# The Five Platonic Solids

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**Abstract**

This entry formalizes that there are exactly five Platonic solids: the tetrahedron, cube, octahedron, dodecahedron, and icosahedron. A Platonic solid is a convex polyhedron whose faces are congruent regular polygons, with the same number of edges meeting at each vertex. Euler's polyhedron formula is used to show that there are at most five Platonic solids, and each of the five, defined as the convex hull of its vertices, is then shown to satisfy the required properties. The formal proof is computational (slow!) and is ported from John Harrison's formalization in HOL Light [1]. Large language models were used to explain the intermediate steps of the HOL Light formalization and to write parts of the more complicated proofs.

## Contents

# 1 Convexity

**theory** *Convex-Euclidean-Space-More*
  **imports** *HOL−Analysis.Starlike*
**begin**

**lemma** *connected-Int-rel-frontier*:
  **assumes** *connected S*
    **and** $S \subseteq$ *affine hull T*
    **and** $S \cap T \neq \{\}$
    **and** $S - T \neq \{\}$
  **shows** $S \cap$ *rel-frontier T* $\neq \{\}$
⟨*proof*⟩

**end**
**theory** *Convex-More*
  **imports** *HOL−Analysis.Starlike*
**begin**

**lemma** *convex-hull-union-explicit*:
  **fixes** *S T* :: *′a::euclidean-space set*
  **assumes** *convex S*
    **and** *convex T*
  **shows** *convex hull* $(S \cup T) =$
      $S \cup T \cup$
      $\{(1 - u) *_R x + u *_R y \mid x\ y\ u.\ x \in S \land y \in T \land 0 \leq u \land u \leq 1\}$
  (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

**lemma** *convex-hull-union-nonempty-explicit*:
  **fixes** *S T* :: *′a::euclidean-space set*
  **assumes** *convex S*
    **and** $S \neq \{\}$
    **and** *convex T*
    **and** $T \neq \{\}$
  **shows** *convex hull* $(S \cup T) =$
      $\{(1 - u) *_R x + u *_R y \mid x\ y\ u.\ x \in S \land y \in T \land 0 \leq u \land u \leq 1\}$
  (**is** - = *?rhs*)
⟨*proof*⟩

**lemma** *convex-hull-isometry*:
  **assumes** *orthogonal-transformation f*
  **shows** $(\lambda x.\ c + f\ x)$ ‘ (*convex hull S*) = *convex hull* $((\lambda x.\ c + f\ x)$ ‘ $S)$
  ⟨*proof*⟩

**end**

## 2 Determinants

**theory** *Determinants-More*
  **imports** *HOL−Analysis.Starlike*
**begin**

**lemma** *hyperplane-subset-imp-scaled*:
  **fixes** $a$ :: $'a$::*euclidean-space*
  **assumes** $\{x.\ a \cdot x = b\} \subseteq \{x.\ a' \cdot x = b'\}$
    **and** $a \neq 0$
    **and** $a' \neq 0$
  **shows** $\exists\, c.\ a' = c *_R a \wedge b' = c * b \wedge c \neq 0$
⟨*proof*⟩

**lemma** *subset-hyperplanes*:
  **fixes** $a$ :: $'a$::*euclidean-space*
  **shows** $(\{x.\ a \cdot x = b\} \subseteq \{x.\ a' \cdot x = b'\}) =$
    $(\{x.\ a \cdot x = b\} = \{\} \vee \{x.\ a' \cdot x = b'\} = UNIV \vee$
    $\{x.\ a \cdot x = b\} = \{x.\ a' \cdot x = b'\})$
  (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

**end**

## 3 Polytopes

**theory** *Polytope-More*
  **imports**
    *Convex-Euclidean-Space-More*
    *HOL−Analysis.Analysis*
**begin**

### 3.1 Invariance under isometries

**lemma** *facet-of-translation-eq*:
  $((+)\ a$ ‘ $T$ *facet-of* $(+)\ a$ ‘ $S) = (T$ *facet-of* $S)$
  ⟨*proof*⟩

**lemma** *facets-of-translation*:
  **shows** $\{F.\ F$ *facet-of* $(+)\ a$ ‘ $S\} = ($‘$)\ ((+)\ a)$ ‘ $\{F.\ F$ *facet-of* $S\}$
⟨*proof*⟩

**lemma** *polyhedron-translation-eq*:
  **shows** *polyhedron* $S$ = *polyhedron* $((+)\ x$ ‘ $S)$
⟨*proof*⟩

**lemma** *face-of-isometry-eq*:
  **assumes** *orthogonal-transformation* $f$

**shows** $((\lambda x.\ c + f\ x)$ ‘ $T$ *face-of* $(\lambda x.\ c + f\ x)$ ‘ $S) \longleftrightarrow T$ *face-of* $S$
$\langle proof \rangle$

**lemma** *extreme-point-of-isometry-eq*:
**assumes** *orthogonal-transformation* $f$
**shows** $(c + f\ T$ *extreme-point-of* $(\lambda x.\ c + f\ x)$ ‘ $S) \longleftrightarrow T$ *extreme-point-of* $S$
$\langle proof \rangle$

**lemma** *aff-dim-isometry-eq*:
**assumes** *orthogonal-transformation* $f$
**shows** *aff-dim* $S =$ *aff-dim* $((\lambda x.\ c + f\ x)$ ‘ $S)$
$\langle proof \rangle$

### 3.2 Distinctness of facets

**lemma** *facets-of-polyhedron-explicit-distinct*:
**fixes** $S :: 'a ::$ *euclidean-space set*
**assumes** *finite*: *finite* $F$
**and** *seq*: $S =$ *affine hull* $S \cap \bigcap F$
**and** *faceq*: $\bigwedge h.\ h \in F \Longrightarrow a\ h \neq 0 \wedge h = \{x.\ a\ h \cdot x \leq b\ h\}$
**and** *psub*: $\bigwedge F'.\ F' \subset F \Longrightarrow S \subset$ *affine hull* $S \cap \bigcap F'$
**and** *h1-in*: $h1 \in F$
**and** *h2-in*: $h2 \in F$
**and** *eq*: $S \cap \{x.\ a\ h1 \cdot x = b\ h1\} = S \cap \{x.\ a\ h2 \cdot x = b\ h2\}$
**shows** $h1 = h2$
$\langle proof \rangle$

### 3.3 Ridges lie in two facets

**lemma** *polyhedron-ridge-two-facets-0*:
**fixes** $p :: 'a ::$ *euclidean-space set*
**assumes** *polyhedron* $p$
**and** $r$ *face-of* $p$
**and** $0 \in r$
**and** *aff-dim* $r =$ *aff-dim* $p - 2$
**shows** $\exists f1\ f2.\ f1$ *face-of* $p \wedge$ *aff-dim* $f1 =$ *aff-dim* $p - 1 \wedge$
$f2$ *face-of* $p \wedge$ *aff-dim* $f2 =$ *aff-dim* $p - 1 \wedge$
$f1 \neq f2 \wedge r \subseteq f1 \wedge r \subseteq f2 \wedge f1 \cap f2 = r \wedge$
$(\forall f.\ f$ *face-of* $p \wedge$ *aff-dim* $f =$ *aff-dim* $p - 1 \wedge r \subseteq f$
$\longrightarrow f = f1 \vee f = f2)$
$\langle proof \rangle$

**lemma** *polyhedron-ridge-two-facets*:
**fixes** $p :: 'a ::$ *euclidean-space set*
**assumes** *polyhedron* $p$
**and** $r$ *face-of* $p$
**and** $r \neq \{\}$
**and** *aff-dim* $r =$ *aff-dim* $p - 2$
**shows** $\exists f1\ f2.\ f1$ *face-of* $p \wedge$ *aff-dim* $f1 =$ *aff-dim* $p - 1 \wedge$
$f2$ *face-of* $p \wedge$ *aff-dim* $f2 =$ *aff-dim* $p - 1 \wedge$

$f1 \neq f2 \wedge r \subseteq f1 \wedge r \subseteq f2 \wedge r = f1 \cap f2 \wedge$
$(\forall f.\ f\ \mathit{face\text{-}of}\ p \wedge \mathit{aff\text{-}dim}\ f = \mathit{aff\text{-}dim}\ p - 1 \wedge r \subseteq f$
$\longrightarrow f = f1 \vee f = f2)$
$\langle proof \rangle$

## 3.4 Lower bounds on face counts

### 3.4.1 Vertices

**lemma** *polytope-vertex-lower-bound*:
**assumes** *polytope p*
**shows** $\mathit{aff\text{-}dim}\ p + 1 \leq \mathit{card}\ \{v.\ v\ \mathit{face\text{-}of}\ p \wedge \mathit{aff\text{-}dim}\ v = 0\}$
$\langle proof \rangle$

### 3.4.2 Facets

**lemma** *polytope-facet-lower-bound-0*:
**assumes** *polytope p*
**assumes** $\mathit{aff\text{-}dim}\ p \neq 0$
**assumes** *origin-in-p*: $0 \in p$
**shows** $\mathit{aff\text{-}dim}\ p + 1 \leq \mathit{card}\ \{f.\ f\ \mathit{facet\text{-}of}\ p\}$
$\langle proof \rangle$

**lemma** *polytope-facet-lower-bound*:
**assumes** *polytope p*
**assumes** $\mathit{aff\text{-}dim}\ p \neq 0$
**shows** $\mathit{aff\text{-}dim}\ p + 1 \leq \mathit{card}\ \{f.\ f\ \mathit{facet\text{-}of}\ p\}$
$\langle proof \rangle$

## 3.5 Double counting

**lemma** *multiple-counting*:
**fixes** $R :: 'a \Rightarrow 'b \Rightarrow \mathit{bool}$
**assumes** *finite s*
**assumes** *finite t*
**assumes** $\bigwedge x.\ x \in s \Longrightarrow \mathit{card}\ \{y \in t.\ R\ x\ y\} = m$
**assumes** $\bigwedge y.\ y \in t \Longrightarrow \mathit{card}\ \{x \in s.\ R\ x\ y\} = n$
**shows** $m * \mathit{card}\ s = n * \mathit{card}\ t$
$\langle proof \rangle$

### 3.5.1 $mF = 2E$

Let F be the number of facets and E be the number of ridges. If there are m ridges per facet, then m*F = 2*E.

**lemma** *facet-ridge-relation*:
**assumes** *polytope p*
**and** $\mathit{aff\text{-}dim}\ p \geq 2$
**and** *edges-per-face*: $\forall f.\ f\ \mathit{face\text{-}of}\ p \wedge \mathit{aff\text{-}dim}\ f = \mathit{aff\text{-}dim}\ p - 1 \longrightarrow$
$\mathit{card}\ \{e.\ e\ \mathit{face\text{-}of}\ p \wedge \mathit{aff\text{-}dim}\ e = \mathit{aff\text{-}dim}\ p - 2 \wedge e \subseteq f\} = m$
**shows** $m * \mathit{card}\ \{f.\ f\ \mathit{face\text{-}of}\ p \wedge \mathit{aff\text{-}dim}\ f = \mathit{aff\text{-}dim}\ p - 1\} =$

$2 * card \{e.\ e\ face\text{-}of\ p \wedge aff\text{-}dim\ e = aff\text{-}dim\ p - 2\}$
⟨proof⟩

### 3.5.2 $nV = 2E$

Let V be the number of vertices and E be the number of edges. If there are n edges per vertex, then n*V = 2*E.

**lemma** *edge-vertex-relation*:
**assumes** *polytope p*
**and** *edges-per-vert*: $\forall v.\ v\ face\text{-}of\ p \wedge aff\text{-}dim\ v = 0 \longrightarrow$
$card\ \{e.\ e\ face\text{-}of\ p \wedge aff\text{-}dim\ e = 1 \wedge v \subseteq e\} = n$
**shows** $n * card\ \{v.\ v\ face\text{-}of\ p \wedge aff\text{-}dim\ v = 0\} =$
$2 * card\ \{e.\ e\ face\text{-}of\ p \wedge aff\text{-}dim\ e = 1\}$
⟨proof⟩

**hide-fact** *multiple-counting*

## 3.6 Face containment

**lemma** *ridge-belongs-to-facet*:
**assumes** *e face-of p*
**and** $aff\text{-}dim\ e = aff\text{-}dim\ p - 2$
**and** $aff\text{-}dim\ p \geq 2$
**and** *polyhedron p*
**obtains** *f* **where** $f\ face\text{-}of\ p\ \ aff\text{-}dim\ f = aff\text{-}dim\ p - 1\ \ e\ face\text{-}of\ f$
⟨proof⟩

**lemma** *peak-belongs-to-ridge*:
**assumes** *v face-of p*
**and** $aff\text{-}dim\ v = aff\text{-}dim\ p - 3$
**and** $aff\text{-}dim\ p \geq 3$
**and** *polyhedron p*
**obtains** *e* **where** $e\ face\text{-}of\ p\ \ aff\text{-}dim\ e = aff\text{-}dim\ p - 2\ \ v\ face\text{-}of\ e$
⟨proof⟩

**end**

# 4 Computational helpers

**theory** *Computation*
**imports**
*Convex-More*
*Determinants-More*
*HOL−Analysis.Cross3*
*HOL−Analysis.Polytope*
*HOL−Eisbach.Eisbach*
*Polytope-More*

*HOL−Examples.Sqrt*
**begin**

## 4.1 Operations in $\mathbb{Q}(\sqrt{5})$

Numbers of the form $a + b\sqrt{5}$, where $a, b \in \mathbb{Q}$

**abbreviation** $\varphi$ :: *real*
**where** $\varphi \equiv (1 + sqrt\ 5)/2$

**lemma** *real-rat5-mul*:
$(a1 + b1 * sqrt\ 5) * (a2 + b2 * sqrt\ 5)$
$= (a1 * a2 + 5 * b1 * b2) + (a1 * b2 + a2 * b1) * sqrt\ 5$
⟨*proof*⟩

**lemma** *real-rat5-inv*:
**assumes** $a\hat{}2 \neq 5 * b\hat{}2$
**shows** *inverse* $(a + b * sqrt\ 5) = a\ /\ (a\hat{}2 - 5 * b\hat{}2) + -b\ /\ (a\hat{}2 - 5 * b\hat{}2)$
$*\ sqrt\ 5$
⟨*proof*⟩

**lemma** *real-rat5-div*:
**assumes** $a2\hat{}2 \neq 5 * b2\hat{}2$
**shows** $(a1 + b1 * sqrt\ 5)\ /\ (a2 + b2 * sqrt\ 5) =$
$((a1 * a2 - 5 * b1 * b2)\ /\ (a2\hat{}2 - 5 * b2\hat{}2)) +$
$((a2 * b1 - a1 * b2)\ /\ (a2\hat{}2 - 5 * b2\hat{}2)) * sqrt\ 5$
⟨*proof*⟩

**lemma** *real-rat5-le-sqrt-cases*:
$x \leq y * sqrt\ 5 \longleftrightarrow$
$(x \leq 0 \wedge 0 \leq y) \vee$
$(0 \leq x \wedge 0 \leq y \wedge x\hat{}2 \leq 5 * y\hat{}2) \vee$
$(x \leq 0 \wedge y \leq 0 \wedge 5 * y\hat{}2 \leq x\hat{}2)$
⟨*proof*⟩

**lemma** *real-rat5-le-comparison*:
$(a1 + b1 * sqrt\ 5) \leq (a2 + b2 * sqrt\ 5) \longleftrightarrow$
$(a1 \leq a2 \wedge b1 \leq b2) \vee$
$(a2 \leq a1 \wedge b1 \leq b2 \wedge (a1 - a2)\hat{}2 \leq 5 * (b2 - b1)\hat{}2) \vee$
$(a1 \leq a2 \wedge b2 \leq b1 \wedge 5 * (b2 - b1)\hat{}2 \leq (a1 - a2)\hat{}2)$
⟨*proof*⟩

Wrapper definition to avoiding oversimplification when applying simp

**definition** *rat5* :: *rat* ⇒ *rat* ⇒ *real*
**where** *rat5 a b* = *real-of-rat a* + *real-of-rat b* ∗ *sqrt 5*

**lemma** *rat5-add*: $rat5\ a1\ b1 + rat5\ a2\ b2 = rat5\ (a1 + a2)\ (b1 + b2)$
⟨*proof*⟩

**lemma** *rat5-sub*: $rat5\ a1\ b1 - rat5\ a2\ b2 = rat5\ (a1 - a2)\ (b1 - b2)$

⟨*proof*⟩

**lemma** *rat5-mul*: *rat5 a1 b1* $*$ *rat5 a2 b2* = *rat5* (*a1* $*$ *a2* + 5 $*$ *b1* $*$ *b2*) (*a1* $*$ *b2* + *b1* $*$ *a2*)
⟨*proof*⟩

**lemma** *rat5-div*:
**assumes** *a2* ≠ *0* ∨ *b2* ≠ *0*
**shows** *rat5 a1 b1* / *rat5 a2 b2* =
*rat5* ((*a1* $*$ *a2* − *5* $*$ *b1* $*$ *b2*) / (*a2*^*2* − *5* $*$ *b2*^*2*))
((*a2* $*$ *b1* − *a1* $*$ *b2*) / (*a2*^*2* − *5* $*$ *b2*^*2*))
⟨*proof*⟩

**lemma** *rat5-le*:
(*rat5 a1 b1* ≤ *rat5 a2 b2*) =
((*a1* ≤ *a2* ∧ *b1* ≤ *b2*) ∨
(*a2* ≤ *a1* ∧ *b1* ≤ *b2* ∧ (*a1* − *a2*)^*2* ≤ *5* $*$ (*b2* − *b1*)^*2*) ∨
(*a1* ≤ *a2* ∧ *b2* ≤ *b1* ∧ *5* $*$ (*b2* − *b1*)^*2* ≤ (*a1* − *a2*)^*2*))
⟨*proof*⟩

**lemma** *rat5-eq*: *rat5 a1 b1* = *rat5 a2 b2* ⟷ *a1* = *a2* ∧ *b1* = *b2*
⟨*proof*⟩

**lemma** *rat5-0-1*:
**shows** *rat5 0 0* = *0* *rat5 1 0* = *1* ⟨*proof*⟩

**lemma** *rat5-eq-0*:
*rat5 a1 b1* = *0* ⟷ *a1* = *0* ∧ *b1* = *0*
⟨*proof*⟩

**lemma** *rat5-eq-1*:
*rat5 a1 b1* = *1* ⟷ *a1* = *1* ∧ *b1* = *0*
⟨*proof*⟩

## 4.2 Operations in $\mathbb{R}^3$

**lemma** *vector3-add*:
**fixes** *x1 x2 x3 y1 y2 y3* :: *real*
**shows** *vector* [*x1*, *x2*, *x3*] + *vector* [*y1*, *y2*, *y3*] =
*vector* [*x1* + *y1*, *x2* + *y2*, *x3* + *y3*]
⟨*proof*⟩

**lemma** *vector3-sub*:
**fixes** *x1 x2 x3 y1 y2 y3* :: *real*
**shows** *vector* [*x1*, *x2*, *x3*] − *vector* [*y1*, *y2*, *y3*] =
*vector* [*x1* − *y1*, *x2* − *y2*, *x3* − *y3*]
⟨*proof*⟩

**unbundle** *cross3-syntax*

**lemma** *vector3-cross*:
**fixes** *x1 x2 x3 y1 y2 y3* :: *real*
**shows** *vector* [*x1*, *x2*, *x3*] × *vector* [*y1*, *y2*, *y3*] =
(*vector* [*x2* ∗ *y3* − *x3* ∗ *y2*, *x3* ∗ *y1* − *x1* ∗ *y3*, *x1* ∗ *y2* − *x2* ∗ *y1*] :: *real*^*3*)
⟨*proof*⟩

**lemma** *vector3-eq-0*:
**fixes** *x1 x2 x3* :: *real*
**shows** (*vector* [*x1*, *x2*, *x3*] :: *real*^*3*) = *0* ⟷ *x1* = *0* ∧ *x2* = *0* ∧ *x3* = *0*
⟨*proof*⟩

**lemma** *vector3-eq*:
**fixes** *x1 x2 x3 y1 y2 y3* :: *real*
**shows** (*vector* [*x1*, *x2*, *x3*] :: *real*^*3*) = *vector* [*y1*, *y2*, *y3*] ⟷ *x1* = *y1* ∧ *x2* = *y2* ∧ *x3* = *y3*
⟨*proof*⟩

**lemma** *vector3-neq*:
**shows** ((*vector* [*x1*, *x2*, *x3*]::*real*^*3*) ≠ *vector* [*y1*, *y2*, *y3*]) =
(*x1* ≠ *y1* ∨ *x2* ≠ *y2* ∨ *x3* ≠ *y3*)
⟨*proof*⟩

**lemma** *vector3-dot*:
**fixes** *x1 x2 x3 y1 y2 y3* :: *real*
**shows** (*vector* [*x1*, *x2*, *x3*] :: *real*^*3*) · *vector* [*y1*, *y2*, *y3*] =
*x1*∗*y1* + *x2*∗*y2* + *x3*∗*y3*
⟨*proof*⟩

## 4.3 Computation of faces

### 4.3.1 2-faces

**lemma** *compute-faces-2*:
**fixes** *F S* :: (*real*^*3*) *set*
**assumes** *finite S*
**shows** (*F face-of* (*convex hull S*) ∧ *aff-dim F* = *2*) =
(∃ *x*∈*S*. ∃ *y*∈*S*. ∃ *z*∈*S*.
*let a* = (*z* − *x*) × (*y* − *x*) *in*
*a* ≠ *0* ∧
(*let b* = *a* · *x in*
((∀ *w*∈*S*. *a* · *w* ≤ *b*) ∨ (∀ *w*∈*S*. *a* · *w* ≥ *b*)) ∧
*F* = *convex hull* (*S* ∩ {*x*. *a* · *x* = *b*})))
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

**lemma** *set-choice-decomposition-1*:
**assumes** *sym-yz*: ⋀*x y z*. *Q x y z* ⟹ *Q x z y*
**and** *sym-xy*: ⋀*x y z*. *Q x y z* ⟹ *Q y x z*

**and** *degen*: $\bigwedge x\ z.\ \neg\ Q\ x\ x\ z$
**shows** $(\exists\, x{\in}insert\ v\ S.\ \exists\, y{\in}insert\ v\ S.\ \exists\, z{\in}insert\ v\ S.\ Q\ x\ y\ z) \longleftrightarrow$
$(\exists\, y{\in}S.\ \exists\, z{\in}S.\ Q\ v\ y\ z) \vee (\exists\, x{\in}S.\ \exists\, y{\in}S.\ \exists\, z{\in}S.\ Q\ x\ y\ z)$
$\langle proof \rangle$

**lemma** *compute-faces-2-step-1*:
**fixes** $F\ S\ v$ **and** $T :: (real^\frown 3)\ set$
**defines** $Q \equiv (\lambda x\ y\ z.\ let\ a = (z - x) \times (y - x)\ in$
$a \neq 0 \wedge$
$(let\ b = a \cdot x\ in$
$((\forall\, w{\in}T.\ a \cdot w \leq b) \vee (\forall\, w{\in}T.\ a \cdot w \geq b)) \wedge$
$F = convex\ hull\ (T \cap \{k.\ a \cdot k = b\})))$
**shows** $(\exists\, x{\in}\ (insert\ v\ S).\ \exists\, y \in (insert\ v\ S).\ \exists\, z \in (insert\ v\ S).\ Q\ x\ y\ z) =$
$((\exists\, y{\in}S.\ \exists\, z{\in}S.\ Q\ v\ y\ z) \vee (\exists\, x{\in}S.\ \exists\, y{\in}S.\ \exists\, z{\in}S.\ Q\ x\ y\ z))$
(**is** *?lhs = ?rhs*)
$\langle proof \rangle$

**lemma** *set-choice-decomposition-2*:
**assumes** *sym*: $\bigwedge x\ y.\ Q\ x\ y \Longrightarrow Q\ y\ x$
**and** *degen*: $\bigwedge x.\ \neg\ Q\ x\ x$
**shows** $(\exists\, y{\in}insert\ u\ S.\ \exists\, z{\in}insert\ u\ S.\ Q\ y\ z) \longleftrightarrow$
$(\exists\, z{\in}S.\ Q\ u\ z) \vee (\exists\, y{\in}S.\ \exists\, z{\in}S.\ Q\ y\ z)$
$\langle proof \rangle$

**lemma** *compute-faces-2-step-2*:
**fixes** $F\ u\ v\ S$ **and** $T :: (real^\frown 3)\ set$
**defines** $Q \equiv (\lambda y\ z.\ let\ a = (z - v) \times (y - v)\ in$
$a \neq 0 \wedge$
$(let\ b = a \cdot v\ in$
$((\forall\, w{\in}T.\ a \cdot w \leq b) \vee (\forall\, w{\in}T.\ a \cdot w \geq b)) \wedge$
$F = convex\ hull\ (T \cap \{k.\ a \cdot k = b\})))$
**shows** $(\exists\, y \in (insert\ u\ S).\ \exists\, z \in (insert\ u\ S).\ Q\ y\ z) =$
$((\exists\, z{\in}S.\ Q\ u\ z) \vee (\exists\, y{\in}S.\ \exists\, z{\in}S.\ Q\ y\ z))$
(**is** *?lhs = ?rhs*)
$\langle proof \rangle$

### 4.3.2 1-faces

**lemma** *compute-faces-1*:
**fixes** $F\ n\ d$ **and** $S :: (real^\frown 3)\ set$
**assumes** $\bigwedge x.\ x{\in}S \Longrightarrow n \cdot x = d$
**and** *finite S*
**and** $n \neq 0$
**shows** $(F\ face\text{-}of\ (convex\ hull\ S) \wedge aff\text{-}dim\ F = 1) =$
$(\exists\, x{\in}S.\ \exists\, y{\in}S.$
$let\ a = n \times (y - x)\ in$
$a \neq 0 \wedge$
$(let\ b = a \cdot x\ in$
$((\forall\, w{\in}S.\ a \cdot w \leq b) \vee (\forall\, w{\in}S.\ a \cdot w \geq b)) \wedge$

$F = convex\ hull\ (S \cap \{w.\ a \cdot w = b\})))$
(**is** $?lhs = ?rhs$)
$\langle proof \rangle$

**lemma** *compute-faces-1-step*:
**fixes** $F\ n\ u\ S$ **and** $T :: (real^\frown 3)\ set$
**defines** $Q \equiv (\lambda x\ y.\ let\ a = n \times (y - x)\ in$
$a \neq 0 \wedge$
$(let\ b = a \cdot x\ in$
$((\forall w \in T.\ a \cdot w \leq b) \vee (\forall w \in T.\ a \cdot w \geq b)) \wedge$
$F = convex\ hull\ (T \cap \{k.\ a \cdot k = b\})))$
**shows** $(\exists y \in (insert\ u\ S).\ \exists z \in (insert\ u\ S).\ Q\ y\ z) =$
$((\exists z \in S.\ Q\ u\ z) \vee (\exists y \in S.\ \exists z \in S.\ Q\ y\ z))$
(**is** $?lhs = ?rhs$)
$\langle proof \rangle$

**method** *facet-edges* **uses** *plane n-nz defs* =
(*simp only*: *compute-faces-1*[*OF plane - n-nz*] *finite-insert finite.emptyI*,
*simp only*: *compute-faces-1-step bex-empty simp-thms*(*31*),
*simp only*: *empty-iff ball-insert bex-empty bex-simps*(*5*) *ball-simps*(*5*) *simp-thms*(*21*,*31*),
*simp add*: *defs vector3-sub vector3-cross vector3-dot vector3-eq-0 Let-def*
*rat5-mul rat5-add rat5-sub rat5-eq rat5-eq-0 rat5-le*)

## 4.4 Full-dimensionality

**lemma** *polytope-3D*:
**fixes** $S :: (real^\frown 3)\ set$
**assumes** $(z - x) \times (y - x) \neq 0$
**and** $\exists w \in S.\ ((z - x) \times (y - x)) \cdot w \neq ((z - x) \times (y - x)) \cdot x$
**shows** $aff\text{-}dim\ (convex\ hull\ (insert\ x\ (insert\ y\ (insert\ z\ S)))) = 3$
(**is** $aff\text{-}dim\ ?hull = 3$)
$\langle proof \rangle$

## 4.5 Relations between faces of different dimensions

These lemmas are used when formalizing the number of edges per face and number of edges per vertex.

