Regular Variation

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Abstract

This entry formalizes regularly varying functions. Roughly speaking, a regularly varying function is a function whose behavior at infinity is similar to that of a power-law function. Regularly varying functions are typically used in extreme value theory, a branch of statistics. We formalize the definition of regularly varying functions and standard results about them, including:

- the uniform convergence theorem,
- Karamata's representation theorem, and
- Karamata's theorem.

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1 Lemmas

theory *Lemmas-Regular-Variation*
imports
HOL−Probability.Probability
Standard-Borel-Spaces.StandardBorel
begin

1.1 General Lemmas

lemma *image-mono-on-continuous-on-Icc*:
fixes $f :: 'a::linear\text{-}continuum\text{-}topology \Rightarrow 'b :: linorder\text{-}topology$
assumes $ab{:}a \leq b$ **and** $mono{:}mono\text{-}on\ \{a..b\}\ f$
and $cont{:}continuous\text{-}on\ \{a..b\}\ f$
shows $f\ `\ \{a..b\} = \{f\ a..f\ b\}$
⟨*proof*⟩

lemma *image-mono-on-continuous-on-Ic*:
fixes $f :: 'a::linear\text{-}continuum\text{-}topology \Rightarrow 'b :: linorder\text{-}topology$
assumes $mono{:}mono\text{-}on\ \{a..\}\ f$
and $cont{:}continuous\text{-}on\ \{a..\}\ f$
and $lim{:}filterlim\ f\ at\text{-}top\ at\text{-}top$
shows $f\ `\ \{a..\} = \{f\ a..\}$
⟨*proof*⟩

lemma *image-mono-on-continuous-on-Ioo*:
fixes $f :: 'a::\{linear\text{-}continuum\text{-}topology\} \Rightarrow 'b :: linorder\text{-}topology$
assumes $ab{:}a \leq b$ **and** $mono{:}strict\text{-}mono\text{-}on\ \{a..b\}\ f$
and $cont{:}continuous\text{-}on\ \{a..b\}\ f$
shows $f\ `\ \{a<..<b\} = \{f\ a<..<f\ b\}$
⟨*proof*⟩

lemma *image-mono-on-continuous-on-Ioc*:
fixes $f :: 'a::\{linear\text{-}continuum\text{-}topology\} \Rightarrow 'b :: linorder\text{-}topology$
assumes $ab{:}a \leq b$ **and** $mono{:}strict\text{-}mono\text{-}on\ \{a..b\}\ f$
and $cont{:}continuous\text{-}on\ \{a..b\}\ f$
shows $f\ `\ \{a<..b\} = \{f\ a<..f\ b\}$
⟨*proof*⟩

lemma *image-mono-on-continuous-on-greaterThan*:
fixes $f :: 'a::\{linear\text{-}continuum\text{-}topology\} \Rightarrow 'b :: linorder\text{-}topology$
assumes $mono{:}strict\text{-}mono\text{-}on\ \{a..\}\ f$

and *cont*:*continuous-on* $\{a..\}$ f
and *lim*:*filterlim* f *at-top at-top*
shows f ‘ $\{a<..\} = \{f\ a<..\}$
⟨*proof*⟩

lemma *image-antimono-on-continuous-on-Icc*:
fixes $f :: 'a::$*linear-continuum-topology* $\Rightarrow$ $'b ::$ *linorder-topology*
assumes *ab*:$a \leq b$ **and** *mono*:*antimono-on* $\{a..b\}$ f
and *cont*:*continuous-on* $\{a..b\}$ f
shows f ‘ $\{a..b\} = \{f\ b..f\ a\}$
⟨*proof*⟩

lemma *image-antimono-on-continuous-on-Ioc*:
fixes $f :: 'a::\{$*linear-continuum-topology*$\}$ $\Rightarrow$ $'b ::$ *linorder-topology*
assumes *ab*:$a \leq b$ **and** *mono*:*strict-antimono-on* $\{a..b\}$ f
and *cont*:*continuous-on* $\{a..b\}$ f
shows f ‘ $\{a<..b\} = \{f\ b..<f\ a\}$
⟨*proof*⟩

lemma *image-mult-greaterThanLessThan*[*simp*]:
fixes $c ::$ *real*
assumes $0 < c$
shows $(*)\ c$ ‘ $\{a<..<b\} = \{c * a<..<c * b\}$
⟨*proof*⟩

lemma *image-times-atLeastLessThan*[*simp*]:
fixes $c ::$ *real*
assumes $0 < c$
shows $(*)\ c$ ‘ $\{a<..b\} = \{c * a<..c * b\}$
⟨*proof*⟩

lemma *image-power-Icc-pos*:
assumes $0 \leq \varrho$
and $0 < a$ **and** $a \leq b$
shows $(\lambda x::$*real*$.\ x$ *powr* $\varrho)$ ‘ $\{a..b\} = \{a$ *powr* $\varrho..b$ *powr* $\varrho\}$
⟨*proof*⟩

lemma *image-inverse-power-Icc-pos*:
assumes $0 \leq \varrho$
and $0 < a$ **and** $a \leq b$
shows $(\lambda x::$*real*$.\ 1\ /\ x$ *powr* $\varrho)$ ‘ $\{a..b\} = \{1\ /\ b$ *powr* $\varrho..1\ /\ a$ *powr* $\varrho\}$
⟨*proof*⟩

lemma *image-power-Icc-neg*:
assumes $\varrho \leq 0$
and $0 < a$ **and** $a \leq b$
shows $(\lambda x::$*real*$.\ x$ *powr* $\varrho)$ ‘ $\{a..b\} = \{b$ *powr* $\varrho..a$ *powr* $\varrho\}$
⟨*proof*⟩

lemma *image-inverse-power-Icc-neg*:
assumes $\varrho \leq 0$
and $0 < a$ **and** $a \leq b$
shows $(\lambda x{::}real.\ 1\ /\ x\ powr\ \varrho)\ `\ \{a..b\} = \{1\ /\ a\ powr\ \varrho..1\ /\ b\ powr\ \varrho\}$
⟨*proof*⟩

lemma *image-times-atLeastLessThan'*[*simp*]:
fixes $c :: real$
assumes $0 < c$
shows $(\lambda x.\ x * c)\ `\ \{a<..b\} = \{c * a<..c * b\}$
⟨*proof*⟩

lemma *bounded-times-comp*:
fixes $f\ g{::}'a \Rightarrow 'b{::}real\text{-}normed\text{-}field$
assumes *bounded* $(f\ `\ S)$
and *bounded* $(g\ `\ S)$
shows *bounded* $((\lambda x.\ f\ x * g\ x)\ `\ S)$
⟨*proof*⟩

lemma *antimono-on-iff-uminus-mono-on*:
fixes $f :: - \Rightarrow - :: ordered\text{-}ab\text{-}group\text{-}add$
shows *antimono-on* $A\ f \longleftrightarrow$ *mono-on* $A\ (\lambda x.\ -f\ x)$
⟨*proof*⟩

lemma *filterlim-Pair-component*:
$LIM\ x\ F.\ f\ x :> G \Longrightarrow LIM\ x\ H.\ g\ x :> I \Longrightarrow LIM\ (x,\ y)\ F \times_F H.\ (f\ x,\ g\ y) :> G \times_F I$
⟨*proof*⟩

lemma *eventually-eventually-prod-filter-principal1*:
eventually $P\ (F \times_F principal\ A) \longleftrightarrow$ *eventually* $(\lambda x.\ \forall y{\in}A.\ P\ (x,\ y))\ F$
⟨*proof*⟩

lemma *eventually-eventually-prod-filter-principal2*:
eventually $P\ (principal\ A \times_F G) \longleftrightarrow$ *eventually* $(\lambda y.\ \forall x{\in}A.\ P\ (x,\ y))\ G$
⟨*proof*⟩

lemma *prod-filter-distrib-right*:
$F \times_F (G \sqcup H) = F \times_F G \sqcup F \times_F H$
⟨*proof*⟩

lemma *suminf-const-infinity*:
fixes $a :: ennreal$
assumes $a{:}0 < a$
shows $(\sum n.\ a) = \top$
⟨*proof*⟩

1.2 Limit

lemma *tendsto-sandwich-mult-const*:
fixes c :: *real*
assumes *ev*: $\bigwedge e$::*real*. $0 < e \Longrightarrow$ *eventually* $(\lambda n.\ f\ e \leq g\ n\ /\ c \wedge g\ n\ /\ c \leq h\ e)$ *net*
and *limf*:$(f \longrightarrow 1)$ (*at-right* 0) **and** *limh*:$(h \longrightarrow 1)$ (*at-right* 0)
shows $(g \longrightarrow c)$ *net*
⟨*proof*⟩

lemma *asymp-equiv-Limsup-cong-ereal*:
assumes *equiv*:$f \sim[F]\ g$
and *nonneg1*:$\forall_F\ x$ *in* $F.\ 0 \leq f\ x$
and *nonneg2*:$\forall_F\ x$ *in* $F.\ 0 \leq g\ x$
shows *Limsup* $F\ (\lambda x.\ ereal\ (f\ x)) = Limsup\ F\ (\lambda x.\ ereal\ (g\ x))$
⟨*proof*⟩

lemma *asymp-equiv-Liminf-cong-ereal*:
assumes *equiv*:$f \sim[F]\ g$
and *nonneg1*:$\forall_F\ x$ *in* $F.\ 0 \leq f\ x$
and *nonneg2*:$\forall_F\ x$ *in* $F.\ 0 \leq g\ x$
shows *Liminf* $F\ (\lambda x.\ ereal\ (f\ x)) = Liminf\ F\ (\lambda x.\ ereal\ (g\ x))$
⟨*proof*⟩

lemma *tendsto-at-topI-sequentially-real'*:
fixes f :: *real* $\Rightarrow$ *real*
assumes *mono*: *mono-on* $\{K..\}\ f$
and *limseq*: $(\lambda n.\ f\ (real\ n)) \longrightarrow y$
shows $(f \longrightarrow y)$ *at-top*
⟨*proof*⟩

lemma *mono-convergent-real-le*:
fixes f :: *real* $\Rightarrow$ *real*
assumes *mono*:*mono-on* $\{K..\}\ f$
and *le*:$\bigwedge r.\ K \leq r \Longrightarrow f\ r \leq A$
shows $\exists\, l.\ (f \longrightarrow l)$ *at-top* $\wedge\ (\forall\, r{\geq}K.\ f\ r \leq l)$
⟨*proof*⟩

lemma *mono-convergent-real-less*:
fixes f :: *real* $\Rightarrow$ *real*
assumes *mono*:*mono-on* $\{K<..\}\ f$
and *less*:$\bigwedge r.\ K < r \Longrightarrow f\ r \leq A$
shows $\exists\, l.\ (f \longrightarrow l)$ *at-top* $\wedge\ (\forall\, r{>}K.\ f\ r \leq l)$
⟨*proof*⟩

lemma *at-top-real-iff-all-sequentially*:
fixes f :: *real* $\Rightarrow$ $'b$ :: *topological-space*
shows $(f \longrightarrow l)$ *at-top*
$\longleftrightarrow\ (\forall\, a.$ *filterlim a at-top sequentially* $\longrightarrow$ *incseq a*
$\longrightarrow (\lambda n.\ (f\ (a\ n))) \longrightarrow l)$

⟨*proof*⟩
lemma *at-top-real-iff-all-sequentially'*:
fixes *f* :: *real* ⇒ *'b* :: *topological-space*
shows (*f* ⟶ *l*) *at-top*
⟷ (∀ *a*. *filterlim a at-top sequentially* ⟶ (∀ *n*. *a n* ≥ *K*)
⟶ *incseq a* ⟶ (λ*n*. (*f* (*a n*))) ⟶ *l*)
⟨*proof*⟩

