Partial Semigroups and Convolution Algebras

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Abstract

Partial Semigroups are relevant to the foundations of quantum mechanics and combinatorics as well as to interval and separation logics. Convolution algebras can be understood either as algebras of generalised binary modalities over ternary Kripke frames, in particular over partial semigroups, or as algebras of quantale-valued functions which are equipped with a convolution-style operation of multiplication that is parametrised by a ternary relation. Convolution algebras provide algebraic semantics for various substructural logics, including categorial, relevance and linear logics, for separation logic and for interval logics; they cover quantitative and qualitative applications. These mathematical components for partial semigroups and convolution algebras provide uniform foundations from which models of computation based on relations, program traces or pomsets, and verification components for separation or interval temporal logics can be built with little effort.

Contents

1 Introductory Remarks **2**

2 Partial Semigroups **3**
2.1 Partial Semigroups . . . 3
2.2 Green's Preorders and Green's Relations . . . 3
2.3 Morphisms . . . 5
2.4 Locally Finite Partial Semigroups . . . 5
2.5 Cancellative Partial Semigroups . . . 5
2.6 Partial Monoids . . . 6
2.7 Cancellative Partial Monoids . . . 7
2.8 Positive Partial Monoids . . . 8
2.9 Positive Cancellative Partial Monoids . . . 8
2.10 From Partial Abelian Semigroups to Partial Abelian Monoids . . . 8
2.11 Alternative Definitions . . . 9
2.12 Product Constructions . . . 10
2.13 Partial Semigroup Actions and Semidirect Products . . . 12

3 Models of Partial Semigroups **13**
3.1 Partial Monoids of Segments and Intervals . . . 13
3.2 Cancellative PAM's of Partial Functions . . . 15
3.3 PAM's of Disjoint Unions of Sets . . . 15

4 Quantales **16**
4.1 Properties of Complete Lattices . . . 16
4.2 Familes of Proto-Quantales . . . 17
4.3 Families of Quantales . . . 19
4.4 Quantales of Booleans and Complete Boolean Algebras . . . 20
4.5 Products of Quantales . . . 21
4.6 Quantale Modules and Semidirect Products . . . 22

5 Binary Modalities and Relational Convolution **26**
5.1 Auxiliary Properties . . . 26
5.2 Binary Modalities . . . 27
5.3 Relational Convolution and Correspondence Theory . . . 29
5.4 Lifting to Function Spaces . . . 33

6 Unary Modalities **34**
6.1 Forward and Backward Diamonds . . . 34
6.2 Forward and Backward Boxes . . . 35
6.3 Symmetries and Dualities . . . 35

7 Liftings of Partial Semigroups **37**
7.1 Relational Semigroups and Partial Semigroups . . . 37
7.2 Liftings of Partial Abelian Semigroups . . . 38

1 Introductory Remarks

These mathematical components supply formal proofs for two articles on *Convolution Algebras* [3] and *Convolution as a Unifying Concept* [2]. They are sparsely documented and referenced; additional information can be found in these articles, and in particular the first one.

The approach generalises previous Isabelle components for covolution algebras that were intended for separation logic and used partial abelian semigroups and monoids for modelling store-heap pairs [1]. Due to the applications in separation logic, a detailed account of cancellative and positive partial abelian monoids has been included, as these structures characterise the heap succinctly. Isabelle verification components based on this approach will be submitted as a separate AFP entry.

Our article on convolution algebras [3] provides a detailed account of convolution-based semantics for Halpern-Shoham-style interval logics [4, 7], interval temporal logics [6] and duration calculi [8] based on partial monoids. While general approaches, including modal algebras over semi-infinite intervals, are supported by the mathematical components provided, additional work on store models and assignments of variables to values is needed in order to build verification components for such interval logics.

Convolution-based liftings of partial semigroups of graphs and partial orders allow formalisations of models of true concurrency such as pomset languages and concurrent Kleene algebras [5] in Isabelle, too. An AFP entry for these is in preparation.

In all these approaches, the main task is to construct suitable partial semigroups or monoids of the computational models intended, for instance, closed intervals over the reals under fusion product, unions of heaplets (i.e. partial functions) provided their domains are disjoint, disjoint unions of graphs as parallel products. Our approach then allows a generic lifting to convolution algebras on suitable function spaces with algebraic properties, for instance of heaplets to the assertion algebra of separation logic with separating conjunction as convolution [1, 2], or

of intervals to algebraic counterparts of interval temporal logics or duration calculi with the chop operation as convolution [3]. We believe that this general construction supports other applications as well—qualitative and quantitative ones.

We would like to thank Alasdair Armstrong for his help with some Isabelle proofs and Tony Hoare for many discussions that helped us shaping the general approach.

2 Partial Semigroups

theory *Partial-Semigroups*
imports *Main*

begin

notation *times* (**infixl** ‹·› *70*)
and *times* (**infixl** ‹⊕› *70*)

2.1 Partial Semigroups

In this context, partiality is modelled by a definedness constraint D instead of a bottom element, which would make the algebra total. This is common practice in mathematics.

class *partial-times* = *times* +
fixes $D :: 'a \Rightarrow 'a \Rightarrow bool$

The definedness constraints for associativity state that the right-hand side of the associativity law is defined if and only if the left-hand side is and that, in this case, both sides are equal. This and slightly different constraints can be found in the literature.

class *partial-semigroup* = *partial-times* +
assumes *add-assocD*: $D\ y\ z \wedge D\ x\ (y \cdot z) \longleftrightarrow D\ x\ y \wedge D\ (x \cdot y)\ z$
and *add-assoc*: $D\ x\ y \wedge D\ (x \cdot y)\ z \Longrightarrow (x \cdot y) \cdot z = x \cdot (y \cdot z)$

Every semigroup is a partial semigroup.

sublocale *semigroup-mult* ⊆ *sg*: *partial-semigroup* - $\lambda x\ y.$ *True*
by *standard* (*simp-all add*: *mult-assoc*)

context *partial-semigroup*
begin

The following abbreviation is useful for sublocale statements.

abbreviation (*input*) $R\ x\ y\ z \equiv D\ y\ z \wedge x = y \cdot z$

lemma *add-assocD-var1*: $D\ y\ z \wedge D\ x\ (y \cdot z) \Longrightarrow D\ x\ y \wedge D\ (x \cdot y)\ z$
by (*simp add*: *add-assocD*)

lemma *add-assocD-var2*: $D\ x\ y \wedge D\ (x \cdot y)\ z \Longrightarrow D\ y\ z \wedge D\ x\ (y \cdot z)$
by (*simp add*: *add-assocD*)

lemma *add-assoc-var*: $D\ y\ z \wedge D\ x\ (y \cdot z) \Longrightarrow (x \cdot y) \cdot z = x \cdot (y \cdot z)$
by (*simp add*: *add-assoc add-assocD*)

2.2 Green's Preorders and Green's Relations

We define the standard Green's preorders and Green's relations. They are usually defined on monoids. On (partial) semigroups, we only obtain transitive relations.

definition *gR-rel* :: *'a* ⇒ *'a* ⇒ *bool* (**infix** ‹$\preceq_R$› *50*) **where**
$x \preceq_R y = (\exists z.\ D\ x\ z \wedge x \cdot z = y)$

definition *strict-gR-rel* :: *'a* ⇒ *'a* ⇒ *bool* (**infix** ‹$\prec_R$› *50*) **where**
$x \prec_R y = (x \preceq_R y \wedge \neg\ y \preceq_R x)$

definition *gL-rel* :: *'a* ⇒ *'a* ⇒ *bool* (**infix** ‹$\preceq_L$› *50*) **where**
$x \preceq_L y = (\exists z.\ D\ z\ x \wedge z \cdot x = y)$

definition *strict-gL-rel* :: *'a* ⇒ *'a* ⇒ *bool* (**infix** ‹$\prec_L$› *50*) **where**
$x \prec_L y = (x \preceq_L y \wedge \neg\ y \preceq_L x)$

definition *gH-rel* :: *'a* ⇒ *'a* ⇒ *bool* (**infix** ‹$\preceq_H$› *50*) **where**
$x \preceq_H y = (x \preceq_L y \wedge x \preceq_R y)$

definition *gJ-rel* :: *'a* ⇒ *'a* ⇒ *bool* (**infix** ‹$\preceq_J$› *50*) **where**
$x \preceq_J y = (\exists v\ w.\ D\ v\ x \wedge D\ (v \cdot x)\ w \wedge (v \cdot x) \cdot w = y)$

definition $gR\ x\ y = (x \preceq_R y \wedge y \preceq_R x)$

definition $gL\ x\ y = (x \preceq_L y \wedge y \preceq_L x)$

definition $gH\ x\ y = (x \preceq_H y \wedge y \preceq_H x)$

definition $gJ\ x\ y = (x \preceq_J y \wedge y \preceq_J x)$

definition *gR-downset* :: *'a* ⇒ *'a set* (‹-↓› [*100*]*100*) **where**
$x{\downarrow} \equiv \{y.\ y \preceq_R x\}$

The following counterexample rules out reflexivity.

lemma $x \preceq_R x$
oops

lemma *gR-rel-trans*: $x \preceq_R y \Longrightarrow y \preceq_R z \Longrightarrow x \preceq_R z$
by (*metis gR-rel-def add-assoc add-assocD-var2*)

lemma *gL-rel-trans*: $x \preceq_L y \Longrightarrow y \preceq_L z \Longrightarrow x \preceq_L z$
by (*metis gL-rel-def add-assocD-var1 add-assoc-var*)

lemma *gR-add-isol*: $D\ z\ y \Longrightarrow x \preceq_R y \Longrightarrow z \cdot x \preceq_R z \cdot y$
apply (*simp add*: *gR-rel-def*)
using *add-assocD-var1 add-assoc-var* **by** *blast*

lemma *gL-add-isor*: $D\ y\ z \Longrightarrow x \preceq_L y \Longrightarrow x \cdot z \preceq_L y \cdot z$
apply (*simp add*: *gL-rel-def*)
by (*metis add-assoc add-assocD-var2*)

definition *annil* :: *'a* ⇒ *bool* **where**
$annil\ x = (\forall y.\ D\ x\ y \wedge x \cdot y = x)$

definition *annir* :: *'a* ⇒ *bool* **where**
$annir\ x = (\forall y.\ D\ y\ x \wedge y \cdot x = x)$

end

2.3 Morphisms

definition *ps-morphism* :: (*'a*::*partial-semigroup* ⇒ *'b*::*partial-semigroup*) ⇒ *bool* **where**
ps-morphism f = (∀ *x y*. *D x y* ⟶ *D* (*f x*) (*f y*) ∧ *f* (*x* · *y*) = (*f x*) · (*f y*))

definition *strong-ps-morphism* :: (*'a*::*partial-semigroup* ⇒ *'b*::*partial-semigroup*) ⇒ *bool* **where**
strong-ps-morphism f = (*ps-morphism f* ∧ (∀ *x y*. *D* (*f x*) (*f y*) ⟶ *D x y*))

2.4 Locally Finite Partial Semigroups

In locally finite partial semigroups, elements can only be split in finitely many ways.

class *locally-finite-partial-semigroup* = *partial-semigroup* +
assumes *loc-fin*: *finite* (*x*↓)

2.5 Cancellative Partial Semigroups

class *cancellative-partial-semigroup* = *partial-semigroup* +
assumes *add-cancl*: *D z x* ⟹ *D z y* ⟹ *z* · *x* = *z* · *y* ⟹ *x* = *y*
and *add-cancr*: *D x z* ⟹ *D y z* ⟹ *x* · *z* = *y* · *z* ⟹ *x* = *y*

begin

lemma *unique-resl*: *D x z* ⟹ *D x z'* ⟹ *x* · *z* = *y* ⟹ *x* · *z'* = *y* ⟹ *z* = *z'*
by (*simp add*: *add-cancl*)

lemma *unique-resr*: *D z x* ⟹ *D z' x* ⟹ *z* · *x* = *y* ⟹ *z'* · *x* = *y* ⟹ *z* = *z'*
by (*simp add*: *add-cancr*)

lemma *gR-rel-mult*: *D x y* ⟹ *x* ≼$_R$ *x* · *y*
using *gR-rel-def* **by** *force*

lemma *gL-rel-mult*: *D x y* ⟹ *y* ≼$_L$ *x* · *y*
using *gL-rel-def* **by** *force*

By cancellation, the element z is uniquely defined for each pair x y, provided it exists. In both cases, z is therefore a function of x and y; it is a quotient or residual of x y.

lemma *quotr-unique*: *x* ≼$_R$ *y* ⟹ (∃!*z*. *D x z* ∧ *y* = *x* · *z*)
using *gR-rel-def add-cancl* **by** *force*

lemma *quotl-unique*: *x* ≼$_L$ *y* ⟹ (∃!*z*. *D z x* ∧ *y* = *z* · *x*)
using *gL-rel-def unique-resr* **by** *force*

definition *rquot y x* = (*THE z*. *D x z* ∧ *x* · *z* = *y*)

definition *lquot y x* = (*THE z*. *D z x* ∧ *z* · *x* = *y*)

lemma *rquot-prop*: *D x z* ∧ *y* = *x* · *z* ⟹ *z* = *rquot y x*
by (*metis* (*mono-tags*, *lifting*) *rquot-def the-equality unique-resl*)

lemma *rquot-mult*: *x* ≼$_R$ *y* ⟹ *z* = *rquot y x* ⟹ *x* · *z* = *y*
using *gR-rel-def rquot-prop* **by** *force*

lemma *rquot-D*: *x* ≼$_R$ *y* ⟹ *z* = *rquot y x* ⟹ *D x z*
using *gR-rel-def rquot-prop* **by** *force*

lemma *add-rquot*: $x \preceq_R y \Longrightarrow (D\ x\ z \wedge x \oplus z = y \longleftrightarrow z = rquot\ y\ x)$
using *gR-rel-def rquot-prop* **by** *fastforce*

lemma *add-canc1*: $D\ x\ y \Longrightarrow rquot\ (x \cdot y)\ x = y$
using *rquot-prop* **by** *simp*

lemma *add-canc2*: $x \preceq_R y \Longrightarrow x \cdot (rquot\ y\ x) = y$
using *gR-rel-def add-canc1* **by** *force*

lemma *add-canc2-prop*: $x \preceq_R y \Longrightarrow rquot\ y\ x \preceq_L y$
using *gL-rel-mult rquot-D rquot-mult* **by** *fastforce*

The next set of lemmas establishes standard Galois connections for cancellative partial semigroups.

lemma *gR-galois-imp1*: $D\ x\ z \Longrightarrow x \cdot z \preceq_R y \Longrightarrow z \preceq_R rquot\ y\ x$
by (*metis gR-rel-def add-assoc add-assocD-var2 rquot-prop*)

lemma *gR-galois-imp21*: $x \preceq_R y \Longrightarrow z \preceq_R rquot\ y\ x \Longrightarrow x \cdot z \preceq_R y$
using *gR-add-isol rquot-D rquot-mult* **by** *fastforce*

lemma *gR-galois-imp22*: $x \preceq_R y \Longrightarrow z \preceq_R rquot\ y\ x \Longrightarrow D\ x\ z$
using *gR-rel-def add-assocD add-canc1* **by** *fastforce*

lemma *gR-galois*: $x \preceq_R y \Longrightarrow (D\ x\ z \wedge x \cdot z \preceq_R y \longleftrightarrow z \preceq_R rquot\ y\ x)$
using *gR-galois-imp1 gR-galois-imp21 gR-galois-imp22* **by** *blast*

lemma *gR-rel-defined*: $x \preceq_R y \Longrightarrow D\ x\ (rquot\ y\ x)$
by (*simp add*: *rquot-D*)

lemma *ex-add-galois*: $D\ x\ z \Longrightarrow (\exists\, y.\ x \cdot z = y \longleftrightarrow rquot\ y\ x = z)$
using *add-canc1* **by** *force*

end

2.6 Partial Monoids

We allow partial monoids with multiple units. This is similar to and inspired by small categories.

