Riesz Representation Theorem

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Abstract

We formalize the Riesz-Markov-Kakutani representation theorem following pp.37-47 of the book *Real and Complex Analysis* by Rudin [1]. This entry also includes formalization of regular measures, tightness of measures, and Urysohn's lemma on locally compact Hausdorff spaces. Roughly speaking, the theorem states that if φ is a positive linear functional from $C(X)$ (the space of continuous functions from X to complex numbers which have compact supports) to complex numbers, then there exists a unique measure μ such that for all $f \in C(X)$,

$$\varphi(f) = \int f \mathrm{d}\mu.$$

Contents

1 Urysohn's Lemma

theory *Urysohn-Locally-Compact-Hausdorff*
  imports *Standard-Borel-Spaces.StandardBorel*
begin

We prove Urysohn's lemma for locally compact Hausdorff space (Lemma 2.12 [1])

1.1 Lemmas for Upper/Lower Semi-Continuous Functions

lemma
  assumes $\bigwedge x.\ x \in topspace\ X \Longrightarrow f\ x = g\ x$
  shows *upper-semicontinuous-map-cong*:
    upper-semicontinuous-map $X\ f \longleftrightarrow$ *upper-semicontinuous-map* $X\ g$ (**is** *?g1*)
    and *lower-semicontinuous-map-cong*:
    lower-semicontinuous-map $X\ f \longleftrightarrow$ *lower-semicontinuous-map* $X\ g$ (**is** *?g2*)
⟨*proof*⟩

lemma *upper-lower-semicontinuous-map-iff-continuous-map*:
  continuous-map X euclidean f $\longleftrightarrow$ *upper-semicontinuous-map X f* $\wedge$ *lower-semicontinuous-map X f*
  ⟨*proof*⟩

lemma [*simp*]:
  shows *upper-semicontinuous-map-const*: *upper-semicontinuous-map* $X\ (\lambda x.\ c)$
    and *lower-semicontinuous-map-const*: *lower-semicontinuous-map* $X\ (\lambda x.\ c)$
  ⟨*proof*⟩

lemma *upper-semicontinuous-map-c-add-iff*:
  fixes c :: *real*
  shows *upper-semicontinuous-map* $X\ (\lambda x.\ c + f\ x) \longleftrightarrow$ *upper-semicontinuous-map* $X\ f$
⟨*proof*⟩

corollary *upper-semicontinuous-map-add-c-iff*:
  fixes c :: *real*
  shows *upper-semicontinuous-map* $X\ (\lambda x.\ f\ x + c) \longleftrightarrow$ *upper-semicontinuous-map* $X\ f$
  ⟨*proof*⟩

lemma *upper-semicontinuous-map-posreal-cmult-iff*:
  fixes c :: *real*
  assumes $c > 0$
  shows *upper-semicontinuous-map* $X\ (\lambda x.\ c * f\ x) \longleftrightarrow$ *upper-semicontinuous-map* $X\ f$
⟨*proof*⟩

lemma *upper-semicontinuous-map-real-cmult*:
fixes $c :: real$
assumes $c \geq 0$ *upper-semicontinuous-map X f*
shows *upper-semicontinuous-map X* $(\lambda x.\ c * f\ x)$
⟨*proof*⟩

lemma *lower-semicontinuous-map-posreal-cmult-iff*:
fixes $c :: real$
assumes $c > 0$
shows *lower-semicontinuous-map X* $(\lambda x.\ c * f\ x) \longleftrightarrow$ *lower-semicontinuous-map X f*
⟨*proof*⟩

lemma *lower-semicontinuous-map-real-cmult*:
fixes $c :: real$
assumes $c \geq 0$ *lower-semicontinuous-map X f*
shows *lower-semicontinuous-map X* $(\lambda x.\ c * f\ x)$
⟨*proof*⟩

lemma *upper-semicontinuous-map-INF*:
fixes f :: - ⇒ - ⇒ *'a* :: {*linorder-topology*, *complete-linorder*}
assumes $\bigwedge i.\ i \in I \Longrightarrow$ *upper-semicontinuous-map X* $(f\ i)$
shows *upper-semicontinuous-map X* $(\lambda x.\ \bigsqcap i{\in}I.\ f\ i\ x)$
⟨*proof*⟩

lemma *upper-semicontinuous-map-cInf*:
fixes f :: - ⇒ - ⇒ *'a* :: {*linorder-topology*, *conditionally-complete-linorder*}
assumes $I \neq \{\}$ $\bigwedge x.\ x \in$ *topspace X* $\Longrightarrow$ *bdd-below* $((\lambda i.\ f\ i\ x)$ ` $I)$
and $\bigwedge i.\ i \in I \Longrightarrow$ *upper-semicontinuous-map X* $(f\ i)$
shows *upper-semicontinuous-map X* $(\lambda x.\ \bigsqcap i{\in}I.\ f\ i\ x)$
⟨*proof*⟩

lemma *lower-semicontinuous-map-Sup*:
fixes f :: - ⇒ - ⇒ *'a* :: {*linorder-topology*, *complete-linorder*}
assumes $\bigwedge i.\ i \in I \Longrightarrow$ *lower-semicontinuous-map X* $(f\ i)$
shows *lower-semicontinuous-map X* $(\lambda x.\ \bigsqcup i{\in}I.\ f\ i\ x)$
⟨*proof*⟩

lemma *indicator-closed-upper-semicontinuous-map*:
assumes *closedin X C*
shows *upper-semicontinuous-map X* (*indicator C* :: - ⇒ *'a* :: {*zero-less-one*, *linorder-topology*})
⟨*proof*⟩

lemma *indicator-open-lower-semicontinuous-map*:
assumes *openin X U*
shows *lower-semicontinuous-map X* (*indicator U* :: - ⇒ *'a* :: {*zero-less-one*,

linorder-topology})
⟨*proof*⟩

lemma *lower-semicontinuous-map-cSup*:
fixes *f* :: - ⇒ - ⇒ *'a* :: {*linorder-topology*, *conditionally-complete-linorder*}
assumes *I* ≠ {} ⋀*x*. *x* ∈ *topspace X* ⟹ *bdd-above* ((λ*i*. *f i x*) ‘ *I*)
and ⋀*i*. *i* ∈ *I* ⟹ *lower-semicontinuous-map X* (*f i*)
shows *lower-semicontinuous-map X* (λ*x*. ⨆ *i*∈*I*. *f i x*)
⟨*proof*⟩

lemma *openin-continuous-map-less*:
assumes *continuous-map X* (*euclidean* :: (*'a* :: {*dense-linorder*, *order-topology*}) *topology*) *f*
and *continuous-map X euclidean g*
shows *openin X* {*x*∈*topspace X*. *f x* < *g x*}
⟨*proof*⟩

corollary *closedin-continuous-map-eq*:
assumes *continuous-map X* (*euclidean* :: (*'a* :: {*dense-linorder*, *order-topology*}) *topology*) *f*
and *continuous-map X euclidean g*
shows *closedin X* {*x*∈*topspace X*. *f x* = *g x*}
⟨*proof*⟩