**corollary** *edge-belongs-to-face*:
**assumes** *e face-of p*
**and** $aff\text{-}dim\ e = 1$
**and** $aff\text{-}dim\ p = 3$
**and** *polyhedron p*
**obtains** *f* **where** *f face-of p* $aff\text{-}dim\ f = 2$ *e face-of f*
$\langle proof \rangle$

**corollary** *vertex-belongs-to-edge*:
**assumes** *v face-of p*
**and** $aff\text{-}dim\ v = 0$
**and** $aff\text{-}dim\ p = 3$

**and** *polyhedron p*
**obtains** *e* **where** *e face-of p aff-dim e* = *1 v face-of e*
⟨*proof*⟩

**lemma** *vertices-of-edge*:
**assumes** $e = convex\ hull\ \{a,\ b\}$
**and** *v face-of e*
**and** *aff-dim v* = *0*
**shows** $v = \{a\} \lor v = \{b\}$
⟨*proof*⟩

**lemma** *vertex-edge-set-eq*:
**assumes** *v face-of S*
**shows** *card* {*e. e face-of S* ∧ *aff-dim e* = *1* ∧ *v* ⊆ *e*} =
*card* {*e. e face-of S* ∧ *aff-dim e* = *1* ∧ *v face-of e*}
⟨*proof*⟩

**lemma** *face-edge-set-eq*:
**assumes** *f face-of S*
**shows** *card* {*e. e face-of S* ∧ *aff-dim e* = *1* ∧ *e* ⊆ *f*} =
*card* {*e. e face-of f* ∧ *aff-dim e* = *1*}
⟨*proof*⟩

**lemma** *set-disj-union-distr*:
**shows** $\{e.\ (P\ e \lor Q\ e) \land R\ e\} = \{e.\ P\ e \land R\ e\} \cup \{e.\ Q\ e \land R\ e\}$
⟨*proof*⟩

**lemma** *v-sing-face-of-e*:
**shows** (*e* = *a* ∧ {*x*} *face-of e*) = (*e* = *a* ∧ *x extreme-point-of a*)
⟨*proof*⟩

**lemma** *card-3-iff2*:
**shows** $(card\ \{x,\ y,\ z\} = 3) = (x \neq y \land x \neq z \land y \neq z)$
⟨*proof*⟩

**lemma** *card-4-iff2*:
**shows** $(card\ \{w,\ x,\ y,\ z\} = 4) = (w \neq x \land w \neq y \land w \neq z \land x \neq y \land x \neq z \land y \neq z)$
⟨*proof*⟩

**lemma** *card-5-iff*:
**shows** $card\ \{u,w,x,y,z\} = 5 \longleftrightarrow (u \neq w \land u \neq x \land u \neq y \land u \neq z \land w \neq x \land w \neq y \land w \neq z \land x \neq y \land y \neq z \land x \neq z)$
⟨*proof*⟩

**method** *edges-per-vertex* =
(*unfold set-disj-union-distr segment-convex-hull*[*symmetric*] *v-sing-face-of-e*;
*simp-all add*: *extreme-point-of-segment vector3-eq rat5-eq*;
*unfold card-3-iff2 card-4-iff2 card-5-iff closed-segment-eq*;

*simp-all add*: *vector3-eq rat5-eq doubleton-eq-iff*)

**lemma** *set-cases*:
**shows** $\{x.\ x = a\} = \{a\}$ $\{x.\ x = a \lor P\ x\} = \mathit{insert}\ a\ \{x.\ P\ x\}$
$\langle proof \rangle$

## 4.6 Congruence

**definition** *congruent* :: $['a::\mathit{euclidean\text{-}space\ set}, 'a\ \mathit{set}] \Rightarrow \mathit{bool}$ (**infixr** ‹(*congruent*)› *50*)
**where** $S$ *congruent* $T \longleftrightarrow (\exists\, c\ f.\ \mathit{orthogonal\text{-}transformation}\ f \land T = (\lambda x.\ c + f\ x)$ ‘ $S)$

**lemma** *congruent-refl*:
**shows** $S$ *congruent* $S$
$\langle proof \rangle$

**lemma** *congruent-sym*:
**shows** $S$ *congruent* $T \longleftrightarrow T$ *congruent* $S$
$\langle proof \rangle$

**lemma** *congruent-trans*:
**assumes** $R$ *congruent* $S$
**assumes** $S$ *congruent* $T$
**shows** $R$ *congruent* $T$
$\langle proof \rangle$

**lemma** *congruent-set*:
**assumes** $\forall\, t\in(\mathit{insert}\ s\ S).\ s$ *congruent* $t$
**assumes** $x \in (\mathit{insert}\ s\ S)$
**assumes** $y \in (\mathit{insert}\ s\ S)$
**shows** $x$ *congruent* $y$
$\langle proof \rangle$

### 4.6.1 Congruence of segments

**lemma** *congruent-segments*:
**fixes** $a\ b\ c\ d :: \mathit{real}\hat{}'n$
**assumes** $\mathit{dist}\ a\ b = \mathit{dist}\ c\ d$
**shows** (*closed-segment* $a$ $b$) *congruent* (*closed-segment* $c$ $d$)
$\langle proof \rangle$

### 4.6.2 Congruence of 2-faces

**lemma** *congruent-simple*:
**assumes** $\exists\, A::\mathit{real}\hat{}3\hat{}3.\ \mathit{orthogonal\text{-}matrix}\ A \land (\lambda x.\ A *v\ x)$ ‘ $S = T$
**shows** (*convex hull* $S$) *congruent* (*convex hull* $T$)
$\langle proof \rangle$

**lemma** *matrix-vector-mul-3*:

**shows** $(vector\ [vector[a11,\ a12,\ a13],$
$\quad vector[a21,\ a22,\ a23],$
$\quad vector[a31,\ a32,\ a33]]::real\hat{\ }3\hat{\ }3)\ *v\ vector\ [x1,\ x2,\ x3] =$
$\quad vector\ [a11 * x1 + a12 * x2 + a13 * x3,$
$\quad a21 * x1 + a22 * x2 + a23 * x3,$
$\quad a31 * x1 + a32 * x2 + a33 * x3]$
$\langle proof \rangle$

**lemma** *matrix-lemma*:
**fixes** $A :: real\hat{\ }3\hat{\ }3$
**shows** $(A *v\ x1 = x2 \wedge A *v\ y1 = y2 \wedge A *v\ z1 = z2) =$
$((vector\ [x1,\ y1,\ z1]::real\hat{\ }3\hat{\ }3)\ *v\ (row\ 1\ A) = vector[x2\$1,\ y2\$1,\ z2\$1] \wedge$
$(vector\ [x1,\ y1,\ z1]::real\hat{\ }3\hat{\ }3)\ *v\ (row\ 2\ A) = vector[x2\$2,\ y2\$2,\ z2\$2] \wedge$
$(vector\ [x1,\ y1,\ z1]::real\hat{\ }3\hat{\ }3)\ *v\ (row\ 3\ A) = vector[x2\$3,\ y2\$3,\ z2\$3])$
$\langle proof \rangle$

**lemma** *matrix-by-cramer-lemma*:
**fixes** $A :: real\hat{\ }3\hat{\ }3$
**assumes** $det\ (vector[x1,\ y1,\ z1]::real\hat{\ }3\hat{\ }3) \neq 0$
**shows** $(A *v\ x1 = x2 \wedge A *v\ y1 = y2 \wedge A *v\ z1 = z2) =$
$(A = (\chi\ m\ k.\ det\ ((\chi\ i\ j.$
$\quad if\ j = k$
$\quad then\ (vector\ [x2\$m,\ y2\$m,\ z2\$m]::real\hat{\ }3)\$i$
$\quad else\ (vector\ [x1,\ y1,\ z1]::real\hat{\ }3\hat{\ }3)\$i\$j)$
$\quad ::real\hat{\ }3\hat{\ }3)\ /$
$\quad det\ (vector\ [x1,\ y1,\ z1]::real\hat{\ }3\hat{\ }3)))$
$\langle proof \rangle$

**definition** $mk\text{-}matrix33 :: (real\hat{\ }3) \Rightarrow (real\hat{\ }3) \Rightarrow (real\hat{\ }3) \Rightarrow (real\hat{\ }3) \Rightarrow (real\hat{\ }3) \Rightarrow (real\hat{\ }3) \Rightarrow real\hat{\ }3\hat{\ }3$
**where** $mk\text{-}matrix33\ x1\ y1\ z1\ x2\ y2\ z2 = (let\ d = det\ (vector[x1,\ y1,\ z1]::real\hat{\ }3\hat{\ }3)$
$in\ vector\ [vector\ [$
$(x2\$1 * y1\$2 * z1\$3 +$
$x1\$2 * y1\$3 * z2\$1 +$
$x1\$3 * y2\$1 * z1\$2 -$
$x2\$1 * y1\$3 * z1\$2 -$
$x1\$2 * y2\$1 * z1\$3 -$
$x1\$3 * y1\$2 * z2\$1)\ /\ d,$
$(x1\$1 * y2\$1 * z1\$3 +$
$x2\$1 * y1\$3 * z1\$1 +$
$x1\$3 * y1\$1 * z2\$1 -$
$x1\$1 * y1\$3 * z2\$1 -$
$x2\$1 * y1\$1 * z1\$3 -$
$x1\$3 * y2\$1 * z1\$1)\ /\ d,$
$(x1\$1 * y1\$2 * z2\$1 +$
$x1\$2 * y2\$1 * z1\$1 +$
$x2\$1 * y1\$1 * z1\$2 -$
$x1\$1 * y2\$1 * z1\$2 -$
$x1\$2 * y1\$1 * z2\$1 -$

$x2\$1 * y1\$2 * z1\$1)$ / $d]$, $vector$ [
$(x2\$2 * y1\$2 * z1\$3 +$
$x1\$2 * y1\$3 * z2\$2 +$
$x1\$3 * y2\$2 * z1\$2 -$
$x2\$2 * y1\$3 * z1\$2 -$
$x1\$2 * y2\$2 * z1\$3 -$
$x1\$3 * y1\$2 * z2\$2)$ / $d$,
$(x1\$1 * y2\$2 * z1\$3 +$
$x2\$2 * y1\$3 * z1\$1 +$
$x1\$3 * y1\$1 * z2\$2 -$
$x1\$1 * y1\$3 * z2\$2 -$
$x2\$2 * y1\$1 * z1\$3 -$
$x1\$3 * y2\$2 * z1\$1)$ / $d$,
$(x1\$1 * y1\$2 * z2\$2 +$
$x1\$2 * y2\$2 * z1\$1 +$
$x2\$2 * y1\$1 * z1\$2 -$
$x1\$1 * y2\$2 * z1\$2 -$
$x1\$2 * y1\$1 * z2\$2 -$
$x2\$2 * y1\$2 * z1\$1)$ / $d]$, $vector$ [
$(x2\$3 * y1\$2 * z1\$3 +$
$x1\$2 * y1\$3 * z2\$3 +$
$x1\$3 * y2\$3 * z1\$2 -$
$x2\$3 * y1\$3 * z1\$2 -$
$x1\$2 * y2\$3 * z1\$3 -$
$x1\$3 * y1\$2 * z2\$3)$ / $d$,
$(x1\$1 * y2\$3 * z1\$3 +$
$x2\$3 * y1\$3 * z1\$1 +$
$x1\$3 * y1\$1 * z2\$3 -$
$x1\$1 * y1\$3 * z2\$3 -$
$x2\$3 * y1\$1 * z1\$3 -$
$x1\$3 * y2\$3 * z1\$1)$ / $d$,
$(x1\$1 * y1\$2 * z2\$3 +$
$x1\$2 * y2\$3 * z1\$1 +$
$x2\$3 * y1\$1 * z1\$2 -$
$x1\$1 * y2\$3 * z1\$2 -$
$x1\$2 * y1\$1 * z2\$3 -$
$x2\$3 * y1\$2 * z1\$1)$ / $d]])$

**lemma** *set3-neqI*:
**assumes** $a \notin \{x, y, z\} \vee b \notin \{x, y, z\} \vee c \notin \{x, y, z\}$
**shows** $\{a, b, c\} \neq \{x, y, z\}$
$\langle proof \rangle$

**lemma** *set4-neqI*:
**assumes** $a \notin \{w, x, y, z\} \vee b \notin \{w, x, y, z\} \vee c \notin \{w, x, y, z\} \vee d \notin \{w, x, y, z\}$
**shows** $\{a, b, c, d\} \neq \{w, x, y, z\}$
$\langle proof \rangle$

**lemma** *orthogonal-matrix33*:
  **fixes** $Q$ :: *real^3^3*
  **shows** (*orthogonal-matrix* $Q$) =
    ($Q\$1\$1 * Q\$1\$1 + Q\$2\$1 * Q\$2\$1 + Q\$3\$1 * Q\$3\$1 = 1 \wedge$
    $Q\$1\$1 * Q\$1\$2 + Q\$2\$1 * Q\$2\$2 + Q\$3\$1 * Q\$3\$2 = 0 \wedge$
    $Q\$1\$1 * Q\$1\$3 + Q\$2\$1 * Q\$2\$3 + Q\$3\$1 * Q\$3\$3 = 0 \wedge$
    $Q\$1\$2 * Q\$1\$1 + Q\$2\$2 * Q\$2\$1 + Q\$3\$2 * Q\$3\$1 = 0 \wedge$
    $Q\$1\$2 * Q\$1\$2 + Q\$2\$2 * Q\$2\$2 + Q\$3\$2 * Q\$3\$2 = 1 \wedge$
    $Q\$1\$2 * Q\$1\$3 + Q\$2\$2 * Q\$2\$3 + Q\$3\$2 * Q\$3\$3 = 0 \wedge$
    $Q\$1\$3 * Q\$1\$1 + Q\$2\$3 * Q\$2\$1 + Q\$3\$3 * Q\$3\$1 = 0 \wedge$
    $Q\$1\$3 * Q\$1\$2 + Q\$2\$3 * Q\$2\$2 + Q\$3\$3 * Q\$3\$2 = 0 \wedge$
    $Q\$1\$3 * Q\$1\$3 + Q\$2\$3 * Q\$2\$3 + Q\$3\$3 * Q\$3\$3 = 1$)
  ⟨*proof*⟩

**lemma** *matrix-by-cramer*:
  **fixes** $A$ :: *real^3^3*
  **assumes** $A$ = *mk-matrix33 x1 y1 z1 x2 y2 z2*
  **assumes** *det* (*vector* [*x1*, *y1*, *z1*]::*real^3^3*) $\neq 0$
  **shows** $A$ **v x1* = *x2* $A$ **v y1* = *y2* $A$ **v z1* = *z2*
  ⟨*proof*⟩

**lemma** *matrix-by-cramer-orthog*:
  **fixes** *x1 y1 z1 x2 y2 z2* :: *real^3*
  **assumes** *orthogonal-matrix* (*mk-matrix33 x1 y1 z1 x2 y2 z2*)
  **assumes** *det* (*vector* [*x1*, *y1*, *z1*]::*real^3^3*) $\neq 0$
  **shows** *orthogonal-matrix* (*mk-matrix33 x1 y1 z1 x2 y2 z2*) $\wedge$
    (**v*) (*mk-matrix33 x1 y1 z1 x2 y2 z2*) ‘ {*x1*, *y1*, *z1*} = {*x2*, *y2*, *z2*}
  ⟨*proof*⟩

Lookup table mapping a face to three vertices on that face that are adjacent (vertices that define a pair of intersecting edges)

**definition** *adjacency-table* :: ((*real*, *3*) *vec set* × ((*real*, *3*) *vec* × (*real*, *3*) *vec* × (*real*, *3*) *vec*)) *list* **where**
  *adjacency-table* = [
    ({*vector*[−1, −1, 1], *vector*[−1, 1, −1], *vector*[1, −1, −1]}, (*vector* [−1, −1, 1], *vector* [−1, 1, −1], *vector* [1, −1, −1])),
    ({*vector*[−1, −1, 1], *vector*[−1, 1, −1], *vector*[1, 1, 1]}, (*vector*[−1, −1, 1], *vector*[−1, 1, − 1], *vector*[1, 1, 1])),
    ({*vector*[−1, −1, 1], *vector*[1, − 1, − 1], *vector*[1, 1, 1]}, (*vector*[−1, −1, 1], *vector*[1, − 1, − 1], *vector*[1, 1, 1])),
    ({*vector*[−1, 1, −1], *vector*[1, − 1, − 1], *vector*[1, 1, 1]}, (*vector*[−1, 1, −1], *vector*[1, − 1, − 1], *vector*[1, 1, 1])),

    ({*vector*[−1, −1, −1], *vector*[−1, −1, 1], *vector*[−1, 1, −1], *vector*[−1, 1, 1]}, (*vector*[−1, −1, −1], *vector*[−1, −1, 1], *vector*[−1, 1, 1])),
    ({*vector*[−1, −1, −1], *vector*[−1, −1, 1], *vector*[1, −1, −1], *vector*[1, −1, 1]}, (*vector*[−1, −1, −1], *vector*[−1, −1, 1], *vector*[1, −1, 1])),
    ({*vector*[−1, −1, −1], *vector*[−1, 1, −1], *vector*[1, −1, −1], *vector*[1, 1,

−1]}, (vector[−1, −1, −1], vector[−1, 1, −1], vector[1, 1, −1])),

({vector[−1, −1, 1], vector[−1, 1, 1], vector[1, −1, 1], vector[1, 1, 1]}, (vector[−1, −1, 1], vector[−1, 1, 1], vector[1, 1, 1])),

({vector[−1, 1, −1], vector[−1, 1, 1], vector[1, 1, −1], vector[1, 1, 1]}, (vector[−1, 1, −1], vector[−1, 1, 1], vector[1, 1, 1])),

({vector[1, −1, −1], vector[1, −1, 1], vector[1, 1, −1], vector[1, 1, 1]}, (vector[1, −1, −1], vector[1, −1, 1], vector[1, 1, 1])),

({vector[−1, 0, 0], vector[0, −1, 0], vector[0, 0, −1]}, (vector[−1, 0, 0], vector[0, −1, 0], vector[0, 0, −1])),

({vector[−1, 0, 0], vector[0, −1, 0], vector[0, 0, 1]}, (vector[−1, 0, 0], vector[0, −1, 0], vector[0, 0, 1])),

({vector[−1, 0, 0], vector[0, 1, 0], vector[0, 0, −1]}, (vector[−1, 0, 0], vector[0, 1, 0], vector[0, 0, −1])),

({vector[−1, 0, 0], vector[0, 1, 0], vector[0, 0, 1]}, (vector[−1, 0, 0], vector[0, 1, 0], vector[0, 0, 1])),

({vector[1, 0, 0], vector[0, −1, 0], vector[0, 0, −1]}, (vector[1, 0, 0], vector[0, −1, 0], vector[0, 0, −1])),

({vector[1, 0, 0], vector[0, −1, 0], vector[0, 0, 1]}, (vector[1, 0, 0], vector[0, −1, 0], vector[0, 0, 1])),

({vector[1, 0, 0], vector[0, 1, 0], vector[0, 0, −1]}, (vector[1, 0, 0], vector[0, 1, 0], vector[0, 0, −1])),

({vector[1, 0, 0], vector[0, 1, 0], vector[0, 0, 1]}, (vector[1, 0, 0], vector[0, 1, 0], vector[0, 0, 1])),

({vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1/2) (1/2)], vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 (−1) 0, rat5 (−1) 0, rat5 (−1) 0], vector[rat5 (−1) 0, rat5 (−1) 0, rat5 1 0]}, (vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1/2) (1/2)], vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 (−1) 0, rat5 (−1) 0, rat5 1 0])),

({vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1/2) (1/2)], vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 (1/2) (−1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 (−1) 0, rat5 1 0, rat5 (−1) 0], vector[rat5 (−1) 0, rat5 1 0, rat5 1 0]}, (vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1/2) (1/2)], vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 (−1) 0, rat5 1 0, rat5 1 0])),

({vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1/2) (1/2)], vector[rat5 (−1) 0, rat5 (−1) 0, rat5 1 0], vector[rat5 (−1) 0, rat5 1 0, rat5 1 0], vector[rat5 0 0, rat5 (−1/2) (1/2), rat5 (1/2) (1/2)], vector[rat5 0 0, rat5 (1/2) (−1/2), rat5 (1/2) (1/2)]}, (vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1/2) (1/2)], vector[rat5 (−1) 0, rat5 (−1) 0, rat5 1 0], vector[rat5 (−1) 0, rat5 1 0, rat5 1 0])),

({vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 (−1) 0, rat5 (−1) 0, rat5 (−1) 0], vector[rat5 (−1) 0, rat5 1 0, rat5 (−1) 0], vector[rat5 0 0, rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2)], vector[rat5 0 0, rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2)]}, (vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 (−1) 0, rat5 (−1) 0, rat5 (−1) 0], vector[rat5 (−1) 0, rat5 1 0, rat5 (−1) 0])), ({vector[rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 (−1) 0, rat5 (−1) 0, rat5 (−1) 0], vector[rat5 1 0, rat5 (−1) 0, rat5 (−1) 0], vector[rat5 0 0, rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2)]}, (vector[rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 1 0, rat5 (−1) 0, rat5 (−1) 0])),

({vector[rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 (−1) 0, rat5 (−1) 0, rat5 1 0], vector[rat5 1 0, rat5 (−1) 0, rat5 1 0], vector[rat5 0 0, rat5 (1/2) (−1/2), rat5 (1/2) (1/2)]}, (vector[rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 1 0, rat5 (−1) 0, rat5 1 0])),

({vector[rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1/2) (1/2)], vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 1 0, rat5 (−1) 0, rat5 (−1) 0], vector[rat5 1 0, rat5 (−1) 0, rat5 1 0]}, (vector[rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 1 0, rat5 (−1) 0, rat5 (−1) 0], vector[rat5 1 0, rat5 (−1) 0, rat5 1 0])),

({vector[rat5 (−1/2) (1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 (1/2) (−1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 (−1) 0, rat5 1 0, rat5 (−1) 0], vector[rat5 1 0, rat5 1 0, rat5 (−1) 0], vector[rat5 0 0, rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2)]}, (vector[rat5 (−1/2) (1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 (1/2) (−1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 1 0, rat5 1 0, rat5 (−1) 0])),

({vector[rat5 (−1/2) (1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 (1/2) (−1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 (−1) 0, rat5 1 0, rat5 1 0], vector[rat5 1 0, rat5 1 0, rat5 1 0], vector[rat5 0 0, rat5 (−1/2) (1/2), rat5 (1/2) (1/2)]}, (vector[rat5 (−1/2) (1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 (1/2) (−1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 1 0, rat5 1 0, rat5 1 0])),

