Riesz Representation Theorem

Michikazu Hirata

February 6, 2026

Abstract

We formalize the Riesz-Markov-Kakutani representation theorem following pp.37-47 of the book *Real and Complex Analysis* by Rudin [1]. This entry also includes formalization of regular measures, tightness of measures, and Urysohn's lemma on locally compact Hausdorff spaces. Roughly speaking, the theorem states that if φ is a positive linear functional from $C(X)$ (the space of continuous functions from X to complex numbers which have compact supports) to complex numbers, then there exists a unique measure μ such that for all $f \in C(X)$,

$$\varphi(f) = \int f \mathrm{d}\mu.$$

Contents

1 Urysohn's Lemma **2**
- 1.1 Lemmas for Upper/Lower Semi-Continuous Functions 2
- 1.2 Urysohn's Lemma 4

2 Regular Measures **4**

3 The Riesz Representation Theorem **8**
- 3.1 Lemmas for Complex-Valued Continuous Maps 8
- 3.2 Compact Supports 9
- 3.3 Positive Linear Functionsls 11
- 3.4 Lemmas for Uniqueness 13
- 3.5 Riesez Representation Theorem for Real Numbers 15
- 3.6 Riesz Representation Theorem for Complex Numbers 15
- 3.7 Other Forms of the Theorem 16
  - 3.7.1 Theorem for Compact Hausdorff Spaces 16
  - 3.7.2 Theorem for Compact Metrizable Spaces 17

1 Urysohn's Lemma

theory *Urysohn-Locally-Compact-Hausdorff*
imports *Standard-Borel-Spaces.StandardBorel*
begin

We prove Urysohn's lemma for locally compact Hausdorff space (Lemma 2.12 [1])

1.1 Lemmas for Upper/Lower Semi-Continuous Functions

lemma
assumes $\bigwedge x.\ x \in \mathit{topspace}\ X \Longrightarrow f\ x = g\ x$
shows *upper-semicontinuous-map-cong*:
$\mathit{upper\text{-}semicontinuous\text{-}map}\ X\ f \longleftrightarrow \mathit{upper\text{-}semicontinuous\text{-}map}\ X\ g$ (**is** *?g1*)
and *lower-semicontinuous-map-cong*:
$\mathit{lower\text{-}semicontinuous\text{-}map}\ X\ f \longleftrightarrow \mathit{lower\text{-}semicontinuous\text{-}map}\ X\ g$ (**is** *?g2*)
⟨*proof*⟩

lemma *upper-lower-semicontinuous-map-iff-continuous-map*:
$\mathit{continuous\text{-}map}\ X\ \mathit{euclidean}\ f \longleftrightarrow \mathit{upper\text{-}semicontinuous\text{-}map}\ X\ f \wedge \mathit{lower\text{-}semicontinuous\text{-}map}\ X\ f$
⟨*proof*⟩

lemma [*simp*]:
shows *upper-semicontinuous-map-const*: $\mathit{upper\text{-}semicontinuous\text{-}map}\ X\ (\lambda x.\ c)$
and *lower-semicontinuous-map-const*: $\mathit{lower\text{-}semicontinuous\text{-}map}\ X\ (\lambda x.\ c)$
⟨*proof*⟩

lemma *upper-semicontinuous-map-c-add-iff*:
fixes $c :: \mathit{real}$
shows $\mathit{upper\text{-}semicontinuous\text{-}map}\ X\ (\lambda x.\ c + f\ x) \longleftrightarrow \mathit{upper\text{-}semicontinuous\text{-}map}\ X\ f$
⟨*proof*⟩

corollary *upper-semicontinuous-map-add-c-iff*:
fixes $c :: \mathit{real}$
shows $\mathit{upper\text{-}semicontinuous\text{-}map}\ X\ (\lambda x.\ f\ x + c) \longleftrightarrow \mathit{upper\text{-}semicontinuous\text{-}map}\ X\ f$
⟨*proof*⟩

lemma *upper-semicontinuous-map-posreal-cmult-iff*:
fixes $c :: \mathit{real}$
assumes $c > 0$
shows $\mathit{upper\text{-}semicontinuous\text{-}map}\ X\ (\lambda x.\ c * f\ x) \longleftrightarrow \mathit{upper\text{-}semicontinuous\text{-}map}\ X\ f$
⟨*proof*⟩

lemma *upper-semicontinuous-map-real-cmult*:
fixes c :: *real*
assumes $c \geq 0$ *upper-semicontinuous-map X f*
shows *upper-semicontinuous-map X* $(\lambda x.\ c * f\ x)$
⟨*proof*⟩

lemma *lower-semicontinuous-map-posreal-cmult-iff*:
fixes c :: *real*
assumes $c > 0$
shows *lower-semicontinuous-map X* $(\lambda x.\ c * f\ x) \longleftrightarrow$ *lower-semicontinuous-map X f*
⟨*proof*⟩

lemma *lower-semicontinuous-map-real-cmult*:
fixes c :: *real*
assumes $c \geq 0$ *lower-semicontinuous-map X f*
shows *lower-semicontinuous-map X* $(\lambda x.\ c * f\ x)$
⟨*proof*⟩

lemma *upper-semicontinuous-map-INF*:
fixes f :: - ⇒ - ⇒ $'a$ :: {*linorder-topology*, *complete-linorder*}
assumes $\bigwedge i.\ i \in I \Longrightarrow$ *upper-semicontinuous-map X* $(f\ i)$
shows *upper-semicontinuous-map X* $(\lambda x.\ \prod i{\in}I.\ f\ i\ x)$
⟨*proof*⟩

lemma *upper-semicontinuous-map-cInf*:
fixes f :: - ⇒ - ⇒ $'a$ :: {*linorder-topology*, *conditionally-complete-linorder*}
assumes $I \neq \{\}$ $\bigwedge x.\ x \in$ *topspace X* $\Longrightarrow$ *bdd-below* $((\lambda i.\ f\ i\ x)\ `\ I)$
and $\bigwedge i.\ i \in I \Longrightarrow$ *upper-semicontinuous-map X* $(f\ i)$
shows *upper-semicontinuous-map X* $(\lambda x.\ \prod i{\in}I.\ f\ i\ x)$
⟨*proof*⟩

lemma *lower-semicontinuous-map-Sup*:
fixes f :: - ⇒ - ⇒ $'a$ :: {*linorder-topology*, *complete-linorder*}
assumes $\bigwedge i.\ i \in I \Longrightarrow$ *lower-semicontinuous-map X* $(f\ i)$
shows *lower-semicontinuous-map X* $(\lambda x.\ \bigsqcup i{\in}I.\ f\ i\ x)$
⟨*proof*⟩

lemma *indicator-closed-upper-semicontinuous-map*:
assumes *closedin X C*
shows *upper-semicontinuous-map X* (*indicator C* :: - ⇒ $'a$ :: {*zero-less-one*, *linorder-topology*})
⟨*proof*⟩

lemma *indicator-open-lower-semicontinuous-map*:
assumes *openin X U*
shows *lower-semicontinuous-map X* (*indicator U* :: - ⇒ $'a$ :: {*zero-less-one*,

linorder-topology})
⟨*proof*⟩

lemma *lower-semicontinuous-map-cSup*:
fixes *f* :: - ⇒ - ⇒ *'a* :: {*linorder-topology*, *conditionally-complete-linorder*}
assumes $I \neq \{\}$ $\bigwedge x.\ x \in$ *topspace X* $\Longrightarrow$ *bdd-above* $((\lambda i.\ f\ i\ x)\ `\ I)$
and $\bigwedge i.\ i \in I \Longrightarrow$ *lower-semicontinuous-map X* (*f i*)
shows *lower-semicontinuous-map X* $(\lambda x.\ \bigsqcup i{\in}I.\ f\ i\ x)$
⟨*proof*⟩

lemma *openin-continuous-map-less*:
assumes *continuous-map X* (*euclidean* :: (*'a* :: {*dense-linorder*, *order-topology*}) *topology*) *f*
and *continuous-map X euclidean g*
shows *openin X* {*x*∈*topspace X*. *f x* < *g x*}
⟨*proof*⟩

corollary *closedin-continuous-map-eq*:
assumes *continuous-map X* (*euclidean* :: (*'a* :: {*dense-linorder*, *order-topology*}) *topology*) *f*
and *continuous-map X euclidean g*
shows *closedin X* {*x*∈*topspace X*. *f x* = *g x*}
⟨*proof*⟩