1.2 Urysohn's Lemma

lemma *locally-compact-Hausdorff-compactin-openin-subset*:
assumes *locally-compact-space X Hausdorff-space X* ∨ *regular-space X*
and *compactin X T openin X V T* ⊆ *V*
shows ∃ *U*. *openin X U* ∧ *compactin X* (*X closure-of U*) ∧ *T* ⊆ *U* ∧ (*X closure-of U*) ⊆ *V*
⟨*proof*⟩

lemma *Urysohn-locally-compact-Hausdorff-closed-compact-support*:
fixes *a b* :: *real* **and** *X* :: *'a topology*
assumes *locally-compact-space X Hausdorff-space X* ∨ *regular-space X*
and *a* ≤ *b closedin X S compactin X T disjnt S T*
obtains *f* **where** *continuous-map X* (*subtopology euclidean* {*a*..*b*}) *f f* ‘ *S* ⊆ {*a*} *f* ‘ *T* ⊆ {*b*} *disjnt* (*X closure-of* {*x*∈*topspace X*. *f x* ≠ *a*}) *S compactin X* (*X closure-of* {*x*∈*topspace X*. *f x* ≠ *a*})
⟨*proof*⟩

end

2 Regular Measures

theory *Regular-Measure*
imports *HOL−Probability.Probability*
Standard-Borel-Spaces.StandardBorel

Urysohn-Locally-Compact-Hausdorff
begin

context *Metric-space*
begin

lemma *nbh-add*: $(\bigcup b\in(\bigcup a\in A.\ mball\ a\ e).\ mball\ b\ f) \subseteq (\bigcup a\in A.\ mball\ a\ (e + f))$
⟨*proof*⟩

lemma *nbh-subset*:
assumes *A*: $A \subseteq M$ **and** *e*: $e > 0$
shows $A \subseteq (\bigcup a\in A.\ mball\ a\ e)$
⟨*proof*⟩

lemma *nbh-decseq*:
assumes *decseq an*
shows $decseq\ (\lambda n.\ \bigcup a\in A.\ mball\ a\ (an\ n))$
⟨*proof*⟩

lemma *nbh-Inter-closure-of*:
assumes *A*: $A \neq \{\}$ $A \subseteq M$
and *an*:$\bigwedge n.\ an\ n > 0$ $decseq\ an$ $an \longrightarrow 0$
shows $(\bigcap n.\ \bigcup a\in A.\ mball\ a\ (an\ n)) = mtopology\ closure\text{-}of\ A$
⟨*proof*⟩

end

lemma(**in** *finite-measure*)
assumes $range\ A \subseteq sets\ M$ $disjoint\text{-}family\ A$
shows *suminf-measure*:$(\sum i.\ measure\ M\ (A\ i)) = measure\ M\ (\bigcup i.\ A\ i)$
and *summable-measure*: $summable\ (\lambda i.\ measure\ M\ (A\ i))$
⟨*proof*⟩

We refer to the lecture note [2].

Inner regular and outer regular with abstract topologies.

definition *inner-regular* :: $'a\ topology \Rightarrow 'a\ measure \Rightarrow bool$ **where**
$inner\text{-}regular\ X\ M \longleftrightarrow sets\ M = sets\ (borel\text{-}of\ X) \wedge (\forall A\in sets\ M.\ M\ A = (\bigsqcup C\in\{C.\ closedin\ X\ C \wedge C \subseteq A\}.\ M\ C))$

definition *outer-regular* :: $'a\ topology \Rightarrow 'a\ measure \Rightarrow bool$ **where**
$outer\text{-}regular\ X\ M \longleftrightarrow sets\ M = sets\ (borel\text{-}of\ X) \wedge (\forall A\in sets\ M.\ M\ A = (\bigsqcap C\in\{C.\ openin\ X\ C \wedge A \subseteq C\}.\ M\ C))$

definition *regular-measure* :: $'a\ topology \Rightarrow 'a\ measure \Rightarrow bool$ **where**
$regular\text{-}measure\ X\ M \longleftrightarrow inner\text{-}regular\ X\ M \wedge outer\text{-}regular\ X\ M$

lemma
shows *inner-reguarI*: $sets\ M = sets\ (borel\text{-}of\ X) \Longrightarrow (\bigwedge A.\ A \in sets\ M$

$\Longrightarrow$ *M A* = ($\bigsqcup$ *C*∈{*C*. *closedin X C* ∧ *C* ⊆ *A*}. *M C*)) $\Longrightarrow$ *inner-regular X M*
and *inner-reguarD*: *inner-regular X M* $\Longrightarrow$ *sets M* = *sets* (*borel-of X*)
inner-regular X M $\Longrightarrow$ *A* ∈ *sets M* $\Longrightarrow$ *M A* = ($\bigsqcup$ *C*∈{*C*. *closedin X C* ∧ *C* ⊆ *A*}. *M C*)
⟨*proof*⟩

lemma
shows *outer-reguarI*: *sets M* = *sets* (*borel-of X*)
$\Longrightarrow$ ($\bigwedge$*A*. *A* ∈ *sets M* $\Longrightarrow$ *M A* = ($\bigsqcap$ *C*∈{*C*. *openin X C* ∧ *A* ⊆ *C*}. *M C*)) $\Longrightarrow$ *outer-regular X M*
and *outer-reguarD*: *outer-regular X M* $\Longrightarrow$ *sets M* = *sets* (*borel-of X*)
outer-regular X M $\Longrightarrow$ *A* ∈ *sets M* $\Longrightarrow$ *M A* = ($\bigsqcap$ *C*∈{*C*. *openin X C* ∧ *A* ⊆ *C*}. *M C*)
⟨*proof*⟩

lemma
shows *regular-measureI*: *inner-regular X M* $\Longrightarrow$ *outer-regular X M* $\Longrightarrow$ *regular-measure X M*
and *regular-measureD*:
regular-measure X M $\Longrightarrow$ *inner-regular X M* *regular-measure X M* $\Longrightarrow$ *outer-regular X M*
⟨*proof*⟩

lemma *inner-regular-finite-measure*:
assumes *finite-measure M*
shows *inner-regular X M* ⟷
sets M = *sets* (*borel-of X*) ∧ (∀ *A*∈*sets M*. *measure M A* = ($\bigsqcup$ *C*∈{*C*. *closedin X C* ∧ *C* ⊆ *A*}. *measure M C*))
⟨*proof*⟩

lemma(**in** *finite-measure*)
shows *inner-regularI*: *sets M* = *sets* (*borel-of X*) $\Longrightarrow$ ($\bigwedge$*A*. *A* ∈ *sets M*
$\Longrightarrow$ *measure M A* = ($\bigsqcup$ *C*∈{*C*. *closedin X C* ∧ *C* ⊆ *A*}. *measure M C*)) $\Longrightarrow$ *inner-regular X M*
and *inner-regularD*:
inner-regular X M $\Longrightarrow$ *A* ∈ *sets M* $\Longrightarrow$ *measure M A* = ($\bigsqcup$ *C*∈{*C*. *closedin X C* ∧ *C* ⊆ *A*}. *measure M C*)
⟨*proof*⟩

lemma *outer-regular-finite-measure*:
assumes *finite-measure M*
shows *outer-regular X M* ⟷ *sets M* = *sets* (*borel-of X*) ∧ (∀ *A*∈*sets M*. *measure M A* = ($\bigsqcap$ *C*∈{*C*. *openin X C* ∧ *A* ⊆ *C*}. *measure M C*))
⟨*proof*⟩

lemma(**in** *finite-measure*)
shows *outer-regularI*: *sets M* = *sets* (*borel-of X*) $\Longrightarrow$ ($\bigwedge$*A*. *A* ∈ *sets M*
$\Longrightarrow$ *measure M A* = ($\bigsqcap$ *C*∈{*C*. *openin X C* ∧ *A* ⊆ *C*}. *measure M C*)) $\Longrightarrow$

outer-regular X M
  and *outer-regularD*: *outer-regular X M* $\Longrightarrow$ *A* $\in$ *sets M*
    $\Longrightarrow$ *measure M A* = ($\bigsqcap$ *C*$\in${*C*. *openin X C* $\wedge$ *A* $\subseteq$ *C*}. *measure M C*)
 ⟨*proof*⟩

