Standard Borel Spaces

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Abstract

This entry includes a formalization of standard Borel spaces and (a variant of) the Borel isomorphism theorem. A separable complete metrizable topological space is called a polish space and a measurable space generated from a polish space is called a standard Borel space. We formalize the notion of standard Borel spaces by establishing set-based metric spaces, and then prove (a variant of) the Borel isomorphism theorem. The theorem states that a standard Borel spaces is either a countable discrete space or isomorphic to $\mathbb{R}$.

Contents

We refer to the HOL-Analysis library, the textbooks by Matsuzaka [2] and Srivastava [3], and the lecture note by Biskup [1].

1 Lemmas

theory *Lemmas-StandardBorel*
imports *HOL−Probability.Probability*
begin

1.1 Lemmas for Abstract Topology

1.1.1 Generated By

lemma *topology-generated-by-sub*:
assumes $\bigwedge U.\ U \in \mathcal{U} \Longrightarrow (\mathit{openin}\ X\ U)$
and $\mathit{openin}\ (\mathit{topology\text{-}generated\text{-}by}\ \mathcal{U})\ U$
shows $\mathit{openin}\ X\ U$
⟨*proof*⟩

lemma *topology-generated-by-open*:
$S = \mathit{topology\text{-}generated\text{-}by}\ \{U \mid U\ .\ \mathit{openin}\ S\ U\}$
⟨*proof*⟩

lemma *topology-generated-by-eq*:
assumes $\bigwedge U.\ U \in \mathcal{U} \Longrightarrow (\mathit{openin}\ (\mathit{topology\text{-}generated\text{-}by}\ \mathcal{O})\ U)$
and $\bigwedge U.\ U \in \mathcal{O} \Longrightarrow (\mathit{openin}\ (\mathit{topology\text{-}generated\text{-}by}\ \mathcal{U})\ U)$
shows $\mathit{topology\text{-}generated\text{-}by}\ \mathcal{O} = \mathit{topology\text{-}generated\text{-}by}\ \mathcal{U}$
⟨*proof*⟩

lemma *topology-generated-by-homeomorphic-spaces*:
assumes $\mathit{homeomorphic\text{-}map}\ X\ Y\ f\ X = \mathit{topology\text{-}generated\text{-}by}\ \mathcal{O}$
shows $Y = \mathit{topology\text{-}generated\text{-}by}\ ((`)\ f\ `\ \mathcal{O})$

⟨*proof*⟩

lemma *open-map-generated-topo*:
assumes $\bigwedge u.\ u \in U \Longrightarrow$ *openin S* (*f* ‘ *u*) *inj-on f* (*topspace* (*topology-generated-by U*))
shows *open-map* (*topology-generated-by U*) *S f*
⟨*proof*⟩

lemma *subtopology-generated-by*:
subtopology (*topology-generated-by* $\mathcal{O}$) $T =$ *topology-generated-by* $\{T \cap U \mid U.\ U \in \mathcal{O}\}$
⟨*proof*⟩

lemma *prod-topology-generated-by*:
topology-generated-by $\{\ U \times V \mid U\ V.\ U \in \mathcal{O} \wedge V \in \mathcal{U}\} =$ *prod-topology* (*topology-generated-by* $\mathcal{O}$) (*topology-generated-by* $\mathcal{U}$)
⟨*proof*⟩

lemma *prod-topology-generated-by-open*:
prod-topology S S′ = *topology-generated-by* $\{U \times V \mid U\ V.$ *openin S U* $\wedge$ *openin S′ V*$\}$
⟨*proof*⟩

lemma *product-topology-cong*:
assumes $\bigwedge i.\ i \in I \Longrightarrow S\ i = K\ i$
shows *product-topology S I* = *product-topology K I*
⟨*proof*⟩

lemma *topology-generated-by-without-empty*:
topology-generated-by $\mathcal{O} =$ *topology-generated-by* $\{\ U \in \mathcal{O}.\ U \neq \{\}\}$
⟨*proof*⟩

lemma *topology-from-bij*:
assumes *bij-betw f A* (*topspace S*)
shows *homeomorphic-map* (*pullback-topology A f S*) *S f topspace* (*pullback-topology A f S*) = *A*
⟨*proof*⟩

lemma *openin-pullback-topology′*:
assumes *bij-betw f A* (*topspace S*)
shows *openin* (*pullback-topology A f S*) *u* $\longleftrightarrow$ (*openin S* (*f* ‘ *u*)) $\wedge\ u \subseteq A$
⟨*proof*⟩

1.1.2 Isolated Point

definition *isolated-points-of* :: *′a topology* ⇒ *′a set* ⇒ *′a set* (**infixr** ‹*isolated′-points′-of*› *80*) **where**
X isolated-points-of A ≡ $\{x \in$ *topspace X* $\cap A.\ x \notin X$ *derived-set-of A*$\}$

lemma *isolated-points-of-eq*:
$X\ \mathit{isolated\text{-}points\text{-}of}\ A = \{x{\in}\mathit{topspace}\ X \cap A.\ \exists\, U.\ x \in U \wedge \mathit{openin}\ X\ U \wedge U \cap (A - \{x\}) = \{\}\}$
⟨*proof*⟩

lemma *in-isolated-points-of*:
$x \in X\ \mathit{isolated\text{-}points\text{-}of}\ A \longleftrightarrow x \in \mathit{topspace}\ X \wedge x \in A \wedge (\exists\, U.\ x \in U \wedge \mathit{openin}\ X\ U \wedge U \cap (A - \{x\}) = \{\})$
⟨*proof*⟩

lemma *derived-set-of-eq*:
$x \in X\ \mathit{derived\text{-}set\text{-}of}\ A \longleftrightarrow x \in X\ \mathit{closure\text{-}of}\ (A - \{x\})$
⟨*proof*⟩

1.1.3 Perfect Set

definition *perfect-set* :: *'a topology* ⇒ *'a set* ⇒ *bool* **where**
$\mathit{perfect\text{-}set}\ X\ A \longleftrightarrow \mathit{closedin}\ X\ A \wedge X\ \mathit{isolated\text{-}points\text{-}of}\ A = \{\}$

abbreviation $\mathit{perfect\text{-}space}\ X \equiv \mathit{perfect\text{-}set}\ X\ (\mathit{topspace}\ X)$

lemma *perfect-space-euclidean*: $\mathit{perfect\text{-}space}\ (\mathit{euclidean} :: {}'a :: \mathit{perfect\text{-}space}\ \mathit{topology})$
⟨*proof*⟩

lemma *perfect-setI*:
assumes $\mathit{closedin}\ X\ A$
and $\bigwedge x\ T.\ [\![x \in A;\ x \in T;\ \mathit{openin}\ X\ T]\!] \Longrightarrow \exists\, y{\neq}x.\ y \in T \wedge y \in A$
shows $\mathit{perfect\text{-}set}\ X\ A$
⟨*proof*⟩

lemma *perfect-spaceI*:
assumes $\bigwedge x\ T.\ [\![x \in T;\ \mathit{openin}\ X\ T]\!] \Longrightarrow \exists\, y{\neq}x.\ y \in T$
shows $\mathit{perfect\text{-}space}\ X$
⟨*proof*⟩

lemma *perfect-setD*:
assumes $\mathit{perfect\text{-}set}\ X\ A$
shows $\mathit{closedin}\ X\ A\ \ A \subseteq \mathit{topspace}\ X\ \ \bigwedge x\ T.\ [\![x \in A;\ x \in T;\ \mathit{openin}\ X\ T]\!] \Longrightarrow \exists\, y{\neq}x.\ y \in T \wedge y \in A$
⟨*proof*⟩

lemma *perfect-space-perfect*:
$\mathit{perfect\text{-}set}\ \mathit{euclidean}\ (\mathit{UNIV} :: {}'a :: \mathit{perfect\text{-}space}\ \mathit{set})$
⟨*proof*⟩

lemma *perfect-set-subtopology*:
assumes $\mathit{perfect\text{-}set}\ X\ A$
shows $\mathit{perfect\text{-}space}\ (\mathit{subtopology}\ X\ A)$

⟨*proof*⟩

1.1.4 Bases and Sub-Bases in Abstract Topology

definition *subbase-in* :: $[{}'a\ topology, {}'a\ set\ set] \Rightarrow bool$ **where**
subbase-in $S\ \mathcal{O} \longleftrightarrow S = topology\text{-}generated\text{-}by\ \mathcal{O}$

definition *base-in* :: $[{}'a\ topology, {}'a\ set\ set] \Rightarrow bool$ **where**
base-in $S\ \mathcal{O} \longleftrightarrow (\forall\, U.\ openin\ S\ U \longleftrightarrow (\exists \mathcal{U}.\ U = \bigcup \mathcal{U} \land \mathcal{U} \subseteq \mathcal{O}))$

lemma *second-countable-base-in*: *second-countable* $S \longleftrightarrow (\exists\, \mathcal{O}.\ countable\ \mathcal{O} \land base\text{-}in\ S\ \mathcal{O})$
⟨*proof*⟩

definition *zero-dimensional* :: $'a\ topology \Rightarrow bool$ **where**
zero-dimensional $S \longleftrightarrow (\exists\, \mathcal{O}.\ base\text{-}in\ S\ \mathcal{O} \land (\forall\, u{\in}\mathcal{O}.\ openin\ S\ u \land closedin\ S\ u))$

lemma *openin-base*:
  assumes $base\text{-}in\ S\ \mathcal{O}\quad U = \bigcup \mathcal{U}$ **and** $\mathcal{U} \subseteq \mathcal{O}$
  shows *openin* $S\ U$
  ⟨*proof*⟩

lemma *base-is-subbase*:
  assumes *base-in* $S\ \mathcal{O}$
  shows *subbase-in* $S\ \mathcal{O}$
  ⟨*proof*⟩

lemma *subbase-in-subset*:
  assumes *subbase-in* $S\ \mathcal{O}$ **and** $U \in \mathcal{O}$
  shows $U \subseteq topspace\ S$
  ⟨*proof*⟩

lemma *subbase-in-openin*:
  assumes *subbase-in* $S\ \mathcal{O}$ **and** $U \in \mathcal{O}$
  shows *openin* $S\ U$
  ⟨*proof*⟩

lemma *base-in-subset*:
  assumes *base-in* $S\ \mathcal{O}$ **and** $U \in \mathcal{O}$
  shows $U \subseteq topspace\ S$
  ⟨*proof*⟩

lemma *base-in-openin*:
  assumes *base-in* $S\ \mathcal{O}$ **and** $U \in \mathcal{O}$
  shows *openin* $S\ U$
  ⟨*proof*⟩

lemma *base-in-def2*:
  assumes $\bigwedge U.\ U \in \mathcal{O} \Longrightarrow openin\ S\ U$

shows *base-in S* $\mathcal{O} \longleftrightarrow (\forall\, U.\ \mathit{openin}\ S\ U \longrightarrow (\forall\, x{\in}U.\ \exists\, W{\in}\mathcal{O}.\ x \in W \wedge W \subseteq U))$
⟨*proof*⟩

lemma *base-in-def2'*:
base-in S $\mathcal{O} \longleftrightarrow (\forall\, b{\in}\mathcal{O}.\ \mathit{openin}\ S\ b) \wedge (\forall\, x.\ \mathit{openin}\ S\ x \longrightarrow (\exists\, B'{\subseteq}\mathcal{O}.\ \bigcup\ B' = x))$
⟨*proof*⟩

corollary *base-in-in-subset*:
assumes *base-in S* $\mathcal{O}$ *openin S u* $x \in u$
shows $\exists\, v{\in}\mathcal{O}.\ x \in v \wedge v \subseteq u$
⟨*proof*⟩

lemma *base-in-without-empty*:
assumes *base-in S* $\mathcal{O}$
shows *base-in S* $\{U \in \mathcal{O}.\ U \neq \{\}\}$
⟨*proof*⟩

lemma *second-countable-ex-without-empty*:
assumes *second-countable S*
shows $\exists\, \mathcal{O}.\ \mathit{countable}\ \mathcal{O} \wedge \mathit{base\text{-}in}\ S\ \mathcal{O} \wedge (\forall\, U{\in}\mathcal{O}.\ U \neq \{\})$
⟨*proof*⟩

lemma *subtopology-subbase-in*:
assumes *subbase-in S* $\mathcal{O}$
shows *subbase-in* (*subtopology S T*) $\{T \cap U \mid U.\ U \in \mathcal{O}\}$
⟨*proof*⟩

lemma *subtopology-base-in*:
assumes *base-in S* $\mathcal{O}$
shows *base-in* (*subtopology S T*) $\{T \cap U \mid U.\ U \in \mathcal{O}\}$
⟨*proof*⟩

lemma *second-countable-subtopology*:
assumes *second-countable S*
shows *second-countable* (*subtopology S T*)
⟨*proof*⟩

lemma *open-map-with-base*:
assumes *base-in S* $\mathcal{O}$ $\bigwedge A.\ A \in \mathcal{O} \Longrightarrow \mathit{openin}\ S'\ (f\ `\ A)$
shows *open-map S S' f*
⟨*proof*⟩

Construct a base from a subbase.

lemma *finite'-intersection-of-idempot* [*simp*]:
finite' intersection-of finite' intersection-of P = finite' intersection-of P
⟨*proof*⟩

lemma *finite′-intersection-of-countable*:
assumes *countable* $\mathcal{O}$
shows *countable* (*Collect* (*finite′ intersection-of* ($\lambda x.\ x \in \mathcal{O}$)))
⟨*proof*⟩

lemma *finite′-intersection-of-openin*:
assumes (*finite′ intersection-of* ($\lambda x.\ x \in \mathcal{O}$)) U
shows *openin* (*topology-generated-by* $\mathcal{O}$) U
⟨*proof*⟩

lemma *topology-generated-by-finite-intersections*:
topology-generated-by $\mathcal{O}$ = *topology-generated-by* (*Collect* (*finite′ intersection-of* ($\lambda x.\ x \in \mathcal{O}$)))
⟨*proof*⟩

lemma *base-from-subbase*:
assumes *subbase-in* S $\mathcal{O}$
shows *base-in* S (*Collect* (*finite′ intersection-of* ($\lambda x.\ x \in \mathcal{O}$)))
⟨*proof*⟩

lemma *countable-base-from-countable-subbase*:
assumes *countable* $\mathcal{O}$ **and** *subbase-in* S $\mathcal{O}$
shows *second-countable* S
⟨*proof*⟩

lemma *prod-topology-second-countable*:
assumes *second-countable* S **and** *second-countable* S'
shows *second-countable* (*prod-topology* S S')
⟨*proof*⟩

