn}(\text{poly } p \ x) = \text{poly-inf } p$

$\bigwedge p \ x. p \in ps \implies x \leq l \implies \text{sgn}(\text{poly } p \ x) = \text{poly-neg-inf } p$ by blast

moreover {

fix p x assume A: $p \in ps \ p \neq 0 \ \text{poly } p \ x = 0$

```

from  $A$  have  $l < x < u$ 
  by (auto simp: not-le[symmetric] dest: lu-props(2,3))
}
note  $A = \text{this}$ 
have  $\bigwedge p. p \in ps \implies p \neq 0 \implies$ 
   $\{x. l < x \wedge x \leq u \wedge \text{poly } p \ x = 0\} = \{x. \text{poly } p \ x = 0\}$ 
  by (auto dest: A)

from prems[OF lu-props(1) this lu-props(2,3)] show thesis .
qed

```

1.2.6 Positivity of polynomials

lemma *poly-pos:*

$(\forall x::\text{real}. \text{poly } p \ x > 0) \iff \text{poly-inf } p = 1 \wedge (\forall x. \text{poly } p \ x \neq 0)$

proof (*intro iffI conjI*)

assume $A: \forall x::\text{real}. \text{poly } p \ x > 0$

have $\bigwedge x. \text{poly } p \ (x::\text{real}) > 0 \implies \text{poly } p \ x \neq 0$ **by** *simp*

with A **show** $\forall x::\text{real}. \text{poly } p \ x \neq 0$ **by** *simp*

from *poly-lim-inf* **obtain** x **where** $\text{sgn } (\text{poly } p \ x) = \text{poly-inf } p$

by (*auto simp: eventually-at-top-linorder*)

with A **show** $\text{poly-inf } p = 1$

by (*simp add: sgn-real-def split: if-split-asm*)

next

assume $\text{poly-inf } p = 1 \wedge (\forall x. \text{poly } p \ x \neq 0)$

hence $A: \text{poly-inf } p = 1$ **and** $B: (\forall x. \text{poly } p \ x \neq 0)$ **by** *simp-all*

from *poly-lim-inf* **obtain** x **where** $C: \text{sgn } (\text{poly } p \ x) = \text{poly-inf } p$

by (*auto simp: eventually-at-top-linorder*)

show $\forall x. \text{poly } p \ x > 0$

proof (*rule ccontr*)

assume $\neg(\forall x. \text{poly } p \ x > 0)$

then obtain x' **where** $\text{poly } p \ x' \leq 0$ **by** (*auto simp: not-less*)

with A **and** C **have** $\text{sgn } (\text{poly } p \ x') \neq \text{sgn } (\text{poly } p \ x)$

by (*auto simp: sgn-real-def split: if-split-asm*)

from *poly-different-sign-imp-root'[OF this]* **and** B

show *False* **by** *blast*

qed

qed

lemma *poly-pos-greater:*

$(\forall x::\text{real}. x > a \implies \text{poly } p \ x > 0) \iff$

$\text{poly-inf } p = 1 \wedge (\forall x. x > a \implies \text{poly } p \ x \neq 0)$

proof (*intro iffI conjI*)

assume $A: \forall x::\text{real}. x > a \implies \text{poly } p \ x > 0$

have $\bigwedge x. \text{poly } p \ (x::\text{real}) > 0 \implies \text{poly } p \ x \neq 0$ **by** *simp*

with A **show** $\forall x::\text{real}. x > a \implies \text{poly } p \ x \neq 0$ **by** *auto*

from *poly-lim-inf* **obtain** x_0 **where**

$\forall x \geq x_0. \text{sgn}(\text{poly } p \ x) = \text{poly-inf } p$
by (*auto simp: eventually-at-top-linorder*)
hence $\text{poly-inf } p = \text{sgn}(\text{poly } p \ (\max x_0 \ (a + 1)))$ **by** *simp*
also from A **have** $\dots = 1$ **by** *force*
finally show $\text{poly-inf } p = 1$.

next
assume $\text{poly-inf } p = 1 \wedge (\forall x. x > a \longrightarrow \text{poly } p \ x \neq 0)$
hence A : $\text{poly-inf } p = 1$ **and**
 B : $(\forall x. x > a \longrightarrow \text{poly } p \ x \neq 0)$ **by** *simp-all*
from *poly-lim-inf* **obtain** x_0 **where**
 C : $\forall x \geq x_0. \text{sgn}(\text{poly } p \ x) = \text{poly-inf } p$
by (*auto simp: eventually-at-top-linorder*)
hence $\text{sgn}(\text{poly } p \ (\max x_0 \ (a+1))) = \text{poly-inf } p$ **by** *simp*
with A **have** D : $\text{sgn}(\text{poly } p \ (\max x_0 \ (a+1))) = 1$ **by** *simp*
show $\forall x. x > a \longrightarrow \text{poly } p \ x > 0$
proof (*rule ccontr*)
assume $\neg(\forall x. x > a \longrightarrow \text{poly } p \ x > 0)$
then obtain x' **where** $x' > a$ $\text{poly } p \ x' \leq 0$ **by** (*auto simp: not-less*)
with A **and** D **have** E : $\text{sgn}(\text{poly } p \ x') \neq \text{sgn}(\text{poly } p \ (\max x_0 \ (a+1)))$
by (*auto simp: sgn-real-def split: if-split-asm*)
show *False*
apply (*cases x' max x_0 (a+1) rule: linorder-cases*)
using $B \ E \ \langle x' > a \rangle$
apply (*force dest!: poly-different-sign-imp-root[of - - p]*)
done

qed
qed

lemma *poly-pos-geq*:
 $(\forall x::\text{real}. x \geq a \longrightarrow \text{poly } p \ x > 0) \longleftrightarrow$
 $\text{poly-inf } p = 1 \wedge (\forall x. x \geq a \longrightarrow \text{poly } p \ x \neq 0)$
proof (*intro iffI conjI*)
assume A : $\forall x::\text{real}. x \geq a \longrightarrow \text{poly } p \ x > 0$
hence $\forall x::\text{real}. x > a \longrightarrow \text{poly } p \ x > 0$ **by** *simp*
also note *poly-pos-greater*
finally have $\text{poly-inf } p = 1 \ (\forall x > a. \text{poly } p \ x \neq 0)$ **by** *simp-all*
moreover from A **have** $\text{poly } p \ a > 0$ **by** *simp*
ultimately show $\text{poly-inf } p = 1 \ \forall x \geq a. \text{poly } p \ x \neq 0$
by (*auto simp: less-eq-real-def*)

next
assume $\text{poly-inf } p = 1 \wedge (\forall x. x \geq a \longrightarrow \text{poly } p \ x \neq 0)$
hence A : $\text{poly-inf } p = 1$ **and**
 B : $\text{poly } p \ a \neq 0$ **and** C : $\forall x > a. \text{poly } p \ x \neq 0$ **by** *simp-all*
from A **and** C **and** *poly-pos-greater* **have** $\forall x > a. \text{poly } p \ x > 0$ **by** *simp*
moreover with $B \ C$ *poly-IVT-pos*[*of a a+1 p*] **have** $\text{poly } p \ a > 0$ **by** *force*
ultimately show $\forall x \geq a. \text{poly } p \ x > 0$ **by** (*auto simp: less-eq-real-def*)

qed

lemma *poly-pos-less*:

$(\forall x::real. x < a \longrightarrow poly\ p\ x > 0) \longleftrightarrow$
 $poly\text{-neg-inf}\ p = 1 \wedge (\forall x. x < a \longrightarrow poly\ p\ x \neq 0)$

proof (*intro iffI conjI*)

assume $A: \forall x::real. x < a \longrightarrow poly\ p\ x > 0$

have $\bigwedge x. poly\ p\ (x::real) > 0 \implies poly\ p\ x \neq 0$ **by** *simp*

with A **show** $\forall x::real. x < a \longrightarrow poly\ p\ x \neq 0$ **by** *auto*

from *poly-lim-neg-inf* **obtain** x_0 **where**

$\forall x \leq x_0. sgn\ (poly\ p\ x) = poly\text{-neg-inf}\ p$

by (*auto simp: eventually-at-bot-linorder*)

hence $poly\text{-neg-inf}\ p = sgn\ (poly\ p\ (min\ x_0\ (a - 1)))$ **by** *simp*

also from A **have** $\dots = 1$ **by** *force*

finally show $poly\text{-neg-inf}\ p = 1$.

next

assume $poly\text{-neg-inf}\ p = 1 \wedge (\forall x. x < a \longrightarrow poly\ p\ x \neq 0)$

hence $A: poly\text{-neg-inf}\ p = 1$ **and**

$B: (\forall x. x < a \longrightarrow poly\ p\ x \neq 0)$ **by** *simp-all*

from *poly-lim-neg-inf* **obtain** x_0 **where**

$C: \forall x \leq x_0. sgn\ (poly\ p\ x) = poly\text{-neg-inf}\ p$

by (*auto simp: eventually-at-bot-linorder*)

hence $sgn\ (poly\ p\ (min\ x_0\ (a - 1))) = poly\text{-neg-inf}\ p$ **by** *simp*

with A **have** $D: sgn\ (poly\ p\ (min\ x_0\ (a - 1))) = 1$ **by** *simp*

show $\forall x. x < a \longrightarrow poly\ p\ x > 0$

proof (*rule ccontr*)

assume $\neg(\forall x. x < a \longrightarrow poly\ p\ x > 0)$

then obtain x' **where** $x' < a$ $poly\ p\ x' \leq 0$ **by** (*auto simp: not-less*)

with A **and** D **have** $E: sgn\ (poly\ p\ x') \neq sgn\ (poly\ p\ (min\ x_0\ (a - 1)))$

by (*auto simp: sgn-real-def split: if-split-asm*)

show *False*

apply (*cases x' min x_0 (a - 1) rule: linorder-cases*)

using $B\ E\ \langle x' < a \rangle$

apply (*auto dest!: poly-different-sign-imp-root[of - - p]*)
done

qed

qed

lemma *poly-pos-leg*:

$(\forall x::real. x \leq a \longrightarrow poly\ p\ x > 0) \longleftrightarrow$
 $poly\text{-neg-inf}\ p = 1 \wedge (\forall x. x \leq a \longrightarrow poly\ p\ x \neq 0)$

proof (*intro iffI conjI*)

assume $A: \forall x::real. x \leq a \longrightarrow poly\ p\ x > 0$

hence $\forall x::real. x < a \longrightarrow poly\ p\ x > 0$ **by** *simp*

also note *poly-pos-less*

finally have $poly\text{-neg-inf}\ p = 1$ ($\forall x < a. poly\ p\ x \neq 0$) **by** *simp-all*

moreover from A **have** $poly\ p\ a > 0$ **by** *simp*

ultimately show $poly\text{-neg-inf}\ p = 1 \wedge \forall x \leq a. poly\ p\ x \neq 0$

by (*auto simp: less-eq-real-def*)

next

assume $poly\text{-neg-inf}\ p = 1 \wedge (\forall x. x \leq a \longrightarrow poly\ p\ x \neq 0)$

hence A : *poly-neg-inf* $p = 1$ and
 B : *poly* $p \ a \neq 0$ and C : $\forall x < a. \text{poly } p \ x \neq 0$ by *simp-all*
from A and C and *poly-pos-less* have $\forall x < a. \text{poly } p \ x > 0$ by *simp*
moreover with $B \ C$ *poly-IVT-neg*[of $a - 1 \ a \ p$] have *poly* $p \ a > 0$ by *force*
ultimately show $\forall x \leq a. \text{poly } p \ x > 0$ by (*auto simp: less-eq-real-def*)
qed

lemma *poly-pos-between-less-less*:

$(\forall x :: \text{real}. a < x \wedge x < b \longrightarrow \text{poly } p \ x > 0) \longleftrightarrow$
 $(a \geq b \vee \text{poly } p \ ((a+b)/2) > 0) \wedge (\forall x. a < x \wedge x < b \longrightarrow \text{poly } p \ x \neq 0)$

proof (*intro iffI conjI*)

assume A : $\forall x. a < x \wedge x < b \longrightarrow \text{poly } p \ x > 0$

have $\bigwedge x. \text{poly } p \ (x :: \text{real}) > 0 \implies \text{poly } p \ x \neq 0$ by *simp*

with A show $\forall x :: \text{real}. a < x \wedge x < b \longrightarrow \text{poly } p \ x \neq 0$ by *auto*

from A show $a \geq b \vee \text{poly } p \ ((a+b)/2) > 0$ by (*cases* $a < b$, *auto*)

next

assume $(b \leq a \vee 0 < \text{poly } p \ ((a+b)/2)) \wedge (\forall x. a < x \wedge x < b \longrightarrow \text{poly } p \ x \neq 0)$

hence A : $b \leq a \vee 0 < \text{poly } p \ ((a+b)/2)$ and

B : $\forall x. a < x \wedge x < b \longrightarrow \text{poly } p \ x \neq 0$ by *simp-all*

show $\forall x. a < x \wedge x < b \longrightarrow \text{poly } p \ x > 0$

proof (*cases* $a \geq b$, *simp*, *clarify*, *rule-tac ccontr*,

simp only: not-le not-less)

fix x assume $a < b \ a < x \ x < b \ \text{poly } p \ x \leq 0$

with B have *poly* $p \ x < 0$ by (*simp add: less-eq-real-def*)

moreover from A and $\langle a < b \rangle$ have *poly* $p \ ((a+b)/2) > 0$ by *simp*

ultimately have *sgn* (*poly* $p \ x$) \neq *sgn* (*poly* $p \ ((a+b)/2)$) by *simp*

thus *False* using B

apply (*cases* $x \ (a+b)/2$ *rule: linorder-cases*)

apply (*drule poly-different-sign-imp-root*[of $- \ - \ p$], *assumption*,
insert $\langle a < b \rangle \ \langle a < x \rangle \ \langle x < b \rangle$, *force*) \square

apply *simp*

apply (*drule poly-different-sign-imp-root*[of $- \ - \ p$], *simp*,
insert $\langle a < b \rangle \ \langle a < x \rangle \ \langle x < b \rangle$, *force*)

done

qed

qed

lemma *poly-pos-between-less-leq*:

$(\forall x :: \text{real}. a < x \wedge x \leq b \longrightarrow \text{poly } p \ x > 0) \longleftrightarrow$

$(a \geq b \vee \text{poly } p \ b > 0) \wedge (\forall x. a < x \wedge x \leq b \longrightarrow \text{poly } p \ x \neq 0)$

proof (*intro iffI conjI*)

assume A : $\forall x. a < x \wedge x \leq b \longrightarrow \text{poly } p \ x > 0$

have $\bigwedge x. \text{poly } p \ (x :: \text{real}) > 0 \implies \text{poly } p \ x \neq 0$ by *simp*

with A show $\forall x :: \text{real}. a < x \wedge x \leq b \longrightarrow \text{poly } p \ x \neq 0$ by *auto*

from A show $a \geq b \vee \text{poly } p \ b > 0$ by (*cases* $a < b$, *auto*)

next

assume $(b \leq a \vee 0 < \text{poly } p \ b) \wedge (\forall x. a < x \wedge x \leq b \longrightarrow \text{poly } p \ x \neq 0)$

hence A : $b \leq a \vee 0 < \text{poly } p \ b$ and B : $\forall x. a < x \wedge x \leq b \longrightarrow \text{poly } p \ x \neq 0$
by *simp-all*

show $\forall x. a < x \wedge x \leq b \longrightarrow \text{poly } p \ x > 0$
proof (*cases* $a \geq b$, *simp*, *clarify*, *rule-tac ccontr*,
simp only: not-le not-less)
fix x **assume** $a < b$ $a < x$ $x \leq b$ $\text{poly } p \ x \leq 0$
with B **have** $\text{poly } p \ x < 0$ **by** (*simp add: less-eq-real-def*)
moreover from A **and** $\langle a < b \rangle$ **have** $\text{poly } p \ b > 0$ **by** *simp*
ultimately have $x < b$ **using** $\langle x \leq b \rangle$ **by** (*auto simp: less-eq-real-def*)
from $\langle \text{poly } p \ x < 0 \rangle$ **and** $\langle \text{poly } p \ b > 0 \rangle$
have $\text{sgn } (\text{poly } p \ x) \neq \text{sgn } (\text{poly } p \ b)$ **by** *simp*
from *poly-different-sign-imp-root*[*OF* $\langle x < b \rangle$ *this*] **and** B **and** $\langle x > a \rangle$
show *False* **by** *auto*
qed
qed

lemma *poly-pos-between-leq-less*:

$(\forall x::\text{real}. a \leq x \wedge x < b \longrightarrow \text{poly } p \ x > 0) \longleftrightarrow$
 $(a \geq b \vee \text{poly } p \ a > 0) \wedge (\forall x. a \leq x \wedge x < b \longrightarrow \text{poly } p \ x \neq 0)$

proof (*intro iffI conjI*)

assume $A: \forall x. a \leq x \wedge x < b \longrightarrow \text{poly } p \ x > 0$
have $\bigwedge x. \text{poly } p \ (x::\text{real}) > 0 \implies \text{poly } p \ x \neq 0$ **by** *simp*
with A **show** $\forall x::\text{real}. a \leq x \wedge x < b \longrightarrow \text{poly } p \ x \neq 0$ **by** *auto*
from A **show** $a \geq b \vee \text{poly } p \ a > 0$ **by** (*cases* $a < b$, *auto*)

next

assume $(b \leq a \vee 0 < \text{poly } p \ a) \wedge (\forall x. a \leq x \wedge x < b \longrightarrow \text{poly } p \ x \neq 0)$
hence $A: b \leq a \vee 0 < \text{poly } p \ a$ **and** $B: \forall x. a \leq x \wedge x < b \longrightarrow \text{poly } p \ x \neq 0$
by *simp-all*

show $\forall x. a \leq x \wedge x < b \longrightarrow \text{poly } p \ x > 0$

proof (*cases* $a \geq b$, *simp*, *clarify*, *rule-tac ccontr*,
simp only: not-le not-less)

fix x **assume** $a < b$ $a \leq x$ $x < b$ $\text{poly } p \ x \leq 0$
with B **have** $\text{poly } p \ x < 0$ **by** (*simp add: less-eq-real-def*)
moreover from A **and** $\langle a < b \rangle$ **have** $\text{poly } p \ a > 0$ **by** *simp*
ultimately have $x > a$ **using** $\langle x \geq a \rangle$ **by** (*auto simp: less-eq-real-def*)
from $\langle \text{poly } p \ x < 0 \rangle$ **and** $\langle \text{poly } p \ a > 0 \rangle$
have $\text{sgn } (\text{poly } p \ a) \neq \text{sgn } (\text{poly } p \ x)$ **by** *simp*
from *poly-different-sign-imp-root*[*OF* $\langle x > a \rangle$ *this*] **and** B **and** $\langle x < b \rangle$
show *False* **by** *auto*

qed

qed

lemma *poly-pos-between-leq-leq*:

$(\forall x::\text{real}. a \leq x \wedge x \leq b \longrightarrow \text{poly } p \ x > 0) \longleftrightarrow$
 $(a > b \vee \text{poly } p \ a > 0) \wedge (\forall x. a \leq x \wedge x \leq b \longrightarrow \text{poly } p \ x \neq 0)$

proof (*intro iffI conjI*)

assume $A: \forall x. a \leq x \wedge x \leq b \longrightarrow \text{poly } p \ x > 0$
have $\bigwedge x. \text{poly } p \ (x::\text{real}) > 0 \implies \text{poly } p \ x \neq 0$ **by** *simp*
with A **show** $\forall x::\text{real}. a \leq x \wedge x \leq b \longrightarrow \text{poly } p \ x \neq 0$ **by** *auto*
from A **show** $a > b \vee \text{poly } p \ a > 0$ **by** (*cases* $a \leq b$, *auto*)

next

```

assume  $(b < a \vee 0 < \text{poly } p \ a) \wedge (\forall x. a \leq x \wedge x \leq b \longrightarrow \text{poly } p \ x \neq 0)$ 
hence  $A: b < a \vee 0 < \text{poly } p \ a$  and  $B: \forall x. a \leq x \wedge x \leq b \longrightarrow \text{poly } p \ x \neq 0$ 
  by simp-all
show  $\forall x. a \leq x \wedge x \leq b \longrightarrow \text{poly } p \ x > 0$ 
proof (cases  $a > b$ , simp, clarify, rule-tac ccontr,
  simp only: not-le not-less)
  fix  $x$  assume  $a \leq b$   $a \leq x$   $x \leq b$   $\text{poly } p \ x \leq 0$ 
  with  $B$  have  $\text{poly } p \ x < 0$  by (simp add: less-eq-real-def)
  moreover from  $A$  and  $\langle a \leq b \rangle$  have  $\text{poly } p \ a > 0$  by simp
  ultimately have  $x > a$  using  $\langle x \geq a \rangle$  by (auto simp: less-eq-real-def)
  from  $\langle \text{poly } p \ x < 0 \rangle$  and  $\langle \text{poly } p \ a > 0 \rangle$ 
    have  $\text{sgn} (\text{poly } p \ a) \neq \text{sgn} (\text{poly } p \ x)$  by simp
  from poly-different-sign-imp-root[OF  $\langle x > a \rangle$  this] and  $B$  and  $\langle x \leq b \rangle$ 
    show False by auto
qed
qed

end

