

The Steiner Deltoid as the Tangent Envelope of Wallace–Simson Lines in Isabelle/HOL

Arthur Freitas Ramos David Barros Hulak
Ruy J. G. B. de Queiroz

October 6, 2026

Abstract

We formalize the classical Steiner deltoid construction in Isabelle/HOL using only the published Archive of Formal Proofs entry for the Wallace–Simson line. In unit-circumcircle coordinates, the deltoid has an explicit complex parametrisation. We prove its derivative formula, characterize the stationary parameters by $m^3 = abc$, and certify them as ordinary cusps using the second and third derivatives. We then prove an equality of sets between the appropriate Wallace–Simson line and the tangent line, for every parameter, including the cusp cases.

The development uses complex coordinates and handles a vertex parameter by cycling to the alternate pair of feet.

Contents

1	Introduction	1
2	Formal setup	2
3	Main result	2
4	The deltoid parametrisation	3

1 Introduction

The Steiner deltoid is the envelope of the Wallace–Simson lines of a fixed triangle as the point on its circumcircle varies. This classical relationship was treated early by Butchart [2] and given a later elementary proof by de Guzmán [4]; computational treatments of envelopes include [1]. The construction is especially transparent after placing the circumcentre at the origin and scaling the circumcircle to the unit circle. If the vertex coordinates are a, b, c and the moving point is m , the deltoid parametrisation

is

$$d(m) = \frac{a + b + c}{2} + m + \frac{abc \overline{m}^2}{2}.$$

The underlying Wallace–Simson theorem is the classical line theorem for the three perpendicular feet [6, 3]; its Isabelle/HOL formalization is available as the published AFP entry [5].

This entry deliberately depends only on the published AFP entry *The Wallace–Simson Line Theorem in Isabelle/HOL*. In particular, the perpendicular feet, circle identities, and line interface are imported from that entry rather than copied from a local development.

2 Formal setup

The definition *steiner_deltoid_point* gives the displayed parametrisation, while *steiner_deltoid* collects its values as m runs over the unit circle. For differential statements, the theory introduces *unit_circle_param* and the real curve *steiner_deltoid_curve*. Its derivative is

$$\gamma'(t) = i N(e^{it}), \quad N(m) = m - abc \overline{m}^2.$$

The tangent direction is defined piecewise: it is $iN(m)$ when $N(m) \neq 0$, and m when $N(m) = 0$. At a stationary parameter the second derivative is $-3m$, so this chosen direction is the same tangent direction. The proof uses the closed form for a perpendicular foot and the published identity for the difference of two feet on a common circle.

3 Main result

The derivative theorem gives $\gamma'(t) = iN(e^{it})$. On the unit circle, the normal vanishes exactly when $m^3 = abc$, and *steiner_deltoid_stationary_parameter_is_ordinary_cusp* proves that the stationary parameters have nonzero, real-linearly independent second and third derivatives. The development separately proves that these two derivatives are perpendicular, a stronger property in this particular curve. Thus the formal cusp claim uses the standard independence criterion rather than merely identifying zeros of the first derivative.

The central theorem *simson_line_eq_steiner_deltoid_tangent_at_all* states the result as an equality of sets. If $m \neq b$, the line through the AB and BC feet is exactly the tangent at $d(m)$; if $m = b$, the line through the BC and CA feet is exactly that tangent. The proof has separate regular and stationary branches. In the regular branch the Simson direction is perpendicular to $N(m)$; in the stationary branch the real ratio of the two feet differences

shows that the line has direction m , the direction of the first nonzero derivative. The parameter-level theorem *simson_line_eq_steiner_deltoid_tangent_all* transports this equality to the real parameter t .

At a stationary parameter, the theorem *steiner_deltoid_cusp_tangent_is_second_derivative* explicitly identifies the piecewise-defined cusp tangent with the line in the direction of $\gamma''(t) = -3m$. The older foot-membership results are named *simson_line_is_tangent_to_steiner_deltoid_at_regular* and *simson_line_is_tangent_to_steiner_deltoid_regular* to make their non-stationary hypotheses visible in the public API.

```

theory Steiner-Deltoid
  imports Simson.Simson HOL-Analysis.Derivative
begin

hide-const (open) Linear-Algebra.collinear

```

4 The deltoid parametrisation

We use circumcentre coordinates with the circumcircle as the unit circle. In these coordinates the classical parametrisation of the Steiner deltoid is

$$d(m) = (a + b + c)/2 + m + abc\overline{m}^2/2.$$

The unit-circle hypothesis turns the last term into the more familiar $abc/(2m^2)$.

```

definition steiner-deltoid-point ::
  complex  $\Rightarrow$  complex  $\Rightarrow$  complex  $\Rightarrow$  complex  $\Rightarrow$  complex
where steiner-deltoid-point  $a\ b\ c\ m =$ 
   $(a + b + c) / 2 + m + a * b * c * (cnj\ m)^2 / 2$ 

```

```

definition steiner-deltoid ::
  complex  $\Rightarrow$  complex  $\Rightarrow$  complex  $\Rightarrow$  complex set
where steiner-deltoid  $a\ b\ c =$ 
   $\{steiner-deltoid-point\ a\ b\ c\ m \mid m. oncircle\ 0\ 1\ m\}$ 

```

```

definition steiner-deltoid-normal ::
  complex  $\Rightarrow$  complex  $\Rightarrow$  complex  $\Rightarrow$  complex  $\Rightarrow$  complex
where steiner-deltoid-normal  $a\ b\ c\ m = m - a * b * c * (cnj\ m)^2$ 

```

For the differential-geometric statements we use the standard real parameter on the unit circle. Written without conjugation, the curve is

$$\gamma(t) = (a + b + c)/2 + e^{it} + abc e^{-2it}/2.$$

At a regular parameter the tangent direction is $iN(m)$. At a stationary parameter we use the first nonzero direction, namely the second derivative $-3m$.

definition *unit-circle-param* :: *real* $\Rightarrow$ *complex*
where *unit-circle-param* *t* = *exp* (*t* *_R *i*)

definition *steiner-deltoid-curve* ::
complex $\Rightarrow$ *complex* $\Rightarrow$ *complex* $\Rightarrow$ *real* $\Rightarrow$ *complex*
where *steiner-deltoid-curve* *a b c t* =
 $(a + b + c) / 2 + \exp (t *_{\mathbf{R}} i) +$
 $a * b * c * (\text{cnj} (\exp (t *_{\mathbf{R}} i)))^2 / 2$

definition *steiner-deltoid-tangent-direction-at* ::
complex $\Rightarrow$ *complex* $\Rightarrow$ *complex* $\Rightarrow$ *complex* $\Rightarrow$ *complex*
where *steiner-deltoid-tangent-direction-at* *a b c m* =
(if *steiner-deltoid-normal* *a b c m* = 0 then *m*
else *i* * *steiner-deltoid-normal* *a b c m*)

definition *steiner-deltoid-tangent* ::
complex $\Rightarrow$ *complex* $\Rightarrow$ *complex* $\Rightarrow$ *real* $\Rightarrow$ *complex set*
where *steiner-deltoid-tangent* *a b c t* =
{*z*. *on-line* (*steiner-deltoid-curve* *a b c t*)
(*steiner-deltoid-curve* *a b c t* +
steiner-deltoid-tangent-direction-at *a b c*
(*unit-circle-param* *t*)) *z*}

definition *steiner-deltoid-tangent-at* ::
complex $\Rightarrow$ *complex* $\Rightarrow$ *complex* $\Rightarrow$ *complex* $\Rightarrow$ *complex set*
where *steiner-deltoid-tangent-at* *a b c m* =
{*z*. *on-line* (*steiner-deltoid-point* *a b c m*)
(*steiner-deltoid-point* *a b c m* +
steiner-deltoid-tangent-direction-at *a b c m*) *z*}

lemma *unit-circle-param-oncircle*:
shows *oncircle* 0 1 (*unit-circle-param* *t*)
unfolding *unit-circle-param-def* *oncircle-def*
by *simp*

lemma *steiner-deltoid-curve-eq-point*:
shows *steiner-deltoid-curve* *a b c t* =
steiner-deltoid-point *a b c* (*unit-circle-param* *t*)
unfolding *steiner-deltoid-curve-def* *steiner-deltoid-point-def*
unit-circle-param-def
by *simp*

lemma *steiner-deltoid-tangent-eq-tangent-at*:
shows *steiner-deltoid-tangent* *a b c t* =
steiner-deltoid-tangent-at *a b c* (*unit-circle-param* *t*)
unfolding *steiner-deltoid-tangent-def* *steiner-deltoid-tangent-at-def*
by (*simp* *add: steiner-deltoid-curve-eq-point*)

lemma *steiner-deltoid-curve-has-vector-derivative*:

shows $((\lambda t. \text{ steiner-deltoid-curve } a \ b \ c \ t)$
 $\text{ has-vector-derivative }$
 $\text{ i } * \text{ steiner-deltoid-normal } a \ b \ c \ (\text{ unit-circle-param } t)) \ (\text{ at } t)$
proof –
have $h1$:
 $((\lambda t. \text{ exp } (t *_{\mathbb{R}} i))$
 $\text{ has-vector-derivative } \text{ exp } (t *_{\mathbb{R}} i) * i) \ (\text{ at } t)$
by $(\text{ rule } \text{ exp-scaleR-has-vector-derivative-right})$
have $hconj$:
 $((\lambda t. \text{ cnj } (\text{ exp } (t *_{\mathbb{R}} i)))$
 $\text{ has-vector-derivative } \text{ cnj } (\text{ exp } (t *_{\mathbb{R}} i) * i)) \ (\text{ at } t)$
by $(\text{ rule } \text{ has-vector-derivative-cnj}[OF \ h1])$
have $hsquare$:
 $((\lambda t. \text{ cnj } (\text{ exp } (t *_{\mathbb{R}} i)) * \text{ cnj } (\text{ exp } (t *_{\mathbb{R}} i)))$
 $\text{ has-vector-derivative }$
 $\text{ cnj } (\text{ exp } (t *_{\mathbb{R}} i)) * \text{ cnj } (\text{ exp } (t *_{\mathbb{R}} i) * i) +$
 $\text{ cnj } (\text{ exp } (t *_{\mathbb{R}} i) * i) * \text{ cnj } (\text{ exp } (t *_{\mathbb{R}} i))) \ (\text{ at } t)$
by $(\text{ rule } \text{ has-vector-derivative-mult}[OF \ hconj \ hconj])$
have $hmul$:
 $((\lambda t. a * b * c *$
 $\text{ cnj } (\text{ exp } (t *_{\mathbb{R}} i)) * \text{ cnj } (\text{ exp } (t *_{\mathbb{R}} i))))$
 $\text{ has-vector-derivative }$
 $a * b * c *$
 $\text{ cnj } (\text{ exp } (t *_{\mathbb{R}} i)) * \text{ cnj } (\text{ exp } (t *_{\mathbb{R}} i) * i) +$
 $\text{ cnj } (\text{ exp } (t *_{\mathbb{R}} i) * i) * \text{ cnj } (\text{ exp } (t *_{\mathbb{R}} i))) \ (\text{ at } t)$
by $(\text{ rule } \text{ has-vector-derivative-mult-right}[OF \ hsquare])$
have $hsecond$:
 $((\lambda t. a * b * c * (\text{ cnj } (\text{ exp } (t *_{\mathbb{R}} i)))^2 / 2)$
 $\text{ has-vector-derivative }$
 $(a * b * c *$
 $\text{ cnj } (\text{ exp } (t *_{\mathbb{R}} i)) * \text{ cnj } (\text{ exp } (t *_{\mathbb{R}} i) * i) +$
 $\text{ cnj } (\text{ exp } (t *_{\mathbb{R}} i) * i) * \text{ cnj } (\text{ exp } (t *_{\mathbb{R}} i)))) / 2) \ (\text{ at } t)$
using $\text{ has-vector-derivative-divide}[\text{ where } a = 2::\text{ complex}, OF \ hmul]$
by $(\text{ simp } \text{ add: power2-eq-square})$
have $hadd$:
 $((\lambda t. (a + b + c) / 2 + \text{ exp } (t *_{\mathbb{R}} i))$
 $\text{ has-vector-derivative }$
 $0 + \text{ exp } (t *_{\mathbb{R}} i) * i) \ (\text{ at } t)$
by $(\text{ rule } \text{ has-vector-derivative-add}[OF \ \text{ has-vector-derivative-const } h1])$
have $hsum0$:
 $((\lambda t. ((a + b + c) / 2 + \text{ exp } (t *_{\mathbb{R}} i)) +$
 $a * b * c * (\text{ cnj } (\text{ exp } (t *_{\mathbb{R}} i)))^2 / 2)$
 $\text{ has-vector-derivative }$
 $(0 + \text{ exp } (t *_{\mathbb{R}} i) * i) +$
 $(a * b * c *$
 $\text{ cnj } (\text{ exp } (t *_{\mathbb{R}} i)) * \text{ cnj } (\text{ exp } (t *_{\mathbb{R}} i) * i) +$
 $\text{ cnj } (\text{ exp } (t *_{\mathbb{R}} i) * i) * \text{ cnj } (\text{ exp } (t *_{\mathbb{R}} i)))) / 2) \ (\text{ at } t)$
by $(\text{ rule } \text{ has-vector-derivative-add}[OF \ hadd \ hsecond])$
have $hsum$:

```

((λt. (a + b + c) / 2 + exp (t *R i) +
  a * b * c * (cnj (exp (t *R i)))2 / 2)
  has-vector-derivative
  exp (t *R i) * i +
  (a * b * c *
    (cnj (exp (t *R i)) * cnj (exp (t *R i) * i) +
      cnj (exp (t *R i) * i) * cnj (exp (t *R i)))) / 2) (at t)
using hsum0 by (simp add: algebra-simps)
show ?thesis
using hsum
by (simp add: steiner-deltoid-curve-def steiner-deltoid-normal-def
  unit-circle-param-def power2-eq-square
  algebra-simps)
qed

