

Automating Public Announcement Logic and the Wise Men Puzzle in Isabelle/HOL

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Abstract

We present a shallow embedding of public announcement logic (PAL) with relativized general knowledge in HOL. We then use PAL to obtain an elegant encoding of the wise men puzzle, which we solve automatically using sledgehammer.

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1 Public Announcement Logic (PAL) in HOL

An earlier encoding and automation of the wise men puzzle, utilizing a shallow embedding of higher-order (multi-)modal logic in HOL, has been presented in [1, 2]. However, this work did not convincingly address the interaction dynamics between the involved agents. Here we therefore extend and adapt the universal (meta-)logical reasoning approach of [1] for public announcement logic (PAL) and we demonstrate how it can be utilized to achieve a convincing encoding and automation of the wise men puzzle in HOL, so that also the interaction dynamics as given in the scenario is adequately addressed. For further background information on the work presented here we refer to [3, 4].

```
theory PAL imports Main begin  
nitpick-params[user-axioms, expect=genuine]
```

Type i is associated with possible worlds

```
typedecl  $i$   
type-synonym  $\sigma = i \Rightarrow \text{bool}$   
type-synonym  $\tau = \sigma \Rightarrow i \Rightarrow \text{bool}$   
type-synonym  $\alpha = i \Rightarrow i \Rightarrow \text{bool}$ 
```

type-synonym $\rho = \alpha \Rightarrow \text{bool}$

Some useful relations (for constraining accessibility relations)

definition *reflexive*:: $\alpha \Rightarrow \text{bool}$

where *reflexive* $R \equiv \forall x. R x x$

definition *symmetric*:: $\alpha \Rightarrow \text{bool}$

where *symmetric* $R \equiv \forall x y. R x y \longrightarrow R y x$

definition *transitive*:: $\alpha \Rightarrow \text{bool}$

where *transitive* $R \equiv \forall x y z. R x y \wedge R y z \longrightarrow R x z$

definition *euclidean*:: $\alpha \Rightarrow \text{bool}$

where *euclidean* $R \equiv \forall x y z. R x y \wedge R x z \longrightarrow R y z$

definition *intersection-rel*:: $\alpha \Rightarrow \alpha \Rightarrow \alpha$

where *intersection-rel* $R Q \equiv \lambda u v. R u v \wedge Q u v$

definition *union-rel*:: $\alpha \Rightarrow \alpha \Rightarrow \alpha$

where *union-rel* $R Q \equiv \lambda u v. R u v \vee Q u v$

definition *sub-rel*:: $\alpha \Rightarrow \alpha \Rightarrow \text{bool}$

where *sub-rel* $R Q \equiv \forall u v. R u v \longrightarrow Q u v$

definition *inverse-rel*:: $\alpha \Rightarrow \alpha$

where *inverse-rel* $R \equiv \lambda u v. R v u$

definition *big-union-rel*:: $\rho \Rightarrow \alpha$

where *big-union-rel* $X \equiv \lambda u v. \exists R. (X R) \wedge (R u v)$

definition *big-intersection-rel*:: $\rho \Rightarrow \alpha$

where *big-intersection-rel* $X \equiv \lambda u v. \forall R. (X R) \longrightarrow (R u v)$

In HOL the transitive closure of a relation can be defined in a single line.

definition *tc*:: $\alpha \Rightarrow \alpha$

where *tc* $R \equiv \lambda x y. \forall Q. \text{transitive } Q \longrightarrow (\text{sub-rel } R Q \longrightarrow Q x y)$

Logical connectives for PAL

abbreviation *patom*:: $\sigma \Rightarrow \tau$ ($\langle^A \rightarrow$ [79]80)

where $^A p \equiv \lambda W w. W w \wedge p w$

abbreviation *ptop*:: τ ($\langle \top \rangle$)

where $\top \equiv \lambda W w. \text{True}$

abbreviation *pneg*:: $\tau \Rightarrow \tau$ ($\langle \neg \rightarrow$ [52]53)

where $\neg \varphi \equiv \lambda W w. \neg(\varphi W w)$

abbreviation *pand*:: $\tau \Rightarrow \tau \Rightarrow \tau$ (**infixr** $\langle \wedge \rangle$ 51)

where $\varphi \wedge \psi \equiv \lambda W w. (\varphi W w) \wedge (\psi W w)$

abbreviation *por*:: $\tau \Rightarrow \tau \Rightarrow \tau$ (**infixr** $\langle \vee \rangle$ 50)

where $\varphi \vee \psi \equiv \lambda W w. (\varphi W w) \vee (\psi W w)$

abbreviation *pimp*:: $\tau \Rightarrow \tau \Rightarrow \tau$ (**infixr** $\langle \rightarrow \rangle$ 49)

