

Steiner’s Line Theorem in Isabelle/HOL

Arthur Freitas Ramos David Barros Hulak
Ruy Jose Guerra Barretto de Queiroz

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Abstract

We formalize Steiner’s line theorem in Isabelle/HOL. For a non-degenerate triangle and a point on its circumcircle, the reflections of that point in the three sidelines are collinear, and the resulting line passes through the orthocenter. The development uses complex coordinates and imports the published Wallace–Simson line entry for its perpendicular-foot and collinearity infrastructure.

1 Introduction

Steiner’s line theorem is the reflection companion to the Wallace–Simson line theorem [1]. Its classical setting goes back to Steiner’s work on triangle and circle geometry [4]; a modern annotated account of related Steiner theorems is given by Ehrmann [2]. Let ABC be a nondegenerate triangle and let M lie on its circumcircle. Reflect M in the sidelines BC , CA , and AB . The three reflected points lie on one line, called the Steiner line of (M) with respect to (ABC) , and this line passes through the orthocenter of (ABC) .

The formalization imports the AFP entry *The Wallace–Simson Line Theorem in Isabelle/HOL* [3]. Each reflection is defined as twice the corresponding perpendicular foot minus (M) , so the collinearity part is obtained by applying a homothety to the Simson line. The orthocenter incidence is then proved by a direct complex-coordinate identity.

2 Main results

The theorem *steiner_line* proves collinearity of the three reflected points. The stronger theorem *steiner_line_through_orthocenter* proves all three cyclic orthocenter/reflection collinearities. This redundant-looking statement is intentional: if M is a vertex, one pair of reflections can coincide, while another cyclic pair still determines the Steiner line. The lemma *orthocenter_is_orthocenter* verifies the geometric meaning of the coordinate

definition: the three vertex-to-orthocenter displacements are perpendicular to the opposite sidelines. Its algebraic statement only assumes the three common-circle conditions; the nondegenerate-triangle and positive-radius hypotheses are introduced by the main theorems. The definitions *steiner_reflect* and *orthocenter* expose the concrete complex-coordinate construction used by both results.

```
theory Steiner
  imports Simson.Simson
begin
```

3 Reflections and the orthocenter

The reflection of a point in a sideline is twice its perpendicular foot, minus the original point.

```
definition steiner-reflect :: complex  $\Rightarrow$  complex  $\Rightarrow$  complex  $\Rightarrow$  complex
  where steiner-reflect a b m = 2 * foot a b m - m
```

In circumcentre coordinates the orthocenter is the sum of the three vertex vectors.

```
definition orthocenter ::
  complex  $\Rightarrow$  complex  $\Rightarrow$  complex  $\Rightarrow$  complex  $\Rightarrow$  complex
  where orthocenter a b c ctr = a + b + c - 2 * ctr
```

