

# Exploring Simplified Variants of Gödel's Ontological Argument in Isabelle/HOL

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## Abstract

Simplified variants of Gödel's ontological argument are explored. Among those is a particularly interesting simplified argument which is (i) valid already in basic modal logics K or KT, (ii) which does not suffer from modal collapse, and (iii) which avoids the rather complex predicates of essence (Ess.) and necessary existence (NE) as used by Gödel.

Whether the presented variants increase or decrease the attractiveness and persuasiveness of the ontological argument is a question I would like to pass on to philosophy and theology.

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## 1 Background Reading

The selected simplified variants of Gödel's ontological argument [5, 7] as presented in §3 have first been extracted from the insights gained in §6 of [4]. These variants are also influenced by the work presented in [1] and they significantly extend the findings from [3]. In §4 we additionally include the sources from [4].

## 2 Higher-Order Modal Logic in HOL (cf. [2] and Fig. 1 in [4]).

```
theory HOML imports Main
begin
nitpick-params[user-axioms,expect=genuine]
```

Type *i* is associated with possible worlds and type *e* with entities:

```
typedecl i — Possible worlds
typedecl e — Individuals
type-synonym  $\sigma = i \Rightarrow bool$  — World-lifted propositions
type-synonym  $\gamma = e \Rightarrow \sigma$  — Lifted predicates
type-synonym  $\mu = \sigma \Rightarrow \sigma$  — Unary modal connectives
type-synonym  $\nu = \sigma \Rightarrow \sigma \Rightarrow \sigma$  — Binary modal connectives
```

Logical connectives (operating on truth-sets):

```
abbreviation c1:: $\sigma$  ( $\perp$ ) where  $\perp \equiv \lambda w. False$ 
abbreviation c2:: $\sigma$  ( $\top$ ) where  $\top \equiv \lambda w. True$ 
abbreviation c3:: $\mu$  ( $\neg$ ) where  $\neg\varphi \equiv \lambda w. \neg(\varphi w)$ 
abbreviation c4:: $\nu$  ( $\wedge$ ) where  $\varphi \wedge \psi \equiv \lambda w. (\varphi w) \wedge (\psi w)$ 
abbreviation c5:: $\nu$  ( $\vee$ ) where  $\varphi \vee \psi \equiv \lambda w. (\varphi w) \vee (\psi w)$ 
abbreviation c6:: $\nu$  ( $\rightarrow$ ) where  $\varphi \rightarrow \psi \equiv \lambda w. (\varphi w) \rightarrow (\psi w)$ 
abbreviation c7:: $\nu$  ( $\leftrightarrow$ ) where  $\varphi \leftrightarrow \psi \equiv \lambda w. (\varphi w) \leftrightarrow (\psi w)$ 
consts R:: $i \Rightarrow i \Rightarrow bool$  ( $r$ ) — Accessibility relation
abbreviation c8:: $\mu$  ( $\Box$ ) where  $\Box\varphi \equiv \lambda w. \forall v. (wrv) \rightarrow (\varphi v)$ 
abbreviation c9:: $\mu$  ( $\Diamond$ ) where  $\Diamond\varphi \equiv \lambda w. \exists v. (wrv) \wedge (\varphi v)$ 
abbreviation c10:: $\gamma \Rightarrow \gamma$  ( $\neg\Phi$ ) where  $\neg\Phi \equiv \lambda x. \lambda w. \neg(\Phi x w)$ 
abbreviation c11:: $\gamma \Rightarrow \gamma$  ( $\rightarrow\Phi$ ) where  $\rightarrow\Phi \equiv \lambda x. \lambda w. \neg(\Phi x w)$ 
abbreviation c12:: $e \Rightarrow e \Rightarrow \sigma$  ( $=$ ) where  $x=y \equiv \lambda w. (x=y)$ 
abbreviation c13:: $e \Rightarrow e \Rightarrow \sigma$  ( $\neq$ ) where  $x \neq y \equiv \lambda w. (x \neq y)$ 
```

Polymorphic possibilist quantification:

**abbreviation**  $q1::('a \Rightarrow \sigma) \Rightarrow \sigma$  ( $\langle \forall \rangle$ ) **where**  $\forall \Phi \equiv \lambda w. \forall x. (\Phi x w)$   
**abbreviation**  $q2$  (**binder** $\langle \forall \rangle$  $[10]11$ ) **where**  $\forall x. \varphi(x) \equiv \forall \varphi$   
**abbreviation**  $q3::('a \Rightarrow \sigma) \Rightarrow \sigma$  ( $\langle \exists \rangle$ ) **where**  $\exists \Phi \equiv \lambda w. \exists x. (\Phi x w)$   
**abbreviation**  $q4$  (**binder** $\langle \exists \rangle$  $[10]11$ ) **where**  $\exists x. \varphi(x) \equiv \exists \varphi$

Actualist quantification for individuals/entities:

**consts**  $existsAt::\gamma$  ( $\langle \text{-@-} \rangle$ )  
**abbreviation**  $q5::\gamma \Rightarrow \sigma$  ( $\langle \forall^E \rangle$ ) **where**  $\forall^E \Phi \equiv \lambda w. \forall x. (x @ w) \longrightarrow (\Phi x w)$   
**abbreviation**  $q6$  (**binder** $\langle \forall^E \rangle$  $[8]9$ ) **where**  $\forall^E x. \varphi(x) \equiv \forall^E \varphi$   
**abbreviation**  $q7::\gamma \Rightarrow \sigma$  ( $\langle \exists^E \rangle$ ) **where**  $\exists^E \Phi \equiv \lambda w. \exists x. (x @ w) \wedge (\Phi x w)$   
**abbreviation**  $q8$  (**binder** $\langle \exists^E \rangle$  $[8]9$ ) **where**  $\exists^E x. \varphi(x) \equiv \exists^E \varphi$

Meta-logical predicate for global validity:

**abbreviation**  $g1::\sigma \Rightarrow bool$  ( $\langle \text{-} \rangle$ ) **where**  $[\psi] \equiv \forall w. \psi w$

