

Public Announcement Logic

Asta Halkjær From

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Abstract

This work is a formalization of public announcement logic with countably many agents. It includes proofs of soundness and completeness for variants of the axiom system $\text{PAL} + \text{DIST!} + \text{NEC!}$ [1]. The completeness proofs build on the Epistemic Logic theory.

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theory PAL imports Epistemic-Logic.Epistemic-Logic begin

1 Syntax

datatype $\langle 'i \text{ pfm} \rangle$
 $= FF \langle \perp \rangle$
 $| Pro' id \langle Pro \rangle$
 $| Dis \langle 'i \text{ pfm} \rangle \langle 'i \text{ pfm} \rangle \langle \mathbf{infixr} \langle \vee \rangle 60 \rangle$
 $| Con \langle 'i \text{ pfm} \rangle \langle 'i \text{ pfm} \rangle \langle \mathbf{infixr} \langle \wedge \rangle 65 \rangle$
 $| Imp \langle 'i \text{ pfm} \rangle \langle 'i \text{ pfm} \rangle \langle \mathbf{infixr} \langle \longrightarrow \rangle 55 \rangle$
 $| K' 'i \langle 'i \text{ pfm} \rangle \langle K \rangle$
 $| Ann \langle 'i \text{ pfm} \rangle \langle 'i \text{ pfm} \rangle \langle [-] \rightarrow [80, 80] 80 \rangle$

abbreviation $PIff :: \langle 'i \text{ pfm} \Rightarrow 'i \text{ pfm} \Rightarrow 'i \text{ pfm} \rangle \langle \mathbf{infixr} \langle \longleftrightarrow \rangle 55 \rangle$ **where**
 $\langle p \longleftrightarrow q \equiv (p \longrightarrow q) \wedge (q \longrightarrow p) \rangle$

abbreviation $PNeg \langle \neg \rightarrow [70] 70 \rangle$ **where**
 $\langle \neg p \equiv p \longrightarrow \perp \rangle$

abbreviation $PL \langle L \rangle$ **where**
 $\langle L i p \equiv (\neg (K i (\neg p))) \rangle$

primrec $anns :: \langle 'i \text{ pfm} \Rightarrow 'i \text{ pfm set} \rangle$ **where**
 $\langle anns \perp = \{\} \rangle$
 $| \langle anns (Pro -) = \{\} \rangle$
 $| \langle anns (p \vee q) = (anns p \cup anns q) \rangle$
 $| \langle anns (p \wedge q) = (anns p \cup anns q) \rangle$
 $| \langle anns (p \longrightarrow q) = (anns p \cup anns q) \rangle$
 $| \langle anns (K i p) = anns p \rangle$
 $| \langle anns ([r] p) = \{r\} \cup anns r \cup anns p \rangle$

2 Semantics

fun

$psemantics :: \langle ('i, 'w) \text{ kripke} \Rightarrow 'w \Rightarrow 'i \text{ pfm} \Rightarrow \text{bool} \rangle \langle \langle -, - \models \rangle \rightarrow [50, 50, 50] 50 \rangle$ **and**

$restrict :: \langle ('i, 'w) \text{ kripke} \Rightarrow 'i \text{ pfm} \Rightarrow ('i, 'w) \text{ kripke} \rangle \langle \langle [-] \rangle [50, 50] 50 \rangle$ **where**

$\langle M, w \models \perp \longleftrightarrow False \rangle$
 $| \langle M, w \models Pro x \longleftrightarrow \pi M w x \rangle$
 $| \langle M, w \models p \vee q \longleftrightarrow M, w \models p \vee M, w \models q \rangle$
 $| \langle M, w \models p \wedge q \longleftrightarrow M, w \models p \wedge M, w \models q \rangle$
 $| \langle M, w \models p \longrightarrow q \longleftrightarrow M, w \models p \longrightarrow M, w \models q \rangle$
 $| \langle M, w \models K i p \longleftrightarrow (\forall v \in \mathcal{W} M \cap \mathcal{K} M i w. M, v \models p) \rangle$
 $| \langle M, w \models [r] p \longleftrightarrow M, w \models r \longrightarrow M[r], w \models p \rangle$
 $| \langle M[r] = M (\mathcal{W} := \{w. w \in \mathcal{W} M \wedge M, w \models r\}) \rangle$

abbreviation $validPStar :: \langle \langle 'i, 'w \rangle kripke \Rightarrow bool \rangle \Rightarrow 'i \text{ pfm set} \Rightarrow 'i \text{ pfm} \Rightarrow bool \rangle$

$\langle \langle -; - \Vdash_{!}^* \rightarrow [50, 50, 50] 50 \rangle \text{ where} \langle P; G \Vdash_{!}^* p \equiv \forall M. P M \longrightarrow (\forall w \in \mathcal{W} M. (\forall q \in G. M, w \Vdash_{!} q) \longrightarrow M, w \Vdash_{!} p) \rangle \rangle$

primrec $static :: \langle 'i \text{ pfm} \Rightarrow bool \rangle \text{ where}$

$\langle static \perp_{!} = True \rangle$
 $| \langle static (Pro_{!} -) = True \rangle$
 $| \langle static (p \vee_{!} q) = (static p \wedge static q) \rangle$
 $| \langle static (p \wedge_{!} q) = (static p \wedge static q) \rangle$
 $| \langle static (p \longrightarrow_{!} q) = (static p \wedge static q) \rangle$
 $| \langle static (K_{!} i p) = static p \rangle$
 $| \langle static ([r]_{!} p) = False \rangle$

primrec $lower :: \langle 'i \text{ pfm} \Rightarrow 'i \text{ fm} \rangle \text{ where}$