The coordinate formula gives the altitude-perpendicularity identities directly from the equal circumradius equations. Thus this algebraic lemma does not need the non-collinearity or positive-radius assumptions; those hypotheses are imposed by the triangle theorems below.

```
lemma orthocenter-is-orthocenter:
  fixes a b c ctr :: complex and r :: real
  assumes ha: oncircle ctr r a and hb: oncircle ctr r b
    and hc: oncircle ctr r c
  shows complex-perpendicular (c - b)
    (orthocenter a b c ctr - a)  $\wedge$ 
    complex-perpendicular (a - c)
    (orthocenter a b c ctr - b)  $\wedge$ 
    complex-perpendicular (b - a)
    (orthocenter a b c ctr - c)
proof -
  have hA: (a - ctr) * cnj (a - ctr) = complex-of-real (r^2)
    using oncircle-norm-sq[OF ha] .
  have hB: (b - ctr) * cnj (b - ctr) = complex-of-real (r^2)
    using oncircle-norm-sq[OF hb] .
  have hC: (c - ctr) * cnj (c - ctr) = complex-of-real (r^2)
    using oncircle-norm-sq[OF hc] .
  have alt-a:
```

```

    complex-perpendicular (c - b)
    (orthocenter a b c ctr - a)
proof -
  have hnorm:
    (c - ctr) * cnj (c - ctr) =
    (b - ctr) * cnj (b - ctr)
  using hB hC by simp
  have eq:
    cnj (c - b) * (orthocenter a b c ctr - a) =
    (cnj (c - ctr) * (b - ctr) -
    cnj (cnj (c - ctr) * (b - ctr))) +
    ((c - ctr) * cnj (c - ctr) -
    (b - ctr) * cnj (b - ctr))
  by (simp add: orthocenter-def complex-cnj-diff complex-cnj-mult
    algebra-simps)
  have hcnj:
    cnj ((cnj (c - ctr) * (b - ctr) -
    cnj (cnj (c - ctr) * (b - ctr))) +
    ((c - ctr) * cnj (c - ctr) -
    (b - ctr) * cnj (b - ctr))) =
    -((cnj (c - ctr) * (b - ctr) -
    cnj (cnj (c - ctr) * (b - ctr))) +
    ((c - ctr) * cnj (c - ctr) -
    (b - ctr) * cnj (b - ctr)))
  using hnorm
  by (simp add: complex-cnj-diff complex-cnj-mult algebra-simps)
  have hzero:
    Re ((cnj (c - ctr) * (b - ctr) -
    cnj (cnj (c - ctr) * (b - ctr))) +
    ((c - ctr) * cnj (c - ctr) -
    (b - ctr) * cnj (b - ctr))) = 0
  using hcnj by (metis eq-minus-cnj-iff-imag)
  show ?thesis
  unfolding complex-perpendicular-def
  using eq hzero
  by (simp only: eq hzero)
qed
have alt-b:
  complex-perpendicular (a - c)
  (orthocenter a b c ctr - b)
proof -
  have hnorm:
    (a - ctr) * cnj (a - ctr) =
    (c - ctr) * cnj (c - ctr)
  using hA hC by simp
  have eq:
    cnj (a - c) * (orthocenter a b c ctr - b) =
    (cnj (a - ctr) * (c - ctr) -
    cnj (cnj (a - ctr) * (c - ctr))) +

```

```

      ((a - ctr) * cnj (a - ctr) -
       (c - ctr) * cnj (c - ctr))
    by (simp add: orthocenter-def complex-cnj-diff complex-cnj-mult
        algebra-simps)
  have hcnj:
    cnj ((cnj (a - ctr) * (c - ctr) -
           cnj (cnj (a - ctr) * (c - ctr))) +
          ((a - ctr) * cnj (a - ctr) -
           (c - ctr) * cnj (c - ctr))) =
    -((cnj (a - ctr) * (c - ctr) -
       cnj (cnj (a - ctr) * (c - ctr))) +
      ((a - ctr) * cnj (a - ctr) -
       (c - ctr) * cnj (c - ctr)))
  using hnorm
  by (simp add: complex-cnj-diff complex-cnj-mult algebra-simps)
  have hzero:
    Re ((cnj (a - ctr) * (c - ctr) -
          cnj (cnj (a - ctr) * (c - ctr))) +
         ((a - ctr) * cnj (a - ctr) -
          (c - ctr) * cnj (c - ctr))) = 0
  using hcnj by (metis eq-minus-cnj-iff-imag)
  show ?thesis
  unfolding complex-perpendicular-def
  using eq hzero
  by (simp only: eq hzero)
qed
have alt-c:
  complex-perpendicular (b - a)
    (orthocenter a b c ctr - c)
proof -
  have hnorm:
    (b - ctr) * cnj (b - ctr) =
    (a - ctr) * cnj (a - ctr)
  using hB hA by simp
  have eq:
    cnj (b - a) * (orthocenter a b c ctr - c) =
    (cnj (b - ctr) * (a - ctr) -
     cnj (cnj (b - ctr) * (a - ctr))) +
    ((b - ctr) * cnj (b - ctr) -
     (a - ctr) * cnj (a - ctr))
  by (simp add: orthocenter-def complex-cnj-diff complex-cnj-mult
      algebra-simps)
  have hcnj:
    cnj ((cnj (b - ctr) * (a - ctr) -
           cnj (cnj (b - ctr) * (a - ctr))) +
          ((b - ctr) * cnj (b - ctr) -
           (a - ctr) * cnj (a - ctr))) =
    -((cnj (b - ctr) * (a - ctr) -
       cnj (cnj (b - ctr) * (a - ctr))) +
      ((b - ctr) * cnj (b - ctr) -
       (a - ctr) * cnj (a - ctr)))

