

# Notes on Gödel’s and Scott’s Variants of the Ontological Argument (Isabelle/HOL dataset)

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## Abstract

Experimental studies with Isabelle/HOL on Kurt Gödel’s modal ontological argument and Dana Scott’s variant of it are presented. They implicitly answer some questions that the authors have received over the last decade(s). In addition, some new results are reported.

Our contribution is explained in full detail in [2]. This document presents the corresponding Isabelle/HOL dataset (which is only slightly modified to meet AFP requirements).

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## 1 Introduction

The Isabelle/HOL dataset associated with [2] is presented. Compared to previous work on Gödel’s modal ontological argument as published in the Archive of Formal Proofs (AFP) [1, 4, 3], our dataset addresses several relevant and in some cases novel aspects, which are combined here in a single publication, including:

1. For the first time, Gödel’s original manuscript [5] has been formalized as closely as possible.
2. The inconsistency of the postulates in Gödel’s original manuscript is explained in detail.
3. Two different ways of eliminating this inconsistency are presented, one of which is novel.
4. Scott’s variant [6] of Gödel’s original manuscript is presented and compared with Gödel’s original variant.
5. In addition to logics S5 and K, the above variants are also studied for logic S4.

6. The above variants are tested for various combinations of of possibilist and actualist quantifiers for individuals (which has not been done systematically in previous publications).
7. Concepts of evil are examined and the derivability of evil is critically questioned.

The purpose of this AFP publication is essentially twofold. One motivation is to make the data sources associated with [2] available in a sustainable, well-maintained way. The other motivation is to support university education in higher-order modal logic by providing a small dataset for reuse that illustrates a systematically explored philosophical argument, emphasizing in particular different notions of quantification.

Compared to [2], the Isabelle sources presented here have been slightly modified to meet some AFP requirements. This concerns the commenting out of calls to sledghemmer (to reduce computational resources) and some minor reformatting (e.g. insertion of new lines). The formalization code itself remains unchanged.

## 2 Interactive and automated theorem proving

### 2.1 SurjectiveCantor.thy (Figure 2 of [2])

The surjective Cantor theorem is used in [2] to illustrate some aspects of interactive and automated theorem proving in Isabelle/HOL as relevant for the paper. To keep the provided data material complete wrt. [2], we include these data sources also here.

```
theory SurjectiveCantor imports Main
begin
```

Surjective Cantor theorem: traditional interactive proof

```
theorem SurjectiveCantor:  $\neg(\exists G.\forall F::'a\Rightarrow bool.\exists X::'a. G X = F)$ 
  <proof>
```

Avoiding proof by contradiction (Fuenmayor & Benzmüller, 2021)

```
theorem SurjectiveCantor':  $\neg(\exists G.\forall F::'a\Rightarrow bool.\exists X::'a. G X = F)$ 
  <proof>
```

Surjective Cantor theorem: automated proof by some internal/external theorem provers

```
theorem SurjectiveCantor'':  $\neg(\exists G.\forall F::'a\Rightarrow bool.\exists X::'a. G X = F)$ 
  nitpick[expect=none] — no counterexample found
  — sledghemmer — most internal provers give up
  — sledghemmer[remote_leo2 remote_leo3] — proof found: leo provers succeed
  <proof>
```

Surjective Cantor theorem (wrong formalization attempt): the types are crucial

**theorem** *SurjectiveCantor'''*:  $\neg(\exists G.\forall F::'b.\exists X::'a. G X = F)$

**nitpick** — counterexample found for card 'a = 1 and card 'b = 1:  $G=(\lambda x. (a1 := b1))$

**nitpick**[*satisfy, expect=genuine*] — model found for card 'a = 1 and card 'b = 2

**nitpick**[*card 'a=2, card 'b=3, expect=none*] — no counterexample found

*<proof>*

**end**

### 3 Mechanization of higher-order modal logic (HOML)

#### 3.1 HOMLinHOL.thy (Figure 3 of [2])

Shallow embedding of higher-order modal logic (HOML) in the classical higher-order logic (HOL) of Isabelle/HOL utilizing the logic-pluralistic LogiKEy methodology. Here logic S5 is introduced.

**theory** *HOMLinHOL* **imports** *Main*

**begin**

— Global parameters setting for the model finder nitpick and the parser; unimport for the reader

**nitpick-params**[*user-axioms, expect=genuine, show-all, format=2, max-genuine=3*]

**declare**[*[syntax-ambiguity-warning=false]*]

— Type *i* is associated with possible worlds and type *e* with entities

**typedecl** *i* — Possible worlds

**typedecl** *e* — Individuals/entities

**type-synonym**  $\sigma = i \Rightarrow \text{bool}$  — World-lifted propositions

**type-synonym**  $\tau = e \Rightarrow \sigma$  — Modal properties

**consts**  $R::i \Rightarrow i \Rightarrow \text{bool}$  (**r**-) — Accessibility relation between worlds

**axiomatization** **where**

*Rrefl*:  $\forall x. \mathbf{r}x \text{ and}$

*Rsymm*:  $\forall x y. \mathbf{r}xy \longrightarrow \mathbf{r}yx \text{ and}$

*Rtrans*:  $\forall x y z. \mathbf{r}xy \wedge \mathbf{r}yz \longrightarrow \mathbf{r}xz$

— Logical connectives (operating on truth-sets)

**abbreviation** *Mbot*:: $\sigma$  ( $\perp$ ) **where**  $\perp \equiv \lambda w. \text{False}$

**abbreviation** *Mtop*:: $\sigma$  ( $\top$ ) **where**  $\top \equiv \lambda w. \text{True}$

**abbreviation** *Mneg*:: $\sigma \Rightarrow \sigma$  ( $\neg$ - [52]53) **where**  $\neg\varphi \equiv \lambda w. \neg(\varphi w)$

**abbreviation** *Mand*:: $\sigma \Rightarrow \sigma \Rightarrow \sigma$  (**infixl**  $\wedge$  50) **where**  $\varphi \wedge \psi \equiv \lambda w. \varphi w \wedge \psi w$

**abbreviation** *Mor*:: $\sigma \Rightarrow \sigma \Rightarrow \sigma$  (**infixl**  $\vee$  49) **where**  $\varphi \vee \psi \equiv \lambda w. \varphi w \vee \psi w$

**abbreviation** *Mimp*:: $\sigma \Rightarrow \sigma \Rightarrow \sigma$  (**infixr**  $\supset$  48) **where**  $\varphi \supset \psi \equiv \lambda w. \varphi w \longrightarrow \psi w$

**abbreviation** *Mequiv*:: $\sigma \Rightarrow \sigma \Rightarrow \sigma$  (**infixl**  $\leftrightarrow$  47) **where**  $\varphi \leftrightarrow \psi \equiv \lambda w. \varphi w \longleftrightarrow \psi w$

**abbreviation** *Mbox*:: $\sigma \Rightarrow \sigma$  ( $\Box$ - [54]55) **where**  $\Box\varphi \equiv \lambda w. \forall v. w \mathbf{r} v \longrightarrow \varphi v$

**abbreviation** *Mdia*:: $\sigma \Rightarrow \sigma$  ( $\Diamond$ - [54]55) **where**  $\Diamond\varphi \equiv \lambda w. \exists v. w \mathbf{r} v \wedge \varphi v$

**abbreviation**  $Mprimeq::'a\Rightarrow'a\Rightarrow\sigma$  ( $-=-$ ) **where**  $x=y \equiv \lambda w. x=y$   
**abbreviation**  $Mprimneg::'a\Rightarrow'a\Rightarrow\sigma$  ( $-\neq-$ ) **where**  $x\neq y \equiv \lambda w. x\neq y$   
**abbreviation**  $Mnegpred::\tau\Rightarrow\tau$  ( $\sim-$ ) **where**  $\sim\Phi \equiv \lambda x.\lambda w. \neg\Phi x w$   
**abbreviation**  $Mconpred::\tau\Rightarrow\tau\Rightarrow\tau$  (**infixl** . 50) **where**  $\Phi.\Psi \equiv \lambda x.\lambda w. \Phi x w \wedge \Psi x w$   
**abbreviation**  $Mexclor::\sigma\Rightarrow\sigma\Rightarrow\sigma$  (**infixl**  $\vee^e$  49) **where**  $\varphi\vee^e\psi \equiv (\varphi \vee \psi) \wedge \neg(\varphi \wedge \psi)$

— Possibilist quantifiers (polymorphic)

**abbreviation**  $Mallposs::('a\Rightarrow\sigma)\Rightarrow\sigma$  ( $\forall$ ) **where**  $\forall\Phi \equiv \lambda w.\forall x. \Phi x w$   
**abbreviation**  $Mallpossb$  (**binder**  $\forall$  [8]9) **where**  $\forall x. \varphi(x) \equiv \forall\varphi$   
**abbreviation**  $Mexiposs::('a\Rightarrow\sigma)\Rightarrow\sigma$  ( $\exists$ ) **where**  $\exists\Phi \equiv \lambda w.\exists x. \Phi x w$   
**abbreviation**  $Mexipossb$  (**binder**  $\exists$  [8]9) **where**  $\exists x. \varphi(x) \equiv \exists\varphi$

— Actualist quantifiers (for individuals/entities)

**consts**  $existsAt::e\Rightarrow\sigma$  ( $-\textcircled{a}-$ )  
**abbreviation**  $Mallact::(e\Rightarrow\sigma)\Rightarrow\sigma$  ( $\forall^E$ ) **where**  $\forall^E\Phi \equiv \lambda w.\forall x. x\textcircled{a}w \longrightarrow \Phi x w$   
**abbreviation**  $Mallactb$  (**binder**  $\forall^E$  [8]9) **where**  $\forall^E x. \varphi(x) \equiv \forall^E\varphi$   
**abbreviation**  $Mexiact::(e\Rightarrow\sigma)\Rightarrow\sigma$  ( $\exists^E$ ) **where**  $\exists^E\Phi \equiv \lambda w.\exists x. x\textcircled{a}w \wedge \Phi x w$   
**abbreviation**  $Mexiactb$  (**binder**  $\exists^E$  [8]9) **where**  $\exists^E x. \varphi(x) \equiv \exists^E\varphi$

— Leibniz equality (polymorphic)

**abbreviation**  $Mleibeq::'a\Rightarrow'a\Rightarrow\sigma$  ( $-\equiv-$ ) **where**  $x\equiv y \equiv \forall P. P x \supset P y$

— Meta-logical predicate for global validity

**abbreviation**  $Mvalid::\sigma\Rightarrow bool$  ( $[-]$ ) **where**  $[\psi] \equiv \forall w. \psi w$

end

### 3.2 TestsHOML.thy (Figure 4 of [2])

Tests and verifications of properties for the embedding of HOML (S5) in HOL.