Barcan and converse Barcan formula:

**lemma** *True nitpick* $[satisfy]$  **oops** — Model found by Nitpick  
**lemma**  $[(\forall^E x. \Box(\varphi x)) \rightarrow \Box(\forall^E x. (\varphi x))]$  **nitpick oops** — Ctm  
**lemma**  $[\Box(\forall^E x. (\varphi x)) \rightarrow (\forall^E x. \Box(\varphi x))]$  **nitpick oops** — Ctm  
**lemma**  $[(\forall x. \Box(\varphi x)) \rightarrow \Box(\forall x. \varphi x)]$  **by simp**  
**lemma**  $[\Box(\forall x. (\varphi x)) \rightarrow (\forall x. \Box(\varphi x))]$  **by simp**  
**end**

### 3 Selected Simplified Ontological Argument

**theory** *SimplifiedOntologicalArgument* **imports**

*HOML*

**begin**

Positive properties:

**consts**  $posProp::\gamma \Rightarrow \sigma$  ( $\langle \mathcal{P} \rangle$ )

An entity  $x$  is God-like if it possesses all positive properties.

**definition**  $G$  ( $\langle \mathcal{G} \rangle$ ) **where**  $\mathcal{G}(x) \equiv \forall \Phi. (\mathcal{P}(\Phi) \rightarrow \Phi(x))$

The axiom's of the simplified variant are presented next; these axioms are further motivated in [4, 1]).

Self-difference is not a positive property (possible alternative: the empty property is not a positive property).

**axiomatization where** *CORO1*:  $[\neg(\mathcal{P}(\lambda x. (x \neq x)))]$

A property entailed by a positive property is positive.

**axiomatization where** *CORO2*:  $[\forall \Phi \Psi. \mathcal{P}(\Phi) \wedge (\forall x. \Phi(x) \rightarrow \Psi(x)) \rightarrow \mathcal{P}(\Psi)]$

Being Godlike is a positive property.

**axiomatization where** *AXIOM3*:  $[\mathcal{P} \mathcal{G}]$

### 3.1 Verifying the Selected Simplified Ontological Argument (version 1)

The existence of a non-exemplified positive property implies that self-difference (or, alternatively, the empty property) is a positive property.

**lemma** *LEMMA1*:  $[(\exists \Phi. (\mathcal{P}(\Phi) \wedge \neg(\exists x. \Phi(x)))) \rightarrow \mathcal{P}(\lambda x. (x \neq x))]$   
**using** *CORO2* **by** *meson*

A non-exemplified positive property does not exist.

**lemma** *LEMMA2*:  $[\neg(\exists \Phi. (\mathcal{P}(\Phi) \wedge \neg(\exists x. \Phi(x))))]$   
**using** *CORO1* *LEMMA1* **by** *blast*

Positive properties are exemplified.

**lemma** *LEMMA3*:  $[\forall \Phi. (\mathcal{P}(\Phi) \rightarrow (\exists x. \Phi(x)))]$   
**using** *LEMMA2* **by** *blast*

There exists a God-like entity.

**theorem** *THEOREM3'*:  $[\exists x. \mathcal{G}(x)]$   
**using** *AXIOM3* *LEMMA3* **by** *auto*

Necessarily, there exists a God-like entity.

**theorem** *THEOREM3*:  $[\Box(\exists x. \mathcal{G}(x))]$   
**using** *THEOREM3'* **by** *simp*

However, the possible existence of Godlike entity is not implied.

**theorem** *CORO*:  $[\Diamond(\exists x. \mathcal{G}(x))]$   
**nitpick** **oops**

### 3.2 Verifying the Selected Simplified Ontological Argument (version 2)

We switch to logic T.

**axiomatization** **where** *T*:  $[\forall \varphi. \Box \varphi \rightarrow \varphi]$   
**lemma** *T'*:  $[\forall \varphi. \varphi \rightarrow \Diamond \varphi]$  **using** *T* **by** *metis*

Positive properties are possibly exemplified.

**theorem** *THEOREM1*:  $[\forall \Phi. \mathcal{P}(\Phi) \rightarrow \Diamond(\exists x. \Phi(x))]$   
**using** *CORO1* *CORO2* *T'* **by** *metis*

Possibly there exists a God-like entity.

**theorem** *CORO*:  $[\Diamond(\exists x. \mathcal{G}(x))]$   
**using** *AXIOM3* *THEOREM1* **by** *auto*

The possible existence of a God-like entity implies the necessary existence of a God-like entity.

**theorem** *THEOREM2*:  $[\Diamond(\exists x. \mathcal{G}(x)) \rightarrow \Box(\exists x. \mathcal{G}(x))]$   
**using** *AXIOM3* *CORO1* *CORO2* **by** *metis*

Necessarily, there exists a God-like entity.

**theorem** *THEO3*:  $[\Box(\exists x. \mathcal{G}(x))]$   
**using** *CORO THEOREM2* **by** *blast*

There exists a God-like entity.

**theorem** *THEO3'*:  $[\exists x. \mathcal{G}(x)]$   
**using** *T THEO3* **by** *metis*

Modal collapse is not implied; nitpick reports a countermodel.

**lemma** *MC*:  $[\forall \Phi. \Phi \rightarrow \Box \Phi]$  **nitpick oops**

Consistency of the theory; nitpick reports a model.

**lemma** *True nitpick[satisfy]* **oops**  
**end**

## 4 Presentation of All Variants as Studied in [4]

### 4.1 Preliminaries: Modal Ultrafilter (Fig. 2 in [4])

**theory** *MFilter* **imports** *HOML*  
**begin**

Some abbreviations for auxiliary operations.

