

Rademacher's Series for the Partition Function

Manuel Eberl

July 8, 2026

Abstract

This entry provides a formalisation of Rademacher's convergent series for the partition function. The partition function $p(n)$ is defined as the number of ways in which a natural number n can be written as the sum of positive integers, without taking the order into account. E.g. $p(3) = 3$ since $3 = 2 + 1 = 1 + 1 + 1$.

Using a contour integration argument based on earlier asymptotic sums for $p(n)$, Rademacher found the remarkable *convergent* series representation

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k \geq 1} A_k(n) \sqrt{k} f(n, k)$$

where

$$A_k(n) = \sum_{\substack{1 \leq h \leq k \\ \gcd(h, k) = 1}} \cos(\pi(s(h, k) - 2nh/k))$$
$$f(x, k) = \frac{d}{dx} \frac{\sinh \left[\frac{\pi}{k} \sqrt{\frac{2}{3} \left(x - \frac{1}{24} \right)} \right]}{\sqrt{x - \frac{1}{24}}}$$

and $s(h, k)$ denotes a Dedekind sum. Decent (albeit not very tight) bounds for the remainder of the truncated sum are also derived.

One consequence of this formula that was also formalised is the asymptotic estimate $p(n) \sim \exp(\pi\sqrt{2/3n})/(4\sqrt{3}n)$.

Rademacher's series also forms the basis for the most efficient known algorithms to compute $p(n)$, but this requires significantly more work and is out of scope for this entry.

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1 Building a path from a list of points

```
theory Polyline_Path
  imports "HOL-Library.Sublist" "Path_Automation.Path_Automation"
begin

lemma weak_wf_pathlist_altdef:
  "weak_wf_pathlist ps  $\longleftrightarrow$  ps  $\neq$  []  $\wedge$  successively ( $\lambda p q.$  pathfinish
p = pathstart q) ps"
  <proof>

lemma path_image_joinpaths_list_subset:
  "ps  $\neq$  []  $\implies$  path_image (joinpaths_list ps)  $\subseteq$  ( $\bigcup_{p \in \text{set } ps} \text{path\_image } p$ )"
  <proof>

definition polyline_path :: "'a :: real_normed_vector list  $\Rightarrow$  real  $\Rightarrow$  'a"
where
  "polyline_path xs =
    (if length xs = 1 then linepath (hd xs) (hd xs) else joinpaths_list
    (map2 linepath xs (tl xs)))"

lemma pathstart_polyline_path:
  assumes "xs  $\neq$  []"
  shows "pathstart (polyline_path xs) = hd xs"
  <proof>

lemma pathfinish_polyline_path:
  assumes "xs  $\neq$  []"
  shows "pathfinish (polyline_path xs) = last xs"
  <proof>

lemma polyline_path_welldefined:
  assumes "xs  $\neq$  []"
  shows "successively ( $\lambda p q.$  pathfinish p = pathstart q) (map2 linepath
xs (tl xs))"
  <proof>

lemma valid_path_polyline_path [simp, intro]: "valid_path (polyline_path
xs)"
  <proof>

lemma path_polyline_path [simp, intro]: "path (polyline_path xs)"
  <proof>

lemma polyline_path_subset_convex:
  assumes "convex A" "set xs  $\subseteq$  A" and [simp]: "xs  $\neq$  []"
```

```

    shows "path_image (polyline_path xs)  $\subseteq$  A"
  <proof>

lemma contour_integral_polyline_path:
  assumes "f contour_integrable_on (polyline_path ps)"
  assumes "ps  $\neq$  []"
  shows "contour_integral (polyline_path ps) f =
        ( $\sum$  i<length ps-1. contour_integral (linepath (ps ! i) (ps
! Suc i)) f)"
  <proof>

end

```

```

theory Rademacher_Series
imports
  "Elliptic_Functions.Dedekind_Eta"
  "Pentagonal_Number_Theorem.Pentagonal_Number_Theorem"
  "Farey_Sequences.Farey_Ford"
  "Bessel.Bessel_Hankel_Integral"
  Polyline_Path
begin

```

1.1 Auxiliary material

```

lemma homotopic_loopsI:
  fixes h :: "real  $\times$  real  $\Rightarrow$  _"
  assumes "continuous_on ({0..1}  $\times$  {0..1}) h"
        "h ' ({0..1}  $\times$  {0..1})  $\subseteq$  s"
        " $\bigwedge$ x. x  $\in$  {0..1}  $\implies$  h (0, x) = p x"
        " $\bigwedge$ x. x  $\in$  {0..1}  $\implies$  h (1, x) = q x"
        " $\bigwedge$ x. x  $\in$  {0..1}  $\implies$  pathfinish (h  $\circ$  Pair x) = pathstart (h
 $\circ$  Pair x)"
  shows "homotopic_loops s p q"
  <proof>

```

```

lemma homotopic_loops_part_circlepath:
  assumes "sphere c r  $\subseteq$  A" and "r  $\geq$  0" and
        "b1 = a1 + 2 * of_int k * pi" and "b2 = a2 + 2 * of_int k *
pi"
  shows "homotopic_loops A (part_circlepath c r a1 b1) (part_circlepath
c r a2 b2)"
  <proof>

```

Any two paths in a simply connected domain are homotopic if they have the same start and end points.

```

lemma simply_connected_imp_homotopic_paths:
  fixes A :: "_ :: real_normed_vector set"
  assumes "simply_connected A"