({vector[rat5 (−1/2) (1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1/2) (1/2)], vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 1 0, rat5 1 0, rat5 (−1) 0], vector[rat5 1 0, rat5 1 0, rat5 1 0]}, (vector[rat5 (−1/2) (1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 1 0, rat5 1 0, rat5 (−1) 0], vector[rat5 1 0, rat5 1 0, rat5 1 0])),

({vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1/2) (1/2)], vector[rat5 1 0, rat5 (−1) 0, rat5 1 0], vector[rat5 1 0, rat5 1 0, rat5 1 0], vector[rat5 0 0, rat5 (−1/2) (1/2), rat5 (1/2) (1/2)], vector[rat5 0 0, rat5 (1/2) (−1/2), rat5 (1/2) (1/2)]}, (vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1/2) (1/2)], vector[rat5 1 0, rat5 (−1) 0, rat5 1 0], vector[rat5 1 0, rat5 1 0, rat5 1 0])),

({vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 1 0, rat5 (−1) 0, rat5 (−1) 0], vector[rat5 1 0, rat5 1 0, rat5 (−1) 0], vector[rat5 0 0, rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2)], vector[rat5 0 0, rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2)]}, (vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 1 0, rat5 (−1) 0, rat5 (−1) 0], vector[rat5 1 0, rat5 1 0, rat5 (−1) 0])),

({vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 1 0], vector[rat5 (−1) 0, rat5 (−1/2) (−1/2), rat5 0 0]}, (vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 1 0], vector[rat5 (−1) 0, rat5 (−1/2) (−1/2), rat5 0 0])), *({vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 1 0], vector[rat5 (−1) 0, rat5 (1/2) (1/2), rat5 0 0]}, (vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 1 0], vector[rat5 (−1) 0, rat5 (1/2) (1/2), rat5 0 0])),*

*({vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 (−1) 0, rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (−1/2) (−1/2)]}, (vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 (−1) 0, rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (−1/2) (−1/2)])),*

*({vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 (−1) 0, rat5 (1/2) (1/2), rat5 0 0], vector[rat5 0 0, rat5 1 0, rat5 (−1/2) (−1/2)]}, (vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 (−1) 0, rat5 (1/2) (1/2), rat5 0 0], vector[rat5 0 0, rat5 1 0, rat5 (−1/2) (−1/2)])),*

*({vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (−1/2) (−1/2)], vector[rat5 0 0, rat5 1 0, rat5 (−1/2) (−1/2)]}, (vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (−1/2) (−1/2)], vector[rat5 0 0, rat5 1 0, rat5 (−1/2) (−1/2)])),*

*({vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 1 0], vector[rat5 (−1) 0, rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (1/2) (1/2)]}, (vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 1 0], vector[rat5 (−1) 0, rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (1/2) (1/2)])),*

*({vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 1 0], vector[rat5 (−1) 0, rat5 (1/2) (1/2), rat5 0 0], vector[rat5 0 0, rat5 1 0, rat5 (1/2) (1/2)]}, (vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 1 0], vector[rat5 (−1) 0, rat5 (1/2) (1/2), rat5 0 0], vector[rat5 0 0, rat5 1 0, rat5 (1/2) (1/2)])),*

*({vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 1 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (1/2) (1/2)], vector[rat5 0 0, rat5 1 0, rat5 (1/2) (1/2)]}, (vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 1 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (1/2) (1/2)], vector[rat5 0 0, rat5 1 0, rat5 (1/2) (1/2)])),*

*({vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 (1/2) (1/2), rat5 0 0, rat5 1 0], vector[rat5 1 0, rat5 (−1/2) (−1/2), rat5 0 0]}, (vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 (1/2) (1/2), rat5 0 0, rat5 1 0], vector[rat5 1 0, rat5 (−1/2) (−1/2), rat5 0 0])),*

*({vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 (1/2) (1/2), rat5 0 0, rat5 1 0], vector[rat5 1 0, rat5 (1/2) (1/2), rat5 0 0]}, (vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 (1/2) (1/2), rat5 0 0, rat5 1 0], vector[rat5 1 0, rat5 (1/2) (1/2), rat5 0 0])),*

*({vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 1 0, rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (−1/2) (−1/2)]}, (vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 1 0, rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (−1/2) (−1/2)])),*

*({vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 1 0, rat5 (1/2) (1/2), rat5 0 0], vector[rat5 0 0, rat5 1 0, rat5 (−1/2) (−1/2)]}, (vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 1 0, rat5 (1/2) (1/2), rat5 0 0], vector[rat5 0 0, rat5 1 0, rat5 (−1/2) (−1/2)])),*

*({vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (−1/2) (−1/2)], vector[rat5 0 0, rat5 1 0, rat5 (−1/2) (−1/2)]}, (vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (−1/2) (−1/2)], vector[rat5 0 0, rat5 1 0, rat5 (−1/2) (−1/2)])),*

*({vector[rat5 (1/2) (1/2), rat5 0 0, rat5 1 0], vector[rat5 1 0, rat5 (−1/2)*

$(-1/2)$, $rat5\ 0\ 0]$, $vector[rat5\ 0\ 0,\ rat5\ (-1)\ 0,\ rat5\ (1/2)\ (1/2)]\}$, $(vector[rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0,\ rat5\ 1\ 0],\ vector[rat5\ 1\ 0,\ rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0],\ vector[rat5\ 0\ 0,\ rat5\ (-1)\ 0,\ rat5\ (1/2)\ (1/2)]))$,

$(\{vector[rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0,\ rat5\ 1\ 0],\ vector[rat5\ 1\ 0,\ rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0],\ vector[rat5\ 0\ 0,\ rat5\ 1\ 0,\ rat5\ (1/2)\ (1/2)]\},\ (vector[rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0,\ rat5\ 1\ 0],\ vector[rat5\ 1\ 0,\ rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0],\ vector[rat5\ 0\ 0,\ rat5\ 1\ 0,\ rat5\ (1/2)\ (1/2)]))$,

$(\{vector[rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0,\ rat5\ 1\ 0],\ vector[rat5\ 0\ 0,\ rat5\ (-1)\ 0,\ rat5\ (1/2)\ (1/2)],\ vector[rat5\ 0\ 0,\ rat5\ 1\ 0,\ rat5\ (1/2)\ (1/2)]\},\ (vector[rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0,\ rat5\ 1\ 0],\ vector[rat5\ 0\ 0,\ rat5\ (-1)\ 0,\ rat5\ (1/2)\ (1/2)],\ vector[rat5\ 0\ 0,\ rat5\ 1\ 0,\ rat5\ (1/2)\ (1/2)]))$,

$(\{vector[rat5\ (-1)\ 0,\ rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0],\ vector[rat5\ 1\ 0,\ rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0],\ vector[rat5\ 0\ 0,\ rat5\ (-1)\ 0,\ rat5\ (-1/2)\ (-1/2)]\},\ (vector[rat5\ (-1)\ 0,\ rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0],\ vector[rat5\ 1\ 0,\ rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0],\ vector[rat5\ 0\ 0,\ rat5\ (-1)\ 0,\ rat5\ (-1/2)\ (-1/2)]))$,

$(\{vector[rat5\ (-1)\ 0,\ rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0],\ vector[rat5\ 1\ 0,\ rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0],\ vector[rat5\ 0\ 0,\ rat5\ (-1)\ 0,\ rat5\ (1/2)\ (1/2)]\},\ (vector[rat5\ (-1)\ 0,\ rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0],\ vector[rat5\ 1\ 0,\ rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0],\ vector[rat5\ 0\ 0,\ rat5\ (-1)\ 0,\ rat5\ (1/2)\ (1/2)]))$,

$(\{vector[rat5\ (-1)\ 0,\ rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0],\ vector[rat5\ 1\ 0,\ rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0],\ vector[rat5\ 0\ 0,\ rat5\ 1\ 0,\ rat5\ (-1/2)\ (-1/2)]\},\ (vector[rat5\ (-1)\ 0,\ rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0],\ vector[rat5\ 1\ 0,\ rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0],\ vector[rat5\ 0\ 0,\ rat5\ 1\ 0,\ rat5\ (-1/2)\ (-1/2)]))$,

$(\{vector[rat5\ (-1)\ 0,\ rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0],\ vector[rat5\ 1\ 0,\ rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0],\ vector[rat5\ 0\ 0,\ rat5\ 1\ 0,\ rat5\ (1/2)\ (1/2)]\},\ (vector[rat5\ (-1)\ 0,\ rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0],\ vector[rat5\ 1\ 0,\ rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0],\ vector[rat5\ 0\ 0,\ rat5\ 1\ 0,\ rat5\ (1/2)\ (1/2)]))$

]

**definition** *adjacent-points* :: $(real,\ 3)\ vec\ set \Rightarrow (real,\ 3)\ vec \times (real,\ 3)\ vec \times (real,\ 3)\ vec$ **where**

*adjacent-points S* = (
  *case map-of adjacency-table S of*
    *Some res* ⇒ *res*
  | *None* ⇒ *undefined*
)

**lemma** *set43-neqI*:
**assumes** $a \notin \{x,\ y,\ z\} \vee b \notin \{x,\ y,\ z\} \vee c \notin \{x,\ y,\ z\} \vee d \notin \{x,\ y,\ z\}$
**shows** $\{a,\ b,\ c,\ d\} \neq \{x,\ y,\ z\}$
$\{x,\ y,\ z\} \neq \{a,\ b,\ c,\ d\}$
$\langle proof \rangle$

**lemma** *solids-vectors-to-rat5*:
**shows** $vector[1,\ 1,\ 1] = vector[rat5\ (1)\ 0,\ rat5\ (1)\ 0,\ rat5\ (1)\ 0]$
$vector[1,\ 1,\ -1] = vector[rat5\ (1)\ 0,\ rat5\ (1)\ 0,\ rat5\ (-1)\ 0]$
$vector[1,\ -1,\ 1] = vector[rat5\ (1)\ 0,\ rat5\ (-1)\ 0,\ rat5\ (1)\ 0]$
$vector[1,\ -1,\ -1] = vector[rat5\ (1)\ 0,\ rat5\ (-1)\ 0,\ rat5\ (-1)\ 0]$
$vector[-1,\ 1,\ 1] = vector[rat5\ (-1)\ 0,\ rat5\ (1)\ 0,\ rat5\ (1)\ 0]$

$vector[-1, 1, -1] = vector[rat5\ (-1)\ 0, rat5\ (1)\ 0, rat5\ (-1)\ 0]$
$vector[-1, -1, 1] = vector[rat5\ (-1)\ 0, rat5\ (-1)\ 0, rat5\ (1)\ 0]$
$vector[-1, -1, -1] = vector[rat5\ (-1)\ 0, rat5\ (-1)\ 0, rat5\ (-1)\ 0]$
$vector[1, 0, 0] = vector[rat5\ (1)\ 0, rat5\ (0)\ 0, rat5\ (0)\ 0]$
$vector[-1, 0, 0] = vector[rat5\ (-1)\ 0, rat5\ (0)\ 0, rat5\ (0)\ 0]$
$vector[0, 0, 1] = vector[rat5\ (0)\ 0, rat5\ (0)\ 0, rat5\ (1)\ 0]$
$vector[0, 0, -1] = vector[rat5\ (0)\ 0, rat5\ (0)\ 0, rat5\ (-1)\ 0]$
$vector[0, 1, 0] = vector[rat5\ (0)\ 0, rat5\ (1)\ 0, rat5\ (0)\ 0]$
$vector[0, -1, 0] = vector[rat5\ (0)\ 0, rat5\ (-1)\ 0, rat5\ (0)\ 0]$
⟨*proof*⟩

**lemma** *set53-neqI*:
**assumes** $a \notin \{x, y, z\} \vee b \notin \{x, y, z\} \vee c \notin \{x, y, z\} \vee d \notin \{x, y, z\} \vee e \notin \{x, y, z\}$
**shows** $\{a, b, c, d,e\} \neq \{x, y, z\}$
$\{x, y, z\} \neq \{a, b, c, d, e\}$
⟨*proof*⟩

**lemma** *set54-neqI*:
**assumes** $a \notin \{w, x, y, z\} \vee b \notin \{w, x, y, z\} \vee c \notin \{w, x, y, z\} \vee d \notin \{w, x, y, z\} \vee e \notin \{w, x, y, z\}$
**shows** $\{a, b, c, d,e\} \neq \{w, x, y, z\}$
$\{w, x, y, z\} \neq \{a, b, c, d, e\}$
⟨*proof*⟩

**lemma** *set5-neqI*:
**assumes** $a \notin \{v, w, x, y, z\} \vee b \notin \{v, w, x, y, z\} \vee c \notin \{v, w, x, y, z\} \vee d \notin \{v, w, x, y, z\} \vee e \notin \{v, w, x, y, z\}$
**shows** $\{a, b, c, d,e\} \neq \{v, w, x, y, z\}$
⟨*proof*⟩

**lemma** *set3-eq-iff*:
**shows** $(\{a, b, c\} = \{d, e, f\}) =$
$(a \in \{d, e, f\} \wedge b \in \{d, e, f\} \wedge c \in \{d, e, f\} \wedge$
$d \in \{a, b, c\} \wedge e \in \{a, b, c\} \wedge f \in \{a, b, c\})$
⟨*proof*⟩

**lemma** *set3-vector-neq*:
**assumes** $(a1 \neq d1 \wedge a1 \neq d2 \wedge a1 \neq d3)$
**shows** $\{(vector\ [a1, b1, c1]::real\hat{\ }3), vector\ [a2, b2, c2], vector\ [a3, b3, c3]\} \neq$
$\{vector\ [d1, e1, f1], vector\ [d2, e2, f2], vector\ [d3, e3, f3]\}$
⟨*proof*⟩

**lemma** *adjacent-points*:
**shows** $adjacent\text{-}points\ \{vector[-1, -1, 1], vector[-1, 1, -1], vector[1, -1, -1]\} = (vector\ [-1, -1, 1], vector\ [-1, 1, -1], vector\ [1, -1, -1])$
$adjacent\text{-}points\ \{vector[-1, -1, 1], vector[-1, 1, -1], vector[1, 1, 1]\} = (vector[-1, -1, 1], vector[-1, 1, -1], vector[1, 1, 1])$
$adjacent\text{-}points\ \{vector[-1, -1, 1], vector[1, -1, -1], vector[1, 1, 1]\} =$

(vector[−1, −1, 1], vector[1, − 1, − 1], vector[1, 1, 1])

adjacent-points {vector[−1, 1, −1], vector[1, − 1, − 1], vector[1, 1, 1]} = (vector[−1, 1, −1], vector[1, − 1, − 1], vector[1, 1, 1])

adjacent-points {vector[−1, −1, −1], vector[−1, −1, 1], vector[−1, 1, −1], vector[−1, 1, 1]} = (vector[−1, −1, −1], vector[−1, −1, 1], vector[−1, 1, 1])

adjacent-points {vector[−1, −1, −1], vector[−1, −1, 1], vector[1, −1, −1], vector[1, −1, 1]} = (vector[−1, −1, −1], vector[−1, −1, 1], vector[1, −1, 1])

adjacent-points {vector[−1, −1, −1], vector[−1, 1, −1], vector[1, −1, −1], vector[1, 1, −1]} = (vector[−1, −1, −1], vector[−1, 1, −1], vector[1, 1, −1])

adjacent-points {vector[−1, −1, 1], vector[−1, 1, 1], vector[1, −1, 1], vector[1, 1, 1]} = (vector[−1, −1, 1], vector[−1, 1, 1], vector[1, 1, 1])

adjacent-points {vector[−1, 1, −1], vector[−1, 1, 1], vector[1, 1, −1], vector[1, 1, 1]} = (vector[−1, 1, −1], vector[−1, 1, 1], vector[1, 1, 1])

adjacent-points {vector[1, −1, −1], vector[1, −1, 1], vector[1, 1, −1], vector[1, 1, 1]} = (vector[1, −1, −1], vector[1, −1, 1], vector[1, 1, 1])

adjacent-points {vector[−1, 0, 0], vector[0, −1, 0], vector[0, 0, −1]} = (vector[−1, 0, 0], vector[0, −1, 0], vector[0, 0, −1])

adjacent-points {vector[−1, 0, 0], vector[0, −1, 0], vector[0, 0, 1]} = (vector[−1, 0, 0], vector[0, −1, 0], vector[0, 0, 1])

adjacent-points {vector[−1, 0, 0], vector[0, 1, 0], vector[0, 0, −1]} = (vector[−1, 0, 0], vector[0, 1, 0], vector[0, 0, −1])

adjacent-points {vector[−1, 0, 0], vector[0, 1, 0], vector[0, 0, 1]} = (vector[−1, 0, 0], vector[0, 1, 0], vector[0, 0, 1])

adjacent-points {vector[1, 0, 0], vector[0, −1, 0], vector[0, 0, −1]} = (vector[1, 0, 0], vector[0, −1, 0], vector[0, 0, −1])

adjacent-points {vector[1, 0, 0], vector[0, −1, 0], vector[0, 0, 1]} = (vector[1, 0, 0], vector[0, −1, 0], vector[0, 0, 1])

adjacent-points {vector[1, 0, 0], vector[0, 1, 0], vector[0, 0, −1]} = (vector[1, 0, 0], vector[0, 1, 0], vector[0, 0, −1])

adjacent-points {vector[1, 0, 0], vector[0, 1, 0], vector[0, 0, 1]} = (vector[1, 0, 0], vector[0, 1, 0], vector[0, 0, 1])

adjacent-points {vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1/2) (1/2)], vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 (−1) 0, rat5 (−1) 0, rat5 (−1) 0], vector[rat5 (−1) 0, rat5 (−1) 0, rat5 1 0]} = (vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1/2) (1/2)], vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 (−1) 0, rat5 (−1) 0, rat5 1 0])

adjacent-points {vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1/2) (1/2)], vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 (1/2) (−1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 (−1) 0, rat5 1 0, rat5 (−1) 0], vector[rat5 (−1) 0, rat5 1 0, rat5 1 0]} = (vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1/2) (1/2)], vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 (−1) 0, rat5 1 0, rat5 1 0])

adjacent-points {vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1/2) (1/2)], vector[rat5 (−1) 0, rat5 (−1) 0, rat5 1 0], vector[rat5 (−1) 0, rat5 1 0, rat5 1 0], vector[rat5 0 0, rat5 (−1/2) (1/2), rat5 (1/2) (1/2)], vector[rat5 0 0, rat5

(1/2) (−1/2), rat5 (1/2) (1/2)]} = (vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1/2) (1/2)], vector[rat5 (−1) 0, rat5 (−1) 0, rat5 1 0], vector[rat5 (−1) 0, rat5 1 0, rat5 1 0])

adjacent-points {vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 (−1) 0, rat5 (−1) 0, rat5 (−1) 0], vector[rat5 (−1) 0, rat5 1 0, rat5 (−1) 0], vector[rat5 0 0, rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2)], vector[rat5 0 0, rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2)]} = (vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 (−1) 0, rat5 (−1) 0, rat5 (−1) 0], vector[rat5 (−1) 0, rat5 1 0, rat5 (−1) 0])

adjacent-points {vector[rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 (−1) 0, rat5 (−1) 0, rat5 (−1) 0], vector[rat5 1 0, rat5 (−1) 0, rat5 (−1) 0], vector[rat5 0 0, rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2)]} = (vector[rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 1 0, rat5 (−1) 0, rat5 (−1) 0])

adjacent-points {vector[rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 (−1) 0, rat5 (−1) 0, rat5 1 0], vector[rat5 1 0, rat5 (−1) 0, rat5 1 0], vector[rat5 0 0, rat5 (1/2) (−1/2), rat5 (1/2) (1/2)]} = (vector[rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 1 0, rat5 (−1) 0, rat5 1 0])

adjacent-points {vector[rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1/2) (1/2)], vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 1 0, rat5 (−1) 0, rat5 (−1) 0], vector[rat5 1 0, rat5 (−1) 0, rat5 1 0]} = (vector[rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 1 0, rat5 (−1) 0, rat5 (−1) 0], vector[rat5 1 0, rat5 (−1) 0, rat5 1 0])

adjacent-points {vector[rat5 (−1/2) (1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 (1/2) (−1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 (−1) 0, rat5 1 0, rat5 (−1) 0], vector[rat5 1 0, rat5 1 0, rat5 (−1) 0], vector[rat5 0 0, rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2)]} = (vector[rat5 (−1/2) (1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 (1/2) (−1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 1 0, rat5 1 0, rat5 (−1) 0])

adjacent-points {vector[rat5 (−1/2) (1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 (1/2) (−1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 (−1) 0, rat5 1 0, rat5 1 0], vector[rat5 1 0, rat5 1 0, rat5 1 0], vector[rat5 0 0, rat5 (−1/2) (1/2), rat5 (1/2) (1/2)]} = (vector[rat5 (−1/2) (1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 (1/2) (−1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 1 0, rat5 1 0, rat5 1 0])

adjacent-points {vector[rat5 (−1/2) (1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1/2) (1/2)], vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 1 0, rat5 1 0, rat5 (−1) 0], vector[rat5 1 0, rat5 1 0, rat5 1 0]} = (vector[rat5 (−1/2) (1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 1 0, rat5 1 0, rat5 (−1) 0], vector[rat5 1 0, rat5 1 0, rat5 1 0])

adjacent-points {vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1/2) (1/2)], vector[rat5 1 0, rat5 (−1) 0, rat5 1 0], vector[rat5 1 0, rat5 1 0, rat5 1 0], vector[rat5 0 0, rat5 (−1/2) (1/2), rat5 (1/2) (1/2)], vector[rat5 0 0, rat5 (1/2) (−1/2), rat5 (1/2) (1/2)]} = (vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1/2) (1/2)], vector[rat5 1 0, rat5 (−1) 0, rat5 1 0], vector[rat5 1 0, rat5 1 0, rat5 1

*0])*

*adjacent-points {vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 1 0, rat5 (−1) 0, rat5 (−1) 0], vector[rat5 1 0, rat5 1 0, rat5 (−1) 0], vector[rat5 0 0, rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2)], vector[rat5 0 0, rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2)]} = (vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 1 0, rat5 (−1) 0, rat5 (−1) 0], vector[rat5 1 0, rat5 1 0, rat5 (−1) 0])*

*adjacent-points {vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 1 0], vector[rat5 (−1) 0, rat5 (−1/2) (−1/2), rat5 0 0]} = (vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 1 0], vector[rat5 (−1) 0, rat5 (−1/2) (−1/2), rat5 0 0])*

*adjacent-points {vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 1 0], vector[rat5 (−1) 0, rat5 (1/2) (1/2), rat5 0 0]} = (vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 1 0], vector[rat5 (−1) 0, rat5 (1/2) (1/2), rat5 0 0])*

*adjacent-points {vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 (−1) 0, rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (−1/2) (−1/2)]} = (vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 (−1) 0, rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (−1/2) (−1/2)])*

*adjacent-points {vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 (−1) 0, rat5 (1/2) (1/2), rat5 0 0], vector[rat5 0 0, rat5 1 0, rat5 (−1/2) (−1/2)]} = (vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 (−1) 0, rat5 (1/2) (1/2), rat5 0 0], vector[rat5 0 0, rat5 1 0, rat5 (−1/2) (−1/2)])*

*adjacent-points {vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (−1/2) (−1/2)], vector[rat5 0 0, rat5 1 0, rat5 (−1/2) (−1/2)]} = (vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (−1/2) (−1/2)], vector[rat5 0 0, rat5 1 0, rat5 (−1/2) (−1/2)])*

*adjacent-points {vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 1 0], vector[rat5 (−1) 0, rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (1/2) (1/2)]} = (vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 1 0], vector[rat5 (−1) 0, rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (1/2) (1/2)])*

*adjacent-points {vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 1 0], vector[rat5 (−1) 0, rat5 (1/2) (1/2), rat5 0 0], vector[rat5 0 0, rat5 1 0, rat5 (1/2) (1/2)]} = (vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 1 0], vector[rat5 (−1) 0, rat5 (1/2) (1/2), rat5 0 0], vector[rat5 0 0, rat5 1 0, rat5 (1/2) (1/2)])*

*adjacent-points {vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 1 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (1/2) (1/2)], vector[rat5 0 0, rat5 1 0, rat5 (1/2) (1/2)]} = (vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 1 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (1/2) (1/2)], vector[rat5 0 0, rat5 1 0, rat5 (1/2) (1/2)])*

*adjacent-points {vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 (1/2) (1/2), rat5 0 0, rat5 1 0], vector[rat5 1 0, rat5 (−1/2) (−1/2), rat5 0 0]} = (vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 (1/2) (1/2), rat5 0 0, rat5 1 0], vector[rat5 1 0, rat5 (−1/2) (−1/2), rat5 0 0])*

*adjacent-points {vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1) 0], vector[rat5*

