The Wallace–Simson Line Theorem in Isabelle/HOL

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Abstract

We formalize the Wallace–Simson line theorem in Isabelle/HOL. Let ABC be a nondegenerate triangle and let M be a point on its circumcircle. Dropping the perpendiculars from M to the three side lines BC, CA, AB gives feet P, Q, R; the theorem asserts that P, Q, R are collinear. The line through them is the *Simson line*, also called the *Wallace line* after William Wallace, whose published work on geometrical porisms is the standard early source for this result. We also prove the converse: if the three feet are collinear, then M lies on the circumcircle of ABC.

The development uses complex coordinates.

Contents

1 Introduction

The Wallace–Simson line is a classical incidence result about a triangle and a point on its circumcircle [1, 2, 4]. Given a nondegenerate triangle ABC and a point M on its circumcircle, the feet of the three perpendiculars from M to the side lines BC, CA, AB are collinear. The line so obtained is usually named after Robert Simson, but it is due to William Wallace [4]. The converse also holds and characterises the circumcircle: the feet are collinear *only* when the point lies on the circumcircle.

Our formalization works in the complex plane. It builds on the published *Complex Geometry* development [3] for its basic complex-plane notions; reducing the incidence statement to complex algebra then turns both directions of the theorem into short computations.

2 Complex-coordinate setup

We identify the Euclidean plane with $\mathbb{C}$. A point z lies on the circle of centre c and radius R when $|z - c| = R$, and a point z lies on the line through a and b when $z = a + t\,(b - a)$ for some real t. Three points a, b, c are collinear when

$$\operatorname{Im}\big(\overline{(b-a)}\,(c-a)\big) = 0,$$

a real multiple of the planar cross product, and two displacement vectors u, v are perpendicular when $\operatorname{Re}(\overline{u}\,v) = 0$, their Euclidean inner product. A basic lemma records that collinearity is equivalent to the ratio $(c-a)/(b-a)$ being real whenever the three points are distinct enough for the ratio to be defined, and this real-ratio form is what the main proofs ultimately use.

3 The foot of a perpendicular

The foot of the perpendicular from m to the line through a and b is

$$\mathrm{foot}(a,b,m) = \frac{a+m}{2} + \frac{(b-a)\,\overline{(m-a)}}{2\,\overline{(b-a)}}.$$

We verify that this closed form is meaningful: $\mathrm{foot}(a,b,m)$ lies on the line ab, and the residual $m - \mathrm{foot}(a,b,m)$ is perpendicular to the direction $b-a$. These properties also uniquely characterise the point on every nondegenerate line. Thus the three points named P, Q, R in the statement really are the feet of the perpendiculars onto the side lines, and the theorem is a statement about that geometric configuration rather than about an unmotivated algebraic expression.

4 The cross-ratio criterion

Four complex numbers w, x, y, z have cross ratio

$$(w,x;y,z) = \frac{(w-y)\,(x-z)}{(w-z)\,(x-y)}.$$

For four suitably distinct points, being concyclic is equivalent to this cross ratio being real. We need the forward direction: if w, x, y, z lie on a circle of centre o and radius $R > 0$, then each point satisfies $\overline{z-o} = R^2/(z-o)$; substituting this into the conjugate of the cross ratio, every factor R^2 and every factor $(\cdot - o)$ cancels, leaving the cross ratio itself, so it equals its own conjugate and is real.

5 The forward theorem

Assume M lies on the circumcircle of ABC. Working relative to the centre o, each pairwise difference of feet on the circle collapses to a clean product; concretely

$$\mathrm{foot}(b,c,m) - \mathrm{foot}(a,b,m) = \frac{(c-a)\,(m-b)}{2\,(m-o)}, \qquad \mathrm{foot}(c,a,m) - \mathrm{foot}(a,b,m) = \frac{(c-b)\,(m-a)}{2\,(m-o)}.$$

Their ratio is

$$\frac{\mathrm{foot}(b,c,m) - \mathrm{foot}(a,b,m)}{\mathrm{foot}(c,a,m) - \mathrm{foot}(a,b,m)} = \frac{(c-a)\,(m-b)}{(c-b)\,(m-a)} = (c,m;a,b),$$

the cross ratio of the four concyclic points c, m, a, b. By the cross-ratio criterion this ratio is real, and reality of the ratio of the two foot differences is exactly the collinearity of the three feet. (The degenerate case $m = a$, where two of the feet coincide, is handled separately.)

