

# Exploring Simplified Variants of Gödel's Ontological Argument in Isabelle/HOL

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## Abstract

Simplified variants of Gödel's ontological argument are explored. Among those is a particularly interesting simplified argument which is (i) valid already in basic modal logics K or KT, (ii) which does not suffer from modal collapse, and (iii) which avoids the rather complex predicates of essence (Ess.) and necessary existence (NE) as used by Gödel.

Whether the presented variants increase or decrease the attractiveness and persuasiveness of the ontological argument is a question I would like to pass on to philosophy and theology.

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## 1 Background Reading

The selected simplified variants of Gödel's ontological argument [5, 7] as presented in §3 have first been extracted from the insights gained in §6 of [4]. These variants are also influenced by the work presented in [1] and they significantly extend the findings from [3]. In §4 we additionally include the sources from [4].

## 2 Higher-Order Modal Logic in HOL (cf. [2] and Fig. 1 in [4]).

```
theory HOML imports Main
begin
nitpick-params[user-axioms,expect=genuine]
```

Type *i* is associated with possible worlds and type *e* with entities:

```
typedecl i — Possible worlds
typedecl e — Individuals
type-synonym  $\sigma = i \Rightarrow bool$  — World-lifted propositions
type-synonym  $\gamma = e \Rightarrow \sigma$  — Lifted predicates
type-synonym  $\mu = \sigma \Rightarrow \sigma$  — Unary modal connectives
type-synonym  $\nu = \sigma \Rightarrow \sigma \Rightarrow \sigma$  — Binary modal connectives
```

Logical connectives (operating on truth-sets):

```
abbreviation c1:: $\sigma$  ( $\perp$ ) where  $\perp \equiv \lambda w. False$ 
abbreviation c2:: $\sigma$  ( $\top$ ) where  $\top \equiv \lambda w. True$ 
abbreviation c3:: $\mu$  ( $\neg$ ) [52]53 where  $\neg\varphi \equiv \lambda w. \neg(\varphi w)$ 
abbreviation c4:: $\nu$  (infix  $\wedge$ ) 50 where  $\varphi \wedge \psi \equiv \lambda w. (\varphi w) \wedge (\psi w)$ 
abbreviation c5:: $\nu$  (infix  $\vee$ ) 49 where  $\varphi \vee \psi \equiv \lambda w. (\varphi w) \vee (\psi w)$ 
abbreviation c6:: $\nu$  (infix  $\rightarrow$ ) 48 where  $\varphi \rightarrow \psi \equiv \lambda w. (\varphi w) \rightarrow (\psi w)$ 
abbreviation c7:: $\nu$  (infix  $\leftrightarrow$ ) 47 where  $\varphi \leftrightarrow \psi \equiv \lambda w. (\varphi w) \leftrightarrow (\psi w)$ 
consts R:: $i \Rightarrow i \Rightarrow bool$  ( $\mathbf{r}$ ) — Accessibility relation
abbreviation c8:: $\mu$  ( $\Box$ ) [54]55 where  $\Box\varphi \equiv \lambda w. \forall v. (w \mathbf{r} v) \rightarrow (\varphi v)$ 
abbreviation c9:: $\mu$  ( $\Diamond$ ) [54]55 where  $\Diamond\varphi \equiv \lambda w. \exists v. (w \mathbf{r} v) \wedge (\varphi v)$ 
abbreviation c10:: $\gamma \Rightarrow \gamma$  ( $\neg$ ) [52]53 where  $\neg\Phi \equiv \lambda x. \lambda w. \neg(\Phi x w)$ 
abbreviation c11:: $\gamma \Rightarrow \gamma$  ( $\rightarrow$ ) where  $\rightarrow\Phi \equiv \lambda x. \lambda w. \neg(\Phi x w)$ 
abbreviation c12:: $e \Rightarrow e \Rightarrow \sigma$  ( $=$ ) where  $x=y \equiv \lambda w. (x=y)$ 
abbreviation c13:: $e \Rightarrow e \Rightarrow \sigma$  ( $\neq$ ) where  $x \neq y \equiv \lambda w. (x \neq y)$ 
```

Polymorphic possibilist quantification:

**abbreviation**  $q1::('a\Rightarrow\sigma)\Rightarrow\sigma$  ( $\langle\forall\rangle$ ) **where**  $\forall\Phi \equiv \lambda w.\forall x.(\Phi x w)$   
**abbreviation**  $q2$  (**binder** $\langle\forall\rangle$ [10]11) **where**  $\forall x. \varphi(x) \equiv \forall\varphi$   
**abbreviation**  $q3::('a\Rightarrow\sigma)\Rightarrow\sigma$  ( $\langle\exists\rangle$ ) **where**  $\exists\Phi \equiv \lambda w.\exists x.(\Phi x w)$   
**abbreviation**  $q4$  (**binder** $\langle\exists\rangle$ [10]11) **where**  $\exists x. \varphi(x) \equiv \exists\varphi$