```

```

      ((b - ctr) * cnj (b - ctr) -
       (a - ctr) * cnj (a - ctr)))
    using hnorm
    by (simp add: complex-cnj-diff complex-cnj-mult algebra-simps)
  have hzero:
    Re ((cnj (b - ctr) * (a - ctr) -
          cnj (cnj (b - ctr) * (a - ctr))) +
         ((b - ctr) * cnj (b - ctr) -
          (a - ctr) * cnj (a - ctr))) = 0
    using hcj by (metis eq-minus-cnj-iff-imag)
  show ?thesis
    unfolding complex-perpendicular-def
    using eq hzero
    by (simp only: eq hzero)
qed
show ?thesis using alt-a alt-b alt-c by blast
qed

lemma steiner-reflect-unfold:
  assumes a ≠ b
  shows steiner-reflect a b m = a + (b - a) * cnj (m - a) / cnj (b - a)
  using assms by (simp add: steiner-reflect-def foot-def field-simps)

lemma collinear-homothety:
  assumes collinear p q r
  shows collinear (2 * p - m) (2 * q - m) (2 * r - m)
  using assms by (simp add: collinear-iff-cross algebra-simps)

lemma im-zero-of-cnj-eq:
  assumes cnj z = z
  shows Im z = 0
  using assms by (metis Reals-cnj-iff complex-is-Real-iff)

```

4 The circle formula for a reflected point

This is the only field calculation needed to place the reflected point in circumcentre coordinates.

```

lemma steiner-reflect-ratio-gen:
  fixes B C M k :: complex
  assumes B0: B ≠ 0 and C0: C ≠ 0 and M0: M ≠ 0
    and BC: B ≠ C and k0: k ≠ 0
  shows (C - B) * (k / M - k / B) / (k / C - k / B)
    = C * (M - B) / M
  using assms by (simp add: divide-simps) algebra

```

```

lemma steiner-reflect-formula:
  fixes b c m ctr :: complex and r :: real
  assumes hb: oncircle ctr r b and hc: oncircle ctr r c

```

and hm : *oncircle* ctr r m **and** pos : $0 < r$ **and** bc : $b \neq c$
shows *steiner-reflect* b c m =

$$ctr + (b - ctr) + (c - ctr) -$$

$$(b - ctr) * (c - ctr) / (m - ctr)$$
proof –
define B **where** $B = b - ctr$
define C **where** $C = c - ctr$
define M **where** $M = m - ctr$
define k **where** $k = \text{complex-of-real } (r^2)$
have $B0$: $B \neq 0$
using *oncircle-ne-centre*[OF hb pos] **by** (*simp add*: $B\text{-def}$)
have $C0$: $C \neq 0$
using *oncircle-ne-centre*[OF hc pos] **by** (*simp add*: $C\text{-def}$)
have $M0$: $M \neq 0$
using *oncircle-ne-centre*[OF hm pos] **by** (*simp add*: $M\text{-def}$)
have BC : $B \neq C$ **using** bc **by** (*simp add*: $B\text{-def}$ $C\text{-def}$)
have $k0$: $k \neq 0$ **using** pos **by** (*simp add*: $k\text{-def}$)
have cc : $cnj (c - ctr) = k / C$
using *cnj-diff-on-circle*[OF hc pos] **by** (*simp add*: $C\text{-def}$ $k\text{-def}$)
have bb : $cnj (b - ctr) = k / B$
using *cnj-diff-on-circle*[OF hb pos] **by** (*simp add*: $B\text{-def}$ $k\text{-def}$)
have mm : $cnj (m - ctr) = k / M$
using *cnj-diff-on-circle*[OF hm pos] **by** (*simp add*: $M\text{-def}$ $k\text{-def}$)
have $diff$: $c - b = C - B$ **by** (*simp add*: $B\text{-def}$ $C\text{-def}$)
have cb : $cnj (c - b) = k / C - k / B$
proof –
have $cnj (c - b) = cnj (c - ctr) - cnj (b - ctr)$
by (*simp only*: *complex-cnj-diff*; *algebra*)
also have $\dots = k / C - k / B$ **by** (*simp only*: cc bb)
finally show *?thesis* .
qed
have $cb\text{-shift}$: $cnj (C - B) = k / C - k / B$
using $diff$ **by** (*simp only*: $diff$)
have mb : $cnj (m - b) = k / M - k / B$
proof –
have $cnj (m - b) = cnj (m - ctr) - cnj (b - ctr)$
by (*simp only*: *complex-cnj-diff*; *algebra*)
also have $\dots = k / M - k / B$ **by** (*simp only*: mm bb)
finally show *?thesis* .
qed
have $ratio$:

$$(c - b) * cnj (m - b) / cnj (c - b) = C * (M - B) / M$$
proof –
have gen :

$$(C - B) * (k / M - k / B) / (k / C - k / B)$$

$$= C * (M - B) / M$$
by (*rule steiner-reflect-ratio-gen*[OF $B0$ $C0$ $M0$ BC $k0$])
have $step$:

$$(c - b) * cnj (m - b) / cnj (c - b) =$$

```

      (C - B) * (k / M - k / B) / (k / C - k / B)
    by (simp only: cb cb-shift mb diff)
  show ?thesis
    using step gen by (simp only: step gen)
qed
have hu: steiner-reflect b c m = b + (c - b) * cnj (m - b) / cnj (c - b)
  by (rule steiner-reflect-unfold[OF bc])
have hsteiner:
  steiner-reflect b c m = b + C * (M - B) / M
  using hu ratio by (simp add: hu ratio)
have frac: C * (M - B) / M = C - B * C / M
  using M0 by (simp add: field-simps M0 algebra-simps)
have hcalc: b + C * (M - B) / M = ctr + B + C - B * C / M
  using frac by (simp add: B-def algebra-simps)
show ?thesis
  using hsteiner hcalc by (simp add: B-def C-def M-def)
qed

```

5 The orthocenter incidence calculation

```

lemma steiner-orthocenter-cross-gen:
  fixes A B C M k :: complex
  assumes A0: A ≠ 0 and B0: B ≠ 0 and C0: C ≠ 0
    and M0: M ≠ 0
    and hA: A * cnj A = k and hB: B * cnj B = k
    and hC: C * cnj C = k and hM: M * cnj M = k
  shows cnj (C + A - C * A / M - (A + B + C)) *
    (B + C - B * C / M - (A + B + C)) =
    (C + A - C * A / M - (A + B + C)) *
    cnj (B + C - B * C / M - (A + B + C))
proof -
  have cA: cnj A = k / A using hA A0 by (simp add: field-simps)
  have cB: cnj B = k / B using hB B0 by (simp add: field-simps)
  have cC: cnj C = k / C using hC C0 by (simp add: field-simps)
  have cM: cnj M = k / M using hM M0 by (simp add: field-simps)
  show ?thesis
    using A0 B0 C0 M0
    by (simp add: cA cB cC cM complex-cnj-diff complex-cnj-mult
      complex-cnj-divide divide-simps field-simps)
qed

```

```

lemma steiner-orthocenter-collinear:
  fixes a b c m ctr :: complex and r :: real
  assumes hncol: ¬ collinear a b c
    and ha: oncircle ctr r a and hb: oncircle ctr r b
    and hc: oncircle ctr r c and hm: oncircle ctr r m
    and pos: 0 < r
  shows collinear (orthocenter a b c ctr)
    (steiner-reflect b c m) (steiner-reflect c a m)