1.2 Urysohn's Lemma

lemma *locally-compact-Hausdorff-compactin-openin-subset*:
assumes *locally-compact-space X Hausdorff-space X* ∨ *regular-space X*
and *compactin X T openin X V T* ⊆ *V*
shows ∃ *U*. *openin X U* ∧ *compactin X* (*X closure-of U*) ∧ *T* ⊆ *U* ∧ (*X closure-of U*) ⊆ *V*
⟨*proof*⟩

lemma *Urysohn-locally-compact-Hausdorff-closed-compact-support*:
fixes *a b* :: *real* **and** *X* :: *'a topology*
assumes *locally-compact-space X Hausdorff-space X* ∨ *regular-space X*
and *a* ≤ *b closedin X S compactin X T disjnt S T*
obtains *f* **where** *continuous-map X* (*subtopology euclidean* {*a*..*b*}) *f f* ` *S* ⊆ {*a*} *f* ` *T* ⊆ {*b*} *disjnt* (*X closure-of* {*x*∈*topspace X*. *f x* ≠ *a*}) *S compactin X* (*X closure-of* {*x*∈*topspace X*. *f x* ≠ *a*})
⟨*proof*⟩

end

2 Regular Measures

theory *Regular-Measure*
imports *HOL−Probability.Probability*
Standard-Borel-Spaces.StandardBorel

Urysohn-Locally-Compact-Hausdorff
begin

context *Metric-space*
begin

lemma *nbh-add*: $(\bigcup b{\in}(\bigcup a{\in}A.\ mball\ a\ e).\ mball\ b\ f) \subseteq (\bigcup a{\in}A.\ mball\ a\ (e + f))$
⟨*proof*⟩

lemma *nbh-subset*:
assumes *A*: $A \subseteq M$ **and** *e*: $e > 0$
shows $A \subseteq (\bigcup a{\in}A.\ mball\ a\ e)$
⟨*proof*⟩

lemma *nbh-decseq*:
assumes *decseq an*
shows *decseq* $(\lambda n.\ \bigcup a{\in}A.\ mball\ a\ (an\ n))$
⟨*proof*⟩

lemma *nbh-Inter-closure-of*:
assumes *A*: $A \neq \{\}$ $A \subseteq M$
and *an*:$\bigwedge n.\ an\ n > 0\ decseq\ an\ an \longrightarrow 0$
shows $(\bigcap n.\ \bigcup a{\in}A.\ mball\ a\ (an\ n)) = mtopology\ closure\text{-}of\ A$
⟨*proof*⟩

end

lemma(**in** *finite-measure*)
assumes *range* $A \subseteq sets\ M$ *disjoint-family A*
shows *suminf-measure*:$(\sum i.\ measure\ M\ (A\ i)) = measure\ M\ (\bigcup i.\ A\ i)$
and *summable-measure*: *summable* $(\lambda i.\ measure\ M\ (A\ i))$
⟨*proof*⟩

We refer to the lecture note [2].

Inner regular and outer regular with abstract topologies.

definition *inner-regular* :: *'a topology* ⇒ *'a measure* ⇒ *bool* **where**
inner-regular X M ⟷ *sets M* = *sets* (*borel-of X*) ∧ (∀ *A*∈*sets M*. *M A* = ($\bigsqcup C{\in}\{C.$ *closedin X C* ∧ *C* ⊆ *A*}. *M C*))

definition *outer-regular* :: *'a topology* ⇒ *'a measure* ⇒ *bool* **where**
outer-regular X M ⟷ *sets M* = *sets* (*borel-of X*) ∧ (∀ *A*∈*sets M*. *M A* = ($\prod C{\in}\{C.$ *openin X C* ∧ *A* ⊆ *C*}. *M C*))

definition *regular-measure* :: *'a topology* ⇒ *'a measure* ⇒ *bool* **where**
regular-measure X M ⟷ *inner-regular X M* ∧ *outer-regular X M*

lemma
shows *inner-reguarI*: *sets M* = *sets* (*borel-of X*) ⟹ ($\bigwedge A.\ A \in sets\ M$

$\Longrightarrow M\ A = (\bigsqcup C{\in}\{C.\ closedin\ X\ C \wedge C \subseteq A\}.\ M\ C)) \Longrightarrow inner\text{-}regular\ X\ M$

and *inner-reguarD*: $inner\text{-}regular\ X\ M \Longrightarrow sets\ M = sets\ (borel\text{-}of\ X)$
$inner\text{-}regular\ X\ M \Longrightarrow A \in sets\ M \Longrightarrow M\ A = (\bigsqcup C{\in}\{C.\ closedin\ X\ C \wedge C \subseteq A\}.\ M\ C)$
⟨*proof*⟩

lemma
shows *outer-reguarI*: $sets\ M = sets\ (borel\text{-}of\ X)$
$\Longrightarrow (\bigwedge A.\ A \in sets\ M \Longrightarrow M\ A = (\bigsqcap C{\in}\{C.\ openin\ X\ C \wedge A \subseteq C\}.\ M\ C)) \Longrightarrow outer\text{-}regular\ X\ M$
and *outer-reguarD*: $outer\text{-}regular\ X\ M \Longrightarrow sets\ M = sets\ (borel\text{-}of\ X)$
$outer\text{-}regular\ X\ M \Longrightarrow A \in sets\ M \Longrightarrow M\ A = (\bigsqcap C{\in}\{C.\ openin\ X\ C \wedge A \subseteq C\}.\ M\ C)$
⟨*proof*⟩

lemma
shows *regular-measureI*: $inner\text{-}regular\ X\ M \Longrightarrow outer\text{-}regular\ X\ M \Longrightarrow regular\text{-}measure\ X\ M$
and *regular-measureD*:
$regular\text{-}measure\ X\ M \Longrightarrow inner\text{-}regular\ X\ M\ regular\text{-}measure\ X\ M \Longrightarrow outer\text{-}regular\ X\ M$
⟨*proof*⟩

lemma *inner-regular-finite-measure*:
assumes *finite-measure M*
shows $inner\text{-}regular\ X\ M \longleftrightarrow$
$sets\ M = sets\ (borel\text{-}of\ X) \wedge (\forall A{\in}sets\ M.\ measure\ M\ A = (\bigsqcup C{\in}\{C.\ closedin\ X\ C \wedge C \subseteq A\}.\ measure\ M\ C))$
⟨*proof*⟩

lemma(**in** *finite-measure*)
shows $inner\text{-}regularI: sets\ M = sets\ (borel\text{-}of\ X) \Longrightarrow (\bigwedge A.\ A \in sets\ M \Longrightarrow measure\ M\ A = (\bigsqcup C{\in}\{C.\ closedin\ X\ C \wedge C \subseteq A\}.\ measure\ M\ C)) \Longrightarrow inner\text{-}regular\ X\ M$
and *inner-regularD*:
$inner\text{-}regular\ X\ M \Longrightarrow A \in sets\ M \Longrightarrow measure\ M\ A = (\bigsqcup C{\in}\{C.\ closedin\ X\ C \wedge C \subseteq A\}.\ measure\ M\ C)$
⟨*proof*⟩