Abstract version of ⟦*sets ?M* = *sets borel*; *emeasure ?M* (*space ?M*) $\neq \infty$; *?B* $\in$ *sets borel*⟧ $\Longrightarrow$ *emeasure ?M ?B* = $\bigsqcup$ (*emeasure ?M* ' {*K*. *K* $\subseteq$ *?B* $\wedge$ *compact K*}) and ⟦*sets ?M* = *sets borel*; *emeasure ?M* (*space ?M*) $\neq \infty$; *?B* $\in$ *sets borel*⟧ $\Longrightarrow$ *emeasure ?M ?B* = $\bigsqcap$ (*emeasure ?M* ' {*U*. *?B* $\subseteq$ *U* $\wedge$ *open U*}).

lemma(**in** *finite-measure*)
 assumes *metrizable-space X sets* (*borel-of X*) = *sets M*
  shows *inner-regular'*:*inner-regular X M*
   and *outer-regular'*:*outer-regular X M*
⟨*proof*⟩

definition *tight-on-set* :: *'a topology* ⇒ *'a measure set* ⇒ *bool* **where**
tight-on-set X Γ ⟷ (∀ *M*∈Γ. *finite-measure M* ∧ *sets* (*borel-of X*) = *sets M*) ∧
          (∀ *e*>*0*. ∃ *K*. *compactin X K* ∧ (∀ *M*∈Γ. *measure M* (*space M* − *K*) < *e*))

abbreviation *tight-on* :: *'a topology* ⇒ *'a measure* ⇒ *bool* **where**
tight-on X M ≡ *tight-on-set X* {*M*}

lemma *tight-on-def*:
tight-on X M ⟷ *finite-measure M* ∧ *sets* (*borel-of X*) = *sets M* ∧
         (∀ *e*>*0*. ∃ *K*. *compactin X K* ∧ *measure M* (*space M* − *K*) < *e*)
 ⟨*proof*⟩

lemma *tight-on-set-subset*: *A* ⊆ *B* ⟹ *tight-on-set X B* ⟹ *tight-on-set X A*
 ⟨*proof*⟩

lemma *tight-on-tight*: *tight-on-set euclidean* (*Mi* ' *UNIV*) ∧ (∀ *i*. *real-distribution* (*Mi i*)) ⟷ *tight Mi*
⟨*proof*⟩

lemma *inner-regular''*:
 assumes *metrizable-space X tight-on X M*
   and [*measurable*]:*A* ∈ *sets M*
  shows *measure M A* = ($\bigsqcup$ *K*∈{*K*. *compactin X K* ∧ *K* ⊆ *A*}. *measure M K*) (**is** - = *?rhs*)
⟨*proof*⟩

lemma(**in** *finite-measure*) *tight-on-compact-space*:
 assumes *metrizable-space X compact-space X sets* (*borel-of X*) = *sets M*
 shows *tight-on X M*
 ⟨*proof*⟩

lemma(**in** *finite-measure*) *tight-on-finite-space*:

assumes *metrizable-space X sets* (*borel-of X*) = *sets M finite* (*space M*)
shows *tight-on X M*
⟨*proof*⟩

lemma(**in** *finite-measure*) *tight-on-Polish*:
assumes *Polish-space X sets* (*borel-of X*) = *sets M*
shows *tight-on X M*
⟨*proof*⟩

corollary(**in** *finite-measure*) *inner-regular-Polish*:
assumes *Polish-space X sets* (*borel-of X*) = *sets M A* ∈ *sets M*
shows *measure M A* = (⨆*K*∈{*K*. *compactin X K* ∧ *K* ⊆ *A*}. *measure M K*)
⟨*proof*⟩

end

3 The Riesz Representation Theorem

theory *Riesz-Representation*
imports *Regular-Measure*
Urysohn-Locally-Compact-Hausdorff
begin

3.1 Lemmas for Complex-Valued Continuous Maps

lemma *continuous-map-Re′*[*simp*,*continuous-intros*]: *continuous-map euclidean euclideanreal Re*
and *continuous-map-Im′*[*simp*,*continuous-intros*]: *continuous-map euclidean euclideanreal Im*
and *continuous-map-complex-of-real′*[*simp*,*continuous-intros*]: *continuous-map euclideanreal euclidean complex-of-real*
⟨*proof*⟩

corollary
assumes *continuous-map X euclidean f*
shows *continuous-map-Re*[*simp*,*continuous-intros*]: *continuous-map X euclideanreal* (λ*x*. *Re* (*f x*))
and *continuous-map-Im*[*simp*,*continuous-intros*]: *continuous-map X euclideanreal* (λ*x*. *Im* (*f x*))
⟨*proof*⟩

lemma *continuous-map-of-real-iff*[*simp*]:
continuous-map X euclidean (λ*x*. *of-real* (*f x*) :: - :: *real-normed-div-algebra*) ⟷ *continuous-map X euclideanreal f*
⟨*proof*⟩

lemma *continuous-map-complex-mult* [*continuous-intros*]:
fixes *f* :: ′*a* ⇒ *complex*

shows ⟦*continuous-map X euclidean f*; *continuous-map X euclidean g*⟧ ⟹ *continuous-map X euclidean* (λ*x*. *f x* ∗ *g x*)
⟨*proof*⟩

lemma *continuous-map-complex-mult-left*:
fixes *f* :: ′*a* ⇒ *complex*
shows *continuous-map X euclidean f* ⟹ *continuous-map X euclidean* (λ*x*. *c* ∗ *f x*)
⟨*proof*⟩

lemma *complex-continuous-map-iff*:
continuous-map X euclidean f ⟷ *continuous-map X euclideanreal* (λ*x*. *Re* (*f x*)) ∧ *continuous-map X euclideanreal* (λ*x*. *Im* (*f x*))
⟨*proof*⟩

lemma *complex-integrable-iff*: *complex-integrable M f* ⟷ *integrable M* (λ*x*. *Re* (*f x*)) ∧ *integrable M* (λ*x*. *Im* (*f x*))
⟨*proof*⟩