class *partial-monoid* = *partial-semigroup* +
fixes $E :: 'a\ set$
assumes *unitl-ex*: $\exists\, e \in E.\ D\ e\ x \wedge e \cdot x = x$
and *unitr-ex*: $\exists\, e \in E.\ D\ x\ e \wedge x \cdot e = x$
and *units-eq*: $e1 \in E \Longrightarrow e2 \in E \Longrightarrow D\ e1\ e2 \Longrightarrow e1 = e2$

Every monoid is a partial monoid.

sublocale *monoid-mult* $\subseteq$ *mon*: *partial-monoid* - $\lambda x\ y.\ True\ \{1\}$
by (*standard*; *simp-all*)

context *partial-monoid*
begin

lemma *units-eq-var*: $e1 \in E \Longrightarrow e2 \in E \Longrightarrow e1 \neq e2 \Longrightarrow \neg\ D\ e1\ e2$
using *units-eq* **by** *force*

In partial monoids, Green's relations become preorders, but need not be partial orders.

sublocale *gR*: *preorder gR-rel strict-gR-rel*
  apply *standard*
  apply (*simp add*: *strict-gR-rel-def*)
  using *gR-rel-def unitr-ex* **apply** *force*
  using *gR-rel-trans* **by** *blast*

sublocale *gL*: *preorder gL-rel strict-gL-rel*
  apply *standard*
  apply (*simp add*: *strict-gL-rel-def*)
  using *gL-rel-def unitl-ex* **apply** *force*
  using *gL-rel-trans* **by** *blast*

lemma $x \preceq_R y \Longrightarrow y \preceq_R x \Longrightarrow x = y$
  oops

lemma *annil* $x \Longrightarrow$ *annil* $y \Longrightarrow x = y$
  oops

lemma *annir* $x \Longrightarrow$ *annir* $y \Longrightarrow x = y$
  oops

end

Next we define partial monoid morphisms.

definition *pm-morphism* :: ('*a*::*partial-monoid* ⇒ '*b*::*partial-monoid*) ⇒ *bool* **where**
  pm-morphism f = (*ps-morphism f* ∧ (∀ *e*. *e* ∈ *E* ⟶ (*f e*) ∈ *E*))

definition *strong-pm-morphism* :: ('*a*::*partial-monoid* ⇒ '*b*::*partial-monoid*) ⇒ *bool* **where**
  strong-pm-morphism f = (*pm-morphism f* ∧ (∀ *e*. (*f e*) ∈ *E* ⟶ *e* ∈ *E*))

Partial Monoids with a single unit form a special case.

class *partial-monoid-one* = *partial-semigroup* + *one* +
  assumes *oneDl*: *D x 1*
  and *oneDr*: *D 1 x*
  and *oner*: $x \cdot 1 = x$
  and *onel*: $1 \cdot x = x$

begin

sublocale *pmo*: *partial-monoid* - - {*1*}
  by *standard* (*simp-all add*: *oneDr onel oneDl oner*)

end

2.7 Cancellative Partial Monoids

class *cancellative-partial-monoid* = *cancellative-partial-semigroup* + *partial-monoid*

begin

lemma *canc-unitr*: *D x e* $\Longrightarrow x \cdot e = x \Longrightarrow e \in E$
  by (*metis add-cancl unitr-ex*)

lemma *canc-unitl*: *D e x* $\Longrightarrow e \cdot x = x \Longrightarrow e \in E$

by (*metis add-cancr unitl-ex*)

end

2.8 Positive Partial Monoids

class *positive-partial-monoid* = *partial-monoid* +
assumes *posl*: $D\ x\ y \Longrightarrow x \cdot y \in E \Longrightarrow x \in E$
and *posr*: $D\ x\ y \Longrightarrow x \cdot y \in E \Longrightarrow y \in E$

begin

lemma *pos-unitl*: $D\ x\ y \Longrightarrow e \in E \Longrightarrow x \cdot y = e \Longrightarrow x = e$
by (*metis posl posr unitr-ex units-eq-var*)

lemma *pos-unitr*: $D\ x\ y \Longrightarrow e \in E \Longrightarrow x \cdot y = e \Longrightarrow y = e$
by (*metis posl posr unitr-ex units-eq-var*)

end

2.9 Positive Cancellative Partial Monoids

class *positive-cancellative-partial-monoid* = *positive-partial-monoid* + *cancellative-partial-monoid*

begin

In positive cancellative monoids, the Green's relations are partial orders.

sublocale *pcpmR*: *order gR-rel strict-gR-rel*
apply *standard*
apply (*clarsimp simp*: *gR-rel-def*)
by (*metis canc-unitr add-assoc add-assocD-var2 pos-unitl*)

sublocale *pcpmL*: *order gL-rel strict-gL-rel*
apply *standard*
apply (*clarsimp simp*: *gL-rel-def*)
by (*metis canc-unitl add-assoc add-assocD-var1 pos-unitr*)

end

2.10 From Partial Abelian Semigroups to Partial Abelian Monoids

Next we define partial abelian semigroups. These are interesting, e.g., for the foundations of quantum mechanics and as resource monoids in separation logic.

class *pas* = *partial-semigroup* +
assumes *add-comm*: $D\ x\ y \Longrightarrow D\ y\ x \wedge x \oplus y = y \oplus x$

begin

lemma *D-comm*: $D\ x\ y \longleftrightarrow D\ y\ x$
by (*auto simp add*: *add-comm*)

lemma *add-comm'*: $D\ x\ y \Longrightarrow x \oplus y = y \oplus x$
by (*auto simp add*: *add-comm*)

lemma *gL-gH-rel*: $(x \preceq_L y) = (x \preceq_H y)$

apply (*simp add*: *gH-rel-def gL-rel-def gR-rel-def*)
using *add-comm* **by** *force*

lemma *gR-gH-rel*: $(x \preceq_R y) = (x \preceq_H y)$
apply (*simp add*: *gH-rel-def gL-rel-def gR-rel-def*)
using *add-comm* **by** *blast*

lemma *annilr*: *annil x* = *annir x*
by (*metis annil-def annir-def add-comm*)

lemma *anni-unique*: *annil x* $\Longrightarrow$ *annil y* $\Longrightarrow$ $x = y$
by (*metis annilr annil-def annir-def*)

end

The following classes collect families of partially ordered abelian semigroups and monoids.

class *locally-finite-pas* = *pas* + *locally-finite-partial-semigroup*

class *pam* = *pas* + *partial-monoid*

class *cancellative-pam* = *pam* + *cancellative-partial-semigroup*

class *positive-pam* = *pam* + *positive-partial-monoid*

class *positive-cancellative-pam* = *positive-pam* + *cancellative-pam*

class *generalised-effect-algebra* = *pas* + *partial-monoid-one*

class *cancellative-pam-one* = *cancellative-pam* + *partial-monoid-one*

class *positive-cancellative-pam-one* = *positive-cancellative-pam* + *cancellative-pam-one*

context *cancellative-pam-one*
begin

lemma *E-eq-one*: $E = \{1\}$
by (*metis oneDr oner unitl-ex units-eq singleton-iff subsetI subset-antisym*)

lemma *one-in-E*: $1 \in E$
by (*simp add*: *E-eq-one*)

end

2.11 Alternative Definitions

PAS's can be axiomatised more compactly as follows.

class *pas-alt* = *partial-times* +
assumes *pas-alt-assoc*: $D\ x\ y \wedge D\ (x \oplus y)\ z \Longrightarrow D\ y\ z \wedge D\ x\ (y \oplus z) \wedge (x \oplus y) \oplus z = x \oplus (y \oplus z)$
and *pas-alt-comm*: $D\ x\ y \Longrightarrow D\ y\ x \wedge x \oplus y = y \oplus x$

sublocale *pas-alt* $\subseteq$ *palt*: *pas*
apply *standard*
using *pas-alt-assoc pas-alt-comm* **by** *blast*+

Positive abelian PAM's can be axiomatised more compactly as well.

class *pam-pos-alt* = *pam* +
  assumes *pos-alt*: $D\ x\ y \Longrightarrow e \in E \Longrightarrow x \oplus y = e \Longrightarrow x = e$

sublocale *pam-pos-alt* $\subseteq$ *ppalt*: *positive-pam*
  apply *standard*
  using *pos-alt* **apply** *force*
  using *add-comm pos-alt* **by** *fastforce*

2.12 Product Constructions

We consider two kinds of product construction. The first one combines partial semigroups with sets, the second one partial semigroups with partial semigroups. The first one is interesting for Separation Logic. Semidirect product constructions are considered later.

instantiation *prod* :: (*type*, *partial-semigroup*) *partial-semigroup*
begin

definition *D-prod x y* = (*fst x* = *fst y* $\wedge$ *D* (*snd x*) (*snd y*))
  for *x y* :: $'a \times 'b$

definition *times-prod* :: $'a \times 'b \Rightarrow 'a \times 'b \Rightarrow 'a \times 'b$ **where**
  times-prod x y = (*fst x*, *snd x* · *snd y*)

instance
  apply (*standard*, *simp-all add*: *D-prod-def times-prod-def*)
  using *partial-semigroup-class.add-assocD* **apply** *force*
  by (*simp add*: *partial-semigroup-class.add-assoc*)

end

instantiation *prod* :: (*type*, *partial-monoid*) *partial-monoid*
begin

definition *E-prod* :: $('a \times 'b)$ *set* **where**
  E-prod = $\{x.\ snd\ x \in E\}$

instance
  apply (*standard*, *simp-all add*: *D-prod-def times-prod-def E-prod-def*)
  using *partial-monoid-class.unitl-ex* **apply** *fastforce*
  using *partial-monoid-class.unitr-ex* **apply** *fastforce*
  by (*simp add*: *partial-monoid-class.units-eq prod-eq-iff*)

end

instance *prod* :: (*type*, *pas*) *pas*
  apply (*standard*, *simp add*: *D-prod-def times-prod-def*)
  using *pas-class.add-comm* **by** *force*

lemma *prod-div1*: (*x1*::$'a$, *y1*::$'b$::*pas*) $\preceq_R$ (*x2*::$'a$, *y2*::$'b$::*pas*) $\Longrightarrow$ *x1* = *x2*
  by (*force simp*: *partial-semigroup-class.gR-rel-def times-prod-def*)

lemma *prod-div2*: (*x1*, *y1*) $\preceq_R$ (*x2*, *y2*) $\Longrightarrow$ *y1* $\preceq_R$ *y2*
  by (*force simp*: *partial-semigroup-class.gR-rel-def D-prod-def times-prod-def*)

lemma *prod-div-eq*: (*x1*, *y1*) $\preceq_R$ (*x2*, *y2*) $\longleftrightarrow$ *x1* = *x2* $\wedge$ *y1* $\preceq_R$ *y2*
  by (*force simp*: *partial-semigroup-class.gR-rel-def D-prod-def times-prod-def*)

instance *prod* :: (*type*, *pam*) *pam*
by *standard*

instance *prod* :: (*type*, *cancellative-pam*) *cancellative-pam*
by (*standard*, *auto simp*: *D-prod-def times-prod-def add-cancr add-cancl*)

lemma *prod-res-eq*: (*x1*, *y1*) $\preceq_R$ (*x2*::'*a*,*y2*::'*b*::*cancellative-pam*)
$\Longrightarrow$ *rquot* (*x2*, *y2*) (*x1*, *y1*) = (*x1*, *rquot y2 y1*)
apply (*clarsimp simp*: *partial-semigroup-class.gR-rel-def D-prod-def times-prod-def rquot-def*)
apply (*rule theI2 conjI*)
apply *force*
using *add-cancl* **apply** *force*
by (*rule the-equality*, *auto simp*: *add-cancl*)

instance *prod* :: (*type*, *positive-pam*) *positive-pam*
apply (*standard*, *simp-all add*: *E-prod-def D-prod-def times-prod-def*)
using *positive-partial-monoid-class.posl* **apply** *blast*
using *positive-partial-monoid-class.posr* **by** *blast*

instance *prod* :: (*type*, *positive-cancellative-pam*) *positive-cancellative-pam* **..**

instance *prod* :: (*type*, *locally-finite-pas*) *locally-finite-pas*
proof (*standard*, *case-tac x*, *clarsimp*)
fix *s* :: '*a* **and** *x* :: '*b*
have *finite* (*x*↓)
by (*simp add*: *loc-fin*)
hence *finite* {*y*. ∃ *z*. *D y z* ∧ *y* ⊕ *z* = *x*}
by (*simp add*: *partial-semigroup-class.gR-downset-def partial-semigroup-class.gR-rel-def*)
hence *finite* {(*s*, *y*)| *y*. ∃ *z*. *D y z* ∧ *y* ⊕ *z* = *x*}
by (*drule-tac f*=λ*y*. (*s*, *y*) **in** *finite-image-set*)
moreover have {*y*. ∃ *z1 z2*. *D y* (*z1*, *z2*) ∧ *y* ⊕ (*z1*, *z2*) = (*s*, *x*)}
⊆ {(*s*, *y*)| *y*. ∃ *z*. *D y z* ∧ *y* ⊕ *z* = *x*}
by (*auto simp*: *D-prod-def times-prod-def*)
ultimately have *finite* {*y*. ∃ *z1 z2*. *D y* (*z1*, *z2*) ∧ *y* ⊕ (*z1*, *z2*) = (*s*, *x*)}
by (*auto intro*: *finite-subset*)
thus *finite* ((*s*, *x*)↓)
by (*simp add*: *partial-semigroup-class.gR-downset-def partial-semigroup-class.gR-rel-def*)
qed

Next we consider products of two partial semigroups.

definition *ps-prod-D* :: '*a* :: *partial-semigroup* × '*b* :: *partial-semigroup* ⇒ '*a* × '*b* ⇒ *bool*
where *ps-prod-D x y* ≡ *D* (*fst x*) (*fst y*) ∧ *D* (*snd x*) (*snd y*)

definition *ps-prod-times* :: '*a* :: *partial-semigroup* × '*b* :: *partial-semigroup* ⇒ '*a* × '*b* ⇒ '*a* × '*b*
where *ps-prod-times x y* = (*fst x* · *fst y*, *snd x* · *snd y*)

interpretation *ps-prod*: *partial-semigroup ps-prod-times ps-prod-D*
apply (*standard*, *simp-all add*: *ps-prod-D-def ps-prod-times-def*)
apply (*meson partial-semigroup-class.add-assocD*)
by (*simp add*: *partial-semigroup-class.add-assoc*)

interpretation *pas-prod*: *pas ps-prod-times ps-prod-D* :: '*a* :: *pas* × '*b* :: *pas* ⇒ '*a* × '*b* ⇒ *bool*
by (*standard*, *clarsimp simp*: *ps-prod-D-def ps-prod-times-def pas-class.add-comm*)

definition *pm-prod-E* :: (*'a* :: *partial-monoid* × *'b* :: *partial-monoid*) *set* **where**
pm-prod-E = {*x*. *fst x* ∈ *E* ∧ *snd x* ∈ *E*}

interpretation *pm-prod*: *partial-monoid ps-prod-times ps-prod-D pm-prod-E*
apply *standard*
apply (*simp-all add*: *ps-prod-times-def ps-prod-D-def pm-prod-E-def*)
apply (*metis partial-monoid-class.unitl-ex prod.collapse*)
apply (*metis partial-monoid-class.unitr-ex prod.collapse*)
by (*simp add*: *partial-monoid-class.units-eq prod.expand*)

interpretation *pam-prod*: *pam ps-prod-times ps-prod-D pm-prod-E* :: (*'a* :: *pam* × *'a* :: *pam*) *set* **..**

2.13 Partial Semigroup Actions and Semidirect Products

(Semi)group actions are a standard mathematical construction. We generalise this to partial semigroups and monoids. We use it to define semidirect products of partial semigroups. A generalisation to wreath products might be added in the future.