```

```

assumes "path p" "path_image p  $\subseteq$  A" "path q" "path_image q  $\subseteq$  A"
assumes "pathstart p = pathstart q" "pathfinish p = pathfinish q"
shows "homotopic_paths A p q"
<proof>

```

```

lemma Re_of_rat [simp]: "Re (of_rat x) = of_rat x"
and Im_of_rat [simp]: "Im (of_rat x) = 0"
<proof>

```

```

lemma tl_eq_Nil_iff: "tl xs = []  $\longleftrightarrow$  length xs  $\leq$  1"
<proof>

```

```

lemma convex_on_cong [cong]:
assumes " $\bigwedge x. x \in A \implies f x = g x$ " "A = B"
shows "convex_on A f  $\longleftrightarrow$  convex_on B g"
<proof>

```

```

lemma concave_on_cong [cong]:
assumes " $\bigwedge x. x \in A \implies f x = g x$ " "A = B"
shows "concave_on A f  $\longleftrightarrow$  concave_on B g"
<proof>

```

```

lemma power_powr: "x > 0  $\implies$  (x ^ n) powr y = x powr (real n * y)"
<proof>

```

An integral of a nonincreasing function gives an upper bound for the corresponding sum.

```

lemma infsum_le_nn_integral_antimono:
fixes f :: "nat  $\Rightarrow$  ennreal" and g :: "real  $\Rightarrow$  ennreal"
assumes le: " $\bigwedge n. f n \leq g$  (real (Suc n))" and antimono: "antimono_on {0..} g"
shows " $(\sum n. f n) \leq (\int^+ x \in \{0..\}. g x \, \partial \text{lborel})$ "
<proof>

```

```

lemma infsum_le_integral_antimono:
fixes f :: "nat  $\Rightarrow$  real" and g :: "real  $\Rightarrow$  real"
assumes nonneg: " $\bigwedge n. f n \geq 0$ "
assumes le: " $\bigwedge n. f n \leq g$  (real (Suc n))"
assumes antimono: "antimono_on {0..} g"
assumes integral: "(g has_integral I) {0..}"
shows "summable f" and " $(\sum n. f n) \leq I$ "
<proof>

```

Approximating the Hurwitz zeta function with the corresponding integral gives the upper bound $\zeta(s, a + 1) \leq a^{1-c}/(c - 1)$.

```
lemma hurwitz_zeta_upper_bound:
  fixes a c :: real
  assumes c: "c > 1" and a: "a > 0"
  shows "Re (hurwitz_zeta (a + 1) (of_real c)) ≤ a powr (1 - c) / (c - 1)"
  <proof>
```

```
lemma sin_abs_real:
  assumes "|x| ≤ pi"
  shows "sin |x| = |sin x :: real|"
  <proof>
```

```
lemma sin_lower_bound_real:
  assumes "x ∈ {0..pi/2}"
  shows "sin x ≥ 2 / pi * x"
  <proof>
```

The following formula expresses the length of a chord of the unit circle in terms of the length of the corresponding arc of the unit circle.

```
lemma dist_cis:
  fixes a b :: real
  assumes "dist a b ≤ 2 * pi"
  shows "dist (cis a) (cis b) = 2 * sin (dist a b / 2)"
  <proof>
```

We can obtain upper and lower bounds for length of a chord in terms of the length of the corresponding arc.

```
lemma dist_cis_ge:
  fixes a b :: real
  assumes "dist a b ≤ pi"
  shows "dist (cis a) (cis b) ≥ 2 / pi * dist a b"
  <proof>
```

```
lemma dist_cis_le:
  fixes a b :: real
  assumes "dist a b ≤ 2 * pi"
  shows "dist (cis a) (cis b) ≤ dist a b"
  <proof>
```

1.2 The integration path

The standard integration contour found in e.g. Apostol [?] is to start at i , follow the Ford circle of $\frac{0}{1}$ until the point of tangency with $\frac{1}{n}$, then continue

to follow the Ford circles up to the point of tangency with the next one and then switch to the next one, until we are on the Ford circle for $\frac{1}{2}$, where we stop at $i + 1$.

The problem with this (which Apostol does not mention at all) is that the first and last arc need special treatment. The easiest solution we could find is that of Kong and Teo [1], who shift the problematic first arc (which has radius $\frac{1}{2}$ and goes from i to $\frac{n+i}{n^2+1}$) to the right by 1 and merge it with the other problematic arc (of radius $\frac{1}{2}$, going from $1 - \frac{n-i}{1+n^2}$ to $1 + i$).

definition `rademacher_vertices` :: "nat \Rightarrow complex list" where
`"rademacher_vertices n =`
`map2 Ford_tanp (fareys n) (tl (fareys n)) @ [Ford_tanp (fareys n`
`! 0) (fareys n ! 1) + 1]"`

lemma

`assumes n: "n > 0"`
`shows hd_rademacher_vertices:`
`"hd (rademacher_vertices n) = (of_nat n + i) / of_nat (n ^ 2`
`+ 1)"`
`and last_rademacher_vertices:`
`"last (rademacher_vertices n) = (of_nat n + i) / of_nat (n ^`
`2 + 1) + 1"`
`and second_to_last_rademacher_vertices:`
`"rademacher_vertices n ! (length (rademacher_vertices n) - 2)`
`=`
`(of_nat n ^ 2 - of_nat n + 1 + i) / (1 + of_nat n ^ 2)"`
`<proof>`

lemma `length_rademacher_vertices:`

`"length (rademacher_vertices n) = 1 + ($\sum_{k=1..n}$ totient k)"`
`<proof>`

lemma `length_rademacher_vertices':`

`"n > 0 \implies length (rademacher_vertices n) = length (fareys n)"`
`<proof>`

lemma `rademacher_vertices_nonempty:` "rademacher_vertices n \neq []"

