

Real-Valued Special Functions:
Upper and Lower Bounds

Lawrence C. Paulson

February 6, 2026

Abstract

This development proves upper and lower bounds for several familiar real-valued functions. For \sin , \cos , \exp and the square root function, it defines and verifies infinite families of upper and lower bounds, mostly based on Taylor series expansions. For \tan^{-1} , \ln and \exp , it verifies a finite collection of upper and lower bounds, originally obtained from the functions' continued fraction expansions using the computer algebra system Maple. A common theme in these proofs is to take the difference between a function and its approximation, which should be zero at one point, and then consider the sign of the derivative.

The immediate purpose of this development is to verify axioms used by MetiTarski [1], an automatic theorem prover for real-valued special functions. Crucial to MetiTarski's operation is the provision of upper and lower bounds for each function of interest.

Contents

1	General Lemmas for Proving Function Inequalities	3
2	Arctan Upper and Lower Bounds	5
2.1	Upper Bound 1	5
2.2	Lower Bound 1	7
2.3	Upper Bound 3	7
2.4	Lower Bound 3	9
2.5	Upper Bound 4	10
2.6	Lower Bound 4	11
3	Exp Upper and Lower Bounds	13
3.1	Taylor Series Bounds	13
3.2	Continued Fraction Bound 2	14
3.3	Continued Fraction Bound 3	15
3.4	Continued Fraction Bound 4	16
3.5	Continued Fraction Bound 5	17
3.5.1	Proving monotonicity via a non-negative derivative . .	18
3.5.2	Results	18
3.6	Continued Fraction Bound 6	18
3.7	Continued Fraction Bound 7	19
3.7.1	Proving monotonicity via a non-negative derivative . .	20
3.7.2	Results	20
4	Log Upper and Lower Bounds	22
4.1	Upper Bound 3	22
4.2	Upper Bound 5	23
4.3	Upper Bound 7	24
4.4	Upper Bound 9	24
4.5	Upper Bound 11	25
4.6	Upper Bound 13	26
4.7	Upper Bound 15	27

5	Sine and Cosine Upper and Lower Bounds	29
5.1	Simple base cases	29
5.2	Taylor series approximants	30
5.3	Inductive proof of sine inequalities	30
5.4	Collecting the results	31
6	Square Root Upper and Lower Bounds	32
6.1	Upper bounds	32
6.2	Lower bounds	33

Chapter 1

General Lemmas for Proving Function Inequalities

```
theory Bounds-Lemmas
imports Complex-Main
```

```
begin
```

These are for functions that are differentiable over a closed interval.

```
lemma gen-lower-bound-increasing:
```

```
  fixes a :: real
```

```
  assumes a ≤ x
```

```
    and  $\bigwedge y. a \leq y \implies y \leq x \implies ((\lambda x. fl\ x - f\ x) \text{ has-real-derivative } g\ y) (at\ y)$ 
```

```
    and  $\bigwedge y. a \leq y \implies y \leq x \implies g\ y \leq 0$ 
```

```
    and fl a = f a
```

```
  shows fl x ≤ f x
```

```
⟨proof⟩
```

```
lemma gen-lower-bound-decreasing:
```

```
  fixes a :: real
```

```
  assumes x ≤ a
```

```
    and  $\bigwedge y. x \leq y \implies y \leq a \implies ((\lambda x. fl\ x - f\ x) \text{ has-real-derivative } g\ y) (at\ y)$ 
```

```
    and  $\bigwedge y. x \leq y \implies y \leq a \implies g\ y \geq 0$ 
```

```
    and fl a = f a
```

```
  shows fl x ≤ f x
```

```
⟨proof⟩
```

```
lemma gen-upper-bound-increasing:
```

```
  fixes a :: real
```

```
  assumes a ≤ x
```

```
    and  $\bigwedge y. a \leq y \implies y \leq x \implies ((\lambda x. fu\ x - f\ x) \text{ has-real-derivative } g\ y) (at\ y)$ 
```

```
    and  $\bigwedge y. a \leq y \implies y \leq x \implies g\ y \geq 0$ 
```

```
    and fu a = f a
```

```
  shows f x ≤ fu x
```

```
⟨proof⟩
```

```
lemma gen-upper-bound-decreasing:  
  fixes  $a :: \text{real}$   
  assumes  $x \leq a$   
    and  $\bigwedge y. x \leq y \implies y \leq a \implies ((\lambda x. fu\ x - f\ x) \text{ has-real-derivative } g\ y) \text{ (at } y)$   
    and  $\bigwedge y. x \leq y \implies y \leq a \implies g\ y \leq 0$   
    and  $fu\ a = f\ a$   
  shows  $f\ x \leq fu\ x$   
  <proof>  
  
end
```

Chapter 2

Arctan Upper and Lower Bounds

```
theory Atan-CF-Bounds
imports Bounds-Lemmas
         HOL-Library.Sum-of-Squares
```

```
begin
```

Covers all bounds used in `arctan-upper.ax`, `arctan-lower.ax` and `arctan-extended.ax`, excepting only `arctan-extended2.ax`, which is used in two `atan-error-analysis` problems.

2.1 Upper Bound 1

```
definition arctan-upper-11 :: real  $\Rightarrow$  real
  where arctan-upper-11  $\equiv \lambda x. -(pi/2) - 1/x$ 
```

```
definition diff-delta-arctan-upper-11 :: real  $\Rightarrow$  real
  where diff-delta-arctan-upper-11  $\equiv \lambda x. 1 / (x^2 * (1 + x^2))$ 
```

```
lemma d-delta-arctan-upper-11:  $x \neq 0 \implies$ 
   $((\lambda x. arctan-upper-11 x - arctan x)$  has-field-derivative diff-delta-arctan-upper-11
   $x)$  (at  $x$ )
  <proof>
```

```
lemma d-delta-arctan-upper-11-pos:  $x \neq 0 \implies diff-delta-arctan-upper-11 x > 0$ 
  <proof>
```

Different proof needed here: they coincide not at zero, but at $(-)$ infinity!

```
lemma arctan-upper-11:
  assumes  $x < 0$ 
  shows  $arctan(x) < arctan-upper-11 x$ 
  <proof>
```

definition *arctan-upper-12* :: *real* \Rightarrow *real*
where *arctan-upper-12* $\equiv \lambda x. 3*x / (x^2 + 3)$

definition *diff-delta-arctan-upper-12* :: *real* \Rightarrow *real*
where *diff-delta-arctan-upper-12* $\equiv \lambda x. -4*x^4 / ((x^2+3)^2 * (1+x^2))$

lemma *d-delta-arctan-upper-12*:
*(($\lambda x. arctan-upper-12 x - arctan x$) has-field-derivative *diff-delta-arctan-upper-12* x) (at x)
 <proof>*

Strict inequalities also possible

lemma *arctan-upper-12*:
assumes $x \leq 0$ **shows** $arctan(x) \leq arctan-upper-12 x$
 <proof>

definition *arctan-upper-13* :: *real* \Rightarrow *real*
where *arctan-upper-13* $\equiv \lambda x. x$

definition *diff-delta-arctan-upper-13* :: *real* \Rightarrow *real*
where *diff-delta-arctan-upper-13* $\equiv \lambda x. x^2 / (1 + x^2)$

lemma *d-delta-arctan-upper-13*:
*(($\lambda x. arctan-upper-13 x - arctan x$) has-field-derivative *diff-delta-arctan-upper-13* x) (at x)
 <proof>*

lemma *arctan-upper-13*:
assumes $x \geq 0$ **shows** $arctan(x) \leq arctan-upper-13 x$
 <proof>

definition *arctan-upper-14* :: *real* \Rightarrow *real*
where *arctan-upper-14* $\equiv \lambda x. pi/2 - 3*x / (1 + 3*x^2)$

definition *diff-delta-arctan-upper-14* :: *real* \Rightarrow *real*
where *diff-delta-arctan-upper-14* $\equiv \lambda x. -4 / ((1 + 3*x^2)^2 * (1+x^2))$

lemma *d-delta-arctan-upper-14*:
*(($\lambda x. arctan-upper-14 x - arctan x$) has-field-derivative *diff-delta-arctan-upper-14* x) (at x)
 <proof>*

lemma *d-delta-arctan-upper-14-neg*: *diff-delta-arctan-upper-14 x < 0*
 <proof>

lemma *lim14*: *(($\lambda x::real. 3 * x / (1 + 3 * x^2)$) $\longrightarrow 0$) at-infinity*
 <proof>

Different proof needed here: they coincide not at zero, but at (+) infinity!