Actualist quantification for individuals/entities:

**consts**  $existsAt::\gamma$  ( $\langle\text{-}\textcircled{\text{A}}\text{-}\rangle$ )  
**abbreviation**  $q5::\gamma\Rightarrow\sigma$  ( $\langle\forall^E\rangle$ ) **where**  $\forall^E\Phi \equiv \lambda w.\forall x.(x\textcircled{\text{A}}w)\longrightarrow(\Phi x w)$   
**abbreviation**  $q6$  (**binder** $\langle\forall^E\rangle$ [8]9) **where**  $\forall^E x. \varphi(x) \equiv \forall^E\varphi$   
**abbreviation**  $q7::\gamma\Rightarrow\sigma$  ( $\langle\exists^E\rangle$ ) **where**  $\exists^E\Phi \equiv \lambda w.\exists x.(x\textcircled{\text{A}}w)\wedge(\Phi x w)$   
**abbreviation**  $q8$  (**binder** $\langle\exists^E\rangle$ [8]9) **where**  $\exists^E x. \varphi(x) \equiv \exists^E\varphi$

Meta-logical predicate for global validity:

**abbreviation**  $g1::\sigma\Rightarrow bool$  ( $\langle\text{-}\rangle$ ) **where**  $[\psi] \equiv \forall w. \psi w$

Barcan and converse Barcan formula:

**lemma** *True nitpick*[satisfy]  $\langle proof \rangle$   
**lemma**  $[(\forall^E x.\Box(\varphi x)) \rightarrow \Box(\forall^E x.(\varphi x))]$  **nitpick**  $\langle proof \rangle$   
**lemma**  $[\Box(\forall^E x.(\varphi x)) \rightarrow (\forall^E x.\Box(\varphi x))]$  **nitpick**  $\langle proof \rangle$   
**lemma**  $[(\forall x.\Box(\varphi x)) \rightarrow \Box(\forall x. \varphi x)]$   $\langle proof \rangle$   
**lemma**  $[\Box(\forall x.(\varphi x)) \rightarrow (\forall x.\Box(\varphi x))]$   $\langle proof \rangle$   
**end**

### 3 Selected Simplified Ontological Argument

**theory** *SimplifiedOntologicalArgument* **imports**  
*HOML*  
**begin**

Positive properties:

**consts**  $posProp::\gamma\Rightarrow\sigma$  ( $\langle\mathcal{P}\rangle$ )

An entity  $x$  is God-like if it possesses all positive properties.

**definition**  $G$  ( $\langle\mathcal{G}\rangle$ ) **where**  $\mathcal{G}(x) \equiv \forall\Phi.(\mathcal{P}(\Phi) \rightarrow \Phi(x))$

The axiom's of the simplified variant are presented next; these axioms are further motivated in [4, 1]).

Self-difference is not a positive property (possible alternative: the empty property is not a positive property).

**axiomatization where** *CORO1*:  $[\neg(\mathcal{P}(\lambda x.(x\neq x)))]$

A property entailed by a positive property is positive.

**axiomatization where** *CORO2*:  $[\forall\Phi\Psi. \mathcal{P}(\Phi) \wedge (\forall x. \Phi(x) \rightarrow \Psi(x)) \rightarrow \mathcal{P}(\Psi)]$

Being Godlike is a positive property.

**axiomatization where** *AXIOM3*:  $[\mathcal{P}\mathcal{G}]$

### 3.1 Verifying the Selected Simplified Ontological Argument (version 1)

The existence of a non-exemplified positive property implies that self-difference (or, alternatively, the empty property) is a positive property.

**lemma** *LEMMA1*:  $[(\exists \Phi. (\mathcal{P}(\Phi) \wedge \neg(\exists x. \Phi(x)))) \rightarrow \mathcal{P}(\lambda x. (x \neq x))]$   
*<proof>*

A non-exemplified positive property does not exist.

**lemma** *LEMMA2*:  $[\neg(\exists \Phi. (\mathcal{P}(\Phi) \wedge \neg(\exists x. \Phi(x))))]$   
*<proof>*

Positive properties are exemplified.

**lemma** *LEMMA3*:  $[\forall \Phi. (\mathcal{P}(\Phi) \rightarrow (\exists x. \Phi(x)))]$   
*<proof>*

There exists a God-like entity.

**theorem** *THEOREM3'*:  $[\exists x. \mathcal{G}(x)]$   
*<proof>*

Necessarily, there exists a God-like entity.

**theorem** *THEOREM3*:  $[\Box(\exists x. \mathcal{G}(x))]$   
*<proof>*

However, the possible existence of Godlike entity is not implied.