`<proof>`

lemma `Im_rademacher_vertices_pos:` "set (rademacher_vertices n) \subseteq {z. Im z > 0}"

`<proof>`

lemma `rademacher_vertices_on_Ford_circle:`

`assumes "Suc j < length (fareys n)"`
`shows "{rademacher_vertices n ! j, rademacher_vertices n ! Suc j}`
 `\subseteq`
`Ford_circle (fareys n ! Suc j)"`

`<proof>`

An interesting detail is that unlike in the typical presentations of the proof, we do not actually integrate along the Ford circles but rather along straight lines connecting the points of tangency. Since the integrand is holomorphic in the entire half plane anyway, this does not matter: the integral along a circular arc is equal to the integral along the line connecting the end points. Choosing a direct line makes some things simpler. For one, the path is very easy and does not require complicated geometric computations.

definition `rademacher_path` :: "nat \Rightarrow real \Rightarrow complex" where
`"rademacher_path n = polyline_path (rademacher_vertices n)"`

lemma

`assumes n: "n > 0"`
`shows pathstart_rademacher_path:`
`"pathstart (rademacher_path n) = (of_nat n + i) / (1 + of_nat`
`n ^ 2)"`
`and pathfinish_rademacher_path:`
`"pathfinish (rademacher_path n) = (of_nat n + i) / (1 + of_nat`
`n ^ 2) + 1"`
`and valid_path_rademacher_path [simp, intro]: "valid_path (rademacher_path`
`n)"`
`and path_rademacher_path [simp, intro]: "path (rademacher_path n)"`
`<proof>`

lemma `rademacher_path_upper_half_plane`: "path_image (rademacher_path
`n) \subseteq {z. Im z > 0}"`
`<proof>`

1.3 Setting the scene

We express the partition number as a contour integral

$$p_n = \int_z^{z+1} \frac{e^{-2ni\pi u}}{\phi(e^{2i\pi u})} du$$

where z is any complex number in the upper half plane.

This works because the line from z to $z + 1$ is mapped to a circle of radius < 1 around the origin by the map $u \mapsto e^{2i\pi u}$, and the integrand, under the same substitution $q := e^{2i\pi u}$, is mapped to $q^{-n+1}/\phi(q)$ so that our integral is extracting the n -th coefficient of the Maclaurin series of ϕ , which is exactly the n -th partition number.

lemma `Partition'_conv_contour_integral`:

`fixes z :: complex and n :: nat`
`assumes z: "Im z > 0"`
`defines "L \equiv linepath z (z + 1)"`
`shows "Partition' n =`
`contour_integral L (λ z. to_nome (-2 * of_nat n * z) / euler_phi`
`(to_nome (2*z)))"`

<proof>

lemmas [simp del] = div_mult_self1 div_mult_self2 div_mult_self3 div_mult_self4

From the functional equation for the Dedekind η function and the connection between the Dedekind η function and Euler's ϕ function, we get a functional equation for ϕ .

theorem euler_phi_functional_equation:

```
  fixes h k H :: int and z :: complex
  assumes z: "Re z > 0" and k: "k > 0" and hk: "coprime h k" and hH:
  "[h * H = -1] (mod k)"
  defines "x  $\equiv$  exp (2*pi*i*h/k - 2*pi*z/k2)" and "x'  $\equiv$  exp (2*pi*i*H/k
  - 2*pi/z)"
  shows "euler_phi x' = cis (pi * dedekind_sum h k) * csqrt (z/k) *
  exp (pi/(12*z) - (pi*z)/(12*k2)) * euler_phi
  x"
```

<proof>

As a corollary, we get a functional equation for the reciprocal of ϕ .

lemma inverse_euler_phi_functional_equation:

```
  defines "F  $\equiv$   $\lambda$ t. 1 / euler_phi t"
  fixes h k H :: int and z :: complex
  assumes z: "Re z > 0" and k: "k > 0" and hk: "coprime h k" and hH:
  "[h * H = -1] (mod k)"
  defines "x  $\equiv$  exp (2*pi*i*h/k - 2*pi*z/k2)" and "x'  $\equiv$  exp (2*pi*i*H/k
  - 2*pi/z)"
  shows "F x = cis (pi * dedekind_sum h k) * csqrt (z/k) * exp (pi/(12*z)
  - (pi*z)/(12*k2)) * F x'"
  <proof>
```

We define two constants that will show up in our error analysis. The first one is $c_3 = \sum_{m=1}^{\infty} p(24m-1) \exp(-2\pi(m-1/24)) \approx 3.6517$.