$\langle lower \perp_{!} = \perp \rangle$
 $| \langle lower (Pro_{!} x) = Pro x \rangle$
 $| \langle lower (p \vee_{!} q) = (lower p \vee lower q) \rangle$
 $| \langle lower (p \wedge_{!} q) = (lower p \wedge lower q) \rangle$
 $| \langle lower (p \longrightarrow_{!} q) = (lower p \longrightarrow lower q) \rangle$
 $| \langle lower (K_{!} i p) = K i (lower p) \rangle$
 $| \langle lower ([r]_{!} p) = undefined \rangle$

primrec $lift :: \langle 'i \text{ fm} \Rightarrow 'i \text{ pfm} \rangle \text{ where}$

$\langle lift \perp = \perp_{!} \rangle$
 $| \langle lift (Pro x) = Pro_{!} x \rangle$
 $| \langle lift (p \vee q) = (lift p \vee_{!} lift q) \rangle$
 $| \langle lift (p \wedge q) = (lift p \wedge_{!} lift q) \rangle$
 $| \langle lift (p \longrightarrow q) = (lift p \longrightarrow_{!} lift q) \rangle$
 $| \langle lift (K i p) = K_{!} i (lift p) \rangle$

lemma $lower\text{-semantics}$:

assumes $\langle static p \rangle$

shows $\langle (M, w \Vdash lower p) \longleftrightarrow (M, w \Vdash_{!} p) \rangle$

$\langle proof \rangle$

lemma $lift\text{-semantics}$: $\langle (M, w \Vdash p) \longleftrightarrow (M, w \Vdash_{!} lift p) \rangle$

$\langle proof \rangle$

lemma $lower\text{-lift}$: $\langle lower (lift p) = p \rangle$

$\langle proof \rangle$

lemma $lift\text{-lower}$: $\langle static p \Longrightarrow lift (lower p) = p \rangle$

$\langle proof \rangle$

3 Soundness of Reduction

primrec $reduce' :: \langle 'i \text{ pfm} \Rightarrow 'i \text{ pfm} \Rightarrow 'i \text{ pfm} \rangle \text{ where}$

$\langle \text{reduce}' r \perp_! = (r \longrightarrow_! \perp_!) \rangle$
 $| \langle \text{reduce}' r (Pro_! x) = (r \longrightarrow_! Pro_! x) \rangle$
 $| \langle \text{reduce}' r (p \vee_! q) = (\text{reduce}' r p \vee_! \text{reduce}' r q) \rangle$
 $| \langle \text{reduce}' r (p \wedge_! q) = (\text{reduce}' r p \wedge_! \text{reduce}' r q) \rangle$
 $| \langle \text{reduce}' r (p \longrightarrow_! q) = (\text{reduce}' r p \longrightarrow_! \text{reduce}' r q) \rangle$
 $| \langle \text{reduce}' r (K_! i p) = (r \longrightarrow_! K_! i (\text{reduce}' r p)) \rangle$
 $| \langle \text{reduce}' r ([p]_! q) = \text{undefined} \rangle$

primrec *reduce* :: $\langle 'i \text{ pfm} \Rightarrow 'i \text{ pfm} \rangle$ **where**

$\langle \text{reduce} \perp_! = \perp_! \rangle$
 $| \langle \text{reduce} (Pro_! x) = Pro_! x \rangle$
 $| \langle \text{reduce} (p \vee_! q) = (\text{reduce} p \vee_! \text{reduce} q) \rangle$
 $| \langle \text{reduce} (p \wedge_! q) = (\text{reduce} p \wedge_! \text{reduce} q) \rangle$
 $| \langle \text{reduce} (p \longrightarrow_! q) = (\text{reduce} p \longrightarrow_! \text{reduce} q) \rangle$
 $| \langle \text{reduce} (K_! i p) = K_! i (\text{reduce} p) \rangle$
 $| \langle \text{reduce} ([r]_! p) = \text{reduce}' (\text{reduce} r) (\text{reduce} p) \rangle$

lemma *static-reduce'*: $\langle \text{static } p \Longrightarrow \text{static } r \Longrightarrow \text{static} (\text{reduce}' r p) \rangle$
 $\langle \text{proof} \rangle$

lemma *static-reduce*: $\langle \text{static} (\text{reduce} p) \rangle$
 $\langle \text{proof} \rangle$

lemma *reduce'-semantics*:

assumes $\langle \text{static } q \rangle$
shows $\langle (M, w \models_! [p]_! q) = (M, w \models_! \text{reduce}' p q) \rangle$
 $\langle \text{proof} \rangle$

lemma *reduce-semantics*: $\langle M, w \models_! p \longleftrightarrow M, w \models_! \text{reduce} p \rangle$
 $\langle \text{proof} \rangle$

4 Chains of Implications

primrec *implyP* :: $\langle 'i \text{ pfm list} \Rightarrow 'i \text{ pfm} \Rightarrow 'i \text{ pfm} \rangle$ (**infixr** $\langle \rightsquigarrow_! \rangle$ 56) **where**

$\langle ([] \rightsquigarrow_! q) = q \rangle$
 $| \langle (p \# ps \rightsquigarrow_! q) = (p \longrightarrow_! ps \rightsquigarrow_! q) \rangle$

lemma *lift-implyP*: $\langle \text{lift} (ps \rightsquigarrow q) = (\text{map lift } ps \rightsquigarrow_! \text{lift } q) \rangle$
 $\langle \text{proof} \rangle$

lemma *reduce-implyP*: $\langle \text{reduce} (ps \rightsquigarrow_! q) = (\text{map reduce } ps \rightsquigarrow_! \text{reduce } q) \rangle$
 $\langle \text{proof} \rangle$