6 The converse

For the converse we drop the assumption that M lies on the circle and assume instead that the three feet are collinear. Using the same closed forms for the foot differences — now keeping the conjugate substitutions symbolic — the collinearity relation $\mathrm{Im}\big(\overline{(P-R)}\,(Q-R)\big) = 0$ becomes, after clearing the three circle denominators $(a-o)$, $(b-o)$, $(c-o)$, a single algebraic identity whose right-hand side factors as

$$R^4\,(a-b)\,(a-c)\,(b-c)\,\big(R^2 - \overline{(m-o)}\,(m-o)\big).$$

The three vertex differences are nonzero because ABC is nondegenerate, and the vanishing of the left-hand side forces $\overline{(m-o)}\,(m-o) = R^2$, that is $|m-o| = R$. Hence M lies on the circumcircle.

7 Formalization notes

The development uses the published `Complex_Geometry` session for its basic collinearity infrastructure. The forward theorem `simson_line` takes a point on the circumcircle and concludes collinearity of the three feet; the converse `simson_line_converse` takes collinearity of the feet and concludes that the point lies on the circumcircle. The theorem `simson_line_iff` combines them into the corresponding characterisation. The only structural hypothesis is that A, B, C are not collinear, together with a positive radius; the distinctness of the vertices is derived from non-collinearity. The closed-form foot is shown to be the unique point on the side line whose displacement from M is perpendicular to that line, so the formal statements precisely capture the Wallace–Simson line theorem and its converse.

theory *Simson-Complex-Geometry*
  imports *Complex-Geometry.Elementary-Complex-Geometry*
begin

lemma *Im-zero-self-cnj*:
  assumes *Im W = 0* **shows** *W = cnj W*
  ⟨*proof*⟩

lemma *im-zero-cross*:
  fixes *A B :: complex*
  assumes *h*: *Im (cnj A ∗ B) = 0*
  shows *cnj A ∗ B = A ∗ cnj B*
⟨*proof*⟩

lemma *collinear-iff-cross*:
  collinear a b c ⟷ Im (cnj (b − a) ∗ (c − a)) = 0
⟨*proof*⟩

8 Circles, lines, and the ordinary complex cross ratio

A point z lies on the circle of centre c and radius R when its distance to the centre equals R.

definition *oncircle* :: *complex* ⇒ *real* ⇒ *complex* ⇒ *bool*
where *oncircle c R z* ⟷ *cmod* (*z* − *c*) = *R*

A point z lies on the line through a and b when it is an affine combination of them with a real parameter.

definition *on-line* :: *complex* ⇒ *complex* ⇒ *complex* ⇒ *bool*
where *on-line a b z* ⟷ (∃ *t*::*real*. *z* = *a* + *of-real t* ∗ (*b* − *a*))

Four points are concyclic when they share a common circle of positive radius; three points are concyclic when they share such a circle.

definition *concyclic* :: *complex* ⇒ *complex* ⇒ *complex* ⇒ *complex* ⇒ *bool*
where *concyclic a b c d* ⟷
(∃ *ctr R*. *0* < *R* ∧ *oncircle ctr R a* ∧ *oncircle ctr R b* ∧ *oncircle ctr R c* ∧ *oncircle ctr R d*)

definition *concyclic3* :: *complex* ⇒ *complex* ⇒ *complex* ⇒ *bool*
where *concyclic3 a b c* ⟷
(∃ *ctr R*. *0* < *R* ∧ *oncircle ctr R a* ∧ *oncircle ctr R b* ∧ *oncircle ctr R c*)

The ordinary cross ratio of four complex numbers.