Abstract version of the theorem $\exists K.$ *topological-basis* $K \wedge$ *countable* $K \wedge (\forall k{\in}K.\ \exists X.\ k = Pi_E\ UNIV\ X \wedge (\forall i.\ open\ (X\ i)) \wedge finite\ \{i.\ X\ i \neq UNIV\})$.

lemma *product-topology-countable-base-in*:
assumes *countable* I **and** $\bigwedge i.\ i \in I \Longrightarrow$ *second-countable* (S i)
shows $\exists \mathcal{O}'.$ *countable* $\mathcal{O}' \wedge$ *base-in* (*product-topology* S I) $\mathcal{O}' \wedge$
$(\forall k \in \mathcal{O}'.\ \exists X.\ k = (\Pi_E\ i{\in}I.\ X\ i) \wedge (\forall i.\ openin\ (S\ i)\ (X\ i)) \wedge finite\ \{i.\ X\ i \neq topspace\ (S\ i)\} \wedge \{i.\ X\ i \neq topspace\ (S\ i)\} \subseteq I)$
⟨*proof*⟩

lemma *product-topology-second-countable*:
assumes *countable* I **and** $\bigwedge i.\ i \in I \Longrightarrow$ *second-countable* (S i)
shows *second-countable* (*product-topology* S I)
⟨*proof*⟩

lemma *second-countable-euclidean*[*simp*]:
second-countable (*euclidean* :: *′a* :: *second-countable-topology topology*)
⟨*proof*⟩

lemma *Cantor-Bendixon*:
assumes $\mathit{second\text{-}countable}\ X$
shows $\exists\, U\, P.\ \mathit{countable}\ U \wedge \mathit{openin}\ X\ U \wedge \mathit{perfect\text{-}set}\ X\ P \wedge U \cup P = \mathit{topspace}\ X \wedge U \cap P = \{\} \wedge (\forall\, a {\neq} \{\}.\ \mathit{openin}\ (\mathit{subtopology}\ X\ P)\ a \longrightarrow \mathit{uncountable}\ a)$
⟨*proof*⟩

1.1.5 Separable Spaces

definition $\mathit{dense\text{-}in} :: [{}'a\ \mathit{topology}, {}'a\ \mathit{set}] \Rightarrow \mathit{bool}$ **where**
$\mathit{dense\text{-}in}\ S\ U \longleftrightarrow (U \subseteq \mathit{topspace}\ S \wedge (\forall\, V.\ \mathit{openin}\ S\ V \longrightarrow V \neq \{\} \longrightarrow U \cap V \neq \{\}))$

lemma *dense-in-def2*:
$\mathit{dense\text{-}in}\ S\ U \longleftrightarrow (U \subseteq \mathit{topspace}\ S \wedge (S\ \mathit{closure\text{-}of}\ U) = \mathit{topspace}\ S)$
⟨*proof*⟩

lemma *dense-in-topspace*[*simp*]: $\mathit{dense\text{-}in}\ S\ (\mathit{topspace}\ S)$
⟨*proof*⟩

lemma *dense-in-subset*:
assumes $\mathit{dense\text{-}in}\ S\ U$
shows $U \subseteq \mathit{topspace}\ S$
⟨*proof*⟩

lemma *dense-in-nonempty*:
assumes $\mathit{topspace}\ S \neq \{\}$ $\mathit{dense\text{-}in}\ S\ U$
shows $U \neq \{\}$
⟨*proof*⟩

lemma *dense-inI*:
assumes $U \subseteq \mathit{topspace}\ S$
and $\bigwedge V.\ \mathit{openin}\ S\ V \Longrightarrow V \neq \{\} \Longrightarrow U \cap V \neq \{\}$
shows $\mathit{dense\text{-}in}\ S\ U$
⟨*proof*⟩

lemma *dense-in-infinite*:
assumes $\mathit{t1\text{-}space}\ X$ $\mathit{infinite}\ (\mathit{topspace}\ X)$ $\mathit{dense\text{-}in}\ X\ U$
shows $\mathit{infinite}\ U$
⟨*proof*⟩

lemma *dense-in-prod*:
assumes $\mathit{dense\text{-}in}\ S\ U$ **and** $\mathit{dense\text{-}in}\ S'\ U'$
shows $\mathit{dense\text{-}in}\ (\mathit{prod\text{-}topology}\ S\ S')\ (U \times U')$
⟨*proof*⟩

lemma *separable-space-def2*:$\mathit{separable\text{-}space}\ S \longleftrightarrow (\exists\, U.\ \mathit{countable}\ U \wedge \mathit{dense\text{-}in}\ S\ U)$
⟨*proof*⟩

lemma *countable-space-separable-space*:
  assumes *countable* (*topspace* S)
  shows *separable-space* S
  ⟨*proof*⟩

lemma *separable-space-prod*:
  assumes *separable-space* S **and** *separable-space* S'
  shows *separable-space* (*prod-topology* S S')
⟨*proof*⟩

lemma *dense-in-product*:
  assumes $\bigwedge i.\ i \in I \Longrightarrow$ *dense-in* $(T\ i)$ $(U\ i)$
  shows *dense-in* (*product-topology* T I) $(\Pi_E\ i{\in}I.\ U\ i)$
⟨*proof*⟩

lemma *separable-countable-product*:
  assumes *countable* I **and** $\bigwedge i.\ i \in I \Longrightarrow$ *separable-space* $(T\ i)$
  shows *separable-space* (*product-topology* T I)
⟨*proof*⟩

lemma *separable-finite-product*:
  assumes *finite* I **and** $\bigwedge i.\ i \in I \Longrightarrow$ *separable-space* $(T\ i)$
  shows *separable-space* (*product-topology* T I)
  ⟨*proof*⟩

1.1.6 G_δ Set

lemma *gdelta-inD*:
  assumes *gdelta-in* S A
  shows $\exists \mathcal{U}.\ \mathcal{U} \neq \{\} \wedge$ *countable* $\mathcal{U} \wedge (\forall\, b{\in}\mathcal{U}.$ *openin* $S\ b) \wedge A = \bigcap \mathcal{U}$
  ⟨*proof*⟩

lemma *gdelta-inD'*:
  assumes *gdelta-in* S A
  shows $\exists\ U.\ (\forall\, n{::}nat.$ *openin* $S\ (U\ n)) \wedge A = \bigcap$ (*range* U)
⟨*proof*⟩

lemma *gdelta-in-continuous-map*:
  assumes *continuous-map* X Y f *gdelta-in* Y a
  shows *gdelta-in* X $(f\ -`\ a \cap$ *topspace* $X)$
⟨*proof*⟩

lemma *g-delta-of-inj-open-map*:
  assumes *open-map* X Y f *inj-on* f (*topspace* X) *gdelta-in* X a
  shows *gdelta-in* Y $(f\ `\ a)$
⟨*proof*⟩

lemma *gdelta-in-prod*:
  assumes *gdelta-in* X A *gdelta-in* Y B

shows *gdelta-in* (*prod-topology X Y*) ($A \times B$)
⟨*proof*⟩

corollary *gdelta-in-prod1*:
assumes *gdelta-in X A*
shows *gdelta-in* (*prod-topology X Y*) ($A \times$ *topspace Y*)
⟨*proof*⟩

corollary *gdelta-in-prod2*:
assumes *gdelta-in Y B*
shows *gdelta-in* (*prod-topology X Y*) (*topspace X* $\times B$)
⟨*proof*⟩

lemma *continuous-map-imp-closed-graph'*:
assumes *continuous-map X Y f Hausdorff-space Y*
shows *closedin* (*prod-topology Y X*) (($\lambda x.$ (*f x*,*x*)) ‘ *topspace X*)
⟨*proof*⟩

1.1.7 Continuous Maps on First Countable Topology

Generalized version of *Metric-space ?M ?d* $\Longrightarrow$ *eventually ?P* (*atin* (*Metric-space.mtopology ?M ?d*) *?a*) = ($\forall \sigma$. *range* $\sigma \subseteq$ *?M* − {*?a*} $\wedge$ *limitin* (*Metric-space.mtopology ?M ?d*) σ *?a sequentially* $\longrightarrow$ ($\forall_F$ *n in sequentially*. *?P* (σ *n*)))

lemma *eventually-atin-sequentially*:
assumes *first-countable X*
shows *eventually P* (*atin X a*) $\longleftrightarrow$ ($\forall \sigma$. *range* $\sigma \subseteq$ *topspace X* − {*a*} $\wedge$ *limitin X* σ *a sequentially* $\longrightarrow$ *eventually* ($\lambda n.$ *P* (σ *n*)) *sequentially*)
⟨*proof*⟩

lemma *continuous-map-iff-limit-seq*:
assumes *first-countable X*
shows *continuous-map X Y f* $\longleftrightarrow$ ($\forall$ *xn x*. *limitin X xn x sequentially* $\longrightarrow$ *limitin Y* ($\lambda n.$ *f* (*xn n*)) (*f x*) *sequentially*)
⟨*proof*⟩

1.1.8 Upper-Semicontinuous Functions

definition *upper-semicontinuous-map* :: [*'a topology*, *'a* ⇒ *'b* :: *linorder-topology*] ⇒ *bool* **where**
upper-semicontinuous-map X f $\longleftrightarrow$ ($\forall a$. *openin X* {*x*∈*topspace X*. *f x* < *a*})

lemma *continuous-upper-semicontinuous*:
assumes *continuous-map X* (*euclidean* :: (*'b* :: *linorder-topology*) *topology*) *f*
shows *upper-semicontinuous-map X f*
⟨*proof*⟩

lemma *upper-semicontinuous-map-iff-closed*:
upper-semicontinuous-map X f $\longleftrightarrow$ ($\forall a$. *closedin X* {*x*∈*topspace X*. *f x* ≥ *a*})
⟨*proof*⟩

lemma *upper-semicontinuous-map-real-iff*:
fixes *f* :: *'a ⇒ real*
shows *upper-semicontinuous-map X f* ⟷ *upper-semicontinuous-map X* (*λx. ereal* (*f x*))
⟨*proof*⟩

1.1.9 Lower-Semicontinuous Functions

definition *lower-semicontinuous-map* :: [*'a topology*, *'a ⇒ 'b* :: *linorder-topology*] ⇒ *bool* **where**
lower-semicontinuous-map X f ⟷ (∀ *a. openin X* {*x*∈*topspace X. a* < *f x*})

lemma *continuous-lower-semicontinuous*:
assumes *continuous-map X* (*euclidean* :: (*'b* :: *linorder-topology*) *topology*) *f*
shows *lower-semicontinuous-map X f*
⟨*proof*⟩

lemma *lower-semicontinuous-map-iff-closed*:
lower-semicontinuous-map X f ⟷ (∀ *a. closedin X* {*x*∈*topspace X. f x* ≤ *a*})
⟨*proof*⟩

lemma *lower-semicontinuous-map-real-iff*:
fixes *f* :: *'a ⇒ real*
shows *lower-semicontinuous-map X f* ⟷ *lower-semicontinuous-map X* (*λx. ereal* (*f x*))
⟨*proof*⟩

1.2 Lemmas for Measure Theory

1.2.1 Lemmas for Measurable Sets

lemma *measurable-preserve-sigma-sets*:
assumes *sets M* = *sigma-sets* Ω *S S* ⊆ *Pow* Ω
⋀*a. a* ∈ *S* ⟹ *f* ‘ *a* ∈ *sets N inj-on f* (*space M*) *f* ‘ *space M* ∈ *sets N*
and *b* ∈ *sets M*
shows *f* ‘ *b* ∈ *sets N*
⟨*proof*⟩

inductive-set *sigma-sets-cinter* :: *'a set ⇒ 'a set set ⇒ 'a set set*
for *sp* :: *'a set* **and** *A* :: *'a set set*
where
Basic-c[*intro*, *simp*]: *a* ∈ *A* ⟹ *a* ∈ *sigma-sets-cinter sp A*
| *Top-c*[*simp*]: *sp* ∈ *sigma-sets-cinter sp A*
| *Inter-c*: (⋀*i*::*nat. a i* ∈ *sigma-sets-cinter sp A*) ⟹ (⋂ *i. a i*) ∈ *sigma-sets-cinter sp A*
| *Union-c*: (⋀*i*::*nat. a i* ∈ *sigma-sets-cinter sp A*) ⟹ (⋃ *i. a i*) ∈ *sigma-sets-cinter sp A*

inductive-set *sigma-sets-cinter-dunion* :: *'a set ⇒ 'a set set ⇒ 'a set set*

for *sp* :: *'a set* **and** *A* :: *'a set set*
where
Basic-cd[*intro*, *simp*]: *a* ∈ *A* ⟹ *a* ∈ *sigma-sets-cinter-dunion sp A*
| *Top-cd*[*simp*]: *sp* ∈ *sigma-sets-cinter-dunion sp A*
| *Inter-cd*: (⋀*i*::*nat*. *a i* ∈ *sigma-sets-cinter-dunion sp A*) ⟹ (⋂*i*. *a i*) ∈ *sigma-sets-cinter-dunion sp A*
| *Union-cd*: (⋀*i*::*nat*. *a i* ∈ *sigma-sets-cinter-dunion sp A*) ⟹ *disjoint-family a* ⟹ (⋃*i*. *a i*) ∈ *sigma-sets-cinter-dunion sp A*

