Class-based Classical Propositional Logic

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Abstract

We formulate classical propositional logic as an axiom class. Our class represents a Hilbert-style proof system with the axioms $\vdash \varphi \to \psi \to \varphi$, $\vdash (\varphi \to \psi \to \chi) \to (\varphi \to \psi) \to \varphi \to \chi$, and $\vdash ((\varphi \to \bot) \to \bot) \to \varphi$ along with the rule *modus ponens* $\vdash \varphi \to \psi \Longrightarrow \vdash \varphi \Longrightarrow \vdash \psi$. In this axiom class we provide lemmas to obtain *Maximally Consistent Sets* via Zorn's lemma. We define the concrete classical propositional calculus inductively and show it instantiates our axiom class. We formulate the usual semantics for the propositional calculus and show strong soundness and completeness. We provide conventional definitions of the other logical connectives and prove various common identities. Finally, we show that the propositional calculus *embeds* into any logic in our axiom class.

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Chapter 1

Implication Logic

theory *Implication-Logic*
  imports *Main*
begin

This theory presents the pure implicational fragment of intuitionistic logic. That is to say, this is the fragment of intuitionistic logic containing *implication only*, and no other connectives nor *falsum* (i.e., $\bot$). We shall refer to this logic as *implication logic* in future discussion.

For further reference see [7].

1.1 Axiomatization

Implication logic can be given by the a Hilbert-style axiom system, following Troelstra and Schwichtenberg [6, §1.3.9, pg. 33].

class *implication-logic* =
  fixes *deduction* :: *'a* ⇒ *bool* (‹⊢ -› [*60*] *55*)
  fixes *implication* :: *'a* ⇒ *'a* ⇒ *'a* (**infixr** ‹→› *70*)
  assumes *axiom-k*: $\vdash \varphi \to \psi \to \varphi$
  assumes *axiom-s*: $\vdash (\varphi \to \psi \to \chi) \to (\varphi \to \psi) \to \varphi \to \chi$
  assumes *modus-ponens*: $\vdash \varphi \to \psi \Longrightarrow \vdash \varphi \Longrightarrow \vdash \psi$

1.2 Common Rules

lemma (**in** *implication-logic*) *trivial-implication*:
  $\vdash \varphi \to \varphi$
  by (*meson axiom-k axiom-s modus-ponens*)

lemma (**in** *implication-logic*) *flip-implication*:
  $\vdash (\varphi \to \psi \to \chi) \to \psi \to \varphi \to \chi$
  by (*meson axiom-k axiom-s modus-ponens*)

lemma (**in** *implication-logic*) *hypothetical-syllogism*:
$\vdash (\psi \to \chi) \to (\varphi \to \psi) \to \varphi \to \chi$
by (*meson axiom-k axiom-s modus-ponens*)

lemma (**in** *implication-logic*) *flip-hypothetical-syllogism*:
$\vdash (\psi \to \varphi) \to (\varphi \to \chi) \to (\psi \to \chi)$
using *modus-ponens flip-implication hypothetical-syllogism* **by** *blast*

lemma (**in** *implication-logic*) *implication-absorption*:
$\vdash (\varphi \to \varphi \to \psi) \to \varphi \to \psi$
by (*meson axiom-k axiom-s modus-ponens*)

1.3 Lists of Assumptions

1.3.1 List Implication

Implication given a list of assumptions can be expressed recursively

primrec (**in** *implication-logic*)
list-implication :: *'a list* ⇒ *'a* ⇒ *'a* (**infix** ‹:→› *80*) **where**
$[] :\to \varphi = \varphi$
| $(\psi \# \Psi) :\to \varphi = \psi \to \Psi :\to \varphi$

1.3.2 Deduction From a List of Assumptions

Deduction from a list of assumptions can be expressed in terms of $(:\to)$.

definition (**in** *implication-logic*) *list-deduction* :: *'a list* ⇒ *'a* ⇒ *bool* (**infix** ‹:⊢› *60*)
where
$\Gamma :\vdash \varphi \equiv \vdash \Gamma :\to \varphi$

1.3.3 List Deduction as Implication Logic

The relation $(:\vdash)$ may naturally be interpreted as a *deduction* predicate for an instance of implication logic for a fixed list of assumptions Γ.

Analogues of the two axioms of implication logic can be naturally stated using list implication.

lemma (**in** *implication-logic*) *list-implication-axiom-k*:
$\vdash \varphi \to \Gamma :\to \varphi$
by (*induct* Γ, (*simp*, *meson axiom-k axiom-s modus-ponens*)+)

lemma (**in** *implication-logic*) *list-implication-axiom-s*:
$\vdash \Gamma :\to (\varphi \to \psi) \to \Gamma :\to \varphi \to \Gamma :\to \psi$
by (*induct* Γ,
(*simp*, *meson axiom-k axiom-s modus-ponens hypothetical-syllogism*)+)

The lemmas $\vdash \varphi \to \Gamma :\to \varphi$ and $\vdash \Gamma :\to (\varphi \to \psi) \to \Gamma :\to \varphi \to \Gamma :\to \psi$ jointly give rise to an interpretation of implication logic, where a list of assumptions Γ play the role of a *background theory* of $(:\vdash)$.

context *implication-logic* **begin**
interpretation *list-deduction-logic*:
  implication-logic $\lambda\ \varphi.\ \Gamma :\vdash \varphi$ $(\to)$
proof qed
 (*meson*
    list-deduction-def
    axiom-k
    axiom-s
    modus-ponens
    list-implication-axiom-k
    list-implication-axiom-s)+
end

The following *weakening* rule can also be derived.

lemma (**in** *implication-logic*) *list-deduction-weaken*:
 $\vdash \varphi \Longrightarrow \Gamma :\vdash \varphi$
 unfolding *list-deduction-def*
 using *modus-ponens list-implication-axiom-k*
 by *blast*

In the case of the empty list, the converse may be established.

lemma (**in** *implication-logic*) *list-deduction-base-theory* [*simp*]:
 $[] :\vdash \varphi \equiv \vdash \varphi$
 unfolding *list-deduction-def*
 by *simp*

lemma (**in** *implication-logic*) *list-deduction-modus-ponens*:
 $\Gamma :\vdash \varphi \to \psi \Longrightarrow \Gamma :\vdash \varphi \Longrightarrow \Gamma :\vdash \psi$
 unfolding *list-deduction-def*
 using *modus-ponens list-implication-axiom-s*
 by *blast*

1.4 The Deduction Theorem

One result in the meta-theory of implication logic is the *deduction theorem*, which is a mechanism for moving antecedents back and forth from collections of assumptions.

To develop the deduction theorem, the following two lemmas generalize $\vdash (\varphi \to \psi \to \chi) \to \psi \to \varphi \to \chi$.

lemma (**in** *implication-logic*) *list-flip-implication1*:
 $\vdash (\varphi \# \Gamma) :\to \chi \to \Gamma :\to (\varphi \to \chi)$
 by (*induct* Γ,
    (*simp*,

meson
axiom-k
axiom-s
modus-ponens
flip-implication
hypothetical-syllogism)+)

lemma (**in** *implication-logic*) *list-flip-implication2*:
$\vdash \Gamma :\rightarrow (\varphi \rightarrow \chi) \rightarrow (\varphi \# \Gamma) :\rightarrow \chi$
by (*induct* Γ,
(*simp*,
meson
axiom-k
axiom-s
modus-ponens
flip-implication
hypothetical-syllogism)+)

Together the two lemmas above suffice to prove a form of the deduction theorem:

theorem (**in** *implication-logic*) *list-deduction-theorem*:
$(\varphi \# \Gamma) :\vdash \psi = \Gamma :\vdash \varphi \rightarrow \psi$
unfolding *list-deduction-def*
by (*metis modus-ponens list-flip-implication1 list-flip-implication2*)

1.5 Monotonic Growth in Deductive Power

In logic, for two sets of assumptions Φ and Ψ, if $\Psi \subseteq \Phi$ then the latter theory Φ is said to be *stronger* than former theory Ψ. In principle, anything a weaker theory can prove a stronger theory can prove. One way of saying this is that deductive power increases monotonically with as the set of underlying assumptions grow.

The monotonic growth of deductive power can be expressed as a meta-theorem in implication logic.

The lemma $\vdash \Gamma :\rightarrow (\varphi \rightarrow \chi) \rightarrow (\varphi \# \Gamma) :\rightarrow \chi$ presents a means of *introducing* assumptions into a list of assumptions when those assumptions have been arrived at by an implication. The next lemma presents a means of *discharging* those assumptions, which can be used in the monotonic growth theorem to be proved.

lemma (**in** *implication-logic*) *list-implication-removeAll*:
$\vdash \Gamma :\rightarrow \psi \rightarrow (removeAll\ \varphi\ \Gamma) :\rightarrow (\varphi \rightarrow \psi)$
proof −
have $\forall\ \psi. \vdash \Gamma :\rightarrow \psi \rightarrow (removeAll\ \varphi\ \Gamma) :\rightarrow (\varphi \rightarrow \psi)$
proof(*induct* Γ)
case *Nil*

```
    then show ?case by (simp, meson axiom-k)
  next
    case (Cons χ Γ)
    assume
      inductive-hypothesis: ∀ ψ. ⊢ Γ :→ ψ → removeAll φ Γ :→ (φ → ψ)
    moreover {
      assume φ ≠ χ
      with inductive-hypothesis
      have ∀ ψ. ⊢ (χ # Γ) :→ ψ → removeAll φ (χ # Γ) :→ (φ → ψ)
        by (simp, meson modus-ponens hypothetical-syllogism)
    }
    moreover {
      fix ψ
      assume φ-equals-χ: φ = χ
      moreover with inductive-hypothesis
      have ⊢ Γ :→ (χ → ψ) → removeAll φ (χ # Γ) :→ (φ → χ → ψ) by simp
      hence ⊢ Γ :→ (χ → ψ) → removeAll φ (χ # Γ) :→ (φ → ψ)
        by (metis
              calculation
              modus-ponens
              implication-absorption
              list-flip-implication1
              list-flip-implication2
              list-implication.simps(2))
      ultimately have ⊢ (χ # Γ) :→ ψ → removeAll φ (χ # Γ) :→ (φ → ψ)
        by (simp,
              metis
                modus-ponens
                hypothetical-syllogism
                list-flip-implication1
                list-implication.simps(2))
    }
    ultimately show ?case by simp
  qed
  thus ?thesis by blast
qed
```

From lemma above presents what is needed to prove that deductive power for lists is monotonic.

```
theorem (in implication-logic) list-implication-monotonic:
  set Σ ⊆ set Γ ⟹ ⊢ Σ :→ φ → Γ :→ φ
proof −
  assume set Σ ⊆ set Γ
  moreover have ∀ Σ φ. set Σ ⊆ set Γ ⟶ ⊢ Σ :→ φ → Γ :→ φ
  proof(induct Γ)
    case Nil
    then show ?case
      by (metis
            list-implication.simps(1)
```

```
        list-implication-axiom-k
        set-empty
        subset-empty)
  next
    case (Cons ψ Γ)
    assume
      inductive-hypothesis: ∀Σ φ. set Σ ⊆ set Γ ⟶ ⊢ Σ :→ φ → Γ :→ φ
    {
      fix Σ
      fix φ
      assume Σ-subset-relation: set Σ ⊆ set (ψ # Γ)
      have ⊢ Σ :→ φ → (ψ # Γ) :→ φ
      proof −
        {
          assume set Σ ⊆ set Γ
          hence ?thesis
            by (metis
                  inductive-hypothesis
                  axiom-k modus-ponens
                  flip-implication
                  list-implication.simps(2))
        }
        moreover {
          let ?Δ = removeAll ψ Σ
          assume ¬ (set Σ ⊆ set Γ)
          hence set ?Δ ⊆ set Γ
            using Σ-subset-relation by auto
          hence ⊢ ?Δ :→ (ψ → φ) → Γ :→ (ψ → φ)
            using inductive-hypothesis by auto
          hence ⊢ ?Δ :→ (ψ → φ) → (ψ # Γ) :→ φ
            by (metis
                  modus-ponens
                  flip-implication
                  list-flip-implication2
                  list-implication.simps(2))
          moreover have ⊢ Σ :→ φ → ?Δ :→ (ψ → φ)
            by (simp add: local.list-implication-removeAll)
          ultimately have ?thesis
            using modus-ponens hypothetical-syllogism by blast
        }
        ultimately show ?thesis by blast
      qed
    }
    thus ?case by simp
  qed
  ultimately show ?thesis by simp
qed
```

A direct consequence is that deduction from lists of assumptions is mono-

tonic as well:

theorem (**in** *implication-logic*) *list-deduction-monotonic*:
set $\Sigma \subseteq$ *set* $\Gamma \Longrightarrow \Sigma :\vdash \varphi \Longrightarrow \Gamma :\vdash \varphi$
unfolding *list-deduction-def*
using *modus-ponens list-implication-monotonic*
by *blast*

1.6 The Deduction Theorem Revisited

The monotonic nature of deduction allows us to prove another form of the deduction theorem, where the assumption being discharged is completely removed from the list of assumptions.

theorem (**in** *implication-logic*) *alternate-list-deduction-theorem*:
$(\varphi \# \Gamma) :\vdash \psi = (\mathit{removeAll}\ \varphi\ \Gamma) :\vdash \varphi \rightarrow \psi$
by (*metis*
list-deduction-def
modus-ponens
filter-is-subset
list-deduction-monotonic
list-deduction-theorem
list-implication-removeAll
removeAll.simps(*2*)
removeAll-filter-not-eq)

1.7 Reflection

In logic the *reflection* principle sometimes refers to when a collection of assumptions can deduce any of its members. It is automatically derivable from $[\![\mathit{set}\ \Sigma \subseteq \mathit{set}\ \Gamma; \Sigma :\vdash \varphi]\!] \Longrightarrow \Gamma :\vdash \varphi$ among the other rules provided.

lemma (**in** *implication-logic*) *list-deduction-reflection*:
$\varphi \in \mathit{set}\ \Gamma \Longrightarrow \Gamma :\vdash \varphi$
by (*metis*
list-deduction-def
insert-subset
list.simps(*15*)
list-deduction-monotonic
list-implication.simps(*2*)
list-implication-axiom-k
order-refl)

1.8 The Cut Rule

Cut is a rule commonly presented in sequent calculi, dating back to Gerhard Gentzen's *Investigations in Logical Deduction* (1935) [4]

The cut rule is not generally necessary in sequent calculi. It can often be shown that the rule can be eliminated without reducing the power of the underlying logic. However, as demonstrated by George Boolos' *Don't Eliminate Cut* (1984) [3], removing the rule can often lead to very inefficient proof systems.

Here the rule is presented just as a meta theorem.

theorem (**in** *implication-logic*) *list-deduction-cut-rule*:
 $(\varphi \# \Gamma) :\vdash \psi \Longrightarrow \Delta :\vdash \varphi \Longrightarrow \Gamma @ \Delta :\vdash \psi$
 by (*metis*
  (*no-types*, *lifting*)
  Un-upper1
  Un-upper2
  list-deduction-modus-ponens
  list-deduction-monotonic
  list-deduction-theorem
  set-append)

The cut rule can also be strengthened to entire lists of propositions.

theorem (**in** *implication-logic*) *strong-list-deduction-cut-rule*:
  $(\Phi @ \Gamma) :\vdash \psi \Longrightarrow \forall\ \varphi \in set\ \Phi.\ \Delta :\vdash \varphi \Longrightarrow \Gamma @ \Delta :\vdash \psi$
proof −
 have $\forall\ \psi.\ (\Phi @ \Gamma :\vdash \psi \longrightarrow (\forall\ \varphi \in set\ \Phi.\ \Delta :\vdash \varphi) \longrightarrow \Gamma @ \Delta :\vdash \psi)$
  proof(*induct* Φ)
   case *Nil*
   then show *?case*
    by (*metis*
      Un-iff
      append.left-neutral
      list-deduction-monotonic
      set-append
      subsetI)
  next
   case (*Cons* χ Φ) **assume** *inductive-hypothesis*:
     $\forall\ \psi.\ \Phi @ \Gamma :\vdash \psi \longrightarrow (\forall \varphi \in set\ \Phi.\ \Delta :\vdash \varphi) \longrightarrow \Gamma @ \Delta :\vdash \psi$
   {
    fix ψ χ
    assume $(\chi \# \Phi) @ \Gamma :\vdash \psi$
    hence *A*: $\Phi @ \Gamma :\vdash \chi \rightarrow \psi$ **using** *list-deduction-theorem* **by** *auto*
    assume $\forall \varphi \in set\ (\chi \# \Phi).\ \Delta :\vdash \varphi$
    hence *B*: $\forall\ \varphi \in set\ \Phi.\ \Delta :\vdash \varphi$
     and *C*: $\Delta :\vdash \chi$ **by** *auto*
    from *A B* **have** $\Gamma @ \Delta :\vdash \chi \rightarrow \psi$ **using** *inductive-hypothesis* **by** *blast*
    with *C* **have** $\Gamma @ \Delta :\vdash \psi$
     by (*meson*
      list.set-intros(*1*)
      list-deduction-cut-rule
      list-deduction-modus-ponens
      list-deduction-reflection)

}
thus *?case* **by** *simp*
qed
moreover assume $(\Phi \mathbin{@} \Gamma) :\vdash \psi$
moreover assume $\forall\ \varphi \in \mathit{set}\ \Phi.\ \Delta :\vdash \varphi$
ultimately show *?thesis* **by** *blast*
qed

1.9 Sets of Assumptions

While deduction in terms of lists of assumptions is straight-forward to define, deduction (and the *deduction theorem*) is commonly given in terms of *sets* of propositions. This formulation is suited to establishing strong completeness theorems and compactness theorems.

The presentation of deduction from a set follows the presentation of list deduction given for $(:\vdash)$.

1.10 Definition of Deduction

Just as deduction from a list $(:\vdash)$ can be defined in terms of $(:\rightarrow)$, deduction from a *set* of assumptions can be expressed in terms of $(:\vdash)$.

definition (**in** *implication-logic*) *set-deduction* :: $'a\ \mathit{set} \Rightarrow 'a \Rightarrow \mathit{bool}$ (**infix** ‹$\Vdash$› *60*)
where
$\Gamma \Vdash \varphi \equiv \exists\ \Psi.\ \mathit{set}\ \Psi\ \subseteq \Gamma \wedge \Psi :\vdash \varphi$

1.10.1 Interpretation as Implication Logic

As in the case of $(:\vdash)$, the relation $(\Vdash)$ may be interpreted as *deduction* predicate for a fixed set of assumptions Γ.

The following lemma is given in order to establish this, which asserts that every implication logic tautology $\vdash \varphi$ is also a tautology for $\Gamma \Vdash \varphi$.

lemma (**in** *implication-logic*) *set-deduction-weaken*:
$\vdash \varphi \Longrightarrow \Gamma \Vdash \varphi$
using *list-deduction-base-theory set-deduction-def* **by** *fastforce*

In the case of the empty set, the converse may be established.

lemma (**in** *implication-logic*) *set-deduction-base-theory*:
$\{\} \Vdash \varphi \equiv \vdash \varphi$
using *list-deduction-base-theory set-deduction-def* **by** *auto*

Next, a form of *modus ponens* is provided for $(\Vdash)$.

lemma (**in** *implication-logic*) *set-deduction-modus-ponens*:

```
  Γ ⊩ φ → ψ ⟹ Γ ⊩ φ ⟹ Γ ⊩ ψ
proof −
  assume Γ ⊩ φ → ψ
  then obtain Φ where A: set Φ ⊆ Γ and B: Φ :⊢ φ → ψ
    using set-deduction-def by blast
  assume Γ ⊩ φ
  then obtain Ψ where C: set Ψ ⊆ Γ and D: Ψ :⊢ φ
    using set-deduction-def by blast
  from B D have Φ @ Ψ :⊢ ψ
    using list-deduction-cut-rule list-deduction-theorem by blast
  moreover from A C have set (Φ @ Ψ) ⊆ Γ by simp
  ultimately show ?thesis
    using set-deduction-def by blast
qed

context implication-logic begin
interpretation set-deduction-logic:
  implication-logic λ φ. Γ ⊩ φ (→)
proof
   fix φ ψ
   show Γ ⊩ φ → ψ → φ  by (metis axiom-k set-deduction-weaken)
next
    fix φ ψ χ
    show Γ ⊩ (φ → ψ → χ) → (φ → ψ) → φ → χ
      by (metis axiom-s set-deduction-weaken)
next
    fix φ ψ
    show Γ ⊩ φ → ψ ⟹ Γ ⊩ φ ⟹ Γ ⊩ ψ
      using set-deduction-modus-ponens by metis
qed
end
```

1.11 The Deduction Theorem

The next result gives the deduction theorem for (⊩).

```
theorem (in implication-logic) set-deduction-theorem:
  insert φ Γ ⊩ ψ = Γ ⊩ φ → ψ
proof −
  have Γ ⊩ φ → ψ ⟹ insert φ Γ ⊩ ψ
    by (metis
          set-deduction-def
          insert-mono
          list.simps(15)
          list-deduction-theorem)
  moreover {
    assume insert φ Γ ⊩ ψ
    then obtain Φ where set Φ ⊆ insert φ Γ and Φ :⊢ ψ
      using set-deduction-def by auto
```

hence *set* (*removeAll* φ Φ) $\subseteq$ Γ **by** *auto*
moreover from ‹Φ :⊢ ψ› **have** *removeAll* φ Φ :⊢ $\varphi \rightarrow \psi$
using *modus-ponens list-implication-removeAll list-deduction-def*
by *blast*
ultimately have $\Gamma \Vdash \varphi \rightarrow \psi$
using *set-deduction-def* **by** *blast*
}
ultimately show *insert* φ $\Gamma \Vdash \psi = \Gamma \Vdash \varphi \rightarrow \psi$ **by** *metis*
qed

1.12 Monotonic Growth in Deductive Power

In contrast to the (:⊢) relation, the proof that the deductive power of (⊩) grows monotonically with its assumptions may be fully automated.

theorem *set-deduction-monotonic*:
$\Sigma \subseteq \Gamma \Longrightarrow \Sigma \Vdash \varphi \Longrightarrow \Gamma \Vdash \varphi$
by (*meson dual-order.trans set-deduction-def*)

1.13 The Deduction Theorem Revisited

As a consequence of the fact that $[\![\Sigma \subseteq \Gamma; \Sigma \Vdash \varphi]\!] \Longrightarrow \Gamma \Vdash \varphi$ is automatically provable, an alternate *deduction theorem* where the discharged assumption is completely removed from the set of assumptions is just a consequence of the more conventional *insert* φ $\Gamma \Vdash \psi = \Gamma \Vdash \varphi \rightarrow \psi$ rule and some basic set identities.

theorem (**in** *implication-logic*) *alternate-set-deduction-theorem*:
insert φ $\Gamma \Vdash \psi = \Gamma - \{\varphi\} \Vdash \varphi \rightarrow \psi$
by (*metis insert-Diff-single set-deduction-theorem*)

1.14 Reflection

Just as in the case of (:⊢), deduction from sets of assumptions makes true the *reflection principle* and is automatically provable.

theorem (**in** *implication-logic*) *set-deduction-reflection*:
$\varphi \in \Gamma \Longrightarrow \Gamma \Vdash \varphi$
by (*metis*
Set.set-insert
list-implication.simps(*1*)
list-implication-axiom-k
set-deduction-theorem
set-deduction-weaken)

1.15 The Cut Rule

The final principle of $(\Vdash)$ presented is the *cut rule*.