lemma *arctan-upper-14*:
assumes $x > 0$
shows $\arctan(x) < \text{arctan-upper-14 } x$
 $\langle \text{proof} \rangle$

2.2 Lower Bound 1

definition *arctan-lower-11* :: *real* \Rightarrow *real*
where *arctan-lower-11* $\equiv \lambda x. -(\pi/2) - 3*x / (1 + 3*x^2)$

lemma *arctan-lower-11*:
assumes $x < 0$
shows $\arctan(x) > \text{arctan-lower-11 } x$
 $\langle \text{proof} \rangle$

abbreviation *arctan-lower-12* $\equiv \text{arctan-upper-13}$

lemma *arctan-lower-12*:
assumes $x \leq 0$
shows $\arctan(x) \geq \text{arctan-lower-12 } x$
 $\langle \text{proof} \rangle$

abbreviation *arctan-lower-13* $\equiv \text{arctan-upper-12}$

lemma *arctan-lower-13*:
assumes $x \geq 0$
shows $\arctan(x) \geq \text{arctan-lower-13 } x$
 $\langle \text{proof} \rangle$

definition *arctan-lower-14* :: *real* \Rightarrow *real*
where *arctan-lower-14* $\equiv \lambda x. \pi/2 - 1/x$

lemma *arctan-lower-14*:
assumes $x > 0$
shows $\arctan(x) > \text{arctan-lower-14 } x$
 $\langle \text{proof} \rangle$

2.3 Upper Bound 3

definition *arctan-upper-31* :: *real* \Rightarrow *real*
where *arctan-upper-31* $\equiv \lambda x. -(\pi/2) - (64 + 735*x^2 + 945*x^4) / (15*x*(15 + 70*x^2 + 63*x^4))$

definition *diff-delta-arctan-upper-31* :: *real* \Rightarrow *real*
where *diff-delta-arctan-upper-31* $\equiv \lambda x. 64 / (x^2 * (15 + 70*x^2 + 63*x^4)^2 * (1 + x^2))$

lemma *d-delta-arctan-upper-31*:

assumes $x \neq 0$
shows $((\lambda x. \arctan\text{-upper-31 } x - \arctan x) \text{ has-field-derivative } \text{diff-delta-arctan-upper-31 } x) \text{ (at } x)$
 $\langle \text{proof} \rangle$

lemma $d\text{-delta-arctan-upper-31-pos: } x \neq 0 \implies \text{diff-delta-arctan-upper-31 } x > 0$
 $\langle \text{proof} \rangle$

lemma $\arctan\text{-upper-31:}$
assumes $x < 0$
shows $\arctan(x) < \arctan\text{-upper-31 } x$
 $\langle \text{proof} \rangle$

definition $\arctan\text{-upper-32} :: \text{real} \Rightarrow \text{real}$
where $\arctan\text{-upper-32} \equiv \lambda x. 7 * (33 * x^4 + 170 * x^2 + 165) * x / (5 * (5 * x^6 + 105 * x^4 + 315 * x^2 + 231))$

definition $\text{diff-delta-arctan-upper-32} :: \text{real} \Rightarrow \text{real}$
where $\text{diff-delta-arctan-upper-32} \equiv \lambda x. -256 * x^{12} / ((5 * x^6 + 105 * x^4 + 315 * x^2 + 231)^2 * (1 + x^2))$

lemma $d\text{-delta-arctan-upper-32:}$
 $((\lambda x. \arctan\text{-upper-32 } x - \arctan x) \text{ has-field-derivative } \text{diff-delta-arctan-upper-32 } x) \text{ (at } x)$
 $\langle \text{proof} \rangle$

lemma $\arctan\text{-upper-32:}$
assumes $x \leq 0$ **shows** $\arctan(x) \leq \arctan\text{-upper-32 } x$
 $\langle \text{proof} \rangle$

definition $\arctan\text{-upper-33} :: \text{real} \Rightarrow \text{real}$
where $\arctan\text{-upper-33} \equiv \lambda x. (64 * x^4 + 735 * x^2 + 945) * x / (15 * (15 * x^4 + 70 * x^2 + 63))$

definition $\text{diff-delta-arctan-upper-33} :: \text{real} \Rightarrow \text{real}$
where $\text{diff-delta-arctan-upper-33} \equiv \lambda x. 64 * x^{10} / ((15 * x^4 + 70 * x^2 + 63)^2 * (1 + x^2))$

lemma $d\text{-delta-arctan-upper-33:}$
 $((\lambda x. \arctan\text{-upper-33 } x - \arctan x) \text{ has-field-derivative } \text{diff-delta-arctan-upper-33 } x) \text{ (at } x)$
 $\langle \text{proof} \rangle$

lemma $\arctan\text{-upper-33:}$
assumes $x \geq 0$ **shows** $\arctan(x) \leq \arctan\text{-upper-33 } x$
 $\langle \text{proof} \rangle$

definition $\arctan\text{-upper-34} :: \text{real} \Rightarrow \text{real}$
where $\arctan\text{-upper-34} \equiv$
 $\lambda x. \pi/2 - (33 + 170 * x^2 + 165 * x^4) * 7 * x / (5 * (5 + 105 * x^2 + 315 * x^4 + 231 * x^6))$

definition *diff-delta-arctan-upper-34* :: *real* \Rightarrow *real*
where *diff-delta-arctan-upper-34* $\equiv \lambda x. -256 / ((5+105*x^2+315*x^4+231*x^6)^2*(1+x^2))$

lemma *d-delta-arctan-upper-34*:
 $((\lambda x. \text{arctan-upper-34 } x - \text{arctan } x)$ has-field-derivative *diff-delta-arctan-upper-34*
 $x)$ (at x)
 \langle *proof* \rangle

lemma *d-delta-arctan-upper-34-pos*: *diff-delta-arctan-upper-34* $x < 0$
 \langle *proof* \rangle

lemma *arctan-upper-34*:
assumes $x > 0$
shows $\text{arctan}(x) < \text{arctan-upper-34 } x$
 \langle *proof* \rangle

2.4 Lower Bound 3

definition *arctan-lower-31* :: *real* \Rightarrow *real*
where *arctan-lower-31* $\equiv \lambda x. -(\text{pi}/2) - (33 + 170*x^2 + 165*x^4)*7*x /$
 $(5*(5 + 105*x^2 + 315*x^4 + 231*x^6))$

lemma *arctan-lower-31*:
assumes $x < 0$
shows $\text{arctan}(x) > \text{arctan-lower-31 } x$
 \langle *proof* \rangle

abbreviation *arctan-lower-32* $\equiv \text{arctan-upper-33}$

lemma *arctan-lower-32*:
assumes $x \leq 0$
shows $\text{arctan}(x) \geq \text{arctan-lower-32 } x$
 \langle *proof* \rangle

abbreviation *arctan-lower-33* $\equiv \text{arctan-upper-32}$

lemma *arctan-lower-33*:
assumes $x \geq 0$
shows $\text{arctan}(x) \geq \text{arctan-lower-33 } x$
 \langle *proof* \rangle

definition *arctan-lower-34* :: *real* \Rightarrow *real*
where *arctan-lower-34* $\equiv \lambda x. \text{pi}/2 - (64 + 735*x^2 + 945*x^4) / (15*x*(15$
 $+ 70*x^2 + 63*x^4))$

lemma *arctan-lower-34*:
assumes $x > 0$
shows $\text{arctan}(x) > \text{arctan-lower-34 } x$
 \langle *proof* \rangle

2.5 Upper Bound 4

definition *arctan-upper-41* :: real \Rightarrow real

where *arctan-upper-41* \equiv

$$\lambda x. \frac{-(\pi/2) - (256 + 5943*x^2 + 19250*x^4 + 15015*x^6) / (35*x*(35 + 315*x^2 + 693*x^4 + 429*x^6))}{}$$

definition *diff-delta-arctan-upper-41* :: real \Rightarrow real

where *diff-delta-arctan-upper-41* $\equiv \lambda x. 256 / (x^2*(35+315*x^2+693*x^4+429*x^6))^2*(1+x^2)$

lemma *d-delta-arctan-upper-41*:

assumes $x \neq 0$

shows $((\lambda x. \text{arctan-upper-41 } x - \text{arctan } x)$ has-field-derivative *diff-delta-arctan-upper-41* $x)$ (at x)
 ⟨proof⟩

lemma *d-delta-arctan-upper-41-pos*: $x \neq 0 \implies \text{diff-delta-arctan-upper-41 } x > 0$