lemma *sigma-sets-cinter-dunion-subset*: *sigma-sets-cinter-dunion sp A* ⊆ *sigma-sets-cinter sp A*
⟨*proof*⟩

lemma *sigma-sets-cinter-into-sp*:
assumes *A* ⊆ *Pow sp x* ∈ *sigma-sets-cinter sp A*
shows *x* ⊆ *sp*
⟨*proof*⟩

lemma *sigma-sets-cinter-dunion-into-sp*:
assumes *A* ⊆ *Pow sp x* ∈ *sigma-sets-cinter-dunion sp A*
shows *x* ⊆ *sp*
⟨*proof*⟩

lemma *sigma-sets-cinter-int*:
assumes *a* ∈ *sigma-sets-cinter sp A b* ∈ *sigma-sets-cinter sp A*
shows *a* ∩ *b* ∈ *sigma-sets-cinter sp A*
⟨*proof*⟩

lemma *sigma-sets-cinter-dunion-int*:
assumes *a* ∈ *sigma-sets-cinter-dunion sp A b* ∈ *sigma-sets-cinter-dunion sp A*
shows *a* ∩ *b* ∈ *sigma-sets-cinter-dunion sp A*
⟨*proof*⟩

lemma *sigma-sets-cinter-un*:
assumes *a* ∈ *sigma-sets-cinter sp A b* ∈ *sigma-sets-cinter sp A*
shows *a* ∪ *b* ∈ *sigma-sets-cinter sp A*
⟨*proof*⟩

lemma *sigma-sets-eq-cinter-dunion*:
assumes *metrizable-space X*
shows *sigma-sets* (*topspace X*) {*U*. *openin X U*} = *sigma-sets-cinter-dunion* (*topspace X*) {*U*. *openin X U*}
⟨*proof*⟩

lemma *sigma-sets-eq-cinter*:
assumes *metrizable-space X*
shows *sigma-sets* (*topspace X*) {*U*. *openin X U*} = *sigma-sets-cinter* (*topspace X*) {*U*. *openin X U*}
⟨*proof*⟩

1.2.2 Measurable Isomorphisms

definition *measurable-isomorphic-map*$::[{}'a\ \mathit{measure},\ {}'b\ \mathit{measure},\ {}'a \Rightarrow {}'b] \Rightarrow \mathit{bool}$
where
$\mathit{measurable\text{-}isomorphic\text{-}map}\ M\ N\ f \longleftrightarrow \mathit{bij\text{-}betw}\ f\ (\mathit{space}\ M)\ (\mathit{space}\ N) \land f \in M \rightarrow_M N \land \mathit{the\text{-}inv\text{-}into}\ (\mathit{space}\ M)\ f \in N \rightarrow_M M$

lemma *measurable-isomorphic-map-sets-cong*:
assumes $\mathit{sets}\ M = \mathit{sets}\ M'$ $\mathit{sets}\ N = \mathit{sets}\ N'$
shows $\mathit{measurable\text{-}isomorphic\text{-}map}\ M\ N\ f \longleftrightarrow \mathit{measurable\text{-}isomorphic\text{-}map}\ M'\ N'\ f$
⟨*proof*⟩

lemma *measurable-isomorphic-map-surj*:
assumes $\mathit{measurable\text{-}isomorphic\text{-}map}\ M\ N\ f$
shows $f\ `\ \mathit{space}\ M = \mathit{space}\ N$
⟨*proof*⟩

lemma *measurable-isomorphic-mapI*:
assumes $\mathit{bij\text{-}betw}\ f\ (\mathit{space}\ M)\ (\mathit{space}\ N)$ $f \in M \rightarrow_M N$ $\mathit{the\text{-}inv\text{-}into}\ (\mathit{space}\ M)\ f \in N \rightarrow_M M$
shows $\mathit{measurable\text{-}isomorphic\text{-}map}\ M\ N\ f$
⟨*proof*⟩

lemma *measurable-isomorphic-map-byWitness*:
assumes $f \in M \rightarrow_M N$ $g \in N \rightarrow_M M$ $\bigwedge x.\ x \in \mathit{space}\ M \Longrightarrow g\ (f\ x) = x$ $\bigwedge x.\ x \in \mathit{space}\ N \Longrightarrow f\ (g\ x) = x$
shows $\mathit{measurable\text{-}isomorphic\text{-}map}\ M\ N\ f$
⟨*proof*⟩

lemma *measurable-isomorphic-map-restrict-space*:
assumes $f \in M \rightarrow_M N$ $\bigwedge A.\ A \in \mathit{sets}\ M \Longrightarrow f\ `\ A \in \mathit{sets}\ N$ $\mathit{inj\text{-}on}\ f\ (\mathit{space}\ M)$
shows $\mathit{measurable\text{-}isomorphic\text{-}map}\ M\ (\mathit{restrict\text{-}space}\ N\ (f\ `\ \mathit{space}\ M))\ f$
⟨*proof*⟩

lemma *measurable-isomorphic-mapD'*:
assumes $\mathit{measurable\text{-}isomorphic\text{-}map}\ M\ N\ f$
shows $\bigwedge A.\ A \in \mathit{sets}\ M \Longrightarrow f\ `\ A \in \mathit{sets}\ N$ $f \in M \rightarrow_M N$
$\exists g.\ \mathit{bij\text{-}betw}\ g\ (\mathit{space}\ N)\ (\mathit{space}\ M) \land g \in N \rightarrow_M M \land (\forall x \in \mathit{space}\ M.\ g\ (f\ x) = x) \land (\forall x\in \mathit{space}\ N.\ f\ (g\ x) = x) \land (\forall A\in\mathit{sets}\ N.\ g\ `\ A \in \mathit{sets}\ M)$
⟨*proof*⟩

lemma *measurable-isomorphic-map-inv*:
assumes $\mathit{measurable\text{-}isomorphic\text{-}map}\ M\ N\ f$
shows $\mathit{measurable\text{-}isomorphic\text{-}map}\ N\ M\ (\mathit{the\text{-}inv\text{-}into}\ (\mathit{space}\ M)\ f)$
⟨*proof*⟩

lemma *measurable-isomorphic-map-comp*:
assumes $\mathit{measurable\text{-}isomorphic\text{-}map}\ M\ N\ f$ **and** $\mathit{measurable\text{-}isomorphic\text{-}map}\ N\ L\ g$

shows *measurable-isomorphic-map M L* $(g \circ f)$
$\langle \mathit{proof} \rangle$

definition *measurable-isomorphic*::[$'$*a measure*, $'$*b measure*] $\Rightarrow$ *bool* (**infixr** ‹*measurable$'$-isomorphic*› *50*) **where**
M measurable-isomorphic N $\longleftrightarrow$ ($\exists f$. *measurable-isomorphic-map M N f*)

lemma *measurable-isomorphic-sets-cong*:
assumes *sets M* = *sets M$'$* *sets N* = *sets N$'$*
shows *M measurable-isomorphic N* $\longleftrightarrow$ *M$'$ measurable-isomorphic N$'$*
$\langle \mathit{proof} \rangle$

lemma *measurable-isomorphicD*:
assumes *M measurable-isomorphic N*
shows $\exists f\ g.\ f \in M \rightarrow_M N \wedge g \in N \rightarrow_M M \wedge (\forall x{\in}\mathit{space}\ M.\ g\ (f\ x) = x) \wedge (\forall y{\in}\mathit{space}\ N.\ f\ (g\ y) = y) \wedge (\forall A{\in}\mathit{sets}\ M.\ f\ `\ A \in \mathit{sets}\ N) \wedge (\forall A{\in}\mathit{sets}\ N.\ g\ `\ A \in \mathit{sets}\ M)$
$\langle \mathit{proof} \rangle$

lemma *measurable-isomorphic-cardinality-eq*:
assumes *M measurable-isomorphic N*
shows *space M* $\approx$ *space N*
$\langle \mathit{proof} \rangle$

lemma *measurable-isomorphic-count-spaces*: *count-space A measurable-isomorphic count-space B* $\longleftrightarrow A \approx B$
$\langle \mathit{proof} \rangle$

lemma *measurable-isomorphic-byWitness*:
assumes $f \in M \rightarrow_M N$ $\bigwedge x.\ x{\in}\mathit{space}\ M \Longrightarrow g\ (f\ x) = x$
and $g \in N \rightarrow_M M$ $\bigwedge y.\ y{\in}\mathit{space}\ N \Longrightarrow f\ (g\ y) = y$
shows *M measurable-isomorphic N*
$\langle \mathit{proof} \rangle$

lemma *measurable-isomorphic-refl*:
M measurable-isomorphic M
$\langle \mathit{proof} \rangle$

lemma *measurable-isomorphic-sym*:
assumes *M measurable-isomorphic N*
shows *N measurable-isomorphic M*
$\langle \mathit{proof} \rangle$

lemma *measurable-isomorphic-trans*:
assumes *M measurable-isomorphic N* **and** *N measurable-isomorphic L*
shows *M measurable-isomorphic L*
$\langle \mathit{proof} \rangle$

lemma *measurable-isomorphic-empty*:

assumes $\mathit{space}\ M = \{\}\ \mathit{space}\ N = \{\}$
shows $M\ \mathit{measurable\text{-}isomorphic}\ N$
$\langle \mathit{proof} \rangle$

lemma *measurable-isomorphic-empty1*:
assumes $\mathit{space}\ M = \{\}\ M\ \mathit{measurable\text{-}isomorphic}\ N$
shows $\mathit{space}\ N = \{\}$
$\langle \mathit{proof} \rangle$

lemma *measurable-ismorphic-empty2*:
assumes $\mathit{space}\ N = \{\}\ M\ \mathit{measurable\text{-}isomorphic}\ N$
shows $\mathit{space}\ M = \{\}$
$\langle \mathit{proof} \rangle$

lemma *measurable-lift-product*:
assumes $\bigwedge i.\ i \in I \Longrightarrow f\ i \in (M\ i) \rightarrow_M (N\ i)$
shows $(\lambda x\ i.\ \mathit{if}\ i \in I\ \mathit{then}\ f\ i\ (x\ i)\ \mathit{else}\ \mathit{undefined}) \in (\Pi_M\ i{\in}I.\ M\ i) \rightarrow_M (\Pi_M\ i{\in}I.\ N\ i)$
$\langle \mathit{proof} \rangle$

lemma *measurable-isomorphic-map-lift-product*:
assumes $\bigwedge i.\ i \in I \Longrightarrow \mathit{measurable\text{-}isomorphic\text{-}map}\ (M\ i)\ (N\ i)\ (h\ i)$
shows $\mathit{measurable\text{-}isomorphic\text{-}map}\ (\Pi_M\ i{\in}I.\ M\ i)\ (\Pi_M\ i{\in}I.\ N\ i)\ (\lambda x\ i.\ \mathit{if}\ i \in I\ \mathit{then}\ h\ i\ (x\ i)\ \mathit{else}\ \mathit{undefined})$
$\langle \mathit{proof} \rangle$

lemma *measurable-isomorphic-lift-product*:
assumes $\bigwedge i.\ i \in I \Longrightarrow (M\ i)\ \mathit{measurable\text{-}isomorphic}\ (N\ i)$
shows $(\Pi_M\ i{\in}I.\ M\ i)\ \mathit{measurable\text{-}isomorphic}\ (\Pi_M\ i{\in}I.\ N\ i)$
$\langle \mathit{proof} \rangle$

https://math24.net/cantor-schroder-bernstein-theorem.html

lemma *Schroeder-Bernstein-measurable′*:
assumes $f\ `\ (\mathit{space}\ M) \in \mathit{sets}\ N\ g\ `\ (\mathit{space}\ N) \in \mathit{sets}\ M$
and $\mathit{measurable\text{-}isomorphic\text{-}map}\ M\ (\mathit{restrict\text{-}space}\ N\ (f\ `\ (\mathit{space}\ M)))\ f$ **and** $\mathit{measurable\text{-}isomorphic\text{-}map}\ N\ (\mathit{restrict\text{-}space}\ M\ (g\ `\ (\mathit{space}\ N)))\ g$
shows $\exists h.\ \mathit{measurable\text{-}isomorphic\text{-}map}\ M\ N\ h$
$\langle \mathit{proof} \rangle$

lemma *Schroeder-Bernstein-measurable*:
assumes $f \in M \rightarrow_M N\ \bigwedge A.\ A \in \mathit{sets}\ M \Longrightarrow f\ `\ A \in \mathit{sets}\ N\ \mathit{inj\text{-}on}\ f\ (\mathit{space}\ M)$
and $g \in N \rightarrow_M M\ \bigwedge A.\ A \in \mathit{sets}\ N \Longrightarrow g\ `\ A \in \mathit{sets}\ M\ \mathit{inj\text{-}on}\ g\ (\mathit{space}\ N)$
shows $\exists h.\ \mathit{measurable\text{-}isomorphic\text{-}map}\ M\ N\ h$
$\langle \mathit{proof} \rangle$

lemma *measurable-isomorphic-from-embeddings*:
assumes $M\ \mathit{measurable\text{-}isomorphic}\ (\mathit{restrict\text{-}space}\ N\ B)\ N\ \mathit{measurable\text{-}isomorphic}\ (\mathit{restrict\text{-}space}\ M\ A)$
and $A \in \mathit{sets}\ M\ B \in \mathit{sets}\ N$

shows *M measurable-isomorphic N*
⟨*proof*⟩

lemma *measurable-isomorphic-antisym*:
assumes *B measurable-isomorphic* (*restrict-space C c*) *A measurable-isomorphic* (*restrict-space B b*)
and *c* ∈ *sets C b* ∈ *sets B C measurable-isomorphic A*
shows *C measurable-isomorphic B*
⟨*proof*⟩

lemma *countable-infinite-isomorphisc-to-nat-index*:
assumes *countable I* **and** *infinite I*
shows (Π_M *x*∈*I*. *M*) *measurable-isomorphic* (Π_M (*x*::*nat*)∈*UNIV*. *M*)
⟨*proof*⟩

lemma *PiM-PiM-isomorphic-to-PiM*:
(Π_M *i*∈*I*. Π_M *j*∈*J*. *M i j*) *measurable-isomorphic* (Π_M (*i*,*j*)∈*I*×*J*. *M i j*)
⟨*proof*⟩

lemma *measurable-isomorphic-map-sigma-sets*:
assumes *sets M* = *sigma-sets* (*space M*) *U measurable-isomorphic-map M N f*
shows *sets N* = *sigma-sets* (*space N*) ((‘) *f* ‘ *U*)
⟨*proof*⟩