First, the weak form of the rule is established.

theorem (**in** *implication-logic*) *set-deduction-cut-rule*:
insert $\varphi\ \Gamma \Vdash \psi \Longrightarrow \Delta \Vdash \varphi \Longrightarrow \Gamma \cup \Delta \Vdash \psi$
proof −
assume *insert* $\varphi\ \Gamma \Vdash \psi$
hence $\Gamma \Vdash \varphi \rightarrow \psi$ **using** *set-deduction-theorem* **by** *auto*
hence $\Gamma \cup \Delta \Vdash \varphi \rightarrow \psi$ **using** *set-deduction-def* **by** *auto*
moreover assume $\Delta \Vdash \varphi$
hence $\Gamma \cup \Delta \Vdash \varphi$ **using** *set-deduction-def* **by** *auto*
ultimately show *?thesis* **using** *set-deduction-modus-ponens* **by** *metis*
qed

Another lemma is shown next in order to establish the strong form of the cut rule. The lemma shows the existence of a *covering list* of assumptions Ψ in the event some set of assumptions Δ proves everything in a finite set of assumptions Φ.

lemma (**in** *implication-logic*) *finite-set-deduction-list-deduction*:
assumes *finite* Φ
and $\forall\ \varphi \in \Phi.\ \Delta \Vdash \varphi$
shows $\exists\, \Psi.\ set\ \Psi \subseteq \Delta \wedge (\forall\, \varphi \in \Phi.\ \Psi :\vdash \varphi)$
using *assms*
proof(*induct* Φ *rule*: *finite-induct*)
case *empty* **thus** *?case* **by** (*metis all-not-in-conv empty-subsetI set-empty*)
next
case (*insert* $\chi\ \Phi$)
assume $\forall\, \varphi \in \Phi.\ \Delta \Vdash \varphi \Longrightarrow \exists\, \Psi.\ set\ \Psi \subseteq \Delta \wedge (\forall\, \varphi \in \Phi.\ \Psi :\vdash \varphi)$
and $\forall\, \varphi \in$ *insert* $\chi\ \Phi.\ \Delta \Vdash \varphi$
hence $\exists\, \Psi.\ set\ \Psi \subseteq \Delta \wedge (\forall\, \varphi \in \Phi.\ \Psi :\vdash \varphi)$ **and** $\Delta \Vdash \chi$ **by** *simp+*
then obtain $\Psi_1\ \Psi_2$ **where**
set $(\Psi_1\ @\ \Psi_2) \subseteq \Delta$
$\forall\, \varphi \in \Phi.\ \Psi_1 :\vdash \varphi$
$\Psi_2 :\vdash \chi$
using *set-deduction-def* **by** *auto*
moreover from *this* **have** $\forall\, \varphi \in$ (*insert* $\chi\ \Phi$). $\Psi_1\ @\ \Psi_2 :\vdash \varphi$
by (*metis*
insert-iff
le-sup-iff
list-deduction-monotonic
order-refl set-append)
ultimately show *?case* **by** *blast*
qed

With $[\![$*finite* Φ; $\forall\, \varphi \in \Phi.\ \Delta \Vdash \varphi]\!] \Longrightarrow \exists\, \Psi.\ set\ \Psi \subseteq \Delta \wedge (\forall\, \varphi \in \Phi.\ \Psi :\vdash \varphi)$ the strengthened form of the cut rule can be given.

theorem (**in** *implication-logic*) *strong-set-deduction-cut-rule*:
assumes $\Phi \cup \Gamma \Vdash \psi$
and $\forall\ \varphi \in \Phi.\ \Delta \Vdash \varphi$
shows $\Gamma \cup \Delta \Vdash \psi$
proof −
obtain Σ **where**
A: *set* $\Sigma\ \subseteq \Phi \cup \Gamma$ **and**
B: $\Sigma :\vdash \psi$
using *assms*(*1*) *set-deduction-def*
by *auto*+
obtain $\Phi'\ \Gamma'$ **where**
C: *set* $\Phi' =$ *set* $\Sigma \cap \Phi$ **and**
D: *set* $\Gamma' =$ *set* $\Sigma \cap \Gamma$
by (*metis inf-sup-aci*(*1*) *inter-set-filter*)+
then have *set* $(\Phi' @ \Gamma') =$ *set* Σ **using** A **by** *auto*
hence E: $\Phi' @ \Gamma' :\vdash \psi$ **using** B *list-deduction-monotonic* **by** *blast*
hence $\forall\ \varphi \in$ *set* $\Phi'.\ \Delta \Vdash \varphi$ **using** *assms*(*2*) C **by** *auto*
from *this* **obtain** Δ' **where** *set* $\Delta' \subseteq \Delta$ **and** $\forall\ \varphi \in$ *set* $\Phi'.\ \Delta' :\vdash \varphi$
using *finite-set-deduction-list-deduction* **by** *blast*
with *strong-list-deduction-cut-rule D E*
have *set* $(\Gamma' @ \Delta') \subseteq \Gamma \cup \Delta$ **and** $\Gamma' @ \Delta' :\vdash \psi$ **by** *auto*
thus *?thesis* **using** *set-deduction-def* **by** *blast*
qed

1.16 Maximally Consistent Sets For Implication Logic

Maximally Consistent Sets are a common construction for proving completeness of logical calculi. For a classic presentation, see Dirk van Dalen's *Logic and Structure* (2013, §1.5, pgs. 42–45) [8].

Maximally consistent sets will form the foundation of all of the model theory we will employ in this text. In fact, apart from classical logic semantics, conventional model theory will not be used at all.

The models we are centrally concerned are derived from maximally consistent sets. These include probability measures used in completeness theorems of probability logic found in §**??**, as well as arbitrage protection and trading strategies stipulated by our formulation of the *Dutch Book Theorem* we present in §**??**.

Since implication logic does not have *falsum*, consistency is defined relative to a formula φ.

definition (**in** *implication-logic*)
formula-consistent :: *'a* ⇒ *'a set* ⇒ *bool* (‹-−*consistent* -› [*100*] *100*)
where
[*simp*]: φ−*consistent* $\Gamma \equiv \neg\ (\Gamma \Vdash \varphi)$

Since consistency is defined relative to some φ, *maximal consistency* is presented as asserting that either ψ or $\psi \to \varphi$ is in the consistent set Γ, for all ψ. This coincides with the traditional definition in classical logic when φ is *falsum*.

definition (**in** *implication-logic*)
formula-maximally-consistent-set-def :: *'a* ⇒ *'a set* ⇒ *bool* (‹-−*MCS* -› [*100*] *100*)
where
[*simp*]: φ−*MCS* $\Gamma \equiv (\varphi$−*consistent* $\Gamma) \wedge (\forall\ \psi.\ \psi \in \Gamma \vee (\psi \to \varphi) \in \Gamma)$

Every consistent set Γ may be extended to a maximally consistent set.

However, no assumption is made regarding the cardinality of the types of an instance of *implication-logic*.

As a result, typical proofs that assume a countable domain are not suitable. Our proof leverages *Zorn's lemma*.

lemma (**in** *implication-logic*) *formula-consistent-extension*:
assumes φ−*consistent* Γ
shows (φ−*consistent* (*insert* ψ Γ)) ∨ (φ−*consistent* (*insert* ($\psi \to \varphi$) Γ))
proof −
{
assume ¬ φ−*consistent insert* ψ Γ
hence $\Gamma \Vdash \psi \to \varphi$
using *set-deduction-theorem*
unfolding *formula-consistent-def*
by *simp*
hence φ−*consistent insert* ($\psi \to \varphi$) Γ
by (*metis Un-absorb assms formula-consistent-def set-deduction-cut-rule*)
}
thus *?thesis* **by** *blast*
qed

theorem (**in** *implication-logic*) *formula-maximally-consistent-extension*:
assumes φ−*consistent* Γ
shows ∃ Ω. (φ−*MCS* Ω) ∧ $\Gamma \subseteq \Omega$
proof −
let *?*Γ-*extensions* = {Σ. (φ−*consistent* Σ) ∧ $\Gamma \subseteq \Sigma$}
have ∃ $\Omega \in$ *?*Γ-*extensions*. ∀$\Sigma \in$ *?*Γ-*extensions*. $\Omega \subseteq \Sigma \longrightarrow \Sigma = \Omega$
proof (*rule subset-Zorn*)
fix $\mathcal{C}$ :: *'a set set*
assume *subset-chain-*$\mathcal{C}$: *subset.chain ?*Γ-*extensions* $\mathcal{C}$
hence $\mathcal{C}$: ∀ $\Sigma \in \mathcal{C}$. $\Gamma \subseteq \Sigma$ ∀ $\Sigma \in \mathcal{C}$. φ−*consistent* Σ
unfolding *subset.chain-def*
by *blast+*
show ∃ $\Omega \in$ *?*Γ-*extensions*. ∀ $\Sigma \in \mathcal{C}$. $\Sigma \subseteq \Omega$
proof *cases*
assume $\mathcal{C} = \{\}$ **thus** *?thesis* **using** *assms* **by** *blast*
next

```
    let ?Ω = ⋃ C
    assume C ≠ {}
    hence Γ ⊆ ?Ω by (simp add: C(1) less-eq-Sup)
    moreover have φ−consistent ?Ω
    proof −
      {
        assume ¬ φ−consistent ?Ω
        then obtain ω where ω:
          finite ω
          ω ⊆ ?Ω
          ¬ φ−consistent ω
          unfolding
            formula-consistent-def
            set-deduction-def
          by auto
        from ω(1) ω(2) have ∃ Σ ∈ C. ω ⊆ Σ
        proof (induct ω rule: finite-induct)
          case empty thus ?case using ‹C ≠ {}› by blast
        next
          case (insert ψ ω)
          from this obtain Σ1 Σ2 where
            Σ1:
                ω ⊆ Σ1
                Σ1 ∈ C
            and Σ2:
                ψ ∈ Σ2
                Σ2 ∈ C
            by auto
          hence Σ1 ⊆ Σ2 ∨ Σ2 ⊆ Σ1
            using subset-chain-C
            unfolding subset.chain-def
            by blast
          hence (insert ψ ω) ⊆ Σ1 ∨ (insert ψ ω) ⊆ Σ2
            using Σ1 Σ2 by blast
          thus ?case using Σ1 Σ2 by blast
        qed
        hence ∃ Σ ∈ C. (φ−consistent Σ) ∧ ¬ (φ−consistent Σ)
          using C(2) ω(3)
          unfolding
            formula-consistent-def
            set-deduction-def
          by auto
        hence False by auto
      }
      thus ?thesis by blast
    qed
    ultimately show ?thesis by blast
  qed
qed
```

```
  then obtain Ω where Ω:
    Ω ∈ ?Γ-extensions
    ∀Σ ∈ ?Γ-extensions. Ω ⊆ Σ ⟶ Σ = Ω
    by auto+
  {
    fix ψ
    have (φ−consistent insert ψ Ω) ∨ (φ−consistent insert (ψ → φ) Ω)
        Γ ⊆ insert ψ Ω
        Γ ⊆ insert (ψ → φ) Ω
      using Ω(1) formula-consistent-extension formula-consistent-def
      by auto
    hence insert ψ Ω ∈ ?Γ-extensions
          ∨ insert (ψ → φ) Ω ∈ ?Γ-extensions
      by blast
    hence ψ ∈ Ω ∨ (ψ → φ) ∈ Ω using Ω(2) by blast
  }
  thus ?thesis
    using Ω(1)
    unfolding formula-maximally-consistent-set-def-def
    by blast
qed
```

Finally, maximally consistent sets contain anything that can be deduced from them, and model a form of *modus ponens*.

```
lemma (in implication-logic) formula-maximally-consistent-set-def-reflection:
  φ−MCS Γ ⟹ ψ ∈ Γ = Γ ⊩ ψ
proof −
  assume φ−MCS Γ
  {
    assume Γ ⊩ ψ
    moreover from ‹φ−MCS Γ› have ψ ∈ Γ ∨ (ψ → φ) ∈ Γ ¬ Γ ⊩ φ
      unfolding
        formula-maximally-consistent-set-def-def
        formula-consistent-def
      by auto
    ultimately have ψ ∈ Γ
      using set-deduction-reflection set-deduction-modus-ponens
      by metis
  }
  thus ψ ∈ Γ = Γ ⊩ ψ
    using set-deduction-reflection
    by metis
qed

theorem (in implication-logic) formula-maximally-consistent-set-def-implication-elimination:
  assumes φ−MCS Ω
  shows (ψ → χ) ∈ Ω ⟹ ψ ∈ Ω ⟹ χ ∈ Ω
  using
    assms
```

formula-maximally-consistent-set-def-reflection
set-deduction-modus-ponens
by *blast*

This concludes our introduction to implication logic.

end

Chapter 2

Classical Propositional Logic

theory *Classical-Logic*
  imports *Implication-Logic*
begin

This theory presents *classical propositional logic*, which is classical logic without quantifiers.

2.1 Axiomatization

Classical propositional logic can be given by the following Hilbert-style axiom system. It is *implication-logic* extended with *falsum* and double negation.

class *classical-logic* = *implication-logic* +
  fixes *falsum* :: $'a$ ($\langle\bot\rangle$)
  assumes *double-negation*: $\vdash (((\varphi \to \bot) \to \bot) \to \varphi)$

In some cases it is useful to assume consistency as an axiom:

class *consistent-classical-logic* = *classical-logic* +
  assumes *consistency*: $\neg \vdash \bot$

2.2 Common Rules

There are many common tautologies in classical logic. Once we have established *completeness* in §3, we will be able to leverage Isabelle/HOL's automation for proving these elementary results.

In order to bootstrap completeness, we develop some common lemmas using classical deduction alone.

lemma (**in** *classical-logic*)
  ex-falso-quodlibet: $\vdash \bot \to \varphi$
  using *axiom-k double-negation modus-ponens hypothetical-syllogism*

by *blast*

lemma (**in** *classical-logic*)
Contraposition: $\vdash ((\varphi \to \bot) \to (\psi \to \bot)) \to \psi \to \varphi$
proof −
have $[\varphi \to \bot, \psi, (\varphi \to \bot) \to (\psi \to \bot)] :\vdash \bot$
using *flip-implication list-deduction-theorem list-implication.simps*(*1*)
unfolding *list-deduction-def*
by *presburger*
hence $[\psi, (\varphi \to \bot) \to (\psi \to \bot)] :\vdash (\varphi \to \bot) \to \bot$
using *list-deduction-theorem* **by** *blast*
hence $[\psi, (\varphi \to \bot) \to (\psi \to \bot)] :\vdash \varphi$
using *double-negation list-deduction-weaken list-deduction-modus-ponens*
by *blast*
thus *?thesis*
using *list-deduction-base-theory list-deduction-theorem* **by** *blast*
qed

lemma (**in** *classical-logic*)
double-negation-converse: $\vdash \varphi \to (\varphi \to \bot) \to \bot$
by (*meson axiom-k modus-ponens flip-implication*)

The following lemma is sometimes referred to as *The Principle of Pseudo-Scotus*[2].

lemma (**in** *classical-logic*)
pseudo-scotus: $\vdash (\varphi \to \bot) \to \varphi \to \psi$
using *ex-falso-quodlibet modus-ponens hypothetical-syllogism* **by** *blast*

Another popular lemma is attributed to Charles Sanders Peirce, and has come to be known as *Peirces Law*[5].

lemma (**in** *classical-logic*) *Peirces-law*:
$\vdash ((\varphi \to \psi) \to \varphi) \to \varphi$
proof −
have $[\varphi \to \bot, (\varphi \to \psi) \to \varphi] :\vdash \varphi \to \psi$
using
pseudo-scotus
list-deduction-theorem
list-deduction-weaken
by *blast*
hence $[\varphi \to \bot, (\varphi \to \psi) \to \varphi] :\vdash \varphi$
by (*meson*
list.set-intros(*1*)
list-deduction-reflection
list-deduction-modus-ponens
set-subset-Cons
subsetCE)
hence $[\varphi \to \bot, (\varphi \to \psi) \to \varphi] :\vdash \bot$
by (*meson*
list.set-intros(*1*)

```
        list-deduction-modus-ponens
        list-deduction-reflection)
  hence [(φ → ψ) → φ] :⊢ (φ → ⊥) → ⊥
    using list-deduction-theorem by blast
  hence [(φ → ψ) → φ] :⊢ φ
    using double-negation
        list-deduction-modus-ponens
        list-deduction-weaken
    by blast
  thus ?thesis
    using list-deduction-def
    by auto
qed

lemma (in classical-logic) excluded-middle-elimination:
  ⊢ (φ → ψ) → ((φ → ⊥) → ψ) → ψ
proof −
  let ?Γ = [ψ → ⊥, φ → ψ, (φ → ⊥) → ψ]
  have ?Γ :⊢ (φ → ⊥) → ψ
       ?Γ :⊢ ψ → ⊥
    by (simp add: list-deduction-reflection)+
  hence ?Γ :⊢ (φ → ⊥) → ⊥
    by (meson
        flip-hypothetical-syllogism
        list-deduction-base-theory
        list-deduction-monotonic
        list-deduction-theorem
        set-subset-Cons)
  hence ?Γ :⊢ φ
    using
      double-negation
      list-deduction-modus-ponens
      list-deduction-weaken
    by blast
  hence ?Γ :⊢ ψ
    by (meson
        list.set-intros(1)
        list-deduction-modus-ponens
        list-deduction-reflection
        set-subset-Cons subsetCE)
  hence [φ → ψ, (φ → ⊥) → ψ] :⊢ ψ
    using
      Peirces-law
      list-deduction-modus-ponens
      list-deduction-theorem
      list-deduction-weaken
    by blast
  thus ?thesis
    unfolding list-deduction-def
```

by *simp*
qed

2.3 Maximally Consistent Sets For Classical Logic

Relativized maximally consistent sets were introduced in §1.16. Often this is exactly what we want in a proof. A completeness theorem typically starts by assuming φ is not provable, then finding a $\varphi{-}MCS\ \Gamma$ which gives rise to a model which does not make φ true.

A more conventional presentation says that Γ is maximally consistent if and only if $\neg\ \Gamma \vdash \bot$ and $\forall \psi.\ \psi \in \Gamma \vee \psi \rightarrow \varphi \in \Gamma$. This conventional presentation will come up when formulating MaxSAT in §**??**. This in turn allows us to formulate MaxSAT completeness for probability inequalities in §**??**, and reduce checking if a strategy will always lose money or if it will always make money if matched to bounded MaxSAT as part of our proof of the *Dutch Book Theorem* in §**??** and §**??** respectively.

definition (**in** *classical-logic*)
  consistent :: *'a set* ⇒ *bool* **where**
    [*simp*]: *consistent* Γ ≡ ⊥−*consistent* Γ

definition (**in** *classical-logic*)
  maximally-consistent-set :: *'a set* ⇒ *bool* (‹*MCS*›) **where**
    [*simp*]: *MCS* Γ ≡ ⊥−*MCS* Γ

lemma (**in** *classical-logic*)
  formula-maximally-consistent-set-def-negation: $\varphi{-}MCS\ \Gamma \Longrightarrow \varphi \rightarrow \bot \in \Gamma$
proof −
  assume $\varphi{-}MCS\ \Gamma$
  {
    assume $\varphi \rightarrow \bot \notin \Gamma$
    hence $(\varphi \rightarrow \bot) \rightarrow \varphi \in \Gamma$
      using ‹$\varphi{-}MCS\ \Gamma$›
      unfolding *formula-maximally-consistent-set-def-def*
      by *blast*
    hence $\Gamma \Vdash (\varphi \rightarrow \bot) \rightarrow \varphi$
      using *set-deduction-reflection*
      by *simp*
    hence $\Gamma \Vdash \varphi$
      using
        Peirces-law
        set-deduction-modus-ponens
        set-deduction-weaken
      by *metis*
    hence *False*
      using ‹$\varphi{-}MCS\ \Gamma$›

```
    unfolding
      formula-maximally-consistent-set-def-def
      formula-consistent-def
    by simp
  }
  thus ?thesis by blast
qed
```

Relative maximal consistency and conventional maximal consistency in fact coincide in classical logic.

```
lemma (in classical-logic)
  formula-maximal-consistency: (∃φ. φ−MCS Γ) = MCS Γ
proof −
  {
    fix φ
    have φ−MCS Γ ⟹ MCS Γ
    proof −
      assume φ−MCS Γ
      have consistent Γ
        using
          ‹φ−MCS Γ›
          ex-falso-quodlibet [where φ=φ]
          set-deduction-weaken [where Γ=Γ]
          set-deduction-modus-ponens
        unfolding
          formula-maximally-consistent-set-def-def
          consistent-def
          formula-consistent-def
        by metis
      moreover {
        fix ψ
        have ψ → ⊥ ∉ Γ ⟹ ψ ∈ Γ
        proof −
          assume ψ → ⊥ ∉ Γ
          hence (ψ → ⊥) → φ ∈ Γ
            using ‹φ−MCS Γ›
            unfolding formula-maximally-consistent-set-def-def
            by blast
          hence Γ ⊩ (ψ → ⊥) → φ
            using set-deduction-reflection
            by simp
          also have Γ ⊩ φ → ⊥
            using ‹φ−MCS Γ›
                  formula-maximally-consistent-set-def-negation
                  set-deduction-reflection
            by simp
          hence Γ ⊩ (ψ → ⊥) → ⊥
            using calculation
                  hypothetical-syllogism
```

```
                [where φ=ψ → ⊥ and ψ=φ and χ=⊥]
              set-deduction-weaken
                [where Γ=Γ]
              set-deduction-modus-ponens
        by metis
      hence Γ ⊩ ψ
        using double-negation
                [where φ=ψ]
              set-deduction-weaken
                [where Γ=Γ]
              set-deduction-modus-ponens
        by metis
      thus ?thesis
        using ‹φ−MCS Γ›
              formula-maximally-consistent-set-def-reflection
        by blast
    qed
   }
   ultimately show ?thesis
    unfolding maximally-consistent-set-def
              formula-maximally-consistent-set-def-def
              formula-consistent-def
              consistent-def
    by blast
  qed
 }
 thus ?thesis
  unfolding maximally-consistent-set-def
  by metis
qed
```

Finally, classical logic allows us to strengthen $[\![\varphi{-}MCS\ \Omega;\ \psi \to \chi \in \Omega;\ \psi \in \Omega]\!] \Longrightarrow \chi \in \Omega$ to a biconditional.

```
lemma (in classical-logic)
 formula-maximally-consistent-set-def-implication:
 assumes φ−MCS Γ
 shows ψ → χ ∈ Γ = (ψ ∈ Γ ⟶ χ ∈ Γ)
proof −
 {
  assume hypothesis: ψ ∈ Γ ⟶ χ ∈ Γ
  {
   assume ψ ∉ Γ
   have ∀ψ. φ → ψ ∈ Γ
    by (meson assms
              formula-maximally-consistent-set-def-negation
              formula-maximally-consistent-set-def-implication-elimination
              formula-maximally-consistent-set-def-reflection
              pseudo-scotus set-deduction-weaken)
   then have ∀χ ψ. insert χ Γ ⊩ ψ ∨ χ → φ ∉ Γ
```

```
      by (meson assms
              axiom-k
              formula-maximally-consistent-set-def-reflection
              set-deduction-modus-ponens
              set-deduction-theorem
              set-deduction-weaken)
    hence ψ → χ ∈ Γ
      by (meson ‹ψ ∉ Γ› assms
              formula-maximally-consistent-set-def-def
              formula-maximally-consistent-set-def-reflection
              set-deduction-theorem)
  }
  moreover {
    assume χ ∈ Γ
    hence ψ → χ ∈ Γ
      by (metis assms
              calculation
              insert-absorb
              formula-maximally-consistent-set-def-reflection
              set-deduction-theorem)
  }
  ultimately have  ψ → χ ∈ Γ using hypothesis by blast
}
thus ?thesis
  using assms
    formula-maximally-consistent-set-def-implication-elimination
  by metis
qed

end
```

Chapter 3

Classical Soundness and Completeness

theory *Classical-Logic-Completeness*
  imports *Classical-Logic*
begin

The following presents soundness completeness of the classical propositional calculus for propositional semantics. The classical propositional calculus is sometimes referred to as the *sentential calculus*. We give a concrete algebraic data type for propositional formulae in §3.1. We inductively define a logical judgement $\vdash_{prop}$ for these formulae. We also define the Tarski truth relation $\models_{prop}$ inductively, which we present in §3.3.