**abbreviation** *a*:: $\gamma \Rightarrow (\gamma \Rightarrow \sigma) \Rightarrow \sigma$  (*ι-ε-ι*) **where**  $x \in S \equiv S x$

**abbreviation** *b*:: $\gamma$  (*ι∅*) **where**  $\emptyset \equiv \lambda x. \perp$

**abbreviation** *c*:: $\gamma$  (*ιU*) **where**  $U \equiv \lambda x. \top$

**abbreviation** *d*:: $\gamma \Rightarrow \gamma \Rightarrow \sigma$  (*ι-⊆-ι*) **where**  $\varphi \subseteq \psi \equiv \forall x. ((\varphi x) \rightarrow (\psi x))$

**abbreviation** *e*:: $\gamma \Rightarrow \gamma \Rightarrow \gamma$  (*ι-∩-ι*) **where**  $\varphi \cap \psi \equiv \lambda x. ((\varphi x) \wedge (\psi x))$

**abbreviation** *f*:: $\gamma \Rightarrow \gamma$  (*ι<sup>-1</sup>-ι*) **where**  ${}^{-1}\psi \equiv \lambda x. \neg(\psi x)$

Definition of modal filter.

**abbreviation** *g*:: $(\gamma \Rightarrow \sigma) \Rightarrow \sigma$  (*ιFilter*)  
**where** *Filter*  $\Phi \equiv (((U \in \Phi) \wedge \neg(\emptyset \in \Phi))$   
 $\wedge (\forall \varphi \psi. (((\varphi \in \Phi) \wedge (\varphi \subseteq \psi)) \rightarrow (\psi \in \Phi))))$   
 $\wedge (\forall \varphi \psi. (((\varphi \in \Phi) \wedge (\psi \in \Phi)) \rightarrow ((\varphi \cap \psi) \in \Phi)))$

Definition of modal ultrafilter .

**abbreviation** *h*:: $(\gamma \Rightarrow \sigma) \Rightarrow \sigma$  (*ιUFilter*) **where**  
*UFilter*  $\Phi \equiv (Filter \Phi) \wedge (\forall \varphi. ((\varphi \in \Phi) \vee (({}^{-1}\varphi) \in \Phi)))$

Modal filter and modal ultrafilter are consistent.

**lemma**  $[\forall \Phi \varphi. ((UFilter \Phi) \rightarrow \neg((\varphi \in \Phi) \wedge (({}^{-1}\varphi) \in \Phi)))]$  **by force**  
**end**

### 4.2 Preliminaries: Further Basic Notions (Fig. 3 in [4])

**theory** *BaseDefs* **imports** *HOML*  
**begin**

Positive properties.

**consts** *posProp*:: $\gamma \Rightarrow \sigma$  ( $\langle \mathcal{P} \rangle$ )

Basic definitions for modal ontological argument.

**abbreviation** *a* ( $\langle \sqsubseteq \rightarrow \rangle$ ) **where**  $X \sqsubseteq Y \equiv \forall^E z. ((X z) \rightarrow (Y z))$

**abbreviation** *b* ( $\langle \Rightarrow \rightarrow \rangle$ ) **where**  $X \Rightarrow Y \equiv \Box(X \sqsubseteq Y)$

**abbreviation** *c* ( $\langle \mathcal{P} \text{os} \rangle$ ) **where**  $\mathcal{P} \text{os } Z \equiv \forall X. ((Z X) \rightarrow (\mathcal{P} X))$

**abbreviation** *d* ( $\langle \sqcap \rightarrow \rangle$ ) **where**  $X \sqcap Z \equiv \Box(\forall^E u. ((X u) \leftrightarrow (\forall Y. ((Z Y) \rightarrow (Y u))))))$

Definition of Godlike.

**definition** *G* ( $\langle \mathcal{G} \rangle$ ) **where**  $\mathcal{G} x \equiv \forall Y. ((\mathcal{P} Y) \rightarrow (Y x))$

Definitions of Essence and Necessary Existence.

**definition** *E* ( $\langle \mathcal{E} \rangle$ ) **where**  $\mathcal{E} Y x \equiv (Y x) \wedge (\forall Z. ((Z x) \rightarrow (Y \Rightarrow Z)))$

**definition** *NE* ( $\langle \mathcal{N}\mathcal{E} \rangle$ ) **where**  $\mathcal{N}\mathcal{E} x \equiv \forall Y. ((\mathcal{E} Y x) \rightarrow \Box(\exists^E Y))$

**end**

### 4.3 Ultrafilter Analysis of Scott's Variant (Fig. 3 in [4])

**theory** *ScottVariant* **imports**

*HOML*

*MFilter*

*BaseDefs*

**begin**

Axioms of Scott's variant.

**axiomatization** **where**

*A1*:  $\lfloor \forall X. ((\neg(\mathcal{P} X)) \leftrightarrow (\mathcal{P}(\neg X))) \rfloor$  **and**

*A2*:  $\lfloor \forall X Y. (((\mathcal{P} X) \wedge (X \Rightarrow Y)) \rightarrow (\mathcal{P} Y)) \rfloor$  **and**

*A3*:  $\lfloor \forall Z. ((\mathcal{P} \text{os } Z) \rightarrow (\forall X. ((X \sqcap Z) \rightarrow (\mathcal{P} X)))) \rfloor$  **and**

*A4*:  $\lfloor \forall X. ((\mathcal{P} X) \rightarrow \Box(\mathcal{P} X)) \rfloor$  **and**

*A5*:  $\lfloor \mathcal{P} \mathcal{N}\mathcal{E} \rfloor$  **and**

*B*:  $\lfloor \forall \varphi. (\varphi \rightarrow \Box \Diamond \varphi) \rfloor$  — Logic KB

**lemma** *B'*:  $\forall x y. \neg(x y) \vee (y x)$  **using** *B* **by** *fastforce*

Necessary existence of a Godlike entity.