**theory** *TestsHOML* **imports** *HOMLinHOL*  
**begin**

— Test for S5 modal logic

**lemma**  $axM$ :  $[\Box\varphi \supset \varphi]$  *<proof>*  
**lemma**  $axD$ :  $[\Box\varphi \supset \Diamond\varphi]$  *<proof>*  
**lemma**  $axB$ :  $[\varphi \supset \Box\Diamond\varphi]$  *<proof>*  
**lemma**  $ax4$ :  $[\Box\varphi \supset \Box\Box\varphi]$  *<proof>*  
**lemma**  $ax5$ :  $[\Diamond\varphi \supset \Box\Diamond\varphi]$  *<proof>*  
**lemma**  $BarcanAct$ :  $[(\forall^E x. \Box(\varphi x)) \supset \Box(\forall^E x. (\varphi x))]$   
**nitpick** $[expect=genuine]$  *<proof>*  
**lemma**  $ConvBarcanAct$ :  $[\Box(\forall^E x. (\varphi x)) \supset (\forall^E x. \Box(\varphi x))]$   
**nitpick** $[expect=genuine]$  *<proof>*  
**lemma**  $BarcanPoss$ :  $[(\forall x. \Box(\varphi x)) \supset \Box(\forall x. \varphi x)]$  *<proof>*  
**lemma**  $ConvBarcanPoss$ :  $[\Box(\forall x. (\varphi x)) \supset (\forall x. \Box(\varphi x))]$  *<proof>*  
**lemma**  $Hilbert-A1$ :  $[A \supset (B \supset A)]$  *<proof>*

```

lemma Hilbert-A2:  $[(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))]$   $\langle proof \rangle$ 
lemma Hilbert-MP: assumes  $[A]$  and  $[A \supset B]$  shows  $[B]$   $\langle proof \rangle$ 
lemma Quant-1: assumes  $[A]$  shows  $[\forall x::'a. A]$   $\langle proof \rangle$ 
lemma ExImPossibilist1:  $[\exists x::e. x = x]$   $\langle proof \rangle$ 
lemma ExImPossibilist2:  $[\exists x::e. x \equiv x]$   $\langle proof \rangle$ 
lemma ExImPossibilist3:  $[\exists x::e. x = t]$   $\langle proof \rangle$ 
lemma ExImPossibilist4:  $[\exists x::'a. x \equiv t::'a]$   $\langle proof \rangle$ 
lemma ExImPossibilist:  $[\exists x::'a. \top]$   $\langle proof \rangle$ 
lemma Quant-2: assumes  $[A]$  shows  $[\forall^E x::e. A]$   $\langle proof \rangle$ 
lemma ExImActualist1:  $[\exists^E x::e. x = x]$ 
  nitpick[card=1,expect=genuine]  $\langle proof \rangle$ 
lemma ExImActualist2:  $[\exists^E x::e. x \equiv x]$ 
  nitpick[card=1,expect=genuine]  $\langle proof \rangle$ 
lemma ExImActualist3:  $[\exists^E x::e. x = t]$ 
  nitpick[card=1,expect=genuine]  $\langle proof \rangle$ 
lemma ExImActualist:  $[\exists^E x::e. \top]$ 
  nitpick[card=1,expect=genuine]  $\langle proof \rangle$ 
lemma EqReft:  $[x = x]$   $\langle proof \rangle$ 
lemma EqSym:  $[(x = y) \leftrightarrow (y = x)]$   $\langle proof \rangle$ 
lemma EqTrans:  $[(x = y) \wedge (y = z) \supset (x = z)]$   $\langle proof \rangle$ 
lemma EQCong:  $[(x = y) \supset ((\varphi x) = (\varphi y))]$   $\langle proof \rangle$ 
lemma EQFuncExt:  $[(\varphi = \psi) \supset (\forall x. ((\varphi x) = (\psi x)))]$   $\langle proof \rangle$ 
lemma EQBoolExt1:  $[(\varphi = \psi) \supset (\varphi \leftrightarrow \psi)]$   $\langle proof \rangle$ 
lemma EQBoolExt2:  $[(\varphi \leftrightarrow \psi) \supset (\varphi = \psi)]$ 
  nitpick[card=2]  $\langle proof \rangle$ 
lemma EQBoolExt3:  $[(\varphi \leftrightarrow \psi)] \longrightarrow [(\varphi = \psi)]$   $\langle proof \rangle$ 
lemma EqPrimLeib:  $[(x = y) \leftrightarrow (x \equiv y)]$   $\langle proof \rangle$ 
lemma Comprehension1:  $[\exists \varphi. \forall x. (\varphi x) \leftrightarrow A]$   $\langle proof \rangle$ 
lemma Comprehension2:  $[\exists \varphi. \forall x. (\varphi x) \leftrightarrow (A x)]$   $\langle proof \rangle$ 
lemma Comprehension3:  $[\exists \varphi. \forall x y. (\varphi x y) \leftrightarrow (A x y)]$   $\langle proof \rangle$ 
lemma ModalCollapse:  $[\forall \varphi. \varphi \supset \Box \varphi]$ 
  nitpick[card=2,expect=genuine]  $\langle proof \rangle$ 
lemma TruePropertyAndSelfIdentity:  $[(\lambda x::e. \top) = (\lambda x. x = x)]$   $\langle proof \rangle$ 
lemma EmptyPropertyAndSelfDifference:  $[(\lambda x::e. \perp) = (\lambda x. x \neq x)]$   $\langle proof \rangle$ 
lemma EmptyProperty2:  $[\exists x. \varphi x] \longrightarrow [\varphi \neq (\lambda x::e. \perp)]$   $\langle proof \rangle$ 
lemma EmptyProperty3:  $[\exists^E x. \varphi x] \longrightarrow [\varphi \neq (\lambda x::e. \perp)]$   $\langle proof \rangle$ 
lemma EmptyProperty4:  $[\varphi \neq (\lambda x::e. \perp)] \longrightarrow [\exists x. \varphi x]$ 
  nitpick[expect=genuine]  $\langle proof \rangle$ 
lemma EmptyProperty5:  $[\varphi \neq (\lambda x::e. \perp)] \longrightarrow [\exists^E x. \varphi x]$ 
  nitpick[expect=genuine]  $\langle proof \rangle$ 

```

end

### 3.3 ModalFilter.thy (Figure 5 of [2])

Set filter and ultrafilter formalized for our modal logic setting.

```

theory ModalFilter imports HOMLinHOL
begin

```

**type-synonym**  $\tau = e \Rightarrow \sigma$   
**abbreviation**  $Element::\tau \Rightarrow (\tau \Rightarrow \sigma) \Rightarrow \sigma$  (**infix**  $\in$  90) **where**  $\varphi \in S \equiv S \ \varphi$   
**abbreviation**  $EmptySet::\tau$  ( $\emptyset$ ) **where**  $\emptyset \equiv \lambda x. \perp$   
**abbreviation**  $UniversalSet::\tau$  ( $\mathbf{U}$ ) **where**  $\mathbf{U} \equiv \lambda x. \top$   
**abbreviation**  $Subset::\tau \Rightarrow \tau \Rightarrow \sigma$  (**infix**  $\subseteq$  80)  
**where**  $\varphi \subseteq \psi \equiv \forall x. ((\varphi \ x) \supset (\psi \ x))$   
**abbreviation**  $SubsetE::\tau \Rightarrow \tau \Rightarrow \sigma$  (**infix**  $\subseteq^E$  80)  
**where**  $\varphi \subseteq^E \psi \equiv \forall^E x. ((\varphi \ x) \supset (\psi \ x))$   
**abbreviation**  $Intersection::\tau \Rightarrow \tau \Rightarrow \tau$  (**infix**  $\sqcap$  91)  
**where**  $\varphi \sqcap \psi \equiv \lambda x. ((\varphi \ x) \wedge (\psi \ x))$   
**abbreviation**  $Inverse::\tau \Rightarrow \tau$  ( $^{-1}$ )  
**where**  $^{-1}\psi \equiv \lambda x. \neg(\psi \ x)$   
**abbreviation**  $Filter \ \Phi \equiv \mathbf{U} \in \Phi \wedge \neg(\emptyset \in \Phi) \wedge$   
 $(\forall \varphi \ \psi. \varphi \in \Phi \wedge \varphi \subseteq^E \psi \supset \psi \in \Phi) \wedge (\forall \varphi \ \psi. \varphi \in \Phi \wedge \psi \in \Phi \supset \varphi \sqcap \psi \in \Phi)$   
**abbreviation**  $UFilter \ \Phi \equiv Filter \ \Phi \wedge (\forall \varphi. \varphi \in \Phi \vee (^{-1}\varphi) \in \Phi)$   
**abbreviation**  $FilterP \ \Phi \equiv \mathbf{U} \in \Phi \wedge \neg(\emptyset \in \Phi) \wedge (\forall \varphi \ \psi. \varphi \in \Phi \wedge \varphi \subseteq \psi \supset \psi \in \Phi) \wedge$   
 $(\forall \varphi \ \psi. \varphi \in \Phi \wedge \psi \in \Phi \supset \varphi \sqcap \psi \in \Phi)$   
**abbreviation**  $UFilterP \ \Phi \equiv FilterP \ \Phi \wedge (\forall \varphi. \varphi \in \Phi \vee (^{-1}\varphi) \in \Phi)$   
  
**end**

## 4 Gödel's ontological argument – 1970 manuscript

### 4.1 GoedelVariantHOML1.thy (Figure 6 of [2])

Gödel's axioms and definitions, as presented in the 1970 manuscript, are inconsistent. Actualist quantifiers (avoiding existential import) are used for quantification over entities, otherwise possibilist quantifiers are used.

**theory** *GoedelVariantHOML1* **imports** *HOMLinHOL*  
**begin**

**consts**  $PositiveProperty::(e \Rightarrow \sigma) \Rightarrow \sigma$  ( $P$ )

**axiomatization where**  $Ax1: [P \ \varphi \wedge P \ \psi \supset P \ (\varphi \ . \ \psi)]$

**axiomatization where**  $Ax2a: [P \ \varphi \vee^e P \ \sim \varphi]$

**definition**  $God \ (G)$  **where**  $G \ x \equiv \forall \varphi. P \ \varphi \supset \varphi \ x$

**abbreviation**  $PropertyInclusion \ (-\supset_N-)$  **where**  $\varphi \supset_N \psi \equiv \Box(\forall^E y. \varphi \ y \supset \psi \ y)$

**definition**  $Essence \ (-Ess.-)$  **where**  $\varphi \ Ess. \ x \equiv \forall \psi. \psi \ x \supset (\varphi \supset_N \psi)$

**axiomatization where**  $Ax2b: [P \ \varphi \supset \Box P \ \varphi]$

**lemma**  $Ax2b'$ :  $[\neg P \ \varphi \supset \Box(\neg P \ \varphi)]$  *<proof>*

**theorem**  $Th1: [G \ x \supset G \ Ess. \ x]$  *<proof>*

**definition** *NecExist* (*E*) **where**  $E\ x \equiv \forall \varphi. \varphi\ Ess.\ x \supset \Box(\exists^E x. \varphi\ x)$

**axiomatization where** *Ax3*:  $[P\ E]$

**theorem** *Th2*:  $[G\ x \supset \Box(\exists^E y. G\ y)]$  *<proof>*

**theorem** *Th3*:  $[\Diamond(\exists^E x. G\ x) \supset \Box(\exists^E y. G\ y)]$   
— sledgehammer(*Th2* *Rsymm*) — Proof found  
*<proof>*

**axiomatization where** *Ax4*:  $[P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi]$

**lemma** *True nitpick*[*satisfy, expect=unknown*] *<proof>*

**lemma** *EmptyEssL*:  $[(\lambda y. \perp)\ Ess.\ x]$  *<proof>*

**theorem** *Inconsistency*: *False*  
— sledgehammer(*Ax2a* *Ax3* *Ax4* *EmptyEssL* *NecExist\_def*) — Proof found  
*<proof>*

**end**

## 4.2 GoedelVariantHOML2.thy (Figure 7 of [2])

After an appropriate modification of the definition of essence in Gödel's 1970 ontological proof, the inconsistency revealed in Figure 6 is avoided, and the argument can be successfully verified in modal logic S5 (indeed, as shown, only symmetry of the accessibility relation is actually needed). Actualist quantifiers (avoiding existential import) are used for quantification over entities, otherwise possibilist quantifiers are used.