**theorem** *CORO*:  $[\Diamond(\exists x. \mathcal{G}(x))]$   
**nitpick** *<proof>*

### 3.2 Verifying the Selected Simplified Ontological Argument (version 2)

We switch to logic T.

**axiomatization where** *T*:  $[\forall \varphi. \Box \varphi \rightarrow \varphi]$

**lemma** *T'*:  $[\forall \varphi. \varphi \rightarrow \Diamond \varphi]$  *<proof>*

Positive properties are possibly exemplified.

**theorem** *THEOREM1*:  $[\forall \Phi. \mathcal{P}(\Phi) \rightarrow \Diamond(\exists x. \Phi(x))]$   
*<proof>*

Possibly there exists a God-like entity.

**theorem** *CORO*:  $[\Diamond(\exists x. \mathcal{G}(x))]$   
*<proof>*

The possible existence of a God-like entity implies the necessary existence of a God-like entity.

**theorem** *THEOREM2*:  $[\Diamond(\exists x. \mathcal{G}(x)) \rightarrow \Box(\exists x. \mathcal{G}(x))]$   
*<proof>*

Necessarily, there exists a God-like entity.

**theorem** *THEO3*:  $[\Box(\exists x. \mathcal{G}(x))]$   
 $\langle proof \rangle$

There exists a God-like entity.

**theorem** *THEO3'*:  $[\exists x. \mathcal{G}(x)]$   
 $\langle proof \rangle$

Modal collapse is not implied; nitpick reports a countermodel.

**lemma** *MC*:  $[\forall \Phi. \Phi \rightarrow \Box \Phi]$  **nitpick**  $\langle proof \rangle$

Consistency of the theory; nitpick reports a model.

**lemma** *True nitpick[satisfy]*  $\langle proof \rangle$   
**end**

## 4 Presentation of All Variants as Studied in [4]

### 4.1 Preliminaries: Modal Ultrafilter (Fig. 2 in [4])

**theory** *MFilter* **imports** *HOML*  
**begin**

Some abbreviations for auxiliary operations.

**abbreviation** *a*:: $\gamma \Rightarrow (\gamma \Rightarrow \sigma) \Rightarrow \sigma$  ( $\langle \cdot \in \cdot \rangle$ ) **where**  $x \in S \equiv S x$

**abbreviation** *b*:: $\gamma$  ( $\langle \emptyset \rangle$ ) **where**  $\emptyset \equiv \lambda x. \perp$

**abbreviation** *c*:: $\gamma$  ( $\langle \mathbf{U} \rangle$ ) **where**  $\mathbf{U} \equiv \lambda x. \top$

**abbreviation** *d*:: $\gamma \Rightarrow \gamma \Rightarrow \sigma$  ( $\langle \subseteq \rangle$ ) **where**  $\varphi \subseteq \psi \equiv \forall x. ((\varphi x) \rightarrow (\psi x))$

**abbreviation** *e*:: $\gamma \Rightarrow \gamma \Rightarrow \gamma$  ( $\langle \neg \rangle$ ) **where**  $\varphi \neg \psi \equiv \lambda x. ((\varphi x) \wedge (\psi x))$

**abbreviation** *f*:: $\gamma \Rightarrow \gamma$  ( $\langle \neg^1 \rangle$ ) **where**  $\neg^1 \psi \equiv \lambda x. \neg(\psi x)$

Definition of modal filter.

**abbreviation** *g*:: $(\gamma \Rightarrow \sigma) \Rightarrow \sigma$  ( $\langle Filter \rangle$ )  
**where** *Filter*  $\Phi \equiv (((\mathbf{U} \in \Phi) \wedge \neg(\emptyset \in \Phi))$   
 $\wedge (\forall \varphi \psi. (((\varphi \in \Phi) \wedge (\varphi \subseteq \psi)) \rightarrow (\psi \in \Phi))))$   
 $\wedge (\forall \varphi \psi. (((\varphi \in \Phi) \wedge (\psi \in \Phi)) \rightarrow ((\varphi \neg \psi) \in \Phi)))$

Definition of modal ultrafilter .

**abbreviation** *h*:: $(\gamma \Rightarrow \sigma) \Rightarrow \sigma$  ( $\langle UFilter \rangle$ ) **where**  
*UFilter*  $\Phi \equiv (Filter \Phi) \wedge (\forall \varphi. ((\varphi \in \Phi) \vee ((\neg^1 \varphi) \in \Phi)))$

Modal filter and modal ultrafilter are consistent.

**lemma**  $[\forall \Phi \varphi. ((UFilter \Phi) \rightarrow \neg((\varphi \in \Phi) \wedge ((\neg^1 \varphi) \in \Phi)))]$   $\langle proof \rangle$   
**end**

### 4.2 Preliminaries: Further Basic Notions (Fig. 3 in [4])

**theory** *BaseDefs* **imports** *HOML*  
**begin**

Positive properties.