5 Proof System

primrec *peval* :: $\langle (id \Rightarrow \text{bool}) \Rightarrow ('i \text{ pfm} \Rightarrow \text{bool}) \Rightarrow 'i \text{ pfm} \Rightarrow \text{bool} \rangle$ **where**

$\langle \text{peval} \text{ - - } \perp_! = \text{False} \rangle$
 $| \langle \text{peval } g \text{ - } (Pro_! x) = g x \rangle$

$\langle \text{peval } g \ h \ (p \vee_! q) = (\text{peval } g \ h \ p \vee \text{peval } g \ h \ q) \rangle$
 $\langle \text{peval } g \ h \ (p \wedge_! q) = (\text{peval } g \ h \ p \wedge \text{peval } g \ h \ q) \rangle$
 $\langle \text{peval } g \ h \ (p \longrightarrow_! q) = (\text{peval } g \ h \ p \longrightarrow \text{peval } g \ h \ q) \rangle$
 $\langle \text{peval } - \ h \ (K_! \ i \ p) = h \ (K_! \ i \ p) \rangle$
 $\langle \text{peval } - \ h \ ([r]_! \ p) = h \ ([r]_! \ p) \rangle$

abbreviation $\langle \text{ptautology } p \equiv \forall g \ h. \ \text{peval } g \ h \ p \rangle$

inductive $\text{PAK} :: \langle ('i \ \text{pfm} \Rightarrow \text{bool}) \Rightarrow ('i \ \text{pfm} \Rightarrow \text{bool}) \Rightarrow 'i \ \text{pfm} \Rightarrow \text{bool} \rangle$

$\langle \langle -, - \vdash_! - \rangle [50, 50, 50] \ 50 \rangle$

for $A \ B :: \langle 'i \ \text{pfm} \Rightarrow \text{bool} \rangle$ **where**

$\text{PA1}: \langle \text{ptautology } p \Longrightarrow A; B \vdash_! p \rangle$
 $\text{PA2}: \langle A; B \vdash_! K_! \ i \ p \wedge_! K_! \ i \ (p \longrightarrow_! q) \longrightarrow_! K_! \ i \ q \rangle$
 $\text{PAx}: \langle A \ p \Longrightarrow A; B \vdash_! p \rangle$
 $\text{PR1}: \langle A; B \vdash_! p \Longrightarrow A; B \vdash_! p \longrightarrow_! q \Longrightarrow A; B \vdash_! q \rangle$
 $\text{PR2}: \langle A; B \vdash_! p \Longrightarrow A; B \vdash_! K_! \ i \ p \rangle$
 $\text{PAnn}: \langle A; B \vdash_! p \Longrightarrow B \ r \Longrightarrow A; B \vdash_! [r]_! \ p \rangle$
 $\text{PFF}: \langle A; B \vdash_! [r]_! \ \perp_! \longleftrightarrow_! (r \longrightarrow_! \perp_!) \rangle$
 $\text{PPro}: \langle A; B \vdash_! [r]_! \ \text{Pro}_! \ x \longleftrightarrow_! (r \longrightarrow_! \text{Pro}_! \ x) \rangle$
 $\text{PDis}: \langle A; B \vdash_! [r]_! \ (p \vee_! q) \longleftrightarrow_! [r]_! \ p \vee_! [r]_! \ q \rangle$
 $\text{PCon}: \langle A; B \vdash_! [r]_! \ (p \wedge_! q) \longleftrightarrow_! [r]_! \ p \wedge_! [r]_! \ q \rangle$
 $\text{PImp}: \langle A; B \vdash_! [r]_! \ (p \longrightarrow_! q) \longleftrightarrow_! ([r]_! \ p \longrightarrow_! [r]_! \ q) \rangle$
 $\text{PK}: \langle A; B \vdash_! [r]_! \ K_! \ i \ p \longleftrightarrow_! (r \longrightarrow_! K_! \ i \ ([r]_! \ p)) \rangle$

abbreviation $\text{PAK-assms} \langle \langle -, -; - \vdash_! - \rangle [50, 50, 50, 50] \ 50 \rangle$ **where**

$\langle A; B; G \vdash_! p \equiv \exists \ \text{qs. set } \text{qs} \subseteq G \wedge (A; B \vdash_! \text{qs} \rightsquigarrow_! p) \rangle$

lemma $\text{eval-peval}: \langle \text{eval } h \ (g \ o \ \text{lift}) \ p = \text{peval } h \ g \ (\text{lift } p) \rangle$
 $\langle \text{proof} \rangle$

lemma $\text{tautology-ptautology}: \langle \text{tautology } p \Longrightarrow \text{ptautology } (\text{lift } p) \rangle$
 $\langle \text{proof} \rangle$

theorem $\text{AK-PAK}: \langle A \ o \ \text{lift} \vdash p \Longrightarrow A; B \vdash_! \text{lift } p \rangle$
 $\langle \text{proof} \rangle$

abbreviation validP

$:: \langle (('i, 'i \ \text{fm} \ \text{set}) \ \text{kripke} \Rightarrow \text{bool}) \Rightarrow 'i \ \text{pfm} \ \text{set} \Rightarrow 'i \ \text{pfm} \Rightarrow \text{bool} \rangle$