```

```

proof –
  have dist:  $a \neq b \wedge a \neq c \wedge b \neq c$ 
    using ncol-distinct[OF hncol] .
  have ab:  $a \neq b$  and ac:  $a \neq c$  and bc:  $b \neq c$ 
    using dist by auto
  have ca:  $c \neq a$  using ac by simp
  define A where  $A = a - ctr$ 
  define B where  $B = b - ctr$ 
  define C where  $C = c - ctr$ 
  define M where  $M = m - ctr$ 
  define k where  $k = \text{complex-of-real } (r^2)$ 
  have A0:  $A \neq 0$ 
    using oncircle-ne-centre[OF ha pos] by (simp add: A-def)
  have B0:  $B \neq 0$ 
    using oncircle-ne-centre[OF hb pos] by (simp add: B-def)
  have C0:  $C \neq 0$ 
    using oncircle-ne-centre[OF hc pos] by (simp add: C-def)
  have M0:  $M \neq 0$ 
    using oncircle-ne-centre[OF hm pos] by (simp add: M-def)
  have hA:  $A * \text{cnj } A = k$ 
    using oncircle-norm-sq[OF ha] by (simp add: A-def k-def)
  have hB:  $B * \text{cnj } B = k$ 
    using oncircle-norm-sq[OF hb] by (simp add: B-def k-def)
  have hC:  $C * \text{cnj } C = k$ 
    using oncircle-norm-sq[OF hc] by (simp add: C-def k-def)
  have hM:  $M * \text{cnj } M = k$ 
    using oncircle-norm-sq[OF hm] by (simp add: M-def k-def)
  have ra: steiner-reflect b c m =
     $ctr + B + C - B * C / M$ 
    using steiner-reflect-formula[OF hb hc hm pos bc]
    by (simp add: B-def C-def M-def)
  have rb: steiner-reflect c a m =
     $ctr + C + A - C * A / M$ 
    using steiner-reflect-formula[OF hc ha hm pos ca]
    by (simp add: A-def C-def M-def)
  have cross:
     $\text{cnj } ((C + A - C * A / M) - (A + B + C)) * ((B + C - B * C / M) - (A + B + C)) =$ 
     $((C + A - C * A / M) - (A + B + C)) * \text{cnj } ((B + C - B * C / M) - (A + B + C))$ 
    by (rule steiner-orthocenter-cross-gen[OF A0 B0 C0 M0 hA hB hC hM])
  have cross':
     $\text{cnj } (\text{steiner-reflect } c a m - \text{orthocenter } a b c ctr) * (\text{steiner-reflect } b c m - \text{orthocenter } a b c ctr) =$ 
     $(\text{steiner-reflect } c a m - \text{orthocenter } a b c ctr) * \text{cnj } (\text{steiner-reflect } b c m - \text{orthocenter } a b c ctr)$ 
    using cross
  by (simp add: ra rb orthocenter-def A-def B-def C-def M-def algebra-simps)
  show ?thesis

```

```

proof –
  have self:
    cnj (cnj (steiner-reflect c a m – orthocenter a b c ctr) *
      (steiner-reflect b c m – orthocenter a b c ctr)) =
    cnj (steiner-reflect c a m – orthocenter a b c ctr) *
      (steiner-reflect b c m – orthocenter a b c ctr)
  using cross' by (simp add: complex-cnj-mult)
  have self':
    cnj (cnj (steiner-reflect b c m – orthocenter a b c ctr) *
      (steiner-reflect c a m – orthocenter a b c ctr)) =
    cnj (steiner-reflect b c m – orthocenter a b c ctr) *
      (steiner-reflect c a m – orthocenter a b c ctr)
  using self by (simp add: complex-cnj-mult algebra-simps)
  have hIm:
    Im (cnj (steiner-reflect b c m – orthocenter a b c ctr) *
      (steiner-reflect c a m – orthocenter a b c ctr)) = 0
  using self' by (rule im-zero-of-cnj-eq)
  show ?thesis using hIm by (simp only: collinear-iff-cross)
qed
qed