**consts** *posProp*:: $\gamma \Rightarrow \sigma$  ( $\langle \mathcal{P} \rangle$ )

Basic definitions for modal ontological argument.

**abbreviation** *a* ( $\langle \sqsubseteq \rightarrow \rangle$ ) **where**  $X \sqsubseteq Y \equiv \forall^{Ez} . ((X z) \rightarrow (Y z))$

**abbreviation** *b* ( $\langle \Rightarrow \rightarrow \rangle$ ) **where**  $X \Rightarrow Y \equiv \Box(X \sqsubseteq Y)$

**abbreviation** *c* ( $\langle \mathcal{P}_{\text{os}} \rangle$ ) **where**  $\mathcal{P}_{\text{os}} Z \equiv \forall X . ((Z X) \rightarrow (\mathcal{P} X))$

**abbreviation** *d* ( $\langle \sqsupset \rightarrow \rangle$ ) **where**  $X \sqsupset Z \equiv \Box(\forall^{Eu} . ((X u) \leftrightarrow (\forall Y . ((Z Y) \rightarrow (Y u))))$

Definition of Godlike.

**definition** *G* ( $\langle \mathcal{G} \rangle$ ) **where**  $\mathcal{G} x \equiv \forall Y . ((\mathcal{P} Y) \rightarrow (Y x))$

Definitions of Essence and Necessary Existence.

**definition** *E* ( $\langle \mathcal{E} \rangle$ ) **where**  $\mathcal{E} Y x \equiv (Y x) \wedge (\forall Z . ((Z x) \rightarrow (Y \Rightarrow Z)))$

**definition** *NE* ( $\langle \mathcal{NE} \rangle$ ) **where**  $\mathcal{NE} x \equiv \forall Y . ((\mathcal{E} Y x) \rightarrow \Box(\exists^E Y))$

**end**

### 4.3 Ultrafilter Analysis of Scott's Variant (Fig. 3 in [4])

**theory** *ScottVariant* **imports**

*HOML*

*MFilter*

*BaseDefs*

**begin**

Axioms of Scott's variant.

**axiomatization** **where**

*A1*:  $\lfloor \forall X . ((\neg(\mathcal{P} X)) \leftrightarrow (\mathcal{P}(\neg X))) \rfloor$  **and**

*A2*:  $\lfloor \forall X Y . (((\mathcal{P} X) \wedge (X \Rightarrow Y)) \rightarrow (\mathcal{P} Y)) \rfloor$  **and**

*A3*:  $\lfloor \forall Z . ((\mathcal{P}_{\text{os}} Z) \rightarrow (\forall X . ((X \sqsupset Z) \rightarrow (\mathcal{P} X)))) \rfloor$  **and**

*A4*:  $\lfloor \forall X . ((\mathcal{P} X) \rightarrow \Box(\mathcal{P} X)) \rfloor$  **and**

*A5*:  $\lfloor \mathcal{P} \mathcal{NE} \rfloor$  **and**

*B*:  $\lfloor \forall \varphi . (\varphi \rightarrow \Box \Diamond \varphi) \rfloor$  — Logic KB

**lemma** *B'*:  $\forall x y . \neg(x y) \vee (y x)$   $\langle \text{proof} \rangle$

Necessary existence of a Godlike entity.

**theorem** *T6*:  $\lfloor \Box(\exists^E \mathcal{G}) \rfloor$

$\langle \text{proof} \rangle$

Existence of a Godlike entity.

**lemma**  $\lfloor \exists^E \mathcal{G} \rfloor$   $\langle \text{proof} \rangle$

Consistency

**lemma** *True nitpick*[*satisfy*]  $\langle \text{proof} \rangle$

Modal collapse: holds.

**lemma** *MC*:  $\lfloor \forall \Phi . (\Phi \rightarrow \Box \Phi) \rfloor$

*<proof>*

Analysis of positive properties using ultrafilters.

**theorem** *U1*: [*UFilter*  $\mathcal{P}$ ] — Proof found by sledgehammer  
*<proof>*

**lemma** *L1*: [ $\forall X Y.((X \Rightarrow Y) \rightarrow (X \sqsubseteq Y))$ ] *<proof>*

**lemma** *L2*: [ $\forall X Y.(((\mathcal{P} X) \wedge (X \sqsubseteq Y)) \rightarrow (\mathcal{P} Y))$ ] *<proof>*

Set of supersets of  $X$ , we call this  $HF X$ .

**abbreviation** *HF* where  $HF X \equiv \lambda Y.(X \sqsubseteq Y)$

*HF*  $\mathcal{G}$  is a filter; hence, *HF*  $\mathcal{G}$  is Hauptfilter of  $\mathcal{G}$ .