where $\varphi \rightarrow \psi \equiv \lambda W w. (\varphi W w) \longrightarrow (\psi W w)$

abbreviation *pequ*:: $\tau \Rightarrow \tau \Rightarrow \tau$ (**infixr** $\langle \leftrightarrow \rangle$ 48)

where $\varphi \leftrightarrow \psi \equiv \lambda W w. (\varphi W w) \longleftrightarrow (\psi W w)$

abbreviation *pknow*:: $\alpha \Rightarrow \tau \Rightarrow \tau$ ($\langle \mathbf{K} - \rightarrow \rangle$)

where $\mathbf{K} r \varphi \equiv \lambda W w. \forall v. (W v \wedge r w v) \longrightarrow (\varphi W w)$

abbreviation *ppal*:: $\tau \Rightarrow \tau \Rightarrow \tau$ ($\langle \mathbf{!} - \rightarrow \rangle$)

where $\mathbf{!} \varphi \psi \equiv \lambda W w. (\varphi W w) \longrightarrow (\psi (\lambda z. W z \wedge \varphi W z) w)$

Global validity of PAL formulas

abbreviation *pvalid*:: $\tau \Rightarrow \text{bool}$ ($\langle \mathbf{!} - \rangle$ [7]8)

where $[\varphi] \equiv \forall W. \forall w. W w \longrightarrow \varphi W w$

Introducing agent knowledge (K), mutual knowledge (E), distributed knowledge (D) and common knowledge (C).

abbreviation $EVR::\rho \Rightarrow \alpha$

where $EVR G \equiv \text{big-union-rel } G$

abbreviation $DIS::\rho \Rightarrow \alpha$

where $DIS G \equiv \text{big-intersection-rel } G$

abbreviation $agtknows::\alpha \Rightarrow \tau \Rightarrow \tau$ ($\langle \mathbf{K}_- _ \rangle$)

where $\mathbf{K}_r \varphi \equiv \mathbf{K} r \varphi$

abbreviation $evrknows::\rho \Rightarrow \tau \Rightarrow \tau$ ($\langle \mathbf{E}_- _ \rangle$)

where $\mathbf{E}_G \varphi \equiv \mathbf{K} (EVR G) \varphi$

abbreviation $disknows :: \rho \Rightarrow \tau \Rightarrow \tau$ ($\langle \mathbf{D}_- _ \rangle$)

where $\mathbf{D}_G \varphi \equiv \mathbf{K} (DIS G) \varphi$

abbreviation $prck::\rho \Rightarrow \tau \Rightarrow \tau \Rightarrow \tau$ ($\langle \mathbf{C}_- (_ | _ | _) \rangle$)

where $\mathbf{C}_G (\varphi | \psi) \equiv \lambda W w. \forall v. (tc (\text{intersection-rel } (EVR G) (\lambda u v. W v \wedge \varphi W v)) w v) \longrightarrow (\psi W v)$

abbreviation $pcmn::\rho \Rightarrow \tau \Rightarrow \tau$ ($\langle \mathbf{C}_- _ \rangle$)

where $\mathbf{C}_G \varphi \equiv \mathbf{C}_G (\top | \varphi)$

Postulating S5 principles for the agent's accessibility relations.

abbreviation $S5Agent::\alpha \Rightarrow \text{bool}$

where $S5Agent i \equiv \text{reflexive } i \wedge \text{transitive } i \wedge \text{euclidean } i$

abbreviation $S5Agents::\rho \Rightarrow \text{bool}$

where $S5Agents A \equiv \forall i. (A i \longrightarrow S5Agent i)$

Introducing "Defs" as the set of the above definitions; useful for convenient unfolding.

named-theorems $Defs$

declare $\text{reflexive-def}[Defs] \text{symmetric-def}[Defs] \text{transitive-def}[Defs]$
 $\text{euclidean-def}[Defs] \text{intersection-rel-def}[Defs] \text{union-rel-def}[Defs]$
 $\text{sub-rel-def}[Defs] \text{inverse-rel-def}[Defs] \text{big-union-rel-def}[Defs]$
 $\text{big-intersection-rel-def}[Defs] \text{tc-def}[Defs]$

Consistency: nitpick reports a model.

lemma $\text{True nitpick [satisfy] oops}$

2 Automating the Wise Men Puzzle

Agents are modeled as accessibility relations.

consts $a::\alpha \ b::\alpha \ c::\alpha$

abbreviation $Agent::\alpha \Rightarrow \text{bool}$ ($\langle \mathcal{A} \rangle$) **where** $\mathcal{A} x \equiv x = a \vee x = b \vee x = c$

axiomatization where $\text{group-S5: } S5Agents \ \mathcal{A}$

Common knowledge: At least one of a, b and c has a white spot.