```

definition *steiner-deltoid-curve-second-derivative* ::
 $\text{complex} \Rightarrow \text{complex} \Rightarrow \text{complex} \Rightarrow \text{real} \Rightarrow \text{complex}$
where *steiner-deltoid-curve-second-derivative* a b c t =
 - unit-circle-param t -
 2 * a * b * c * (cnj (unit-circle-param t))²

definition *steiner-deltoid-curve-third-derivative* ::
 $\text{complex} \Rightarrow \text{complex} \Rightarrow \text{complex} \Rightarrow \text{real} \Rightarrow \text{complex}$
where *steiner-deltoid-curve-third-derivative* a b c t =
 - i * unit-circle-param t +
 4 * i * a * b * c * (cnj (unit-circle-param t))²

definition *complex-real-linearly-independent* ::
 $\text{complex} \Rightarrow \text{complex} \Rightarrow \text{bool}$
where *complex-real-linearly-independent* u v $\longleftrightarrow$
 $\neg \text{collinear } 0 \ u \ v$

definition *ordinary-cusp-parameter* ::
 $\text{complex} \Rightarrow \text{complex} \Rightarrow \text{complex} \Rightarrow \text{real} \Rightarrow \text{bool}$
where *ordinary-cusp-parameter* a b c t $\longleftrightarrow$
 vector-derivative (λu. *steiner-deltoid-curve* a b c u) (at t) = 0
 $\wedge$ *steiner-deltoid-curve-second-derivative* a b c t $\neq 0$
 $\wedge$ *steiner-deltoid-curve-third-derivative* a b c t $\neq 0$
 $\wedge$ *complex-real-linearly-independent*
 (*steiner-deltoid-curve-second-derivative* a b c t)
 (*steiner-deltoid-curve-third-derivative* a b c t)

lemma *steiner-deltoid-curve-first-derivative-has-vector-derivative*:
shows ((λt. i *
steiner-deltoid-normal a b c (unit-circle-param t))
 has-vector-derivative
steiner-deltoid-curve-second-derivative a b c t) (at t)
proof -
have h1:

```

      ((λt. exp (t *R i))
        has-vector-derivative exp (t *R i) * i) (at t)
    by (rule exp-scaleR-has-vector-derivative-right)
  have hconj:
    ((λt. conj (exp (t *R i)))
      has-vector-derivative conj (exp (t *R i) * i)) (at t)
    by (rule has-vector-derivative-conj[OF h1])
  have hsquare:
    ((λt. conj (exp (t *R i)) *
      conj (exp (t *R i)))
      has-vector-derivative
      conj (exp (t *R i)) * conj (exp (t *R i) * i) +
      conj (exp (t *R i) * i) *
      conj (exp (t *R i))) (at t)
    by (rule has-vector-derivative-mult[OF hconj hconj])
  have hmul:
    ((λt. a * b * c *
      (conj (exp (t *R i)) * conj (exp (t *R i))))
      has-vector-derivative
      a * b * c *
      (conj (exp (t *R i)) * conj (exp (t *R i) * i) +
      conj (exp (t *R i) * i) *
      conj (exp (t *R i)))) (at t)
    by (rule has-vector-derivative-mult-right[OF hsquare])
  have hnormal:
    ((λt. exp (t *R i) - a * b * c *
      (conj (exp (t *R i)) * conj (exp (t *R i))))
      has-vector-derivative
      exp (t *R i) * i -
      a * b * c *
      (conj (exp (t *R i)) * conj (exp (t *R i) * i) +
      conj (exp (t *R i) * i) *
      conj (exp (t *R i)))) (at t)
    by (rule has-vector-derivative-diff[OF h1 hmul])
  have hi:
    ((λt. i *
      (exp (t *R i) - a * b * c *
      (conj (exp (t *R i)) * conj (exp (t *R i))))
      has-vector-derivative
      i *
      (exp (t *R i) * i -
      a * b * c *
      (conj (exp (t *R i)) * conj (exp (t *R i) * i) +
      conj (exp (t *R i) * i) *
      conj (exp (t *R i)))) (at t)
    by (rule has-vector-derivative-mult-right[OF hnormal])
  show ?thesis
  using hi
  by (simp add: steiner-deltoid-normal-def)

```

$\text{steiner-deltoid-curve-second-derivative-def}$
 $\text{unit-circle-param-def}$ power2-eq-square algebra-simps)
qed

lemma $\text{steiner-deltoid-curve-second-derivative-has-vector-derivative}$:

shows $((\lambda t. \text{steiner-deltoid-curve-second-derivative } a \ b \ c \ t)$
 $\text{has-vector-derivative}$
 $\text{steiner-deltoid-curve-third-derivative } a \ b \ c \ t) \ (at \ t)$

proof –

have $h1$:

$((\lambda t. \text{exp } (t *_{\mathbb{R}} i))$
 $\text{has-vector-derivative } \text{exp } (t *_{\mathbb{R}} i) * i) \ (at \ t)$
by $(\text{rule } \text{exp-scaleR-has-vector-derivative-right})$

have $hconj$:

$((\lambda t. \text{cnj } (\text{exp } (t *_{\mathbb{R}} i)))$
 $\text{has-vector-derivative } \text{cnj } (\text{exp } (t *_{\mathbb{R}} i) * i)) \ (at \ t)$
by $(\text{rule } \text{has-vector-derivative-cnj}[OF \ h1])$

have $hsquare$:

$((\lambda t. \text{cnj } (\text{exp } (t *_{\mathbb{R}} i)) *$
 $\text{cnj } (\text{exp } (t *_{\mathbb{R}} i)))$
 $\text{has-vector-derivative}$
 $\text{cnj } (\text{exp } (t *_{\mathbb{R}} i) * \text{cnj } (\text{exp } (t *_{\mathbb{R}} i) * i) +$
 $\text{cnj } (\text{exp } (t *_{\mathbb{R}} i) * i) * \text{cnj } (\text{exp } (t *_{\mathbb{R}} i))) \ (at \ t)$
by $(\text{rule } \text{has-vector-derivative-mult}[OF \ hconj \ hconj])$

have $hmul$:

$((\lambda t. 2 * a * b * c * \text{cnj } (\text{exp } (t *_{\mathbb{R}} i)) * \text{cnj } (\text{exp } (t *_{\mathbb{R}} i))))$
 $\text{has-vector-derivative}$
 $2 * a * b * c * (\text{cnj } (\text{exp } (t *_{\mathbb{R}} i)) * \text{cnj } (\text{exp } (t *_{\mathbb{R}} i) * i) +$
 $\text{cnj } (\text{exp } (t *_{\mathbb{R}} i) * i) * \text{cnj } (\text{exp } (t *_{\mathbb{R}} i)))) \ (at \ t)$
by $(\text{rule } \text{has-vector-derivative-mult-right}[OF \ hsquare])$

have $hneg$:

$((\lambda t. - \text{exp } (t *_{\mathbb{R}} i))$
 $\text{has-vector-derivative } -(\text{exp } (t *_{\mathbb{R}} i) * i)) \ (at \ t)$
by $(\text{rule } \text{has-vector-derivative-minus}[OF \ h1])$

have $hsum$:

$((\lambda t. - \text{exp } (t *_{\mathbb{R}} i) - 2 * a * b * c * \text{cnj } (\text{exp } (t *_{\mathbb{R}} i)) * \text{cnj } (\text{exp } (t *_{\mathbb{R}} i))))$
 $\text{has-vector-derivative}$
 $-(\text{exp } (t *_{\mathbb{R}} i) * i) -$
 $2 * a * b * c * (\text{cnj } (\text{exp } (t *_{\mathbb{R}} i)) * \text{cnj } (\text{exp } (t *_{\mathbb{R}} i) * i) +$
 $\text{cnj } (\text{exp } (t *_{\mathbb{R}} i) * i) * \text{cnj } (\text{exp } (t *_{\mathbb{R}} i)))) \ (at \ t)$
by $(\text{rule } \text{has-vector-derivative-diff}[OF \ hneg \ hmul])$

show $?thesis$

using *hsum*
by (*simp add: steiner-deltoid-curve-second-derivative-def*
steiner-deltoid-curve-third-derivative-def
unit-circle-param-def power2-eq-square algebra-simps)
qed

lemma *steiner-deltoid-point-mem*:
assumes *oncircle 0 1 m*
shows *steiner-deltoid-point a b c m ∈ steiner-deltoid a b c*
using *assms* **by** (*auto simp add: steiner-deltoid-def*)

lemma *unit-cn timer*:
assumes *oncircle 0 1 z*
shows *cnj z = 1 / z*
using *cnj-diff-on-circle[OF assms zero-less-one]* **by** *simp*

lemma *unit-nonzero*:
assumes *oncircle 0 1 z*
shows *z ≠ 0*
using *oncircle-ne-centre[OF assms zero-less-one]* **by** *simp*

lemma *steiner-deltoid-normal-eq-zero-iff-cube*:
assumes *hm: oncircle 0 1 m*
shows *steiner-deltoid-normal a b c m = 0 ⟷*
 $m^3 = a * b * c$
proof –
have *hcm: cnj m = 1 / m* **using** *unit-cn timer[OF hm]* .
have *hm0: m ≠ 0* **using** *unit-nonzero[OF hm]* .
show *?thesis*
unfolding *steiner-deltoid-normal-def*
using *hm0*
by (*simp add: hcm field-simps; algebra*)
qed