**lemma** *F1*: [*Filter* (*HF*  $\mathcal{G}$ )] *<proof>*

**lemma** *F2*: [*UFilter* (*HF*  $\mathcal{G}$ )] *<proof>*

T6 follows directly from F1.

**theorem** *T6again*: [ $\Box(\exists^E \mathcal{G})$ ] *<proof>*  
**end**

#### 4.4 Ultrafilter Variant (Fig. 5 in [4])

**theory** *UFilterVariant* **imports**

*HOML*

*MFilter*

*BaseDefs*

**begin**

Axiom's of ultrafilter variant.

**axiomatization** where

*U1*: [*UFilter*  $\mathcal{P}$ ] **and**

*A2*: [ $\forall X Y.(((\mathcal{P} X) \wedge (X \Rightarrow Y)) \rightarrow (\mathcal{P} Y))$ ] **and**

*A3*: [ $\forall \mathcal{Z}.((\mathcal{P}_{\text{os}} \mathcal{Z}) \rightarrow (\forall X.((X \sqcap \mathcal{Z}) \rightarrow (\mathcal{P} X))))$ ]

Necessary existence of a Godlike entity.

**theorem** *T6*: [ $\Box(\exists^E \mathcal{G})$ ] — Proof also found by sledgehammer  
*<proof>*

Checking for consistency.

**lemma** *True* **nitpick**[*satisfy*] *<proof>*

Checking for modal collapse.

**lemma** *MC*: [ $\forall \Phi.(\Phi \rightarrow \Box \Phi)$ ] **nitpick** *<proof>*  
**end**

## 4.5 Simplified Variant (Fig. 6 in [4])

**theory** *SimpleVariant* **imports**

*HOML*

*MFilter*

*BaseDefs*

**begin**

Axiom's of new, simplified variant.

**axiomatization where**

*A1'*:  $\lceil \neg(\mathcal{P}(\lambda x.(x \neq x))) \rceil$  **and**

*A2'*:  $\lceil \forall X Y.((\mathcal{P} X) \wedge ((X \sqsubseteq Y) \vee (X \Rightarrow Y))) \rightarrow (\mathcal{P} Y) \rceil$  **and**

*A3*:  $\lceil \forall \mathcal{Z}.((\mathcal{P} \text{os } \mathcal{Z}) \rightarrow (\forall X.((X \sqcap \mathcal{Z}) \rightarrow (\mathcal{P} X)))) \rceil$

**lemma** *T2*:  $\lceil \mathcal{P} \mathcal{G} \rceil$  *<proof>*

**lemma** *L1*:  $\lceil \mathcal{P}(\lambda x.(x = x)) \rceil$  *<proof>*

Necessary existence of a Godlike entity.

**theorem** *T6*:  $\lceil \Box(\exists^E \mathcal{G}) \rceil$  — Proof found by sledgehammer  
*<proof>*

**lemma** *True nitpick*[*satisfy*] *<proof>*

Modal collapse and monotheism: not implied.

**lemma** *MC*:  $\lceil \forall \Phi.(\Phi \rightarrow \Box \Phi) \rceil$  **nitpick** *<proof>*

**lemma** *MT*:  $\lceil \forall x y.(((\mathcal{G} x) \wedge (\mathcal{G} y)) \rightarrow (x = y)) \rceil$

**nitpick** *<proof>*

Gödel's A1, A4, A5: not implied anymore.

**lemma** *A1*:  $\lceil \forall X.((\neg(\mathcal{P} X)) \leftrightarrow (\mathcal{P}(\neg X))) \rceil$  **nitpick** *<proof>*

**lemma** *A4*:  $\lceil \forall X.((\mathcal{P} X) \rightarrow \Box(\mathcal{P} X)) \rceil$  **nitpick** *<proof>*

**lemma** *A5*:  $\lceil \mathcal{P} \mathcal{N}\mathcal{E} \rceil$  **nitpick** *<proof>*

Checking filter and ultrafilter properties.

**theorem** *F1*:  $\lceil \text{Filter } \mathcal{P} \rceil$  *<proof>*

**theorem** *U1*:  $\lceil \text{UFilter } \mathcal{P} \rceil$  **nitpick** *<proof>*

**end**

## 4.6 Simplified Variant with Axiom T2 (Fig. 7 in [4])

**theory** *SimpleVariantPG* **imports**

*HOML*

*MFilter*

*BaseDefs*

**begin**

Axiom's of simplified variant with A3 replaced.