⟨proof⟩

lemma *arctan-upper-41*:

assumes $x < 0$

shows $\text{arctan}(x) < \text{arctan-upper-41 } x$
 ⟨proof⟩

definition *arctan-upper-42* :: real \Rightarrow real

where *arctan-upper-42* \equiv

$$\lambda x. (15159*x^6 + 147455*x^4 + 345345*x^2 + 225225)*x / (35*(35*x^8 + 1260*x^6 + 6930*x^4 + 12012*x^2 + 6435))$$

definition *diff-delta-arctan-upper-42* :: real \Rightarrow real

where *diff-delta-arctan-upper-42* \equiv

$$\lambda x. -16384*x^16 / ((35*x^8 + 1260*x^6 + 6930*x^4 + 12012*x^2 + 6435)^2*(1+x^2))$$

lemma *d-delta-arctan-upper-42*:

$((\lambda x. \text{arctan-upper-42 } x - \text{arctan } x)$ has-field-derivative *diff-delta-arctan-upper-42* $x)$ (at x)
 ⟨proof⟩

lemma *arctan-upper-42*:

assumes $x \leq 0$ **shows** $\text{arctan}(x) \leq \text{arctan-upper-42 } x$
 ⟨proof⟩

definition *arctan-upper-43* :: real \Rightarrow real

where *arctan-upper-43* \equiv

$$\lambda x. (256*x^6 + 5943*x^4 + 19250*x^2 + 15015)*x / (35 * (35*x^6 + 315*x^4 + 693*x^2 + 429))$$

definition *diff-delta-arctan-upper-43* :: real \Rightarrow real

where *diff-delta-arctan-upper-43* $\equiv \lambda x. 256*x^14 / ((35*x^6 + 315*x^4 + 693*x^2 + 429)^2*(1+x^2))$

lemma *d-delta-arctan-upper-43*:

$((\lambda x. \text{arctan-upper-43 } x - \arctan x)$ has-field-derivative *diff-delta-arctan-upper-43*
 $x)$ (at x)
 \langle proof \rangle

lemma *arctan-upper-43*:

assumes $x \geq 0$ **shows** $\arctan(x) \leq \text{arctan-upper-43 } x$
 \langle proof \rangle

definition *arctan-upper-44* :: *real* \Rightarrow *real*

where *arctan-upper-44* \equiv

$$\lambda x. \pi/2 - (15159 + 147455*x^2 + 345345*x^4 + 225225*x^6)*x / \\ (35*(35 + 1260*x^2 + 6930*x^4 + 12012*x^6 + 6435*x^8))$$

definition *diff-delta-arctan-upper-44* :: *real* \Rightarrow *real*

where *diff-delta-arctan-upper-44* \equiv

$$\lambda x. -16384 / ((35 + 1260*x^2 + 6930*x^4 + 12012*x^6 + 6435*x^8)^2*(1 + x^2))$$

lemma *d-delta-arctan-upper-44*:

$((\lambda x. \text{arctan-upper-44 } x - \arctan x)$ has-field-derivative *diff-delta-arctan-upper-44*
 $x)$ (at x)
 \langle proof \rangle

lemma *d-delta-arctan-upper-44-pos*: *diff-delta-arctan-upper-44* $x < 0$

\langle proof \rangle

lemma *arctan-upper-44*:

assumes $x > 0$
shows $\arctan(x) < \text{arctan-upper-44 } x$
 \langle proof \rangle

2.6 Lower Bound 4

definition *arctan-lower-41* :: *real* \Rightarrow *real*

where *arctan-lower-41* \equiv

$$\lambda x. -(pi/2) - (15159 + 147455*x^2 + 345345*x^4 + 225225*x^6)*x / \\ (35*(35 + 1260*x^2 + 6930*x^4 + 12012*x^6 + 6435*x^8))$$

lemma *arctan-lower-41*:

assumes $x < 0$

shows $\arctan(x) > \text{arctan-lower-41 } x$

\langle proof \rangle

abbreviation *arctan-lower-42* \equiv *arctan-upper-43*

lemma *arctan-lower-42*:

assumes $x \leq 0$

shows $\arctan(x) \geq \text{arctan-lower-42 } x$

\langle proof \rangle

abbreviation $\text{arctan-lower-43} \equiv \text{arctan-upper-42}$

lemma arctan-lower-43 :

assumes $x \geq 0$

shows $\text{arctan}(x) \geq \text{arctan-lower-43 } x$

$\langle \text{proof} \rangle$

definition $\text{arctan-lower-44} :: \text{real} \Rightarrow \text{real}$

where $\text{arctan-lower-44} \equiv$

$$\lambda x. \pi/2 - (256 + 5943*x^2 + 19250*x^4 + 15015*x^6) / (35*x*(35 + 315*x^2 + 693*x^4 + 429*x^6))$$

lemma arctan-lower-44 :

assumes $x > 0$

shows $\text{arctan}(x) > \text{arctan-lower-44 } x$

$\langle \text{proof} \rangle$

end

Chapter 3

Exp Upper and Lower Bounds

```
theory Exp-Bounds
imports Bounds-Lemmas
         HOL-Library.Sum-of-Squares
         Sturm-Sequences.Sturm
```

```
begin
```

Covers all bounds used in `exp-upper.ax`, `exp-lower.ax` and `exp-extended.ax`.

3.1 Taylor Series Bounds

exp-positive is the theorem $0 \leq \exp ?x$

exp-lower-taylor-1 is the theorem $1 + ?x \leq \exp ?x$

All even approximants are lower bounds.

```
lemma exp-lower-taylor-even:
```

```
fixes x::real
```

```
shows even n  $\implies (\sum m < n. (x \wedge m) / (\text{fact } m)) \leq \exp x$ 
```

```
<proof>
```

```
lemma exp-upper-taylor-even:
```

```
fixes x::real
```

```
assumes n: even n
```

```
and pos:  $(\sum m < n. ((-x) \wedge m) / (\text{fact } m)) > 0$  (is ?sum > 0)
```

```
shows  $\exp x \leq \text{inverse } ?sum$ 
```

```
<proof>
```

3 if the previous lemma is expressed in terms of $(2::'a) * m$.

```
lemma exp-lower-taylor-3:
```

```
fixes x::real
```

shows $1 + x + (1/2)*x^2 + (1/6)*x^3 + (1/24)*x^4 + (1/120)*x^5 \leq \exp x$
 <proof>

lemma *exp-lower-taylor-3-cubed*:

fixes $x::real$
shows $(1 + x/3 + (1/2)*(x/3)^2 + (1/6)*(x/3)^3 + (1/24)*(x/3)^4 + (1/120)*(x/3)^5)^3 \leq \exp x$
 <proof>

lemma *exp-lower-taylor-2*:

fixes $x::real$
shows $1 + x + (1/2)*x^2 + (1/6)*x^3 \leq \exp x$
 <proof>

lemma *exp-upper-bound-case-3*:

fixes $x::real$
assumes $x \leq 3.19$
shows $\exp x \leq 2304 / (-(x^3) + 6*x^2 - 24*x + 48)^2$
 <proof>

lemma *exp-upper-bound-case-5*:

fixes $x::real$
assumes $x \leq 6.36$
shows $\exp x \leq 21743271936 / (-(x^3) + 12*x^2 - 96*x + 384)^4$
 <proof>

3.2 Continued Fraction Bound 2

definition *exp-cf2* :: $real \Rightarrow real$

where $exp-cf2 \equiv \lambda x. (x^2 + 6*x + 12) / (x^2 - 6*x + 12)$

lemma *denom-cf2-pos*: **fixes** $x::real$ **shows** $x^2 - 6 * x + 12 > 0$
 <proof>

lemma *numer-cf2-pos*: **fixes** $x::real$ **shows** $x^2 + 6 * x + 12 > 0$
 <proof>

lemma *exp-cf2-pos*: $exp-cf2 x > 0$
 <proof>

definition *diff-delta-lnexp-cf2* :: $real \Rightarrow real$

where $diff-delta-lnexp-cf2 \equiv \lambda x. -(x^4) / ((x^2 - 6*x + 12) * (x^2 + 6*x + 12))$

lemma *d-delta-lnexp-cf2-nonpos*: $diff-delta-lnexp-cf2 x \leq 0$
 <proof>

lemma *d-delta-lnexp-cf2*:

$((\lambda x. \ln (\exp\text{-cf2 } x) - x) \text{ has-field-derivative } \text{diff-delta-lnexp-cf2 } x) \text{ (at } x)$
 $\langle \text{proof} \rangle$