The most significant results here are the *embedding theorems*. These theorems show that the propositional calculus can be embedded in any logic extending *classical-logic*. These theorems are proved in §3.5.

3.1 Syntax

Here we provide the usual language for formulae in the propositional calculus. It contains *falsum* $\bot$, implication ($\rightarrow$), and a way of constructing *atomic* propositions $\lambda\ \varphi\ .\ \langle\ \varphi\ \rangle$. Defining the language is straight-forward using an algebraic data type.

datatype *'a classical-propositional-formula* =
    Falsum (‹$\bot$›)
  | *Proposition* *'a* (‹$\langle$ - $\rangle$› [*45*])
  | *Implication*
      'a classical-propositional-formula
      'a classical-propositional-formula (**infixr** ‹$\rightarrow$› *70*)

3.2 Propositional Calculus

In this section we recursively define what a proof is in the classical propositional calculus. We provide the familiar K and S axioms, as well as *double negation* and *modus ponens*.

named-theorems *classical-propositional-calculus*
Rules for the Propositional Calculus

inductive *classical-propositional-calculus* ::
'a classical-propositional-formula ⇒ *bool* (‹$\vdash_{prop}$ -› [*60*] *55*)
where
axiom-k [*classical-propositional-calculus*]:
$\vdash_{prop} \varphi \longrightarrow \psi \longrightarrow \varphi$
| *axiom-s* [*classical-propositional-calculus*]:
$\vdash_{prop} (\varphi \longrightarrow \psi \longrightarrow \chi) \longrightarrow (\varphi \longrightarrow \psi) \longrightarrow \varphi \longrightarrow \chi$
| *double-negation* [*classical-propositional-calculus*]:
$\vdash_{prop} ((\varphi \longrightarrow \bot) \longrightarrow \bot) \longrightarrow \varphi$
| *modus-ponens* [*classical-propositional-calculus*]:
$\vdash_{prop} \varphi \longrightarrow \psi \Longrightarrow \vdash_{prop} \varphi \Longrightarrow \vdash_{prop} \psi$

Our proof system for our propositional calculus is trivially an instance of *classical-logic*. The introduction rules for $\vdash_{prop}$ naturally reflect the axioms of the classical logic axiom class.

instantiation *classical-propositional-formula*
:: (*type*) *classical-logic*
begin
definition [*simp*]: $\bot = \bot$
definition [*simp*]: $\vdash \varphi = \vdash_{prop} \varphi$
definition [*simp*]: $\varphi \longrightarrow \psi = \varphi \longrightarrow \psi$
instance by *standard* (*simp add*: *classical-propositional-calculus*)+
end

3.3 Propositional Semantics

Below we give the typical definition of the Tarski truth relation $\models_{prop}$.

primrec *classical-propositional-semantics* ::
'a set ⇒ *'a classical-propositional-formula* ⇒ *bool*
(**infix** ‹$\models_{prop}$› *65*)
where
$\mathfrak{M} \models_{prop} \langle\ p\ \rangle = (p \in \mathfrak{M})$
| $\mathfrak{M} \models_{prop} \varphi \longrightarrow \psi = (\mathfrak{M} \models_{prop} \varphi \longrightarrow \mathfrak{M} \models_{prop} \psi)$
| $\mathfrak{M} \models_{prop} \bot = \mathit{False}$

Soundness of our calculus for these semantics is trivial.

theorem *classical-propositional-calculus-soundness*:
$\vdash_{prop} \varphi \Longrightarrow \mathfrak{M} \models_{prop} \varphi$
by (*induct rule*: *classical-propositional-calculus.induct*, *simp*+)

3.4 Soundness and Completeness Proofs

definition *strong-classical-propositional-deduction* ::
  'a classical-propositional-formula set
    ⇒ *'a classical-propositional-formula* ⇒ *bool*
  (**infix** ‹$\Vdash_{prop}$› *65*)
  where
    [*simp*]: $\Gamma \Vdash_{prop} \varphi \equiv \Gamma \Vdash \varphi$

definition *strong-classical-propositional-tarski-truth* ::
  'a classical-propositional-formula set
    ⇒ *'a classical-propositional-formula* ⇒ *bool*
  (**infix** ‹$\models\!\models_{prop}$› *65*)
  where
    [*simp*]: $\Gamma \models\!\models_{prop} \varphi \equiv \forall\ \mathfrak{M}.(\forall\ \gamma \in \Gamma.\ \mathfrak{M} \models_{prop} \gamma) \longrightarrow \mathfrak{M} \models_{prop} \varphi$

definition *theory-propositions* ::
  'a classical-propositional-formula set ⇒ *'a set* (‹$\{\!|$ - $|\!\}$› [*50*])
  where
    [*simp*]: $\{\!|\ \Gamma\ |\!\} = \{p\ .\ \Gamma \Vdash_{prop} \langle\ p\ \rangle\}$

Below we give the main lemma for completeness: the *truth lemma*. This proof connects the maximally consistent sets developed in §1.16 and §2.3 with the semantics given in §3.3.

All together, the technique we are using essentially follows the approach by Blackburn et al. [1, §4.2, pgs. 196-201].

lemma *truth-lemma*:
  assumes *MCS* Γ
  shows $\Gamma \Vdash_{prop} \varphi \equiv \{\!|\ \Gamma\ |\!\} \models_{prop} \varphi$
proof (*induct* φ)
  case *Falsum*
  then show *?case* **using** *assms* **by** *auto*
next
  case (*Proposition x*)
  then show *?case* **by** *simp*
next
  case (*Implication* ψ χ)
  thus *?case*
    unfolding *strong-classical-propositional-deduction-def*
    by (*metis*
        assms
        maximally-consistent-set-def
        formula-maximally-consistent-set-def-implication
        classical-propositional-semantics.simps(*2*)
        implication-classical-propositional-formula-def
        set-deduction-modus-ponens
        set-deduction-reflection)
qed

Here the truth lemma above is combined with $\varphi{-}consistent\ \Gamma \Longrightarrow \exists\,\Omega.\ \varphi{-}MCS\ \Omega \wedge \Gamma \subseteq \Omega$ proven in §3.3. These theorems together give rise to strong completeness for the propositional calculus.

theorem *classical-propositional-calculus-strong-soundness-and-completeness*:
$\Gamma \Vdash_{prop} \varphi = \Gamma \Vdash\!\models_{prop} \varphi$
proof −
have *soundness*: $\Gamma \Vdash_{prop} \varphi \Longrightarrow \Gamma \Vdash\!\models_{prop} \varphi$
proof −
assume $\Gamma \Vdash_{prop} \varphi$
from *this* **obtain** Γ' **where** Γ': *set* $\Gamma' \subseteq \Gamma$ $\Gamma' :\vdash \varphi$
by (*simp add*: *set-deduction-def*, *blast*)
{
fix $\mathfrak{M}$
assume $\forall\ \gamma \in \Gamma.\ \mathfrak{M} \models_{prop} \gamma$
hence $\forall\ \gamma \in$ *set* $\Gamma'.\ \mathfrak{M} \models_{prop} \gamma$ **using** $\Gamma'(1)$ **by** *auto*
hence $\forall\ \varphi.\ \Gamma' :\vdash \varphi \longrightarrow \mathfrak{M} \models_{prop} \varphi$
proof (*induct* Γ')
case *Nil*
then show *?case*
by (*simp add*:
classical-propositional-calculus-soundness
list-deduction-def)
next
case (*Cons* ψ Γ')
thus *?case* **using** *list-deduction-theorem* **by** *fastforce*
qed
with $\Gamma'(2)$ **have** $\mathfrak{M} \models_{prop} \varphi$ **by** *blast*
}
thus $\Gamma \Vdash\!\models_{prop} \varphi$
using *strong-classical-propositional-tarski-truth-def* **by** *blast*
qed
have *completeness*: $\Gamma \Vdash\!\models_{prop} \varphi \Longrightarrow \Gamma \Vdash_{prop} \varphi$
proof (*erule contrapos-pp*)
assume $\neg\ \Gamma \Vdash_{prop} \varphi$
hence $\exists\ \mathfrak{M}.\ (\forall\ \gamma \in \Gamma.\ \mathfrak{M} \models_{prop} \gamma) \wedge \neg\ \mathfrak{M} \models_{prop} \varphi$
proof −
from ‹$\neg\ \Gamma \Vdash_{prop} \varphi$› **obtain** Ω **where** Ω: $\Gamma \subseteq \Omega$ $\varphi{-}MCS\ \Omega$
by (*meson*
formula-consistent-def
formula-maximally-consistent-extension
strong-classical-propositional-deduction-def)
hence $(\varphi \rightarrow \bot) \in \Omega$
using *formula-maximally-consistent-set-def-negation* **by** *blast*
hence $\neg$ ⦃ Ω ⦄ $\models_{prop} \varphi$
using Ω
formula-consistent-def
formula-maximal-consistency
formula-maximally-consistent-set-def-def
truth-lemma

```
      unfolding strong-classical-propositional-deduction-def
      by blast
    moreover have ∀ γ ∈ Γ. {⦇ Ω ⦈} ⊨_prop γ
      using
        formula-maximal-consistency
        truth-lemma
        Ω
        set-deduction-reflection
      unfolding strong-classical-propositional-deduction-def
      by blast
    ultimately show ?thesis by auto
  qed
  thus ¬ Γ ⊭_prop φ
    unfolding strong-classical-propositional-tarski-truth-def
    by simp
 qed
 from soundness completeness show Γ ⊩_prop φ = Γ ⊨_prop φ
  by linarith
qed
```

For our applications in §sec:propositional-embedding, we will only need a weaker form of soundness and completeness rather than the stronger form proved above.

```
theorem classical-propositional-calculus-soundness-and-completeness:
 ⊢_prop φ = (∀ 𝔐. 𝔐 ⊨_prop φ)
 using classical-propositional-calculus-soundness [where φ=φ]
       classical-propositional-calculus-strong-soundness-and-completeness
          [where φ=φ and Γ={}]
       strong-classical-propositional-deduction-def
          [where φ=φ and Γ={}]
       strong-classical-propositional-tarski-truth-def
          [where φ=φ and Γ={}]
       deduction-classical-propositional-formula-def [where φ=φ]
       set-deduction-base-theory [where φ=φ]
 by metis

instantiation classical-propositional-formula
 :: (type) consistent-classical-logic
begin
instance by standard
 (simp add: classical-propositional-calculus-soundness-and-completeness)
end
```

3.5 Embedding Theorem For the Propositional Calculus

A recurring technique to prove theorems in logic moving forward is *embed* our theorem into the classical propositional calculus.

Using our embedding, we can leverage completeness to turn our problem into semantics and dispatch to Isabelle/HOL's classical theorem provers.

In future work we may make a tactic for this, but for now we just manually leverage the technique throughout our subsequent proofs.

primrec (**in** *classical-logic*)
  classical-propositional-formula-embedding
  :: *'a classical-propositional-formula* ⇒ *'a* (‹⦇ - ⦈› [*50*]) **where**
    ⦇ ⟨ *p* ⟩ ⦈ = *p*
  | ⦇ φ → ψ ⦈ = ⦇ φ ⦈ → ⦇ ψ ⦈
  | ⦇ ⊥ ⦈ = ⊥

theorem (**in** *classical-logic*) *propositional-calculus*:
  $\vdash_{prop}$ φ ⟹ ⊢ ⦇ φ ⦈
  by (*induct rule*: *classical-propositional-calculus.induct*,
      (*simp add*: *axiom-k axiom-s double-negation modus-ponens*)+)

The following theorem in particular shows that it suffices to prove theorems using classical semantics to prove theorems about the logic under investigation.

theorem (**in** *classical-logic*) *propositional-semantics*:
  ∀ 𝔐. 𝔐 $\models_{prop}$ φ ⟹ ⊢ ⦇ φ ⦈
  by (*simp add*:
      classical-propositional-calculus-soundness-and-completeness
      propositional-calculus)

end

Chapter 4

List Utility Theorems

theory *List-Utilities*
  imports
    HOL−Combinatorics.List-Permutation
begin

Throughout our work it will be necessary to reuse common lemmas regarding lists and multisets. These results are proved in the following section and reused by subsequent lemmas and theorems.

4.1 Multisets

lemma *mset-removeAll* [*simp*]:
  ‹*mset* (*removeAll x xs*) = *filter-mset* ((≠) *x*) (*mset xs*)›
  by (*induction xs*) *auto*

lemma *length-sub-mset*:
  assumes *mset* Ψ ⊆# *mset* Γ
      and *length* Ψ >= *length* Γ
    shows *mset* Ψ = *mset* Γ
  using *assms*
  by (*metis*
        append-Nil2
        append-eq-append-conv
        leD
        linorder-neqE-nat
        mset-le-perm-append
        perm-length
        size-mset
        size-mset-mono)

lemma *set-exclusion-mset-simplify*:
  assumes ¬ (∃ ψ ∈ *set* Ψ. ψ ∈ *set* Σ)
      and *mset* Ψ ⊆# *mset* (Σ @ Γ)
    shows *mset* Ψ ⊆# *mset* Γ

using *assms*
proof (*induct* Σ)
  case *Nil*
  then show *?case* **by** *simp*
next
  case (*Cons* σ Σ)
  then show *?case*
    by (*cases* $\sigma \in$ *set* Ψ,
        fastforce,
        metis
          add.commute
          add-mset-add-single
          diff-single-trivial
          in-multiset-in-set
          mset.simps(*2*)
          notin-set-remove1
          remove-hd
          subset-eq-diff-conv
          union-code
          append-Cons)
qed

lemma *image-mset-cons-homomorphism*:
  image-mset mset (*image-mset* ((#) φ) Φ) = *image-mset* ((+) {# φ #}) (*image-mset mset* Φ)
  by (*induct* Φ, *simp*+)

lemma *image-mset-append-homomorphism*:
  image-mset mset (*image-mset* ((@) Δ) Φ) = *image-mset* ((+) (*mset* Δ)) (*image-mset mset* Φ)
  by (*induct* Φ, *simp*+)

lemma *image-mset-add-collapse*:
  fixes *A B* :: *'a multiset*
  shows *image-mset* ((+) *A*) (*image-mset* ((+) *B*) *X*) = *image-mset* ((+) (*A* + *B*)) *X*
  by (*induct X*, *simp*, *simp*)

lemma *remove1-remdups-removeAll*: *remove1 x* (*remdups A*) = *remdups* (*removeAll x A*)
proof (*induct A*)
  case *Nil*
  then show *?case* **by** *simp*
next
  case (*Cons a A*)
  then show *?case*
    by (*cases a* = *x*, (*simp add*: *Cons*)+)
qed

```
lemma mset-remdups:
  assumes mset A = mset B
  shows mset (remdups A) = mset (remdups B)
proof −
  have ∀ B. mset A = mset B ⟶ mset (remdups A) = mset (remdups B)
  proof (induct A)
    case Nil
    then show ?case by simp
  next
    case (Cons a A)
    {
      fix B
      assume mset (a # A) = mset B
      hence mset A = mset (remove1 a B)
        by (metis add-mset-add-mset-same-iff
                  list.set-intros(1)
                  mset.simps(2)
                  mset-eq-setD
                  perm-remove)
      hence mset (remdups A) = mset (remdups (remove1 a B))
        using Cons.hyps by blast
      hence mset (remdups (a # (remdups A))) = mset (remdups (a # (remdups
(remove1 a B))))
        by (metis mset-eq-setD set-eq-iff-mset-remdups-eq list.simps(15))
      hence   mset (remdups (a # (removeAll a (remdups A))))
            = mset (remdups (a # (removeAll a (remdups (remove1 a B)))))
      by (metis insert-Diff-single list.set(2) set-eq-iff-mset-remdups-eq set-removeAll)
      hence   mset (remdups (a # (remdups (removeAll a A))))
            = mset (remdups (a # (remdups (removeAll a (remove1 a B)))))
      by (metis distinct-remdups distinct-remove1-removeAll remove1-remdups-removeAll)
      hence mset (remdups (remdups (a # A))) = mset (remdups (remdups (a #
(remove1 a B))))
        by (metis ‹mset A = mset (remove1 a B)›
                  list.set(2)
                  mset-eq-setD
                  set-eq-iff-mset-remdups-eq)
      hence mset (remdups (a # A)) = mset (remdups (a # (remove1 a B)))
        by (metis remdups-remdups)
      hence mset (remdups (a # A)) = mset (remdups B)
       using ‹mset (a # A) = mset B› mset-eq-setD set-eq-iff-mset-remdups-eq by
blast
    }
    then show ?case by simp
  qed
  thus ?thesis using assms by blast
qed

lemma mset-mset-map-snd-remdups:
  assumes mset (map mset A) = mset (map mset B)
```

```
  shows mset (map (mset ∘ (map snd) ∘ remdups) A) = mset (map (mset ∘ (map
snd) ∘ remdups) B)
proof −
  {
    fix B :: ('a × 'b) list list
    fix b :: ('a × 'b) list
    assume b ∈ set B
    hence   mset (map (mset ∘ (map snd) ∘ remdups) (b # (remove1 b B)))
          = mset (map (mset ∘ (map snd) ∘ remdups) B)
    proof (induct B)
      case Nil
      then show ?case by simp
    next
      case (Cons b' B)
      then show ?case
      by (cases b = b', simp+)
    qed
  }
  note ♢ = this
  have
    ∀ B :: ('a × 'b) list list.
     mset (map mset A) = mset (map mset B)
       ⟶ mset (map (mset ∘ (map snd) ∘ remdups) A) = mset (map (mset ∘ (map
snd) ∘ remdups) B)
  proof (induct A)
    case Nil
    then show ?case by simp
  next
    case (Cons a A)
    {
      fix B
      assume ♠: mset (map mset (a # A)) = mset (map mset B)
      hence mset a ∈# mset (map mset B)
        by (simp,
            metis ♠
                  image-set
                  list.set-intros(1)
                  list.simps(9)
                  mset-eq-setD)
      from this obtain b where †:
        b ∈ set B
        mset a = mset b
        by auto
      with ♠ have mset (map mset A) = mset (remove1 (mset b) (map mset B))
        by (simp add: union-single-eq-diff)
      moreover have mset B = mset (b # remove1 b B) using † by simp
      hence mset (map mset B) = mset (map mset (b # (remove1 b B)))
        by (simp,
            metis image-mset-add-mset
```

```
            mset.simps(2)
            mset-remove1)
    ultimately have mset (map mset A) = mset (map mset (remove1 b B))
      by simp
    hence   mset (map (mset ∘ (map snd) ∘ remdups) A)
          = mset (map (mset ∘ (map snd) ∘ remdups) (remove1 b B))
      using Cons.hyps by blast
     moreover have (mset ∘ (map snd) ∘ remdups) a = (mset ∘ (map snd) ∘
remdups) b
      using †(2) mset-remdups by fastforce
    ultimately have
       mset (map (mset ∘ (map snd) ∘ remdups) (a # A))
      = mset (map (mset ∘ (map snd) ∘ remdups) (b # (remove1 b B)))
      by simp
    moreover have
       mset (map (mset ∘ (map snd) ∘ remdups) (b # (remove1 b B)))
      = mset (map (mset ∘ (map snd) ∘ remdups) B)
      using †(1) ♢ by blast
    ultimately have
       mset (map (mset ∘ (map snd) ∘ remdups) (a # A))
      = mset (map (mset ∘ (map snd) ∘ remdups) B)
      by simp
  }
  then show ?case by blast
 qed
 thus ?thesis using assms by blast
qed

lemma mset-remdups-append-msub:
 mset (remdups A) ⊆# mset (remdups (B @ A))
proof −
 have ∀ B. mset (remdups A) ⊆# mset (remdups (B @ A))
 proof (induct A)
  case Nil
  then show ?case by simp
 next
  case (Cons a A)
  {
   fix B
   have †: mset (remdups (B @ (a # A))) = mset (remdups (a # (B @ A)))
    by (induct B, simp+)
   have mset (remdups (a # A)) ⊆# mset (remdups (B @ (a # A)))
   proof (cases a ∈ set B ∧ a ∉ set A)
    case True
    hence †: mset (remove1 a (remdups (B @ A))) = mset (remdups ((removeAll
a B) @ A))
     by (simp add: remove1-remdups-removeAll)
    hence   (add-mset a (mset (remdups A)) ⊆# mset (remdups (B @ A)))
          = (mset (remdups A) ⊆# mset (remdups ((removeAll a B) @ A)))
```

```
      using True
      by (simp add: insert-subset-eq-iff)
    then show ?thesis
      by (metis † Cons True
                Un-insert-right
                list.set(2)
                mset.simps(2)
                mset-subset-eq-insertD
                remdups.simps(2)
                set-append
                set-eq-iff-mset-remdups-eq
                set-mset-mset set-remdups)
  next
    case False
    then show ?thesis using † Cons by simp
  qed
 }
 thus ?case by blast
qed
thus ?thesis by blast
qed
```