3.2 Compact Supports

definition *has-compact-support-on* :: (′*a* ⇒ ′*b* :: *monoid-add*) ⇒ ′*a topology* ⇒ *bool*
(**infix** *has*′*-compact*′*-support*′*-on 60*) **where**
f has-compact-support-on X ⟷ *compactin X* (*X closure-of support-on* (*topspace X*) *f*)

lemma *has-compact-support-on-iff*:
f has-compact-support-on X ⟷ *compactin X* (*X closure-of* {*x*∈*topspace X*. *f x* ≠ *0*})
⟨*proof*⟩

lemma *has-compact-support-on-zero*[*simp*]: (λ*x*. *0*) *has-compact-support-on X*
⟨*proof*⟩

lemma *has-compact-support-on-compact-space*[*simp*]: *compact-space X* ⟹ *f has-compact-support-on X*
⟨*proof*⟩

lemma *has-compact-support-on-add*[*simp*,*intro*!]:
assumes *f has-compact-support-on X g has-compact-support-on X*
shows (λ*x*. *f x* + *g x*) *has-compact-support-on X*
⟨*proof*⟩

lemma *has-compact-support-on-sum*:
assumes *finite I* ⋀*i*. *i* ∈ *I* ⟹ *f i has-compact-support-on X*
shows (λ*x*. (∑ *i*∈*I*. *f i x*)) *has-compact-support-on X*
⟨*proof*⟩

lemma *has-compact-support-on-mult-left*:
fixes g :: - ⇒ - :: *mult-zero*
assumes g *has-compact-support-on* X
shows $(\lambda x.\ f\ x * g\ x)$ *has-compact-support-on* X
⟨*proof*⟩

lemma *has-compact-support-on-mult-right*:
fixes f :: - ⇒ - :: *mult-zero*
assumes f *has-compact-support-on* X
shows $(\lambda x.\ f\ x * g\ x)$ *has-compact-support-on* X
⟨*proof*⟩

lemma *has-compact-support-on-uminus-iff* [*simp*]:
fixes f :: - ⇒ - :: *group-add*
shows $(\lambda x.\ -\ f\ x)$ *has-compact-support-on* $X \longleftrightarrow f$ *has-compact-support-on* X
⟨*proof*⟩

lemma *has-compact-support-on-diff* [*simp*,*intro*!]:
fixes f :: - ⇒ - :: *group-add*
shows f *has-compact-support-on* $X \Longrightarrow g$ *has-compact-support-on* X
$\Longrightarrow (\lambda x.\ f\ x\ -\ g\ x)$ *has-compact-support-on* X
⟨*proof*⟩

lemma *has-compact-support-on-max*[*simp*,*intro*!]:
assumes f *has-compact-support-on* X g *has-compact-support-on* X
shows $(\lambda x.\ max\ (f\ x)\ (g\ x))$ *has-compact-support-on* X
⟨*proof*⟩

lemma *has-compact-support-on-ext-iff* [*iff*]:
$(\lambda x{\in}topspace\ X.\ f\ x)$ *has-compact-support-on* $X \longleftrightarrow f$ *has-compact-support-on* X
⟨*proof*⟩

lemma *has-compact-support-on-of-real-iff* [*iff*]:
$(\lambda x.\ of\text{-}real\ (f\ x))$ *has-compact-support-on* $X = f$ *has-compact-support-on* X
⟨*proof*⟩

lemma *has-compact-support-on-complex-iff*:
f *has-compact-support-on* $X \longleftrightarrow$
$(\lambda x.\ Re\ (f\ x))$ *has-compact-support-on* $X \land (\lambda x.\ Im\ (f\ x))$ *has-compact-support-on*
X
⟨*proof*⟩

lemma [*simp*]:
assumes f *has-compact-support-on* X
shows *has-compact-support-on-Re*:$(\lambda x.\ Re\ (f\ x))$ *has-compact-support-on* X
and *has-compact-support-on-Im*:$(\lambda x.\ Im\ (f\ x))$ *has-compact-support-on* X
⟨*proof*⟩

3.3 Positive Linear Functionsls

definition *positive-linear-functional-on-CX* :: $'a$ *topology* $\Rightarrow$ $(('a \Rightarrow 'b$ :: {*ring*, *order*, *topological-space*}) $\Rightarrow$ $'b) \Rightarrow$ *bool*
where *positive-linear-functional-on-CX X* $\varphi \equiv$
$(\forall f.$ *continuous-map X euclidean f* $\longrightarrow$ *f has-compact-support-on X*
$\longrightarrow (\forall x \in$ *topspace X*. $f\,x \geq 0) \longrightarrow \varphi\ (\lambda x \in$ *topspace X*. $f\,x) \geq 0) \wedge$
$(\forall f\ a.$ *continuous-map X euclidean f* $\longrightarrow$ *f has-compact-support-on X*
$\longrightarrow \varphi\ (\lambda x \in$ *topspace X*. $a * f\,x) = a * \varphi\ (\lambda x \in$ *topspace X*. $f\,x)) \wedge$
$(\forall f\ g.$ *continuous-map X euclidean f* $\longrightarrow$ *f has-compact-support-on X*
$\longrightarrow$ *continuous-map X euclidean g* $\longrightarrow$ *g has-compact-support-on X*
$\longrightarrow \varphi\ (\lambda x \in$ *topspace X*. $f\,x + g\,x) = \varphi\ (\lambda x \in$ *topspace X*. $f\,x) + \varphi\ (\lambda x \in$ *topspace X*. $g\,x))$

lemma
assumes *positive-linear-functional-on-CX X* φ
shows *pos-lin-functional-on-CX-pos*:
$\bigwedge f.$ *continuous-map X euclidean f* $\Longrightarrow$ *f has-compact-support-on X*
$\Longrightarrow (\bigwedge x.\ x \in$ *topspace X* $\Longrightarrow f\,x \geq 0) \Longrightarrow \varphi\ (\lambda x \in$ *topspace X*. $f\,x) \geq 0$
and *pos-lin-functional-on-CX-lin*:
$\bigwedge f\ a.$ *continuous-map X euclidean f* $\Longrightarrow$ *f has-compact-support-on X*
$\Longrightarrow \varphi\ (\lambda x \in$ *topspace X*. $a * f\,x) = a * \varphi\ (\lambda x \in$ *topspace X*. $f\,x)$
$\bigwedge f\ g.$ *continuous-map X euclidean f* $\Longrightarrow$ *f has-compact-support-on X*
$\Longrightarrow$ *continuous-map X euclidean g* $\Longrightarrow$ *g has-compact-support-on X*
$\Longrightarrow \varphi\ (\lambda x \in$ *topspace X*. $f\ x + g\ x) = \varphi\ (\lambda x \in$ *topspace X*. $f\ x) + \varphi\ (\lambda x \in$ *topspace X*. $g\ x)$
⟨*proof*⟩

lemma *pos-lin-functional-on-CX-pos-complex*:
assumes *positive-linear-functional-on-CX X* φ
shows *continuous-map X euclidean f* $\Longrightarrow$ *f has-compact-support-on X*
$\Longrightarrow (\bigwedge x.\ x \in$ *topspace X* $\Longrightarrow$ *Re* $(f\,x) \geq 0) \Longrightarrow (\bigwedge x.\ x \in$ *topspace X* $\Longrightarrow f\ x \in \mathbb{R})$
$\Longrightarrow \varphi\ (\lambda x \in$ *topspace X*. $f\,x) \geq 0$
⟨*proof*⟩