1.3 Integrals on Sets

lemma *set-borel-measurable-iff-restrict-space*:
fixes *f* :: *'a* ⇒ *'b*::*real-normed-vector*
assumes [*measurable*]:*A* ∈ *sets M*
shows *set-borel-measurable M A f* ⟷ *f* ∈ *borel-measurable* (*restrict-space M A*)
⟨*proof*⟩

lemma *set-borel-measurable-add*:
fixes *f* :: - ⇒ - :: {*second-countable-topology*, *topological-monoid-add*, *real-vector*}
assumes *set-borel-measurable M A f*
and *set-borel-measurable M A g*
shows *set-borel-measurable M A* (λ*x*. *f x* + *g x*)
⟨*proof*⟩

lemma *set-borel-measurable-scaleR*:
fixes *g* :: - ⇒ - :: {*second-countable-topology*, *real-normed-vector*}
assumes *set-borel-measurable M A f*
and *set-borel-measurable M A g*
shows *set-borel-measurable M A* (λ*x*. *f x* ∗$_R$ *g x*)
⟨*proof*⟩

lemma *set-borel-measurable-times*:
fixes *f* :: *'a* ⇒ *'b*::{*second-countable-topology*, *real-normed-algebra*}
assumes *set-borel-measurable M A f*
and *set-borel-measurable M A g*
shows *set-borel-measurable M A* (λ*x*. *f x* ∗ *g x*)
⟨*proof*⟩

lemma *set-borel-measurable-restrict-space-iff*:
fixes *f* :: *'a* ⇒ *'b*::*real-normed-vector*
assumes [*measurable*]:*A* ∈ *sets M*
shows *set-borel-measurable* (*restrict-space M A*) *A f* ⟷ *set-borel-measurable M A f*
⟨*proof*⟩

lemma *set-borel-measurable-comp'*:
fixes *f* :: *'a* ⇒ *'b*::*real-normed-vector*
assumes [*measurable*]: *A* ∈ *sets M*
and *set-borel-measurable N* (*g* ‘ *A*) *f*
and [*measurable*]: *g* ∈ *measurable M N*

shows *set-borel-measurable M A* ($\lambda x.$ *f* (*g x*))
⟨*proof*⟩

lemma *set-borel-measurable-comp*:
fixes *f* :: - ⇒ - ::*real-normed-vector* **and** *g* :: - ⇒ -::*real-vector*
assumes [*measurable*]: *A* ∈ *sets M*
and *set-borel-measurable borel* (*g* ' *A*) *f*
and *set-borel-measurable M A g*
shows *set-borel-measurable M A* ($\lambda x.$ *f* (*g x*))
⟨*proof*⟩

lemma *set-integrable-set-borel-measurable*:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second-countable-topology*}
assumes *set-integrable M S f*
shows *set-borel-measurable M S f*
⟨*proof*⟩

lemma *set-integrable-uminus*:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second-countable-topology*}
shows *set-integrable M S* ($\lambda x.$ − *f x*) ⟷ *set-integrable M S f*
⟨*proof*⟩

lemma *set-integrable-const*:
fixes *a* :: *'b*::{*banach*, *second-countable-topology*}
assumes *emeasure M S* < ⊤ *S* ∈ *sets M*
shows *set-integrable M S* ($\lambda x.$ *a*)
⟨*proof*⟩

lemma *set-integrable-zero*[*simp*]: *set-integrable M S* ($\lambda x.$ *0*)
⟨*proof*⟩

lemma *set-integral-cong*:
assumes ⋀*x*. *x* ∈ *A* ⟹ *f x* = *g x*
shows (∫ *x*∈*A*. *f x* ∂*M*) = (∫ *x*∈*A*. *g x* ∂*M*)
⟨*proof*⟩

lemma *set-integrable-bounded*:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second-countable-topology*}
assumes *set-borel-measurable M S f*
and *S*:*emeasure M S* < ⊤ *S* ∈ *sets M*
and *le*:⋀*x*. *x* ∈ *S* ⟹ *norm* (*f x*) ≤ *K*
shows *set-integrable M S f*
⟨*proof*⟩

lemma *set-integral-subset-AE*:
fixes *f* :: - ⇒ *real*
assumes *integ*:*set-integrable M B f*
and *AB*:*A* ∈ *sets M A* ⊆ *B*
and *nonneg*:*AE x*∈ *B* − *A in M*. *0* ≤ *f x*

shows $(\int x{\in}A.\ f\ x\ \partial M) \leq (\int x{\in}B.\ f\ x\ \partial M)$
⟨*proof*⟩

lemma *set-integral-subset*:
fixes $f :: - \Rightarrow real$
assumes *integ*:*set-integrable M B f*
and *AB*:$A \in sets\ M\ A \subseteq B$
and *nonneg*:$\bigwedge x.\ x{\in}\ B - A \Longrightarrow 0 \leq f\ x$
shows $(\int x{\in}A.\ f\ x\ \partial M) \leq (\int x{\in}B.\ f\ x\ \partial M)$
⟨*proof*⟩

lemma *set-integral-empty*[*simp*]:$(\int x{\in}\{\}.\ f\ x\ \partial M) = 0$
⟨*proof*⟩

lemma *set-integral-null-sets*:
assumes $N \in null\text{-}sets\ M$
shows $(\int x{\in}N.\ f\ x\ \partial M) = 0$
⟨*proof*⟩

corollary *LBINT-singleton*[*simp*]:$(\int x{\in}\{a\}.\ f\ x\ \partial lborel) = 0$
⟨*proof*⟩

lemma *not-set-integrable-integral-eq*:
assumes ¬ *set-integrable M A f*
shows $(\int x{\in}A.\ f\ x\ \partial M) = 0$
⟨*proof*⟩

The following is more general than *set-integrable lborel* {*?x*} *?f*.

lemma *set-borel-integrable-singleton'*[*simp*]: *set-integrable lborel* {*a*} *f*
⟨*proof*⟩

lemma *set-integrable-Ioc-eq-Icc*:
fixes *a* :: - :: *linorder-topology*
fixes *f* :: - ⇒ - :: {*banach*, *second-countable-topology*}
shows *set-integrable lborel* {*a*<..*b*} *f* ⟷ *set-integrable lborel* {*a*..*b*} *f*
⟨*proof*⟩

lemma *LBINT-Ioc-eq-Icc*:
fixes *f* :: - :: *linorder-topology* ⇒ - :: {*banach*, *second-countable-topology*}
shows $(\int x{\in}\{a<..b\}.\ f\ x\ \partial lborel) = (\int x{\in}\{a..b\}.\ f\ x\ \partial lborel)$
⟨*proof*⟩

lemma *set-integrable-atLeast-eq*:
fixes *a* :: - :: *linorder-topology*
fixes *f* :: - ⇒ - :: {*banach*, *second-countable-topology*}
shows *set-integrable lborel* {*a*<..} *f* ⟷ *set-integrable lborel* {*a*..} *f*
⟨*proof*⟩

lemma *set-integral-atLeast-eq*:

fixes $f :: real \Rightarrow - :: \{banach, second\text{-}countable\text{-}topology\}$
shows $(\int x{\in}\{a{<}..\}.\ f\ x\ \partial lborel) = (\int x{\in}\{a..\}.\ f\ x\ \partial lborel)$
⟨*proof*⟩

lemma *set-nn-integral-atLeast-eq*:
fixes $a :: - :: linorder\text{-}topology$
shows $(\int^+ x{\in}\{a{<}..\}.\ f\ x\ \partial lborel) = (\int^+ x{\in}\{a..\}.\ f\ x\ \partial lborel)$
⟨*proof*⟩

lemma *set-integrable-atMost-eq*:
fixes $b :: - :: linorder\text{-}topology$
fixes $f :: - \Rightarrow - :: \{banach, second\text{-}countable\text{-}topology\}$
shows *set-integrable lborel* $\{..{<}b\}\ f \longleftrightarrow$ *set-integrable lborel* $\{..b\}\ f$
⟨*proof*⟩

lemma *set-integral-atMost-eq*:
fixes $f :: real \Rightarrow - :: \{banach, second\text{-}countable\text{-}topology\}$
shows $(\int x{\in}\{..{<}b\}.\ f\ x\ \partial lborel) = (\int x{\in}\{..b\}.\ f\ x\ \partial lborel)$
⟨*proof*⟩

lemma *set-nn-integral-atMost-eq*:
fixes $b :: - :: linorder\text{-}topology$
shows $(\int^+ x{\in}\{..{<}b\}.\ f\ x\ \partial lborel) = (\int^+ x{\in}\{..b\}.\ f\ x\ \partial lborel)$
⟨*proof*⟩

lemma *AE-False-iff-False*:
assumes $0 < emeasure\ M\ A$
shows $(AE\ x{\in}A\ in\ M.\ False) \longleftrightarrow False$
⟨*proof*⟩

lemma *nn-integral-less'*:
assumes [*measurable*]: $f \in borel\text{-}measurable\ M\ g \in borel\text{-}measurable\ M$
assumes *f*: $(\int^+ x.\ f\ x\ \partial M) \neq \infty$
and *ord1*: $AE\ x \in A\ in\ M.\ f\ x < g\ x$ **and** *ord2*:$AE\ x\ in\ M.\ f\ x \leq g\ x$
and *pos*:$0 < emeasure\ M\ A$
shows $(\int^+ x.\ f\ x\ \partial M) < (\int^+ x.\ g\ x\ \partial M)$
⟨*proof*⟩

lemma *set-nn-integral-less'*:
fixes $f :: - \Rightarrow ennreal$
assumes $(\lambda x.\ f\ x * indicator\ A\ x) \in borel\text{-}measurable\ M$
and $(\lambda x.\ g\ x * indicator\ A\ x) \in borel\text{-}measurable\ M$
and $A \in sets\ M$
and $B \subseteq A$
and *le*:$AE\ x{\in}B\ in\ M.\ f\ x < g\ x\ AE\ x{\in}A\ in\ M.\ f\ x \leq g\ x$
and $(\int^+ x{\in}A.\ f\ x\ \partial M) \neq \infty$
and $0 < emeasure\ M\ B$
shows $(\int^+ x{\in}A.\ f\ x\ \partial M) < (\int^+ x{\in}A.\ g\ x\ \partial M)$
⟨*proof*⟩

lemma *set-nn-integral-less*:
fixes f :: - $\Rightarrow$ *ennreal*
assumes $(\lambda x.\ f\ x * \mathit{indicator}\ A\ x) \in \mathit{borel\text{-}measurable}\ M$
and $(\lambda x.\ g\ x * \mathit{indicator}\ A\ x) \in \mathit{borel\text{-}measurable}\ M$
and $AE\ x{\in}A\ \mathit{in}\ M.\ f\ x < g\ x$
and $(\int^+ x{\in}A.\ f\ x\ \partial M) \neq \infty$
and $0 < \mathit{emeasure}\ M\ A$
shows $(\int^+ x{\in}A.\ f\ x\ \partial M) < (\int^+ x{\in}A.\ g\ x\ \partial M)$
⟨*proof*⟩

lemma *set-nn-integral-zero-less*:
assumes [*measurable*]:$(\lambda x.\ g\ x * \mathit{indicator}\ A\ x) \in \mathit{borel\text{-}measurable}\ M$
and *pos*:$AE\ x{\in}A\ \mathit{in}\ M.\ 0 < g\ x$
and *meas*:$0 < \mathit{emeasure}\ M\ A$
shows $0 < (\int^+ x{\in}A.\ g\ x\ \partial M)$
⟨*proof*⟩

lemma *set-integral-zero-less*:
fixes f :: - $\Rightarrow$ *real*
assumes *integ*:*set-integrable M A f*
and *le*:$AE\ x{\in}A\ \mathit{in}\ M.\ 0 < f\ x$
and *meas*:$0 < \mathit{emeasure}\ M\ A$
shows $0 < (\int x{\in}A.\ f\ x\ \partial M)$
⟨*proof*⟩

lemma *set-integral-zero-less′*:
fixes f :: - $\Rightarrow$ *real*
assumes *integ*:*set-integrable M A f*
and *le*:$AE\ x{\in}B\ \mathit{in}\ M.\ 0 < f\ x$
and *le2*:$AE\ x{\in}A\ \mathit{in}\ M.\ 0 \leq f\ x$
and [*measurable*]:$A \in \mathit{sets}\ M$
and AB:$B \subseteq A$
and *meas*:$0 < \mathit{emeasure}\ M\ B$
shows $0 < (\int x{\in}A.\ f\ x\ \partial M)$
⟨*proof*⟩

