Stirling's formula

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Abstract

This work contains a proof of Stirling's formula both for the factorial $n! \sim \sqrt{2\pi n}(n/e)^n$ on natural numbers and the real Gamma function $\Gamma(x) \sim \sqrt{2\pi/x}(x/e)^x$. The proof is based on work by Graham Jameson [3].

This is then extended to the full asymptotic expansion

$$\log\Gamma(z) = \left(z - \tfrac{1}{2}\right)\log z - z + \tfrac{1}{2}\log(2\pi) + \sum_{k=1}^{n-1} \frac{B_{k+1}}{k(k+1)} z^{-k} - \frac{1}{n}\int_0^\infty B_n([t])(t+z)^{-n}\,\mathrm{d}t$$

uniformly for all complex $z \neq 0$ in the cone $\arg(z) \leq \alpha$ for any $\alpha \in (0, \pi)$, with which the above asymptotic relation for Γ is also extended to complex arguments.

Contents

1 Stirling's Formula

theory *Stirling-Formula*
imports
HOL−Analysis.Analysis
Landau-Symbols.Landau-More
HOL−Real-Asymp.Real-Asymp
begin

context
begin

First, we define the S_n^* from Jameson's article:

qualified definition S' :: *nat* ⇒ *real* ⇒ *real* **where**
$S'\ n\ x = 1/(2*x) + (\sum r=1..<n.\ 1/(of\text{-}nat\ r+x)) + 1\ /(2*(n + x))$

Next, the trapezium (also called T in Jameson's article):

qualified definition T :: *real* ⇒ *real* **where**
$T\ x = 1/(2*x) + 1/(2*(x+1))$

Now we define The difference $\Delta(x)$:

qualified definition D :: *real* ⇒ *real* **where**
$D\ x = T\ x - ln\ (x + 1) + ln\ x$

qualified lemma *S′-telescope-trapezium*:
assumes $n > 0$
shows $S'\ n\ x = (\sum r<n.\ T\ (of\text{-}nat\ r + x))$
⟨*proof*⟩ **lemma** *stirling-trapezium*:
assumes x: $(x{::}real) > 0$
shows $D\ x \in \{0\ ..\ 1/(12*x\hat{\ }2) - 1/(12 * (x+1)\hat{\ }2)\}$
⟨*proof*⟩

The following functions correspond to the $p_n(x)$ from the article. The special case $n = 0$ would not, strictly speaking, be necessary, but it allows some theorems to work even without the precondition $n \neq 0$:

qualified definition p :: *nat* ⇒ *real* ⇒ *real* **where**
$p\ n\ x = (if\ n = 0\ then\ 1/x\ else\ (\sum r<n.\ D\ (real\ r + x)))$

We can write the Digamma function in terms of S':

qualified lemma *S′-LIMSEQ-Digamma*:
assumes x: $x \neq 0$
shows $(\lambda n.\ ln\ (real\ n) - S'\ n\ x - 1/(2*x)) \longrightarrow Digamma\ x$
⟨*proof*⟩

Moreover, we can give an expansion of S' with the p as variation terms.

qualified lemma *S′-approx*:
$S'\ n\ x = ln\ (real\ n + x) - ln\ x + p\ n\ x$
⟨*proof*⟩

We define the limit of the p (simply called $p(x)$ in Jameson's article):

qualified definition P :: *real* ⇒ *real* **where**
$P\ x = (\sum n.\ D\ (real\ n + x))$

qualified lemma *D-summable*:
assumes x: $x > 0$

shows *summable* $(\lambda n.\ D\ (real\ n + x))$
⟨*proof*⟩ **lemma** *p-LIMSEQ*:
assumes *x*: $x > 0$
shows $(\lambda n.\ p\ n\ x) \longrightarrow P\ x$
⟨*proof*⟩

This gives us an expansion of the Digamma function:

lemma *Digamma-approx*:
assumes *x*: $(x :: real) > 0$
shows $Digamma\ x = ln\ x - 1\ /\ (2 * x) - P\ x$
⟨*proof*⟩

Next, we derive some bounds on P:

qualified lemma *p-ge-0*: $x > 0 \Longrightarrow p\ n\ x \geq 0$
⟨*proof*⟩ **lemma** *P-ge-0*: $x > 0 \Longrightarrow P\ x \geq 0$
⟨*proof*⟩ **lemma** *p-upper-bound*:
assumes $x > 0\ n > 0$
shows $p\ n\ x \leq 1/(12*x\hat{\ }2)$
⟨*proof*⟩ **lemma** *P-upper-bound*:
assumes $x > 0$
shows $P\ x \leq 1/(12*x\hat{\ }2)$
⟨*proof*⟩

We can now use this approximation of the Digamma function to approximate *ln-Gamma* (since the former is the derivative of the latter). We therefore define the function g from Jameson's article, which measures the difference between *ln-Gamma* and its approximation:

qualified definition $g :: real \Rightarrow real$ **where**
$g\ x = ln\text{-}Gamma\ x - (x - 1/2) * ln\ x + x$

qualified lemma *DERIV-g*: $x > 0 \Longrightarrow (g\ has\text{-}field\text{-}derivative\ {-P\ x})\ (at\ x)$
⟨*proof*⟩ **lemma** *isCont-P*:
assumes $x > 0$
shows *isCont P x*
⟨*proof*⟩ **lemma** *P-continuous-on* [*THEN continuous-on-subset*]: *continuous-on* {*0*<..} *P*
⟨*proof*⟩ **lemma** *P-integrable*:
assumes *a*: $a > 0$
shows *P integrable-on* {*a*..}
⟨*proof*⟩ **definition** $c :: real$ **where** $c = integral\ \{1..\}\ (\lambda x.\ {-P\ x}) + g\ 1$