1.2.3 Borel Spaces Genereted from Abstract Topologies

definition *borel-of* :: *'a topology* ⇒ *'a measure* **where**
borel-of X ≡ *sigma* (*topspace X*) {*U*. *openin X U*}

lemma *emeasure-borel-of*: *emeasure* (*borel-of X*) *A* = *0*
⟨*proof*⟩

lemma *borel-of-euclidean*: *borel-of euclidean* = *borel*
⟨*proof*⟩

lemma *space-borel-of*: *space* (*borel-of X*) = *topspace X*
⟨*proof*⟩

lemma *sets-borel-of*: *sets* (*borel-of X*) = *sigma-sets* (*topspace X*) {*U*. *openin X U*}
⟨*proof*⟩

lemma *sets-borel-of-closed*: *sets* (*borel-of X*) = *sigma-sets* (*topspace X*) {*U*. *closedin X U*}
⟨*proof*⟩

lemma *borel-of-open*:
assumes *openin X U*
shows *U* ∈ *sets* (*borel-of X*)

⟨*proof*⟩

lemma *borel-of-closed*:
assumes *closedin X U*
shows *U* $\in$ *sets* (*borel-of X*)
⟨*proof*⟩

lemma(**in** *Metric-space*) *nbh-sets*[*measurable*]: ($\bigcup$ *a*$\in$*A*. *mball a e*) $\in$ *sets* (*borel-of mtopology*)
⟨*proof*⟩

lemma *borel-of-gdelta-in*:
assumes *gdelta-in X U*
shows *U* $\in$ *sets* (*borel-of X*)
⟨*proof*⟩

lemma *borel-of-subtopology*:
borel-of (*subtopology X U*) = *restrict-space* (*borel-of X*) *U*
⟨*proof*⟩

lemma *sets-borel-of-discrete-topology*: *sets* (*borel-of* (*discrete-topology I*)) = *sets* (*count-space I*)
⟨*proof*⟩

lemma *continuous-map-measurable*:
assumes *continuous-map X Y f*
shows *f* $\in$ *borel-of X* $\rightarrow_M$ *borel-of Y*
⟨*proof*⟩

lemma *upper-semicontinuous-map-measurable*:
fixes *f* :: *'a* $\Rightarrow$ *'b* :: {*linorder-topology*, *second-countable-topology*}
assumes *upper-semicontinuous-map X f*
shows *f* $\in$ *borel-measurable* (*borel-of X*)
⟨*proof*⟩

lemma *lower-semicontinuous-map-measurable*:
fixes *f* :: *'a* $\Rightarrow$ *'b* :: {*linorder-topology*, *second-countable-topology*}
assumes *lower-semicontinuous-map X f*
shows *f* $\in$ *borel-measurable* (*borel-of X*)
⟨*proof*⟩

lemma *open-map-preserves-sets*:
assumes *open-map S T f inj-on f* (*topspace S*) *A* $\in$ *sets* (*borel-of S*)
shows *f* ‘ *A* $\in$ *sets* (*borel-of T*)
⟨*proof*⟩

lemma *open-map-preserves-sets'*:
assumes *open-map S* (*subtopology T* (*f* ‘ (*topspace S*))) *f inj-on f* (*topspace S*) *f* ‘ (*topspace S*) $\in$ *sets* (*borel-of T*) *A* $\in$ *sets* (*borel-of S*)

shows f ‘ $A \in$ *sets* (*borel-of* T)
⟨*proof*⟩

Abstract topology version of *open* = *generate-topology ?X* $\Longrightarrow$ *borel* = *sigma UNIV ?X*.

lemma *borel-of-second-countable′*:
assumes *second-countable* S **and** *subbase-in* S $\mathcal{U}$
shows *borel-of* S = *sigma* (*topspace* S) $\mathcal{U}$
⟨*proof*⟩

Abstract topology version *borel* $\bigotimes_M$ *borel* = *borel*.

lemma *borel-of-prod*:
assumes *second-countable* S **and** *second-countable* S'
shows *borel-of* S $\bigotimes_M$ *borel-of* S' = *borel-of* (*prod-topology* S S')
⟨*proof*⟩

lemma *product-borel-of-measurable*:
assumes $i \in I$
shows ($\lambda x.\ x\ i$) $\in$ (*borel-of* (*product-topology* S I)) $\to_M$ *borel-of* (S i)
⟨*proof*⟩

Abstract topology version of *sets* (Pi_M *UNIV* (λ-. *borel*)) $\subseteq$ *sets borel*

lemma *sets-PiM-subset-borel-of*:
sets (Π_M $i{\in}I$. *borel-of* (S i)) $\subseteq$ *sets* (*borel-of* (*product-topology* S I))
⟨*proof*⟩

Abstract topology version of *sets* (Pi_M *UNIV* (λi. *borel*)) = *sets borel*.

lemma *sets-PiM-equal-borel-of*:
assumes *countable* I **and** $\bigwedge i.\ i \in I \Longrightarrow$ *second-countable* (S i)
shows *sets* (Π_M $i{\in}I$. *borel-of* (S i)) = *sets* (*borel-of* (*product-topology* S I))
⟨*proof*⟩

lemma *homeomorphic-map-borel-isomorphic*:
assumes *homeomorphic-map* X Y f
shows *measurable-isomorphic-map* (*borel-of* X) (*borel-of* Y) f
⟨*proof*⟩

lemma *homeomorphic-space-measurable-isomorphic*:
assumes S *homeomorphic-space* T
shows *borel-of* S *measurable-isomorphic* *borel-of* T
⟨*proof*⟩

lemma *measurable-isomorphic-borel-map*:
assumes *sets* M = *sets* (*borel-of* S) **and** f: *measurable-isomorphic-map* M N f
shows $\exists\, S'$. *homeomorphic-map* S S' f $\wedge$ *sets* N = *sets* (*borel-of* S')
⟨*proof*⟩

lemma *measurable-isomorphic-borels*:

assumes *sets* M = *sets* (*borel-of* S) M *measurable-isomorphic* N
shows $\exists\, S'$. S *homeomorphic-space* $S' \wedge$ *sets* N = *sets* (*borel-of* S')
⟨*proof*⟩

end

1.3 Lemmas for Abstract Metric Spaces

theory *Set-Based-Metric-Space*
imports *Lemmas-StandardBorel*
begin

We prove additional lemmas related to set-based metric spaces.

1.3.1 Basic Lemmas

lemma
assumes *Metric-space* M d $\bigwedge x\ y.\ x \in M \Longrightarrow y \in M \Longrightarrow d\ x\ y = d'\ x\ y$
and $\bigwedge x\ y.\ d'\ x\ y = d'\ y\ x$ $\bigwedge x\ y.\ d'\ x\ y \geq 0$
shows *Metric-space-eq*: *Metric-space* M d'
and *Metric-space-eq-mtopology*: *Metric-space.mtopology* M d = *Metric-space.mtopology* M d'
and *Metric-space-eq-mcomplete*: *Metric-space.mcomplete* M d $\longleftrightarrow$ *Metric-space.mcomplete* M d'
⟨*proof*⟩

context *Metric-space*
begin

lemma *mtopology-base-in-balls*: *base-in mtopology* $\{$*mball* $a\ \varepsilon \mid a\ \varepsilon.\ a \in M \wedge \varepsilon > 0\}$
⟨*proof*⟩

lemma *closedin-metric2*: *closedin mtopology* $C \longleftrightarrow C \subseteq M \wedge (\forall x.\ x \in C \longleftrightarrow (\forall \varepsilon{>}0.$ *mball* $x\ \varepsilon \cap C \neq \{\}))$
⟨*proof*⟩

lemma *openin-mtopology2*:
openin mtopology $U \longleftrightarrow U \subseteq M \wedge (\forall xn\ x.$ *limitin mtopology* $xn\ x$ *sequentially* $\wedge\ x \in U \longrightarrow (\exists N.\ \forall n{\geq}N.\ xn\ n \in U))$
⟨*proof*⟩

lemma *closure-of-mball*: *mtopology closure-of mball* $a\ e \subseteq$ *mcball* $a\ e$
⟨*proof*⟩

lemma *interior-of-mcball*: *mball* $a\ e \subseteq$ *mtopology interior-of mcball* $a\ e$
⟨*proof*⟩

lemma *isolated-points-of-mtopology*:

mtopology isolated-points-of A = {*x*∈*M*∩*A*. ∀ *xn*. *range xn* ⊆ *A* ∧ *limitin mtopology xn x sequentially* ⟶ (∃ *no*. ∀ *n*≥*no*. *xn n* = *x*)}
⟨*proof*⟩

lemma *perfect-set-mball-infinite*:
assumes *perfect-set mtopology A a* ∈ *A e* > *0*
shows *infinite* (*mball a e*)
⟨*proof*⟩

lemma *MCauchy-dist-Cauchy*:
assumes *MCauchy xn MCauchy yn*
shows *Cauchy* (λ*n*. *d* (*xn n*) (*yn n*))
⟨*proof*⟩

1.3.2 Dense in Metric Spaces

abbreviation *mdense* ≡ *dense-in mtopology*

https://people.bath.ac.uk/mw2319/ma30252/sec-dense.html

lemma *mdense-def*:
mdense U ⟷ *U* ⊆ *M* ∧ (∀ *x*∈*M*. ∀ *ε*>*0*. *mball x ε* ∩ *U* ≠ {})
⟨*proof*⟩

corollary *mdense-balls-cover*:
assumes *mdense U* **and** *e* > *0*
shows (⋃ *u*∈*U*. *mball u e*) = *M*
⟨*proof*⟩

lemma *mdense-empty-iff*: *mdense* {} ⟷ *M* = {}
⟨*proof*⟩

lemma *mdense-M*: *mdense M*
⟨*proof*⟩

lemma *mdense-def2*:
mdense U ⟷ *U* ⊆ *M* ∧ (∀ *x*∈*M*. ∀ *ε*>*0*.∃ *y*∈*U*. *d x y* < *ε*)
⟨*proof*⟩

lemma *mdense-def3*:
mdense U ⟷ *U* ⊆ *M* ∧ (∀ *x*∈*M*. ∃ *xn*. *range xn* ⊆ *U* ∧ *limitin mtopology xn x sequentially*)
⟨*proof*⟩

Diameter

definition *mdiameter* :: *'a set* ⇒ *ennreal* **where**
mdiameter A ≡ ⨆ {*ennreal* (*d x y*) | *x y*. *x* ∈ *A* ∩ *M* ∧ *y* ∈ *A* ∩ *M*}

lemma *mdiameter-empty*[*simp*]:
mdiameter {} = *0*

⟨*proof*⟩

lemma *mdiameter-def2*:
assumes $A \subseteq M$
shows $\mathit{mdiameter}\ A = \bigsqcup\ \{\mathit{ennreal}\ (d\ x\ y) \mid x\ y.\ x \in A \wedge y \in A\}$
⟨*proof*⟩

lemma *mdiameter-subset*:
assumes $A \subseteq B$
shows $\mathit{mdiameter}\ A \le \mathit{mdiameter}\ B$
⟨*proof*⟩

lemma *mdiameter-cball-leq*: $\mathit{mdiameter}\ (\mathit{mcball}\ a\ \varepsilon) \le \mathit{ennreal}\ (2 * \varepsilon)$
⟨*proof*⟩

lemma *mdiameter-ball-leq*:
$\mathit{mdiameter}\ (\mathit{mball}\ a\ \varepsilon) \le \mathit{ennreal}\ (2 * \varepsilon)$
⟨*proof*⟩

lemma *mdiameter-is-sup*:
assumes $x \in A \cap M\ y \in A \cap M$
shows $d\ x\ y \le \mathit{mdiameter}\ A$
⟨*proof*⟩

lemma *mdiameter-is-sup'*:
assumes $x \in A \cap M\ y \in A \cap M\ \mathit{mdiameter}\ A \le \mathit{ennreal}\ r\ r \ge 0$
shows $d\ x\ y \le r$
⟨*proof*⟩

lemma *mdiameter-le*:
assumes $\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow d\ x\ y \le r$
shows $\mathit{mdiameter}\ A \le r$
⟨*proof*⟩

lemma *mdiameter-eq-closure*: $\mathit{mdiameter}\ (\mathit{mtopology}\ \mathit{closure\text{-}of}\ A) = \mathit{mdiameter}\ A$
⟨*proof*⟩

lemma *mbounded-finite-mdiameter*: $\mathit{mbounded}\ A \longleftrightarrow A \subseteq M \wedge \mathit{mdiameter}\ A < \infty$
⟨*proof*⟩

Distance between a point and a set.