lemma *steiner-deltoid-normal-nonzero-iff-cube-neg*:
assumes *hm: oncircle 0 1 m*
shows *steiner-deltoid-normal a b c m ≠ 0 ⟷*
 $m^3 ≠ a * b * c$
using *steiner-deltoid-normal-eq-zero-iff-cube[OF hm]* **by** *blast*

theorem *steiner-deltoid-curve-derivative-zero-iff*:
shows *vector-derivative (λt. steiner-deltoid-curve a b c t) (at t) = 0*
 $⟷ (unit-circle-param t)^3 = a * b * c$
proof –
have *hderiv*:
 $vector-derivative (λt. steiner-deltoid-curve a b c t) (at t) =$
 $i * steiner-deltoid-normal a b c (unit-circle-param t)$
by (*rule vector-derivative-at*
[OF steiner-deltoid-curve-has-vector-derivative])

```

have hnormal:
  steiner-deltoid-normal a b c (unit-circle-param t) = 0
   $\longleftrightarrow$  (unit-circle-param t)  $^3$  = a * b * c
by (rule steiner-deltoid-normal-eq-zero-iff-cube
  [OF unit-circle-param-oncircle])
show ?thesis
  using hderiv hnormal by simp
qed

lemma steiner-deltoid-curve-second-derivative-stationary:
  fixes a b c :: complex and t :: real
  assumes hcube: (unit-circle-param t)  $^3$  = a * b * c
  shows steiner-deltoid-curve-second-derivative a b c t =
    -3 * unit-circle-param t
proof -
  have hm: oncircle 0 1 (unit-circle-param t)
  by (rule unit-circle-param-oncircle)
  have hnormal:
    steiner-deltoid-normal a b c (unit-circle-param t) = 0
  using steiner-deltoid-normal-eq-zero-iff-cube[OF hm] hcube by blast
  have hprod:
    a * (b * (c * (cnj (unit-circle-param t)) $^2$ )) =
      unit-circle-param t
  using hnormal by (simp add: steiner-deltoid-normal-def; algebra)
  show ?thesis
    unfolding steiner-deltoid-curve-second-derivative-def
    using hprod by (simp add: hprod; algebra)
qed

lemma steiner-deltoid-curve-third-derivative-stationary:
  fixes a b c :: complex and t :: real
  assumes hcube: (unit-circle-param t)  $^3$  = a * b * c
  shows steiner-deltoid-curve-third-derivative a b c t =
    3 * i * unit-circle-param t
proof -
  have hm: oncircle 0 1 (unit-circle-param t)
  by (rule unit-circle-param-oncircle)
  have hnormal:
    steiner-deltoid-normal a b c (unit-circle-param t) = 0
  using steiner-deltoid-normal-eq-zero-iff-cube[OF hm] hcube by blast
  have hprod:
    a * (b * (c * (cnj (unit-circle-param t)) $^2$ )) =
      unit-circle-param t
  using hnormal by (simp add: steiner-deltoid-normal-def; algebra)
  show ?thesis
    unfolding steiner-deltoid-curve-third-derivative-def
    using hprod by (simp add: hprod algebra-simps; algebra)
qed

```

```

lemma complex-perpendicular-cusp-directions:
  assumes hm: oncircle 0 1 m
  shows complex-perpendicular  $(-3 * m) (3 * i * m)$ 
proof -
  have hnorm:  $m * \text{cnj } m = 1$ 
    using oncircle-norm-sq[OF hm] by simp
  show ?thesis
    unfolding complex-perpendicular-def
    using hnorm by (simp add: algebra-simps)
qed

lemma complex-perpendicular-imp-real-linearly-independent:
  fixes u v :: complex
  assumes hperp: complex-perpendicular u v
    and hu:  $u \neq 0$  and hv:  $v \neq 0$ 
  shows complex-real-linearly-independent u v
proof -
  show ?thesis
    unfolding complex-real-linearly-independent-def
  proof
    assume hcol: collinear 0 u v
    have him:
       $\text{Im } (\text{cnj } u * v) = 0$ 
      using collinear-iff-cross[of 0 u v] hcol by simp
    have hRe:  $\text{Re } (\text{cnj } u * v) = 0$ 
      using hperp unfolding complex-perpendicular-def .
    have  $\text{cnj } u * v = 0$ 
      using hRe him by (simp add: complex-eq-iff)
    then show False using hu hv by simp
  qed
qed

theorem steiner-deltoid-stationary-parameter-is-ordinary-cusp:
  fixes a b c :: complex and t :: real
  assumes hcube:  $(\text{unit-circle-param } t) ^ 3 = a * b * c$ 
  shows ordinary-cusp-parameter a b c t
proof -
  have hm: oncircle 0 1 (unit-circle-param t)
    by (rule unit-circle-param-oncircle)
  have hm0: unit-circle-param t  $\neq 0$ 
    using unit-nonzero[OF hm] .
  have hzero:
     $\text{vector-derivative } (\lambda u. \text{steiner-deltoid-curve } a b c u)$ 
       $(\text{at } t) = 0$ 
    using steiner-deltoid-curve-derivative-zero-iff[of a b c t]
      hcube by blast
  have hsecond:
     $\text{steiner-deltoid-curve-second-derivative } a b c t \neq 0$ 
    using steiner-deltoid-curve-second-derivative-stationary[OF hcube] hm0

```

```

    by simp
  have hthird:
    steiner-deltoid-curve-third-derivative a b c t ≠ 0
  using steiner-deltoid-curve-third-derivative-stationary[OF hcube] hm0
  by simp
  have hperp:
    complex-perpendicular
      (steiner-deltoid-curve-second-derivative a b c t)
      (steiner-deltoid-curve-third-derivative a b c t)
  using complex-perpendicular-cusp-directions[OF hm]
    steiner-deltoid-curve-second-derivative-stationary[OF hcube]
    steiner-deltoid-curve-third-derivative-stationary[OF hcube]
  by simp
  have hind:
    complex-real-linearly-independent
      (steiner-deltoid-curve-second-derivative a b c t)
      (steiner-deltoid-curve-third-derivative a b c t)
  by (rule complex-perpendicular-imp-real-linearly-independent
    [OF hperp hsecond hthird])
  show ?thesis
    unfolding ordinary-cusp-parameter-def
    using hzero hsecond hthird hind by blast
qed

```

```

lemma foot-unit:
  fixes a b m :: complex
  assumes ha: oncircle 0 1 a and hb: oncircle 0 1 b
    and ab: a ≠ b
  shows foot a b m = (a + b + m - a * b * cnj m) / 2
proof -
  have h:
    foot a b m =
      0 + (a - 0 + (m - 0)) / 2 +
      (b - 0) * (complex-of-real (12) -
        cnj (m - 0) * (a - 0)) /
      (2 * complex-of-real (12))
  by (rule foot-general[OF ha hb zero-less-one ab])
  have foot a b m =
    0 + (a - 0 + (m - 0)) / 2 +
    (b - 0) * (complex-of-real (12) -
      cnj (m - 0) * (a - 0)) /
    (2 * complex-of-real (12))
  using h .
  also have ... = (a + b + m - a * b * cnj m) / 2
  by (simp add: algebra-simps; algebra)
  finally show ?thesis .
qed

```

```

lemma deltoid-point-minus-foot:

```

```

fixes  $a\ b\ c\ m :: \text{complex}$ 
assumes  $ha: \text{oncircle } 0\ 1\ a$  and  $hb: \text{oncircle } 0\ 1\ b$ 
and  $hc: \text{oncircle } 0\ 1\ c$  and  $hm: \text{oncircle } 0\ 1\ m$ 
and  $ab: a \neq b$ 
shows  $\text{steiner-deltoid-point } a\ b\ c\ m - \text{foot } a\ b\ m =$ 
 $(c + m + a * b * \text{cnj } m + a * b * c * (\text{cnj } m)^2) / 2$ 
using  $\text{foot-unit}[OF\ ha\ hb\ ab]$ 
by ( $\text{simp add: steiner-deltoid-point-def algebra-simps}$ )

lemma feet-difference-unit:
fixes  $a\ b\ c\ m :: \text{complex}$ 
assumes  $ha: \text{oncircle } 0\ 1\ a$  and  $hb: \text{oncircle } 0\ 1\ b$ 
and  $hc: \text{oncircle } 0\ 1\ c$  and  $hm: \text{oncircle } 0\ 1\ m$ 
and  $ab: a \neq b$  and  $bc: b \neq c$ 
shows  $\text{foot } b\ c\ m - \text{foot } a\ b\ m =$ 
 $(c - a) * (1 - b * \text{cnj } m) / 2$ 
proof -
have  $hd:$ 
 $\text{foot } b\ c\ m - \text{foot } a\ b\ m =$ 
 $(c - a) * (m - b) / (2 * (m - 0))$ 
by ( $\text{rule foot-diff-PR-circ}[OF\ ha\ hb\ hc\ hm\ \text{zero-less-one } ab\ bc]$ )
have  $hp:$ 
 $\text{foot } b\ c\ m = (b + c + m - b * c * \text{cnj } m) / 2$ 
by ( $\text{rule foot-unit}[OF\ hb\ hc\ bc]$ )
have  $hq:$ 
 $\text{foot } a\ b\ m = (a + b + m - a * b * \text{cnj } m) / 2$ 
by ( $\text{rule foot-unit}[OF\ ha\ hb\ ab]$ )
show  $?thesis$  using  $hd\ hp\ hq$  by ( $\text{simp add: algebra-simps}$ )
qed

```

```

lemma cusped-feet-difference-ratio-real:
fixes  $a\ b\ c\ m :: \text{complex}$ 
assumes  $ha: \text{oncircle } 0\ 1\ a$  and  $hb: \text{oncircle } 0\ 1\ b$ 
and  $hc: \text{oncircle } 0\ 1\ c$  and  $hm: \text{oncircle } 0\ 1\ m$ 
and  $ab: a \neq b$  and  $bc: b \neq c$ 
and  $hcube: m^3 = a * b * c$ 
shows  $(\text{foot } b\ c\ m - \text{foot } a\ b\ m) / m \in \mathbb{R}$ 
proof -
have  $hca: \text{cnj } a = 1 / a$  using  $\text{unit-cnj}[OF\ ha]$  .
have  $hcb: \text{cnj } b = 1 / b$  using  $\text{unit-cnj}[OF\ hb]$  .
have  $hcc: \text{cnj } c = 1 / c$  using  $\text{unit-cnj}[OF\ hc]$  .
have  $hcm: \text{cnj } m = 1 / m$  using  $\text{unit-cnj}[OF\ hm]$  .
have  $ha0: a \neq 0$  using  $\text{unit-nonzero}[OF\ ha]$  .
have  $hb0: b \neq 0$  using  $\text{unit-nonzero}[OF\ hb]$  .
have  $hc0: c \neq 0$  using  $\text{unit-nonzero}[OF\ hc]$  .
have  $hm0: m \neq 0$  using  $\text{unit-nonzero}[OF\ hm]$  .
have  $hfeet:$ 
 $\text{foot } b\ c\ m - \text{foot } a\ b\ m =$ 
 $(c - a) * (m - b) / (2 * (m - 0))$ 