**theory** *GoedelVariantHOML2* **imports** *HOMLinHOL* *ModalFilter*  
**begin**

**consts** *PositiveProperty*:: $(e \Rightarrow \sigma) \Rightarrow \sigma$  (*P*)

**axiomatization where** *Ax1*:  $[P\ \varphi \wedge P\ \psi \supset P\ (\varphi . \psi)]$

**axiomatization where** *Ax2a*:  $[P\ \varphi \vee^e P\ \sim\varphi]$

**definition** *God* (*G*) **where**  $G\ x \equiv \forall \varphi. P\ \varphi \supset \varphi\ x$

**abbreviation** *PropertyInclusion* ( $-\supset_N-$ ) **where**  $\varphi \supset_N \psi \equiv \Box(\forall^E y. \varphi\ y \supset \psi\ y)$

**definition** *Essence* ( $-Ess.-$ ) **where**  $\varphi\ Ess.\ x \equiv \varphi\ x \wedge (\forall \psi. \psi\ x \supset (\varphi \supset_N \psi))$

**axiomatization where** *Ax2b*:  $[P\ \varphi \supset \Box P\ \varphi]$

**lemma** *Ax2b'*:  $[\neg P \varphi \supset \Box(\neg P \varphi)]$   $\langle$ *proof* $\rangle$

**theorem** *Th1*:  $[G x \supset G \text{Ess. } x]$   $\langle$ *proof* $\rangle$

**definition** *NecExist* (*E*) **where**  $E x \equiv \forall \varphi. \varphi \text{Ess. } x \supset \Box(\exists^E x. \varphi x)$

**axiomatization where** *Ax3*:  $[P E]$

**theorem** *Th2*:  $[G x \supset \Box(\exists^E y. G y)]$   $\langle$ *proof* $\rangle$

**theorem** *Th3*:  $[\Diamond(\exists^E x. G x) \supset \Box(\exists^E y. G y)]$   
— sledgehammer(*Th2* *Rsymm*) — Proof found  
 $\langle$ *proof* $\rangle$

**axiomatization where** *Ax4*:  $[P \varphi \wedge (\varphi \supset_N \psi) \supset P \psi]$

**lemma** *True nitpick*[*satisfy,card=1,eval=[P (λx.⊥)]*]  $\langle$ *proof* $\rangle$

**abbreviation** *PosProps*  $\Phi \equiv \forall \varphi. \Phi \varphi \supset P \varphi$

**abbreviation** *ConjOfPropsFrom*  $\varphi \Phi \equiv \Box(\forall^E z. \varphi z \leftrightarrow (\forall \psi. \Phi \psi \supset \psi z))$

**axiomatization where** *Ax1Gen*:  $[(\text{PosProps } \Phi \wedge \text{ConjOfPropsFrom } \varphi \Phi) \supset P \varphi]$

**lemma** *L*:  $[P G]$   $\langle$ *proof* $\rangle$

**theorem** *Th4*:  $[\Diamond(\exists^E x. G x)]$   $\langle$ *proof* $\rangle$

**theorem** *Th5*:  $[\Box(\exists^E x. G x)]$   $\langle$ *proof* $\rangle$

**lemma** *MC*:  $[\varphi \supset \Box\varphi]$   
— sledgehammer(*Ax2a Ax2b Th5 God\_def Rsymm*) — Proof found  
 $\langle$ *proof* $\rangle$

**lemma** *PosProps*:  $[P (\lambda x. \top) \wedge P (\lambda x. x = x)]$   $\langle$ *proof* $\rangle$

**lemma** *NegProps*:  $[\neg P (\lambda x. \perp) \wedge \neg P (\lambda x. x \neq x)]$   $\langle$ *proof* $\rangle$

**lemma** *UniqueEss1*:  $[\varphi \text{Ess. } x \wedge \psi \text{Ess. } x \supset \Box(\forall^E y. \varphi y \leftrightarrow \psi y)]$   $\langle$ *proof* $\rangle$

**lemma** *UniqueEss2*:  $[\varphi \text{Ess. } x \wedge \psi \text{Ess. } x \supset \Box(\varphi \equiv \psi)]$  **nitpick**[*card i=1*]  $\langle$ *proof* $\rangle$

**lemma** *UniqueEss3*:  $[\varphi \text{Ess. } x \supset \Box(\forall^E y. \varphi y \supset y \equiv x)]$   $\langle$ *proof* $\rangle$

**lemma** *Monotheism*:  $[G x \wedge G y \supset x \equiv y]$   $\langle$ *proof* $\rangle$

**lemma** *Filter*:  $[Filter P]$   $\langle$ *proof* $\rangle$

**lemma** *UltraFilter*:  $[UltraFilter P]$   $\langle$ *proof* $\rangle$

**lemma** *True nitpick*[*satisfy,card=1,eval=[P (λx.⊥)]*]  $\langle$ *proof* $\rangle$

**end**

### 4.3 GoedelVariantHOML3.thy (Figure 8 of [2])

After an appropriate modification of the notion of necessary property inclusion in Gödel's 1970 ontological proof, the inconsistency revealed in Figure

6 is avoided, and the argument can be successfully verified in modal logic S5 (indeed, as shown, only symmetry of the accessibility relation is actually needed). Actualist quantifiers (avoiding existential import) are used for quantification over entities, otherwise possibilist quantifiers are used.

**theory** *GoedelVariantHOML3* **imports** *HOMLinHOL* *ModalFilter*  
**begin**

**consts** *PositiveProperty*:: $(e \Rightarrow \sigma) \Rightarrow \sigma$  (*P*)

**axiomatization where** *Ax1*:  $[P \varphi \wedge P \psi \supset P (\varphi \cdot \psi)]$

**axiomatization where** *Ax2a*:  $[P \varphi \vee^e P \sim \varphi]$

**definition** *God* (*G*) **where**  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$

**abbreviation** *PropertyInclusion* ( $-\supset_N$ ) **where**  $\varphi \supset_N \psi \equiv \Box(\varphi \neq (\lambda x. \perp) \wedge (\forall^E y. \varphi y \supset \psi y))$

**definition** *Essence* ( $-Ess.$ ) **where**  $\varphi Ess. x \equiv \forall \psi. \psi x \supset (\varphi \supset_N \psi)$

**axiomatization where** *Ax2b*:  $[P \varphi \supset \Box P \varphi]$

**lemma** *Ax2b'*:  $[\neg P \varphi \supset \Box(\neg P \varphi)]$  *<proof>*

**theorem** *Th1*:  $[G x \supset G Ess. x]$  *<proof>*

**definition** *NecExist* (*E*) **where**  $E x \equiv \forall \varphi. \varphi Ess. x \supset \Box(\exists^E x. \varphi x)$

**axiomatization where** *Ax3*:  $[P E]$

**theorem** *Th2*:  $[G x \supset \Box(\exists^E y. G y)]$  *<proof>*

**theorem** *Th3*:  $[\Diamond(\exists^E x. G x) \supset \Box(\exists^E y. G y)]$   
— sledgehammer(*Th2* *Rsymm*) — Proof found  
*<proof>*

**axiomatization where** *Ax4*:  $[P \varphi \wedge (\varphi \supset_N \psi) \supset P \psi]$

**lemma** *True nitpick*[*satisfy, card=1, eval=[P (\lambda x. \perp)]*] *<proof>*

**abbreviation** *PosProps*  $\Phi \equiv \forall \varphi. \Phi \varphi \supset P \varphi$

**abbreviation** *ConjOfPropsFrom*  $\varphi \Phi \equiv \Box(\forall^E z. \varphi z \leftrightarrow (\forall \psi. \Phi \psi \supset \psi z))$

**axiomatization where** *Ax1Gen*:  $[(PosProps \Phi \wedge ConjOfPropsFrom \varphi \Phi) \supset P \varphi]$

**lemma** *L*:  $[P G]$  *<proof>*

**theorem** *Th4*:  $[\Diamond(\exists^E x. G x)]$   
— sledgehammer[*timeout=200*](*Ax2a* *L* *Ax1Gen*) *<proof>*

**axiomatization where** *Th4*:  $[\Diamond(\exists^E x. G x)]$

**theorem** *Th5*:  $[\Box(\exists^E x. G x)]$  *<proof>*

**lemma** *MC*:  $[\varphi \supset \Box\varphi]$

— sledgehammer(Ax2a Ax2b Th5 God\_def Rsymm) — Proof found  
*<proof>*

**lemma** *PosProps*:  $[P(\lambda x. \top) \wedge P(\lambda x. x = x)]$  *<proof>*

**lemma** *NegProps*:  $[\neg P(\lambda x. \perp) \wedge \neg P(\lambda x. x \neq x)]$  *<proof>*

**lemma** *UniqueEss1*:  $[\varphi \text{ Ess. } x \wedge \psi \text{ Ess. } x \supset \Box(\forall^E y. \varphi y \leftrightarrow \psi y)]$  *<proof>*

**lemma** *UniqueEss2*:  $[\varphi \text{ Ess. } x \wedge \psi \text{ Ess. } x \supset \Box(\varphi \equiv \psi)]$  *<proof>*

**lemma** *UniqueEss3*:  $[\varphi \text{ Ess. } x \supset \Box(\forall^E y. \varphi y \supset y \equiv x)]$  *<proof>*

**lemma** *Monotheism*:  $[G x \wedge G y \supset x \equiv y]$  *<proof>*

**lemma** *Filter*:  $[Filter P]$  *<proof>*

**lemma** *UltraFilter*:  $[UltraFilter P]$  *<proof>*

**lemma** *True nitpick* $[satisfy, card=1, eval=[P(\lambda x. \top)]]$  *<proof>*

**end**

#### 4.4 ThereIsNoEvil1.thy (Figure 10 of [2])

Importing Gödel's modified axioms from Figure 7 we can prove that necessarily there exists no entity that possesses all non-positive (=negative) properties.