We can now give bounds on g:

qualified lemma *g-bounds*:
assumes *x*: $x \geq 1$
shows $g\ x \in \{c..c + 1/(12*x)\}$
⟨*proof*⟩

Finally, we have bounds on *ln-Gamma*, *Gamma*, and *fact*.

qualified lemma *ln-Gamma-bounds-aux*:
  $x \geq 1 \Longrightarrow$ *ln-Gamma* $x \geq c + (x - 1/2) * \mathit{ln}\ x - x$
  $x \geq 1 \Longrightarrow$ *ln-Gamma* $x \leq c + (x - 1/2) * \mathit{ln}\ x - x + 1/(12{*}x)$
  ⟨*proof*⟩ **lemma** *Gamma-bounds-aux*:
  assumes x: $x \geq 1$
  shows *Gamma* $x \geq \mathit{exp}\ c * x\ \mathit{powr}\ (x - 1/2)\ /\ \mathit{exp}\ x$
  Gamma $x \leq \mathit{exp}\ c * x\ \mathit{powr}\ (x - 1/2)\ /\ \mathit{exp}\ x * \mathit{exp}\ (1/(12{*}x))$
⟨*proof*⟩ **lemma** *Gamma-asymp-equiv-aux*:
  Gamma $\sim$[*at-top*] $(\lambda x.\ \mathit{exp}\ c * x\ \mathit{powr}\ (x - 1/2)\ /\ \mathit{exp}\ x)$
⟨*proof*⟩
  include *asymp-equiv-syntax*
  ⟨*proof*⟩ **lemma** *exp-1-powr-real* [*simp*]: $\mathit{exp}\ (1{::}\mathit{real})\ \mathit{powr}\ x = \mathit{exp}\ x$
  ⟨*proof*⟩ **lemma** *fact-asymp-equiv-aux*:
  fact $\sim$[*at-top*] $(\lambda x.\ \mathit{exp}\ c * \mathit{sqrt}\ (\mathit{real}\ x) * (\mathit{real}\ x\ /\ \mathit{exp}\ 1)\ \mathit{powr}\ \mathit{real}\ x)$
⟨*proof*⟩
  include *asymp-equiv-syntax*
  ⟨*proof*⟩

We cal also bound *Digamma* above and below.

lemma *Digamma-plus-1-gt-ln*:
  assumes x: $x > (0 :: \mathit{real})$
  shows *Digamma* $(x + 1) > \mathit{ln}\ x$
⟨*proof*⟩

lemma *Digamma-less-ln*:
  assumes x: $x > (0 :: \mathit{real})$
  shows *Digamma* $x < \mathit{ln}\ x$
⟨*proof*⟩

We still need to determine the constant term c, which we do using Wallis' product formula:

$$\prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1} = \frac{\pi}{2}$$

qualified lemma *powr-mult-2*: $(x{::}\mathit{real}) > 0 \Longrightarrow x\ \mathit{powr}\ (y * 2) = (x\hat{\ }2)\ \mathit{powr}\ y$
  ⟨*proof*⟩ **lemma** *exp-mult-2*: $\mathit{exp}\ (y * 2 :: \mathit{real}) = \mathit{exp}\ y * \mathit{exp}\ y$
  ⟨*proof*⟩ **lemma** *exp-c*: $\mathit{exp}\ c = \mathit{sqrt}\ (2{*}\mathit{pi})$
⟨*proof*⟩
  include *asymp-equiv-syntax*
  ⟨*proof*⟩ **lemma** *c*: $c = \mathit{ln}\ (2{*}\mathit{pi})\ /\ 2$
⟨*proof*⟩

This gives us the final bounds:

theorem *Gamma-bounds*:
  assumes $x \geq 1$
  shows *Gamma* $x \geq \mathit{sqrt}\ (2{*}\mathit{pi}/x) * (x\ /\ \mathit{exp}\ 1)\ \mathit{powr}\ x$ (**is** *?th1*)
  Gamma $x \leq \mathit{sqrt}\ (2{*}\mathit{pi}/x) * (x\ /\ \mathit{exp}\ 1)\ \mathit{powr}\ x * \mathit{exp}\ (1\ /\ (12 * x))$ (**is** *?th2*)

⟨*proof*⟩

theorem *ln-Gamma-bounds*:
  assumes $x \geq 1$
  shows $ln\text{-}Gamma\ x \geq ln\ (2*pi/x)\ /\ 2 + x * ln\ x - x$ (**is** *?th1*)
  $ln\text{-}Gamma\ x \leq ln\ (2*pi/x)\ /\ 2 + x * ln\ x - x + 1/(12*x)$ (**is** *?th2*)
⟨*proof*⟩

theorem *fact-bounds*:
  assumes $n > 0$
  shows $(fact\ n :: real) \geq sqrt\ (2*pi*n) * (n\ /\ exp\ 1)\ \hat{}\ n$ (**is** *?th1*)
  $(fact\ n :: real) \leq sqrt\ (2*pi*n) * (n\ /\ exp\ 1)\ \hat{}\ n * exp\ (1\ /\ (12 * n))$ (**is** *?th2*)
⟨*proof*⟩