definition *d-set* :: *'a set* ⇒ *'a* ⇒ *real* **where**
$\mathit{d\text{-}set}\ A \equiv (\lambda x.\ \mathit{if}\ A \neq \{\} \wedge A \subseteq M \wedge x \in M\ \mathit{then}\ \mathit{Inf}\ \{d\ x\ y\ |y.\ y \in A\}\ \mathit{else}\ 0)$

lemma *d-set-nonneg*[*simp*]:
$\mathit{d\text{-}set}\ A\ x \ge 0$
⟨*proof*⟩

lemma *d-set-bdd-below*[*simp*]:
$\mathit{bdd\text{-}below}\ \{d\ x\ y\ |y.\ y \in A\}$
⟨*proof*⟩

lemma *d-set-singleton*[*simp*]:
$x \in M \Longrightarrow y \in M \Longrightarrow \mathit{d\text{-}set}\ \{y\}\ x = d\ x\ y$
⟨*proof*⟩

lemma *d-set-empty*[*simp*]:
$\mathit{d\text{-}set}\ \{\}\ x = \mathit{0}$
⟨*proof*⟩

lemma *d-set-notin*:
$x \notin M \Longrightarrow \mathit{d\text{-}set}\ A\ x = \mathit{0}$
⟨*proof*⟩

lemma *d-set-inA*:
assumes $x \in A$
shows $\mathit{d\text{-}set}\ A\ x = \mathit{0}$
⟨*proof*⟩

lemma *d-set-nzeroD*:
assumes $\mathit{d\text{-}set}\ A\ x \neq \mathit{0}$
shows $A \subseteq M\ x \notin A\ A \neq \{\}$
⟨*proof*⟩

lemma *d-set-antimono*:
assumes $A \subseteq B\ A \neq \{\}\ B \subseteq M$
shows $\mathit{d\text{-}set}\ B\ x \leq \mathit{d\text{-}set}\ A\ x$
⟨*proof*⟩

lemma *d-set-bounded*:
assumes $\bigwedge y.\ y \in A \Longrightarrow d\ x\ y < K\ K > \mathit{0}$
shows $\mathit{d\text{-}set}\ A\ x < K$
⟨*proof*⟩

lemma *d-set-tr*:
assumes $x \in M\ y \in M$
shows $\mathit{d\text{-}set}\ A\ x \leq d\ x\ y + \mathit{d\text{-}set}\ A\ y$
⟨*proof*⟩

lemma *d-set-abs-le*:
assumes $x \in M\ y \in M$
shows $|\mathit{d\text{-}set}\ A\ x - \mathit{d\text{-}set}\ A\ y| \leq d\ x\ y$
⟨*proof*⟩

lemma *d-set-inA-le*:
assumes $y \in A$

shows $\mathit{d\text{-}set}\ A\ x \le d\ x\ y$
⟨*proof*⟩

lemma *d-set-ball-empty*:
assumes $A \neq \{\}\ A \subseteq M\ e > 0\ x \in M\ \mathit{mball}\ x\ e \cap A = \{\}$
shows $\mathit{d\text{-}set}\ A\ x \ge e$
⟨*proof*⟩

lemma *d-set-closed-pos*:
assumes $\mathit{closedin}\ \mathit{mtopology}\ A\ A \neq \{\}\ x \in M\ x \notin A$
shows $\mathit{d\text{-}set}\ A\ x > 0$
⟨*proof*⟩

lemma *gdelta-in-closed*:
assumes $\mathit{closedin}\ \mathit{mtopology}\ M$
shows $\mathit{gdelta\text{-}in}\ \mathit{mtopology}\ M$
⟨*proof*⟩

Oscillation

definition $\mathit{osc\text{-}on} :: [\prime b\ \mathit{set}, \prime b\ \mathit{topology}, \prime b \Rightarrow \prime a, \prime b] \Rightarrow \mathit{ennreal}$ **where**
$\mathit{osc\text{-}on}\ A\ X\ f \equiv (\lambda y.\ \bigsqcap\ \{\mathit{mdiameter}\ (f\ \text{`}\ (A \cap U))\ |U.\ y \in U \wedge \mathit{openin}\ X\ U\})$

abbreviation $\mathit{osc}\ X \equiv \mathit{osc\text{-}on}\ (\mathit{topspace}\ X)\ X$

lemma *osc-def*: $\mathit{osc}\ X\ f = (\lambda y.\ \bigsqcap\ \{\mathit{mdiameter}\ (f\ \text{`}\ U)\ |U.\ y \in U \wedge \mathit{openin}\ X\ U\})$
⟨*proof*⟩

lemma *osc-on-less-iff*:
$\mathit{osc\text{-}on}\ A\ X\ f\ x < t \longleftrightarrow (\exists\, v.\ x \in v \wedge \mathit{openin}\ X\ v \wedge \mathit{mdiameter}\ (f\ \text{`}\ (A \cap v)) < t)$
⟨*proof*⟩

lemma *osc-less-iff*:
$\mathit{osc}\ X\ f\ x < t \longleftrightarrow (\exists\, v.\ x \in v \wedge \mathit{openin}\ X\ v \wedge \mathit{mdiameter}\ (f\ \text{`}\ v) < t)$
⟨*proof*⟩

end

definition $\mathit{mdist\text{-}set} :: \prime a\ \mathit{metric} \Rightarrow \prime a\ \mathit{set} \Rightarrow \prime a \Rightarrow \mathit{real}$ **where**
$\mathit{mdist\text{-}set}\ m \equiv \mathit{Metric\text{-}space.d\text{-}set}\ (\mathit{mspace}\ m)\ (\mathit{mdist}\ m)$

lemma(**in** *Metric-space*) *mdist-set-Self*: $\mathit{mdist\text{-}set}\ \mathit{Self} = \mathit{d\text{-}set}$
⟨*proof*⟩

lemma *mdist-set-nonneg*[*simp*]: $\mathit{mdist\text{-}set}\ m\ A\ x \ge 0$
⟨*proof*⟩

lemma *mdist-set-singleton*[*simp*]:
$x \in \mathit{mspace}\ m \Longrightarrow y \in \mathit{mspace}\ m \Longrightarrow \mathit{mdist\text{-}set}\ m\ \{y\}\ x = \mathit{mdist}\ m\ x\ y$
⟨*proof*⟩

lemma *mdist-set-empty*[*simp*]: *mdist-set m* $\{\}$ *x* = *0*
⟨*proof*⟩

lemma *mdist-set-inA*:
assumes $x \in A$
shows *mdist-set m A x* = *0*
⟨*proof*⟩

lemma *mdist-set-nzeroD*:
assumes *mdist-set m A x* $\neq$ *0*
shows $A \subseteq$ *mspace m* $x \notin A$ $A \neq \{\}$
⟨*proof*⟩

lemma *mdist-set-antimono*:
assumes $A \subseteq B$ $A \neq \{\}$ $B \subseteq$ *mspace m*
shows *mdist-set m B x* $\leq$ *mdist-set m A x*
⟨*proof*⟩

lemma *mdist-set-bounded*:
assumes $\bigwedge y.\ y \in A \Longrightarrow$ *mdist m x y* $<$ *K* $K > 0$
shows *mdist-set m A x* $<$ *K*
⟨*proof*⟩

lemma *mdist-set-tr*:
assumes $x \in$ *mspace m* $y \in$ *mspace m*
shows *mdist-set m A x* $\leq$ *mdist m x y* + *mdist-set m A y*
⟨*proof*⟩

lemma *mdist-set-abs-le*:
assumes $x \in$ *mspace m* $y \in$ *mspace m*
shows $|$*mdist-set m A x* $-$ *mdist-set m A y*$|$ $\leq$ *mdist m x y*
⟨*proof*⟩

lemma *mdist-set-inA-le*:
assumes $y \in A$
shows *mdist-set m A x* $\leq$ *mdist m x y*
⟨*proof*⟩

lemma *mdist-set-ball-empty*:
assumes $A \neq \{\}$ $A \subseteq$ *mspace m* $e > 0$ $x \in$ *mspace m* *mball-of m x e* $\cap A = \{\}$
shows *mdist-set m A x* $\geq e$
⟨*proof*⟩

lemma *mdist-set-closed-pos*:
assumes *closedin* (*mtopology-of m*) *A* $A \neq \{\}$ $x \in$ *mspace m* $x \notin A$
shows *mdist-set m A x* $>$ *0*
⟨*proof*⟩

lemma *mdist-set-uniformly-continuous*: *uniformly-continuous-map m euclidean-metric* (*mdist-set m A*)
⟨*proof*⟩

lemma *uniformly-continuous-map-add*:
fixes $f :: {'a} \Rightarrow {'b}::\mathit{real\text{-}normed\text{-}vector}$
assumes *uniformly-continuous-map m euclidean-metric f uniformly-continuous-map m euclidean-metric g*
shows *uniformly-continuous-map m euclidean-metric* $(\lambda x.\ f\ x + g\ x)$
⟨*proof*⟩

lemma *uniformly-continuous-map-real-divide*:
fixes $f :: {'a} \Rightarrow \mathit{real}$
assumes *uniformly-continuous-map m euclidean-metric f uniformly-continuous-map m euclidean-metric g*
and $\bigwedge x.\ x \in \mathit{mspace}\ m \Longrightarrow g\ x \neq 0$ $\bigwedge x.\ x \in \mathit{mspace}\ m \Longrightarrow |g\ x| \geq a$ $a > 0$ $\bigwedge x.\ x \in \mathit{mspace}\ m \Longrightarrow |g\ x| < \mathit{Kg}$
and $\bigwedge x.\ x \in \mathit{mspace}\ m \Longrightarrow |f\ x| < \mathit{Kf}$
shows *uniformly-continuous-map m euclidean-metric* $(\lambda x.\ f\ x\ /\ g\ x)$
⟨*proof*⟩

lemma
assumes $e > 0$
shows *uniformly-continuous-map-from-capped-metric*:*uniformly-continuous-map* (*capped-metric e m1*) *m2 f* $\longleftrightarrow$ *uniformly-continuous-map m1 m2 f* (**is** *?g1*)
and *uniformly-continuous-map-to-capped-metric*:*uniformly-continuous-map m1* (*capped-metric e m2*) *f* $\longleftrightarrow$ *uniformly-continuous-map m1 m2 f* (**is** *?g2*)
⟨*proof*⟩

lemma *Urysohn-lemma-uniform*:
assumes *closedin* (*mtopology-of m*) *T closedin* (*mtopology-of m*) *U* $T \cap U = \{\}$ $\bigwedge x\ y.\ x \in T \Longrightarrow y \in U \Longrightarrow \mathit{mdist}\ m\ x\ y \geq e$ $e > 0$
obtains $f :: {'a} \Rightarrow \mathit{real}$
where *uniformly-continuous-map m euclidean-metric f*
$\bigwedge x.\ f\ x \geq 0$ $\bigwedge x.\ f\ x \leq 1$ $\bigwedge x.\ x \in T \Longrightarrow f\ x = 1$ $\bigwedge x.\ x \in U \Longrightarrow f\ x = 0$
⟨*proof*⟩

Open maps

lemma *Metric-space-open-map-from-dist*:
assumes $f \in \mathit{mspace}\ m1 \rightarrow \mathit{mspace}\ m2$
and $\bigwedge x\ \varepsilon.\ x \in \mathit{mspace}\ m1 \Longrightarrow \varepsilon > 0 \Longrightarrow \exists \delta{>}0.\ \forall y{\in}\mathit{mspace}\ m1.\ \mathit{mdist}\ m2\ (f\ x)\ (f\ y) < \delta \longrightarrow \mathit{mdist}\ m1\ x\ y < \varepsilon$
shows *open-map* (*mtopology-of m1*) (*subtopology* (*mtopology-of m2*) (*f ‘ mspace m1*)) *f*
⟨*proof*⟩

1.3.3 Separability in Metric Spaces

context *Metric-space*

begin

For a metric space M, M is separable iff M is second countable.

lemma *generated-by-countable-balls*:
  assumes *countable U* **and** *mdense U*
  shows *mtopology = topology-generated-by {mball y (1 / real n) | y n. y ∈ U}*
⟨*proof*⟩

lemma *separable-space-imp-second-countable*:
  assumes *separable-space mtopology*
  shows *second-countable mtopology*
⟨*proof*⟩

corollary *separable-space-iff-second-countable*:
  separable-space mtopology ⟷ second-countable mtopology
  ⟨*proof*⟩

lemma *Lindelof-mdiameter*:
  assumes *separable-space mtopology 0 < e*
  shows *∃ U. countable U ∧ ⋃ U = M ∧ (∀ u∈U. mdiameter u < ennreal e)*
⟨*proof*⟩

end

lemma *metrizable-space-separable-iff-second-countable*:
  assumes *metrizable-space X*
  shows *separable-space X ⟷ second-countable X*
⟨*proof*⟩

abbreviation *mdense-of m U ≡ dense-in (mtopology-of m) U*

lemma *mdense-of-def*: *mdense-of m U ⟷ (U ⊆ mspace m ∧ (∀ x∈mspace m. ∀ ε>0. mball-of m x ε ∩ U ≠ {}))*
  ⟨*proof*⟩

lemma *mdense-of-def2*: *mdense-of m U ⟷ (U ⊆ mspace m ∧ (∀ x∈mspace m. ∀ ε>0. ∃ y∈U. mdist m x y < ε))*
  ⟨*proof*⟩

lemma *mdense-of-def3*: *mdense-of m U ⟷ (U ⊆ mspace m ∧ (∀ x∈mspace m. ∃ xn. range xn ⊆ U ∧ limitin (mtopology-of m) xn x sequentially))*
  ⟨*proof*⟩