**theorem** *T6*:  $\lfloor \Box(\exists^E \mathcal{G}) \rfloor$

**proof** —

**have** *T1*:  $\lfloor \forall X. ((\mathcal{P} X) \rightarrow \Diamond(\exists^E X)) \rfloor$  **using** *A1 A2* **by** *blast*

**have** *T2*:  $\lfloor \mathcal{P} \mathcal{G} \rfloor$  **by** (*metis A3 G-def*)

**have** *T3*:  $\lfloor \Diamond(\exists^E \mathcal{G}) \rfloor$  **using** *T1 T2* **by** *simp*

**have** *T4*:  $\lfloor \forall^E x. ((\mathcal{G} x) \rightarrow (\mathcal{E} \mathcal{G} x)) \rfloor$  **unfolding** *G-def E-def* **using** *A1 A4* **by** *metis*

**have** *T5*:  $\lfloor (\Diamond(\exists^E \mathcal{G})) \rightarrow \Box(\exists^E \mathcal{G}) \rfloor$  **by** (*smt A5 G-def B' NE-def T4*)

**thus** *?thesis* **using** *T3* **by** *blast qed*

Existence of a Godlike entity.

**lemma**  $[\exists^E \mathcal{G}]$  **using**  $A1 A2 B' T6$  **by** *blast*

Consistency

**lemma** *True nitpick[satisfy] oops* — Model found.

Modal collapse: holds.

**lemma** *MC*:  $[\forall \Phi.(\Phi \rightarrow \Box \Phi)]$

**proof** — {

**fix**  $w$  **fix**  $Q$

**have**  $1$ :  $\forall x.((\mathcal{G} x w) \longrightarrow (\forall Z.((Z x) \rightarrow \Box(\forall^E z.((\mathcal{G} z) \rightarrow (Z z)))))) w$

**by** (*metis A1 A4 G-def*)

**have**  $2$ :  $(\exists x. \mathcal{G} x w) \longrightarrow ((Q \rightarrow \Box(\forall^E z.((\mathcal{G} z) \rightarrow Q))) w)$

**using**  $1$  **by** *force*

**have**  $3$ :  $(Q \rightarrow \Box Q) w$  **using**  $B' T6 2$  **by** *blast*}

**thus** *?thesis* **by** *auto qed*

Analysis of positive properties using ultrafilters.

**theorem** *U1*:  $[UFilter \mathcal{P}]$  — Proof found by sledgehammer

**proof** —

**have**  $1$ :  $[(U \in \mathcal{P}) \wedge \neg(\emptyset \in \mathcal{P})]$  **using**  $A1 A2$  **by** *blast*

**have**  $2$ :  $[\forall \varphi \psi.(((\varphi \in \mathcal{P}) \wedge (\varphi \subseteq \psi)) \rightarrow (\psi \in \mathcal{P}))]$  **by** (*smt A2 B' MC*)

**have**  $3$ :  $[\forall \varphi \psi.(((\varphi \in \mathcal{P}) \wedge (\psi \in \mathcal{P})) \rightarrow ((\varphi \sqcap \psi) \in \mathcal{P}))]$  **by** (*metis A1 A2 G-def B' T6*)

**have**  $4$ :  $[\forall \varphi.((\varphi \in \mathcal{P}) \vee ((^{-1}\varphi) \in \mathcal{P}))]$  **using**  $A1$  **by** *blast*

**thus** *?thesis* **using**  $1 2 3 4$  **by** *simp qed*

**lemma** *L1*:  $[\forall X Y.((X \Rightarrow Y) \rightarrow (X \sqsubseteq Y))]$  **by** (*metis A1 A2 MC*)

**lemma** *L2*:  $[\forall X Y.(((\mathcal{P} X) \wedge (X \sqsubseteq Y)) \rightarrow (\mathcal{P} Y))]$  **by** (*smt A2 B' MC*)

Set of supersets of X, we call this HF X.

**abbreviation** *HF* **where**  $HF X \equiv \lambda Y.(X \sqsubseteq Y)$

*HF*  $\mathcal{G}$  is a filter; hence, *HF*  $\mathcal{G}$  is Hauptfilter of  $\mathcal{G}$ .

**lemma** *F1*:  $[Filter (HF \mathcal{G})]$  **by** (*metis A2 B' T6 U1*)

**lemma** *F2*:  $[UFilter (HF \mathcal{G})]$  **by** (*smt A1 F1 G-def*)

T6 follows directly from F1.

**theorem** *T6again*:  $[\Box(\exists^E \mathcal{G})]$  **using** *F1* **by** *simp*

**end**

#### 4.4 Ultrafilter Variant (Fig. 5 in [4])

**theory** *UFilterVariant* **imports**

*HOML*

*MFilter*

*BaseDefs*

**begin**

Axiom's of ultrafilter variant.

**axiomatization where**

*U1*: [*UFilter*  $\mathcal{P}$ ] **and**  
*A2*: [ $\forall X Y.((\mathcal{P} X) \wedge (X \Rightarrow Y)) \rightarrow (\mathcal{P} Y)$ ] **and**  
*A3*: [ $\forall \mathcal{Z}.((\mathcal{P} \text{os } \mathcal{Z}) \rightarrow (\forall X.(X \sqcap \mathcal{Z}) \rightarrow (\mathcal{P} X)))$ ]

Necessary existence of a Godlike entity.

**theorem T6**: [ $\Box(\exists^E \mathcal{G})$ ] — Proof also found by sledgehammer

**proof** –

**have** *T1*: [ $\forall X.((\mathcal{P} X) \rightarrow \Diamond(\exists^E X))$ ] **by** (*metis A2 U1*)  
**have** *T2*: [ $\mathcal{P} \mathcal{G}$ ] **by** (*metis A3 G-def*)  
**have** *T3*: [ $\Diamond(\exists^E \mathcal{G})$ ] **using** *T1 T2* **by** *simp*  
**have** *T5*: [ $(\Diamond(\exists^E \mathcal{G})) \rightarrow \Box(\exists^E \mathcal{G})$ ] **by** (*metis A2 G-def T2 U1*)  
**thus** *?thesis* **using** *T3* **by** *blast qed*

Checking for consistency.

**lemma True nitpick**[*satisfy*] **oops** — Model found

Checking for modal collapse.

**lemma MC**: [ $\forall \Phi.(\Phi \rightarrow \Box\Phi)$ ] **nitpick oops** — Countermodel  
**end**

## 4.5 Simplified Variant (Fig. 6 in [4])

**theory SimpleVariant imports**

*HOML*  
*MFilter*  
*BaseDefs*

**begin**

Axiom's of new, simplified variant.