$\langle \langle -, - \Vdash_! - \rangle [50, 50, 50] \ 50 \rangle$

where $\langle P; G \Vdash_! p \equiv P; G \Vdash_! \star p \rangle$

lemma set-map-inv :

assumes $\langle \text{set } \text{xs} \subseteq f \ ' \ X \rangle$

shows $\langle \exists \ \text{ys. set } \text{ys} \subseteq X \wedge \text{map } f \ \text{ys} = \text{xs} \rangle$

$\langle \text{proof} \rangle$

lemma $\text{strong-static-completeness'}$:

assumes $\langle \text{static } p \rangle$ **and** $\langle \forall q \in G. \ \text{static } q \rangle$ **and** $\langle P; G \Vdash_! p \rangle$

and $\langle P; \text{lower } ' \ G \Vdash_! \star \text{lower } p \Longrightarrow A \ o \ \text{lift}; \text{lower } ' \ G \vdash \text{lower } p \rangle$

shows $\langle A; B; G \vdash_! p \rangle$
 $\langle \text{proof} \rangle$

theorem *strong-static-completeness*:

assumes $\langle \text{static } p \rangle$ **and** $\langle \forall q \in G. \text{static } q \rangle$ **and** $\langle P; G \Vdash_! p \rangle$
and $\langle \bigwedge G p. P; G \Vdash p \implies A \text{ o lift}; G \vdash p \rangle$
shows $\langle A; B; G \vdash_! p \rangle$
 $\langle \text{proof} \rangle$

corollary *static-completeness'*:

assumes $\langle \text{static } p \rangle$ **and** $\langle P; \{\} \Vdash_! \star p \rangle$
and $\langle P; \{\} \Vdash \text{lower } p \implies A \text{ o lift} \vdash \text{lower } p \rangle$
shows $\langle A; B \vdash_! p \rangle$
 $\langle \text{proof} \rangle$

corollary *static-completeness*:

assumes $\langle \text{static } p \rangle$ **and** $\langle P; \{\} \Vdash_! \star p \rangle$ **and** $\langle \bigwedge p. P; \{\} \Vdash p \implies A \text{ o lift} \vdash p \rangle$
shows $\langle A; B \vdash_! p \rangle$
 $\langle \text{proof} \rangle$

corollary

assumes $\langle \text{static } p \rangle$ $\langle (\lambda-. \text{True}); \{\} \Vdash_! p \rangle$
shows $\langle A; B \vdash_! p \rangle$
 $\langle \text{proof} \rangle$

6 Soundness

lemma *peval-semantic*s:

$\langle \text{peval } (\text{val } w) (\lambda q. (\mathcal{W} = W, \mathcal{K} = r, \pi = \text{val}), w \Vdash_! q) p = ((\mathcal{W} = W, \mathcal{K} = r, \pi = \text{val}), w \Vdash_! p) \rangle$
 $\langle \text{proof} \rangle$

lemma *ptautology*:

assumes $\langle \text{ptautology } p \rangle$
shows $\langle M, w \Vdash_! p \rangle$
 $\langle \text{proof} \rangle$

theorem *soundness_P*:

assumes
 $\langle \bigwedge M p w. A p \implies P M \implies w \in \mathcal{W} M \implies M, w \Vdash_! p \rangle$
 $\langle \bigwedge M r. P M \implies B r \implies P (M[r!]) \rangle$
shows $\langle A; B \vdash_! p \implies P M \implies w \in \mathcal{W} M \implies M, w \Vdash_! p \rangle$
 $\langle \text{proof} \rangle$

corollary $\langle (\lambda-. \text{False}); B \vdash_! p \implies w \in \mathcal{W} M \implies M, w \Vdash_! p \rangle$
 $\langle \text{proof} \rangle$

7 Strong Soundness

lemma *ptautology-imply-superset*:

assumes $\langle \text{set } ps \subseteq \text{set } qs \rangle$

shows $\langle \text{ptautology } (ps \rightsquigarrow! r \longrightarrow! qs \rightsquigarrow! r) \rangle$

$\langle \text{proof} \rangle$

lemma *PK-imply-weaken*:

assumes $\langle A; B \vdash! ps \rightsquigarrow! q \rangle \langle \text{set } ps \subseteq \text{set } ps' \rangle$

shows $\langle A; B \vdash! ps' \rightsquigarrow! q \rangle$

$\langle \text{proof} \rangle$

lemma *implyP-append*: $\langle (ps @ ps' \rightsquigarrow! q) = (ps \rightsquigarrow! ps' \rightsquigarrow! q) \rangle$

$\langle \text{proof} \rangle$

lemma *PK-ImpI*:

assumes $\langle A; B \vdash! p \# G \rightsquigarrow! q \rangle$

shows $\langle A; B \vdash! G \rightsquigarrow! (p \longrightarrow! q) \rangle$

$\langle \text{proof} \rangle$

corollary *soundness-implyP*:

assumes

$\langle \bigwedge M p w. A p \Longrightarrow P M \Longrightarrow w \in \mathcal{W} M \Longrightarrow M, w \models! p \rangle$

$\langle \bigwedge M r. P M \Longrightarrow B r \Longrightarrow P (M[r!]) \rangle$

shows $\langle A; B \vdash! qs \rightsquigarrow! p \Longrightarrow P M \Longrightarrow w \in \mathcal{W} M \Longrightarrow \forall q \in \text{set } qs. M, w \models! q \Longrightarrow M, w \models! p \rangle$

$\langle \text{proof} \rangle$

theorem *strong-soundnessP*:

assumes

$\langle \bigwedge M w p. A p \Longrightarrow P M \Longrightarrow w \in \mathcal{W} M \Longrightarrow M, w \models! p \rangle$