```

by (rule foot-diff-PR-circ[OF ha hb hc hm zero-less-one ab bc])
 have hratio:

$$\frac{(\text{foot } b \ c \ m - \text{foot } a \ b \ m)}{m} = \frac{(c - a) * (m - b)}{(2 * (m * m))}$$

 using hfeet hm0
 by (simp add: power2-eq-square field-simps; algebra)
 have hconj-num:

$$\text{cnj } ((c - a) * (m - b)) = \frac{(c - a) * (m - b)}{(a * b * c * m)}$$

 using hca hcb hcc hcm ha0 hb0 hc0 hm0
 by (simp add: complex-cnj-mult field-simps; algebra)
 have hconj-ratio:

$$\text{cnj } ((c - a) * (m - b) / (2 * (m * m))) = \frac{(c - a) * (m - b)}{(2 * (m * m))}$$

 using hconj-num hcm ha0 hb0 hc0 hm0 hcube
 by (simp add: complex-cnj-divide complex-cnj-mult field-simps; algebra)
 have hreal:

$$\text{cnj } ((\text{foot } b \ c \ m - \text{foot } a \ b \ m) / m) = \frac{(\text{foot } b \ c \ m - \text{foot } a \ b \ m)}{m}$$

 using hratio hconj-ratio by simp
 show ?thesis using hreal by (simp add: Reals-cnj-iff)
 qed

lemma deltoid-cross-identity:

fixes a b c m :: complex
 assumes ha: oncircle 0 1 a and hb: oncircle 0 1 b
 and hc: oncircle 0 1 c and hm: oncircle 0 1 m
 shows

$$\begin{aligned} & \text{cnj } ((c + m + a * b * \text{cnj } m + \\ & \quad a * b * c * (\text{cnj } m)^2) / 2) * \\ & ((c - a) * (1 - b * \text{cnj } m) / 2) = \\ & ((c + m + a * b * \text{cnj } m + \\ & \quad a * b * c * (\text{cnj } m)^2) / 2) * \\ & \text{cnj } ((c - a) * (1 - b * \text{cnj } m) / 2) \end{aligned}$$

 proof -
 have hca: cnj a = 1 / a using unit-cnj[OF ha] .
 have hcb: cnj b = 1 / b using unit-cnj[OF hb] .
 have hcc: cnj c = 1 / c using unit-cnj[OF hc] .
 have hcm: cnj m = 1 / m using unit-cnj[OF hm] .
 have ha0: a ≠ 0 using unit-nonzero[OF ha] .
 have hb0: b ≠ 0 using unit-nonzero[OF hb] .
 have hc0: c ≠ 0 using unit-nonzero[OF hc] .
 have hm0: m ≠ 0 using unit-nonzero[OF hm] .
 show ?thesis
 by (simp add: hca hcb hcc hcm ha0 hb0 hc0 hm0 field-simps; algebra)
 qed

lemma deltoid-point-collinear:

fixes a b c m :: complex

```

assumes hncol:  $\neg \text{collinear } a \ b \ c$ 
and ha: oncircle 0 1 a and hb: oncircle 0 1 b
and hc: oncircle 0 1 c and hm: oncircle 0 1 m
shows collinear (foot a b m)
      (steiner-deltoid-point a b c m) (foot b c m)
proof -
obtain ab:  $a \neq b$  and ac:  $a \neq c$  and bc:  $b \neq c$ 
using ncol-distinct[OF hncol] by blast
have hX:
  steiner-deltoid-point a b c m - foot a b m =
    (c + m + a * b * cnj m +
     a * b * c * (cnj m)2) / 2
by (rule deltoid-point-minus-foot[OF ha hb hc hm ab])
have hY:
  foot b c m - foot a b m =
    (c - a) * (1 - b * cnj m) / 2
by (rule feet-difference-unit[OF ha hb hc hm ab bc])
have hlineeq:
  cnj (steiner-deltoid-point a b c m - foot a b m) *
    (foot b c m - foot a b m) =
    (steiner-deltoid-point a b c m - foot a b m) *
      cnj (foot b c m - foot a b m)
proof -
show ?thesis
using deltoid-cross-identity[OF ha hb hc hm] by (simp add: hX hY)
qed
have hself:
  cnj (cnj (steiner-deltoid-point a b c m - foot a b m) *
    (foot b c m - foot a b m)) =
  cnj (steiner-deltoid-point a b c m - foot a b m) *
    (foot b c m - foot a b m)
using hlineeq by (simp add: complex-cnj-mult)
have him:
  Im (cnj (steiner-deltoid-point a b c m - foot a b m) *
    (foot b c m - foot a b m)) = 0
using hself by (metis Reals-cnj-iff complex-is-Real-iff)
have hiff:
  collinear (foot a b m)
    (steiner-deltoid-point a b c m) (foot b c m)  $\longleftrightarrow$ 
  Im (cnj (steiner-deltoid-point a b c m - foot a b m) *
    (foot b c m - foot a b m)) = 0
by (rule collinear-iff-cross)
show ?thesis using hiff him by simp
qed

```

```

theorem simson-line-contains-steiner-deltoid-point-pair:
fixes a b c m :: complex
assumes hncol:  $\neg \text{collinear } a \ b \ c$ 
and ha: oncircle 0 1 a and hb: oncircle 0 1 b

```

and hc : *oncircle* 0 1 c **and** hm : *oncircle* 0 1 m
and mb : $m \neq b$
shows *on-line* (*foot* a b m) (*foot* b c m)
(*steiner-deltoid-point* a b c m)
proof –
obtain ab : $a \neq b$ **and** ac : $a \neq c$ **and** bc : $b \neq c$
using *ncol-distinct*[*OF* $hncol$] **by** *blast*
have $hm0$: $m \neq 0$ **using** *unit-nonzero*[*OF* hm] .
have $hfeet$:
foot b c m – *foot* a b m =
 $(c - a) * (m - b) / (2 * (m - 0))$
by (*rule foot-diff-PR-circ*[*OF* ha hb hc hm *zero-less-one* ab bc])
have hne : *foot* b c m $\neq$ *foot* a b m
proof
assume *foot* b c m = *foot* a b m
then have *foot* b c m – *foot* a b m = 0 **by** *simp*
with $hfeet$ **have** $(c - a) * (m - b) / (2 * (m - 0)) = 0$
by *simp*
then show *False* **using** ac mb $hm0$ **by** *simp*
qed
have hne' : *foot* a b m $\neq$ *foot* b c m **using** hne **by** *auto*
have $hcol$:
collinear (*foot* a b m)
(*steiner-deltoid-point* a b c m) (*foot* b c m)
by (*rule deltoid-point-collinear*[*OF* $hncol$ ha hb hc hm])
have $hratio$ -eq:
collinear (*foot* a b m)
(*steiner-deltoid-point* a b c m) (*foot* b c m) =
 $((\text{steiner-deltoid-point } a \ b \ c \ m - \text{foot } a \ b \ m) /$
 $(\text{foot } b \ c \ m - \text{foot } a \ b \ m) \in \mathbb{R})$
by (*rule collinear-iff-real-ratio*[*OF* hne'])
have $hratio$:
 $((\text{steiner-deltoid-point } a \ b \ c \ m - \text{foot } a \ b \ m) /$
 $(\text{foot } b \ c \ m - \text{foot } a \ b \ m) \in \mathbb{R})$
using $hratio$ -eq $hcol$ **by** *simp*
have $hlineiff$:
on-line (*foot* a b m) (*foot* b c m)
(*steiner-deltoid-point* a b c m) =
 $((\text{steiner-deltoid-point } a \ b \ c \ m - \text{foot } a \ b \ m) /$
 $(\text{foot } b \ c \ m - \text{foot } a \ b \ m) \in \mathbb{R})$
by (*rule on-line-iff*[*OF* hne'])
show *?thesis* **using** $hlineiff$ $hratio$ **by** *simp*
qed

theorem *simson-line-contains-steiner-deltoid-point*:
fixes a b c m :: *complex*
assumes $hncol$: $\neg$ *collinear* a b c
and ha : *oncircle* 0 1 a **and** hb : *oncircle* 0 1 b
and hc : *oncircle* 0 1 c **and** hm : *oncircle* 0 1 m

```

shows ( $m \neq b \longrightarrow$ 
   $on-line\ (foot\ a\ b\ m)\ (foot\ b\ c\ m)$ 
   $(steiner-deltoid-point\ a\ b\ c\ m)) \wedge$ 
  ( $m \neq c \longrightarrow$ 
   $on-line\ (foot\ b\ c\ m)\ (foot\ c\ a\ m)$ 
   $(steiner-deltoid-point\ a\ b\ c\ m)) \wedge$ 
  ( $m \neq a \longrightarrow$ 
   $on-line\ (foot\ c\ a\ m)\ (foot\ a\ b\ m)$ 
   $(steiner-deltoid-point\ a\ b\ c\ m))$ 
proof -
  have hfirst:
     $m \neq b \longrightarrow$ 
     $on-line\ (foot\ a\ b\ m)\ (foot\ b\ c\ m)$ 
     $(steiner-deltoid-point\ a\ b\ c\ m)$ 
  proof (rule impI)
    assume mb:  $m \neq b$ 
    show  $on-line\ (foot\ a\ b\ m)\ (foot\ b\ c\ m)$ 
     $(steiner-deltoid-point\ a\ b\ c\ m)$ 
    by (rule simson-line-contains-steiner-deltoid-point-pair
      [OF hncol ha hb hc hm mb])
  qed
  have hncol-bca:  $\neg collinear\ b\ c\ a$ 
    using hncol by (metis collinear-cycle)
  have hsecond:
     $m \neq c \longrightarrow$ 
     $on-line\ (foot\ b\ c\ m)\ (foot\ c\ a\ m)$ 
     $(steiner-deltoid-point\ a\ b\ c\ m)$ 
  proof (rule impI)
    assume mc:  $m \neq c$ 
    have hrot:
       $on-line\ (foot\ b\ c\ m)\ (foot\ c\ a\ m)$ 
       $(steiner-deltoid-point\ b\ c\ a\ m)$ 
    by (rule simson-line-contains-steiner-deltoid-point-pair
      [OF hncol-bca hb hc ha hm mc])
    show  $on-line\ (foot\ b\ c\ m)\ (foot\ c\ a\ m)$ 
     $(steiner-deltoid-point\ a\ b\ c\ m)$ 
    using hrot
    by (simp add: steiner-deltoid-point-def ac-simps)
  qed
  have hncol-cab:  $\neg collinear\ c\ a\ b$ 
    using hncol by (metis collinear-cycle)
  have hthird:
     $m \neq a \longrightarrow$ 
     $on-line\ (foot\ c\ a\ m)\ (foot\ a\ b\ m)$ 
     $(steiner-deltoid-point\ a\ b\ c\ m)$ 
  proof (rule impI)
    assume ma:  $m \neq a$ 
    have hrot:
       $on-line\ (foot\ c\ a\ m)\ (foot\ a\ b\ m)$ 

```

```

      (steiner-deltoid-point c a b m)
    by (rule simson-line-contains-steiner-deltoid-point-pair
        [OF hncol-cab hc ha hb hm ma])
  show on-line (foot c a m) (foot a b m)
    (steiner-deltoid-point a b c m)
  using hrot
  by (simp add: steiner-deltoid-point-def ac-simps)
qed
show ?thesis using hfirst hsecond hthird by blast
qed

```

lemma *deltoid-normal-cross-identity*:

```

  fixes a b c m :: complex
  assumes ha: oncircle 0 1 a and hb: oncircle 0 1 b
    and hc: oncircle 0 1 c and hm: oncircle 0 1 m
  shows
    cnj (cnj ((c - a) * (1 - b * cnj m) / 2) *
      (m - a * b * c * (cnj m)2)) =
    -(cnj ((c - a) * (1 - b * cnj m) / 2) *
      (m - a * b * c * (cnj m)2))
proof -
  have hca: cnj a = 1 / a using unit-cnj[OF ha] .
  have hcb: cnj b = 1 / b using unit-cnj[OF hb] .
  have hcc: cnj c = 1 / c using unit-cnj[OF hc] .
  have hcm: cnj m = 1 / m using unit-cnj[OF hm] .
  have ha0: a ≠ 0 using unit-nonzero[OF ha] .
  have hb0: b ≠ 0 using unit-nonzero[OF hb] .
  have hc0: c ≠ 0 using unit-nonzero[OF hc] .
  have hm0: m ≠ 0 using unit-nonzero[OF hm] .
  show ?thesis
    by (simp add: hca hcb hcc hcm ha0 hb0 hc0 hm0 field-simps; algebra)
qed