Upper bound for non-positive x

lemma *ln-exp-cf2-upper-bound-neg*:
assumes $x \leq 0$
shows $x \leq \ln (\exp\text{-cf2 } x)$
 $\langle \text{proof} \rangle$

theorem *exp-cf2-upper-bound-neg*: $x \leq 0 \implies \exp(x) \leq \exp\text{-cf2 } x$
 $\langle \text{proof} \rangle$

Lower bound for non-negative x

lemma *ln-exp-cf2-lower-bound-pos*:
assumes $0 \leq x$
shows $\ln (\exp\text{-cf2 } x) \leq x$
 $\langle \text{proof} \rangle$

theorem *exp-cf2-lower-bound-pos*: $0 \leq x \implies \exp\text{-cf2 } x \leq \exp x$
 $\langle \text{proof} \rangle$

3.3 Continued Fraction Bound 3

This bound crosses the X-axis twice, causing complications.

definition *numer-cf3* :: $real \Rightarrow real$
where $\text{numer-cf3} \equiv \lambda x. x^3 + 12*x^2 + 60*x + 120$

definition *exp-cf3* :: $real \Rightarrow real$
where $\text{exp-cf3} \equiv \lambda x. \text{numer-cf3 } x / \text{numer-cf3 } (-x)$

lemma *numer-cf3-pos*: $-4.64 \leq x \implies \text{numer-cf3 } x > 0$
 $\langle \text{proof} \rangle$

lemma *exp-cf3-pos*: $\text{numer-cf3 } x > 0 \implies \text{numer-cf3 } (-x) > 0 \implies \text{exp-cf3 } x > 0$
 $\langle \text{proof} \rangle$

definition *diff-delta-lnexp-cf3* :: $real \Rightarrow real$
where $\text{diff-delta-lnexp-cf3} \equiv \lambda x. (x^6) / (\text{numer-cf3 } (-x) * \text{numer-cf3 } x)$

lemma *d-delta-lnexp-cf3-nonneg*: $\text{numer-cf3 } x > 0 \implies \text{numer-cf3 } (-x) > 0 \implies$
 $\text{diff-delta-lnexp-cf3 } x \geq 0$
 $\langle \text{proof} \rangle$

lemma *d-delta-lnexp-cf3*:
assumes $\text{numer-cf3 } x > 0 \text{ numer-cf3 } (-x) > 0$
shows $((\lambda x. \ln (\exp\text{-cf3 } x) - x) \text{ has-field-derivative } \text{diff-delta-lnexp-cf3 } x) \text{ (at } x)$
 $\langle \text{proof} \rangle$

lemma *numer-cf3-mono*: $y \leq x \implies \text{numer-cf3 } y \leq \text{numer-cf3 } x$
 ⟨proof⟩

Upper bound for non-negative x

lemma *ln-exp-cf3-upper-bound-nonneg*:
assumes $x0$: $0 \leq x$ **and** $xless$: $\text{numer-cf3 } (-x) > 0$
shows $x \leq \ln (\text{exp-cf3 } x)$
 ⟨proof⟩

theorem *exp-cf3-upper-bound-pos*: $0 \leq x \implies \text{numer-cf3 } (-x) > 0 \implies \text{exp } x \leq \text{exp-cf3 } x$
 ⟨proof⟩

corollary $0 \leq x \implies x \leq 4.64 \implies \text{exp } x \leq \text{exp-cf3 } x$
 ⟨proof⟩

Lower bound for negative x, provided $0 < \text{exp-cf3 } x$

lemma *ln-exp-cf3-lower-bound-neg*:
assumes $x0$: $x \leq 0$ **and** $xgtr$: $\text{numer-cf3 } x > 0$
shows $\ln (\text{exp-cf3 } x) \leq x$
 ⟨proof⟩

theorem *exp-cf3-lower-bound-pos*:
assumes $x \leq 0$ **shows** $\text{exp-cf3 } x \leq \text{exp } x$
 ⟨proof⟩

3.4 Continued Fraction Bound 4

Here we have the extended exp continued fraction bounds

definition *numer-cf4* :: $\text{real} \Rightarrow \text{real}$
where $\text{numer-cf4} \equiv \lambda x. x^4 + 20*x^3 + 180*x^2 + 840*x + 1680$

definition *exp-cf4* :: $\text{real} \Rightarrow \text{real}$
where $\text{exp-cf4} \equiv \lambda x. \text{numer-cf4 } x / \text{numer-cf4 } (-x)$

lemma *numer-cf4-pos*: **fixes** $x::\text{real}$ **shows** $\text{numer-cf4 } x > 0$
 ⟨proof⟩

lemma *exp-cf4-pos*: $\text{exp-cf4 } x > 0$
 ⟨proof⟩

definition *diff-delta-lnexp-cf4* :: $\text{real} \Rightarrow \text{real}$
where $\text{diff-delta-lnexp-cf4} \equiv \lambda x. -(x^8) / (\text{numer-cf4 } (-x) * \text{numer-cf4 } x)$

lemma *d-delta-lnexp-cf4-nonpos*: $\text{diff-delta-lnexp-cf4 } x \leq 0$
 ⟨proof⟩

lemma *d-delta-lnexp-cf4*:

$((\lambda x. \ln (\exp\text{-cf}_4 x) - x)$ has-field-derivative $\text{diff-delta-lnexp-cf}_4 x$) (at x)
 $\langle \text{proof} \rangle$

Upper bound for non-positive x

lemma $\text{ln-exp-cf}_4\text{-upper-bound-neg}$:
assumes $x \leq 0$
shows $x \leq \ln (\exp\text{-cf}_4 x)$
 $\langle \text{proof} \rangle$

theorem $\text{exp-cf}_4\text{-upper-bound-neg}$: $x \leq 0 \implies \exp(x) \leq \exp\text{-cf}_4 x$
 $\langle \text{proof} \rangle$

Lower bound for non-negative x

lemma $\text{ln-exp-cf}_4\text{-lower-bound-pos}$:
assumes $0 \leq x$
shows $\ln (\exp\text{-cf}_4 x) \leq x$
 $\langle \text{proof} \rangle$

theorem $\text{exp-cf}_4\text{-lower-bound-pos}$: $0 \leq x \implies \exp\text{-cf}_4 x \leq \exp x$
 $\langle \text{proof} \rangle$

3.5 Continued Fraction Bound 5

This bound crosses the X-axis twice, causing complications.

definition $\text{numer-cf}_5 :: \text{real} \Rightarrow \text{real}$
where $\text{numer-cf}_5 \equiv \lambda x. x^5 + 30*x^4 + 420*x^3 + 3360*x^2 + 15120*x + 30240$

definition $\text{exp-cf}_5 :: \text{real} \Rightarrow \text{real}$
where $\text{exp-cf}_5 \equiv \lambda x. \text{numer-cf}_5 x / \text{numer-cf}_5 (-x)$

lemma $\text{numer-cf}_5\text{-pos}$: $-7.293 \leq x \implies \text{numer-cf}_5 x > 0$
 $\langle \text{proof} \rangle$

lemma $\text{exp-cf}_5\text{-pos}$: $\text{numer-cf}_5 x > 0 \implies \text{numer-cf}_5 (-x) > 0 \implies \text{exp-cf}_5 x > 0$
 $\langle \text{proof} \rangle$

definition $\text{diff-delta-lnexp-cf}_5 :: \text{real} \Rightarrow \text{real}$
where $\text{diff-delta-lnexp-cf}_5 \equiv \lambda x. (x^{\wedge}10) / (\text{numer-cf}_5 (-x) * \text{numer-cf}_5 x)$

lemma $d\text{-delta-lnexp-cf}_5\text{-nonneg}$: $\text{numer-cf}_5 x > 0 \implies \text{numer-cf}_5 (-x) > 0 \implies \text{diff-delta-lnexp-cf}_5 x \geq 0$
 $\langle \text{proof} \rangle$

lemma $d\text{-delta-lnexp-cf}_5$:
assumes $\text{numer-cf}_5 x > 0$ $\text{numer-cf}_5 (-x) > 0$
shows $((\lambda x. \ln (\exp\text{-cf}_5 x) - x)$ has-field-derivative $\text{diff-delta-lnexp-cf}_5 x$) (at x)
 $\langle \text{proof} \rangle$

3.5.1 Proving monotonicity via a non-negative derivative

definition *numer-cf5-deriv* :: *real* \Rightarrow *real*

where *numer-cf5-deriv* $\equiv \lambda x. 5*x^4 + 120*x^3 + 1260*x^2 + 6720*x + 15120$

lemma *numer-cf5-deriv*:

shows (*numer-cf5* has-field-derivative *numer-cf5-deriv* *x*) (at *x*)