These constants will ultimately not be very interesting since the error bound we obtain is very bad anyway.

definition c3 :: real where

```
"c3 =
  ( $\sum$  m. (if m = 0 then 0 else real (Partition' (24*m-1))) * exp (-2*pi/24)
  ^ (24*m-1))"
```

lemma sums_c3:

```
"( $\lambda$ m. (if m = 0 then 0 else real (Partition' (24*m-1))) * exp (-2*pi/24)
  ^ (24*m-1))
  sums c3"
```

<proof>

lemma c3_nonneg: "c3 \geq 0"

<proof>

The second one is $c_1 = 2^{7/4}c_3 + 2^{3/4}\pi \exp(\pi/12) \approx 19.15$.

```

definition c1 :: real where
  "c1 = c3 * 2 powr (7/4) + pi * exp (pi/12) * 2 powr (3/4)"

```

```

lemma c1_pos: "c1 > 0"
  ⟨proof⟩

```

We set up one locale to analyse the series for a fixed argument n and a fixed series cutoff N , both positive.

```

locale rademacher_series =
  fixes n N :: nat
  assumes N: "N > 0"
  assumes n: "n > 0"
begin

```

```

definition f :: "complex ⇒ complex"
  where "f = (λz. to_nome (-2 * of_nat n * z) / euler_phi (to_nome (2*z)))"
definition fs :: "complex list" where "fs = fareys N"
definition xs :: "complex list" where "xs = rademacher_vertices N"
definition M :: nat where "M = length xs - 1"

```

```

definition X :: "int ⇒ real" where "X k = pi / of_int k * sqrt (2/3 *
(n - 1 / 24))"

```

```

definition A0 :: "nat ⇒ complex" where
  "A0 j = (let (h, k) = quotient_of (fareys N ! Suc j) in
    (pi powr (5/2) / (3 * sqrt 6) * of_int k powr (-5/2) * X
k powr (-3/2)) *R
    (cis (pi * (dedekind_sum h k - 2 * real n * of_int h /
of_int k)) *
      Bessel_I (3/2) (of_real (X k))))"

```

This is the integral along an individual segment.

```

definition I where "I j = contour_integral (linepath (xs ! j) (xs ! Suc
j)) f"

```

```

lemma M_eq_length: "M = length (fareys N) - 1"
  ⟨proof⟩

```

```

definition c4 where "c4 = c3 * exp (2*n*pi) * 2 powr (7/4)"

```

```

definition c5 where "c5 = pi * exp (pi/12) * exp (2*n*pi) * 2 powr (3/4)"

```

```

end

```

1.4 Estimating the error of individual summands

Next, we set up a locale to look at one particular summand in the series. In most cases, this corresponds directly to three successive Farey fractions and

an integral between the two points of tangency of their Ford circles. The only exception is the last summand, which requires some extra care.

```

locale rademacher_series_summand = rademacher_series +
  fixes j :: nat
  assumes j: "j < M"
begin

```

The Farey fraction of the Ford circle we are integrating over:

```

definition h where "h = fst (quotient_of (fareys N ! (j+1)))"
definition k where "k = snd (quotient_of (fareys N ! (j+1)))"
definition H where "H = k - modular_inverse k h"

```

```

lemma fareys_j: "fareys N ! Suc j = of_int h / of_int k"
  <proof>

```

The Farey fractions of the previous and next Ford circle:

```

definition h1 where "h1 = fst (quotient_of (fareys N ! j))"
definition k1 where "k1 = snd (quotient_of (fareys N ! j))"

```

```

lemma hk1: "coprime h1 k1"
  <proof>

```

```

definition h2 where "h2 = fst (quotient_of (fareys N ! (j+2)))"
definition k2 where "k2 = snd (quotient_of (fareys N ! (j+2)))"

```

```

lemma hk2: "coprime h2 k2"
  <proof>

```

```

lemma k_pos: "k > 0" "k1 > 0" "k2 > 0"
  <proof>

```

```

lemma h: "h ∈ {0..k}" and k: "k ≤ N"
  <proof>

```

```

lemma hk: "coprime h k"
  <proof>

```

```

lemma j_less_fareysN: "j < length (fareys N)"
  <proof>

```

```

lemma hH: "[h * H = -1] (mod k)"
  <proof>

```

```

lemma H: "H ∈ {0..k}"
  <proof>

```

The images of the points of tangency under the transformation $\tau \mapsto -ik^2(\tau - h/k)$.

```

definition z1 :: "complex" where "z1 = Radem_trans (of_int h / of_int
k) (xs ! j)"
definition z2 :: "complex" where "z2 = Radem_trans (of_int h / of_int
k) (xs ! Suc j)"

lemma Re_z1_pos: "Re z1 > 0" and Re_z2_pos: "Re z2 > 0"
<proof>

lemma z1_on_sphere: "z1 ∈ sphere (1/2) (1/2)"
and z2_on_sphere: "z2 ∈ sphere (1/2) (1/2)"
<proof>

lemma
shows Im_z1_pos: "Im z1 > 0"
and Im_z2_neg: "Im z2 < 0"
and norm_z1_le: "norm z1 ≤ sqrt 2 * k / N"
and norm_z2_le: "norm z2 ≤ sqrt 2 * k / N"
<proof>

lemma Re_z1_le: "Re z1 ≤ 1" and Re_z2_le: "Re z2 ≤ 1"
<proof>

definition L where "L = linepath z1 z2"

lemma Re_L: "closed_segment z1 z2 ⊆ {z. Re z ∈ {0..1}}"
<proof>

lemma norm_L:
assumes "z ∈ closed_segment z1 z2"
shows "norm z ≤ sqrt 2 * k / N"
<proof>

lemma L_in_cball: "closed_segment z1 z2 ⊆ cball (1/2) (1/2)"
<proof>

definition ψ where "ψ = (λw::complex. csqrt w * exp (pi/(12*w) - pi*w/(12*k^2)))"
definition F where "F = (λw::complex. 1 / euler_phi (exp (2*pi*i*H/k -
2*pi/w)))"
definition I1 where "I1 = contour_integral L (λw. ψ w * exp (2*n*pi*w/k^2))"
definition I2 where "I2 = contour_integral L (λw. ψ w * exp (2*n*pi*w/k^2)
* (F w - 1))"

definition C where "C a b = part_circlepath (1/2) (1/2) a b"
definition J where "J a b = contour_integral (C a b) (λw. ψ w * exp (2*n*pi*w/k^2))"

lemma Re_pos_imp_nz: "Re z > 0 ⇒ z ≠ 0" for z