lemma *set-integral-less′*:
fixes f :: - $\Rightarrow$ *real*
assumes *f*:*set-integrable M A f*
and *g*:*set-integrable M A g*
and [*measurable*]:$A \in \mathit{sets}\ M$
and AB:$B \subseteq A$
and *le*:$AE\ x{\in}B\ \mathit{in}\ M.\ f\ x < g\ x$ **and** *le2*:$AE\ x{\in}A\ \mathit{in}\ M.\ f\ x \leq g\ x$
and *meas*:$0 < \mathit{emeasure}\ M\ B$
shows $(\int x{\in}A.\ f\ x\ \partial M) < (\int x{\in}A.\ g\ x\ \partial M)$
⟨*proof*⟩

lemma *set-integral-less*:

fixes f :: - $\Rightarrow$ *real*
assumes f:*set-integrable M A f*
and g:*set-integrable M A g*
and *le*:*AE* $x \in A$ *in M*. $f\ x < g\ x$
and *meas*:$0 <$ *emeasure M A*
shows $(\int x \in A.\ f\ x\ \partial M) < (\int x \in A.\ g\ x\ \partial M)$
⟨*proof*⟩

corollary *set-itegral-strict-mono-Icc*:
fixes f :: *real* $\Rightarrow$ *real*
fixes $a\ b$ :: *real*
assumes *integ*:*set-integrable lborel* $\{a..b\}$ f
and *le*:*AE* $x \in \{a<..b\}$ *in lborel*. $0 < f\ x$
shows *strict-mono-on* $\{a..b\}$ $(\lambda x.\ (\int t \in \{a<..x\}.\ f\ t\ \partial lborel))$
⟨*proof*⟩

corollary *set-itegral-tail-strict-antimono*:
fixes f :: *real* $\Rightarrow$ *real*
assumes *integ*:*set-integrable lborel* $\{a..\}$ f
and *le*:*AE* $x \in \{a<..\}$ *in lborel*. $0 < f\ x$
shows *strict-antimono-on* $\{a..\}$ $(\lambda x.\ (\int t \in \{x<..\}.\ f\ t\ \partial lborel))$
⟨*proof*⟩

1.4 Integral of Composite Functions $g(\int_{t \in A_x} f(t)\,\mathrm{d}t)$

lemma *continuous-on-LBINT-Ioc*:
fixes f :: *real* $\Rightarrow$ *real*
assumes f: $\bigwedge b.\ a \le b \Longrightarrow$ *set-integrable lborel* $\{a..b\}$ f
shows *continuous-on UNIV* $(\lambda b.\ LBINT\ x{:}\{a<..b\}.\ f\ x)$
⟨*proof*⟩

corollary *continuous-on-LBINT-tail*:
fixes f :: *real* $\Rightarrow$ *real*
assumes f: *set-integrable lborel* $\{a..\}$ f
shows *continuous-on* $\{a..\}$ $(\lambda b.\ LBINT\ x{:}\{b<..\}.\ f\ x)$
⟨*proof*⟩

corollary *continuous-on-LBINT-com*:
fixes f :: *real* $\Rightarrow$ *real*
assumes *integrable lborel f*
shows *isCont* $(\lambda b.\ LBINT\ x{:}\{..b\}.\ f\ x)\ x$
⟨*proof*⟩

lemma *sets-restrict-space-sigma-sets*:
assumes *sets M* = *sigma-sets* Ω *E*
and *A*:$A \in$ *sets M* **and** *Ao*:$A \subseteq \Omega$
shows *sets* (*restrict-space M A*) = *sigma-sets A* $((\cap)\ A\ `\ E)$
⟨*proof*⟩

lemma *Int-stable-image-inter*:
assumes *Int-stable A*
shows *Int-stable* ((∩) *a* ‘ *A*)
⟨*proof*⟩

lemma
fixes *f* **and** *g*:: *real* ⇒ *real* **and** *A B*
defines *F* ≡ λ*x*::*real*. (∫ *t*∈*A x*. *f t* ∂*lborel*)
defines *M* ≡ *density* (*restrict-space lborel B*) *f*
assumes *f-pos*:*AE x* ∈ *B in lborel*. 0 ≤ *f x*
and *f-int*:⋀*x*. *x* ∈ *X* ⟹ *set-integrable lborel* (*A x*) *f*
and *F-meas*:*F* ∈ (*restrict-space lborel B*) →*M restrict-space borel C*
and *B*:*F* ‘*B* = *C* **and** [*measurable*]:*B* ∈ *sets borel C* ∈ *sets borel* ⋀*x*. *A x* ∈ *sets borel*
and *AB*:*B* ⊆ *A a* **and** *a*:*a* ∈ *X*
and *meq*:*restrict-space lborel C* = *distr M* (*restrict-space borel C*) *F*
shows *set-integral-compose-integral*:
g ∈ *borel-measurable* (*restrict-space borel C*)
⟹ (∫ *t*∈*B*. *g* (*F t*) ∗ *f t* ∂*lborel*) = (∫ *t*∈*C*. *g t* ∂*lborel*)
and *set-integrable-compose-integral*:
set-integrable lborel C g
⟹ *set-integrable lborel B* (λ*t*. *g* (*F t*) ∗ *f t*)
⟨*proof*⟩

lemma
fixes *f* **and** *g*:: *real* ⇒ *real* **and** *a b c*::*real*
defines *F* ≡ λ*x*. (∫ *t*∈{*a*<..*x*}. *f t* ∂*lborel*)
assumes *f-pos*:⋀*x*. *a* < *x* ⟹ 0 < *f x*
and *f-int*:⋀*d*. *a* ≤ *d* ⟹ *set-integrable lborel* {*a*..*d*} *f*
and *ab*[*arith*]:*a* ≤ *b* **and** *bc*[*arith*]:*b* ≤ *c*
shows *set-integral-compose-integral-Ioc*:
g ∈ *borel-measurable* (*restrict-space borel* {*F b*<..*F c*})
⟹ (∫ *t*∈{*b*<..*c*}. *g* (*F t*) ∗ *f t* ∂*lborel*) = (∫ *t*∈{*F b*<..*F c*}. *g t* ∂*lborel*)
and *set-integrable-compose-integral-Ioc*:
set-integrable lborel {*F b*<..*F c*} *g*
⟹ *set-integrable lborel* {*b*<..*c*} (λ*t*. *g* (*F t*) ∗ *f t*)
⟨*proof*⟩

lemma
fixes *f* **and** *g*:: *real* ⇒ *real* **and** *a b c*::*real*
defines *F* ≡ λ*x*. (∫ *t*∈{*x*<..}. *f t* ∂*lborel*)
assumes *f-pos*:⋀*x*. *a* < *x* ⟹ 0 < *f x*
and *f-int*:*set-integrable lborel* {*a*..} *f*
and *ab*[*arith*]:*a* ≤ *b* **and** *bc*[*arith*]:*b* ≤ *c*
shows *set-integral-compose-integral-tail*:
g ∈ *borel-measurable* (*restrict-space borel* {*F c*<..*F b*})
⟹ (∫ *t*∈{*b*<..*c*}. *g* (*F t*) ∗ *f t* ∂*lborel*) = (∫ *t*∈{*F c*<..*F b*}. *g t* ∂*lborel*)
and *set-integrable-compose-integral-tail*:

set-integrable lborel $\{F\ c<..F\ b\}\ g$
$\Longrightarrow$ *set-integrable lborel* $\{b<..c\}\ (\lambda t.\ g\ (F\ t) * f\ t)$
⟨*proof*⟩

1.5 Convergence Theorems with Real Index

lemma *borel-measurable-LIMSEQ-metric-at-top*:
fixes f :: *real* $\Rightarrow$ $'a \Rightarrow$ $'b$ :: *metric-space*
assumes [*measurable*]: $\bigwedge r.\ K \leq r \Longrightarrow f\ r \in$ *borel-measurable M*
assumes *lim*: $\bigwedge x.\ x \in$ *space M* $\Longrightarrow ((\lambda r.\ f\ r\ x) \longrightarrow g\ x)$ *at-top*
shows $g \in$ *borel-measurable M*
⟨*proof*⟩

lemma *nn-integral-at-top-real*:
assumes *mono-on* $\{K..\}\ f$ $\bigwedge r.\ r \geq K \Longrightarrow f\ r \in$ *borel-measurable M*
and *u*: $\bigwedge x.\ ((\lambda r{::}real.\ f\ r\ x) \longrightarrow u\ x)$ *at-top*
shows $((\lambda r.\ integral^N\ M\ (f\ r)) \longrightarrow integral^N\ M\ u)$ *at-top*
⟨*proof*⟩

theorem *nn-integral-dominated-convergence-at-top*:
assumes
$\bigwedge r{::}real.\ r \geq K \Longrightarrow u\ r \in$ *borel-measurable M*
$u' \in$ *borel-measurable M* $w \in$ *borel-measurable M*
and $\bigwedge r.\ r \geq K \Longrightarrow AE\ x\ in\ M.\ u\ r\ x \leq w\ x$
and $(\int^+ x.\ w\ x\ \partial M) < \infty$
and $AE\ x\ in\ M.\ ((\lambda r.\ u\ r\ x) \longrightarrow u'\ x)$ *at-top*
shows $((\lambda r.\ (\int^+ x.\ u\ r\ x\ \partial M)) \longrightarrow (\int^+ x.\ u'\ x\ \partial M))$ *at-top*
⟨*proof*⟩

lemma
fixes f :: *real* $\Rightarrow$ $'a \Rightarrow$ *real*
assumes $\bigwedge r.\ K \leq r \Longrightarrow$ *integrable M* $(f\ r)$ **and** *mono*: $AE\ x\ in\ M.$ *mono-on* $\{K..\}\ (\lambda r.\ f\ r\ x)$
and $AE\ x\ in\ M.\ ((\lambda r.\ f\ r\ x) \longrightarrow u\ x)$ *at-top*
and $((\lambda r.\ integral^L\ M\ (f\ r)) \longrightarrow x)$ *at-top*
and $u{:}u \in$ *borel-measurable M*
shows *integrable-monotone-convergence-real*: *integrable M u* (**is** *?g1*)
and *integral-monotone-convergence-real*: $integral^L\ M\ u = x$ (**is** *?g2*)
and *has-bochner-integral-monotone-convergence-real*: *has-bochner-integral M u x* (**is** *?g3*)
⟨*proof*⟩

lemma
fixes f :: $'a \Rightarrow$ $'b{::}\{banach,\ second\text{-}countable\text{-}topology\}$ **and** w :: $'a \Rightarrow$ *real*
assumes $f \in$ *borel-measurable M* $\bigwedge r{::}real.\ K \leq r \Longrightarrow s\ r \in$ *borel-measurable M*
integrable M w
assumes *lim*: $AE\ x\ in\ M.\ ((\lambda r.\ s\ r\ x) \longrightarrow f\ x)$ *at-top*
assumes *bound*: $\bigwedge r.\ K \leq r \Longrightarrow AE\ x\ in\ M.\ norm\ (s\ r\ x) \leq w\ x$
shows *integrable-dominated-convergence-real*: *integrable M f*

and *integrable-dominated-convergence-real2*: $\bigwedge r.\ K \leq r \Longrightarrow$ *integrable* $M\ (s\ r)$
and *integral-dominated-convergence-real*: $((\lambda r.\ integral^L\ M\ (s\ r)) \longrightarrow integral^L\ M\ f)$ *at-top*
⟨*proof*⟩

corollary
fixes f :: *real* ⇒ *real*
assumes *f-nonneg*: $\bigwedge x.\ K < x \Longrightarrow 0 \leq f\ x$
and *integrable*: $\bigwedge r.\ K < r \Longrightarrow$ *set-integrable* $M\ \{K<..r\}\ f$
and *le*:$\bigwedge r.\ K < r \Longrightarrow (\int x{\in}\{K<..r\}.\ f\ x\ \partial M) \leq G$
shows *interval-integral-inf*:
$((\lambda r.\ (\int x{\in}\{K<..r\}.\ f\ x\ \partial M)) \longrightarrow (\int x{\in}\{K<..\}.\ f\ x\ \partial M))$ *at-top*
and *interval-integral-inf-integrable*:
set-integrable $M\ \{K<..\}\ f$
and *measurable-integral-inf*:
set-borel-measurable $M\ \{K<..\}\ f$
⟨*proof*⟩

corollary *interval-integral-inf2*:
fixes f :: *real* ⇒ *real*
assumes *f-nonneg*: $\bigwedge x.\ K < x \Longrightarrow 0 \leq f\ x$
and *integrable*: $\bigwedge r.\ K < r \Longrightarrow$ *set-integrable* $M\ \{K<..r\}\ f$
and *integrable2*:*set-integrable* $M\ \{K<..\}\ f$
shows $((\lambda r.\ (\int x{\in}\{K<..r\}.\ f\ x\ \partial M)) \longrightarrow (\int x{\in}\{K<..\}.\ f\ x\ \partial M))$ *at-top*
⟨*proof*⟩