```

lemma *real-complex-mult*:

```

  fixes p q :: complex
  assumes p ∈ ℝ and q ∈ ℝ
  shows p * q ∈ ℝ
proof -
  have hp: cnj p = p by (metis Reals-cnj-iff assms(1))
  have hq: cnj q = q by (metis Reals-cnj-iff assms(2))
  show ?thesis
    using hp hq by (simp add: Reals-cnj-iff)
qed

```

lemma *real-ratio-of-perpendicular*:

```

  fixes u n :: complex
  assumes hperp: complex-perpendicular u n and hn: n ≠ 0
  shows u / (i * n) ∈ ℝ
proof -

```

```

have hRe: Re (cnj u * n) = 0
  using hperp unfolding complex-perpendicular-def .
have hsum:
  cnj u * n + u * cnj n = 0
proof -
  have h:
    cnj u * n + u * cnj n = 2 * Re (cnj u * n)
  using cnj-add-mult-eq-Re[of cnj u cnj n]
  by (simp only: complex-cnj-cnj)
  have hzero: complex-of-real (2 * Re (cnj u * n)) = 0
  using hRe by simp
  show ?thesis
    by (rule trans[OF h hzero])
qed
have hrat:
  cnj (u / (i * n)) = u / (i * n)
  using hsum hn
  by (simp add: complex-cnj-divide field-simps; algebra)
show ?thesis using hrat by (simp add: Reals-cnj-iff)
qed

```

```

lemma on-line-of-collinear-perpendicular:
  fixes x y z n :: complex
  assumes hcol: collinear x z y
  and hxy: x ≠ y
  and hperp: complex-perpendicular (y - x) n
  and hn: n ≠ 0
  shows on-line x (x + i * n) z
proof -
  have hyx: y - x ≠ 0 using hxy by simp
  have hratio1:
    (z - x) / (y - x) ∈ ℝ
  using collinear-iff-real-ratio[OF hxy] hcol by simp
  have hratio2:
    (y - x) / (i * n) ∈ ℝ
  by (rule real-ratio-of-perpendicular[OF hperp hn])
  have hprod:
    ((z - x) / (y - x)) * ((y - x) / (i * n)) ∈ ℝ
  by (rule real-complex-mult[OF hratio1 hratio2])
  have hratio:
    (z - x) / (i * n) =
      ((z - x) / (y - x)) * ((y - x) / (i * n))
  using hyx hn by (simp add: field-simps hyx hn; algebra)
  have hreal: (z - x) / (i * n) ∈ ℝ
  by (metis hratio hprod)
  have hbase: x ≠ x + i * n using hn by simp
  show ?thesis
    using on-line-iff[OF hbase] hreal by simp
qed

```

lemma *on-line-of-collinear-real-direction*:
fixes $x\ y\ z\ d :: \text{complex}$
assumes $hcol: \text{collinear } x\ z\ y$
and $hxy: x \neq y$
and $hratio2: (y - x) / d \in \mathbb{R}$
and $hd: d \neq 0$
shows *on-line* $x\ (x + d)\ z$
proof –
have $hyx: y - x \neq 0$ **using** hxy **by** *simp*
have $hratio1:$
 $(z - x) / (y - x) \in \mathbb{R}$
using *collinear-iff-real-ratio*[*OF* hxy] $hcol$ **by** *simp*
have $hprod:$
 $((z - x) / (y - x)) * ((y - x) / d) \in \mathbb{R}$
by (*rule* *real-complex-mult*[*OF* $hratio1\ hratio2$])
have $hratio:$
 $(z - x) / d =$
 $((z - x) / (y - x)) * ((y - x) / d)$
using $hyx\ hd$ **by** (*simp* *add: field-simps* $hyx\ hd$; *algebra*)
have $hreal: (z - x) / d \in \mathbb{R}$
by (*metis* $hratio\ hprod$)
have $hbase: x \neq x + d$ **using** hd **by** *simp*
show *?thesis*
using *on-line-iff*[*OF* $hbase$] $hreal$ **by** *simp*
qed

lemma *on-line-change-base*:
fixes $x\ y\ z\ d :: \text{complex}$
assumes $hy: \text{on-line } x\ (x + d)\ y$
and $hz: \text{on-line } x\ (x + d)\ z$
shows *on-line* $z\ (z + d)\ y$
proof –
obtain $r :: \text{real}$ **where**
 $yr: y = x + \text{complex-of-real } r * d$
using hy **unfolding** *on-line-def* **by** (*auto* *simp* *add: algebra-simps*)
obtain $s :: \text{real}$ **where**
 $zs: z = x + \text{complex-of-real } s * d$
using hz **unfolding** *on-line-def* **by** (*auto* *simp* *add: algebra-simps*)
have $hyz: y = z + \text{complex-of-real } (r - s) * d$
using $yr\ zs$ **by** (*simp* *add: algebra-simps*)
show *?thesis*
unfolding *on-line-def*
apply (*rule* *exI*[*of* - $r - s$])
using hyz **by** (*simp* *add: algebra-simps*)
qed

lemma *on-line-set-eq-of-distinct*:
fixes $a\ b\ c\ d :: \text{complex}$

```

assumes hab:  $a \neq b$  and hcd:  $c \neq d$ 
and hc: on-line  $a\ b\ c$  and hd: on-line  $a\ b\ d$ 
shows  $\{z. \text{on-line } a\ b\ z\} = \{z. \text{on-line } c\ d\ z\}$ 
proof -
obtain  $r\ s :: \text{real}$  where
  hcr:  $c = a + \text{complex-of-real } r * (b - a)$ 
and hds:  $d = a + \text{complex-of-real } s * (b - a)$ 
using hc hd unfolding on-line-def by blast
have hrs:  $r \neq s$ 
proof
  assume  $r = s$ 
  with hcr hds have  $c = d$  by simp
  with hcd show False by simp
qed
have hsr:  $s - r \neq 0$  using hrs by simp
have hdc:  $d - c = \text{complex-of-real } (s - r) * (b - a)$ 
  using hcr hds by (simp add: algebra-simps; algebra)
show  $\{z. \text{on-line } a\ b\ z\} = \{z. \text{on-line } c\ d\ z\}$ 
proof (rule set-eqI)
  fix  $z$ 
  show  $z \in \{z. \text{on-line } a\ b\ z\} \longleftrightarrow$ 
     $z \in \{z. \text{on-line } c\ d\ z\}$ 
proof
  assume hz:  $z \in \{z. \text{on-line } a\ b\ z\}$ 
  then obtain  $t :: \text{real}$  where
    hzt:  $z = a + \text{complex-of-real } t * (b - a)$ 
  unfolding on-line-def by blast
have hztc:  $z - c = \text{complex-of-real } (t - r) * (b - a)$ 
  using hzt hcr by (simp add: algebra-simps; algebra)
have hscale:
   $\text{complex-of-real } ((t - r) / (s - r)) * (d - c) =$ 
   $\text{complex-of-real } (t - r) * (b - a)$ 
proof -
have hsr0:  $\text{complex-of-real } (s - r) \neq 0$  using hsr by simp
have hrewrite:
   $\text{complex-of-real } ((t - r) / (s - r)) * (d - c) =$ 
   $(\text{complex-of-real } (t - r) / \text{complex-of-real } (s - r)) *$ 
   $(\text{complex-of-real } (s - r) * (b - a))$ 
using hdc by (simp add: of-real-divide)
show ?thesis using hrewrite hsr0
  by (simp add: field-simps; algebra)
qed
have hzd:
   $z = c + \text{complex-of-real } ((t - r) / (s - r)) * (d - c)$ 
using hztc hscale by (simp add: algebra-simps; algebra)
show  $z \in \{z. \text{on-line } c\ d\ z\}$ 
unfolding on-line-def using hzd by blast
next
  assume hz:  $z \in \{z. \text{on-line } c\ d\ z\}$ 

```

then obtain $t :: \text{real}$ where
 $hzt: z = c + \text{complex-of-real } t * (d - c)$
 unfolding on-line-def by blast
 have $hzb:$
 $z = a + \text{complex-of-real } (r + t * (s - r)) * (b - a)$
 using hzt hcr hdc
 by (simp add: algebra-simps of-real-mult; algebra)
 show $z \in \{z. \text{on-line } a \ b \ z\}$
 unfolding on-line-def using hzb by blast
 qed
 qed
 qed

lemma on-line-eq-of-real-scaled-direction:
 fixes $x \ d :: \text{complex}$ and $r :: \text{real}$
 assumes $hd: d \neq 0$ and $hr: r \neq 0$
 shows $\{z. \text{on-line } x \ (x + d) \ z\} =$
 $\{z. \text{on-line } x \ (x + \text{complex-of-real } r * d) \ z\}$
 proof -
 have $hbase: x \neq x + d$ using hd by simp
 have $hscaled: x \neq x + \text{complex-of-real } r * d$
 using hd hr by simp
 have $hx: \text{on-line } x \ (x + d) \ x$
 unfolding on-line-def
 by (rule exI[of - 0]) simp
 have $hxr: \text{on-line } x \ (x + d) \ (x + \text{complex-of-real } r * d)$
 unfolding on-line-def
 by (rule exI[of - r]) simp
 show ?thesis
 by (rule on-line-set-eq-of-distinct[OF $hbase$ $hscaled$ hx hxr])
 qed

theorem steiner-deltoid-cusp-tangent-is-second-derivative:
 fixes $a \ b \ c :: \text{complex}$ and $t :: \text{real}$
 assumes $hcube: (\text{unit-circle-param } t) ^ 3 = a * b * c$
 shows $\text{steiner-deltoid-tangent-at } a \ b \ c \ (\text{unit-circle-param } t) =$
 $\{z. \text{on-line } (\text{steiner-deltoid-point } a \ b \ c \ (\text{unit-circle-param } t))$
 $(\text{steiner-deltoid-point } a \ b \ c \ (\text{unit-circle-param } t) +$
 $\text{steiner-deltoid-curve-second-derivative } a \ b \ c \ t) \ z\}$
 proof -
 have $hm: \text{oncircle } 0 \ 1 \ (\text{unit-circle-param } t)$
 by (rule unit-circle-param-oncircle)
 have $hm0: \text{unit-circle-param } t \neq 0$
 using unit-nonzero[OF hm] .
 have $hminus: (-3::\text{real}) \neq 0$ by simp
 have $hnormal0:$
 $\text{steiner-deltoid-normal } a \ b \ c \ (\text{unit-circle-param } t) = 0$
 using steiner-deltoid-normal-eq-zero-iff-cube[OF hm] $hcube$ by blast
 have $hdirection:$

