

Steiner’s Line Theorem in Isabelle/HOL

Arthur Freitas Ramos David Barros Hulak
Ruy Jose Guerra Barretto de Queiroz

October 6, 2026

Abstract

We formalize Steiner’s line theorem in Isabelle/HOL. For a non-degenerate triangle and a point on its circumcircle, the reflections of that point in the three sidelines are collinear, and the resulting line passes through the orthocenter. The development uses complex coordinates and imports the published Wallace–Simson line entry for its perpendicular-foot and collinearity infrastructure.

1 Introduction

Steiner’s line theorem is the reflection companion to the Wallace–Simson line theorem [1]. Its classical setting goes back to Steiner’s work on triangle and circle geometry [4]; a modern annotated account of related Steiner theorems is given by Ehrmann [2]. Let ABC be a nondegenerate triangle and let M lie on its circumcircle. Reflect M in the sidelines BC , CA , and AB . The three reflected points lie on one line, called the Steiner line of (M) with respect to (ABC) , and this line passes through the orthocenter of (ABC) .

The formalization imports the AFP entry *The Wallace–Simson Line Theorem in Isabelle/HOL* [3]. Each reflection is defined as twice the corresponding perpendicular foot minus (M) , so the collinearity part is obtained by applying a homothety to the Simson line. The orthocenter incidence is then proved by a direct complex-coordinate identity.

2 Main results

The theorem *steiner_line* proves collinearity of the three reflected points. The stronger theorem *steiner_line_through_orthocenter* proves all three cyclic orthocenter/reflection collinearities. This redundant-looking statement is intentional: if M is a vertex, one pair of reflections can coincide, while another cyclic pair still determines the Steiner line. The lemma *orthocenter_is_orthocenter* verifies the geometric meaning of the coordinate

definition: the three vertex-to-orthocenter displacements are perpendicular to the opposite sidelines. Its algebraic statement only assumes the three common-circle conditions; the nondegenerate-triangle and positive-radius hypotheses are introduced by the main theorems. The definitions *steiner_reflect* and *orthocenter* expose the concrete complex-coordinate construction used by both results.

```
theory Steiner
  imports Simson.Simson
begin
```

3 Reflections and the orthocenter

The reflection of a point in a sideline is twice its perpendicular foot, minus the original point.

```
definition steiner-reflect :: complex  $\Rightarrow$  complex  $\Rightarrow$  complex  $\Rightarrow$  complex
  where steiner-reflect a b m = 2 * foot a b m - m
```

In circumcentre coordinates the orthocenter is the sum of the three vertex vectors.

```
definition orthocenter ::
  complex  $\Rightarrow$  complex  $\Rightarrow$  complex  $\Rightarrow$  complex  $\Rightarrow$  complex
  where orthocenter a b c ctr = a + b + c - 2 * ctr
```

The coordinate formula gives the altitude-perpendicularity identities directly from the equal circumradius equations. Thus this algebraic lemma does not need the non-collinearity or positive-radius assumptions; those hypotheses are imposed by the triangle theorems below.

```
lemma orthocenter-is-orthocenter:
  fixes a b c ctr :: complex and r :: real
  assumes ha: oncircle ctr r a and hb: oncircle ctr r b
  and hc: oncircle ctr r c
  shows complex-perpendicular (c - b)
    (orthocenter a b c ctr - a)  $\wedge$ 
    complex-perpendicular (a - c)
    (orthocenter a b c ctr - b)  $\wedge$ 
    complex-perpendicular (b - a)
    (orthocenter a b c ctr - c)
  <proof>
```

```
lemma steiner-reflect-unfold:
  assumes a  $\neq$  b
  shows steiner-reflect a b m = a + (b - a) * cnj (m - a) / cnj (b - a)
  <proof>
```

```
lemma collinear-homothety:
```

assumes *collinear* $p\ q\ r$
shows *collinear* $(2 * p - m)\ (2 * q - m)\ (2 * r - m)$
 $\langle proof \rangle$

lemma *im-zero-of-cnj-eq*:
assumes *cnj* $z = z$
shows *Im* $z = 0$
 $\langle proof \rangle$

4 The circle formula for a reflected point

This is the only field calculation needed to place the reflected point in circumcentre coordinates.

lemma *steiner-reflect-ratio-gen*:
fixes $B\ C\ M\ k :: complex$
assumes $B0: B \neq 0$ **and** $C0: C \neq 0$ **and** $M0: M \neq 0$
and $BC: B \neq C$ **and** $k0: k \neq 0$
shows $(C - B) * (k / M - k / B) / (k / C - k / B)$
 $= C * (M - B) / M$
 $\langle proof \rangle$

lemma *steiner-reflect-formula*:
fixes $b\ c\ m\ ctr :: complex$ **and** $r :: real$
assumes $hb: oncircle\ ctr\ r\ b$ **and** $hc: oncircle\ ctr\ r\ c$
and $hm: oncircle\ ctr\ r\ m$ **and** $pos: 0 < r$ **and** $bc: b \neq c$
shows *steiner-reflect* $b\ c\ m =$
 $ctr + (b - ctr) + (c - ctr) -$
 $(b - ctr) * (c - ctr) / (m - ctr)$
 $\langle proof \rangle$