```

2 Proof of Sturm's Theorem

theory *Sturm-Theorem*

imports *HOL-Computational-Algebra.Polynomial*

Lib/Sturm-Library *HOL-Computational-Algebra.Field-as-Ring*

begin

2.1 Sign changes of polynomial sequences

For a given sequence of polynomials, this function computes the number of sign changes of the sequence of polynomials evaluated at a given position x . A sign change is a change from a negative value to a positive one or vice versa; zeros in the sequence are ignored.

definition *sign-changes where*

sign-changes ps $(x::\text{real}) =$

$\text{length} (\text{remdups-adj} (\text{filter} (\lambda x. x \neq 0) (\text{map} (\lambda p. \text{sgn} (\text{poly } p \ x)) \ ps))) - 1$

The number of sign changes of a sequence distributes over a list in the sense that the number of sign changes of a sequence $p_1, \dots, p_i, \dots, p_n$ at x is the same as the sum of the sign changes of the sequence p_1, \dots, p_i and p_i, \dots, p_n as long as $p_i(x) \neq 0$.

lemma *sign-changes-distrib:*

$\text{poly } p \ x \neq 0 \implies$

$\text{sign-changes} (ps_1 @ [p] @ ps_2) \ x =$

$\text{sign-changes} (ps_1 @ [p]) \ x + \text{sign-changes} ([p] @ ps_2) \ x$

by (*simp add: sign-changes-def sgn-zero-iff, subst remdups-adj-append, simp*)

The following two congruences state that the number of sign changes is the same if all the involved signs are the same.

lemma *sign-changes-cong*:
assumes $\text{length } ps = \text{length } ps'$
assumes $\forall i < \text{length } ps. \text{sgn } (\text{poly } (ps!i) x) = \text{sgn } (\text{poly } (ps'!i) y)$
shows $\text{sign-changes } ps x = \text{sign-changes } ps' y$
proof –
from *assms*(2) **have** $A: \text{map } (\lambda p. \text{sgn } (\text{poly } p x)) ps = \text{map } (\lambda p. \text{sgn } (\text{poly } p y)) ps'$
proof (*induction rule: list-induct2[OF assms(1)]*)
case 1
then show ?*case* **by** *simp*
next
case (2 *p ps p' ps'*)
from 2(3)
have $\forall i < \text{length } ps. \text{sgn } (\text{poly } (ps ! i) x) = \text{sgn } (\text{poly } (ps' ! i) y)$ **by** *auto*
from 2(2)[*OF this*] 2(3) **show** ?*case* **by** *auto*
qed
show ?*thesis* **unfolding** *sign-changes-def* **by** (*simp add: A*)
qed

lemma *sign-changes-cong'*:
assumes $\forall p \in \text{set } ps. \text{sgn } (\text{poly } p x) = \text{sgn } (\text{poly } p y)$
shows $\text{sign-changes } ps x = \text{sign-changes } ps y$
using *assms* **by** (*intro sign-changes-cong, simp-all*)

For a sequence of polynomials of length 3, if the first and the third polynomial have opposite and nonzero sign at some x , the number of sign changes is always 1, irrespective of the sign of the second polynomial.

lemma *sign-changes-sturm-triple*:
assumes $\text{poly } p x \neq 0$ **and** $\text{sgn } (\text{poly } r x) = - \text{sgn } (\text{poly } p x)$
shows $\text{sign-changes } [p, q, r] x = 1$
unfolding *sign-changes-def* **by** (*insert assms, auto simp: sgn-real-def*)

Finally, we define two additional functions that count the sign changes “at infinity”.

definition *sign-changes-inf* **where**
 $\text{sign-changes-inf } ps = \text{length } (\text{remdups-adj } (\text{filter } (\lambda x. x \neq 0) (\text{map } \text{poly-inf } ps))) - 1$

definition *sign-changes-neg-inf* **where**
 $\text{sign-changes-neg-inf } ps = \text{length } (\text{remdups-adj } (\text{filter } (\lambda x. x \neq 0) (\text{map } \text{poly-neg-inf } ps))) - 1$

2.2 Definition of Sturm sequences locale

We first define the notion of a “Quasi-Sturm sequence”, which is a weakening of a Sturm sequence that captures the properties that are fulfilled by a nonempty suffix of a Sturm sequence:

- The sequence is nonempty.
- The last polynomial does not change its sign.
- If the middle one of three adjacent polynomials has a root at x , the other two have opposite and nonzero signs at x .

locale *quasi-sturm-seq* =
fixes $ps :: (\text{real poly}) \text{ list}$
assumes *last-ps-sgn-const*[simp]:
 $\bigwedge x y. \text{sgn} (\text{poly} (\text{last } ps) x) = \text{sgn} (\text{poly} (\text{last } ps) y)$
assumes *ps-not-Nil*[simp]: $ps \neq []$
assumes *signs*: $\bigwedge i x. \llbracket i < \text{length } ps - 2; \text{poly} (ps ! (i+1)) x = 0 \rrbracket$
 $\implies (\text{poly} (ps ! (i+2)) x) * (\text{poly} (ps ! i) x) < 0$

Now we define a Sturm sequence p_1, \dots, p_n of a polynomial p in the following way:

- The sequence contains at least two elements.
- p is the first polynomial, i. e. $p_1 = p$.
- At any root x of p , p_2 and p have opposite sign left of x and the same sign right of x in some neighbourhood around x .
- The first two polynomials in the sequence have no common roots.
- If the middle one of three adjacent polynomials has a root at x , the other two have opposite and nonzero signs at x .

locale *sturm-seq* = *quasi-sturm-seq* +
fixes $p :: \text{real poly}$
assumes *hd-ps-p*[simp]: $\text{hd } ps = p$
assumes *length-ps-ge-2*[simp]: $\text{length } ps \geq 2$
assumes *deriv*: $\bigwedge x_0. \text{poly } p x_0 = 0 \implies$
 $\text{eventually } (\lambda x. \text{sgn} (\text{poly} (p * ps!1) x) =$
 $(\text{if } x > x_0 \text{ then } 1 \text{ else } -1)) (\text{at } x_0)$
assumes *p-squarefree*: $\bigwedge x. \neg(\text{poly } p x = 0 \wedge \text{poly} (ps!1) x = 0)$
begin

Any Sturm sequence is obviously a Quasi-Sturm sequence.

lemma *quasi-sturm-seq*: *quasi-sturm-seq* $ps ..$
end

Any suffix of a Quasi-Sturm sequence is again a Quasi-Sturm sequence.

lemma *quasi-sturm-seq-Cons*:
assumes *quasi-sturm-seq* $(p\#ps)$ **and** $ps \neq []$
shows *quasi-sturm-seq* ps

```

proof (unfold-locales)
  show  $ps \neq []$  by fact
next
  from assms(1) interpret quasi-sturm-seq p#ps .
  fix  $x y$ 
  from last-ps-sgn-const and  $\langle ps \neq [] \rangle$ 
    show  $sgn (poly (last ps) x) = sgn (poly (last ps) y)$  by simp-all
next
  from assms(1) interpret quasi-sturm-seq p#ps .
  fix  $i x$ 
  assume  $i < length ps - 2$  and  $poly (ps ! (i+1)) x = 0$ 
  with signs[of i+1]
    show  $poly (ps ! (i+2)) x * poly (ps ! i) x < 0$  by simp
qed

```

2.3 Auxiliary lemmas about roots and sign changes

```

lemma sturm-adjacent-root-aux:
  assumes  $i < length (ps :: real poly list) - 1$ 
  assumes  $poly (ps ! i) x = 0$  and  $poly (ps ! (i + 1)) x = 0$ 
  assumes  $\bigwedge i x. [i < length ps - 2; poly (ps ! (i+1)) x = 0]$ 
     $\implies sgn (poly (ps ! (i+2)) x) = - sgn (poly (ps ! i) x)$ 
  shows  $\forall j \leq i+1. poly (ps ! j) x = 0$ 
using assms
proof (induction i)
  case 0 thus ?case by (clarsimp, rename-tac j, case-tac j, simp-all)
next
  case (Suc i)
    from Suc.prems(1,2)
      have  $sgn (poly (ps ! (i + 2)) x) = - sgn (poly (ps ! i) x)$ 
      by (intro assms(4)) simp-all
    with Suc.prems(3) have  $poly (ps ! i) x = 0$  by (simp add: sgn-zero-iff)
    with Suc.prems have  $\forall j \leq i+1. poly (ps ! j) x = 0$ 
      by (intro Suc.IH, simp-all)
    with Suc.prems(3) show ?case
      by (clarsimp, rename-tac j, case-tac j = Suc (Suc i), simp-all)
qed

```

This function splits the sign list of a Sturm sequence at a position x that is not a root of p into a list of sublists such that the number of sign changes within every sublist is constant in the neighbourhood of x , thus proving that the total number is also constant.

```

fun split-sign-changes where
  split-sign-changes [p] (x :: real) = [[p]] |
  split-sign-changes [p,q] x = [[p,q]] |
  split-sign-changes (p#q#r#ps) x =
    (if  $poly p x \neq 0 \wedge poly q x = 0$  then
      [p,q,r] # split-sign-changes (r#ps) x
    else

```

$[p,q] \# \text{split-sign-changes } (q\#r\#ps) x$

lemma (in *quasi-sturm-seq*) *split-sign-changes-subset*[*dest*]:
 $ps' \in \text{set } (\text{split-sign-changes } ps x) \implies \text{set } ps' \subseteq \text{set } ps$
apply (*insert ps-not-Nil*)
apply (*induction ps x rule: split-sign-changes.induct*)
apply (*simp, simp, rename-tac p q r ps x,*
case-tac poly p x $\neq 0 \wedge$ poly q x = 0, auto)
done

A custom induction rule for *split-sign-changes* that uses the fact that all the intermediate parameters in calls of *split-sign-changes* are quasi-Sturm sequences.

lemma (in *quasi-sturm-seq*) *split-sign-changes-induct*:
 $\llbracket \bigwedge p x. P [p] x; \bigwedge p q x. \text{quasi-sturm-seq } [p,q] \implies P [p,q] x;$
 $\bigwedge p q r ps x. \text{quasi-sturm-seq } (p\#q\#r\#ps) \implies$
 $\llbracket \text{poly } p x \neq 0 \implies \text{poly } q x = 0 \implies P (r\#ps) x;$
 $\text{poly } q x \neq 0 \implies P (q\#r\#ps) x;$
 $\text{poly } p x = 0 \implies P (q\#r\#ps) x \rrbracket$
 $\implies P (p\#q\#r\#ps) x \rrbracket \implies P ps x$

proof *goal-cases*
case *prems: 1*
have *quasi-sturm-seq ps ..*
with *prems show ?thesis*
proof (*induction ps x rule: split-sign-changes.induct*)
case ($\exists p q r ps x$)
show *?case*
proof (*rule 3(5)[OF 3(6)]*)
assume *A: poly p x $\neq 0$ poly q x = 0*
from *3(6) have quasi-sturm-seq (r#ps)*
by (*force dest: quasi-sturm-seq-Cons*)
with *3 A show P (r # ps) x by blast*
next
assume *A: poly q x $\neq 0$*
from *3(6) have quasi-sturm-seq (q#r#ps)*
by (*force dest: quasi-sturm-seq-Cons*)
with *3 A show P (q # r # ps) x by blast*
next
assume *A: poly p x = 0*
from *3(6) have quasi-sturm-seq (q#r#ps)*
by (*force dest: quasi-sturm-seq-Cons*)
with *3 A show P (q # r # ps) x by blast*
qed
qed *simp-all*
qed

The total number of sign changes in the split list is the same as the number of sign changes in the original list.

lemma (in *quasi-sturm-seq*) *split-sign-changes-correct*:

```

assumes poly (hd ps)  $x_0 \neq 0$ 
defines sign-changes'  $\equiv \lambda ps x.$ 
     $\sum ps' \leftarrow \text{split-sign-changes } ps x. \text{ sign-changes } ps' x$ 
shows sign-changes' ps  $x_0 = \text{sign-changes } ps x_0$ 
using assms(1)
proof (induction  $x_0$  rule: split-sign-changes-induct)
case ( $\exists p q r ps x_0$ )
  hence poly  $p x_0 \neq 0$  by simp
  note  $IH = \exists(2,3,4)$ 
  show ?case
  proof (cases poly  $q x_0 = 0$ )
    case True
      from  $\exists$  interpret quasi-sturm-seq  $p\#q\#r\#ps$  by simp
      from signs[of 0] and True have
        sgn-r-x0: poly  $r x_0 * \text{poly } p x_0 < 0$  by simp
      with  $\exists$  have poly  $r x_0 \neq 0$  by force
      from sign-changes-distrib[OF this, of [ $p,q$ ] ps]
      have sign-changes ( $p\#q\#r\#ps$ )  $x_0 =$ 
        sign-changes ( $[p, q, r]$ )  $x_0 + \text{sign-changes } (r \# ps) x_0$  by simp
      also have sign-changes ( $r\#ps$ )  $x_0 = \text{sign-changes}' (r\#ps) x_0$ 
        using  $\langle \text{poly } q x_0 = 0 \rangle \langle \text{poly } p x_0 \neq 0 \rangle \exists(5) \langle \text{poly } r x_0 \neq 0 \rangle$ 
        by (intro  $IH(1)$ [symmetric], simp-all)
      finally show ?thesis unfolding sign-changes'-def
        using True  $\langle \text{poly } p x_0 \neq 0 \rangle$  by simp
    next
      case False
        from sign-changes-distrib[OF this, of [ $p$ ]  $r\#ps$ ]
        have sign-changes ( $p\#q\#r\#ps$ )  $x_0 =$ 
          sign-changes ( $[p,q]$ )  $x_0 + \text{sign-changes } (q\#r\#ps) x_0$  by simp
        also have sign-changes ( $q\#r\#ps$ )  $x_0 = \text{sign-changes}' (q\#r\#ps) x_0$ 
          using  $\langle \text{poly } q x_0 \neq 0 \rangle \langle \text{poly } p x_0 \neq 0 \rangle \exists(5)$ 
          by (intro  $IH(2)$ [symmetric], simp-all)
        finally show ?thesis unfolding sign-changes'-def
          using False by simp
      qed
  qed (simp-all add: sign-changes-def sign-changes'-def)

```

We now prove that if $p(x) \neq 0$, the number of sign changes of a Sturm sequence of p at x is constant in a neighbourhood of x .

lemma (in *quasi-sturm-seq*) *split-sign-changes-correct-nbh*:

```

assumes poly (hd ps)  $x_0 \neq 0$ 
defines sign-changes'  $\equiv \lambda x_0 ps x.$ 
     $\sum ps' \leftarrow \text{split-sign-changes } ps x_0. \text{ sign-changes } ps' x$ 
shows eventually ( $\lambda x. \text{sign-changes}' x_0 ps x = \text{sign-changes } ps x$ ) (at  $x_0$ )
proof (rule eventually-mono)
  show eventually ( $\lambda x. \forall p \in \{p \in \text{set } ps. \text{poly } p x_0 \neq 0\}. \text{sgn } (\text{poly } p x) = \text{sgn } (\text{poly } p x_0)$ ) (at  $x_0$ )
  by (rule eventually-ball-finite, auto intro: poly-neighbourhood-same-sign)
next

```

```

fix x
show ( $\forall p \in \{p \in \text{set } ps. \text{poly } p \ x_0 \neq 0\}. \text{sgn } (\text{poly } p \ x) = \text{sgn } (\text{poly } p \ x_0) \implies$ 
   $\text{sign-changes}' \ x_0 \ ps \ x = \text{sign-changes } ps \ x$ )
proof -
  fix x assume nbh:  $\forall p \in \{p \in \text{set } ps. \text{poly } p \ x_0 \neq 0\}. \text{sgn } (\text{poly } p \ x) = \text{sgn } (\text{poly } p \ x_0)$ 
  thus  $\text{sign-changes}' \ x_0 \ ps \ x = \text{sign-changes } ps \ x$  using assms(1)
  proof (induction  $x_0$  rule: split-sign-changes-induct)
  case ( $\exists p \ q \ r \ ps \ x_0$ )
    hence  $\text{poly } p \ x_0 \neq 0$  by simp
    note  $IH = \mathcal{I}(2,3,4)$ 
    show ?case
    proof (cases  $\text{poly } q \ x_0 = 0$ )
      case True
        from  $\mathcal{I}$  interpret quasi-sturm-seq  $p\#q\#r\#ps$  by simp
        from signs[of 0] and True have
           $\text{sgn-r-x0: poly } r \ x_0 * \text{poly } p \ x_0 < 0$  by simp
        with  $\mathcal{I}$  have  $\text{poly } r \ x_0 \neq 0$  by force
        with nbh  $\mathcal{I}(5)$  have  $\text{poly } r \ x \neq 0$  by (auto simp: sgn-zero-iff)
        from sign-changes-distrib[OF this, of [p,q] ps]
          have  $\text{sign-changes } (p\#q\#r\#ps) \ x =$ 
             $\text{sign-changes } ([p, q, r]) \ x + \text{sign-changes } (r \# ps) \ x$  by simp
        also have  $\text{sign-changes } (r\#ps) \ x = \text{sign-changes}' \ x_0 \ (r\#ps) \ x$ 
          using  $\langle \text{poly } q \ x_0 = 0 \rangle$  nbh  $\langle \text{poly } p \ x_0 \neq 0 \rangle$   $\mathcal{I}(5)$   $\langle \text{poly } r \ x_0 \neq 0 \rangle$ 
          by (intro IH(1)[symmetric], simp-all)
        finally show ?thesis unfolding sign-changes'-def
          using True  $\langle \text{poly } p \ x_0 \neq 0 \rangle$  by simp
      case False
        with nbh  $\mathcal{I}(5)$  have  $\text{poly } q \ x \neq 0$  by (auto simp: sgn-zero-iff)
        from sign-changes-distrib[OF this, of [p] r#ps]
          have  $\text{sign-changes } (p\#q\#r\#ps) \ x =$ 
             $\text{sign-changes } ([p,q]) \ x + \text{sign-changes } (q\#r\#ps) \ x$  by simp
        also have  $\text{sign-changes } (q\#r\#ps) \ x = \text{sign-changes}' \ x_0 \ (q\#r\#ps) \ x$ 
          using  $\langle \text{poly } q \ x_0 \neq 0 \rangle$  nbh  $\langle \text{poly } p \ x_0 \neq 0 \rangle$   $\mathcal{I}(5)$ 
          by (intro IH(2)[symmetric], simp-all)
        finally show ?thesis unfolding sign-changes'-def
          using False by simp
    qed
  qed (simp-all add: sign-changes-def sign-changes'-def)
qed
qed