1.3.4 Compact Metric Spaces

context *Metric-space*
begin

lemma *mtotally-bounded-eq-compact-closedin*:

assumes *mcomplete closedin mtopology S*
shows *mtotally-bounded S* $\longleftrightarrow$ *S* $\subseteq$ *M* $\wedge$ *compactin mtopology S*
⟨*proof*⟩

lemma *mtotally-bounded-def2*: *mtotally-bounded S* $\longleftrightarrow$ ($\forall\,\varepsilon$>*0*. $\exists$*K*. *finite K* $\wedge$ *K* $\subseteq$ *M* $\wedge$ *S* $\subseteq$ ($\bigcup$*x*$\in$*K*. *mball x* ε))
⟨*proof*⟩

lemma *compact-space-imp-separable*:
assumes *compact-space mtopology*
shows *separable-space mtopology*
⟨*proof*⟩

lemma *separable-space-cfunspace*:
assumes *separable-space mtopology mcomplete*
and *metrizable-space X compact-space X*
shows *separable-space* (*mtopology-of* (*cfunspace X Self*))
⟨*proof*⟩

end

context *Submetric*
begin

lemma *separable-sub*:
assumes *separable-space mtopology*
shows *separable-space sub.mtopology*
⟨*proof*⟩

end

1.3.5 Discrete Distance

lemma(**in** *discrete-metric*) *separable-space-iff*: *separable-space disc.mtopology* $\longleftrightarrow$ *countable M*
⟨*proof*⟩

1.3.6 Binary Product Metric Spaces

We define the L^1-distance. L^1-distance and L^2 distance (Euclid distance) generate the same topological space.

definition *prod-dist-L1* $\equiv$ λ*d1 d2* (*x*,*y*) (*x'*,*y'*). *d1 x x'* + *d2 y y'*

context *Metric-space12*
begin

lemma *prod-L1-metric*: *Metric-space* (*M1* $\times$ *M2*) (*prod-dist-L1 d1 d2*)
⟨*proof*⟩

sublocale *Prod-metric-L1*: *Metric-space M1* × *M2 prod-dist-L1 d1 d2*
⟨*proof*⟩

lemma *prod-dist-L1-geq*:
shows *d1 x y* ≤ *prod-dist-L1 d1 d2* (*x*,*x′*) (*y*,*y′*)
d2 x′ y′ ≤ *prod-dist-L1 d1 d2* (*x*,*x′*) (*y*,*y′*)
⟨*proof*⟩

lemma *prod-dist-L1-ball*:
assumes (*x*,*x′*) ∈ *Prod-metric-L1*.*mball* (*a*,*a′*) *ε*
shows *x* ∈ *M1*.*mball a ε*
and *x′* ∈ *M2*.*mball a′ ε*
⟨*proof*⟩

lemma *prod-dist-L1-ball′*:
assumes *z* ∈ *Prod-metric-L1*.*mball a ε*
shows *fst z* ∈ *M1*.*mball* (*fst a*) *ε*
and *snd z* ∈ *M2*.*mball* (*snd a*) *ε*
⟨*proof*⟩

lemma *prod-dist-L1-ball1′*: *Prod-metric-L1*.*mball* (*a1*,*a2*) (*min e1 e2*) ⊆ *M1*.*mball a1 e1* × *M2*.*mball a2 e2*
⟨*proof*⟩

lemma *prod-dist-L1-ball1*:
assumes *b1* ∈ *M1*.*mball a1 e1 b2* ∈ *M2*.*mball a2 e2*
shows ∃ *e12*>*0*. *Prod-metric-L1*.*mball* (*b1*,*b2*) *e12* ⊆ *M1*.*mball a1 e1* × *M2*.*mball a2 e2*
⟨*proof*⟩

lemma *prod-dist-L1-ball2′*:
M1.*mball a1 e1* × *M2*.*mball a2 e2* ⊆ *Prod-metric-L1*.*mball* (*a1*,*a2*) (*e1* + *e2*)
⟨*proof*⟩

lemma *prod-dist-L1-ball2*:
assumes (*b1*,*b2*) ∈ *Prod-metric-L1*.*mball* (*a1*,*a2*) *e12*
shows ∃ *e1*>*0*. ∃ *e2*>*0*. *M1*.*mball b1 e1* × *M2*.*mball b2 e2* ⊆ *Prod-metric-L1*.*mball* (*a1*,*a2*) *e12*
⟨*proof*⟩

lemma *prod-dist-L1-mtopology*:
Prod-metric-L1.*mtopology* = *prod-topology M1*.*mtopology M2*.*mtopology*
⟨*proof*⟩

lemma *prod-dist-L1-limitin-iff*: *limitin Prod-metric-L1*.*mtopology zn z sequentially* ⟷ *limitin M1*.*mtopology* (*λn*. *fst* (*zn n*)) (*fst z*) *sequentially* ∧ *limitin M2*.*mtopology* (*λn*. *snd* (*zn n*)) (*snd z*) *sequentially*
⟨*proof*⟩

lemma *prod-dist-L1-MCauchy-iff*: $\mathit{Prod\text{-}metric\text{-}L1.MCauchy}\ \mathit{zn} \longleftrightarrow \mathit{M1.MCauchy}\ (\lambda n.\ \mathit{fst}\ (\mathit{zn}\ n)) \wedge \mathit{M2.MCauchy}\ (\lambda n.\ \mathit{snd}\ (\mathit{zn}\ n))$
$\langle \mathit{proof} \rangle$

end

1.3.7 Sum Metric Spaces

locale *Sum-metric* =
fixes $I :: {}'i\ \mathit{set}$
and $\mathit{Mi} :: {}'i \Rightarrow {}'a\ \mathit{set}$
and $\mathit{di} :: {}'i \Rightarrow {}'a \Rightarrow {}'a \Rightarrow \mathit{real}$
assumes *Mi-disj*: $\mathit{disjoint\text{-}family\text{-}on}\ \mathit{Mi}\ I$
and *d-nonneg*: $\bigwedge i\ x\ y.\ 0 \le \mathit{di}\ i\ x\ y$
and *d-bounded*: $\bigwedge i\ x\ y.\ \mathit{di}\ i\ x\ y < 1$
and *Md-metric*: $\bigwedge i.\ i \in I \Longrightarrow \mathit{Metric\text{-}space}\ (\mathit{Mi}\ i)\ (\mathit{di}\ i)$
begin

abbreviation $M \equiv \bigcup i{\in}I.\ \mathit{Mi}\ i$

lemma *Mi-inj-on*:
assumes $i \in I\ j \in I\ a \in \mathit{Mi}\ i\ a \in \mathit{Mi}\ j$
shows $i = j$
$\langle \mathit{proof} \rangle$

definition $\mathit{sum\text{-}dist} :: [{}'a, {}'a] \Rightarrow \mathit{real}$ **where**
$\mathit{sum\text{-}dist}\ x\ y \equiv (\mathit{if}\ x \in M \wedge y \in M\ \mathit{then}\ (\mathit{if}\ \exists i{\in}I.\ x \in \mathit{Mi}\ i \wedge y \in \mathit{Mi}\ i\ \mathit{then}\ \mathit{di}\ (\mathit{THE}\ i.\ i \in I \wedge x \in \mathit{Mi}\ i \wedge y \in \mathit{Mi}\ i)\ x\ y\ \mathit{else}\ 1)\ \mathit{else}\ 0)$

lemma *sum-dist-simps*:
shows $\bigwedge i.\ [\![i \in I;\ x \in \mathit{Mi}\ i;\ y \in \mathit{Mi}\ i]\!] \Longrightarrow \mathit{sum\text{-}dist}\ x\ y = \mathit{di}\ i\ x\ y$
and $\bigwedge i\ j.\ [\![i \in I;\ j \in I;\ i \neq j;\ x \in \mathit{Mi}\ i;\ y \in \mathit{Mi}\ j]\!] \Longrightarrow \mathit{sum\text{-}dist}\ x\ y = 1$
and $\bigwedge i.\ [\![i \in I;\ y \in M;\ x \in \mathit{Mi}\ i;\ y \notin \mathit{Mi}\ i]\!] \Longrightarrow \mathit{sum\text{-}dist}\ x\ y = 1$
and $\bigwedge i.\ [\![i \in I;\ x \in M;\ y \in \mathit{Mi}\ i;\ x \notin \mathit{Mi}\ i]\!] \Longrightarrow \mathit{sum\text{-}dist}\ x\ y = 1$
and $x \notin M \Longrightarrow \mathit{sum\text{-}dist}\ x\ y = 0\ \ y \notin M \Longrightarrow \mathit{sum\text{-}dist}\ x\ y = 0$
$\langle \mathit{proof} \rangle$

lemma *sum-dist-if-less1*:
assumes $i \in I\ x \in \mathit{Mi}\ i\ y \in M\ \mathit{sum\text{-}dist}\ x\ y < 1$
shows $y \in \mathit{Mi}\ i$
$\langle \mathit{proof} \rangle$

lemma *inM-cases*:
assumes $x \in M\ y \in M$
and $\bigwedge i.\ [\![i \in I;\ x \in \mathit{Mi}\ i;\ y \in \mathit{Mi}\ i]\!] \Longrightarrow P\ x\ y$
and $\bigwedge i\ j.\ [\![i \in I;\ j \in I;\ i \neq j;\ x \in \mathit{Mi}\ i;\ y \in \mathit{Mi}\ j;\ x \neq y]\!] \Longrightarrow P\ x\ y$
shows $P\ x\ y\ \langle \mathit{proof} \rangle$

sublocale *Sum-metric*: *Metric-space M sum-dist*

⟨*proof*⟩

lemma *sum-dist-le1*: *sum-dist x y* $\leq$ *1*
⟨*proof*⟩

lemma *sum-dist-ball-eq-ball*:
assumes *i* $\in$ *I e* $\leq$ *1 x* $\in$ *Mi i*
shows *Metric-space.mball* (*Mi i*) (*di i*) *x e* = *Sum-metric.mball x e*
⟨*proof*⟩

lemma *ball-le-sum-dist-ball*:
assumes *i* $\in$ *I*
shows *Metric-space.mball* (*Mi i*) (*di i*) *x e* $\subseteq$ *Sum-metric.mball x e*
⟨*proof*⟩

lemma *openin-mtopology-iff*:
openin Sum-metric.mtopology U $\longleftrightarrow$ *U* $\subseteq$ *M* $\wedge$ ($\forall$ *i*$\in$*I*. *openin* (*Metric-space.mtopology* (*Mi i*) (*di i*)) (*U* $\cap$ *Mi i*))
⟨*proof*⟩

corollary *openin-mtopology-Mi*:
assumes *i* $\in$ *I*
shows *openin Sum-metric.mtopology* (*Mi i*)
⟨*proof*⟩

corollary *subtopology-mtopology-Mi*:
assumes *i* $\in$ *I*
shows *subtopology Sum-metric.mtopology* (*Mi i*) = *Metric-space.mtopology* (*Mi i*) (*di i*)
⟨*proof*⟩

lemma *limitin-Mi-limitin-M*:
assumes *i* $\in$ *I limitin* (*Metric-space.mtopology* (*Mi i*) (*di i*)) *xn x sequentially*
shows *limitin Sum-metric.mtopology xn x sequentially*
⟨*proof*⟩

lemma *limitin-M-limitin-Mi*:
assumes *limitin Sum-metric.mtopology xn x sequentially*
shows $\exists$ *i*$\in$*I*. *limitin* (*Metric-space.mtopology* (*Mi i*) (*di i*)) *xn x sequentially*
⟨*proof*⟩

lemma *MCauchy-Mi-MCauchy-M*:
assumes *i* $\in$ *I Metric-space.MCauchy* (*Mi i*) (*di i*) *xn*
shows *Sum-metric.MCauchy xn*
⟨*proof*⟩

lemma *MCauchy-M-MCauchy-Mi*:
assumes *Sum-metric.MCauchy xn*

shows $\exists\, m.\ \exists\, i{\in}I.$ *Metric-space.MCauchy* (*Mi i*) (*di i*) ($\lambda n.$ *xn* ($n + m$))
⟨*proof*⟩

lemma *separable-Mi-separable-M*:
assumes *countable I* $\bigwedge i.\ i \in I \Longrightarrow$ *separable-space* (*Metric-space.mtopology* (*Mi i*) (*di i*))
shows *separable-space Sum-metric.mtopology*
⟨*proof*⟩

lemma *separable-M-separable-Mi*:
assumes *separable-space Sum-metric.mtopology* $\bigwedge i.\ i \in I$
shows *separable-space* (*Metric-space.mtopology* (*Mi i*) (*di i*))
⟨*proof*⟩

lemma *mcomplete-Mi-mcomplete-M*:
assumes $\bigwedge i.\ i \in I \Longrightarrow$ *Metric-space.mcomplete* (*Mi i*) (*di i*)
shows *Sum-metric.mcomplete*
⟨*proof*⟩

lemma *mcomplete-M-mcomplete-Mi*:
assumes *Sum-metric.mcomplete* $i \in I$
shows *Metric-space.mcomplete* (*Mi i*) (*di i*)
⟨*proof*⟩

end

lemma *sum-metricI*:
fixes *Si*
assumes *disjoint-family-on Si I*
and $\bigwedge i\ x\ y.\ i \notin I \Longrightarrow 0 \le di\ i\ x\ y$
and $\bigwedge i\ x\ y.\ di\ i\ x\ y < 1$
and $\bigwedge i.\ i \in I \Longrightarrow$ *Metric-space* (*Si i*) (*di i*)
shows *Sum-metric I Si di*
⟨*proof*⟩

end

1.3.8 Product Metric Spaces

theory *Set-Based-Metric-Product*
imports *Set-Based-Metric-Space*
begin

lemma *nsum-of-r'*:
fixes r :: *real*
assumes $r{:}0 < r\ r < 1$
shows $(\sum n.\ r\hat{\ }(n + k) * K) = r\hat{\ }k\ /\ (1 - r) * K$