**axiomatization where**

*A1'*:  $\lceil \neg(\mathcal{P}(\lambda x.(x \neq x))) \rceil$  **and**

*A2'*:  $\lceil \forall X Y.((\mathcal{P} X) \wedge ((X \sqsubseteq Y) \vee (X \Rightarrow Y))) \rightarrow (\mathcal{P} Y) \rceil$  **and**

*T2*:  $[\mathcal{P} \mathcal{G}]$

Necessary existence of a Godlike entity.

**theorem** *T6*:  $[\Box(\exists^E \mathcal{G})]$  — Proof found by sledgehammer  
*<proof>*

**lemma** *True nitpick*[*satisfy*] *<proof>*

Modal collapse and Monotheism: not implied.

**lemma** *MC*:  $[\forall \Phi.(\Phi \rightarrow \Box \Phi)]$  **nitpick** *<proof>*

**lemma** *MT*:  $[\forall x y.(((\mathcal{G} x) \wedge (\mathcal{G} y)) \rightarrow (x=y))]$  **nitpick** *<proof>*  
**end**

#### 4.7 Simplified Variant with Simple Entailment in Logic K (Fig. 8 in [4])

**theory** *SimpleVariantSE* **imports**  
*HOML*  
*MFilter*  
*BaseDefs*  
**begin**

Axiom's of new variant based on ultrafilters.

**axiomatization where**

*A1'*:  $[\neg(\mathcal{P}(\lambda x.(x \neq x)))]$  **and**

*A2''*:  $[\forall X Y.(((\mathcal{P} X) \wedge (X \sqsubseteq Y)) \rightarrow (\mathcal{P} Y))]$  **and**

*T2*:  $[\mathcal{P} \mathcal{G}]$

Necessary existence of a Godlike entity.

**theorem** *T6*:  $[\Box(\exists^E \mathcal{G})]$  *<proof>*

**theorem** *T7*:  $[\exists^E \mathcal{G}]$  *<proof>*

Possible existence of a Godlike: has counterodel.

**lemma** *T3*:  $[\Diamond(\exists^E \mathcal{G})]$  **nitpick** *<proof>*

**lemma** *T3'*: **assumes** *T*:  $[\forall \varphi.((\Box \varphi) \rightarrow \varphi)]$

**shows**  $[\Diamond(\exists^E \mathcal{G})]$

*<proof>*

**end**

#### 4.8 Simplified Variant with Simple Entailment in Logic T (Fig. 9 in [4])

**theory** *SimpleVariantSEinT* **imports**  
*HOML*  
*MFilter*  
*BaseDefs*  
**begin**

Axiom's of new variant based on ultrafilters.

**axiomatization where**

$A1'$ :  $\lceil \neg(\mathcal{P}(\lambda x.(x \neq x))) \rceil$  **and**

$A2''$ :  $\lceil \forall X Y.((\mathcal{P} X) \wedge (X \sqsubseteq Y)) \rightarrow (\mathcal{P} Y) \rceil$  **and**

$T2$ :  $\lceil \mathcal{P} \mathcal{G} \rceil$

Modal Logic T.

**axiomatization where**  $T$ :  $\lceil \forall \varphi.((\Box \varphi) \rightarrow \varphi) \rceil$

**lemma**  $T'$ :  $\lceil \forall \varphi.(\varphi \rightarrow (\Diamond \varphi)) \rceil$   $\langle proof \rangle$

Necessary existence of a Godlike entity.

**theorem**  $T6$ :  $\lceil \Box(\exists^E \mathcal{G}) \rceil$  — Proof found by sledgehammer  
 $\langle proof \rangle$

T6 again, with an alternative, simpler proof.

**theorem**  $T6again$ :  $\lceil \Box(\exists^E \mathcal{G}) \rceil$

$\langle proof \rangle$

**end**

#### 4.9 Hauptfiltervariant (Fig. 10 in [4])

**theory** *SimpleVariantHF* **imports**

*HOML*

*MFilter*

*BaseDefs*

**begin**

Definition: Set of supersets of  $X$ , we call this  $\mathcal{HF} X$ .

**abbreviation**  $HF::\gamma \Rightarrow (\gamma \Rightarrow \sigma)$  **where**  $HF X \equiv \lambda Y.(X \sqsubseteq Y)$

Postulate:  $\mathcal{HF} \mathcal{G}$  is a filter; i.e., Hauptfilter of  $\mathcal{G}$ .

**axiomatization where**  $F1$ :  $\lceil Filter (HF \mathcal{G}) \rceil$

Necessary existence of a Godlike entity.

**theorem**  $T6$ :  $\lceil \Box(\exists^E \mathcal{G}) \rceil$   $\langle proof \rangle$

**theorem**  $T6again$ :  $\lceil \Box(\exists^E \mathcal{G}) \rceil$

$\langle proof \rangle$

Possible existence of Godlike entity not implied.

