Ramsey Number Bounds

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Abstract

Ramsey's theorem [1] implies that for any given natural numbers k and l, there exists some $R(k,l)$ such that a graph having at least $R(k,l)$ vertices must have either a clique of cardinality k or an anticlique (independent set) of cardinality l. Equivalently, for a *complete* graph of size $R(k,l)$, every red/blue colouring of the edges must yield an entirely red k-clique or an entirely blue l-clique. Although $R(k,l)$ is for practical purposes impossible to calculate from k and l, some upper and lower bounds are known. The celebrated probabilistic argument by Paul Erdős is formalised here, with various of its consequences.

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1 Lower bounds for Ramsey numbers

Probabilistic proofs of lower bounds for Ramsey numbers. Variations and strengthenings of the classical Erdős–Szekeres upper bound, which is proved in the original Ramsey theory Also a number of simple properties of Ramsey numbers, including the equivalence of the clique/anticlique and edge colouring definitions.

theory *Ramsey-Bounds*
  imports
    HOL−Library.Ramsey
    HOL−Library.Infinite-Typeclass
    HOL−Probability.Probability
    Undirected-Graph-Theory.Undirected-Graph-Basics
begin

1.1 Preliminaries

Elementary facts involving binomial coefficients

lemma *choose-two-real*: *of-nat* (*n choose 2*) = *real n* ∗ (*real n* − *1*) / *2*
⟨*proof*⟩

lemma *add-choose-le-power*: (*n* + *k*) *choose n* ≤ *Suc k* ^ *n*
⟨*proof*⟩

lemma *choose-le-power*: *m choose k* ≤ (*Suc m* − *k*) ^ *k*
⟨*proof*⟩

lemma *sum-nsets-one*: $(\sum U{\in}[V]^{Suc\ 0}.\ f\ U) = (\sum x{\in}V.\ f\ \{x\})$
⟨*proof*⟩

1.2 Relating cliques to graphs; Ramsey numbers

When talking about Ramsey numbers, sometimes cliques are best, sometimes colour maps

lemma *nsets2-eq-all-edges*: $[A]^2$ = *all-edges A*
⟨*proof*⟩

lemma *indep-eq-clique-compl*: *indep R E* = *clique R* (*all-edges R* − *E*)
⟨*proof*⟩

lemma *all-edges-subset-iff-clique*: *all-edges K* ⊆ *E* ⟷ *clique K E*
⟨*proof*⟩

definition *clique-indep* ≡ λ*m n K E*. *card K* = *m* ∧ *clique K E* ∨ *card K* = *n* ∧ *indep K E*

lemma *clique-all-edges-iff*: *clique K* (*E* ∩ *all-edges K*) ⟷ *clique K E*

⟨*proof*⟩

lemma *indep-all-edges-iff*: *indep K* (*E* ∩ *all-edges K*) ⟷ *indep K E*
⟨*proof*⟩

lemma *clique-indep-all-edges-iff*: *clique-indep s t K* (*E* ∩ *all-edges K*) = *clique-indep s t K E*
⟨*proof*⟩

identifying Ramsey numbers (possibly not the minimum) for a given type and pair of integers

definition *is-clique-RN* **where**
is-clique-RN ≡ λ*U*::'*a itself*. λ*m n r*.
(∀ *V*::'*a set*. ∀ *E*. *finite V* ⟶ *card V* ≥ *r* ⟶ (∃ *K*⊆*V*. *clique-indep m n K E*))

could be generalised to allow e.g. any hereditarily finite set

abbreviation *is-Ramsey-number* :: [*nat*,*nat*,*nat*] ⇒ *bool* **where**
is-Ramsey-number m n r ≡ *partn-lst* {..<*r*} [*m*,*n*] *2*

lemma *is-clique-RN-imp-partn-lst*:
fixes *U* :: '*a itself*
assumes *r*: *is-clique-RN U m n r* **and** *inf*: *infinite* (*UNIV*::'*a set*)
shows *partn-lst* {..<*r*} [*m*,*n*] *2*
⟨*proof*⟩

lemma *partn-lst-imp-is-clique-RN*:
fixes *U* :: '*a itself*
assumes *partn-lst* {..<*r*} [*m*,*n*] *2*
shows *is-clique-RN U m n r*
⟨*proof*⟩