**theory** *ThereIsNoEvil1* **imports** *GoedelVariantHOML2*  
**begin**

**definition** *Evil* (*Evil*) **where**  $Evil\ x \equiv \forall \varphi. \neg P\ \varphi \supset \varphi\ x$

**theorem** *NecNoEvil*:  $[\Box(\neg(\exists^E x. Evil\ x))]$

— sledgehammer(Ax2a Ax4 Evil\_def) — Proof found  
*<proof>*

**end**

#### 4.5 ThereIsNoEvil2.thy (Figure 11 of [2])

Importing Gödel's modified axioms from Figure 8 we can prove that necessarily there exists no entity that possesses all non-positive (=negative) properties.

**theory** *ThereIsNoEvil2* **imports** *GoedelVariantHOML3*  
**begin**

**definition** *Evil* (*Evil*) **where**  $Evil\ x \equiv \forall \varphi. \neg P\ \varphi \supset \varphi\ x$

**theorem** *NecNoEvil*:  $[\Box(\neg(\exists^E x. Evil\ x))]$

— sledgehammer(Ax1Gen Ax2a Evil\_def) *<proof>*

end

## 5 Scott's variant

### 5.1 ScottVariantHOML.thy (Figure 12 of [2])

Verification of Scott's variant of Gödel's ontological argument. Actualist quantifiers (avoiding existential import) are used for quantification over entities, otherwise possibilist quantifiers are used.

**theory** *ScottVariantHOML* **imports** *HOMLinHOL ModalFilter*  
**begin**

**consts** *PositiveProperty*:: $(e \Rightarrow \sigma) \Rightarrow \sigma$  (*P*)

**axiomatization where** *A1*:  $\lceil \neg P \varphi \leftrightarrow P \sim \varphi \rceil$

**axiomatization where** *A2*:  $\lceil P \varphi \wedge \Box(\forall^E y. \varphi y \supset \psi y) \supset P \psi \rceil$

**theorem** *T1*:  $\lceil P \varphi \supset \Diamond(\exists^E x. \varphi x) \rceil$  *<proof>*

**definition** *God* (*G*) **where**  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$

**axiomatization where** *A3*:  $\lceil P G \rceil$

**theorem** *Coro*:  $\lceil \Diamond(\exists^E x. G x) \rceil$  *<proof>*

**axiomatization where** *A4*:  $\lceil P \varphi \supset \Box P \varphi \rceil$

**definition** *Ess* (*-Ess.*) **where**  $\varphi \text{Ess. } x \equiv \varphi x \wedge (\forall \psi. \psi x \supset \Box(\forall^E y. \varphi y \supset \psi y))$

**theorem** *T2*:  $\lceil G x \supset G \text{Ess. } x \rceil$  *<proof>*

**definition** *NecExist* (*NE*) **where**  $NE x \equiv \forall \varphi. \varphi \text{Ess. } x \supset \Box(\exists^E x. \varphi x)$

**axiomatization where** *A5*:  $\lceil P NE \rceil$

**lemma** *True nitpick*[*satisfy, card=1, eval=[P (λx. ⊤)]*] *<proof>*

**theorem** *T3*:  $\lceil \Box(\exists^E x. G x) \rceil$   
— *sledgehammer*(*A5 Coro God\_def NecExist\_def Rsymm T2*) — *Proof found*  
*<proof>*

**lemma** *MC*:  $\lceil \varphi \supset \Box \varphi \rceil$   
— *sledgehammer*(*A1 A4 God\_def Rsymm T3*) — *Proof found*  
*<proof>*

**lemma** *PosProps*:  $\lceil P (\lambda x. \top) \wedge P (\lambda x. x = x) \rceil$  *<proof>*

```

lemma NegProps: [ $\neg P(\lambda x. \perp) \wedge \neg P(\lambda x. x \neq x)$ ] <proof>
lemma UniqueEss1: [ $\varphi \text{ Ess. } x \wedge \psi \text{ Ess. } x \supset \Box(\forall^E y. \varphi y \leftrightarrow \psi y)$ ] <proof>
lemma UniqueEss2: [ $\varphi \text{ Ess. } x \wedge \psi \text{ Ess. } x \supset \Box(\varphi \equiv \psi)$ ] nitpick[card i=1] <proof>
lemma UniqueEss3: [ $\varphi \text{ Ess. } x \supset \Box(\forall^E y. \varphi y \supset y \equiv x)$ ] <proof>
lemma Monotheism: [ $G x \wedge G y \supset x \equiv y$ ] <proof>
lemma Filter: [Filter P] <proof>
lemma UltraFilter: [UFilter P] <proof>
lemma True nitpick[satisfy,card=1,eval=[P (\lambda x. \perp)]] <proof>

end

```

## 5.2 HOMLinHOLOnlyK.thy (slight variation of Figure 3 of [2])

Shallow embedding of higher-order modal logic (HOML) in the classical higher-order logic (HOL) of Isabelle/HOL utilizing the LogiKEy methodology. Here logic K is introduced.

```

theory HOMLinHOLOnlyK imports Main
begin

```

— Global parameters setting for the model finder nitpick and the parser; unimport for the reader

```

nitpick-params[user-axioms,expect=genuine,show-all,format=2,max-genuine=3]
declare[[syntax-ambiguity-warning=false]]

```

— Type *i* is associated with possible worlds and type *e* with entities

```

typedecl i — Possible worlds

```

```

typedecl e — Individuals/entities

```

```

type-synonym  $\sigma = i \Rightarrow \text{bool}$  — World-lifted propositions

```

```

type-synonym  $\tau = e \Rightarrow \sigma$  — modal properties

```

```

consts R::i⇒i⇒bool (-r-) — Accessibility relation between worlds

```

— Logical connectives (operating on truth-sets)

```

abbreviation Mbot::σ (⊥) where  $\perp \equiv \lambda w. \text{False}$ 

```

```

abbreviation Mtop::σ (⊤) where  $\top \equiv \lambda w. \text{True}$ 

```

```

abbreviation Mneg::σ⇒σ (¬- [52]53) where  $\neg\varphi \equiv \lambda w. \neg(\varphi w)$ 

```

```

abbreviation Mand::σ⇒σ⇒σ (infixl ∧ 50) where  $\varphi \wedge \psi \equiv \lambda w. \varphi w \wedge \psi w$ 

```

```

abbreviation Mor::σ⇒σ⇒σ (infixl ∨ 49) where  $\varphi \vee \psi \equiv \lambda w. \varphi w \vee \psi w$ 

```

```

abbreviation Mimp::σ⇒σ⇒σ (infixr ⊃ 48) where  $\varphi \supset \psi \equiv \lambda w. \varphi w \longrightarrow \psi w$ 

```

```

abbreviation Mequiv::σ⇒σ⇒σ (infixl ↔ 47) where  $\varphi \leftrightarrow \psi \equiv \lambda w. \varphi w \longleftrightarrow \psi w$ 

```

```

abbreviation Mbox::σ⇒σ (□- [54]55) where  $\Box\varphi \equiv \lambda w. \forall v. w \mathbf{r} v \longrightarrow \varphi v$ 

```

```

abbreviation Mdia::σ⇒σ (◇- [54]55) where  $\Diamond\varphi \equiv \lambda w. \exists v. w \mathbf{r} v \wedge \varphi v$ 

```

```

abbreviation Mprimeq::'a⇒'a⇒σ (-=) where  $x=y \equiv \lambda w. x=y$ 

```

```

abbreviation Mprimneg::'a⇒'a⇒σ (-≠) where  $x \neq y \equiv \lambda w. x \neq y$ 

```

```

abbreviation Mnegpred::τ⇒τ (¬~) where  $\sim\Phi \equiv \lambda x. \lambda w. \neg\Phi x w$ 

```

```

abbreviation Mconpred::τ⇒τ⇒τ (infixl . 50) where  $\Phi.\Psi \equiv \lambda x. \lambda w. \Phi x w \wedge \Psi$ 

```

$x w$

**abbreviation**  $Mexclor::\sigma\Rightarrow\sigma\Rightarrow\sigma$  (**infixl**  $\vee^e$  49) **where**  $\varphi\vee^e\psi\equiv(\varphi\vee\psi)\wedge\neg(\varphi\wedge\psi)$

— Possibilist quantifiers (polymorphic)

**abbreviation**  $Mallposs::('a\Rightarrow\sigma)\Rightarrow\sigma$  ( $\forall$ ) **where**  $\forall\Phi\equiv\lambda w.\forall x.\Phi x w$

**abbreviation**  $Mallpossb$  (**binder**  $\forall$  [8]9) **where**  $\forall x.\varphi(x)\equiv\forall\varphi$

**abbreviation**  $Mexiposs::('a\Rightarrow\sigma)\Rightarrow\sigma$  ( $\exists$ ) **where**  $\exists\Phi\equiv\lambda w.\exists x.\Phi x w$

**abbreviation**  $Mexipossb$  (**binder**  $\exists$  [8]9) **where**  $\exists x.\varphi(x)\equiv\exists\varphi$

— Actualist quantifiers (for individuals/entities)

**consts**  $existsAt::e\Rightarrow\sigma$  ( $-\textcircled{a}$ -)

**abbreviation**  $Mallact::(e\Rightarrow\sigma)\Rightarrow\sigma$  ( $\forall^E$ ) **where**  $\forall^E\Phi\equiv\lambda w.\forall x.x\textcircled{a}w\longrightarrow\Phi x w$

**abbreviation**  $Mallactb$  (**binder**  $\forall^E$  [8]9) **where**  $\forall^E x.\varphi(x)\equiv\forall^E\varphi$

**abbreviation**  $Mexiact::(e\Rightarrow\sigma)\Rightarrow\sigma$  ( $\exists^E$ ) **where**  $\exists^E\Phi\equiv\lambda w.\exists x.x\textcircled{a}w\wedge\Phi x w$

**abbreviation**  $Mexiactb$  (**binder**  $\exists^E$  [8]9) **where**  $\exists^E x.\varphi(x)\equiv\exists^E\varphi$

— Leibniz equality (polymorphic)

**abbreviation**  $Mleibeq::'a\Rightarrow'a\Rightarrow\sigma$  ( $-\equiv-$ ) **where**  $x\equiv y\equiv\forall P.P x \supset P y$

— Meta-logical predicate for global validity

**abbreviation**  $Mvalid::\sigma\Rightarrow bool$  ( $[-]$ ) **where**  $[\psi]\equiv\forall w.\psi w$

**end**

### 5.3 ScottVariantHOMLinK.thy (Figure 13 of [2])

Scott's variant of Gödel's argument fails for base logic K (but only it the last step).