4.2 List Mapping

The following notation for permutations is slightly nicer when formatted in LaTeX.

```
notation perm (infix ‹⇌› 50)

lemma map-monotonic:
  assumes mset A ⊆# mset B
  shows mset (map f A) ⊆# mset (map f B)
  by (simp add: assms image-mset-subseteq-mono)

lemma perm-map-perm-list-exists:
  assumes A ⇌ map f B
  shows ∃ B'. A = map f B' ∧ B' ⇌ B
proof −
  have ∀ B. A ⇌ map f B ⟶ (∃ B'. A = map f B' ∧ B' ⇌ B)
  proof (induct A)
    case Nil
    then show ?case by simp
  next
    case (Cons a A)
    {
      fix B
      assume a # A ⇌ map f B
      from this obtain b where b:
        b ∈ set B
```

```
      f b = a
      by (metis
          (full-types)
          imageE
          list.set-intros(1)
          set-map
          set-mset-mset)
    hence A ⇌ (remove1 (f b) (map f B))
        B ⇌ b # remove1 b B
      by (metis
          ‹a # A ⇌ map f B›
          perm-remove-perm
          remove-hd,
        meson b(1) perm-remove)
    hence A ⇌ (map f (remove1 b B))
      by (metis (no-types)
          list.simps(9)
          mset-map
          mset-remove1
          remove-hd)
    from this obtain B' where B':
      A = map f B'
      B' ⇌ (remove1 b B)
      using Cons.hyps by blast
    with b have a # A = map f (b # B')
      by simp
    moreover have B ⇌ b # B'
      by (metis B'(2) ‹mset B = mset (b # remove1 b B)› mset.simps(2))
    ultimately have ∃B'. a # A = map f B' ∧ B' ⇌ B
      by (meson perm-sym)
  }
  thus ?case by blast
 qed
 with assms show ?thesis by blast
qed

lemma mset-sub-map-list-exists:
 assumes mset Φ ⊆# mset (map f Γ)
 shows ∃ Φ'. mset Φ' ⊆# mset Γ ∧ Φ = (map f Φ')
proof −
 have ∀ Φ. mset Φ ⊆# mset (map f Γ)
          ⟶ (∃ Φ'. mset Φ' ⊆# mset Γ ∧ Φ = (map f Φ'))
 proof (induct Γ)
  case Nil
  then show ?case by simp
 next
  case (Cons γ Γ)
  {
   fix Φ
```

assume *mset* $\Phi \subseteq\#$ *mset* (*map* f (γ # Γ))
have $\exists \Phi'$. *mset* $\Phi' \subseteq\#$ *mset* (γ # Γ) $\wedge$ Φ = *map* f Φ'
proof *cases*
  assume $f\ \gamma \in$ *set* Φ
  hence $f\ \gamma$ # (*remove1* ($f\ \gamma$) Φ) $\rightleftharpoons$ Φ
    by *force*
  with ‹*mset* $\Phi \subseteq\#$ *mset* (*map* f (γ # Γ))›
  have *mset* (*remove1* ($f\ \gamma$) Φ) $\subseteq\#$ *mset* (*map* f Γ)
    by (*metis*
        insert-subset-eq-iff
        list.simps(*9*)
        mset.simps(*2*)
        mset-remove1
        remove-hd)
  from *this Cons* **obtain** Φ' **where** Φ':
    mset $\Phi' \subseteq\#$ *mset* Γ
    remove1 ($f\ \gamma$) Φ = *map* f Φ'
    by *blast*
  hence *mset* (γ # Φ') $\subseteq\#$ *mset* (γ # Γ)
    and $f\ \gamma$ # (*remove1* ($f\ \gamma$) Φ) = *map* f (γ # Φ')
    by *simp+*
  hence $\Phi \rightleftharpoons$ *map* f (γ # Φ')
    using ‹$f\ \gamma \in$ *set* Φ› *perm-remove*
    by *metis*
  from *this* **obtain** Φ'' **where** Φ'':
    Φ = *map* f Φ''
    $\Phi'' \rightleftharpoons \gamma$ # Φ'
    using *perm-map-perm-list-exists*
    by *blast*
  hence *mset* $\Phi'' \subseteq\#$ *mset* (γ # Γ)
    by (*metis* ‹*mset* (γ # Φ') $\subseteq\#$ *mset* (γ # Γ)›)
  thus *?thesis* **using** Φ'' **by** *blast*
next
  assume $f\ \gamma \notin$ *set* Φ
  have *mset* Φ − {#$f\ \gamma$#} = *mset* Φ
    by (*metis* (*no-types*)
        ‹$f\ \gamma \notin$ *set* Φ›
        diff-single-trivial
        set-mset-mset)
  moreover
  have *mset* (*map* f (γ # Γ))
        = *add-mset* ($f\ \gamma$) (*image-mset* f (*mset* Γ))
    by *simp*
  ultimately have *mset* $\Phi \subseteq\#$ *mset* (*map* f Γ)
    by (*metis* (*no-types*)
        Diff-eq-empty-iff-mset
        ‹*mset* $\Phi \subseteq\#$ *mset* (*map* f (γ # Γ))›
        add-mset-add-single
        cancel-ab-semigroup-add-class.diff-right-commute

```
        diff-diff-add mset-map)
    with Cons show ?thesis
     by (metis
        mset-le-perm-append
        perm-append-single
        subset-mset.order-trans)
  qed
 }
 thus ?case using Cons by blast
qed
thus ?thesis using assms by blast
qed
```

4.3 Laws for Searching a List

```
lemma find-Some-predicate:
 assumes find P Ψ = Some ψ
 shows P ψ
 using assms
proof (induct Ψ)
 case Nil
 then show ?case by simp
next
 case (Cons ω Ψ)
 then show ?case by (cases P ω, fastforce+)
qed

lemma find-Some-set-membership:
 assumes find P Ψ = Some ψ
 shows ψ ∈ set Ψ
 using assms
proof (induct Ψ)
 case Nil
 then show ?case by simp
next
 case (Cons ω Ψ)
 then show ?case by (cases P ω, fastforce+)
qed
```

4.4 Permutations

```
lemma perm-count-list:
 assumes Φ ⇌ Ψ
 shows count-list Φ φ = count-list Ψ φ
 using assms
proof (induct Φ arbitrary: Ψ)
 case Nil
 then show ?case
```

```
    by blast
next
  case (Cons χ Φ Ψ)
  hence ♢: count-list Φ φ = count-list (remove1 χ Ψ) φ
    by (metis mset-remove1 remove-hd)
  show ?case
  proof cases
    assume χ = φ
    hence count-list (χ # Φ) φ = count-list Φ φ + 1 by simp
    with ♢ have count-list (χ # Φ) φ = count-list (remove1 χ Ψ) φ + 1
      by simp
    moreover
    have χ ∈ set Ψ
      by (metis Cons.prems list.set-intros(1) set-mset-mset)
    hence count-list (remove1 χ Ψ) φ + 1 = count-list Ψ φ
      using ‹χ = φ›
      by (induct Ψ, simp, auto)
    ultimately show ?thesis by simp
  next
    assume χ ≠ φ
    with ♢ have count-list (χ # Φ) φ = count-list (remove1 χ Ψ) φ
      by simp
    moreover have count-list (remove1 χ Ψ) φ = count-list Ψ φ
      using ‹χ ≠ φ›
      by (induct Ψ, simp+)
    ultimately show ?thesis by simp
  qed
qed

lemma count-list-append:
  count-list (A @ B) a = count-list A a + count-list B a
  by (induct A, simp, simp)

lemma concat-remove1:
  assumes Ψ ∈ set L
  shows concat L ⇌ Ψ @ concat (remove1 Ψ L)
    using assms
    by (induct L, simp, simp, metis)

lemma concat-set-membership-mset-containment:
  assumes concat Γ ⇌ Λ
  and     Φ ∈ set Γ
  shows   mset Φ ⊆# mset Λ
  using assms
  by (induct Γ, simp, simp, metis concat-remove1 mset-le-perm-append)

lemma (in comm-monoid-add) perm-list-summation:
  assumes Ψ ⇌ Φ
  shows (∑ ψ'←Ψ. f ψ') = (∑ φ'←Φ. f φ')
```

```
  using assms
proof (induct Ψ arbitrary: Φ)
  case Nil
  then show ?case by auto
next
  case (Cons ψ Ψ Φ)
  hence (∑ψ' ← Ψ. f ψ') = (∑φ' ← (remove1 ψ Φ). f φ')
    by (metis mset-remove1 remove-hd)
  moreover have ψ ∈ set Φ
    by (metis Cons.prems list.set-intros(1) set-mset-mset)
  hence (∑φ' ← (ψ # (remove1 ψ Φ)). f φ') = (∑φ'←Φ. f φ')
  proof (induct Φ)
    case Nil
    then show ?case by auto
  next
    case (Cons φ Φ)
    show ?case
    proof cases
      assume φ = ψ
      then show ?thesis by simp
    next
      assume φ ≠ ψ
      hence ψ ∈ set Φ
        using Cons.prems by auto
      hence (∑φ' ← (ψ # (remove1 ψ Φ)). f φ') = (∑φ'←Φ. f φ')
        using Cons.hyps by blast
      hence (∑φ'←(φ # Φ). f φ')
              = (∑φ' ← (ψ # φ # (remove1 ψ Φ)). f φ')
        by (simp add: add.left-commute)
      moreover
      have (ψ # (φ # (remove1 ψ Φ))) = (ψ # (remove1 ψ (φ # Φ)))
        using ‹φ ≠ ψ› by simp
      ultimately show ?thesis
        by simp
    qed
  qed
  ultimately show ?case
    by simp
qed
```

4.5 List Duplicates

```
primrec duplicates :: 'a list ⇒ 'a set
  where
    duplicates [] = {}
  | duplicates (x # xs) =
      (if (x ∈ set xs)
       then insert x (duplicates xs)
       else duplicates xs)
```

lemma *duplicates-subset*:
duplicates $\Phi \subseteq$ *set* Φ
by (*induct* Φ, *simp*, *auto*)

lemma *duplicates-alt-def*:
duplicates xs = $\{x.$ *count-list xs x* $\geq$ *2*$\}$
proof (*induct xs*)
case *Nil*
then show *?case* **by** *simp*
next
case (*Cons x xs*)
assume *inductive-hypothesis*: *duplicates xs* = $\{x.$ *2* $\leq$ *count-list xs x*$\}$
then show *?case*
proof *cases*
assume *x* $\in$ *set xs*
hence *count-list* (*x* # *xs*) *x* $\geq$ *2*
by (*simp*, *induct xs*, *simp*, *simp*, *blast*)
hence $\{y.$ *2* $\leq$ *count-list* (*x* # *xs*) *y*$\}$
= *insert x* $\{y.$ *2* $\leq$ *count-list xs y*$\}$
by (*simp*, *blast*)
thus *?thesis* **using** *inductive-hypothesis* ‹*x* $\in$ *set xs*›
by *simp*
next
assume *x* $\notin$ *set xs*
hence $\{y.$ *2* $\leq$ *count-list* (*x* # *xs*) *y*$\}$ = $\{y.$ *2* $\leq$ *count-list xs y*$\}$
by (*simp*, *auto*)
thus *?thesis* **using** *inductive-hypothesis* ‹*x* $\notin$ *set xs*›
by *simp*
qed
qed

4.6 List Subtraction

primrec *list-subtract* :: *'a list* $\Rightarrow$ *'a list* $\Rightarrow$ *'a list* (**infixl** ‹$\ominus$› *70*)
where
xs $\ominus$ [] = *xs*
| *xs* $\ominus$ (*y* # *ys*) = (*remove1 y* (*xs* $\ominus$ *ys*))

lemma *list-subtract-mset-homomorphism* [*simp*]:
mset (*A* $\ominus$ *B*) = *mset A* $-$ *mset B*
by (*induct B*, *simp*, *simp*)

lemma *list-subtract-empty* [*simp*]:
[] $\ominus$ Φ = []
by (*induct* Φ, *simp*, *simp*)

lemma *list-subtract-remove1-cons-perm*:
$\Phi \ominus (\varphi$ # $\Lambda) \rightleftharpoons$ (*remove1* φ Φ) $\ominus$ Λ

by (*induct* Λ, *simp*, *simp add*: *add-mset-commute*)

lemma *list-subtract-cons*:
assumes $\varphi \notin$ *set* Λ
shows $(\varphi \# \Phi) \ominus \Lambda = \varphi \# (\Phi \ominus \Lambda)$
using *assms*
by (*induct* Λ, *simp*, *simp*, *blast*)

lemma *list-subtract-cons-absorb*:
assumes *count-list* $\Phi\ \varphi \geq$ *count-list* $\Lambda\ \varphi$
shows $\varphi \# (\Phi \ominus \Lambda) \rightleftharpoons (\varphi \# \Phi) \ominus \Lambda$
using *assms*
proof (*induct* Λ *arbitrary*: Φ)
case *Nil*
then show *?case* **using** *list-subtract-cons* **by** *fastforce*
next
case (*Cons* $\psi\ \Lambda\ \Phi$)
then show *?case*
proof *cases*
assume $\varphi = \psi$
hence $\varphi \in$ *set* Φ
using *Cons.prems count-notin* **by** *force*
hence $\Phi \rightleftharpoons \varphi \# (remove1\ \psi\ \Phi)$
unfolding $\langle\varphi = \psi\rangle$
by *force*
thus *?thesis* **using** *perm-count-list*
by (*metis*
(*no-types*, *lifting*)
Cons.hyps
Cons.prems
$\langle\varphi = \psi\rangle$
add-le-cancel-right
add-mset-diff-bothsides
count-list.simps(*2*)
list-subtract-mset-homomorphism
mset.simps(*2*))
next
assume $\varphi \neq \psi$
hence *count-list* $(\psi \# \Lambda)\ \varphi =$ *count-list* $\Lambda\ \varphi$
by *simp*
moreover have *count-list* $\Phi\ \varphi =$ *count-list* $(remove1\ \psi\ \Phi)\ \varphi$
proof (*induct* Φ)
case *Nil*
then show *?case* **by** *simp*
next
case (*Cons* $\varphi'\ \Phi$)
show *?case*
proof *cases*
assume $\varphi' = \varphi$

```
      with ‹φ ≠ ψ›
      have count-list (φ' # Φ) φ = 1 + count-list Φ φ
          count-list (remove1 ψ (φ' # Φ)) φ
            = 1 + count-list (remove1 ψ Φ) φ
        by simp+
      with Cons show ?thesis by linarith
    next
      assume φ' ≠ φ
      with Cons show ?thesis by simp
    qed
  qed
  ultimately show ?thesis
    using ‹count-list (ψ # Λ) φ ≤ count-list Φ φ›
    by (metis
        Cons.hyps
        ‹φ ≠ ψ›
        list-subtract-remove1-cons-perm
        mset.simps(2)
        remove1.simps(2))
 qed
qed

lemma list-subtract-cons-remove1-perm:
 assumes φ ∈ set Λ
 shows (φ # Φ) ⊖ Λ ⇌ Φ ⊖ (remove1 φ Λ)
 using assms
 by (metis
      list-subtract-mset-homomorphism
      list-subtract-remove1-cons-perm
      perm-remove
      remove-hd)

lemma list-subtract-removeAll-perm:
 assumes count-list Φ φ ≤ count-list Λ φ
 shows Φ ⊖ Λ ⇌ (removeAll φ Φ) ⊖ (removeAll φ Λ)
 using assms
proof (induct Φ arbitrary: Λ)
 case Nil
 then show ?case by auto
next
 case (Cons ξ Φ Λ)
 hence Φ ⊖ Λ ⇌ (removeAll φ Φ) ⊖ (removeAll φ Λ)
   by (metis add-leE count-list.simps(2))
 show ?case
 proof cases
   assume ξ = φ
   hence count-list Φ φ < count-list Λ φ
     using ‹count-list (ξ # Φ) φ ≤ count-list Λ φ›
     by auto
```

```
  hence count-list Φ φ ≤ count-list (remove1 φ Λ) φ
    by (induct Λ, simp, auto)
  hence Φ ⊖ (remove1 φ Λ)
          ⇌ removeAll φ Φ ⊖ removeAll φ (remove1 φ Λ)
    using Cons.hyps by blast
  hence Φ ⊖ (remove1 φ Λ) ⇌ removeAll φ Φ ⊖ removeAll φ Λ
    by (simp add: filter-remove1 removeAll-filter-not-eq)
  moreover have φ ∈ set Λ and φ ∈ set (φ # Φ)
    using ‹ξ = φ›
          ‹count-list (ξ # Φ) φ ≤ count-list Λ φ›
          gr-implies-not0
    by fastforce+
  hence (φ # Φ) ⊖ Λ ⇌ (remove1 φ (φ # Φ)) ⊖ (remove1 φ Λ)
    by (metis list-subtract-cons-remove1-perm remove-hd)

  hence (φ # Φ) ⊖ Λ ⇌ Φ ⊖ (remove1 φ Λ) by simp
  ultimately show ?thesis using ‹ξ = φ› by auto
next
  assume ξ ≠ φ
  show ?thesis
  proof cases
    assume ξ ∈ set Λ
    hence (ξ # Φ) ⊖ Λ ⇌ Φ ⊖ remove1 ξ Λ
      by (meson list-subtract-cons-remove1-perm)
    moreover have count-list Λ φ = count-list (remove1 ξ Λ) φ
      by (metis
            count-list.simps(2)
            ‹ξ ≠ φ›
            ‹ξ ∈ set Λ›
            perm-count-list
            perm-remove)
    hence count-list Φ φ ≤ count-list (remove1 ξ Λ) φ
      using ‹ξ ≠ φ› ‹count-list (ξ # Φ) φ ≤ count-list Λ φ› by auto
    hence Φ ⊖ remove1 ξ Λ
            ⇌ (removeAll φ Φ) ⊖ (removeAll φ (remove1 ξ Λ))
      using Cons.hyps by blast
    moreover
    have (removeAll φ Φ) ⊖ (removeAll φ (remove1 ξ Λ)) ⇌
          (removeAll φ Φ) ⊖ (remove1 ξ (removeAll φ Λ))
      by (induct Λ,
            simp,
            metis
              ‹ξ ≠ φ›
              list-subtract.simps(2)
              mset-remove1
              remove1.simps(2)
              removeAll.simps(2))
    then have (removeAll φ Φ) ⊖ (removeAll φ (remove1 ξ Λ)) ⇌
          (removeAll φ (ξ # Φ)) ⊖ (removeAll φ Λ)
```

```
      using ‹ξ ∈ set Λ› ‹ξ ≠ φ› by (auto intro!: multiset-eqI)
    ultimately show ?thesis
      by presburger
  next
    assume ξ ∉ set Λ
    hence (ξ # Φ) ⊖ Λ ⇌ ξ # (Φ ⊖ Λ)
      by fastforce
    hence (ξ # Φ) ⊖ Λ ⇌ ξ # ((removeAll φ Φ) ⊖ (removeAll φ Λ))
      using ‹Φ ⊖ Λ ⇌ removeAll φ Φ ⊖ removeAll φ Λ›
      by simp
    hence (ξ # Φ) ⊖ Λ ⇌ (ξ # (removeAll φ Φ)) ⊖ (removeAll φ Λ)
      by (simp add: ‹ξ ∉ set Λ› list-subtract-cons)
    thus ?thesis using ‹ξ ≠ φ› by auto
  qed
 qed
qed

lemma list-subtract-permute:
 assumes Φ ⇌ Ψ
 shows Φ ⊖ Λ ⇌ Ψ ⊖ Λ
 using assms
 by simp

lemma append-perm-list-subtract-intro:
 assumes A ⇌ B @ C
 shows A ⊖ C ⇌ B
proof −
 from ‹A ⇌ B @ C› have mset (A ⊖ C) = mset B
   by simp
 thus ?thesis by blast
qed

lemma list-subtract-concat:
 assumes Ψ ∈ set ℒ
 shows concat (ℒ ⊖ [Ψ]) ⇌ (concat ℒ) ⊖ Ψ
 using assms
 by (simp add: concat-remove1)

lemma (in comm-monoid-add) listSubstract-multisubset-list-summation:
 assumes mset Ψ ⊆# mset Φ
 shows (∑ψ←Ψ. f ψ) + (∑φ'←(Φ ⊖ Ψ). f φ') = (∑φ'←Φ. f φ')
proof −
 have ∀ Φ. mset Ψ ⊆# mset Φ
       ⟶ (∑ψ'←Ψ. f ψ') + (∑φ'←(Φ ⊖ Ψ). f φ') = (∑φ'←Φ. f φ')
 proof(induct Ψ)
   case Nil
   then show ?case
     by simp
 next
```

```
    case (Cons ψ Ψ)
    {
      fix Φ
      assume hypothesis: mset (ψ # Ψ) ⊆# mset Φ
      hence mset Ψ ⊆# mset (remove1 ψ Φ)
        by (metis append-Cons mset-le-perm-append perm-remove-perm remove-hd)
      hence
        (∑ψ' ← Ψ. f ψ') + (∑φ'←((remove1 ψ Φ) ⊖ Ψ). f φ')
              = (∑φ'← (remove1 ψ Φ). f φ')
        using Cons.hyps by blast
      moreover have (remove1 ψ Φ) ⊖ Ψ ⇌ Φ ⊖ (ψ # Ψ)
        by (meson list-subtract-remove1-cons-perm perm-sym)
      hence (∑φ'←((remove1 ψ Φ) ⊖ Ψ). f φ') = (∑φ'←(Φ ⊖ (ψ # Ψ)). f φ')
        using perm-list-summation by blast
      ultimately have
        (∑ψ' ← Ψ. f ψ') + (∑φ'←(Φ ⊖ (ψ # Ψ)). f φ')
              = (∑φ'← (remove1 ψ Φ). f φ')
        by simp
      hence
        (∑ψ' ← (ψ # Ψ). f ψ') + (∑φ'←(Φ ⊖ (ψ # Ψ)). f φ')
              = (∑φ'← (ψ # (remove1 ψ Φ)). f φ')
        by (simp add: add.assoc)
      moreover have ψ ∈ set Φ
        by (metis
              append-Cons
              hypothesis
              list.set-intros(1)
              mset-le-perm-append
              perm-set-eq)
      hence (ψ # (remove1 ψ Φ)) ⇌ Φ
        by auto
      hence (∑φ' ← (ψ # (remove1 ψ Φ)). f φ') = (∑φ'←Φ. f φ')
        using perm-list-summation by blast
      ultimately have
        (∑ψ' ← (ψ # Ψ). f ψ') + (∑φ'←(Φ ⊖ (ψ # Ψ)). f φ')
              = (∑φ'←Φ. f φ')
        by simp
    }
    then show ?case
      by blast
  qed
  with assms show ?thesis by blast
qed

lemma list-subtract-set-difference-lower-bound:
  set Γ − set Φ ⊆ set (Γ ⊖ Φ)
  using subset-Diff-insert
  by (induct Φ, simp, fastforce)
```

```
lemma list-subtract-set-trivial-upper-bound:
  set (Γ ⊖ Φ) ⊆ set Γ
    by (induct Φ,
        simp,
        simp,
        meson
          dual-order.trans
          set-remove1-subset)

lemma list-subtract-msub-eq:
  assumes mset Φ ⊆# mset Γ
    and length (Γ ⊖ Φ) = m
   shows length Γ = m + length Φ
  using assms
proof −
  have ∀ Γ. mset Φ ⊆# mset Γ
        ⟶ length (Γ ⊖ Φ) = m −−> length Γ = m + length Φ
  proof (induct Φ)
    case Nil
    then show ?case by simp
  next
    case (Cons φ Φ)
    {
      fix Γ :: 'a list
      assume mset (φ # Φ) ⊆# mset Γ
             length (Γ ⊖ (φ # Φ)) = m
      moreover from this have
        mset Φ ⊆# mset (remove1 φ Γ)
        mset (Γ ⊖ (φ # Φ)) = mset ((remove1 φ Γ) ⊖ Φ)
        by (metis
              append-Cons
              mset-le-perm-append
              perm-remove-perm
              remove-hd,
            simp)
      ultimately have length (remove1 φ Γ) = m + length Φ
        using Cons.hyps
        by (metis mset-eq-length)
      hence length (φ # (remove1 φ Γ)) = m + length (φ # Φ)
        by simp
      moreover have φ ∈ set Γ
        by (metis
              ‹mset (Γ ⊖ (φ # Φ)) = mset (remove1 φ Γ ⊖ Φ)›
              ‹mset (φ # Φ) ⊆# mset Γ›
              ‹mset Φ ⊆# mset (remove1 φ Γ)›
              add-diff-cancel-left'
              add-right-cancel
              eq-iff
              impossible-Cons
```