lemma *outer-regular-finite-measure*:
assumes *finite-measure M*
shows $outer\text{-}regular\ X\ M \longleftrightarrow sets\ M = sets\ (borel\text{-}of\ X) \wedge (\forall A{\in}sets\ M.\ measure\ M\ A = (\bigsqcap C{\in}\{C.\ openin\ X\ C \wedge A \subseteq C\}.\ measure\ M\ C))$
⟨*proof*⟩

lemma(**in** *finite-measure*)
shows $outer\text{-}regularI: sets\ M = sets\ (borel\text{-}of\ X) \Longrightarrow (\bigwedge A.\ A \in sets\ M \Longrightarrow measure\ M\ A = (\bigsqcap C{\in}\{C.\ openin\ X\ C \wedge A \subseteq C\}.\ measure\ M\ C)) \Longrightarrow$

outer-regular X M
and *outer-regularD*: *outer-regular X M* $\Longrightarrow$ *A* $\in$ *sets M*
$\Longrightarrow$ *measure M A* = ($\bigsqcap$ *C*$\in${*C*. *openin X C* $\wedge$ *A* $\subseteq$ *C*}. *measure M C*)
⟨*proof*⟩

Abstract version of ⟦*sets ?M* = *sets borel*; *emeasure ?M* (*space ?M*) $\neq \infty$; *?B* $\in$ *sets borel*⟧ $\Longrightarrow$ *emeasure ?M ?B* = $\bigsqcup$ (*emeasure ?M* ‘ {*K*. *K* $\subseteq$ *?B* $\wedge$ *compact K*}) and ⟦*sets ?M* = *sets borel*; *emeasure ?M* (*space ?M*) $\neq \infty$; *?B* $\in$ *sets borel*⟧ $\Longrightarrow$ *emeasure ?M ?B* = $\bigsqcap$ (*emeasure ?M* ‘ {*U*. *?B* $\subseteq$ *U* $\wedge$ *open U*}).

lemma(**in** *finite-measure*)
assumes *metrizable-space X sets* (*borel-of X*) = *sets M*
shows *inner-regular′*:*inner-regular X M*
and *outer-regular′*:*outer-regular X M*
⟨*proof*⟩

definition *tight-on-set* :: *′a topology* $\Rightarrow$ *′a measure set* $\Rightarrow$ *bool* **where**
tight-on-set X Γ $\longleftrightarrow$ ($\forall$ *M*$\in$Γ. *finite-measure M* $\wedge$ *sets* (*borel-of X*) = *sets M*) $\wedge$
($\forall$ *e*>*0*. $\exists$ *K*. *compactin X K* $\wedge$ ($\forall$ *M*$\in$Γ. *measure M* (*space M* − *K*) < *e*))

abbreviation *tight-on* :: *′a topology* $\Rightarrow$ *′a measure* $\Rightarrow$ *bool* **where**
tight-on X M $\equiv$ *tight-on-set X* {*M*}

lemma *tight-on-def*:
tight-on X M $\longleftrightarrow$ *finite-measure M* $\wedge$ *sets* (*borel-of X*) = *sets M* $\wedge$
($\forall$ *e*>*0*. $\exists$ *K*. *compactin X K* $\wedge$ *measure M* (*space M* − *K*) < *e*)
⟨*proof*⟩

lemma *tight-on-set-subset*: *A* $\subseteq$ *B* $\Longrightarrow$ *tight-on-set X B* $\Longrightarrow$ *tight-on-set X A*
⟨*proof*⟩

lemma *tight-on-tight*: *tight-on-set euclidean* (*Mi* ‘ *UNIV*) $\wedge$ ($\forall$ *i*. *real-distribution* (*Mi i*)) $\longleftrightarrow$ *tight Mi*
⟨*proof*⟩

lemma *inner-regular′′*:
assumes *metrizable-space X tight-on X M*
and [*measurable*]:*A* $\in$ *sets M*
shows *measure M A* = ($\bigsqcup$ *K*$\in${*K*. *compactin X K* $\wedge$ *K* $\subseteq$ *A*}. *measure M K*)
(**is** - = *?rhs*)
⟨*proof*⟩

lemma(**in** *finite-measure*) *tight-on-compact-space*:
assumes *metrizable-space X compact-space X sets* (*borel-of X*) = *sets M*
shows *tight-on X M*
⟨*proof*⟩

lemma(**in** *finite-measure*) *tight-on-finite-space*:

assumes *metrizable-space X sets* (*borel-of X*) = *sets M finite* (*space M*)
shows *tight-on X M*
⟨*proof*⟩

lemma(**in** *finite-measure*) *tight-on-Polish*:
assumes *Polish-space X sets* (*borel-of X*) = *sets M*
shows *tight-on X M*
⟨*proof*⟩

corollary(**in** *finite-measure*) *inner-regular-Polish*:
assumes *Polish-space X sets* (*borel-of X*) = *sets M A* ∈ *sets M*
shows *measure M A* = (⨆ *K*∈{*K*. *compactin X K* ∧ *K* ⊆ *A*}. *measure M K*)
⟨*proof*⟩

end

3 The Riesz Representation Theorem

theory *Riesz-Representation*
imports *Regular-Measure*
Urysohn-Locally-Compact-Hausdorff
begin

3.1 Lemmas for Complex-Valued Continuous Maps

lemma *continuous-map-Re'*[*simp*,*continuous-intros*]: *continuous-map euclidean euclideanreal Re*
and *continuous-map-Im'*[*simp*,*continuous-intros*]: *continuous-map euclidean euclideanreal Im*
and *continuous-map-complex-of-real'*[*simp*,*continuous-intros*]: *continuous-map euclideanreal euclidean complex-of-real*
⟨*proof*⟩

corollary
assumes *continuous-map X euclidean f*
shows *continuous-map-Re*[*simp*,*continuous-intros*]: *continuous-map X euclideanreal* (λ*x*. *Re* (*f x*))
and *continuous-map-Im*[*simp*,*continuous-intros*]: *continuous-map X euclideanreal* (λ*x*. *Im* (*f x*))
⟨*proof*⟩

lemma *continuous-map-of-real-iff*[*simp*]:
continuous-map X euclidean (λ*x*. *of-real* (*f x*) :: - :: *real-normed-div-algebra*) ⟷ *continuous-map X euclideanreal f*
⟨*proof*⟩

lemma *continuous-map-complex-mult* [*continuous-intros*]:
fixes *f* :: *'a* ⇒ *complex*

shows $[\![$*continuous-map X euclidean f*; *continuous-map X euclidean g*$]\!] \Longrightarrow$ *continuous-map X euclidean* $(\lambda x.\ f\ x * g\ x)$
⟨*proof*⟩

lemma *continuous-map-complex-mult-left*:
fixes $f :: 'a \Rightarrow$ *complex*
shows *continuous-map X euclidean f* $\Longrightarrow$ *continuous-map X euclidean* $(\lambda x.\ c * f\ x)$
⟨*proof*⟩

lemma *complex-continuous-map-iff*:
continuous-map X euclidean f $\longleftrightarrow$ *continuous-map X euclideanreal* $(\lambda x.\ Re\ (f\ x)) \wedge$ *continuous-map X euclideanreal* $(\lambda x.\ Im\ (f\ x))$
⟨*proof*⟩

lemma *complex-integrable-iff*: *complex-integrable M f* $\longleftrightarrow$ *integrable M* $(\lambda x.\ Re\ (f\ x)) \wedge$ *integrable M* $(\lambda x.\ Im\ (f\ x))$
⟨*proof*⟩