First we define the (left) action of a partial semigroup on a set. A right action could be defined in a similar way, but we do not pursue this at the moment.

locale *partial-sg-laction* =
fixes *Dla* :: *'a*::*partial-semigroup* ⇒ *'b* ⇒ *bool*
and *act* :: *'a*::*partial-semigroup* ⇒ *'b* ⇒ *'b* (‹*α*›)
assumes *act-assocD*: *D x y* ∧ *Dla* (*x* · *y*) *p* ⟷ *Dla y p* ∧ *Dla x* (*α y p*)
and *act-assoc*: *D x y* ∧ *Dla* (*x* · *y*) *p* ⟹ *α* (*x* · *y*) *p* = *α x* (*α y p*)

Next we define the action of a partial semigroup on another partial semigroup. In the tradition of semigroup theory we use addition as a non-commutative operation for the second semigroup.

locale *partial-sg-sg-laction* = *partial-sg-laction* +
assumes *act-distribD*: *D* (*p*::*'b*::*partial-semigroup*) *q* ∧ *Dla* (*x*::*'a*::*partial-semigroup*) (*p* ⊕ *q*) ⟷ *Dla x p* ∧ *Dla x q* ∧ *D* (*α x p*) (*α x q*)
and *act-distrib*: *D p q* ∧ *Dla x* (*p* ⊕ *q*) ⟹ *α x* (*p* ⊕ *q*) = (*α x p*) ⊕ (*α x q*)

begin

Next we define the semidirect product as a partial operation and show that the semidirect product of two partial semigroups forms a partial semigroup.

definition *sd-D* :: (*'a* × *'b*) ⇒ (*'a* × *'b*) ⇒ *bool* **where**
sd-D x y ≡ *D* (*fst x*) (*fst y*) ∧ *Dla* (*fst x*) (*snd y*) ∧ *D* (*snd x*) (*α* (*fst x*) (*snd y*))

definition *sd-prod* :: (*'a* × *'b*) ⇒ (*'a* × *'b*) ⇒ (*'a* × *'b*) **where**
sd-prod x y = ((*fst x*) · (*fst y*), (*snd x*) ⊕ (*α* (*fst x*) (*snd y*)))

sublocale *dp-semigroup*: *partial-semigroup sd-prod sd-D*
apply *unfold-locales*
apply (*simp-all add*: *sd-prod-def sd-D-def*)
apply (*clarsimp*, *metis act-assoc act-assocD act-distrib act-distribD add-assocD*)
by (*clarsimp*, *metis act-assoc act-assocD act-distrib act-distribD add-assoc add-assocD*)

end

Finally we define the semigroup action for two partial monoids and show that the semidirect product of two partial monoids is a partial monoid.

locale *partial-mon-sg-laction* = *partial-sg-sg-laction Dla*

for *Dla* :: *'a*::*partial-monoid* ⇒ *'b*::*partial-semigroup* ⇒ *bool* +
assumes *act-unitl*: $e \in E \Longrightarrow Dla\ e\ p \wedge \alpha\ e\ p = p$

locale *partial-mon-mon-laction* = *partial-mon-sg-laction* - *Dla*
for *Dla* :: *'a*::*partial-monoid* ⇒ *'b*::*partial-monoid* ⇒ *bool* +
assumes *act-annir*: $e \in Ea \Longrightarrow Dla\ x\ e \wedge \alpha\ x\ e = e$

begin

definition *sd-E* :: (*'a* × *'b*) *set* **where**
sd-E = {*x*. *fst x* ∈ *E* ∧ *snd x* ∈ *E*}

sublocale *dp-semigroup* : *partial-monoid sd-prod sd-D sd-E*
apply *unfold-locales*
apply (*simp-all add*: *sd-prod-def sd-D-def sd-E-def*)
apply (*metis act-annir eq-fst-iff eq-snd-iff mem-Collect-eq partial-monoid-class.unitl-ex*)
apply (*metis act-annir eq-fst-iff eq-snd-iff partial-monoid-class.unitr-ex*)
by (*metis act-annir partial-monoid-class.units-eq prod-eqI*)

end

end

3 Models of Partial Semigroups

theory *Partial-Semigroup-Models*
imports *Partial-Semigroups*

begin

So far this section collects three models that we need for applications. Other interesting models might be added in the future. These might include binary relations, formal power series and matrices, paths in graphs under fusion, program traces with alternating state and action symbols under fusion, partial orders under series and parallel products.

3.1 Partial Monoids of Segments and Intervals

Segments of a partial order are sub partial orders between two points. Segments generalise intervals in that intervals are segments in linear orders. We formalise segments and intervals as pairs, where the first coordinate is smaller than the second one. Algebras of segments and intervals are interesting in Rota's work on the foundations of combinatorics as well as for interval logics and duration calculi.

First we define the subtype of ordered pairs of one single type.

typedef *'a dprod* = {(*x*::*'a*, *y*::*'a*). *True*}
by *simp*

setup-lifting *type-definition-dprod*

Such pairs form partial semigroups and partial monoids with respect to fusion.

instantiation *dprod* :: (*type*) *partial-semigroup*
begin

lift-definition *D-dprod* :: *'a dprod* ⇒ *'a dprod* ⇒ *bool* **is** λ*x y*. (*snd x* = *fst y*) **.**

lift-definition *times-dprod* :: *'a dprod* ⇒ *'a dprod* ⇒ *'a dprod* **is** λ*x y*. (*fst x*, *snd y*)
by *simp*

instance
by *standard* (*transfer*, *force*)+

end

instantiation *dprod* :: (*type*) *partial-monoid*
begin

lift-definition *E-dprod* :: *'a dprod set* **is** {*x*. *fst x* = *snd x*}
by *simp*

instance
by *standard* (*transfer*,*force*)+

end

Next we define the type of segments.

typedef (**overloaded**) *'a segment* = {*x*::(*'a*::*order* × *'a*::*order*). *fst x* ≤ *snd x*}
by *force*

setup-lifting *type-definition-segment*

Segments form partial monoids as well.

instantiation *segment* :: (*order*) *partial-monoid*
begin

lift-definition *E-segment* :: *'a segment set* **is** {*x*. *fst x* = *snd x*}
by *simp*

lift-definition *D-segment* :: *'a*::*order segment* ⇒ *'a segment* ⇒ *bool*
is λ*x y*. (*snd x* = *fst y*) **.**

lift-definition *times-segment* :: *'a*::*order segment* ⇒ *'a segment* ⇒ *'a segment*
is λ*x y*. *if snd x* = *fst y then* (*fst x*, *snd y*) *else x*
by *auto*

instance
by *standard* (*transfer*, *force*)+

end

Next we define the function segm that maps segments-as-pairs to segments-as-sets.

definition *segm* :: *'a*::*order segment* ⇒ *'a set* **where**
segm x = {*y*. *fst* (*Rep-segment x*) ≤ *y* ∧ *y* ≤ *snd* (*Rep-segment x*)}

thm *Rep-segment*

lemma *segm-sub-morph*: *snd* (*Rep-segment x*) = *fst* (*Rep-segment y*) ⟹ *segm x* ∪ *segm y* ≤ *segm* (*x* · *y*)
apply (*simp add*: *segm-def times-segment.rep-eq*, *safe*)

using *Rep-segment dual-order.trans* **apply** *blast*
by (*metis* (*mono-tags*, *lifting*) *Rep-segment dual-order.trans mem-Collect-eq*)

The function segm is not generally a morphism.

lemma *snd* (*Rep-segment x*) = *fst* (*Rep-segment y*) ⟹ *segm x* ∪ *segm y* = *segm* (*x* · *y*)
oops

Intervals are segments over orders that satisfy Halpern and Shoham's linear order property. This is still more general than linearity of the poset.

class *lip-order* = *order* +
assumes *lip*: *x* ≤ *y* ⟹ (∀ *v w*. (*x* ≤ *v* ∧ *v* ≤ *y* ∧ *x* ≤ *w* ∧ *w* ≤ *y* ⟶ *v* ≤ *w* ∨ *w* ≤ *v*))

The function segm is now a morphism.

lemma *segm-morph*: *snd* (*Rep-segment x*::('*a*::*lip-order* × '*a*::*lip-order*)) = *fst* (*Rep-segment y*)
⟹ *segm x* ∪ *segm y* = *segm* (*x* · *y*)
apply (*simp add*: *segm-def times-segment-def*)
apply (*transfer*, *clarsimp simp add*: *Abs-segment-inverse lip*, *safe*)
apply *force*+
by (*meson lip order-trans*)

3.2 Cancellative PAM's of Partial Functions

We show that partial functions under disjoint union form a positive cancellative PAM. This is interesting for modeling the heap in separation logic.

type-synonym '*a pfun* = '*a* ⇒ '*a option*

definition *ortho* :: '*a pfun* ⇒ '*a pfun* ⇒ *bool*
where *ortho f g* ≡ *dom f* ∩ *dom g* = {}

lemma *pfun-comm*: *ortho x y* ⟹ *x* ++ *y* = *y* ++ *x*
by (*force simp*: *ortho-def intro*!: *map-add-comm*)

lemma *pfun-canc*: *ortho z x* ⟹ *ortho z y* ⟹ *z* ++ *x* = *z* ++ *y* ⟹ *x* = *y*
apply (*auto simp*: *ortho-def map-add-def option.case-eq-if fun-eq-iff*)
by (*metis domIff dom-restrict option.collapse restrict-map-def*)

interpretation *pfun*: *positive-cancellative-pam-one map-add ortho* {*Map.empty*} *Map.empty*
apply (*standard*, *auto simp*: *ortho-def pfun-canc*)
by (*simp-all add*: *inf-commute map-add-comm ortho-def pfun-canc*)

3.3 PAM's of Disjoint Unions of Sets

This simple disjoint union construction underlies important compositions of graphs or partial orders, in particular in the context of complete joins and disjoint unions of graphs and of series and parallel products of partial orders.

instantiation *set* :: (*type*) *pas*
begin

definition *D-set* :: '*a set* ⇒ '*a set* ⇒ *bool* **where**
D-set x y ≡ *x* ∩ *y* = {}

definition *times-set* :: '*a set* ⇒ '*a set* ⇒ '*a set* **where**
times-set x y = *x* ∪ *y*

instance
by *standard* (*auto simp*: *D-set-def times-set-def*)

end

instantiation *set* :: (*type*) *pam*
begin

definition *E-set* :: *'a set set* **where**
E-set = {{}}

instance
by *standard* (*auto simp*: *D-set-def times-set-def E-set-def*)

end

end

4 Quantales

This entry will be merged eventually with other quantale entries and become a standalone one.

theory *Quantales*
imports *Main*

begin

notation *sup* (**infixl** ‹⊔› *60*)
and *inf* (**infixl** ‹⊓› *55*)
and *top* (‹⊤›)
and *bot* (‹⊥›)
and *relcomp* (**infixl** ‹;› *70*)
and *times* (**infixl** ‹·› *70*)
and *Sup* (‹⨆ -› [*900*] *900*)
and *Inf* (‹⨅ -› [*900*] *900*)

4.1 Properties of Complete Lattices

lemma (**in** *complete-lattice*) *Sup-sup-pred*: $x \sqcup \bigsqcup \{y.\ P\ y\} = \bigsqcup \{y.\ y = x \vee P\ y\}$
apply (*rule order.antisym*)
apply (*simp add*: *Collect-mono Sup-subset-mono Sup-upper*)
using *Sup-least Sup-upper sup.coboundedI2* **by** *force*

lemma (**in** *complete-lattice*) *sup-Sup*: $x \sqcup y = \bigsqcup \{x,y\}$
by *simp*

lemma (**in** *complete-lattice*) *sup-Sup-var*: $x \sqcup y = \bigsqcup \{z.\ z \in \{x,y\}\}$
by (*metis Collect-mem-eq sup-Sup*)

lemma (**in** *complete-boolean-algebra*) *shunt1*: $x \sqcap y \leq z \longleftrightarrow x \leq -y \sqcup z$
proof *standard*
assume $x \sqcap y \leq z$
hence $-y \sqcup (x \sqcap y) \leq -y \sqcup z$
using *sup.mono* **by** *blast*

hence $-y \sqcup x \leq -y \sqcup z$
by (*simp add*: *sup-inf-distrib1*)
thus $x \leq -y \sqcup z$
by *simp*
next
assume $x \leq -y \sqcup z$
hence $x \sqcap y \leq (-y \sqcup z) \sqcap y$
using *inf-mono* **by** *auto*
thus $x \sqcap y \leq z$
using *inf.boundedE inf-sup-distrib2* **by** *auto*
qed

lemma (**in** *complete-boolean-algebra*) *meet-shunt*: $x \sqcap y = \bot \longleftrightarrow x \leq -y$
by (*metis bot-least inf-absorb2 inf-compl-bot-left2 shunt1 sup-absorb1*)

lemma (**in** *complete-boolean-algebra*) *join-shunt*: $x \sqcup y = \top \longleftrightarrow -x \leq y$
by (*metis compl-sup compl-top-eq double-compl meet-shunt*)

4.2 Familes of Proto-Quantales

Proto-Quanales are complete lattices equipped with an operation of composition or multiplication that need not be associative.