1.6 Change of Variables w.r.t. Lebesgue Integration on a Set

lemma *lebesgue-integral-change-of-variables-real1*:
fixes f :: *real* ⇒ *real* **and** g :: *real*⇒ *real*
assumes S[*measurable*]: $S \in$ *sets borel*
and g[*measurable*]:$g \in$ *borel-measurable* (*restrict-space borel* S)
and g'[*measurable*]:$g' \in$ *borel-measurable* (*restrict-space borel* S)
and *der-g*: $\bigwedge x.\ x \in S \Longrightarrow (g$ *has-field-derivative* $g'\ x)$ (*at* x *within* S)
and *inj*: *inj-on* $g\ S$
and *integrable*: *set-integrable lborel* $(g\ `\ S)\ f$
shows $(\int x{\in}\ g\ `\ S.\ f\ x\ \partial lborel) = (\int x{\in}S.\ |g'\ x| * f\ (g\ x)\ \partial lborel)$
⟨*proof*⟩

lemma *lebesgue-integral-change-of-variables-real2*:
fixes f :: *real* ⇒ *real* **and** g :: *real*⇒ *real*
assumes S[*measurable*]: $S \in$ *sets borel*
and f:*set-borel-measurable borel* $(g\ `\ S)\ f$
and *der-g*: $\bigwedge x.\ x \in S \Longrightarrow (g$ *has-field-derivative* $g'\ x)$ (*at* x *within* S)
and *inj*: *inj-on* $g\ S$
and *integrable*:*set-integrable lborel* $S\ (\lambda x.\ |g'\ x| * f\ (g\ x))$
shows $(\int x{\in}\ g\ `\ S.\ f\ x\ \partial lborel) = (\int x{\in}S.\ |g'\ x| * f\ (g\ x)\ \partial lborel)$
⟨*proof*⟩

lemma *lebesgue-integrable-change-of-variables-real*:
fixes $f :: real \Rightarrow real$ **and** $g :: real \Rightarrow real$
assumes $S[measurable]$: $S \in sets\ borel$
and f:*set-borel-measurable borel* $(g\ `\ S)\ f$
and $g[measurable]$:$g \in borel\text{-}measurable\ (restrict\text{-}space\ borel\ S)$
and $g'[measurable]$:$g' \in borel\text{-}measurable\ (restrict\text{-}space\ borel\ S)$
and *der-g*: $\bigwedge x.\ x \in S \Longrightarrow (g\ has\text{-}field\text{-}derivative\ g'\ x)\ (at\ x\ within\ S)$
and *inj*: *inj-on g S*
shows *set-integrable lborel* $(g\ `\ S)\ f$
$\longleftrightarrow$ *set-integrable lborel S* $(\lambda x.\ |g'\ x| * f\ (g\ x))$
⟨*proof*⟩

lemma *nn-integral-change-of-variables-real1*:
fixes $f :: real \Rightarrow real$ **and** $g :: real \Rightarrow real$
assumes $S[measurable]$: $S \in sets\ borel$
and $g[measurable]$:$g \in borel\text{-}measurable\ (restrict\text{-}space\ borel\ S)$
and $g'[measurable]$:$g' \in borel\text{-}measurable\ (restrict\text{-}space\ borel\ S)$
and *der-g*: $\bigwedge x.\ x \in S \Longrightarrow (g\ has\text{-}field\text{-}derivative\ g'\ x)\ (at\ x\ within\ S)$
and *inj*: *inj-on g S*
and *integrable*: *set-integrable lborel* $(g\ `\ S)\ f$
and *nonneg*:*AE* $x \in S$ *in lborel.* $0 \le f\ (g\ x)$ *AE* $x \in g\ `\ S$ *in lborel.* $0 \le f\ x$
shows $(\int^+ x \in g\ `\ S.\ ennreal\ (f\ x)\ \partial lborel) = (\int^+ x \in S.\ ennreal\ (|g'\ x| * f\ (g\ x))\ \partial lborel)$
⟨*proof*⟩

lemma *nn-integral-change-of-variables-real1'*:
fixes $f :: real \Rightarrow real$ **and** $g :: real \Rightarrow real$
assumes $S \in sets\ borel$
and $g \in borel\text{-}measurable\ (restrict\text{-}space\ borel\ S)$
and $g' \in borel\text{-}measurable\ (restrict\text{-}space\ borel\ S)$
and $\bigwedge x.\ x \in S \Longrightarrow (g\ has\text{-}field\text{-}derivative\ g'\ x)\ (at\ x\ within\ S)$
and *inj-on g S*
and *set-integrable lborel* $(g\ `\ S)\ f$
and *nonneg*:$\bigwedge x.\ x \in S \Longrightarrow 0 \le f\ (g\ x)$
shows $(\int^+ x \in g\ `\ S.\ ennreal\ (f\ x)\ \partial lborel) = (\int^+ x \in S.\ ennreal\ (|g'\ x| * f\ (g\ x))\ \partial lborel)$
⟨*proof*⟩

lemma *nn-integral-change-of-variables-real2*:
fixes $f :: real \Rightarrow real$ **and** $g :: real \Rightarrow real$
assumes $S[measurable]$: $S \in sets\ borel$
and f:*set-borel-measurable borel* $(g\ `\ S)\ f$
and $g[measurable]$:$g \in borel\text{-}measurable\ (restrict\text{-}space\ borel\ S)$
and $g'[measurable]$:$g' \in borel\text{-}measurable\ (restrict\text{-}space\ borel\ S)$
and *der-g*: $\bigwedge x.\ x \in S \Longrightarrow (g\ has\text{-}field\text{-}derivative\ g'\ x)\ (at\ x\ within\ S)$
and *inj*: *inj-on g S*
and *integrable*:*set-integrable lborel S* $(\lambda x.\ |g'\ x| * f\ (g\ x))$
and *nonneg*:*AE* $x \in S$ *in lborel.* $0 \le f\ (g\ x)$ *AE* $x \in g\ `\ S$ *in lborel.* $0 \le f\ x$

shows $(\int^+ x \in g\ `\ S.\ ennreal\ (f\ x)\ \partial lborel) = (\int^+ x \in S.\ ennreal\ (|g'\ x| * f\ (g\ x))\ \partial lborel)$
⟨*proof*⟩

lemma *nn-integral-change-of-variables-real2'*:
fixes $f :: real \Rightarrow real$ **and** $g :: real \Rightarrow real$
assumes S[*measurable*]: $S \in sets\ borel$
and f:*set-borel-measurable borel* $(g\ `\ S)\ f$
and g[*measurable*]:$g \in borel\text{-}measurable\ (restrict\text{-}space\ borel\ S)$
and g'[*measurable*]:$g' \in borel\text{-}measurable\ (restrict\text{-}space\ borel\ S)$
and *der-g*: $\bigwedge x.\ x \in S \Longrightarrow (g\ has\text{-}field\text{-}derivative\ g'\ x)\ (at\ x\ within\ S)$
and *inj*: *inj-on g S*
and *integrable*:*set-integrable lborel S* $(\lambda x.\ |g'\ x| * f\ (g\ x))$
and *nonneg*:$\bigwedge x.\ x \in S \Longrightarrow 0 \le f\ (g\ x)$
shows $(\int^+ x \in g\ `\ S.\ ennreal\ (f\ x)\ \partial lborel) = (\int^+ x \in S.\ ennreal\ (|g'\ x| * f\ (g\ x))\ \partial lborel)$
⟨*proof*⟩

1.7 Integral of Power Functions

lemma *set-integrable-inverse*:
fixes $a\ b\ c :: real$
assumes *a-pos*:$0 < a$
shows *set-integrable lborel* $\{a..b\}\ (\lambda x.\ c\ /\ x)$
⟨*proof*⟩

lemma *set-integrable-inverse'*:
fixes $a\ b\ c :: real$
assumes $0 < a$
shows *set-integrable lborel* $\{a<..b\}\ (\lambda x.\ c\ /\ x)$
⟨*proof*⟩

corollary *set-integral-inverse*:
fixes $a\ b\ c :: real$
assumes *a-pos*:$0 < a$
and *a-le-b*:$a \le b$
shows $(\int x \in \{a<..b\}.\ c\ /\ x\ \partial lborel) = c * (ln\ b - ln\ a)$
⟨*proof*⟩

lemma *set-integrable-powr-Icc*:
fixes $c :: real$
assumes c:$0 < c$
shows *set-integrable lborel* $\{c..d\}\ (\lambda x.\ x\ powr\ a)$
⟨*proof*⟩

lemma *set-integral-powr-Icc*:
fixes $c :: real$
assumes a:$-1 \neq a$
and c:$0 < c$

and *cd*:$c \leq d$
shows $(\int x{\in}\{c..d\}.\ x\ powr\ a\ \partial lborel) = d\ powr\ (a + 1)\ /\ (a + 1) - c\ powr\ (a + 1)\ /\ (a + 1)$
⟨*proof*⟩

corollary *set-integrable-powr-Ioc*:
fixes c :: *real*
assumes *c*:$0 < c$
shows *set-integrable lborel* $\{c<..d\}$ $(\lambda x.\ x\ powr\ a)$
⟨*proof*⟩

corollary *set-integral-powr-Ioc*:
fixes c :: *real*
shows $a \neq -1 \Longrightarrow 0 < c \Longrightarrow c \leq d \Longrightarrow$
$(\int x{\in}\{c<..d\}.\ x\ powr\ a\ \partial lborel) = d\ powr\ (a + 1)\ /\ (a + 1) - c\ powr\ (a + 1)\ /\ (a + 1)$
⟨*proof*⟩

lemma
fixes c :: *real*
assumes *a*:$-1 < a$
shows *set-integrable-powr-from-0*:*set-integrable lborel* $\{0<..c\}$ $(\lambda x.\ x\ powr\ a)$
and *set-integral-powr-from-0*:$0 \leq c \Longrightarrow$
$(\int x{\in}\{0<..c\}.\ x\ powr\ a\ \partial lborel) = c\ powr\ (a + 1)\ /\ (a + 1)$
⟨*proof*⟩

lemma
fixes c :: *real*
assumes *a*:$a < -1$
and *c*:$0 < c$
shows *set-integrable-powr-inf*:*set-integrable lborel* $\{c<..\}$ $(\lambda x.\ x\ powr\ a)$
and *set-integral-powr-inf*:$(\int x{\in}\{c<..\}.\ x\ powr\ a\ \partial lborel) = -(c\ powr\ (a + 1)\ /\ (a + 1))$
⟨*proof*⟩

