

Two theorems about the geometry of the critical points of a complex polynomial

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February 6, 2026

Abstract

This entry formalises two well-known results about the geometric relation between the *roots* of a complex polynomial and its *critical points*, i.e. the roots of its derivative.

The first of these is the *GauSS–Lucas Theorem*: The critical points of a complex polynomial lie inside the convex hull of its roots.

The second one is *Jensen’s Theorem*: Every non-real critical point of a real polynomial lies inside a disc between two conjugate roots. These discs are called the *Jensen discs*: the Jensen disc of a pair of conjugate roots $a \pm bi$ is the smallest disc that contains both of them, i.e. the disc with centre a and radius b .

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1 Missing Library Material

```
theory Polynomial_Crit_Geometry_Library
imports
  "HOL-Computational_Algebra.Computational_Algebra"
  "HOL-Library.FuncSet"
  "Polynomial_Interpolation.Ring_Hom_Poly"
begin
```

1.1 Multisets

```
lemma size_repeat_mset [simp]: "size (repeat_mset n A) = n * size A"
  <proof>
```

```
lemma count_image_mset_inj:
  "inj f  $\implies$  count (image_mset f A) (f x) = count A x"
  <proof>
```

```
lemma count_le_size: "count A x  $\leq$  size A"
  <proof>
```

```
lemma image_mset_cong_simp:
  "M = M'  $\implies$  ( $\bigwedge x. x \in\# M \implies f x = g x$ )  $\implies$  {#f x. x  $\in\# M$ #} = {#g
x. x  $\in\# M'$ #}"
  <proof>
```

```
lemma sum_mset_nonneg:
  fixes A :: "'a :: ordered_comm_monoid_add multiset"
  assumes " $\bigwedge x. x \in\# A \implies x \geq 0$ "
  shows "sum_mset A  $\geq 0$ "
  <proof>
```

```
lemma sum_mset_pos:
  fixes A :: "'a :: ordered_comm_monoid_add multiset"
  assumes "A  $\neq$  {#}"
  assumes " $\bigwedge x. x \in\# A \implies x > 0$ "
  shows "sum_mset A  $> 0$ "
  <proof>
```

1.2 Polynomials

```
lemma order_pos_iff: "p  $\neq 0 \implies$  order x p  $> 0 \iff$  poly p x = 0"
  <proof>
```

```
lemma order_prod_mset:
  "0  $\notin\# P \implies$  order x (prod_mset P) = sum_mset (image_mset ( $\lambda p. \text{order}
x p$ ) P)"
  <proof>
```

```
lemma order_prod:
```

```
"( $\bigwedge x. x \in I \implies f x \neq 0$ )  $\implies$  order x (prod f I) = ( $\sum i \in I. \text{order } x$ 
(f i))"
<proof>
```

```
lemma order_linear_factor:
  assumes "a  $\neq$  0  $\vee$  b  $\neq$  0"
  shows "order x [:a, b:] = (if b * x + a = 0 then 1 else 0)"
<proof>
```

```
lemma order_linear_factor' [simp]:
  assumes "a  $\neq$  0  $\vee$  b  $\neq$  0" "b * x + a = 0"
  shows "order x [:a, b:] = 1"
<proof>
```

```
lemma degree_prod_mset_eq: "0  $\notin$  # P  $\implies$  degree (prod_mset P) = ( $\sum p \in \#P. \text{degree } p$ )"
  for P :: "'a::idom poly multiset"
<proof>
```

```
lemma degree_prod_list_eq: "0  $\notin$  set ps  $\implies$  degree (prod_list ps) = ( $\sum p \leftarrow ps. \text{degree } p$ )"
  for ps :: "'a::idom poly list"
<proof>
```

```
lemma order_conv_multiplicity:
  assumes "p  $\neq$  0"
  shows "order x p = multiplicity [:-x, 1:] p"
<proof>
```

1.3 Polynomials over algebraically closed fields

```
lemma irreducible_alg_closed_imp_degree_1:
  assumes "irreducible (p :: 'a :: alg_closed_field poly)"
  shows "degree p = 1"
<proof>
```

```
lemma prime_poly_alg_closedE:
  assumes "prime (q :: 'a :: {alg_closed_field, field_gcd} poly)"
  obtains c where "q = [:-c, 1]" "poly q c = 0"
<proof>
```

```
lemma prime_factors_alg_closed_poly_bij_betw:
  assumes "p  $\neq$  (0 :: 'a :: {alg_closed_field, field_gcd} poly)"
  shows "bij_betw ( $\lambda x. [:-x, 1:]$ ) {x. poly p x = 0} (prime_factors p)"
<proof>
```

```
lemma alg_closed_imp_factorization':
  assumes "p  $\neq$  (0 :: 'a :: alg_closed_field poly)"
  shows "p = smult (lead_coeff p) ( $\prod [x \mid \text{poly } p x = 0. [:-x, 1:]$ ) ^ order
```