**theory** *ScottVariantHOMLinK* **imports** *HOMLinHOLonlyK*

**begin**

**consts**  $PositiveProperty::(e\Rightarrow\sigma)\Rightarrow\sigma$  ( $P$ )

**axiomatization** **where**  $A1: [\neg P \varphi \leftrightarrow P \sim\varphi]$

**axiomatization** **where**  $A2: [P \varphi \wedge \Box(\forall^E y.\varphi y \supset \psi y) \supset P \psi]$

**theorem**  $T1: [P \varphi \supset \Diamond(\exists^E x.\varphi x)]$   $\langle proof \rangle$

**definition**  $God$  ( $G$ ) **where**  $G x \equiv \forall\varphi.P \varphi \supset \varphi x$

**axiomatization** **where**  $A3: [P G]$

**theorem**  $Coro: [\Diamond(\exists^E x.G x)]$  **nitpick**[*satisfy,eval=G*]  $\langle proof \rangle$

**axiomatization** **where**  $A4: [P \varphi \supset \Box P \varphi]$

**definition**  $Ess$  ( $-Ess.-$ ) **where**  $\varphi Ess. x \equiv \varphi x \wedge (\forall\psi.\psi x \supset \Box(\forall^E y.\varphi y \supset \psi$

y))

**theorem** *T2*:  $[G x \supset G \text{Ess. } x]$   $\langle \text{proof} \rangle$

**definition** *NecExist* (*NE*) **where**  $NE x \equiv \forall \varphi. \varphi \text{Ess. } x \supset \Box(\exists^E x. \varphi x)$

**axiomatization where** *A5*:  $[P \text{NE}]$

**lemma** *True nitpick* $[satisfy, card=1, eval=[P (\lambda x. \top)]]$   $\langle \text{proof} \rangle$

**theorem** *T3*:  $[\Box(\exists^E x. G x)]$  **nitpick** $[card e=1, card i=2, eval=G]$   $\langle \text{proof} \rangle$

**lemma** *MC*:  $[\varphi \supset \Box \varphi]$  **nitpick** $[card e=1, card i=2, eval=G]$   $\langle \text{proof} \rangle$

**end**

#### 5.4 ScottVariantHOMLAndersonQuant.thy (Figure 15 of [2])

Verification of Scott's variant of Gödel's argument with a mixed use of actualist and possibilist quantifiers for entities; cf. Footnote 20 in [2].

**theory** *ScottVariantHOMLAndersonQuant* **imports** *HOMLinHOL ModalFilter*  
**begin**

**consts** *PositiveProperty*:: $(e \Rightarrow \sigma) \Rightarrow \sigma$  (*P*)

**axiomatization where** *A1*:  $[\neg P \varphi \leftrightarrow P \sim \varphi]$

**axiomatization where** *A2*:  $[P \varphi \wedge \Box(\forall y. \varphi y \supset \psi y) \supset P \psi]$

**theorem** *T1*:  $[P \varphi \supset \Diamond(\exists x. \varphi x)]$   $\langle \text{proof} \rangle$

**definition** *God* (*G*) **where**  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$

**axiomatization where** *A3*:  $[P G]$

**theorem** *Coro*:  $[\Diamond(\exists x. G x)]$   $\langle \text{proof} \rangle$

**axiomatization where** *A4*:  $[P \varphi \supset \Box P \varphi]$

**definition** *Ess* (*-Ess.-*) **where**  $\varphi \text{Ess. } x \equiv \varphi x \wedge (\forall \psi. \psi x \supset \Box(\forall y::e. \varphi y \supset \psi y))$

**theorem** *T2*:  $[G x \supset G \text{Ess. } x]$   $\langle \text{proof} \rangle$

**definition** *NecExist* (*NE*) **where**  $NE x \equiv \forall \varphi. \varphi \text{Ess. } x \supset \Box(\exists^E x. \varphi x)$

**axiomatization where** *A5*:  $[P \text{NE}]$

**lemma** *True nitpick* $[satisfy, card=1, eval=[P (\lambda x. \top)]]$   $\langle \text{proof} \rangle$

**theorem** *T3*:  $[\Box(\exists^E x. G x)]$   
 — sledgehammer(A5 Coro God\_def NecExist\_def Rsymm T2) — Proof found  
 $\langle proof \rangle$

**lemma** *MC*:  $[\varphi \supset \Box\varphi]$   
 — sledgehammer(A1 A4 God\_def Rsymm T3) — Proof found  
 $\langle proof \rangle$

**lemma** *PosProps*:  $[P(\lambda x. \top) \wedge P(\lambda x. x = x)] \langle proof \rangle$

**lemma** *NegProps*:  $[\neg P(\lambda x. \perp) \wedge \neg P(\lambda x. x \neq x)] \langle proof \rangle$

**lemma** *UniqueEss1*:  $[\varphi \text{ Ess. } x \wedge \psi \text{ Ess. } x \supset \Box(\forall y. \varphi y \leftrightarrow \psi y)] \langle proof \rangle$

**lemma** *UniqueEss2*:  $[\varphi \text{ Ess. } x \wedge \psi \text{ Ess. } x \supset \Box(\varphi = \psi)]$  **nitpick**[*card i=2*]  $\langle proof \rangle$

**lemma** *UniqueEss3*:  $[\varphi \text{ Ess. } x \supset \Box(\forall y. \varphi y \supset y \equiv x)] \langle proof \rangle$

**lemma** *Monotheism*:  $[G x \wedge G y \supset x \equiv y] \langle proof \rangle$

**lemma** *Filter*:  $[Filter P] \langle proof \rangle$

**lemma** *UltraFilter*:  $[UltraFilter P] \langle proof \rangle$

**lemma** *True* **nitpick**[*satisfy, card=1, eval=[P(\lambda x. \perp)]*]  $\langle proof \rangle$

end

## 6 Appendix

### 6.1 GoedelVariantHOML1poss.thy (Figure 16 of [2])

Gödel's axioms and definitions, as presented in the 1970 manuscript, are inconsistent. In contrast to Figure 6 we here use only possibilist quantifiers and still derive falsity.

**theory** *GoedelVariantHOML1poss* **imports** *HOMLinHOL*  
**begin**

**consts** *PositiveProperty*:  $(e \Rightarrow \sigma) \Rightarrow \sigma$  (*P*)

**axiomatization** **where** *Ax1*:  $[P \varphi \wedge P \psi \supset P(\varphi \cdot \psi)]$

**axiomatization** **where** *Ax2a*:  $[P \varphi \vee^e P \sim \varphi]$

**definition** *God* (*G*) **where**  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$

**abbreviation** *PropertyInclusion* ( $-\supset_N-$ ) **where**  $\varphi \supset_N \psi \equiv \Box(\forall y::e. \varphi y \supset \psi y)$

**definition** *Essence* ( $-Ess.-$ ) **where**  $\varphi \text{ Ess. } x \equiv \forall \psi. \psi x \supset (\varphi \supset_N \psi)$

**axiomatization** **where** *Ax2b*:  $[P \varphi \supset \Box P \varphi]$

**lemma** *Ax2b'*:  $[\neg P \varphi \supset \Box(\neg P \varphi)] \langle proof \rangle$

**theorem** *Th1*:  $[G x \supset G \text{ Ess. } x] \langle proof \rangle$

**definition** *NecExist* (*E*) **where**  $E\ x \equiv \forall \varphi. \varphi\ \text{Ess.}\ x \supset \Box(\exists x. \varphi\ x)$

**axiomatization where** *Ax3*:  $[P\ E]$

**theorem** *Th2*:  $[G\ x \supset \Box(\exists y. G\ y)]$  *<proof>*

**theorem** *Th3*:  $[\Diamond(\exists x. G\ x) \supset \Box(\exists y. G\ y)]$   
— sledgehammer(*Th2* *Rsymm*) — Proof found  
*<proof>*

**axiomatization where** *Ax4*:  $[P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi]$

**lemma** *True nitpick*[*satisfy, expect=unknown*] *<proof>*

**lemma** *EmptyEssL*:  $[(\lambda y. \perp)\ \text{Ess.}\ x]$  *<proof>*

**theorem** *Inconsistency*: *False*  
— sledgehammer(*Ax2a* *Ax3* *Ax4* *EmptyEssL* *NecExist\_def*) — Proof found  
*<proof>*

**end**

## 6.2 GoedelVariantHOML2poss.thy (Figure 17 of [2])

After an appropriate modification of the definition of essence, the inconsistency revealed in Figure 16 is avoided, and the argument can be successfully verified in modal logic S5 (indeed, as shown, only the modal schema B is actually needed). In contrast to Figure 7 we here use only possibilist quantifiers to obtain these results.

**theory** *GoedelVariantHOML2poss* **imports** *HOMLinHOL* *ModalFilter*  
**begin**

**consts** *PositiveProperty*:: $(e \Rightarrow \sigma) \Rightarrow \sigma$  (*P*)

**axiomatization where** *Ax1*:  $[P\ \varphi \wedge P\ \psi \supset P\ (\varphi \cdot \psi)]$

**axiomatization where** *Ax2a*:  $[P\ \varphi \vee^e P\ \sim\varphi]$

**definition** *God* (*G*) **where**  $G\ x \equiv \forall \varphi. P\ \varphi \supset \varphi\ x$

**abbreviation** *PropertyInclusion* ( $-\supset_N-$ ) **where**  $\varphi \supset_N \psi \equiv \Box(\forall y::e. \varphi\ y \supset \psi\ y)$

**definition** *Essence* ( $-\text{Ess.}-$ ) **where**  $\varphi\ \text{Ess.}\ x \equiv \varphi\ x \wedge (\forall \psi. \psi\ x \supset (\varphi \supset_N \psi))$

**axiomatization where** *Ax2b*:  $[P\ \varphi \supset \Box P\ \varphi]$

**lemma** *Ax2b'*:  $[\neg P\ \varphi \supset \Box(\neg P\ \varphi)]$  *<proof>*

**theorem** *Th1*:  $[G\ x \supset G\ \text{Ess.}\ x]$   $\langle\text{proof}\rangle$

**definition** *NecExist* (*E*) **where**  $E\ x \equiv \forall \varphi. \varphi\ \text{Ess.}\ x \supset \Box(\exists x. \varphi\ x)$

**axiomatization** **where** *Ax3*:  $[P\ E]$

**theorem** *Th2*:  $[G\ x \supset \Box(\exists y. G\ y)]$   $\langle\text{proof}\rangle$

**theorem** *Th3*:  $[\Diamond(\exists x. G\ x) \supset \Box(\exists y. G\ y)]$   
— sledgehammer(*Th2* *Rsymm*) — Proof found  
 $\langle\text{proof}\rangle$

**axiomatization** **where** *Ax4*:  $[P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi]$

**lemma** *True nitpick* $[\text{satisfy}, \text{card}=1, \text{eval}=[P\ (\lambda x. \perp)]]$   $\langle\text{proof}\rangle$

**abbreviation** *PosProps*  $\Phi \equiv \forall \varphi. \Phi\ \varphi \supset P\ \varphi$

**abbreviation** *ConjOfPropsFrom*  $\varphi\ \Phi \equiv \Box(\forall z. \varphi\ z \leftrightarrow (\forall \psi. \Phi\ \psi \supset \psi\ z))$