theorem *ln-fact-bounds*:
  assumes $n > 0$
  shows $ln\ (fact\ n :: real) \geq ln\ (2*pi*n)/2 + n * ln\ n - n$ (**is** *?th1*)
  $ln\ (fact\ n :: real) \leq ln\ (2*pi*n)/2 + n * ln\ n - n + 1/(12*real\ n)$ (**is** *?th2*)
⟨*proof*⟩

theorem *Gamma-asymp-equiv*:
  $Gamma \sim[at\text{-}top]\ (\lambda x.\ sqrt\ (2*pi/x) * (x\ /\ exp\ 1)\ powr\ x :: real)$
⟨*proof*⟩

theorem *fact-asymp-equiv*:
  $fact \sim[at\text{-}top]\ (\lambda n.\ sqrt\ (2*pi*n) * (n\ /\ exp\ 1)\ \hat{}\ n :: real)$
⟨*proof*⟩

corollary *stirling-tendsto-sqrt-pi*:
  $(\lambda n.\ fact\ n\ /\ (sqrt\ (2 * n) * (n\ /\ exp\ 1)\ \hat{}\ n)) \longrightarrow sqrt\ pi$
⟨*proof*⟩

end

end

2 Complete asymptotics of the logarithmic Gamma function

theory *Gamma-Asymptotics*
imports
  HOL−Complex-Analysis.Complex-Analysis
  Bernoulli.Bernoulli-FPS
  Bernoulli.Periodic-Bernpoly
  Stirling-Formula
begin

2.1 Auxiliary Facts

lemma *stirling-limit-aux1*:
$((\lambda y.\ Ln\ (1 + z * of\text{-}real\ y)\ /\ of\text{-}real\ y) \longrightarrow z)\ (at\text{-}right\ 0)$ **for** $z :: complex$
⟨*proof*⟩

lemma *stirling-limit-aux2*:
$((\lambda y.\ y * Ln\ (1 + z\ /\ of\text{-}real\ y)) \longrightarrow z)\ at\text{-}top$ **for** $z :: complex$
⟨*proof*⟩

lemma *Union-atLeastAtMost*:
assumes $N > 0$
shows $(\bigcup n\in\{0..<N\}.\ \{real\ n..real\ (n + 1)\}) = \{0..real\ N\}$
⟨*proof*⟩

2.2 Cones in the complex plane

definition $complex\text{-}cone :: real \Rightarrow real \Rightarrow complex\ set$ **where**
$complex\text{-}cone\ a\ b = \{z.\ \exists y\in\{a..b\}.\ z = rcis\ (norm\ z)\ y\}$

abbreviation $complex\text{-}cone' :: real \Rightarrow complex\ set$ **where**
$complex\text{-}cone'\ a \equiv complex\text{-}cone\ (-a)\ a$

lemma *zero-in-complex-cone* [*simp*, *intro*]: $a \leq b \Longrightarrow 0 \in complex\text{-}cone\ a\ b$
⟨*proof*⟩

lemma *complex-coneE*:
assumes $z \in complex\text{-}cone\ a\ b$
obtains $r\ \alpha$ **where** $r \geq 0\ \ \alpha \in \{a..b\}\ \ z = rcis\ r\ \alpha$
⟨*proof*⟩

lemma *arg-cis* [*simp*]:
assumes $x \in \{-pi<..pi\}$
shows $Arg\ (cis\ x) = x$
⟨*proof*⟩

lemma *arg-mult-of-real-left* [*simp*]:
assumes $r > 0$
shows $Arg\ (of\text{-}real\ r * z) = Arg\ z$
⟨*proof*⟩

lemma *arg-mult-of-real-right* [*simp*]:
assumes $r > 0$
shows $Arg\ (z * of\text{-}real\ r) = Arg\ z$
⟨*proof*⟩

lemma *arg-rcis* [*simp*]:
assumes $x \in \{-pi<..pi\}\ \ r > 0$
shows $Arg\ (rcis\ r\ x) = x$
⟨*proof*⟩

lemma *rcis-in-complex-cone* [*intro*]:
assumes $\alpha \in \{a..b\}$ $r \geq 0$
shows *rcis r* $\alpha \in$ *complex-cone a b*
⟨*proof*⟩

lemma *arg-imp-in-complex-cone*:
assumes *Arg z* $\in \{a..b\}$
shows $z \in$ *complex-cone a b*
⟨*proof*⟩

lemma *complex-cone-altdef*:
assumes $-pi < a$ $a \leq b$ $b \leq pi$
shows *complex-cone a b* = *insert 0* {*z. Arg z* $\in \{a..b\}$}
⟨*proof*⟩

lemma *nonneg-of-real-in-complex-cone* [*simp*, *intro*]:
assumes $x \geq 0$ $a \leq 0$ $0 \leq b$
shows *of-real x* $\in$ *complex-cone a b*
⟨*proof*⟩

lemma *one-in-complex-cone* [*simp*, *intro*]: $a \leq 0 \Longrightarrow 0 \leq b \Longrightarrow 1 \in$ *complex-cone a b*
⟨*proof*⟩

lemma *of-nat-in-complex-cone* [*simp*, *intro*]: $a \leq 0 \Longrightarrow 0 \leq b \Longrightarrow$ *of-nat n* $\in$ *complex-cone a b*
⟨*proof*⟩