**axiomatization where**

*A1'*: [ $\neg(\mathcal{P}(\lambda x.(x \neq x)))$ ] **and**  
*A2'*: [ $\forall X Y.((\mathcal{P} X) \wedge ((X \sqsubseteq Y) \vee (X \Rightarrow Y))) \rightarrow (\mathcal{P} Y)$ ] **and**  
*A3*: [ $\forall \mathcal{Z}.((\mathcal{P} \text{os } \mathcal{Z}) \rightarrow (\forall X.(X \sqcap \mathcal{Z}) \rightarrow (\mathcal{P} X)))$ ]

**lemma T2**: [ $\mathcal{P} \mathcal{G}$ ] **by** (*metis A3 G-def*) — From A3

**lemma L1**: [ $\mathcal{P}(\lambda x.(x=x))$ ] **by** (*metis A2' A3*)

Necessary existence of a Godlike entity.

**theorem T6**: [ $\Box(\exists^E \mathcal{G})$ ] — Proof found by sledgehammer

**proof** –

**have** *T1*: [ $\forall X.((\mathcal{P} X) \rightarrow \Diamond(\exists^E X))$ ] **by** (*metis A1' A2'*)  
**have** *T3*: [ $\Diamond(\exists^E \mathcal{G})$ ] **using** *T1 T2* **by** *simp*  
**have** *T5*: [ $(\Diamond(\exists^E \mathcal{G})) \rightarrow \Box(\exists^E \mathcal{G})$ ] **by** (*metis A1' A2' T2*)  
**thus** *?thesis* **using** *T3* **by** *blast qed*

**lemma** *True nitpick[satisfy] oops* — Consistency: model found

Modal collapse and monotheism: not implied.

**lemma** *MC*:  $[\forall \Phi.(\Phi \rightarrow \Box \Phi)]$  **nitpick oops** — Countermodel

**lemma** *MT*:  $[\forall x y.(((\mathcal{G} x) \wedge (\mathcal{G} y)) \rightarrow (x=y))]$

**nitpick oops** — Countermodel.

Gödel's A1, A4, A5: not implied anymore.

**lemma** *A1*:  $[\forall X.((\neg(\mathcal{P} X)) \leftrightarrow (\mathcal{P}(\neg X)))]$  **nitpick oops** — Countermodel

**lemma** *A4*:  $[\forall X.((\mathcal{P} X) \rightarrow \Box(\mathcal{P} X))]$  **nitpick oops** — Countermodel

**lemma** *A5*:  $[\mathcal{P} \mathcal{N}\mathcal{E}]$  **nitpick oops** — Countermodel

Checking filter and ultrafilter properties.

**theorem** *F1*:  $[\text{Filter } \mathcal{P}]$  **oops** — Proof found by sledgehammer, but reconstruction timeout

**theorem** *U1*:  $[\text{UFilter } \mathcal{P}]$  **nitpick oops** — Countermodel

**end**

## 4.6 Simplified Variant with Axiom T2 (Fig. 7 in [4])

**theory** *SimpleVariantPG* **imports**

*HOML*

*MFilter*

*BaseDefs*

**begin**

Axiom's of simplified variant with A3 replaced.

**axiomatization where**

*A1'*:  $[\neg(\mathcal{P}(\lambda x.(x \neq x)))]$  **and**

*A2'*:  $[\forall X Y.(((\mathcal{P} X) \wedge ((X \sqsubseteq Y) \vee (X \ni Y))) \rightarrow (\mathcal{P} Y))]$  **and**

*T2*:  $[\mathcal{P} \mathcal{G}]$

Necessary existence of a Godlike entity.

**theorem** *T6*:  $[\Box(\exists^E \mathcal{G})]$  — Proof found by sledgehammer

**proof** —

**have** *T1*:  $[\forall X.((\mathcal{P} X) \rightarrow \Diamond(\exists^E X))]$  **by** (*metis A1' A2'*)

**have** *T3*:  $[\Diamond(\exists^E \mathcal{G})]$  **using** *T1 T2* **by** *simp*

**have** *T5*:  $[(\Diamond(\exists^E \mathcal{G})) \rightarrow \Box(\exists^E \mathcal{G})]$  **by** (*metis A1' A2' T2*)

**thus** *?thesis* **using** *T3* **by** *blast qed*

**lemma** *True nitpick[satisfy] oops* — Consistency: model found

Modal collapse and Monotheism: not implied.

**lemma** *MC*:  $[\forall \Phi.(\Phi \rightarrow \Box \Phi)]$  **nitpick oops** — Countermodel

**lemma** *MT*:  $[\forall x y.(((\mathcal{G} x) \wedge (\mathcal{G} y)) \rightarrow (x=y))]$  **nitpick oops** — Countermodel

**end**

## 4.7 Simplified Variant with Simple Entailment in Logic K (Fig. 8 in [4])

**theory** *SimpleVariantSE* **imports**  
*HOML*  
*MFilter*  
*BaseDefs*  
**begin**

Axiom's of new variant based on ultrafilters.

**axiomatization where**

$A1'$ :  $\lceil \neg(\mathcal{P}(\lambda x.(x \neq x))) \rceil$  **and**  
 $A2''$ :  $\lceil \forall X Y.((\mathcal{P} X) \wedge (X \sqsubseteq Y)) \rightarrow (\mathcal{P} Y) \rceil$  **and**  
 $T2$ :  $\lceil \mathcal{P} \mathcal{G} \rceil$

Necessary existence of a Godlike entity.

**theorem**  $T6$ :  $\lceil \Box(\exists^E \mathcal{G}) \rceil$  **using**  $A1' A2'' T2$  **by** *blast*  
**theorem**  $T7$ :  $\lceil \exists^E \mathcal{G} \rceil$  **using**  $A1' A2'' T2$  **by** *blast*

Possible existence of a Godlike: has counterodel.