lemma *positive-linear-functional-on-CX-compact*:
assumes *compact-space X*
shows *positive-linear-functional-on-CX X* $\varphi \longleftrightarrow$
$(\forall f.$ *continuous-map X euclidean f* $\longrightarrow (\forall x \in$ *topspace X*. $f\,x \geq 0) \longrightarrow \varphi\ (\lambda x \in$ *topspace X*. $f\,x) \geq 0) \wedge$
$(\forall f\ a.$ *continuous-map X euclidean f* $\longrightarrow \varphi\ (\lambda x \in$ *topspace X*. $a * f\,x) = a * \varphi\ (\lambda x \in$ *topspace X*. $f\,x)) \wedge$
$(\forall f\ g.$ *continuous-map X euclidean f* $\longrightarrow$ *continuous-map X euclidean g*
$\longrightarrow \varphi\ (\lambda x \in$ *topspace X*. $f\,x + g\,x) = \varphi\ (\lambda x \in$ *topspace X*. $f\,x) + \varphi\ (\lambda x \in$ *topspace X*. $g\,x))$
⟨*proof*⟩

lemma
assumes *positive-linear-functional-on-CX X* φ *compact-space X*

shows *pos-lin-functional-on-CX-compact-pos*:
$\bigwedge f.$ *continuous-map X euclidean f*
$\Longrightarrow (\bigwedge x.\ x \in topspace\ X \Longrightarrow f\ x \geq 0) \Longrightarrow \varphi\ (\lambda x \in topspace\ X.\ f\ x) \geq 0$
and *pos-lin-functional-on-CX-compact-lin*:
$\bigwedge f\ a.$ *continuous-map X euclidean f*
$\Longrightarrow \varphi\ (\lambda x \in topspace\ X.\ a * f\ x) = a * \varphi\ (\lambda x \in topspace\ X.\ f\ x)$
$\bigwedge f\ g.$ *continuous-map X euclidean f* $\Longrightarrow$ *continuous-map X euclidean g*
$\Longrightarrow \varphi\ (\lambda x \in topspace\ X.\ f\ x + g\ x) = \varphi\ (\lambda x \in topspace\ X.\ f\ x) + \varphi\ (\lambda x \in topspace\ X.\ g\ x)$
⟨*proof*⟩

lemma *pos-lin-functional-on-CX-diff*:
fixes f :: - ⇒ - :: {*real-normed-vector*, *ring-1*}
assumes *positive-linear-functional-on-CX X* φ
and *cont*:*continuous-map X euclidean f continuous-map X euclidean g*
and *csupp*: *f has-compact-support-on X g has-compact-support-on X*
shows $\varphi\ (\lambda x \in topspace\ X.\ f\ x - g\ x) = \varphi\ (\lambda x \in topspace\ X.\ f\ x) - \varphi\ (\lambda x \in topspace\ X.\ g\ x)$
⟨*proof*⟩

lemma *pos-lin-functional-on-CX-compact-diff*:
fixes f :: - ⇒ - :: {*real-normed-vector*, *ring-1*}
assumes *positive-linear-functional-on-CX X* φ *compact-space X*
and *continuous-map X euclidean f continuous-map X euclidean g*
shows $\varphi\ (\lambda x \in topspace\ X.\ f\ x - g\ x) = \varphi\ (\lambda x \in topspace\ X.\ f\ x) - \varphi\ (\lambda x \in topspace\ X.\ g\ x)$
⟨*proof*⟩

lemma *pos-lin-functional-on-CX-mono*:
fixes f :: - ⇒ - :: {*real-normed-vector*, *ring-1*, *ordered-ab-group-add*}
assumes *positive-linear-functional-on-CX X* φ
and *mono*:$\bigwedge x.\ x \in topspace\ X \Longrightarrow f\ x \leq g\ x$
and *cont*:*continuous-map X euclidean f continuous-map X euclidean g*
and *csupp*: *f has-compact-support-on X g has-compact-support-on X*
shows $\varphi\ (\lambda x \in topspace\ X.\ f\ x) \leq \varphi\ (\lambda x \in topspace\ X.\ g\ x)$
⟨*proof*⟩

lemma *pos-lin-functional-on-CX-compact-mono*:
fixes f :: - ⇒ - :: {*real-normed-vector*, *ring-1*, *ordered-ab-group-add*}
assumes *positive-linear-functional-on-CX X* φ *compact-space X*
and $\bigwedge x.\ x \in topspace\ X \Longrightarrow f\ x \leq g\ x$
and *continuous-map X euclidean f continuous-map X euclidean g*
shows $\varphi\ (\lambda x \in topspace\ X.\ f\ x) \leq \varphi\ (\lambda x \in topspace\ X.\ g\ x)$
⟨*proof*⟩

lemma *pos-lin-functional-on-CX-zero*:
assumes *positive-linear-functional-on-CX X* φ
shows $\varphi\ (\lambda x \in topspace\ X.\ 0) = 0$
⟨*proof*⟩

lemma *pos-lin-functional-on-CX-uminus*:
fixes f :: - ⇒ - :: {*real-normed-vector*, *ring-1*}
assumes *positive-linear-functional-on-CX X* φ
and *continuous-map X euclidean f*
and *csupp*: *f has-compact-support-on X*
shows φ (λx∈*topspace X*. − *f x*) = − φ (λx∈*topspace X*. *f x*)
⟨*proof*⟩

lemma *pos-lin-functional-on-CX-compact-uminus*:
fixes f :: - ⇒ - :: {*real-normed-vector*, *ring-1*}
assumes *positive-linear-functional-on-CX X* φ *compact-space X*
and *continuous-map X euclidean f*
shows φ (λx∈*topspace X*. − *f x*) = − φ (λx∈*topspace X*. *f x*)
⟨*proof*⟩

lemma *pos-lin-functional-on-CX-sum*:
fixes f :: - ⇒ - ⇒ - :: {*real-normed-vector*}
assumes *positive-linear-functional-on-CX X* φ
and *finite I* $\bigwedge i.\ i \in I \Longrightarrow$ *continuous-map X euclidean* (*f i*)
and $\bigwedge i.\ i \in I \Longrightarrow$ *f i has-compact-support-on X*
shows φ (λx∈*topspace X*. ($\sum i$∈*I*. *f i x*)) = ($\sum i$∈*I*. φ (λx∈*topspace X*. *f i x*))
⟨*proof*⟩

lemma *pos-lin-functional-on-CX-pos-is-real*:
fixes f :: - ⇒ *complex*
assumes *positive-linear-functional-on-CX X* φ
and *continuous-map X euclidean f f has-compact-support-on X*
and $\bigwedge x.\ x \in$ *topspace X* $\Longrightarrow f\ x \in \mathbb{R}$
shows φ (λx∈*topspace X*. *f x*) $\in \mathbb{R}$
⟨*proof*⟩

lemma
fixes φ *X*
defines $\varphi' \equiv$ (λf. *Re* (φ (λx∈*topspace X*. *complex-of-real* (*f x*))))
assumes *plf*:*positive-linear-functional-on-CX X* φ
shows *pos-lin-functional-on-CX-complex-decompose*:
$\bigwedge f$. *continuous-map X euclidean f* $\Longrightarrow$ *f has-compact-support-on X*
$\Longrightarrow$ φ (λx∈*topspace X*. *f x*)
= *complex-of-real* (φ' (λx∈*topspace X*. *Re* (*f x*))) + i ∗ *complex-of-real* (φ'
(λx∈*topspace X*. *Im* (*f x*)))
and *pos-lin-functional-on-CX-complex-decompose-plf*:
positive-linear-functional-on-CX X φ'
⟨*proof*⟩