2.3 Another integral representation of the Beta function

lemma *complex-cone-inter-nonpos-Reals*:
assumes $-pi < a$ $a \leq b$ $b < pi$
shows *complex-cone a b* $\cap \mathbb{R}_{\leq 0} = \{0\}$
⟨*proof*⟩

theorem
assumes *a*: $a > 0$ **and** *b*: $b > (0 :: real)$
shows *has-integral-Beta-real′*:
((λ*u. u powr* (*b* − *1*) / (*1* + *u*) *powr* (*a* + *b*)) *has-integral Beta a b*) {*0*<..}
and *Beta-conv-nn-integral*:
Beta a b = (∫⁺*u. ennreal* (*indicator* {*0*<..} *u* ∗ *u powr* (*b* − *1*) / (*1* + *u*) *powr* (*a* + *b*)) ∂*lborel*)
⟨*proof*⟩

lemma *has-integral-Beta2*:
fixes *a* :: *real*
assumes $a < -1/2$
shows ((λ*x.* (*1* + *x* ^ *2*) *powr a*) *has-integral Beta* (− *a* − *1* / *2*) (*1* / *2*) /

2) {*0*<..}
⟨*proof*⟩

lemma *has-integral-Beta3*:
fixes *a b* :: *real*
assumes *a* < −*1*/*2* **and** *b* > *0*
shows ((λ*x*. (*b* + *x* ^ *2*) *powr a*) *has-integral*
Beta (−*a* − *1*/*2*) (*1*/*2*) / *2* ∗ *b powr* (*a* + *1*/*2*)) {*0*<..}
⟨*proof*⟩

2.4 Asymptotics of the real $\log \Gamma$ function and its derivatives

This is the error term that occurs in the expansion of *ln-Gamma*. It can be shown to be of order $O(s^{-n})$.

definition *stirling-integral* :: *nat* ⇒ *'a* :: {*real-normed-div-algebra*, *banach*} ⇒ *'a*
where
stirling-integral n s =
lim (λ*N*. *integral* {*0*..*N*} (λ*x*. *of-real* (*pbernpoly n x*) / (*of-real x* + *s*) ^ *n*))

context
fixes *s* :: *complex* **assumes** *s*: $s \notin \mathbb{R}_{\leq 0}$
fixes *approx* :: *nat* ⇒ *complex*
defines *approx* ≡ (λ*N*.
(∑*n* = *1*..<*N*. *s* / *of-nat n* − *ln* (*1* + *s* / *of-nat n*)) − (*euler-mascheroni* ∗ *s* + *ln s*) − —⟶ *ln-Gamma s*
(*ln-Gamma* (*of-nat N*) − *ln* (*2* ∗ *pi* / *of-nat N*) / *2* − *of-nat N* ∗ *ln* (*of-nat N*) + *of-nat N*) − —⟶ *0*
s ∗ (*harm* (*N* − *1*) − *ln* (*of-nat* (*N* − *1*)) − *euler-mascheroni*) + —⟶ *0*
s ∗ (*ln* (*of-nat N* + *s*) − *ln* (*of-nat* (*N* − *1*))) − —⟶ *0*
(*1*/*2*) ∗ (*ln* (*of-nat N* + *s*) − *ln* (*of-nat N*)) + —⟶ *0*
of-nat N ∗ (*ln* (*of-nat N* + *s*) − *ln* (*of-nat N*)) − —⟶ *s*
(*s* − *1*/*2*) ∗ *ln s* − *ln* (*2* ∗ *pi*) / *2*)
begin

qualified lemma
assumes *N*: *N* > *0*
shows *integrable-pbernpoly-1*:
(λ*x*. *of-real* (−*pbernpoly 1 x*) / (*of-real x* + *s*)) *integrable-on* {*0*..*real N*}
and *integral-pbernpoly-1-aux*:
integral {*0*..*real N*} (λ*x*. −*of-real* (*pbernpoly 1 x*) / (*of-real x* + *s*)) = *approx N*
and *has-integral-pbernpoly-1*:
((λ*x*. *pbernpoly 1 x* /(*x* + *s*)) *has-integral*
(∑*m*<*N*. (*of-nat m* + *1* / *2* + *s*) ∗ (*ln* (*of-nat m* + *s*) −
ln (*of-nat m* + *1* + *s*)) + *1*)) {*0*..*real N*}
⟨*proof*⟩

lemma *integrable-ln-Gamma-aux*:
shows (λ*x*. *of-real* (*pbernpoly n x*) / (*of-real x* + *s*) ^ *n*) *integrable-on* {*0*..*real*

N}
⟨*proof*⟩

This following proof is based on "Rudiments of the theory of the gamma function" by Bruce Berndt [1].