```

lemma (*in quasi-sturm-seq*) *hd-nonzero-imp-sign-changes-const-aux*:
assumes $\text{poly } (\text{hd } ps) \ x_0 \neq 0$ **and** $ps' \in \text{set } (\text{split-sign-changes } ps \ x_0)$
shows *eventually* $(\lambda x. \text{sign-changes } ps' \ x = \text{sign-changes } ps' \ x_0)$ (*at* x_0)
using *assms*

proof (*induction* x_0 *rule: split-sign-changes-induct*)
case (1 p x)
thus ?*case* **by** (*simp add: sign-changes-def*)
next
case (2 p q x_0)
hence [*simp*]: $ps' = [p, q]$ **by** *simp*
from 2 **have** *poly* p $x_0 \neq 0$ **by** *simp*
from 2(1) **interpret** *quasi-sturm-seq* [p, q] .
from *poly-neighbourhood-same-sign*[*OF* \langle *poly* p $x_0 \neq 0$ \rangle]
have *eventually* ($\lambda x. \text{sgn}(\text{poly } p \ x) = \text{sgn}(\text{poly } p \ x_0)$) (*at* x_0) .
moreover from *last-ps-sgn-const*
have *sgn-q*: $\bigwedge x. \text{sgn}(\text{poly } q \ x) = \text{sgn}(\text{poly } q \ x_0)$ **by** *simp*
ultimately have A : *eventually* ($\lambda x. \forall p \in \text{set}[p, q]. \text{sgn}(\text{poly } p \ x) =$
 $\text{sgn}(\text{poly } p \ x_0)$) (*at* x_0) **by** *simp*
thus ?*case* **by** (*force intro: eventually-mono*[*OF* A]
sign-changes-cong')
next
case (3 p q r ps'' x_0)
hence *p-not-0*: *poly* p $x_0 \neq 0$ **by** *simp*
note *sturm* = 3(1)
note *IH* = 3(2,3)
note *ps''-props* = 3(6)
show ?*case*
proof (*cases* *poly* q $x_0 = 0$)
case *True*
note *q-0* = *this*
from *sturm* **interpret** *quasi-sturm-seq* $p \# q \# r \# ps''$.
from *signs*[*of* 0] **and** *q-0*
have *signs'*: *poly* r $x_0 * \text{poly } p \ x_0 < 0$ **by** *simp*
with *p-not-0* **have** *r-not-0*: *poly* r $x_0 \neq 0$ **by** *force*
show ?*thesis*
proof (*cases* $ps' \in \text{set}(\text{split-sign-changes } (r \# ps'') \ x_0)$)
case *True*
show ?*thesis* **by** (*rule* *IH*(1), *fact*, *fact*, *simp add: r-not-0*, *fact*)
next
case *False*
with *ps''-props* *p-not-0* *q-0* **have** *ps'-props*: $ps' = [p, q, r]$ **by** *simp*
from *signs*[*of* 0] **and** *q-0*
have *sgn-r*: *poly* r $x_0 * \text{poly } p \ x_0 < 0$ **by** *simp*
from *p-not-0* *sgn-r*
have A : *eventually* ($\lambda x. \text{sgn}(\text{poly } p \ x) = \text{sgn}(\text{poly } p \ x_0) \wedge$
 $\text{sgn}(\text{poly } r \ x) = \text{sgn}(\text{poly } r \ x_0)$) (*at* x_0)
by (*intro eventually-conj poly-neighbourhood-same-sign*,
simp-all add: r-not-0)
show ?*thesis*
proof (*rule eventually-mono*[*OF* A], *clarify*,
subst ps'-props, *subst sign-changes-sturm-triple*)
fix x **assume** A : $\text{sgn}(\text{poly } p \ x) = \text{sgn}(\text{poly } p \ x_0)$
and B : $\text{sgn}(\text{poly } r \ x) = \text{sgn}(\text{poly } r \ x_0)$

```

have prod-neg:  $\bigwedge a (b::real). \llbracket a>0; b>0; a*b<0 \rrbracket \implies False$ 
              $\bigwedge a (b::real). \llbracket a<0; b<0; a*b<0 \rrbracket \implies False$ 
  by (drule mult-pos-pos, simp, simp,
      drule mult-neg-neg, simp, simp)
from A and  $\langle poly\ p\ x_0 \neq 0 \rangle$  show  $poly\ p\ x \neq 0$ 
  by (force simp: sgn-zero-iff)

with sgn-r p-not-0 r-not-0 A B
  have  $poly\ r\ x * poly\ p\ x < 0 \implies poly\ r\ x \neq 0$ 
  by (metis sgn-less sgn-mult, metis sgn-0-0)
with sgn-r show  $sgn\ r' : sgn\ (poly\ r\ x) = -\ sgn\ (poly\ p\ x)$ 
  apply (simp add: sgn-real-def not-le not-less
            split: if-split-asm, intro conjI impI)
  using prod-neg[of  $poly\ r\ x\ poly\ p\ x$ ] apply force+
  done

show  $1 = sign\ changes\ ps'\ x_0$ 
  by (subst ps'-props, subst sign-changes-sturm-triple,
      fact, metis A B sgn-r', simp)
qed
qed
next
case False
  note q-not-0 = this
  show ?thesis
  proof (cases  $ps' \in set\ (split\ sign\ changes\ (q \# r \# ps''))\ x_0$ )
    case True
      show ?thesis by (rule IH(2), fact, simp add: q-not-0, fact)
  next
  case False
    with ps''-props and q-not-0 have  $ps' = [p, q]$  by simp
    hence [simp]:  $\forall p \in set\ ps'. poly\ p\ x_0 \neq 0$ 
      using q-not-0 p-not-0 by simp
    show ?thesis
    proof (rule eventually-mono)
      fix x assume  $\forall p \in set\ ps'. sgn\ (poly\ p\ x) = sgn\ (poly\ p\ x_0)$ 
      thus  $sign\ changes\ ps'\ x = sign\ changes\ ps'\ x_0$ 
        by (rule sign-changes-cong')
    next
      show eventually  $(\lambda x. \forall p \in set\ ps'. sgn\ (poly\ p\ x) = sgn\ (poly\ p\ x_0))\ (at\ x_0)$ 
        by (force intro: eventually-ball-finite
            poly-neighbourhood-same-sign)
    qed
  qed
qed
qed
qed

```

lemma (in *quasi-sturm-seq*) *hd-nonzero-imp-sign-changes-const*:
assumes *poly* (*hd ps*) $x_0 \neq 0$
shows *eventually* ($\lambda x. \text{sign-changes } ps \ x = \text{sign-changes } ps \ x_0$) (*at* x_0)
proof –
let $?pss = \text{split-sign-changes } ps \ x_0$
let $?f = \lambda pss \ x. \sum ps' \leftarrow pss. \text{sign-changes } ps' \ x$
{
fix *pss* **assume** $\bigwedge ps'. ps' \in \text{set } pss \implies$
eventually ($\lambda x. \text{sign-changes } ps' \ x = \text{sign-changes } ps' \ x_0$) (*at* x_0)
hence *eventually* ($\lambda x. ?f \ pss \ x = ?f \ pss \ x_0$) (*at* x_0)
proof (*induction pss*)
case (*Cons ps' pss*)
then show *?case*
apply (*rule eventually-mono[OF eventually-conj]*)
apply (*auto simp add: Cons.prem*)
done
qed *simp*
}
note *A = this[of ?pss]*
have *B: eventually* ($\lambda x. ?f \ ?pss \ x = ?f \ ?pss \ x_0$) (*at* x_0)
by (*rule A, rule hd-nonzero-imp-sign-changes-const-aux[OF assms], simp*)
note *C = split-sign-changes-correct-nbh[OF assms]*
note *D = split-sign-changes-correct[OF assms]*
note *E = eventually-conj[OF B C]*
show *?thesis by* (*rule eventually-mono[OF E], auto simp: D*)
qed

lemma (in *sturm-seq*) *p-nonzero-imp-sign-changes-const*:
poly p x₀ ≠ 0 \implies
eventually ($\lambda x. \text{sign-changes } ps \ x = \text{sign-changes } ps \ x_0$) (*at* x_0)
using *hd-nonzero-imp-sign-changes-const by simp*

If x is a root of p and p is not the zero polynomial, the number of sign changes of a Sturm chain of p decreases by 1 at x .

lemma (in *sturm-seq*) *p-zero*:
assumes *poly p x₀ = 0 p ≠ 0*
shows *eventually* ($\lambda x. \text{sign-changes } ps \ x =$
sign-changes } ps \ x₀ + (if $x < x_0$ then 1 else 0)) (*at* x_0)
proof –
from *ps-first-two* **obtain** $q \ ps'$ **where** [*simp*]: $ps = p \# q \# ps'$.
hence $ps!1 = q$ **by** *simp*
have *eventually* ($\lambda x. x \neq x_0$) (*at* x_0)
by (*simp add: eventually-at, rule exI[of - 1], simp*)
moreover from *p-squarefree* **and** *assms(1)* **have** *poly q x₀ ≠ 0* **by** *simp*
{
have *A: quasi-sturm-seq ps ..*
with *quasi-sturm-seq-Cons[of p q # ps']*
interpret *quasi-sturm-seq q # ps'* **by** *simp*
from $\langle \text{poly } q \ x_0 \neq 0 \rangle$ **have** *eventually* ($\lambda x. \text{sign-changes } (q \# ps') \ x =$

```

      sign-changes (q#ps^) x0 (at x0)
    using hd-nonzero-imp-sign-changes-const[where x0=x0] by simp
  }
  moreover note poly-neighbourhood-without-roots[OF assms(2)] deriv[OF assms(1)]
  ultimately
    have A: eventually (λx. x ≠ x0 ∧ poly p x ≠ 0 ∧
      sgn (poly (p*ps!1) x) = (if x > x0 then 1 else -1) ∧
      sign-changes (q#ps^) x = sign-changes (q#ps^) x0 (at x0)
      by (simp only: ⟨ps!1 = q⟩, intro eventually-conj)
  show ?thesis
  proof (rule eventually-mono[OF A], clarify, goal-cases)
    case prems: (1 x)
    from zero-less-mult-pos have zero-less-mult-pos':
      ∧ a b. [(0::real) < a*b; 0 < b] ⇒ 0 < a
      by (subgoal-tac a*b = b*a, auto)
    from prems have poly q x ≠ 0 and q-sgn: sgn (poly q x) =
      (if x < x0 then -sgn (poly p x) else sgn (poly p x))
      by (auto simp add: sgn-real-def elim: linorder-neqE-linordered-idom
        dest: mult-neg-neg zero-less-mult-pos
        zero-less-mult-pos' split: if-split-asm)
    from sign-changes-distrib[OF ⟨poly q x ≠ 0⟩, of [p] ps^]
      have sign-changes ps x = sign-changes [p,q] x + sign-changes (q#ps^) x
      by simp
    also from q-sgn and ⟨poly p x ≠ 0⟩
      have sign-changes [p,q] x = (if x < x0 then 1 else 0)
      by (simp add: sign-changes-def sgn-zero-iff split: if-split-asm)
    also note prems(4)
    also from assms(1) have sign-changes (q#ps^) x0 = sign-changes ps x0
      by (simp add: sign-changes-def)
    finally show ?case by simp
  qed
qed

```

With these two results, we can now show that if p is nonzero, the number of roots in an interval of the form $(a; b]$ is the difference of the sign changes of a Sturm sequence of p at a and b .

First, however, we prove the following auxiliary lemma that shows that if a function $f : \mathbb{R} \rightarrow \mathbb{N}$ is locally constant at any $x \in (a; b]$, it is constant across the entire interval $(a; b]$:

lemma *count-roots-between-aux*:

assumes $a \leq b$

assumes $\forall x::\text{real}. a < x \wedge x \leq b \longrightarrow \text{eventually } (\lambda \xi. f \xi = (f x::\text{nat})) \text{ (at } x)$

shows $\forall x. a < x \wedge x \leq b \longrightarrow f x = f b$

proof (*clarify*)

fix x **assume** $x > a \wedge x \leq b$

with *assms* **have** $\forall x'. x \leq x' \wedge x' \leq b \longrightarrow$

$\text{eventually } (\lambda \xi. f \xi = f x') \text{ (at } x')$ **by** *auto*

from *fun-eq-in-ivl*[OF ⟨ $x \leq b$ ⟩ *this*] **show** $f x = f b$.

qed

Now we can prove the actual root-counting theorem:

theorem (in *sturm-seq*) *count-roots-between*:

assumes [*simp*]: $p \neq 0 \ a \leq b$

shows $\text{sign-changes } ps \ a - \text{sign-changes } ps \ b =$
 $\text{card } \{x. x > a \wedge x \leq b \wedge \text{poly } p \ x = 0\}$

proof –

have $\text{sign-changes } ps \ a - \text{int } (\text{sign-changes } ps \ b) =$
 $\text{card } \{x. x > a \wedge x \leq b \wedge \text{poly } p \ x = 0\}$ **using** $\langle a \leq b \rangle$

proof (*induction card* $\{x. x > a \wedge x \leq b \wedge \text{poly } p \ x = 0\}$ *arbitrary: a b*
rule: less-induct)

case (*less a b*)

show *?case*

proof (*cases* $\exists x. a < x \wedge x \leq b \wedge \text{poly } p \ x = 0$)

case *False*

hence *no-roots*: $\{x. a < x \wedge x \leq b \wedge \text{poly } p \ x = 0\} = \{\}$

(**is** *?roots=-*) **by** *auto*

hence *card-roots*: $\text{card } ?\text{roots} = (0::\text{int})$ **by** (*subst no-roots, simp*)

show *?thesis*

proof (*simp only: card-roots eq-iff-diff-eq-0[symmetric] of-nat-eq-iff,*
cases poly p a = 0)

case *False*

with *no-roots* **show** $\text{sign-changes } ps \ a = \text{sign-changes } ps \ b$

by (*force intro: fun-eq-in-ivl* $\langle a \leq b \rangle$

p-nonzero-imp-sign-changes-const)

next

case *True*

have $A: \forall x. a < x \wedge x \leq b \longrightarrow \text{sign-changes } ps \ x = \text{sign-changes } ps \ b$

apply (*rule count-roots-between-aux, fact, clarify*)

apply (*rule p-nonzero-imp-sign-changes-const*)

apply (*insert False, simp*)

done

have *eventually* $(\lambda x. x > a \longrightarrow$

$\text{sign-changes } ps \ x = \text{sign-changes } ps \ a)$ (*at a*)

apply (*rule eventually-mono* [*OF p-zero* [*OF* $\langle \text{poly } p \ a = 0 \rangle \langle p \neq$

$0 \rangle$]])

apply *force*

done

then obtain δ **where** *δ -props*:

$\delta > 0 \ \forall x. x > a \wedge x < a + \delta \longrightarrow$

$\text{sign-changes } ps \ x = \text{sign-changes } ps \ a$

by (*auto simp: eventually-at dist-real-def*)

show $\text{sign-changes } ps \ a = \text{sign-changes } ps \ b$

proof (*cases a = b*)

case *False*

define x **where** $x = \min (a + \delta / 2) \ b$

with *False* **have** $a < x \ x < a + \delta \ x \leq b$

using $\langle \delta > 0 \rangle \ \langle a \leq b \rangle$ **by** *simp-all*

from *δ -props* $\langle a < x \rangle \ \langle x < a + \delta \rangle$

have *sign-changes ps a = sign-changes ps x* **by** *simp*
also from $A \langle a < x \rangle \langle x \leq b \rangle$ **have** $\dots = \text{sign-changes ps } b$
by *blast*
finally show *?thesis* .
qed *simp*
qed

next

case *True*
from *poly-roots-finite[OF assms(1)]*
have *fin: finite {x. x > a ∧ x ≤ b ∧ poly p x = 0}*
by (*force intro: finite-subset*)
from *True* **have** $\{x. x > a \wedge x \leq b \wedge \text{poly } p \ x = 0\} \neq \{\}$ **by** *blast*
with *fin* **have** *card-greater-0:*
 $\text{card } \{x. x > a \wedge x \leq b \wedge \text{poly } p \ x = 0\} > 0$ **by** *fastforce*

define x_2 **where** $x_2 = \text{Min } \{x. x > a \wedge x \leq b \wedge \text{poly } p \ x = 0\}$
from *Min-in[OF fin]* **and** *True*
have $x_2\text{-props: } x_2 > a \ x_2 \leq b \ \text{poly } p \ x_2 = 0$
unfolding $x_2\text{-def}$ **by** *blast+*
from *Min-le[OF fin]* $x_2\text{-props}$
have $x_2\text{-le: } \bigwedge x'. \llbracket x' > a; x' \leq b; \text{poly } p \ x' = 0 \rrbracket \implies x_2 \leq x'$
unfolding $x_2\text{-def}$ **by** *simp*

have *left: {x. a < x ∧ x ≤ x₂ ∧ poly p x = 0} = {x_{2}}}*
using $x_2\text{-props } x_2\text{-le}$ **by** *force*
hence [*simp*]: $\text{card } \{x. a < x \wedge x \leq x_2 \wedge \text{poly } p \ x = 0\} = 1$ **by** *simp*

from *p-zero[OF ⟨poly p x₂ = 0⟩ ⟨p ≠ 0⟩,*
unfolded eventually-at dist-real-def] **guess** $\varepsilon \dots$
hence $\varepsilon\text{-props: } \varepsilon > 0$
 $\forall x. x \neq x_2 \wedge |x - x_2| < \varepsilon \implies$
 $\text{sign-changes ps } x = \text{sign-changes ps } x_2 +$
 $(\text{if } x < x_2 \text{ then } 1 \text{ else } 0)$ **by** *auto*
define x_1 **where** $x_1 = \max (x_2 - \varepsilon / 2) a$
have $|x_1 - x_2| < \varepsilon$ **using** $\langle \varepsilon > 0 \rangle$ $x_2\text{-props}$ **by** (*simp add: x₁-def*)
hence $\text{sign-changes ps } x_1 =$
 $(\text{if } x_1 < x_2 \text{ then } \text{sign-changes ps } x_2 + 1 \text{ else } \text{sign-changes ps } x_2)$
using $\varepsilon\text{-props}(2)$ **by** (*cases x₁ = x₂, auto*)
hence $\text{sign-changes ps } x_1 - \text{sign-changes ps } x_2 = 1$
unfolding $x_1\text{-def}$ **using** $x_2\text{-props } \langle \varepsilon > 0 \rangle$ **by** *simp*

also have $x_2 \notin \{x. a < x \wedge x \leq x_1 \wedge \text{poly } p \ x = 0\}$
unfolding $x_1\text{-def}$ **using** $\langle \varepsilon > 0 \rangle$ **by** *force*
with *left* **have** $\{x. a < x \wedge x \leq x_1 \wedge \text{poly } p \ x = 0\} = \{\}$ **by** *force*
with *less(1)[of a x₁]* **have** $\text{sign-changes ps } x_1 = \text{sign-changes ps } a$
unfolding $x_1\text{-def } \langle \varepsilon > 0 \rangle$ **by** (*force simp: card-greater-0*)

finally have *signs-left:*

sign-changes ps a - int (sign-changes ps x₂) = 1 by simp

have $\{x. x > a \wedge x \leq b \wedge \text{poly } p x = 0\} =$
 $\{x. a < x \wedge x \leq x_2 \wedge \text{poly } p x = 0\} \cup$
 $\{x. x_2 < x \wedge x \leq b \wedge \text{poly } p x = 0\}$ **using** *x₂-props* **by** *auto*

also note *left*

finally have *A*: $\text{card } \{x. x_2 < x \wedge x \leq b \wedge \text{poly } p x = 0\} + 1 =$
 $\text{card } \{x. a < x \wedge x \leq b \wedge \text{poly } p x = 0\}$ **using** *fin* **by** *simp*

hence $\text{card } \{x. x_2 < x \wedge x \leq b \wedge \text{poly } p x = 0\} <$
 $\text{card } \{x. a < x \wedge x \leq b \wedge \text{poly } p x = 0\}$ **by** *simp*

from *less(1)[OF this x₂-props(2)]* **and** *A*

have *signs-right*: $\text{sign-changes ps } x_2 - \text{int (sign-changes ps } b) + 1 =$
 $\text{card } \{x. a < x \wedge x \leq b \wedge \text{poly } p x = 0\}$ **by** *simp*

from *signs-left* **and** *signs-right* **show** *?thesis* **by** *simp*

qed

qed

thus *?thesis* **by** *simp*

qed

By applying this result to a sufficiently large upper bound, we can effectively count the number of roots “between *a* and infinity”, i. e. the roots greater than *a*:

lemma (*in sturm-seq*) *count-roots-above*:

assumes $p \neq 0$

shows $\text{sign-changes ps } a - \text{sign-changes-inf ps} =$
 $\text{card } \{x. x > a \wedge \text{poly } p x = 0\}$

proof –

have $p \in \text{set ps}$ **using** *hd-in-set[OF ps-not-Nil]* **by** *simp*

have *finite (set ps)* **by** *simp*

from *polys-inf-sign-thresholds[OF this]* **guess** *l u* .

note *lu-props = this*

let *?u = max a u*

{fix *x* **assume** $\text{poly } p x = 0$ **hence** $x \leq ?u$

using *lu-props(3)[OF <p ∈ set ps>, of x]* $\langle p \neq 0 \rangle$

by (*cases u ≤ x, auto simp: sgn-zero-iff*)

} note [*simp*] = *this*

from *lu-props*

have $\text{map } (\lambda p. \text{sgn (poly } p ?u)) \text{ ps} = \text{map poly-inf ps}$ **by** *simp*

hence $\text{sign-changes ps } a - \text{sign-changes-inf ps} =$

$\text{sign-changes ps } a - \text{sign-changes ps } ?u$

by (*simp-all only: sign-changes-def sign-changes-inf-def*)

also from *count-roots-between[OF assms]* *lu-props*

have ... = $\text{card } \{x. a < x \wedge x \leq ?u \wedge \text{poly } p x = 0\}$ **by** *simp*

also have $\{x. a < x \wedge x \leq ?u \wedge \text{poly } p x = 0\} = \{x. a < x \wedge \text{poly } p x = 0\}$

using *lu-props* **by** *auto*

finally show *?thesis* .

qed

The same works analogously for the number of roots below a and the total number of roots.

lemma (in *sturm-seq*) *count-roots-below*:

assumes $p \neq 0$

shows $\text{sign-changes-neg-inf } ps - \text{sign-changes } ps \ a =$
 $\text{card } \{x. x \leq a \wedge \text{poly } p \ x = 0\}$

proof –

have $p \in \text{set } ps$ **using** *hd-in-set[OF ps-not-Nil]* **by** *simp*

have *finite (set ps)* **by** *simp*

from *polys-inf-sign-thresholds[OF this]* **guess** $l \ u$.

note $lu\text{-props} = \text{this}$

let $?l = \min \ a \ l$

{fix x **assume** $\text{poly } p \ x = 0$ **hence** $x > ?l$

using $lu\text{-props}(4)[OF \langle p \in \text{set } ps \rangle, \text{of } x] \langle p \neq 0 \rangle$

by (*cases* $l < x$, *auto simp: sgn-zero-iff*)