class *proto-near-quantale* = *complete-lattice* + *times* +
assumes *Sup-distr*: $\bigsqcup X \cdot y = \bigsqcup \{x \cdot y \mid x.\ x \in X\}$

begin

lemma *mult-botl* [*simp*]: $\bot \cdot x = \bot$
using *Sup-distr*[**where** X={}] **by** *auto*

lemma *sup-distr*: $(x \sqcup y) \cdot z = (x \cdot z) \sqcup (y \cdot z)$
using *Sup-distr*[**where** X={x, y}] **by** (*fastforce intro*!: *Sup-eqI*)

lemma *mult-isor*: $x \leq y \Longrightarrow x \cdot z \leq y \cdot z$
by (*metis sup.absorb-iff1 sup-distr*)

definition *bres* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixr** $\leftarrow$ *60*) **where**
$x \rightarrow z = \bigsqcup \{y.\ x \cdot y \leq z\}$

definition *fres* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixl** $\leftarrow$ *60*) **where**
$z \leftarrow y = \bigsqcup \{x.\ x \cdot y \leq z\}$

lemma *bres-galois-imp*: $x \cdot y \leq z \longrightarrow y \leq x \rightarrow z$
by (*simp add*: *Sup-upper bres-def*)

lemma *fres-galois*: $x \cdot y \leq z \longleftrightarrow x \leq z \leftarrow y$
proof
show $x \cdot y \leq z \Longrightarrow x \leq z \leftarrow y$
by (*simp add*: *Sup-upper fres-def*)
next
assume $x \leq z \leftarrow y$
hence $x \cdot y \leq \bigsqcup \{x.\ x \cdot y \leq z\} \cdot y$
by (*simp add*: *fres-def mult-isor*)
also have ... $= \bigsqcup \{x \cdot y \mid x.\ x \cdot y \leq z\}$
by (*simp add*: *Sup-distr*)

also have $... \leq z$
by (*rule Sup-least*, *auto*)
finally show $x \cdot y \leq z$ **.**
qed

end

class *proto-pre-quantale* = *proto-near-quantale* +
assumes *Sup-subdistl*: $\bigsqcup\{x \cdot y \mid y\ .\ y \in Y\} \leq x \cdot \bigsqcup Y$

begin

lemma *sup-subdistl*: $(x \cdot y) \sqcup (x \cdot z) \leq x \cdot (y \sqcup z)$
using *Sup-subdistl*[**where** Y=$\{y, z\}$] *Sup-le-iff* **by** *auto*

lemma *mult-isol*: $x \leq y \Longrightarrow z \cdot x \leq z \cdot y$
by (*metis le-iff-sup le-sup-iff sup-subdistl*)

end

class *weak-proto-quantale* = *proto-near-quantale* +
assumes *weak-Sup-distl*: $Y \neq \{\} \Longrightarrow x \cdot \bigsqcup Y = \bigsqcup\{x \cdot y \mid y.\ y \in Y\}$

begin

subclass *proto-pre-quantale*
proof *standard*
have *a*: $\bigwedge x\ Y.\ Y = \{\} \Longrightarrow \bigsqcup\{x \cdot y \mid y.\ y \in Y\} \leq x \cdot \bigsqcup Y$
by *simp*
have *b*: $\bigwedge x\ Y.\ Y \neq \{\} \Longrightarrow \bigsqcup\{x \cdot y \mid y.\ y \in Y\} \leq x \cdot \bigsqcup Y$
by (*simp add*: *weak-Sup-distl*)
show $\bigwedge x\ Y.\ \bigsqcup\{x \cdot y \mid y.\ y \in Y\} \leq x \cdot \bigsqcup Y$
using *a b* **by** *blast*
qed

lemma *sup-distl*: $x \cdot (y \sqcup z) = (x \cdot y) \sqcup (x \cdot z)$
using *weak-Sup-distl*[**where** Y=$\{y, z\}$] **by** (*fastforce intro*!: *Sup-eqI*)

lemma $y \leq x \rightarrow z \longrightarrow x \cdot y \leq z$
oops

end

class *proto-quantale* = *proto-near-quantale* +
assumes *Sup-distl*: $x \cdot \bigsqcup Y = \bigsqcup\{x \cdot y \mid y.\ y \in Y\}$

begin

subclass *weak-proto-quantale*
by *standard* (*simp add*: *Sup-distl*)

lemma *bres-galois*: $x \cdot y \leq z \longleftrightarrow y \leq x \rightarrow z$
proof
show $x \cdot y \leq z \Longrightarrow y \leq x \rightarrow z$
by (*simp add*: *Sup-upper bres-def*)

next
assume $y \leq x \rightarrow z$
hence $x \cdot y \leq x \cdot \bigsqcup \{y.\ x \cdot y \leq z\}$
by (*simp add*: *bres-def mult-isol*)
also have ... $= \bigsqcup \{x \cdot y \mid y.\ x \cdot y \leq z\}$
by (*simp add*: *Sup-distl*)
also have ... $\leq z$
by (*rule Sup-least*, *auto*)
finally show $x \cdot y \leq z$ **.**
qed

end

4.3 Families of Quantales

class *near-quantale* = *proto-near-quantale* + *semigroup-mult*

class *unital-near-quantale* = *near-quantale* + *monoid-mult*

begin

definition *iter* :: $'a \Rightarrow 'a$ **where**
iter $x \equiv \bigsqcap \{y.\ \exists\, i.\ y = x \,\hat{}\, i\}$

lemma *iter-ref* [*simp*]: *iter* $x \leq 1$
apply (*simp add*: *iter-def*)
by (*metis* (*mono-tags*, *lifting*) *Inf-lower local.power.power-0 mem-Collect-eq*)

lemma *le-top*: $x \leq \top \cdot x$
by (*metis mult-isor mult.monoid-axioms top-greatest monoid.left-neutral*)

end

class *pre-quantale* = *proto-pre-quantale* + *semigroup-mult*

subclass (**in** *pre-quantale*) *near-quantale* **..**

class *unital-pre-quantale* = *pre-quantale* + *monoid-mult*

subclass (**in** *unital-pre-quantale*) *unital-near-quantale* **..**

class *weak-quantale* = *weak-proto-quantale* + *semigroup-mult*

subclass (**in** *weak-quantale*) *pre-quantale* **..**

The following counterexample shows an important consequence of weakness: the absence of right annihilation.

lemma (**in** *weak-quantale*) $x \cdot \bot = \bot$
oops

class *unital-weak-quantale* = *weak-quantale* + *monoid-mult*

lemma (**in** *unital-weak-quantale*) $x \cdot \bot = \bot$
oops

subclass (**in** *unital-weak-quantale*) *unital-pre-quantale* **..**

class *quantale* = *proto-quantale* + *semigroup-mult*

begin

subclass *weak-quantale* **..**

lemma *mult-botr* [*simp*]: $x \cdot \bot = \bot$
using *Sup-distl*[**where** Y={}] **by** *auto*

end

class *unital-quantale* = *quantale* + *monoid-mult*

subclass (**in** *unital-quantale*) *unital-weak-quantale* **..**

class *ab-quantale* = *quantale* + *ab-semigroup-mult*

begin

lemma *bres-fres-eq*: $x \rightarrow y = y \leftarrow x$
by (*simp add*: *fres-def bres-def mult-commute*)

end

class *ab-unital-quantale* = *ab-quantale* + *unital-quantale*

class *distrib-quantale* = *quantale* + *complete-distrib-lattice*

class *bool-quantale* = *quantale* + *complete-boolean-algebra*

class *distrib-unital-quantale* = *unital-quantale* + *complete-distrib-lattice*

class *bool-unital-quantale* = *unital-quantale* + *complete-boolean-algebra*

class *distrib-ab-quantale* = *distrib-quantale* + *ab-quantale*

class *bool-ab-quantale* = *bool-quantale* + *ab-quantale*

class *distrib-ab-unital-quantale* = *distrib-quantale* + *unital-quantale*

class *bool-ab-unital-quantale* = *bool-ab-quantale* + *unital-quantale*

4.4 Quantales of Booleans and Complete Boolean Algebras

instantiation *bool* :: *bool-ab-unital-quantale*
begin

definition *one-bool* = *True*

definition *times-bool* = ($\lambda x\ y.\ x \wedge y$)

instance
by *standard* (*auto simp*: *times-bool-def one-bool-def*)

end

context *complete-distrib-lattice*
begin

interpretation *cdl-quantale*: *distrib-quantale* - - - - - - - - *inf*
by *standard* (*simp-all add*: *inf.assoc Setcompr-eq-image Sup-inf inf-Sup*)

end

context *complete-boolean-algebra*
begin

interpretation *cba-quantale*: *bool-ab-unital-quantale inf* - - - - - - - - - - *top*
apply *standard*
apply (*simp add*: *inf.assoc*)
apply (*simp add*: *inf.commute*)
apply (*simp add*: *Setcompr-eq-image Sup-inf*)
apply (*simp add*: *Setcompr-eq-image inf-Sup*)
by *simp-all*

In this setting, residuation can be translated like classical implication.

lemma *cba-bres1*: $x \sqcap y \leq z \longleftrightarrow x \leq$ *cba-quantale.bres y z*
using *cba-quantale.bres-galois inf.commute* **by** *fastforce*

lemma *cba-bres2*: $x \leq -y \sqcup z \longleftrightarrow x \leq$ *cba-quantale.bres y z*
using *cba-bres1 shunt1* **by** *auto*

lemma *cba-bres-prop*: *cba-quantale.bres x y* $= -x \sqcup y$
using *cba-bres2 order.eq-iff* **by** *blast*

end

Other models will follow.

4.5 Products of Quantales

definition *Inf-prod X* $= (\bigsqcap\{fst\ x\ |x.\ x \in X\}, \bigsqcap\{snd\ x\ |x.\ \ x \in X\})$

definition *inf-prod x y* $= (fst\ x \sqcap fst\ y,\ snd\ x \sqcap snd\ y)$

definition *bot-prod* $= (bot,bot)$

definition *Sup-prod X* $= (\bigsqcup\{fst\ x\ |x.\ x \in X\}, \bigsqcup\{snd\ x\ |x.\ \ x \in X\})$

definition *sup-prod x y* $= (fst\ x \sqcup fst\ y,\ snd\ x \sqcup snd\ y)$

definition *top-prod* $= (top,top)$

definition *less-eq-prod x y* $\equiv$ *less-eq* $(fst\ x)\ (fst\ y) \wedge$ *less-eq* $(snd\ x)\ (snd\ y)$

definition *less-prod x y* $\equiv$ *less-eq* $(fst\ x)\ (fst\ y) \wedge$ *less-eq* $(snd\ x)\ (snd\ y) \wedge x \neq y$

definition *times-prod′ x y* $= (fst\ x \cdot fst\ y,\ snd\ x \cdot snd\ y)$

definition *one-prod* = (*1*,*1*)

interpretation *prod*: *complete-lattice Inf-prod Sup-prod inf-prod less-eq-prod less-prod sup-prod bot-prod top-prod* :: (*'a*::*complete-lattice* × *'b*::*complete-lattice*)
apply *standard*
apply (*simp-all add*: *Inf-prod-def Sup-prod-def inf-prod-def sup-prod-def bot-prod-def top-prod-def less-eq-prod-def less-prod-def Sup-distl Sup-distr*)
apply *force+*
apply (*rule conjI*, (*rule Inf-lower*, *force*)+)
apply (*rule conjI*, (*rule Inf-greatest*, *force*)+)
apply (*rule conjI*, (*rule Sup-upper*, *force*)+)
by (*rule conjI*, (*rule Sup-least*, *force*)+)

interpretation *prod*: *proto-near-quantale Inf-prod Sup-prod inf-prod less-eq-prod less-prod sup-prod bot-prod top-prod* :: (*'a*::*proto-near-quantale* × *'b*::*proto-near-quantale*) *times-prod'*
apply (*standard*, *simp add*: *times-prod'-def Sup-prod-def*)
by (*rule conjI*, (*simp add*: *Sup-distr*, *clarify*, *metis*)+)

interpretation *prod*: *proto-quantale Inf-prod Sup-prod inf-prod less-eq-prod less-prod sup-prod bot-prod top-prod* :: (*'a*::*proto-quantale* × *'b*::*proto-quantale*) *times-prod'*
apply (*standard*, *simp add*: *times-prod'-def Sup-prod-def less-eq-prod-def*)
by (*rule conjI*, (*simp add*: *Sup-distl*, *metis*)+)

interpretation *prod*: *unital-quantale one-prod times-prod' Inf-prod Sup-prod inf-prod less-eq-prod less-prod sup-prod bot-prod top-prod* :: (*'a*::*unital-quantale* × *'b*::*unital-quantale*)
by *standard* (*simp-all add*: *one-prod-def times-prod'-def mult.assoc*)

4.6 Quantale Modules and Semidirect Products

Quantale modules are extensions of semigroup actions in that a quantale acts on a complete lattice.

locale *unital-quantale-module* =
fixes *act* :: *'a*::*unital-quantale* ⇒ *'b*::*complete-lattice* ⇒ *'b* (‹*α*›)
assumes *act1*: *α* (*x* · *y*) *p* = *α x* (*α y p*)
and *act2* [*simp*]: *α 1 p* = *p*
and *act3*: *α* (⨆*X*) *p* = ⨆{*α x p* |*x*. *x* ∈ *X*}
and *act4*: *α x* (⨆*P*) = ⨆{*α x p* |*p*. *p* ∈ *P*}

context *unital-quantale-module*
begin

Actions are morphisms. The curried notation is particularly convenient for this.

lemma *act-morph1*: *α* (*x* · *y*) = (*α x*) ∘ (*α y*)
by (*simp add*: *fun-eq-iff act1*)

lemma *act-morph2*: *α 1* = *id*
by (*simp add*: *fun-eq-iff*)

lemma *emp-act*: *α* (⨆{}) *p* = ⊥
by (*simp only*: *act3*, *force*)

lemma *emp-act-var*: *α* ⊥ *p* = ⊥
using *emp-act* **by** *auto*

lemma *act-emp*: *α x* (⨆{}) = ⊥

by (*simp only*: *act4*, *force*)

lemma *act-emp-var*: $\alpha\ x\ \bot = \bot$
using *act-emp* **by** *auto*

lemma *act-sup-distl*: $\alpha\ x\ (p \sqcup q) = (\alpha\ x\ p) \sqcup (\alpha\ x\ q)$
proof−
have $\alpha\ x\ (p \sqcup q) = \alpha\ x\ (\bigsqcup\{p,q\})$
by *simp*
also have ... $= \bigsqcup\{\alpha\ x\ y\ |y.\ y \in \{p,q\}\}$
by (*rule act4*)
also have ... $= \bigsqcup\{v.\ v = \alpha\ x\ p \vee v = \alpha\ x\ q\}$
by (*metis empty-iff insert-iff*)
finally show *?thesis*
by (*metis Collect-disj-eq Sup-insert ccpo-Sup-singleton insert-def singleton-conv*)
qed

lemma *act-sup-distr*: $\alpha\ (x \sqcup y)\ p = (\alpha\ x\ p) \sqcup (\alpha\ y\ p)$
proof−
have $\alpha\ (x \sqcup y)\ p\ = \alpha\ (\bigsqcup\{x,y\})\ p$
by *simp*
also have ... $= \bigsqcup\{\alpha\ v\ p\ |v.\ v \in \{x,y\}\}$
by (*rule act3*)
also have ... $= \bigsqcup\{v.\ v = \alpha\ x\ p \vee v = \alpha\ y\ p\}$
by (*metis empty-iff insert-iff*)
finally show *?thesis*
by (*metis Collect-disj-eq Sup-insert ccpo-Sup-singleton insert-def singleton-conv*)
qed

lemma *act-sup-distr-var*: $\alpha\ (x \sqcup y) = (\alpha\ x) \sqcup (\alpha\ y)$
by (*simp add*: *fun-eq-iff act-sup-distr*)

Next we define the semidirect product of a unital quantale and a complete lattice.