**lemma**  $T3$ :  $\lceil \Diamond(\exists^E \mathcal{G}) \rceil$  **nitpick**  $\langle proof \rangle$

Axiom T enforces possible existence of Godlike entity.

**axiomatization**

**lemma**  $T3$ : **assumes**  $T$ :  $\lceil \forall \varphi.((\Box \varphi) \rightarrow \varphi) \rceil$

**shows**  $\lceil \Diamond(\exists^E \mathcal{G}) \rceil$   $\langle proof \rangle$

**lemma** *True* **nitpick** $[satisfy]$   $\langle proof \rangle$

Modal collapse: not implied anymore.

```

lemma MC:  $[\forall \Phi. (\Phi \rightarrow \Box \Phi)]$  nitpick  $\langle proof \rangle$ 
lemma MT:  $[\forall x y. ((\mathcal{G} x) \wedge (\mathcal{G} y)) \rightarrow (x=y)]$ 
nitpick  $\langle proof \rangle$ 
end

```

#### 4.10 Formal Study of Version No.2 of Gödel's Argument as Reported by Kanckos and Lethen, 2019 [6] (Fig. 11 in [4])

```

theory KanckosLethenNo2Possibilist imports
  HOML
  MFilter
  BaseDefs
begin

```

Axioms of Version No. 2 [6].

```

abbreviation delta ( $\langle \Delta \rangle$ ) where  $\Delta A \equiv \lambda x. (\forall \psi. ((A \psi) \rightarrow (\psi x)))$ 
abbreviation N ( $\langle \mathcal{N} \rangle$ ) where  $\mathcal{N} \varphi \equiv \lambda x. (\Box(\varphi x))$ 

```

**axiomatization where**

*Axiom1*:  $[\forall \varphi \psi. (((\mathcal{P} \varphi) \wedge (\Box(\forall x. ((\varphi x) \rightarrow (\psi x)))))) \rightarrow (\mathcal{P} \psi)]$  **and** — The  $\Box$  can be omitted here; the proofs still work.

*Axiom2*:  $[\forall A. (\Box((\forall \varphi. ((A \varphi) \rightarrow (\mathcal{P} \varphi))) \rightarrow (\mathcal{P} (\Delta A))))]$  **and** — The  $\Box$  can be omitted here; the proofs still work.

*Axiom3*:  $[\forall \varphi. ((\mathcal{P} \varphi) \rightarrow (\mathcal{P} (\mathcal{N} \varphi)))]$  **and**

*Axiom4*:  $[\forall \varphi. ((\mathcal{P} \varphi) \rightarrow (\neg(\mathcal{P}(\neg\varphi))))]$  **and**

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*axB*:  $[\forall \varphi. (\varphi \rightarrow \Box \Diamond \varphi)]$  **and** *axM*:  $[\forall \varphi. ((\Box \varphi) \rightarrow \varphi)]$  **and** *ax4*:  $[\forall \varphi. ((\Box \varphi) \rightarrow (\Box \Box \varphi))]$

Sahlqvist correspondences: they are better suited for proof automation.

```

lemma axB':  $\forall x y. \neg(xry) \vee (yrx)$   $\langle proof \rangle$ 

```

```

lemma axM':  $\forall x. (xx)$   $\langle proof \rangle$ 

```

```

lemma ax4':  $\forall x y z. (((xy) \wedge (yz)) \rightarrow (xz))$   $\langle proof \rangle$ 

```

Proofs for all theorems for No.2 from [6].

```

theorem Theorem0:  $[\forall \varphi \psi. ((\forall Q. ((Q \varphi) \rightarrow (Q \psi))) \rightarrow ((\mathcal{P} \varphi) \rightarrow (\mathcal{P} \psi)))]$ 
 $\langle proof \rangle$ 

```

```

theorem Theorem1:  $[\mathcal{P} \mathcal{G}]$   $\langle proof \rangle$ 

```

```

theorem Theorem2:  $[\forall x. ((\mathcal{G} x) \rightarrow (\exists y. \mathcal{G} y))]$   $\langle proof \rangle$ 

```

```

theorem Theorem3a:  $[\mathcal{P} (\lambda x. (\exists y. \mathcal{G} y))]$   $\langle proof \rangle$ 

```

```

theorem Theorem3b:  $[\Box(\mathcal{P} (\lambda x. (\Box(\exists y. \mathcal{G} y))))]$   $\langle proof \rangle$ 

```

```

theorem Theorem4:  $[\forall x. \Box((\mathcal{G} x) \rightarrow ((\mathcal{P} (\lambda x. (\Box(\exists y. \mathcal{G} y)))) \rightarrow (\Box(\exists y. \mathcal{G} y))))]$ 
 $\langle proof \rangle$ 

```

```

theorem Theorem5:  $[\forall x. \Box((\mathcal{G} x) \rightarrow (\Box(\exists y. \mathcal{G} y)))]$   $\langle proof \rangle$ 