3.2 Compact Supports

definition *has-compact-support-on* :: $('a \Rightarrow 'b ::$ *monoid-add*$) \Rightarrow 'a$ *topology* $\Rightarrow$ *bool*
(**infix** ‹*has'-compact'-support'-on*› *60*) **where**
f has-compact-support-on X $\longleftrightarrow$ *compactin X* (*X closure-of support-on* (*topspace X*) *f*)

lemma *has-compact-support-on-iff*:
f has-compact-support-on X $\longleftrightarrow$ *compactin X* (*X closure-of* $\{x \in$ *topspace X*. *f x* $\neq 0\}$)
⟨*proof*⟩

lemma *has-compact-support-on-zero*[*simp*]: $(\lambda x.\ 0)$ *has-compact-support-on X*
⟨*proof*⟩

lemma *has-compact-support-on-compact-space*[*simp*]: *compact-space X* $\Longrightarrow$ *f has-compact-support-on X*
⟨*proof*⟩

lemma *has-compact-support-on-add*[*simp*,*intro*!]:
assumes *f has-compact-support-on X g has-compact-support-on X*
shows $(\lambda x.\ f\ x + g\ x)$ *has-compact-support-on X*
⟨*proof*⟩

lemma *has-compact-support-on-sum*:
assumes *finite I* $\bigwedge i.\ i \in I \Longrightarrow$ *f i has-compact-support-on X*
shows $(\lambda x.\ (\sum i \in I.\ f\ i\ x))$ *has-compact-support-on X*
⟨*proof*⟩

lemma *has-compact-support-on-mult-left*:
fixes g :: - $\Rightarrow$ - :: *mult-zero*
assumes g *has-compact-support-on* X
shows $(\lambda x.\ f\ x * g\ x)$ *has-compact-support-on* X
⟨*proof*⟩

lemma *has-compact-support-on-mult-right*:
fixes f :: - $\Rightarrow$ - :: *mult-zero*
assumes f *has-compact-support-on* X
shows $(\lambda x.\ f\ x * g\ x)$ *has-compact-support-on* X
⟨*proof*⟩

lemma *has-compact-support-on-uminus-iff*[*simp*]:
fixes f :: - $\Rightarrow$ - :: *group-add*
shows $(\lambda x.\ -f\ x)$ *has-compact-support-on* $X \longleftrightarrow f$ *has-compact-support-on* X
⟨*proof*⟩

lemma *has-compact-support-on-diff*[*simp*,*intro*!]:
fixes f :: - $\Rightarrow$ - :: *group-add*
shows f *has-compact-support-on* $X \Longrightarrow g$ *has-compact-support-on* X
$\Longrightarrow (\lambda x.\ f\ x - g\ x)$ *has-compact-support-on* X
⟨*proof*⟩

lemma *has-compact-support-on-max*[*simp*,*intro*!]:
assumes f *has-compact-support-on* X g *has-compact-support-on* X
shows $(\lambda x.\ max\ (f\ x)\ (g\ x))$ *has-compact-support-on* X
⟨*proof*⟩

lemma *has-compact-support-on-ext-iff*[*iff*]:
$(\lambda x{\in}topspace\ X.\ f\ x)$ *has-compact-support-on* $X \longleftrightarrow f$ *has-compact-support-on* X
⟨*proof*⟩

lemma *has-compact-support-on-of-real-iff*[*iff*]:
$(\lambda x.\ of\text{-}real\ (f\ x))$ *has-compact-support-on* $X = f$ *has-compact-support-on* X
⟨*proof*⟩

lemma *has-compact-support-on-complex-iff*:
f *has-compact-support-on* $X \longleftrightarrow$
$(\lambda x.\ Re\ (f\ x))$ *has-compact-support-on* $X \wedge (\lambda x.\ Im\ (f\ x))$ *has-compact-support-on* X
⟨*proof*⟩

lemma [*simp*]:
assumes f *has-compact-support-on* X
shows *has-compact-support-on-Re*:$(\lambda x.\ Re\ (f\ x))$ *has-compact-support-on* X
and *has-compact-support-on-Im*:$(\lambda x.\ Im\ (f\ x))$ *has-compact-support-on* X
⟨*proof*⟩

3.3 Positive Linear Functionsls

definition *positive-linear-functional-on-CX* :: *'a topology* ⇒ ((*'a* ⇒ *'b* :: {*ring*, *order*, *topological-space*}) ⇒ *'b*) ⇒ *bool*
where *positive-linear-functional-on-CX X φ* ≡
(∀*f*. *continuous-map X euclidean f* ⟶ *f has-compact-support-on X*
⟶ (∀ *x*∈*topspace X*. *f x* ≥ *0*) ⟶ *φ* (λ*x*∈*topspace X*. *f x*) ≥ *0*) ∧
(∀*f a*. *continuous-map X euclidean f* ⟶ *f has-compact-support-on X*
⟶ *φ* (λ*x*∈*topspace X*. *a* ∗ *f x*) = *a* ∗ *φ* (λ*x*∈*topspace X*. *f x*)) ∧
(∀*f g*. *continuous-map X euclidean f* ⟶ *f has-compact-support-on X*
⟶ *continuous-map X euclidean g* ⟶ *g has-compact-support-on X*
⟶ *φ* (λ*x*∈*topspace X*. *f x* + *g x*) = *φ* (λ*x*∈*topspace X*. *f x*) + *φ* (λ*x*∈*topspace X*. *g x*))

lemma
assumes *positive-linear-functional-on-CX X φ*
shows *pos-lin-functional-on-CX-pos*:
⋀*f*. *continuous-map X euclidean f* ⟹ *f has-compact-support-on X*
⟹ (⋀*x*. *x*∈*topspace X* ⟹ *f x* ≥ *0*) ⟹ *φ* (λ*x*∈*topspace X*. *f x*) ≥ *0*
and *pos-lin-functional-on-CX-lin*:
⋀*f a*. *continuous-map X euclidean f* ⟹ *f has-compact-support-on X*
⟹ *φ* (λ*x*∈*topspace X*. *a* ∗ *f x*) = *a* ∗ *φ* (λ*x*∈*topspace X*. *f x*)
⋀*f g*. *continuous-map X euclidean f* ⟹ *f has-compact-support-on X*
⟹ *continuous-map X euclidean g* ⟹ *g has-compact-support-on X*
⟹ *φ* (λ*x*∈*topspace X*. *f x* + *g x*) = *φ* (λ*x*∈*topspace X*. *f x*) + *φ* (λ*x*∈*topspace X*. *g x*)
⟨*proof*⟩

lemma *pos-lin-functional-on-CX-pos-complex*:
assumes *positive-linear-functional-on-CX X φ*
shows *continuous-map X euclidean f* ⟹ *f has-compact-support-on X*
⟹ (⋀*x*. *x*∈*topspace X* ⟹ *Re* (*f x*) ≥ *0*) ⟹ (⋀*x*. *x* ∈ *topspace X* ⟹ *f x* ∈ ℝ)
⟹ *φ* (λ*x*∈*topspace X*. *f x*) ≥ *0*
⟨*proof*⟩

lemma *positive-linear-functional-on-CX-compact*:
assumes *compact-space X*
shows *positive-linear-functional-on-CX X φ* ⟷
(∀*f*. *continuous-map X euclidean f* ⟶ (∀ *x*∈*topspace X*. *f x* ≥ *0*) ⟶ *φ* (λ*x*∈*topspace X*. *f x*) ≥ *0*) ∧
(∀*f a*. *continuous-map X euclidean f* ⟶ *φ* (λ*x*∈*topspace X*. *a* ∗ *f x*) = *a* ∗ *φ* (λ*x*∈*topspace X*. *f x*)) ∧
(∀*f g*. *continuous-map X euclidean f* ⟶ *continuous-map X euclidean g*
⟶ *φ* (λ*x*∈*topspace X*. *f x* + *g x*) = *φ* (λ*x*∈*topspace X*. *f x*) + *φ* (λ*x*∈*topspace X*. *g x*))
⟨*proof*⟩