list-subtract-mset-homomorphism
mset-subset-eq-exists-conv
remove1-idem size-mset)
hence *length* (φ # (*remove1* φ Γ)) = *length* Γ
by (*metis*
One-nat-def
Suc-pred
length-Cons
length-pos-if-in-set
length-remove1)
ultimately have *length* Γ = m + *length* (φ # Φ) **by** *simp*
}
thus *?case* **by** *blast*
qed
thus *?thesis* **using** *assms* **by** *blast*
qed

lemma *list-subtract-not-member*:
assumes $b \notin$ *set* A
shows $A \ominus B = A \ominus$ (*remove1* b B)
using *assms*
by (*induct* B,
simp,
simp,
metis
add-mset-add-single
diff-subset-eq-self
insert-DiffM2
insert-subset-eq-iff
list-subtract-mset-homomorphism
remove1-idem
set-mset-mset)

lemma *list-subtract-monotonic*:
assumes *mset* $A \subseteq\#$ *mset* B
shows *mset* ($A \ominus C$) $\subseteq\#$ *mset* ($B \ominus C$)
by (*simp*,
meson
assms
subset-eq-diff-conv
subset-mset.dual-order.refl
subset-mset.order-trans)

lemma *map-list-subtract-mset-containment*:
mset ((*map* f A) $\ominus$ (*map* f B)) $\subseteq\#$ *mset* (*map* f ($A \ominus B$))
by (*induct* B, *simp*, *simp*,
metis
diff-subset-eq-self
diff-zero

```
    image-mset-add-mset
    image-mset-subseteq-mono
    image-mset-union
    subset-eq-diff-conv
    subset-eq-diff-conv)

lemma map-list-subtract-mset-equivalence:
  assumes mset B ⊆# mset A
  shows mset ((map f A) ⊖ (map f B)) = mset (map f (A ⊖ B))
  using assms
  by (induct B, simp, simp add: image-mset-Diff)

lemma msub-list-subtract-elem-cons-msub:
  assumes mset Ξ ⊆# mset Γ
      and ψ ∈ set (Γ ⊖ Ξ)
    shows mset (ψ # Ξ) ⊆# mset Γ
proof −
  have ∀ Γ. mset Ξ ⊆# mset Γ
          ⟶ ψ ∈ set (Γ ⊖ Ξ) −−> mset (ψ # Ξ) ⊆# mset Γ
  proof(induct Ξ)
    case Nil
    then show ?case by simp
  next
    case (Cons ξ Ξ)
    {
      fix Γ
      assume
        mset (ξ # Ξ) ⊆# mset Γ
        ψ ∈ set (Γ ⊖ (ξ # Ξ))
      hence
         ξ ∈ set Γ
         mset Ξ ⊆# mset (remove1 ξ Γ)
         ψ ∈ set ((remove1 ξ Γ) ⊖ Ξ)
        by (simp,
            metis
              ex-mset
              list.set-intros(1)
              mset.simps(2)
              mset-eq-setD
              subset-mset.le-iff-add
              union-mset-add-mset-left,
            metis
              list-subtract.simps(1)
              list-subtract.simps(2)
              list-subtract-monotonic
              remove-hd,
            simp,
            metis
              list-subtract-remove1-cons-perm
```

```
        perm-set-eq)
    with Cons.hyps have
      mset Γ = mset (ξ # (remove1 ξ Γ))
      mset (ψ # Ξ) ⊆# mset (remove1 ξ Γ)
      by (simp, blast)
    hence mset (ψ # ξ # Ξ) ⊆# mset Γ
      by (simp,
          metis
            add-mset-commute
            mset-subset-eq-add-mset-cancel)
  }
  then show ?case by auto
 qed
 thus ?thesis using assms by blast
qed
```

4.7 Tuple Lists

```
lemma remove1-pairs-list-projections-fst:
 assumes (γ,σ) ∈# mset Φ
 shows mset (map fst (remove1 (γ, σ) Φ)) = mset (map fst Φ) − {# γ #}
using assms
proof (induct Φ)
 case Nil
 then show ?case by simp
next
 case (Cons φ Φ)
 assume (γ, σ) ∈# mset (φ # Φ)
 show ?case
 proof (cases φ = (γ, σ))
  assume φ = (γ, σ)
  then show ?thesis by simp
 next
  assume φ ≠ (γ, σ)
  then have add-mset φ (mset Φ − {#(γ, σ)#})
        = add-mset φ (mset Φ) − {#(γ, σ)#}
    by force
  then have add-mset (fst φ) (image-mset fst (mset Φ − {#(γ, σ)#}))
          = add-mset (fst φ) (image-mset fst (mset Φ)) − {#γ#}
    by (metis (no-types) Cons.prems
                         add-mset-remove-trivial
                         fst-conv
                         image-mset-add-mset
                         insert-DiffM mset.simps(2))
  with ‹φ ≠ (γ, σ)› show ?thesis
    by simp
 qed
qed
```

lemma *remove1-pairs-list-projections-snd*:
assumes $(\gamma,\sigma) \in\# \ mset\ \Phi$
shows $mset\ (map\ snd\ (remove1\ (\gamma,\ \sigma)\ \Phi)) = mset\ (map\ snd\ \Phi) - \{\#\ \sigma\ \#\}$
using *assms*
proof (*induct* Φ)
case *Nil*
then show *?case* **by** *simp*
next
case (*Cons* $\varphi\ \Phi$)
assume $(\gamma,\ \sigma) \in\# \ mset\ (\varphi\ \#\ \Phi)$
show *?case*
proof (*cases* $\varphi = (\gamma,\ \sigma)$)
assume $\varphi = (\gamma,\ \sigma)$
then show *?thesis* **by** *simp*
next
assume $\varphi \neq (\gamma,\ \sigma)$
then have *add-mset* (*snd* φ) (*image-mset snd* (*mset* $\Phi - \{\#(\gamma,\ \sigma)\#\}$))
$=$ *image-mset snd* (*mset* $(\varphi\ \#\ \Phi) - \{\#(\gamma,\ \sigma)\#\}$)
by *auto*
moreover have *add-mset* (*snd* φ) (*image-mset snd* (*mset* Φ))
$=$ *add-mset* σ (*image-mset snd* (*mset* $(\varphi\ \#\ \Phi) - \{\#(\gamma,\ \sigma)\#\}$))
by (*metis* (*no-types*)
Cons.prems
image-mset-add-mset
insert-DiffM
mset.simps(*2*)
snd-conv)
ultimately
have *add-mset* (*snd* φ) (*image-mset snd* (*mset* $\Phi - \{\#(\gamma,\ \sigma)\#\}$))
$=$ *add-mset* (*snd* φ) (*image-mset snd* (*mset* Φ)) $- \{\#\sigma\#\}$
by *simp*
with ‹$\varphi \neq (\gamma,\ \sigma)$› **show** *?thesis*
by *simp*
qed
qed

lemma *triple-list-exists*:
assumes $mset\ (map\ snd\ \Psi) \subseteq\#\ mset\ \Sigma$
and $mset\ \Sigma \subseteq\#\ mset\ (map\ snd\ \Delta)$
shows $\exists\ \Omega.\ map\ (\lambda\ (\psi,\ \sigma,\ \text{-}).\ (\psi,\ \sigma))\ \Omega = \Psi\ \wedge$
$mset\ (map\ (\lambda\ (\text{-},\ \sigma,\ \gamma).\ (\gamma,\ \sigma))\ \Omega) \subseteq\#\ mset\ \Delta$
using *assms*(*1*)
proof (*induct* Ψ)
case *Nil*
then show *?case* **by** *fastforce*
next
case (*Cons* $\psi\ \Psi$)
from *Cons* **obtain** Ω **where** Ω:
$map\ (\lambda\ (\psi,\ \sigma,\ \text{-}).\ (\psi,\ \sigma))\ \Omega = \Psi$

mset (*map* (λ (-, σ, γ). (γ, σ)) Ω) ⊆# *mset* Δ
by (*metis*
(*no-types*, *lifting*)
diff-subset-eq-self
list.set-intros(*1*)
remove1-pairs-list-projections-snd
remove-hd
set-mset-mset
subset-mset.dual-order.trans
surjective-pairing)
let *?*Δ_Ω = *map* (λ (-, σ, γ). (γ, σ)) Ω
let *?ψ* = *fst* ψ
let *?σ* = *snd* ψ
from *Cons.prems* **have** *add-mset ?σ* (*image-mset snd* (*mset* Ψ)) ⊆# *mset* Σ
by *simp*
then have *mset* Σ − {#*?σ*#} − *image-mset snd* (*mset* Ψ)
≠ *mset* Σ − *image-mset snd* (*mset* Ψ)
by (*metis*
(*no-types*)
insert-subset-eq-iff
mset-subset-eq-insertD
multi-drop-mem-not-eq
subset-mset.diff-add
subset-mset-def)
hence *?σ* ∈# *mset* Σ − *mset* (*map snd* Ψ)
using *diff-single-trivial* **by** *fastforce*
have *mset* (*map snd* (ψ # Ψ)) ⊆# *mset* (*map snd* Δ)
by (*meson*
Cons.prems
‹*mset* Σ ⊆# *mset* (*map snd* Δ)›
subset-mset.dual-order.trans)
then have
mset (*map snd* Δ) − *mset* (*map snd* (ψ # Ψ)) + ({#} + {#*snd* ψ#})
= *mset* (*map snd* Δ) + ({#} + {#*snd* ψ#})
− *add-mset* (*snd* ψ) (*mset* (*map snd* Ψ))
by (*metis*
(*no-types*)
list.simps(*9*)
mset.simps(*2*)
mset-subset-eq-multiset-union-diff-commute)
then have
mset (*map snd* Δ) − *mset* (*map snd* (ψ # Ψ)) + ({#} + {#*snd* ψ#})
= *mset* (*map snd* Δ) − *mset* (*map snd* Ψ)
by *auto*
hence *?σ* ∈# *mset* (*map snd* Δ) − *mset* (*map snd* Ψ)
using *add-mset-remove-trivial-eq* **by** *fastforce*
moreover have *snd* ∘ (λ (ψ, σ, -). (ψ, σ)) = *snd* ∘ (λ (-, σ, γ). (γ, σ))
by *auto*
hence *map snd* (*?*Δ_Ω) = *map snd* (*map* (λ (ψ, σ, -). (ψ, σ)) Ω)

by *fastforce*
hence *map snd* (*?*Δ_Ω) = *map snd* Ψ
using $\Omega(1)$ **by** *simp*
ultimately have *?σ* ∈# *mset* (*map snd* Δ) − *mset* (*map snd ?*Δ_Ω)
by *simp*
hence *?σ* ∈# *image-mset snd* (*mset* Δ − *mset ?*Δ_Ω)
using $\Omega(2)$ **by** (*metis image-mset-Diff mset-map*)
hence *?σ* ∈ *snd* ‘ *set-mset* (*mset* Δ − *mset ?*Δ_Ω)
by (*metis in-image-mset*)
from *this* **obtain** ϱ **where** ϱ:
snd ϱ = *?σ* ϱ ∈# *mset* Δ − *mset ?*Δ_Ω
using *imageE* **by** *auto*
from *this* **obtain** γ **where**
(γ, *?σ*) = ϱ
by (*metis prod.collapse*)
with $\varrho(2)$ **have** γ: (γ, *?σ*) ∈# *mset* Δ − *mset ?*Δ_Ω **by** *auto*
let *?*Ω = (*?ψ*, *?σ*, γ) # Ω
have *map* (λ (ψ, σ, -). (ψ, σ)) *?*Ω = ψ # Ψ
using $\Omega(1)$ **by** *simp*
moreover
have *A*: (γ, *snd* ψ) = (*case* (*snd* ψ, γ) *of* (a, c) ⇒ (c, a))
by *auto*
have *B*: *mset* (*map* (λ(b, a, c). (c, a)) Ω)
+ {# *case* (*snd* ψ, γ) *of* (a, c) ⇒ (c, a) #}
= *mset* (*map* (λ(b, a, c). (c, a)) ((*fst* ψ, *snd* ψ, γ) # Ω))
by *simp*
obtain *mm*
:: (′*c* × ′*a*) *multiset*
⇒ (′*c* × ′*a*) *multiset*
⇒ (′*c* × ′*a*) *multiset*
where ∀ *x0 x1*. (∃ *v2*. *x0* = *x1* + *v2*) = (*x0* = *x1* + *mm x0 x1*)
by *moura*
then have *mset* Δ = *mset* (*map* (λ(b, a, c). (c, a)) Ω)
+ *mm* (*mset* Δ) (*mset* (*map* (λ(b, a, c). (c, a)) Ω))
by (*metis* $\Omega(2)$ *subset-mset.le-iff-add*)
then have *mset* (*map* (λ (-, σ, γ). (γ, σ)) *?*Ω) ⊆# *mset* Δ
using *A B* **by**
(*metis*
γ
add-diff-cancel-left′
single-subset-iff
subset-mset.add-le-cancel-left)
ultimately show *?case* **by** *meson*
qed

4.8 List Intersection

primrec *list-intersect* :: ′*a list* => ′*a list* => ′*a list* (**infixl** ‹∩› *60*)
where

```
  - ∩ [] = []
| xs ∩ (y # ys) =
      (if (y ∈ set xs)
       then (y # (remove1 y xs ∩ ys))
       else (xs ∩ ys))

lemma list-intersect-mset-homomorphism [simp]:
  mset (Φ ∩ Ψ) = mset Φ ∩# mset Ψ
proof −
  have ∀ Φ. mset (Φ ∩ Ψ) = mset Φ ∩# mset Ψ
  proof (induct Ψ)
    case Nil
    then show ?case by simp
  next
    case (Cons ψ Ψ)
    {
      fix Φ
      have mset (Φ ∩ ψ # Ψ) = mset Φ ∩# mset (ψ # Ψ)
        using Cons.hyps
        by (cases ψ ∈ set Φ,
            simp add: inter-add-right2,
            simp add: inter-add-right1)
    }
    then show ?case by blast
  qed
  thus ?thesis by simp
qed

lemma list-intersect-left-empty [simp]: [] ∩ Φ = [] by (induct Φ, simp+)

lemma list-diff-intersect-comp:
  mset Φ = mset (Φ ⊖ Ψ) + mset (Φ ∩ Ψ)
  by (metis
      diff-intersect-left-idem
      list-intersect-mset-homomorphism
      list-subtract-mset-homomorphism
      subset-mset.inf-le1
      subset-mset.le-imp-diff-is-add)

lemma list-intersect-left-project: mset (Φ ∩ Ψ) ⊆# mset Φ
  by simp

lemma list-intersect-right-project: mset (Φ ∩ Ψ) ⊆# mset Ψ
  by simp

end
```

Chapter 5

Classical Logic Connectives

theory *Classical-Connectives*
  imports
    Classical-Logic-Completeness
    List-Utilities
begin

Here we define the usual connectives for classical logic.

unbundle *no funcset-syntax*

5.1 Verum

definition (**in** *classical-logic*) *verum* :: *'a* (‹⊤›)
  where
    $\top = \bot \rightarrow \bot$

lemma (**in** *classical-logic*) *verum-tautology* [*simp*]: $\vdash \top$
  by (*metis list-implication.simps*(*1*) *list-implication-axiom-k verum-def*)

lemma *verum-semantics* [*simp*]:
  $\mathfrak{M} \models_{prop} \top$
  unfolding *verum-def* **by** *simp*

lemma (**in** *classical-logic*) *verum-embedding* [*simp*]:
  $(\!| \top |\!) = \top$
  by (*simp add*: *classical-logic-class.verum-def verum-def*)

5.2 Conjunction

definition (**in** *classical-logic*)
  conjunction :: *'a* ⇒ *'a* ⇒ *'a* (**infixr** ‹⊓› *67*)
  where
    $\varphi \sqcap \psi = (\varphi \rightarrow \psi \rightarrow \bot) \rightarrow \bot$

primrec (**in** *classical-logic*)
arbitrary-conjunction :: *'a list* ⇒ *'a* (‹⨅›)
where
⨅ [] = ⊤
| ⨅ (φ # Φ) = φ ⊓ ⨅ Φ

lemma (**in** *classical-logic*) *conjunction-introduction*:
⊢ φ → ψ → (φ ⊓ ψ)
by (*metis*
modus-ponens
conjunction-def
list-flip-implication1
list-implication.simps(*1*)
list-implication.simps(*2*))

lemma (**in** *classical-logic*) *conjunction-left-elimination*:
⊢ (φ ⊓ ψ) → φ
by (*metis* (*full-types*)
Peirces-law
pseudo-scotus
conjunction-def
list-deduction-base-theory
list-deduction-modus-ponens
list-deduction-theorem
list-deduction-weaken)

lemma (**in** *classical-logic*) *conjunction-right-elimination*:
⊢ (φ ⊓ ψ) → ψ
by (*metis* (*full-types*)
axiom-k
Contraposition
modus-ponens
conjunction-def
flip-hypothetical-syllogism
flip-implication)

lemma (**in** *classical-logic*) *conjunction-embedding* [*simp*]:
⦇ φ ⊓ ψ ⦈ = ⦇ φ ⦈ ⊓ ⦇ ψ ⦈
unfolding *conjunction-def classical-logic-class.conjunction-def*
by *simp*

lemma *conjunction-semantics* [*simp*]:
𝔐 ⊨*prop* φ ⊓ ψ = (𝔐 ⊨*prop* φ ∧ 𝔐 ⊨*prop* ψ)
unfolding *conjunction-def* **by** *simp*

5.3 Biconditional

definition (**in** *classical-logic*) *biconditional* :: *'a* ⇒ *'a* ⇒ *'a* (**infixr** ‹↔› *75*)
where

$\varphi \leftrightarrow \psi = (\varphi \rightarrow \psi) \sqcap (\psi \rightarrow \varphi)$

lemma (**in** *classical-logic*) *biconditional-introduction*:
$\vdash (\varphi \rightarrow \psi) \rightarrow (\psi \rightarrow \varphi) \rightarrow (\varphi \leftrightarrow \psi)$
by (*simp add*: *biconditional-def conjunction-introduction*)

lemma (**in** *classical-logic*) *biconditional-left-elimination*:
$\vdash (\varphi \leftrightarrow \psi) \rightarrow \varphi \rightarrow \psi$
by (*simp add*: *biconditional-def conjunction-left-elimination*)

lemma (**in** *classical-logic*) *biconditional-right-elimination*:
$\vdash (\varphi \leftrightarrow \psi) \rightarrow \psi \rightarrow \varphi$
by (*simp add*: *biconditional-def conjunction-right-elimination*)

lemma (**in** *classical-logic*) *biconditional-embedding* [*simp*]:
$(\!| \varphi \leftrightarrow \psi |\!) = (\!| \varphi |\!) \leftrightarrow (\!| \psi |\!)$
unfolding *biconditional-def classical-logic-class.biconditional-def*
by *simp*

lemma *biconditional-semantics* [*simp*]:
$\mathfrak{M} \models_{prop} \varphi \leftrightarrow \psi = (\mathfrak{M} \models_{prop} \varphi \longleftrightarrow \mathfrak{M} \models_{prop} \psi)$
unfolding *biconditional-def*
by (*simp*, *blast*)