**lemma**  $T3$ :  $\lceil \Diamond(\exists^E \mathcal{G}) \rceil$  **nitpick oops** — Countermodel

**lemma**  $T3'$ : **assumes**  $T$ :  $\lceil \forall \varphi.((\Box \varphi) \rightarrow \varphi) \rceil$   
**shows**  $\lceil \Diamond(\exists^E \mathcal{G}) \rceil$   
**using**  $A1' A2'' T2 T$  **by** *metis*  
**end**

## 4.8 Simplified Variant with Simple Entailment in Logic T (Fig. 9 in [4])

**theory** *SimpleVariantSEinT* **imports**  
*HOML*  
*MFilter*  
*BaseDefs*  
**begin**

Axiom's of new variant based on ultrafilters.

**axiomatization where**

$A1'$ :  $\lceil \neg(\mathcal{P}(\lambda x.(x \neq x))) \rceil$  **and**  
 $A2''$ :  $\lceil \forall X Y.((\mathcal{P} X) \wedge (X \sqsubseteq Y)) \rightarrow (\mathcal{P} Y) \rceil$  **and**  
 $T2$ :  $\lceil \mathcal{P} \mathcal{G} \rceil$

Modal Logic T.

**axiomatization where**  $T$ :  $\lceil \forall \varphi.((\Box \varphi) \rightarrow \varphi) \rceil$   
**lemma**  $T'$ :  $\lceil \forall \varphi.(\varphi \rightarrow (\Diamond \varphi)) \rceil$  **by** (*metis*  $T$ )

Necessary existence of a Godlike entity.

**theorem**  $T6$ :  $\lceil \Box(\exists^E \mathcal{G}) \rceil$  — Proof found by sledgehammer  
**proof** —

**have**  $T1$ :  $[\forall X.((\mathcal{P} X) \rightarrow (\diamond(\exists^E X)))]$  **by** (*metis A1' A2'' T'*)  
**have**  $T3$ :  $[\diamond(\exists^E \mathcal{G})]$  **by** (*metis T1 T2*)  
**have**  $T5$ :  $[(\diamond(\exists^E \mathcal{G})) \rightarrow \square(\exists^E \mathcal{G})]$  **by** (*metis A1' A2'' T2*)  
**thus** *?thesis* **using**  $T3$  **by** *simp qed*

T6 again, with an alternative, simpler proof.

**theorem**  $T6$ *again*:  $[\square(\exists^E \mathcal{G})]$   
**proof** –  
**have**  $L1$ :  $[(\exists X.((\mathcal{P} X) \wedge \neg(\exists^E X))) \rightarrow (\mathcal{P}(\lambda x.(x \neq x)))]$   
**by** (*smt A2''*)  
**have**  $L2$ :  $[\neg(\exists X.((\mathcal{P} X) \wedge \neg(\exists^E X)))]$  **by** (*metis L1 A1'*)  
**have**  $T1'$ :  $[\forall X.((\mathcal{P} X) \rightarrow (\exists^E X))]$  **by** (*metis L2*)  
**have**  $T3'$ :  $[\exists^E \mathcal{G}]$  **by** (*metis T1' T2*)  
**have**  $L3$ :  $[\diamond(\exists^E \mathcal{G})]$  **by** (*metis T3' T'*) — not needed  
**thus** *?thesis* **using**  $T3'$  **by** *simp qed*  
**end**

#### 4.9 Hauptfiltervariant (Fig. 10 in [4])

**theory** *SimpleVariantHF* **imports**

*HOML*

*MFilter*

*BaseDefs*

**begin**

Definition: Set of supersets of  $X$ , we call this  $\mathcal{HF} X$ .

**abbreviation**  $HF::\gamma \Rightarrow (\gamma \Rightarrow \sigma)$  **where**  $HF X \equiv \lambda Y.(X \sqsubseteq Y)$

Postulate:  $\mathcal{HF} \mathcal{G}$  is a filter; i.e., Hauptfilter of  $\mathcal{G}$ .

**axiomatization where**  $F1$ :  $[Filter (HF \mathcal{G})]$

Necessary existence of a Godlike entity.

**theorem**  $T6$ :  $[\square(\exists^E \mathcal{G})]$  **using**  $F1$  **by** *auto* — Proof found

**theorem**  $T6$ *again*:  $[\square(\exists^E \mathcal{G})]$

**proof** –

**have**  $T3'$ :  $[\exists^E \mathcal{G}]$  **using**  $F1$  **by** *auto*

**have**  $T6$ :  $[\square(\exists^E \mathcal{G})]$  **using**  $T3'$  **by** *blast*

**thus** *?thesis* **by** *simp qed*

Possible existence of Godlike entity not implied.

**lemma**  $T3$ :  $[\diamond(\exists^E \mathcal{G})]$  **nitpick oops** — Countermodel

Axiom T enforces possible existence of Godlike entity.

**axiomatization**

**lemma**  $T3$ : **assumes**  $T$ :  $[\forall \varphi.((\square \varphi) \rightarrow \varphi)]$

**shows**  $[\diamond(\exists^E \mathcal{G})]$  **using**  $F1 T$  **by** *auto*

**lemma** *True* **nitpick**[*satisfy*] **oops** — Consistency: model found

Modal collapse: not implied anymore.

**lemma** *MC*:  $[\forall \Phi. (\Phi \rightarrow \Box \Phi)]$  **nitpick oops** — Countermodel  
**lemma** *MT*:  $[\forall x y. ((\mathcal{G} x) \wedge (\mathcal{G} y)) \rightarrow (x=y)]$   
**nitpick oops** — Countermodel  
**end**

#### 4.10 Formal Study of Version No.2 of Gödel's Argument as Reported by Kanckos and Lethen, 2019 [6] (Fig. 11 in [4])

**theory** *KanckosLethenNo2Possibilist* **imports**  
*HOML*  
*MFilter*  
*BaseDefs*  
**begin**

Axioms of Version No. 2 [6].