consts $ws::\alpha \Rightarrow \sigma$

axiomatization where $WM1: [\mathbf{C}_{\mathcal{A}} ({}^A ws a \vee {}^A ws b \vee {}^A ws c)]$

Common knowledge: If x does not have a white spot then y knows this.

axiomatization where

WM2ab: $[\mathbf{C}_A (\neg({}^A ws\ a) \rightarrow (\mathbf{K}_b (\neg({}^A ws\ a))))]$ **and**
WM2ac: $[\mathbf{C}_A (\neg({}^A ws\ a) \rightarrow (\mathbf{K}_c (\neg({}^A ws\ a))))]$ **and**
WM2ba: $[\mathbf{C}_A (\neg({}^A ws\ b) \rightarrow (\mathbf{K}_a (\neg({}^A ws\ b))))]$ **and**
WM2bc: $[\mathbf{C}_A (\neg({}^A ws\ b) \rightarrow (\mathbf{K}_c (\neg({}^A ws\ b))))]$ **and**
WM2ca: $[\mathbf{C}_A (\neg({}^A ws\ c) \rightarrow (\mathbf{K}_a (\neg({}^A ws\ c))))]$ **and**
WM2cb: $[\mathbf{C}_A (\neg({}^A ws\ c) \rightarrow (\mathbf{K}_b (\neg({}^A ws\ c))))]$

Positive introspection principles are implied.

lemma *WM2ab'*: $[\mathbf{C}_A (({}^A ws\ a) \rightarrow \mathbf{K}_b ({}^A ws\ a))]$
using *WM2ab group-S5 unfolding Defs by metis*
lemma *WM2ac'*: $[\mathbf{C}_A (({}^A ws\ a) \rightarrow \mathbf{K}_c ({}^A ws\ a))]$
using *WM2ac group-S5 unfolding Defs by metis*
lemma *WM2ba'*: $[\mathbf{C}_A (({}^A ws\ b) \rightarrow \mathbf{K}_a ({}^A ws\ b))]$
using *WM2ba group-S5 unfolding Defs by metis*
lemma *WM2bc'*: $[\mathbf{C}_A (({}^A ws\ b) \rightarrow \mathbf{K}_c ({}^A ws\ b))]$
using *WM2bc group-S5 unfolding Defs by metis*
lemma *WM2ca'*: $[\mathbf{C}_A (({}^A ws\ c) \rightarrow \mathbf{K}_a ({}^A ws\ c))]$
using *WM2ca group-S5 unfolding Defs by metis*
lemma *WM2cb'*: $[\mathbf{C}_A (({}^A ws\ c) \rightarrow \mathbf{K}_b ({}^A ws\ c))]$
using *WM2cb group-S5 unfolding Defs by metis*

Automated solutions of the Wise Men Puzzle.

theorem *whitespot-c*: $[[!\neg\mathbf{K}_a({}^A ws\ a)]([!\neg\mathbf{K}_b({}^A ws\ b)](\mathbf{K}_c ({}^A ws\ c)))]$
using *WM1 WM2ba WM2ca WM2cb unfolding Defs by (smt (verit))*

For the following, alternative formulation a proof is found by sledgehammer, while the reconstruction of this proof using trusted methods (often) fails; this hints at further opportunities to improve the reasoning tools in Isabelle/HOL.

theorem *whitespot-c'*:
 $[[!\neg((\mathbf{K}_a ({}^A ws\ a)) \vee (\mathbf{K}_a (\neg({}^A ws\ a))))]([!\neg((\mathbf{K}_b ({}^A ws\ b)) \vee (\mathbf{K}_b (\neg({}^A ws\ b))))](\mathbf{K}_c ({}^A ws\ c)))]$
using *WM1 WM2ab WM2ac WM2ba WM2bc WM2ca WM2cb unfolding Defs*
— *sledgehammer by (smt (verit))*
oops

Consistency: nitpick reports a model.

lemma *True nitpick [satisfy] oops*
end

References

- [1] C. Benzmüller. Universal (meta-)logical reasoning: Recent successes. *Science of Computer Programming*, 172:48–62, 2019.

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- [3] C. Benzmüller and S. Reiche. Modeling and automating public announcement logic with relativized common knowledge as a fragment of HOL in LogiKEY. Technical Report arXiv:2111.01654, CoRR, 2021.
- [4] S. Reiche and C. Benzmüller. Public announcement logic in HOL. In M. A. Martins and S. Igor, editors, *Dynamic Logic. New Trends and Applications. DaLi 2020*, volume 12569 of *Lecture Notes in Computer Science*. Springer, Cham, 2020.