(1/2) (1/2), rat5 0 0, rat5 1 0], vector[rat5 1 0, rat5 (1/2) (1/2), rat5 0 0]} = (vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 (1/2) (1/2), rat5 0 0, rat5 1 0], vector[rat5 1 0, rat5 (1/2) (1/2), rat5 0 0])

adjacent-points {vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 1 0, rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (−1/2) (−1/2)]} = (vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 1 0, rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (−1/2) (−1/2)])

adjacent-points {vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 1 0, rat5 (1/2) (1/2), rat5 0 0], vector[rat5 0 0, rat5 1 0, rat5 (−1/2) (−1/2)]} = (vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 1 0, rat5 (1/2) (1/2), rat5 0 0], vector[rat5 0 0, rat5 1 0, rat5 (−1/2) (−1/2)])

adjacent-points {vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (−1/2) (−1/2)], vector[rat5 0 0, rat5 1 0, rat5 (−1/2) (−1/2)]} = (vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1) 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (−1/2) (−1/2)], vector[rat5 0 0, rat5 1 0, rat5 (−1/2) (−1/2)])

adjacent-points {vector[rat5 (1/2) (1/2), rat5 0 0, rat5 1 0], vector[rat5 1 0, rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (1/2) (1/2)]} = (vector[rat5 (1/2) (1/2), rat5 0 0, rat5 1 0], vector[rat5 1 0, rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (1/2) (1/2)])

adjacent-points {vector[rat5 (1/2) (1/2), rat5 0 0, rat5 1 0], vector[rat5 1 0, rat5 (1/2) (1/2), rat5 0 0], vector[rat5 0 0, rat5 1 0, rat5 (1/2) (1/2)]} = (vector[rat5 (1/2) (1/2), rat5 0 0, rat5 1 0], vector[rat5 1 0, rat5 (1/2) (1/2), rat5 0 0], vector[rat5 0 0, rat5 1 0, rat5 (1/2) (1/2)])

adjacent-points {vector[rat5 (1/2) (1/2), rat5 0 0, rat5 1 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (1/2) (1/2)], vector[rat5 0 0, rat5 1 0, rat5 (1/2) (1/2)]} = (vector[rat5 (1/2) (1/2), rat5 0 0, rat5 1 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (1/2) (1/2)], vector[rat5 0 0, rat5 1 0, rat5 (1/2) (1/2)])

adjacent-points {vector[rat5 (−1) 0, rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 1 0, rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (−1/2) (−1/2)]} = (vector[rat5 (−1) 0, rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 1 0, rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (−1/2) (−1/2)])

adjacent-points {vector[rat5 (−1) 0, rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 1 0, rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (1/2) (1/2)]} = (vector[rat5 (−1) 0, rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 1 0, rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 0 0, rat5 (−1) 0, rat5 (1/2) (1/2)])

adjacent-points {vector[rat5 (−1) 0, rat5 (1/2) (1/2), rat5 0 0], vector[rat5 1 0, rat5 (1/2) (1/2), rat5 0 0], vector[rat5 0 0, rat5 1 0, rat5 (−1/2) (−1/2)]} = (vector[rat5 (−1) 0, rat5 (1/2) (1/2), rat5 0 0], vector[rat5 1 0, rat5 (1/2) (1/2), rat5 0 0], vector[rat5 0 0, rat5 1 0, rat5 (−1/2) (−1/2)])

adjacent-points {vector[rat5 (−1) 0, rat5 (1/2) (1/2), rat5 0 0], vector[rat5 1 0, rat5 (1/2) (1/2), rat5 0 0], vector[rat5 0 0, rat5 1 0, rat5 (1/2) (1/2)]} = (vector[rat5 (−1) 0, rat5 (1/2) (1/2), rat5 0 0], vector[rat5 1 0, rat5 (1/2) (1/2), rat5 0 0], vector[rat5 0 0, rat5 1 0, rat5 (1/2) (1/2)])

⟨proof⟩

**definition** *mk-matrix33-ap* :: (real^3) set ⇒ (real^3) set ⇒ real^3^3 **where**
*mk-matrix33-ap f1 f2 = (let vs = adjacent-points f1; us = adjacent-points f2 in*

*mk-matrix33* (*fst vs*) (*fst* (*snd vs*)) (*snd* (*snd vs*))
(*fst us*) (*fst* (*snd us*)) (*snd* (*snd us*)))

**method** *show-congruence* =
(((*erule forw-subst*)+)?;
(*match* **conclusion in** *convex hull* (*s*::(*real*^*3*) *set*) *congruent convex hull* (*t*::(*real*^*3*) *set*) **for** *s t* ⇒
‹(*rule congruent-simple*), (*rule exI*[*where x*=*mk-matrix33-ap s t*])›);
(*simp only*: *mk-matrix33-ap-def Let-def fst-conv snd-conv adjacent-points*);
(*rule matrix-by-cramer-orthog* | *simp add*: *matrix-by-cramer det-3 rat5-eq-0 rat5-add rat5-mul rat5-sub*);
(*simp only*: *mk-matrix33-def vector-3 det-3 rat5-add rat5-mul rat5-sub rat5-eq*);
(*simp add*: *matrix-vector-mul-3 orthogonal-matrix33 rat5-add rat5-mul rat5-sub rat5-div rat5-eq*
*rat5-eq-0 rat5-eq-1 Let-def*; *fast*))

## 4.7 Equiangularity

vangle is from Manuel Eberl's AFP entry on triangles (see https://www.isa-afp.org/thys/Triangle/Angles.html)

**definition** *vangle* :: *'a*::*real-inner* ⇒ *'a* ⇒ *real* **where**
*vangle x y* = (*if x* = *0* ∨ *y* = *0 then pi* / *2 else arccos* (*x* · *y* / (*norm x* ∗ *norm y*)))

**lemma** *vangle-commute*:
**shows** *vangle x y* = *vangle y x*
⟨*proof*⟩

**lemma** *isometry-maintains-angles-forw*:
**fixes** *c*
**assumes** *orthogonal-transformation h*
**defines** *g* ≡ (λ*x*. *c* + *h x*)
**shows** ((*e1 face-of f1* ∧ *aff-dim e1* = *1*) ∧
(*e2 face-of f1* ∧ *aff-dim e2* = *1*) ∧
*e1* = *convex hull* {*v*, *a*} ∧ *e2* = *convex hull* {*v*, *b*} ∧
*vangle* (*a* − *v*) (*b* − *v*) = *t*) ⟹
((*g* ' *e1 face-of g* ' *f1* ∧ *aff-dim* (*g* ' *e1*) = *1*) ∧
(*g* ' *e2 face-of g* ' *f1* ∧ *aff-dim* (*g* ' *e2*) = *1*) ∧
*g* ' *e1* = *convex hull* {*g v*, *g a*} ∧ *g* ' *e2* = *convex hull* {*g v*, *g b*} ∧
*vangle* (*g a* − *g v*) (*g b* − *g v*) = *t*)
(**is** *?lhs* ⟹ *?rhs*)
⟨*proof*⟩

**definition** *equiangular* :: *'a*::*euclidean-space set* ⇒ *real* ⇒ *bool*
**where** *equiangular f t* ⟷
(∀ *e1 e2 v*. (*e1 face-of f* ∧ *aff-dim e1* = *1*) ∧ (*e2 face-of f* ∧ *aff-dim e2* = *1*) ∧
*e1* ≠ *e2* ∧ *v extreme-point-of e1* ∧ *v extreme-point-of e2* ⟶
(∃ *a b*. *e1* = *convex hull* {*v*, *a*} ∧ *e2* = *convex hull* {*v*, *b*} ∧

$vangle\ (a - v)\ (b - v) = t))$

**lemma** *equiangular-congruent-forw*:
  **assumes** *f1 congruent f2*
  **assumes** *equiangular f1 t*
  **shows** *equiangular f2 t*
⟨*proof*⟩

**lemma** *equiangular-congruent*:
  **assumes** *f1 congruent f2*
  **shows** *equiangular f1 t* $\longleftrightarrow$ *equiangular f2 t*
  ⟨*proof*⟩

**end**

# 5 Tetrahedron

**theory** *Tetrahedron*
  **imports** *Computation*
**begin**

## 5.1 Definition

**definition** *std-tetrahedron* :: (*real^3*) *set*
  **where** *std-tetrahedron* ≡ *convex hull*
    {*vector* [*1*, *1*, *1*], *vector* [−*1*, −*1*, *1*],
     *vector* [−*1*, *1*, −*1*], *vector* [*1*, −*1*, −*1*]}

**lemma** *tetrahedron-fulldim*:
  **shows** *aff-dim std-tetrahedron* = *3*
  ⟨*proof*⟩

**lemma** *tetrahedron-polyhedron*:
  **shows** *polyhedron std-tetrahedron*
  ⟨*proof*⟩

## 5.2 Facets, edges, and vertices

**lemma** *tetrahedron-facets*:
  *f face-of std-tetrahedron* ∧ *aff-dim f* = *2* $\longleftrightarrow$
   *f* = *convex hull* {*vector*[−*1*, −*1*, *1*], *vector*[− *1*, *1*, − *1*], *vector*[*1*, − *1*, − *1*]}
∨
   *f* = *convex hull* {*vector*[−*1*, −*1*, *1*], *vector*[− *1*, *1*, − *1*], *vector*[*1*, *1*, *1*]} ∨
   *f* = *convex hull* {*vector*[−*1*, −*1*, *1*], *vector*[*1*, − *1*, − *1*], *vector*[*1*, *1*, *1*]} ∨
   *f* = *convex hull* {*vector*[−*1*, *1*, −*1*], *vector*[*1*, − *1*, − *1*], *vector*[*1*, *1*, *1*]}
  ⟨*proof*⟩

**lemma** *tetrahedron-facet1-edges*:
  **defines** *v1* ≡ *vector*[−*1*, −*1*, *1*] :: *real^3*

**and** *v2* ≡ *vector*[−*1*, *1*, −*1*] :: *real^3*
**and** *v3* ≡ *vector*[*1*, −*1*, −*1*] :: *real^3*
**and** *n* ≡ *vector*[−*4*, −*4*, −*4*] :: *real^3*
**shows** (*e face-of* (*convex hull* {*v1*, *v2*, *v3*}) ∧ *aff-dim e* = *1*) =
(*e* = *convex hull* {*v1*, *v2*} ∨ *e* = *convex hull* {*v1*, *v3*} ∨ *e* = *convex hull* {*v2*, *v3*})
⟨*proof*⟩

**lemma** *tetrahedron-facet2-edges*:
**defines** *v1* ≡ *vector*[−*1*,−*1*,*1*] :: *real^3*
**and** *v2* ≡ *vector*[−*1*,*1*,−*1*] :: *real^3*
**and** *v3* ≡ *vector*[*1*,*1*,*1*] :: *real^3*
**and** *n* ≡ *vector*[−*4*, *4*, *4*] :: *real^3*
**shows** (*e face-of* (*convex hull* {*v1*, *v2*, *v3*}) ∧ *aff-dim e* = *1*) =
(*e* = *convex hull* {*v1*, *v2*} ∨ *e* = *convex hull* {*v1*, *v3*} ∨ *e* = *convex hull* {*v2*, *v3*})
⟨*proof*⟩

**lemma** *tetrahedron-facet3-edges*:
**defines** *v1* ≡ *vector*[−*1*,−*1*,*1*] :: *real^3*
**and** *v2* ≡ *vector*[*1*,−*1*,−*1*] :: *real^3*
**and** *v3* ≡ *vector*[*1*,*1*,*1*] :: *real^3*
**and** *n* ≡ *vector*[*4*, −*4*, *4*] :: *real^3*
**shows** (*e face-of* (*convex hull* {*v1*, *v2*, *v3*}) ∧ *aff-dim e* = *1*) =
(*e* = *convex hull* {*v1*, *v2*} ∨ *e* = *convex hull* {*v1*, *v3*} ∨ *e* = *convex hull* {*v2*, *v3*})
⟨*proof*⟩

**lemma** *tetrahedron-facet4-edges*:
**defines** *v1* ≡ *vector*[−*1*,*1*,−*1*] :: *real^3*
**and** *v2* ≡ *vector*[*1*,−*1*,−*1*] :: *real^3*
**and** *v3* ≡ *vector*[*1*,*1*,*1*] :: *real^3*
**and** *n* ≡ *vector*[*4*, *4*, −*4*] :: *real^3*
**shows** (*e face-of* (*convex hull* {*v1*, *v2*, *v3*}) ∧ *aff-dim e* = *1*) =
(*e* = *convex hull* {*v1*, *v2*} ∨ *e* = *convex hull* {*v1*, *v3*} ∨ *e* = *convex hull* {*v2*, *v3*})
⟨*proof*⟩

**lemmas** *tetrahedron-facet-edges* =
*tetrahedron-facet1-edges tetrahedron-facet2-edges*
*tetrahedron-facet3-edges tetrahedron-facet4-edges*

**lemma** *tetrahedron-edges-per-face*:
**assumes** *f face-of std-tetrahedron*
**and** *aff-dim f* = *2*
**shows** *card* {*e*. *e face-of std-tetrahedron* ∧ *aff-dim e* = *1* ∧ *e* ⊆ *f*} = *3*
⟨*proof*⟩

**lemma** *tetrahedron-edges*:

$e$ *face-of std-tetrahedron* $\wedge$ *aff-dim* $e = 1 \longleftrightarrow$
$e =$ *convex hull* $\{$*vector* $[-1, -1, 1]$, *vector* $[-1, 1, -1]\} \vee$
$e =$ *convex hull* $\{$*vector* $[-1, -1, 1]$, *vector* $[1, -1, -1]\} \vee$
$e =$ *convex hull* $\{$*vector* $[-1, -1, 1]$, *vector* $[1, 1, 1]\} \vee$
$e =$ *convex hull* $\{$*vector* $[-1, 1, -1]$, *vector* $[1, -1, -1]\} \vee$
$e =$ *convex hull* $\{$*vector* $[-1, 1, -1]$, *vector* $[1, 1, 1]\} \vee$
$e =$ *convex hull* $\{$*vector* $[1, -1, -1]$, *vector* $[1, 1, 1]\}$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
$\langle proof \rangle$

**lemma** *tetrahedron-vertices*:
**shows** ($v$ *face-of std-tetrahedron* $\wedge$ *aff-dim* $v = 0$) $=$
($v = \{$*vector* $[-\ 1, -\ 1, 1]\} \vee$
$v = \{$*vector* $[-\ 1, 1, -\ 1]\} \vee$
$v = \{$*vector* $[1, -\ 1, -\ 1]\} \vee$
$v = \{$*vector* $[1, 1, 1]\}$)
(**is** *?lhs* $=$ *?rhs*)
$\langle proof \rangle$

**lemma** *tetrahedron-edges-per-vertex*:
**assumes** $v$ *face-of std-tetrahedron*
**and** *aff-dim* $v = 0$
**shows** *card* $\{e.\ e$ *face-of std-tetrahedron* $\wedge$ *aff-dim* $e = 1 \wedge v \subseteq e\} = 3$
$\langle proof \rangle$

## 5.3 Regularity

**lemma** *tetrahedron-congruent-edges*:
**assumes** *e1 face-of std-tetrahedron* $\wedge$ *aff-dim e1* $= 1$
**and** *e2 face-of std-tetrahedron* $\wedge$ *aff-dim e2* $= 1$
**shows** *e1 congruent e2*
$\langle proof \rangle$

**lemma** *tetrahedron-congruent-faces*:
**assumes** *f1 face-of std-tetrahedron* $\wedge$ *aff-dim f1* $= 2$
**and** *f2 face-of std-tetrahedron* $\wedge$ *aff-dim f2* $= 2$
**shows** *f1 congruent f2*
$\langle proof \rangle$

**lemma** *tetrahedron-facet1-equiangular*:
**defines** *v1* $\equiv$ *vector*$[-1, -1, 1]$ :: *real^3*
**defines** *v2* $\equiv$ *vector*$[-1, 1, -1]$ :: *real^3*
**defines** *v3* $\equiv$ *vector*$[1, -1, -1]$ :: *real^3*
**shows** *equiangular* (*convex hull* $\{$*v1*, *v2*, *v3*$\}$) (*pi* / *3*)
$\langle proof \rangle$

**lemma** *tetrahedron-equiangular*:
**assumes** $f$ *face-of std-tetrahedron* $\wedge$ *aff-dim* $f = 2$
**shows** *equiangular* $f$ (*pi* / *3*)

⟨*proof*⟩

**end**

# 6 Cube

**theory** *Cube*
  **imports** *Computation*
**begin**

## 6.1 Definition

**definition** *std-cube* :: (*real*, *3*) *vec set*
  **where** *std-cube* ≡ *convex hull*
    {*vector* [*1*, *1*, *1*], *vector* [*1*, *1*, −*1*],
     *vector* [*1*, −*1*, *1*], *vector* [*1*, −*1*, −*1*],
     *vector* [−*1*, *1*, *1*], *vector* [−*1*, *1*, −*1*],
     *vector* [−*1*, −*1*, *1*], *vector* [−*1*, −*1*, −*1*]}

**lemma** *cube-fulldim*:
  **shows** *aff-dim std-cube* = *3*
  ⟨*proof*⟩

**lemma** *cube-polyhedron*:
  **shows** *polyhedron std-cube*
  ⟨*proof*⟩

## 6.2 Facets, edges, and vertices

**lemma** *cube-facets*:
  **fixes** *f* :: (*real^3*) *set*
  **shows** *f face-of std-cube* ∧ *aff-dim f* = *2* ⟷
    *f* = *convex hull* {*vector*[− *1*, − *1*, − *1*], *vector*[− *1*, − *1*, *1*], *vector*[− *1*, *1*, − *1*], *vector*[− *1*, *1*, *1*]} ∨
    *f* = *convex hull* {*vector*[− *1*, − *1*, − *1*], *vector*[− *1*, − *1*, *1*], *vector*[*1*, − *1*, − *1*], *vector*[*1*, − *1*, *1*]} ∨
    *f* = *convex hull* {*vector*[− *1*, − *1*, − *1*], *vector*[− *1*, *1*, − *1*], *vector*[*1*, − *1*, − *1*], *vector*[*1*, *1*, − *1*]} ∨
    *f* = *convex hull* {*vector*[− *1*, − *1*, *1*], *vector*[− *1*, *1*, *1*], *vector*[*1*, − *1*, *1*], *vector*[*1*, *1*, *1*]} ∨
    *f* = *convex hull* {*vector*[− *1*, *1*, − *1*], *vector*[− *1*, *1*, *1*], *vector*[*1*, *1*, − *1*], *vector*[*1*, *1*, *1*]} ∨
    *f* = *convex hull* {*vector*[*1*, − *1*, − *1*], *vector*[*1*, − *1*, *1*], *vector*[*1*, *1*, − *1*], *vector*[*1*, *1*, *1*]}
  ⟨*proof*⟩

**lemma** *cube-facet1-edges*:
  **defines** *v1* ≡ *vector*[−*1*, −*1*, −*1*] :: *real^3*
    **and** *v2* ≡ *vector*[−*1*, −*1*, *1*] :: *real^3*

**and** $v3 \equiv vector[-1, 1, -1] :: real^3$
**and** $v4 \equiv vector[-1, 1, 1] :: real^3$
**and** $n \equiv vector[1, 0, 0] :: real^3$
**shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1, v2, v3, v4\}) \wedge aff\text{-}dim\ e = 1) =$
$(e = convex\ hull\ \{v1 ,v2\} \vee$
$e = convex\ hull\ \{v1, v3\} \vee$
$e = convex\ hull\ \{v2, v4\} \vee$
$e = convex\ hull\ \{v3, v4\})$
$\langle proof \rangle$

**lemma** *cube-facet2-edges*:
**defines** $v1 \equiv vector[-1, -1, -1] :: real^3$
**and** $v2 \equiv vector[-1, -1, 1] :: real^3$
**and** $v3 \equiv vector[1, -1, -1] :: real^3$
**and** $v4 \equiv vector[1, -1, 1] :: real^3$
**and** $n \equiv vector[0, 1, 0] :: real^3$
**shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1, v2, v3, v4\}) \wedge aff\text{-}dim\ e = 1) =$
$(e = convex\ hull\ \{v1 ,v2\} \vee$
$e = convex\ hull\ \{v1, v3\} \vee$
$e = convex\ hull\ \{v2, v4\} \vee$
$e = convex\ hull\ \{v3, v4\})$
$\langle proof \rangle$

**lemma** *cube-facet3-edges*:
**defines** $v1 \equiv vector[-1, -1, -1] :: real^3$
**and** $v2 \equiv vector[-1, 1, -1] :: real^3$
**and** $v3 \equiv vector[1, -1, -1] :: real^3$
**and** $v4 \equiv vector[1, 1, -1] :: real^3$
**and** $n \equiv vector[0, 0, 1] :: real^3$
**shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1, v2, v3, v4\}) \wedge aff\text{-}dim\ e = 1) =$
$(e = convex\ hull\ \{v1 ,v2\} \vee$
$e = convex\ hull\ \{v1, v3\} \vee$
$e = convex\ hull\ \{v2, v4\} \vee$
$e = convex\ hull\ \{v3, v4\})$
$\langle proof \rangle$

**lemma** *cube-facet4-edges*:
**defines** $v1 \equiv vector[-1, -1, 1] :: real^3$
**and** $v2 \equiv vector[-1, 1, 1] :: real^3$
**and** $v3 \equiv vector[1, -1, 1] :: real^3$
**and** $v4 \equiv vector[1, 1, 1] :: real^3$
**and** $n \equiv vector[0, 0, 1] :: real^3$
**shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1, v2, v3, v4\}) \wedge aff\text{-}dim\ e = 1) =$
$(e = convex\ hull\ \{v1 ,v2\} \vee$
$e = convex\ hull\ \{v1, v3\} \vee$
$e = convex\ hull\ \{v2, v4\} \vee$
$e = convex\ hull\ \{v3, v4\})$
$\langle proof \rangle$

**lemma** *cube-facet5-edges*:
**defines** $v1 \equiv vector[-1, 1, -1] :: real\hat{}3$
**and** $v2 \equiv vector[-1, 1, 1] :: real\hat{}3$
**and** $v3 \equiv vector[1, 1, -1] :: real\hat{}3$
**and** $v4 \equiv vector[1, 1, 1] :: real\hat{}3$
**and** $n \equiv vector[0, 1, 0] :: real\hat{}3$
**shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1, v2, v3, v4\}) \wedge aff\text{-}dim\ e = 1) =$
$(e = convex\ hull\ \{v1 ,v2\} \vee$
$e = convex\ hull\ \{v1, v3\} \vee$
$e = convex\ hull\ \{v2, v4\} \vee$
$e = convex\ hull\ \{v3, v4\})$
⟨*proof*⟩

**lemma** *cube-facet6-edges*:
**defines** $v1 \equiv vector[1, -1, -1] :: real\hat{}3$
**and** $v2 \equiv vector[1, -1, 1] :: real\hat{}3$
**and** $v3 \equiv vector[1, 1, -1] :: real\hat{}3$
**and** $v4 \equiv vector[1, 1, 1] :: real\hat{}3$
**and** $n \equiv vector[1, 0, 0] :: real\hat{}3$
**shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1, v2, v3, v4\}) \wedge aff\text{-}dim\ e = 1) =$
$(e = convex\ hull\ \{v1 ,v2\} \vee$
$e = convex\ hull\ \{v1, v3\} \vee$
$e = convex\ hull\ \{v2, v4\} \vee$
$e = convex\ hull\ \{v3, v4\})$
⟨*proof*⟩

**lemmas** *cube-facet-edges* =
*cube-facet1-edges cube-facet2-edges cube-facet3-edges*
*cube-facet4-edges cube-facet5-edges cube-facet6-edges*

**lemma** *cube-edges-per-face*:
**assumes** *f face-of std-cube*
**and** $aff\text{-}dim\ f = 2$
**shows** $card\ \{e.\ e\ face\text{-}of\ std\text{-}cube \wedge aff\text{-}dim\ e = 1 \wedge e \subseteq f\} = 4$
⟨*proof*⟩

**lemma** *cube-edges*:
$e\ face\text{-}of\ std\text{-}cube \wedge aff\text{-}dim\ e = 1 \longleftrightarrow$
$e = convex\ hull\ \{vector[- 1, - 1, - 1], vector[- 1, - 1, 1]\} \vee$
$e = convex\ hull\ \{vector[- 1, - 1, - 1], vector[- 1, 1, - 1]\} \vee$
$e = convex\ hull\ \{vector[- 1, - 1, - 1], vector[1, - 1, - 1]\} \vee$
$e = convex\ hull\ \{vector[- 1, - 1, 1], vector[- 1, 1, 1]\} \vee$
$e = convex\ hull\ \{vector[- 1, - 1, 1], vector[1, - 1, 1]\} \vee$
$e = convex\ hull\ \{vector[- 1, 1, - 1], vector[- 1, 1, 1]\} \vee$
$e = convex\ hull\ \{vector[- 1, 1, - 1], vector[1, 1, - 1]\} \vee$
$e = convex\ hull\ \{vector[- 1, 1, 1], vector[1, 1, 1]\} \vee$
$e = convex\ hull\ \{vector[1, - 1, - 1], vector[1, - 1, 1]\} \vee$
$e = convex\ hull\ \{vector[1, - 1, - 1], vector[1, 1, - 1]\} \vee$
$e = convex\ hull\ \{vector[1, - 1, 1], vector[1, 1, 1]\} \vee$

$e = convex\ hull\ \{vector[1,\ 1,\ -\ 1],\ vector[1,\ 1,\ 1]\}$
(**is** $?lhs \longleftrightarrow\ ?rhs$)
$\langle proof \rangle$

**lemma** *cube-vertices*:
**shows** $(v\ face\text{-}of\ std\text{-}cube \wedge aff\text{-}dim\ v = 0) =$
$(v = \{vector\ [1,\ 1,\ 1]\} \vee$
$v = \{vector\ [1,\ 1,\ -\ 1]\} \vee$
$v = \{vector\ [1,\ -1,\ 1]\} \vee$
$v = \{vector\ [1,\ -1,\ -1]\} \vee$
$v = \{vector\ [-1,\ 1,\ 1]\} \vee$
$v = \{vector\ [-1,\ 1,\ -1]\} \vee$
$v = \{vector\ [-1,\ -1,\ 1]\} \vee$
$v = \{vector\ [-1,\ -1,\ -1]\})$
(**is** $?lhs = ?rhs$)
$\langle proof \rangle$