lemma
assumes *positive-linear-functional-on-CX X φ compact-space X*

shows *pos-lin-functional-on-CX-compact-pos*:
$\bigwedge f.$ *continuous-map X euclidean f*
$\Longrightarrow (\bigwedge x.\ x \in topspace\ X \Longrightarrow f\ x \geq 0) \Longrightarrow \varphi\ (\lambda x \in topspace\ X.\ f\ x) \geq 0$
and *pos-lin-functional-on-CX-compact-lin*:
$\bigwedge f\ a.$ *continuous-map X euclidean f*
$\Longrightarrow \varphi\ (\lambda x \in topspace\ X.\ a * f\ x) = a * \varphi\ (\lambda x \in topspace\ X.\ f\ x)$
$\bigwedge f\ g.$ *continuous-map X euclidean f* $\Longrightarrow$ *continuous-map X euclidean g*
$\Longrightarrow \varphi\ (\lambda x \in topspace\ X.\ f\ x + g\ x) = \varphi\ (\lambda x \in topspace\ X.\ f\ x) + \varphi\ (\lambda x \in topspace\ X.\ g\ x)$
⟨*proof*⟩

lemma *pos-lin-functional-on-CX-diff*:
fixes $f :: - \Rightarrow -$ :: {*real-normed-vector*, *ring-1*}
assumes *positive-linear-functional-on-CX X* φ
and *cont*:*continuous-map X euclidean f continuous-map X euclidean g*
and *csupp*: *f has-compact-support-on X g has-compact-support-on X*
shows $\varphi\ (\lambda x \in topspace\ X.\ f\ x - g\ x) = \varphi\ (\lambda x \in topspace\ X.\ f\ x) - \varphi\ (\lambda x \in topspace\ X.\ g\ x)$
⟨*proof*⟩

lemma *pos-lin-functional-on-CX-compact-diff*:
fixes $f :: - \Rightarrow -$ :: {*real-normed-vector*, *ring-1*}
assumes *positive-linear-functional-on-CX X* φ *compact-space X*
and *continuous-map X euclidean f continuous-map X euclidean g*
shows $\varphi\ (\lambda x \in topspace\ X.\ f\ x - g\ x) = \varphi\ (\lambda x \in topspace\ X.\ f\ x) - \varphi\ (\lambda x \in topspace\ X.\ g\ x)$
⟨*proof*⟩

lemma *pos-lin-functional-on-CX-mono*:
fixes $f :: - \Rightarrow -$ :: {*real-normed-vector*, *ring-1*, *ordered-ab-group-add*}
assumes *positive-linear-functional-on-CX X* φ
and *mono*:$\bigwedge x.\ x \in topspace\ X \Longrightarrow f\ x \leq g\ x$
and *cont*:*continuous-map X euclidean f continuous-map X euclidean g*
and *csupp*: *f has-compact-support-on X g has-compact-support-on X*
shows $\varphi\ (\lambda x \in topspace\ X.\ f\ x) \leq \varphi\ (\lambda x \in topspace\ X.\ g\ x)$
⟨*proof*⟩

lemma *pos-lin-functional-on-CX-compact-mono*:
fixes $f :: - \Rightarrow -$ :: {*real-normed-vector*, *ring-1*, *ordered-ab-group-add*}
assumes *positive-linear-functional-on-CX X* φ *compact-space X*
and $\bigwedge x.\ x \in topspace\ X \Longrightarrow f\ x \leq g\ x$
and *continuous-map X euclidean f continuous-map X euclidean g*
shows $\varphi\ (\lambda x \in topspace\ X.\ f\ x) \leq \varphi\ (\lambda x \in topspace\ X.\ g\ x)$
⟨*proof*⟩

lemma *pos-lin-functional-on-CX-zero*:
assumes *positive-linear-functional-on-CX X* φ
shows $\varphi\ (\lambda x \in topspace\ X.\ 0) = 0$
⟨*proof*⟩

lemma *pos-lin-functional-on-CX-uminus*:
fixes $f :: - \Rightarrow - :: \{real\text{-}normed\text{-}vector, ring\text{-}1\}$
assumes *positive-linear-functional-on-CX X* φ
and *continuous-map X euclidean f*
and *csupp*: *f has-compact-support-on X*
shows $\varphi\ (\lambda x{\in}topspace\ X.\ -\ f\ x) = -\ \varphi\ (\lambda x{\in}topspace\ X.\ f\ x)$
⟨*proof*⟩

lemma *pos-lin-functional-on-CX-compact-uminus*:
fixes $f :: - \Rightarrow - :: \{real\text{-}normed\text{-}vector, ring\text{-}1\}$
assumes *positive-linear-functional-on-CX X* φ *compact-space X*
and *continuous-map X euclidean f*
shows $\varphi\ (\lambda x{\in}topspace\ X.\ -\ f\ x) = -\ \varphi\ (\lambda x{\in}topspace\ X.\ f\ x)$
⟨*proof*⟩

lemma *pos-lin-functional-on-CX-sum*:
fixes $f :: - \Rightarrow - \Rightarrow\ - :: \{real\text{-}normed\text{-}vector\}$
assumes *positive-linear-functional-on-CX X* φ
and *finite I* $\bigwedge i.\ i \in I \Longrightarrow$ *continuous-map X euclidean* $(f\ i)$
and $\bigwedge i.\ i \in I \Longrightarrow$ *f i has-compact-support-on X*
shows $\varphi\ (\lambda x{\in}topspace\ X.\ (\sum i{\in}I.\ f\ i\ x)) = (\sum i{\in}I.\ \varphi\ (\lambda x{\in}topspace\ X.\ f\ i\ x))$
⟨*proof*⟩

lemma *pos-lin-functional-on-CX-pos-is-real*:
fixes $f :: - \Rightarrow complex$
assumes *positive-linear-functional-on-CX X* φ
and *continuous-map X euclidean f f has-compact-support-on X*
and $\bigwedge x.\ x \in topspace\ X \Longrightarrow f\ x \in \mathbb{R}$
shows $\varphi\ (\lambda x{\in}topspace\ X.\ f\ x) \in \mathbb{R}$
⟨*proof*⟩

lemma
fixes $\varphi\ X$
defines $\varphi' \equiv (\lambda f.\ Re\ (\varphi\ (\lambda x{\in}topspace\ X.\ complex\text{-}of\text{-}real\ (f\ x))))$
assumes *plf*:*positive-linear-functional-on-CX X* φ
shows *pos-lin-functional-on-CX-complex-decompose*:
$\bigwedge f.$ *continuous-map X euclidean f* $\Longrightarrow$ *f has-compact-support-on X*
$\Longrightarrow \varphi\ (\lambda x{\in}topspace\ X.\ f\ x)$
$= complex\text{-}of\text{-}real\ (\varphi'\ (\lambda x{\in}topspace\ X.\ Re\ (f\ x))) + \mathrm{i} * complex\text{-}of\text{-}real\ (\varphi'$
$(\lambda x{\in}topspace\ X.\ Im\ (f\ x)))$
and *pos-lin-functional-on-CX-complex-decompose-plf*:
positive-linear-functional-on-CX X φ'
⟨*proof*⟩