$\langle \bigwedge M r. P M \Longrightarrow B r \Longrightarrow P (M[r!]) \rangle$

shows $\langle A; B; G \vdash! p \Longrightarrow P; G \models! \star p \rangle$

$\langle \text{proof} \rangle$

8 Completeness

lemma *ConE*:

assumes $\langle A; B \vdash! p \wedge! q \rangle$

shows $\langle A; B \vdash! p \rangle \langle A; B \vdash! q \rangle$

$\langle \text{proof} \rangle$

lemma *Iff-Dis*:

assumes $\langle A; B \vdash! p \longleftrightarrow! p' \rangle \langle A; B \vdash! q \longleftrightarrow! q' \rangle$

shows $\langle A; B \vdash! ((p \vee! q) \longleftrightarrow! (p' \vee! q')) \rangle$

$\langle \text{proof} \rangle$

lemma *Iff-Con*:

assumes $\langle A; B \vdash! p \longleftrightarrow! p' \rangle \langle A; B \vdash! q \longleftrightarrow! q' \rangle$

shows $\langle A; B \vdash_! (p \wedge_! q) \longleftrightarrow_! (p' \wedge_! q') \rangle$
 $\langle proof \rangle$

lemma *Iff-Imp*:

assumes $\langle A; B \vdash_! p \longleftrightarrow_! p' \rangle$ $\langle A; B \vdash_! q \longleftrightarrow_! q' \rangle$
shows $\langle A; B \vdash_! ((p \longrightarrow_! q) \longleftrightarrow_! (p' \longrightarrow_! q')) \rangle$
 $\langle proof \rangle$

lemma *Iff-sym*: $\langle (A; B \vdash_! p \longleftrightarrow_! q) = (A; B \vdash_! q \longleftrightarrow_! p) \rangle$
 $\langle proof \rangle$

lemma *Iff-Iff*:

assumes $\langle A; B \vdash_! p \longleftrightarrow_! p' \rangle$ $\langle A; B \vdash_! p \longleftrightarrow_! q \rangle$
shows $\langle A; B \vdash_! p' \longleftrightarrow_! q \rangle$
 $\langle proof \rangle$

lemma *K'-A2'*: $\langle A; B \vdash_! K_! i (p \longrightarrow_! q) \longrightarrow_! K_! i p \longrightarrow_! K_! i q \rangle$
 $\langle proof \rangle$

lemma *K'-map*:

assumes $\langle A; B \vdash_! p \longrightarrow_! q \rangle$
shows $\langle A; B \vdash_! K_! i p \longrightarrow_! K_! i q \rangle$
 $\langle proof \rangle$

lemma *ConI*:

assumes $\langle A; B \vdash_! p \rangle$ $\langle A; B \vdash_! q \rangle$
shows $\langle A; B \vdash_! p \wedge_! q \rangle$
 $\langle proof \rangle$

lemma *Iff-wk*:

assumes $\langle A; B \vdash_! p \longleftrightarrow_! q \rangle$
shows $\langle A; B \vdash_! (r \longrightarrow_! p) \longleftrightarrow_! (r \longrightarrow_! q) \rangle$
 $\langle proof \rangle$

lemma *Iff-reduce'*:

assumes $\langle static\ p \rangle$
shows $\langle A; B \vdash_! [r]_! p \longleftrightarrow_! reduce'\ r\ p \rangle$
 $\langle proof \rangle$

lemma *Iff-Ann1*:

assumes r : $\langle A; B \vdash_! r \longleftrightarrow_! r' \rangle$ **and** $\langle static\ p \rangle$
shows $\langle A; B \vdash_! [r]_! p \longleftrightarrow_! [r']_! p \rangle$
 $\langle proof \rangle$

lemma *Iff-Ann2*:

assumes $\langle A; B \vdash_! p \longleftrightarrow_! p' \rangle$ **and** $\langle B\ r \rangle$
shows $\langle A; B \vdash_! [r]_! p \longleftrightarrow_! [r]_! p' \rangle$
 $\langle proof \rangle$

lemma *Iff-reduce*:

assumes $\langle \forall r \in \text{anns } p. B \ r \rangle$
shows $\langle A; B \vdash_! p \longleftrightarrow_! \text{reduce } p \rangle$
\langle proof \rangle

lemma *anns-implyP [simp]*:

$\langle \text{anns } (ps \rightsquigarrow_! q) = \text{anns } q \cup (\bigcup p \in \text{set } ps. \text{anns } p) \rangle$
\langle proof \rangle

lemma *strong-completeness_P'*:

assumes $\langle P; G \Vdash_! p \rangle$
and $\langle \forall r \in \text{anns } p. B \ r \rangle \langle \forall q \in G. \forall r \in \text{anns } q. B \ r \rangle$
and $\langle P; \text{lower } \text{'reduce'} \text{' } G \Vdash_\star \text{lower } (\text{reduce } p) \implies$
 $\text{A o lift; lower } \text{'reduce'} \text{' } G \vdash \text{lower } (\text{reduce } p) \rangle$
shows $\langle A; B; G \vdash_! p \rangle$
\langle proof \rangle

theorem *strong-completeness_P*:

assumes $\langle P; G \Vdash_! p \rangle$
and $\langle \forall r \in \text{anns } p. B \ r \rangle \langle \forall q \in G. \forall r \in \text{anns } q. B \ r \rangle$
and $\langle \bigwedge G \ p. P; G \Vdash_\star p \implies A \text{ o lift; } G \vdash p \rangle$
shows $\langle A; B; G \vdash_! p \rangle$
\langle proof \rangle