} note $[simp] = \text{this}$

from $lu\text{-props}$

have $\text{map } (\lambda p. \text{sgn } (\text{poly } p \ ?l)) \ ps = \text{map } \text{poly-neg-inf } ps$ **by** *simp*

hence $\text{sign-changes-neg-inf } ps - \text{sign-changes } ps \ a =$

$\text{sign-changes } ps \ ?l - \text{sign-changes } ps \ a$

by (*simp-all only: sign-changes-def sign-changes-neg-inf-def*)

also from *count-roots-between[OF assms]* $lu\text{-props}$

have $\dots = \text{card } \{x. ?l < x \wedge x \leq a \wedge \text{poly } p \ x = 0\}$ **by** *simp*

also have $\{x. ?l < x \wedge x \leq a \wedge \text{poly } p \ x = 0\} = \{x. a \geq x \wedge \text{poly } p \ x = 0\}$

using $lu\text{-props}$ **by** *auto*

finally show *?thesis* .

qed

lemma (in *sturm-seq*) *count-roots*:

assumes $p \neq 0$

shows $\text{sign-changes-neg-inf } ps - \text{sign-changes-inf } ps =$
 $\text{card } \{x. \text{poly } p \ x = 0\}$

proof –

have *finite (set ps)* **by** *simp*

from *polys-inf-sign-thresholds[OF this]* **guess** $l \ u$.

note $lu\text{-props} = \text{this}$

from $lu\text{-props}$

have $\text{map } (\lambda p. \text{sgn } (\text{poly } p \ l)) \ ps = \text{map } \text{poly-neg-inf } ps$

$\text{map } (\lambda p. \text{sgn } (\text{poly } p \ u)) \ ps = \text{map } \text{poly-inf } ps$ **by** *simp-all*

hence $\text{sign-changes-neg-inf } ps - \text{sign-changes-inf } ps =$

$\text{sign-changes } ps \ l - \text{sign-changes } ps \ u$

by (*simp-all only: sign-changes-def sign-changes-inf-def*

sign-changes-neg-inf-def)

also from *count-roots-between[OF assms]* $lu\text{-props}$

have $\dots = \text{card } \{x. l < x \wedge x \leq u \wedge \text{poly } p \ x = 0\}$ **by** *simp*

also have $\{x. l < x \wedge x \leq u \wedge \text{poly } p \ x = 0\} = \{x. \text{poly } p \ x = 0\}$

using $lu\text{-props}$ *assms* **by** *simp*

finally show *?thesis* .
qed

2.4 Constructing Sturm sequences

2.5 The canonical Sturm sequence

In this subsection, we will present the canonical Sturm sequence construction for a polynomial p without multiple roots that is very similar to the Euclidean algorithm:

$$p_i = \begin{cases} p & \text{for } i = 1 \\ p' & \text{for } i = 2 \\ -p_{i-2} \bmod p_{i-1} & \text{otherwise} \end{cases}$$

We break off the sequence at the first constant polynomial.

function *sturm-aux* **where**
sturm-aux ($p :: \text{real poly}$) $q =$
(if degree $q = 0$ *then* $[p,q]$ *else* $p \# \text{sturm-aux } q \text{ } (- (p \bmod q))$
by (*pat-completeness, simp-all*)
termination by (*relation measure (degree \circ snd),*
simp-all add: o-def degree-mod-less')

definition *sturm* **where** $\text{sturm } p = \text{sturm-aux } p \text{ } (p\text{deriv } p)$

Next, we show some simple facts about this construction:

lemma *sturm-0[simp]*: $\text{sturm } 0 = [0,0]$
by (*unfold sturm-def, subst sturm-aux.simps, simp*)

lemma *[simp]*: $\text{sturm-aux } p \ q = [] \longleftrightarrow \text{False}$
by (*induction p q rule: sturm-aux.induct, subst sturm-aux.simps, auto*)

lemma *sturm-neq-Nil[simp]*: $\text{sturm } p \neq []$ **unfolding** *sturm-def* **by** *simp*

lemma *[simp]*: $\text{hd } (\text{sturm } p) = p$
unfolding *sturm-def* **by** (*subst sturm-aux.simps, simp*)

lemma *[simp]*: $p \in \text{set } (\text{sturm } p)$
using *hd-in-set[OF sturm-neq-Nil]* **by** *simp*

lemma *[simp]*: $\text{length } (\text{sturm } p) \geq 2$

proof–

{**fix** q **have** $\text{length } (\text{sturm-aux } p \ q) \geq 2$
by (*induction p q rule: sturm-aux.induct, subst sturm-aux.simps, auto*)

}

thus *?thesis* **unfolding** *sturm-def* .

qed

```

lemma [simp]: degree (last (sturm p)) = 0
proof -
  {fix q have degree (last (sturm-aux p q)) = 0
    by (induction p q rule: sturm-aux.induct, subst sturm-aux.simps, simp)
  }
  thus ?thesis unfolding sturm-def .
qed

```

```

lemma [simp]: sturm-aux p q ! 0 = p
  by (subst sturm-aux.simps, simp)
lemma [simp]: sturm-aux p q ! Suc 0 = q
  by (subst sturm-aux.simps, simp)

```

```

lemma [simp]: sturm p ! 0 = p
  unfolding sturm-def by simp
lemma [simp]: sturm p ! Suc 0 = pderiv p
  unfolding sturm-def by simp

```

```

lemma sturm-indices:
  assumes  $i < \text{length } (\text{sturm } p) - 2$ 
  shows  $\text{sturm } p!(i+2) = -(\text{sturm } p!i \text{ mod } \text{sturm } p!(i+1))$ 
proof -
  {fix ps q
  have  $\llbracket ps = \text{sturm-aux } p \ q; i < \text{length } ps - 2 \rrbracket$ 
     $\implies ps!(i+2) = -(ps!i \text{ mod } ps!(i+1))$ 
  proof (induction p q arbitrary: ps i rule: sturm-aux.induct)
  case (1 p q)
  show ?case
  proof (cases i = 0)
  case False
  then obtain  $i'$  where [simp]:  $i = \text{Suc } i'$  by (cases i, simp-all)
  hence  $\text{length } ps \geq 4$  using 1 by simp
  with 1(2) have deg:  $\text{degree } q \neq 0$ 
    by (subst (asm) sturm-aux.simps, simp split: if-split-asm)
  with 1(2) obtain  $ps'$  where [simp]:  $ps = p \# ps'$ 
    by (subst (asm) sturm-aux.simps, simp)
  with 1(2) deg have  $ps': ps' = \text{sturm-aux } q \ (-p \text{ mod } q)$ 
    by (subst (asm) sturm-aux.simps, simp)
  from  $\langle \text{length } ps \geq 4 \rangle$  and  $\langle ps = p \# ps' \rangle$  1(3) False
  have  $i - 1 < \text{length } ps' - 2$  by simp
  from 1(1)[OF deg ps' this]
  show ?thesis by simp
  next
  case True
  with 1(3) have  $\text{length } ps \geq 3$  by simp
  with 1(2) have  $\text{degree } q \neq 0$ 
    by (subst (asm) sturm-aux.simps, simp split: if-split-asm)
  }
  }

```

```

with 1(2) have [simp]: sturm-aux p q ! Suc (Suc 0) = -(p mod q)
  by (subst sturm-aux.simps, simp)
from True have ps!i = p ps!(i+1) = q ps!(i+2) = -(p mod q)
  by (simp-all add: 1(2))
thus ?thesis by simp
qed
qed}
from this[OF sturm-def assms] show ?thesis .
qed

```

If the Sturm sequence construction is applied to polynomials p and q , the greatest common divisor of p and q a divisor of every element in the sequence. This is obvious from the similarity to Euclid's algorithm for computing the GCD.

lemma *sturm-aux-gcd*: $r \in \text{set } (\text{sturm-aux } p \ q) \implies \text{gcd } p \ q \ \text{dvd } r$

proof (*induction p q rule: sturm-aux.induct*)

case (1 p q)

show ?case

proof (*cases r = p*)

case False

with 1(2) **have** $r: r \in \text{set } (\text{sturm-aux } q \ (- (p \ \text{mod } q)))$

by (subst (asm) sturm-aux.simps, simp split: if-split-asm,
subst sturm-aux.simps, simp)

show ?thesis

proof (*cases degree q = 0*)

case False

hence $q \neq 0$ **by** force

with 1(1) [OF False r] **show** ?thesis

by (simp add: gcd-mod-right ac-simps)

next

case True

with 1(2) **and** $\langle r \neq p \rangle$ **have** $r = q$

by (subst (asm) sturm-aux.simps, simp)

thus ?thesis **by** simp

qed

qed simp

qed

lemma *sturm-gcd*: $r \in \text{set } (\text{sturm } p) \implies \text{gcd } p \ (\text{pderiv } p) \ \text{dvd } r$

unfolding sturm-def **by** (rule sturm-aux-gcd)

If two adjacent polynomials in the result of the canonical Sturm chain construction both have a root at some x , this x is a root of all polynomials in the sequence.

lemma *sturm-adjacent-root-propagate-left*:

assumes $i < \text{length } (\text{sturm } (p :: \text{real poly})) - 1$

assumes $\text{poly } (\text{sturm } p \ ! \ i) \ x = 0$

and $\text{poly } (\text{sturm } p \ ! \ (i + 1)) \ x = 0$

shows $\forall j \leq i+1. \text{poly} (\text{sturm } p ! j) x = 0$
using *assms(2)*
proof (*intro sturm-adjacent-root-aux[OF assms(1,2,3)], goal-cases*)
case *prems: (1 i x)*
let $?p = \text{sturm } p ! i$
let $?q = \text{sturm } p ! (i + 1)$
let $?r = \text{sturm } p ! (i + 2)$
from *sturm-indices[OF prems(2)]* **have** $?p = ?p \text{ div } ?q * ?q - ?r$
by (*simp add: div-mult-mod-eq*)
hence $\text{poly } ?p x = \text{poly} (?p \text{ div } ?q * ?q - ?r) x$ **by** *simp*
hence $\text{poly } ?p x = -\text{poly } ?r x$ **using** *prems(3)* **by** *simp*
thus $?case$ **by** (*simp add: sgn-minus*)
qed

Consequently, if this is the case in the canonical Sturm chain of p , p must have multiple roots.

lemma *sturm-adjacent-root-not-squarefree*:
assumes $i < \text{length} (\text{sturm } (p :: \text{real poly})) - 1$
 $\text{poly} (\text{sturm } p ! i) x = 0 \text{ poly} (\text{sturm } p ! (i + 1)) x = 0$
shows $\neg \text{rsquarefree } p$
proof –
from *sturm-adjacent-root-propagate-left[OF assms]*
have $\text{poly } p x = 0 \text{ poly} (\text{pderiv } p) x = 0$ **by** *auto*
thus $?thesis$ **by** (*auto simp: rsquarefree-roots*)
qed

Since the second element of the sequence is chosen to be the derivative of p , p_1 and p_2 fulfil the property demanded by the definition of a Sturm sequence that they locally have opposite sign left of a root x of p and the same sign to the right of x .

lemma *sturm-firsttwo-signs-aux*:
assumes $(p :: \text{real poly}) \neq 0 \ q \neq 0$
assumes *q-pderiv*:
 $\text{eventually } (\lambda x. \text{sgn} (\text{poly } q x) = \text{sgn} (\text{poly} (\text{pderiv } p) x)) \text{ (at } x_0)$
assumes *p-0*: $\text{poly } p (x_0 :: \text{real}) = 0$
shows $\text{eventually } (\lambda x. \text{sgn} (\text{poly} (p*q) x) = (\text{if } x > x_0 \text{ then } 1 \text{ else } -1)) \text{ (at } x_0)$
proof –
have *A*: $\text{eventually } (\lambda x. \text{poly } p x \neq 0 \wedge \text{poly } q x \neq 0 \wedge$
 $\text{sgn} (\text{poly } q x) = \text{sgn} (\text{poly} (\text{pderiv } p) x)) \text{ (at } x_0)$
using $\langle p \neq 0 \rangle \ \langle q \neq 0 \rangle$
by (*intro poly-neighbourhood-same-sign q-pderiv*
 $\text{poly-neighbourhood-without-roots eventually-conj}$)
then obtain ε **where** ε -*props*: $\varepsilon > 0 \ \forall x. x \neq x_0 \wedge |x - x_0| < \varepsilon \longrightarrow$
 $\text{poly } p x \neq 0 \wedge \text{poly } q x \neq 0 \wedge \text{sgn} (\text{poly} (\text{pderiv } p) x) = \text{sgn} (\text{poly } q x)$
by (*auto simp: eventually-at dist-real-def*)
have *sqr-pos*: $\bigwedge x :: \text{real}. x \neq 0 \implies \text{sgn } x * \text{sgn } x = 1$
by (*auto simp: sgn-real-def*)
show $?thesis$

proof (*simp only: eventually-at dist-real-def, rule exI[of - ε],
intro conjI, fact ⟨ε > 0⟩, clarify*)
fix x **assume** $x \neq x_0 \mid x - x_0 \mid < \varepsilon$
with ε -props **have** [simp]: $\text{poly } p \ x \neq 0 \ \text{poly } q \ x \neq 0$
 $\text{sgn}(\text{poly}(pderiv\ p)\ x) = \text{sgn}(\text{poly } q\ x)$ **by** *auto*
show $\text{sgn}(\text{poly}(p * q)\ x) = (\text{if } x > x_0 \text{ then } 1 \text{ else } -1)$
proof (*cases* $x \geq x_0$)
case *True*
with $\langle x \neq x_0 \rangle$ **have** $x > x_0$ **by** *simp*
from *poly-MVT[OF this, of p]* **guess** $\xi \ ..$
note ξ -props = *this*
with $\langle |x - x_0| < \varepsilon \rangle \langle \text{poly } p \ x_0 = 0 \rangle \langle x > x_0 \rangle$ ε -props
have $|\xi - x_0| < \varepsilon \ \text{sgn}(\text{poly } p \ x) = \text{sgn}(x - x_0) * \text{sgn}(\text{poly } q \ \xi)$
by (*auto simp add: q-pderiv sgn-mult*)
moreover from ξ -props ε -props $\langle |x - x_0| < \varepsilon \rangle$
have $\forall t. \xi \leq t \wedge t \leq x \longrightarrow \text{poly } q \ t \neq 0$ **by** *auto*
hence $\text{sgn}(\text{poly } q \ \xi) = \text{sgn}(\text{poly } q \ x)$ **using** ξ -props ε -props
by (*intro no-roots-inbetween-imp-same-sign, simp-all*)
ultimately show ?thesis **using** *True* $\langle x \neq x_0 \rangle$ ε -props ξ -props
by (*auto simp: sgn-mult sqr-pos*)
next
case *False*
hence $x < x_0$ **by** *simp*
hence $\text{sgn}(\text{sgn}(x - x_0)) = -1$ **by** *simp*
from *poly-MVT[OF ⟨x < x_0⟩, of p]* **guess** $\xi \ ..$
note ξ -props = *this*
with $\langle |x - x_0| < \varepsilon \rangle \langle \text{poly } p \ x_0 = 0 \rangle \langle x < x_0 \rangle$ ε -props
have $|\xi - x_0| < \varepsilon \ \text{poly } p \ x = (x - x_0) * \text{poly}(pderiv\ p)\ \xi$
 $\text{poly } p \ \xi \neq 0$ **by** (*auto simp: field-simps*)
hence $\text{sgn}(\text{poly } p \ x) = \text{sgn}(x - x_0) * \text{sgn}(\text{poly } q \ \xi)$
using ε -props ξ -props **by** (*auto simp: q-pderiv sgn-mult*)
moreover from ξ -props ε -props $\langle |x - x_0| < \varepsilon \rangle$
have $\forall t. x \leq t \wedge t \leq \xi \longrightarrow \text{poly } q \ t \neq 0$ **by** *auto*
hence $\text{sgn}(\text{poly } q \ \xi) = \text{sgn}(\text{poly } q \ x)$ **using** ξ -props ε -props
by (*rule-tac sym, intro no-roots-inbetween-imp-same-sign, simp-all*)
ultimately show ?thesis **using** *False* $\langle x \neq x_0 \rangle$
by (*auto simp: sgn-mult sqr-pos*)
qed
qed
qed

lemma *sturm-firsttwo-signs*:

fixes $ps :: \text{real poly list}$

assumes *squarefree: rsquarefree p*

assumes *p-0: poly p (x_0::real) = 0*

shows *eventually* $(\lambda x. \text{sgn}(\text{poly}(p * \text{sturm } p \ ! \ 1)\ x) =$
 $(\text{if } x > x_0 \text{ then } 1 \text{ else } -1))$ (at x_0)

proof –

from *assms* **have** [simp]: $p \neq 0$ **by** (*auto simp add: rsquarefree-roots*)

with *squarefree p-0* **have** [*simp*]: $pderiv\ p \neq 0$
by (*auto simp add: rsquarefree-roots*)
from *assms* **show** *?thesis*
by (*intro sturm-firsttwo-signs-aux,*
simp-all add: rsquarefree-roots)
qed

The construction also obviously fulfils the property about three adjacent polynomials in the sequence.

lemma *sturm-signs*:

assumes *squarefree: rsquarefree p*
assumes *i-in-range: i < length (sturm (p :: real poly)) - 2*
assumes *q-0: poly (sturm p ! (i+1)) x = 0 (is poly ?q x = 0)*
shows *poly (sturm p ! (i+2)) x * poly (sturm p ! i) x < 0*
*(is poly ?p x * poly ?r x < 0)*

proof–

from *sturm-indices[OF i-in-range]*
have $sturm\ p\ !\ (i+2) = - (sturm\ p\ !\ i\ mod\ sturm\ p\ !\ (i+1))$
(is ?r = - (?p mod ?q)) .
hence $-?r = ?p\ mod\ ?q$ **by** *simp*
with *div-mult-mod-eq[of ?p ?q]* **have** $?p\ div\ ?q * ?q - ?r = ?p$ **by** *simp*
hence $poly\ (?p\ div\ ?q)\ x * poly\ ?q\ x - poly\ ?r\ x = poly\ ?p\ x$
by (*metis poly-diff poly-mult*)
with *q-0* **have** $r-x: poly\ ?r\ x = -poly\ ?p\ x$ **by** *simp*
moreover **have** $sqr-pos: \bigwedge x::real. x \neq 0 \implies x * x > 0$ **apply** (*case-tac x ≥ 0*)
by (*simp-all add: mult-neg-neg*)
from *sturm-adjacent-root-not-squarefree[of i p]* *assms r-x*
have $poly\ ?p\ x * poly\ ?p\ x > 0$ **by** (*force intro: sqr-pos*)
ultimately show $poly\ ?r\ x * poly\ ?p\ x < 0$ **by** *simp*
qed

Finally, if p contains no multiple roots, *sturm p*, i.e. the canonical Sturm sequence for p , is a Sturm sequence and can be used to determine the number of roots of p .

lemma *sturm-seq-sturm[simp]*:

assumes *rsquarefree p*
shows *sturm-seq (sturm p) p*

proof

show $sturm\ p \neq []$ **by** *simp*
show $hd\ (sturm\ p) = p$ **by** *simp*
show $length\ (sturm\ p) \geq 2$ **by** *simp*
from *assms* **show** $\bigwedge x. \neg (poly\ p\ x = 0 \wedge poly\ (sturm\ p\ !\ 1)\ x = 0)$
by (*simp add: rsquarefree-roots*)

next

fix $x :: real$ **and** $y :: real$
have $degree\ (last\ (sturm\ p)) = 0$ **by** *simp*
then obtain c **where** $last\ (sturm\ p) = [:c:]$
by (*cases last (sturm p), simp split: if-split-asm*)
thus $\bigwedge x\ y. sgn\ (poly\ (last\ (sturm\ p))\ x) =$

```

      sgn (poly (last (sturm p)) y) by simp
next
from sturm-firsttwo-signs[OF assms]
  show  $\bigwedge x_0. \text{poly } p \ x_0 = 0 \implies$ 
    eventually ( $\lambda x. \text{sgn } (\text{poly } (p * \text{sturm } p \ ! \ 1) \ x) =$ 
      (if  $x > x_0$  then 1 else -1)) (at  $x_0$ ) by simp
next
from sturm-signs[OF assms]
  show  $\bigwedge i \ x. \llbracket i < \text{length } (\text{sturm } p) - 2; \text{poly } (\text{sturm } p \ ! \ (i + 1)) \ x = 0 \rrbracket$ 
     $\implies \text{poly } (\text{sturm } p \ ! \ (i + 2)) \ x * \text{poly } (\text{sturm } p \ ! \ i) \ x < 0$  by simp
qed

```

2.5.1 Canonical squarefree Sturm sequence

The previous construction does not work for polynomials with multiple roots, but we can simply “divide away” multiple roots by dividing p by the GCD of p and p' . The resulting polynomial has the same roots as p , but with multiplicity 1, allowing us to again use the canonical construction.