3.4 Lemmas for Uniqueness

lemma *rep-measures-real-unique*:
assumes *locally-compact-space X Hausdorff-space X*

assumes N: *subalgebra N* (*borel-of X*)
$\bigwedge f$. *continuous-map X euclideanreal f* $\Longrightarrow$ *f has-compact-support-on X* $\Longrightarrow$ *integrable N f*
$\bigwedge A$. *A*$\in$*sets N* $\Longrightarrow$ *emeasure N A* = ($\bigsqcap C\in\{C.$ *openin X C* $\wedge$ *A* $\subseteq$ *C*$\}$. *emeasure N C*)
$\bigwedge A$. *openin X A* $\Longrightarrow$ *emeasure N A* = ($\bigsqcup K\in\{K.$ *compactin X K* $\wedge$ *K* $\subseteq$ *A*$\}$. *emeasure N K*)
$\bigwedge A$. *A*$\in$*sets N* $\Longrightarrow$ *emeasure N A* $<\infty\Longrightarrow$ *emeasure N A* = ($\bigsqcup K\in\{K.$ *compactin X K* $\wedge$ *K* $\subseteq$ *A*$\}$. *emeasure N K*)
$\bigwedge K$. *compactin X K* $\Longrightarrow$ *N K* $<\infty$
assumes M: *subalgebra M* (*borel-of X*)
$\bigwedge f$. *continuous-map X euclideanreal f* $\Longrightarrow$ *f has-compact-support-on X* $\Longrightarrow$ *integrable M f*
$\bigwedge A$. *A*$\in$*sets M* $\Longrightarrow$ *emeasure M A* = ($\bigsqcap C\in\{C.$ *openin X C* $\wedge$ *A* $\subseteq$ *C*$\}$. *emeasure M C*)
$\bigwedge A$. *openin X A* $\Longrightarrow$ *emeasure M A* = ($\bigsqcup K\in\{K.$ *compactin X K* $\wedge$ *K* $\subseteq$ *A*$\}$. *emeasure M K*)
$\bigwedge A$. *A*$\in$*sets M* $\Longrightarrow$ *emeasure M A* $<\infty\Longrightarrow$ *emeasure M A* = ($\bigsqcup K\in\{K.$ *compactin X K* $\wedge$ *K* $\subseteq$ *A*$\}$. *emeasure M K*)
$\bigwedge K$. *compactin X K* $\Longrightarrow$ *M K* $<\infty$
and *sets-eq*: *sets N* = *sets M*
and *integ-eq*: $\bigwedge f$. *continuous-map X euclideanreal f* $\Longrightarrow$ *f has-compact-support-on X* $\Longrightarrow$ ($\int x.$ *f x* ∂N) = ($\int x.$ *f x* ∂M)
shows *N* = *M*
⟨*proof*⟩

lemma *rep-measures-complex-unique*:
fixes *X* :: *'a topology*
assumes *locally-compact-space X Hausdorff-space X*
assumes N: *subalgebra N* (*borel-of X*)
$\bigwedge f$. *continuous-map X euclidean f* $\Longrightarrow$ *f has-compact-support-on X* $\Longrightarrow$ *complex-integrable N f*
$\bigwedge A$. *A*$\in$*sets N* $\Longrightarrow$ *emeasure N A* = ($\bigsqcap C\in\{C.$ *openin X C* $\wedge$ *A* $\subseteq$ *C*$\}$. *emeasure N C*)
$\bigwedge A$. *openin X A* $\Longrightarrow$ *emeasure N A* = ($\bigsqcup K\in\{K.$ *compactin X K* $\wedge$ *K* $\subseteq$ *A*$\}$. *emeasure N K*)
$\bigwedge A$. *A*$\in$*sets N* $\Longrightarrow$ *emeasure N A* $<\infty\Longrightarrow$ *emeasure N A* = ($\bigsqcup K\in\{K.$ *compactin X K* $\wedge$ *K* $\subseteq$ *A*$\}$. *emeasure N K*)
$\bigwedge K$. *compactin X K* $\Longrightarrow$ *N K* $<\infty$
assumes M: *subalgebra M* (*borel-of X*)
$\bigwedge f$. *continuous-map X euclidean f* $\Longrightarrow$ *f has-compact-support-on X* $\Longrightarrow$ *complex-integrable M f*
$\bigwedge A$. *A*$\in$*sets M* $\Longrightarrow$ *emeasure M A* = ($\bigsqcap C\in\{C.$ *openin X C* $\wedge$ *A* $\subseteq$ *C*$\}$. *emeasure M C*)
$\bigwedge A$. *openin X A* $\Longrightarrow$ *emeasure M A* = ($\bigsqcup K\in\{K.$ *compactin X K* $\wedge$ *K* $\subseteq$ *A*$\}$. *emeasure M K*)
$\bigwedge A$. *A*$\in$*sets M* $\Longrightarrow$ *emeasure M A* $<\infty\Longrightarrow$ *emeasure M A* = ($\bigsqcup K\in\{K.$ *compactin X K* $\wedge$ *K* $\subseteq$ *A*$\}$. *emeasure M K*)
$\bigwedge K$. *compactin X K* $\Longrightarrow$ *M K* $<\infty$

and *sets-eq*: *sets N = sets M*
and *integ-eq*: $\bigwedge f::'a \Rightarrow complex.\ continuous\text{-}map\ X\ euclidean\ f \Longrightarrow f\ has\text{-}compact\text{-}support\text{-}on\ X \Longrightarrow (\int x.\ f\ x\ \partial N) = (\int x.\ f\ x\ \partial M)$
shows $N = M$
⟨*proof*⟩

lemma *finite-tight-measure-eq*:
assumes *locally-compact-space X metrizable-space X tight-on X N tight-on X M*
and *integ-eq*: $\bigwedge f.\ continuous\text{-}map\ X\ euclideanreal\ f \Longrightarrow f \in topspace\ X \to \{0..1\} \Longrightarrow (\int x.\ f\ x\ \partial N) = (\int x.\ f\ x\ \partial M)$
shows $N = M$
⟨*proof*⟩

3.5 Riesz Representation Theorem for Real Numbers

theorem *Riesz-representation-real-complete*:
fixes $X :: 'a\ topology$ **and** $\varphi :: ('a \Rightarrow real) \Rightarrow real$
assumes *lh:locally-compact-space X Hausdorff-space X*
and *plf:positive-linear-functional-on-CX X φ*
shows $\exists M.\ \exists!N.\ sets\ N = M \wedge subalgebra\ N\ (borel\text{-}of\ X)$
$\wedge\ (\forall A{\in}sets\ N.\ emeasure\ N\ A = (\prod C{\in}\{C.\ openin\ X\ C \wedge A \subseteq C\}.\ emeasure\ N\ C))$
$\wedge\ (\forall A.\ openin\ X\ A \longrightarrow emeasure\ N\ A = (\bigsqcup K{\in}\{K.\ compactin\ X\ K \wedge K \subseteq A\}.\ emeasure\ N\ K))$
$\wedge\ (\forall A{\in}sets\ N.\ emeasure\ N\ A < \infty \longrightarrow emeasure\ N\ A = (\bigsqcup K{\in}\{K.\ compactin\ X\ K \wedge K \subseteq A\}.\ emeasure\ N\ K))$
$\wedge\ (\forall K.\ compactin\ X\ K \longrightarrow emeasure\ N\ K < \infty)$
$\wedge\ (\forall f.\ continuous\text{-}map\ X\ euclideanreal\ f \longrightarrow f\ has\text{-}compact\text{-}support\text{-}on\ X \longrightarrow \varphi\ (\lambda x{\in}topspace\ X.\ f\ x) = (\int x.\ f\ x\ \partial N))$
$\wedge\ (\forall f.\ continuous\text{-}map\ X\ euclideanreal\ f \longrightarrow f\ has\text{-}compact\text{-}support\text{-}on\ X \longrightarrow integrable\ N\ f)$
$\wedge\ complete\text{-}measure\ N$
⟨*proof*⟩