```

6 Steiner's line theorem

The reflections in the three sidelines are collinear.

```

theorem steiner-line:
  fixes a b c m ctr :: complex and r :: real
  assumes hncol:  $\neg$  collinear a b c
    and ha: oncircle ctr r a and hb: oncircle ctr r b
    and hc: oncircle ctr r c and hm: oncircle ctr r m
    and pos:  $0 < r$ 
  shows collinear (steiner-reflect b c m)
    (steiner-reflect c a m) (steiner-reflect a b m)
proof –
  have hsim: collinear (foot b c m) (foot c a m) (foot a b m)
    by (rule simson-line[OF hncol ha hb hc hm pos])
  have hhom:
    collinear ( $2 * \text{foot } b \ c \ m - m$ ) ( $2 * \text{foot } c \ a \ m - m$ )
      ( $2 * \text{foot } a \ b \ m - m$ )
    by (rule collinear-homothety[OF hsim])
  thus ?thesis by (simp add: steiner-reflect-def)
qed

```

The Steiner line passes through the orthocenter.

```

theorem steiner-line-through-orthocenter:
  fixes a b c m ctr :: complex and r :: real
  assumes hncol:  $\neg$  collinear a b c
    and ha: oncircle ctr r a and hb: oncircle ctr r b
    and hc: oncircle ctr r c and hm: oncircle ctr r m

```

```

and pos:  $0 < r$ 
shows collinear (orthocenter a b c ctr)
  (steiner-reflect b c m) (steiner-reflect c a m)  $\wedge$ 
  collinear (orthocenter a b c ctr)
  (steiner-reflect c a m) (steiner-reflect a b m)  $\wedge$ 
  collinear (orthocenter a b c ctr)
  (steiner-reflect a b m) (steiner-reflect b c m)
proof –
  have hncol-bca:  $\neg$  collinear b c a
    using hncol by (metis collinear-cycle)
  have hncol-cab:  $\neg$  collinear c a b
    using hncol by (metis collinear-cycle)
  have hbc-ca:
    collinear (orthocenter a b c ctr)
    (steiner-reflect b c m) (steiner-reflect c a m)
    by (rule steiner-orthocenter-collinear[OF hncol ha hb hc hm pos])
  have hca-ab':
    collinear (orthocenter b c a ctr)
    (steiner-reflect c a m) (steiner-reflect a b m)
    by (rule steiner-orthocenter-collinear[OF hncol-bca hb hc ha hm pos])
  have hca-ab:
    collinear (orthocenter a b c ctr)
    (steiner-reflect c a m) (steiner-reflect a b m)
    using hca-ab'
    by (simp add: orthocenter-def add.commute add.left-commute add.assoc)
  have hab-bc':
    collinear (orthocenter c a b ctr)
    (steiner-reflect a b m) (steiner-reflect b c m)
    by (rule steiner-orthocenter-collinear[OF hncol-cab hc ha hb hm pos])
  have hab-bc:
    collinear (orthocenter a b c ctr)
    (steiner-reflect a b m) (steiner-reflect b c m)
    using hab-bc'
    by (simp add: orthocenter-def add.commute add.left-commute add.assoc)
  show ?thesis using hbc-ca hca-ab hab-bc by blast
qed

end

```

References

- [1] H. S. M. Coxeter and S. L. Greitzer. *Geometry Revisited*. Mathematical Association of America, 1967. DOI: <https://doi.org/10.5948/UPO9780883859346>.
- [2] J.-P. Ehrmann. Steiner’s theorems on the complete quadrilateral. *Forum Geometricorum*, 4:35–52, 2004.

- [3] A. F. Ramos, D. B. Hulak, and R. J. G. B. de Queiroz. The wallace–simson line theorem in isabelle/hol. *Archive of Formal Proofs*, August 2026. <https://isa-afp.org/entries/Simson.html>, Formal proof development.
- [4] J. Steiner. Einige geometrischen betrachtungen. *Journal für die reine und angewandte Mathematik*, 1:161–184, 1826. Available at <https://eudml.org/doc/183024>.