3.4 Lemmas for Uniqueness

lemma *rep-measures-real-unique*:
assumes *locally-compact-space X Hausdorff-space X*

assumes N: *subalgebra* N (*borel-of* X)
$\bigwedge f.$ *continuous-map* X *euclideanreal* $f \Longrightarrow f$ *has-compact-support-on* $X \Longrightarrow$ *integrable* $N\, f$
$\bigwedge A.$ $A \in$*sets* $N \Longrightarrow$ *emeasure* N A = ($\bigsqcap C\in\{C.$ *openin* X $C \wedge A \subseteq C\}$. *emeasure* N C)
$\bigwedge A.$ *openin* X $A \Longrightarrow$ *emeasure* N A = ($\bigsqcup K\in\{K.$ *compactin* X $K \wedge K \subseteq A\}$. *emeasure* N K)
$\bigwedge A.$ $A \in$*sets* $N \Longrightarrow$ *emeasure* N $A < \infty \Longrightarrow$ *emeasure* N A = ($\bigsqcup K\in\{K.$ *compactin* X $K \wedge K \subseteq A\}$. *emeasure* N K)
$\bigwedge K.$ *compactin* X $K \Longrightarrow N\ K < \infty$
assumes M: *subalgebra* M (*borel-of* X)
$\bigwedge f.$ *continuous-map* X *euclideanreal* $f \Longrightarrow f$ *has-compact-support-on* $X \Longrightarrow$ *integrable* $M\, f$
$\bigwedge A.$ $A \in$*sets* $M \Longrightarrow$ *emeasure* M A = ($\bigsqcap C\in\{C.$ *openin* X $C \wedge A \subseteq C\}$. *emeasure* M C)
$\bigwedge A.$ *openin* X $A \Longrightarrow$ *emeasure* M A = ($\bigsqcup K\in\{K.$ *compactin* X $K \wedge K \subseteq A\}$. *emeasure* M K)
$\bigwedge A.$ $A \in$*sets* $M \Longrightarrow$ *emeasure* M $A < \infty \Longrightarrow$ *emeasure* M A = ($\bigsqcup K\in\{K.$ *compactin* X $K \wedge K \subseteq A\}$. *emeasure* M K)
$\bigwedge K.$ *compactin* X $K \Longrightarrow M\ K < \infty$
and *sets-eq*: *sets* N = *sets* M
and *integ-eq*: $\bigwedge f.$ *continuous-map* X *euclideanreal* $f \Longrightarrow f$ *has-compact-support-on* $X \Longrightarrow (\int x.\ f\ x\ \partial N) = (\int x.\ f\ x\ \partial M)$
shows $N = M$
⟨*proof*⟩

lemma *rep-measures-complex-unique*:
fixes X :: *'a topology*
assumes *locally-compact-space* X *Hausdorff-space* X
assumes N: *subalgebra* N (*borel-of* X)
$\bigwedge f.$ *continuous-map* X *euclidean* $f \Longrightarrow f$ *has-compact-support-on* $X \Longrightarrow$ *complex-integrable* $N\, f$
$\bigwedge A.$ $A \in$*sets* $N \Longrightarrow$ *emeasure* N A = ($\bigsqcap C\in\{C.$ *openin* X $C \wedge A \subseteq C\}$. *emeasure* N C)
$\bigwedge A.$ *openin* X $A \Longrightarrow$ *emeasure* N A = ($\bigsqcup K\in\{K.$ *compactin* X $K \wedge K \subseteq A\}$. *emeasure* N K)
$\bigwedge A.$ $A \in$*sets* $N \Longrightarrow$ *emeasure* N $A < \infty \Longrightarrow$ *emeasure* N A = ($\bigsqcup K\in\{K.$ *compactin* X $K \wedge K \subseteq A\}$. *emeasure* N K)
$\bigwedge K.$ *compactin* X $K \Longrightarrow N\ K < \infty$
assumes M: *subalgebra* M (*borel-of* X)
$\bigwedge f.$ *continuous-map* X *euclidean* $f \Longrightarrow f$ *has-compact-support-on* $X \Longrightarrow$ *complex-integrable* $M\, f$
$\bigwedge A.$ $A \in$*sets* $M \Longrightarrow$ *emeasure* M A = ($\bigsqcap C\in\{C.$ *openin* X $C \wedge A \subseteq C\}$. *emeasure* M C)
$\bigwedge A.$ *openin* X $A \Longrightarrow$ *emeasure* M A = ($\bigsqcup K\in\{K.$ *compactin* X $K \wedge K \subseteq A\}$. *emeasure* M K)
$\bigwedge A.$ $A \in$*sets* $M \Longrightarrow$ *emeasure* M $A < \infty \Longrightarrow$ *emeasure* M A = ($\bigsqcup K\in\{K.$ *compactin* X $K \wedge K \subseteq A\}$. *emeasure* M K)
$\bigwedge K.$ *compactin* X $K \Longrightarrow M\ K < \infty$

and *sets-eq*: *sets* N = *sets* M
and *integ-eq*: $\bigwedge f$::$'a \Rightarrow$ *complex*. *continuous-map* X *euclidean* $f \Longrightarrow f$ *has-compact-support-on* X
$\Longrightarrow (\int x.\ f\ x\ \partial N) = (\int x.\ f\ x\ \partial M)$
shows $N = M$
⟨*proof*⟩

lemma *finite-tight-measure-eq*:
assumes *locally-compact-space* X *metrizable-space* X *tight-on* X N *tight-on* X M
and *integ-eq*: $\bigwedge f$. *continuous-map* X *euclideanreal* $f \Longrightarrow f \in$ *topspace* $X \rightarrow \{0..1\} \Longrightarrow (\int x.\ f\ x\ \partial N) = (\int x.\ f\ x\ \partial M)$
shows $N = M$
⟨*proof*⟩