definition *sturm-squarefree* **where**
 $\text{sturm-squarefree } p = \text{sturm } (p \ \text{div } (\text{gcd } p \ (p\text{deriv } p)))$

lemma *sturm-squarefree-not-Nil*[simp]: $\text{sturm-squarefree } p \neq []$
by (simp add: sturm-squarefree-def)

lemma *sturm-seq-sturm-squarefree*:
assumes [simp]: $p \neq 0$
defines [simp]: $p' \equiv p \ \text{div } \text{gcd } p \ (p\text{deriv } p)$
shows *sturm-seq* (sturm-squarefree p) p'
proof
have *rsquarefree* p'
proof (subst *rsquarefree-roots*, clarify)
fix x **assume** $\text{poly } p' \ x = 0$ $\text{poly } (p\text{deriv } p') \ x = 0$
hence $[-x, 1:] \ \text{dvd } \text{gcd } p' \ (p\text{deriv } p')$ **by** (simp add: poly-eq-0-iff-dvd)
also from *poly-div-gcd-squarefree(1)*[OF *assms(1)*]
have $\text{gcd } p' \ (p\text{deriv } p') = 1$ **by** simp
finally show *False* **by** (simp add: poly-eq-0-iff-dvd[symmetric])
qed

from *sturm-seq-sturm*[OF $\langle \text{rsquarefree } p' \rangle$]
interpret *sturm-seq*: *sturm-seq* sturm-squarefree $p \ p'$
by (simp add: sturm-squarefree-def)

show $\bigwedge x \ y. \text{sgn } (\text{poly } (\text{last } (\text{sturm-squarefree } p)) \ x) =$
 $\text{sgn } (\text{poly } (\text{last } (\text{sturm-squarefree } p)) \ y)$ **by** simp
show *sturm-squarefree* $p \neq []$ **by** simp
show $\text{hd } (\text{sturm-squarefree } p) = p'$ **by** (simp add: sturm-squarefree-def)
show $\text{length } (\text{sturm-squarefree } p) \geq 2$ **by** simp

```

have [simp]: sturm-squarefree p ! 0 = p'
           sturm-squarefree p ! Suc 0 = pderiv p'
by (simp-all add: sturm-squarefree-def)

from ⟨rsquarefree p⟩
show  $\bigwedge x. \neg (poly\ p'\ x = 0 \wedge poly\ (sturm-squarefree\ p\ !\ 1)\ x = 0)$ 
by (simp add: rsquarefree-roots)

from sturm-seq.signs show  $\bigwedge i\ x. \llbracket i < length\ (sturm-squarefree\ p) - 2;$ 
            $poly\ (sturm-squarefree\ p\ !\ (i + 1))\ x = 0 \rrbracket$ 
            $\implies poly\ (sturm-squarefree\ p\ !\ (i + 2))\ x *$ 
            $poly\ (sturm-squarefree\ p\ !\ i)\ x < 0 .$ 

from sturm-seq.deriv show  $\bigwedge x_0. poly\ p'\ x_0 = 0 \implies$ 
           eventually  $(\lambda x. sgn\ (poly\ (p' * sturm-squarefree\ p\ !\ 1)\ x) =$ 
            $(if\ x > x_0\ then\ 1\ else\ -1))\ (at\ x_0) .$ 
qed

```

2.5.2 Optimisation for multiple roots

We can also define the following non-canonical Sturm sequence that is obtained by taking the canonical Sturm sequence of p (possibly with multiple roots) and then dividing the entire sequence by the GCD of p and its derivative.

definition *sturm-squarefree'* **where**
sturm-squarefree' $p = (let\ d = gcd\ p\ (pderiv\ p)$
 in $map\ (\lambda p'. p' \div d)\ (sturm\ p))$

This construction also has all the desired properties:

lemma *sturm-squarefree'-adjacent-root-propagate-left*:
assumes $p \neq 0$
assumes $i < length\ (sturm-squarefree'\ (p :: real\ poly)) - 1$
assumes $poly\ (sturm-squarefree'\ p\ !\ i)\ x = 0$
and $poly\ (sturm-squarefree'\ p\ !\ (i + 1))\ x = 0$
shows $\forall j \leq i+1. poly\ (sturm-squarefree'\ p\ !\ j)\ x = 0$
proof (*intro sturm-adjacent-root-aux[OF assms(2,3,4)], goal-cases*)
case *prems*: $(1\ i\ x)$
define q **where** $q = sturm\ p\ !\ i$
define r **where** $r = sturm\ p\ !\ (Suc\ i)$
define s **where** $s = sturm\ p\ !\ (Suc\ (Suc\ i))$
define d **where** $d = gcd\ p\ (pderiv\ p)$
define $q'\ r'\ s'$ **where** $q' = q \div d$ **and** $r' = r \div d$ **and** $s' = s \div d$
from $\langle p \neq 0 \rangle$ **have** $d \neq 0$ **unfolding** *d-def* **by** *simp*
from *prems*(1) **have** *i-in-range*: $i < length\ (sturm\ p) - 2$
unfolding *sturm-squarefree'-def* *Let-def* **by** *simp*
have [simp]: $d \ dvd\ q\ d\ dvd\ r\ d\ dvd\ s$ **unfolding** *q-def* *r-def* *s-def* *d-def*
using *i-in-range* **by** (*auto intro: sturm-gcd*)

hence *qrs-simps*: $q = q' * d r = r' * d s = s' * d$
unfolding *q'-def r'-def s'-def* **by** (*simp-all*)
with *prems(2) i-in-range* **have** $r'-0: \text{poly } r' x = 0$
unfolding *r'-def r-def d-def sturm-squarefree'-def Let-def* **by** *simp*
hence $r-0: \text{poly } r x = 0$ **by** (*simp add: ⟨r = r' * d⟩*)
from *sturm-indices[OF i-in-range]* **have** $q = q \text{ div } r * r - s$
unfolding *q-def r-def s-def* **by** (*simp add: div-mult-mod-eq*)
hence $q' = (q \text{ div } r * r - s) \text{ div } d$ **by** (*simp add: q'-def*)
also have $\dots = (q \text{ div } r * r) \text{ div } d - s'$
by (*simp add: s'-def poly-div-diff-left*)
also have $\dots = q \text{ div } r * r' - s'$
using *dvd-div-mult[OF ⟨d dvd r⟩, of q div r]*
by (*simp add: algebra-simps r'-def*)
also have $q \text{ div } r = q' \text{ div } r'$ **by** (*simp add: qrs-simps ⟨d ≠ 0⟩*)
finally have $\text{poly } q' x = \text{poly } (q' \text{ div } r' * r' - s') x$ **by** *simp*
also from $r'-0$ **have** $\dots = -\text{poly } s' x$ **by** *simp*
finally have $\text{poly } s' x = -\text{poly } q' x$ **by** *simp*
thus *?case using i-in-range*
unfolding *q'-def s'-def q-def s-def sturm-squarefree'-def Let-def*
by (*simp add: d-def sgn-minus*)

qed

lemma *sturm-squarefree'-adjacent-roots*:

assumes $p \neq 0$
 $i < \text{length } (\text{sturm-squarefree}' (p :: \text{real poly})) - 1$
 $\text{poly } (\text{sturm-squarefree}' p ! i) x = 0$
 $\text{poly } (\text{sturm-squarefree}' p ! (i + 1)) x = 0$

shows *False*

proof –

define d **where** $d = \text{gcd } p (\text{pderiv } p)$
from *sturm-squarefree'-adjacent-root-propagate-left[OF assms]*
have $\text{poly } (\text{sturm-squarefree}' p ! 0) x = 0$
 $\text{poly } (\text{sturm-squarefree}' p ! 1) x = 0$ **by** *auto*
hence $\text{poly } (p \text{ div } d) x = 0$ $\text{poly } (\text{pderiv } p \text{ div } d) x = 0$
using *assms(2)*
unfolding *sturm-squarefree'-def Let-def d-def* **by** *auto*
moreover from *div-gcd-coprime assms(1)*
have *coprime (p div d) (pderiv p div d)* **unfolding** *d-def* **by** *auto*
ultimately show *False using coprime-imp-no-common-roots* **by** *auto*

qed

lemma *sturm-squarefree'-signs*:

assumes $p \neq 0$
assumes *i-in-range*: $i < \text{length } (\text{sturm-squarefree}' (p :: \text{real poly})) - 2$
assumes $q-0$: $\text{poly } (\text{sturm-squarefree}' p ! (i+1)) x = 0$ (**is** $\text{poly } ?q x = 0$)
shows $\text{poly } (\text{sturm-squarefree}' p ! (i+2)) x *$
 $\text{poly } (\text{sturm-squarefree}' p ! i) x < 0$
(**is** $\text{poly } ?r x * \text{poly } ?p x < 0$)

proof –

```

define  $d$  where  $d = \text{gcd } p \text{ (pderiv } p)$ 
with  $\langle p \neq 0 \rangle$  have [simp]:  $d \neq 0$  by simp
from poly-div-gcd-squarefree(1)[OF  $\langle p \neq 0 \rangle$ ]
  coprime-imp-no-common-roots
  have rsquarefree: rsquarefree ( $p \text{ div } d$ )
  by (auto simp: rsquarefree-roots d-def)

from i-in-range have i-in-range':  $i < \text{length } (\text{sturm } p) - 2$ 
  unfolding sturm-squarefree'-def by simp
hence  $d \text{ dvd } (\text{sturm } p ! i)$  (is  $d \text{ dvd } ?p'$ )
   $d \text{ dvd } (\text{sturm } p ! (\text{Suc } i))$  (is  $d \text{ dvd } ?q'$ )
   $d \text{ dvd } (\text{sturm } p ! (\text{Suc } (\text{Suc } i)))$  (is  $d \text{ dvd } ?r'$ )
  unfolding d-def by (auto intro: sturm-gcd)
hence pqr-simps:  $?p' = ?p * d \text{ ?}q' = ?q * d \text{ ?}r' = ?r * d$ 
  unfolding sturm-squarefree'-def Let-def d-def using i-in-range'
  by (auto simp: dvd-div-mult-self)
with q-0 have q'-0: poly  $?q' x = 0$  by simp
from sturm-indices[OF i-in-range']
  have  $\text{sturm } p ! (i+2) = - (\text{sturm } p ! i \text{ mod } \text{sturm } p ! (i+1))$  .
hence  $-?r' = ?p' \text{ mod } ?q'$  by simp
with div-mult-mod-eq[of  $?p' ?q'$ ] have  $?p' \text{ div } ?q' * ?q' - ?r' = ?p'$  by simp
hence  $d * (?p \text{ div } ?q * ?q - ?r) = d * ?p$  by (simp add: pqr-simps algebra-simps)
hence  $?p \text{ div } ?q * ?q - ?r = ?p$  by simp
hence poly ( $?p \text{ div } ?q$ )  $x * \text{poly } ?q x - \text{poly } ?r x = \text{poly } ?p x$ 
  by (metis poly-diff poly-mult)
with q-0 have r-x: poly  $?r x = -\text{poly } ?p x$  by simp

from sturm-squarefree'-adjacent-roots[OF  $\langle p \neq 0 \rangle$ ] i-in-range q-0
  have poly  $?p x \neq 0$  by force
moreover have sqr-pos:  $\bigwedge x :: \text{real}. x \neq 0 \implies x * x > 0$  apply (case-tac  $x \geq 0$ )
  by (simp-all add: mult-neg-neg)
ultimately show ?thesis using r-x by simp
qed

```

This approach indeed also yields a valid squarefree Sturm sequence for the polynomial $p/\text{gcd}(p, p')$.

```

lemma sturm-seq-sturm-squarefree':
  assumes ( $p :: \text{real poly}$ )  $\neq 0$ 
  defines  $d \equiv \text{gcd } p \text{ (pderiv } p)$ 
  shows sturm-seq (sturm-squarefree'  $p$ ) ( $p \text{ div } d$ )
    (is sturm-seq  $?ps' ?p'$ )
proof
  show  $?ps' \neq []$  hd  $?ps' = ?p' 2 \leq \text{length } ?ps'$ 
    by (simp-all add: sturm-squarefree'-def d-def hd-map)

from assms have  $d \neq 0$  by simp
  {
    have  $d \text{ dvd last } (\text{sturm } p)$  unfolding d-def
      by (rule sturm-gcd, simp)
  }

```

```

hence *:  $last (sturm p) = last ?ps' * d$ 
  by (simp add: sturm-squarefree'-def last-map d-def dvd-div-mult-self)
then have  $last ?ps' dvd last (sturm p)$  by simp
with *  $dvd-imp-degree-le[OF this]$  have  $degree (last ?ps') \leq degree (last (sturm p))$ 
  using  $\langle d \neq 0 \rangle$  by (cases last ?ps' = 0) auto
hence  $degree (last ?ps') = 0$  by simp
then obtain  $c$  where  $last ?ps' = [:c]$ 
  by (cases last ?ps', simp split: if-split-asm)
thus  $\bigwedge x y. sgn (poly (last ?ps') x) = sgn (poly (last ?ps') y)$  by simp
}

have squarefree:  $rsquarefree ?p'$  using  $\langle p \neq 0 \rangle$ 
  by (subst rsquarefree-roots, unfold d-def,
    intro allI coprime-imp-no-common-roots poly-div-gcd-squarefree)
have [simp]:  $sturm-squarefree' p ! Suc 0 = pderiv p div d$ 
  unfolding sturm-squarefree'-def Let-def sturm-def d-def
  by (subst sturm-aux.simps, simp)
have coprime:  $coprime ?p' (pderiv p div d)$ 
  unfolding d-def using div-gcd-coprime  $\langle p \neq 0 \rangle$  by blast
thus squarefree':
   $\bigwedge x. \neg (poly (p div d) x = 0 \wedge poly (sturm-squarefree' p ! 1) x = 0)$ 
  using coprime-imp-no-common-roots by simp

from sturm-squarefree'-signs[OF  $\langle p \neq 0 \rangle$ ]
  show  $\bigwedge i x. \llbracket i < length ?ps' - 2; poly (?ps' ! (i + 1)) x = 0 \rrbracket$ 
     $\implies poly (?ps' ! (i + 2)) x * poly (?ps' ! i) x < 0$  .

have [simp]:  $?p' \neq 0$  using squarefree by (simp add: rsquarefree-def)
have A:  $?p' = ?ps' ! 0 pderiv p div d = ?ps' ! 1$ 
  by (simp-all add: sturm-squarefree'-def Let-def d-def sturm-def,
    subst sturm-aux.simps, simp)
have [simp]:  $?ps' ! 0 \neq 0$  using squarefree
  by (auto simp: A rsquarefree-def)

fix  $x_0 :: real$ 
assume  $poly ?p' x_0 = 0$ 
hence  $poly p x_0 = 0$  using poly-div-gcd-squarefree(2)[OF  $\langle p \neq 0 \rangle$ ]
  unfolding d-def by simp
hence  $pderiv p \neq 0$  using  $\langle p \neq 0 \rangle$  by (auto dest: pderiv-iszero)
with  $\langle p \neq 0 \rangle \langle poly p x_0 = 0 \rangle$ 
  have A:  $eventually (\lambda x. sgn (poly (p * pderiv p) x) =$ 
     $(if x_0 < x then 1 else -1)) (at x_0)$ 
  by (intro sturm-firsttwo-signs-aux, simp-all)
note  $ev = eventually-conj[OF A poly-neighbourhood-without-roots[OF  $\langle d \neq 0 \rangle$ ]]$ 

show  $eventually (\lambda x. sgn (poly (p div d * sturm-squarefree' p ! 1) x) =$ 
   $(if x_0 < x then 1 else -1)) (at x_0)$ 
proof (rule eventually-mono[OF ev], goal-cases)

```

```

have [intro]:
   $\bigwedge a (b::real). b \neq 0 \implies a < 0 \implies a / (b * b) < 0$ 
   $\bigwedge a (b::real). b \neq 0 \implies a > 0 \implies a / (b * b) > 0$ 
  by ((case-tac b > 0,
        auto simp: mult-neg-neg field-simps) [])+
case prems: (1 x)
hence [simp]: poly d x * poly d x > 0
  by (cases poly d x > 0, auto simp: mult-neg-neg)
from poly-div-gcd-squarefree-aux(2)[OF ⟨pderiv p ≠ 0⟩]
  have poly (p div d) x = 0  $\longleftrightarrow$  poly p x = 0 by (simp add: d-def)
moreover have d dvd p d dvd pderiv p unfolding d-def by simp-all
ultimately show ?case using prems
  by (auto simp: sgn-real-def poly-div not-less[symmetric]
        zero-less-divide-iff split: if-split-asm)

```

qed
qed

This construction is obviously more expensive to compute than the one that *first* divides p by $\gcd(p, p')$ and *then* applies the canonical construction. In this construction, we *first* compute the canonical Sturm sequence of p as if it had no multiple roots and *then* divide by the GCD. However, it can be seen quite easily that unless x is a multiple root of p , i.e. as long as $\gcd(P, P') \neq 0$, the number of sign changes in a sequence of polynomials does not actually change when we divide the polynomials by $\gcd(p, p')$. Therefore we can use the canonical Sturm sequence even in the non-square-free case as long as the borders of the interval we are interested in are not multiple roots of the polynomial.

lemma *sign-changes-mult-aux*:

```

assumes d ≠ (0::real)
shows length (remdups-adj (filter (λx. x ≠ 0) (map ((* d ∘ f) xs))) =
        length (remdups-adj (filter (λx. x ≠ 0) (map f xs)))
proof –
from assms have inj: inj ((* d) by (auto intro: injI)
from assms have [simp]: filter (λx. ((* d ∘ f) x ≠ 0) = filter (λx. f x ≠ 0)
        filter ((λx. x ≠ 0) ∘ f) = filter (λx. f x ≠ 0)
  by (simp-all add: o-def)
have filter (λx. x ≠ 0) (map ((* d ∘ f) xs) =
        map ((* d ∘ f) (filter (λx. ((* d ∘ f) x ≠ 0) xs)
  by (simp add: filter-map o-def)
thus ?thesis using remdups-adj-map-injective[OF inj] assms
  by (simp add: filter-map map-map[symmetric] del: map-map)
qed

```

lemma *sturm-sturm-squarefree'-same-sign-changes*:

```

fixes p :: real poly
defines ps ≡ sturm p and ps' ≡ sturm-squarefree' p
shows poly p x ≠ 0 ∨ poly (pderiv p) x ≠ 0  $\implies$ 
        sign-changes ps' x = sign-changes ps x

```

$p \neq 0 \implies \text{sign-changes-inf } ps' = \text{sign-changes-inf } ps$
 $p \neq 0 \implies \text{sign-changes-neg-inf } ps' = \text{sign-changes-neg-inf } ps$

proof –

define d **where** $d = \text{gcd } p \text{ (pderiv } p)$
define p' **where** $p' = p \text{ div } d$
define s' **where** $s' = \text{poly-inf } d$
define s'' **where** $s'' = \text{poly-neg-inf } d$

{
fix $x :: \text{real}$ **and** $q :: \text{real poly}$
assume $q \in \text{set } ps$
hence $d \text{ dvd } q$ **unfolding** $d\text{-def } ps\text{-def}$ **using** sturm-gcd **by** simp
hence $q\text{-prod}: q = (q \text{ div } d) * d$ **unfolding** $p'\text{-def } d\text{-def}$
by $(\text{simp add: algebra-simps dvd-mult-div-cancel})$

have $\text{poly } q \ x = \text{poly } d \ x * \text{poly } (q \text{ div } d) \ x$ **by** $(\text{subst } q\text{-prod}, \text{simp})$
hence $s1: \text{sgn } (\text{poly } q \ x) = \text{sgn } (\text{poly } d \ x) * \text{sgn } (\text{poly } (q \text{ div } d) \ x)$
by $(\text{subst } q\text{-prod}, \text{simp add: sgn-mult})$
from poly-inf-mult **have** $s2: \text{poly-inf } q = s' * \text{poly-inf } (q \text{ div } d)$
unfolding $s'\text{-def}$ **by** $(\text{subst } q\text{-prod}, \text{simp})$
from poly-inf-mult **have** $s3: \text{poly-neg-inf } q = s'' * \text{poly-neg-inf } (q \text{ div } d)$
unfolding $s''\text{-def}$ **by** $(\text{subst } q\text{-prod}, \text{simp})$
note $s1 \ s2 \ s3$
}

note $\text{signs} = \text{this}$

{
fix $f :: \text{real poly} \implies \text{real}$ **and** $s :: \text{real}$
assume $f: \bigwedge q. q \in \text{set } ps \implies f \ q = s * f \ (q \text{ div } d)$ **and** $s: s \neq 0$
hence $\text{inverse } s \neq 0$ **by** simp
{fix q **assume** $q \in \text{set } ps$
hence $f \ (q \text{ div } d) = \text{inverse } s * f \ q$
by $(\text{subst } f[\text{of } q], \text{simp-all add: } s)$
} **note** $f' = \text{this}$
have $\text{length } (\text{remdups-adj } [x \leftarrow \text{map } f \ (\text{map } (\lambda q. q \text{ div } d) \ ps). \ x \neq 0]) - 1 =$
 $\text{length } (\text{remdups-adj } [x \leftarrow \text{map } (\lambda q. f \ (q \text{ div } d)) \ ps . \ x \neq 0]) - 1$
by $(\text{simp only: sign-changes-def o-def map-map})$
also have $\text{map } (\lambda q. q \text{ div } d) \ ps = ps'$
by $(\text{simp add: ps-def ps'\text{-def sturm-squarefree'\text{-def Let-def } d\text{-def})}$
also from f' **have** $\text{map } (\lambda q. f \ (q \text{ div } d)) \ ps =$
 $\text{map } (\lambda x. ((*)(\text{inverse } s) \circ f) \ x) \ ps$ **by** (simp add: o-def)
also note $\text{sign-changes-mult-aux}[OF \langle \text{inverse } s \neq 0 \rangle, \text{of } f \ ps]$
finally have
 $\text{length } (\text{remdups-adj } [x \leftarrow \text{map } f \ ps' . \ x \neq 0]) - 1 =$
 $\text{length } (\text{remdups-adj } [x \leftarrow \text{map } f \ ps . \ x \neq 0]) - 1$ **by** simp
}

note $\text{length-remdups-adj} = \text{this}$

{

```

fix  $x$  assume  $A$ :  $\text{poly } p \ x \neq 0 \vee \text{poly } (\text{pderiv } p) \ x \neq 0$ 
have  $d \ \text{dvd} \ p \ d \ \text{dvd} \ \text{pderiv } p$  unfolding  $d\text{-def}$  by  $\text{simp-all}$ 
with  $A$  have  $\text{sgn } (\text{poly } d \ x) \neq 0$ 
  by  $(\text{auto simp add: sgn-zero-iff elim: dvdE})$ 
thus  $\text{sign-changes } ps' \ x = \text{sign-changes } ps \ x$  using  $\text{signs}(1)$ 
  unfolding  $\text{sign-changes-def}$ 
  by  $(\text{intro length-remdups-adj}[of \ \lambda q. \ \text{sgn } (\text{poly } q \ x)], \text{simp-all})$ 
}