```

<proof>

lemma *holomorphic_ψ* [*holomorphic_intros*]:
 assumes " $(\bigwedge z. z \in A \implies z \notin \mathbb{R}_{\leq 0})$ "
 shows " ψ *holomorphic_on* A"
<proof>

lemma *holomorphic_F* [*holomorphic_intros*]: "*F* *holomorphic_on* {*z*. *Re z* > 0}"
<proof>

definition *c2* :: *complex* where "*c2* = $i * k \text{ powr } (-5/2) * \text{cis } (\pi * (\text{dedekind_sum } h \ k - 2*n*h/k))$ "

lemma *I_conv_I12*: "*I j* = *c2* * (*I1* + *I2*)"
<proof>

lemma *I2_bound*: "*norm I2* ≤ *c4* * (*k/N*) *powr* (3/2)"
<proof>

lemma *I1_conv_Bessel_I*:
 defines "*V* ≡ $(\pi \text{ powr } (5/2) / (3 * \text{sqrt } 6) * X \ k \ \text{powr } (-3/2)) *_R (i * \text{Bessel_I } (3/2) \ (\text{of_real } (X \ k)))$ "
 shows "*dist I1* (*-V*) ≤ *c5* * (*k / N*) *powr* (3/2)"
<proof>

We now obtain an estimate for the difference between a Rademacher summand and the corresponding segment of the contour integral.

lemma *I_conv_A0*:
 "*dist (I j)* (*A0 j*) ≤ *c1* * $\exp (2*n*\pi) * (N \ \text{powr } (-3/2) / k)$ "
<proof>

end

1.5 Obtaining the full error estimate

definition *rademacher_aux1'* :: "*real* ⇒ *int* ⇒ *complex*" where
 "*rademacher_aux1'* *n k* =
 $(\sum h \mid h \in \{1..k\} \wedge \text{coprime } h \ k. \text{cis } (\pi * (\text{dedekind_sum } h \ k - 2 * n * \text{of_int } h / \text{of_int } k)))$ "

definition *rademacher_aux1* :: "*real* ⇒ *int* ⇒ *real*" where
 "*rademacher_aux1* *n k* =
 $(\sum h \mid h \in \{1..k\} \wedge \text{coprime } h \ k. \text{cos } (\pi * (\text{dedekind_sum } h \ k - 2 * n * \text{of_int } h / \text{of_int } k)))$ "

```

definition rademacher_aux2 :: "real  $\Rightarrow$  real  $\Rightarrow$  real" where
  "rademacher_aux2 n k = pi / k * sqrt (2/3 * (n - 1/24))"

definition rademacher_aux3 :: "real  $\Rightarrow$  real  $\Rightarrow$  real" where
  "rademacher_aux3 z k = pi powr (7/2) * (3*z) powr (-3/2) / k^3 * Bessel_I
  (3/2) z"

lemma Bessel_I_three_halves_real:
  assumes "z > (0 :: real)"
  shows "Bessel_I (3 / 2) z = sqrt (2 * z / pi) * (cosh z / z - sinh
  z / z ^ 2)"
  <proof>

lemma has_field_derivative_rademacher_aux3:
  assumes "x > 1 / (24 :: real)" "k > 0"
  defines "y  $\equiv$  sqrt (x - 1 / 24)"
  defines "z  $\equiv$  rademacher_aux2 x k"
  shows "(( $\lambda$ x. sinh (pi / k * (2/3 * (x - 1 / 24))) powr (1/2)) * (x
  - 1 / 24) powr (-1/2))
  has_field_derivative rademacher_aux3 z k) (at x within A)"
  <proof>

definition rademacher_aux4 :: "real  $\Rightarrow$  real  $\Rightarrow$  real"
  where "rademacher_aux4 x k = rademacher_aux3 (rademacher_aux2 x k)
  k"

The following is the usual presentation in the literature of the factor in the
Rademacher sum.

lemma rademacher_aux4_altdef:
  assumes "x > (1 / 24 :: real)" "k > 0"
  shows "rademacher_aux4 x k =
  deriv ( $\lambda$ x. sinh (pi / k * (2/3 * (x - 1/24))) powr (1/2))
  * (x - 1/24) powr (-1/2)) x"
  <proof>

definition rademacher_term :: "nat  $\Rightarrow$  nat  $\Rightarrow$  real" where
  "rademacher_term n k = 1 / (sqrt 2 * pi) * sqrt k * rademacher_aux4
  n k * rademacher_aux1 n k"
thm rademacher_aux2_def

definition rademacher_sum :: "nat  $\Rightarrow$  nat  $\Rightarrow$  real" where
  "rademacher_sum n N = ( $\sum$  k=1..N. rademacher_term n k)"

definition rademacher_remainder :: "nat  $\Rightarrow$  nat  $\Rightarrow$  real" where
  "rademacher_remainder n N = ( $\sum$  k. rademacher_term n (N+k+1))"

context rademacher_series
begin