(**is** *?lhs* = -)
⟨*proof*⟩

lemma *nsum-of-r-leq*:
fixes r :: *real* **and** a :: *nat* ⇒ *real*
assumes r:$0 < r$ $r < 1$
and a:$\bigwedge n.\ 0 \leq a\ n$ $\bigwedge n.\ a\ n \leq K$
shows $0 \leq (\sum n.\ r\hat{}(n + k) * a\ (n + l))$ $(\sum n.\ r\hat{}(n + k) * a\ (n + l)) \leq r\hat{}k\ /\ (1 - r) * K$
⟨*proof*⟩

lemma *nsum-of-r-le*:
fixes r :: *real* **and** a :: *nat* ⇒ *real*
assumes r:$0 < r$ $r < 1$
and a:$\bigwedge n.\ 0 \leq a\ n$ $\bigwedge n.\ a\ n \leq K$ $\exists\, n' \geq l.\ a\ n' < K$
shows $(\sum n.\ r\hat{}(n + k) * a\ (n + l)) < r\hat{}k\ /\ (1 - r) * K$
⟨*proof*⟩

definition *product-dist′* :: [*real*, ′*i set*, *nat* ⇒ ′*i*, ′*i* ⇒ ′*a set*, ′*i* ⇒ ′*a* ⇒ ′*a* ⇒ *real*] ⇒ (′*i* ⇒ ′*a*) ⇒ (′*i* ⇒ ′*a*) ⇒ *real* **where**
product-dist-def: $product\text{-}dist'\ r\ I\ g\ Mi\ di \equiv (\lambda x\ y.\ if\ x \in (\Pi_E\ i{\in}I.\ Mi\ i) \wedge y \in (\Pi_E\ i{\in}I.\ Mi\ i)\ then\ (\sum n.\ if\ g\ n \in I\ then\ r\hat{}n * di\ (g\ n)\ (x\ (g\ n))\ (y\ (g\ n))\ else\ 0)\ else\ 0)$

$d(x, y) = \sum_{n\in\mathbb{N}} r^n * d_{g_I(i)}(x_{g_I(i)}, y_{g_I(i)})$.

locale *Product-metric* =
fixes r :: *real*
and I :: ′*i set*
and f :: ′*i* ⇒ *nat*
and g :: *nat* ⇒ ′*i*
and *Mi* :: ′*i* ⇒ ′*a set*
and *di* :: ′*i* ⇒ ′*a* ⇒ ′*a* ⇒ *real*
and K :: *real*
assumes r: $0 < r$ $r < 1$
and I: *countable* I
and *gf-comp-id* : $\bigwedge i.\ i \in I \Longrightarrow g\ (f\ i) = i$
and *gf-if-finite*: *finite* $I \Longrightarrow$ *bij-betw* $f\ I$ {..< *card* I}
finite $I \Longrightarrow$ *bij-betw* g {..< *card* I} I
and *gf-if-infinite*: *infinite* $I \Longrightarrow$ *bij-betw* $f\ I$ *UNIV*
infinite $I \Longrightarrow$ *bij-betw* g *UNIV* I
$\bigwedge n.$ *infinite* $I \Longrightarrow f\ (g\ n) = n$
and *Md-metric*: $\bigwedge i.\ i \in I \Longrightarrow$ *Metric-space* (*Mi* i) (*di* i)
and *di-nonneg*: $\bigwedge i\ x\ y.\ 0 \leq di\ i\ x\ y$
and *di-bounded*: $\bigwedge i\ x\ y.\ di\ i\ x\ y \leq K$
and *K-pos* : $0 < K$

lemma *from-nat-into-to-nat-on-product-metric-pair*:
assumes *countable* I
shows $\bigwedge i.\ i \in I \Longrightarrow$ *from-nat-into* I (*to-nat-on* $I\ i$) $= i$

and *finite I* $\Longrightarrow$ *bij-betw* (*to-nat-on I*) *I* {..< *card I*}
and *finite I* $\Longrightarrow$ *bij-betw* (*from-nat-into I*) {..< *card I*} *I*
and *infinite I* $\Longrightarrow$ *bij-betw* (*to-nat-on I*) *I UNIV*
and *infinite I* $\Longrightarrow$ *bij-betw* (*from-nat-into I*) *UNIV I*
and $\bigwedge$*n. infinite I* $\Longrightarrow$ *to-nat-on I* (*from-nat-into I n*) = *n*
⟨*proof*⟩

lemma *product-metric-pair-finite-nat*:
bij-betw id {..*n*} {..< *card* {..*n*}} *bij-betw id* {..< *card* {..*n*}} {..*n*}
⟨*proof*⟩

lemma *product-metric-pair-finite-nat′*:
bij-betw id {..<*n*} {..< *card* {..<*n*}} *bij-betw id* {..< *card* {..<*n*}} {..<*n*}
⟨*proof*⟩

context *Product-metric*
begin

abbreviation *product-dist* $\equiv$ *product-dist′ r I g Mi di*

lemma *nsum-of-rK*: ($\sum$ *n. r*^(*n* + *k*)∗*K*) = *r*^*k* / (*1* − *r*) ∗ *K*
⟨*proof*⟩

lemma *i-min*:
assumes *i* $\in$ *I g n* = *i*
shows *f i* $\leq$ *n*
⟨*proof*⟩

lemma *g-surj*:
assumes *i* $\in$ *I*
shows $\exists$ *n. g n* = *i*
⟨*proof*⟩

lemma *product-dist-summable′*[*simp*]:
summable (λ*n. r*^*n* ∗ *di* (*g n*) (*x* (*g n*)) (*y* (*g n*)))
⟨*proof*⟩

lemma *product-dist-summable*[*simp*]:
summable (λ*n. if g n* $\in$ *I then r*^*n* ∗ *di* (*g n*) (*x* (*g n*)) (*y* (*g n*)) *else 0*)
⟨*proof*⟩

lemma *summable-rK*[*simp*]: *summable* (λ*n. r*^*n* ∗ *K*)
⟨*proof*⟩

lemma *Product-metric*: *Metric-space* (Π_E *i*$\in$*I. Mi i*) *product-dist*
⟨*proof*⟩

sublocale *Product-metric*: *Metric-space* Π_E *i*$\in$*I. Mi i product-dist*
⟨*proof*⟩

lemma *product-dist-leqr*: *product-dist x y* ≤ *1* / (*1* − *r*) ∗ *K*
⟨*proof*⟩

lemma *product-dist-geq*:
  assumes *i* ∈ *I* **and** *g n* = *i x* ∈ (Π_E *i*∈*I*. *Mi i*) *y* ∈ (Π_E *i*∈*I*. *Mi i*)
    shows *di i* (*x i*) (*y i*) ≤ (*1*/*r*)^*n* ∗ *product-dist x y*
      (**is** *?lhs* ≤ *?rhs*)
⟨*proof*⟩

lemma *limitin-M-iff-limitin-Mi*:
  shows *limitin Product-metric.mtopology xn x sequentially* ⟷ (∃ *N*. ∀ *n*≥*N*. (∀ *i*∈*I*. *xn n i* ∈ *Mi i*) ∧ (∀ *i*. *i*∉*I* ⟶ *xn n i* = *undefined*)) ∧ (∀ *i*∈*I*. *limitin* (*Metric-space.mtopology* (*Mi i*) (*di i*)) (λ*n*. *xn n i*) (*x i*) *sequentially*) ∧ *x* ∈ (Π_E *i*∈*I*. *Mi i*)
⟨*proof*⟩

lemma *Product-metric-mtopology-eq*: *product-topology* (λ*i*. *Metric-space.mtopology* (*Mi i*) (*di i*)) *I* = *Product-metric.mtopology*
⟨*proof*⟩

corollary *separable-Mi-separable-M*:
  assumes ⋀*i*. *i* ∈ *I* ⟹ *separable-space* (*Metric-space.mtopology* (*Mi i*) (*di i*))
  shows *separable-space Product-metric.mtopology*
  ⟨*proof*⟩

lemma *mcomplete-Mi-mcomplete-M*:
  assumes ⋀*i*. *i* ∈ *I* ⟹ *Metric-space.mcomplete* (*Mi i*) (*di i*)
  shows *Product-metric.mcomplete*
⟨*proof*⟩

end

lemma *product-metricI*:
  assumes *0* < *r r* < *1* *countable I* ⋀*i*. *i* ∈ *I* ⟹ *Metric-space* (*Mi i*) (*di i*)
    and ⋀*i x y*. *0* ≤ *di i x y* ⋀*i x y*. *di i x y* ≤ *K 0* < *K*
   shows *Product-metric r I* (*to-nat-on I*) (*from-nat-into I*) *Mi di K*
  ⟨*proof*⟩

lemma *product-metric-natI*:
  assumes *0* < *r r* < *1* ⋀*n*. *Metric-space* (*Mi n*) (*di n*)
    and ⋀*i x y*. *0* ≤ *di i x y* ⋀*i x y*. *di i x y* ≤ *K 0* < *K*
   shows *Product-metric r UNIV id id Mi di K*
  ⟨*proof*⟩

end

2 Abstract Polish Spaces

theory *Abstract-Metrizable-Topology*
  imports *Set-Based-Metric-Product*
begin

2.1 Polish Spaces

definition *Polish-space X* ≡ *completely-metrizable-space X* ∧ *separable-space X*

lemma(**in** *Metric-space*) *Polish-space-mtopology*:
  assumes *mcomplete separable-space mtopology*
  shows *Polish-space mtopology*
  ⟨*proof*⟩

lemma
  assumes *Polish-space X*
  shows *Polish-space-imp-completely-metrizable-space*: *completely-metrizable-space X*
    and *Polish-space-imp-metrizable-space*: *metrizable-space X*
    and *Polish-space-imp-second-countable*: *second-countable X*
    and *Polish-space-imp-separable-space*: *separable-space X*
  ⟨*proof*⟩

lemma *Polish-space-closedin*:
  assumes *Polish-space X closedin X A*
  shows *Polish-space* (*subtopology X A*)
  ⟨*proof*⟩

lemma *Polish-space-gdelta-in*:
  assumes *Polish-space X gdelta-in X A*
  shows *Polish-space* (*subtopology X A*)
  ⟨*proof*⟩

corollary *Polish-space-openin*:
  assumes *Polish-space X openin X A*
  shows *Polish-space* (*subtopology X A*)
  ⟨*proof*⟩

lemma *homeomorphic-Polish-space-aux*:
  assumes *Polish-space X X homeomorphic-space Y*
  shows *Polish-space Y*
  ⟨*proof*⟩

corollary *homeomorphic-Polish-space*:
  assumes *X homeomorphic-space Y*
  shows *Polish-space X* ⟷ *Polish-space Y*
  ⟨*proof*⟩

lemma *Polish-space-euclidean*[*simp*]: *Polish-space* (*euclidean* :: (*'a* :: *polish-space*)

topology)
⟨*proof*⟩

lemma *Polish-space-countable*[*simp*]:
Polish-space (*euclidean* :: *'a* :: {*countable*,*discrete-topology*} *topology*)
⟨*proof*⟩

lemma *Polish-space-discrete-topology*: *Polish-space* (*discrete-topology I*) $\longleftrightarrow$ *countable I*
⟨*proof*⟩

lemma *Polish-space-prod*:
assumes *Polish-space X* **and** *Polish-space Y*
shows *Polish-space* (*prod-topology X Y*)
⟨*proof*⟩

lemma *Polish-space-product*:
assumes *countable I* **and** $\bigwedge i.\ i \in I \Longrightarrow$ *Polish-space* (*S i*)
shows *Polish-space* (*product-topology S I*)
⟨*proof*⟩

lemma(**in** *Product-metric*) *Polish-spaceI*:
assumes $\bigwedge i.\ i \in I \Longrightarrow$ *separable-space* (*Metric-space.mtopology* (*Mi i*) (*di i*))
and $\bigwedge i.\ i \in I \Longrightarrow$ *Metric-space.mcomplete* (*Mi i*) (*di i*)
shows *Polish-space Product-metric.mtopology*
⟨*proof*⟩

lemma(**in** *Sum-metric*) *Polish-spaceI*:
assumes *countable I*
and $\bigwedge i.\ i \in I \Longrightarrow$ *separable-space* (*Metric-space.mtopology* (*Mi i*) (*di i*))
and $\bigwedge i.\ i \in I \Longrightarrow$ *Metric-space.mcomplete* (*Mi i*) (*di i*)
shows *Polish-space Sum-metric.mtopology*
⟨*proof*⟩

lemma *compact-metrizable-imp-Polish-space*:
assumes *metrizable-space X compact-space X*
shows *Polish-space X*
⟨*proof*⟩

2.2 Extended Reals and Non-Negative Extended Reals

lemma *Polish-space-ereal*:*Polish-space* (*euclidean* :: *ereal topology*)
⟨*proof*⟩

corollary *Polish-space-ennreal*:*Polish-space* (*euclidean* :: *ennreal topology*)
⟨*proof*⟩

2.3 Continuous Embddings

abbreviation *Hilbert-cube-topology* :: (*nat* ⇒ *real*) *topology* **where**

Hilbert-cube-topology ≡ (*product-topology* (λ*n*. *top-of-set* {*0..1*}) *UNIV*)

lemma *topspace-Hilbert-cube*: *topspace Hilbert-cube-topology* = (Π_E *x*∈*UNIV*. {*0..1*})
⟨*proof*⟩

lemma *Polish-space-Hilbert-cube*: *Polish-space Hilbert-cube-topology*
⟨*proof*⟩

abbreviation *Cantor-space-topology* :: (*nat* ⇒ *real*) *topology* **where**
Cantor-space-topology ≡ (*product-topology* (λ*n*. *top-of-set* {*0*,*1*}) *UNIV*)

lemma *topspace-Cantor-space*:
topspace Cantor-space-topology = (Π_E *x*∈*UNIV*. {*0*,*1*})
⟨*proof*⟩

lemma *Polish-space-Cantor-space*: *Polish-space Cantor-space-topology*
⟨*proof*⟩

corollary *completely-metrizable-space-homeo-image-gdelta-in*:
assumes *completely-metrizable-space X completely-metrizable-space Y B* ⊆ *topspace Y X homeomorphic-space subtopology Y B*
shows *gdelta-in Y B*
⟨*proof*⟩