lemma *tendsto-of-real-0-I*:
  $(f \longrightarrow 0)\ G \Longrightarrow ((\lambda x.\ (\textit{of-real}\ (f\ x))) \longrightarrow (0 :: 'a::\textit{real-normed-div-algebra}))\ G$
  ⟨*proof*⟩ **lemma** *integral-pbernpoly-1*:
  $(\lambda N.\ \textit{integral}\ \{0..\textit{real}\ N\}\ (\lambda x.\ \textit{pbernpoly}\ 1\ x\ /\ (x + s)))$
  $\longrightarrow -\textit{ln-Gamma}\ s - s + (s - 1\ /\ 2) * \textit{ln}\ s + \textit{ln}\ (2 * \textit{pi})\ /\ 2$
⟨*proof*⟩ **lemma** *pbernpoly-integral-conv-pbernpoly-integral-Suc*:
  assumes $n \geq 1$
  shows $\textit{integral}\ \{0..\textit{real}\ N\}\ (\lambda x.\ \textit{pbernpoly}\ n\ x\ /\ (x + s)\ \hat{}\ n) =$
  $\textit{of-real}\ (\textit{pbernpoly}\ (\textit{Suc}\ n)\ (\textit{real}\ N))\ /\ (\textit{of-nat}\ (\textit{Suc}\ n) * (s + \textit{of-nat}\ N)\ \hat{}\ n) -$
  $\textit{of-real}\ (\textit{bernoulli}\ (\textit{Suc}\ n))\ /\ (\textit{of-nat}\ (\textit{Suc}\ n) * s\ \hat{}\ n) + \textit{of-nat}\ n\ /\ \textit{of-nat}\ (\textit{Suc}\ n) *$
  $\textit{integral}\ \{0..\textit{real}\ N\}\ (\lambda x.\ \textit{of-real}\ (\textit{pbernpoly}\ (\textit{Suc}\ n)\ x)\ /\ (\textit{of-real}\ x + s)\ \hat{}\ \textit{Suc}\ n)$
⟨*proof*⟩

lemma *pbernpoly-over-power-tendsto-0*:
  assumes $n > 0$
  shows $(\lambda x.\ \textit{of-real}\ (\textit{pbernpoly}\ (\textit{Suc}\ n)\ (\textit{real}\ x))\ /\ (\textit{of-nat}\ (\textit{Suc}\ n) * (s + \textit{of-nat}\ x)\ \hat{}\ n)) \longrightarrow 0$
⟨*proof*⟩

lemma *convergent-stirling-integral*:
  assumes $n > 0$
  shows $\textit{convergent}\ (\lambda N.\ \textit{integral}\ \{0..\textit{real}\ N\}$
  $(\lambda x.\ \textit{of-real}\ (\textit{pbernpoly}\ n\ x)\ /\ (\textit{of-real}\ x + s)\ \hat{}\ n))$ (**is** $\textit{convergent}\ (?f\ n)$)
⟨*proof*⟩

lemma *stirling-integral-conv-stirling-integral-Suc*:
  assumes $n > 0$
  shows $\textit{stirling-integral}\ n\ s =$
  $\textit{of-nat}\ n\ /\ \textit{of-nat}\ (\textit{Suc}\ n) * \textit{stirling-integral}\ (\textit{Suc}\ n)\ s -$
  $\textit{of-real}\ (\textit{bernoulli}\ (\textit{Suc}\ n))\ /\ (\textit{of-nat}\ (\textit{Suc}\ n) * s\ \hat{}\ n)$
⟨*proof*⟩

lemma *stirling-integral-1-unfold*:
  assumes $m > 0$
  shows $\textit{stirling-integral}\ 1\ s = \textit{stirling-integral}\ m\ s\ /\ \textit{of-nat}\ m -$
  $(\sum k{=}1..{<}m.\ \textit{of-real}\ (\textit{bernoulli}\ (\textit{Suc}\ k))\ /\ (\textit{of-nat}\ k * \textit{of-nat}\ (\textit{Suc}\ k) * s\ \hat{}\ k))$
⟨*proof*⟩

lemma *ln-Gamma-stirling-complex*:

assumes $m > 0$
shows *ln-Gamma* $s = (s - 1 / 2) * ln\ s - s + ln\ (2 * pi) / 2 +$
$(\sum k{=}1..{<}m.$ *of-real* (*bernoulli* (*Suc k*)) / (*of-nat k* $*$ *of-nat* (*Suc k*) $*$ s ^ k)) $-$
stirling-integral m s / *of-nat m*
⟨*proof*⟩

lemma *LIMSEQ-stirling-integral*:
$n > 0 \Longrightarrow$ ($\lambda x.$ *integral* $\{0..real\ x\}$ ($\lambda x.$ *of-real* (*pbernpoly n x*) / (*of-real x* + *s*) ^ *n*))
$\longrightarrow$ *stirling-integral n s* ⟨*proof*⟩

end

lemmas *has-integral-of-real* = *has-integral-linear*[*OF* - *bounded-linear-of-real*, *unfolded o-def*]
lemmas *integral-of-real* = *integral-linear*[*OF* - *bounded-linear-of-real*, *unfolded o-def*]

lemma *integrable-ln-Gamma-aux-real*:
assumes $0 < s$
shows ($\lambda x.$ *pbernpoly n x* / ($x + s$) ^ *n*) *integrable-on* $\{0..real\ N\}$
⟨*proof*⟩

lemma
assumes $x > 0$ $n > 0$
shows *stirling-integral-complex-of-real*:
stirling-integral n (*complex-of-real x*) = *of-real* (*stirling-integral n x*)
and *LIMSEQ-stirling-integral-real*:
($\lambda N.$ *integral* $\{0..real\ N\}$ ($\lambda t.$ *pbernpoly n t* / ($t + x$) ^ *n*))
$\longrightarrow$ *stirling-integral n x*
and *stirling-integral-real-convergent*:
convergent ($\lambda N.$ *integral* $\{0..real\ N\}$ ($\lambda t.$ *pbernpoly n t* / ($t + x$) ^ *n*))
⟨*proof*⟩