definition *complex-cross-ratio* :: *complex* ⇒ *complex* ⇒ *complex* ⇒ *complex* ⇒ *complex*
where *complex-cross-ratio w x y z* = (*w* − *y*) ∗ (*x* − *z*) / ((*w* − *z*) ∗ (*x* − *y*))

9 Elementary facts about circles

lemma *oncircle-norm-sq*:
assumes *oncircle c R z*
shows (*z* − *c*) ∗ *cnj* (*z* − *c*) = *complex-of-real* (R^2)
⟨*proof*⟩

lemma *oncircle-ne-centre*:
assumes *oncircle c R z* **and** *0* < *R*
shows *z* ≠ *c*
⟨*proof*⟩

lemma *cnj-diff-on-circle*:
assumes *oncircle c R z* **and** *0* < *R*
shows *cnj* (*z* − *c*) = *complex-of-real* (R^2) / (*z* − *c*)
⟨*proof*⟩

10 The cross-ratio criterion

The algebraic heart of the criterion: replacing each of four points by the reciprocal expression that describes a point on a circle leaves the cross ratio unchanged.

lemma *complex-cross-ratio-reciprocal-aux*:
  fixes $A\ B\ C\ D\ k$ :: *complex*
  assumes $A \neq 0$ **and** $B \neq 0$ **and** $C \neq 0$ **and** $D \neq 0$
    and $A \neq D$ **and** $B \neq C$ **and** $k \neq 0$
  shows $(k/A - k/C) * (k/B - k/D) / ((k/A - k/D) * (k/B - k/C))$
      $= (A - C) * (B - D) / ((A - D) * (B - C))$
  ⟨*proof*⟩

If four points lie on a common circle of positive radius (and the two relevant differences are nonzero), then their cross ratio is real.

lemma *concyclic-complex-cross-ratio-real*:
  assumes *a*: *oncircle ctr R a* **and** *b*: *oncircle ctr R b*
    and *c*: *oncircle ctr R c* **and** *d*: *oncircle ctr R d*
    and *pos*: $0 < R$ **and** *ad*: $a \neq d$ **and** *bc*: $b \neq c$
  shows *complex-cross-ratio a b c d* $\in \mathbb{R}$
⟨*proof*⟩

If three points lie on a circle and the cross ratio of a fourth point with them is real, the fourth point lies on the same circle.

lemma *complex-cross-ratio-real-on-circle*:
  assumes *hc*: *oncircle ctr R c* **and** *hp*: *oncircle ctr R p* **and** *hq*: *oncircle ctr R q*
    and *pos*: $0 < R$
    and *cp*: $c \neq p$ **and** *cq*: $c \neq q$ **and** *pq*: $p \neq q$ **and** *mq*: $m \neq q$
    and *crreal*: *complex-cross-ratio m c p q* $\in \mathbb{R}$
  shows *oncircle ctr R m*
⟨*proof*⟩

11 Collinearity gives real ratios

lemma *on-line-ratio-real*:
  assumes *on-line a b z* **and** $b \neq z$
  shows $(a - z) / (b - z) \in \mathbb{R}$
⟨*proof*⟩

end

theory *Simson*
  imports *Simson-Complex-Geometry*
begin

12 Perpendicularity and feet of perpendiculars

A point z lies on the circle of centre c and radius R when its distance to the centre equals R.

Two displacement vectors are perpendicular when the real part of $cnj\ u * v$ (their Euclidean inner product) vanishes.

definition *complex-perpendicular* :: *complex* ⇒ *complex* ⇒ *bool*
where *complex-perpendicular u v* ⟷ *Re* (*cnj u* ∗ *v*) = *0*

The foot of the perpendicular from m to the line through a and b, in closed complex form.

definition *foot* :: *complex* ⇒ *complex* ⇒ *complex* ⇒ *complex*
where *foot a b m* = (*a* + *m*)/*2* + (*b* − *a*) ∗ *cnj* (*m* − *a*) / (*2* ∗ *cnj* (*b* − *a*))

13 Collinearity as a real ratio

Three distinct-enough points are collinear exactly when the ratio of the two displacement vectors is real.

