

A simple proof that π is irrational

Manuel Eberl

September 13, 2024

Abstract

This entry provides a formalisation of Niven's famously short one-page proof that π is irrational. The proof uses only elementary algebra and analysis.

The intrinsic de Bruijn factor, i.e. the file size ratio between the gzipped Isabelle sources and a gzipped L^AT_EX version of the original paper's content, is roughly 4 despite the original paper's terse presentation.

Contents

1	A short proof of the irrationality of π	2
1.1	Auxiliary material	2
1.2	Main proof	2

1 A short proof of the irrationality of π

```
theory Pi_Irrational
imports
  "HOL-Analysis.Analysis"
  "Polynomial_Interpolation.Ring_Hom_Poly"
begin
```

1.1 Auxiliary material

```
lemma fact_dvd_pochhammer:
  assumes "m ≤ n + 1"
  shows "fact m dvd pochhammer (int n - int m + 1) m"
⟨proof⟩
```

```
lemma factor_dvd_higher_pderiv:
  fixes p :: "'a :: idom poly"
  assumes "p ^ n dvd q" "i < n"
  shows "p dvd (pderiv ^^ i) q"
⟨proof⟩
```

```
lemma fact_dvd_higher_pderiv:
  "[fact n :: int:] dvd (pderiv ^^ n) p"
⟨proof⟩
```

```
lemma higher_pderiv_eq_0_iff:
  fixes p :: "'a::{comm_semiring_1,semiring_no_zero_divisors,semiring_char_0}
poly"
  shows "(pderiv ^^ n) p = 0 ⟷ p = 0 ∨ n > degree p"
⟨proof⟩
```

```
lemma higher_pderiv_pcompose_linear:
  shows "(pderiv ^^ n) (pcompose p [:a, b:]) = smult (b ^ n) (pcompose
((pderiv ^^ n) p) [:a, b:])"
⟨proof⟩
```

```
lemma power_over_fact_tendsto_0:
  "(λn. (x :: real) ^ n / fact n) ⟶ 0"
⟨proof⟩
```

1.2 Main proof

```
locale pi_rational =
  fixes a b :: int
  assumes ab: "a / b = pi"
  assumes b: "b > 0"
begin
```

```
context
  fixes n :: nat
```

```

    assumes n: "n > 1"
begin

definition f :: "real poly" where
  "f = smult (1/fact n) ([:0, of_int a, -of_int b:] ^ n)"

lemma f_mirror: "f ◦p [:pi, -1:] = f"
  ⟨proof⟩

lemma degree_f [simp]: "degree f = 2 * n"
  ⟨proof⟩

definition F :: "real poly" where
  "F = (∑ j ≤ n. (-1)^j * (pderiv ^^ (2*j)) f)"

lemma F_mirror: "F ◦p [:pi, -1:] = F"
  ⟨proof⟩

lemma poly_F_pi: "poly F pi = poly F 0"
  ⟨proof⟩

lemma F_int: "poly F 0 ∈ ℤ"
  ⟨proof⟩

lemma antideriv:
  "((λx. poly (pderiv F) x * sin x - poly F x * cos x)
    has_field_derivative (poly f x * sin x)) (at x within A)"
  ⟨proof⟩

lemma bound: "pi / 2 * (a * pi) ^ n / fact n ≥ 1"
  ⟨proof⟩

end

lemma absurd: False
  ⟨proof⟩

end

theorem pi_irrational: "pi ∉ ℚ"
  ⟨proof⟩

end

```

References

- [1] I. Niven. A simple proof that π is irrational. *Bulletin of the American Mathematical Society*, 53(6):509, 1947.