$\{z. \text{on-line } (\text{steiner-deltoid-point } a \ b \ c \ (\text{unit-circle-param } t))$
 $\quad (\text{steiner-deltoid-point } a \ b \ c \ (\text{unit-circle-param } t) +$
 $\quad \text{unit-circle-param } t) \ z\} =$
 $\{z. \text{on-line } (\text{steiner-deltoid-point } a \ b \ c \ (\text{unit-circle-param } t))$
 $\quad (\text{steiner-deltoid-point } a \ b \ c \ (\text{unit-circle-param } t) +$
 $\quad \text{complex-of-real } (-3) * \text{unit-circle-param } t) \ z\}$
by (rule on-line-eq-of-real-scaled-direction[OF hm0 hminus])
have htangent:
 $\text{steiner-deltoid-tangent-at } a \ b \ c \ (\text{unit-circle-param } t) =$
 $\{z. \text{on-line } (\text{steiner-deltoid-point } a \ b \ c \ (\text{unit-circle-param } t))$
 $\quad (\text{steiner-deltoid-point } a \ b \ c \ (\text{unit-circle-param } t) +$
 $\quad \text{unit-circle-param } t) \ z\}$
unfolding steiner-deltoid-tangent-at-def
 $\text{steiner-deltoid-tangent-direction-at-def}$
using hnormal0 **by** simp
show ?thesis
using htangent hdirection
 $\text{steiner-deltoid-curve-second-derivative-stationary}[OF hcube]$
by (simp add: algebra-simps)
qed

theorem simson-line-perpendicular-to-steiner-deltoid-normal:

fixes $a \ b \ c \ m :: \text{complex}$
assumes hncol: $\neg \text{collinear } a \ b \ c$
and ha: $\text{oncircle } 0 \ 1 \ a$ **and** hb: $\text{oncircle } 0 \ 1 \ b$
and hc: $\text{oncircle } 0 \ 1 \ c$ **and** hm: $\text{oncircle } 0 \ 1 \ m$
shows complex-perpendicular
 $(\text{foot } b \ c \ m - \text{foot } a \ b \ m)$
 $(\text{steiner-deltoid-normal } a \ b \ c \ m)$

proof –

obtain ab: $a \neq b$ **and** bc: $b \neq c$
using ncol-distinct[OF hncol] **by** blast
have hfeet:
 $\text{foot } b \ c \ m - \text{foot } a \ b \ m =$
 $(c - a) * (1 - b * \text{cnj } m) / 2$
by (rule feet-difference-unit[OF ha hb hc hm ab bc])

have hnormal:

$\text{steiner-deltoid-normal } a \ b \ c \ m =$
 $m - a * b * c * (\text{cnj } m)^2$
by (simp add: steiner-deltoid-normal-def)

have hcnj:

$\text{cnj } (\text{cnj } (\text{foot } b \ c \ m - \text{foot } a \ b \ m) * \text{steiner-deltoid-normal } a \ b \ c \ m) =$
 $-(\text{cnj } (\text{foot } b \ c \ m - \text{foot } a \ b \ m) * \text{steiner-deltoid-normal } a \ b \ c \ m)$

proof –

show ?thesis

using deltoid-normal-cross-identity[OF ha hb hc hm]
by (simp add: hfeet hnormal)

```

qed
show ?thesis
  unfolding complex-perpendicular-def
  using hcnj by (metis eq-minus-cnj-iff-imag)
qed

theorem simson-line-is-tangent-to-steiner-deltoid-at-regular:
  fixes a b c m :: complex
  assumes hncol:  $\neg$  collinear a b c
    and ha: oncircle 0 1 a and hb: oncircle 0 1 b
    and hc: oncircle 0 1 c and hm: oncircle 0 1 m
    and hnormal: steiner-deltoid-normal a b c m  $\neq$  0
  shows (m  $\neq$  b  $\longrightarrow$ 
    foot a b m  $\in$  steiner-deltoid-tangent-at a b c m  $\wedge$ 
    foot b c m  $\in$  steiner-deltoid-tangent-at a b c m  $\wedge$ 
    foot a b m  $\neq$  foot b c m)  $\wedge$ 
    (m = b  $\longrightarrow$ 
    foot b c m  $\in$  steiner-deltoid-tangent-at a b c m  $\wedge$ 
    foot c a m  $\in$  steiner-deltoid-tangent-at a b c m  $\wedge$ 
    foot b c m  $\neq$  foot c a m)
proof -
  obtain ab: a  $\neq$  b and ac: a  $\neq$  c and bc: b  $\neq$  c
  using ncol-distinct[OF hncol] by blast
  have ca: c  $\neq$  a using ac by simp
  have hm0: m  $\neq$  0 using unit-nonzero[OF hm] .
  have hcol:
    collinear (foot a b m)
      (steiner-deltoid-point a b c m) (foot b c m)
  by (rule deltoid-point-collinear[OF hncol ha hb hc hm])
  have hperp:
    complex-perpendicular
      (foot b c m - foot a b m)
      (steiner-deltoid-normal a b c m)
  by (rule simson-line-perpendicular-to-steiner-deltoid-normal
    [OF hncol ha hb hc hm])
  have hfirst:
    m  $\neq$  b  $\longrightarrow$ 
    foot a b m  $\in$  steiner-deltoid-tangent-at a b c m  $\wedge$ 
    foot b c m  $\in$  steiner-deltoid-tangent-at a b c m  $\wedge$ 
    foot a b m  $\neq$  foot b c m
  proof (rule impI)
    assume mb: m  $\neq$  b
    have hne-rev: foot b c m  $\neq$  foot a b m
    proof
      assume foot b c m = foot a b m
      then have hdiff: foot b c m - foot a b m = 0 by simp
      have hfeet:
        foot b c m - foot a b m =
          (c - a) * (m - b) / (2 * (m - 0))
    
```

```

    by (rule foot-diff-PR-circ[OF ha hb hc hm zero-less-one ab bc])
  have hz:  $(c - a) * (m - b) / (2 * (m - 0)) = 0$ 
    using hdiff hfeet by (metis)
  then show False using ac mb hm0 by (simp add: ac mb hm0)
qed
have hne:  $\text{foot } a \ b \ m \neq \text{foot } b \ c \ m$  using hne-rev by auto
have hline-d:
  on-line (foot a b m)
    (foot a b m + i * steiner-deltoid-normal a b c m)
    (steiner-deltoid-point a b c m)
  by (rule on-line-of-collinear-perpendicular
    [OF hcol hne hperp hnormal])
have hline-ab:
  on-line (foot a b m)
    (foot a b m + i * steiner-deltoid-normal a b c m)
    (foot a b m)
  by (rule on-line-of-collinear-perpendicular
    [OF collinear-aac hne hperp hnormal])
have hline-bc:
  on-line (foot a b m)
    (foot a b m + i * steiner-deltoid-normal a b c m)
    (foot b c m)
  by (rule on-line-of-collinear-perpendicular
    [OF collinear-refl hne hperp hnormal])
have htan-ab:
  on-line (steiner-deltoid-point a b c m)
    (steiner-deltoid-point a b c m +
      i * steiner-deltoid-normal a b c m)
    (foot a b m)
  by (rule on-line-change-base[OF hline-ab hline-d])
have htan-bc:
  on-line (steiner-deltoid-point a b c m)
    (steiner-deltoid-point a b c m +
      i * steiner-deltoid-normal a b c m)
    (foot b c m)
  by (rule on-line-change-base[OF hline-bc hline-d])
have hab-mem:
  foot a b m  $\in$  steiner-deltoid-tangent-at a b c m
  using htan-ab
  by (simp add: steiner-deltoid-tangent-at-def
    steiner-deltoid-tangent-direction-at-def hnormal)
have hbc-mem:
  foot b c m  $\in$  steiner-deltoid-tangent-at a b c m
  using htan-bc
  by (simp add: steiner-deltoid-tangent-at-def
    steiner-deltoid-tangent-direction-at-def hnormal)
show foot a b m  $\in$  steiner-deltoid-tangent-at a b c m  $\wedge$ 
  foot b c m  $\in$  steiner-deltoid-tangent-at a b c m  $\wedge$ 
  foot a b m  $\neq$  foot b c m

```

```

    using hab-mem hbc-mem hne by blast
  qed
  have hncol-bca:  $\neg \text{collinear } b \ c \ a$ 
  using hncol by (metis collinear-cycle)
  have hsecond:
     $m = b \longrightarrow$ 
     $\text{foot } b \ c \ m \in \text{steiner-deltoid-tangent-at } a \ b \ c \ m \wedge$ 
     $\text{foot } c \ a \ m \in \text{steiner-deltoid-tangent-at } a \ b \ c \ m \wedge$ 
     $\text{foot } b \ c \ m \neq \text{foot } c \ a \ m$ 
  proof (rule impI)
    assume mb:  $m = b$ 
    have hcol-rot:
       $\text{collinear } (\text{foot } b \ c \ m)$ 
       $(\text{steiner-deltoid-point } a \ b \ c \ m) (\text{foot } c \ a \ m)$ 
    proof -
      have hrot:
         $\text{collinear } (\text{foot } b \ c \ m)$ 
         $(\text{steiner-deltoid-point } b \ c \ a \ m) (\text{foot } c \ a \ m)$ 
        by (rule deltoid-point-collinear[OF hncol-bca hb hc ha hm])
      show ?thesis using hrot
        by (simp add: steiner-deltoid-point-def ac-simps)
    qed
    have hnormal-rot:
       $\text{steiner-deltoid-normal } b \ c \ a \ m =$ 
       $\text{steiner-deltoid-normal } a \ b \ c \ m$ 
      by (simp add: steiner-deltoid-normal-def ac-simps)
    have hperp-rot:
       $\text{complex-perpendicular}$ 
       $(\text{foot } c \ a \ m - \text{foot } b \ c \ m)$ 
       $(\text{steiner-deltoid-normal } b \ c \ a \ m)$ 
      by (rule simson-line-perpendicular-to-steiner-deltoid-normal
        [OF hncol-bca hb hc ha hm])
    have hperp2:
       $\text{complex-perpendicular}$ 
       $(\text{foot } c \ a \ m - \text{foot } b \ c \ m)$ 
       $(\text{steiner-deltoid-normal } a \ b \ c \ m)$ 
      using hperp-rot hnormal-rot by simp
    have hfeet:
       $\text{foot } c \ a \ m - \text{foot } b \ c \ m =$ 
       $(a - b) * (m - c) / (2 * (m - 0))$ 
      by (rule foot-diff-PR-circ[OF hb hc ha hm zero-less-one bc ca])
    have hne-rev:  $\text{foot } c \ a \ m \neq \text{foot } b \ c \ m$ 
    proof
      assume foot c a m = foot b c m
      then have hdiff:  $\text{foot } c \ a \ m - \text{foot } b \ c \ m = 0$  by simp
      have hz:  $(a - b) * (m - c) / (2 * (m - 0)) = 0$ 
      using hdiff hfeet by (metis)
      then show False using ab bc hm0 mb by (simp add: ab bc hm0 mb)
    qed
  qed

```

```

have hne: foot b c m  $\neq$  foot c a m using hne-rev by auto
have hline-d:
  on-line (foot b c m)
    (foot b c m + i * steiner-deltoid-normal a b c m)
    (steiner-deltoid-point a b c m)
  by (rule on-line-of-collinear-perpendicular
    [OF hcol-rot hne hperp2 hnormal])
have hline-bc:
  on-line (foot b c m)
    (foot b c m + i * steiner-deltoid-normal a b c m)
    (foot b c m)
  by (rule on-line-of-collinear-perpendicular
    [OF collinear-aac hne hperp2 hnormal])
have hline-ca:
  on-line (foot b c m)
    (foot b c m + i * steiner-deltoid-normal a b c m)
    (foot c a m)
  by (rule on-line-of-collinear-perpendicular
    [OF collinear-refl hne hperp2 hnormal])
have htan-bc:
  on-line (steiner-deltoid-point a b c m)
    (steiner-deltoid-point a b c m +
      i * steiner-deltoid-normal a b c m)
    (foot b c m)
  by (rule on-line-change-base[OF hline-bc hline-d])
have htan-ca:
  on-line (steiner-deltoid-point a b c m)
    (steiner-deltoid-point a b c m +
      i * steiner-deltoid-normal a b c m)
    (foot c a m)
  by (rule on-line-change-base[OF hline-ca hline-d])
have hbc-mem:
  foot b c m  $\in$  steiner-deltoid-tangent-at a b c m
  using htan-bc
  by (simp add: steiner-deltoid-tangent-at-def
    steiner-deltoid-tangent-direction-at-def hnormal)
have hca-mem:
  foot c a m  $\in$  steiner-deltoid-tangent-at a b c m
  using htan-ca
  by (simp add: steiner-deltoid-tangent-at-def
    steiner-deltoid-tangent-direction-at-def hnormal)
show foot b c m  $\in$  steiner-deltoid-tangent-at a b c m  $\wedge$ 
  foot c a m  $\in$  steiner-deltoid-tangent-at a b c m  $\wedge$ 
  foot b c m  $\neq$  foot c a m
  using hbc-mem hca-mem hne by blast
qed
show ?thesis using hfirst hsecond by blast
qed