lemma *collinear-iff-real-ratio*:
assumes *a* ≠ *c* **shows** *collinear a b c* = ((*b* − *a*)/(*c* − *a*) ∈ ℝ)
⟨*proof*⟩

Degenerate instances of collinearity, and its cyclic symmetry.

lemma *collinear-refl*: *collinear a b b*
⟨*proof*⟩

lemma *collinear-aba*: *collinear a b a*
⟨*proof*⟩

lemma *collinear-aac*: *collinear a a c*
⟨*proof*⟩

lemma *collinear-cycle*:
assumes *collinear a b c* **shows** *collinear b c a*
⟨*proof*⟩

A non-collinear triple has three distinct vertices.

lemma *ncol-distinct*:
assumes ¬ *collinear a b c* **shows** *a* ≠ *b* ∧ *a* ≠ *c* ∧ *b* ≠ *c*
⟨*proof*⟩

14 Points on a circle

On a circle, the squared distance to the centre is the constant R^2.

The converse: the norm identity characterises membership of a circle of positive radius.

lemma *oncircle-of-norm-sq*:
  fixes *z ctr* :: *complex* **and** *R* :: *real*
  assumes *h*: $(z - ctr) * cnj\ (z - ctr) = complex\text{-}of\text{-}real\ (R\hat{\ }2)$ **and** *pos*: $0 < R$
  shows *oncircle ctr R z*
⟨*proof*⟩

15 The foot of a perpendicular

Two elementary conjugate identities.

lemma *of-real-Re-eq*: $complex\text{-}of\text{-}real\ (Re\ z) = (z + cnj\ z)/2$
  ⟨*proof*⟩

lemma *Re-sub-cnj-half-zero*: $Re\ ((z - cnj\ z)/2) = 0$
  ⟨*proof*⟩

The parameter of the foot along the line, before any circle assumption.

lemma *foot-ratio-gen*:
  fixes *a b m p q* :: *complex*
  assumes $q \neq 0$ **and** $b - a \neq 0$
  shows $((a + m)/2 + (b - a)*p/(2*q) - a)/(b - a) = ((m - a)/(b - a) + p/q)/2$
  ⟨*proof*⟩

A point lies on a line exactly when the displacement ratio is real.

lemma *on-line-iff*:
  assumes $a \neq b$ **shows** *on-line a b z* $= ((z - a)/(b - a) \in \mathbb{R})$
⟨*proof*⟩

The foot lies on its line.

lemma *foot-on-line*:
  assumes $a \neq b$ **shows** *on-line a b* (*foot a b m*)
⟨*proof*⟩

The segment from m to its foot is perpendicular to the line.

lemma *foot-perp*:
  assumes $a \neq b$ **shows** *complex-perpendicular* $(b - a)$ (*foot a b m* $- m$)
⟨*proof*⟩

The preceding two properties uniquely characterise the perpendicular foot on every nondegenerate line.

lemma *foot-unique*:
  assumes *ab*: $a \neq b$
    and *zline*: *on-line a b z*
    and *zperp*: *complex-perpendicular* $(b - a)\ (z - m)$
  shows $z =$ *foot a b m*
⟨*proof*⟩

16 Feet of the three perpendiculars from a point on the circle

Rewriting the foot when a and b lie on the circle: a pure field identity supporting the geometric rewrite that follows.

lemma *foot-general-gen*:
fixes $A\ B\ mu\ k$ :: *complex*
assumes $A \neq 0$ **and** $B \neq 0$ **and** $A \neq B$ **and** $k \neq 0$
shows $(B - A)*(mu - k/A)/(2*(k/B - k/A)) = B*(k - mu*A)/(2*k)$
⟨*proof*⟩

Closed form for the foot when both line endpoints lie on the circle of centre *ctr* and radius r.