<proof>

lemma *numer-cf5-deriv-pos*: *numer-cf5-deriv* *x* ≥ 0

<proof>

lemma *numer-cf5-mono*: $y \leq x \implies \text{numer-cf5 } y \leq \text{numer-cf5 } x$

<proof>

3.5.2 Results

Upper bound for non-negative *x*

lemma *ln-exp-cf5-upper-bound-nonneg*:

assumes *x0*: $0 \leq x$ **and** *xless*: *numer-cf5* $(-x) > 0$

shows $x \leq \ln (\text{exp-cf5 } x)$

<proof>

theorem *exp-cf5-upper-bound-pos*: $0 \leq x \implies \text{numer-cf5 } (-x) > 0 \implies \text{exp } x \leq \text{exp-cf5 } x$

<proof>

corollary $0 \leq x \implies x \leq 7.293 \implies \text{exp } x \leq \text{exp-cf5 } x$

<proof>

Lower bound for negative *x*, provided $0 < \text{exp-cf5 } x$

lemma *ln-exp-cf5-lower-bound-neg*:

assumes *x0*: $x \leq 0$ **and** *xgtr*: *numer-cf5* $x > 0$

shows $\ln (\text{exp-cf5 } x) \leq x$

<proof>

theorem *exp-cf5-lower-bound-pos*:

assumes $x \leq 0$ **shows** $\text{exp-cf5 } x \leq \text{exp } x$

<proof>

3.6 Continued Fraction Bound 6

definition *numer-cf6* :: *real* \Rightarrow *real*

where *numer-cf6* $\equiv \lambda x. x^6 + 42*x^5 + 840*x^4 + 10080*x^3 + 75600*x^2 + 332640*x + 665280$

definition *exp-cf6* :: *real* \Rightarrow *real*

where *exp-cf6* $\equiv \lambda x. \text{numer-cf6 } x / \text{numer-cf6 } (-x)$

lemma *numer-cf6-pos*: **fixes** $x::real$ **shows** $numer-cf6\ x > 0$
 ⟨*proof*⟩

lemma *exp-cf6-pos*: $exp-cf6\ x > 0$
 ⟨*proof*⟩

definition *diff-delta-lnexp-cf6* :: $real \Rightarrow real$
where $diff-delta-lnexp-cf6 \equiv \lambda x. -(x^{12}) / (numer-cf6\ (-x) * numer-cf6\ x)$

lemma *d-delta-lnexp-cf6-nonpos*: $diff-delta-lnexp-cf6\ x \leq 0$
 ⟨*proof*⟩

lemma *d-delta-lnexp-cf6*:
 (($\lambda x. \ln (exp-cf6\ x) - x$) *has-field-derivative* $diff-delta-lnexp-cf6\ x$) (at x)
 ⟨*proof*⟩

Upper bound for non-positive x

lemma *ln-exp-cf6-upper-bound-neg*:
assumes $x \leq 0$
shows $x \leq \ln (exp-cf6\ x)$
 ⟨*proof*⟩

theorem *exp-cf6-upper-bound-neg*: $x \leq 0 \implies exp(x) \leq exp-cf6\ x$
 ⟨*proof*⟩

Lower bound for non-negative x

lemma *ln-exp-cf6-lower-bound-pos*:
assumes $0 \leq x$
shows $\ln (exp-cf6\ x) \leq x$
 ⟨*proof*⟩

theorem *exp-cf6-lower-bound-pos*: $0 \leq x \implies exp-cf6\ x \leq exp\ x$
 ⟨*proof*⟩

3.7 Continued Fraction Bound 7

This bound crosses the X-axis twice, causing complications.

definition *numer-cf7* :: $real \Rightarrow real$
where $numer-cf7 \equiv \lambda x. x^7 + 56*x^6 + 1512*x^5 + 25200*x^4 + 277200*x^3 + 1995840*x^2 + 8648640*x + 17297280$

definition *exp-cf7* :: $real \Rightarrow real$
where $exp-cf7 \equiv \lambda x. numer-cf7\ x / numer-cf7\ (-x)$

lemma *numer-cf7-pos*: $-9.943 \leq x \implies numer-cf7\ x > 0$
 ⟨*proof*⟩

lemma *exp-cf7-pos*: $numer-cf7\ x > 0 \implies numer-cf7\ (-x) > 0 \implies exp-cf7\ x > 0$

<proof>

definition *diff-delta-lnexp-cf7* :: *real* \Rightarrow *real*

where *diff-delta-lnexp-cf7* $\equiv \lambda x. (x^{14}) / (\text{numer-cf7 } (-x) * \text{numer-cf7 } x)$

lemma *d-delta-lnexp-cf7-nonneg*: *numer-cf7* $x > 0 \implies \text{numer-cf7 } (-x) > 0 \implies \text{diff-delta-lnexp-cf7 } x \geq 0$

<proof>

lemma *d-delta-lnexp-cf7*:

assumes *numer-cf7* $x > 0$ *numer-cf7* $(-x) > 0$

shows $((\lambda x. \ln (\text{exp-cf7 } x) - x)$ *has-field-derivative* *diff-delta-lnexp-cf7* $x)$ (*at* x)

<proof>

3.7.1 Proving monotonicity via a non-negative derivative

definition *numer-cf7-deriv* :: *real* \Rightarrow *real*

where *numer-cf7-deriv* $\equiv \lambda x. 7*x^6 + 336*x^5 + 7560*x^4 + 100800*x^3 + 831600*x^2 + 3991680*x + 8648640$

lemma *numer-cf7-deriv*:

shows (*numer-cf7* *has-field-derivative* *numer-cf7-deriv* x) (*at* x)

<proof>

lemma *numer-cf7-deriv-pos*: *numer-cf7-deriv* $x \geq 0$

<proof>

lemma *numer-cf7-mono*: $y \leq x \implies \text{numer-cf7 } y \leq \text{numer-cf7 } x$

<proof>

3.7.2 Results

Upper bound for non-negative x

lemma *ln-exp-cf7-upper-bound-nonneg*:

assumes $x0: 0 \leq x$ **and** $xless: \text{numer-cf7 } (-x) > 0$

shows $x \leq \ln (\text{exp-cf7 } x)$

<proof>

theorem *exp-cf7-upper-bound-pos*: $0 \leq x \implies \text{numer-cf7 } (-x) > 0 \implies \text{exp } x \leq \text{exp-cf7 } x$

<proof>

corollary $0 \leq x \implies x \leq 9.943 \implies \text{exp } x \leq \text{exp-cf7 } x$

<proof>

Lower bound for negative x, provided $0 < \text{exp-cf7 } x$

lemma *ln-exp-cf7-lower-bound-neg*:

assumes $x0: x \leq 0$ **and** $xgtr: \text{numer-cf7 } x > 0$

shows $\ln (\text{exp-cf7 } x) \leq x$

<proof>

theorem *exp-cf7-lower-bound-pos:*

assumes $x \leq 0$ **shows** $\exp\text{-cf7 } x \leq \exp x$

<proof>

end

Chapter 4

Log Upper and Lower Bounds

theory *Log-CF-Bounds*
imports *Bounds-Lemmas*

begin

theorem *ln-upper-1*: $0 < x \implies \ln(x::\text{real}) \leq x - 1$
<proof>

definition *ln-lower-1* :: *real* \Rightarrow *real*
where *ln-lower-1* $\equiv \lambda x. 1 - (\text{inverse } x)$

corollary *ln-lower-1*: $0 < x \implies \text{ln-lower-1 } x \leq \ln x$
<proof>

theorem *ln-lower-1-eq*: $0 < x \implies \text{ln-lower-1 } x = (x - 1)/x$
<proof>

4.1 Upper Bound 3

definition *ln-upper-3* :: *real* \Rightarrow *real*
where *ln-upper-3* $\equiv \lambda x. (x + 5) * (x - 1) / (2 * (2 * x + 1))$

definition *diff-delta-ln-upper-3* :: *real* \Rightarrow *real*
where *diff-delta-ln-upper-3* $\equiv \lambda x. (x - 1)^3 / ((2 * x + 1)^2 * x)$

lemma *d-delta-ln-upper-3*: $x > 0 \implies$
 $((\lambda x. \text{ln-upper-3 } x - \ln x) \text{ has-field-derivative } \text{diff-delta-ln-upper-3 } x) \text{ (at } x)$
<proof>

Strict inequalities also possible

lemma *ln-upper-3-pos*:
assumes $1 \leq x$ **shows** $\ln(x) \leq \text{ln-upper-3 } x$