```

theorem *simson-line-eq-steiner-deltoid-tangent-at:*
fixes $a\ b\ c\ m :: \text{complex}$
assumes $hncol: \neg \text{collinear } a\ b\ c$
and $ha: \text{oncircle } 0\ 1\ a$ **and** $hb: \text{oncircle } 0\ 1\ b$
and $hc: \text{oncircle } 0\ 1\ c$ **and** $hm: \text{oncircle } 0\ 1\ m$
and $hnormal: \text{steiner-deltoid-normal } a\ b\ c\ m \neq 0$
shows $(m \neq b \longrightarrow$
 $\{z. \text{on-line } (\text{foot } a\ b\ m) (\text{foot } b\ c\ m) z\} =$
 $\text{steiner-deltoid-tangent-at } a\ b\ c\ m) \wedge$
 $(m = b \longrightarrow$
 $\{z. \text{on-line } (\text{foot } b\ c\ m) (\text{foot } c\ a\ m) z\} =$
 $\text{steiner-deltoid-tangent-at } a\ b\ c\ m)$
proof –
have $hAt:$
 $(m \neq b \longrightarrow$
 $\text{foot } a\ b\ m \in \text{steiner-deltoid-tangent-at } a\ b\ c\ m \wedge$
 $\text{foot } b\ c\ m \in \text{steiner-deltoid-tangent-at } a\ b\ c\ m \wedge$
 $\text{foot } a\ b\ m \neq \text{foot } b\ c\ m) \wedge$
 $(m = b \longrightarrow$
 $\text{foot } b\ c\ m \in \text{steiner-deltoid-tangent-at } a\ b\ c\ m \wedge$
 $\text{foot } c\ a\ m \in \text{steiner-deltoid-tangent-at } a\ b\ c\ m \wedge$
 $\text{foot } b\ c\ m \neq \text{foot } c\ a\ m)$
by $(\text{rule } \text{simson-line-is-tangent-to-steiner-deltoid-at-regular}$
 $[\text{OF } hncol\ ha\ hb\ hc\ hm\ hnormal])$
have $hdir:$
 $\text{steiner-deltoid-point } a\ b\ c\ m \neq$
 $\text{steiner-deltoid-point } a\ b\ c\ m +$
 $\text{steiner-deltoid-tangent-direction-at } a\ b\ c\ m$
using $hnormal$
by $(\text{simp add: steiner-deltoid-tangent-direction-at-def } hnormal)$
have $hfirst:$
 $m \neq b \longrightarrow$
 $\{z. \text{on-line } (\text{foot } a\ b\ m) (\text{foot } b\ c\ m) z\} =$
 $\text{steiner-deltoid-tangent-at } a\ b\ c\ m$
proof $(\text{rule } \text{impI})$
assume $mb: m \neq b$
have $hne: \text{foot } a\ b\ m \neq \text{foot } b\ c\ m$
using $hAt\ mb$ **by** blast
have $hab\text{-mem}: \text{foot } a\ b\ m \in \text{steiner-deltoid-tangent-at } a\ b\ c\ m$
using $hAt\ mb$ **by** blast
have $hbc\text{-mem}: \text{foot } b\ c\ m \in \text{steiner-deltoid-tangent-at } a\ b\ c\ m$
using $hAt\ mb$ **by** blast
have $hset:$
 $\{z. \text{on-line } (\text{steiner-deltoid-point } a\ b\ c\ m)$
 $(\text{steiner-deltoid-point } a\ b\ c\ m +$
 $\text{steiner-deltoid-tangent-direction-at } a\ b\ c\ m) z\} =$
 $\{z. \text{on-line } (\text{foot } a\ b\ m) (\text{foot } b\ c\ m) z\}$
proof $(\text{rule } \text{on-line-set-eq-of-distinct}[\text{OF } hdir\ hne])$
show $\text{on-line } (\text{steiner-deltoid-point } a\ b\ c\ m)$

```

      (steiner-deltoid-point a b c m +
       steiner-deltoid-tangent-direction-at a b c m)
      (foot a b m)
    using hab-mem by (simp add: steiner-deltoid-tangent-at-def)
  show on-line (steiner-deltoid-point a b c m)
    (steiner-deltoid-point a b c m +
     steiner-deltoid-tangent-direction-at a b c m)
    (foot b c m)
  using hbc-mem by (simp add: steiner-deltoid-tangent-at-def)
qed
have hset-rev:
  {z. on-line (foot a b m) (foot b c m) z} =
  {z. on-line (steiner-deltoid-point a b c m)
    (steiner-deltoid-point a b c m +
     steiner-deltoid-tangent-direction-at a b c m) z}
  using hset by (rule sym)
show {z. on-line (foot a b m) (foot b c m) z} =
  steiner-deltoid-tangent-at a b c m
  using hset-rev
  by (simp add: steiner-deltoid-tangent-at-def)
qed
have hsecond:
  m = b  $\longrightarrow$ 
  {z. on-line (foot b c m) (foot c a m) z} =
  steiner-deltoid-tangent-at a b c m
proof (rule impI)
  assume mb: m = b
  have hne: foot b c m  $\neq$  foot c a m
  using hAt mb by blast
  have hbc-mem: foot b c m  $\in$  steiner-deltoid-tangent-at a b c m
  using hAt mb by blast
  have hca-mem: foot c a m  $\in$  steiner-deltoid-tangent-at a b c m
  using hAt mb by blast
  have hset:
    {z. on-line (steiner-deltoid-point a b c m)
      (steiner-deltoid-point a b c m +
       steiner-deltoid-tangent-direction-at a b c m) z} =
    {z. on-line (foot b c m) (foot c a m) z}
  proof (rule on-line-set-eq-of-distinct[OF hdir hne])
    show on-line (steiner-deltoid-point a b c m)
      (steiner-deltoid-point a b c m +
       steiner-deltoid-tangent-direction-at a b c m)
      (foot b c m)
    using hbc-mem by (simp add: steiner-deltoid-tangent-at-def)
  show on-line (steiner-deltoid-point a b c m)
    (steiner-deltoid-point a b c m +
     steiner-deltoid-tangent-direction-at a b c m)
    (foot c a m)
  using hca-mem by (simp add: steiner-deltoid-tangent-at-def)

```

```

qed
have hset-rev:
  {z. on-line (foot b c m) (foot c a m) z} =
  {z. on-line (steiner-deltoid-point a b c m)
    (steiner-deltoid-point a b c m +
     steiner-deltoid-tangent-direction-at a b c m) z}
  using hset by (rule sym)
show {z. on-line (foot b c m) (foot c a m) z} =
  steiner-deltoid-tangent-at a b c m
  using hset-rev
  by (simp add: steiner-deltoid-tangent-at-def)
qed
show ?thesis using hfirst hsecond by blast
qed

theorem simson-line-eq-steiner-deltoid-tangent-at-all:
  fixes a b c m :: complex
  assumes hncol:  $\neg$  collinear a b c
  and ha: oncircle 0 1 a and hb: oncircle 0 1 b
  and hc: oncircle 0 1 c and hm: oncircle 0 1 m
  shows (m  $\neq$  b  $\longrightarrow$ 
    {z. on-line (foot a b m) (foot b c m) z} =
    steiner-deltoid-tangent-at a b c m)  $\wedge$ 
    (m = b  $\longrightarrow$ 
    {z. on-line (foot b c m) (foot c a m) z} =
    steiner-deltoid-tangent-at a b c m)
  proof -
    obtain ab: a  $\neq$  b and ac: a  $\neq$  c and bc: b  $\neq$  c
    using ncol-distinct[OF hncol] by blast
    have ca: c  $\neq$  a using ac by simp
    have hm0: m  $\neq$  0 using unit-nonzero[OF hm] .
    show ?thesis
    proof (cases m  $\wedge$  3 = a * b * c)
    case True
    have hcube: m  $\wedge$  3 = a * b * c using True .
    have hnormal0:
      steiner-deltoid-normal a b c m = 0
    using steiner-deltoid-normal-eq-zero-iff-cube[OF hm] hcube by blast
    have hdir:
      steiner-deltoid-point a b c m  $\neq$ 
      steiner-deltoid-point a b c m + m
    using hm0 by simp
    have hfirst:
      m  $\neq$  b  $\longrightarrow$ 
      {z. on-line (foot a b m) (foot b c m) z} =
      steiner-deltoid-tangent-at a b c m
    proof (rule impI)
      assume mb: m  $\neq$  b
      have hfeet:

```

```

    foot b c m - foot a b m =
      (c - a) * (m - b) / (2 * (m - 0))
  by (rule foot-diff-PR-circ[OF ha hb hc hm zero-less-one ab bc])
have hne-rev: foot b c m ≠ foot a b m
proof
  assume foot b c m = foot a b m
  then have hdiff: foot b c m - foot a b m = 0 by simp
  have hz: (c - a) * (m - b) / (2 * (m - 0)) = 0
    using hdiff hfeet by (metis)
  then show False using ac mb hm0 by (simp add: ac mb hm0)
qed
have hne: foot a b m ≠ foot b c m using hne-rev by auto
have hcol:
  collinear (foot a b m)
    (steiner-deltoid-point a b c m) (foot b c m)
  by (rule deltoid-point-collinear[OF hncol ha hb hc hm])
have hratio:
  (foot b c m - foot a b m) / m ∈ ℝ
  by (rule cusp-feet-difference-ratio-real
    [OF ha hb hc hm ab bc hcube])
have hline-d:
  on-line (foot a b m) (foot a b m + m)
    (steiner-deltoid-point a b c m)
  by (rule on-line-of-collinear-real-direction
    [OF hcol hne hratio hm0])
have hline-ab:
  on-line (foot a b m) (foot a b m + m) (foot a b m)
  by (rule on-line-of-collinear-real-direction
    [OF collinear-aac hne hratio hm0])
have hline-bc:
  on-line (foot a b m) (foot a b m + m) (foot b c m)
  by (rule on-line-of-collinear-real-direction
    [OF collinear-refl hne hratio hm0])
have htan-ab:
  on-line (steiner-deltoid-point a b c m)
    (steiner-deltoid-point a b c m + m) (foot a b m)
  by (rule on-line-change-base[OF hline-ab hline-d])
have htan-bc:
  on-line (steiner-deltoid-point a b c m)
    (steiner-deltoid-point a b c m + m) (foot b c m)
  by (rule on-line-change-base[OF hline-bc hline-d])
have hset:
  {z. on-line (steiner-deltoid-point a b c m)
    (steiner-deltoid-point a b c m + m) z} =
  {z. on-line (foot a b m) (foot b c m) z}
proof (rule on-line-set-eq-of-distinct[OF hdir hne])
  show on-line (steiner-deltoid-point a b c m)
    (steiner-deltoid-point a b c m + m) (foot a b m)
    using htan-ab .