definition *sd-prod x y* = (*fst x* · *fst y*, *snd x* ⊔ α (*fst x*) (*snd y*))

lemma *sd-distr-aux*:
$\bigsqcup\{snd\ x\ |x.\ x \in X\} \sqcup \bigsqcup\{\alpha\ (fst\ x)\ p\ |x.\ x \in X\} = \bigsqcup\{snd\ x \sqcup \alpha\ (fst\ x)\ p\ |x.\ x \in X\}$
proof (*rule antisym*, *rule sup-least*)
show $\bigsqcup\{snd\ x\ |x.\ x \in X\} \leq \bigsqcup\{snd\ x \sqcup \alpha\ (fst\ x)\ p\ |x.\ x \in X\}$
proof (*rule Sup-least*)
fix $x :: 'b$
assume $x \in \{snd\ x\ |x.\ x \in X\}$
hence $\exists\, b\ pa.\ x \sqcup b = snd\ pa \sqcup \alpha\ (fst\ pa)\ p \wedge pa \in X$
by *blast*
hence $\exists\, b.\ x \sqcup b \in \{snd\ pa \sqcup \alpha\ (fst\ pa)\ p\ |pa.\ pa \in X\}$
by *blast*
thus $x \leq \bigsqcup\{snd\ pa \sqcup \alpha\ (fst\ pa)\ p\ |pa.\ pa \in X\}$
by (*meson Sup-upper sup.bounded-iff*)
qed
next
show $\bigsqcup\{\alpha\ (fst\ x)\ p\ |x.\ x \in X\} \leq \bigsqcup\{snd\ x \sqcup \alpha\ (fst\ x)\ p\ |x.\ x \in X\}$
proof (*rule Sup-least*)
fix $x :: 'b$
assume $x \in \{\alpha\ (fst\ x)\ p\ |x.\ x \in X\}$
then have $\exists\, b\ pa.\ b \sqcup x = snd\ pa \sqcup \alpha\ (fst\ pa)\ p \wedge pa \in X$

by *blast*
then have $\exists b.\ b \sqcup x \in \{snd\ pa \sqcup \alpha\ (fst\ pa)\ p\ |pa.\ pa \in X\}$
by *blast*
then show $x \leq \bigsqcup\{snd\ pa \sqcup \alpha\ (fst\ pa)\ p\ |pa.\ pa \in X\}$
by (*meson Sup-upper le-supE*)
qed
next
show $\bigsqcup\{snd\ x \sqcup \alpha\ (fst\ x)\ p\ |x.\ x \in X\} \leq \bigsqcup\{snd\ x\ |x.\ x \in X\} \sqcup \bigsqcup\{\alpha\ (fst\ x)\ p\ |x.\ x \in X\}$
apply (*rule Sup-least*)
apply *safe*
apply (*rule sup-least*)
apply (*metis* (*mono-tags*, *lifting*) *Sup-upper mem-Collect-eq sup.coboundedI1*)
by (*metis* (*mono-tags*, *lifting*) *Sup-upper mem-Collect-eq sup.coboundedI2*)
qed

lemma *sd-distr*: *sd-prod* (*Sup-prod X*) *y* = *Sup-prod* {*sd-prod x y* |*x*. *x* ∈ *X*}
proof −
have *sd-prod* (*Sup-prod X*) *y* = *sd-prod* (⨆{*fst x* |*x*. *x* ∈ *X*}, ⨆{*snd x* |*x*. *x* ∈ *X*}) *y*
by (*simp add*: *Sup-prod-def*)
also have
... = ((⨆{*fst x* |*x*. *x* ∈ *X*}) · *fst y*, (⨆{*snd x* |*x*. *x* ∈ *X*}) ⊔ (α (⨆{*fst x* | *x*. *x* ∈ *X*}) (*snd y*)))
by (*simp add*: *sd-prod-def*)
also have
... = (⨆{*fst x* · *fst y*|*x*. *x* ∈ *X*}, (⨆{*snd x* |*x*. *x* ∈ *X*}) ⊔ (α (⨆{*fst x* | *x*. *x* ∈ *X*}) (*snd y*)))
by (*simp add*: *Sup-distr*)
also have
... = (⨆{*fst x* · *fst y*|*x*. *x* ∈ *X*}, (⨆ {*snd x* |*x*. *x* ∈ *X*}) ⊔ (⨆{α (*fst x*) (*snd y*) | *x*. *x* ∈ *X*}))
by (*simp add*: *act3*)
also have ... = (⨆{*fst x* · *fst y*|*x*. *x* ∈ *X*}, ⨆{*snd x* ⊔ (α (*fst x*) (*snd y*)) | *x*. *x* ∈ *X*})
by (*simp only*: *sd-distr-aux*)
also have ... = *Sup-prod* {(*fst x* · *fst y*, *snd x* ⊔ (α (*fst x*) (*snd y*))) |*x*. *x* ∈ *X*}
by (*simp add*: *Sup-prod-def*, *metis*)
finally show *?thesis*
by (*simp add*: *sd-prod-def*)
qed

lemma *sd-distl-aux*: *Y* ≠ {} ⟹ *p* ⊔ (⨆{α *x* (*snd y*) |*y*. *y* ∈ *Y*}) = ⨆{*p* ⊔ α *x* (*snd y*) |*y*. *y* ∈ *Y*}
proof (*rule antisym*, *rule sup-least*)
show *Y* ≠ {} ⟹ *p* ≤ ⨆{*p* ⊔ α *x* (*snd y*) |*y*. *y* ∈ *Y*}
proof −
assume *Y* ≠ {}
hence ∃ *b*. *b* ∈ {*p* ⊔ α *x* (*snd pa*) |*pa*. *pa* ∈ *Y*} ∧ *p* ≤ *b*
by *fastforce*
thus *p* ≤ ⨆{*p* ⊔ α *x* (*snd pa*) |*pa*. *pa* ∈ *Y*}
by (*meson Sup-upper2*)
qed
next
show *Y* ≠ {} ⟹ ⨆{α *x* (*snd y*) |*y*. *y* ∈ *Y*} ≤ ⨆{*p* ⊔ α *x* (*snd y*) |*y*. *y* ∈ *Y*}
apply (*rule Sup-least*)
proof −
fix *xa* :: *'b*
assume *xa* ∈ {α *x* (*snd y*) |*y*. *y* ∈ *Y*}
then have ∃ *b*. (∃ *pa*. *b* = *p* ⊔ α *x* (*snd pa*) ∧ *pa* ∈ *Y*) ∧ *xa* ≤ *b*
by *fastforce*
then have ∃ *b*. *b* ∈ {*p* ⊔ α *x* (*snd pa*) |*pa*. *pa* ∈ *Y*} ∧ *xa* ≤ *b*

by *blast*
then show $xa \leq \bigsqcup\{p \sqcup \alpha\ x\ (snd\ pa)\ |pa.\ pa \in Y\}$
by (*meson Sup-upper2*)
qed
next
show $Y \neq \{\} \Longrightarrow \bigsqcup\{p \sqcup \alpha\ x\ (snd\ y)\ |y.\ y \in Y\} \leq p \sqcup \bigsqcup\{\alpha\ x\ (snd\ y)\ |y.\ y \in Y\}$
apply (*rule Sup-least*)
apply *safe*
by (*metis* (*mono-tags*, *lifting*) *Sup-le-iff le-sup-iff mem-Collect-eq sup-ge1 sup-ge2*)
qed

lemma *sd-distl*: $Y \neq \{\} \Longrightarrow sd\text{-}prod\ x\ (Sup\text{-}prod\ Y) = Sup\text{-}prod\ \{sd\text{-}prod\ x\ y\ |y.\ y \in Y\}$
proof −
assume *a*: $Y \neq \{\}$
have $sd\text{-}prod\ x\ (Sup\text{-}prod\ Y) = sd\text{-}prod\ x\ (\bigsqcup\{fst\ y\ |y.\ y \in Y\}, \bigsqcup\{snd\ y\ |y.\ y \in Y\})$
by (*simp add*: *Sup-prod-def*)
also have $... = ((fst\ x) \cdot (\bigsqcup\{fst\ y\ |y.\ y \in Y\}), (snd\ x \sqcup (\alpha\ (fst\ x)\ (\bigsqcup\{snd\ y\ |y.\ y \in Y\}))))$
by (*simp add*: *sd-prod-def*)
also have $... = (\bigsqcup\{fst\ x \cdot fst\ y\ |y.\ y \in Y\}, (snd\ x \sqcup (\alpha\ (fst\ x)\ (\bigsqcup\{snd\ y\ |y.\ y \in Y\}))))$
by (*simp add*: *Sup-distl*)
also have $... = (\bigsqcup\{fst\ x \cdot fst\ y\ |y.\ y \in Y\}, (snd\ x \sqcup (\bigsqcup\{\alpha\ (fst\ x)\ (snd\ y)\ |y.\ y \in Y\})))$
by (*simp add*: *act4*, *meson*)
also have $... = (\bigsqcup\{fst\ x \cdot fst\ y\ |y.\ y \in Y\}, \bigsqcup\{snd\ x \sqcup (\alpha\ (fst\ x)\ (snd\ y))\ |y.\ y \in Y\})$
using *a sd-distl-aux* **by** *blast*
also have $... = Sup\text{-}prod\ \{(fst\ x \cdot fst\ y, snd\ x \sqcup (\alpha\ (fst\ x)\ (snd\ y)))\ |y.\ y \in Y\}$
by (*simp add*: *Sup-prod-def*, *metis*)
finally show *?thesis*
by (*simp add*: *sd-prod-def*)
qed

definition $sd\text{-}unit = (1, \bot)$

lemma *sd-unitl* [*simp*]: $sd\text{-}prod\ sd\text{-}unit\ x = x$
by (*simp add*: *sd-prod-def sd-unit-def*)

lemma *sd-unitr* [*simp*]: $sd\text{-}prod\ x\ sd\text{-}unit = x$
apply (*simp add*: *sd-prod-def sd-unit-def*)
using *act-emp* **by** *force*

The following counterexamples rule out that semidirect products of quantales and complete lattices form quantales. The reason is that the right annihilation law fails.

lemma $sd\text{-}prod\ x\ (Sup\text{-}prod\ Y) = Sup\text{-}prod\ \{sd\text{-}prod\ x\ y\ |y.\ y \in Y\}$
oops

lemma $sd\text{-}prod\ x\ bot\text{-}prod = bot\text{-}prod$
oops

However we can show that semidirect products of (unital) quantales with complete lattices form weak (unital) quantales.

interpretation *dp-quantale*: *weak-quantale sd-prod Inf-prod Sup-prod inf-prod less-eq-prod less-prod sup-prod bot-prod top-prod*
apply *standard*
apply (*simp-all add*: *sd-distl sd-distr*)
apply (*simp-all add*: *sd-prod-def Inf-prod-def Sup-prod-def bot-prod-def sup-prod-def top-prod-def inf-prod-def less-eq-prod-def less-prod-def*)

by (*rule conjI*, *simp add*: *mult.assoc*, *simp add*: *act1 act-sup-distl sup-assoc*)

interpretation *dpu-quantale*: *unital-weak-quantale sd-unit sd-prod Inf-prod Sup-prod inf-prod less-eq-prod less-prod sup-prod bot-prod top-prod*
by (*standard*; *simp-all*)

end

end

5 Binary Modalities and Relational Convolution

theory *Binary-Modalities*
imports *Quantales*

begin

5.1 Auxiliary Properties

lemma *SUP-is-Sup*: $(SUP\ f{\in}F.\ f\ y) = \bigsqcup\{(f{::}'a \Rightarrow 'b{::}proto\text{-}near\text{-}quantale)\ y\ |f.\ f \in F\}$
proof (*rule antisym*)
fix $f{::}'a \Rightarrow 'b{::}proto\text{-}near\text{-}quantale$
have $f \in F \Longrightarrow f\ y \in \{f\ y\ |f.\ f \in F\}$
by (*simp add*: *Setcompr-eq-image*)
hence $f \in F \Longrightarrow f\ y \leq \bigsqcup\{f\ y\ |f.\ f \in F\}$
by (*simp add*: *Sup-upper*)
thus $(SUP\ f{\in}F.\ f\ y) \leq \bigsqcup\{(f{::}'a \Rightarrow 'b{::}proto\text{-}near\text{-}quantale)\ y\ |f.\ f \in F\}$
by (*simp add*: *Setcompr-eq-image*)
next
fix x
have $x \in \{f\ y\ |f.\ f \in F\} \Longrightarrow x \leq (SUP\ f{\in}F.\ f\ y)$
using *mk-disjoint-insert* **by** *force*
thus $Sup\ \{(f{::}'a \Rightarrow 'b{::}proto\text{-}near\text{-}quantale)\ y\ |f.\ f \in F\} \leq (SUP\ f{\in}F.\ f\ y)$
by (*simp add*: *Setcompr-eq-image*)
qed

lemma *bmod-auxl*: $\{x \cdot g\ z\ |x.\ \exists f.\ x = f\ y \wedge f \in F\} = \{f\ y \cdot g\ z\ |f.\ f \in F\}$
by *force*

lemma *bmod-auxr*: $\{f\ y \cdot x\ |x.\ \exists g.\ x = g\ z \wedge g \in G\} = \{f\ y \cdot g\ z\ |g.\ g \in G\}$
by *force*

lemma *bmod-assoc-aux1*:
$\bigsqcup\{\bigsqcup\{(f :: 'a \Rightarrow 'b{::}proto\text{-}near\text{-}quantale)\ u \cdot g\ v \cdot h\ w\ |u\ v.\ R\ y\ u\ v\}\ |y\ w.\ R\ x\ y\ w\}$
$= \bigsqcup\{(f\ u \cdot g\ v) \cdot h\ w\ |u\ v\ y\ w.\ R\ y\ u\ v \wedge R\ x\ y\ w\}$
apply (*rule antisym*)
apply (*rule Sup-least*, *safe*)
apply (*intro Sup-least Sup-upper*, *force*)
apply (*rule Sup-least*, *safe*)
by (*rule Sup-upper2*, *auto*)+

lemma *bmod-assoc-aux2*:
$\bigsqcup\{\bigsqcup\{(f{::}'a \Rightarrow 'b{::}proto\text{-}near\text{-}quantale)\ u \cdot g\ v \cdot h\ w\ |v\ w.\ R\ y\ v\ w\}\ |u\ y.\ R\ x\ u\ y\}$
$= \bigsqcup\{f\ u \cdot g\ v \cdot h\ w\ |u\ v\ w\ y.\ R\ y\ v\ w \wedge R\ x\ u\ y\}$
apply (*rule antisym*)

apply (*rule Sup-least*, *safe*)
apply (*intro Sup-least Sup-upper*, *force*)
apply (*rule Sup-least*, *safe*)
by (*rule Sup-upper2*, *auto*)+

5.2 Binary Modalities

Most of the development in the papers mentioned in the introduction generalises to proto-near-quantales. Binary modalities are interesting for various substructural logics over ternary Kripke frames. They also arise, e.g., as chop modalities in interval logics or as separation conjunction in separation logic. Binary modalities can be understood as a convolution operation parametrised by a ternary operation. Our development yields a unifying framework.