1.8 Uniform Convergence

corollary *uniform-limit-uminus-iff*:
fixes f :: - ⇒ - :: *real-normed-vector*
shows *uniform-limit A* $(\lambda i\ x.\ -\ fi\ i\ x)$ $(\lambda x.\ -\ f\ x)$ *F*
$\longleftrightarrow$ *uniform-limit A fi f F*
⟨*proof*⟩

lemma *uniform-lim-mult-bdd1*:
fixes f :: - ⇒ - :: *real-normed-field*
assumes *ul*:*uniform-limit A fi f F*
and *bdd*:*bounded* $(g\ `\ A)$
shows *uniform-limit A* $(\lambda i\ x.\ g\ x * fi\ i\ x)$ $(\lambda x.\ g\ x * f\ x)$ *F*
⟨*proof*⟩

lemma *uconvergence-on-bdd-times2*:
fixes f :: - $\Rightarrow$ - :: *real-normed-field*
assumes *uniform-limit A fi f F*
and *bounded* (g ' A)
shows *uniform-limit A* ($\lambda i\ x.$ *fi i x* $*$ $g\ x$) ($\lambda x.$ $f\ x * g\ x$) F
⟨*proof*⟩

lemma *uniform-limit-eventually-in*:
assumes *lim*: *uniform-limit A g g′ F*
and *e*:$0 < e$
and *d*: $\bigwedge x\ y.\ x \in A \Longrightarrow dist\ (g'\ x)\ y < e \Longrightarrow y \in B$
shows $\forall_F\ t$ *in F*. $\forall\, y{\in}A.\ g\ t\ y \in B$
⟨*proof*⟩

lemma *mono-on-UCT*:
fixes *Ui* :: - $\Rightarrow$ *real* $\Rightarrow$ *real*
assumes *ab*:$a \leq b$
and *mono*:$\forall_F\ i$ *in F*. *mono-on* $\{a..b\}$ (*Ui i*) **and** *monoU*:*mono-on* $\{a..b\}$ *U*
and *U*:*continuous-on* $\{a..b\}$ *U*
and *lim*: $\bigwedge x.\ a \leq x \Longrightarrow x \leq b \Longrightarrow ((\lambda i.\ Ui\ i\ x) \longrightarrow U\ x)\ F$
shows *uniform-limit* $\{a..b\}$ *Ui U F*
⟨*proof*⟩

lemma *mono-on-UCT-atLeast*:
fixes f :: *′a* $\Rightarrow$ *real* $\Rightarrow$ *real* **and** g :: *real* $\Rightarrow$ *real* **and** F :: *′a filter*
assumes *mono*: $\forall_F\ i$ *in F*. *mono-on* $\{a..\}$ ($f\ i$)
and *bdd*: $\forall_F\ i$ *in F*. $\forall\, x{\geq}a.\ f\ i\ x \leq B$
and *gmono*: *mono-on* $\{a..\}$ g
and *gcont*: *continuous-on* $\{a..\}$ g
and *glim*: ($g \longrightarrow B$) *at-top*
and *lim*: $\bigwedge x.\ a \leq x \Longrightarrow ((\lambda i.\ f\ i\ x) \longrightarrow g\ x)\ F$
shows *uniform-limit* $\{a..\}$ f g F
⟨*proof*⟩

lemma *antimono-on-UCT*:
fixes *Ui* :: - $\Rightarrow$ *real* $\Rightarrow$ *real*
assumes *ab*:$a \leq b$
and *mono*:$\forall_F\ i$ *in F*. *antimono-on* $\{a..b\}$ (*Ui i*) **and** *monoU*:*antimono-on* $\{a..b\}$ *U*
and *U*:*continuous-on* $\{a..b\}$ *U*
and *lim*: $\bigwedge x.\ a \leq x \Longrightarrow x \leq b \Longrightarrow ((\lambda i.\ Ui\ i\ x) \longrightarrow U\ x)\ F$
shows *uniform-limit* $\{a..b\}$ *Ui U F*
⟨*proof*⟩

end

2 Regularly Varying Functions

theory *Regular-Variation*
imports
Lemmas-Regular-Variation
Cauchy-Functional-Equation.Cauchy-Functional-Equation
begin

We refer to Chapter 1 of the book by Bingham et al. [1], and Chapter 2 of the book by Resnick [3].

2.1 Definition and Basic Properties

2.1.1 Regularly Varying Functions

definition *regularly-varying* :: $(real \Rightarrow real) \Rightarrow real \Rightarrow bool$ **where**
regularly-varying $U\ \varrho \longleftrightarrow$
$U \in$ *borel-measurable* (*restrict-space borel* $\{0<..\}$) $\wedge$
U ' $\{0<..\} \subseteq \{0<..\} \wedge$
$(\forall x{>}0.\ ((\lambda t.\ U\ (t * x)\ /\ U\ t) \longrightarrow x\ powr\ \varrho)\ at\text{-}top)$

lemma *regularly-varyingI*:
assumes $U \in$ *borel-measurable* (*restrict-space borel* $\{0<..\}$)
and $\bigwedge r.\ 0 < r \Longrightarrow 0 < U\ r$
and $\bigwedge x.\ 0 < x \Longrightarrow ((\lambda t.\ U\ (t * x)\ /\ U\ t) \longrightarrow x\ powr\ \varrho)\ at\text{-}top$
shows *regularly-varying* $U\ \varrho$
⟨*proof*⟩

lemma *regularly-varyingI''*:
assumes *set-borel-measurable borel* $\{0<..\}\ U$
and $\bigwedge r.\ 0 < r \Longrightarrow 0 < U\ r$
and $\bigwedge x.\ 0 < x \Longrightarrow ((\lambda t.\ U\ (t * x)\ /\ U\ t) \longrightarrow x\ powr\ \varrho)\ at\text{-}top$
shows *regularly-varying* $U\ \varrho$
⟨*proof*⟩

lemma *regularly-varyingI-total*:
assumes $U \in$ *borel-measurable borel*
and $\bigwedge r.\ 0 < r \Longrightarrow 0 < U\ r$
and $\bigwedge x.\ 0 < x \Longrightarrow ((\lambda t.\ U\ (t * x)\ /\ U\ t) \longrightarrow x\ powr\ \varrho)\ at\text{-}top$
shows *regularly-varying* $U\ \varrho$
⟨*proof*⟩

lemma
assumes *regularly-varying* $U\ \varrho$
shows *regularly-varying-meas'*:*set-borel-measurable borel* $\{0<..\}\ U$
and *regularly-varying-meas*:$U \in$ *borel-measurable* (*restrict-space borel* $\{0<..\}$)
and *regularly-varying-pos*:$\bigwedge r.\ 0 < r \Longrightarrow 0 < U\ r$
and *regularly-varying-lim*:$\bigwedge x.\ x > 0 \Longrightarrow ((\lambda t.\ U\ (t * x)\ /\ U\ t) \longrightarrow x\ powr\ \varrho)\ at\text{-}top$
⟨*proof*⟩

lemma *regularly-varying-meas-log-exp*:
assumes *regularly-varying U ϱ*
shows $(\lambda t.\ ln\ (U\ (exp\ t))) \in$ *borel-measurable borel*
⟨*proof*⟩

lemma
assumes *regularly-varying U ϱ*
shows *regularly-varying-meas-o*:
$\bigwedge a.\ 0 \leq a \Longrightarrow (\lambda x.$ *indicat-real* $\{a<..\}\ x * U\ x) \in$ *borel-measurable borel*
and *regularly-varying-meas-c*:
$\bigwedge a.\ 0 < a \Longrightarrow (\lambda x.$ *indicat-real* $\{a..\}\ x * U\ x) \in$ *borel-measurable borel*
and *regularly-varying-meas-Ioo*:
$\bigwedge a\ b.\ 0 \leq a \Longrightarrow (\lambda x.$ *indicat-real* $\{a<..<b\}\ x * U\ x) \in$ *borel-measurable borel*
and *regularly-varying-meas-Ico*:
$\bigwedge a\ b.\ 0 < a \Longrightarrow (\lambda x.$ *indicat-real* $\{a..<b\}\ x * U\ x) \in$ *borel-measurable borel*
and *regularly-varying-meas-Ioc*:
$\bigwedge a\ b.\ 0 \leq a \Longrightarrow (\lambda x.$ *indicat-real* $\{a<..b\}\ x * U\ x) \in$ *borel-measurable borel*
and *regularly-varying-meas-Icc*:
$\bigwedge a\ b.\ 0 < a \Longrightarrow (\lambda x.$ *indicat-real* $\{a..b\}\ x * U\ x) \in$ *borel-measurable borel*
and *regularly-varying-meas-o-ennreal*:
$\bigwedge a.\ 0 \leq a \Longrightarrow (\lambda x.$ *ennreal* $(U\ x) *$ *indicator* $\{a<..\}\ x) \in$ *borel-measurable borel*
and *regularly-varying-meas-c-ennreal*:
$\bigwedge a.\ 0 < a \Longrightarrow (\lambda x.$ *ennreal* $(U\ x) *$ *indicator* $\{a..\}\ x) \in$ *borel-measurable borel*
and *regularly-varying-meas-Ioo-ennreal*:
$\bigwedge a\ b.\ 0 \leq a \Longrightarrow (\lambda x.$ *ennreal* $(U\ x) *$ *indicator* $\{a<..<b\}\ x) \in$ *borel-measurable borel*
and *regularly-varying-meas-Ico-ennreal*:
$\bigwedge a\ b.\ 0 < a \Longrightarrow (\lambda x.$ *ennreal* $(U\ x) *$ *indicator* $\{a..<b\}\ x) \in$ *borel-measurable borel*
and *regularly-varying-meas-Ioc-ennreal*:
$\bigwedge a\ b.\ 0 \leq a \Longrightarrow (\lambda x.$ *ennreal* $(U\ x) *$ *indicator* $\{a<..b\}\ x) \in$ *borel-measurable borel*
and *regularly-varying-meas-Icc-ennreal*:
$\bigwedge a\ b.\ 0 < a \Longrightarrow (\lambda x.$ *ennreal* $(U\ x) *$ *indicator* $\{a..b\}\ x) \in$ *borel-measurable borel*
⟨*proof*⟩

lemma *regularly-varying-asymp-equiv*:
assumes *regularly-varying U ϱ*
and *x*:$0 < x$
shows $(\lambda t.\ U\ (t * x)) \sim[at\text{-}top]\ (\lambda t.\ x\ powr\ \varrho * U\ t)$
⟨*proof*⟩

lemma *regularly-varying-def2*:
regularly-varying U ϱ $\longleftrightarrow$
$U \in$ *borel-measurable* (*restrict-space borel* $\{0<..\}) \wedge$
U ‘ $\{0<..\} \subseteq \{0<..\} \wedge$

$(\forall x{>}0.\ (\lambda t.\ U\ (t * x)) \sim[at\text{-}top]\ (\lambda t.\ x\ powr\ \varrho * U\ t))$
⟨*proof*⟩

lemma *regularly-varying-cong*:
assumes $\bigwedge x.\ 0 < x \Longrightarrow U\ x = T\ x$
shows *regularly-varying* $U\ \varrho \longleftrightarrow$ *regularly-varying* $T\ \varrho$
⟨*proof*⟩

lemma *regularly-varying-ln-exp-iff*:
regularly-varying $U\ \varrho \longleftrightarrow$
$U \in$ *borel-measurable* (*restrict-space borel* $\{0{<}..\}) \wedge$
U ‘ $\{0{<}..\} \subseteq \{0{<}..\} \wedge$
$(\forall x.\ ((\lambda t.\ ln\ (U\ (exp\ (t + x))) - ln\ (U\ (exp\ t))) \longrightarrow \varrho * x)\ at\text{-}top)$
⟨*proof*⟩

lemma *regularly-varying-ln-exp*:
assumes *regularly-varying* $U\ \varrho$
shows $((\lambda t.\ ln\ (U\ (exp\ (t + x))) - ln\ (U\ (exp\ t))) \longrightarrow \varrho * x)\ at\text{-}top$
⟨*proof*⟩