<proof>

lemma *ln-upper-3-neg*:

assumes $0 < x$ **and** $x1: x \leq 1$ **shows** $\ln(x) \leq \text{ln-upper-3 } x$
<proof>

theorem *ln-upper-3*: $0 < x \implies \ln(x) \leq \text{ln-upper-3 } x$
<proof>

definition *ln-lower-3* :: *real* \Rightarrow *real*
where *ln-lower-3* $\equiv \lambda x. - \text{ln-upper-3 } (inverse\ x)$

corollary *ln-lower-3*: $0 < x \implies \text{ln-lower-3 } x \leq \ln\ x$
<proof>

theorem *ln-lower-3-eq*: $0 < x \implies \text{ln-lower-3 } x = (1/2) * (1 + 5*x) * (x - 1) / (x * (2 + x))$
<proof>

4.2 Upper Bound 5

definition *ln-upper-5* :: *real* \Rightarrow *real*
where *ln-upper-5* $x \equiv (x^2 + 19*x + 10) * (x - 1) / (3 * (3*x^2 + 6*x + 1))$

definition *diff-delta-ln-upper-5* :: *real* \Rightarrow *real*
where *diff-delta-ln-upper-5* $\equiv \lambda x. (x - 1)^5 / ((3*x^2 + 6*x + 1)^2 * x)$

lemma *d-delta-ln-upper-5*: $x > 0 \implies$
 $((\lambda x. \text{ln-upper-5 } x - \ln\ x)$ *has-field-derivative* *diff-delta-ln-upper-5* $x)$ (at x)
<proof>

lemma *ln-upper-5-pos*:
assumes $1 \leq x$ **shows** $\ln(x) \leq \text{ln-upper-5 } x$
<proof>

lemma *ln-upper-5-neg*:
assumes $0 < x$ **and** $x1: x \leq 1$ **shows** $\ln(x) \leq \text{ln-upper-5 } x$
<proof>

theorem *ln-upper-5*: $0 < x \implies \ln(x) \leq \text{ln-upper-5 } x$
<proof>

definition *ln-lower-5* :: *real* \Rightarrow *real*
where *ln-lower-5* $\equiv \lambda x. - \text{ln-upper-5 } (inverse\ x)$

corollary *ln-lower-5*: $0 < x \implies \text{ln-lower-5 } x \leq \ln\ x$
<proof>

theorem *ln-lower-5-eq*: $0 < x \implies$

ln-lower-5 $x = (1/3)*(10*x^2 + 19*x + 1)*(x - 1) / (x*(x^2 + 6*x + 3))$
 ⟨proof⟩

4.3 Upper Bound 7

definition *ln-upper-7* :: *real* ⇒ *real*

where *ln-upper-7* $x \equiv (3*x^3 + 131*x^2 + 239*x + 47)*(x - 1) / (12*(4*x^3 + 18*x^2 + 12*x + 1))$

definition *diff-delta-ln-upper-7* :: *real* ⇒ *real*

where *diff-delta-ln-upper-7* $\equiv \lambda x. (x - 1)^7 / ((4*x^3 + 18*x^2 + 12*x + 1)^2 * x)$

lemma *d-delta-ln-upper-7*: $x > 0 \implies$

$((\lambda x. \text{ln-upper-7 } x - \text{ln } x) \text{ has-field-derivative } \text{diff-delta-ln-upper-7 } x) \text{ (at } x)$
 ⟨proof⟩

lemma *ln-upper-7-pos*:

assumes $1 \leq x$ **shows** $\text{ln}(x) \leq \text{ln-upper-7 } x$
 ⟨proof⟩

lemma *ln-upper-7-neg*:

assumes $0 < x$ **and** $x \leq 1$ **shows** $\text{ln}(x) \leq \text{ln-upper-7 } x$
 ⟨proof⟩

theorem *ln-upper-7*: $0 < x \implies \text{ln}(x) \leq \text{ln-upper-7 } x$

⟨proof⟩

definition *ln-lower-7* :: *real* ⇒ *real*

where *ln-lower-7* $\equiv \lambda x. - \text{ln-upper-7 } (inverse\ x)$

corollary *ln-lower-7*: $0 < x \implies \text{ln-lower-7 } x \leq \text{ln } x$

⟨proof⟩

theorem *ln-lower-7-eq*: $0 < x \implies$

$\text{ln-lower-7 } x = (1/12)*(47*x^3 + 239*x^2 + 131*x + 3)*(x - 1) / (x*(x^3 + 12*x^2 + 18*x + 4))$

⟨proof⟩

4.4 Upper Bound 9

definition *ln-upper-9* :: *real* ⇒ *real*

where *ln-upper-9* $x \equiv (6*x^4 + 481*x^3 + 1881*x^2 + 1281*x + 131)*(x - 1) /$

$$(30 * (5*x^4 + 40*x^3 + 60*x^2 + 20*x + 1))$$

definition *diff-delta-ln-upper-9* :: *real* ⇒ *real*

where *diff-delta-ln-upper-9* $\equiv \lambda x. (x - 1)^9 / (((5*x^4 + 40*x^3 + 60*x^2 + 20*x + 1) * 30))$

+ 20*x + 1)^2) * x)

lemma *d-delta-ln-upper-9*: $x > 0 \implies$

(($\lambda x. \ln\text{-upper-9 } x - \ln x$) has-field-derivative *diff-delta-ln-upper-9* x) (at x)
<proof>

lemma *ln-upper-9-pos*:

assumes $1 \leq x$ **shows** $\ln(x) \leq \ln\text{-upper-9 } x$
<proof>

lemma *ln-upper-9-neg*:

assumes $0 < x$ **and** $x \leq 1$ **shows** $\ln(x) \leq \ln\text{-upper-9 } x$
<proof>

theorem *ln-upper-9*: $0 < x \implies \ln(x) \leq \ln\text{-upper-9 } x$

<proof>

definition *ln-lower-9* :: *real* \Rightarrow *real*

where *ln-lower-9* $\equiv \lambda x. - \ln\text{-upper-9 } (inverse\ x)$

corollary *ln-lower-9*: $0 < x \implies \ln\text{-lower-9 } x \leq \ln x$

<proof>

theorem *ln-lower-9-eq*: $0 < x \implies$

$\ln\text{-lower-9 } x = (1/30)*(6 + 481*x + 1881*x^2 + 1281*x^3 + 131*x^4)*(x - 1) /$
 $(x*(5 + 40*x + 60*x^2 + 20*x^3 + x^4))$

<proof>

4.5 Upper Bound 11

Extended bounds start here

definition *ln-upper-11* :: *real* \Rightarrow *real*

where *ln-upper-11* $x \equiv$
 $(5*x^5 + 647*x^4 + 4397*x^3 + 6397*x^2 + 2272*x + 142) * (x - 1) /$
 $(30*(6*x^5 + 75*x^4 + 200*x^3 + 150*x^2 + 30*x + 1))$

definition *diff-delta-ln-upper-11* :: *real* \Rightarrow *real*

where *diff-delta-ln-upper-11* $\equiv \lambda x. (x - 1)^{11} / ((6*x^5 + 75*x^4 + 200*x^3 + 150*x^2 + 30*x + 1)^2 * x)$

lemma *d-delta-ln-upper-11*: $x > 0 \implies$

(($\lambda x. \ln\text{-upper-11 } x - \ln x$) has-field-derivative *diff-delta-ln-upper-11* x) (at x)
<proof>

lemma *ln-upper-11-pos*:

assumes $1 \leq x$ **shows** $\ln(x) \leq \text{ln-upper-11 } x$
 ⟨proof⟩

lemma *ln-upper-11-neg*:

assumes $0 < x$ **and** $x1: x \leq 1$ **shows** $\ln(x) \leq \text{ln-upper-11 } x$
 ⟨proof⟩

theorem *ln-upper-11*: $0 < x \implies \ln(x) \leq \text{ln-upper-11 } x$
 ⟨proof⟩

definition *ln-lower-11* :: *real* \Rightarrow *real*

where *ln-lower-11* $\equiv \lambda x. - \text{ln-upper-11 } (inverse\ x)$

corollary *ln-lower-11*: $0 < x \implies \text{ln-lower-11 } x \leq \ln\ x$
 ⟨proof⟩

theorem *ln-lower-11-eq*: $0 < x \implies$

$$\text{ln-lower-11 } x = (1/30) * (142 * x^5 + 2272 * x^4 + 6397 * x^3 + 4397 * x^2 + 647 * x + 5) * (x - 1) / (x * (x^5 + 30 * x^4 + 150 * x^3 + 200 * x^2 + 75 * x + 6))$$