**abbreviation** *delta* ( $\langle \Delta \rangle$ ) **where**  $\Delta A \equiv \lambda x. (\forall \psi. ((A \psi) \rightarrow (\psi x)))$   
**abbreviation** *N* ( $\langle \mathcal{N} \rangle$ ) **where**  $\mathcal{N} \varphi \equiv \lambda x. (\Box(\varphi x))$

**axiomatization where**

*Axiom1*:  $[\forall \varphi \psi. (((\mathcal{P} \varphi) \wedge (\Box(\forall x. ((\varphi x) \rightarrow (\psi x)))))) \rightarrow (\mathcal{P} \psi)]$  **and** — The  $\Box$  can be omitted here; the proofs still work.

*Axiom2*:  $[\forall A. (\Box((\forall \varphi. ((A \varphi) \rightarrow (\mathcal{P} \varphi))) \rightarrow (\mathcal{P} (\Delta A))))]$  **and** — The  $\Box$  can be omitted here; the proofs still work.

*Axiom3*:  $[\forall \varphi. ((\mathcal{P} \varphi) \rightarrow (\mathcal{P} (\mathcal{N} \varphi)))]$  **and**

*Axiom4*:  $[\forall \varphi. ((\mathcal{P} \varphi) \rightarrow (\neg(\mathcal{P}(\neg\varphi))))]$  **and**

— Logic S5

*axB*:  $[\forall \varphi. (\varphi \rightarrow \Box \Diamond \varphi)]$  **and** *axM*:  $[\forall \varphi. ((\Box \varphi) \rightarrow \varphi)]$  **and** *ax4*:  $[\forall \varphi. ((\Box \varphi) \rightarrow (\Box \Box \varphi))]$

Sahlqvist correspondences: they are better suited for proof automation.

**lemma** *axB'*:  $\forall x y. \neg(xry) \vee (yrx)$  **using** *axB* **by** *fastforce*

**lemma** *axM'*:  $\forall x. (xxx)$  **using** *axM* **by** *blast*

**lemma** *ax4'*:  $\forall x y z. (((xy) \wedge (ygz)) \rightarrow (xgz))$  **using** *ax4* **by** *auto*

Proofs for all theorems for No.2 from [6].

**theorem** *Theorem0*:  $[\forall \varphi \psi. ((\forall Q. ((Q \varphi) \rightarrow (Q \psi))) \rightarrow ((\mathcal{P} \varphi) \rightarrow (\mathcal{P} \psi)))]$   
**by** *auto* — not needed

**theorem** *Theorem1*:  $[\mathcal{P} \mathcal{G}]$  **unfolding** *G-def* **using** *Axiom2 axM* **by** *blast*

**theorem** *Theorem2*:  $[\forall x. ((\mathcal{G} x) \rightarrow (\exists y. \mathcal{G} y))]$  **by** *blast* — not needed

**theorem** *Theorem3a*:  $[\mathcal{P} (\lambda x. (\exists y. \mathcal{G} y))]$  **by** (*metis (no-types, lifting) Axiom1 Theorem1*)

**theorem** *Theorem3b*:  $[\Box(\mathcal{P} (\lambda x. (\Box(\exists y. \mathcal{G} y))))]$  **by** (*smt Axiom1 G-def Theorem3a Axiom3 Theorem1 axB'*)

**theorem** *Theorem4*:  $[\forall x. \Box((\mathcal{G} x) \rightarrow ((\mathcal{P} (\lambda x. (\Box(\exists y. \mathcal{G} y)))) \rightarrow (\Box(\exists y. \mathcal{G} y))))]$   
**using** *G-def* **by** *fastforce* — not needed

**theorem** *Theorem5*:  $[\forall x. \Box((\mathcal{G} x) \rightarrow (\Box(\exists y. \mathcal{G} y)))]$  **by** (*smt (verit) G-def Theorem3a Theorem3b*) — not needed

**theorem** *Theorem6*:  $[\Box((\exists y. \mathcal{G} y) \rightarrow (\Box(\exists y. \mathcal{G} y)))]$  **by** (*smt G-def Theorem3a Theorem3b*)

**theorem** *Theorem7*:  $[\Box((\Diamond(\exists y. \mathcal{G} y)) \rightarrow (\Box(\exists y. \mathcal{G} y)))]$  **using** *Theorem6 axB'* **by** *blast*

**theorem** *Theorem8*:  $[\Box(\exists y. \mathcal{G} y)]$  **by** (*metis Axiom1 Axiom4 Theorem1 Theorem7 axB'*)

**theorem** *Theorem9*:  $[\forall \varphi. ((\mathcal{P} \varphi) \rightarrow \Diamond(\exists x. \varphi x))]$  **using** *Axiom1 Axiom4 axM'* **by** *metis*

Short proof of Theorem8; analogous to the one presented in Sec. 7 of Benzmüller 2020.

**theorem**  $[\Box(\exists y. \mathcal{G} y)]$  — Note: this version of the proof uses only *axB'* and *axM'*.  
**proof** —

**have** *L1*:  $[(\exists X. ((\mathcal{P} X) \wedge \neg(\exists X))) \rightarrow (\mathcal{P}(\lambda x. (x \neq x)))]$  **using** *Axiom1 Axiom3 axB'* **by** *blast* — Use *metis* here if  $\Box$  is omitted in *Axiom1* and *Axiom2*

**have** *L2*:  $[\neg(\mathcal{P}(\lambda x. (x \neq x)))]$  **using** *Axiom1 Axiom4* **by** *metis*

**have** *L3*:  $[\neg(\exists X. ((\mathcal{P} X) \wedge \neg(\exists X)))]$  **using** *L1 L2* **by** *blast*

**have** *T2*:  $[\mathcal{P} \mathcal{G}]$  **by** (*smt Axiom1 Axiom2 G-def axM'*)

**have** *T3*:  $[\exists y. \mathcal{G} y]$  **using** *L3 T2* **by** *blast*

**have** *T6*:  $[\Box(\exists y. \mathcal{G} y)]$  **by** (*simp add: T3*)

**thus** *?thesis* **by** *blast qed*

**theorem** *T5*:  $[(\Diamond(\exists y. \mathcal{G} y)) \rightarrow \Box(\exists y. \mathcal{G} y)]$  — Obvious: If we can prove Theorem8, then we also have *T5*.