```

```

assume  $p \neq 0$ 
hence  $d \neq 0$  unfolding  $d\text{-def}$  by  $\text{simp}$ 
hence  $s' \neq 0 \ s'' \neq 0$  unfolding  $s'\text{-def } s''\text{-def}$  by  $\text{simp-all}$ 
from  $\text{length-remdups-adj}[of \ \text{poly-inf } s', \ \text{OF } \text{signs}(2) \ \langle s' \neq 0 \rangle]$ 
  show  $\text{sign-changes-inf } ps' = \text{sign-changes-inf } ps$ 
  unfolding  $\text{sign-changes-inf-def}$  .
from  $\text{length-remdups-adj}[of \ \text{poly-neg-inf } s'', \ \text{OF } \text{signs}(3) \ \langle s'' \neq 0 \rangle]$ 
  show  $\text{sign-changes-neg-inf } ps' = \text{sign-changes-neg-inf } ps$ 
  unfolding  $\text{sign-changes-neg-inf-def}$  .
qed

```

2.6 Root-counting functions

With all these results, we can now define functions that count roots in bounded and unbounded intervals:

definition *count-roots-between* **where**
count-roots-between $p \ a \ b = (\text{if } a \leq b \wedge p \neq 0 \text{ then}$
 ($\text{let } ps = \text{sturm-squarefree } p$
 in $\text{sign-changes } ps \ a - \text{sign-changes } ps \ b$) *else* 0)

definition *count-roots* **where**
count-roots $p = (\text{if } (p::\text{real poly}) = 0 \text{ then } 0 \text{ else}$
 ($\text{let } ps = \text{sturm-squarefree } p$
 in $\text{sign-changes-neg-inf } ps - \text{sign-changes-inf } ps$))

definition *count-roots-above* **where**
count-roots-above $p \ a = (\text{if } (p::\text{real poly}) = 0 \text{ then } 0 \text{ else}$
 ($\text{let } ps = \text{sturm-squarefree } p$
 in $\text{sign-changes } ps \ a - \text{sign-changes-inf } ps$))

definition *count-roots-below* **where**
count-roots-below $p \ a = (\text{if } (p::\text{real poly}) = 0 \text{ then } 0 \text{ else}$
 ($\text{let } ps = \text{sturm-squarefree } p$
 in $\text{sign-changes-neg-inf } ps - \text{sign-changes } ps \ a$))

lemma *count-roots-between-correct*:

$\text{count-roots-between } p \ a \ b = \text{card } \{x. a < x \wedge x \leq b \wedge \text{poly } p \ x = 0\}$

proof (*cases* $p \neq 0 \wedge a \leq b$)

case *False*

```

note False' = this
hence  $\text{card } \{x. a < x \wedge x \leq b \wedge \text{poly } p \ x = 0\} = 0$ 
proof (cases  $a < b$ )
  case True
    with False have [simp]:  $p = 0$  by simp
    have subset:  $\{a < .. < b\} \subseteq \{x. a < x \wedge x \leq b \wedge \text{poly } p \ x = 0\}$  by auto
    from infinite-Ioo[OF True] have  $\neg \text{finite } \{a < .. < b\}$  .
    hence  $\neg \text{finite } \{x. a < x \wedge x \leq b \wedge \text{poly } p \ x = 0\}$ 
      using finite-subset[OF subset] by blast
    thus ?thesis by simp
  next
    case False
      with False' show ?thesis by (auto simp: not-less card-eq-0-iff)
    qed
  thus ?thesis unfolding count-roots-between-def Let-def using False by auto
next
  case True
    hence  $p \neq 0 \wedge a \leq b$  by simp-all
    define p' where  $p' = p \ \text{div} \ (\text{gcd } p \ (\text{pderiv } p))$ 
    from poly-div-gcd-squarefree(1)[OF <p ≠ 0>] have  $p' \neq 0$ 
      unfolding p'-def by clarsimp

    from sturm-seq-sturm-squarefree[OF <p ≠ 0>]
      interpret sturm-seq sturm-squarefree p p'
      unfolding p'-def .
    from poly-roots-finite[OF <p' ≠ 0>]
      have finite  $\{x. a < x \wedge x \leq b \wedge \text{poly } p' \ x = 0\}$  by fast
    have count-roots-between  $p \ a \ b = \text{card } \{x. a < x \wedge x \leq b \wedge \text{poly } p' \ x = 0\}$ 
      unfolding count-roots-between-def Let-def
      using True count-roots-between[OF <p' ≠ 0> <a ≤ b>] by simp
    also from poly-div-gcd-squarefree(2)[OF <p ≠ 0>]
      have  $\{x. a < x \wedge x \leq b \wedge \text{poly } p' \ x = 0\} =$ 
         $\{x. a < x \wedge x \leq b \wedge \text{poly } p \ x = 0\}$  unfolding p'-def by blast
    finally show ?thesis .
  qed

lemma count-roots-correct:
  fixes  $p :: \text{real poly}$ 
  shows count-roots  $p = \text{card } \{x. \text{poly } p \ x = 0\}$  (is  $= \text{card } ?S$ )
proof (cases  $p = 0$ )
  case True
    with finite-subset[of  $\{0 < .. < 1\} ?S$ ]
      have  $\neg \text{finite } \{x. \text{poly } p \ x = 0\}$  by (auto simp: infinite-Ioo)
      thus ?thesis by (simp add: count-roots-def True)
  next
    case False
      define p' where  $p' = p \ \text{div} \ (\text{gcd } p \ (\text{pderiv } p))$ 
      from poly-div-gcd-squarefree(1)[OF <p ≠ 0>] have  $p' \neq 0$ 
        unfolding p'-def by clarsimp

```

```

from sturm-seq-sturm-squarefree[OF ‹ $p \neq 0$ ›]
  interpret sturm-seq sturm-squarefree  $p$   $p'$ 
  unfolding  $p'$ -def .
from count-roots[OF ‹ $p' \neq 0$ ›]
  have count-roots  $p = \text{card } \{x. \text{poly } p' x = 0\}$ 
  unfolding count-roots-def Let-def by (simp add: ‹ $p \neq 0$ ›)
also from poly-div-gcd-squarefree(2)[OF ‹ $p \neq 0$ ›]
  have  $\{x. \text{poly } p' x = 0\} = \{x. \text{poly } p x = 0\}$  unfolding  $p'$ -def by blast
finally show ?thesis .
qed

```

lemma count-roots-above-correct:

```

fixes  $p :: \text{real poly}$ 
shows count-roots-above  $p$   $a = \text{card } \{x. x > a \wedge \text{poly } p x = 0\}$ 
  (is - = card ?S)
proof (cases  $p = 0$ )
  case True
  with finite-subset[of  $\{a <..<a+1\}$  ?S]
  have  $\neg \text{finite } \{x. x > a \wedge \text{poly } p x = 0\}$  by (auto simp: infinite-Ioo subset-eq)
  thus ?thesis by (simp add: count-roots-above-def True)
next
  case False
  define  $p'$  where  $p' = p \text{ div } (\text{gcd } p (\text{pderiv } p))$ 
  from poly-div-gcd-squarefree(1)[OF ‹ $p \neq 0$ ›] have  $p' \neq 0$ 
  unfolding  $p'$ -def by clarsimp

```

```

from sturm-seq-sturm-squarefree[OF ‹ $p \neq 0$ ›]
  interpret sturm-seq sturm-squarefree  $p$   $p'$ 
  unfolding  $p'$ -def .
from count-roots-above[OF ‹ $p' \neq 0$ ›]
  have count-roots-above  $p$   $a = \text{card } \{x. x > a \wedge \text{poly } p' x = 0\}$ 
  unfolding count-roots-above-def Let-def by (simp add: ‹ $p \neq 0$ ›)
also from poly-div-gcd-squarefree(2)[OF ‹ $p \neq 0$ ›]
  have  $\{x. x > a \wedge \text{poly } p' x = 0\} = \{x. x > a \wedge \text{poly } p x = 0\}$ 
  unfolding  $p'$ -def by blast
finally show ?thesis .
qed

```

lemma count-roots-below-correct:

```

fixes  $p :: \text{real poly}$ 
shows count-roots-below  $p$   $a = \text{card } \{x. x \leq a \wedge \text{poly } p x = 0\}$ 
  (is - = card ?S)
proof (cases  $p = 0$ )
  case True
  with finite-subset[of  $\{a - 1 <..<a\}$  ?S]
  have  $\neg \text{finite } \{x. x \leq a \wedge \text{poly } p x = 0\}$  by (auto simp: infinite-Ioo subset-eq)
  thus ?thesis by (simp add: count-roots-below-def True)
next

```

```

case False
define  $p'$  where  $p' = p \text{ div } (\text{gcd } p \text{ (pderiv } p))$ 
from poly-div-gcd-squarefree(1)[OF  $\langle p \neq 0 \rangle$ ] have  $p' \neq 0$ 
  unfolding  $p'$ -def by clarsimp

from sturm-seq-sturm-squarefree[OF  $\langle p \neq 0 \rangle$ ]
  interpret sturm-seq sturm-squarefree  $p \ p'$ 
  unfolding  $p'$ -def .
from count-roots-below[OF  $\langle p' \neq 0 \rangle$ ]
  have count-roots-below  $p \ a = \text{card } \{x. x \leq a \wedge \text{poly } p' \ x = 0\}$ 
  unfolding count-roots-below-def Let-def by (simp add:  $\langle p \neq 0 \rangle$ )
also from poly-div-gcd-squarefree(2)[OF  $\langle p \neq 0 \rangle$ ]
  have  $\{x. x \leq a \wedge \text{poly } p' \ x = 0\} = \{x. x \leq a \wedge \text{poly } p \ x = 0\}$ 
  unfolding  $p'$ -def by blast
finally show ?thesis .
qed

```

The optimisation explained above can be used to prove more efficient code equations that use the more efficient construction in the case that the interval borders are not multiple roots:

```

lemma count-roots-between[code]:
  count-roots-between  $p \ a \ b =$ 
    (let  $q = \text{pderiv } p$ 
      in if  $a > b \vee p = 0$  then  $0$ 
      else if  $(\text{poly } p \ a \neq 0 \vee \text{poly } q \ a \neq 0) \wedge (\text{poly } p \ b \neq 0 \vee \text{poly } q \ b \neq 0)$ 
        then (let  $ps = \text{sturm } p$ 
          in sign-changes  $ps \ a - \text{sign-changes } ps \ b$ )
        else (let  $ps = \text{sturm-squarefree } p$ 
          in sign-changes  $ps \ a - \text{sign-changes } ps \ b$ ))
proof (cases  $a > b \vee p = 0$ )
  case True
    thus ?thesis by (auto simp add: count-roots-between-def Let-def)
next
  case False
    note False1 = this
    hence  $a \leq b \ p \neq 0$  by simp-all
    thus ?thesis
    proof (cases  $(\text{poly } p \ a \neq 0 \vee \text{poly } (\text{pderiv } p) \ a \neq 0) \wedge$ 
       $(\text{poly } p \ b \neq 0 \vee \text{poly } (\text{pderiv } p) \ b \neq 0)$ )
    case False
      thus ?thesis using False1
      by (auto simp add: Let-def count-roots-between-def)
    next
    case True
      hence A: poly  $p \ a \neq 0 \vee \text{poly } (\text{pderiv } p) \ a \neq 0$  and
        B: poly  $p \ b \neq 0 \vee \text{poly } (\text{pderiv } p) \ b \neq 0$  by auto
      define  $d$  where  $d = \text{gcd } p \ (\text{pderiv } p)$ 
      from  $\langle p \neq 0 \rangle$  have [simp]:  $p \ \text{div} \ d \neq 0$ 
      using poly-div-gcd-squarefree(1)[OF  $\langle p \neq 0 \rangle$ ] by (auto simp add: d-def)

```

```

from sturm-seq-sturm-squarefree'[OF ⟨p ≠ 0⟩]
  interpret sturm-seq sturm-squarefree' p p div d
  unfolding sturm-squarefree'-def Let-def d-def .
note count-roots-between-correct
also have {x. a < x ∧ x ≤ b ∧ poly p x = 0} =
  {x. a < x ∧ x ≤ b ∧ poly (p div d) x = 0}
  unfolding d-def using poly-div-gcd-squarefree(2)[OF ⟨p ≠ 0⟩] by simp
also note count-roots-between[OF ⟨p div d ≠ 0⟩ ⟨a ≤ b⟩, symmetric]
also note sturm-sturm-squarefree'-same-sign-changes(1)[OF A]
also note sturm-sturm-squarefree'-same-sign-changes(1)[OF B]
finally show ?thesis using True False by (simp add: Let-def)
qed
qed

```

```

lemma count-roots-code[code]:
  count-roots (p::real poly) =
    (if p = 0 then 0
     else let ps = sturm p
          in sign-changes-neg-inf ps - sign-changes-inf ps)
proof (cases p = 0, simp add: count-roots-def)
case False
  define d where d = gcd p (pderiv p)
  from ⟨p ≠ 0⟩ have [simp]: p div d ≠ 0
    using poly-div-gcd-squarefree(1)[OF ⟨p ≠ 0⟩] by (auto simp add: d-def)
  from sturm-seq-sturm-squarefree'[OF ⟨p ≠ 0⟩]
    interpret sturm-seq sturm-squarefree' p p div d
    unfolding sturm-squarefree'-def Let-def d-def .

  note count-roots-correct
  also have {x. poly p x = 0} = {x. poly (p div d) x = 0}
    unfolding d-def using poly-div-gcd-squarefree(2)[OF ⟨p ≠ 0⟩] by simp
  also note count-roots[OF ⟨p div d ≠ 0⟩, symmetric]
  also note sturm-sturm-squarefree'-same-sign-changes(2)[OF ⟨p ≠ 0⟩]
  also note sturm-sturm-squarefree'-same-sign-changes(3)[OF ⟨p ≠ 0⟩]
  finally show ?thesis using False unfolding Let-def by simp
qed

```

```

lemma count-roots-above-code[code]:
  count-roots-above p a =
    (let q = pderiv p
     in if p = 0 then 0
        else if poly p a ≠ 0 ∨ poly q a ≠ 0
            then (let ps = sturm p
                  in sign-changes ps a - sign-changes-inf ps)
            else (let ps = sturm-squarefree p
                  in sign-changes ps a - sign-changes-inf ps))
proof (cases p = 0)

```

```

case True
  thus ?thesis by (auto simp add: count-roots-above-def Let-def)
next
case False
  note False1 = this
  hence  $p \neq 0$  by simp-all
  thus ?thesis
  proof (cases (poly p a  $\neq$  0  $\vee$  poly (pderiv p) a  $\neq$  0))
  case False
    thus ?thesis using False1
    by (auto simp add: Let-def count-roots-above-def)
  next
  case True
    hence  $A: \text{poly } p \ a \neq 0 \vee \text{poly } (pderiv \ p) \ a \neq 0$  by simp
    define d where  $d = \text{gcd } p \ (pderiv \ p)$ 
    from  $\langle p \neq 0 \rangle$  have [simp]:  $p \ \text{div} \ d \neq 0$ 
      using poly-div-gcd-squarefree(1)[OF  $\langle p \neq 0 \rangle$ ] by (auto simp add: d-def)
    from sturm-seq-sturm-squarefree'[OF  $\langle p \neq 0 \rangle$ ]
      interpret sturm-seq sturm-squarefree' p p div d
      unfolding sturm-squarefree'-def Let-def d-def .
    note count-roots-above-correct
    also have  $\{x. a < x \wedge \text{poly } p \ x = 0\} =$ 
       $\{x. a < x \wedge \text{poly } (p \ \text{div} \ d) \ x = 0\}$ 
      unfolding d-def using poly-div-gcd-squarefree(2)[OF  $\langle p \neq 0 \rangle$ ] by simp
    also note count-roots-above[OF  $\langle p \ \text{div} \ d \neq 0 \rangle$ , symmetric]
    also note sturm-sturm-squarefree'-same-sign-changes(1)[OF A]
    also note sturm-sturm-squarefree'-same-sign-changes(2)[OF  $\langle p \neq 0 \rangle$ ]
    finally show ?thesis using True False by (simp add: Let-def)
  qed
qed

```

lemma count-roots-below-code[code]:

```

count-roots-below p a =
  (let q = pderiv p
   in if p = 0 then 0
   else if poly p a  $\neq$  0  $\vee$  poly q a  $\neq$  0
     then (let ps = sturm p
            in sign-changes-neg-inf ps - sign-changes ps a)
     else (let ps = sturm-squarefree p
            in sign-changes-neg-inf ps - sign-changes ps a))
proof (cases p = 0)

```

```

case True
  thus ?thesis by (auto simp add: count-roots-below-def Let-def)
next
case False
  note False1 = this
  hence  $p \neq 0$  by simp-all
  thus ?thesis
  proof (cases (poly p a  $\neq$  0  $\vee$  poly (pderiv p) a  $\neq$  0))

```

```

case False
  thus ?thesis using False1
    by (auto simp add: Let-def count-roots-below-def)
next
case True
  hence A: poly p a ≠ 0 ∨ poly (pderiv p) a ≠ 0 by simp
  define d where d = gcd p (pderiv p)
  from ⟨p ≠ 0⟩ have [simp]: p div d ≠ 0
    using poly-div-gcd-squarefree(1)[OF ⟨p ≠ 0⟩] by (auto simp add: d-def)
  from sturm-seq-sturm-squarefree'[OF ⟨p ≠ 0⟩]
    interpret sturm-seq sturm-squarefree' p p div d
    unfolding sturm-squarefree'-def Let-def d-def .
  note count-roots-below-correct
  also have {x. x ≤ a ∧ poly p x = 0} =
    {x. x ≤ a ∧ poly (p div d) x = 0}
    unfolding d-def using poly-div-gcd-squarefree(2)[OF ⟨p ≠ 0⟩] by simp
  also note count-roots-below[OF ⟨p div d ≠ 0⟩, symmetric]
  also note sturm-sturm-squarefree'-same-sign-changes(1)[OF A]
  also note sturm-sturm-squarefree'-same-sign-changes(3)[OF ⟨p ≠ 0⟩]
  finally show ?thesis using True False by (simp add: Let-def)
qed
qed
end

```

3 The “sturm” proof method

```

theory Sturm-Method
imports Sturm-Theorem
begin

```

3.1 Preliminary lemmas

In this subsection, we prove lemmas that reduce root counting and related statements to simple, computable expressions using the *count-roots* function family.

lemma *poly-card-roots-less-leq*:

```

card {x. a < x ∧ x ≤ b ∧ poly p x = 0} = count-roots-between p a b
by (simp add: count-roots-between-correct)

```

lemma *poly-card-roots-leq-leq*:

```

card {x. a ≤ x ∧ x ≤ b ∧ poly p x = 0} =
  ( count-roots-between p a b +
    (if (a ≤ b ∧ poly p a = 0 ∧ p ≠ 0) ∨ (a = b ∧ p = 0) then 1 else 0))