2.1.2 Slowly Varying Functions

abbreviation *slowly-varying* $L \equiv$ *regularly-varying* $L\ 0$

lemma *slowly-varyingI*:
assumes $L \in$ *borel-measurable* (*restrict-space borel* $\{0{<}..\}$)
and $\bigwedge r.\ 0 < r \Longrightarrow 0 < L\ r$
and $\bigwedge x.\ 0 < x \Longrightarrow ((\lambda t.\ L\ (t * x)\ /\ L\ t) \longrightarrow 1)\ at\text{-}top$
shows *slowly-varying* L
⟨*proof*⟩

lemma *slowly-varyingI-total*:
assumes $L \in$ *borel-measurable borel*
and $\bigwedge r.\ 0 < r \Longrightarrow 0 < L\ r$
and $\bigwedge x.\ 0 < x \Longrightarrow ((\lambda t.\ L\ (t * x)\ /\ L\ t) \longrightarrow 1)\ at\text{-}top$
shows *slowly-varying* L
⟨*proof*⟩

lemma *slowly-varying-lim*:
slowly-varying $L \Longrightarrow 0 < x \Longrightarrow ((\lambda t.\ L\ (t * x)\ /\ L\ t) \longrightarrow 1)\ at\text{-}top$
⟨*proof*⟩

lemma *slowly-varying-asymp-equiv*:
assumes *slowly-varying* L
and $x{:}0 < x$
shows $(\lambda t.\ L\ (t * x)) \sim[at\text{-}top]\ (\lambda t.\ L\ t)$
⟨*proof*⟩

lemma *slowly-varying-def2*:

slowly-varying $L \longleftrightarrow$
$L \in$ *borel-measurable* (*restrict-space borel* $\{0<..\}$) $\wedge$
L ‘ $\{0<..\} \subseteq \{0<..\} \wedge$
$(\forall x{>}0.\ (\lambda t.\ L\ (t * x)) \sim[\textit{at-top}]\ (\lambda t.\ L\ t))$
⟨*proof*⟩

lemma *slowly-varying-ln-exp*:
assumes *slowly-varying* L
shows $((\lambda t.\ ln\ (L\ (exp\ (t + x))) - ln\ (L\ (exp\ t))) \longrightarrow 0)$ *at-top*
⟨*proof*⟩

lemma *slowly-varying-const*[*simp*]: $0 < c \Longrightarrow$ *slowly-varying* $(\lambda x.\ c)$
⟨*proof*⟩

lemma *regularly-varying-powr*[*simp*]: *regularly-varying* $(\lambda x.\ x\ powr\ \varrho)\ \varrho$
⟨*proof*⟩

lemma *regularly-varying-id*[*simp*]: *regularly-varying* $(\lambda x.\ x)\ 1$
⟨*proof*⟩

lemma *slowly-varying-1*: *slowly-varying* $(\lambda x.\ 1)$
⟨*proof*⟩

lemma *regularly-varying-mult*:
assumes U:*regularly-varying* $U\ \varrho$
and V:*regularly-varying* $V\ \eta$
shows *regularly-varying* $(\lambda x.\ U\ x * V\ x)\ (\varrho + \eta)$
⟨*proof*⟩

corollary
shows *regularly-varying-cmult*:
$0 < c \Longrightarrow$ *regularly-varying* $U\ \varrho \Longrightarrow$ *regularly-varying* $(\lambda x.\ c * U\ x)\ \varrho$
and *regularly-varying-multc*:
$0 < c \Longrightarrow$ *regularly-varying* $U\ \varrho \Longrightarrow$ *regularly-varying* $(\lambda x.\ U\ x * c)\ \varrho$
⟨*proof*⟩

lemma *regularly-varying-inverse*:
assumes *regularly-varying* $U\ \varrho$
shows *regularly-varying* $(\lambda x.\ 1\ /\ U\ x)\ (-\ \varrho)$
⟨*proof*⟩

corollary *regularly-varying-inverse-iff*:
regularly-varying $(\lambda x.\ 1\ /\ U\ x)\ (-\ \varrho) \longleftrightarrow$ *regularly-varying* $U\ \varrho$
⟨*proof*⟩

lemma *regularly-varying-divide*:
assumes U:*regularly-varying* $U\ \varrho$
and V:*regularly-varying* $V\ \eta$
shows *regularly-varying* $(\lambda x.\ U\ x\ /\ V\ x)\ (\varrho - \eta)$

⟨*proof*⟩

lemma *regularly-varying-comp-cmult*:
assumes *rv*:*regularly-varying U ϱ*
and *c*:*0 < c*
shows *regularly-varying* (*λx. U* (*c ∗ x*)) *ϱ*
⟨*proof*⟩

lemma *regularly-varying-comp-multc*:
assumes *regularly-varying U ϱ*
and *0 < c*
shows *regularly-varying* (*λx. U* (*x ∗ c*)) *ϱ*
⟨*proof*⟩

lemma *regularly-varying-iff-slowly-varying-divide-powr*:
regularly-varying U ϱ ⟷ slowly-varying (*λx. U x / x powr ϱ*)
⟨*proof*⟩

corollary *regularly-varying-decomp*:
assumes *regularly-varying U ϱ*
obtains *L* **where**
slowly-varying L
⋀x. 0 < x ⟹ U x = x powr ϱ ∗ L x
⟨*proof*⟩

lemma *slowly-varying-ln*:*slowly-varying* (*λx. ln* (*1 + x*))
⟨*proof*⟩

Exercise 2.8 of the book [3].

lemma *slowly-varying-tendsto-gr-le-eq*:
fixes *U* :: *'a* :: *linordered-field ⇒ 'b*::{*real-normed-div-algebra*, *field*}
shows (*∀ x>1. ((λt. U* (*t ∗ x*) */ U t*) *⟶⟶ 1*) *at-top*)
⟷ (*∀ x. 0 < x ⟶ x < 1 ⟶* ((*λt. U* (*t ∗ x*) */ U t*) *⟶⟶ 1*) *at-top*)
⟨*proof*⟩

lemma *slowly-varying-iff-all-gr-1*:
assumes *L ∈ borel-measurable* (*restrict-space borel* {*0<..*})
and *⋀x. 0 < x ⟹ 0 < L x*
shows *slowly-varying L ⟷* (*∀ x>1. ((λt. L* (*t ∗ x*) */ L t*) *⟶⟶ 1*) *at-top*)
⟨*proof*⟩

lemma *slowly-varying-iff-ne-1*:
assumes *L-meas*:*L ∈ borel-measurable* (*restrict-space borel* {*0<..*})
and *L-pos*:*⋀x. 0 < x ⟹ 0 < L x*
and *L-mono*:*mono-on* {*0<..*} *L*
shows *slowly-varying L ⟷* (*∃ x≠1. x > 0 ∧ ((λt. L* (*t ∗ x*) */ L t*) *⟶⟶ 1*) *at-top*)
⟨*proof*⟩

2.2 Uniform Convergence Theorem on Intervals

theorem *slowly-varying-UCT*:
assumes *sv*:*slowly-varying L*
and *a*: $0 < a$
shows *uniform-limit* $\{a..b\}$ $(\lambda t\ x.\ L\ (t * x)\ /\ L\ t)$ $(\lambda x.\ 1)$ *at-top*
⟨*proof*⟩

corollary *regularly-varying-UCT-interval*:
assumes *sv*:*regularly-varying U* ϱ
and *a*: $0 < a$
shows *uniform-limit* $\{a..b\}$ $(\lambda t\ x.\ U\ (t * x)\ /\ U\ t)$ $(\lambda x.\ x\ powr\ \varrho)$ *at-top*
⟨*proof*⟩

corollary *regularly-varying-UCT-interval-ln-exp*:
assumes *rv*:*regularly-varying U* ϱ
defines $h \equiv \lambda x.\ ln\ (U\ (exp\ x))$
shows *uniform-limit* $\{a..b\}$ $(\lambda t\ x.\ h\ (t + x) - h\ t)$ $(\lambda x.\ \varrho * x)$ *at-top*
⟨*proof*⟩

corollary *slowly-varying-UCT-interval-ln-exp*:
assumes *sv*:*slowly-varying L*
defines $h \equiv \lambda x.\ ln\ (L\ (exp\ x))$
shows *uniform-limit* $\{a..b\}$ $(\lambda t\ x.\ h\ (t + x) - h\ t)$ $(\lambda x.\ 0)$ *at-top*
⟨*proof*⟩

corollary *regularly-varying-compose*:
assumes *rv1*:*regularly-varying U* ϱ
and *rv2*:*regularly-varying V* η
and *lim*:*filterlim U at-top at-top*
shows *regularly-varying* $(\lambda x.\ V\ (U\ x))$ $(\varrho * \eta)$
⟨*proof*⟩

lemma
assumes *sv*:*slowly-varying L*
defines $h \equiv \lambda x.\ ln\ (L\ (exp\ x))$
shows *slowly-varying-eventually-bounded-interval*:
$\forall_F\ x$ *in at-top*. $\forall\, a{\geq}x.\ \forall\, b{\geq}a.$ *bounded* $(L\ `\ \{a..b\})$
and *slowly-varying-eventually-bounded-interval-strong*:
$\forall_F\ x$ *in at-top*. $\forall\, a{\geq}x.\ \forall\, b{\geq}a.\ \exists\, A{>}0.\ \exists\, B.\ L\ `\ \{a..b\} \subseteq \{A..B\}$
and *slowly-varying-eventually-bounded-interval-ln-exp*:
$\forall_F\ x$ *in at-top*. $\forall\, a{\geq}x.\ \forall\, b{\geq}a.$ *bounded* $(h\ `\ \{a..b\})$
⟨*proof*⟩

lemma
assumes *rv*:*regularly-varying U* ϱ
defines $h \equiv \lambda x.\ ln\ (U\ (exp\ x))$
shows *regularly-varying-eventually-bounded-interval*:
$\forall_F\ x$ *in at-top*. $\forall\, a{\geq}x.\ \forall\, b.$ *bounded* $(U\ `\ \{a..b\})$

and *regularly-varying-eventually-bounded-interval-strong*:
$\forall_F\ x\ in\ at\text{-}top.\ \forall\, a{\geq}x.\ \forall\, b.\ \exists\, A{>}0.\ \exists\, B.\ U\ `\ \{a..b\} \subseteq \{A..B\}$
and *regularly-varying-eventually-bounded-interval-ln-exp*:
$\forall_F\ x\ in\ at\text{-}top.\ \forall\, a{\geq}x.\ \forall\, b.\ bounded\ (h\ `\ \{a..b\})$
and *regularly-varying-eventually-locally-integrable*:
$\forall_F\ x\ in\ at\text{-}top.\ \forall\, a{\geq}x.\ \forall\, b.\ set\text{-}integrable\ lborel\ \{a..b\}\ U$
and *regularly-varying-eventually-locally-integrable-ln-exp*:
$\forall_F\ x\ in\ at\text{-}top.\ \forall\, a{\geq}x.\ \forall\, b.\ set\text{-}integrable\ lborel\ \{a..b\}\ h$
⟨*proof*⟩

2.3 Weakening the Definition of Regularly Varying Functions

We prove Proposition 2.3 in [3]. In the definition of a regularly varying function, $U(tx)/U(t)$ needs to converge to x^ρ as $t \to \infty$. However, we can weaken this assumption to: $U(tx)/U(t)$ converges to some function h.

lemma
assumes [*measurable*]:$U \in borel\text{-}measurable\ (restrict\text{-}space\ borel\ \{0{<}..\})$
and U:$\bigwedge x.\ 0 < x \Longrightarrow 0 < U\ x$
and *lim*:$\bigwedge x.\ x > 0 \Longrightarrow ((\lambda t.\ U\ (t * x)\ /\ U\ t) \longrightarrow h\ x)\ at\text{-}top$
shows *divide-limit-power*:$\bigwedge x.\ x > 0 \Longrightarrow h\ x = x\ powr\ ln\ (h\ (exp\ 1))$
and *divide-limit-imp-regularly-varying*:$regularly\text{-}varying\ U\ (ln\ (h\ (exp\ 1)))$
and *divide-limit-imp-regularly-varying-ex*:$\exists\, \varrho.\ regularly\text{-}varying\ U\ \varrho$
⟨*proof*⟩

lemma
assumes *mono*:$mono\text{-}on\ \{0{<}..\}\ U$ **and** *U-pos*: $\bigwedge x.\ 0 < x \Longrightarrow 0 < U\ x$
and *b-pos*:$\forall_F\ n\ in\ sequentially.\ 0 < b\ n$
and *l-pos*:$\forall_F\ n\ in\ sequentially.\ 0 < l\ n$
and *lim-b*:$filterlim\ b\ at\text{-}top\ sequentially$
and *l-equiv*:$l \sim[sequentially]\ (\lambda n.\ l\ (Suc\ n))$
and *lim*:$\bigwedge x.\ 0 < x \Longrightarrow (\lambda n.\ l\ n * U\ (b\ n * x)) \longrightarrow h\ x$
and *h-pos*: $\bigwedge x.\ 0 < x \Longrightarrow 0 < h\ x$
shows *regularly-varying-sequentialy-lim*:
$\bigwedge x.\ 0 < x \Longrightarrow ((\lambda t.\ U\ (t * x)\ /\ U\ t) \longrightarrow h\ x\ /\ h\ 1)\ at\text{-}top$
and *regularly-varying-sequentialy*:$regularly\text{-}varying\ U\ (ln\ (h\ (exp\ 1)\ /\ h\ 1))$
and *regularly-varying-sequentialy-ex*:$\exists\, \varrho.\ regularly\text{-}varying\ U\ \varrho$
⟨*proof*⟩

lemma *regularly-varying-integral-infinity*:
assumes *rv*:$regularly\text{-}varying\ U\ \varrho$
assumes *rho*:$-\ 1 < \varrho$
shows $(\int^+ t{\in}\{a{<}..\}.\ ennreal\ (U\ t)\ \partial lborel) = \top$
⟨*proof*⟩