5.4 Negation

definition (**in** *classical-logic*) *negation* :: $'a \Rightarrow 'a$ $(\langle\sim\rangle)$
where
$\sim \varphi = \varphi \rightarrow \bot$

lemma (**in** *classical-logic*) *negation-introduction*:
$\vdash (\varphi \rightarrow \bot) \rightarrow \sim \varphi$
unfolding *negation-def*
by (*metis axiom-k modus-ponens implication-absorption*)

lemma (**in** *classical-logic*) *negation-elimination*:
$\vdash \sim \varphi \rightarrow (\varphi \rightarrow \bot)$
unfolding *negation-def*
by (*metis axiom-k modus-ponens implication-absorption*)

lemma (**in** *classical-logic*) *negation-embedding* [*simp*]:
$(\!| \sim \varphi |\!) = \sim (\!| \varphi |\!)$
by (*simp add*:
classical-logic-class.negation-def
negation-def)

lemma *negation-semantics* [*simp*]:
$\mathfrak{M} \models_{prop} \sim \varphi = (\neg\ \mathfrak{M} \models_{prop} \varphi)$
unfolding *negation-def*

by *simp*

5.5 Disjunction

definition (**in** *classical-logic*) *disjunction* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixr** ‹⊔› *67*)
where
$\varphi \sqcup \psi = (\varphi \to \bot) \to \psi$

primrec (**in** *classical-logic*) *arbitrary-disjunction* :: $'a\ list \Rightarrow 'a$ (‹$\bigsqcup$ ›)
where
$\bigsqcup\ [] = \bot$
| $\bigsqcup\ (\varphi\ \#\ \Phi) = \varphi \sqcup \bigsqcup\ \Phi$

lemma (**in** *classical-logic*) *disjunction-elimination*:
$\vdash (\varphi \to \chi) \to (\psi \to \chi) \to (\varphi \sqcup \psi) \to \chi$
proof −
let *?*$\Gamma = [\varphi \to \chi, \psi \to \chi, \varphi \sqcup \psi]$
have *?*$\Gamma :\vdash (\varphi \to \bot) \to \chi$
unfolding *disjunction-def*
by (*metis hypothetical-syllogism*
list-deduction-def
list-implication.simps(*1*)
list-implication.simps(*2*)
set-deduction-base-theory
set-deduction-theorem
set-deduction-weaken)
hence *?*$\Gamma :\vdash \chi$
using *excluded-middle-elimination*
list-deduction-modus-ponens
list-deduction-theorem
list-deduction-weaken
by *blast*
thus *?thesis*
unfolding *list-deduction-def*
by *simp*
qed

lemma (**in** *classical-logic*) *disjunction-left-introduction*:
$\vdash \varphi \to (\varphi \sqcup \psi)$
unfolding *disjunction-def*
by (*metis modus-ponens*
pseudo-scotus
flip-implication)

lemma (**in** *classical-logic*) *disjunction-right-introduction*:
$\vdash \psi \to (\varphi \sqcup \psi)$
unfolding *disjunction-def*
using *axiom-k*
by *simp*

lemma (**in** *classical-logic*) *disjunction-embedding* [*simp*]:
⦇ $\varphi \sqcup \psi$ ⦈ = ⦇ φ ⦈ $\sqcup$ ⦇ ψ ⦈
unfolding *disjunction-def classical-logic-class.disjunction-def*
by *simp*

lemma *disjunction-semantics* [*simp*]:
$\mathfrak{M} \models_{prop} \varphi \sqcup \psi = (\mathfrak{M} \models_{prop} \varphi \vee \mathfrak{M} \models_{prop} \psi)$
unfolding *disjunction-def*
by (*simp*, *blast*)

5.6 Mutual Exclusion

primrec (**in** *classical-logic*) *exclusive* :: *'a list* ⇒ *'a* (‹$\coprod$›)
where
$\coprod$ [] = $\top$
| $\coprod$ (φ # Φ) = $\sim$ ($\varphi \sqcap \bigsqcup \Phi$) $\sqcap \coprod \Phi$

5.7 Subtraction

definition (**in** *classical-logic*) *subtraction* :: *'a* ⇒ *'a* ⇒ *'a* (**infixl** ‹\› *69*)
where $\varphi \setminus \psi = \varphi \sqcap \sim \psi$

lemma (**in** *classical-logic*) *subtraction-embedding* [*simp*]:
⦇ $\varphi \setminus \psi$ ⦈ = ⦇ φ ⦈ \ ⦇ ψ ⦈
unfolding *subtraction-def classical-logic-class.subtraction-def*
by *simp*

5.8 Negated Lists

definition (**in** *classical-logic*) *map-negation* :: *'a list* ⇒ *'a list* (‹$\sim$›)
where [*simp*]: $\sim \Phi \equiv$ *map* $\sim \Phi$

5.9 Common (& Uncommon) Identities

5.9.1 Biconditional Equivalence Relation

lemma (**in** *classical-logic*) *biconditional-reflection*:
$\vdash \varphi \leftrightarrow \varphi$
by (*meson*
axiom-k
modus-ponens
biconditional-introduction
implication-absorption)

lemma (**in** *classical-logic*) *biconditional-symmetry*:
$\vdash (\varphi \leftrightarrow \psi) \leftrightarrow (\psi \leftrightarrow \varphi)$

by (*metis* (*full-types*) *modus-ponens*
biconditional-def
conjunction-def
flip-hypothetical-syllogism
flip-implication)

lemma (**in** *classical-logic*) *biconditional-symmetry-rule*:
$\vdash \varphi \leftrightarrow \psi \Longrightarrow \vdash \psi \leftrightarrow \varphi$
by (*meson modus-ponens*
biconditional-introduction
biconditional-left-elimination
biconditional-right-elimination)

lemma (**in** *classical-logic*) *biconditional-transitivity*:
$\vdash (\varphi \leftrightarrow \psi) \rightarrow (\psi \leftrightarrow \chi) \rightarrow (\varphi \leftrightarrow \chi)$
proof −
have $\forall\ \mathfrak{M}.\ \mathfrak{M} \models_{prop} (\langle\varphi\rangle \leftrightarrow \langle\psi\rangle) \rightarrow (\langle\psi\rangle \leftrightarrow \langle\chi\rangle) \rightarrow (\langle\varphi\rangle \leftrightarrow \langle\chi\rangle)$
by *simp*
hence $\vdash$ ⦇ $(\langle\varphi\rangle \leftrightarrow \langle\psi\rangle) \rightarrow (\langle\psi\rangle \leftrightarrow \langle\chi\rangle) \rightarrow (\langle\varphi\rangle \leftrightarrow \langle\chi\rangle)$ ⦈
using *propositional-semantics* **by** *blast*
thus *?thesis* **by** *simp*
qed

lemma (**in** *classical-logic*) *biconditional-transitivity-rule*:
$\vdash \varphi \leftrightarrow \psi \Longrightarrow \vdash \psi \leftrightarrow \chi \Longrightarrow \vdash \varphi \leftrightarrow \chi$
using *modus-ponens biconditional-transitivity* **by** *blast*

5.9.2 Biconditional Weakening

lemma (**in** *classical-logic*) *biconditional-weaken*:
assumes $\Gamma \Vdash \varphi \leftrightarrow \psi$
shows $\Gamma \Vdash \varphi = \Gamma \Vdash \psi$
by (*metis assms*
biconditional-left-elimination
biconditional-right-elimination
set-deduction-modus-ponens
set-deduction-weaken)

lemma (**in** *classical-logic*) *list-biconditional-weaken*:
assumes $\Gamma :\vdash \varphi \leftrightarrow \psi$
shows $\Gamma :\vdash \varphi = \Gamma :\vdash \psi$
by (*metis assms*
biconditional-left-elimination
biconditional-right-elimination
list-deduction-modus-ponens
list-deduction-weaken)

lemma (**in** *classical-logic*) *weak-biconditional-weaken*:
assumes $\vdash \varphi \leftrightarrow \psi$

shows $\vdash \varphi = \vdash \psi$
by (*metis assms*
biconditional-left-elimination
biconditional-right-elimination
modus-ponens)

5.9.3 Conjunction Identities

lemma (**in** *classical-logic*) *conjunction-negation-identity*:
$\vdash \sim (\varphi \sqcap \psi) \leftrightarrow (\varphi \rightarrow \psi \rightarrow \bot)$
by (*metis Contraposition*
double-negation-converse
modus-ponens
biconditional-introduction
conjunction-def
negation-def)

lemma (**in** *classical-logic*) *conjunction-set-deduction-equivalence* [*simp*]:
$\Gamma \Vdash \varphi \sqcap \psi = (\Gamma \Vdash \varphi \wedge \Gamma \Vdash \psi)$
by (*metis set-deduction-weaken* [**where** Γ=Γ]
set-deduction-modus-ponens [**where** Γ=Γ]
conjunction-introduction
conjunction-left-elimination
conjunction-right-elimination)

lemma (**in** *classical-logic*) *conjunction-list-deduction-equivalence* [*simp*]:
$\Gamma :\vdash \varphi \sqcap \psi = (\Gamma :\vdash \varphi \wedge \Gamma :\vdash \psi)$
by (*metis list-deduction-weaken* [**where** Γ=Γ]
list-deduction-modus-ponens [**where** Γ=Γ]
conjunction-introduction
conjunction-left-elimination
conjunction-right-elimination)

lemma (**in** *classical-logic*) *weak-conjunction-deduction-equivalence* [*simp*]:
$\vdash \varphi \sqcap \psi = (\vdash \varphi \wedge \vdash \psi)$
by (*metis conjunction-set-deduction-equivalence set-deduction-base-theory*)

lemma (**in** *classical-logic*) *conjunction-set-deduction-arbitrary-equivalence* [*simp*]:
$\Gamma \Vdash \bigsqcap \Phi = (\forall\ \varphi \in \mathit{set}\ \Phi.\ \Gamma \Vdash \varphi)$
by (*induct* Φ, *simp add*: *set-deduction-weaken*, *simp*)

lemma (**in** *classical-logic*) *conjunction-list-deduction-arbitrary-equivalence* [*simp*]:
$\Gamma :\vdash \bigsqcap \Phi = (\forall\ \varphi \in \mathit{set}\ \Phi.\ \Gamma :\vdash \varphi)$
by (*induct* Φ, *simp add*: *list-deduction-weaken*, *simp*)

lemma (**in** *classical-logic*) *weak-conjunction-deduction-arbitrary-equivalence* [*simp*]:
$\vdash \bigsqcap \Phi = (\forall\ \varphi \in \mathit{set}\ \Phi.\ \vdash \varphi)$
by (*induct* Φ, *simp*+)

lemma (**in** *classical-logic*) *conjunction-commutativity*:
 $\vdash (\psi \sqcap \varphi) \leftrightarrow (\varphi \sqcap \psi)$
 by (*metis*
  (*full-types*)
  modus-ponens
  biconditional-introduction
  conjunction-def
  flip-hypothetical-syllogism
  flip-implication)

lemma (**in** *classical-logic*) *conjunction-associativity*:
 $\vdash ((\varphi \sqcap \psi) \sqcap \chi) \leftrightarrow (\varphi \sqcap (\psi \sqcap \chi))$
proof −
 have $\forall\ \mathfrak{M}.\ \mathfrak{M} \models_{prop} (((\langle\varphi\rangle \sqcap \langle\psi\rangle) \sqcap \langle\chi\rangle) \leftrightarrow (\langle\varphi\rangle \sqcap (\langle\psi\rangle \sqcap \langle\chi\rangle)))$
  by *simp*
 hence $\vdash$ ⦇ $(((\langle\varphi\rangle \sqcap \langle\psi\rangle) \sqcap \langle\chi\rangle) \leftrightarrow (\langle\varphi\rangle \sqcap (\langle\psi\rangle \sqcap \langle\chi\rangle)))$ ⦈
  using *propositional-semantics* **by** *blast*
 thus *?thesis* **by** *simp*
qed

lemma (**in** *classical-logic*) *arbitrary-conjunction-antitone*:
 $set\ \Phi \subseteq set\ \Psi \Longrightarrow \vdash \bigsqcap \Psi \rightarrow \bigsqcap \Phi$
proof −
 have $\forall\ \Phi.\ set\ \Phi \subseteq set\ \Psi \longrightarrow \vdash \bigsqcap \Psi \rightarrow \bigsqcap \Phi$
 proof (*induct* Ψ)
  case *Nil*
  then show *?case*
   by (*simp add*: *pseudo-scotus verum-def*)
 next
  case (*Cons* ψ Ψ)
  {
   fix Φ
   assume $set\ \Phi \subseteq set\ (\psi\ \#\ \Psi)$
   have $\vdash \bigsqcap (\psi\ \#\ \Psi) \rightarrow \bigsqcap \Phi$
   proof (*cases* $\psi \in set\ \Phi$)
    assume $\psi \in set\ \Phi$
     have $\forall\ \varphi \in set\ \Phi.\ \vdash \bigsqcap \Phi \leftrightarrow (\varphi \sqcap \bigsqcap (removeAll\ \varphi\ \Phi))$
    proof (*induct* Φ)
     case *Nil*
     then show *?case* **by** *simp*
    next
     case (*Cons* χ Φ)
     {
      fix φ
      assume $\varphi \in set\ (\chi\ \#\ \Phi)$
      have $\vdash \bigsqcap (\chi\ \#\ \Phi) \leftrightarrow (\varphi \sqcap \bigsqcap (removeAll\ \varphi\ (\chi\ \#\ \Phi)))$
      proof *cases*
       assume $\varphi \in set\ \Phi$
       hence $\vdash \bigsqcap \Phi \leftrightarrow (\varphi \sqcap \bigsqcap (removeAll\ \varphi\ \Phi))$

```
  using Cons.hyps ‹φ ∈ set Φ›
  by auto
moreover
have ⊢ (⨅ Φ ↔ (φ ⊓ ⨅ (removeAll φ Φ))) →
       (χ ⊓ ⨅ Φ) ↔ (φ ⊓ χ ⊓ ⨅ (removeAll φ Φ))
proof −
  have ∀ 𝔐. 𝔐 ⊨_prop
         (⟨⨅ Φ⟩ ↔ (⟨φ⟩ ⊓ ⟨⨅ (removeAll φ Φ)⟩))
           → (⟨χ⟩ ⊓ ⟨⨅ Φ⟩)
                ↔ (⟨φ⟩ ⊓ ⟨χ⟩ ⊓ ⟨⨅ (removeAll φ Φ)⟩)
    by auto
  hence ⊢ ⦇ (⟨⨅ Φ⟩ ↔ (⟨φ⟩ ⊓ ⟨⨅ (removeAll φ Φ)⟩))
              → (⟨χ⟩ ⊓ ⟨⨅ Φ⟩)
                   ↔ (⟨φ⟩ ⊓ ⟨χ⟩ ⊓ ⟨⨅ (removeAll φ Φ)⟩) ⦈
    using propositional-semantics by blast
  thus ?thesis by simp
qed
ultimately have ⊢ ⨅ (χ # Φ) ↔ (φ ⊓ χ ⊓ ⨅ (removeAll φ Φ))
  using modus-ponens by auto
show ?thesis
proof cases
  assume φ = χ
  moreover
  {
    fix φ
    have ⊢ (χ ⊓ φ) → (χ ⊓ χ ⊓ φ)
      unfolding conjunction-def
      by (meson
          axiom-s
          double-negation
          modus-ponens
          flip-hypothetical-syllogism
          flip-implication)
  } note tautology = this
  from ‹⊢ ⨅ (χ # Φ) ↔ (φ ⊓ χ ⊓ ⨅ (removeAll φ Φ))›
       ‹φ = χ›
  have ⊢ (χ ⊓ ⨅ (removeAll χ Φ)) → (χ ⊓ ⨅ Φ)
    unfolding biconditional-def
    by (simp, metis tautology hypothetical-syllogism modus-ponens)
  moreover
  from ‹⊢ ⨅ (χ # Φ) ↔ (φ ⊓ χ ⊓ ⨅ (removeAll φ Φ))›
       ‹φ = χ›
  have ⊢ (χ ⊓ ⨅ Φ) → (χ ⊓ ⨅ (removeAll χ Φ))
    unfolding biconditional-def
    by (simp,
        metis conjunction-right-elimination
              hypothetical-syllogism
              modus-ponens)
  ultimately show ?thesis
```

```
        unfolding biconditional-def
        by simp
      next
        assume φ ≠ χ
        then show ?thesis
          using ‹⊢ ⨅ (χ # Φ) ↔ (φ ⊓ χ ⊓ ⨅ (removeAll φ Φ))›
          by simp
      qed
    next
      assume φ ∉ set Φ
      hence φ = χ χ ∉ set Φ
        using ‹φ ∈ set (χ # Φ)› by auto
      then show ?thesis
        using biconditional-reflection
        by simp
    qed
  }
  thus ?case by blast
qed
hence ⊢ (ψ ⊓ ⨅ (removeAll ψ Φ)) → ⨅ Φ
  using modus-ponens biconditional-right-elimination ‹ψ ∈ set Φ›
  by blast
moreover
from ‹ψ ∈ set Φ› ‹set Φ ⊆ set (ψ # Ψ)› Cons.hyps
have ⊢ ⨅ Ψ → ⨅ (removeAll ψ Φ)
  by (simp add: subset-insert-iff insert-absorb)
hence ⊢ (ψ ⊓ ⨅ Ψ) → (ψ ⊓ ⨅ (removeAll ψ Φ))
  unfolding conjunction-def
  using
    modus-ponens
    hypothetical-syllogism
    flip-hypothetical-syllogism
  by meson
ultimately have ⊢ (ψ ⊓ ⨅ Ψ) → ⨅ Φ
  using modus-ponens hypothetical-syllogism
  by blast
thus ?thesis
  by simp
next
assume ψ ∉ set Φ
hence ⊢ ⨅ Ψ → ⨅ Φ
  using Cons.hyps ‹set Φ ⊆ set (ψ # Ψ)›
  by auto
hence ⊢ (ψ ⊓ ⨅ Ψ) → ⨅ Φ
  unfolding conjunction-def
  by (metis
      modus-ponens
      conjunction-def
      conjunction-right-elimination
```

```
            hypothetical-syllogism)
      thus ?thesis
        by simp
    qed
  }
  thus ?case by blast
 qed
 thus set Φ ⊆ set Ψ ⟹ ⊢ ⨅ Ψ → ⨅ Φ by blast
qed

lemma (in classical-logic) arbitrary-conjunction-remdups:
 ⊢ (⨅ Φ) ↔ ⨅ (remdups Φ)
 by (simp add: arbitrary-conjunction-antitone biconditional-def)

lemma (in classical-logic) curry-uncurry:
 ⊢ (φ → ψ → χ) ↔ ((φ ⊓ ψ) → χ)
proof −
 have ∀ 𝔐. 𝔐 ⊨_prop (⟨φ⟩ → ⟨ψ⟩ → ⟨χ⟩) ↔ ((⟨φ⟩ ⊓ ⟨ψ⟩) → ⟨χ⟩)
   by auto
 hence ⊢ ⦇ (⟨φ⟩ → ⟨ψ⟩ → ⟨χ⟩) ↔ ((⟨φ⟩ ⊓ ⟨ψ⟩) → ⟨χ⟩) ⦈
  using propositional-semantics by blast
 thus ?thesis by simp
qed

lemma (in classical-logic) list-curry-uncurry:
 ⊢ (Φ :→ χ) ↔ (⨅ Φ → χ)
proof (induct Φ)
 case Nil
 have ⊢ χ ↔ (⊤ → χ)
  unfolding biconditional-def
            conjunction-def
            verum-def
  using
    axiom-k
    ex-falso-quodlibet
    modus-ponens
    conjunction-def
    excluded-middle-elimination
    set-deduction-base-theory
    conjunction-set-deduction-equivalence
  by metis
 with Nil show ?case
  by simp
next
 case (Cons φ Φ)
 have ⊢ ((φ # Φ) :→ χ) ↔ (φ → (Φ :→ χ))
  by (simp add: biconditional-reflection)
 with Cons have ⊢ ((φ # Φ) :→ χ) ↔ (φ → ⨅ Φ → χ)
  by (metis modus-ponens
```

biconditional-def
hypothetical-syllogism
list-implication.simps(*2*)
weak-conjunction-deduction-equivalence)
with *curry-uncurry* [**where** *?φ*=*φ* **and** *?ψ*=$\bigsqcap$ Φ **and** *?χ*=*χ*]
show *?case*
unfolding *biconditional-def*
by (*simp*, *metis modus-ponens hypothetical-syllogism*)
qed