⟨proof⟩

4.6 Upper Bound 13

definition *ln-upper-13* :: *real* \Rightarrow *real*

where *ln-upper-13* $x \equiv (353 + 8389 * x + 20149 * x^4 + 50774 * x^3 + 38524 * x^2 + 1921 * x^5 + 10 * x^6) * (x - 1) / (70 * (1 + 42 * x + 525 * x^4 + 700 * x^3 + 315 * x^2 + 126 * x^5 + 7 * x^6))$

definition *diff-delta-ln-upper-13* :: *real* \Rightarrow *real*

where *diff-delta-ln-upper-13* $\equiv \lambda x. (x - 1)^{13} / ((1 + 42 * x + 525 * x^4 + 700 * x^3 + 315 * x^2 + 126 * x^5 + 7 * x^6)^2 * x)$

lemma *d-delta-ln-upper-13*: $x > 0 \implies$

$((\lambda x. \text{ln-upper-13 } x - \ln\ x)$ *has-field-derivative* *diff-delta-ln-upper-13* $x)$ (at x)
 ⟨proof⟩

lemma *ln-upper-13-pos*:

assumes $1 \leq x$ **shows** $\ln(x) \leq \text{ln-upper-13 } x$
 ⟨proof⟩

lemma *ln-upper-13-neg*:

assumes $0 < x$ **and** $x1: x \leq 1$ **shows** $\ln(x) \leq \text{ln-upper-13 } x$
 ⟨proof⟩

theorem *ln-upper-13*: $0 < x \implies \ln(x) \leq \text{ln-upper-13 } x$
 ⟨proof⟩

definition *ln-lower-13* :: real \Rightarrow real

where *ln-lower-13* $\equiv \lambda x. - \text{ln-upper-13}$ (inverse x)

corollary *ln-lower-13*: $0 < x \implies \text{ln-lower-13 } x \leq \ln x$

<proof>

theorem *ln-lower-13-eq*: $0 < x \implies$

$$\begin{aligned} \text{ln-lower-13 } x = & (1/70) * (10 + 1921 * x + 20149 * x^2 + 50774 * x^3 + 38524 * x^4 \\ & + 8389 * x^5 + 353 * x^6) * (x - 1) / \\ & (x * (7 + 126 * x + 525 * x^2 + 700 * x^3 + 315 * x^4 + 42 * x^5 + \\ & x^6)) \end{aligned}$$

<proof>

4.7 Upper Bound 15

definition *ln-upper-15* :: real \Rightarrow real

where *ln-upper-15* $x \equiv$

$$\begin{aligned} & (1487 + 49199 * x + 547235 * x^4 + 718735 * x^3 + 334575 * x^2 + \\ & 141123 * x^5 + 35 * x^7 + 9411 * x^6) * (x - 1) / \\ & (280 * (1 + 56 * x + 2450 * x^4 + 1960 * x^3 + 588 * x^2 + 1176 * x^5 + \\ & 8 * x^7 + 196 * x^6)) \end{aligned}$$

definition *diff-delta-ln-upper-15* :: real \Rightarrow real

where *diff-delta-ln-upper-15*

$$\equiv \lambda x. (x - 1)^{15} / ((1 + 56 * x + 2450 * x^4 + 1960 * x^3 + 588 * x^2 + 8 * x^7 + 196 * x^6 + 1176 * x^5)^2 * x)$$

lemma *d-delta-ln-upper-15*: $x > 0 \implies$

$((\lambda x. \text{ln-upper-15 } x - \ln x)$ has-field-derivative $\text{diff-delta-ln-upper-15 } x)$ (at x)

<proof>

lemma *ln-upper-15-pos*:

assumes $1 \leq x$ **shows** $\ln(x) \leq \text{ln-upper-15 } x$

<proof>

lemma *ln-upper-15-neg*:

assumes $0 < x$ **and** $x \leq 1$ **shows** $\ln(x) \leq \text{ln-upper-15 } x$

<proof>

theorem *ln-upper-15*: $0 < x \implies \ln(x) \leq \text{ln-upper-15 } x$

<proof>

definition *ln-lower-15* :: real \Rightarrow real

where *ln-lower-15* $\equiv \lambda x. - \text{ln-upper-15}$ (inverse x)

corollary *ln-lower-15*: $0 < x \implies \text{ln-lower-15 } x \leq \ln x$

<proof>

theorem *ln-lower-15-eq*: $0 < x \implies$

$$\begin{aligned} \text{ln-lower-15 } x = & (1/280) * (35 + 9411 * x + 141123 * x^2 + 547235 * x^3 + \\ & 718735 * x^4 + 334575 * x^5 + 49199 * x^6 + 1487 * x^7) * (x - 1) / \\ & (x * (8 + 196 * x + 1176 * x^2 + 2450 * x^3 + 1960 * x^4 + 588 * x^5 \\ & + 56 * x^6 + x^7)) \\ & \langle \text{proof} \rangle \end{aligned}$$

end

Chapter 5

Sine and Cosine Upper and Lower Bounds

```
theory Sin-Cos-Bounds
imports Bounds-Lemmas
```

```
begin
```

5.1 Simple base cases

Upper bound for $0 \leq x$

```
lemma sin-le-arg:
  fixes  $x :: real$ 
  shows  $0 \leq x \implies \sin x \leq x$ 
  <proof>
```

```
lemma cos-ge-1-arg:
  fixes  $x :: real$ 
  assumes  $0 \leq x$ 
  shows  $1 - x \leq \cos x$ 
  <proof>
```

lemmas *sin-Taylor-0-upper-bound-pos* = *sin-le-arg* — MetiTarski bound

```
lemma cos-Taylor-1-lower-bound:
  fixes  $x :: real$ 
  assumes  $0 \leq x$ 
  shows  $(1 - x^2 / 2) \leq \cos x$ 
  <proof>
```

```
lemma sin-Taylor-1-lower-bound:
  fixes  $x :: real$ 
  assumes  $0 \leq x$ 
  shows  $(x - x^3 / 6) \leq \sin x$ 
```

<proof>

5.2 Taylor series approximants

definition *sinpoly* :: [nat,real] \Rightarrow real
where *sinpoly* n = ($\lambda x. \sum_{k < n}. \text{sin-coeff } k * x \wedge k$)

definition *cospoly* :: [nat,real] \Rightarrow real
where *cospoly* n = ($\lambda x. \sum_{k < n}. \text{cos-coeff } k * x \wedge k$)

lemma *sinpoly-Suc*: *sinpoly* (Suc n) = ($\lambda x. \text{sinpoly } n x + \text{sin-coeff } n * x \wedge n$)
<proof>

lemma *cospoly-Suc*: *cospoly* (Suc n) = ($\lambda x. \text{cospoly } n x + \text{cos-coeff } n * x \wedge n$)
<proof>

lemma *sinpoly-minus [simp]*: *sinpoly* n (-x) = - *sinpoly* n x
<proof>

lemma *cospoly-minus [simp]*: *cospoly* n (-x) = *cospoly* n x
<proof>

lemma *d-sinpoly-cospoly*:
(*sinpoly* (Suc n) has-field-derivative *cospoly* n x) (at x)
<proof>

lemma *d-cospoly-sinpoly*:
(*cospoly* (Suc n) has-field-derivative -*sinpoly* n x) (at x)
<proof>

5.3 Inductive proof of sine inequalities

lemma *sinpoly-lb-imp-cospoly-ub*:
assumes *x0*: $0 \leq x$ and *k0*: $k > 0$ and $\bigwedge x. 0 \leq x \implies \text{sinpoly } (k - 1) x \leq \text{sin } x$
shows $\text{cos } x \leq \text{cospoly } k x$
<proof>

lemma *cospoly-ub-imp-sinpoly-ub*:
assumes *x0*: $0 \leq x$ and *k0*: $k > 0$ and $\bigwedge x. 0 \leq x \implies \text{cos } x \leq \text{cospoly } (k - 1) x$
shows $\text{sin } x \leq \text{sinpoly } k x$
<proof>

lemma *sinpoly-ub-imp-cospoly-lb*:
assumes *x0*: $0 \leq x$ and *k0*: $k > 0$ and $\bigwedge x. 0 \leq x \implies \text{sin } x \leq \text{sinpoly } (k - 1) x$
shows $\text{cospoly } k x \leq \text{cos } x$
<proof>

lemma *cospoly-lb-imp-sinpoly-lb*:

assumes $x0: 0 \leq x$ **and** $k0: k > 0$ **and** $\bigwedge x. 0 \leq x \implies \text{cospoly } (k - 1) x \leq \text{cos } x$
shows $\text{sinpoly } k x \leq \text{sin } x$
<proof>

lemma

assumes $0 \leq x$
shows *sinpoly-lower-nonneg*: $\text{sinpoly } (4 * \text{Suc } n) x \leq \text{sin } x$ (**is** ?th1)
and *sinpoly-upper-nonneg*: $\text{sin } x \leq \text{sinpoly } (\text{Suc } (\text{Suc } (4 * n))) x$ (**is** ?th2)
<proof>