We would prefer a notation that is more similar to our articles, that is, $f *_R g$, but we don' know how we could index an infix operator by a variable in Isabelle.

definition *bmod-comp* :: ('*a* ⇒ '*b* ⇒ '*c* ⇒ *bool*) ⇒ ('*b* ⇒ '*d*::*proto-near-quantale*) ⇒ ('*c* ⇒ '*d*) ⇒ '*a* ⇒ '*d* (‹⊗›) **where**
⊗ *R f g x* = ⨆{*f y* · *g z* |*y z*. *R x y z*}

definition *bmod-bres* :: ('*c* ⇒ '*b* ⇒ '*a* ⇒ *bool*) ⇒ ('*b* ⇒ '*d*::*proto-near-quantale*) ⇒ ('*c* ⇒ '*d*) ⇒ '*a* ⇒ '*d* (‹◁›) **where**
◁ *R f g x* = ⨅{(*f y*) → (*g z*) |*y z*. *R z y x*}

definition *bmod-fres* :: ('*b* ⇒ '*a* ⇒ '*c* ⇒ *bool*) ⇒ ('*b* ⇒ '*d*::*proto-near-quantale*) ⇒ ('*c* ⇒ '*d*) ⇒ '*a* ⇒ '*d* (‹▷›)**where**
▷ *R f g x* = ⨅{(*f y*) ← (*g z*) |*y z*. *R y x z*}

lemma *bmod-un-rel*: ⊗ (*R* ⊔ *S*) = ⊗ *R* ⊔ ⊗ *S*
apply (*clarsimp simp*: *fun-eq-iff bmod-comp-def Sup-union-distrib*[*symmetric*] *Collect-disj-eq*[*symmetric*])
by (*metis* (*no-types*, *lifting*))

lemma *bmod-Un-rel*: ⊗ (⨆ℛ) *f g x* = ⨆{⊗ *R f g x* |*R*. *R* ∈ ℛ}
apply (*simp add*: *bmod-comp-def*)
apply (*rule antisym*)
apply (*rule Sup-least*, *safe*)
apply (*rule Sup-upper2*, *force*)
apply (*rule Sup-upper*, *force*)
apply (*rule Sup-least*, *safe*)+
by (*metis* (*mono-tags*, *lifting*) *Sup-upper mem-Collect-eq*)

lemma *bmod-sup-fun1*: ⊗ *R* (*f* ⊔ *g*) = ⊗ *R f* ⊔ ⊗ *R g*
apply (*clarsimp simp add*: *fun-eq-iff bmod-comp-def sup-distr*)
apply (*rule antisym*)
apply (*intro Sup-least*, *safe*)
apply (*rule sup-least*)
apply (*intro le-supI1 Sup-upper*, *force*)
apply (*intro le-supI2 Sup-upper*, *force*)
apply (*rule sup-least*)
by (*intro Sup-least*, *safe*, *rule Sup-upper2*, *force*, *simp*)+

lemma *bmod-Sup-fun1*: ⊗ *R* (⨆ℱ) *g x* = ⨆{⊗ *R f g x* |*f*. *f* ∈ ℱ}
proof −
have ⊗ *R* (⨆{*f*. *f* ∈ ℱ}) *g x* = ⨆{⨆{*f y* |*f*. *f* ∈ ℱ} · *g z* |*y z*. *R x y z*}
by (*simp add*: *bmod-comp-def SUP-is-Sup*)
also have ... = ⨆{⨆{*f y* · *g z* |*f*. *f* ∈ ℱ} |*y z*. *R x y z*}

by (*simp add*: *Sup-distr bmod-auxl*)
also have ... = $\bigsqcup\{\bigsqcup\{f\ y \cdot g\ z\ |y\ z.\ R\ x\ y\ z\}\ |f.\ f \in \mathcal{F}\}$
apply (*rule antisym*)
by ((*rule Sup-least*, *safe*)+ , (*rule Sup-upper2*, *force*, *rule Sup-upper*, *force*))+
finally show *?thesis*
by (*simp add*: *bmod-comp-def*)
qed

lemma *bmod-sup-fun2*: $\otimes\ R\ (f::'a \Rightarrow 'b::weak\text{-}proto\text{-}quantale)\ (g \sqcup h) = \otimes\ R\ f\ g \sqcup \otimes\ R\ f\ h$
apply (*clarsimp simp add*: *fun-eq-iff bmod-comp-def sup-distl*)
apply (*rule antisym*)
apply (*intro Sup-least*, *safe*)
apply (*rule sup-least*)
apply (*intro le-supI1 Sup-upper*, *force*)
apply (*intro le-supI2 Sup-upper*, *force*)
apply (*rule sup-least*)
by (*intro Sup-least*, *safe*, *rule Sup-upper2*, *force*, *simp*)+

lemma *bmod-Sup-fun2-weak*:
assumes $\mathcal{G} \neq \{\}$
shows $\otimes\ R\ f\ (\bigsqcup\mathcal{G})\ x = \bigsqcup\{\otimes\ R\ f\ (g::'a \Rightarrow 'b::weak\text{-}proto\text{-}quantale)\ x\ |g.\ g \in \mathcal{G}\}$
proof −
have *set*: $\bigwedge z.\ \{g\ z\ |g::'a \Rightarrow 'b.\ g \in \mathcal{G}\} \neq \{\}$
using *assms* **by** *blast*
have $\otimes\ R\ f\ (\bigsqcup\{g.\ g \in \mathcal{G}\})\ x = \bigsqcup\{f\ y \cdot \bigsqcup\{g\ z\ |g.\ g \in \mathcal{G}\}\ |y\ z.\ R\ x\ y\ z\}$
by (*simp add*: *bmod-comp-def SUP-is-Sup*)
also have ... = $\bigsqcup\{\bigsqcup\{f\ y \cdot g\ z\ |g.\ g \in \mathcal{G}\}\ |y\ z.\ R\ x\ y\ z\}$
by (*simp add*: *weak-Sup-distl*[*OF set*] *bmod-auxr*)
also have ... = $\bigsqcup\{\bigsqcup\{f\ y \cdot g\ z\ |y\ z.\ R\ x\ y\ z\}\ |g.\ g \in \mathcal{G}\}$
apply (*rule antisym*)
by ((*rule Sup-least*, *safe*)+, (*rule Sup-upper2*, *force*, *rule Sup-upper*, *force*))+
finally show *?thesis*
by (*auto simp*: *bmod-comp-def*)
qed

lemma *bmod-Sup-fun2*: $\otimes\ R\ f\ (\bigsqcup\mathcal{G})\ x = \bigsqcup\{\otimes\ R\ f\ (g::'a \Rightarrow 'b::proto\text{-}quantale)\ x\ |g.\ g \in \mathcal{G}\}$
proof −
have $\otimes\ R\ f\ (\bigsqcup\{g.\ g \in \mathcal{G}\})\ x = \bigsqcup\{f\ y \cdot \bigsqcup\{g\ z\ |g.\ g \in \mathcal{G}\}\ |y\ z.\ R\ x\ y\ z\}$
by (*simp add*: *bmod-comp-def SUP-is-Sup*)
also have ... = $\bigsqcup\{\bigsqcup\{f\ y \cdot g\ z\ |g.\ g \in \mathcal{G}\}\ |y\ z.\ R\ x\ y\ z\}$
by (*simp add*: *Sup-distl bmod-auxr*)
also have ... = $\bigsqcup\{\bigsqcup\{f\ y \cdot g\ z\ |y\ z.\ R\ x\ y\ z\}\ |g.\ g \in \mathcal{G}\}$
apply (*rule antisym*)
by ((*rule Sup-least*, *safe*)+, (*rule Sup-upper2*, *force*, *rule Sup-upper*, *force*))+
finally show *?thesis*
by (*auto simp*: *bmod-comp-def*)
qed

lemma *bmod-comp-bres-galois*: $(\forall x.\ \otimes\ R\ f\ g\ x \leq h\ x) \longleftrightarrow (\forall x.\ g\ x \leq \triangleleft\ R\ f\ h\ x)$
oops

The following Galois connection requires functions into proto-quantales.

lemma *bmod-comp-bres-galois*: $(\forall x.\ \otimes\ R\ (f::'a \Rightarrow 'b::proto\text{-}quantale)\ g\ x \leq h\ x) \longleftrightarrow (\forall x.\ g\ x \leq \triangleleft\ R\ f\ h\ x)$
proof −

have $(\forall\, x.\ \otimes\ R\ f\ g\ x \leq h\ x) \longleftrightarrow (\forall\, x\ y\ z.\ R\ x\ y\ z \longrightarrow (f\ y) \cdot (g\ z) \leq h\ x)$
apply (*simp add*: *bmod-comp-def*, *standard*, *safe*)
apply (*metis* (*mono-tags*, *lifting*) *CollectI Sup-le-iff*)
by (*rule Sup-least*, *force*)
also have ... $\longleftrightarrow$ $(\forall\, x\ y\ z.\ R\ x\ y\ z \longrightarrow g\ z \leq (f\ y) \rightarrow (h\ x))$
by (*simp add*: *bres-galois*)
finally show *?thesis*
apply (*simp add*: *fun-eq-iff bmod-bres-def*)
apply *standard*
using *le-Inf-iff* **apply** *fastforce*
by (*metis* (*mono-tags*, *lifting*) *CollectI le-Inf-iff*)
qed

lemma *bmod-comp-fres-galois*: $(\forall\, x.\ \otimes\ R\ f\ g\ x \leq h\ x) \longleftrightarrow (\forall\, x.\ f\ x \leq \triangleright\ R\ h\ g\ x)$
proof −
have $(\forall\, x.\ \otimes\ R\ f\ g\ x \leq h\ x) \longleftrightarrow (\forall\, x\ y\ z.\ R\ x\ y\ z \longrightarrow (f\ y) \cdot (g\ z) \leq h\ x)$
apply (*simp add*: *bmod-comp-def*, *standard*, *safe*)
apply (*metis* (*mono-tags*, *lifting*) *CollectI Sup-le-iff*)
by (*rule Sup-least*, *force*)
also have ... $\longleftrightarrow$ $(\forall\, x\ y\ z.\ R\ x\ y\ z \longrightarrow f\ y \leq (h\ x) \leftarrow (g\ z))$
by (*simp add*: *fres-galois*)
finally show *?thesis*
apply (*simp add*: *bmod-fres-def fun-eq-iff*)
apply *standard*
using *le-Inf-iff* **apply** *fastforce*
by (*metis* (*mono-tags*, *lifting*) *CollectI le-Inf-iff*)
qed

5.3 Relational Convolution and Correspondence Theory

We now fix a ternary relation ρ and can then hide the parameter in a convolution-style notation.

class *rel-magma* =
fixes $\varrho :: 'a \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$

begin

definition *times-rel-fun* :: $('a \Rightarrow 'b{::}proto\text{-}near\text{-}quantale) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b$ (**infix** $\langle\star\rangle$ *70*) **where**
$f \star g = \otimes\ \varrho\ f\ g$

lemma *rel-fun-Sup-distl-weak*:
$G \neq \{\} \Longrightarrow (f{::}'a \Rightarrow 'b{::}weak\text{-}proto\text{-}quantale) \star \bigsqcup G = \bigsqcup \{f \star g\ |g.\ g \in G\}$
proof−
fix $f :: 'a \Rightarrow 'b$ **and** $G :: ('a \Rightarrow 'b)\ set$
assume *a*: $G \neq \{\}$
show $f \star \bigsqcup G = \bigsqcup \{f \star g\ |g.\ g \in G\}$
apply (*clarsimp simp*: *fun-eq-iff times-rel-fun-def bmod-Sup-fun2-weak*[*OF a*])
apply (*rule antisym*)
apply (*rule Sup-least*, *safe*)
apply (*rule SUP-upper2*, *force*+)
apply (*rule SUP-least*, *safe*)
by (*rule Sup-upper2*, *force*+)
qed

lemma *rel-fun-Sup-distl*: $(f{::}'a \Rightarrow 'b{::}proto\text{-}quantale) \star \bigsqcup G = \bigsqcup \{f \star g\ |g.\ g \in G\}$
apply (*clarsimp simp*: *fun-eq-iff times-rel-fun-def bmod-Sup-fun2*)

apply (*rule antisym*)
apply (*rule Sup-least*, *safe*)
apply (*rule SUP-upper2*, *force*+)
apply (*rule SUP-least*, *safe*)
by (*rule Sup-upper2*, *force*+)

lemma *rel-fun-Sup-distr*: $\bigsqcup G \star (f::'a \Rightarrow 'b::\textit{proto-near-quantale}) = \bigsqcup \{g \star f \mid g.\ g \in G\}$
apply (*clarsimp simp*: *fun-eq-iff times-rel-fun-def bmod-Sup-fun1*)
apply (*rule antisym*)
apply (*rule Sup-least*, *safe*)
apply (*rule SUP-upper2*, *force*+)
apply (*rule SUP-least*, *safe*)
by (*rule Sup-upper2*, *force*+)

end

class *rel-semigroup* = *rel-magma* +
assumes *rel-assoc*: $(\exists\, y.\ \varrho\ y\ u\ v \wedge \varrho\ x\ y\ w) \longleftrightarrow (\exists\, z.\ \varrho\ z\ v\ w \wedge \varrho\ x\ u\ z)$

begin

Nitpick produces counterexamples even for weak quantales. Hence one cannot generally lift functions into weak quantales to weak quantales.

lemma *bmod-assoc*: $\otimes\ \varrho\ (\otimes\ \varrho\ (f::'a \Rightarrow 'b::\textit{weak-quantale})\ g)\ h\ x = \otimes\ \varrho\ f\ (\otimes\ \varrho\ g\ h)\ x$

oops

lemma *bmod-assoc*: $\otimes\ \varrho\ (\otimes\ \varrho\ (f::'a \Rightarrow 'b::\textit{quantale})\ g)\ h\ x = \otimes\ \varrho\ f\ (\otimes\ \varrho\ g\ h)\ x$
proof −
have $\otimes\ \varrho\ (\otimes\ \varrho\ f\ g)\ h\ x = \bigsqcup \{\bigsqcup \{f\ u \cdot g\ v \cdot h\ z \mid u\ v.\ \varrho\ y\ u\ v\} \mid y\ z.\ \varrho\ x\ y\ z\}$
apply (*simp add*: *bmod-comp-def Sup-distr*)
apply (*rule antisym*)
by (*intro Sup-least Sup-upper*, *safe*, *intro exI conjI*, *simp-all*, *rule-tac f* = *Sup* **and** *g* = *Sup* **in** *cong*, *auto*)+
also have ... $= \bigsqcup \{f\ u \cdot g\ v \cdot h\ z \mid u\ v\ y\ z.\ \varrho\ y\ u\ v \wedge \varrho\ x\ y\ z\}$
by (*simp add*: *bmod-assoc-aux1*)
also have ... $= \bigsqcup \{f\ u \cdot g\ v \cdot h\ z \mid u\ v\ z\ y.\ \varrho\ y\ v\ z \wedge \varrho\ x\ u\ y\}$
apply (*rule antisym*)
apply (*rule Sup-least*, *rule Sup-upper*, *safe*)
using *rel-assoc* **apply** *force*
apply (*rule Sup-least*, *rule Sup-upper*, *safe*)
using *rel-assoc* **by** *blast*
also have ... $= \bigsqcup \{\bigsqcup \{f\ u \cdot g\ v \cdot h\ z \mid v\ z.\ \varrho\ y\ v\ z\} \mid u\ y.\ \varrho\ x\ u\ y\}$
by (*simp add*: *bmod-assoc-aux2*)
also have ... $= \bigsqcup \{f\ u \cdot \bigsqcup \{g\ v \cdot h\ z \mid v\ z.\ \varrho\ y\ v\ z\} \mid u\ y.\ \varrho\ x\ u\ y\}$
apply (*simp add*: *Sup-distl mult.assoc*)
apply (*rule antisym*)
by (*intro Sup-least Sup-upper*, *safe*, *intro exI conjI*, *simp-all*, *rule-tac f* = *Sup* **and** *g* = *Sup* **in** *cong*, *auto*)+
finally show *?thesis*
by (*auto simp*: *bmod-comp-def*)
qed

lemma *rel-fun-assoc*: $((f :: 'a \Rightarrow 'b::\textit{quantale}) \star g) \star h = f \star (g \star h)$
by (*simp add*: *times-rel-fun-def fun-eq-iff bmod-assoc*[*symmetric*])

end

lemma $\otimes\ R\ (\otimes\ R\ f\ f)\ f\ x = \otimes\ R\ f\ (\otimes\ R\ f\ f)\ x$

oops

class *rel-monoid* = *rel-semigroup* +
  fixes ξ :: $'a$ *set*
  assumes *unitl-ex*: $\exists\, e \in \xi.\ \varrho\ x\ e\ x$
  and *unitr-ex*: $\exists\, e \in \xi.\ \varrho\ x\ x\ e$
  and *unitl-eq*: $e \in \xi \Longrightarrow \varrho\ x\ e\ y \Longrightarrow x = y$
  and *unitr-eq*: $e \in \xi \Longrightarrow \varrho\ x\ y\ e \Longrightarrow x = y$

begin

lemma *xi-prop*: $e1 \in \xi \Longrightarrow e2 \in \xi \Longrightarrow e1 \neq e2 \Longrightarrow \neg\ \varrho\ x\ e1\ e2 \wedge \neg\ \varrho\ x\ e2\ e1$
  using *unitl-eq unitr-eq* **by** *blast*

definition *pid* :: $'a\ \Rightarrow\ 'b$::*unital-weak-quantale* ($\langle\delta\rangle$) **where**
  $\delta\ x = (\mathit{if}\ x \in \xi\ \mathit{then}\ 1\ \mathit{else}\ \bot)$