$x\ p)$ "
 $\langle proof \rangle$

1.4 Complex polynomials and conjugation

lemma *complex_poly_real_coeffsE*:
 assumes "set (coeffs p) $\subseteq \mathbb{R}$ "
 obtains p' where " $p = \text{map_poly complex_of_real } p'$ "
 $\langle proof \rangle$

lemma *order_map_poly_cnj*:
 assumes " $p \neq 0$ "
 shows " $\text{order } x (\text{map_poly } \text{cnj } p) = \text{order } (\text{cnj } x) p$ "
 $\langle proof \rangle$

1.5 n -ary product rule for the derivative

lemma *has_field_derivative_prod_mset [derivative_intros]*:
 assumes " $\bigwedge x. x \in \# A \implies (f\ x \text{ has_field_derivative } f'\ x) \text{ (at } z)$ "
 shows " $((\lambda u. \prod_{x \in \# A}. f\ x\ u) \text{ has_field_derivative } (\sum_{x \in \# A}. f'\ x * (\prod_{y \in \# A - \{x\}}. f\ y\ z))) \text{ (at } z)$ "
 $\langle proof \rangle$

lemma *has_field_derivative_prod [derivative_intros]*:
 assumes " $\bigwedge x. x \in A \implies (f\ x \text{ has_field_derivative } f'\ x) \text{ (at } z)$ "
 shows " $((\lambda u. \prod_{x \in A}. f\ x\ u) \text{ has_field_derivative } (\sum_{x \in A}. f'\ x * (\prod_{y \in A - \{x\}}. f\ y\ z))) \text{ (at } z)$ "
 $\langle proof \rangle$

lemma *has_field_derivative_prod_mset'*:
 assumes " $\bigwedge x. x \in \# A \implies f\ x\ z \neq 0$ "
 assumes " $\bigwedge x. x \in \# A \implies (f\ x \text{ has_field_derivative } f'\ x) \text{ (at } z)$ "
 defines " $P \equiv (\lambda A\ u. \prod_{x \in \# A}. f\ x\ u)$ "
 shows " $(P\ A \text{ has_field_derivative } (P\ A\ z * (\sum_{x \in \# A}. f'\ x / f\ x\ z))) \text{ (at } z)$ "
 $\langle proof \rangle$

lemma *has_field_derivative_prod'*:
 assumes " $\bigwedge x. x \in A \implies f\ x\ z \neq 0$ "
 assumes " $\bigwedge x. x \in A \implies (f\ x \text{ has_field_derivative } f'\ x) \text{ (at } z)$ "
 defines " $P \equiv (\lambda A\ u. \prod_{x \in A}. f\ x\ u)$ "
 shows " $(P\ A \text{ has_field_derivative } (P\ A\ z * (\sum_{x \in A}. f'\ x / f\ x\ z))) \text{ (at } z)$ "
 $\langle proof \rangle$

1.6 Facts about complex numbers

lemma *Re_sum_mset*: " $\text{Re } (\text{sum_mset } X) = (\sum_{x \in \# X}. \text{Re } x)$ "
 $\langle proof \rangle$

```

lemma Im_sum_mset: "Im (sum_mset X) = ( $\sum_{x \in \#X}. \text{Im } x$ )"
  <proof>

lemma Re_sum_mset': "Re ( $\sum_{x \in \#X}. f\ x$ ) = ( $\sum_{x \in \#X}. \text{Re } (f\ x)$ )"
  <proof>

lemma Im_sum_mset': "Im ( $\sum_{x \in \#X}. f\ x$ ) = ( $\sum_{x \in \#X}. \text{Im } (f\ x)$ )"
  <proof>

lemma inverse_complex_altdef: "inverse z = cnj z / norm z ^ 2"
  <proof>

end

theory Polynomial_Crit_Geometry
imports
  "HOL-Computational_Algebra.Computational_Algebra"
  "HOL-Analysis.Analysis"
  Polynomial_Crit_Geometry_Library
begin

```

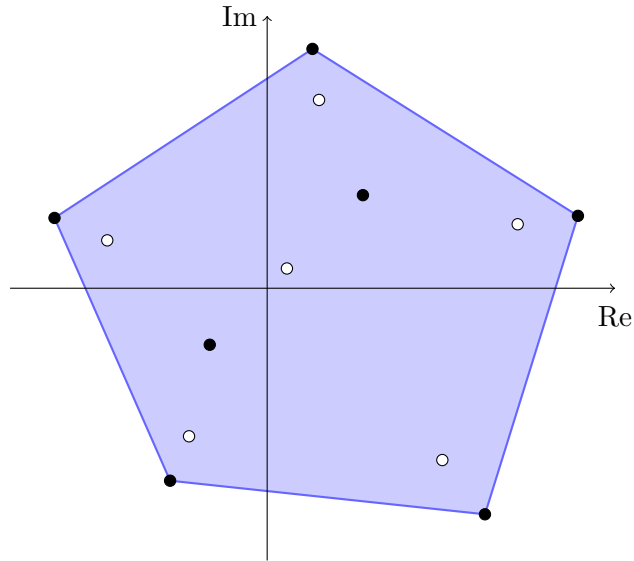


Figure 1: Example for the GauSS–Lucas Theorem: The roots (●) and critical points (○) of $x^7 - 2x^6 + x^5 + x^4 - (1 + i)x^3 - 15ix^2 - 4(1 - i)x - 7$. The critical points all lie inside the convex hull of the roots (□).