All complete graphs of a given cardinality are the same

lemma *is-clique-RN-any-type*:
assumes *is-clique-RN* (*U*::'*a itself*) *m n r infinite* (*UNIV*::'*a set*)
shows *is-clique-RN* (*V*::'*b*::*infinite itself*) *m n r*
⟨*proof*⟩

lemma *is-Ramsey-number-le*:
assumes *is-Ramsey-number m n r* **and** *le*: *m'* ≤ *m n'* ≤ *n*
shows *is-Ramsey-number m' n' r*
⟨*proof*⟩

definition *RN* **where**
RN ≡ λ*m n*. *LEAST r*. *is-Ramsey-number m n r*

lemma *is-Ramsey-number-RN*: *partn-lst* {..< (*RN m n*)} [*m*,*n*] *2*
⟨*proof*⟩

lemma *RN-le*: $[\![$*is-Ramsey-number m n r*$]\!] \Longrightarrow$ *RN m n* $\leq$ *r*
⟨*proof*⟩

lemma *RN-le-ES*: *RN i j* $\leq$ *ES 2 i j*
⟨*proof*⟩

lemma *RN-mono*:
assumes $m' \leq m$ $n' \leq n$
shows *RN* m' n' $\leq$ *RN m n*
⟨*proof*⟩

lemma *indep-iff-clique* [*simp*]: $K \subseteq V \Longrightarrow$ *indep K* (*all-edges V* $-$ *E*) $\longleftrightarrow$ *clique K E*
⟨*proof*⟩

lemma *clique-iff-indep* [*simp*]: $K \subseteq V \Longrightarrow$ *clique K* (*all-edges V* $-$ *E*) $\longleftrightarrow$ *indep K E*
⟨*proof*⟩

lemma *is-Ramsey-number-commute-aux*:
assumes *is-Ramsey-number m n r*
shows *is-Ramsey-number n m r*
⟨*proof*⟩

1.3 Elementary properties of Ramsey numbers

lemma *is-Ramsey-number-commute*: *is-Ramsey-number m n r* $\longleftrightarrow$ *is-Ramsey-number n m r*
⟨*proof*⟩

lemma *RN-commute-aux*: *RN n m* $\leq$ *RN m n*
⟨*proof*⟩

lemma *RN-commute*: *RN m n* $=$ *RN n m*
⟨*proof*⟩

lemma *RN-le-choose*: *RN k l* $\leq$ (*k*+*l choose k*)
⟨*proof*⟩

lemma *RN-le-choose'*: *RN k l* $\leq$ (*k*+*l choose l*)
⟨*proof*⟩

lemma *RN-0* [*simp*]: *RN 0 m* $=$ *0*
⟨*proof*⟩

lemma *RN-1* [*simp*]:
assumes *m*>*0* **shows** *RN* (*Suc 0*) *m* $=$ *Suc 0*
⟨*proof*⟩

lemma *RN-0′* [*simp*]: *RN m 0* = *0* **and** *RN-1′* [*simp*]: *m*>*0* ⟹ *RN m* (*Suc 0*) = *Suc 0*
⟨*proof*⟩

lemma *is-clique-RN-2*: *is-clique-RN TYPE*(*nat*) *2 m m*
⟨*proof*⟩

lemma *RN-2* [*simp*]:
shows *RN 2 m* = *m*
⟨*proof*⟩

lemma *RN-2′* [*simp*]:
shows *RN m 2* = *m*
⟨*proof*⟩

lemma *RN-3plus*:
assumes *k* ≥ *3*
shows *RN k m* ≥ *m*
⟨*proof*⟩

lemma *RN-3plus′*:
assumes *k* ≥ *3*
shows *RN m k* ≥ *m*
⟨*proof*⟩

lemma *clique-iff*: *F* ⊆ *all-edges K* ⟹ *clique K F* ⟷ *F* = *all-edges K*
⟨*proof*⟩

lemma *indep-iff*: *F* ⊆ *all-edges K* ⟹ *indep K F* ⟷ *F* = {}
⟨*proof*⟩

lemma *all-edges-empty-iff*: *all-edges K* = {} ⟷ (∃ *v*. *K* ⊆ {*v*})
⟨*proof*⟩

lemma *Ramsey-number-zero*: ¬ *is-Ramsey-number* (*Suc m*) (*Suc n*) *0*
⟨*proof*⟩

1.4 The product lower bound

lemma *Ramsey-number-times-lower*: ¬ *is-clique-RN* (*TYPE*(*nat*∗*nat*)) (*Suc m*) (*Suc n*) (*m*∗*n*)
⟨*proof*⟩

theorem *RN-times-lower*:
shows *RN* (*Suc m*) (*Suc n*) > *m*∗*n*
⟨*proof*⟩

corollary *RN-times-lower′*:

shows $[\![m>0;\ n>0]\!] \Longrightarrow RN\ m\ n > (m-1)*(n-1)$
⟨*proof*⟩

lemma *RN-eq-0-iff*: $RN\ m\ n = 0 \longleftrightarrow m=0 \lor n=0$
⟨*proof*⟩

lemma *RN-gt1*:
assumes $2 \le k$ $3 \le l$ **shows** $k < RN\ k\ l$
⟨*proof*⟩

lemma *RN-gt2*:
assumes $2 \le k$ $3 \le l$ **shows** $k < RN\ l\ k$
⟨*proof*⟩

1.5 A variety of upper bounds, including a stronger Erdős–Szekeres

lemma *RN-1-le*: $RN\ (Suc\ 0)\ l \le Suc\ 0$
⟨*proof*⟩

lemma *is-Ramsey-number-add*:
assumes $i>1$ $j>1$
and *n1*: *is-Ramsey-number* $(i - 1)\ j\ n1$
and *n2*: *is-Ramsey-number* $i\ (j - 1)\ n2$
shows *is-Ramsey-number* $i\ j\ (n1+n2)$
⟨*proof*⟩

lemma *RN-le-add-RN-RN*:
assumes $i>1$ $j>1$
shows $RN\ i\ j \le RN\ (i - Suc\ 0)\ j + RN\ i\ (j - Suc\ 0)$
⟨*proof*⟩

Cribbed from Bhavik Mehta

lemma *RN-le-choose-strong*: $RN\ k\ l \le (k + l - 2)$ *choose* $(k - 1)$
⟨*proof*⟩

lemma *RN-le-power2*: $RN\ i\ j \le 2\ \hat{}\ (i+j-2)$
⟨*proof*⟩

lemma *RN-le-power4*: $RN\ i\ i \le 4\ \hat{}\ (i-1)$
⟨*proof*⟩

Bhavik Mehta again

lemma *RN-le-argpower*: $RN\ i\ j \le j\ \hat{}\ (i-1)$
⟨*proof*⟩

lemma *RN-le-argpower'*: $RN\ j\ i \le j\ \hat{}\ (i-1)$
⟨*proof*⟩

1.6 Probabilistic lower bounds: the main theorem and applications

General probabilistic setup, omitting the actual probability calculation. Andrew Thomason's proof (private communication)

theorem *Ramsey-number-lower-gen*:
fixes n k::*nat* **and** p::*real*
assumes n: $(n \text{ choose } k) * p \mathbin{\hat{}} (k \text{ choose } 2) + (n \text{ choose } l) * (1 - p) \mathbin{\hat{}} (l \text{ choose } 2) < 1$
assumes *p01*: $0<p$ $p<1$
shows $\neg$ *is-Ramsey-number k l n*
⟨*proof*⟩

Andrew's calculation for the Ramsey lower bound. Symmetric, so works for both colours

lemma *Ramsey-lower-calc*:
fixes s::*nat* **and** t::*nat* **and** p::*real*
assumes $s \geq 3$ $t \geq 3$ $n > 4$
and n: *real* $n \leq \exp ((\text{real } s - 1) * (\text{real } t - 1) / (2*(s+t)))$
defines $p \equiv \text{real } s / (\text{real } s + \text{real } t)$
shows $(n \text{ choose } s) * p \mathbin{\hat{}} (s \text{ choose } 2) < 1/2$
⟨*proof*⟩

Andrew Thomason's specific example

corollary *Ramsey-number-lower-off-diag*:
fixes n k::*nat*
assumes $k \geq 3$ $l \geq 3$ **and** n: *real* $n \leq \exp ((\text{real } k - 1) * (\text{real } l - 1) / (2*(k+l)))$
shows $\neg$ *is-Ramsey-number k l n*
⟨*proof*⟩

theorem *RN-lower-off-diag*:
assumes $s \geq 3$ $t \geq 3$
shows $RN\ s\ t > \exp ((\text{real } s - 1) * (\text{real } t - 1) / (2*(s+t)))$
⟨*proof*⟩

The original Ramsey number lower bound, by Erdős

proposition *Ramsey-number-lower*:
fixes n s::*nat*
assumes $s \geq 3$ **and** n: *real* $n \leq 2 \text{ powr } (s/2)$
shows $\neg$ *is-Ramsey-number s s n*
⟨*proof*⟩

theorem *RN-lower*:
assumes $k \geq 3$
shows $RN\ k\ k > 2 \text{ powr } (k/2)$
⟨*proof*⟩

and trivially, off the diagonal too

corollary *RN-lower-nodiag*:
assumes $k \geq 3$ $l \geq k$
shows *RN k l* $>$ *2 powr* $(k/2)$
⟨*proof*⟩

lemma *powr-half-ge*:
fixes *x*::*real*
assumes $x \geq 4$
shows $x \leq$ *2 powr* $(x/2)$
⟨*proof*⟩

corollary *RN-lower-self*:
assumes $k \geq 3$
shows *RN k k* $> k$
⟨*proof*⟩

end

References

[1] B. Bollobás. *Graph Theory: An Introductory Course*. Springer, 1979.