Due to the absence of right annihilation, the right unit law fails for functions into weak quantales.

lemma *bmod-onel*: $\otimes\ \varrho\ f\ (\delta$::$'a \Rightarrow\ 'b$::*unital-weak-quantale*$)\ x = f\ x$

  oops

A unital quantale is required for this lifting.

lemma *bmod-onel*: $\otimes\ \varrho\ f\ (\delta$::$'a \Rightarrow\ 'b$::*unital-quantale*$)\ x = f\ x$
  apply (*simp add*: *bmod-comp-def pid-def*)
  apply (*rule antisym*)
  apply (*rule Sup-least*, *safe*)
  apply (*simp add*: *bres-galois*)
  using *unitr-eq* **apply** *fastforce*
  apply (*metis bot.extremum*)
  by (*metis* (*mono-tags*, *lifting*) *Sup-upper mem-Collect-eq unitr-ex*)

lemma *bmod-oner*: $\otimes\ \varrho\ \delta\ f\ x = f\ x$
  apply (*simp add*: *bmod-comp-def pid-def*)
  apply (*rule antisym*)
  apply (*rule Sup-least*, *safe*)
  apply (*simp add*: *fres-galois*)
  using *unitl-eq* **apply** *fastforce*
  apply (*metis bot.extremum*)
  by (*metis* (*mono-tags*, *lifting*) *Sup-upper mem-Collect-eq unitl-ex*)

lemma *pid-unitl* [*simp*]: $\delta \star f = f$
  by (*simp add*: *fun-eq-iff times-rel-fun-def bmod-oner*)

lemma *pid-unitr* [*simp*]: $f \star (\delta$::$'a \Rightarrow\ 'b$::*unital-quantale*$) = f$
  by (*simp add*: *fun-eq-iff times-rel-fun-def bmod-onel*)

lemma *bmod-assoc-weak-aux*:
  $f\ u \cdot \bigsqcup\{g\ v \cdot h\ z\ |v\ z.\ \varrho\ y\ v\ z\} = \bigsqcup\{(f$::$'a \Rightarrow\ 'b$::*weak-quantale*$)\ u \cdot g\ v \cdot h\ z\ |v\ z.\ \varrho\ y\ v\ z\}$
  apply (*subst weak-Sup-distl*)

using *unitl-ex* **apply** *force*
apply *simp*
by (*metis* (*no-types*, *lifting*) *mult.assoc*)

lemma *bmod-assoc-weak*: $\otimes\ \varrho\ (\otimes\ \varrho\ (f{::}'a \Rightarrow 'b{::}\textit{weak-quantale})\ g)\ h\ x = \otimes\ \varrho\ f\ (\otimes\ \varrho\ g\ h)\ x$
proof −
have $\otimes\ \varrho\ (\otimes\ \varrho\ f\ g)\ h\ x = \bigsqcup\{\bigsqcup\{f\ u \cdot g\ v \cdot h\ z\ |u\ v.\ \varrho\ y\ u\ v\}\ |y\ z.\ \varrho\ x\ y\ z\}$
apply (*simp add*: *bmod-comp-def Sup-distr*)
apply (*rule antisym*)
by (*intro Sup-least Sup-upper*, *safe*, *intro exI conjI*, *simp-all*, *rule-tac* $f = \textit{Sup}$ **and** $g = \textit{Sup}$ **in** *cong*, *auto*)+
also have ... $= \bigsqcup\{f\ u \cdot g\ v \cdot h\ z\ |u\ v\ y\ z.\ \varrho\ y\ u\ v \wedge \varrho\ x\ y\ z\}$
by (*simp add*: *bmod-assoc-aux1*)
also have ... $= \bigsqcup\{f\ u \cdot g\ v \cdot h\ z\ |u\ v\ z\ y.\ \varrho\ y\ v\ z \wedge \varrho\ x\ u\ y\}$
apply (*rule antisym*)
apply (*rule Sup-least*, *rule Sup-upper*, *safe*)
using *rel-assoc* **apply** *force*
apply (*rule Sup-least*, *rule Sup-upper*, *safe*)
using *rel-assoc* **by** *blast*
also have ... $= \bigsqcup\{\bigsqcup\{f\ u \cdot g\ v \cdot h\ z\ |v\ z.\ \varrho\ y\ v\ z\}\ |u\ y.\ \varrho\ x\ u\ y\}$
by (*simp add*: *bmod-assoc-aux2*)
also have ... $= \bigsqcup\{f\ u \cdot \bigsqcup\{g\ v \cdot h\ z\ |v\ z.\ \varrho\ y\ v\ z\}\ |u\ y.\ \varrho\ x\ u\ y\}$
by (*simp add*: *bmod-assoc-weak-aux*)
finally show *?thesis*
by (*auto simp*: *bmod-comp-def*)
qed

lemma *rel-fun-assoc-weak*: $((f :: 'a \Rightarrow 'b{::}\textit{weak-quantale}) \star g) \star h = f \star (g \star h)$
by (*simp add*: *times-rel-fun-def fun-eq-iff bmod-assoc-weak*[*symmetric*])

end

lemma (**in** *rel-semigroup*) $\exists\ id.\ \forall f\ x.\ (\otimes\ \varrho\ f\ id\ x = f\ x \wedge \otimes\ \varrho\ id\ f\ x = f\ x)$

oops

class *rel-ab-semigroup* = *rel-semigroup* +
assumes *rel-comm*: $\varrho\ x\ y\ z \Longrightarrow\ \varrho\ x\ z\ y$

begin

lemma *bmod-comm*: $\otimes\ \varrho\ (f{::}'a \Rightarrow 'b{::}\textit{ab-quantale})\ g = \otimes\ \varrho\ g\ f$
by (*simp add*: *fun-eq-iff bmod-comp-def mult.commute*, *meson rel-comm*)

lemma $\otimes\ \varrho\ f\ g = \otimes\ \varrho\ g\ f$
oops

lemma *bmod-bres-fres-eq*: $\lhd\ \varrho\ (f{::}'a \Rightarrow 'b{::}\textit{ab-quantale})\ g = \rhd\ \varrho\ g\ f$
by (*simp add*: *fun-eq-iff bmod-bres-def bmod-fres-def bres-fres-eq*, *meson rel-comm*)

lemma *rel-fun-comm*: $(f :: 'a \Rightarrow 'b{::}\textit{ab-quantale}) \star g = g \star f$
by (*simp add*: *times-rel-fun-def bmod-comm*)

end

class *rel-ab-monoid* = *rel-ab-semigroup* + *rel-monoid*

5.4 Lifting to Function Spaces

We lift by interpretation, since we need sort instantiations to be used for functions from PAM's to Quantales. Both instantiations cannot be used in Isabelle at the same time.

interpretation *rel-fun*: *weak-proto-quantale Inf Sup inf less-eq less sup bot top* :: *'a*::*rel-magma* ⇒ *'b*::*weak-proto-quantale times-rel-fun*
by *standard* (*simp-all add*: *rel-fun-Sup-distr rel-fun-Sup-distl-weak*)

interpretation *rel-fun*: *proto-quantale Inf Sup inf less-eq less sup bot top* :: *'a*::*rel-magma* ⇒ *'b*::*proto-quantale times-rel-fun*
by *standard* (*simp add*: *rel-fun-Sup-distl*)

Nitpick shows that the lifting of weak quantales to weak quantales does not work for relational semigroups, because associativity fails.

interpretation *rel-fun*: *weak-quantale times-rel-fun Inf Sup inf less-eq less sup bot top*::*'a*::*rel-semigroup* ⇒ *'b*::*weak-quantale*

oops

Associativity is obtained when lifting from relational monoids, but the right unit law doesn't hold in the quantale on the function space, according to our above results. Hence the lifting results into a non-unital quantale, in which only the left unit law holds, as shown above. We don't provide a special class for such quantales. Hence we lift only to non-unital quantales.

interpretation *rel-fun*: *weak-quantale times-rel-fun Inf Sup inf less-eq less sup bot top*::*'a*::*rel-monoid* ⇒ *'b*::*unital-weak-quantale*
by *standard* (*simp-all add*: *rel-fun-assoc-weak*)

interpretation *rel-fun2*: *quantale times-rel-fun Inf Sup inf less-eq less sup bot top*::*'a*::*rel-semigroup* ⇒ *'b*::*quantale*
by *standard* (*simp add*: *rel-fun-assoc*)

interpretation *rel-fun2*: *distrib-quantale Inf Sup inf less-eq less sup bot top*::*'a*::*rel-semigroup* ⇒ *'b*::*distrib-quantale times-rel-fun* **..**

interpretation *rel-fun2*: *bool-quantale minus uminus inf less-eq less sup bot* ‹*top*::*'a*::*rel-semigroup* ⇒ *'b*::*bool-quantale*› *Inf Sup times-rel-fun* **..**

interpretation *rel-fun2*: *unital-quantale pid times-rel-fun Inf Sup inf less-eq less sup bot top*::*'a*::*rel-monoid* ⇒ *'b*::*unital-quantale*
by (*standard*; *simp*)

interpretation *rel-fun2*: *distrib-unital-quantale Inf Sup inf less-eq less sup bot top*::*'a*::*rel-monoid* ⇒ *'b*::*distrib-unital-quantale pid times-rel-fun* **..**

interpretation *rel-fun2*: *bool-unital-quantale minus uminus inf less-eq less sup bot* ‹*top*::*'a*::*rel-monoid* ⇒ *'b*::*bool-unital-quantale*› *Inf Sup pid times-rel-fun* **..**

interpretation *rel-fun*: *ab-quantale times-rel-fun Inf Sup inf less-eq less sup bot top*::*'a*::*rel-ab-semigroup* ⇒ *'b*::*ab-quantale*
by *standard* (*simp add*: *rel-fun-comm*)

interpretation *rel-fun*: *ab-unital-quantale times-rel-fun Inf Sup inf less-eq less sup bot top*::*'a*::*rel-ab-monoid* ⇒ *'b*::*ab-unital-quantale pid* **..**

interpretation *rel-fun2*: *distrib-ab-unital-quantale Inf Sup inf less-eq less sup bot top*::*'a*::*rel-ab-monoid* ⇒ *'b*::*distrib-ab-unital-quantale times-rel-fun pid* **..**

interpretation *rel-fun2*: *bool-ab-unital-quantale times-rel-fun Inf Sup inf less-eq less sup bot top*::*'a*::*rel-ab-monoid* ⇒ *'b*::*bool-ab-unital-quantale minus uminus pid* **..**

end

6 Unary Modalities

theory *Unary-Modalities*
imports *Binary-Modalities*
begin

Unary modalites arise as specialisations of the binary ones; and as generalisations of the standard (multi-)modal operators from predicates to functions into complete lattices. They are interesting, for instance, in combination with partial semigroups or monoids, for modelling the Halpern-Shoham modalities in interval logics.

6.1 Forward and Backward Diamonds

definition *fdia* :: (*'a* × *'b*) *set* ⇒ (*'b* ⇒ *'c*::*complete-lattice*) ⇒ *'a* ⇒ *'c* (‹(|-⟩ - -)› [*61*,*81*] *82*) **where**
(|*R*⟩ *f x*) = ⨆{*f y*|*y*. (*x*,*y*) ∈ *R*}

definition *bdia* :: (*'a* × *'b*) *set* ⇒ (*'a* ⇒ *'c*::*complete-lattice*) ⇒ *'b* ⇒ *'c* (‹(⟨-| - -)› [*61*,*81*] *82*)**where**
(⟨*R*| *f y*) = ⨆{*f x* |*x*. (*x*,*y*) ∈ *R*}

definition *c1* :: *'a* ⇒ *'b*::*unital-quantale* **where**
c1 x = *1*

The relationship with binary modalities is as follows.

lemma *fdia-bmod-comp*: (|*R*⟩ *f x*) = ⊗ (λ*x y z*. (*x*,*y*) ∈ *R*) *f c1 x*
by (*simp add*: *fdia-def bmod-comp-def c1-def*)

lemma *bdia-bmod-comp*: (⟨*R*| *f x*) = ⊗ (λ*y x z*. (*x*,*y*) ∈ *R*) *f c1 x*
by (*simp add*: *bdia-def bmod-comp-def c1-def*)

lemma *bmod-fdia-comp*: ⊗ *R f g x* = |{(*x*,(*y*,*z*)) |*x y z*. *R x y z*}⟩ (λ(*x*,*y*). (*f x*) · (*g y*)) *x*
by (*simp add*: *fdia-def bmod-comp-def*)

lemma *bmod-fdia-comp-var*:
⊗ *R f g x* = |{(*x*,(*y*,*z*)) |*x y z*. *R x y z*}⟩ (λ(*x*,*y*). (λ(*v*,*w*).(*v* · *w*)) (*f x*,*g y*)) *x*
by (*simp add*: *fdia-def bmod-comp-def*)

lemma *fdia-im*: (|*R*⟩ *f x*) = ⨆(*f* ‘ *R* `` {*x*})
apply (*simp add*: *fdia-def*)
apply (*rule antisym*)
apply (*intro Sup-least*, *clarsimp simp*: *SUP-upper*)
by (*intro SUP-least Sup-upper*, *force*)

lemma *fdia-un-rel*: *fdia* (*R* ∪ *S*) = *fdia R* ⊔ *fdia S*

apply (*simp add*: *fun-eq-iff*)
by (*clarsimp simp*: *fun-eq-iff fdia-im SUP-union Un-Image*)

lemma *fdia-Un-rel*: $(|\bigcup \mathcal{R}\rangle\ f\ x) = \bigsqcup \{|R\rangle\ f\ x\ |R.\ R \in \mathcal{R}\}$
apply (*simp add*: *fdia-im*)
apply (*rule antisym*)
apply (*intro SUP-least*, *safe*)
apply (*rule Sup-upper2*, *force*)
apply (*rule SUP-upper*, *simp*)
apply (*rule Sup-least*)
by (*clarsimp simp*: *Image-mono SUP-subset-mono Sup-upper*)

lemma *fdia-sup-fun*: *fdia R* $(f \sqcup g)$ = *fdia R f* $\sqcup$ *fdia R g*
by (*simp add*: *fun-eq-iff fdia-im complete-lattice-class.SUP-sup-distrib*)

lemma *fdia-Sup-fun*: $(|R\rangle\ (\bigsqcup \mathcal{F})\ x) = \bigsqcup \{|R\rangle\ f\ x\ |f.\ f \in \mathcal{F}\}$
apply (*simp add*: *fdia-im*)
apply (*rule antisym*)
apply (*rule SUP-least*)+
apply (*rule Sup-upper2*, *force*)
apply (*rule SUP-upper*, *simp*)
apply (*rule Sup-least*, *safe*)
apply (*rule SUP-least*)
by (*simp add*: *SUP-upper2*)

lemma *fdia-seq*: *fdia* $(R\ ;\ S)\ f\ x$ = *fdia R* (*fdia S f*) *x*
by (*simp add*: *fdia-im relcomp-Image*, *metis Image-eq-UN SUP-UNION*)

lemma *fdia-Id* [*simp*]: $(|Id\rangle\ f\ x) = f\ x$
by (*simp add*: *fdia-def*)