2.3.1 Embedding into Hilbert Cube

lemma *embedding-into-Hilbert-cube*:
assumes *metrizable-space X separable-space X*
shows ∃ *A* ⊆ *topspace Hilbert-cube-topology*. *X homeomorphic-space* (*subtopology Hilbert-cube-topology A*)
⟨*proof*⟩

corollary *embedding-into-Hilbert-cube-gdelta-in*:
assumes *Polish-space X*
shows ∃ *A*. *gdelta-in Hilbert-cube-topology A* ∧ *X homeomorphic-space* (*subtopology Hilbert-cube-topology A*)
⟨*proof*⟩

2.3.2 Embedding from Cantor Space

lemma *embedding-from-Cantor-space*:
assumes *Polish-space X uncountable* (*topspace X*)
shows ∃ *A*. *gdelta-in X A* ∧ *Cantor-space-topology homeomorphic-space* (*subtopology X A*)
⟨*proof*⟩

2.4 Borel Spaces generated from Polish Spaces

lemma *closedin-clopen-topology*:
assumes *Polish-space X closedin X a*

shows $\exists X'$. *Polish-space* $X' \wedge (\forall u.$ *openin* $X\ u \longrightarrow$ *openin* $X'\ u) \wedge$ *topspace* X = *topspace* $X' \wedge$ *sets* (*borel-of* X) = *sets* (*borel-of* X') $\wedge$ *openin* $X'\ a \wedge$ *closedin* $X'\ a$
⟨*proof*⟩

lemma *Polish-space-union-Polish*:
fixes X :: *nat* ⇒ *'a topology*
assumes $\bigwedge n$. *Polish-space* $(X\ n)$ $\bigwedge n$. *topspace* $(X\ n) = Xt$ $\bigwedge x\ y.\ x \in Xt \Longrightarrow y \in Xt \Longrightarrow x \neq y \Longrightarrow \exists Ox\ Oy.\ (\forall n.$ *openin* $(X\ n)\ Ox) \wedge (\forall n.$ *openin* $(X\ n)\ Oy) \wedge x \in Ox \wedge y \in Oy \wedge$ *disjnt* $Ox\ Oy$
defines $Xun \equiv$ *topology-generated-by* $(\bigcup n.\ \{u.$ *openin* $(X\ n)\ u\})$
shows *Polish-space Xun*
⟨*proof*⟩

lemma *sets-clopen-topology*:
assumes *Polish-space* $X\ a \in$ *sets* (*borel-of* X)
shows $\exists X'$. *Polish-space* $X' \wedge (\forall u.$ *openin* $X\ u \longrightarrow$ *openin* $X'\ u) \wedge$ *topspace* X = *topspace* $X' \wedge$ *sets* (*borel-of* X) = *sets* (*borel-of* X') $\wedge$ *openin* $X'\ a \wedge$ *closedin* $X'\ a$
⟨*proof*⟩

end

3 Standard Borel Spaces

3.1 Standard Borel Spaces

theory *StandardBorel*
imports *Abstract-Metrizable-Topology*
begin

locale *standard-borel* =
fixes M :: *'a measure*
assumes *Polish-space*: $\exists S$. *Polish-space* $S \wedge$ *sets* M = *sets* (*borel-of* S)
begin

lemma *singleton-sets*:
assumes $x \in$ *space* M
shows $\{x\} \in$ *sets* M
⟨*proof*⟩

corollary *countable-sets*:
assumes $A \subseteq$ *space* M *countable* A
shows $A \in$ *sets* M
⟨*proof*⟩

lemma *standard-borel-restrict-space*:
assumes $A \in$ *sets* M
shows *standard-borel* (*restrict-space* $M\ A$)

⟨*proof*⟩

end

locale *standard-borel-ne* = *standard-borel* +
  assumes *space-ne*: *space* $M \neq \{\}$
begin

lemma *standard-borel-ne-restrict-space*:
  assumes $A \in$ *sets* M $A \neq \{\}$
  shows *standard-borel-ne* (*restrict-space* M A)
  ⟨*proof*⟩

lemma *standard-borel*: *standard-borel* M
  ⟨*proof*⟩

end

lemma *standard-borel-sets*:
  assumes *standard-borel* M **and** *sets* M = *sets* N
  shows *standard-borel* N
  ⟨*proof*⟩

lemma *standard-borel-ne-sets*:
  assumes *standard-borel-ne* M **and** *sets* M = *sets* N
  shows *standard-borel-ne* N
  ⟨*proof*⟩

lemma *pair-standard-borel*:
  assumes *standard-borel* M *standard-borel* N
  shows *standard-borel* ($M \bigotimes_M N$)
⟨*proof*⟩

lemma *pair-standard-borel-ne*:
  assumes *standard-borel-ne* M *standard-borel-ne* N
  shows *standard-borel-ne* ($M \bigotimes_M N$)
  ⟨*proof*⟩

lemma *product-standard-borel*:
  assumes *countable* I
      and $\bigwedge i.\ i \in I \Longrightarrow$ *standard-borel* (M i)
    shows *standard-borel* ($\Pi_M\ i{\in}I.\ M\ i$)
⟨*proof*⟩

lemma *product-standard-borel-ne*:
  assumes *countable* I
      and $\bigwedge i.\ i \in I \Longrightarrow$ *standard-borel-ne* (M i)
    shows *standard-borel-ne* ($\Pi_M\ i{\in}I.\ M\ i$)
  ⟨*proof*⟩

lemma *closed-set-standard-borel*[*simp*]:
  fixes *U* :: *'a* :: *topological-space set*
  assumes *Polish-space* (*euclidean* :: *'a topology*) *closed U*
  shows *standard-borel* (*restrict-space borel U*)
  ⟨*proof*⟩

lemma *closed-set-standard-borel-ne*[*simp*]:
  fixes *U* :: *'a* :: *topological-space set*
  assumes *Polish-space* (*euclidean* :: *'a topology*) *closed U U* ≠ {}
  shows *standard-borel-ne* (*restrict-space borel U*)
  ⟨*proof*⟩

lemma *open-set-standard-borel*[*simp*]:
  fixes *U* :: *'a* :: *topological-space set*
  assumes *Polish-space* (*euclidean* :: *'a topology*) *open U*
  shows *standard-borel* (*restrict-space borel U*)
  ⟨*proof*⟩

lemma *open-set-standard-borel-ne*[*simp*]:
  fixes *U* :: *'a* :: *topological-space set*
  assumes *Polish-space* (*euclidean* :: *'a topology*) *open U U* ≠ {}
  shows *standard-borel-ne* (*restrict-space borel U*)
  ⟨*proof*⟩

lemma *standard-borel-ne-borel*[*simp*]: *standard-borel-ne* (*borel* :: (*'a* :: *polish-space*) *measure*)
  and *standard-borel-ne-lborel*[*simp*]: *standard-borel-ne lborel*
  ⟨*proof*⟩

lemma *count-space-standard'*[*simp*]:
  assumes *countable I*
  shows *standard-borel* (*count-space I*)
  ⟨*proof*⟩

lemma *count-space-standard-ne*[*simp*]: *standard-borel-ne* (*count-space* (*UNIV* :: (- :: *countable*) *set*))
  ⟨*proof*⟩

corollary *measure-pmf-standard-borel-ne*[*simp*]: *standard-borel-ne* (*measure-pmf* (*p* :: (- :: *countable*) *pmf*))
  ⟨*proof*⟩

corollary *measure-spmf-standard-borel-ne*[*simp*]: *standard-borel-ne* (*measure-spmf* (*p* :: (- :: *countable*) *spmf*))
  ⟨*proof*⟩

corollary *countable-standard-ne*[*simp*]:
 standard-borel-ne (*borel* :: *'a* :: {*countable*,*t2-space*} *measure*)

⟨*proof*⟩

lemma(**in** *standard-borel*) *countable-discrete-space*:
assumes *countable* (*space M*)
shows *sets M* = *Pow* (*space M*)
⟨*proof*⟩

lemma(**in** *standard-borel*) *measurable-isomorphic-standard*:
assumes *M measurable-isomorphic N*
shows *standard-borel N*
⟨*proof*⟩

lemma(**in** *standard-borel-ne*) *measurable-isomorphic-standard-ne*:
assumes *M measurable-isomorphic N*
shows *standard-borel-ne N*
⟨*proof*⟩

lemma(**in** *standard-borel*) *standard-borel-embed-measure*:
assumes *inj-on f* (*space M*)
shows *standard-borel* (*embed-measure M f*)
⟨*proof*⟩

corollary(**in** *standard-borel-ne*) *standard-borel-ne-embed-measure*:
assumes *inj-on f* (*space M*)
shows *standard-borel-ne* (*embed-measure M f*)
⟨*proof*⟩

lemma
shows *standard-ne-ereal*: *standard-borel-ne* (*borel* :: *ereal measure*)
and *standard-ne-ennreal*: *standard-borel-ne* (*borel* :: *ennreal measure*)
⟨*proof*⟩

Cantor space $\mathscr{C}$

definition *Cantor-space* :: (*nat* ⇒ *real*) *measure* **where**
Cantor-space ≡ (Π_M *i*∈ *UNIV*. *restrict-space borel* {*0*,*1*})

lemma *Cantor-space-standard-ne*: *standard-borel-ne Cantor-space*
⟨*proof*⟩

lemma *Cantor-space-borel*:
sets (*borel-of Cantor-space-topology*) = *sets Cantor-space*
(**is** *?lhs* = -)
⟨*proof*⟩

Hilbert cube $\mathscr{H}$

definition *Hilbert-cube* :: (*nat* ⇒ *real*) *measure* **where**
Hilbert-cube ≡ (Π_M *i*∈ *UNIV*. *restrict-space borel* {*0*..*1*})

lemma *Hilbert-cube-standard-ne*: *standard-borel-ne Hilbert-cube*

⟨*proof*⟩

lemma *Hilbert-cube-borel*:
sets (*borel-of Hilbert-cube-topology*) = *sets Hilbert-cube* (**is** *?lhs* = -)
⟨*proof*⟩

3.2 Isomorphism between $\mathscr{C}$ and $\mathscr{H}$

lemma *Cantor-space-isomorphic-to-Hilbert-cube*:
Cantor-space measurable-isomorphic Hilbert-cube
⟨*proof*⟩

3.3 Final Results

lemma(**in** *standard-borel*) *embedding-into-Hilbert-cube*:
∃ *A* ∈ *sets Hilbert-cube*. *M measurable-isomorphic* (*restrict-space Hilbert-cube A*)
⟨*proof*⟩

lemma(**in** *standard-borel*) *embedding-from-Cantor-space*:
assumes *uncountable* (*space M*)
shows ∃ *A* ∈ *sets M*. *Cantor-space measurable-isomorphic* (*restrict-space M A*)
⟨*proof*⟩

corollary(**in** *standard-borel*) *uncountable-isomorphic-to-Hilbert-cube*:
assumes *uncountable* (*space M*)
shows *Hilbert-cube measurable-isomorphic M*
⟨*proof*⟩

corollary(**in** *standard-borel*) *uncountable-isomorphic-to-real*:
assumes *uncountable* (*space M*)
shows *M measurable-isomorphic* (*borel* :: *real measure*)
⟨*proof*⟩

lemma(**in** *standard-borel*) *isomorphic-subset-real*:
assumes *A* ∈ *sets* (*borel* :: *real measure*) *uncountable A*
obtains *B* **where** *B* ∈ *sets borel B* ⊆ *A M measurable-isomorphic restrict-space borel B*
⟨*proof*⟩

lemma(**in** *standard-borel*) *countable-isomorphic-to-subset-real*:
assumes *countable* (*space M*)
obtains *A* :: *real set*
where *countable A A* ∈ *sets borel M measurable-isomorphic restrict-space borel A*
⟨*proof*⟩

theorem *Borel-isomorphism-theorem*:
assumes *standard-borel M standard-borel N*
shows *space M* ≈ *space N* ⟷ *M measurable-isomorphic N*
⟨*proof*⟩

definition *to-real-on* :: *'a measure* ⇒ *'a* ⇒ *real* **where**
to-real-on M ≡ (*if uncountable* (*space M*) *then* (*SOME f*. *measurable-isomorphic-map M* (*borel* :: *real measure*) *f*) *else* (*real* ∘ *to-nat-on* (*space M*)))

definition *from-real-into* :: *'a measure* ⇒ *real* ⇒ *'a* **where**
from-real-into M ≡ (*if uncountable* (*space M*) *then the-inv-into* (*space M*) (*to-real-on M*) *else* (*λr*. *from-nat-into* (*space M*) (*nat* ⌊*r*⌋)))

context *standard-borel*
begin

abbreviation *to-real* ≡ *to-real-on M*
abbreviation *from-real* ≡ *from-real-into M*

lemma *to-real-def-countable*:
  assumes *countable* (*space M*)
  shows *to-real* = (*λr*. *real* (*to-nat-on* (*space M*) *r*))
  ⟨*proof*⟩

lemma *from-real-def-countable*:
  assumes *countable* (*space M*)
  shows *from-real* = (*λr*. *from-nat-into* (*space M*) (*nat* ⌊*r*⌋))
  ⟨*proof*⟩

lemma *from-real-to-real*[*simp*]:
  assumes *x* ∈ *space M*
  shows *from-real* (*to-real x*) = *x*
⟨*proof*⟩

lemma *to-real-measurable*[*measurable*]:
 to-real ∈ *M* $\rightarrow_M$ *borel*
⟨*proof*⟩

lemma *from-real-measurable'*:
  assumes *space M* ≠ {}
  shows *from-real* ∈ *borel* $\rightarrow_M$ *M*
⟨*proof*⟩

lemma *to-real-from-real*:
  assumes *uncountable* (*space M*)
  shows *to-real* (*from-real r*) = *r*
⟨*proof*⟩

end

lemma(**in** *standard-borel-ne*) *from-real-measurable*[*measurable*]: *from-real* ∈ *borel* $\rightarrow_M$ *M*
  ⟨*proof*⟩

end

References

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