**proof** —

**have** *L1*:  $[(\exists X. ((\mathcal{P} X) \wedge \neg(\exists X))) \rightarrow (\mathcal{P}(\lambda x. (x \neq x)))]$  **using** *Axiom1 Axiom3 axB'* **by** *blast* — Use *metis* here if  $\Box$  is omitted in *Axiom1* and *Axiom2*

**have** *L2*:  $[\neg(\mathcal{P}(\lambda x. (x \neq x)))]$  **using** *Axiom1 Axiom4* **by** *metis*

**have** *L3*:  $[\neg(\exists X. ((\mathcal{P} X) \wedge \neg(\exists X)))]$  **using** *L1 L2* **by** *blast*

**have** *T2*:  $[\mathcal{P} \mathcal{G}]$  **by** (*smt Axiom1 Axiom2 G-def axM'*)

**have** *T3*:  $[\exists y. \mathcal{G} y]$  **using** *L3 T2* **by** *blast*

**have** *T6*:  $[\Box(\exists y. \mathcal{G} y)]$  **by** (*simp add: T3*)

**thus** *?thesis* **by** *blast qed*

Another short proof of Theorem8.

**theorem**  $[\Box(\exists y. \mathcal{G} y)]$  — Note: fewer assumptions used in some cases than in [6].

**proof** —

**have** *T1*:  $[\mathcal{P} \mathcal{G}]$  **unfolding** *G-def* **using** *Axiom2 axM* **by** *blast*

**have** *T3a*:  $[\mathcal{P}(\lambda x. (\exists y. \mathcal{G} y))]$  **by** (*metis (no-types, lifting) Axiom1 T1*)

**have** *T3b*:  $[\Box(\mathcal{P}(\lambda x. (\Box(\exists y. \mathcal{G} y))))]$  **by** (*smt Axiom1 G-def T3a Axiom3 T1 axB'*)

**have** *T6*:  $[\Box((\exists y. \mathcal{G} y) \rightarrow (\Box(\exists y. \mathcal{G} y)))]$  **by** (*smt G-def T3a T3b*)

**have** *T7*:  $[\Box((\Diamond(\exists y. \mathcal{G} y)) \rightarrow (\Box(\exists y. \mathcal{G} y)))]$  **using** *T6 axB'* **by** *blast*

**thus** *?thesis* **by** (*smt Axiom1 Axiom4 T3b axB'*) **qed**

Are the axioms of the simplified versions implied?

Actualist version of the axioms.

**lemma** *A1'*:  $[\neg(\mathcal{P}(\lambda x. (x \neq x)))]$  **using** *Theorem9* **by** *blast*

**lemma A2'**:  $[\forall X Y.((\mathcal{P} X) \wedge ((X \sqsubseteq Y) \vee (X \Rightarrow Y))) \rightarrow (\mathcal{P} Y)]$  **nitpick oops**  
 — Countermodel

**lemma A3**:  $[\forall \mathcal{Z}.((\mathcal{P} \text{os } \mathcal{Z}) \rightarrow (\forall X.((X \sqcap \mathcal{Z}) \rightarrow (\mathcal{P} X))))]$  **nitpick oops** —  
 Countermodel

Possibilist version of the axioms.

**abbreviation a** ( $\langle \cdot \sqsubseteq^p \cdot \rangle$ ) **where**  $X \sqsubseteq^p Y \equiv \forall z.((X z) \rightarrow (Y z))$

**abbreviation b** ( $\langle \cdot \Rightarrow^p \cdot \rangle$ ) **where**  $X \Rightarrow^p Y \equiv \Box(X \sqsubseteq^p Y)$

**abbreviation d** ( $\langle \cdot \sqcap^p \cdot \rangle$ ) **where**  $X \sqcap^p \mathcal{Z} \equiv \Box(\forall u.((X u) \leftrightarrow (\forall Y.((\mathcal{Z} Y) \rightarrow (Y u))))))$

**lemma A1'P**:  $[\neg(\mathcal{P}(\lambda x.(x \neq x)))]$  **using Theorem9 by blast**

**lemma A2'P**:  $[\forall X Y.((\mathcal{P} X) \wedge ((X \sqsubseteq^p Y) \vee (X \Rightarrow^p Y))) \rightarrow (\mathcal{P} Y)]$  **oops** — no  
 answer, yet by sledgehammer and nitpick

**lemma A2'aP**:  $[\forall X Y.((\mathcal{P} X) \wedge (X \Rightarrow^p Y)) \rightarrow (\mathcal{P} Y)]$  **using Axiom1 axM' by**  
*metis*

**lemma A2'bP**:  $[\forall X Y.((\mathcal{P} X) \wedge (X \sqsubseteq^p Y)) \rightarrow (\mathcal{P} Y)]$  **oops** — no answer, yet  
 by sledgehammer and nitpick

**lemma A3P**:  $[\forall \mathcal{Z}.((\mathcal{P} \text{os } \mathcal{Z}) \rightarrow (\forall X.((X \sqcap^p \mathcal{Z}) \rightarrow (\mathcal{P} X))))]$

**by** (*smt (verit, del-insts) Axiom1 Axiom2 axM'*) — proof found

Are Axiom2 and A3 equivalent? Only when assuming Axiom1 and axiom  
 M.

**lemma**  $[\forall A.(\Box((\forall \varphi.((A \varphi) \rightarrow (\mathcal{P} \varphi))) \rightarrow (\mathcal{P}(\Delta A))))] \equiv [\forall \mathcal{Z}.((\mathcal{P} \text{os } \mathcal{Z}) \rightarrow$   
 $(\forall X.((X \sqcap^p \mathcal{Z}) \rightarrow (\mathcal{P} X))))]$

**by** (*smt (verit, ccfv-threshold) Axiom1 axM'*) — proof found  
**end**

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