**lemma** *cube-edges-per-vertex*:
**assumes** $v\ face\text{-}of\ std\text{-}cube$
**and** $aff\text{-}dim\ v = 0$
**shows** $card\ \{e.\ e\ face\text{-}of\ std\text{-}cube \wedge aff\text{-}dim\ e = 1 \wedge v \subseteq e\} = 3$
$\langle proof \rangle$

## 6.3 Regularity

**lemma** *cube-congruent-edges*:
**assumes** $e1\ face\text{-}of\ std\text{-}cube \wedge aff\text{-}dim\ e1 = 1$
**and** $e2\ face\text{-}of\ std\text{-}cube \wedge aff\text{-}dim\ e2 = 1$
**shows** $e1\ congruent\ e2$
$\langle proof \rangle$

**lemma** *cube-congruent-faces*:
**assumes** $f1\ face\text{-}of\ std\text{-}cube \wedge aff\text{-}dim\ f1 = 2$
**and** $f2\ face\text{-}of\ std\text{-}cube \wedge aff\text{-}dim\ f2 = 2$
**shows** $f1\ congruent\ f2$
$\langle proof \rangle$

**lemma** *cube-facet1-equiangular*:
**defines** $v1 \equiv vector[-1,\ -1,\ -1] :: real\hat{\ }3$
**defines** $v2 \equiv vector[-1,\ -1,\ 1] :: real\hat{\ }3$
**defines** $v3 \equiv vector[-1,\ 1,\ -1] :: real\hat{\ }3$
**defines** $v4 \equiv vector[-1,\ 1,\ 1] :: real\hat{\ }3$
**shows** $equiangular\ (convex\ hull\ \{v1,\ v2,\ v3,\ v4\})\ (pi\ /\ 2)$
$\langle proof \rangle$

**lemma** *cube-equiangular*:
**assumes** $f\ face\text{-}of\ std\text{-}cube \wedge aff\text{-}dim\ f = 2$
**shows** $equiangular\ f\ (pi\ /\ 2)$
$\langle proof \rangle$

**end**

# 7 Octahedron

**theory** *Octahedron*
**imports** *Computation*
**begin**

## 7.1 Definition

**definition** *std-octahedron* :: (*real*, *3*) *vec set*
**where** *std-octahedron* ≡ *convex hull*
{*vector* [*1*, *0*, *0*], *vector* [−*1*, *0*, *0*],
*vector* [*0*, *0*, *1*], *vector* [*0*, *0*, −*1*],
*vector* [*0*, *1*, *0*], *vector* [*0*, −*1*, *0*]}

**lemma** *octahedron-fulldim*:
**shows** *aff-dim std-octahedron* = *3*
⟨*proof*⟩

**lemma** *octahedron-polyhedron*:
**shows** *polyhedron std-octahedron*
⟨*proof*⟩

## 7.2 Facets, edges, and vertices

**lemma** *octahedron-facets*:
**fixes** *f* :: (*real*^*3*) *set*
**shows** *f face-of std-octahedron* ∧ *aff-dim f* = *2* ⟷
*f* = *convex hull* {*vector*[− *1*, *0*, *0*], *vector*[*0*, − *1*, *0*], *vector*[*0*, *0*, − *1*]} ∨
*f* = *convex hull* {*vector*[− *1*, *0*, *0*], *vector*[*0*, − *1*, *0*], *vector*[*0*, *0*, *1*]} ∨
*f* = *convex hull* {*vector*[− *1*, *0*, *0*], *vector*[*0*, *1*, *0*], *vector*[*0*, *0*, − *1*]} ∨
*f* = *convex hull* {*vector*[− *1*, *0*, *0*], *vector*[*0*, *1*, *0*], *vector*[*0*, *0*, *1*]} ∨
*f* = *convex hull* {*vector*[*1*, *0*, *0*], *vector*[*0*, − *1*, *0*], *vector*[*0*, *0*, − *1*]} ∨
*f* = *convex hull* {*vector*[*1*, *0*, *0*], *vector*[*0*, − *1*, *0*], *vector*[*0*, *0*, *1*]} ∨
*f* = *convex hull* {*vector*[*1*, *0*, *0*], *vector*[*0*, *1*, *0*], *vector*[*0*, *0*, − *1*]} ∨
*f* = *convex hull* {*vector*[*1*, *0*, *0*], *vector*[*0*, *1*, *0*], *vector*[*0*, *0*, *1*]}
⟨*proof*⟩

**lemma** *octahedron-facet1-edges*:
**defines** *v1* ≡ *vector*[−*1*, *0*, *0*] :: *real*^*3*
**and** *v2* ≡ *vector*[*0*, −*1*, *0*] :: *real*^*3*
**and** *v3* ≡ *vector*[*0*, *0*, −*1*] :: *real*^*3*
**and** *n* ≡ *vector*[*1*, *1*, *1*] :: *real*^*3*
**shows** (*e face-of* (*convex hull* {*v1*, *v2*, *v3*}) ∧ *aff-dim e* = *1*) =
(*e* = *convex hull* {*v1*, *v2*} ∨ *e* = *convex hull* {*v1*, *v3*} ∨ *e* = *convex hull* {*v2*, *v3*})
⟨*proof*⟩

**lemma** *octahedron-facet2-edges*:
  **defines** $v1 \equiv vector[-1, 0, 0] :: real\hat{}3$
    **and** $v2 \equiv vector[0, -1, 0] :: real\hat{}3$
    **and** $v3 \equiv vector[0, 0, 1] :: real\hat{}3$
    **and** $n \equiv vector[-1, -1, 1] :: real\hat{}3$
  **shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1, v2, v3\}) \wedge aff\text{-}dim\ e = 1) =$
    $(e = convex\ hull\ \{v1, v2\} \vee e = convex\ hull\ \{v1, v3\} \vee e = convex\ hull\ \{v2, v3\})$
$\langle proof \rangle$

**lemma** *octahedron-facet3-edges*:
  **defines** $v1 \equiv vector[-1, 0, 0] :: real\hat{}3$
    **and** $v2 \equiv vector[0, 1, 0] :: real\hat{}3$
    **and** $v3 \equiv vector[0, 0, -1] :: real\hat{}3$
    **and** $n \equiv vector[-1, 1, -1] :: real\hat{}3$
  **shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1, v2, v3\}) \wedge aff\text{-}dim\ e = 1) =$
    $(e = convex\ hull\ \{v1, v2\} \vee e = convex\ hull\ \{v1, v3\} \vee e = convex\ hull\ \{v2, v3\})$
$\langle proof \rangle$

**lemma** *octahedron-facet4-edges*:
  **defines** $v1 \equiv vector[-1, 0, 0] :: real\hat{}3$
    **and** $v2 \equiv vector[0, 1, 0] :: real\hat{}3$
    **and** $v3 \equiv vector[0, 0, 1] :: real\hat{}3$
    **and** $n \equiv vector[1, -1, -1] :: real\hat{}3$
  **shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1, v2, v3\}) \wedge aff\text{-}dim\ e = 1) =$
    $(e = convex\ hull\ \{v1, v2\} \vee e = convex\ hull\ \{v1, v3\} \vee e = convex\ hull\ \{v2, v3\})$
$\langle proof \rangle$

**lemma** *octahedron-facet5-edges*:
  **defines** $v1 \equiv vector[1, 0, 0] :: real\hat{}3$
    **and** $v2 \equiv vector[0, -1, 0] :: real\hat{}3$
    **and** $v3 \equiv vector[0, 0, -1] :: real\hat{}3$
    **and** $n \equiv vector[1, -1, -1] :: real\hat{}3$
  **shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1, v2, v3\}) \wedge aff\text{-}dim\ e = 1) =$
    $(e = convex\ hull\ \{v1, v2\} \vee e = convex\ hull\ \{v1, v3\} \vee e = convex\ hull\ \{v2, v3\})$
$\langle proof \rangle$

**lemma** *octahedron-facet6-edges*:
  **defines** $v1 \equiv vector[1, 0, 0] :: real\hat{}3$
    **and** $v2 \equiv vector[0, -1, 0] :: real\hat{}3$
    **and** $v3 \equiv vector[0, 0, 1] :: real\hat{}3$
    **and** $n \equiv vector[-1, 1, -1] :: real\hat{}3$
  **shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1, v2, v3\}) \wedge aff\text{-}dim\ e = 1) =$
    $(e = convex\ hull\ \{v1, v2\} \vee e = convex\ hull\ \{v1, v3\} \vee e = convex\ hull\ \{v2, v3\})$

⟨proof⟩

**lemma** *octahedron-facet7-edges*:
  **defines** $v1 \equiv vector[1,\ 0,\ 0]$ :: $real^\frown 3$
    **and** $v2 \equiv vector[0,\ 1,\ 0]$ :: $real^\frown 3$
    **and** $v3 \equiv vector[0,\ 0,\ -1]$ :: $real^\frown 3$
    **and** $n \equiv vector[-1,\ -1,\ 1]$ :: $real^\frown 3$
  **shows** ($e$ *face-of* (*convex hull* $\{v1,\ v2,\ v3\}$) $\wedge$ *aff-dim* $e = 1$) =
        ($e$ = *convex hull* $\{v1,\ v2\}$ $\vee$ $e$ = *convex hull* $\{v1,\ v3\}$ $\vee$ $e$ = *convex hull* $\{v2,\ v3\}$)
⟨proof⟩

**lemma** *octahedron-facet8-edges*:
  **defines** $v1 \equiv vector[1,\ 0,\ 0]$ :: $real^\frown 3$
    **and** $v2 \equiv vector[0,\ 1,\ 0]$ :: $real^\frown 3$
    **and** $v3 \equiv vector[0,\ 0,\ 1]$ :: $real^\frown 3$
    **and** $n \equiv vector[1,\ 1,\ 1]$ :: $real^\frown 3$
  **shows** ($e$ *face-of* (*convex hull* $\{v1,\ v2,\ v3\}$) $\wedge$ *aff-dim* $e = 1$) =
        ($e$ = *convex hull* $\{v1,\ v2\}$ $\vee$ $e$ = *convex hull* $\{v1,\ v3\}$ $\vee$ $e$ = *convex hull* $\{v2,\ v3\}$)
⟨proof⟩

**lemmas** *octahedron-facet-edges* =
  *octahedron-facet1-edges octahedron-facet2-edges*
  *octahedron-facet3-edges octahedron-facet4-edges octahedron-facet5-edges*
  *octahedron-facet6-edges octahedron-facet7-edges octahedron-facet8-edges*

**lemma** *octahedron-edges-per-face*:
  **assumes** *f face-of std-octahedron*
    **and** *aff-dim* $f = 2$
  **shows** *card* $\{e.\ e$ *face-of std-octahedron* $\wedge$ *aff-dim* $e = 1 \wedge e \subseteq f\} = 3$
  ⟨proof⟩

**lemma** *octahedron-edges*:
  *e face-of std-octahedron* $\wedge$ *aff-dim* $e = 1 \longleftrightarrow$
   $e$ = *convex hull* $\{vector[-\ 1,\ 0,\ 0],\ vector[0,\ -\ 1,\ 0]\}$ $\vee$
   $e$ = *convex hull* $\{vector[-\ 1,\ 0,\ 0],\ vector[0,\ 1,\ 0]\}$ $\vee$
   $e$ = *convex hull* $\{vector[-\ 1,\ 0,\ 0],\ vector[0,\ 0,\ -\ 1]\}$ $\vee$
   $e$ = *convex hull* $\{vector[-\ 1,\ 0,\ 0],\ vector[0,\ 0,\ 1]\}$ $\vee$
   $e$ = *convex hull* $\{vector[1,\ 0,\ 0],\ vector[0,\ -\ 1,\ 0]\}$ $\vee$
   $e$ = *convex hull* $\{vector[1,\ 0,\ 0],\ vector[0,\ 1,\ 0]\}$ $\vee$
   $e$ = *convex hull* $\{vector[1,\ 0,\ 0],\ vector[0,\ 0,\ -\ 1]\}$ $\vee$
   $e$ = *convex hull* $\{vector[1,\ 0,\ 0],\ vector[0,\ 0,\ 1]\}$ $\vee$
   $e$ = *convex hull* $\{vector[0,\ -\ 1,\ 0],\ vector[0,\ 0,\ -\ 1]\}$ $\vee$
   $e$ = *convex hull* $\{vector[0,\ -\ 1,\ 0],\ vector[0,\ 0,\ 1]\}$ $\vee$
   $e$ = *convex hull* $\{vector[0,\ 1,\ 0],\ vector[0,\ 0,\ -\ 1]\}$ $\vee$
   $e$ = *convex hull* $\{vector[0,\ 1,\ 0],\ vector[0,\ 0,\ 1]\}$
  (**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨proof⟩

**lemma** *octahedron-vertices*:
  **shows** ($v$ *face-of std-octahedron* $\wedge$ *aff-dim* $v = 0$) =
    ($v = \{vector\ [1,\ 0,\ 0]\}$ $\vee$
     $v = \{vector\ [-1,\ 0,\ 0]\}$ $\vee$
     $v = \{vector\ [0,\ 0,\ 1]\}$ $\vee$
     $v = \{vector\ [0,\ 0,\ -1]\}$ $\vee$
     $v = \{vector\ [0,\ 1,\ 0]\}$ $\vee$
     $v = \{vector\ [0,\ -1,\ 0]\}$)
  (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

**lemma** *octahedron-edges-per-vertex*:
  **assumes** $v$ *face-of std-octahedron*
    **and** *aff-dim* $v = 0$
  **shows** *card* $\{e.\ e$ *face-of std-octahedron* $\wedge$ *aff-dim* $e = 1 \wedge v \subseteq e\} = 4$
  ⟨*proof*⟩

### 7.3 Regularity

**lemma** *octahedron-congruent-edges*:
  **assumes** *e1 face-of std-octahedron* $\wedge$ *aff-dim e1* = *1*
    **and** *e2 face-of std-octahedron* $\wedge$ *aff-dim e2* = *2*
  **shows** *e1 congruent e2*
⟨*proof*⟩

**lemma** *octahedron-congruent-faces*:
  **assumes** *f1 face-of std-octahedron* $\wedge$ *aff-dim f1* = *2*
    **and** *f2 face-of std-octahedron* $\wedge$ *aff-dim f2* = *2*
   **shows** *f1 congruent f2*
⟨*proof*⟩

**lemma** *octahedron-facet1-equiangular*:
  **defines** $v1 \equiv vector[-1,\ 0,\ 0]$ :: *real^3*
  **defines** $v2 \equiv vector[0,\ -1,\ 0]$ :: *real^3*
  **defines** $v3 \equiv vector[0,\ 0,\ -1]$ :: *real^3*
  **shows** *equiangular* (*convex hull* $\{v1,\ v2,\ v3\}$) (*pi* / *3*)
  ⟨*proof*⟩

**lemma** *octahedron-equiangular*:
  **assumes** $f$ *face-of std-octahedron* $\wedge$ *aff-dim* $f = 2$
  **shows** *equiangular* $f$ (*pi* / *3*)
⟨*proof*⟩

**end**

## 8 Dodecahedron

**theory** *Dodecahedron*

**imports**
*Computation*
*HOL−Library.Quadratic-Discriminant*
**begin**

## 8.1 Definition

**definition** *std-dodecahedron* :: (*real*, *3*) *vec set*
**where** *std-dodecahedron* ≡ *convex hull*
{*vector* [*1*, *1*, *1*], *vector* [*1*, *1*, −*1*],
*vector* [*1*, −*1*, *1*], *vector* [*1*, −*1*, −*1*],
*vector* [−*1*, *1*, *1*], *vector* [−*1*, *1*, −*1*],
*vector* [−*1*, −*1*, *1*], *vector* [−*1*, −*1*, −*1*],
*vector* [*0*, *inverse* φ, φ], *vector*[ *0*, *inverse* φ, −φ],
*vector* [*0*, −*inverse* φ, φ], *vector* [*0*, −*inverse* φ, −φ],
*vector* [*inverse* φ, φ, *0*], *vector* [*inverse* φ, −φ, *0*],
*vector* [−*inverse* φ, φ, *0*], *vector* [−*inverse* φ, −φ, *0*],
*vector* [φ, *0*, *inverse* φ], *vector* [−φ, *0*, *inverse* φ],
*vector* [φ, *0*, −*inverse* φ], *vector* [−φ, *0*, −*inverse* φ]}

**lemma** *std-dodecahedron-explicit*:
**shows** *std-dodecahedron* = *convex hull*
{ *vector*[*1*, *1*, *1*],
*vector*[*1*, *1*, −*1*],
*vector*[*1*, −*1*, *1*],
*vector*[*1*, −*1*, −*1*],
*vector*[−*1*, *1*, *1*],
*vector*[−*1*, *1*, −*1*],
*vector*[−*1*, −*1*, *1*],
*vector*[−*1*, −*1*, −*1*],
*vector*[*0*, −*1* / *2* + *1* / *2* ∗ *sqrt*(*5*), *1* / *2* + *1* / *2* ∗ *sqrt*(*5*)],
*vector*[*0*, −*1* / *2* + *1* / *2* ∗ *sqrt*(*5*), −*1* / *2* + −*1* / *2* ∗ *sqrt*(*5*)],
*vector*[*0*, *1* / *2* + −*1* / *2* ∗ *sqrt*(*5*), *1* / *2* + *1* / *2* ∗ *sqrt*(*5*)],
*vector*[*0*, *1* / *2* + −*1* / *2* ∗ *sqrt*(*5*), −*1* / *2* + −*1* / *2* ∗ *sqrt*(*5*)],
*vector*[−*1* / *2* + *1* / *2* ∗ *sqrt*(*5*), *1* / *2* + *1* / *2* ∗ *sqrt*(*5*), *0*],
*vector*[−*1* / *2* + *1* / *2* ∗ *sqrt*(*5*), −*1* / *2* + −*1* / *2* ∗ *sqrt*(*5*), *0*],
*vector*[*1* / *2* + −*1* / *2* ∗ *sqrt*(*5*), *1* / *2* + *1* / *2* ∗ *sqrt*(*5*), *0*],
*vector*[*1* / *2* + −*1* / *2* ∗ *sqrt*(*5*), −*1* / *2* + −*1* / *2* ∗ *sqrt*(*5*), *0*],
*vector*[*1* / *2* + *1* / *2* ∗ *sqrt*(*5*), *0*, −*1* / *2* + *1* / *2* ∗ *sqrt*(*5*)],
*vector*[−*1* / *2* + −*1* / *2* ∗ *sqrt*(*5*), *0*, −*1* / *2* + *1* / *2* ∗ *sqrt*(*5*)],
*vector*[*1* / *2* + *1* / *2* ∗ *sqrt*(*5*), *0*, *1* / *2* + −*1* / *2* ∗ *sqrt*(*5*)],
*vector*[−*1* / *2* + −*1* / *2* ∗ *sqrt*(*5*), *0*, *1* / *2* + −*1* / *2* ∗ *sqrt*(*5*)]}
⟨*proof*⟩

**lemma** *std-dodecahedron-eq*:
**shows** *std-dodecahedron* = *convex hull*
{ *vector*[*rat5 1 0*, *rat5 1 0*, *rat5 1 0*],
*vector*[*rat5 1 0*, *rat5 1 0*, *rat5* (−*1*) *0*],
*vector*[*rat5 1 0*, *rat5* (−*1*) *0*, *rat5 1 0*],

vector[rat5 1 0, rat5 (−1) 0, rat5 (−1) 0],
vector[rat5 (−1) 0, rat5 1 0, rat5 1 0],
vector[rat5 (−1) 0, rat5 1 0, rat5 (−1) 0],
vector[rat5 (−1) 0, rat5 (−1) 0, rat5 1 0],
vector[rat5 (−1) 0, rat5 (−1) 0, rat5 (−1) 0],
vector[rat5 0 0, rat5 (−1/2) (1/2), rat5 (1/2) (1/2)],
vector[rat5 0 0, rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2)],
vector[rat5 0 0, rat5 (1/2) (−1/2), rat5 (1/2) (1/2)],
vector[rat5 0 0, rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2)],
vector[rat5 (−1/2) (1/2), rat5 (1/2) (1/2), rat5 0 0],
vector[rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2), rat5 0 0],
vector[rat5 (1/2) (−1/2), rat5 (1/2) (1/2), rat5 0 0],
vector[rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2), rat5 0 0],
vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1/2) (1/2)],
vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1/2) (1/2)],
vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (1/2) (−1/2)],
vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (1/2) (−1/2)]}
⟨proof⟩

**lemma** *dodecahedron-fulldim*:
**shows** *aff-dim std-dodecahedron = 3*
⟨proof⟩

**lemma** *dodecahedron-polyhedron*:
**shows** *polyhedron std-dodecahedron*
⟨proof⟩

## 8.2 Facets, edges, and vertices

**lemma** *dodecahedron-facets*:
**fixes** *f* :: (*real^3*) *set*
**shows** *f face-of std-dodecahedron* ∧ *aff-dim f* = *2* ⟷
f = convex hull {vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1/2) (1/2)], vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 (−1) 0, rat5 (−1) 0, rat5 (−1) 0], vector[rat5 (−1) 0, rat5 (−1) 0, rat5 1 0]} ∨
f = convex hull {vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1/2) (1/2)], vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 (1/2) (−1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 (−1) 0, rat5 1 0, rat5 (−1) 0], vector[rat5 (−1) 0, rat5 1 0, rat5 1 0]} ∨
f = convex hull {vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1/2) (1/2)], vector[rat5 (−1) 0, rat5 (−1) 0, rat5 1 0], vector[rat5 (−1) 0, rat5 1 0, rat5 1 0], vector[rat5 0 0, rat5 (−1/2) (1/2), rat5 (1/2) (1/2)], vector[rat5 0 0, rat5 (1/2) (−1/2), rat5 (1/2) (1/2)]} ∨
f = convex hull {vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 (−1) 0, rat5 (−1) 0, rat5 (−1) 0], vector[rat5 (−1) 0, rat5 1 0, rat5 (−1) 0], vector[rat5 0 0, rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2)], vector[rat5 0 0, rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2)]} ∨
f = convex hull {vector[rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2), rat5 0 0],

vector[rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 (−1) 0, rat5 (−1) 0, rat5 (−1) 0], vector[rat5 1 0, rat5 (−1) 0, rat5 (−1) 0], vector[rat5 0 0, rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2)]} ∨

f = convex hull {vector[rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 (−1) 0, rat5 (−1) 0, rat5 1 0], vector[rat5 1 0, rat5 (−1) 0, rat5 1 0], vector[rat5 0 0, rat5 (1/2) (−1/2), rat5 (1/2) (1/2)]} ∨

f = convex hull {vector[rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1/2) (1/2)], vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 1 0, rat5 (−1) 0, rat5 (−1) 0], vector[rat5 1 0, rat5 (−1) 0, rat5 1 0]} ∨

f = convex hull {vector[rat5 (−1/2) (1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 (1/2) (−1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 (−1) 0, rat5 1 0, rat5 (−1) 0], vector[rat5 1 0, rat5 1 0, rat5 (−1) 0], vector[rat5 0 0, rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2)]} ∨

f = convex hull {vector[rat5 (−1/2) (1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 (1/2) (−1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 (−1) 0, rat5 1 0, rat5 1 0], vector[rat5 1 0, rat5 1 0, rat5 1 0], vector[rat5 0 0, rat5 (−1/2) (1/2), rat5 (1/2) (1/2)]} ∨

f = convex hull {vector[rat5 (−1/2) (1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1/2) (1/2)], vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 1 0, rat5 1 0, rat5 (−1) 0], vector[rat5 1 0, rat5 1 0, rat5 1 0]} ∨

f = convex hull {vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1/2) (1/2)], vector[rat5 1 0, rat5 (−1) 0, rat5 1 0], vector[rat5 1 0, rat5 1 0, rat5 1 0], vector[rat5 0 0, rat5 (−1/2) (1/2), rat5 (1/2) (1/2)], vector[rat5 0 0, rat5 (1/2) (−1/2), rat5 (1/2) (1/2)]} ∨

f = convex hull {vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 1 0, rat5 (−1) 0, rat5 (−1) 0], vector[rat5 1 0, rat5 1 0, rat5 (−1) 0], vector[rat5 0 0, rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2)], vector[rat5 0 0, rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2)]}
⟨proof⟩

**lemma** *dodecahedron-facet1-edges*:
**defines** v1 ≡ vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1/2) (1/2)] :: real^3
**and** v2 ≡ vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (1/2) (−1/2)] :: real^3
**and** v3 ≡ vector[rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2), rat5 0 0] :: real^3
**and** v4 ≡ vector[rat5 (−1) 0, rat5 (−1) 0, rat5 (−1) 0] :: real^3
**and** v5 ≡ vector[rat5 (−1) 0, rat5 (−1) 0, rat5 1 0] :: real^3
**and** n ≡ vector[rat5 (−2) 0, rat5 1 (−1), rat5 0 0] :: real^3
**shows** (e face-of (convex hull {v1, v2, v3, v4, v5}) ∧ aff-dim e = 1) =
(e = convex hull {v1, v2} ∨ e = convex hull {v1, v5} ∨ e = convex hull {v2, v4} ∨ e = convex hull {v3, v4} ∨ e = convex hull {v3, v5})
⟨proof⟩