**axiomatization** **where** *Ax1Gen*:  $[(\text{PosProps}\ \Phi \wedge \text{ConjOfPropsFrom}\ \varphi\ \Phi) \supset P\ \varphi]$

**lemma** *L*:  $[P\ G]$   $\langle\text{proof}\rangle$

**theorem** *Th4*:  $[\Diamond(\exists x. G\ x)]$   $\langle\text{proof}\rangle$

**theorem** *Th5*:  $[\Box(\exists x. G\ x)]$   $\langle\text{proof}\rangle$

**lemma** *MC*:  $[\varphi \supset \Box\varphi]$   
— sledgehammer(*Ax2a* *Ax2b* *Th5* *God\_def* *Rsymm*) — Proof found  
 $\langle\text{proof}\rangle$

**lemma** *PosProps*:  $[P\ (\lambda x. \top) \wedge P\ (\lambda x. x = x)]$   $\langle\text{proof}\rangle$

**lemma** *NegProps*:  $[\neg P(\lambda x. \perp) \wedge \neg P(\lambda x. x \neq x)]$   $\langle\text{proof}\rangle$

**lemma** *UniqueEss1*:  $[\varphi\ \text{Ess.}\ x \wedge \psi\ \text{Ess.}\ x \supset \Box(\forall y. \varphi\ y \leftrightarrow \psi\ y)]$   $\langle\text{proof}\rangle$

**lemma** *UniqueEss2*:  $[\varphi\ \text{Ess.}\ x \wedge \psi\ \text{Ess.}\ x \supset \Box(\varphi \equiv \psi)]$  **nitpick** $[\text{card}\ i=2]$   $\langle\text{proof}\rangle$

**lemma** *UniqueEss3*:  $[\varphi\ \text{Ess.}\ x \supset \Box(\forall y. \varphi\ y \supset y \equiv x)]$   $\langle\text{proof}\rangle$

**lemma** *Monotheism*:  $[G\ x \wedge G\ y \supset x \equiv y]$   $\langle\text{proof}\rangle$

**lemma** *Filter*:  $[FilterP\ P]$   $\langle\text{proof}\rangle$

**lemma** *UltraFilter*:  $[UltraFilterP\ P]$   $\langle\text{proof}\rangle$

**lemma** *True nitpick* $[\text{satisfy}, \text{card}=1, \text{eval}=[P\ (\lambda x. \perp)]]$   $\langle\text{proof}\rangle$

**end**

### 6.3 GoedelVariantHOML3poss.thy (Figure 18 of [2])

After an appropriate modification of the definition of necessary property implication, the inconsistency shown in Figure 16 is avoided, and the argument can be successfully verified. As shown here, this still holds when using

possibilist quantifiers only.

**theory** *GoedelVariantHOML3poss* **imports** *HOMLinHOL ModalFilter*  
**begin**

**consts** *PositiveProperty*:  $(e \Rightarrow \sigma) \Rightarrow \sigma$  (*P*)

**axiomatization where** *Ax1*:  $[P \varphi \wedge P \psi \supset P (\varphi \cdot \psi)]$

**axiomatization where** *Ax2a*:  $[P \varphi \vee^e P \sim \varphi]$

**definition** *God* (*G*) **where**  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$

**abbreviation** *PropertyInclusion* ( $-\supset_N$ -) **where**  $\varphi \supset_N \psi \equiv \Box(\varphi \neq (\lambda x. \perp) \wedge (\forall y::e. \varphi y \supset \psi y))$

**definition** *Essence* ( $-Ess.$ -) **where**  $\varphi Ess. x \equiv \forall \psi. \psi x \supset (\varphi \supset_N \psi)$

**axiomatization where** *Ax2b*:  $[P \varphi \supset \Box P \varphi]$

**lemma** *Ax2b'*:  $[\neg P \varphi \supset \Box(\neg P \varphi)]$  *<proof>*

**theorem** *Th1*:  $[G x \supset G Ess. x]$  *<proof>*

**definition** *NecExist* (*E*) **where**  $E x \equiv \forall \varphi. (\varphi Ess. x) \supset \Box(\exists x. \varphi x)$

**axiomatization where** *Ax3*:  $[P E]$

**theorem** *Th2*:  $[G x \supset \Box(\exists y. G y)]$  *<proof>*

**theorem** *Th3*:  $[\Diamond(\exists x. G x) \supset \Box(\exists y. G y)]$   
— sledgehammer(*Th2* *Rsymm*) — *Proof found*  
*<proof>*

**axiomatization where** *Ax4*:  $[P \varphi \wedge (\varphi \supset_N \psi) \supset P \psi]$

**lemma** *True nitpick*[*satisfy, card=1, eval=[P (\lambda x. \perp)]*] *<proof>*

**abbreviation** *PosProps*  $\Phi \equiv \forall \varphi. \Phi \varphi \supset P \varphi$

**abbreviation** *ConjOfPropsFrom*  $\varphi \Phi \equiv \Box(\forall z. \varphi z \leftrightarrow (\forall \psi. \Phi \psi \supset \psi z))$

**axiomatization where** *Ax1Gen*:  $[(PosProps \Phi \wedge ConjOfPropsFrom \varphi \Phi) \supset P \varphi]$

**lemma** *L*:  $[P G]$  *<proof>*

**theorem** *Th4*:  $[\Diamond(\exists x. G x)]$   
— sledgehammer[*timeout=200*](*Ax2a* *L* *Ax1Gen*) *<proof>*

**axiomatization where** *Th4*:  $[\Diamond(\exists x. G x)]$

**theorem** *Th5*:  $[\Box(\exists x. G x)]$   $\langle proof \rangle$

**lemma** *MC*:  $[\varphi \supset \Box\varphi]$

— sledgehammer(Ax2a Ax2b Th5 God\_def Rsymm) — Proof found  
 $\langle proof \rangle$

**lemma** *PosProps*:  $[P(\lambda x. \top) \wedge P(\lambda x. x = x)]$   $\langle proof \rangle$

**lemma** *NegProps*:  $[\neg P(\lambda x. \perp) \wedge \neg P(\lambda x. x \neq x)]$   $\langle proof \rangle$

**lemma** *UniqueEss1*:  $[\varphi \text{ Ess. } x \wedge \psi \text{ Ess. } x \supset \Box(\forall y. \varphi y \leftrightarrow \psi y)]$   $\langle proof \rangle$

**lemma** *UniqueEss2*:  $[\varphi \text{ Ess. } x \wedge \psi \text{ Ess. } x \supset \Box(\varphi \equiv \psi)]$   $\langle proof \rangle$

**lemma** *UniqueEss3*:  $[\varphi \text{ Ess. } x \supset \Box(\forall y. \varphi y \supset y \equiv x)]$   $\langle proof \rangle$

**lemma** *Monotheism*:  $[G x \wedge G y \supset x \equiv y]$   $\langle proof \rangle$

**lemma** *Filter*:  $[FilterP P]$   $\langle proof \rangle$

**lemma** *UltraFilter*:  $[UltraFilterP P]$   $\langle proof \rangle$

**lemma** *True nitpick* $[satisfy, card=1, eval=[P(\lambda x. \top)]]$   $\langle proof \rangle$

**end**

## 6.4 ScottVariantHOMLposs.thy (Figure 19 of [2])

Scott's variant of Gödel's ontological proof is still valid when using possibilist quantifiers only.

**theory** *ScottVariantHOMLposs* **imports** *HOMLinHOL ModalFilter*  
**begin**

**consts** *PositiveProperty*:: $(e \Rightarrow \sigma) \Rightarrow \sigma$  (*P*)

**axiomatization** **where** *A1*:  $[\neg P \varphi \leftrightarrow P \sim \varphi]$

**axiomatization** **where** *A2*:  $[P \varphi \wedge \Box(\forall y. \varphi y \supset \psi y) \supset P \psi]$

**theorem** *T1*:  $[P \varphi \supset \Diamond(\exists x. \varphi x)]$   $\langle proof \rangle$

**definition** *God* (*G*) **where**  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$

**axiomatization** **where** *A3*:  $[P G]$

**theorem** *Coro*:  $[\Diamond(\exists x. G x)]$   $\langle proof \rangle$

**axiomatization** **where** *A4*:  $[P \varphi \supset \Box P \varphi]$

**definition** *Ess* (*-Ess.*) **where**  $\varphi \text{ Ess. } x \equiv \varphi x \wedge (\forall \psi. \psi x \supset \Box(\forall y::e. \varphi y \supset \psi y))$

**theorem** *T2*:  $[G x \supset G \text{ Ess. } x]$   $\langle proof \rangle$

**definition** *NecExist* (*NE*) **where**  $NE x \equiv \forall \varphi. \varphi \text{ Ess. } x \supset \Box(\exists x. \varphi x)$

**axiomatization** **where** *A5*:  $[P NE]$

**lemma** *True nitpick*[*satisfy,card=1,eval=[P (λx.⊥)]*] *<proof>*

**theorem** *T3*:  $[\Box(\exists x. G x)]$

— sledgehammer(A5 Coro God\_def NecExist\_def Rsymm T2) — Proof found  
*<proof>*

**lemma** *MC*:  $[\varphi \supset \Box\varphi]$

— sledgehammer(A1 A4 God\_def Rsymm T3) — Proof found  
*<proof>*

**lemma** *PosProps*:  $[P (\lambda x. \top) \wedge P (\lambda x. x = x)]$  *<proof>*

**lemma** *NegProps*:  $[\neg P(\lambda x. \perp) \wedge \neg P(\lambda x. x \neq x)]$  *<proof>*

**lemma** *UniqueEss1*:  $[\varphi \text{ Ess. } x \wedge \psi \text{ Ess. } x \supset \Box(\forall y. \varphi y \leftrightarrow \psi y)]$  *<proof>*

**lemma** *UniqueEss2*:  $[\varphi \text{ Ess. } x \wedge \psi \text{ Ess. } x \supset \Box(\varphi \equiv \psi)]$  **nitpick**[*card i=2*] *<proof>*

**lemma** *UniqueEss3*:  $[\varphi \text{ Ess. } x \supset \Box(\forall y. \varphi y \supset y \equiv x)]$  *<proof>*

**lemma** *Monotheism*:  $[G x \wedge G y \supset x \equiv y]$  *<proof>*

**lemma** *Filter*:  $[FilterP P]$  *<proof>*

**lemma** *UltraFilter*:  $[UltraFilterP P]$  *<proof>*

**lemma** *True nitpick*[*satisfy,card=1,eval=[P (λx.⊥)]*] *<proof>*

end

## 6.5 EvilDerivable.thy (Figure 20 of [2])

The necessary existence of an Evil-like entity proved from (controversially) modified assumptions. By rejecting Gödel’s assumptions and instead postulating corresponding negative versions of them, as shown in the Figure 20, the necessary existence of Evil becomes derivable. The non-positive properties of this Evil-like entity are however identical to the positive properties of Gödel’s God-like entity.