3.5 Riesez Representation Theorem for Real Numbers

theorem *Riesz-representation-real-complete*:
fixes X :: $'a$ *topology* **and** φ :: ($'a \Rightarrow$ *real*) $\Rightarrow$ *real*
assumes *lh*:*locally-compact-space* X *Hausdorff-space* X
and *plf*:*positive-linear-functional-on-CX* X φ
shows $\exists M.\ \exists!N.$ *sets* $N = M \wedge$ *subalgebra* N (*borel-of* X)
$\wedge$ ($\forall A \in$*sets* N. *emeasure* N A = ($\bigsqcap C \in \{C.$ *openin* X $C \wedge A \subseteq C\}$. *emeasure* N C))
$\wedge$ ($\forall A$. *openin* X $A \longrightarrow$ *emeasure* N A = ($\bigsqcup K \in \{K.$ *compactin* X $K \wedge K \subseteq A\}$. *emeasure* N K))
$\wedge$ ($\forall A \in$*sets* N. *emeasure* N $A < \infty$
$\longrightarrow$ *emeasure* N A = ($\bigsqcup K \in \{K.$ *compactin* X $K \wedge K \subseteq A\}$. *emeasure* N K))
$\wedge$ ($\forall K$. *compactin* X $K \longrightarrow$ *emeasure* N $K < \infty$)
$\wedge$ ($\forall f$. *continuous-map* X *euclideanreal* $f \longrightarrow f$ *has-compact-support-on* $X \longrightarrow \varphi$ ($\lambda x \in$*topspace* X. $f\ x$) = ($\int x.\ f\ x\ \partial N$))
$\wedge$ ($\forall f$. *continuous-map* X *euclideanreal* $f \longrightarrow f$ *has-compact-support-on* $X \longrightarrow$ *integrable* N f)
$\wedge$ *complete-measure* N
⟨*proof*⟩

3.6 Riesz Representation Theorem for Complex Numbers

theorem *Riesz-representation-complex-complete*:
fixes X :: $'a$ *topology* **and** φ :: ($'a \Rightarrow$ *complex*) $\Rightarrow$ *complex*
assumes *lh*:*locally-compact-space* X *Hausdorff-space* X
and *plf*:*positive-linear-functional-on-CX* X φ
shows $\exists M.\ \exists!N.$ *sets* $N = M \wedge$ *subalgebra* N (*borel-of* X)
$\wedge$ ($\forall A \in$*sets* N. *emeasure* N A = ($\bigsqcap C \in \{C.$ *openin* X $C \wedge A \subseteq C\}$. *emeasure* N C))
$\wedge$ ($\forall A$. *openin* X $A \longrightarrow$ *emeasure* N A = ($\bigsqcup K \in \{K.$ *compactin* X $K \wedge K \subseteq A\}$. *emeasure* N K))
$\wedge$ ($\forall A \in$*sets* N. *emeasure* N $A < \infty \longrightarrow$ *emeasure* N A = ($\bigsqcup K \in \{K.$ *compactin* X $K \wedge K \subseteq A\}$. *emeasure* N K))
$\wedge$ ($\forall K$. *compactin* X $K \longrightarrow$ *emeasure* N $K < \infty$)

$\wedge$ ($\forall f$. *continuous-map X euclidean f* $\longrightarrow$ *f has-compact-support-on X* $\longrightarrow$ φ ($\lambda x\in$*topspace X*. *f x*) = ($\int$ *x*. *f x* ∂N))
$\wedge$ ($\forall f$. *continuous-map X euclidean f* $\longrightarrow$ *f has-compact-support-on X* $\longrightarrow$ *complex-integrable N f*)
$\wedge$ *complete-measure N*
⟨*proof*⟩

3.7 Other Forms of the Theorem

In the case when the representation measure is on X.

theorem *Riesz-representation-real*:
assumes *lh*:*locally-compact-space X Hausdorff-space X*
and *positive-linear-functional-on-CX X* φ
shows $\exists!N$. *sets N* = *sets* (*borel-of X*)
$\wedge$ ($\forall A\in$*sets N*. *emeasure N A* = ($\bigsqcap C\in\{C$. *openin X C* $\wedge$ $A \subseteq C\}$. *emeasure N C*))
$\wedge$ ($\forall A$. *openin X A* $\longrightarrow$ *emeasure N A* = ($\bigsqcup K\in\{K$. *compactin X K* $\wedge$ $K \subseteq A\}$. *emeasure N K*))
$\wedge$ ($\forall A\in$*sets N*. *emeasure N A* $< \infty$ $\longrightarrow$ *emeasure N A* = ($\bigsqcup K\in\{K$. *compactin X K* $\wedge$ $K \subseteq A\}$. *emeasure N K*))
$\wedge$ ($\forall K$. *compactin X K* $\longrightarrow$ *emeasure N K* $< \infty$)
$\wedge$ ($\forall f$. *continuous-map X euclideanreal f* $\longrightarrow$ *f has-compact-support-on X* $\longrightarrow$ φ ($\lambda x\in$*topspace X*. *f x*) = ($\int$ *x*. *f x* ∂N))
$\wedge$ ($\forall f$. *continuous-map X euclideanreal f* $\longrightarrow$ *f has-compact-support-on X* $\longrightarrow$ *integrable N f*)
⟨*proof*⟩

theorem *Riesz-representation-complex*:
fixes X :: *'a topology* **and** φ :: (*'a* $\Rightarrow$ *complex*) $\Rightarrow$ *complex*
assumes *lh*:*locally-compact-space X Hausdorff-space X*
and *positive-linear-functional-on-CX X* φ
shows $\exists!N$. *sets N* = *sets* (*borel-of X*)
$\wedge$ ($\forall A\in$*sets N*. *emeasure N A* = ($\bigsqcap C\in\{C$. *openin X C* $\wedge$ $A \subseteq C\}$. *emeasure N C*))
$\wedge$ ($\forall A$. *openin X A* $\longrightarrow$ *emeasure N A* = ($\bigsqcup K\in\{K$. *compactin X K* $\wedge$ $K \subseteq A\}$. *emeasure N K*))
$\wedge$ ($\forall A\in$*sets N*. *emeasure N A* $< \infty$ $\longrightarrow$ *emeasure N A* = ($\bigsqcup K\in\{K$. *compactin X K* $\wedge$ $K \subseteq A\}$. *emeasure N K*))
$\wedge$ ($\forall K$. *compactin X K* $\longrightarrow$ *emeasure N K* $< \infty$)
$\wedge$ ($\forall f$. *continuous-map X euclidean f* $\longrightarrow$ *f has-compact-support-on X* $\longrightarrow$ φ ($\lambda x\in$*topspace X*. *f x*) = ($\int$ *x*. *f x* ∂N))
$\wedge$ ($\forall f$. *continuous-map X euclidean f* $\longrightarrow$ *f has-compact-support-on X* $\longrightarrow$ *complex-integrable N f*)
⟨*proof*⟩

3.7.1 Theorem for Compact Hausdorff Spaces

theorem *Riesz-representation-real-compact-Hausdorff*:
fixes X :: *'a topology* **and** φ :: (*'a* $\Rightarrow$ *real*) $\Rightarrow$ *real*

assumes *lh*:*compact-space X Hausdorff-space X*
and *positive-linear-functional-on-CX X φ*
shows ∃!*N*. *sets N* = *sets* (*borel-of X*) ∧ *finite-measure N*
∧ (∀ *A*∈*sets N*. *emeasure N A* = (⨅ *C*∈{*C*. *openin X C* ∧ *A* ⊆ *C*}. *emeasure N C*))
∧ (∀ *A*. *openin X A* ⟶ *emeasure N A* = (⨆ *K*∈{*K*. *compactin X K* ∧ *K* ⊆ *A*}. *emeasure N K*))
∧ (∀ *A*∈*sets N*. *emeasure N A* < ∞ ⟶ *emeasure N A* = (⨆ *K*∈{*K*. *compactin X K* ∧ *K* ⊆ *A*}. *emeasure N K*))
∧ (∀ *K*. *compactin X K* ⟶ *emeasure N K* < ∞)
∧ (∀ *f*. *continuous-map X euclideanreal f* ⟶ φ (λ*x*∈*topspace X*. *f x*) = (∫ *x*. *f x* ∂*N*))
∧ (∀ *f*. *continuous-map X euclideanreal f* ⟶ *integrable N f*)
⟨*proof*⟩