**lemma** *dodecahedron-facet2-edges*:
**defines** v1 ≡ vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (−1/2) (1/2)] :: real^3
**and** v2 ≡ vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (1/2) (−1/2)] :: real^3
**and** v3 ≡ vector[rat5 (1/2) (−1/2), rat5 (1/2) (1/2), rat5 0 0] :: real^3

**and** $v4 \equiv vector[rat5\ (-1)\ 0,\ rat5\ 1\ 0,\ rat5\ (-1)\ 0] :: real\hat{\ }3$
**and** $v5 \equiv vector[rat5\ (-1)\ 0,\ rat5\ 1\ 0,\ rat5\ 1\ 0] :: real\hat{\ }3$
**and** $n \equiv vector[rat5\ 2\ 0,\ rat5\ 1\ (-1),\ rat5\ 0\ 0] :: real\hat{\ }3$
**shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1,\ v2,\ v3,\ v4,\ v5\}) \wedge aff\text{-}dim\ e = 1) =$
$(e = convex\ hull\ \{v1,\ v2\} \vee e = convex\ hull\ \{v1,\ v5\} \vee e = convex\ hull\ \{v2,\ v4\} \vee e = convex\ hull\ \{v3,\ v4\} \vee e = convex\ hull\ \{v3,\ v5\})$
$\langle proof \rangle$

**lemma** *dodecahedron-facet3-edges*:
**defines** $v1 \equiv vector[rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0,\ rat5\ (-1/2)\ (1/2)] :: real\hat{\ }3$
**and** $v2 \equiv vector[rat5\ (-1)\ 0,\ rat5\ (-1)\ 0,\ rat5\ 1\ 0] :: real\hat{\ }3$
**and** $v3 \equiv vector[rat5\ (-1)\ 0,\ rat5\ 1\ 0,\ rat5\ 1\ 0] :: real\hat{\ }3$
**and** $v4 \equiv vector[rat5\ 0\ 0,\ rat5\ (-1/2)\ (1/2),\ rat5\ (1/2)\ (1/2)] :: real\hat{\ }3$
**and** $v5 \equiv vector[rat5\ 0\ 0,\ rat5\ (1/2)\ (-1/2),\ rat5\ (1/2)\ (1/2)] :: real\hat{\ }3$
**and** $n \equiv vector[rat5\ (-3)\ 1,\ rat5\ 0\ 0,\ rat5\ (-1)\ 1] :: real\hat{\ }3$
**shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1,\ v2,\ v3,\ v4,\ v5\}) \wedge aff\text{-}dim\ e = 1) =$
$(e = convex\ hull\ \{v1,\ v2\} \vee e = convex\ hull\ \{v1,\ v3\} \vee e = convex\ hull\ \{v2,\ v5\} \vee e = convex\ hull\ \{v3,\ v4\} \vee e = convex\ hull\ \{v4,\ v5\})$
$\langle proof \rangle$

**lemma** *dodecahedron-facet4-edges*:
**defines** $v1 \equiv vector[rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0,\ rat5\ (1/2)\ (-1/2)] :: real\hat{\ }3$
**and** $v2 \equiv vector[rat5\ (-1)\ 0,\ rat5\ (-1)\ 0,\ rat5\ (-1)\ 0] :: real\hat{\ }3$
**and** $v3 \equiv vector[rat5\ (-1)\ 0,\ rat5\ 1\ 0,\ rat5\ (-1)\ 0] :: real\hat{\ }3$
**and** $v4 \equiv vector[rat5\ 0\ 0,\ rat5\ (-1/2)\ (1/2),\ rat5\ (-1/2)\ (-1/2)] :: real\hat{\ }3$
**and** $v5 \equiv vector[rat5\ 0\ 0,\ rat5\ (1/2)\ (-1/2),\ rat5\ (-1/2)\ (-1/2)] :: real\hat{\ }3$
**and** $n \equiv vector[rat5\ 3\ (-1),\ rat5\ 0\ 0,\ rat5\ (-1)\ 1] :: real\hat{\ }3$
**shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1,\ v2,\ v3,\ v4,\ v5\}) \wedge aff\text{-}dim\ e = 1) =$
$(e = convex\ hull\ \{v1,\ v2\} \vee e = convex\ hull\ \{v1,\ v3\} \vee e = convex\ hull\ \{v2,\ v5\} \vee e = convex\ hull\ \{v3,\ v4\} \vee e = convex\ hull\ \{v4,\ v5\})$
$\langle proof \rangle$

**lemma** *dodecahedron-facet5-edges*:
**defines** $v1 \equiv vector[rat5\ (-1/2)\ (1/2),\ rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0] :: real\hat{\ }3$
**and** $v2 \equiv vector[rat5\ (1/2)\ (-1/2),\ rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0] :: real\hat{\ }3$
**and** $v3 \equiv vector[rat5\ (-1)\ 0,\ rat5\ (-1)\ 0,\ rat5\ (-1)\ 0] :: real\hat{\ }3$
**and** $v4 \equiv vector[rat5\ 1\ 0,\ rat5\ (-1)\ 0,\ rat5\ (-1)\ 0] :: real\hat{\ }3$
**and** $v5 \equiv vector[rat5\ 0\ 0,\ rat5\ (1/2)\ (-1/2),\ rat5\ (-1/2)\ (-1/2)] :: real\hat{\ }3$
**and** $n \equiv vector[rat5\ 0\ 0,\ rat5\ 1\ (-1),\ rat5\ (-3)\ 1] :: real\hat{\ }3$
**shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1,\ v2,\ v3,\ v4,\ v5\}) \wedge aff\text{-}dim\ e = 1) =$
$(e = convex\ hull\ \{v1,\ v2\} \vee e = convex\ hull\ \{v1,\ v4\} \vee e = convex\ hull\ \{v2,\ v3\} \vee e = convex\ hull\ \{v3,\ v5\} \vee e = convex\ hull\ \{v4,\ v5\})$
$\langle proof \rangle$

**lemma** *dodecahedron-facet6-edges*:
**defines** $v1 \equiv vector[rat5\ (-1/2)\ (1/2),\ rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0] :: real\hat{\ }3$
**and** $v2 \equiv vector[rat5\ (1/2)\ (-1/2),\ rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0] :: real\hat{\ }3$
**and** $v3 \equiv vector[rat5\ (-1)\ 0,\ rat5\ (-1)\ 0,\ rat5\ 1\ 0] :: real\hat{\ }3$
**and** $v4 \equiv vector[rat5\ 1\ 0,\ rat5\ (-1)\ 0,\ rat5\ 1\ 0] :: real\hat{\ }3$

**and** *v5* ≡ *vector*[*rat5 0 0*, *rat5* (*1/2*) (−*1/2*), *rat5* (*1/2*) (*1/2*)] :: *real^3*
**and** *n* ≡ *vector*[*rat5 0 0*, *rat5* (−*1*) *1*, *rat5* (−*3*) *1*] :: *real^3*
**shows** (*e face-of* (*convex hull* {*v1*, *v2*, *v3*, *v4*, *v5*}) ∧ *aff-dim e* = *1*) =
(*e* = *convex hull* {*v1*, *v2*} ∨ *e* = *convex hull* {*v1*, *v4*} ∨ *e* = *convex hull* {*v2*, *v3*} ∨ *e* = *convex hull* {*v3*, *v5*} ∨ *e* = *convex hull* {*v4*, *v5*})
⟨*proof*⟩

**lemma** *dodecahedron-facet7-edges*:
**defines** *v1* ≡ *vector*[*rat5* (−*1/2*) (*1/2*), *rat5* (−*1/2*) (−*1/2*), *rat5 0 0*] :: *real^3*
**and** *v2* ≡ *vector*[*rat5* (*1/2*) (*1/2*), *rat5 0 0*, *rat5* (−*1/2*) (*1/2*)] :: *real^3*
**and** *v3* ≡ *vector*[*rat5* (*1/2*) (*1/2*), *rat5 0 0*, *rat5* (*1/2*) (−*1/2*)] :: *real^3*
**and** *v4* ≡ *vector*[*rat5 1 0*, *rat5* (−*1*) *0*, *rat5* (−*1*) *0*] :: *real^3*
**and** *v5* ≡ *vector*[*rat5 1 0*, *rat5* (−*1*) *0*, *rat5 1 0*] :: *real^3*
**and** *n* ≡ *vector*[*rat5* (−*2*) *0*, *rat5* (−*1*) *1*, *rat5 0 0*] :: *real^3*
**shows** (*e face-of* (*convex hull* {*v1*, *v2*, *v3*, *v4*, *v5*}) ∧ *aff-dim e* = *1*) =
(*e* = *convex hull* {*v1*, *v4*} ∨ *e* = *convex hull* {*v1*, *v5*} ∨ *e* = *convex hull* {*v2*, *v3*} ∨ *e* = *convex hull* {*v2*, *v5*} ∨ *e* = *convex hull* {*v3*, *v4*})
⟨*proof*⟩

**lemma** *dodecahedron-facet8-edges*:
**defines** *v1* ≡ *vector*[*rat5* (−*1/2*) (*1/2*), *rat5* (*1/2*) (*1/2*), *rat5 0 0*] :: *real^3*
**and** *v2* ≡ *vector*[*rat5* (*1/2*) (−*1/2*), *rat5* (*1/2*) (*1/2*), *rat5 0 0*] :: *real^3*
**and** *v3* ≡ *vector*[*rat5* (−*1*) *0*, *rat5 1 0*, *rat5* (−*1*) *0*] :: *real^3*
**and** *v4* ≡ *vector*[*rat5 1 0*, *rat5 1 0*, *rat5* (−*1*) *0*] :: *real^3*
**and** *v5* ≡ *vector*[*rat5 0 0*, *rat5* (−*1/2*) (*1/2*), *rat5* (−*1/2*) (−*1/2*)] :: *real^3*
**and** *n* ≡ *vector*[*rat5 0 0*, *rat5 1* (−*1*), *rat5 3* (−*1*)] :: *real^3*
**shows** (*e face-of* (*convex hull* {*v1*, *v2*, *v3*, *v4*, *v5*}) ∧ *aff-dim e* = *1*) =
(*e* = *convex hull* {*v1*, *v2*} ∨ *e* = *convex hull* {*v1*, *v4*} ∨ *e* = *convex hull* {*v2*, *v3*} ∨ *e* = *convex hull* {*v3*, *v5*} ∨ *e* = *convex hull* {*v4*, *v5*})
⟨*proof*⟩

**lemma** *dodecahedron-facet9-edges*:
**defines** *v1* ≡ *vector*[*rat5* (−*1/2*) (*1/2*), *rat5* (*1/2*) (*1/2*), *rat5 0 0*] :: *real^3*
**and** *v2* ≡ *vector*[*rat5* (*1/2*) (−*1/2*), *rat5* (*1/2*) (*1/2*), *rat5 0 0*] :: *real^3*
**and** *v3* ≡ *vector*[*rat5* (−*1*) *0*, *rat5 1 0*, *rat5 1 0*] :: *real^3*
**and** *v4* ≡ *vector*[*rat5 1 0*, *rat5 1 0*, *rat5 1 0*] :: *real^3*
**and** *v5* ≡ *vector*[*rat5 0 0*, *rat5* (−*1/2*) (*1/2*), *rat5* (*1/2*) (*1/2*)] :: *real^3*
**and** *n* ≡ *vector*[*rat5 0 0*, *rat5* (−*1*) *1*, *rat5 3* (−*1*)] :: *real^3*
**shows** (*e face-of* (*convex hull* {*v1*, *v2*, *v3*, *v4*, *v5*}) ∧ *aff-dim e* = *1*) =
(*e* = *convex hull* {*v1*, *v2*} ∨ *e* = *convex hull* {*v1*, *v4*} ∨ *e* = *convex hull* {*v2*, *v3*} ∨ *e* = *convex hull* {*v3*, *v5*} ∨ *e* = *convex hull* {*v4*, *v5*})
⟨*proof*⟩

**lemma** *dodecahedron-facet10-edges*:
**defines** *v1* ≡ *vector*[*rat5* (−*1/2*) (*1/2*), *rat5* (*1/2*) (*1/2*), *rat5 0 0*] :: *real^3*
**and** *v2* ≡ *vector*[*rat5* (*1/2*) (*1/2*), *rat5 0 0*, *rat5* (−*1/2*) (*1/2*)] :: *real^3*
**and** *v3* ≡ *vector*[*rat5* (*1/2*) (*1/2*), *rat5 0 0*, *rat5* (*1/2*) (−*1/2*)] :: *real^3*
**and** *v4* ≡ *vector*[*rat5 1 0*, *rat5 1 0*, *rat5* (−*1*) *0*] :: *real^3*
**and** *v5* ≡ *vector*[*rat5 1 0*, *rat5 1 0*, *rat5 1 0*] :: *real^3*

**and** $n \equiv vector[rat5\ 2\ 0,\ rat5\ (-1)\ 1,\ rat5\ 0\ 0]$ :: $real\hat{\ }3$
**shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1,\ v2,\ v3,\ v4,\ v5\}) \wedge aff\text{-}dim\ e = 1) =$
$(e = convex\ hull\ \{v1,\ v4\} \vee e = convex\ hull\ \{v1,\ v5\} \vee e = convex\ hull\ \{v2,\ v3\} \vee e = convex\ hull\ \{v2,\ v5\} \vee e = convex\ hull\ \{v3,\ v4\})$
$\langle proof \rangle$

**lemma** *dodecahedron-facet11-edges*:
**defines** $v1 \equiv vector[rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0,\ rat5\ (-1/2)\ (1/2)]$ :: $real\hat{\ }3$
**and** $v2 \equiv vector[rat5\ 1\ 0,\ rat5\ (-1)\ 0,\ rat5\ 1\ 0]$ :: $real\hat{\ }3$
**and** $v3 \equiv vector[rat5\ 1\ 0,\ rat5\ 1\ 0,\ rat5\ 1\ 0]$ :: $real\hat{\ }3$
**and** $v4 \equiv vector[rat5\ 0\ 0,\ rat5\ (-1/2)\ (1/2),\ rat5\ (1/2)\ (1/2)]$ :: $real\hat{\ }3$
**and** $v5 \equiv vector[rat5\ 0\ 0,\ rat5\ (1/2)\ (-1/2),\ rat5\ (1/2)\ (1/2)]$ :: $real\hat{\ }3$
**and** $n \equiv vector[rat5\ (-3)\ 1,\ rat5\ 0\ 0,\ rat5\ 1\ (-1)]$ :: $real\hat{\ }3$
**shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1,\ v2,\ v3,\ v4,\ v5\}) \wedge aff\text{-}dim\ e = 1) =$
$(e = convex\ hull\ \{v1,\ v2\} \vee e = convex\ hull\ \{v1,\ v3\} \vee e = convex\ hull\ \{v2,\ v5\} \vee e = convex\ hull\ \{v3,\ v4\} \vee e = convex\ hull\ \{v4,\ v5\})$
$\langle proof \rangle$

**lemma** *dodecahedron-facet12-edges*:
**defines** $v1 \equiv vector[rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0,\ rat5\ (1/2)\ (-1/2)]$ :: $real\hat{\ }3$
**and** $v2 \equiv vector[rat5\ 1\ 0,\ rat5\ (-1)\ 0,\ rat5\ (-1)\ 0]$ :: $real\hat{\ }3$
**and** $v3 \equiv vector[rat5\ 1\ 0,\ rat5\ 1\ 0,\ rat5\ (-1)\ 0]$ :: $real\hat{\ }3$
**and** $v4 \equiv vector[rat5\ 0\ 0,\ rat5\ (-1/2)\ (1/2),\ rat5\ (-1/2)\ (-1/2)]$ :: $real\hat{\ }3$
**and** $v5 \equiv vector[rat5\ 0\ 0,\ rat5\ (1/2)\ (-1/2),\ rat5\ (-1/2)\ (-1/2)]$ :: $real\hat{\ }3$
**and** $n \equiv vector[rat5\ 3\ (-1),\ rat5\ 0\ 0,\ rat5\ 1\ (-1)]$ :: $real\hat{\ }3$
**shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1,\ v2,\ v3,\ v4,\ v5\}) \wedge aff\text{-}dim\ e = 1) =$
$(e = convex\ hull\ \{v1,\ v2\} \vee e = convex\ hull\ \{v1,\ v3\} \vee e = convex\ hull\ \{v2,\ v5\} \vee e = convex\ hull\ \{v3,\ v4\} \vee e = convex\ hull\ \{v4,\ v5\})$
$\langle proof \rangle$

**lemmas** *dodecahedron-facet-edges* =
*dodecahedron-facet1-edges dodecahedron-facet2-edges dodecahedron-facet3-edges dodecahedron-facet4-edges dodecahedron-facet5-edges dodecahedron-facet6-edges dodecahedron-facet7-edges dodecahedron-facet8-edges dodecahedron-facet9-edges dodecahedron-facet10-edges dodecahedron-facet11-edges dodecahedron-facet12-edges*

**lemma** *dodecahedron-edges-per-face*:
**assumes** $f\ face\text{-}of\ std\text{-}dodecahedron$
**and** $aff\text{-}dim\ f = 2$
**shows** $card\ \{e.\ e\ face\text{-}of\ std\text{-}dodecahedron \wedge aff\text{-}dim\ e = 1 \wedge e \subseteq f\} = 5$
$\langle proof \rangle$

**lemma** *dodecahedron-edges*:
$e\ face\text{-}of\ std\text{-}dodecahedron \wedge aff\text{-}dim\ e = 1 \longleftrightarrow$
$e = convex\ hull\ \{vector[rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0,\ rat5\ (-1/2)\ (1/2)],\ vector[rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0,\ rat5\ (1/2)\ (-1/2)]\} \vee$
$e = convex\ hull\ \{vector[rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0,\ rat5\ (-1/2)\ (1/2)],\ vector[rat5\ (-1)\ 0,\ rat5\ (-1)\ 0,\ rat5\ 1\ 0]\} \vee$
$e = convex\ hull\ \{vector[rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0,\ rat5\ (-1/2)\ (1/2)],$

vector[rat5 (−1) 0, rat5 1 0, rat5 1 0]} ∨

e = convex hull {vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 (−1) 0, rat5 (−1) 0, rat5 (−1) 0]} ∨

e = convex hull {vector[rat5 (−1/2) (−1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 (−1) 0, rat5 1 0, rat5 (−1) 0]} ∨

e = convex hull {vector[rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2), rat5 0 0]} ∨

e = convex hull {vector[rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 1 0, rat5 (−1) 0, rat5 (−1) 0]} ∨

e = convex hull {vector[rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 1 0, rat5 (−1) 0, rat5 1 0]} ∨

e = convex hull {vector[rat5 (−1/2) (1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 (1/2) (−1/2), rat5 (1/2) (1/2), rat5 0 0]} ∨

e = convex hull {vector[rat5 (−1/2) (1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 1 0, rat5 1 0, rat5 (−1) 0]} ∨

e = convex hull {vector[rat5 (−1/2) (1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 1 0, rat5 1 0, rat5 1 0]} ∨

e = convex hull {vector[rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 (−1) 0, rat5 (−1) 0, rat5 (−1) 0]} ∨

e = convex hull {vector[rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2), rat5 0 0], vector[rat5 (−1) 0, rat5 (−1) 0, rat5 1 0]} ∨

e = convex hull {vector[rat5 (1/2) (−1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 (−1) 0, rat5 1 0, rat5 (−1) 0]} ∨

e = convex hull {vector[rat5 (1/2) (−1/2), rat5 (1/2) (1/2), rat5 0 0], vector[rat5 (−1) 0, rat5 1 0, rat5 1 0]} ∨

e = convex hull {vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1/2) (1/2)], vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (1/2) (−1/2)]} ∨

e = convex hull {vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1/2) (1/2)], vector[rat5 1 0, rat5 (−1) 0, rat5 1 0]} ∨

e = convex hull {vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (−1/2) (1/2)], vector[rat5 1 0, rat5 1 0, rat5 1 0]} ∨

e = convex hull {vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 1 0, rat5 (−1) 0, rat5 (−1) 0]} ∨

e = convex hull {vector[rat5 (1/2) (1/2), rat5 0 0, rat5 (1/2) (−1/2)], vector[rat5 1 0, rat5 1 0, rat5 (−1) 0]} ∨

e = convex hull {vector[rat5 (−1) 0, rat5 (−1) 0, rat5 (−1) 0], vector[rat5 0 0, rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2)]} ∨

e = convex hull {vector[rat5 (−1) 0, rat5 (−1) 0, rat5 1 0], vector[rat5 0 0, rat5 (1/2) (−1/2), rat5 (1/2) (1/2)]} ∨

e = convex hull {vector[rat5 (−1) 0, rat5 1 0, rat5 (−1) 0], vector[rat5 0 0, rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2)]} ∨

e = convex hull {vector[rat5 (−1) 0, rat5 1 0, rat5 1 0], vector[rat5 0 0, rat5 (−1/2) (1/2), rat5 (1/2) (1/2)]} ∨

e = convex hull {vector[rat5 1 0, rat5 (−1) 0, rat5 (−1) 0], vector[rat5 0 0, rat5 (1/2) (−1/2), rat5 (−1/2) (−1/2)]} ∨

e = convex hull {vector[rat5 1 0, rat5 (−1) 0, rat5 1 0], vector[rat5 0 0, rat5 (1/2) (−1/2), rat5 (1/2) (1/2)]} ∨

e = convex hull {vector[rat5 1 0, rat5 1 0, rat5 (−1) 0], vector[rat5 0 0, rat5 (−1/2) (1/2), rat5 (−1/2) (−1/2)]} ∨

$e = convex\ hull\ \{vector[rat5\ 1\ 0,\ rat5\ 1\ 0,\ rat5\ 1\ 0],\ vector[rat5\ 0\ 0,\ rat5\ (-1/2)\ (1/2),\ rat5\ (1/2)\ (1/2)]\} \lor$
$e = convex\ hull\ \{vector[rat5\ 0\ 0,\ rat5\ (-1/2)\ (1/2),\ rat5\ (-1/2)\ (-1/2)],\ vector[rat5\ 0\ 0,\ rat5\ (1/2)\ (-1/2),\ rat5\ (-1/2)\ (-1/2)]\} \lor$
$e = convex\ hull\ \{vector[rat5\ 0\ 0,\ rat5\ (-1/2)\ (1/2),\ rat5\ (1/2)\ (1/2)],\ vector[rat5\ 0\ 0,\ rat5\ (1/2)\ (-1/2),\ rat5\ (1/2)\ (1/2)]\}$
(**is** $?lhs \longleftrightarrow ?rhs$)
$\langle proof \rangle$

**lemma** *dodecahedron-vertices*:
**shows** $(v\ face\text{-}of\ std\text{-}dodecahedron \land aff\text{-}dim\ v = 0) =$
$(v = \{vector[rat5\ 1\ 0,\ rat5\ 1\ 0,\ rat5\ 1\ 0]\} \lor$
$v = \{vector[rat5\ 1\ 0,\ rat5\ 1\ 0,\ rat5\ (-1)\ 0]\} \lor$
$v = \{vector[rat5\ 1\ 0,\ rat5\ (-1)\ 0,\ rat5\ 1\ 0]\} \lor$
$v = \{vector[rat5\ 1\ 0,\ rat5\ (-1)\ 0,\ rat5\ (-1)\ 0]\} \lor$
$v = \{vector[rat5\ (-1)\ 0,\ rat5\ 1\ 0,\ rat5\ 1\ 0]\} \lor$
$v = \{vector[rat5\ (-1)\ 0,\ rat5\ 1\ 0,\ rat5\ (-1)\ 0]\} \lor$
$v = \{vector[rat5\ (-1)\ 0,\ rat5\ (-1)\ 0,\ rat5\ 1\ 0]\} \lor$
$v = \{vector[rat5\ (-1)\ 0,\ rat5\ (-1)\ 0,\ rat5\ (-1)\ 0]\} \lor$
$v = \{vector[rat5\ 0\ 0,\ rat5\ (-1/2)\ (1/2),\ rat5\ (1/2)\ (1/2)]\} \lor$
$v = \{vector[rat5\ 0\ 0,\ rat5\ (-1/2)\ (1/2),\ rat5\ (-1/2)\ (-1/2)]\} \lor$
$v = \{vector[rat5\ 0\ 0,\ rat5\ (1/2)\ (-1/2),\ rat5\ (1/2)\ (1/2)]\} \lor$
$v = \{vector[rat5\ 0\ 0,\ rat5\ (1/2)\ (-1/2),\ rat5\ (-1/2)\ (-1/2)]\} \lor$
$v = \{vector[rat5\ (-1/2)\ (1/2),\ rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0]\} \lor$
$v = \{vector[rat5\ (-1/2)\ (1/2),\ rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0]\} \lor$
$v = \{vector[rat5\ (1/2)\ (-1/2),\ rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0]\} \lor$
$v = \{vector[rat5\ (1/2)\ (-1/2),\ rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0]\} \lor$
$v = \{vector[rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0,\ rat5\ (-1/2)\ (1/2)]\} \lor$
$v = \{vector[rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0,\ rat5\ (-1/2)\ (1/2)]\} \lor$
$v = \{vector[rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0,\ rat5\ (1/2)\ (-1/2)]\} \lor$
$v = \{vector[rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0,\ rat5\ (1/2)\ (-1/2)]\})$
(**is** $?lhs = ?rhs$)
$\langle proof \rangle$