2.4 Karamata's Representation Theorem

Lemma 1.3.1 in [1].

lemma *Karamata-representation-ln-exp-slowly-varying*:
fixes L
defines $h \equiv (\lambda x.\ ln\ (L\ (exp\ x)))$
assumes *sv*:*slowly-varying* L
and A: $0 < A$
and *bdd*:$\bigwedge a\ b.\ A \leq a \Longrightarrow \exists\, X{>}0.\ \exists\, Y.\ L\ `\ \{a..b\} \subseteq \{X..Y\}$
shows $\exists\, c\ e\ C.$
$(\forall\, x \geq ln\ A.\ h\ x = c\ x + (\int u{\in}\{ln\ A{<}..x\}.\ e\ u\ \partial lborel))$
$\wedge\ (\forall\, x \geq ln\ A.\ set\text{-}integrable\ lborel\ \{ln\ A{<}..x\}\ e)$
$\wedge\ e \in borel\text{-}measurable\ borel$
$\wedge\ (e \longrightarrow 0)\ at\text{-}top$
$\wedge\ bounded\ (e\ `\ \{ln\ A..\})$
$\wedge\ c \in borel\text{-}measurable\ borel$
$\wedge\ (c \longrightarrow C)\ at\text{-}top$
⟨*proof*⟩

lemma *Karamata-representation-ln-exp2*:
fixes U
defines $h \equiv (\lambda x.\ ln\ (U\ (exp\ x)))$
assumes *rv*:*regularly-varying* $U\ \varrho$
and A: $0 < A$
and *bdd*:$\bigwedge a\ b.\ A \leq a \Longrightarrow \exists\, X{>}0.\ \exists\, Y.\ U\ `\ \{a..b\} \subseteq \{X..Y\}$
shows $\exists\, c\ e\ C.$
$(\forall\, x \geq ln\ A.\ h\ x = c\ x + \varrho * ln\ A + (\int u{\in}\{ln\ A{<}..x\}.\ \varrho + \ e\ u\ \partial lborel))$
$\wedge\ (\forall\, x \geq ln\ A.\ set\text{-}integrable\ lborel\ \{ln\ A{<}..x\}\ e)$
$\wedge\ e \in borel\text{-}measurable\ borel$
$\wedge\ (e \longrightarrow 0)\ at\text{-}top$
$\wedge\ bounded\ (e\ `\ \{ln\ A..\})$
$\wedge\ c \in borel\text{-}measurable\ borel$
$\wedge\ (c \longrightarrow C)\ at\text{-}top$
⟨*proof*⟩

lemma *Karamata-representation-ln-exp*:
fixes U
defines $h \equiv (\lambda x.\ ln\ (U\ (exp\ x)))$
assumes *rv*:*regularly-varying* $U\ \varrho$
shows $\exists\, A{>}0.\ \exists\, c\ e\ C.$
$(\forall\, x \geq ln\ A.\ h\ x = c\ x + \varrho * ln\ A + (\int u{\in}\{ln\ A{<}..x\}.\ \varrho + \ e\ u\ \partial lborel))$
$\wedge\ (\forall\, x \geq ln\ A.\ set\text{-}integrable\ lborel\ \{ln\ A{<}..x\}\ e)$
$\wedge\ e \in borel\text{-}measurable\ borel$
$\wedge\ (e \longrightarrow 0)\ at\text{-}top$
$\wedge\ bounded\ (e\ `\ \{ln\ A..\})$
$\wedge\ c \in borel\text{-}measurable\ borel$
$\wedge\ (c \longrightarrow C)\ at\text{-}top$
⟨*proof*⟩

This equation for *ln* is useful for transforming equations in proofs.

lemma *Karamata-representation-ex1*:
assumes *rv*:*regularly-varying* $U\ \varrho$

and $A{:}0 < A$
and $bdd{:}\bigwedge a\ b.\ A \leq a \Longrightarrow \exists\, X{>}0.\ \exists\, Y.\ U\ `\ \{a..b\} \subseteq \{X..Y\}$
shows $\exists\, c\ r\ C.$
$(\forall\, x \geq A.\ U\ x = c\ x * exp\ (\int t{\in}\{A{<}..x\}.\ r\ t\ /\ t\ \partial lborel))$
$\wedge\ (\forall\, x \geq A.\ set\text{-}integrable\ lborel\ \{A{<}..x\}\ (\lambda t.\ r\ t\ /\ t))$
$\wedge\ r \in borel\text{-}measurable\ borel$
$\wedge\ (r \longrightarrow \varrho)\ at\text{-}top$
$\wedge\ bounded\ (r\ `\ \{A..\})$
$\wedge\ c \in borel\text{-}measurable\ borel$
$\wedge\ (\forall\, x.\ 0 < c\ x)$
$\wedge\ (c \longrightarrow C)\ at\text{-}top$
$\wedge\ 0 < C$
⟨*proof*⟩

lemma *Karamata-representation-ex*:
assumes *rv*:*regularly-varying U ϱ*
shows $\exists\, A{>}0.\ \exists\, c\ r\ C.$
$(\forall\, x \geq A.\ U\ x = c\ x * exp\ (\int t{\in}\{A{<}..x\}.\ r\ t\ /\ t\ \partial lborel))$
$\wedge\ (\forall\, x \geq A.\ set\text{-}integrable\ lborel\ \{A{<}..x\}\ (\lambda t.\ r\ t\ /\ t))$
$\wedge\ r \in borel\text{-}measurable\ borel$
$\wedge\ (r \longrightarrow \varrho)\ at\text{-}top$
$\wedge\ bounded\ (r\ `\ \{A..\})$
$\wedge\ c \in borel\text{-}measurable\ borel$
$\wedge\ (\forall\, x.\ 0 < c\ x)$
$\wedge\ (c \longrightarrow C)\ at\text{-}top$
$\wedge\ 0 < C$
⟨*proof*⟩

corollary *Karamata-representation-ex-total*:
assumes *rv*:*regularly-varying U ϱ*
shows $\exists\, c\ r\ C.$
$(\forall\, x.\ U\ x = c\ x * exp\ (\int t{\in}\{0{<}..x\}.\ r\ t\ /\ t\ \partial lborel))$
$\wedge\ (\forall\, x\ y.\ set\text{-}integrable\ lborel\ \{..y\}\ (\lambda t.\ r\ t\ /\ t))$
$\wedge\ r \in borel\text{-}measurable\ borel$
$\wedge\ (r \longrightarrow \varrho)\ at\text{-}top$
$\wedge\ bounded\ (range\ r)$
$\wedge\ c \in borel\text{-}measurable\ (restrict\text{-}space\ borel\ \{0{<}..\})$
$\wedge\ (\forall\, x{>}0.\ 0 < c\ x)$
$\wedge\ (c \longrightarrow C)\ at\text{-}top$
$\wedge\ 0 < C$
⟨*proof*⟩

lemma *Karamata-representation-iff*:
assumes *U-meas*:$U \in borel\text{-}measurable\ (restrict\text{-}space\ borel\ \{0{<}..\})$
and *U-pos*:$\bigwedge x.\ 0 < x \Longrightarrow 0 < U\ x$
shows *regularly-varying U ϱ* $\longleftrightarrow$
$(\exists\, A{>}0.\ \exists\, c\ r\ C.$
$(\forall\, x \geq A.\ U\ x = c\ x * exp\ (\int t{\in}\{A{<}..x\}.\ r\ t\ /\ t\ \partial lborel))$
$\wedge\ (\forall\, x \geq A.\ set\text{-}integrable\ lborel\ \{A{<}..x\}\ (\lambda t.\ r\ t\ /\ t))$

$\wedge\ (r \longrightarrow \varrho)$ *at-top*
$\wedge\ (c \longrightarrow C)$ *at-top*
$\wedge\ 0 < C)$
⟨*proof*⟩

2.4.1 Theorems Obtained from Karamata's Representation Theorem

lemma
assumes *rv*:*regularly-varying U* ϱ
shows *regularly-varying-pos-at-top*:*0* $<$ $\varrho \Longrightarrow$ *filterlim U at-top at-top*
and *regularly-varying-neg-at-top*:$\varrho < 0 \Longrightarrow (U \longrightarrow 0)$ *at-top*
⟨*proof*⟩

2.4.2 Uniform Convergence Theorem for $\rho \neq 0$

corollary *regularly-varying-pos-UCT*:
assumes *rv*:*regularly-varying U* ϱ
and *bdd*:$\bigwedge x.\ 0 < x \Longrightarrow$ *bounded* $(U$ ‘ $\{0<..x\})$
and *rho*:*0* $<$ ϱ
and *a*: *0* $<$ *a*
shows *uniform-limit* $\{0<..a\}$ $(\lambda t\ x.\ U\ (t * x)\ /\ U\ t)$ $(\lambda x.\ x\ powr\ \varrho)$ *at-top*
⟨*proof*⟩

corollary *regularly-varying-neg-UCT*:
assumes *rv*:*regularly-varying U* ϱ
and *rho*:$\varrho < 0$
and *a*: *0* $<$ *a*
shows *uniform-limit* $\{a..\}$ $(\lambda t\ x.\ U\ (t * x)\ /\ U\ t)$ $(\lambda x.\ x\ powr\ \varrho)$ *at-top*
⟨*proof*⟩

2.4.3 Potter Bounds

corollary
assumes *rv*:*regularly-varying U* ϱ
and *A*:*1* $<$ *A* **and** *d*:*0* $<$ *d*
shows *Potter-bounds1*:
$\exists\, T{>}0.\ \forall\, x{\geq}T.\ \forall\, y{\geq}x.\ U\ y\ /\ U\ x \leq A * (y\ /\ x)\ powr\ (\varrho + d)$ (**is** *?g1*)
and *Potter-bounds2*:
$\exists\, T{>}0.\ \forall\, y{\geq}T.\ \forall\, x{\geq}y.\ U\ y\ /\ U\ x \leq A * (y\ /\ x)\ powr\ (\varrho - d)$ (**is** *?g2*)
and *Potter-bounds*:
$\exists\, T{>}0.\ \forall\, x{\geq}T.\ \forall\, y{\geq}T.\ U\ y\ /\ U\ x \leq A * max\ ((y\ /\ x)\ powr\ (\varrho + d))\ ((y\ /\ x)\ powr\ (\varrho - d))$
(**is** *?g3*)
⟨*proof*⟩

corollary
assumes *rv*:*regularly-varying U* ϱ
and *A*:*A* $<$ *1* **and** *d*:*0* $<$ *d*
shows *Potter-bounds-lower1*:

$\exists\, T{>}0.\ \forall\, x{\geq}T.\ \forall\, y{\geq}x.\ A * (y\ /\ x)\ powr\ (\varrho - d) \leq U\ y\ /\ U\ x$ (**is** *?g1*)
and *Potter-bounds-lower2*:
$\exists\, T{>}0.\ \forall\, y{\geq}T.\ \forall\, x{\geq}y.\ A * (y\ /\ x)\ powr\ (\varrho + d) \leq U\ y\ /\ U\ x$ (**is** *?g2*)
and *Potter-bounds-lower*:
$\exists\, T{>}0.\ \forall\, x{\geq}T.\ \forall\, y{\geq}T.\ A * min\ ((y\ /\ x)\ powr\ (\varrho + d))\ ((y\ /\ x)\ powr\ (\varrho - d))$
$\leq U\ y\ /\ U\ x$
(**is** *?g3*)
⟨*proof*⟩

2.5 Karamata's Theorem

2.5.1 Direct Half

lemma *Karamata-theorem-half1*:
assumes *rv*: *regularly-varying U ϱ*
and *a*: $0 \leq a$ **and** *rho*:$-\ 1 < \varrho$
and *integrable*:$\bigwedge c\ d.\ a \leq c \Longrightarrow$ *set-integrable lborel* $\{c{<}..d\}$ *U*
shows $((\lambda t.\ (\int x{\in}\{a{<}..t\}.\ U\ x\ \partial lborel)\ /\ (t * U\ t)) \longrightarrow 1\ /\ (\varrho + 1))$ *at-top*
⟨*proof*⟩

lemma *Karamata-theorem-half-uminus1*:
assumes *rv*:*regularly-varying U* $(-\ 1)$
and *a*: $0 \leq a$
and *integrable*:$\bigwedge c\ d.\ a \leq c \Longrightarrow$ *set-integrable lborel* $\{c{<}..d\}$ *U*
shows *filterlim* $(\lambda t.\ (\int x{\in}\{a{<}..t\}.\ U\ x\ \partial lborel)\ /\ (t * U\ t))$ *at-top at-top*
⟨*proof*⟩

corollary *Karamata-theorem-half1-inverse*:
assumes *rv*: *regularly-varying U ϱ*
and *rho*:$-\ 1 \leq \varrho$ **and** *a*: $0 \leq a$
and *integrable*:$\bigwedge c\ d.\ a \leq c \Longrightarrow$ *set-integrable lborel* $\{c{<}..d\}$ *U*
shows $((\lambda t.\ (t * U\ t)\ /\ (\int x{\in}\{a{<}..t\}.\ U\ x\ \partial lborel)) \longrightarrow \varrho + 1)$ *at-top*
⟨*proof*⟩

lemma *Karamata-theorem-half1-ex*:
assumes *rv*: *regularly-varying U ϱ*
and *rho*:$-\ 1 \leq \varrho$
shows $\exists\, a.\ ((\lambda t.\ (t * U\ t)\ /\ (\int x{\in}\{a{<}..t\}.\ U\ x\ \partial lborel)) \longrightarrow \varrho + 1)$ *at-top*
⟨*proof*⟩

lemma *regularly-varying-integral-limit*:
assumes *rv*: *regularly-varying U ϱ*
and *rho*:$-\ 1 \leq \varrho$ **and** *a*: $0 \leq a$
and *integrable*:$\bigwedge c\ d.\ a \leq c \Longrightarrow$ *set-integrable lborel* $\{c{<}..d\}$ *U*
and *x*: $0 < x$
shows $((\lambda t.\ (\int s{\in}\{a{<}..t * x\}.\ U\ s\ \partial lborel)\ /\ (\int s{\in}\{a{<}..t\}.\ U\ s\ \partial lborel))$
$\longrightarrow x\ powr\ (\varrho + 1))$ *at-top*
(**is** $(?g \longrightarrow x\ powr\ (\varrho + 1))$ *at-top*)
⟨*proof*⟩

lemma
assumes *rv*: *regularly-varying U ϱ*
and *rho*:*ϱ < − 1*
shows *Karamata-theorem-half2*:*((λt. (∫ x∈{t<..}. U x ∂lborel) / (t ∗ U t))*
⟶ − 1 / (ϱ + 1)) at-top
and *regularly-varying-tail-integrable*:*∃ T>0. set-integrable lborel {T<..} U*
⟨*proof*⟩

lemma *Karamata-theorem-half-uminus2-at-top*:
assumes *rv*: *regularly-varying U (− 1)*
and *a*:*0 ≤ a*
and *integ*:*set-integrable lborel {a<..} U*
shows *filterlim (λt. (∫ x∈{t<..}. U x ∂lborel) / (t ∗ U t)) at-top at-top*
⟨*proof*⟩

corollary *Karamata-theorem-half-uminus2*:
assumes *rv*: *regularly-varying U (− 1)*
and *a*:*0 ≤ a*
and *integ*:*set-integrable lborel {a<..} U*
shows *((λt. (t ∗ U t) / (∫ x∈{t<..}. U x ∂lborel)) ⟶ 0) at-top*
⟨*proof*⟩

lemma *regularly-varying-integral-tail-limit*:
assumes *rv*: *regularly-varying U ϱ*
and *rho*:*ϱ < − 1*
and *x*: *0 < x*
shows *((λt. (∫ s∈{t ∗ x<..}. U s ∂lborel) / (∫ s∈{t<..}. U s ∂lborel))*
⟶ x powr (ϱ + 1)) at-top
⟨*proof*⟩

lemma *regularly-varying-integral-tail-limit2*:
assumes *rv*: *regularly-varying U (− 1)*
and *integ*:*set-integrable lborel {a<..} U*
and *x*:*0 < x*
shows *((λt. (∫ s∈{t ∗ x<..}. U s ∂lborel) / (∫ s∈{t<..}. U s ∂lborel)) ⟶ 1)*
at-top
⟨*proof*⟩

lemma *regularly-varying-integral-measurable*:
assumes *rv*: *regularly-varying U ϱ*
and *a*:*0 ≤ a*
shows *(λx. (∫ s∈{a<..x}. U s ∂lborel)) ∈ borel-measurable borel*
⟨*proof*⟩

lemma *regularly-varying-tail-integral-measurable*:
assumes *rv*: *regularly-varying U ϱ*
shows *(λx. (∫ s∈{x<..}. U s ∂lborel)) ∈ borel-measurable (restrict-space borel*
{0<..})

⟨*proof*⟩

lemma *regularly-varying-integral*:
assumes *rv*:*regularly-varying U ϱ*
and *rho*:− *1* ≤ *ϱ*
and *integrable*:⋀*x*. *set-integrable lborel* {*0*<..*x*} *U*
shows *regularly-varying* (λ*x*. (∫ *t*∈{*0*<..*x*}. *U t ∂lborel*)) (*ϱ* + *1*)
⟨*proof*⟩

lemma *regularly-varying-tail-integral*:
assumes *rv*:*regularly-varying U ϱ*
and *rho*:*ϱ* ≤ − *1*
and *integrable*:*set-integrable lborel* {*0*<..} *U*
shows *regularly-varying* (λ*x*. (∫ *t*∈{*x*<..}. *U t ∂lborel*)) (*ϱ* + *1*)
⟨*proof*⟩

2.5.2 Converse Half

lemma
assumes *U-pos*:⋀*r*. *0* < *r* ⟹ *0* < *U r*
and *U-meas*:*U* ∈ *borel-measurable* (*restrict-space borel* {*0*<..})
and *rho*:− *1* < *ϱ* **and** *a*[*arith*]:*0* ≤ *a*
and *lim*: ((λ*t*. (*t* ∗ *U t*) / (∫ *x*∈{*a*<..*t*}. *U x ∂lborel*)) ⟶ *ϱ* + *1*) *at-top*
shows *Karamata-theorem-converse1*:*regularly-varying U ϱ*
and *Karamata-theorem-converse1-integrable*:
⋀*c d*. *a* ≤ *c* ⟹ *set-integrable lborel* {*c*..*d*} *U*
⟨*proof*⟩

lemma
assumes *U-pos*:⋀*r*. *0* < *r* ⟹ *0* < *U r*
and *U-meas*:*U* ∈ *borel-measurable* (*restrict-space borel* {*0*<..})
and *rho*:− *1* > *ϱ*
and *lim*: ((λ*t*. *t* ∗ *U t* / (∫ *x*∈{*t*<..}. *U x ∂lborel*)) ⟶ − (*ϱ* + *1*)) *at-top*
shows *Karamata-theorem-converse2*:*regularly-varying U ϱ*
and *Karamata-theorem-converse2-itegrable*: ∃ *a*>*0*. *set-integrable lborel* {*a*..} *U*
⟨*proof*⟩

2.6 Properties Related to Derivative

Proposition 2.5 (a) [3].

lemma *deriv-ratio-regularly-varying-uniform*:
fixes *U U′* :: *real* ⇒ *real* **and** *ϱ a b* :: *real*
assumes *t0*: *0* < *t0*
and *deriv*: ⋀*t*. *t0* < *t* ⟹ (*U has-real-derivative U′ t*) (*at t*)
and *U′cont*: *continuous-on* {*t0*<..} *U′*
and *Upos*: ⋀*t*. *t0* < *t* ⟹ *0* < *U t*
and *lim*: ((λ*t*. *t* ∗ *U′ t* / *U t*) ⟶ *ϱ*) *at-top*
and *a*: *0* < *a*
shows *uniform-limit* {*a*..*b*} (λ*t x*. *U* (*t* ∗ *x*) / *U t*) (λ*x*. *x powr ϱ*) *at-top*

⟨*proof*⟩

corollary *deriv-ratio-regularly-varying*:
fixes $U\ U' :: real \Rightarrow real$ **and** $\gamma\ x :: real$
assumes $0 < t0$
and $\bigwedge t.\ t0 < t \Longrightarrow (U\ has\text{-}real\text{-}derivative\ U'\ t)\ (at\ t)$
and *continuous-on* $\{t0<..\}\ U'$
and $\bigwedge t.\ t0 < t \Longrightarrow 0 < U\ t$
and $((\lambda t.\ t * U'\ t\ /\ U\ t) \longrightarrow \gamma)\ at\text{-}top$
and $0 < x$
shows $((\lambda t.\ U\ (t * x)\ /\ U\ t) \longrightarrow x\ powr\ \gamma)\ at\text{-}top$
⟨*proof*⟩

lemma *deriv-ratio-regularly-varying-uminus*:
fixes $U\ u :: real \Rightarrow real$ **and** $\gamma\ L\ x :: real$
assumes *t0*: $0 < t0$
and *Uderiv*: $\bigwedge t.\ t0 < t \Longrightarrow (U\ has\text{-}real\text{-}derivative\ u\ t)\ (at\ t)$
and *ucont*: *continuous-on* $\{t0<..\}\ u$
and *Ult*: $\bigwedge t.\ t0 < t \Longrightarrow U\ t < L$
and *lim*: $((\lambda t.\ t * u\ t\ /\ (L - U\ t)) \longrightarrow -\gamma)\ at\text{-}top$
and *x*: $0 < x$
shows $((\lambda t.\ (L - U\ (t * x))\ /\ (L - U\ t)) \longrightarrow x\ powr\ \gamma)\ at\text{-}top$
⟨*proof*⟩

Corollary 1.1.10 [2].

lemma *deriv-diff-ratio-regularly-varying*:
fixes $U\ u\ u' :: real \Rightarrow real$ **and** $\gamma\ x :: real$
assumes *t0*: $0 < t0$
and *U-deriv*: $\bigwedge t.\ t0 < t \Longrightarrow (U\ has\text{-}real\text{-}derivative\ u\ t)\ (at\ t)$
and *u-deriv*: $\bigwedge t.\ t0 < t \Longrightarrow (u\ has\text{-}real\text{-}derivative\ u'\ t)\ (at\ t)$
and *u'-cont*: *continuous-on* $\{t0<..\}\ u'$
and *u-pos*: $\bigwedge t.\ t0 < t \Longrightarrow 0 < u\ t$
and *lim*: $((\lambda t.\ t * u'\ t\ /\ u\ t) \longrightarrow \gamma - 1)\ at\text{-}top$
and *x*: $0 < x$
shows $((\lambda t.\ (U\ (t * x) - U\ t)\ /\ (t * u\ t))$
$\longrightarrow (if\ \gamma = 0\ then\ ln\ x\ else\ (x\ powr\ \gamma - 1)\ /\ \gamma))\ at\text{-}top$
⟨*proof*⟩

end

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