lemma *foot-general*:
fixes $a\ b\ m\ ctr$ :: *complex* **and** r :: *real*
assumes *ha*: *oncircle ctr r a* **and** *hb*: *oncircle ctr r b* **and** *pos*: $0 < r$ **and** *dab*: $a \neq b$
shows *foot a b m* $= ctr + (a - ctr + (m - ctr))/2 + (b - ctr)*($*complex-of-real* $(r\hat{}2) -$ *cnj* $(m - ctr)*(a - ctr))/(2*$*complex-of-real* $(r\hat{}2))$
⟨*proof*⟩

The difference of the feet on BC and AB.

lemma *foot-diff-PR*:
fixes $a\ b\ c\ m\ ctr$ :: *complex* **and** r :: *real*
assumes *ha*: *oncircle ctr r a* **and** *hb*: *oncircle ctr r b* **and** *hc*: *oncircle ctr r c*
and *pos*: $0 < r$ **and** *dab*: $a \neq b$ **and** *dbc*: $b \neq c$
shows *foot b c m* $-$ *foot a b m* $= (c - a)*($*complex-of-real* $(r\hat{}2) -$ *cnj* $(m - ctr)*(b - ctr))/(2*$*complex-of-real* $(r\hat{}2))$
⟨*proof*⟩

The difference of the feet on CA and AB.

lemma *foot-diff-QR*:
fixes $a\ b\ c\ m\ ctr$ :: *complex* **and** r :: *real*
assumes *ha*: *oncircle ctr r a* **and** *hb*: *oncircle ctr r b* **and** *hc*: *oncircle ctr r c*
and *pos*: $0 < r$ **and** *dab*: $a \neq b$ **and** *dca*: $c \neq a$
shows *foot c a m* $-$ *foot a b m* $= (c - b)*($*complex-of-real* $(r\hat{}2) -$ *cnj* $(m - ctr)*(a - ctr))/(2*$*complex-of-real* $(r\hat{}2))$
⟨*proof*⟩

On the circle, *cor* $(r^2) -$ *cnj* $(m - ctr) * (z - ctr)$ factors through $m - z$.

lemma *circle-factor*:
assumes *hm*: *oncircle ctr r m* **and** *pos*: $0 < r$
shows *complex-of-real* $(r\hat{}2) -$ *cnj* $(m - ctr)*(z - ctr) =$ *complex-of-real* $(r\hat{}2)*(m - z)/(m - ctr)$
⟨*proof*⟩

When m too lies on the circle, the two foot differences take a particularly clean form.

lemma *foot-diff-PR-circ*:
  assumes *ha*: *oncircle ctr r a* **and** *hb*: *oncircle ctr r b* **and** *hc*: *oncircle ctr r c*
    and *hm*: *oncircle ctr r m* **and** *pos*: $0 < r$ **and** *dab*: $a \neq b$ **and** *dbc*: $b \neq c$
  shows *foot b c m* − *foot a b m* = $(c - a)*(m - b)/(2*(m - ctr))$
⟨*proof*⟩

lemma *foot-diff-QR-circ*:
  assumes *ha*: *oncircle ctr r a* **and** *hb*: *oncircle ctr r b* **and** *hc*: *oncircle ctr r c*
    and *hm*: *oncircle ctr r m* **and** *pos*: $0 < r$ **and** *dab*: $a \neq b$ **and** *dca*: $c \neq a$
  shows *foot c a m* − *foot a b m* = $(c - b)*(m - a)/(2*(m - ctr))$
⟨*proof*⟩

17 The Wallace-Simson line

Forward direction: if m lies on the circumcircle of the nondegenerate triangle a, b, c, then the three feet of the perpendiculars from m to the side lines are collinear. The key step is that the ratio of two foot differences equals a cross ratio of four concyclic points, hence is real.

theorem *simson-line*:
  fixes *a b c m ctr* :: *complex* **and** *r* :: *real*
  assumes *hncol*: ¬ *collinear a b c*
    and *ha*: *oncircle ctr r a* **and** *hb*: *oncircle ctr r b* **and** *hc*: *oncircle ctr r c*
    and *hm*: *oncircle ctr r m* **and** *pos*: $0 < r$
  shows *collinear* (*foot b c m*) (*foot c a m*) (*foot a b m*)
⟨*proof*⟩