3.6 Riesz Representation Theorem for Complex Numbers

theorem *Riesz-representation-complex-complete*:
fixes $X :: 'a\ topology$ **and** $\varphi :: ('a \Rightarrow complex) \Rightarrow complex$
assumes *lh:locally-compact-space X Hausdorff-space X*
and *plf:positive-linear-functional-on-CX X φ*
shows $\exists M.\ \exists!N.\ sets\ N = M \wedge subalgebra\ N\ (borel\text{-}of\ X)$
$\wedge\ (\forall A{\in}sets\ N.\ emeasure\ N\ A = (\prod C{\in}\{C.\ openin\ X\ C \wedge A \subseteq C\}.\ emeasure\ N\ C))$
$\wedge\ (\forall A.\ openin\ X\ A \longrightarrow emeasure\ N\ A = (\bigsqcup K{\in}\{K.\ compactin\ X\ K \wedge K \subseteq A\}.\ emeasure\ N\ K))$
$\wedge\ (\forall A{\in}sets\ N.\ emeasure\ N\ A < \infty \longrightarrow emeasure\ N\ A = (\bigsqcup K{\in}\{K.\ compactin\ X\ K \wedge K \subseteq A\}.\ emeasure\ N\ K))$
$\wedge\ (\forall K.\ compactin\ X\ K \longrightarrow emeasure\ N\ K < \infty)$

$\wedge$ ($\forall f$. *continuous-map X euclidean f* $\longrightarrow$ *f has-compact-support-on X* $\longrightarrow$ φ ($\lambda x{\in}$*topspace X*. *f x*) = ($\int$ *x*. *f x* ∂N))
$\wedge$ ($\forall f$. *continuous-map X euclidean f* $\longrightarrow$ *f has-compact-support-on X* $\longrightarrow$ *complex-integrable N f*)
$\wedge$ *complete-measure N*
⟨*proof*⟩

3.7 Other Forms of the Theorem

In the case when the representation measure is on X.

theorem *Riesz-representation-real*:
assumes *lh*:*locally-compact-space X Hausdorff-space X*
and *positive-linear-functional-on-CX X* φ
shows $\exists!N$. *sets N* = *sets* (*borel-of X*)
$\wedge$ ($\forall A{\in}$*sets N*. *emeasure N A* = ($\prod C{\in}\{C$. *openin X C* $\wedge$ *A* $\subseteq$ *C*$\}$. *emeasure N C*))
$\wedge$ ($\forall A$. *openin X A* $\longrightarrow$ *emeasure N A* = ($\bigsqcup K{\in}\{K$. *compactin X K* $\wedge$ *K* $\subseteq$ *A*$\}$. *emeasure N K*))
$\wedge$ ($\forall A{\in}$*sets N*. *emeasure N A* $<$ ∞ $\longrightarrow$ *emeasure N A* = ($\bigsqcup K{\in}\{K$. *compactin X K* $\wedge$ *K* $\subseteq$ *A*$\}$. *emeasure N K*))
$\wedge$ ($\forall K$. *compactin X K* $\longrightarrow$ *emeasure N K* $<$ ∞)
$\wedge$ ($\forall f$. *continuous-map X euclideanreal f* $\longrightarrow$ *f has-compact-support-on X* $\longrightarrow$ φ ($\lambda x{\in}$*topspace X*. *f x*) = ($\int$ *x*. *f x* ∂N))
$\wedge$ ($\forall f$. *continuous-map X euclideanreal f* $\longrightarrow$ *f has-compact-support-on X* $\longrightarrow$ *integrable N f*)
⟨*proof*⟩

theorem *Riesz-representation-complex*:
fixes *X* :: *'a topology* **and** φ :: (*'a* $\Rightarrow$ *complex*) $\Rightarrow$ *complex*
assumes *lh*:*locally-compact-space X Hausdorff-space X*
and *positive-linear-functional-on-CX X* φ
shows $\exists!N$. *sets N* = *sets* (*borel-of X*)
$\wedge$ ($\forall A{\in}$*sets N*. *emeasure N A* = ($\prod C{\in}\{C$. *openin X C* $\wedge$ *A* $\subseteq$ *C*$\}$. *emeasure N C*))
$\wedge$ ($\forall A$. *openin X A* $\longrightarrow$ *emeasure N A* = ($\bigsqcup K{\in}\{K$. *compactin X K* $\wedge$ *K* $\subseteq$ *A*$\}$. *emeasure N K*))
$\wedge$ ($\forall A{\in}$*sets N*. *emeasure N A* $<$ ∞ $\longrightarrow$ *emeasure N A* = ($\bigsqcup K{\in}\{K$. *compactin X K* $\wedge$ *K* $\subseteq$ *A*$\}$. *emeasure N K*))
$\wedge$ ($\forall K$. *compactin X K* $\longrightarrow$ *emeasure N K* $<$ ∞)
$\wedge$ ($\forall f$. *continuous-map X euclidean f* $\longrightarrow$ *f has-compact-support-on X* $\longrightarrow$ φ ($\lambda x{\in}$*topspace X*. *f x*) = ($\int$ *x*. *f x* ∂N))
$\wedge$ ($\forall f$. *continuous-map X euclidean f* $\longrightarrow$ *f has-compact-support-on X* $\longrightarrow$ *complex-integrable N f*)
⟨*proof*⟩

3.7.1 Theorem for Compact Hausdorff Spaces

theorem *Riesz-representation-real-compact-Hausdorff*:
fixes *X* :: *'a topology* **and** φ :: (*'a* $\Rightarrow$ *real*) $\Rightarrow$ *real*

assumes *lh*:*compact-space X Hausdorff-space X*
and *positive-linear-functional-on-CX X* φ
shows $\exists!N.$ *sets N* = *sets* (*borel-of X*) $\wedge$ *finite-measure N*
$\wedge$ ($\forall A{\in}$*sets N*. *emeasure N A* = ($\prod C{\in}\{C.$ *openin X C* $\wedge$ *A* $\subseteq$ *C*$\}$. *emeasure N C*))
$\wedge$ ($\forall A.$ *openin X A* $\longrightarrow$ *emeasure N A* = ($\bigsqcup K{\in}\{K.$ *compactin X K* $\wedge$ *K* $\subseteq$ *A*$\}$. *emeasure N K*))
$\wedge$ ($\forall A{\in}$*sets N*. *emeasure N A* $<$ ∞ $\longrightarrow$ *emeasure N A* = ($\bigsqcup K{\in}\{K.$ *compactin X K* $\wedge$ *K* $\subseteq$ *A*$\}$. *emeasure N K*))
$\wedge$ ($\forall K.$ *compactin X K* $\longrightarrow$ *emeasure N K* $<$ ∞)
$\wedge$ ($\forall f.$ *continuous-map X euclideanreal f* $\longrightarrow$ φ ($\lambda x{\in}$*topspace X*. *f x*) = ($\int x.$ *f x* ∂N))
$\wedge$ ($\forall f.$ *continuous-map X euclideanreal f* $\longrightarrow$ *integrable N f*)
⟨*proof*⟩

theorem *Riesz-representation-complex-compact-Hausdorff*:
fixes *X* :: *'a topology* **and** φ :: (*'a* $\Rightarrow$ *complex*) $\Rightarrow$ *complex*
assumes *lh*:*compact-space X Hausdorff-space X*
and *positive-linear-functional-on-CX X* φ
shows $\exists!N.$ *sets N* = *sets* (*borel-of X*) $\wedge$ *finite-measure N*
$\wedge$ ($\forall A{\in}$*sets N*. *emeasure N A* = ($\prod C{\in}\{C.$ *openin X C* $\wedge$ *A* $\subseteq$ *C*$\}$. *emeasure N C*))
$\wedge$ ($\forall A.$ *openin X A* $\longrightarrow$ *emeasure N A* = ($\bigsqcup K{\in}\{K.$ *compactin X K* $\wedge$ *K* $\subseteq$ *A*$\}$. *emeasure N K*))
$\wedge$ ($\forall A{\in}$*sets N*. *emeasure N A* $<$ ∞ $\longrightarrow$ *emeasure N A* = ($\bigsqcup K{\in}\{K.$ *compactin X K* $\wedge$ *K* $\subseteq$ *A*$\}$. *emeasure N K*))
$\wedge$ ($\forall K.$ *compactin X K* $\longrightarrow$ *emeasure N K* $<$ ∞)
$\wedge$ ($\forall f.$ *continuous-map X euclidean f* $\longrightarrow$ φ ($\lambda x{\in}$*topspace X*. *f x*) = ($\int x.$ *f x* ∂N))
$\wedge$ ($\forall f.$ *continuous-map X euclidean f* $\longrightarrow$ *complex-integrable N f*)
⟨*proof*⟩