theorem *main_P*:

assumes $\langle \bigwedge M \ w \ p. A \ p \implies P \ M \implies w \in \mathcal{W} \ M \implies M, w \Vdash_! p \rangle$
and $\langle \bigwedge M \ r. P \ M \implies B \ r \implies P \ (M[r!]) \rangle$
and $\langle \forall r \in \text{anns } p. B \ r \rangle \langle \forall q \in G. \forall r \in \text{anns } q. B \ r \rangle$
and $\langle \bigwedge G \ p. P; G \Vdash_\star p \implies A \text{ o lift; } G \vdash p \rangle$
shows $\langle P; G \Vdash_! p \longleftrightarrow A; B; G \vdash_! p \rangle$
\langle proof \rangle

corollary *strong-completeness_{PB}*:

assumes $\langle P; G \Vdash_! p \rangle$
and $\langle \bigwedge G \ p. P; G \Vdash_\star p \implies A \text{ o lift; } G \vdash p \rangle$
shows $\langle A; (\lambda-. \text{True}); G \vdash_! p \rangle$
\langle proof \rangle

corollary *completeness_P'*:

assumes $\langle P; \{\} \Vdash_! p \rangle$
and $\langle \forall r \in \text{anns } p. B \ r \rangle$
and $\langle \bigwedge p. P; \{\} \Vdash \text{lower } p \implies A \text{ o lift } \vdash \text{lower } p \rangle$
shows $\langle A; B \vdash_! p \rangle$
\langle proof \rangle

corollary *completeness_P*:

assumes $\langle P; \{\} \Vdash_! p \rangle$
and $\langle \forall r \in \text{anns } p. B \ r \rangle$
and $\langle \bigwedge p. P; \{\} \Vdash p \implies A \text{ o lift } \vdash p \rangle$

shows $\langle A; B \vdash_! p \rangle$
 $\langle proof \rangle$

corollary *completeness_{PA}*:
assumes $\langle (\lambda-. True); \{\} \Vdash_! p \rangle$
shows $\langle A; (\lambda-. True) \vdash_! p \rangle$
 $\langle proof \rangle$

9 System PAL + K

abbreviation *SystemPK* $(\langle - \vdash_{!K} - \rangle [50, 50] 50)$ **where**
 $\langle G \vdash_{!K} p \equiv (\lambda-. False); (\lambda-. True); G \vdash_! p \rangle$

lemma *strong-soundness_{PK}*: $\langle G \vdash_{!K} p \implies (\lambda-. True); G \Vdash_{! \star} p \rangle$
 $\langle proof \rangle$

abbreviation *validPK* $(\langle - \Vdash_{!K} - \rangle [50, 50] 50)$ **where**
 $\langle G \Vdash_{!K} p \equiv (\lambda-. True); G \Vdash_! p \rangle$

lemma *strong-completeness_{PK}*:
assumes $\langle G \Vdash_{!K} p \rangle$
shows $\langle G \vdash_{!K} p \rangle$
 $\langle proof \rangle$

theorem *main_{PK}*: $\langle G \Vdash_{!K} p \longleftrightarrow G \vdash_{!K} p \rangle$
 $\langle proof \rangle$

corollary $\langle G \Vdash_{!K} p \implies (\lambda-. True); G \Vdash_{! \star} p \rangle$
 $\langle proof \rangle$

10 System PAL + T

Also known as System PAL + M

inductive *AxPT* :: $\langle 'i \text{ pfm} \Rightarrow \text{bool} \rangle$ **where**
 $\langle AxPT (K! i p \longrightarrow_! p) \rangle$

abbreviation *SystemPT* $(\langle - \vdash_{!T} - \rangle [50, 50] 50)$ **where**
 $\langle G \vdash_{!T} p \equiv AxPT; (\lambda-. True); G \vdash_! p \rangle$

lemma *soundness-AxPT*: $\langle AxPT p \implies \text{reflexive } M \implies w \in \mathcal{W} M \implies M, w \Vdash_! p \rangle$
 $\langle proof \rangle$

lemma *reflexive-restrict*: $\langle \text{reflexive } M \implies \text{reflexive } (M[r!]) \rangle$
 $\langle proof \rangle$

lemma *strong-soundness_{PT}*: $\langle G \vdash_{!T} p \implies \text{reflexive}; G \Vdash_{! \star} p \rangle$

<proof>

lemma *AxT-AxPT*: $\langle AxT = AxPT \text{ o lift} \rangle$

<proof>

abbreviation *validPT* ($\langle \cdot \Vdash_{!T} \rightarrow [50, 50] 50 \rangle$) **where**

$\langle G \Vdash_{!T} p \equiv \text{reflexive}; G \Vdash_{!} p \rangle$

lemma *strong-completeness_{PT}*:

assumes $\langle G \Vdash_{!T} p \rangle$

shows $\langle G \vdash_{!T} p \rangle$

<proof>

theorem *main_{PT}*: $\langle G \Vdash_{!T} p \longleftrightarrow G \vdash_{!T} p \rangle$

<proof>

corollary $\langle G \Vdash_{!T} p \implies \text{reflexive}; G \Vdash_{!} \star p \rangle$

<proof>

11 System PAL + KB

inductive *AxPB* :: $\langle 'i \text{ pfm} \Rightarrow \text{bool} \rangle$ **where**

$\langle AxPB (p \longrightarrow_{!} K_{!} i (L_{!} i p)) \rangle$

abbreviation *SystemPKB* ($\langle \cdot \vdash_{!KB} \rightarrow [50, 50] 50 \rangle$) **where**