```

We now combine the individual estimates to get an estimate for the full partial sum that corresponds to summing over all Farey fractions with denominator up to N .

```

lemma remainder_bound_complex:
  "dist (of_nat (Partition' n))
    ((1 / (sqrt 2 * pi)) *R
      (∑ k=1..N. (sqrt (real k) * rademacher_aux4 n (real k)) *R rademacher_aux1'
n k))
    ≤ c1 * exp (2 * n * pi) / sqrt N"
⟨proof⟩

```

Since we know that $p(n)$ is an integer, we can easily project the right-hand side to the reals, which turns the exponential into a cosine.

```

lemma remainder_bound_real:
  "dist (of_nat (Partition' n)) (rademacher_sum n N) ≤ c1 * exp (2 * n
* pi) / sqrt N"
⟨proof⟩

```

end

In particular, noting that the error vanishes as $N \rightarrow \infty$, we obtain our main result: a closed-form expression for $P(n)$ as an infinite series.

Note that while it would seem that the result above gives us more precise information, namely an explicit error for the partial sums, that error bound is extremely bad due to the $\exp(2n\pi)$ term. That error bound would suggest that one would have to sum $N \approx \exp(4n\pi)$ terms in order to compute $p(n)$, whereas in reality, $N \approx \sqrt{n}$ suffices. One can prove this more precise error bound by directly bounding the tail sums of the series.

```

theorem Rademacher_series:
  fixes n :: nat
  assumes n: "n > 0"
  shows "(λk. rademacher_term n (Suc k)) sums (real (Partition' n))"
⟨proof⟩

```

```

hide_const (open) c1
hide_const (open) c3

```

```

lemma summable_rademacher:
  assumes "n > 0"
  shows "summable (λk. rademacher_term n (N+k+1))"
⟨proof⟩

```

```

lemma rademacher_remainder_altdef:
  assumes "n > 0"
  shows "rademacher_remainder n N = real (Partition' n) - rademacher_sum
n N"
⟨proof⟩

```

1.6 Bounding the remainder

The exponential sum $A_k(n)$ is a sum of at most k exponentials with imaginary coefficients, so it is clearly bounded by k . Much better bounds hold; in fact, its asymptotics are $|A_k(n)| = k^{1/2+o(1)}$, and the bound $|A_k(n)| \leq 2k^{5/6}$ holds for all n and k (see e.g. Lehmer [2]).

lemma rademacher_aux1_bound: " $|rademacher_aux1\ n\ k| \leq real\ k$ "
 $\langle proof \rangle$

lemma rademacher_aux4_eq:
assumes " $n \geq 1$ " " $k > 0$ "
shows " $rademacher_aux4\ n\ k =$
 $(6 * (n - 1/24))\ powr\ -(3/4) * pi^2 * k\ powr\ (-3/2) * Bessel_I\ (3/2)\ (rademacher_aux2\ n\ k)$ "
 $\langle proof \rangle$

definition rademacher_aux5 :: "real \Rightarrow real" where
"rademacher_aux5 x = 4 / sqrt pi * (cosh x / x ^ 2 - sinh x / x ^ 3)"

lemma rademacher_aux5_eq:
assumes " $x > (0 :: real)$ "
shows " $rademacher_aux5\ x = Bessel_Clifford\ (3/2)\ (x ^ 2 / 4)$ "
 $\langle proof \rangle$

lemma rademacher_aux5_mono: " $0 < x \implies x \leq y \implies rademacher_aux5\ x \leq rademacher_aux5\ y$ "
 $\langle proof \rangle$

lemma rademacher_aux5_nonneg: " $0 < x \implies rademacher_aux5\ x \geq 0$ "
 $\langle proof \rangle$

We derive the obvious bound for the remainder term: let

$$g(x) = I_{3/2}(x)x^{-3/2} = \frac{4}{\sqrt{\pi}} \left(\frac{\cosh x}{x^2} - \frac{\sinh x}{x^3} \right)$$

and $h(n, k) = \frac{\pi}{k} \sqrt{\frac{2}{3}(n - \frac{1}{24})}$. Then the k -th term of the Rademacher series is bounded by:

$$\frac{4\pi^{5/2}}{3\sqrt{3}} k^{-3/2} g(h(n, k))$$

Note that this bound (and all the bounds that will follow) is far from tight. For better bounds, see Lehmer [2]. He uses additional knowledge about the growth of the exponential sum $A_k(n)$ to estimate it with $\leq 2k^{5/6}$ rather than our crude $\leq k$.

lemma rademacher_term_bound:
assumes n : " $n > 0$ " and k : " $k > 0$ "

shows " $|rademacher_term\ n\ k| \leq$
 $1/4 * pi\ powr\ (5/2) * (3*k)\ powr\ (-3/2) * rademacher_aux5$
 $(rademacher_aux2\ n\ k)$ "
 <proof>

It is now also easy to see that the Rademacher series is converges absolutely.

lemma norm_summable_rademacher:
assumes $n: "n > 0"$
shows " $summable\ (\lambda k. norm\ (rademacher_term\ n\ (N + k + 1)))$ "
 <proof>

Summing our bound for the terms over all k gives us the following remainder bound:

$$|R_N(n)| \leq \frac{\pi^{5/2}}{6\sqrt{3}} N^{-1/2} g(h(n, N + 1))$$

To achieve this, we use the monotonicity of the Bessel–Clifford function to bound the sum by a multiple of the Hurwitz zeta function $\zeta(s, a) = \sum_{n \geq 1} (n + a)^{-s}$ and then bound the Hurwitz zeta function with the corresponding integral $\int_{0^\infty} (x + a)^{-s} dx = a^{1-s}/(s - 1)$.