lemma *ln-Gamma-stirling-real*:
assumes $x > (0 :: real)$ $m > (0::nat)$
shows *ln-Gamma* $x = (x - 1 / 2) * ln\ x - x + ln\ (2 * pi) / 2 +$
$(\sum k = 1..{<}m.$ *bernoulli* (*Suc k*) / (*of-nat k* $*$ *of-nat* (*Suc k*) $*$ x ^ k))
$-$
stirling-integral m x / *of-nat m*
⟨*proof*⟩

lemma *stirling-integral-bound-aux*:
assumes *n*: $n > (1::nat)$
obtains *c* **where** $\bigwedge s.$ *Re* $s > 0 \Longrightarrow$ *norm* (*stirling-integral n s*) $\leq$ *c* / *Re s* ^ ($n - 1$)
⟨*proof*⟩

lemma *stirling-integral-bound-aux-integral1*:
fixes $a\ b\ c$ :: *real* **and** n :: *nat*
assumes $a \geq 0$ $b > 0$ $c \geq 0$ $n > 1$ $l < a - b$ $r > a + b$
shows $((\lambda x.\ c\ /\ max\ b\ |x - a|\ \hat{}\ n)$ *has-integral*
$2{*}c{*}(n\ /\ (n - 1))/b\hat{}(n{-}1) - c/(n{-}1) * (1/(a{-}l)\hat{}(n{-}1) + 1/(r{-}a)\hat{}(n{-}1)))$
$\{l..r\}$
⟨*proof*⟩

lemma *stirling-integral-bound-aux-integral2*:
fixes $a\ b\ c$ :: *real* **and** n :: *nat*
assumes $a \geq 0$ $b > 0$ $c \geq 0$ $n > 1$
obtains I **where** $((\lambda x.\ c\ /\ max\ b\ |x - a|\ \hat{}\ n)$ *has-integral* $I)$ $\{l..r\}$
$I \leq 2 * c * (n\ /\ (n - 1))\ /\ b\ \hat{}(n{-}1)$
⟨*proof*⟩

lemma *stirling-integral-bound-aux'*:
assumes n: $n > (1{::}nat)$ **and** α: $\alpha \in \{0{<}..{<}pi\}$
obtains c **where** $\bigwedge s{::}complex.\ s \in$ *complex-cone'* $\alpha - \{0\} \Longrightarrow$
norm (*stirling-integral* n s) $\leq c$ / *norm* $s\ \hat{}\ (n - 1)$
⟨*proof*⟩

lemma *stirling-integral-bound*:
assumes $n > 0$
obtains c **where**
$\bigwedge s.\ Re\ s > 0 \Longrightarrow$ *norm* (*stirling-integral* n s) $\leq c$ / *Re* $s\ \hat{}\ n$
⟨*proof*⟩

lemma *stirling-integral-bound'*:
assumes $n > 0$ **and** $\alpha \in \{0{<}..{<}pi\}$
obtains c **where**
$\bigwedge s{::}complex.\ s \in$ *complex-cone'* $\alpha - \{0\} \Longrightarrow$ *norm* (*stirling-integral* n s) $\leq c$ /
norm $s\ \hat{}\ n$
⟨*proof*⟩

lemma *stirling-integral-holomorphic* [*holomorphic-intros*]:
assumes m: $m > 0$ **and** $A \cap \mathbb{R}_{\leq 0} = \{\}$
shows *stirling-integral* m *holomorphic-on* A
⟨*proof*⟩

lemma *stirling-integral-continuous-on-complex* [*continuous-intros*]:
assumes m: $m > 0$ **and** $A \cap \mathbb{R}_{\leq 0} = \{\}$
shows *continuous-on* A (*stirling-integral* m :: - $\Rightarrow$ *complex*)
⟨*proof*⟩

lemma *has-field-derivative-stirling-integral-complex*:
fixes x :: *complex*
assumes $x \notin \mathbb{R}_{\leq 0}$ $n > 0$
shows (*stirling-integral* n *has-field-derivative* *deriv* (*stirling-integral* n) x) (*at*

x)
⟨*proof*⟩

lemma
assumes *n*: $n > 0$ **and** $x > 0$
shows *deriv-stirling-integral-complex-of-real*:
(*deriv* ^^ *j*) (*stirling-integral n*) (*complex-of-real x*) =
complex-of-real ((*deriv* ^^ *j*) (*stirling-integral n*) *x*) (**is** *?lhs x* = *?rhs x*)
and *differentiable-stirling-integral-real*:
(*deriv* ^^ *j*) (*stirling-integral n*) *field-differentiable at x* (**is** *?thesis2*)
⟨*proof*⟩

Unfortunately, asymptotic power series cannot, in general, be differentiated. However, since *ln-Gamma* is holomorphic on the entire positive real half-space, we can differentiate its asymptotic expansion after all.