```

```

show on-line (steiner-deltoid-point a b c m)
  (steiner-deltoid-point a b c m + m) (foot b c m)
using htan-bc .
qed
have hset-rev:
  {z. on-line (foot a b m) (foot b c m) z} =
  {z. on-line (steiner-deltoid-point a b c m)
    (steiner-deltoid-point a b c m + m) z}
using hset by (rule sym)
show {z. on-line (foot a b m) (foot b c m) z} =
  steiner-deltoid-tangent-at a b c m
using hset-rev
by (simp add: steiner-deltoid-tangent-at-def
  steiner-deltoid-tangent-direction-at-def hnormal0)
qed
have hncol-bca:  $\neg$  collinear b c a
using hncol by (metis collinear-cycle)
have hsecond:
   $m = b \longrightarrow$ 
  {z. on-line (foot b c m) (foot c a m) z} =
  steiner-deltoid-tangent-at a b c m
proof (rule impI)
assume mb:  $m = b$ 
have hfeet:
  foot c a m - foot b c m =
   $(a - b) * (m - c) / (2 * (m - 0))$ 
by (rule foot-diff-PR-circ[OF hb hc ha hm zero-less-one bc ca])
have hne-rev: foot c a m  $\neq$  foot b c m
proof
assume foot c a m = foot b c m
then have hdiff: foot c a m - foot b c m = 0 by simp
have hz:  $(a - b) * (m - c) / (2 * (m - 0)) = 0$ 
using hdiff hfeet by (metis)
then show False using ab bc hm0 mb by (simp add: ab bc hm0 mb)
qed
have hne: foot b c m  $\neq$  foot c a m using hne-rev by auto
have hcol:
  collinear (foot b c m)
  (steiner-deltoid-point a b c m) (foot c a m)
proof -
have hrot:
  collinear (foot b c m)
  (steiner-deltoid-point b c a m) (foot c a m)
by (rule deltoid-point-collinear[OF hncol-bca hb hc ha hm])
show ?thesis using hrot
by (simp add: steiner-deltoid-point-def ac-simps)
qed
have hcube-rot:  $m \wedge 3 = b * c * a$ 
using hcube by (simp add: ac-simps)

```

```

have hratio:
  (foot c a m - foot b c m) / m ∈ ℝ
by (rule cusp-feet-difference-ratio-real
  [OF hb hc ha hm bc ca hcube-rot])
have hline-d:
  on-line (foot b c m) (foot b c m + m)
  (steiner-deltoid-point a b c m)
by (rule on-line-of-collinear-real-direction
  [OF hcol hne hratio hm0])
have hline-bc:
  on-line (foot b c m) (foot b c m + m) (foot b c m)
by (rule on-line-of-collinear-real-direction
  [OF collinear-aac hne hratio hm0])
have hline-ca:
  on-line (foot b c m) (foot b c m + m) (foot c a m)
by (rule on-line-of-collinear-real-direction
  [OF collinear-refl hne hratio hm0])
have htan-bc:
  on-line (steiner-deltoid-point a b c m)
  (steiner-deltoid-point a b c m + m) (foot b c m)
by (rule on-line-change-base[OF hline-bc hline-d])
have htan-ca:
  on-line (steiner-deltoid-point a b c m)
  (steiner-deltoid-point a b c m + m) (foot c a m)
by (rule on-line-change-base[OF hline-ca hline-d])
have hset:
  {z. on-line (steiner-deltoid-point a b c m)
    (steiner-deltoid-point a b c m + m) z} =
  {z. on-line (foot b c m) (foot c a m) z}
proof (rule on-line-set-eq-of-distinct[OF hdir hne])
  show on-line (steiner-deltoid-point a b c m)
    (steiner-deltoid-point a b c m + m) (foot b c m)
  using htan-bc .
  show on-line (steiner-deltoid-point a b c m)
    (steiner-deltoid-point a b c m + m) (foot c a m)
  using htan-ca .
qed
have hset-rev:
  {z. on-line (foot b c m) (foot c a m) z} =
  {z. on-line (steiner-deltoid-point a b c m)
    (steiner-deltoid-point a b c m + m) z}
  using hset by (rule sym)
show {z. on-line (foot b c m) (foot c a m) z} =
  steiner-deltoid-tangent-at a b c m
  using hset-rev
  by (simp add: steiner-deltoid-tangent-at-def
    steiner-deltoid-tangent-direction-at-def hnormal0)
qed
show ?thesis using hfirst hsecond by blast

```

```

next
  case False
  have hnormal:
    steiner-deltoid-normal a b c m  $\neq 0$ 
  using steiner-deltoid-normal-nonzero-iff-cube-neq[OF hm]
    False by blast
  show ?thesis
    by (rule simson-line-eq-steiner-deltoid-tangent-at
      [OF hncol ha hb hc hm hnormal])
qed
qed

theorem simson-line-eq-steiner-deltoid-tangent-all:
  fixes a b c :: complex and t :: real
  assumes hncol:  $\neg \text{collinear } a \ b \ c$ 
    and ha: oncircle 0 1 a and hb: oncircle 0 1 b
    and hc: oncircle 0 1 c
  shows (unit-circle-param t  $\neq b \longrightarrow$ 
    {z. on-line (foot a b (unit-circle-param t))
      (foot b c (unit-circle-param t)) z} =
      steiner-deltoid-tangent a b c t)  $\wedge$ 
    (unit-circle-param t = b  $\longrightarrow$ 
      {z. on-line (foot b c (unit-circle-param t))
        (foot c a (unit-circle-param t)) z} =
        steiner-deltoid-tangent a b c t)
  proof -
    have hm: oncircle 0 1 (unit-circle-param t)
    by (rule unit-circle-param-oncircle)
    have hAt:
      (unit-circle-param t  $\neq b \longrightarrow$ 
        {z. on-line (foot a b (unit-circle-param t))
          (foot b c (unit-circle-param t)) z} =
          steiner-deltoid-tangent-at a b c (unit-circle-param t))  $\wedge$ 
        (unit-circle-param t = b  $\longrightarrow$ 
          {z. on-line (foot b c (unit-circle-param t))
            (foot c a (unit-circle-param t)) z} =
            steiner-deltoid-tangent-at a b c (unit-circle-param t))
      by (rule simson-line-eq-steiner-deltoid-tangent-at-all
        [OF hncol ha hb hc hm])
    have heq:
      steiner-deltoid-tangent a b c t =
      steiner-deltoid-tangent-at a b c (unit-circle-param t)
    by (rule steiner-deltoid-tangent-eq-tangent-at)
    show ?thesis using hAt heq by blast
  qed
qed

```

```

theorem simson-line-is-tangent-to-steiner-deltoid-regular:
  fixes a b c :: complex and t :: real
  assumes hncol:  $\neg \text{collinear } a \ b \ c$ 

```

```

and ha: oncircle 0 1 a and hb: oncircle 0 1 b
and hc: oncircle 0 1 c
and hregular: (unit-circle-param t)  $\wedge^3 \neq a * b * c$ 
shows (unit-circle-param t  $\neq b \longrightarrow$ 
  foot a b (unit-circle-param t)  $\in$  steiner-deltoid-tangent a b c t  $\wedge$ 
  foot b c (unit-circle-param t)  $\in$  steiner-deltoid-tangent a b c t  $\wedge$ 
  foot a b (unit-circle-param t)  $\neq$  foot b c (unit-circle-param t))  $\wedge$ 
  (unit-circle-param t = b  $\longrightarrow$ 
    foot b c (unit-circle-param t)  $\in$  steiner-deltoid-tangent a b c t  $\wedge$ 
    foot c a (unit-circle-param t)  $\in$  steiner-deltoid-tangent a b c t  $\wedge$ 
    foot b c (unit-circle-param t)  $\neq$  foot c a (unit-circle-param t))
proof -
  have hm: oncircle 0 1 (unit-circle-param t)
  by (rule unit-circle-param-oncircle)
  have hnormal:
    steiner-deltoid-normal a b c (unit-circle-param t)  $\neq 0$ 
  using steiner-deltoid-normal-nonzero-iff-cube-neq[OF hm]
  hregular by blast
  have hAt:
    (unit-circle-param t  $\neq b \longrightarrow$ 
      foot a b (unit-circle-param t)  $\in$ 
        steiner-deltoid-tangent-at a b c (unit-circle-param t)  $\wedge$ 
      foot b c (unit-circle-param t)  $\in$ 
        steiner-deltoid-tangent-at a b c (unit-circle-param t)  $\wedge$ 
      foot a b (unit-circle-param t)  $\neq$ 
        foot b c (unit-circle-param t))  $\wedge$ 
    (unit-circle-param t = b  $\longrightarrow$ 
      foot b c (unit-circle-param t)  $\in$ 
        steiner-deltoid-tangent-at a b c (unit-circle-param t)  $\wedge$ 
      foot c a (unit-circle-param t)  $\in$ 
        steiner-deltoid-tangent-at a b c (unit-circle-param t)  $\wedge$ 
      foot b c (unit-circle-param t)  $\neq$ 
        foot c a (unit-circle-param t))
  by (rule simson-line-is-tangent-to-steiner-deltoid-at-regular
    [OF hncol ha hb hc hm hnormal])
  have heq:
    steiner-deltoid-tangent a b c t =
      steiner-deltoid-tangent-at a b c (unit-circle-param t)
  by (rule steiner-deltoid-tangent-eq-tangent-at)
  show ?thesis using hAt heq by blast
qed

```

```

theorem simson-line-meets-steiner-deltoid:
  fixes a b c m :: complex
  assumes hncol:  $\neg$  collinear a b c
  and ha: oncircle 0 1 a and hb: oncircle 0 1 b
  and hc: oncircle 0 1 c and hm: oncircle 0 1 m
  shows steiner-deltoid-point a b c m  $\in$  steiner-deltoid a b c  $\wedge$ 
    ((m  $\neq b \longrightarrow$ 

```

```

      on-line (foot a b m) (foot b c m)
      (steiner-deltoid-point a b c m)) ∧
(m ≠ c →
  on-line (foot b c m) (foot c a m)
  (steiner-deltoid-point a b c m)) ∧
(m ≠ a →
  on-line (foot c a m) (foot a b m)
  (steiner-deltoid-point a b c m)))
using steiner-deltoid-point-mem[OF hm]
simson-line-contains-steiner-deltoid-point[OF hncol ha hb hc hm]
by blast

end

```

References

- [1] F. Botana and J. L. Valcarce. Automatic determination of envelopes and other derived curves within a graphic environment. *Mathematics and Computers in Simulation*, 67:3–13, 2004. DOI: <https://doi.org/10.1016/j.matcom.2004.05.004>.
- [2] J. H. Butchart. The deltoid regarded as the envelope of simson lines. *The American Mathematical Monthly*, 46(2):85–86, 1939.
- [3] H. S. M. Coxeter and S. L. Greitzer. *Geometry Revisited*, volume 19 of *New Mathematical Library*. Mathematical Association of America, 1967.
- [4] M. de Guzmán. The envelope of the wallace–simson lines of a triangle: A simple proof of the steiner theorem on the deltoid. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, 95(1):57–64, 2001.
- [5] A. F. Ramos, D. B. Hulak, and R. J. G. B. de Queiroz. The wallace–simson line theorem in isabelle/hol. *Archive of Formal Proofs*, 2026. <https://www.isa-afp.org/entries/Simson.html>.
- [6] W. Wallace. Some geometrical porisms, with examples of their application to the solution of problems. *Transactions of the Royal Society of Edinburgh*, 4:107–134, 1798.