3.7.2 Theorem for Compact Metrizable Spaces

theorem *Riesz-representation-real-compact-metrizable*:
fixes *X* :: *'a topology* **and** φ :: (*'a* $\Rightarrow$ *real*) $\Rightarrow$ *real*
assumes *lh*:*compact-space X metrizable-space X*
and *plf*:*positive-linear-functional-on-CX X* φ
shows $\exists!N.$ *sets N* = *sets* (*borel-of X*) $\wedge$ *finite-measure N*
$\wedge$ ($\forall f.$ *continuous-map X euclideanreal f* $\longrightarrow$ φ ($\lambda x{\in}$*topspace X*. *f x*) = ($\int x.$ *f x* ∂N))
⟨*proof*⟩

theorem *Riesz-representation-real-compact-metrizable-le1*:
fixes *X* :: *'a topology* **and** φ :: (*'a* $\Rightarrow$ *real*) $\Rightarrow$ *real*
assumes *lh*:*compact-space X metrizable-space X*
and *plf*:*positive-linear-functional-on-CX X* φ
shows $\exists!N.$ *sets N* = *sets* (*borel-of X*) $\wedge$ *finite-measure N*
$\wedge$ ($\forall f.$ *continuous-map X euclideanreal f* $\longrightarrow$ *f* $\in$ *topspace X* $\to$ $\{0..1\}$

$\longrightarrow \varphi\ (\lambda x{\in}topspace\ X.\ f\ x) = (\int x.\ f\ x\ \partial N))$
⟨*proof*⟩

theorem *Riesz-representation-complex-compact-metrizable*:
fixes $X :: 'a\ topology$ **and** $\varphi :: ('a \Rightarrow complex) \Rightarrow complex$
assumes *lh*:*compact-space X metrizable-space X*
and *plf*:*positive-linear-functional-on-CX X* φ
shows $\exists!N.\ sets\ N = sets\ (borel\text{-}of\ X) \wedge finite\text{-}measure\ N$
$\wedge\ (\forall f.\ continuous\text{-}map\ X\ euclidean\ f \longrightarrow \varphi\ (\lambda x{\in}topspace\ X.\ f\ x) = (\int x.$
$f\ x\ \partial N))$
⟨*proof*⟩

theorem *Riesz-representation-real-compact-metrizable-subprob*:
fixes $X :: 'a\ topology$ **and** $\varphi :: ('a \Rightarrow real) \Rightarrow real$
assumes *lh*:*compact-space X metrizable-space X*
and *plf*:*positive-linear-functional-on-CX X* φ
and *le1*: $\varphi\ (\lambda x{\in}topspace\ X.\ 1) \leq 1$ **and** *ne*: $X \neq trivial\text{-}topology$
shows $\exists!N.\ sets\ N = sets\ (borel\text{-}of\ X) \wedge subprob\text{-}space\ N$
$\wedge\ (\forall f.\ continuous\text{-}map\ X\ euclideanreal\ f \longrightarrow \varphi\ (\lambda x{\in}topspace\ X.\ f\ x) =$
$(\int x.\ f\ x\ \partial N))$
⟨*proof*⟩

theorem *Riesz-representation-real-compact-metrizable-subprob-le1*:
fixes $X :: 'a\ topology$ **and** $\varphi :: ('a \Rightarrow real) \Rightarrow real$
assumes *lh*:*compact-space X metrizable-space X*
and *plf*:*positive-linear-functional-on-CX X* φ
and *le1*: $\varphi\ (\lambda x{\in}topspace\ X.\ 1) \leq 1$ **and** *ne*: $X \neq trivial\text{-}topology$
shows $\exists!N.\ sets\ N = sets\ (borel\text{-}of\ X) \wedge subprob\text{-}space\ N$
$\wedge\ (\forall f.\ continuous\text{-}map\ X\ euclideanreal\ f \longrightarrow f \in topspace\ X \to \{0..1\}$
$\longrightarrow \varphi\ (\lambda x{\in}topspace\ X.\ f\ x) = (\int x.\ f\ x\ \partial N))$
⟨*proof*⟩

theorem *Riesz-representation-real-compact-metrizable-prob*:
fixes $X :: 'a\ topology$ **and** $\varphi :: ('a \Rightarrow real) \Rightarrow real$
assumes *lh*:*compact-space X metrizable-space X*
and *plf*:*positive-linear-functional-on-CX X* φ
and $\varphi\ (\lambda x{\in}topspace\ X.\ 1) = 1$
shows $\exists!N.\ sets\ N = sets\ (borel\text{-}of\ X) \wedge prob\text{-}space\ N$
$\wedge\ (\forall f.\ continuous\text{-}map\ X\ euclideanreal\ f \longrightarrow \varphi\ (\lambda x{\in}topspace\ X.\ f\ x) =$
$(\int x.\ f\ x\ \partial N))$
⟨*proof*⟩

theorem *Riesz-representation-complex-compact-metrizable-subprob*:
fixes $X :: 'a\ topology$ **and** $\varphi :: ('a \Rightarrow complex) \Rightarrow complex$
assumes *lh*:*compact-space X metrizable-space X*
and *plf*:*positive-linear-functional-on-CX X* φ
and *le1*: $Re\ (\varphi\ (\lambda x{\in}topspace\ X.\ 1)) \leq 1$ **and** *ne*: $X \neq trivial\text{-}topology$
shows $\exists!N.\ sets\ N = sets\ (borel\text{-}of\ X) \wedge subprob\text{-}space\ N$
$\wedge\ (\forall f.\ continuous\text{-}map\ X\ euclidean\ f \longrightarrow \varphi\ (\lambda x{\in}topspace\ X.\ f\ x) = (\int x.$

$f\ x\ \partial N))$
⟨*proof*⟩

theorem *Riesz-representation-complex-compact-metrizable-prob*:
fixes $X :: 'a$ *topology* **and** $\varphi :: ('a \Rightarrow complex) \Rightarrow complex$
assumes *lh*:*compact-space X metrizable-space X*
and *plf*:*positive-linear-functional-on-CX X* φ
and $Re\ (\varphi\ (\lambda x{\in}topspace\ X.\ 1)) = 1$
shows $\exists!N.\ sets\ N = sets\ (borel\text{-}of\ X) \wedge prob\text{-}space\ N$
$\wedge\ (\forall f.\ continuous\text{-}map\ X\ euclidean\ f \longrightarrow \varphi\ (\lambda x{\in}topspace\ X.\ f\ x) = (\int x.$
$f\ x\ \partial N))$
⟨*proof*⟩

end

References

[1] W. Rudin. *Real and Complex Analysis, 3rd Ed.* McGraw-Hill, Inc., USA, 1987.

[2] O. van Gaans. Probability measures on metric spaces. https://www.math.leidenuniv.nl/~vangaans/jancol1.pdf. Accessed: March 2. 2023.