5.9.4 Disjunction Identities

lemma (**in** *classical-logic*) *bivalence*:
$\vdash \sim \varphi \sqcup \varphi$
by (*simp add*: *double-negation disjunction-def negation-def*)

lemma (**in** *classical-logic*) *implication-equivalence*:
$\vdash (\sim \varphi \sqcup \psi) \leftrightarrow (\varphi \rightarrow \psi)$
by (*metis double-negation-converse*
modus-ponens
biconditional-introduction
bivalence
disjunction-def
flip-hypothetical-syllogism
negation-def)

lemma (**in** *classical-logic*) *disjunction-commutativity*:
$\vdash (\psi \sqcup \varphi) \leftrightarrow (\varphi \sqcup \psi)$
by (*meson modus-ponens*
biconditional-introduction
disjunction-elimination
disjunction-left-introduction
disjunction-right-introduction)

lemma (**in** *classical-logic*) *disjunction-associativity*:
$\vdash ((\varphi \sqcup \psi) \sqcup \chi) \leftrightarrow (\varphi \sqcup (\psi \sqcup \chi))$
proof −
have $\forall\ \mathfrak{M}.\ \mathfrak{M} \models_{prop} ((\langle\varphi\rangle \sqcup \langle\psi\rangle) \sqcup \langle\chi\rangle) \leftrightarrow (\langle\varphi\rangle \sqcup (\langle\psi\rangle \sqcup \langle\chi\rangle))$
by *simp*
hence $\vdash$ ⦇ $((\langle\varphi\rangle \sqcup \langle\psi\rangle) \sqcup \langle\chi\rangle) \leftrightarrow (\langle\varphi\rangle \sqcup (\langle\psi\rangle \sqcup \langle\chi\rangle))$ ⦈
using *propositional-semantics* **by** *blast*
thus *?thesis* **by** *simp*
qed

lemma (**in** *classical-logic*) *arbitrary-disjunction-monotone*:
set Φ ⊆ *set* Ψ ⟹ $\vdash \bigsqcup \Phi \rightarrow \bigsqcup \Psi$
proof −
have $\forall\ \Phi.$ *set* Φ ⊆ *set* Ψ ⟶ $\vdash \bigsqcup \Phi \rightarrow \bigsqcup \Psi$
proof (*induct* Ψ)

```
  case Nil
  then show ?case using verum-def verum-tautology by auto
next
  case (Cons ψ Ψ)
  {
    fix Φ
    assume set Φ ⊆ set (ψ # Ψ)
    have ⊢ ⨆ Φ → ⨆ (ψ # Ψ)
    proof cases
      assume ψ ∈ set Φ
      have ∀ φ ∈ set Φ. ⊢ ⨆ Φ ↔ (φ ⊔ ⨆ (removeAll φ Φ))
      proof (induct Φ)
        case Nil
        then show ?case by simp
      next
        case (Cons χ Φ)
        {
          fix φ
          assume φ ∈ set (χ # Φ)
          have ⊢ ⨆ (χ # Φ) ↔ (φ ⊔ ⨆ (removeAll φ (χ # Φ)))
          proof cases
            assume φ ∈ set Φ
            hence ⊢ ⨆ Φ ↔ (φ ⊔ ⨆ (removeAll φ Φ))
              using Cons.hyps ‹φ ∈ set Φ›
              by auto
            moreover
            have ⊢ (⨆ Φ ↔ (φ ⊔ ⨆ (removeAll φ Φ))) →
                  (χ ⊔ ⨆ Φ) ↔ (φ ⊔ χ ⊔ ⨆ (removeAll φ Φ))
            proof −
              have ∀ 𝔐. 𝔐 ⊨_prop
                    (⟨⨆ Φ⟩ ↔ (⟨φ⟩ ⊔ ⟨⨆ (removeAll φ Φ)⟩))
                     → (⟨χ⟩ ⊔ ⟨⨆ Φ⟩)
                          ↔ (⟨φ⟩ ⊔ ⟨χ⟩ ⊔ ⟨⨆ (removeAll φ Φ)⟩)
                by auto
              hence ⊢ ⦇ (⟨⨆ Φ⟩ ↔ (⟨φ⟩ ⊔ ⟨⨆ (removeAll φ Φ)⟩))
                        → (⟨χ⟩ ⊔ ⟨⨆ Φ⟩)
                           ↔ (⟨φ⟩ ⊔ ⟨χ⟩ ⊔ ⟨⨆ (removeAll φ Φ)⟩) ⦈
                using propositional-semantics by blast
              thus ?thesis by simp
            qed
            ultimately have ⊢ ⨆ (χ # Φ) ↔ (φ ⊔ χ ⊔ ⨆ (removeAll φ Φ))
              using modus-ponens by auto
            show ?thesis
            proof cases
              assume φ = χ
              then show ?thesis
                using ‹⊢ ⨆ (χ # Φ) ↔ (φ ⊔ χ ⊔ ⨆ (removeAll φ Φ))›
                unfolding biconditional-def
                by (simp add: disjunction-def,
```

```
            meson
              axiom-k
              modus-ponens
              flip-hypothetical-syllogism
              implication-absorption)
      next
        assume φ ≠ χ
        then show ?thesis
          using ‹⊢ ⨆ (χ # Φ) ↔ (φ ⊔ χ ⊔ ⨆ (removeAll φ Φ))›
          by simp
      qed
    next
      assume φ ∉ set Φ
      hence φ = χ χ ∉ set Φ
        using ‹φ ∈ set (χ # Φ)› by auto
      then show ?thesis
        using biconditional-reflection
        by simp
    qed
  }
  thus ?case by blast
qed
hence ⊢ ⨆ Φ → (ψ ⊔ ⨆ (removeAll ψ Φ))
  using modus-ponens biconditional-left-elimination ‹ψ ∈ set Φ›
  by blast
moreover
from ‹ψ ∈ set Φ› ‹set Φ ⊆ set (ψ # Ψ)› Cons.hyps
have ⊢ ⨆ (removeAll ψ Φ) → ⨆ Ψ
  by (simp add: subset-insert-iff insert-absorb)
hence ⊢ (ψ ⊔ ⨆ (removeAll ψ Φ)) → ⨆ (ψ # Ψ)
  using
    modus-ponens
    disjunction-def
    hypothetical-syllogism
  by fastforce
ultimately show ?thesis
  by (simp, metis modus-ponens hypothetical-syllogism)
next
assume ψ ∉ set Φ
hence ⊢ ⨆ Φ → ⨆ Ψ
  using Cons.hyps ‹set Φ ⊆ set (ψ # Ψ)›
  by auto
then show ?thesis
  by (metis
      arbitrary-disjunction.simps(2)
      disjunction-def
      list-deduction-def
      list-deduction-theorem
      list-deduction-weaken
```

```
            list-implication.simps(1)
            list-implication.simps(2))
    qed
   }
   then show ?case by blast
  qed
  thus set Φ ⊆ set Ψ ⟹ ⊢ ⨆ Φ → ⨆ Ψ by blast
qed

lemma (in classical-logic) arbitrary-disjunction-remdups:
  ⊢ (⨆ Φ) ↔ ⨆ (remdups Φ)
  by (simp add: arbitrary-disjunction-monotone biconditional-def)

lemma (in classical-logic) arbitrary-disjunction-exclusion-MCS:
  assumes MCS Ω
  shows ⨆ Ψ ∉ Ω ≡ ∀ ψ ∈ set Ψ. ψ ∉ Ω
proof (induct Ψ)
  case Nil
  then show ?case
    using
      assms
      formula-consistent-def
      formula-maximally-consistent-set-def-def
      maximally-consistent-set-def
      set-deduction-reflection
    by (simp, blast)
next
  case (Cons ψ Ψ)
  have ⨆ (ψ # Ψ) ∉ Ω = (ψ ∉ Ω ∧ ⨆ Ψ ∉ Ω)
    by (simp add: disjunction-def,
        meson
          assms
          formula-consistent-def
          formula-maximally-consistent-set-def-def
          formula-maximally-consistent-set-def-implication
          maximally-consistent-set-def
          set-deduction-reflection)
  thus ?case using Cons.hyps by simp
qed

lemma (in classical-logic) contra-list-curry-uncurry:
  ⊢ (Φ :→ χ) ↔ (∼ χ → ⨆ (∼ Φ))
proof (induct Φ)
  case Nil
  then show ?case
    by (simp,
          metis
            biconditional-introduction
            bivalence
```

disjunction-def
double-negation-converse
modus-ponens
negation-def)
next
case (*Cons* φ Φ)
hence $\vdash (\bigsqcap\ \Phi \to \chi) \leftrightarrow (\sim \chi \to \bigsqcup\ (\sim \Phi))$
by (*metis*
biconditional-symmetry-rule
biconditional-transitivity-rule
list-curry-uncurry)
have $\vdash (\bigsqcap\ (\varphi \#\ \Phi) \to \chi) \leftrightarrow (\sim \chi \to \bigsqcup\ (\sim (\varphi \#\ \Phi)))$
proof −
have $\vdash (\bigsqcap\ \Phi \to \chi) \leftrightarrow (\sim \chi \to \bigsqcup\ (\sim \Phi))$
$\to ((\varphi \sqcap \bigsqcap\ \Phi) \to \chi) \leftrightarrow (\sim \chi \to (\sim \varphi \sqcup \bigsqcup\ (\sim \Phi)))$
proof −
have
$\forall\ \mathfrak{M}.\ \mathfrak{M} \models_{prop}$
$(\langle \bigsqcap\ \Phi \rangle \to \langle \chi \rangle) \leftrightarrow (\sim \langle \chi \rangle \to \langle \bigsqcup\ (\sim \Phi) \rangle)$
$\to ((\langle \varphi \rangle \sqcap \langle \bigsqcap\ \Phi \rangle) \to \langle \chi \rangle) \leftrightarrow (\sim \langle \chi \rangle \to (\sim \langle \varphi \rangle \sqcup \langle \bigsqcup\ (\sim \Phi) \rangle))$
by *auto*
hence
$\vdash (\!|\ (\langle \bigsqcap\ \Phi \rangle \to \langle \chi \rangle) \leftrightarrow (\sim \langle \chi \rangle \to \langle \bigsqcup\ (\sim \Phi) \rangle)$
$\to ((\langle \varphi \rangle \sqcap \langle \bigsqcap\ \Phi \rangle) \to \langle \chi \rangle) \leftrightarrow (\sim \langle \chi \rangle \to (\sim \langle \varphi \rangle \sqcup \langle \bigsqcup\ (\sim \Phi) \rangle))\ |\!)$
using *propositional-semantics* **by** *blast*
thus *?thesis* **by** *simp*
qed
thus *?thesis*
using ‹$\vdash (\bigsqcap\ \Phi \to \chi) \leftrightarrow (\sim \chi \to \bigsqcup\ (\sim \Phi))$› *modus-ponens* **by** *auto*
qed
then show *?case*
using *biconditional-transitivity-rule list-curry-uncurry* **by** *blast*
qed

5.9.5 Monotony of Conjunction and Disjunction

lemma (**in** *classical-logic*) *conjunction-monotonic-identity*:
$\vdash (\varphi \to \psi) \to (\varphi \sqcap \chi) \to (\psi \sqcap \chi)$
unfolding *conjunction-def*
using *modus-ponens*
flip-hypothetical-syllogism
by *blast*

lemma (**in** *classical-logic*) *conjunction-monotonic*:
assumes $\vdash \varphi \to \psi$
shows $\vdash (\varphi \sqcap \chi) \to (\psi \sqcap \chi)$
using *assms*
modus-ponens
conjunction-monotonic-identity

by *blast*

lemma (**in** *classical-logic*) *disjunction-monotonic-identity*:
$\vdash (\varphi \to \psi) \to (\varphi \sqcup \chi) \to (\psi \sqcup \chi)$
unfolding *disjunction-def*
using *modus-ponens*
flip-hypothetical-syllogism
by *blast*

lemma (**in** *classical-logic*) *disjunction-monotonic*:
assumes $\vdash \varphi \to \psi$
shows $\vdash (\varphi \sqcup \chi) \to (\psi \sqcup \chi)$
using *assms*
modus-ponens
disjunction-monotonic-identity
by *blast*

5.9.6 Distribution Identities

lemma (**in** *classical-logic*) *conjunction-distribution*:
$\vdash ((\psi \sqcup \chi) \sqcap \varphi) \leftrightarrow ((\psi \sqcap \varphi) \sqcup (\chi \sqcap \varphi))$
proof −
have $\forall\ \mathfrak{M}.\ \mathfrak{M} \models_{prop} ((\langle\psi\rangle \sqcup \langle\chi\rangle) \sqcap \langle\varphi\rangle) \leftrightarrow ((\langle\psi\rangle \sqcap \langle\varphi\rangle) \sqcup (\langle\chi\rangle \sqcap \langle\varphi\rangle))$
by *auto*
hence $\vdash (\!| ((\langle\psi\rangle \sqcup \langle\chi\rangle) \sqcap \langle\varphi\rangle) \leftrightarrow ((\langle\psi\rangle \sqcap \langle\varphi\rangle) \sqcup (\langle\chi\rangle \sqcap \langle\varphi\rangle)) |\!)$
using *propositional-semantics* **by** *blast*
thus *?thesis* **by** *simp*
qed

lemma (**in** *classical-logic*) *subtraction-distribution*:
$\vdash ((\psi \sqcup \chi) \setminus \varphi) \leftrightarrow ((\psi \setminus \varphi) \sqcup (\chi \setminus \varphi))$
by (*simp add*: *conjunction-distribution subtraction-def*)

lemma (**in** *classical-logic*) *conjunction-arbitrary-distribution*:
$\vdash (\bigsqcup\ \Psi \sqcap \varphi) \leftrightarrow \bigsqcup\ [\psi \sqcap \varphi.\ \psi \leftarrow \Psi]$
proof (*induct* Ψ)
case *Nil*
then show *?case*
by (*simp add*: *ex-falso-quodlibet*
biconditional-def
conjunction-left-elimination)
next
case (*Cons* ψ Ψ)
have $\vdash (\bigsqcup\ (\psi\ \#\ \Psi) \sqcap \varphi) \leftrightarrow ((\psi \sqcap \varphi) \sqcup ((\bigsqcup\ \Psi) \sqcap \varphi))$
using *conjunction-distribution* **by** *auto*
moreover
from *Cons* **have**
$\vdash ((\psi \sqcap \varphi) \sqcup ((\bigsqcup\ \Psi) \sqcap \varphi)) \leftrightarrow ((\psi \sqcap \varphi) \sqcup (\bigsqcup\ [\psi \sqcap \varphi.\ \psi \leftarrow \Psi]))$
unfolding *disjunction-def biconditional-def*

by (*simp*, *meson modus-ponens hypothetical-syllogism*)
ultimately show *?case*
by (*simp*, *metis biconditional-transitivity-rule*)
qed

lemma (**in** *classical-logic*) *subtraction-arbitrary-distribution*:
$\vdash (\bigsqcup\ \Psi \setminus \varphi) \leftrightarrow \bigsqcup\ [\psi \setminus \varphi.\ \psi \leftarrow \Psi]$
by (*simp add*: *conjunction-arbitrary-distribution subtraction-def*)

lemma (**in** *classical-logic*) *disjunction-distribution*:
$\vdash (\varphi \sqcup (\psi \sqcap \chi)) \leftrightarrow ((\varphi \sqcup \psi) \sqcap (\varphi \sqcup \chi))$
proof −
have $\forall\ \mathfrak{M}.\ \mathfrak{M} \models_{prop} (\langle\varphi\rangle \sqcup (\langle\psi\rangle \sqcap \langle\chi\rangle)) \leftrightarrow ((\langle\varphi\rangle \sqcup \langle\psi\rangle) \sqcap (\langle\varphi\rangle \sqcup \langle\chi\rangle))$
by *auto*
hence $\vdash (\!|\ (\langle\varphi\rangle \sqcup (\langle\psi\rangle \sqcap \langle\chi\rangle)) \leftrightarrow ((\langle\varphi\rangle \sqcup \langle\psi\rangle) \sqcap (\langle\varphi\rangle \sqcup \langle\chi\rangle))\ |\!)$
using *propositional-semantics* **by** *blast*
thus *?thesis* **by** *simp*
qed

lemma (**in** *classical-logic*) *implication-distribution*:
$\vdash (\varphi \rightarrow (\psi \sqcap \chi)) \leftrightarrow ((\varphi \rightarrow \psi) \sqcap (\varphi \rightarrow \chi))$
proof −
have $\forall\ \mathfrak{M}.\ \mathfrak{M} \models_{prop} (\langle\varphi\rangle \rightarrow (\langle\psi\rangle \sqcap \langle\chi\rangle)) \leftrightarrow ((\langle\varphi\rangle \rightarrow \langle\psi\rangle) \sqcap (\langle\varphi\rangle \rightarrow \langle\chi\rangle))$
by *auto*
hence $\vdash (\!|\ (\langle\varphi\rangle \rightarrow (\langle\psi\rangle \sqcap \langle\chi\rangle)) \leftrightarrow ((\langle\varphi\rangle \rightarrow \langle\psi\rangle) \sqcap (\langle\varphi\rangle \rightarrow \langle\chi\rangle))\ |\!)$
using *propositional-semantics* **by** *blast*
thus *?thesis* **by** *simp*
qed

lemma (**in** *classical-logic*) *list-implication-distribution*:
$\vdash (\Phi :\rightarrow (\psi \sqcap \chi)) \leftrightarrow ((\Phi :\rightarrow \psi) \sqcap (\Phi :\rightarrow \chi))$
proof (*induct* Φ)
case *Nil*
then show *?case*
by (*simp add*: *biconditional-reflection*)
next
case (*Cons* φ Φ)
hence $\vdash (\varphi \#\ \Phi) :\rightarrow (\psi \sqcap \chi) \leftrightarrow (\varphi \rightarrow (\Phi :\rightarrow \psi \sqcap \Phi :\rightarrow \chi))$
by (*metis*
modus-ponens
biconditional-def
hypothetical-syllogism
list-implication.simps(*2*)
weak-conjunction-deduction-equivalence)
moreover
have $\vdash (\varphi \rightarrow (\Phi :\rightarrow \psi \sqcap \Phi :\rightarrow \chi)) \leftrightarrow (((\varphi \#\ \Phi) :\rightarrow \psi) \sqcap ((\varphi \#\ \Phi) :\rightarrow \chi))$
using *implication-distribution* **by** *auto*
ultimately show *?case*
by (*simp*, *metis biconditional-transitivity-rule*)

qed

lemma (**in** *classical-logic*) *biconditional-conjunction-weaken*:
$\vdash (\alpha \leftrightarrow \beta) \rightarrow ((\gamma \sqcap \alpha) \leftrightarrow (\gamma \sqcap \beta))$
proof −
have $\forall\ \mathfrak{M}.\ \mathfrak{M} \models_{prop} (\langle\alpha\rangle \leftrightarrow \langle\beta\rangle) \rightarrow ((\langle\gamma\rangle \sqcap \langle\alpha\rangle) \leftrightarrow (\langle\gamma\rangle \sqcap \langle\beta\rangle))$
by *auto*
hence $\vdash$ ⦇ $(\langle\alpha\rangle \leftrightarrow \langle\beta\rangle) \rightarrow ((\langle\gamma\rangle \sqcap \langle\alpha\rangle) \leftrightarrow (\langle\gamma\rangle \sqcap \langle\beta\rangle))$ ⦈
using *propositional-semantics* **by** *blast*
thus *?thesis* **by** *simp*
qed

lemma (**in** *classical-logic*) *biconditional-conjunction-weaken-rule*:
$\vdash (\alpha \leftrightarrow \beta) \Longrightarrow \vdash (\gamma \sqcap \alpha) \leftrightarrow (\gamma \sqcap \beta)$
using *modus-ponens biconditional-conjunction-weaken* **by** *blast*

lemma (**in** *classical-logic*) *disjunction-arbitrary-distribution*:
$\vdash (\varphi \sqcup \bigsqcap \Psi) \leftrightarrow \bigsqcap [\varphi \sqcup \psi.\ \psi \leftarrow \Psi]$
proof (*induct* Ψ)
case *Nil*
then show *?case*
unfolding *disjunction-def biconditional-def*
using *axiom-k modus-ponens verum-tautology*
by (*simp*, *blast*)
next
case (*Cons* ψ Ψ)
have $\vdash (\varphi \sqcup \bigsqcap (\psi \# \Psi)) \leftrightarrow ((\varphi \sqcup \psi) \sqcap (\varphi \sqcup \bigsqcap \Psi))$
by (*simp add*: *disjunction-distribution*)
moreover
from *biconditional-conjunction-weaken-rule*
Cons
have $\vdash ((\varphi \sqcup \psi) \sqcap \varphi \sqcup \bigsqcap \Psi) \leftrightarrow \bigsqcap (map\ (\lambda\ \chi\ .\ \varphi \sqcup \chi)\ (\psi \# \Psi))$
by *simp*
ultimately show *?case*
by (*metis biconditional-transitivity-rule*)
qed

lemma (**in** *classical-logic*) *list-implication-arbitrary-distribution*:
$\vdash (\Phi :\rightarrow \bigsqcap \Psi) \leftrightarrow \bigsqcap [\Phi :\rightarrow \psi.\ \psi \leftarrow \Psi]$
proof (*induct* Ψ)
case *Nil*
then show *?case*
by (*simp add*: *biconditional-def*,
meson
axiom-k
modus-ponens
list-implication-axiom-k
verum-tautology)
next

case (*Cons* ψ Ψ)
have $\vdash \Phi :\rightarrow \bigsqcap (\psi \# \Psi) \leftrightarrow (\Phi :\rightarrow \psi \sqcap \Phi :\rightarrow \bigsqcap \Psi)$
using *list-implication-distribution*
by *fastforce*
moreover
from *biconditional-conjunction-weaken-rule*
Cons
have $\vdash (\Phi :\rightarrow \psi \sqcap \Phi :\rightarrow \bigsqcap \Psi) \leftrightarrow \bigsqcap [\Phi :\rightarrow \psi.\ \psi \leftarrow (\psi \# \Psi)]$
by *simp*
ultimately show *?case*
by (*metis biconditional-transitivity-rule*)
qed

lemma (**in** *classical-logic*) *implication-arbitrary-distribution*:
$\vdash (\varphi \rightarrow \bigsqcap \Psi) \leftrightarrow \bigsqcap [\varphi \rightarrow \psi.\ \psi \leftarrow \Psi]$
using *list-implication-arbitrary-distribution* [**where** *?*Φ=$[\varphi]$]
by *simp*

5.9.7 Negation

lemma (**in** *classical-logic*) *double-negation-biconditional*:
$\vdash \sim (\sim \varphi) \leftrightarrow \varphi$
unfolding *biconditional-def negation-def*
by (*simp add*: *double-negation double-negation-converse*)

lemma (**in** *classical-logic*) *double-negation-elimination* [*simp*]:
$\Gamma \Vdash \sim (\sim \varphi) = \Gamma \Vdash \varphi$
using
set-deduction-weaken
biconditional-weaken
double-negation-biconditional
by *metis*

lemma (**in** *classical-logic*) *alt-double-negation-elimination* [*simp*]:
$\Gamma \Vdash (\varphi \rightarrow \bot) \rightarrow \bot \equiv \Gamma \Vdash \varphi$
using *double-negation-elimination*
unfolding *negation-def*
by *auto*

lemma (**in** *classical-logic*) *base-double-negation-elimination* [*simp*]:
$\vdash \sim (\sim \varphi) = \vdash \varphi$
by (*metis double-negation-elimination set-deduction-base-theory*)

lemma (**in** *classical-logic*) *alt-base-double-negation-elimination* [*simp*]:
$\vdash (\varphi \rightarrow \bot) \rightarrow \bot \equiv \vdash \varphi$
using *base-double-negation-elimination*
unfolding *negation-def*
by *auto*