18 The converse

A field identity that clears the three circle denominators from the collinearity relation of the feet. It is stated over an arbitrary field so that the algebraic simplification can be discharged uniformly.

lemma *converse-key-gen*:
  fixes *A B C ca cb cc R D cm* :: *'a::field*
  assumes *A0*: $A \neq 0$ **and** *B0*: $B \neq 0$ **and** *C0*: $C \neq 0$
    and *hA*: $A * ca = R$ **and** *hB*: $B * cb = R$ **and** *hC*: $C * cc = R$
  shows
    $A * B * C *$
      $((cc - ca) * (R - D * cb) * (C - B) * (R - cm * A) -$
      $(C - A) * (R - cm * B) * (cc - cb) * (R - D * ca))$
    $= R\hat{\ }2 * (A - B) * (A - C) * (B - C) * (R - cm * D)$
⟨*proof*⟩

Specialised to the circle, with the conjugate substitutions folded back into differences of the original points.

lemma *converse-key-complex*:
  fixes *a b c m ctr* :: *complex* **and** *r* :: *real*

assumes *Ane*: $a - ctr \neq 0$ **and** *Bne*: $b - ctr \neq 0$ **and** *Cne*: $c - ctr \neq 0$
  and *hA*: $(a - ctr)*cnj\ (a - ctr) = complex\text{-}of\text{-}real\ (r\hat{}2)$
  and *hB*: $(b - ctr)*cnj\ (b - ctr) = complex\text{-}of\text{-}real\ (r\hat{}2)$
  and *hC*: $(c - ctr)*cnj\ (c - ctr) = complex\text{-}of\text{-}real\ (r\hat{}2)$
shows
$$\begin{aligned}
&(a - ctr) * (b - ctr) * (c - ctr) * \\
&\quad (cnj\ (c - a) * \\
&\qquad (complex\text{-}of\text{-}real\ (r\hat{}2) - (m - ctr) * cnj\ (b - ctr)) * \\
&\qquad (c - b) * \\
&\qquad (complex\text{-}of\text{-}real\ (r\hat{}2) - cnj\ (m - ctr) * (a - ctr)) - \\
&\quad (c - a) * \\
&\qquad (complex\text{-}of\text{-}real\ (r\hat{}2) - cnj\ (m - ctr) * (b - ctr)) * \\
&\qquad cnj\ (c - b) * \\
&\qquad (complex\text{-}of\text{-}real\ (r\hat{}2) - (m - ctr) * cnj\ (a - ctr))) \\
&= (complex\text{-}of\text{-}real\ (r\hat{}2))\hat{}2 * (a - b) * (a - c) * (b - c) * \\
&\quad (complex\text{-}of\text{-}real\ (r\hat{}2) - cnj\ (m - ctr) * (m - ctr))
\end{aligned}$$
⟨*proof*⟩

Converse direction: if the three feet of the perpendiculars from m to the side lines of the nondegenerate triangle are collinear, then m lies on the circumcircle.

theorem *simson-line-converse*:
  fixes *a b c m ctr* :: *complex* **and** *r* :: *real*
  assumes *hncol*: ¬ *collinear a b c*
    and *ha*: *oncircle ctr r a* **and** *hb*: *oncircle ctr r b* **and** *hc*: *oncircle ctr r c*
    and *pos*: $0 < r$
    and *hcol*: *collinear* (*foot b c m*) (*foot c a m*) (*foot a b m*)
  shows *oncircle ctr r m*
⟨*proof*⟩

The two directions combine into the usual characterisation of the circumcircle by collinearity of the three perpendicular feet.

theorem *simson-line-iff*:
  fixes *a b c m ctr* :: *complex* **and** *r* :: *real*
  assumes *hncol*: ¬ *collinear a b c*
    and *ha*: *oncircle ctr r a* **and** *hb*: *oncircle ctr r b* **and** *hc*: *oncircle ctr r c*
    and *pos*: $0 < r$
  shows
    oncircle ctr r m ⟷
      collinear (*foot b c m*) (*foot c a m*) (*foot a b m*)
⟨*proof*⟩

end

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