5 The orthocenter incidence calculation

lemma *steiner-orthocenter-cross-gen*:
fixes $A\ B\ C\ M\ k :: complex$
assumes $A0: A \neq 0$ **and** $B0: B \neq 0$ **and** $C0: C \neq 0$
and $M0: M \neq 0$
and $hA: A * cnj\ A = k$ **and** $hB: B * cnj\ B = k$
and $hC: C * cnj\ C = k$ **and** $hM: M * cnj\ M = k$
shows $cnj\ (C + A - C * A / M - (A + B + C)) *$
 $(B + C - B * C / M - (A + B + C)) =$
 $(C + A - C * A / M - (A + B + C)) *$
 $cnj\ (B + C - B * C / M - (A + B + C))$
 $\langle proof \rangle$

lemma *steiner-orthocenter-collinear*:
fixes $a\ b\ c\ m\ ctr :: complex$ **and** $r :: real$
assumes $hncol: \neg collinear\ a\ b\ c$
and $ha: oncircle\ ctr\ r\ a$ **and** $hb: oncircle\ ctr\ r\ b$

and $hc: \text{oncircle } ctr \ r \ c$ **and** $hm: \text{oncircle } ctr \ r \ m$
and $pos: 0 < r$
shows $\text{collinear } (\text{orthocenter } a \ b \ c \ ctr)$
 $(\text{steiner-reflect } b \ c \ m) (\text{steiner-reflect } c \ a \ m)$
 $\langle \text{proof} \rangle$

6 Steiner's line theorem

The reflections in the three sidelines are collinear.

theorem *steiner-line*:
fixes $a \ b \ c \ m \ ctr :: \text{complex}$ **and** $r :: \text{real}$
assumes $hncol: \neg \text{collinear } a \ b \ c$
and $ha: \text{oncircle } ctr \ r \ a$ **and** $hb: \text{oncircle } ctr \ r \ b$
and $hc: \text{oncircle } ctr \ r \ c$ **and** $hm: \text{oncircle } ctr \ r \ m$
and $pos: 0 < r$
shows $\text{collinear } (\text{steiner-reflect } b \ c \ m)$
 $(\text{steiner-reflect } c \ a \ m) (\text{steiner-reflect } a \ b \ m)$
 $\langle \text{proof} \rangle$

The Steiner line passes through the orthocenter.

theorem *steiner-line-through-orthocenter*:
fixes $a \ b \ c \ m \ ctr :: \text{complex}$ **and** $r :: \text{real}$
assumes $hncol: \neg \text{collinear } a \ b \ c$
and $ha: \text{oncircle } ctr \ r \ a$ **and** $hb: \text{oncircle } ctr \ r \ b$
and $hc: \text{oncircle } ctr \ r \ c$ **and** $hm: \text{oncircle } ctr \ r \ m$
and $pos: 0 < r$
shows $\text{collinear } (\text{orthocenter } a \ b \ c \ ctr)$
 $(\text{steiner-reflect } b \ c \ m) (\text{steiner-reflect } c \ a \ m) \wedge$
 $\text{collinear } (\text{orthocenter } a \ b \ c \ ctr)$
 $(\text{steiner-reflect } c \ a \ m) (\text{steiner-reflect } a \ b \ m) \wedge$
 $\text{collinear } (\text{orthocenter } a \ b \ c \ ctr)$
 $(\text{steiner-reflect } a \ b \ m) (\text{steiner-reflect } b \ c \ m)$
 $\langle \text{proof} \rangle$

end

References

- [1] H. S. M. Coxeter and S. L. Greitzer. *Geometry Revisited*. Mathematical Association of America, 1967. DOI: <https://doi.org/10.5948/UPO9780883859346>.
- [2] J.-P. Ehrmann. Steiner's theorems on the complete quadrilateral. *Forum Geometricorum*, 4:35–52, 2004.
- [3] A. F. Ramos, D. B. Hulak, and R. J. G. B. de Queiroz. The wallace–simson line theorem in isabelle/hol. *Archive of Formal Proofs*, August

2026. <https://isa-afp.org/entries/Simson.html>, Formal proof development.

- [4] J. Steiner. Einige geometrischen betrachtungen. *Journal für die reine und angewandte Mathematik*, 1:161–184, 1826. Available at <https://eudml.org/doc/183024>.