5.9.8 Mutual Exclusion Identities

lemma (**in** *classical-logic*) *exclusion-contrapositive-equivalence*:
$\vdash (\varphi \to \gamma) \leftrightarrow \sim (\varphi \sqcap \sim \gamma)$
proof −
have $\forall\ \mathfrak{M}.\ \mathfrak{M} \models_{prop} (\langle\varphi\rangle \to \langle\gamma\rangle) \leftrightarrow \sim (\langle\varphi\rangle \sqcap \sim \langle\gamma\rangle)$
by *auto*
hence $\vdash (\!| (\langle\varphi\rangle \to \langle\gamma\rangle) \leftrightarrow \sim (\langle\varphi\rangle \sqcap \sim \langle\gamma\rangle) |\!)$
using *propositional-semantics* **by** *blast*
thus *?thesis* **by** *simp*
qed

lemma (**in** *classical-logic*) *disjuction-exclusion-equivalence*:
$\Gamma \Vdash \sim (\psi \sqcap \bigsqcup \Phi) \equiv \forall\ \varphi \in set\ \Phi.\ \Gamma \Vdash \sim (\psi \sqcap \varphi)$
proof (*induct* Φ)
case *Nil*
then show *?case*
by (*simp add*:
conjunction-right-elimination
negation-def
set-deduction-weaken)
next
case (*Cons* φ Φ)
have $\vdash \sim (\psi \sqcap \bigsqcup (\varphi \# \Phi)) \leftrightarrow \sim (\psi \sqcap (\varphi \sqcup \bigsqcup \Phi))$
by (*simp add*: *biconditional-reflection*)
moreover have $\vdash \sim (\psi \sqcap (\varphi \sqcup \bigsqcup \Phi)) \leftrightarrow (\sim (\psi \sqcap \varphi) \sqcap \sim (\psi \sqcap \bigsqcup \Phi))$
proof −
have $\forall\ \mathfrak{M}.\ \mathfrak{M} \models_{prop} \sim (\langle\psi\rangle \sqcap (\langle\varphi\rangle \sqcup \langle\bigsqcup \Phi\rangle))$
$\leftrightarrow (\sim (\langle\psi\rangle \sqcap \langle\varphi\rangle) \sqcap \sim (\langle\psi\rangle \sqcap \langle\bigsqcup \Phi\rangle))$
by *auto*
hence $\vdash (\!| \sim (\langle\psi\rangle \sqcap (\langle\varphi\rangle \sqcup \langle\bigsqcup \Phi\rangle))$
$\leftrightarrow (\sim (\langle\psi\rangle \sqcap \langle\varphi\rangle) \sqcap \sim (\langle\psi\rangle \sqcap \langle\bigsqcup \Phi\rangle)) |\!)$
using *propositional-semantics* **by** *blast*
thus *?thesis* **by** *simp*
qed
ultimately
have $\vdash \sim (\psi \sqcap \bigsqcup (\varphi \# \Phi)) \leftrightarrow (\sim (\psi \sqcap \varphi) \sqcap \sim (\psi \sqcap \bigsqcup \Phi))$
by *simp*
hence $\Gamma \Vdash \sim (\psi \sqcap \bigsqcup (\varphi \# \Phi)) = (\Gamma \Vdash \sim (\psi \sqcap \varphi)$
$\wedge\ (\forall \varphi \in set\ \Phi.\ \Gamma \Vdash \sim (\psi \sqcap \varphi)))$
using *set-deduction-weaken* [**where** Γ=Γ]
conjunction-set-deduction-equivalence [**where** Γ=Γ]
Cons.hyps
biconditional-def
set-deduction-modus-ponens
by *metis*
thus $\Gamma \Vdash \sim (\psi \sqcap \bigsqcup (\varphi \# \Phi)) = (\forall \varphi \in set\ (\varphi \# \Phi).\ \Gamma \Vdash \sim (\psi \sqcap \varphi))$
by *simp*
qed

lemma (**in** *classical-logic*) *exclusive-elimination1*:
assumes $\Gamma \Vdash \coprod \Phi$
shows $\forall\ \varphi \in set\ \Phi.\ \forall\ \psi \in set\ \Phi.\ (\varphi \neq \psi) \longrightarrow \Gamma \Vdash \sim (\varphi \sqcap \psi)$
using *assms*
proof (*induct* Φ)
case *Nil*
thus *?case* **by** *auto*
next
case (*Cons* χ Φ)
assume $\Gamma \Vdash \coprod (\chi \# \Phi)$
hence $\Gamma \Vdash \coprod \Phi$ **by** *simp*
hence $\forall \varphi \in set\ \Phi.\ \forall \psi \in set\ \Phi.\ \varphi \neq \psi \longrightarrow \Gamma \Vdash \sim (\varphi \sqcap \psi)$
using *Cons.hyps* **by** *blast*
moreover have $\Gamma \Vdash \sim (\chi \sqcap \bigsqcup \Phi)$
using ‹$\Gamma \Vdash \coprod (\chi \# \Phi)$› *conjunction-set-deduction-equivalence* **by** *auto*
hence $\forall\ \varphi \in set\ \Phi.\ \Gamma \Vdash \sim (\chi \sqcap \varphi)$
using *disjuction-exclusion-equivalence* **by** *auto*
moreover {
fix φ
have $\vdash \sim (\chi \sqcap \varphi) \rightarrow \sim (\varphi \sqcap \chi)$
unfolding *negation-def*
conjunction-def
using *modus-ponens flip-hypothetical-syllogism flip-implication* **by** *blast*
}
with ‹$\forall\ \varphi \in set\ \Phi.\ \Gamma \Vdash \sim (\chi \sqcap \varphi)$› **have** $\forall\ \varphi \in set\ \Phi.\ \Gamma \Vdash \sim (\varphi \sqcap \chi)$
using *set-deduction-weaken* [**where** Γ=Γ]
set-deduction-modus-ponens [**where** Γ=Γ]
by *blast*
ultimately
show $\forall \varphi \in set\ (\chi \# \Phi).\ \forall \psi \in set\ (\chi \# \Phi).\ \varphi \neq \psi \longrightarrow \Gamma \Vdash \sim (\varphi \sqcap \psi)$
by *simp*
qed

lemma (**in** *classical-logic*) *exclusive-elimination2*:
assumes $\Gamma \Vdash \coprod \Phi$
shows $\forall\ \varphi \in duplicates\ \Phi.\ \Gamma \Vdash \sim \varphi$
using *assms*
proof (*induct* Φ)
case *Nil*
then show *?case* **by** *simp*
next
case (*Cons* φ Φ)
assume $\Gamma \Vdash \coprod (\varphi \# \Phi)$
hence $\Gamma \Vdash \coprod \Phi$ **by** *simp*
hence $\forall \varphi \in duplicates\ \Phi.\ \Gamma \Vdash \sim \varphi$ **using** *Cons.hyps* **by** *auto*
show *?case*
proof *cases*
assume $\varphi \in set\ \Phi$
moreover {

```
    fix φ ψ χ
    have ⊢ ∼ (φ ⊓ (ψ ⊔ χ)) ↔ (∼ (φ ⊓ ψ) ⊓ ∼ (φ ⊓ χ))
    proof −
      have ∀ 𝔐. 𝔐 ⊨_prop ∼ (⟨φ⟩ ⊓ (⟨ψ⟩ ⊔ ⟨χ⟩))
                    ↔ (∼ (⟨φ⟩ ⊓ ⟨ψ⟩) ⊓ ∼ (⟨φ⟩ ⊓ ⟨χ⟩))
        by auto
      hence ⊢ ⦇ ∼ (⟨φ⟩ ⊓ (⟨ψ⟩ ⊔ ⟨χ⟩)) ↔ (∼ (⟨φ⟩ ⊓ ⟨ψ⟩) ⊓ ∼ (⟨φ⟩ ⊓ ⟨χ⟩)) ⦈
        using propositional-semantics by blast
      thus ?thesis by simp
    qed
    hence Γ ⊩ ∼ (φ ⊓ (ψ ⊔ χ)) ≡ Γ ⊩ ∼ (φ ⊓ ψ) ⊓ ∼ (φ ⊓ χ)
      using set-deduction-weaken
            biconditional-weaken by presburger
  }
  moreover
  have ⊢ ∼ (φ ⊓ φ) ↔ ∼ φ
  proof −
    have ∀ 𝔐. 𝔐 ⊨_prop ∼ (⟨φ⟩ ⊓ ⟨φ⟩) ↔ ∼ ⟨φ⟩
      by auto
    hence ⊢ ⦇ ∼ (⟨φ⟩ ⊓ ⟨φ⟩) ↔ ∼ ⟨φ⟩ ⦈
      using propositional-semantics by blast
    thus ?thesis by simp
  qed
  hence Γ ⊩ ∼ (φ ⊓ φ) ≡ Γ ⊩ ∼ φ
    using set-deduction-weaken
          biconditional-weaken by presburger
  moreover have Γ ⊩ ∼ (φ ⊓ ⨆ Φ) using ‹Γ ⊩ ⨅ (φ # Φ)› by simp
  ultimately have Γ ⊩ ∼ φ by (induct Φ, simp, simp, blast)
  thus ?thesis using ‹φ ∈ set Φ› ‹∀φ∈duplicates Φ. Γ ⊩ ∼ φ› by simp
 next
  assume φ ∉ set Φ
  hence duplicates (φ # Φ) = duplicates Φ by simp
  then show ?thesis using ‹∀φ∈duplicates Φ. Γ ⊩ ∼ φ›
    by auto
 qed
qed

lemma (in classical-logic) exclusive-equivalence:
  Γ ⊩ ⨅ Φ =
    ((∀φ∈duplicates Φ. Γ ⊩ ∼ φ) ∧
      (∀ φ ∈ set Φ. ∀ ψ ∈ set Φ. (φ ≠ ψ) ⟶ Γ ⊩ ∼ (φ ⊓ ψ)))
proof −
 {
  assume ∀φ∈duplicates Φ. Γ ⊩ ∼ φ
         ∀ φ ∈ set Φ. ∀ ψ ∈ set Φ. (φ ≠ ψ) ⟶ Γ ⊩ ∼ (φ ⊓ ψ)
  hence Γ ⊩ ⨅ Φ
  proof (induct Φ)
    case Nil
    then show ?case
```

```
    by (simp add: set-deduction-weaken)
next
  case (Cons φ Φ)
  assume A: ∀φ∈duplicates (φ # Φ). Γ ⊩ ∼ φ
    and B: ∀χ∈set (φ # Φ). ∀ψ∈set (φ # Φ). χ ≠ ψ ⟶ Γ ⊩ ∼ (χ ⊓ ψ)
  hence C: Γ ⊩ ⨅ Φ using Cons.hyps by simp
  then show ?case
  proof cases
    assume φ ∈ duplicates (φ # Φ)
    moreover from this have Γ ⊩ ∼ φ using A by auto
    moreover have duplicates Φ ⊆ set Φ by (induct Φ, simp, auto)
    ultimately have φ ∈ set Φ by (metis duplicates.simps(2) subsetCE)
    hence ⊢ ∼ φ ↔ ∼(φ ⊓ ⨆ Φ)
    proof (induct Φ)
      case Nil
      then show ?case by simp
    next
      case (Cons ψ Φ)
      assume φ ∈ set (ψ # Φ)
      then show ⊢ ∼ φ ↔ ∼ (φ ⊓ ⨆ (ψ # Φ))
      proof −
        {
          assume φ = ψ
          hence ?thesis
          proof −
            have ∀ 𝔐. 𝔐 ⊨_prop ∼ ⟨φ⟩ ↔ ∼ (⟨φ⟩ ⊓ (⟨ψ⟩ ⊔ ⟨⨆ Φ⟩))
              using ‹φ = ψ› by auto
            hence ⊢ ⦇ ∼ ⟨φ⟩ ↔ ∼ (⟨φ⟩ ⊓ (⟨ψ⟩ ⊔ ⟨⨆ Φ⟩)) ⦈
              using propositional-semantics by blast
            thus ?thesis by simp
          qed
        }
        moreover
        {
          assume φ ≠ ψ
          hence φ ∈ set Φ using ‹φ ∈ set (ψ # Φ)› by auto
          hence ⊢ ∼ φ ↔ ∼ (φ ⊓ ⨆ Φ) using Cons.hyps by auto
          moreover have ⊢ (∼ φ ↔ ∼ (φ ⊓ ⨆ Φ))
                              → (∼ φ ↔ ∼ (φ ⊓ (ψ ⊔ ⨆ Φ)))
          proof −
            have ∀ 𝔐. 𝔐 ⊨_prop (∼ ⟨φ⟩ ↔ ∼ (⟨φ⟩ ⊓ ⟨⨆ Φ⟩)) →
                                (∼ ⟨φ⟩ ↔ ∼ (⟨φ⟩ ⊓ (⟨ψ⟩ ⊔ ⟨⨆ Φ⟩))))
              by auto
            hence ⊢ ⦇ (∼ ⟨φ⟩ ↔ ∼ (⟨φ⟩ ⊓ ⟨⨆ Φ⟩))
                      → (∼ ⟨φ⟩ ↔ ∼ (⟨φ⟩ ⊓ (⟨ψ⟩ ⊔ ⟨⨆ Φ⟩))) ⦈
              using propositional-semantics by blast
            thus ?thesis by simp
          qed
          ultimately have ?thesis using modus-ponens by simp
```

```
        }
        ultimately show ?thesis by auto
      qed
    qed
    with ‹Γ ⊩ ∼ φ› have Γ ⊩ ∼(φ ⊓ ⨆ Φ)
      using biconditional-weaken set-deduction-weaken by blast
    with ‹Γ ⊩ ⨅ Φ› show ?thesis by simp
  next
    assume φ ∉ duplicates (φ # Φ)
    hence φ ∉ set Φ by auto
    with B have ∀ψ∈set Φ. Γ ⊩ ∼ (φ ⊓ ψ) by (simp, metis)
    hence Γ ⊩ ∼ (φ ⊓ ⨆ Φ)
      by (simp add: disjuction-exclusion-equivalence)
    with ‹Γ ⊩ ⨅ Φ› show ?thesis by simp
  qed
 qed
}
thus ?thesis
 by (metis exclusive-elimination1 exclusive-elimination2)
qed
```

5.9.9 Miscellaneous Disjunctive Normal Form Identities

```
lemma (in classical-logic) map-negation-list-implication:
 ⊢ ((∼ Φ) :→ (∼ φ)) ↔ (φ → ⨆ Φ)
proof (induct Φ)
 case Nil
 then show ?case
  unfolding
    biconditional-def
    map-negation-def
    negation-def
  using
    conjunction-introduction
    modus-ponens
    trivial-implication
  by simp
next
 case (Cons ψ Φ)
 have ⊢ (∼ Φ :→ ∼ φ ↔ (φ → ⨆ Φ))
       → (∼ ψ → ∼ Φ :→ ∼ φ) ↔ (φ → (ψ ⊔ ⨆ Φ))
 proof −
  have ∀𝔐. 𝔐 ⊨_prop (⟨∼ Φ :→ ∼ φ⟩ ↔ (⟨φ⟩ → ⟨⨆ Φ⟩)) →
                    (∼ ⟨ψ⟩ → ⟨∼ Φ :→ ∼ φ⟩) ↔ (⟨φ⟩ → (⟨ψ⟩ ⊔ ⟨⨆ Φ⟩))
    by fastforce
  hence ⊢ ⦇ (⟨∼ Φ :→ ∼ φ⟩ ↔ (⟨φ⟩ → ⟨⨆ Φ⟩)) →
            (∼ ⟨ψ⟩ → ⟨∼ Φ :→ ∼ φ⟩) ↔ (⟨φ⟩ → (⟨ψ⟩ ⊔ ⟨⨆ Φ⟩)) ⦈
    using propositional-semantics by blast
  thus ?thesis
```

```
    by simp
  qed
  with Cons show ?case
    by (metis
        map-negation-def
        list.simps(9)
        arbitrary-disjunction.simps(2)
        modus-ponens
        list-implication.simps(2))
qed


lemma (in classical-logic) conj-dnf-distribute:
  ⊢ ⨆ (map (⨅ ∘ (λ φs. φ # φs)) Λ) ↔ (φ ⊓ ⨆ (map ⨅ Λ))
proof(induct Λ)
  case Nil
  have ⊢ ⊥ ↔ (φ ⊓ ⊥)
  proof −
    let ?φ = ⊥ ↔ (⟨φ⟩ ⊓ ⊥)
    have ∀ 𝔐. 𝔐 ⊨_prop ?φ by fastforce
    hence ⊢ ⦇ ?φ ⦈ using propositional-semantics by blast
    thus ?thesis by simp
  qed
  then show ?case by simp
next
  case (Cons Ψ Λ)
  assume ⊢ ⨆ (map (⨅ ∘ (λ φs. φ # φs)) Λ) ↔ (φ ⊓ ⨆ (map ⨅ Λ))
    (is ⊢ ?A ↔ (φ ⊓ ?B))
  moreover
  have ⊢ (?A ↔ (φ ⊓ ?B)) → (((φ ⊓ ⨅ Ψ) ⊔ ?A) ↔ (φ ⊓ ⨅ Ψ ⊔ ?B))
  proof −
    let ?φ = (⟨?A⟩ ↔ (⟨φ⟩ ⊓ ⟨?B⟩)) →
             ((((⟨φ⟩ ⊓ ⟨⨅ Ψ⟩) ⊔ ⟨?A⟩) ↔ (⟨φ⟩ ⊓ ⟨⨅ Ψ⟩ ⊔ ⟨?B⟩))
    have ∀ 𝔐. 𝔐 ⊨_prop ?φ by fastforce
    hence ⊢ ⦇ ?φ ⦈ using propositional-semantics by blast
    thus ?thesis
      by simp
  qed
  ultimately have ⊢ ((φ ⊓ ⨅ Ψ) ⊔ ?A) ↔ (φ ⊓ ⨅ Ψ ⊔ ?B)
    using modus-ponens
    by blast
  moreover
  have map (⨅ ∘ (λ φs. φ # φs)) Λ = map (λΨ. φ ⊓ ⨅ Ψ) Λ by simp
  ultimately show ?case by simp
qed

lemma (in classical-logic) append-dnf-distribute:
  ⊢ ⨆ (map (⨅ ∘ (λ Ψ. Φ @ Ψ)) Λ) ↔ (⨅ Φ ⊓ ⨆ (map ⨅ Λ))
proof(induct Φ)
```

case *Nil*
have $\vdash \bigsqcup\ (map\ \bigsqcap\ \Lambda) \leftrightarrow (\top \sqcap \bigsqcup\ (map\ \bigsqcap\ \Lambda))$
(**is** $\vdash$ *?A* $\leftrightarrow$ $(\top \sqcap$ *?A*$)$)
proof −
let *?φ* $= \langle$*?A*$\rangle \leftrightarrow ((\bot \to \bot) \sqcap \langle$*?A*$\rangle)$
have $\forall\, \mathfrak{M}.\ \mathfrak{M} \models_{prop}$ *?φ* **by** *simp*
hence $\vdash$ ⦇ *?φ* ⦈ **using** *propositional-semantics* **by** *blast*
thus *?thesis*
unfolding *verum-def*
by *simp*
qed
then show *?case* **by** *simp*
next
case (*Cons φ* Φ)
have $\vdash \bigsqcup\ (map\ (\bigsqcap \circ (@)\ \Phi)\ \Lambda) \leftrightarrow (\bigsqcap\ \Phi \sqcap \bigsqcup\ (map\ \bigsqcap\ \Lambda))$
$= \vdash \bigsqcup\ (map\ \bigsqcap\ (map\ ((@)\ \Phi)\ \Lambda)) \leftrightarrow (\bigsqcap\ \Phi \sqcap \bigsqcup\ (map\ \bigsqcap\ \Lambda))$
by *simp*
with *Cons* **have**
$\vdash \bigsqcup\ (map\ \bigsqcap\ (map\ (\lambda\ \Psi.\ \Phi\ @\ \Psi)\ \Lambda)) \leftrightarrow (\bigsqcap\ \Phi \sqcap \bigsqcup\ (map\ \bigsqcap\ \Lambda))$
(**is** $\vdash \bigsqcup\ (map\ \bigsqcap$ *?A*$) \leftrightarrow$ (*?B* $\sqcap$ *?C*))
by *meson*
moreover have $\vdash \bigsqcup\ (map\ \bigsqcap$ *?A*$) \leftrightarrow$ (*?B* $\sqcap$ *?C*)
$\to (\bigsqcup\ (map\ (\bigsqcap \circ (\lambda\ \varphi s.\ \varphi\ \#\ \varphi s))$ *?A*$) \leftrightarrow (\varphi \sqcap \bigsqcup\ (map\ \bigsqcap$ *?A*$)))$
$\to \bigsqcup\ (map\ (\bigsqcap \circ (\lambda\ \varphi s.\ \varphi\ \#\ \varphi s))$ *?A*$) \leftrightarrow ((\varphi \sqcap$ *?B*$) \sqcap$ *?C*
proof −
let *?φ* $= \langle \bigsqcup\ (map\ \bigsqcap$ *?A*$)\rangle \leftrightarrow (\langle$*?B*$\rangle \sqcap \langle$*?C*$\rangle)$
$\to (\langle \bigsqcup\ (map\ (\bigsqcap \circ (\lambda\ \varphi s.\ \varphi\ \#\ \varphi s))$ *?A*$)\rangle \leftrightarrow (\langle\varphi\rangle \sqcap \langle \bigsqcup\ (map\ \bigsqcap$ *?A*$)\rangle))$
$\to \langle \bigsqcup\ (map\ (\bigsqcap \circ (\lambda\ \varphi s.\ \varphi\ \#\ \varphi s))$ *?A*$)\rangle \leftrightarrow ((\langle\varphi\rangle \sqcap \langle$*?B*$\rangle) \sqcap \langle$*?C*$\rangle)$
have $\forall\, \mathfrak{M}.\ \mathfrak{M} \models_{prop}$ *?φ* **by** *simp*
hence $\vdash$ ⦇ *?φ* ⦈ **using** *propositional-semantics* **by** *blast*
thus *?thesis*
by *simp*
qed
ultimately have $\vdash \bigsqcup\ (map\ (\bigsqcap \circ (\lambda\ \varphi s.\ \varphi\ \#\ \varphi s))$ *?A*$) \leftrightarrow ((\varphi \sqcap$ *?B*$) \sqcap$ *?C*
using *modus-ponens conj-dnf-distribute*
by *blast*
moreover
have $\bigsqcap \circ (@)\ (\varphi\ \#\ \Phi) = \bigsqcap \circ (\#)\ \varphi \circ (@)\ \Phi$ **by** *auto*
hence
$\vdash \bigsqcup\ (map\ (\bigsqcap \circ (@)\ (\varphi\ \#\ \Phi))\ \Lambda) \leftrightarrow (\bigsqcap\ (\varphi\ \#\ \Phi) \sqcap$ *?C*$)$
$= \vdash \bigsqcup\ (map\ (\bigsqcap \circ (\#)\ \varphi)$ *?A*$) \leftrightarrow ((\varphi \sqcap$ *?B*$) \sqcap$ *?C*
by *simp*
ultimately show *?case* **by** *meson*
qed

unbundle *funcset-syntax*

end

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