**theory** *EvilDerivable* **imports** *HOMLinHOL ModalFilter*  
**begin**

**consts** *PositiveProperty*:: $(e \Rightarrow \sigma) \Rightarrow \sigma$  (*P*)

**definition** *Evil* (*Evil*) **where**  $Evil\ x \equiv \forall \varphi. \neg P\ \varphi \supset \varphi\ x$

**definition** *Essence* (*-Ess.-*) **where**  $\varphi\ \text{Ess. } x \equiv \varphi\ x \wedge (\forall \psi. \psi\ x \supset \Box(\forall^E y. \varphi\ y \supset \psi\ y))$

**definition** *NecExist* (*E*) **where**  $E\ x \equiv \forall \varphi. \varphi\ \text{Ess. } x \supset \Box(\exists^E x. \varphi\ x)$

**axiomatization** **where** *A1*:  $[\neg P\ \varphi \leftrightarrow P\ \sim\varphi]$

**axiomatization** **where** *A2*:  $[\neg P\ \varphi \wedge \Box(\forall^E y. \varphi\ y \supset \psi\ y) \supset \neg P\ \psi]$

**axiomatization** **where** *A4*:  $[\neg P\ Evil]$

**axiomatization where**  $A3: [\neg P \varphi \supset \Box (\neg P \varphi)]$

**axiomatization where**  $A5: [\neg P E]$

**lemma** *True nitpick*[*satisfy, card i=1, eval=[P (\lambda x. \perp)], eval=[P (\lambda x. \top)]*] *<proof>*

**theorem**  $T1: [\neg P \varphi \supset \Diamond (\exists^E x. \varphi x)]$  *<proof>*

**theorem**  $T2: [\Diamond (\exists^E x. Evil x)]$  *<proof>*

**theorem**  $T3: [Evil x \supset Evil Ess. x]$  *<proof>*

**theorem**  $T4: [\Diamond (\exists^E x. Evil x) \supset \Box (\exists^E y. Evil y)]$  *<proof>*

**theorem**  $T5: [\Box (\exists^E x. Evil x)]$  *<proof>*

**lemma** *MC*:  $[\varphi \supset \Box \varphi]$

— *sledgehammer*(A1 A3 T5 Evil\_def Rsymm) *<proof>*

**lemma** *PosProps*:  $[P (\lambda x. \perp) \wedge P (\lambda x. x \neq x)]$  *<proof>*

**lemma** *NegProps*:  $[\neg P (\lambda x. \top) \wedge \neg P (\lambda x. x = x)]$  *<proof>*

**lemma** *UniqueEss1*:  $[\varphi Ess. x \wedge \psi Ess. x \supset \Box (\forall^E y. \varphi y \leftrightarrow \psi y)]$  *<proof>*

**lemma** *UniqueEss2*:  $[\varphi Ess. x \wedge \psi Ess. x \supset \Box (\varphi = \psi)]$  **nitpick**[*card i=2*] *<proof>*

**lemma** *Monoevilism*:  $[Evil x \wedge Evil y \supset x \equiv y]$  *<proof>*

**lemma** *Filter*:  $[Filter (\lambda \varphi. \neg P \varphi)]$  *<proof>*

**lemma** *UltraFilter*:  $[UltraFilter (\lambda \varphi. \neg P \varphi)]$  *<proof>*

end

## 7 Further Appendices

### 7.1 GoedelVariantHOML1AndersonQuan.thy

The same as GoedelVariantHOML1, but now for a mixed use of actualist and possibilist quantifiers for entities; cf. Footnote 20 in [2].

**theory** *GoedelVariantHOML1AndersonQuant* **imports** *HOMLinHOL*  
**begin**

**consts** *PositiveProperty*:: $(e \Rightarrow \sigma) \Rightarrow \sigma$  (*P*)

**axiomatization where**  $Ax1: [P \varphi \wedge P \psi \supset P (\varphi . \psi)]$

**axiomatization where**  $Ax2a: [P \varphi \vee^e P \sim \varphi]$

**definition** *God* (*G*) **where**  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$

**abbreviation** *PropertyInclusion* ( $-\supset_N$ -) **where**  $\varphi \supset_N \psi \equiv \Box (\forall y::e. \varphi y \supset \psi y)$

**definition** *Essence* (-Ess.-) **where**  $\varphi \text{ Ess. } x \equiv \forall \psi. \psi x \supset (\varphi \supset_N \psi)$

**axiomatization where** *Ax2b*:  $[P \varphi \supset \Box P \varphi]$

**lemma** *Ax2b'*:  $[\neg P \varphi \supset \Box(\neg P \varphi)]$  *<proof>*

**theorem** *Th1*:  $[G x \supset G \text{ Ess. } x]$  *<proof>*

**definition** *NecExist* (*E*) **where**  $E x \equiv \forall \varphi. \varphi \text{ Ess. } x \supset \Box(\exists^E x. \varphi x)$

**axiomatization where** *Ax3*:  $[P E]$

**theorem** *Th2*:  $[G x \supset \Box(\exists^E y. G y)]$  *<proof>*

**theorem** *Th3*:  $[\Diamond(\exists^E x. G x) \supset \Box(\exists^E y. G y)]$   
— sledgehammer(*Th2 Rsymm*) — Proof found  
*<proof>*

**axiomatization where** *Ax4*:  $[P \varphi \wedge (\varphi \supset_N \psi) \supset P \psi]$

**lemma** *True nitpick*[*satisfy, expect=unknown*] *<proof>*

**lemma** *EmptyEssL*:  $[(\lambda y. \perp) \text{ Ess. } x]$  *<proof>*

**theorem** *Inconsistency: False*  
— sledgehammer(*Ax2a Ax3 Ax4 EmptyEssL NecExist\_def*) — Proof found  
*<proof>*

**end**

## 7.2 GoedelVariantHOML2AndersonQuan.thy

The same as GoedelVariantHOML2, but now for a mixed use of actualist and possibilist quantifiers for entities; cf. Footnote 20 in [2].

**theory** *GoedelVariantHOML2AndersonQuant* **imports** *HOMLinHOL ModalFilter*  
**begin**

**consts** *PositiveProperty*:  $(e \Rightarrow \sigma) \Rightarrow \sigma$  (*P*)

**axiomatization where** *Ax1*:  $[P \varphi \wedge P \psi \supset P (\varphi \cdot \psi)]$

**axiomatization where** *Ax2a*:  $[P \varphi \vee^e P \sim \varphi]$

**definition** *God* (*G*) **where**  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$

**abbreviation** *PropertyInclusion* ( $-\supset_N-$ ) **where**  $\varphi \supset_N \psi \equiv \Box(\forall y::e. \varphi y \supset \psi y)$

**definition** *Essence* (-Ess.-) **where**  $\varphi \text{ Ess. } x \equiv \varphi x \wedge (\forall \psi. \psi x \supset (\varphi \supset_N \psi))$

**axiomatization where**  $Ax2b$ :  $[P \varphi \supset \Box P \varphi]$

**lemma**  $Ax2b'$ :  $[\neg P \varphi \supset \Box(\neg P \varphi)]$   $\langle proof \rangle$

**theorem**  $Th1$ :  $[G x \supset G \text{Ess. } x]$   $\langle proof \rangle$

**definition**  $NecExist (E)$  **where**  $E x \equiv \forall \varphi. \varphi \text{Ess. } x \supset \Box(\exists^E x. \varphi x)$

**axiomatization where**  $Ax3$ :  $[P E]$

**theorem**  $Th2$ :  $[G x \supset \Box(\exists^E y. G y)]$   $\langle proof \rangle$

**theorem**  $Th3$ :  $[\Diamond(\exists^E x. G x) \supset \Box(\exists^E y. G y)]$   
— sledgehammer( $Th2$   $Rsymm$ ) — Proof found  
 $\langle proof \rangle$

**axiomatization where**  $Ax4$ :  $[P \varphi \wedge (\varphi \supset_N \psi) \supset P \psi]$

**lemma**  $True$  **nitpick** $[satisfy, card=1, eval=[P (\lambda x. \perp)]]$   $\langle proof \rangle$

**abbreviation**  $PosProps \Phi \equiv \forall \varphi. \Phi \varphi \supset P \varphi$

**abbreviation**  $ConjOfPropsFrom \varphi \Phi \equiv \Box(\forall z. \varphi z \leftrightarrow (\forall \psi. \Phi \psi \supset \psi z))$

**axiomatization where**  $Ax1Gen$ :  $[(PosProps \Phi \wedge ConjOfPropsFrom \varphi \Phi) \supset P \varphi]$

**lemma**  $L$ :  $[P G]$   $\langle proof \rangle$

**theorem**  $Th4$ :  $[\Diamond(\exists^E x. G x)]$   $\langle proof \rangle$

**theorem**  $Th5$ :  $[\Box(\exists^E x. G x)]$   $\langle proof \rangle$

**lemma**  $MC$ :  $[\varphi \supset \Box \varphi]$   
— sledgehammer( $Ax2a$   $Ax2b$   $Th5$   $God\_def$   $Rsymm$ ) — proof found  
 $\langle proof \rangle$

**lemma**  $PosProps$ :  $[P (\lambda x. \top) \wedge P (\lambda x. x = x)]$   $\langle proof \rangle$

**lemma**  $NegProps$ :  $[\neg P (\lambda x. \perp) \wedge \neg P (\lambda x. x \neq x)]$   $\langle proof \rangle$

**lemma**  $UniqueEss1$ :  $[\varphi \text{Ess. } x \wedge \psi \text{Ess. } x \supset \Box(\forall y. \varphi y \leftrightarrow \psi y)]$   $\langle proof \rangle$

**lemma**  $UniqueEss2$ :  $[\varphi \text{Ess. } x \wedge \psi \text{Ess. } x \supset \Box(\varphi \equiv \psi)]$  **nitpick** $[card i=2]$   $\langle proof \rangle$

**lemma**  $UniqueEss3$ :  $[\varphi \text{Ess. } x \supset \Box(\forall y. \varphi y \supset y \equiv x)]$   $\langle proof \rangle$

**lemma**  $Monotheism$ :  $[G x \wedge G y \supset x \equiv y]$   $\langle proof \rangle$

**lemma**  $Filter$ :  $[Filter P]$   $\langle proof \rangle$

**lemma**  $UltraFilter$ :  $[UltraFilter P]$   $\langle proof \rangle$

**lemma**  $True$  **nitpick** $[satisfy, card=1, eval=[P (\lambda x. \perp)]]$   $\langle proof \rangle$

**end**

### 7.3 GoedelVariantHOML3AndersonQuan.thy

The same as GoedelVariantHOML3, but now for a mixed use of actualist and possibilist quantifiers for entities; cf. Footnote 20 in [2].