Below, we will also give a better bound that uses the convexity of the Bessel–Clifford rather than its monotonicity.

theorem rademacher_remainder_bound:
assumes $n: "n > 0"$ **and** $N: "N > 0"$
defines " $g \equiv (\lambda k. rademacher_aux5\ (rademacher_aux2\ n\ k))$ "
shows " $|rademacher_remainder\ n\ N| \leq$
 $1/2 * pi\ powr\ (5/2) * 3\ powr\ (-3/2) * g\ (N+1) * N\ powr\ (-1/2)$ "
 <proof>

1.7 Asymptotics

For fixed n and $N \rightarrow \infty$, the remainder $R_N(n)$ is $O(N^{-1/2})$.

lemma rademacher_remainder_bigo:
assumes $n: "n > 0"$
shows " $rademacher_remainder\ n \in O(\lambda N. real\ N\ powr\ (-1/2))$ "
 <proof>

For fixed N and $n \rightarrow \infty$, the remainder $R_N(n)$ is $O(\exp(\frac{\pi}{N+1} \sqrt{\frac{2}{3}n}))$.

lemma rademacher_remainder_bigo':
assumes $N: "N > (0::nat)"$
shows " $(\lambda n. rademacher_remainder\ n\ N) \in O(\lambda n. exp\ (pi / (N+1) * sqrt$
 $(2 / 3 * n)))$ "
 <proof>

The asymptotics of the leading term as $n \rightarrow \infty$ are $\exp(\pi\sqrt{2/3n})/(4\sqrt{3n})$. Therefore, this is also the asymptotic behaviour of $p(n)$.

lemma rademacher_term_1_asymp_equiv:

```

"( $\lambda n$ . rademacher_term n 1)  $\sim$  [at_top]
  ( $\lambda n$ . exp (pi * sqrt (2/3 * real n)) / (4 * sqrt 3 * real n))"
<proof>

```

The following lemma shows the asymptotic error made by approximating $p(n)$ by the leading term of the Rademacher formula.

```

theorem Partition'_leading_term:
  " $(\lambda n$ . Partition' n - rademacher_sum n 1)  $\in$   $O(\lambda n$ . exp (pi / sqrt 6 *
  sqrt (real n)))"
<proof>

```

A direct consequence of this is the asymptotic formula for $p(n)$:

$$p(n) \sim \frac{1}{4\sqrt{3n}} \exp(\pi\sqrt{2/3n})$$

Note that we had already derived the non-asymptotic upper bound

$$p(n) \leq \frac{\pi}{\sqrt{6(n-1)}} \exp(\pi\sqrt{2/3n})$$

by more elementary means.

```

theorem Partition'_asyp_equiv:
  " $(\lambda n$ . real (Partition' n))  $\sim$  [at_top]
  ( $\lambda n$ . exp (pi * sqrt (2/3 * real n)) / (4 * sqrt 3 * real n))"
<proof>

```

1.8 Concrete error bounds

In this section, we will derive some stronger concrete bounds for the remainder of the Rademacher series. These are still fairly elementary bounds that can be obtained using only the obvious bound $\sum_{h=1}^k \cos(\dots) \leq k$ for the exponential sum.

```

context
  fixes g :: "real  $\Rightarrow$  real" and h :: "nat  $\Rightarrow$  nat  $\Rightarrow$  real"
  defines "g  $\equiv$  rademacher_aux5"
  defines "h  $\equiv$  ( $\lambda n N$ . rademacher_aux2 (real n) (real N))"
begin

```

We first define some constants that will occur in the rest of our analysis.

```

definition rademacher_bound_const1 :: real where
  "rademacher_bound_const1 = 2 * pi ^ 2 * 3 powr (-5/2)"

```

```

definition rademacher_bound_const3 :: "real  $\Rightarrow$  real" where
  "rademacher_bound_const3 c =
  1/10 * pi powr (5/2) * 3 powr (-3/2) * (g (sqrt(2/3) * pi/c) - 4
  / 3 / sqrt pi)"

```

definition `rademacher_bound_const2` :: "nat \Rightarrow nat \Rightarrow real" where
`"rademacher_bound_const2 N n =`
`1/10 * pi powr (5/2) * 3 powr (-3/2) * (N+1)^2 * (g (h n (N+1)) -`
`4 / 3 / sqrt pi)"`

lemma `rademacher_bound_const1_nonneg`: "rademacher_bound_const1 \geq 0"
`\langle proof \rangle`

lemma `rademacher_bound_const2_nonneg`:
assumes `N: "n > 0" and n: "n > 0"`
shows `"rademacher_bound_const2 N n \geq 0"`
`\langle proof \rangle`

lemma `rademacher_bound_const3_nonneg`:
assumes `c: "c > 0"`
shows `"rademacher_bound_const3 c \geq 0"`
`\langle proof \rangle`

lemmas `c1s_nonneg =`
`rademacher_bound_const1_nonneg rademacher_bound_const2_nonneg rademacher_bound_const3_nonneg`

notation `rademacher_bound_const1` ("C1")
notation `rademacher_bound_const2` ("C2")
notation `rademacher_bound_const3` ("C3")