$\langle G \vdash_{!KB} p \equiv AxPB; (\lambda \cdot. \text{True}); G \vdash_{!} p \rangle$

lemma *soundness-AxPB*: $\langle AxPB p \implies \text{symmetric } M \implies w \in \mathcal{W} M \implies M, w \Vdash_{!} p \rangle$

<proof>

lemma *symmetric-restrict*: $\langle \text{symmetric } M \implies \text{symmetric } (M[r!]) \rangle$

<proof>

lemma *strong-soundness_{PKB}*: $\langle G \vdash_{!KB} p \implies \text{symmetric}; G \Vdash_{!} \star p \rangle$

<proof>

lemma *AxB-AxPB*: $\langle AxB = AxPB \text{ o lift} \rangle$

<proof>

abbreviation *validPKB* ($\langle \cdot \Vdash_{!KB} \rightarrow [50, 50] 50 \rangle$) **where**

$\langle G \Vdash_{!KB} p \equiv \text{symmetric}; G \Vdash_{!} p \rangle$

lemma *strong-completeness_{PKB}*:

assumes $\langle G \Vdash_{!KB} p \rangle$

shows $\langle G \vdash_{!KB} p \rangle$

<proof>

theorem *main_{PKB}*: $\langle G \Vdash_{!KB} p \longleftrightarrow G \vdash_{!KB} p \rangle$

$\langle \text{proof} \rangle$

corollary $\langle G \models_{!KB} p \implies \text{symmetric}; G \models_{! \star} p \rangle$
 $\langle \text{proof} \rangle$

12 System PAL + K4

inductive $AxP4 :: \langle 'i \text{ pfm} \Rightarrow \text{bool} \rangle$ **where**
 $\langle AxP4 (K! i p \longrightarrow! K! i (K! i p)) \rangle$

abbreviation $SystemPK4 (\langle - \vdash_{!K4} - \rangle [50, 50] 50)$ **where**
 $\langle G \vdash_{!K4} p \equiv AxP4; (\lambda-. \text{True}); G \vdash! p \rangle$

lemma *pos-introspection*:

assumes $\langle \text{transitive } M \rangle \langle w \in \mathcal{W} M \rangle$

shows $\langle M, w \models! (K! i p \longrightarrow! K! i (K! i p)) \rangle$

$\langle \text{proof} \rangle$

lemma *soundness-AxP4*: $\langle AxP4 p \implies \text{transitive } M \implies w \in \mathcal{W} M \implies M, w \models! p \rangle$

$\langle \text{proof} \rangle$

lemma *transitive-restrict*: $\langle \text{transitive } M \implies \text{transitive } (M[r!]) \rangle$

$\langle \text{proof} \rangle$

lemma *strong-soundness_{PK4}*: $\langle G \vdash_{!K4} p \implies \text{transitive}; G \models_{! \star} p \rangle$

$\langle \text{proof} \rangle$

lemma *Ax4-AxP4*: $\langle Ax4 = AxP4 \text{ o lift} \rangle$

$\langle \text{proof} \rangle$

abbreviation $validPK4 (\langle - \models_{!K4} - \rangle [50, 50] 50)$ **where**

$\langle G \models_{!K4} p \equiv \text{transitive}; G \models! p \rangle$

lemma *strong-completeness_{PK4}*:

assumes $\langle G \models_{!K4} p \rangle$

shows $\langle G \vdash_{!K4} p \rangle$

$\langle \text{proof} \rangle$

theorem *main_{PK4}*: $\langle G \models_{!K4} p \longleftrightarrow G \vdash_{!K4} p \rangle$

$\langle \text{proof} \rangle$

corollary $\langle G \models_{!K4} p \implies \text{transitive}; G \models_{! \star} p \rangle$

$\langle \text{proof} \rangle$

13 System PAL + K5

inductive $AxP5 :: \langle 'i \text{ pfm} \Rightarrow \text{bool} \rangle$ **where**

$\langle AxP5 (L_! i p \longrightarrow_! K_! i (L_! i p)) \rangle$

abbreviation *SystemPK5* ($\langle \cdot \vdash_{!K5} \cdot \rangle [50, 50] 50$) **where**
 $\langle G \vdash_{!K5} p \equiv AxP5; (\lambda \cdot. True); G \vdash_! p \rangle$

lemma *soundness-AxP5*: $\langle AxP5 p \implies Euclidean M \implies w \in \mathcal{W} M \implies M, w \models_! p \rangle$
 $\langle proof \rangle$

lemma *Euclidean-restrict*: $\langle Euclidean M \implies Euclidean (M[r!]) \rangle$
 $\langle proof \rangle$

lemma *strong-soundness_{PK5}*: $\langle G \vdash_{!K5} p \implies Euclidean; G \models_{! \star} p \rangle$
 $\langle proof \rangle$

lemma *Ax5-AxP5*: $\langle Ax5 = AxP5 \text{ o lift} \rangle$
 $\langle proof \rangle$

abbreviation *validPK5* ($\langle \cdot \models_{!K5} \cdot \rangle [50, 50] 50$) **where**
 $\langle G \models_{!K5} p \equiv Euclidean; G \models_! p \rangle$

lemma *strong-completeness_{PK5}*:
assumes $\langle G \models_{!K5} p \rangle$
shows $\langle G \vdash_{!K5} p \rangle$
 $\langle proof \rangle$

theorem *main_{PK5}*: $\langle G \models_{!K5} p \longleftrightarrow G \vdash_{!K5} p \rangle$
 $\langle proof \rangle$

corollary $\langle G \models_{!K5} p \implies Euclidean; G \models_{! \star} p \rangle$
 $\langle proof \rangle$