6.2 Forward and Backward Boxes

definition *fbox* :: $('a \times 'b)$ *set* $\Rightarrow$ $('b \Rightarrow 'c$::*complete-lattice*$) \Rightarrow 'a \Rightarrow 'c$ ($\langle|\text{-}]$ - -› [*61*,*81*] *82*) **where**
$(|R]\ f\ x) = \bigsqcap \{f\ y|y.\ (x,y) \in R\}$

definition *bbox* :: $('a \times 'b)$ *set* $\Rightarrow$ $('a \Rightarrow 'c$::*complete-lattice*$) \Rightarrow 'b \Rightarrow 'c$ ($\langle[\text{-}|$ - -› [*61*,*81*] *82*)**where**
$([R|\ f\ y) = \bigsqcap \{f\ x\ |x.\ (x,y) \in R\}$

6.3 Symmetries and Dualities

lemma *fdia-fbox-demorgan*: $(|R\rangle\ (f$::$'b \Rightarrow 'c$::*complete-boolean-algebra*$)\ x) = -\ |R]\ (\lambda y.\ -f\ y)\ x$
apply (*simp add*: *fbox-def fdia-def*)
apply (*rule antisym*)
apply (*rule Sup-least*)
apply (*simp add*: *Inf-lower compl-le-swap1*)
apply (*simp add*: *uminus-Inf*)
apply (*rule SUP-least*; *intro Sup-upper*)
by *auto*

lemma *fbox-fdia-demorgan*: $(|R]\ (f$::$'b \Rightarrow 'c$::*complete-boolean-algebra*$)\ x) = -\ |R\rangle\ (\lambda y.\ -f\ y)\ x$
apply (*simp add*: *fbox-def fdia-def*)
apply (*rule antisym*)
apply (*simp add*: *uminus-Sup*)
apply (*rule INF-greatest*; *rule Inf-lower*)
apply *auto*[*1*]

apply (*rule Inf-greatest*)
by (*simp add*: *Sup-upper compl-le-swap2*)

lemma *bdia-bbox-demorgan*: $(\langle R|\ (f::'b \Rightarrow 'c::complete\text{-}boolean\text{-}algebra)\ x) = -\ [R|\ (\lambda y.\ -f\ y)\ x$
apply (*simp add*: *bbox-def bdia-def*)
apply (*rule antisym*)
apply (*rule Sup-least*)
apply (*simp add*: *Inf-lower compl-le-swap1*)
apply (*simp add*: *uminus-Inf*)
apply (*rule SUP-least*; *intro Sup-upper*)
by *auto*

lemma *bbox-bdia-demorgan*: $(\ [R|\ (f::'b \Rightarrow 'c::complete\text{-}boolean\text{-}algebra)\ x) = -\ \langle R|\ (\lambda y.\ -f\ y)\ x$
apply (*simp add*: *bbox-def bdia-def*)
apply (*rule antisym*)
apply (*simp add*: *uminus-Sup*)
apply (*rule INF-greatest*; *rule Inf-lower*)
apply *auto*[*1*]
apply (*rule Inf-greatest*)
by (*simp add*: *Sup-upper compl-le-swap2*)

lemma *fdia-bdia-conv*: $(\ |R\rangle\ f\ x) = \langle converse\ R|\ f\ x$
by (*simp add*: *fdia-def bdia-def*)

lemma *fbox-bbox-conv*: $(\ |R]\ f\ x) = [converse\ R|\ f\ x$
by (*simp add*: *fbox-def bbox-def*)

lemma *fdia-bbox-galois*: $(\forall x.\ (\ |R\rangle\ f\ x) \leq g\ x) \longleftrightarrow (\forall x.\ f\ x \leq [R|\ g\ x)$
apply (*standard*, *simp-all add*: *fdia-def bbox-def*)
apply *safe*
apply (*rule Inf-greatest*)
apply (*force simp*: *Sup-le-iff*)
apply (*rule Sup-least*)
by (*force simp*: *le-Inf-iff*)

lemma *bdia-fbox-galois*: $(\forall x.\ (\langle R|\ f\ x) \leq g\ x) \longleftrightarrow (\forall x.\ f\ x \leq |R]\ g\ x)$
apply (*standard*, *simp-all add*: *bdia-def fbox-def*)
apply *safe*
apply (*rule Inf-greatest*)
apply (*force simp*: *Sup-le-iff*)
apply (*rule Sup-least*)
by (*force simp*: *le-Inf-iff*)

lemma *dia-conjugate*:
$(\forall x.\ (\ |R\rangle\ (f::'b \Rightarrow 'c::complete\text{-}boolean\text{-}algebra)\ x) \sqcap g\ x = \bot) \longleftrightarrow (\forall x.\ f\ x \sqcap (\langle R|\ g\ x) = \bot)$
by (*simp add*: *meet-shunt fdia-bbox-galois bdia-bbox-demorgan*)

lemma *box-conjugate*:
$(\forall x.\ (\ |R]\ (f::'b \Rightarrow 'c::complete\text{-}boolean\text{-}algebra)\ x) \sqcup g\ x = \top) \longleftrightarrow (\forall x.\ f\ x \sqcup ([R|\ g\ x) = \top)$
proof−
have $(\forall x.\ (\ |R]\ f\ x) \sqcup g\ x = \top) \longleftrightarrow (\forall x.\ -g\ x \leq |R]\ f\ x)$
by (*simp add*: *join-shunt sup-commute*)
also have ... $\longleftrightarrow (\forall x.\ -g\ x \leq -\ |R\rangle\ (\lambda y.\ -f\ y)\ x)$
by (*simp add*: *fbox-fdia-demorgan*)
also have ... $\longleftrightarrow (\forall x.\ (\ |R\rangle\ (\lambda y.\ -f\ y)\ x) \leq g\ x)$

by *simp*
also have ... $\longleftrightarrow$ $(\forall\, x.\ -f\ x \leq [R|\ g\ x)$
by (*simp add*: *fdia-bbox-galois*)
finally show *?thesis*
by (*simp add*: *join-shunt*)
qed

end

7 Liftings of Partial Semigroups

theory *Partial-Semigroup-Lifting*
imports *Partial-Semigroups Binary-Modalities*

begin

First we show that partial semigroups are instances of relational semigroups. Then we extend the lifting results for relational semigroups to partial semigroups.

7.1 Relational Semigroups and Partial Semigroups

Every partial semigroup is a relational partial semigroup.

context *partial-semigroup*
begin

sublocale *rel-partial-semigroup*: *rel-semigroup R*
by *standard* (*metis add-assoc add-assocD*)

end

Every partial monoid is a relational monoid.

context *partial-monoid*
begin

sublocale *rel-partial-monoid*: *rel-monoid R E*
apply *standard*
apply (*metis unitl-ex*)
apply (*metis unitr-ex*)
apply (*metis add-assocD-var1 unitl-ex units-eq-var*)
by (*metis add-assocD-var2 unitr-ex units-eq-var*)

end

Every PAS is a relational abelian semigroup.

context *pas*
begin

sublocale *rel-pas*: *rel-ab-semigroup R*
apply *standard*
using *add-comm* **by** *blast*

end

Every PAM is a relational abelian monoid.

context *pam*
begin

sublocale *rel-pam*: *rel-ab-monoid R E* **..**

end

7.2 Liftings of Partial Abelian Semigroups

Functions from partial semigroups into weak quantales form weak proto-quantales.

instantiation *fun* :: (*partial-semigroup*, *weak-quantale*) *weak-proto-quantale*
begin

definition *times-fun* :: $('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b$ **where**
times-fun ≡ *rel-partial-semigroup.times-rel-fun*

The following counterexample shows that the associativity law may fail in convolution algebras of functions from partial semigroups into weak quantales.

lemma (*rel-partial-semigroup.times-rel-fun* (*rel-partial-semigroup.times-rel-fun f f*) *f*) *x* =
(*rel-partial-semigroup.times-rel-fun* (*f*::'*a*::*partial-semigroup* ⇒ '*b*::*weak-quantale*) (*rel-partial-semigroup.times-rel-fun f f*)) *x*

oops

lemma *rel-partial-semigroup.times-rel-fun* (*rel-partial-semigroup.times-rel-fun f g*) *h* =
rel-partial-semigroup.times-rel-fun (*f*::'*a*::*partial-semigroup* ⇒ '*b*::*weak-quantale*) (*rel-partial-semigroup.times-rel-fun g h*)

oops

instance
by *standard* (*simp-all add*: *times-fun-def rel-partial-semigroup.rel-fun-Sup-distr rel-magma.rel-fun-Sup-distl-weak*)

end

Functions from partial semigroups into quantales form quantales.

instance *fun* :: (*partial-semigroup*, *quantale*) *quantale*
by *standard* (*simp-all add*: *times-fun-def rel-partial-semigroup.rel-fun-assoc rel-magma.rel-fun-Sup-distl*)

The following counterexample shows that the right unit law may fail in convolution algebras of functions from partial monoids into weak unital quantales.

lemma (*rel-partial-semigroup.times-rel-fun* (*f*::'*a*::*partial-monoid* ⇒ '*b*::*unital-weak-quantale*) *rel-partial-monoid.pid*) *x* = *f x*

oops

Functions from partial monoids into unital quantales form unital quantales.

instantiation *fun* :: (*partial-monoid*, *unital-quantale*) *unital-quantale*
begin

definition *one-fun* :: $'a \Rightarrow 'b$ **where**
one-fun ≡ *rel-partial-monoid.pid*

instance

by *standard* (*simp-all add*: *one-fun-def times-fun-def*)

end

These lifting results extend to PASs and PAMs as expected.

instance *fun* :: (*pam*, *ab-quantale*) *ab-quantale*
by *standard* (*simp-all add*: *times-fun-def rel-pas.rel-fun-comm*)

instance *fun* :: (*pam*, *bool-ab-quantale*) *bool-ab-quantale* **..**

instance *fun* :: (*pam*, *bool-ab-unital-quantale*) *bool-ab-unital-quantale* **..**

sublocale *ab-quantale* < *abq*: *pas* (∗) λ- -. *True*
apply *standard*
apply (*simp-all add*: *mult-assoc*)
by (*simp add*: *mult-commute*)

Finally we prove some identities that hold in function spaces.

lemma *times-fun-var*: (*f* ∗ *g*) *x* = ⨆{*f y* ∗ *g z* | *y z*. *R x y z*}
by (*simp add*: *times-fun-def rel-partial-semigroup.times-rel-fun-def bmod-comp-def*)

lemma *times-fun-var2*: (*f* ∗ *g*) = (λ*x*. ⨆{*f y* ∗ *g z* | *y z*. *R x y z*})
by (*auto simp*: *times-fun-var*)

lemma *one-fun-var1* [*simp*]: *x* ∈ *E* ⟹ *1 x* = *1*
by (*simp add*: *one-fun-def rel-partial-monoid.pid-def*)

lemma *one-fun-var2* [*simp*]: *x* ∉ *E* ⟹ *1 x* = ⊥
by (*simp add*: *one-fun-def rel-partial-monoid.pid-def*)

lemma *times-fun-canc*: (*f* ∗ *g*) *x* = ⨆{*f y* ∗ *g* (*rquot x y*) | *y*. *y* ≼$_R$ *x*}
apply (*rule antisym*)
apply (*simp add*: *times-fun-var*, *intro Sup-subset-mono*, *simp add*: *Collect-mono-iff*)
using *gR-rel-mult add-canc1* **apply** *force*
apply (*simp add*: *times-fun-var*, *intro Sup-subset-mono*, *simp add*: *Collect-mono-iff*)
using *gR-rel-defined add-canc2* **by** *fastforce*

lemma *times-fun-prod*: (*f* ∗ *g*) = (λ(*x*, *y*). ⨆{*f* (*x*, *y1*) ∗ *g* (*x*, *y2*) | *y1 y2*. *R y y1 y2*})
by (*auto simp*: *times-fun-var2 times-prod-def D-prod-def*)

lemma *one-fun-prod1* [*simp*]: *y* ∈ *E* ⟹ *1* (*x*, *y*) = *1*
by (*simp add*: *E-prod-def*)

lemma *one-fun-prod2* [*simp*]: *y* ∉ *E* ⟹ *1* (*x*, *y*) = ⊥
by (*simp add*: *E-prod-def*)

lemma *fres-galois-funI*: ∀ *x*. (*f* ∗ *g*) *x* ≤ *h x* ⟹ *f x* ≤ (*h* ← *g*) *x*
by (*meson fres-galois le-funD le-funI*)

lemma *times-fun-prod-canc*: (*f* ∗ *g*) (*x*, *y*) = ⨆{*f* (*x*, *z*) ∗ *g* (*x*, *rquot y z*) | *z*. *z* ≼$_R$ *y*}
apply (*simp add*: *times-fun-prod*)
by (*metis* (*no-types*, *lifting*) *gR-rel-defined gR-rel-mult add-canc1 add-canc2*)

The following statement shows, in a generalised setting, that the magic wand operator of separation logic can be lifted from the heap subtraction operation generalised to a cancellative PAM.

lemma *fres-lift*: (*fres f g*) (*x*::'*b*::*cancellative-pam*) = $\bigsqcap\{(f\ y) \leftarrow (g\ z) \mid y\ z\ .\ z \preceq_R y \wedge x = rquot\ y\ z\}$
proof (*rule antisym*)
{ fix *h y z*
assume *assms*: $h \cdot g \leq f\ z \preceq_R y\ x = rquot\ y\ z$
moreover hence *D z x*
using *add-rquot* **by** *blast*
moreover hence $h\ x \cdot g\ z \leq (h \cdot g)\ (x \oplus z)$
using *add-comm* **by** (*auto simp add*: *times-fun-var intro*!: *Sup-upper*)
moreover hence $(h * g)\ (x \oplus z) \leq f\ (z \oplus x)$
by (*simp add*: ‹*D z x*› *calculation*(*1*) *le-funD add-comm*)
ultimately have $h\ x \leq (f\ (z \oplus x)) \leftarrow (g\ z)$
by (*auto simp*: *fres-def intro*: *Sup-upper*)
from *this* **and** *assms* **have** $h\ (rquot\ y\ z) \leq (f\ y) \leftarrow (g\ z)$
by (*simp add*: *add-canc2*)
}
thus $(f \leftarrow g)\ x \leq \bigsqcap\{(f\ y) \leftarrow (g\ z)\ |y\ z.\ z \preceq_R y \wedge x = rquot\ y\ z\}$
by (*clarsimp simp*: *fres-def intro*!: *Inf-greatest SUP-least*)
next
have $\bigsqcap\{(f\ y) \leftarrow (g\ z)\ |y\ z.\ z \preceq_R y \wedge x = rquot\ y\ z\} \leq Sup\{x.\ x \cdot g \leq f\}\ x$
apply (*clarsimp simp*: *times-fun-var intro*!: *SUP-upper le-funI Sup-least*)
apply (*simp add*: *fres-galois*)
apply (*intro Inf-lower*)
apply *safe*
by (*metis gR-rel-mult add-canc1 add-comm*)
thus $\bigsqcap\{(f\ y) \leftarrow (g\ z)\ |y\ z.\ z \preceq_R y \wedge x = rquot\ y\ z\} \leq (f \leftarrow g)\ x$
by (*simp add*: *fres-def*)
qed

end

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