For all $n, N > 0$, we have $|R_N(n)| \leq C_1 N^{-1/2} + C_2(N, n) N^{-5/2}$. This bound is obtained by estimating the individual terms in the same way as before: Let $g(x) = C_{3/2}(x^2/4)$ and $z = \frac{\pi}{k} \sqrt{\frac{2}{3}(n - \frac{1}{2})}$. Then the k -th term is bounded by:

$$\frac{1}{4} \pi^{5/2} (3k)^{-3/2} g(z/k)$$

We want to bound the sum $\sum_{k=N+1}^{\infty}$, so as a first simplification, we use the convexity of $C_{3/2}$ to bound $g(z/k)$ in terms of $g(0)$ and $g(z/(N+1))$. This gives us an estimate of the form

$$\frac{1}{2} C_1 k^{-3/2} + \frac{5}{2} * C_2 N, n k^{-7/2}$$

Summing this for all k from $N+1$ to ∞ , we obtain the estimate

$$\frac{1}{2} C_1 \zeta(\frac{3}{2}, N+1) + \frac{5}{2} * C_2 N, n \zeta(\frac{7}{2}, N+1)$$

in terms of the Hurwitz zeta function $\zeta(s, a) = \sum_{k=0}^{\infty} (k+a)^{-s}$.

By bounding $\zeta(s, a)$ with $\int_0^{\infty} (x+a-1)^{-s} dx$, we obtain the estimate $\zeta(s, a) \leq (a-1)^{1-s}/(s-1)$. In our case, we obtain the estimate $C_1 N^{-1/2} + C_2(N, n) N^{-5/2}$ as desired.

lemma `rademacher_remainder_bound_strong`:
assumes `n: "n > 0" and N: "N > 0"`

shows " $|rademacher_remainder\ n\ N| \leq$
 $C1 * N\ powr\ (-1/2) + C2\ N\ n * N\ powr\ (-5/2)$ "

<proof>

We rephrase the above estimate specifically for the situation where we have $N + 1 \geq c\sqrt{n - \frac{1}{24}}$. In this case we get:

$$|R_N(n)| \leq C_1 N^{-1/2} + C_3(c)(N + 1)^2 N^{-5/2}$$

lemma rademacher_remainder_bound_concrete_strong:

assumes n : " $n > 0$ " and N : " $N > 0$ "

assumes c : " $real\ (N+1) \geq c * sqrt\ (n - 1/24)$ " " $c > 0$ "

shows " $|rademacher_remainder\ n\ N| \leq$

$C1 * real\ N\ powr\ (-1/2) + C3\ c * (real\ N + 1) ^ 2 * real\ N\ powr\ (-5/2)$ "

<proof>

To make this result more applicable, we now deduce the following result, which allows us to establish, for any given $c > 0$, a threshold from which on it is sufficient to use $c\sqrt{n}$ terms of the Rademacher series.

Let $a = C_1 c^{-1/2}$ and $b = C_3(c)c^{-5/2}$. Let y be the unique positive real root of $y(a + b(c + y^2)^2) - \varepsilon$. Then for all $n \geq y^{-4}$ and $N \geq c\sqrt{n}$ we have $|R_N(n)| < \varepsilon$.

theorem rademacher_remainder_bound_concrete_strong':

assumes ε : " $\varepsilon > 0$ " and c : " $c > 0$ " " $real\ N \geq c * sqrt\ n$ "

assumes " $y > 0$ "

defines " $a \equiv C1 * c\ powr\ (-1/2)$ " and " $b \equiv C3\ c * c\ powr\ (-5/2)$ "

assumes " $y * (a + b * (c + y^2) ^ 2) < \varepsilon$ " and " $real\ n \geq 1 / y ^ 4$ "

shows " $|rademacher_remainder\ n\ N| < \varepsilon$ "

<proof>

no_notation rademacher_bound_const1 ("C1")

no_notation rademacher_bound_const2 ("C2")

no_notation rademacher_bound_const3 ("C3")

end

end

theory Rademacher_Series_Concrete_Bounds

imports Rademacher_Series "HOL-Decision_Procs.Approximation"

begin

Concretely, for $\varepsilon = \frac{1}{2}$, $c = 2$ works for all $n \geq 13$, $c = 1$ works for all $n \geq 87$, and $c = \frac{1}{2}$ works for all $n \geq 5682$.

Of course, in practice, one will presumably want to add a bit more leeway to account for rounding errors that occur during the evaluation. However, it is

probably a good idea to keep this leeway as small as possible, since adding a few more bits of precision during the evaluation is cheaper than computing more terms of the series.

Note again that these bounds are far from tight.

```
corollary rademacher_remainder_bound_concrete_onehalf_onehalf:  
  assumes "real N ≥ sqrt n / 2" "n ≥ 5682"  
  shows   "|rademacher_remainder n N| < 1 / 2"  
  <proof>
```

```
corollary rademacher_remainder_bound_concrete_onehalf_1:  
  assumes "real N ≥ sqrt n" "n ≥ 87"  
  shows   "|rademacher_remainder n N| < 1 / 2"  
  <proof>
```

```
corollary rademacher_remainder_bound_concrete_onehalf_2:  
  assumes "real N ≥ 2 * sqrt n" "n ≥ 13"  
  shows   "|rademacher_remainder n N| < 1 / 2"  
  <proof>
```

end

References

- [1] Z.-Y. Kong and L.-P. Teo. Rademacher’s formula for the partition function, 2023.
- [2] D. H. Lehmer. On the series for the partition function. *Transactions of the American Mathematical Society*, 43(2):271–295, 1938.