theorem *Riesz-representation-complex-compact-Hausdorff*:
fixes *X* :: *'a topology* **and** φ :: (*'a* ⇒ *complex*) ⇒ *complex*
assumes *lh*:*compact-space X Hausdorff-space X*
and *positive-linear-functional-on-CX X φ*
shows ∃!*N*. *sets N* = *sets* (*borel-of X*) ∧ *finite-measure N*
∧ (∀ *A*∈*sets N*. *emeasure N A* = (⨅ *C*∈{*C*. *openin X C* ∧ *A* ⊆ *C*}. *emeasure N C*))
∧ (∀ *A*. *openin X A* ⟶ *emeasure N A* = (⨆ *K*∈{*K*. *compactin X K* ∧ *K* ⊆ *A*}. *emeasure N K*))
∧ (∀ *A*∈*sets N*. *emeasure N A* < ∞ ⟶ *emeasure N A* = (⨆ *K*∈{*K*. *compactin X K* ∧ *K* ⊆ *A*}. *emeasure N K*))
∧ (∀ *K*. *compactin X K* ⟶ *emeasure N K* < ∞)
∧ (∀ *f*. *continuous-map X euclidean f* ⟶ φ (λ*x*∈*topspace X*. *f x*) = (∫ *x*. *f x* ∂*N*))
∧ (∀ *f*. *continuous-map X euclidean f* ⟶ *complex-integrable N f*)
⟨*proof*⟩

3.7.2 Theorem for Compact Metrizable Spaces

theorem *Riesz-representation-real-compact-metrizable*:
fixes *X* :: *'a topology* **and** φ :: (*'a* ⇒ *real*) ⇒ *real*
assumes *lh*:*compact-space X metrizable-space X*
and *plf*:*positive-linear-functional-on-CX X φ*
shows ∃!*N*. *sets N* = *sets* (*borel-of X*) ∧ *finite-measure N*
∧ (∀ *f*. *continuous-map X euclideanreal f* ⟶ φ (λ*x*∈*topspace X*. *f x*) = (∫ *x*. *f x* ∂*N*))
⟨*proof*⟩

theorem *Riesz-representation-real-compact-metrizable-le1*:
fixes *X* :: *'a topology* **and** φ :: (*'a* ⇒ *real*) ⇒ *real*
assumes *lh*:*compact-space X metrizable-space X*
and *plf*:*positive-linear-functional-on-CX X φ*
shows ∃!*N*. *sets N* = *sets* (*borel-of X*) ∧ *finite-measure N*
∧ (∀ *f*. *continuous-map X euclideanreal f* ⟶ *f* ∈ *topspace X* → {*0..1*}

⟶ φ (λx∈*topspace* X. f x) = (∫ x. f x ∂N))
⟨*proof*⟩

theorem *Riesz-representation-complex-compact-metrizable*:
fixes X :: 'a *topology* **and** φ :: ('a ⇒ *complex*) ⇒ *complex*
assumes *lh*:*compact-space* X *metrizable-space* X
and *plf*:*positive-linear-functional-on-CX* X φ
shows ∃!N. *sets* N = *sets* (*borel-of* X) ∧ *finite-measure* N
∧ (∀ f. *continuous-map* X *euclidean* f ⟶ φ (λx∈*topspace* X. f x) = (∫ x. f x ∂N))
⟨*proof*⟩

theorem *Riesz-representation-real-compact-metrizable-subprob*:
fixes X :: 'a *topology* **and** φ :: ('a ⇒ *real*) ⇒ *real*
assumes *lh*:*compact-space* X *metrizable-space* X
and *plf*:*positive-linear-functional-on-CX* X φ
and *le1*: φ (λx∈*topspace* X. 1) ≤ 1 **and** *ne*: X ≠ *trivial-topology*
shows ∃!N. *sets* N = *sets* (*borel-of* X) ∧ *subprob-space* N
∧ (∀ f. *continuous-map* X *euclideanreal* f ⟶ φ (λx∈*topspace* X. f x) = (∫ x. f x ∂N))
⟨*proof*⟩

theorem *Riesz-representation-real-compact-metrizable-subprob-le1*:
fixes X :: 'a *topology* **and** φ :: ('a ⇒ *real*) ⇒ *real*
assumes *lh*:*compact-space* X *metrizable-space* X
and *plf*:*positive-linear-functional-on-CX* X φ
and *le1*: φ (λx∈*topspace* X. 1) ≤ 1 **and** *ne*: X ≠ *trivial-topology*
shows ∃!N. *sets* N = *sets* (*borel-of* X) ∧ *subprob-space* N
∧ (∀ f. *continuous-map* X *euclideanreal* f ⟶ f ∈ *topspace* X → {0..1} ⟶ φ (λx∈*topspace* X. f x) = (∫ x. f x ∂N))
⟨*proof*⟩

theorem *Riesz-representation-real-compact-metrizable-prob*:
fixes X :: 'a *topology* **and** φ :: ('a ⇒ *real*) ⇒ *real*
assumes *lh*:*compact-space* X *metrizable-space* X
and *plf*:*positive-linear-functional-on-CX* X φ
and φ (λx∈*topspace* X. 1) = 1
shows ∃!N. *sets* N = *sets* (*borel-of* X) ∧ *prob-space* N
∧ (∀ f. *continuous-map* X *euclideanreal* f ⟶ φ (λx∈*topspace* X. f x) = (∫ x. f x ∂N))
⟨*proof*⟩

theorem *Riesz-representation-complex-compact-metrizable-subprob*:
fixes X :: 'a *topology* **and** φ :: ('a ⇒ *complex*) ⇒ *complex*
assumes *lh*:*compact-space* X *metrizable-space* X
and *plf*:*positive-linear-functional-on-CX* X φ
and *le1*: *Re* (φ (λx∈*topspace* X. 1)) ≤ 1 **and** *ne*: X ≠ *trivial-topology*
shows ∃!N. *sets* N = *sets* (*borel-of* X) ∧ *subprob-space* N
∧ (∀ f. *continuous-map* X *euclidean* f ⟶ φ (λx∈*topspace* X. f x) = (∫ x.

f x ∂*N*))
⟨*proof*⟩

theorem *Riesz-representation-complex-compact-metrizable-prob*:
fixes *X* :: *'a topology* **and** *φ* :: (*'a* ⇒ *complex*) ⇒ *complex*
assumes *lh*:*compact-space X metrizable-space X*
and *plf*:*positive-linear-functional-on-CX X φ*
and *Re* (*φ* (*λx*∈*topspace X*. *1*)) = *1*
shows ∃!*N*. *sets N* = *sets* (*borel-of X*) ∧ *prob-space N*
∧ (∀*f*. *continuous-map X euclidean f* ⟶ *φ* (*λx*∈*topspace X*. *f x*) = (∫ *x*. *f x* ∂*N*))
⟨*proof*⟩

end

References

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