**lemma** *dodecahedron-edges-per-vertex*:
**assumes** $v\ face\text{-}of\ std\text{-}dodecahedron$
**and** $aff\text{-}dim\ v = 0$
**shows** $card\ \{e.\ e\ face\text{-}of\ std\text{-}dodecahedron \land aff\text{-}dim\ e = 1 \land v \subseteq e\} = 3$
$\langle proof \rangle$

## 8.3 Regularity

**lemma** *dodecahedron-congruent-edges*:
**assumes** $e1\ face\text{-}of\ std\text{-}dodecahedron \land aff\text{-}dim\ e1 = 1$
**and** $e2\ face\text{-}of\ std\text{-}dodecahedron \land aff\text{-}dim\ e2 = 1$
**shows** $e1\ congruent\ e2$
$\langle proof \rangle$

**lemma** *dodecahedron-congruent-faces*:

**assumes** *f1 face-of std-dodecahedron* $\wedge$ *aff-dim f1* = *2*
**and** *f2 face-of std-dodecahedron* $\wedge$ *aff-dim f2* = *2*
**shows** *f1 congruent f2*
$\langle proof \rangle$

**lemma** *dodecahedron-facet1-equiangular*:
**defines** *v1* $\equiv$ *vector*[*rat5* (−*1*/*2*) (−*1*/*2*), *rat5 0 0*, *rat5* (−*1*/*2*) (*1*/*2*)] :: *real^3*
**defines** *v2* $\equiv$ *vector*[*rat5* (−*1*/*2*) (−*1*/*2*), *rat5 0 0*, *rat5* (*1*/*2*) (−*1*/*2*)] :: *real^3*
**defines** *v3* $\equiv$ *vector*[*rat5* (*1*/*2*) (−*1*/*2*), *rat5* (−*1*/*2*) (−*1*/*2*), *rat5 0 0*] :: *real^3*
**defines** *v4* $\equiv$ *vector*[*rat5* (−*1*) *0*, *rat5* (−*1*) *0*, *rat5* (−*1*) *0*] :: *real^3*
**defines** *v5* $\equiv$ *vector*[*rat5* (−*1*) *0*, *rat5* (−*1*) *0*, *rat5 1 0*] :: *real^3*
**shows** *equiangular* (*convex hull* {*v1*, *v2*, *v3*, *v4*, *v5*}) (*3* ∗ *pi* / *5*)
$\langle proof \rangle$

**lemma** *dodecahedron-equiangular*:
**assumes** *f face-of std-dodecahedron* $\wedge$ *aff-dim f* = *2*
**shows** *equiangular f* (*3* ∗ *pi* / *5*)
$\langle proof \rangle$

**end**

# 9 Icosahedron

**theory** *Icosahedron*
**imports**
*Computation*
**begin**

## 9.1 Definition

**definition** *std-icosahedron* :: (*real*, *3*) *vec set*
**where** *std-icosahedron* $\equiv$ *convex hull*
{*vector* [*0*, *1*, $\varphi$], *vector* [*0*, *1*, −$\varphi$],
*vector* [*0*, −*1*, $\varphi$], *vector* [*0*, −*1*, −$\varphi$],
*vector* [*1*, $\varphi$, *0*], *vector* [*1*, −$\varphi$, *0*],
*vector* [−*1*, $\varphi$, *0*], *vector* [−*1*, −$\varphi$, *0*],
*vector* [$\varphi$, *0*, *1*], *vector* [−$\varphi$, *0*, *1*],
*vector* [$\varphi$, *0*, −*1*], *vector* [−$\varphi$, *0*, −*1*]}

**lemma** *std-icosahedron-eq*:
**shows** *std-icosahedron* = *convex hull*
{ *vector*[*rat5 0 0*, *rat5 1 0*, *rat5* (*1*/*2*) (*1*/*2*)],
*vector*[*rat5 0 0*, *rat5 1 0*, *rat5* (−*1*/*2*) (−*1*/*2*)],
*vector*[*rat5 0 0*, *rat5* (−*1*) *0*, *rat5* (*1*/*2*) (*1*/*2*)],
*vector*[*rat5 0 0*, *rat5* (−*1*) *0*, *rat5* (−*1*/*2*) (−*1*/*2*)],
*vector*[*rat5 1 0*, *rat5* (*1*/*2*) (*1*/*2*), *rat5 0 0*],
*vector*[*rat5 1 0*, *rat5* (−*1*/*2*) (−*1*/*2*), *rat5 0 0*],
*vector*[*rat5* (−*1*) *0*, *rat5* (*1*/*2*) (*1*/*2*), *rat5 0 0*],
*vector*[*rat5* (−*1*) *0*, *rat5* (−*1*/*2*) (−*1*/*2*), *rat5 0 0*],

*vector*[*rat5* (*1/2*) (*1/2*), *rat5 0 0*, *rat5 1 0*],
*vector*[*rat5* (−*1/2*) (−*1/2*), *rat5 0 0*, *rat5 1 0*],
*vector*[*rat5* (*1/2*) (*1/2*), *rat5 0 0*, *rat5* (−*1*) *0*],
*vector*[*rat5* (−*1/2*) (−*1/2*), *rat5 0 0*, *rat5* (−*1*) *0*]}
⟨*proof*⟩

**lemma** *icosahedron-fulldim*:
**shows** *aff-dim std-icosahedron* = *3*
⟨*proof*⟩

**lemma** *icosahedron-polyhedron*:
**shows** *polyhedron std-icosahedron*
⟨*proof*⟩

## 9.2 Facets, edges, and vertices

**lemma** *icosahedron-facets*:
**fixes** *f* :: (*real^3*) *set*
**shows** *f face-of std-icosahedron* ∧ *aff-dim f* = *2* ⟷
*f* = *convex hull* {*vector*[*rat5* (−*1/2*) (−*1/2*), *rat5 0 0*, *rat5* (−*1*) *0*], *vector*[*rat5* (−*1/2*) (−*1/2*), *rat5 0 0*, *rat5 1 0*], *vector*[*rat5* (−*1*) *0*, *rat5* (−*1/2*) (−*1/2*), *rat5 0 0*]} ∨
*f* = *convex hull* {*vector*[*rat5* (−*1/2*) (−*1/2*), *rat5 0 0*, *rat5* (−*1*) *0*], *vector*[*rat5* (−*1/2*) (−*1/2*), *rat5 0 0*, *rat5 1 0*], *vector*[*rat5* (−*1*) *0*, *rat5* (*1/2*) (*1/2*), *rat5 0 0*]} ∨
*f* = *convex hull* {*vector*[*rat5* (−*1/2*) (−*1/2*), *rat5 0 0*, *rat5* (−*1*) *0*], *vector*[*rat5* (−*1*) *0*, *rat5* (−*1/2*) (−*1/2*), *rat5 0 0*], *vector*[*rat5 0 0*, *rat5* (−*1*) *0*, *rat5* (−*1/2*) (−*1/2*)]} ∨
*f* = *convex hull* {*vector*[*rat5* (−*1/2*) (−*1/2*), *rat5 0 0*, *rat5* (−*1*) *0*], *vector*[*rat5* (−*1*) *0*, *rat5* (*1/2*) (*1/2*), *rat5 0 0*], *vector*[*rat5 0 0*, *rat5 1 0*, *rat5* (−*1/2*) (−*1/2*)]} ∨
*f* = *convex hull* {*vector*[*rat5* (−*1/2*) (−*1/2*), *rat5 0 0*, *rat5* (−*1*) *0*], *vector*[*rat5 0 0*, *rat5* (−*1*) *0*, *rat5* (−*1/2*) (−*1/2*)], *vector*[*rat5 0 0*, *rat5 1 0*, *rat5* (−*1/2*) (−*1/2*)]} ∨
*f* = *convex hull* {*vector*[*rat5* (−*1/2*) (−*1/2*), *rat5 0 0*, *rat5 1 0*], *vector*[*rat5* (−*1*) *0*, *rat5* (−*1/2*) (−*1/2*), *rat5 0 0*], *vector*[*rat5 0 0*, *rat5* (−*1*) *0*, *rat5* (*1/2*) (*1/2*)]} ∨
*f* = *convex hull* {*vector*[*rat5* (−*1/2*) (−*1/2*), *rat5 0 0*, *rat5 1 0*], *vector*[*rat5* (−*1*) *0*, *rat5* (*1/2*) (*1/2*), *rat5 0 0*], *vector*[*rat5 0 0*, *rat5 1 0*, *rat5* (*1/2*) (*1/2*)]} ∨
*f* = *convex hull* {*vector*[*rat5* (−*1/2*) (−*1/2*), *rat5 0 0*, *rat5 1 0*], *vector*[*rat5 0 0*, *rat5* (−*1*) *0*, *rat5* (*1/2*) (*1/2*)], *vector*[*rat5 0 0*, *rat5 1 0*, *rat5* (*1/2*) (*1/2*)]} ∨
*f* = *convex hull* {*vector*[*rat5* (*1/2*) (*1/2*), *rat5 0 0*, *rat5* (−*1*) *0*], *vector*[*rat5* (*1/2*) (*1/2*), *rat5 0 0*, *rat5 1 0*], *vector*[*rat5 1 0*, *rat5* (−*1/2*) (−*1/2*), *rat5 0 0*]} ∨
*f* = *convex hull* {*vector*[*rat5* (*1/2*) (*1/2*), *rat5 0 0*, *rat5* (−*1*) *0*], *vector*[*rat5* (*1/2*) (*1/2*), *rat5 0 0*, *rat5 1 0*], *vector*[*rat5 1 0*, *rat5* (*1/2*) (*1/2*), *rat5 0 0*]} ∨
*f* = *convex hull* {*vector*[*rat5* (*1/2*) (*1/2*), *rat5 0 0*, *rat5* (−*1*) *0*], *vector*[*rat5*

*1 0*, *rat5* (−*1*/*2*) (−*1*/*2*), *rat5 0 0*], *vector*[*rat5 0 0*, *rat5* (−*1*) *0*, *rat5* (−*1*/*2*) (−*1*/*2*)]} ∨
*f* = *convex hull* {*vector*[*rat5* (*1*/*2*) (*1*/*2*), *rat5 0 0*, *rat5* (−*1*) *0*], *vector*[*rat5 1 0*, *rat5* (*1*/*2*) (*1*/*2*), *rat5 0 0*], *vector*[*rat5 0 0*, *rat5 1 0*, *rat5* (−*1*/*2*) (−*1*/*2*)]} ∨
*f* = *convex hull* {*vector*[*rat5* (*1*/*2*) (*1*/*2*), *rat5 0 0*, *rat5* (−*1*) *0*], *vector*[*rat5 0 0*, *rat5* (−*1*) *0*, *rat5* (−*1*/*2*) (−*1*/*2*)], *vector*[*rat5 0 0*, *rat5 1 0*, *rat5* (−*1*/*2*) (−*1*/*2*)]} ∨
*f* = *convex hull* {*vector*[*rat5* (*1*/*2*) (*1*/*2*), *rat5 0 0*, *rat5 1 0*], *vector*[*rat5 1 0*, *rat5* (−*1*/*2*) (−*1*/*2*), *rat5 0 0*], *vector*[*rat5 0 0*, *rat5* (−*1*) *0*, *rat5* (*1*/*2*) (*1*/*2*)]} ∨
*f* = *convex hull* {*vector*[*rat5* (*1*/*2*) (*1*/*2*), *rat5 0 0*, *rat5 1 0*], *vector*[*rat5 1 0*, *rat5* (*1*/*2*) (*1*/*2*), *rat5 0 0*], *vector*[*rat5 0 0*, *rat5 1 0*, *rat5* (*1*/*2*) (*1*/*2*)]} ∨
*f* = *convex hull* {*vector*[*rat5* (*1*/*2*) (*1*/*2*), *rat5 0 0*, *rat5 1 0*], *vector*[*rat5 0 0*, *rat5* (−*1*) *0*, *rat5* (*1*/*2*) (*1*/*2*)], *vector*[*rat5 0 0*, *rat5 1 0*, *rat5* (*1*/*2*) (*1*/*2*)]} ∨
*f* = *convex hull* {*vector*[*rat5* (−*1*) *0*, *rat5* (−*1*/*2*) (−*1*/*2*), *rat5 0 0*], *vector*[*rat5 1 0*, *rat5* (−*1*/*2*) (−*1*/*2*), *rat5 0 0*], *vector*[*rat5 0 0*, *rat5* (−*1*) *0*, *rat5* (−*1*/*2*) (−*1*/*2*)]} ∨
*f* = *convex hull* {*vector*[*rat5* (−*1*) *0*, *rat5* (−*1*/*2*) (−*1*/*2*), *rat5 0 0*], *vector*[*rat5 1 0*, *rat5* (−*1*/*2*) (−*1*/*2*), *rat5 0 0*], *vector*[*rat5 0 0*, *rat5* (−*1*) *0*, *rat5* (*1*/*2*) (*1*/*2*)]} ∨
*f* = *convex hull* {*vector*[*rat5* (−*1*) *0*, *rat5* (*1*/*2*) (*1*/*2*), *rat5 0 0*], *vector*[*rat5 1 0*, *rat5* (*1*/*2*) (*1*/*2*), *rat5 0 0*], *vector*[*rat5 0 0*, *rat5 1 0*, *rat5* (−*1*/*2*) (−*1*/*2*)]} ∨
*f* = *convex hull* {*vector*[*rat5* (−*1*) *0*, *rat5* (*1*/*2*) (*1*/*2*), *rat5 0 0*], *vector*[*rat5 1 0*, *rat5* (*1*/*2*) (*1*/*2*), *rat5 0 0*], *vector*[*rat5 0 0*, *rat5 1 0*, *rat5* (*1*/*2*) (*1*/*2*)]}
⟨*proof*⟩

**lemma** *icosahedron-facet1-edges*:
**defines** *v1* ≡ *vector*[*rat5* (−*1*/*2*) (−*1*/*2*), *rat5 0 0*, *rat5* (−*1*) *0*] :: *real^3*
**and** *v2* ≡ *vector*[*rat5* (−*1*/*2*) (−*1*/*2*), *rat5 0 0*, *rat5 1 0*] :: *real^3*
**and** *v3* ≡ *vector*[*rat5* (−*1*) *0*, *rat5* (−*1*/*2*) (−*1*/*2*), *rat5 0 0*] :: *real^3*
**and** *n* ≡ *vector*[*rat5 1 1*, *rat5* (−*1*) *1*, *rat5 0 0*] :: *real^3*
**shows** (*e face-of* (*convex hull* {*v1*, *v2*, *v3*}) ∧ *aff-dim e* = *1*) =
(*e* = *convex hull* {*v1*, *v2*} ∨ *e* = *convex hull* {*v1*, *v3*} ∨ *e* = *convex hull* {*v2*, *v3*})
⟨*proof*⟩

**lemma** *icosahedron-facet2-edges*:
**defines** *v1* ≡ *vector*[*rat5* (−*1*/*2*) (−*1*/*2*), *rat5 0 0*, *rat5* (−*1*) *0*] :: *real^3*
**and** *v2* ≡ *vector*[*rat5* (−*1*/*2*) (−*1*/*2*), *rat5 0 0*, *rat5 1 0*] :: *real^3*
**and** *v3* ≡ *vector*[*rat5* (−*1*) *0*, *rat5* (*1*/*2*) (*1*/*2*), *rat5 0 0*] :: *real^3*
**and** *n* ≡ *vector*[*rat5* (−*1*) (−*1*), *rat5* (−*1*) *1*, *rat5 0 0*] :: *real^3*
**shows** (*e face-of* (*convex hull* {*v1*, *v2*, *v3*}) ∧ *aff-dim e* = *1*) =
(*e* = *convex hull* {*v1*, *v2*} ∨ *e* = *convex hull* {*v1*, *v3*} ∨ *e* = *convex hull* {*v2*, *v3*})
⟨*proof*⟩

**lemma** *icosahedron-facet3-edges*:
**defines** $v1 \equiv vector[rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0,\ rat5\ (-1)\ 0] :: real\hat{}3$
**and** $v2 \equiv vector[rat5\ (-1)\ 0,\ rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0] :: real\hat{}3$
**and** $v3 \equiv vector[rat5\ 0\ 0,\ rat5\ (-1)\ 0,\ rat5\ (-1/2)\ (-1/2)] :: real\hat{}3$
**and** $n \equiv vector[rat5\ 2\ 0,\ rat5\ 2\ 0,\ rat5\ 2\ 0] :: real\hat{}3$
**shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1,\ v2,\ v3\}) \wedge aff\text{-}dim\ e = 1) =$
$(e = convex\ hull\ \{v1,\ v2\} \vee e = convex\ hull\ \{v1,\ v3\} \vee e = convex\ hull\ \{v2,\ v3\})$
$\langle proof \rangle$

**lemma** *icosahedron-facet4-edges*:
**defines** $v1 \equiv vector[rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0,\ rat5\ (-1)\ 0] :: real\hat{}3$
**and** $v2 \equiv vector[rat5\ (-1)\ 0,\ rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0] :: real\hat{}3$
**and** $v3 \equiv vector[rat5\ 0\ 0,\ rat5\ 1\ 0,\ rat5\ (-1/2)\ (-1/2)] :: real\hat{}3$
**and** $n \equiv vector[rat5\ (-2)\ 0,\ rat5\ 2\ 0,\ rat5\ (-2)\ 0] :: real\hat{}3$
**shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1,\ v2,\ v3\}) \wedge aff\text{-}dim\ e = 1) =$
$(e = convex\ hull\ \{v1,\ v2\} \vee e = convex\ hull\ \{v1,\ v3\} \vee e = convex\ hull\ \{v2,\ v3\})$
$\langle proof \rangle$

**lemma** *icosahedron-facet5-edges*:
**defines** $v1 \equiv vector[rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0,\ rat5\ (-1)\ 0] :: real\hat{}3$
**and** $v2 \equiv vector[rat5\ 0\ 0,\ rat5\ (-1)\ 0,\ rat5\ (-1/2)\ (-1/2)] :: real\hat{}3$
**and** $v3 \equiv vector[rat5\ 0\ 0,\ rat5\ 1\ 0,\ rat5\ (-1/2)\ (-1/2)] :: real\hat{}3$
**and** $n \equiv vector[rat5\ (-1)\ 1,\ rat5\ 0\ 0,\ rat5\ 1\ 1] :: real\hat{}3$
**shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1,\ v2,\ v3\}) \wedge aff\text{-}dim\ e = 1) =$
$(e = convex\ hull\ \{v1,\ v2\} \vee e = convex\ hull\ \{v1,\ v3\} \vee e = convex\ hull\ \{v2,\ v3\})$
$\langle proof \rangle$

**lemma** *icosahedron-facet6-edges*:
**defines** $v1 \equiv vector[rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0,\ rat5\ 1\ 0] :: real\hat{}3$
**and** $v2 \equiv vector[rat5\ (-1)\ 0,\ rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0] :: real\hat{}3$
**and** $v3 \equiv vector[rat5\ 0\ 0,\ rat5\ (-1)\ 0,\ rat5\ (1/2)\ (1/2)] :: real\hat{}3$
**and** $n \equiv vector[rat5\ (-2)\ 0,\ rat5\ (-2)\ 0,\ rat5\ 2\ 0] :: real\hat{}3$
**shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1,\ v2,\ v3\}) \wedge aff\text{-}dim\ e = 1) =$
$(e = convex\ hull\ \{v1,\ v2\} \vee e = convex\ hull\ \{v1,\ v3\} \vee e = convex\ hull\ \{v2,\ v3\})$
$\langle proof \rangle$

**lemma** *icosahedron-facet7-edges*:
**defines** $v1 \equiv vector[rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0,\ rat5\ 1\ 0] :: real\hat{}3$
**and** $v2 \equiv vector[rat5\ (-1)\ 0,\ rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0] :: real\hat{}3$
**and** $v3 \equiv vector[rat5\ 0\ 0,\ rat5\ 1\ 0,\ rat5\ (1/2)\ (1/2)] :: real\hat{}3$
**and** $n \equiv vector[rat5\ 2\ 0,\ rat5\ (-2)\ 0,\ rat5\ (-2)\ 0] :: real\hat{}3$
**shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1,\ v2,\ v3\}) \wedge aff\text{-}dim\ e = 1) =$
$(e = convex\ hull\ \{v1,\ v2\} \vee e = convex\ hull\ \{v1,\ v3\} \vee e = convex\ hull\ \{v2,\ v3\})$
$\langle proof \rangle$

**lemma** *icosahedron-facet8-edges*:
  **defines** $v1 \equiv vector[rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0,\ rat5\ 1\ 0] :: real\^{}3$
    **and** $v2 \equiv vector[rat5\ 0\ 0,\ rat5\ (-1)\ 0,\ rat5\ (1/2)\ (1/2)] :: real\^{}3$
    **and** $v3 \equiv vector[rat5\ 0\ 0,\ rat5\ 1\ 0,\ rat5\ (1/2)\ (1/2)] :: real\^{}3$
    **and** $n \equiv vector[rat5\ 1\ (-1),\ rat5\ 0\ 0,\ rat5\ 1\ 1] :: real\^{}3$
  **shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1,\ v2,\ v3\}) \wedge aff\text{-}dim\ e = 1) =$
    $(e = convex\ hull\ \{v1,\ v2\} \vee e = convex\ hull\ \{v1,\ v3\} \vee e = convex\ hull\ \{v2,\ v3\})$
$\langle proof \rangle$

**lemma** *icosahedron-facet9-edges*:
  **defines** $v1 \equiv vector[rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0,\ rat5\ (-1)\ 0] :: real\^{}3$
    **and** $v2 \equiv vector[rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0,\ rat5\ 1\ 0] :: real\^{}3$
    **and** $v3 \equiv vector[rat5\ 1\ 0,\ rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0] :: real\^{}3$
    **and** $n \equiv vector[rat5\ 1\ 1,\ rat5\ 1\ (-1),\ rat5\ 0\ 0] :: real\^{}3$
  **shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1,\ v2,\ v3\}) \wedge aff\text{-}dim\ e = 1) =$
    $(e = convex\ hull\ \{v1,\ v2\} \vee e = convex\ hull\ \{v1,\ v3\} \vee e = convex\ hull\ \{v2,\ v3\})$
$\langle proof \rangle$

**lemma** *icosahedron-facet10-edges*:
  **defines** $v1 \equiv vector[rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0,\ rat5\ (-1)\ 0] :: real\^{}3$
    **and** $v2 \equiv vector[rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0,\ rat5\ 1\ 0] :: real\^{}3$
    **and** $v3 \equiv vector[rat5\ 1\ 0,\ rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0] :: real\^{}3$
    **and** $n \equiv vector[rat5\ (-1)\ (-1),\ rat5\ 1\ (-1),\ rat5\ 0\ 0] :: real\^{}3$
  **shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1,\ v2,\ v3\}) \wedge aff\text{-}dim\ e = 1) =$
    $(e = convex\ hull\ \{v1,\ v2\} \vee e = convex\ hull\ \{v1,\ v3\} \vee e = convex\ hull\ \{v2,\ v3\})$
$\langle proof \rangle$

**lemma** *icosahedron-facet11-edges*:
  **defines** $v1 \equiv vector[rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0,\ rat5\ (-1)\ 0] :: real\^{}3$
    **and** $v2 \equiv vector[rat5\ 1\ 0,\ rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0] :: real\^{}3$
    **and** $v3 \equiv vector[rat5\ 0\ 0,\ rat5\ (-1)\ 0,\ rat5\ (-1/2)\ (-1/2)] :: real\^{}3$
    **and** $n \equiv vector[rat5\ 2\ 0,\ rat5\ (-2)\ 0,\ rat5\ (-2)\ 0] :: real\^{}3$
  **shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1,\ v2,\ v3\}) \wedge aff\text{-}dim\ e = 1) =$
    $(e = convex\ hull\ \{v1,\ v2\} \vee e = convex\ hull\ \{v1,\ v3\} \vee e = convex\ hull\ \{v2,\ v3\})$
$\langle proof \rangle$

**lemma** *icosahedron-facet12-edges*:
  **defines** $v1 \equiv vector[rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0,\ rat5\ (-1)\ 0] :: real\^{}3$
    **and** $v2 \equiv vector[rat5\ 1\ 0,\ rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0] :: real\^{}3$
    **and** $v3 \equiv vector[rat5\ 0\ 0,\ rat5\ 1\ 0,\ rat5\ (-1/2)\ (-1/2)] :: real\^{}3$
    **and** $n \equiv vector[rat5\ (-2)\ 0,\ rat5\ (-2)\ 0,\ rat5\ 2\ 0] :: real\^{}3$
  **shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1,\ v2,\ v3\}) \wedge aff\text{-}dim\ e = 1) =$
    $(e = convex\ hull\ \{v1,\ v2\} \vee e = convex\ hull\ \{v1,\ v3\} \vee e = convex\ hull\ \{v2,\ v3\})$

⟨*proof*⟩

**lemma** *icosahedron-facet13-edges*:
**defines** $v1 \equiv vector[rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0,\ rat5\ (-1)\ 0] :: real^\frown 3$
**and** $v2 \equiv vector[rat5\ 0\ 0,\ rat5\ (-1)\ 0,\ rat5\ (-1/2)\ (-1/2)] :: real^\frown 3$
**and** $v3 \equiv vector[rat5\ 0\ 0,\ rat5\ 1\ 0,\ rat5\ (-1/2)\ (-1/2)] :: real^\frown 3$
**and** $n \equiv vector[rat5\ (-1)\ 1,\ rat5\ 0\ 0,\ rat5\ (-1)\ (-1)] :: real^\frown 3$
**shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1,\ v2,\ v3\}) \wedge aff\text{-}dim\ e = 1) =$
$(e = convex\ hull\ \{v1,\ v2\} \vee e = convex\ hull\ \{v1,\ v3\} \vee e = convex\ hull\ \{v2,\ v3\})$
⟨*proof*⟩