To do this, we use an ad-hoc version of the more general approach outlined in Erdelyi's "Asymptotic Expansions" for holomorphic functions: We bound the value of the j-th derivative of the remainder term at some value x by applying Cauchy's integral formula along a circle centred at x with radius $\frac{1}{2}x$.

lemma *deriv-stirling-integral-real-bound*:
assumes *m*: $m > 0$
shows (*deriv* ^^ *j*) (*stirling-integral m*) ∈ *O*(λ*x*::*real*. *1* / *x* ^ (*m* + *j*))
⟨*proof*⟩

definition *stirling-sum* **where**
stirling-sum j m x =
(−*1*) ^ *j* ∗ (∑ *k* = *1*..<*m*. (*of-real* (*bernoulli* (*Suc k*)) ∗ *pochhammer* (*of-nat* *k*) *j* / (*of-nat k* ∗
of-nat (*Suc k*))) ∗ *inverse x* ^ (*k* + *j*))

definition *stirling-sum′* **where**
stirling-sum′ j m x =
(−*1*) ^ (*Suc j*) ∗ (∑ *k*≤*m*. (*of-real* (*bernoulli′ k*) ∗
pochhammer (*of-nat* (*Suc k*)) (*j* − *1*) ∗ *inverse x* ^ (*k* + *j*)))

lemma *stirling-sum-complex-of-real*:
stirling-sum j m (*complex-of-real x*) = *complex-of-real* (*stirling-sum j m x*)
⟨*proof*⟩

lemma *stirling-sum′-complex-of-real*:
stirling-sum′ j m (*complex-of-real x*) = *complex-of-real* (*stirling-sum′ j m x*)
⟨*proof*⟩

lemma *has-field-derivative-stirling-sum-complex* [*derivative-intros*]:
Re x > *0* ⟹ (*stirling-sum j m has-field-derivative stirling-sum* (*Suc j*) *m x*) (*at*

x)
⟨*proof*⟩

lemma *has-field-derivative-stirling-sum-real* [*derivative-intros*]:
x > (*0*::*real*) ⟹ (*stirling-sum j m has-field-derivative stirling-sum* (*Suc j*) *m x*) (*at x*)
⟨*proof*⟩

lemma *has-field-derivative-stirling-sum′-complex* [*derivative-intros*]:
assumes *j* > *0 Re x* > *0*
shows (*stirling-sum′ j m has-field-derivative stirling-sum′* (*Suc j*) *m x*) (*at x*)
⟨*proof*⟩

lemma *has-field-derivative-stirling-sum′-real* [*derivative-intros*]:
assumes *j* > *0 x* > (*0*::*real*)
shows (*stirling-sum′ j m has-field-derivative stirling-sum′* (*Suc j*) *m x*) (*at x*)
⟨*proof*⟩

lemma *higher-deriv-stirling-sum-complex*:
Re x > *0* ⟹ (*deriv* ^^ *i*) (*stirling-sum j m*) *x* = *stirling-sum* (*i* + *j*) *m x*
⟨*proof*⟩

definition *Polygamma-approx* :: *nat* ⇒ *nat* ⇒ *′a* ⇒ *′a* :: {*real-normed-field*, *ln*}
where
Polygamma-approx j m =
(*deriv* ^^ *j*) (λ*x*. (*x* − *1* / *2*) ∗ *ln x* − *x* + *of-real* (*ln* (*2* ∗ *pi*)) / *2* + *stirling-sum 0 m x*)

lemma *Polygamma-approx-Suc*: *Polygamma-approx* (*Suc j*) *m* = *deriv* (*Polygamma-approx j m*)
⟨*proof*⟩

lemma *Polygamma-approx-0*:
Polygamma-approx 0 m x = (*x* − *1*/*2*) ∗ *ln x* − *x* + *of-real* (*ln* (*2*∗*pi*)) / *2* + *stirling-sum 0 m x*
⟨*proof*⟩

lemma *Polygamma-approx-1-complex*:
Re x > *0* ⟹
Polygamma-approx (*Suc 0*) *m x* = *ln x* − *1* / (*2*∗*x*) + *stirling-sum* (*Suc 0*) *m x*
⟨*proof*⟩

lemma *Polygamma-approx-1-real*:
x > (*0* :: *real*) ⟹
Polygamma-approx (*Suc 0*) *m x* = *ln x* − *1* / (*2*∗*x*) + *stirling-sum* (*Suc 0*) *m x*
⟨*proof*⟩

lemma *stirling-sum-2-conv-stirling-sum′-1*:
fixes x :: $'a$ :: {*real-div-algebra*, *field-char-0*}
assumes $m > 0$ $x \neq 0$
shows *stirling-sum′ 1 m x* $= 1 \ / \ x + 1 \ / \ (2 * x \hat{} 2) +$ *stirling-sum 2 m x*
⟨*proof*⟩

lemma *Polygamma-approx-2-real*:
assumes $x > (0$::*real*$)$ $m > 0$
shows *Polygamma-approx* (*Suc* (*Suc 0*)) *m x* = *stirling-sum′ 1 m x*
⟨*proof*⟩

lemma *Polygamma-approx-2-complex*:
assumes *Re x* > 0 $m > 0$
shows *Polygamma-approx* (*Suc* (*Suc 0*)) *m x* = *stirling-sum′ 1 m x*
⟨*proof*⟩

lemma *Polygamma-approx-ge-2-real*:
assumes $x > (0$::*real*$)$ $m > 0$
shows *Polygamma-approx* (*Suc* (*Suc j*)) *m x* = *stirling-sum′* (*Suc j*) *m x*
⟨*proof*⟩

lemma *Polygamma-approx-ge-2-complex*:
assumes *Re x* > 0 $m > 0$
shows *Polygamma-approx* (*Suc* (*Suc j*)) *m x* = *stirling-sum′* (*Suc j*) *m x*
⟨*proof*⟩