```

```

theorem Theorem6:  $[\Box((\exists y. \mathcal{G} y) \rightarrow (\Box(\exists y. \mathcal{G} y)))]$   $\langle proof \rangle$ 

```

```

theorem Theorem7:  $[\Box((\Diamond(\exists y. \mathcal{G} y)) \rightarrow (\Box(\exists y. \mathcal{G} y)))]$   $\langle proof \rangle$ 

```

```

theorem Theorem8:  $[\Box(\exists y. \mathcal{G} y)]$   $\langle proof \rangle$ 

```

```

theorem Theorem9:  $[\forall \varphi. ((\mathcal{P} \varphi) \rightarrow \Diamond(\exists x. \varphi x))]$   $\langle proof \rangle$ 

```

Short proof of Theorem8; analogous to the one presented in Sec. 7 of Benzmüller 2020.

**theorem**  $[\Box(\exists y. \mathcal{G} y)]$  — Note: this version of the proof uses only  $axB'$  and  $axM'$ .  
 $\langle proof \rangle$

**theorem**  $T5: [(\Diamond(\exists y. \mathcal{G} y)) \rightarrow \Box(\exists y. \mathcal{G} y)]$  — Obvious: If we can prove Theorem8, then we also have T5.  
 $\langle proof \rangle$

Another short proof of Theorem8.

**theorem**  $[\Box(\exists y. \mathcal{G} y)]$  — Note: fewer assumptions used in some cases than in [6].  
 $\langle proof \rangle$

Are the axioms of the simplified versions implied?

Actualist version of the axioms.

**lemma**  $A1'$ :  $[\neg(\mathcal{P}(\lambda x.(x \neq x)))]$   $\langle proof \rangle$

**lemma**  $A2'$ :  $[\forall X Y.((\mathcal{P} X) \wedge ((X \sqsubseteq Y) \vee (X \ni Y))) \rightarrow (\mathcal{P} Y)]$  **nitpick**  $\langle proof \rangle$

**lemma**  $A3$ :  $[\forall \mathcal{Z}.((\mathcal{P} \text{os } \mathcal{Z}) \rightarrow (\forall X.((X \sqcap \mathcal{Z}) \rightarrow (\mathcal{P} X))))]$  **nitpick**  $\langle proof \rangle$

Possibilist version of the axioms.

**abbreviation**  $a$  ( $\langle \neg \sqsubseteq^p \rangle$ ) **where**  $X \sqsubseteq^p Y \equiv \forall z.((X z) \rightarrow (Y z))$

**abbreviation**  $b$  ( $\langle \ni^p \rangle$ ) **where**  $X \ni^p Y \equiv \Box(X \sqsubseteq^p Y)$

**abbreviation**  $d$  ( $\langle \sqcap^p \rangle$ ) **where**  $X \sqcap^p \mathcal{Z} \equiv \Box(\forall u.((X u) \leftrightarrow (\forall Y.((\mathcal{Z} Y) \rightarrow (Y u)))))$

**lemma**  $A1'P$ :  $[\neg(\mathcal{P}(\lambda x.(x \neq x)))]$   $\langle proof \rangle$

**lemma**  $A2'P$ :  $[\forall X Y.((\mathcal{P} X) \wedge ((X \sqsubseteq^p Y) \vee (X \ni^p Y))) \rightarrow (\mathcal{P} Y)]$   $\langle proof \rangle$

**lemma**  $A2'aP$ :  $[\forall X Y.((\mathcal{P} X) \wedge (X \ni^p Y)) \rightarrow (\mathcal{P} Y)]$   $\langle proof \rangle$

**lemma**  $A2'bP$ :  $[\forall X Y.((\mathcal{P} X) \wedge (X \sqsubseteq^p Y)) \rightarrow (\mathcal{P} Y)]$   $\langle proof \rangle$

**lemma**  $A3P$ :  $[\forall \mathcal{Z}.((\mathcal{P} \text{os } \mathcal{Z}) \rightarrow (\forall X.((X \sqcap^p \mathcal{Z}) \rightarrow (\mathcal{P} X))))]$   
 $\langle proof \rangle$

Are Axiom2 and A3 equivalent? Only when assuming Axiom1 and axiom M.

**lemma**  $[\forall A.(\Box((\forall \varphi.((A \varphi) \rightarrow (\mathcal{P} \varphi))) \rightarrow (\mathcal{P} (\Delta A))))] \equiv [\forall \mathcal{Z}.((\mathcal{P} \text{os } \mathcal{Z}) \rightarrow (\forall X.((X \sqcap^p \mathcal{Z}) \rightarrow (\mathcal{P} X))))]$   
 $\langle proof \rangle$

**end**

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