```

proof (cases (a ≤ b ∧ poly p a = 0 ∧ p ≠ 0) ∨ (a = b ∧ p = 0))

case False

note False' = this

thus ?thesis

```

proof (cases p = 0)
  case False
    with False' have poly p a ≠ 0 ∨ a > b by auto
    hence {x. a ≤ x ∧ x ≤ b ∧ poly p x = 0} =
      {x. a < x ∧ x ≤ b ∧ poly p x = 0}
    by (auto simp: less-eq-real-def)
    thus ?thesis using poly-card-roots-less-leq False'
      by (auto split: if-split-asm)
  next
    case True
      have {x. a ≤ x ∧ x ≤ b} = {a..b}
        {x. a < x ∧ x ≤ b} = {a<..b} by auto
      with True False have card {x. a < x ∧ x ≤ b} = 0 card {x. a ≤ x ∧ x ≤
b} = 0
        by (auto simp add: card-eq-0-iff infinite-Ioc infinite-Icc)
      with True False show ?thesis
        using count-roots-between-correct by simp
    qed
  next
    case True
      note True' = this
      have fin: finite {x. a ≤ x ∧ x ≤ b ∧ poly p x = 0}
      proof (cases p = 0)
        case True
          with True' have a = b by simp
          hence {x. a ≤ x ∧ x ≤ b ∧ poly p x = 0} = {b} using True by auto
          thus ?thesis by simp
        next
          case False
            from poly-roots-finite[OF this] show ?thesis by fast
          qed
        with True have {x. a ≤ x ∧ x ≤ b ∧ poly p x = 0} =
          insert a {x. a < x ∧ x ≤ b ∧ poly p x = 0} by auto
        hence card {x. a ≤ x ∧ x ≤ b ∧ poly p x = 0} =
          Suc (card {x. a < x ∧ x ≤ b ∧ poly p x = 0}) using fin by force
        thus ?thesis using True count-roots-between-correct by simp
      qed

```

lemma poly-card-roots-less-less:

$$\begin{aligned}
& \text{card } \{x. a < x \wedge x < b \wedge \text{poly } p \ x = 0\} = \\
& \quad (\text{count-roots-between } p \ a \ b - \\
& \quad \quad (\text{if } \text{poly } p \ b = 0 \wedge a < b \wedge p \neq 0 \text{ then } 1 \text{ else } 0))
\end{aligned}$$

proof (cases poly p b = 0 ∧ a < b ∧ p ≠ 0)

```

case False
  note False' = this
  show ?thesis
  proof (cases p = 0)
    case True
      have [simp]: {x. a < x ∧ x < b} = {a<..b}

```

$\{x. a < x \wedge x \leq b\} = \{a <..b\}$ **by** *auto*
with *True False* **have** $\text{card } \{x. a < x \wedge x \leq b\} = 0$ $\text{card } \{x. a < x \wedge x < b\} = 0$
by (*auto simp add: card-eq-0-iff infinite-Ioo infinite-Ioc*)
with *True False'* **show** *?thesis*
by (*auto simp: count-roots-between-correct*)
next
case *False*
with *False'* **have** $\{x. a < x \wedge x < b \wedge \text{poly } p \ x = 0\} =$
 $\{x. a < x \wedge x \leq b \wedge \text{poly } p \ x = 0\}$
by (*auto simp: less-eq-real-def*)
thus *?thesis* **using** *poly-card-roots-less-leq False* **by** *auto*
qed
next
case *True*
with *poly-roots-finite*
have *fin: finite* $\{x. a < x \wedge x < b \wedge \text{poly } p \ x = 0\}$ **by** *fast*
from *True* **have** $\{x. a < x \wedge x \leq b \wedge \text{poly } p \ x = 0\} =$
 $\text{insert } b \ \{x. a < x \wedge x < b \wedge \text{poly } p \ x = 0\}$ **by** *auto*
hence *Suc* $(\text{card } \{x. a < x \wedge x < b \wedge \text{poly } p \ x = 0\}) =$
 $\text{card } \{x. a < x \wedge x \leq b \wedge \text{poly } p \ x = 0\}$ **using** *fin* **by** *force*
also note *count-roots-between-correct[symmetric]*
finally show *?thesis* **using** *True* **by** *simp*
qed

lemma *poly-card-roots-leq-less:*

$\text{card } \{x::\text{real}. a \leq x \wedge x < b \wedge \text{poly } p \ x = 0\} =$
 $(\text{count-roots-between } p \ a \ b +$
 $(\text{if } p \neq 0 \wedge a < b \wedge \text{poly } p \ a = 0 \text{ then } 1 \text{ else } 0) -$
 $(\text{if } p \neq 0 \wedge a < b \wedge \text{poly } p \ b = 0 \text{ then } 1 \text{ else } 0))$

proof (*cases* $p = 0 \vee a \geq b$)

case *True*

note *True' = this*

show *?thesis*

proof (*cases* $a \geq b$)

case *False*

hence $\{x. a < x \wedge x \leq b\} = \{a <..b\}$

$\{x. a \leq x \wedge x < b\} = \{a..<b\}$ **by** *auto*

with *True False* **have** $\text{card } \{x. a < x \wedge x \leq b\} = 0$ $\text{card } \{x. a \leq x \wedge x < b\} = 0$
by (*auto simp add: card-eq-0-iff infinite-Ico infinite-Ioc*)

with *False True'* **show** *?thesis*

by (*simp add: count-roots-between-correct*)

next

case *True*

with *True'* **have** $\{x. a \leq x \wedge x < b \wedge \text{poly } p \ x = 0\} =$
 $\{x. a < x \wedge x \leq b \wedge \text{poly } p \ x = 0\}$

by (*auto simp: less-eq-real-def*)

thus *?thesis* **using** *poly-card-roots-less-leq True* **by** *simp*

qed
next
case *False*
let $?A = \{x. a \leq x \wedge x < b \wedge \text{poly } p \ x = 0\}$
let $?B = \{x. a < x \wedge x \leq b \wedge \text{poly } p \ x = 0\}$
let $?C = \{x. x = b \wedge \text{poly } p \ x = 0\}$
let $?D = \{x. x = a \wedge \text{poly } p \ a = 0\}$
have *CD-if*: $?C = (\text{if } \text{poly } p \ b = 0 \text{ then } \{b\} \text{ else } \{\})$
 $?D = (\text{if } \text{poly } p \ a = 0 \text{ then } \{a\} \text{ else } \{\})$ **by** *auto*
from *False poly-roots-finite*
have [*simp*]: *finite ?A finite ?B finite ?C finite ?D*
by (*fast, fast, simp-all*)

from *False* **have** $?A = (?B \cup ?D) - ?C$ **by** (*auto simp: less-eq-real-def*)
with *False* **have** $\text{card } ?A = \text{card } ?B + (\text{if } \text{poly } p \ a = 0 \text{ then } 1 \text{ else } 0) -$
 $(\text{if } \text{poly } p \ b = 0 \text{ then } 1 \text{ else } 0)$ **by** (*auto simp: CD-if*)
also note *count-roots-between-correct[symmetric]*
finally show *?thesis* **using** *False* **by** *simp*
qed

lemma *poly-card-roots*:
 $\text{card } \{x::\text{real}. \text{poly } p \ x = 0\} = \text{count-roots } p$
using *count-roots-correct* **by** *simp*

lemma *poly-no-roots*:
 $(\forall x. \text{poly } p \ x \neq 0) \longleftrightarrow (p \neq 0 \wedge \text{count-roots } p = 0)$
by (*auto simp: count-roots-correct dest: poly-roots-finite*)

lemma *poly-pos*:
 $(\forall x. \text{poly } p \ x > 0) \longleftrightarrow ($
 $p \neq 0 \wedge \text{poly-inf } p = 1 \wedge \text{count-roots } p = 0)$
by (*simp only: Let-def poly-pos poly-no-roots, blast*)

lemma *poly-card-roots-greater*:
 $\text{card } \{x::\text{real}. x > a \wedge \text{poly } p \ x = 0\} = \text{count-roots-above } p \ a$
using *count-roots-above-correct* **by** *simp*

lemma *poly-card-roots-leq*:
 $\text{card } \{x::\text{real}. x \leq a \wedge \text{poly } p \ x = 0\} = \text{count-roots-below } p \ a$
using *count-roots-below-correct* **by** *simp*

lemma *poly-card-roots-geq*:
 $\text{card } \{x::\text{real}. x \geq a \wedge \text{poly } p \ x = 0\} = ($
 $\text{count-roots-above } p \ a + (\text{if } \text{poly } p \ a = 0 \wedge p \neq 0 \text{ then } 1 \text{ else } 0))$

proof (*cases poly p a = 0 ∧ p ≠ 0*)

case *False*

hence $\text{card } \{x. x \geq a \wedge \text{poly } p \ x = 0\} = \text{card } \{x. x > a \wedge \text{poly } p \ x = 0\}$

proof (*cases rule: disjE*)

```

assume  $p = 0$ 
have  $\neg \text{finite } \{a <..<a+1\}$ 
  by (metis infinite-Ioo less-add-one)
moreover have  $\{a <..<a+1\} \subseteq \{x. x \geq a \wedge \text{poly } p x = 0\}$ 
   $\{a <..<a+1\} \subseteq \{x. x > a \wedge \text{poly } p x = 0\}$ 
  using  $\langle p = 0 \rangle$  by auto
ultimately have  $\neg \text{finite } \{x. x \geq a \wedge \text{poly } p x = 0\}$ 
   $\neg \text{finite } \{x. x > a \wedge \text{poly } p x = 0\}$ 
  by (auto dest!: finite-subset[of {a <..<a+1}] simp: infinite-Ioo)
thus ?thesis by simp
next
assume  $\text{poly } p a \neq 0$ 
hence  $\{x. x \geq a \wedge \text{poly } p x = 0\} = \{x. x > a \wedge \text{poly } p x = 0\}$ 
  by (auto simp: less-eq-real-def)
thus ?thesis by simp
qed auto
thus ?thesis using False
  by (auto intro: poly-card-roots-greater)
next
case True
hence finite  $\{x. x > a \wedge \text{poly } p x = 0\}$  using poly-roots-finite by force
moreover have  $\{x. x \geq a \wedge \text{poly } p x = 0\} =$ 
   $\text{insert } a \{x. x > a \wedge \text{poly } p x = 0\}$  using True by auto
ultimately have  $\text{card } \{x. x \geq a \wedge \text{poly } p x = 0\} =$ 
   $\text{Suc } (\text{card } \{x. x > a \wedge \text{poly } p x = 0\})$ 
  using card-insert-disjoint by auto
thus ?thesis using True by (auto intro: poly-card-roots-greater)
qed

lemma poly-card-roots-less:
 $\text{card } \{x::\text{real}. x < a \wedge \text{poly } p x = 0\} =$ 
  (count-roots-below p a - (if poly p a = 0 \wedge p \neq 0 then 1 else 0))
proof (cases poly p a = 0 \wedge p \neq 0)
case False
hence  $\text{card } \{x. x < a \wedge \text{poly } p x = 0\} = \text{card } \{x. x \leq a \wedge \text{poly } p x = 0\}$ 
proof (cases rule: disjE)
  assume  $p = 0$ 
  have  $\neg \text{finite } \{a - 1 <..<a\}$ 
    by (metis infinite-Ioo diff-add-cancel less-add-one)
  moreover have  $\{a - 1 <..<a\} \subseteq \{x. x \leq a \wedge \text{poly } p x = 0\}$ 
     $\{a - 1 <..<a\} \subseteq \{x. x < a \wedge \text{poly } p x = 0\}$ 
    using  $\langle p = 0 \rangle$  by auto
  ultimately have  $\neg \text{finite } \{x. x \leq a \wedge \text{poly } p x = 0\}$ 
     $\neg \text{finite } \{x. x < a \wedge \text{poly } p x = 0\}$ 
    by (auto dest: finite-subset[of {a - 1 <..<a}] simp: infinite-Ioo)
  thus ?thesis by simp
next
assume  $\text{poly } p a \neq 0$ 
hence  $\{x. x < a \wedge \text{poly } p x = 0\} = \{x. x \leq a \wedge \text{poly } p x = 0\}$ 

```

by (auto simp: less-eq-real-def)
 thus ?thesis by simp
 qed auto
 thus ?thesis using False
 by (auto intro: poly-card-roots-leq)
 next
 case True
 hence finite $\{x. x < a \wedge \text{poly } p \ x = 0\}$ using poly-roots-finite by force
 moreover have $\{x. x \leq a \wedge \text{poly } p \ x = 0\} =$
 $\text{insert } a \ \{x. x < a \wedge \text{poly } p \ x = 0\}$ using True by auto
 ultimately have Suc (card $\{x. x < a \wedge \text{poly } p \ x = 0\}$) =
 (card $\{x. x \leq a \wedge \text{poly } p \ x = 0\}$)
 using card-insert-disjoint by auto
 also note count-roots-below-correct[symmetric]
 finally show ?thesis using True by simp
 qed

lemma poly-no-roots-less-leq:
 $(\forall x. a < x \wedge x \leq b \longrightarrow \text{poly } p \ x \neq 0) \longleftrightarrow$
 $((a \geq b \vee (p \neq 0 \wedge \text{count-roots-between } p \ a \ b = 0)))$
 by (auto simp: count-roots-between-correct card-eq-0-iff not-le
 dest: poly-roots-finite)

lemma poly-pos-between-less-leq:
 $(\forall x. a < x \wedge x \leq b \longrightarrow \text{poly } p \ x > 0) \longleftrightarrow$
 $((a \geq b \vee (p \neq 0 \wedge \text{poly } p \ b > 0 \wedge \text{count-roots-between } p \ a \ b = 0)))$
 by (simp only: poly-pos-between-less-leq Let-def
 poly-no-roots-less-leq, blast)

lemma poly-no-roots-leq-leq:
 $(\forall x. a \leq x \wedge x \leq b \longrightarrow \text{poly } p \ x \neq 0) \longleftrightarrow$
 $((a > b \vee (p \neq 0 \wedge \text{poly } p \ a \neq 0 \wedge \text{count-roots-between } p \ a \ b = 0)))$
 apply (intro iffI)
 apply (force simp add: count-roots-between-correct card-eq-0-iff)
 apply (elim conjE disjE, simp, intro allI)
 apply (rename-tac x, case-tac x = a)
 apply (auto simp add: count-roots-between-correct card-eq-0-iff
 dest: poly-roots-finite)

done

lemma poly-pos-between-leq-leq:
 $(\forall x. a \leq x \wedge x \leq b \longrightarrow \text{poly } p \ x > 0) \longleftrightarrow$
 $((a > b \vee (p \neq 0 \wedge \text{poly } p \ a > 0 \wedge$
 $\text{count-roots-between } p \ a \ b = 0)))$
 by (simp only: poly-pos-between-leq-leq Let-def poly-no-roots-leq-leq, force)

lemma *poly-no-roots-less-less*:
 $(\forall x. a < x \wedge x < b \longrightarrow \text{poly } p \ x \neq 0) \longleftrightarrow$
 $((a \geq b \vee p \neq 0 \wedge \text{count-roots-between } p \ a \ b =$
 $(\text{if } \text{poly } p \ b = 0 \text{ then } 1 \text{ else } 0)))$

proof (*standard, goal-cases*)
case *A*: 1
show *?case*
proof (*cases* $a \geq b$)
case *True*
with *A* **show** *?thesis* **by** *simp*
next
case *False*
with *A* **have** [*simp*]: $p \neq 0$ **using** *dense*[*of a b*] **by** *auto*
have *B*: $\{x. a < x \wedge x \leq b \wedge \text{poly } p \ x = 0\} =$
 $\{x. a < x \wedge x < b \wedge \text{poly } p \ x = 0\} \cup$
 $(\text{if } \text{poly } p \ b = 0 \text{ then } \{b\} \text{ else } \{\})$ **using** *A False* **by** *auto*
have *count-roots-between* $p \ a \ b =$
 $\text{card } \{x. a < x \wedge x < b \wedge \text{poly } p \ x = 0\} +$
 $(\text{if } \text{poly } p \ b = 0 \text{ then } 1 \text{ else } 0)$
by (*subst count-roots-between-correct, subst B, subst card-Un-disjoint,*
rule finite-subset[OF - poly-roots-finite], blast, simp-all)
also from *A* **have** $\{x. a < x \wedge x < b \wedge \text{poly } p \ x = 0\} = \{\}$ **by** *simp*
finally show *?thesis* **by** *auto*
qed
next
case *prems*: 2
hence $\text{card } \{x. a < x \wedge x < b \wedge \text{poly } p \ x = 0\} = 0$
by (*subst poly-card-roots-less-less, auto simp: count-roots-between-def*)
thus *?case* **using** *prems*
by (*cases* $p = 0$, *simp, subst (asm) card-eq-0-iff,*
auto dest: poly-roots-finite)
qed

lemma *poly-pos-between-less-less*:
 $(\forall x. a < x \wedge x < b \longrightarrow \text{poly } p \ x > 0) \longleftrightarrow$
 $((a \geq b \vee (p \neq 0 \wedge \text{poly } p \ ((a+b)/2) > 0 \wedge$
 $\text{count-roots-between } p \ a \ b = (\text{if } \text{poly } p \ b = 0 \text{ then } 1 \text{ else } 0))))$

by (*simp only: poly-pos-between-less-less Let-def*
poly-no-roots-less-less, blast)

lemma *poly-no-roots-leq-less*:
 $(\forall x. a \leq x \wedge x < b \longrightarrow \text{poly } p \ x \neq 0) \longleftrightarrow$
 $((a \geq b \vee p \neq 0 \wedge \text{poly } p \ a \neq 0 \wedge \text{count-roots-between } p \ a \ b =$
 $(\text{if } a < b \wedge \text{poly } p \ b = 0 \text{ then } 1 \text{ else } 0)))$

proof (*standard, goal-cases*)
case *prems*: 1
hence $\forall x. a < x \wedge x < b \longrightarrow \text{poly } p \ x \neq 0$ **by** *simp*
thus *?case* **using** *prems* **by** (*subst (asm) poly-no-roots-less-less, auto*)

next

case *prems*: 2

hence $(b \leq a \vee p \neq 0 \wedge \text{count-roots-between } p \ a \ b =$
 $(\text{if } \text{poly } p \ b = 0 \text{ then } 1 \text{ else } 0))$ **by** *auto*

thus ?*case using prems unfolding Let-def*

by (*subst (asm) poly-no-roots-less-less[symmetric, unfolded Let-def]*),
auto split: if-split-asm simp: less-eq-real-def)

qed

lemma *poly-pos-between-leq-less*:

$(\forall x. a \leq x \wedge x < b \longrightarrow \text{poly } p \ x > 0) \longleftrightarrow$
 $((a \geq b \vee (p \neq 0 \wedge \text{poly } p \ a > 0 \wedge \text{count-roots-between } p \ a \ b =$
 $(\text{if } a < b \wedge \text{poly } p \ b = 0 \text{ then } 1 \text{ else } 0))))$

by (*simp only: poly-pos-between-leq-less Let-def*
poly-no-roots-leq-less, force)

lemma *poly-no-roots-greater*:

$(\forall x. x > a \longrightarrow \text{poly } p \ x \neq 0) \longleftrightarrow$
 $((p \neq 0 \wedge \text{count-roots-above } p \ a = 0))$

proof –

have $\forall x. \neg a < x \implies \text{False}$ **by** (*metis gt-ex*)

thus ?*thesis by (auto simp: count-roots-above-correct card-eq-0-iff*
intro: poly-roots-finite)

qed

lemma *poly-pos-greater*:

$(\forall x. x > a \longrightarrow \text{poly } p \ x > 0) \longleftrightarrow ($
 $p \neq 0 \wedge \text{poly-inf } p = 1 \wedge \text{count-roots-above } p \ a = 0)$

unfolding *Let-def*

by (*subst poly-pos-greater, subst poly-no-roots-greater, force*)

lemma *poly-no-roots-leq*:

$(\forall x. x \leq a \longrightarrow \text{poly } p \ x \neq 0) \longleftrightarrow$
 $(p \neq 0 \wedge \text{count-roots-below } p \ a = 0)$

by (*auto simp: Let-def count-roots-below-correct card-eq-0-iff*
intro: poly-roots-finite)

lemma *poly-pos-leq*:

$(\forall x. x \leq a \longrightarrow \text{poly } p \ x > 0) \longleftrightarrow$
 $(p \neq 0 \wedge \text{poly-neg-inf } p = 1 \wedge \text{count-roots-below } p \ a = 0)$

by (*simp only: poly-pos-leq Let-def poly-no-roots-leq, blast*)

lemma *poly-no-roots-geq*:

$(\forall x. x \geq a \longrightarrow \text{poly } p \ x \neq 0) \longleftrightarrow$
 $(p \neq 0 \wedge \text{poly } p \ a \neq 0 \wedge \text{count-roots-above } p \ a = 0)$

proof (*standard, goal-cases*)

case *prems: 1*
hence $\forall x > a. \text{poly } p \ x \neq 0$ **by** *simp*
thus *?case using prems by (subst (asm) poly-no-roots-greater, auto)*
next
case *prems: 2*
hence $(p \neq 0 \wedge \text{count-roots-above } p \ a = 0)$ **by** *simp*
thus *?case using prems*
by *(subst (asm) poly-no-roots-greater[symmetric, unfolded Let-def], auto simp: less-eq-real-def)*

qed

lemma *poly-pos-geq:*

$(\forall x. x \geq a \longrightarrow \text{poly } p \ x > 0) \longleftrightarrow$
 $(p \neq 0 \wedge \text{poly-inf } p = 1 \wedge \text{poly } p \ a \neq 0 \wedge \text{count-roots-above } p \ a = 0)$
by *(simp only: poly-pos-geq Let-def poly-no-roots-geq, blast)*

lemma *poly-no-roots-less:*

$(\forall x. x < a \longrightarrow \text{poly } p \ x \neq 0) \longleftrightarrow$
 $((p \neq 0 \wedge \text{count-roots-below } p \ a = (\text{if } \text{poly } p \ a = 0 \text{ then } 1 \text{ else } 0)))$

proof *(standard, goal-cases)*

case *prems: 1*

hence $\{x. x \leq a \wedge \text{poly } p \ x = 0\} = (\text{if } \text{poly } p \ a = 0 \text{ then } \{a\} \text{ else } \{\})$
by *(auto simp: less-eq-real-def)*

moreover **have** $\forall x. \neg x < a \implies \text{False}$ **by** *(metis lt-ex)*

ultimately show *?case using prems by (auto simp: count-roots-below-correct)*

next

case *prems: 2*

have $A: \{x. x \leq a \wedge \text{poly } p \ x = 0\} = \{x. x < a \wedge \text{poly } p \ x = 0\} \cup$
 $(\text{if } \text{poly } p \ a = 0 \text{ then } \{a\} \text{ else } \{\})$ **by** *(auto simp: less-eq-real-def)*

have $\text{count-roots-below } p \ a = \text{card } \{x. x < a \wedge \text{poly } p \ x = 0\} +$
 $(\text{if } \text{poly } p \ a = 0 \text{ then } 1 \text{ else } 0)$ **using** *prems*

by *(subst count-roots-below-correct, subst A, subst card-Un-disjoint, auto intro: poly-roots-finite)*

with *prems* **have** $\text{card } \{x. x < a \wedge \text{poly } p \ x = 0\} = 0$ **by** *simp*

thus *?case using prems*

by *(subst (asm) card-eq-0-iff, auto intro: poly-roots-finite)*

qed

lemma *poly-pos-less:*

$(\forall x. x < a \longrightarrow \text{poly } p \ x > 0) \longleftrightarrow$
 $(p \neq 0 \wedge \text{poly-neg-inf } p = 1 \wedge \text{count-roots-below } p \ a =$
 $(\text{if } \text{poly } p \ a = 0 \text{ then } 1 \text{ else } 0))$
by *(simp only: poly-pos-less Let-def poly-no-roots-less, blast)*

lemmas *sturm-card-substs = poly-card-roots poly-card-roots-less-leq*
poly-card-roots-leq-less poly-card-roots-less-less poly-card-roots-leq-leq
poly-card-roots-less poly-card-roots-leq poly-card-roots-greater
poly-card-roots-geq

lemmas *sturm-prop-substs* = *poly-no-roots poly-no-roots-less-leq poly-no-roots-leq-leq poly-no-roots-less-less poly-no-roots-leq-less poly-no-roots-leq poly-no-roots-less poly-no-roots-geq poly-no-roots-greater poly-pos poly-pos-greater poly-pos-geq poly-pos-less poly-pos-leq poly-pos-between-leq-less poly-pos-between-less-leq poly-pos-between-leq-leq poly-pos-between-less-less*

3.2 Reification

This subsection defines a number of equations to automatically convert statements about roots of polynomials into a canonical form so that they can be proven using the above substitutions.

definition *PR-TAG* $x \equiv x$

lemma *sturm-id-PR-prio0*:

$$\begin{aligned} \{x::real. P x\} &= \{x::real. (PR-TAG P) x\} \\ (\forall x::real. f x < g x) &= (\forall x::real. PR-TAG (\lambda x. f x < g x) x) \\ (\forall x::real. P x) &= (\forall x::real. \neg(PR-TAG (\lambda x. \neg P x)) x) \\ \text{by } &(\text{simp-all add: } PR-TAG\text{-def}) \end{aligned}$$

lemma *sturm-id-PR-prio1*:

$$\begin{aligned} \{x::real. x < a \wedge P x\} &= \{x::real. x < a \wedge (PR-TAG P) x\} \\ \{x::real. x \leq a \wedge P x\} &= \{x::real. x \leq a \wedge (PR-TAG P) x\} \\ \{x::real. x \geq b \wedge P x\} &= \{x::real. x \geq b \wedge (PR-TAG P) x\} \\ \{x::real. x > b \wedge P x\} &= \{x::real. x > b \wedge (PR-TAG P) x\} \\ (\forall x::real < a. f x < g x) &= (\forall x::real < a. PR-TAG (\lambda x. f x < g x) x) \\ (\forall x::real \leq a. f x < g x) &= (\forall x::real \leq a. PR-TAG (\lambda x. f x < g x) x) \\ (\forall x::real > a. f x < g x) &= (\forall x::real > a. PR-TAG (\lambda x. f x < g x) x) \\ (\forall x::real \geq a. f x < g x) &= (\forall x::real \geq a. PR-TAG (\lambda x. f x < g x) x) \\ (\forall x::real < a. P x) &= (\forall x::real < a. \neg(PR-TAG (\lambda x. \neg P x)) x) \\ (\forall x::real > a. P x) &= (\forall x::real > a. \neg(PR-TAG (\lambda x. \neg P x)) x) \\ (\forall x::real \leq a. P x) &= (\forall x::real \leq a. \neg(PR-TAG (\lambda x. \neg P x)) x) \\ (\forall x::real \geq a. P x) &= (\forall x::real \geq a. \neg(PR-TAG (\lambda x. \neg P x)) x) \\ \text{by } &(\text{simp-all add: } PR-TAG\text{-def}) \end{aligned}$$

lemma *sturm-id-PR-prio2*:

$$\begin{aligned} \{x::real. x > a \wedge x \leq b \wedge P x\} &= \\ \{x::real. x > a \wedge x \leq b \wedge PR-TAG P x\} & \\ \{x::real. x \geq a \wedge x \leq b \wedge P x\} &= \\ \{x::real. x \geq a \wedge x \leq b \wedge PR-TAG P x\} & \\ \{x::real. x \geq a \wedge x < b \wedge P x\} &= \\ \{x::real. x \geq a \wedge x < b \wedge PR-TAG P x\} & \\ \{x::real. x > a \wedge x < b \wedge P x\} &= \\ \{x::real. x > a \wedge x < b \wedge PR-TAG P x\} & \\ (\forall x::real. a < x \wedge x \leq b \longrightarrow f x < g x) &= \\ (\forall x::real. a < x \wedge x \leq b \longrightarrow PR-TAG (\lambda x. f x < g x) x) & \\ (\forall x::real. a \leq x \wedge x \leq b \longrightarrow f x < g x) &= \end{aligned}$$

$(\forall x::real. a \leq x \wedge x \leq b \longrightarrow PR\text{-TAG } (\lambda x. f x < g x) x)$
 $(\forall x::real. a < x \wedge x < b \longrightarrow f x < g x) =$
 $(\forall x::real. a < x \wedge x < b \longrightarrow PR\text{-TAG } (\lambda x. f x < g x) x)$
 $(\forall x::real. a \leq x \wedge x < b \longrightarrow f x < g x) =$
 $(\forall x::real. a \leq x \wedge x < b \longrightarrow PR\text{-TAG } (\lambda x. f x < g x) x)$
 $(\forall x::real. a < x \wedge x \leq b \longrightarrow P x) =$
 $(\forall x::real. a < x \wedge x \leq b \longrightarrow \neg(PR\text{-TAG } (\lambda x. \neg P x) x))$
 $(\forall x::real. a \leq x \wedge x \leq b \longrightarrow P x) =$
 $(\forall x::real. a \leq x \wedge x \leq b \longrightarrow \neg(PR\text{-TAG } (\lambda x. \neg P x) x))$
 $(\forall x::real. a \leq x \wedge x < b \longrightarrow P x) =$
 $(\forall x::real. a \leq x \wedge x < b \longrightarrow \neg(PR\text{-TAG } (\lambda x. \neg P x) x))$
 $(\forall x::real. a < x \wedge x < b \longrightarrow P x) =$
 $(\forall x::real. a < x \wedge x < b \longrightarrow \neg(PR\text{-TAG } (\lambda x. \neg P x) x))$
by (*simp-all add: PR-TAG-def*)

lemma *PR-TAG-intro-prio0*:

fixes $P :: real \Rightarrow bool$ **and** $f :: real \Rightarrow real$

shows

$PR\text{-TAG } P = P' \Longrightarrow PR\text{-TAG } (\lambda x. \neg(\neg P x)) = P'$
 $\llbracket PR\text{-TAG } P = (\lambda x. poly p x = 0); PR\text{-TAG } Q = (\lambda x. poly q x = 0) \rrbracket$
 $\Longrightarrow PR\text{-TAG } (\lambda x. P x \wedge Q x) = (\lambda x. poly (gcd p q) x = 0)$ **and**
 $\llbracket PR\text{-TAG } P = (\lambda x. poly p x = 0); PR\text{-TAG } Q = (\lambda x. poly q x = 0) \rrbracket$
 $\Longrightarrow PR\text{-TAG } (\lambda x. P x \vee Q x) = (\lambda x. poly (p*q) x = 0)$ **and**

$\llbracket PR\text{-TAG } f = (\lambda x. poly p x); PR\text{-TAG } g = (\lambda x. poly q x) \rrbracket$
 $\Longrightarrow PR\text{-TAG } (\lambda x. f x = g x) = (\lambda x. poly (p-q) x = 0)$
 $\llbracket PR\text{-TAG } f = (\lambda x. poly p x); PR\text{-TAG } g = (\lambda x. poly q x) \rrbracket$
 $\Longrightarrow PR\text{-TAG } (\lambda x. f x \neq g x) = (\lambda x. poly (p-q) x \neq 0)$
 $\llbracket PR\text{-TAG } f = (\lambda x. poly p x); PR\text{-TAG } g = (\lambda x. poly q x) \rrbracket$
 $\Longrightarrow PR\text{-TAG } (\lambda x. f x < g x) = (\lambda x. poly (q-p) x > 0)$
 $\llbracket PR\text{-TAG } f = (\lambda x. poly p x); PR\text{-TAG } g = (\lambda x. poly q x) \rrbracket$
 $\Longrightarrow PR\text{-TAG } (\lambda x. f x \leq g x) = (\lambda x. poly (q-p) x \geq 0)$

$PR\text{-TAG } f = (\lambda x. poly p x) \Longrightarrow PR\text{-TAG } (\lambda x. -f x) = (\lambda x. poly (-p) x)$
 $\llbracket PR\text{-TAG } f = (\lambda x. poly p x); PR\text{-TAG } g = (\lambda x. poly q x) \rrbracket$
 $\Longrightarrow PR\text{-TAG } (\lambda x. f x + g x) = (\lambda x. poly (p+q) x)$
 $\llbracket PR\text{-TAG } f = (\lambda x. poly p x); PR\text{-TAG } g = (\lambda x. poly q x) \rrbracket$
 $\Longrightarrow PR\text{-TAG } (\lambda x. f x - g x) = (\lambda x. poly (p-q) x)$
 $\llbracket PR\text{-TAG } f = (\lambda x. poly p x); PR\text{-TAG } g = (\lambda x. poly q x) \rrbracket$
 $\Longrightarrow PR\text{-TAG } (\lambda x. f x * g x) = (\lambda x. poly (p*q) x)$
 $PR\text{-TAG } f = (\lambda x. poly p x) \Longrightarrow PR\text{-TAG } (\lambda x. (f x) \widehat{n}) = (\lambda x. poly (p \widehat{n}) x)$
 $PR\text{-TAG } (\lambda x. poly p x :: real) = (\lambda x. poly [0,1:] x)$
 $PR\text{-TAG } (\lambda x. a::real) = (\lambda x. poly [:a:] x)$
by (*simp-all add: PR-TAG-def poly-eq-0-iff-dvd field-simps*)

lemma *PR-TAG-intro-prio1*:

fixes $f :: \text{real} \Rightarrow \text{real}$

shows

$PR\text{-TAG } f = (\lambda x. \text{poly } p \ x) \Longrightarrow PR\text{-TAG } (\lambda x. f \ x = 0) = (\lambda x. \text{poly } p \ x = 0)$
 $PR\text{-TAG } f = (\lambda x. \text{poly } p \ x) \Longrightarrow PR\text{-TAG } (\lambda x. f \ x \neq 0) = (\lambda x. \text{poly } p \ x \neq 0)$
 $PR\text{-TAG } f = (\lambda x. \text{poly } p \ x) \Longrightarrow PR\text{-TAG } (\lambda x. 0 = f \ x) = (\lambda x. \text{poly } p \ x = 0)$
 $PR\text{-TAG } f = (\lambda x. \text{poly } p \ x) \Longrightarrow PR\text{-TAG } (\lambda x. 0 \neq f \ x) = (\lambda x. \text{poly } p \ x \neq 0)$
 $PR\text{-TAG } f = (\lambda x. \text{poly } p \ x) \Longrightarrow PR\text{-TAG } (\lambda x. f \ x \geq 0) = (\lambda x. \text{poly } p \ x \geq 0)$
 $PR\text{-TAG } f = (\lambda x. \text{poly } p \ x) \Longrightarrow PR\text{-TAG } (\lambda x. f \ x > 0) = (\lambda x. \text{poly } p \ x > 0)$
 $PR\text{-TAG } f = (\lambda x. \text{poly } p \ x) \Longrightarrow PR\text{-TAG } (\lambda x. f \ x \leq 0) = (\lambda x. \text{poly } (-p) \ x \geq 0)$
 $PR\text{-TAG } f = (\lambda x. \text{poly } p \ x) \Longrightarrow PR\text{-TAG } (\lambda x. f \ x < 0) = (\lambda x. \text{poly } (-p) \ x > 0)$
 $PR\text{-TAG } f = (\lambda x. \text{poly } p \ x) \Longrightarrow$
 $PR\text{-TAG } (\lambda x. 0 \leq f \ x) = (\lambda x. \text{poly } (-p) \ x \leq 0)$
 $PR\text{-TAG } f = (\lambda x. \text{poly } p \ x) \Longrightarrow$
 $PR\text{-TAG } (\lambda x. 0 < f \ x) = (\lambda x. \text{poly } (-p) \ x < 0)$
 $PR\text{-TAG } f = (\lambda x. \text{poly } p \ x)$
 $\Longrightarrow PR\text{-TAG } (\lambda x. a * f \ x) = (\lambda x. \text{poly } (\text{smult } a \ p) \ x)$
 $PR\text{-TAG } f = (\lambda x. \text{poly } p \ x)$
 $\Longrightarrow PR\text{-TAG } (\lambda x. f \ x * a) = (\lambda x. \text{poly } (\text{smult } a \ p) \ x)$
 $PR\text{-TAG } f = (\lambda x. \text{poly } p \ x)$
 $\Longrightarrow PR\text{-TAG } (\lambda x. f \ x / a) = (\lambda x. \text{poly } (\text{smult } (\text{inverse } a) \ p) \ x)$
 $PR\text{-TAG } (\lambda x. x \hat{\ } n :: \text{real}) = (\lambda x. \text{poly } (\text{monom } 1 \ n) \ x)$

by (*simp-all add: PR-TAG-def field-simps poly-monom*)

lemma *PR-TAG-intro-prio2*:

$PR\text{-TAG } (\lambda x. 1 / b) = (\lambda x. \text{inverse } b)$
 $PR\text{-TAG } (\lambda x. a / b) = (\lambda x. a / b)$
 $PR\text{-TAG } (\lambda x. a / b * x \hat{\ } n :: \text{real}) = (\lambda x. \text{poly } (\text{monom } (a/b) \ n) \ x)$
 $PR\text{-TAG } (\lambda x. x \hat{\ } n * a / b :: \text{real}) = (\lambda x. \text{poly } (\text{monom } (a/b) \ n) \ x)$
 $PR\text{-TAG } (\lambda x. a * x \hat{\ } n :: \text{real}) = (\lambda x. \text{poly } (\text{monom } a \ n) \ x)$
 $PR\text{-TAG } (\lambda x. x \hat{\ } n * a :: \text{real}) = (\lambda x. \text{poly } (\text{monom } a \ n) \ x)$
 $PR\text{-TAG } (\lambda x. x \hat{\ } n / a :: \text{real}) = (\lambda x. \text{poly } (\text{monom } (\text{inverse } a) \ n) \ x)$

$PR\text{-TAG } (\lambda x. f \ x \hat{\ } (\text{Suc } (\text{Suc } 0)) :: \text{real}) = (\lambda x. \text{poly } p \ x)$
 $\Longrightarrow PR\text{-TAG } (\lambda x. f \ x * f \ x :: \text{real}) = (\lambda x. \text{poly } p \ x)$
 $PR\text{-TAG } (\lambda x. (f \ x) \hat{\ } \text{Suc } n :: \text{real}) = (\lambda x. \text{poly } p \ x)$
 $\Longrightarrow PR\text{-TAG } (\lambda x. (f \ x) \hat{\ } n * f \ x :: \text{real}) = (\lambda x. \text{poly } p \ x)$
 $PR\text{-TAG } (\lambda x. (f \ x) \hat{\ } \text{Suc } n :: \text{real}) = (\lambda x. \text{poly } p \ x)$
 $\Longrightarrow PR\text{-TAG } (\lambda x. f \ x * (f \ x) \hat{\ } n :: \text{real}) = (\lambda x. \text{poly } p \ x)$
 $PR\text{-TAG } (\lambda x. (f \ x) \hat{\ } (m+n) :: \text{real}) = (\lambda x. \text{poly } p \ x)$
 $\Longrightarrow PR\text{-TAG } (\lambda x. (f \ x) \hat{\ } m * (f \ x) \hat{\ } n :: \text{real}) = (\lambda x. \text{poly } p \ x)$

by (*simp-all add: PR-TAG-def field-simps poly-monom power-add*)

lemma *sturm-meta-spec*: $(\bigwedge x :: \text{real}. P \ x) \Longrightarrow P \ x$ **by** *simp*

lemma *sturm-imp-conv*:

$(a < x \longrightarrow x < b \longrightarrow c) \longleftrightarrow (a < x \wedge x < b \longrightarrow c)$
 $(a \leq x \longrightarrow x < b \longrightarrow c) \longleftrightarrow (a \leq x \wedge x < b \longrightarrow c)$
 $(a < x \longrightarrow x \leq b \longrightarrow c) \longleftrightarrow (a < x \wedge x \leq b \longrightarrow c)$
 $(a \leq x \longrightarrow x \leq b \longrightarrow c) \longleftrightarrow (a \leq x \wedge x \leq b \longrightarrow c)$

```

(x < b → a < x → c) ↔ (a < x ∧ x < b → c)
(x < b → a ≤ x → c) ↔ (a ≤ x ∧ x < b → c)
(x ≤ b → a < x → c) ↔ (a < x ∧ x ≤ b → c)
(x ≤ b → a ≤ x → c) ↔ (a ≤ x ∧ x ≤ b → c)
by auto

```

3.3 Setup for the “sturm” method

ML-file \langle *sturm.ML* \rangle

```

method-setup sturm =  $\langle$ 
  Scan.succeed (fn ctxt => SIMPLE-METHOD' (Sturm.sturm-tac ctxt true))
 $\rangle$ 

```

end

```

theory Sturm
imports Sturm-Method
begin

```

end

4 Example usage of the “sturm” method

```

theory Sturm-Ex
imports ../Sturm
begin

```

In this section, we give a variety of statements about real polynomials that can be proven by the *sturm* method.

```

lemma
   $\forall x::real. x^2 + 1 \neq 0$ 
by sturm

```

```

lemma
  fixes  $x :: real$ 
  shows  $x^2 + 1 \neq 0$  by sturm

```

```

lemma  $(x::real) > 1 \implies x^3 > 1$  by sturm

```

```

lemma  $\forall x::real. x*x \neq -1$  by sturm

```

schematic-goal *A*:

```

card { $x::real. -0.010831 < x \wedge x < 0.010831 \wedge$ 
   $1/120*x^5 + 1/24*x^4 + 1/6*x^3 - 49/16777216*x^2 - 17/2097152*x =$ 
  0}
  = ?n
by sturm

```

lemma *card* { $x::real. x^3 + x = 2*x^2 \wedge x^3 - 6*x^2 + 11*x = 6$ } = 1
by *sturm*

schematic-goal *card* { $x::real. x^3 + x = 2*x^2 \vee x^3 - 6*x^2 + 11*x = 6$ }
= ?*n* **by** *sturm*

lemma
card { $x::real. -0.010831 < x \wedge x < 0.010831 \wedge$
poly [$0, -17/2097152, -49/16777216, 1/6, 1/24, 1/120$]:] $x = 0$ } = 3
by *sturm*

lemma $\forall x::real. x*x \neq 0 \vee x*x - 1 \neq 2*x$ **by** *sturm*

lemma $(x::real)*x+1 \neq 0 \wedge (x^2+1)*(x^2+2) \neq 0$ **by** *sturm*

3 examples related to continued fraction approximants to exp: LCP

lemma fixes $x::real$
shows $-7.29347719 \leq x \implies 0 < x^5 + 30*x^4 + 420*x^3 + 3360*x^2 +$
 $15120*x + 30240$
by *sturm*

lemma fixes $x::real$
shows $0 < x^6 + 42*x^5 + 840*x^4 + 10080*x^3 + 75600*x^2 + 332640*x$
 $+ 665280$
by *sturm*

schematic-goal *card* { $x::real. x^7 + 56*x^6 + 1512*x^5 + 25200*x^4 + 277200*x^3$
 $+ 1995840*x^2 + 8648640*x = -17297280$ } = ?*n*
by *sturm*

end