lemma *Polygamma-approx-complex-of-real*:
assumes $x > 0$ $m > 0$
shows *Polygamma-approx j m* (*complex-of-real x*) = *of-real* (*Polygamma-approx j m x*)
⟨*proof*⟩

lemma *higher-deriv-Polygamma-approx* [*simp*]:
(*deriv* ^^ *j*) (*Polygamma-approx i m*) = *Polygamma-approx* (*j* + *i*) *m*
⟨*proof*⟩

lemma *stirling-sum-holomorphic* [*holomorphic-intros*]:
$0 \notin A \Longrightarrow$ *stirling-sum j m holomorphic-on A*
⟨*proof*⟩

lemma *Polygamma-approx-holomorphic* [*holomorphic-intros*]:
Polygamma-approx j m holomorphic-on {*s*. *Re s* > 0}
⟨*proof*⟩

lemma *higher-deriv-lnGamma-stirling*:
assumes *m*: $m > 0$
shows (λx::*real*. (*deriv* ^^ *j*) *ln-Gamma x* − *Polygamma-approx j m x*) ∈ $O(\lambda x.\ 1 \ / \ x \hat{} (m + j))$

⟨*proof*⟩

lemma *Polygamma-approx-1-real'*:
assumes *x*: (*x*::*real*) > *0* **and** *m*: *m* > *0*
shows *Polygamma-approx 1 m x = ln x* − (∑ *k* = *Suc 0..m. bernoulli' k* ∗ *inverse x ^ k / real k*)
⟨*proof*⟩

theorem
assumes *m*: *m* > *0*
shows *ln-Gamma-real-asymptotics*:
(λ*x. ln-Gamma x* − ((*x* − *1* / *2*) ∗ *ln x* − *x* + *ln* (*2* ∗ *pi*) / *2* +
(∑ *k* = *1..<m. bernoulli* (*Suc k*) / (*real k* ∗ *real* (*Suc k*)) / *x^k*)))
∈ *O*(λ*x. 1* / *x ^ m*) (**is** *?th1*)
and *Digamma-real-asymptotics*:
(λ*x. Digamma x* − (*ln x* − (∑ *k=1..m. bernoulli' k* / *real k* / *x ^ k*)))
∈ *O*(λ*x. 1* / (*x ^ Suc m*)) (**is** *?th2*)
and *Polygamma-real-asymptotics*: *j* > *0* ⟹
(λ*x. Polygamma j x* − (− *1*) ^ *Suc j* ∗ (∑ *k*≤*m. bernoulli' k* ∗
pochhammer (*real* (*Suc k*)) (*j* − *1*) / *x* ^ (*k* + *j*)))
∈ *O*(λ*x. 1* / *x* ^ (*m*+*j*+*1*)) (**is** - ⟹ *?th3*)
⟨*proof*⟩

2.5 Asymptotics of the complex Gamma function

The m-th order remainder of Stirling's formula for $\log\Gamma$ is $O(s^{-m})$ uniformly over any complex cone $\text{Arg}(z) \leq \alpha$, $z \neq 0$ for any angle $\alpha \in (0, \pi)$. This means that there is bounded by cz^{-m} for some constant c for all z in this cone.

context
fixes *F* **and** α
assumes α: α ∈ {*0*<..<*pi*}
defines *F* ≡ *principal* (*complex-cone'* α − {*0*})
begin

lemma *stirling-integral-bigo*:
fixes *m* :: *nat*
assumes *m*: *m* > *0*
shows *stirling-integral m* ∈ *O*[*F*](λ*s. 1* / *s ^ m*)
⟨*proof*⟩

end

The following is a more explicit statement of this:

theorem *ln-Gamma-complex-asymptotics-explicit*:
fixes *m* :: *nat* **and** α :: *real*
assumes *m* > *0* **and** α ∈ {*0*<..<*pi*}
obtains *C* :: *real* **and** *R* :: *complex* ⇒ *complex*

where $\forall\, s{::}complex.\ s \notin \mathbb{R}_{\leq 0} \longrightarrow$
$ln\text{-}Gamma\ s = (s - 1/2) * ln\ s - s + ln\ (2 * pi)\ /\ 2 +$
$(\sum k{=}1..{<}m.\ bernoulli\ (k{+}1)\ /\ (k * (k{+}1) * s\ \hat{}\ k)) - R\ s$
and $\forall\, s.\ s \neq 0 \wedge |Arg\ s| \leq \alpha \longrightarrow norm\ (R\ s) \leq C\ /\ norm\ s\ \hat{}\ m$
⟨*proof*⟩

Lastly, we can also derive the asymptotics of Γ itself:

$$\Gamma(z) \sim \sqrt{2\pi/z}\left(\frac{z}{e}\right)^z$$

uniformly for $|z| \to \infty$ within the cone $\mathrm{Arg}(z) \leq \alpha$ for $\alpha \in (0, \pi)$:

context
fixes F **and** α
assumes α: $\alpha \in \{0{<}..{<}pi\}$
defines $F \equiv inf\ at\text{-}infinity\ (principal\ (complex\text{-}cone'\ \alpha))$
begin

lemma *Gamma-complex-asymp-equiv*:
$Gamma \sim[F]\ (\lambda s.\ sqrt\ (2 * pi) * (s\ /\ exp\ 1)\ powr\ s\ /\ s\ powr\ (1\ /\ 2))$
⟨*proof*⟩

end

end

References

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