**lemma** *icosahedron-facet14-edges*:
**defines** $v1 \equiv vector[rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0,\ rat5\ 1\ 0] :: real^\frown 3$
**and** $v2 \equiv vector[rat5\ 1\ 0,\ rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0] :: real^\frown 3$
**and** $v3 \equiv vector[rat5\ 0\ 0,\ rat5\ (-1)\ 0,\ rat5\ (1/2)\ (1/2)] :: real^\frown 3$
**and** $n \equiv vector[rat5\ (-2)\ 0,\ rat5\ 2\ 0,\ rat5\ (-2)\ 0] :: real^\frown 3$
**shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1,\ v2,\ v3\}) \wedge aff\text{-}dim\ e = 1) =$
$(e = convex\ hull\ \{v1,\ v2\} \vee e = convex\ hull\ \{v1,\ v3\} \vee e = convex\ hull\ \{v2,\ v3\})$
⟨*proof*⟩

**lemma** *icosahedron-facet15-edges*:
**defines** $v1 \equiv vector[rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0,\ rat5\ 1\ 0] :: real^\frown 3$
**and** $v2 \equiv vector[rat5\ 1\ 0,\ rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0] :: real^\frown 3$
**and** $v3 \equiv vector[rat5\ 0\ 0,\ rat5\ 1\ 0,\ rat5\ (1/2)\ (1/2)] :: real^\frown 3$
**and** $n \equiv vector[rat5\ 2\ 0,\ rat5\ 2\ 0,\ rat5\ 2\ 0] :: real^\frown 3$
**shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1,\ v2,\ v3\}) \wedge aff\text{-}dim\ e = 1) =$
$(e = convex\ hull\ \{v1,\ v2\} \vee e = convex\ hull\ \{v1,\ v3\} \vee e = convex\ hull\ \{v2,\ v3\})$
⟨*proof*⟩

**lemma** *icosahedron-facet16-edges*:
**defines** $v1 \equiv vector[rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0,\ rat5\ 1\ 0] :: real^\frown 3$
**and** $v2 \equiv vector[rat5\ 0\ 0,\ rat5\ (-1)\ 0,\ rat5\ (1/2)\ (1/2)] :: real^\frown 3$
**and** $v3 \equiv vector[rat5\ 0\ 0,\ rat5\ 1\ 0,\ rat5\ (1/2)\ (1/2)] :: real^\frown 3$
**and** $n \equiv vector[rat5\ 1\ (-1),\ rat5\ 0\ 0,\ rat5\ (-1)\ (-1)] :: real^\frown 3$
**shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1,\ v2,\ v3\}) \wedge aff\text{-}dim\ e = 1) =$
$(e = convex\ hull\ \{v1,\ v2\} \vee e = convex\ hull\ \{v1,\ v3\} \vee e = convex\ hull\ \{v2,\ v3\})$
⟨*proof*⟩

**lemma** *icosahedron-facet17-edges*:
**defines** $v1 \equiv vector[rat5\ (-1)\ 0,\ rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0] :: real^\frown 3$
**and** $v2 \equiv vector[rat5\ 1\ 0,\ rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0] :: real^\frown 3$
**and** $v3 \equiv vector[rat5\ 0\ 0,\ rat5\ (-1)\ 0,\ rat5\ (-1/2)\ (-1/2)] :: real^\frown 3$
**and** $n \equiv vector[rat5\ 0\ 0,\ rat5\ 1\ 1,\ rat5\ (-1)\ 1] :: real^\frown 3$
**shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1,\ v2,\ v3\}) \wedge aff\text{-}dim\ e = 1) =$
$(e = convex\ hull\ \{v1,\ v2\} \vee e = convex\ hull\ \{v1,\ v3\} \vee e = convex\ hull$

$\{v2,\ v3\})$
$\langle proof \rangle$

**lemma** *icosahedron-facet18-edges*:
**defines** $v1 \equiv vector[rat5\ (-1)\ 0,\ rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0] :: real\hat{\ }3$
**and** $v2 \equiv vector[rat5\ 1\ 0,\ rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0] :: real\hat{\ }3$
**and** $v3 \equiv vector[rat5\ 0\ 0,\ rat5\ (-1)\ 0,\ rat5\ (1/2)\ (1/2)] :: real\hat{\ }3$
**and** $n \equiv vector[rat5\ 0\ 0,\ rat5\ (-1)\ (-1),\ rat5\ (-1)\ 1] :: real\hat{\ }3$
**shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1,\ v2,\ v3\}) \wedge aff\text{-}dim\ e = 1) =$
$(e = convex\ hull\ \{v1,\ v2\} \vee e = convex\ hull\ \{v1,\ v3\} \vee e = convex\ hull$
$\{v2,\ v3\})$
$\langle proof \rangle$

**lemma** *icosahedron-facet19-edges*:
**defines** $v1 \equiv vector[rat5\ (-1)\ 0,\ rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0] :: real\hat{\ }3$
**and** $v2 \equiv vector[rat5\ 1\ 0,\ rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0] :: real\hat{\ }3$
**and** $v3 \equiv vector[rat5\ 0\ 0,\ rat5\ 1\ 0,\ rat5\ (-1/2)\ (-1/2)] :: real\hat{\ }3$
**and** $n \equiv vector[rat5\ 0\ 0,\ rat5\ 1\ 1,\ rat5\ 1\ (-1)] :: real\hat{\ }3$
**shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1,\ v2,\ v3\}) \wedge aff\text{-}dim\ e = 1) =$
$(e = convex\ hull\ \{v1,\ v2\} \vee e = convex\ hull\ \{v1,\ v3\} \vee e = convex\ hull$
$\{v2,\ v3\})$
$\langle proof \rangle$

**lemma** *icosahedron-facet20-edges*:
**defines** $v1 \equiv vector[rat5\ (-1)\ 0,\ rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0] :: real\hat{\ }3$
**and** $v2 \equiv vector[rat5\ 1\ 0,\ rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0] :: real\hat{\ }3$
**and** $v3 \equiv vector[rat5\ 0\ 0,\ rat5\ 1\ 0,\ rat5\ (1/2)\ (1/2)] :: real\hat{\ }3$
**and** $n \equiv vector[rat5\ 0\ 0,\ rat5\ (-1)\ (-1),\ rat5\ 1\ (-1)] :: real\hat{\ }3$
**shows** $(e\ face\text{-}of\ (convex\ hull\ \{v1,\ v2,\ v3\}) \wedge aff\text{-}dim\ e = 1) =$
$(e = convex\ hull\ \{v1,\ v2\} \vee e = convex\ hull\ \{v1,\ v3\} \vee e = convex\ hull$
$\{v2,\ v3\})$
$\langle proof \rangle$

**lemmas** *icosahedron-facet-edges* =
*icosahedron-facet1-edges icosahedron-facet2-edges icosahedron-facet3-edges*
*icosahedron-facet4-edges icosahedron-facet5-edges icosahedron-facet6-edges*
*icosahedron-facet7-edges icosahedron-facet8-edges icosahedron-facet9-edges*
*icosahedron-facet10-edges icosahedron-facet11-edges icosahedron-facet12-edges*
*icosahedron-facet13-edges icosahedron-facet14-edges icosahedron-facet15-edges*
*icosahedron-facet16-edges icosahedron-facet17-edges icosahedron-facet18-edges*
*icosahedron-facet19-edges icosahedron-facet20-edges*

**lemma** *icosahedron-edges-per-face*:
**assumes** $f\ face\text{-}of\ std\text{-}icosahedron$
**and** $aff\text{-}dim\ f = 2$
**shows** $card\ \{e.\ e\ face\text{-}of\ std\text{-}icosahedron \wedge aff\text{-}dim\ e = 1 \wedge e \subseteq f\} = 3$
$\langle proof \rangle$

**lemma** *icosahedron-edges*:

$e$ *face-of std-icosahedron* $\wedge$ *aff-dim* $e = 1 \longleftrightarrow$
$e =$ *convex hull* $\{$*vector*[*rat5* $(-1/2)$ $(-1/2)$, *rat5* $0$ $0$, *rat5* $(-1)$ $0$], *vector*[*rat5* $(-1/2)$ $(-1/2)$, *rat5* $0$ $0$, *rat5* $1$ $0$]$\}$ $\vee$
$e =$ *convex hull* $\{$*vector*[*rat5* $(-1/2)$ $(-1/2)$, *rat5* $0$ $0$, *rat5* $(-1)$ $0$], *vector*[*rat5* $(-1)$ $0$, *rat5* $(-1/2)$ $(-1/2)$, *rat5* $0$ $0$]$\}$ $\vee$
$e =$ *convex hull* $\{$*vector*[*rat5* $(-1/2)$ $(-1/2)$, *rat5* $0$ $0$, *rat5* $(-1)$ $0$], *vector*[*rat5* $(-1)$ $0$, *rat5* $(1/2)$ $(1/2)$, *rat5* $0$ $0$]$\}$ $\vee$
$e =$ *convex hull* $\{$*vector*[*rat5* $(-1/2)$ $(-1/2)$, *rat5* $0$ $0$, *rat5* $(-1)$ $0$], *vector*[*rat5* $0$ $0$, *rat5* $(-1)$ $0$, *rat5* $(-1/2)$ $(-1/2)$]$\}$ $\vee$
$e =$ *convex hull* $\{$*vector*[*rat5* $(-1/2)$ $(-1/2)$, *rat5* $0$ $0$, *rat5* $(-1)$ $0$], *vector*[*rat5* $0$ $0$, *rat5* $1$ $0$, *rat5* $(-1/2)$ $(-1/2)$]$\}$ $\vee$
$e =$ *convex hull* $\{$*vector*[*rat5* $(-1/2)$ $(-1/2)$, *rat5* $0$ $0$, *rat5* $1$ $0$], *vector*[*rat5* $(-1)$ $0$, *rat5* $(-1/2)$ $(-1/2)$, *rat5* $0$ $0$]$\}$ $\vee$
$e =$ *convex hull* $\{$*vector*[*rat5* $(-1/2)$ $(-1/2)$, *rat5* $0$ $0$, *rat5* $1$ $0$], *vector*[*rat5* $(-1)$ $0$, *rat5* $(1/2)$ $(1/2)$, *rat5* $0$ $0$]$\}$ $\vee$
$e =$ *convex hull* $\{$*vector*[*rat5* $(-1/2)$ $(-1/2)$, *rat5* $0$ $0$, *rat5* $1$ $0$], *vector*[*rat5* $0$ $0$, *rat5* $(-1)$ $0$, *rat5* $(1/2)$ $(1/2)$]$\}$ $\vee$
$e =$ *convex hull* $\{$*vector*[*rat5* $(-1/2)$ $(-1/2)$, *rat5* $0$ $0$, *rat5* $1$ $0$], *vector*[*rat5* $0$ $0$, *rat5* $1$ $0$, *rat5* $(1/2)$ $(1/2)$]$\}$ $\vee$
$e =$ *convex hull* $\{$*vector*[*rat5* $(1/2)$ $(1/2)$, *rat5* $0$ $0$, *rat5* $(-1)$ $0$], *vector*[*rat5* $(1/2)$ $(1/2)$, *rat5* $0$ $0$, *rat5* $1$ $0$]$\}$ $\vee$
$e =$ *convex hull* $\{$*vector*[*rat5* $(1/2)$ $(1/2)$, *rat5* $0$ $0$, *rat5* $(-1)$ $0$], *vector*[*rat5* $1$ $0$, *rat5* $(-1/2)$ $(-1/2)$, *rat5* $0$ $0$]$\}$ $\vee$
$e =$ *convex hull* $\{$*vector*[*rat5* $(1/2)$ $(1/2)$, *rat5* $0$ $0$, *rat5* $(-1)$ $0$], *vector*[*rat5* $1$ $0$, *rat5* $(1/2)$ $(1/2)$, *rat5* $0$ $0$]$\}$ $\vee$
$e =$ *convex hull* $\{$*vector*[*rat5* $(1/2)$ $(1/2)$, *rat5* $0$ $0$, *rat5* $(-1)$ $0$], *vector*[*rat5* $0$ $0$, *rat5* $(-1)$ $0$, *rat5* $(-1/2)$ $(-1/2)$]$\}$ $\vee$
$e =$ *convex hull* $\{$*vector*[*rat5* $(1/2)$ $(1/2)$, *rat5* $0$ $0$, *rat5* $(-1)$ $0$], *vector*[*rat5* $0$ $0$, *rat5* $1$ $0$, *rat5* $(-1/2)$ $(-1/2)$]$\}$ $\vee$
$e =$ *convex hull* $\{$*vector*[*rat5* $(1/2)$ $(1/2)$, *rat5* $0$ $0$, *rat5* $1$ $0$], *vector*[*rat5* $1$ $0$, *rat5* $(-1/2)$ $(-1/2)$, *rat5* $0$ $0$]$\}$ $\vee$
$e =$ *convex hull* $\{$*vector*[*rat5* $(1/2)$ $(1/2)$, *rat5* $0$ $0$, *rat5* $1$ $0$], *vector*[*rat5* $1$ $0$, *rat5* $(1/2)$ $(1/2)$, *rat5* $0$ $0$]$\}$ $\vee$
$e =$ *convex hull* $\{$*vector*[*rat5* $(1/2)$ $(1/2)$, *rat5* $0$ $0$, *rat5* $1$ $0$], *vector*[*rat5* $0$ $0$, *rat5* $(-1)$ $0$, *rat5* $(1/2)$ $(1/2)$]$\}$ $\vee$
$e =$ *convex hull* $\{$*vector*[*rat5* $(1/2)$ $(1/2)$, *rat5* $0$ $0$, *rat5* $1$ $0$], *vector*[*rat5* $0$ $0$, *rat5* $1$ $0$, *rat5* $(1/2)$ $(1/2)$]$\}$ $\vee$
$e =$ *convex hull* $\{$*vector*[*rat5* $(-1)$ $0$, *rat5* $(-1/2)$ $(-1/2)$, *rat5* $0$ $0$], *vector*[*rat5* $1$ $0$, *rat5* $(-1/2)$ $(-1/2)$, *rat5* $0$ $0$]$\}$ $\vee$
$e =$ *convex hull* $\{$*vector*[*rat5* $(-1)$ $0$, *rat5* $(-1/2)$ $(-1/2)$, *rat5* $0$ $0$], *vector*[*rat5* $0$ $0$, *rat5* $(-1)$ $0$, *rat5* $(-1/2)$ $(-1/2)$]$\}$ $\vee$
$e =$ *convex hull* $\{$*vector*[*rat5* $(-1)$ $0$, *rat5* $(-1/2)$ $(-1/2)$, *rat5* $0$ $0$], *vector*[*rat5* $0$ $0$, *rat5* $(-1)$ $0$, *rat5* $(1/2)$ $(1/2)$]$\}$ $\vee$
$e =$ *convex hull* $\{$*vector*[*rat5* $(-1)$ $0$, *rat5* $(1/2)$ $(1/2)$, *rat5* $0$ $0$], *vector*[*rat5* $1$ $0$, *rat5* $(1/2)$ $(1/2)$, *rat5* $0$ $0$]$\}$ $\vee$
$e =$ *convex hull* $\{$*vector*[*rat5* $(-1)$ $0$, *rat5* $(1/2)$ $(1/2)$, *rat5* $0$ $0$], *vector*[*rat5* $0$ $0$, *rat5* $1$ $0$, *rat5* $(-1/2)$ $(-1/2)$]$\}$ $\vee$
$e =$ *convex hull* $\{$*vector*[*rat5* $(-1)$ $0$, *rat5* $(1/2)$ $(1/2)$, *rat5* $0$ $0$], *vector*[*rat5* $0$ $0$, *rat5* $1$ $0$, *rat5* $(1/2)$ $(1/2)$]$\}$ $\vee$

$e = convex\ hull\ \{vector[rat5\ 1\ 0,\ rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0],\ vector[rat5\ 0\ 0,\ rat5\ (-1)\ 0,\ rat5\ (-1/2)\ (-1/2)]\} \vee$
$e = convex\ hull\ \{vector[rat5\ 1\ 0,\ rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0],\ vector[rat5\ 0\ 0,\ rat5\ (-1)\ 0,\ rat5\ (1/2)\ (1/2)]\} \vee$
$e = convex\ hull\ \{vector[rat5\ 1\ 0,\ rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0],\ vector[rat5\ 0\ 0,\ rat5\ 1\ 0,\ rat5\ (-1/2)\ (-1/2)]\} \vee$
$e = convex\ hull\ \{vector[rat5\ 1\ 0,\ rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0],\ vector[rat5\ 0\ 0,\ rat5\ 1\ 0,\ rat5\ (1/2)\ (1/2)]\} \vee$
$e = convex\ hull\ \{vector[rat5\ 0\ 0,\ rat5\ (-1)\ 0,\ rat5\ (-1/2)\ (-1/2)],\ vector[rat5\ 0\ 0,\ rat5\ 1\ 0,\ rat5\ (-1/2)\ (-1/2)]\} \vee$
$e = convex\ hull\ \{vector[rat5\ 0\ 0,\ rat5\ (-1)\ 0,\ rat5\ (1/2)\ (1/2)],\ vector[rat5\ 0\ 0,\ rat5\ 1\ 0,\ rat5\ (1/2)\ (1/2)]\}$
(**is** $?lhs \longleftrightarrow ?rhs$)
$\langle proof \rangle$

**lemma** *icosahedron-vertices*:
**shows** $(v\ face\text{-}of\ std\text{-}icosahedron \wedge aff\text{-}dim\ v = 0) =$
$(v = \{vector[rat5\ 0\ 0,\ rat5\ 1\ 0,\ rat5\ (1/2)\ (1/2)]\} \vee$
$v = \{vector[rat5\ 0\ 0,\ rat5\ 1\ 0,\ rat5\ (-1/2)\ (-1/2)]\} \vee$
$v = \{vector[rat5\ 0\ 0,\ rat5\ (-1)\ 0,\ rat5\ (1/2)\ (1/2)]\} \vee$
$v = \{vector[rat5\ 0\ 0,\ rat5\ (-1)\ 0,\ rat5\ (-1/2)\ (-1/2)]\} \vee$
$v = \{vector[rat5\ 1\ 0,\ rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0]\} \vee$
$v = \{vector[rat5\ 1\ 0,\ rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0]\} \vee$
$v = \{vector[rat5\ (-1)\ 0,\ rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0]\} \vee$
$v = \{vector[rat5\ (-1)\ 0,\ rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0]\} \vee$
$v = \{vector[rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0,\ rat5\ 1\ 0]\} \vee$
$v = \{vector[rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0,\ rat5\ 1\ 0]\} \vee$
$v = \{vector[rat5\ (1/2)\ (1/2),\ rat5\ 0\ 0,\ rat5\ (-1)\ 0]\} \vee$
$v = \{vector[rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0,\ rat5\ (-1)\ 0]\})$
(**is** $?lhs = ?rhs$)
$\langle proof \rangle$

**lemma** *icosahedron-edges-per-vertex*:
**assumes** $v\ face\text{-}of\ std\text{-}icosahedron$
**and** $aff\text{-}dim\ v = 0$
**shows** $card\ \{e.\ e\ face\text{-}of\ std\text{-}icosahedron \wedge aff\text{-}dim\ e = 1 \wedge v \subseteq e\} = 5$
$\langle proof \rangle$

## 9.3 Regularity

**lemma** *icosahedron-congruent-edges*:
**assumes** $e1\ face\text{-}of\ std\text{-}icosahedron \wedge aff\text{-}dim\ e1 = 1$
**and** $e2\ face\text{-}of\ std\text{-}icosahedron \wedge aff\text{-}dim\ e2 = 1$
**shows** $e1\ congruent\ e2$
$\langle proof \rangle$

**lemma** *icosahedron-congruent-faces*:
**assumes** $f1\ face\text{-}of\ std\text{-}icosahedron \wedge aff\text{-}dim\ f1 = 2$
**and** $f2\ face\text{-}of\ std\text{-}icosahedron \wedge aff\text{-}dim\ f2 = 2$

**shows** *f1 congruent f2*
⟨*proof*⟩

**lemma** *icosahedron-facet1-equiangular*:
**defines** $v1 \equiv vector[rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0,\ rat5\ (-1)\ 0] :: real\hat{}3$
**defines** $v2 \equiv vector[rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0,\ rat5\ 1\ 0] :: real\hat{}3$
**defines** $v3 \equiv vector[rat5\ (-1)\ 0,\ rat5\ (-1/2)\ (-1/2),\ rat5\ 0\ 0] :: real\hat{}3$
**shows** *equiangular* (*convex hull* {*v1*, *v2*, *v3*}) (*pi* / *3*)
⟨*proof*⟩

**lemma** *icosahedron-equiangular*:
**assumes** *f face-of std-icosahedron* ∧ *aff-dim f* = *2*
**shows** *equiangular f* (*pi* / *3*)
⟨*proof*⟩

**end**

# 10 Main result

**theory** *Platonic-Solids*
**imports**
*Euler-Polyhedron-Formula.Euler-Formula*
*Polytope-More*
*Tetrahedron*
*Cube*
*Octahedron*
*Dodecahedron*
*Icosahedron*
**begin**

## 10.1 Possible Schläfli symbols

**lemma** *platonic-solid-limits*:
**fixes** $p :: (real\hat{}3)\ set$
**assumes** *is-polytope*: *polytope p*
**and** *aff-dim-p*: *aff-dim p* = *3*
**and** *edges-per-face*: ∀*f*. *f face-of p* ∧ *aff-dim f* = *2* ⟶
*card* {*e*. *e face-of p* ∧ *aff-dim e* = *1* ∧ *e* ⊆ *f*} = *m*
**and** *edges-per-vert*: ∀*v*. *v face-of p* ∧ *aff-dim v* = *0* ⟶
*card* {*e*. *e face-of p* ∧ *aff-dim e* = *1* ∧ *v* ⊆ *e*} = *n*
**shows** (*m* = *3* ∧ *n* = *3*) ∨
(*m* = *4* ∧ *n* = *3*) ∨
(*m* = *3* ∧ *n* = *4*) ∨
(*m* = *5* ∧ *n* = *3*) ∨
(*m* = *3* ∧ *n* = *5*)
⟨*proof*⟩

## 10.2 Exactly five Platonic solids

**theorem** *platonic-solids-full*:
  **shows** $(\exists\, p{::}(real\hat{\ }3)\ set.\ polytope\ p \land aff\text{-}dim\ p = 3 \land$
$(\forall f.\ f\ face\text{-}of\ p \land aff\text{-}dim\ f = 2 \longrightarrow$
$card\ \{e.\ e\ face\text{-}of\ p \land aff\text{-}dim\ e = 1 \land e \subseteq f\} = m) \land$
$(\forall v.\ v\ face\text{-}of\ p \land aff\text{-}dim\ v = 0 \longrightarrow$
$card\ \{e.\ e\ face\text{-}of\ p \land aff\text{-}dim\ e = 1 \land v \subseteq e\} = n) \land$
$(\forall f1\ f2.\ f1\ face\text{-}of\ p \land aff\text{-}dim\ f1 = 2 \land$
$f2\ face\text{-}of\ p \land aff\text{-}dim\ f2 = 2 \longrightarrow f1\ congruent\ f2) \land$
$(\forall e1\ e2.\ e1\ face\text{-}of\ p \land aff\text{-}dim\ e1 = 1 \land$
$e2\ face\text{-}of\ p \land aff\text{-}dim\ e2 = 1 \longrightarrow e1\ congruent\ e2) \land$
$(\exists t.\ \forall f.\ f\ face\text{-}of\ p \land aff\text{-}dim\ f = 2 \longrightarrow equiangular\ f\ t)) =$
$((m = 3 \land n = 3) \lor$
$(m = 4 \land n = 3) \lor$
$(m = 3 \land n = 4) \lor$
$(m = 5 \land n = 3) \lor$
$(m = 3 \land n = 5))$
  (**is** *?lhs = ?rhs*)
⟨*proof*⟩

This is the original top-level statement of the HOL Light formalization, kept here for parity. It omits the regularity condition on $p$.

**theorem** *platonic-solids*:
  **shows** $(\exists\, p{::}(real\hat{\ }3)\ set.\ polytope\ p \land aff\text{-}dim\ p = 3 \land$
$(\forall f.\ f\ face\text{-}of\ p \land aff\text{-}dim\ f = 2 \longrightarrow$
$card\ \{e.\ e\ face\text{-}of\ p \land aff\text{-}dim\ e = 1 \land e \subseteq f\} = m) \land$
$(\forall v.\ v\ face\text{-}of\ p \land aff\text{-}dim\ v = 0 \longrightarrow$
$card\ \{e.\ e\ face\text{-}of\ p \land aff\text{-}dim\ e = 1 \land v \subseteq e\} = n)) =$
$((m = 3 \land n = 3) \lor$
$(m = 4 \land n = 3) \lor$
$(m = 3 \land n = 4) \lor$
$(m = 5 \land n = 3) \lor$
$(m = 3 \land n = 5))$
  (**is** *?lhs = ?rhs*)
⟨*proof*⟩

**end**

# References


[1] J. Harrison. Proof of the Platonic solids theorem, accessed 2025. HOL Light repository.