14 System PAL + S4

abbreviation *SystemPS4* ($\langle \cdot \vdash_{!S4} \cdot \rangle [50, 50] 50$) **where**
 $\langle G \vdash_{!S4} p \equiv AxPT \oplus AxP4; (\lambda \cdot. True); G \vdash_! p \rangle$

lemma *soundness-AxPT4*: $\langle (AxPT \oplus AxP4) p \implies refltrans M \implies w \in \mathcal{W} M \implies M, w \models_! p \rangle$
 $\langle proof \rangle$

lemma *refltrans-restrict*: $\langle refltrans M \implies refltrans (M[r!]) \rangle$
 $\langle proof \rangle$

lemma *strong-soundness_{PS4}*: $\langle G \vdash_{!S4} p \implies refltrans; G \models_{! \star} p \rangle$
 $\langle proof \rangle$

lemma *AxT4-AxPT4*: $\langle (AxT \oplus Ax4) = (AxPT \oplus AxP4) \text{ o lift} \rangle$
 $\langle proof \rangle$

abbreviation $validPS_4$ ($\langle \vdash_{!S_4} \rightarrow [50, 50] 50 \rangle$) **where**

$\langle G \Vdash_{!S_4} p \equiv refltrans; G \Vdash_{!} p \rangle$

theorem $strong-completeness_{PS_4}$:

assumes $\langle G \Vdash_{!S_4} p \rangle$

shows $\langle G \vdash_{!S_4} p \rangle$

$\langle proof \rangle$

theorem $main_{PS_4}$: $\langle G \Vdash_{!S_4} p \longleftrightarrow G \vdash_{!S_4} p \rangle$

$\langle proof \rangle$

corollary $\langle G \Vdash_{!S_4} p \implies refltrans; G \Vdash_{!} p \rangle$

$\langle proof \rangle$

15 System PAL + S5

abbreviation $SystemPS5$ ($\langle \vdash_{!S_5} \rightarrow [50, 50] 50 \rangle$) **where**

$\langle G \vdash_{!S_5} p \equiv AxPT \oplus AxPB \oplus AxP_4; (\lambda \cdot True); G \vdash_{!} p \rangle$

abbreviation $AxPTB_4$:: $\langle 'i pfm \Rightarrow bool \rangle$ **where**

$\langle AxPTB_4 \equiv AxPT \oplus AxPB \oplus AxP_4 \rangle$

lemma $soundness-AxPTB_4$: $\langle AxPTB_4 p \implies equivalence M \implies w \in \mathcal{W} M \implies$

$M, w \Vdash_{!} p \rangle$

$\langle proof \rangle$

lemma $equivalence-restrict$: $\langle equivalence M \implies equivalence (M[r!]) \rangle$

$\langle proof \rangle$

lemma $strong-soundness_{PS_5}$: $\langle G \vdash_{!S_5} p \implies equivalence; G \Vdash_{!} p \rangle$

$\langle proof \rangle$

lemma $AxTB_4-AxPTB_4$: $\langle AxTB_4 = AxPTB_4 \text{ o lift} \rangle$

$\langle proof \rangle$

abbreviation $validPS5$ ($\langle \vdash_{!S_5} \rightarrow [50, 50] 50 \rangle$) **where**

$\langle G \Vdash_{!S_5} p \equiv equivalence; G \Vdash_{!} p \rangle$

theorem $strong-completeness_{PS_5}$:

assumes $\langle G \Vdash_{!S_5} p \rangle$

shows $\langle G \vdash_{!S_5} p \rangle$

$\langle proof \rangle$

theorem $main_{PS_5}$: $\langle G \Vdash_{!S_5} p \longleftrightarrow G \vdash_{!S_5} p \rangle$

$\langle proof \rangle$

corollary $\langle G \Vdash_{!S_5} p \implies equivalence; G \Vdash_{!} p \rangle$

$\langle proof \rangle$

16 System PAL + S5'

abbreviation *SystemPS5'* ($\langle \vdash_{S5'} \rightarrow [50, 50] 50 \rangle$) **where**
 $\langle G \vdash_{S5'} p \equiv AxPT \oplus AxP5; (\lambda \cdot True); G \vdash_! p \rangle$

abbreviation *AxPT5* :: $\langle 'i pfm \Rightarrow bool \rangle$ **where**
 $\langle AxPT5 \equiv AxPT \oplus AxP5 \rangle$

lemma *soundness-AxPT5*: $\langle AxPT5 p \Longrightarrow equivalence M \Longrightarrow w \in \mathcal{W} M \Longrightarrow M, w \models_! p \rangle$
 $\langle proof \rangle$

lemma *strong-soundness_{PS5'}*: $\langle G \vdash_{S5'} p \Longrightarrow equivalence; G \models_{! \star} p \rangle$
 $\langle proof \rangle$

lemma *AxT5-AxPT5*: $\langle AxT5 = AxPT5 \text{ o lift} \rangle$
 $\langle proof \rangle$

theorem *strong-completeness_{PS5'}*:
assumes $\langle G \models_{S5} p \rangle$
shows $\langle G \vdash_{S5'} p \rangle$
 $\langle proof \rangle$

theorem *main_{PS5'}*: $\langle G \models_{S5} p \longleftrightarrow G \vdash_{S5'} p \rangle$
 $\langle proof \rangle$

end

References

- [1] Y. Wang and Q. Cao. On axiomatizations of public announcement logic. *Synthese*, 190(Supplement-1):103–134, 2013.