2 The GauSS–Lucas Theorem

The following result is known as the *GauSS–Lucas Theorem*: The critical points of a non-constant complex polynomial lie inside the convex hull of its roots.

The proof is relatively straightforward by writing the polynomial in the form

$$p(x) = \prod_{i=1}^n (x - x_i)^{a_i} ,$$

from which we get the derivative

$$p'(x) = p(x) \cdot \sum_{i=1}^n \frac{a_i}{x - x_i} .$$

With some more calculations, one can then see that every root x of p' can be written as

$$x = \sum_{i=1}^n \frac{u_i}{U} \cdot x_i$$

where $u_i = \frac{a_i}{|x - x_i|^2}$ and $U = \sum_{i=1}^n u_i$.

theorem `pderiv_roots_in_convex_hull`:

```
fixes p :: "complex poly"  
assumes "degree p  $\neq$  0"  
shows "{z. poly (pderiv p) z = 0}  $\subseteq$  convex hull {z. poly p z = 0}"  
<proof>
```

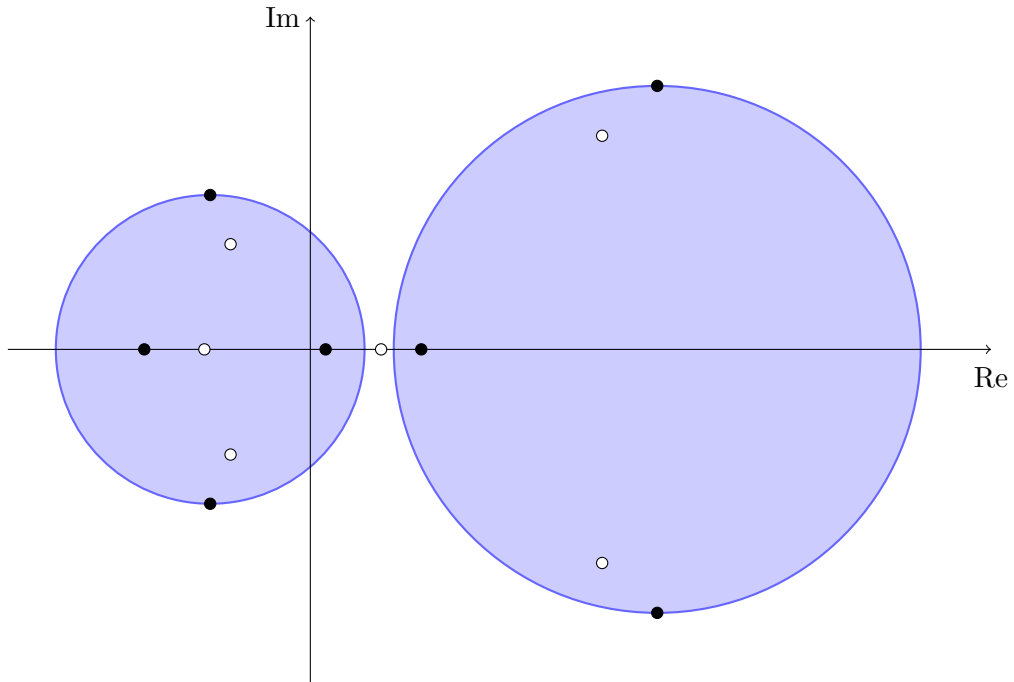


Figure 2: Example for Jensen's Theorem: The roots (\bullet) and critical points (\circ) of the polynomial $x^7 - 3x^6 + 2x^5 + 8x^4 + 10x^3 - 10x + 1$. It can be seen that all the non-real critical points lie inside a Jensen disc (\circ), whereas there can be real critical points that do *not* lie inside a Jensen disc.

3 Jensen's Theorem

For each root w of a real polynomial p , the Jensen disc of w is the smallest disc containing both w and \bar{w} , i.e. the disc with centre $\text{Re}(w)$ and radius $|\text{Im}(w)|$.

We now show that if p is a real polynomial, every non-real critical point of p lies inside a Jensen disc of one of its non-real roots.

definition `jensen_disc` :: "complex \Rightarrow complex set" where
`"jensen_disc w = cball (of_real (Re w)) |Im w|"`

theorem `pderiv_root_in_jensen_disc`:
`fixes p :: "complex poly"`
`assumes "set (coeffs p) \subseteq \mathbb{R} " and "degree p \neq 0"`
`assumes "poly (pderiv p) z = 0" and "z \notin \mathbb{R} "`
`shows " $\exists w. w \notin \mathbb{R} \wedge \text{poly } p w = 0 \wedge z \in \text{jensen_disc } w$ "`
`<proof>`

end

References

- [1] F. Enescu. Math 4444/6444 Polynomials 2, Lecture Notes, Lecture 9.
<https://math.gsu.edu/fenescu/fall2010/lec9-polyn-2010.pdf>, 2010.
- [2] M. Marden. *Geometry of Polynomials*. American Mathematical Society
Mathematical Surveys. American Mathematical Society, 1966.