5.4 Collecting the results

corollary *sinpoly-upper-nonpos*:

$x \leq 0 \implies \text{sin } x \leq \text{sinpoly } (4 * \text{Suc } n) x$
<proof>

corollary *sinpoly-lower-nonpos*:

$x \leq 0 \implies \text{sinpoly } (\text{Suc } (\text{Suc } (4 * n))) x \leq \text{sin } x$
<proof>

corollary *cospoly-lower-nonneg*:

$0 \leq x \implies \text{cospoly } (\text{Suc } (\text{Suc } (\text{Suc } (4 * n)))) x \leq \text{cos } x$
<proof>

lemma *cospoly-lower*:

$\text{cospoly } (\text{Suc } (\text{Suc } (\text{Suc } (4 * n)))) x \leq \text{cos } x$
<proof>

lemma *cospoly-upper-nonneg*:

assumes $0 \leq x$
shows $\text{cos } x \leq \text{cospoly } (\text{Suc } (4 * n)) x$
<proof>

lemma *cospoly-upper*:

$\text{cos } x \leq \text{cospoly } (\text{Suc } (4 * n)) x$
<proof>

end

Chapter 6

Square Root Upper and Lower Bounds

```
theory Sqrt-Bounds
imports Bounds-Lemmas
        HOL-Library.Sum-of-Squares
```

```
begin
```

Covers all bounds used in sqrt-upper.ax, sqrt-lower.ax and sqrt-extended.ax.

6.1 Upper bounds

```
primrec sqrtu :: [real,nat]  $\Rightarrow$  real where
  sqrtu x 0 = (x+1)/2
| sqrtu x (Suc n) = (sqrtu x n + x/sqrtu x n) / 2
```

```
lemma sqrtu-upper:  $x \leq \text{sqrtu } x \ n \ ^2$ 
<proof>
```

```
lemma sqrtu-numeral:
  sqrtu x (numeral n) = (sqrtu x (pred-numeral n) + x/sqrtu x (pred-numeral n))
/ 2
<proof>
```

```
lemma sqrtu-gt-0:  $x \geq 0 \implies \text{sqrtu } x \ n > 0$ 
<proof>
```

```
theorem gen-sqrt-upper:  $0 \leq x \implies \text{sqrt } x \leq \text{sqrtu } x \ n$ 
<proof>
```

```
lemma sqrt-upper-bound-0:
  assumes  $x \geq 0$  shows  $\text{sqrt } x \leq (x+1)/2$  (is -  $\leq$  ?rhs)
<proof>
```

lemma *sqrt-upper-bound-1*:

assumes $x \geq 0$ **shows** $\text{sqrt } x \leq (1/4)*(x^2+6*x+1) / (x+1)$ (**is - ≤ ?rhs**)
{proof}

lemma *sqrtu-2-eq*:

$\text{sqrtu } x^2 = (1/8)*(x^4 + 28*x^3 + 70*x^2 + 28*x + 1) / ((x + 1)*(x^2 + 6*x + 1))$
{proof}

lemma *sqrt-upper-bound-2*:

assumes $x \geq 0$
shows $\text{sqrt } x \leq (1/8)*(x^4 + 28*x^3 + 70*x^2 + 28*x + 1) / ((x + 1)*(x^2 + 6*x + 1))$
{proof}

lemma *sqrtu-4-eq*:

$x \geq 0 \implies$

$\text{sqrtu } x^4 = (1/32)*(225792840*x^6+64512240*x^5+601080390*x^8+471435600*x^7+496*x+1+35960$
 $/ ((x+1)*(x^2+6*x+1)*(x^4+28*x^3+70*x^2+28*x+1)*(1820*x^6+8008*x^5+x^8+120*x^7+$
{proof}

lemma *sqrt-upper-bound-4*:

assumes $x \geq 0$

shows $\text{sqrt } x \leq (1/32)*(225792840*x^6+64512240*x^5+601080390*x^8+471435600*x^7+496*x+1+35960$
 $/ ((x+1)*(x^2+6*x+1)*(x^4+28*x^3+70*x^2+28*x+1)*(1820*x^6+8008*x^5+x^8+120*x^7+$
{proof}

lemma *gen-sqrt-upper-scaled*:

assumes $0 \leq x < u$

shows $\text{sqrt } x \leq \text{sqrtu } (x*u^2) n / u$
{proof}

lemma *sqrt-upper-bound-2-small*:

assumes $0 \leq x$

shows $\text{sqrt } x \leq (1/32)*(65536*x^4 + 114688*x^3 + 17920*x^2 + 448*x + 1) / ((16*x + 1)*(256*x^2 + 96*x + 1))$
{proof}

lemma *sqrt-upper-bound-2-large*:

assumes $0 \leq x$

shows $\text{sqrt } x \leq (1/32)*(65536 + 114688*x + 17920*x^2 + 448*x^3 + x^4) / ((x + 16)*(256 + 96*x + x^2))$
{proof}

6.2 Lower bounds

lemma *sqrt-lower-bound-id*:

assumes $0 \leq x \leq 1$

shows $x \leq \text{sqrt } x$

<proof>

definition *sqrtn* :: [real,nat] \Rightarrow real **where**
sqrtn x n = x/sqrt x n

lemma *sqrtn-lower*: $0 \leq x \implies \text{sqrtn } x \text{ n}^2 \leq x$
<proof>

theorem *gen-sqrtn-lower*: $0 \leq x \implies \text{sqrtn } x \text{ n} \leq \text{sqrt } x$
<proof>

lemma *sqrtn-lower-bound-0*:
assumes $x \geq 0$ **shows** $2*x/(x+1) \leq \text{sqrt } x$ (**is ?lhs \leq -**)
<proof>

lemma *sqrtn-lower-bound-1*:
assumes $x \geq 0$ **shows** $4*x*(x+1) / (x^2+6*x+1) \leq \text{sqrt } x$ (**is ?lhs \leq -**)
<proof>

lemma *sqrtn-2-eq*: *sqrtn* x 2 =
 $8*x*(x+1)*(x^2+6*x+1) / (x^4+28*x^3+70*x^2+28*x+1)$
<proof>

lemma *sqrtn-lower-bound-2*:
assumes $x \geq 0$
shows $8*x*(x+1)*(x^2+6*x+1) / (x^4+28*x^3+70*x^2+28*x+1) \leq \text{sqrt } x$
<proof>

lemma *sqrtn-4-eq*: $x \geq 0 \implies$
sqrtn x 4
 $= (32*x*(x+1)*(x^2+6*x+1)*(x^4+28*x^3+70*x^2+28*x+1)*(1820*x^6+8008*x^5+x^8+120*x^7-1) / (225792840*x^6+64512240*x^5+601080390*x^8+471435600*x^7+496*x+1+35960*x^2+906192*x^3-1))$
<proof>

lemma *sqrtn-lower-bound-4*:
assumes $x \geq 0$
shows $(32*x*(x+1)*(x^2+6*x+1)*(x^4+28*x^3+70*x^2+28*x+1)*(1820*x^6+8008*x^5+x^8+120*x^7-1) / (225792840*x^6+64512240*x^5+601080390*x^8+471435600*x^7+496*x+1+35960*x^2+906192*x^3-1)) \leq \text{sqrt } x$
<proof>

lemma *gen-sqrtn-lower-scaled*:
assumes $0 \leq x < u$
shows *sqrtn* (x*u²) n / u $\leq \text{sqrt } x$
<proof>

lemma *sqrtn-lower-bound-2-small*:
assumes $0 \leq x$

shows $32*x*(16*x + 1)*(256*x^2 + 96*x + 1) / (65536*x^4 + 114688*x^3 + 17920*x^2 + 448*x + 1) \leq \text{sqrt } x$
(proof)

lemma *sqrt-lower-bound-2-large*:

assumes $0 \leq x$

shows $32*x*(x + 16)*(x^2 + 96*x + 256) / (x^4 + 448*x^3 + 17920*x^2 + 114688*x + 65536) \leq \text{sqrt } x$
(proof)

end

Bibliography

- [1] B. Akbarpour and L. Paulson. MetiTarski: An automatic theorem prover for real-valued special functions. 44(3):175–205, Mar. 2010.