**theory** *GoedelVariantHOML3AndersonQuant* **imports** *HOMLinHOL ModalFilter*  
**begin**

**consts** *PositiveProperty*:  $(e \Rightarrow \sigma) \Rightarrow \sigma$  (*P*)

**axiomatization where** *Ax1*:  $[P \varphi \wedge P \psi \supset P (\varphi \cdot \psi)]$

**axiomatization where** *Ax2a*:  $[P \varphi \vee^e P \sim \varphi]$

**definition** *God* (*G*) **where**  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$

**abbreviation** *PropertyInclusion* ( $-\supset_N$ -) **where**  $\varphi \supset_N \psi \equiv \Box(\varphi \neq (\lambda x. \perp) \wedge (\forall y::e. \varphi y \supset \psi y))$

**definition** *Essence* ( $-Ess.$ -) **where**  $\varphi Ess. x \equiv \forall \psi. \psi x \supset (\varphi \supset_N \psi)$

**axiomatization where** *Ax2b*:  $[P \varphi \supset \Box P \varphi]$

**lemma** *Ax2b'*:  $[\neg P \varphi \supset \Box(\neg P \varphi)]$  *<proof>*

**theorem** *Th1*:  $[G x \supset G Ess. x]$  *<proof>*

**definition** *NecExist* (*E*) **where**  $E x \equiv \forall \varphi. \varphi Ess. x \supset \Box(\exists^E x. \varphi x)$

**axiomatization where** *Ax3*:  $[P E]$

**theorem** *Th2*:  $[G x \supset \Box(\exists^E y. G y)]$  *<proof>*

**theorem** *Th3*:  $[\Diamond(\exists^E x. G x) \supset \Box(\exists^E y. G y)]$   
— sledgehammer(Th2 Rsymm) — Proof found  
*<proof>*

**axiomatization where** *Ax4*:  $[P \varphi \wedge (\varphi \supset_N \psi) \supset P \psi]$

**lemma** *True nitpick*[*satisfy, card=1, eval=[P (\lambda x. \perp)]*] *<proof>*

**abbreviation** *PosProps*  $\Phi \equiv \forall \varphi. \Phi \varphi \supset P \varphi$

**abbreviation** *ConjOfPropsFrom*  $\varphi \Phi \equiv \Box(\forall^E z. \varphi z \leftrightarrow (\forall \psi. \Phi \psi \supset \psi z))$

**axiomatization where** *Ax1Gen*:  $[(PosProps \Phi \wedge ConjOfPropsFrom \varphi \Phi) \supset P \varphi]$

**lemma** *L*:  $[P G]$  *<proof>*

**theorem** *Th4*:  $[\Diamond(\exists^E x. G x)]$   
— sledgehammer[timeout=200](Ax2a L Ax1Gen) *<proof>*

**axiomatization where** *Th4*:  $[\Diamond(\exists^E x. G x)]$

**theorem** *Th5*:  $[\Box(\exists^E x. G x)]$  *<proof>*

**lemma** *MC*:  $[\varphi \supset \Box\varphi]$

— sledgehammer(Ax2a Ax2b Th5 God\_def Rsymm) — Proof found  
*<proof>*

**lemma** *PosProps*:  $[P(\lambda x. \top) \wedge P(\lambda x. x = x)]$  *<proof>*

**lemma** *NegProps*:  $[\neg P(\lambda x. \perp) \wedge \neg P(\lambda x. x \neq x)]$  *<proof>*

**lemma** *UniqueEss1*:  $[\varphi \text{ Ess. } x \wedge \psi \text{ Ess. } x \supset \Box(\forall y. \varphi y \leftrightarrow \psi y)]$  *<proof>*

**lemma** *UniqueEss2*:  $[\varphi \text{ Ess. } x \wedge \psi \text{ Ess. } x \supset \Box(\varphi \equiv \psi)]$  *<proof>*

**lemma** *UniqueEss3*:  $[\varphi \text{ Ess. } x \supset \Box(\forall y. \varphi y \supset y \equiv x)]$  *<proof>*

**lemma** *Monotheism*:  $[G x \wedge G y \supset x \equiv y]$  *<proof>*

**lemma** *Filter*:  $[Filter P]$  *<proof>*

**lemma** *UltraFilter*:  $[UltraFilter P]$  *<proof>*

**lemma** *True nitpick* $[satisfy, card=1, eval=[P(\lambda x. \top)]]$  *<proof>*

**end**

## 7.4 HOMLinHOLonlyS4.thy (slight variation of Figure 3 of [2])

Shallow embedding of higher-order modal logic (HOML) in the classical higher-order logic (HOL) of Isabelle/HOL utilizing the LogiKEY methodology. Here logic S4 is introduced.

**theory** *HOMLinHOLonlyS4* **imports** *Main*  
**begin**

— Global parameters setting for the model finder nitpick and the parser; unimport for the reader

**nitpick-params** $[user-axioms, expect=genuine, show-all, format=2, max-genuine=3]$   
**declare** $[[syntax-ambiguity-warning=false]]$

— Type *i* is associated with possible worlds and type *e* with entities:

**typedecl** *i* — Possible worlds

**typedecl** *e* — Individuals/entities

**type-synonym**  $\sigma = i \Rightarrow bool$  — World-lifted propositions

**type-synonym**  $\tau = e \Rightarrow \sigma$  — modal properties

**consts** *R*:  $i \Rightarrow i \Rightarrow bool$  (**-r-**) — Accessibility relation between worlds

**axiomatization where**

*Rrefl*:  $\forall x. \mathit{rx} \text{ and}$

*Rtrans*:  $\forall x y z. \mathit{xy} \wedge \mathit{yz} \longrightarrow \mathit{xz}$

— Logical connectives (operating on truth-sets)

**abbreviation** *Mbot*:  $\sigma$  ( $\perp$ ) **where**  $\perp \equiv \lambda w. False$

**abbreviation**  $Mtop::\sigma (\top)$  **where**  $\top \equiv \lambda w. True$   
**abbreviation**  $Mneg::\sigma \Rightarrow \sigma (\neg - [52]53)$  **where**  $\neg \varphi \equiv \lambda w. \neg(\varphi w)$   
**abbreviation**  $Mand::\sigma \Rightarrow \sigma \Rightarrow \sigma$  (**infixl**  $\wedge$  50) **where**  $\varphi \wedge \psi \equiv \lambda w. \varphi w \wedge \psi w$   
**abbreviation**  $Mor::\sigma \Rightarrow \sigma \Rightarrow \sigma$  (**infixl**  $\vee$  49) **where**  $\varphi \vee \psi \equiv \lambda w. \varphi w \vee \psi w$   
**abbreviation**  $Mimp::\sigma \Rightarrow \sigma \Rightarrow \sigma$  (**infixr**  $\supset$  48) **where**  $\varphi \supset \psi \equiv \lambda w. \varphi w \longrightarrow \psi w$   
**abbreviation**  $Mequiv::\sigma \Rightarrow \sigma \Rightarrow \sigma$  (**infixl**  $\leftrightarrow$  47) **where**  $\varphi \leftrightarrow \psi \equiv \lambda w. \varphi w \longleftrightarrow \psi w$   
**abbreviation**  $Mbox::\sigma \Rightarrow \sigma$  ( $\Box - [54]55$ ) **where**  $\Box \varphi \equiv \lambda w. \forall v. w \mathbf{r} v \longrightarrow \varphi v$   
**abbreviation**  $Mdia::\sigma \Rightarrow \sigma$  ( $\Diamond - [54]55$ ) **where**  $\Diamond \varphi \equiv \lambda w. \exists v. w \mathbf{r} v \wedge \varphi v$   
**abbreviation**  $Mprimeq::'a \Rightarrow 'a \Rightarrow \sigma$  ( $- = -$ ) **where**  $x = y \equiv \lambda w. x = y$   
**abbreviation**  $Mprimneg::'a \Rightarrow 'a \Rightarrow \sigma$  ( $- \neq -$ ) **where**  $x \neq y \equiv \lambda w. x \neq y$   
**abbreviation**  $Mnegpred::\tau \Rightarrow \tau$  ( $\sim -$ ) **where**  $\sim \Phi \equiv \lambda x. \lambda w. \neg \Phi x w$   
**abbreviation**  $Mconpred::\tau \Rightarrow \tau \Rightarrow \tau$  (**infixl** . 50) **where**  $\Phi . \Psi \equiv \lambda x. \lambda w. \Phi x w \wedge \Psi x w$   
**abbreviation**  $Mexclor::\sigma \Rightarrow \sigma \Rightarrow \sigma$  (**infixl**  $\vee^e$  49) **where**  $\varphi \vee^e \psi \equiv (\varphi \vee \psi) \wedge \neg(\varphi \wedge \psi)$

— Possibilist quantifiers (polymorphic)

**abbreviation**  $Mallposs::('a \Rightarrow \sigma) \Rightarrow \sigma$  ( $\forall^E$ ) **where**  $\forall^E \Phi \equiv \lambda w. \forall x. \Phi x w$   
**abbreviation**  $Mallpossb$  (**binder**  $\forall^E$  [8]9) **where**  $\forall^E x. \varphi(x) \equiv \forall \varphi$   
**abbreviation**  $Mexiposs::('a \Rightarrow \sigma) \Rightarrow \sigma$  ( $\exists^E$ ) **where**  $\exists^E \Phi \equiv \lambda w. \exists x. \Phi x w$   
**abbreviation**  $Mexiposb$  (**binder**  $\exists^E$  [8]9) **where**  $\exists^E x. \varphi(x) \equiv \exists \varphi$

— Actualist quantifiers (for individuals/entities)

**consts**  $existsAt::e \Rightarrow \sigma$  ( $- @ -$ )  
**abbreviation**  $Mallact::(e \Rightarrow \sigma) \Rightarrow \sigma$  ( $\forall^E$ ) **where**  $\forall^E \Phi \equiv \lambda w. \forall x. x @ w \longrightarrow \Phi x w$   
**abbreviation**  $Mallactb$  (**binder**  $\forall^E$  [8]9) **where**  $\forall^E x. \varphi(x) \equiv \forall^E \varphi$   
**abbreviation**  $Mexiact::(e \Rightarrow \sigma) \Rightarrow \sigma$  ( $\exists^E$ ) **where**  $\exists^E \Phi \equiv \lambda w. \exists x. x @ w \wedge \Phi x w$   
**abbreviation**  $Mexiactb$  (**binder**  $\exists^E$  [8]9) **where**  $\exists^E x. \varphi(x) \equiv \exists^E \varphi$

— Leibniz equality (polymorphic)

**abbreviation**  $Mleibeq::'a \Rightarrow 'a \Rightarrow \sigma$  ( $- \equiv -$ ) **where**  $x \equiv y \equiv \forall P. P x \supset P y$

— Meta-logical predicate for global validity

**abbreviation**  $Mvalid::\sigma \Rightarrow bool$  ( $[-]$ ) **where**  $[\psi] \equiv \forall w. \psi w$

end

## 7.5 TestsHOMLinS4.thy

Tests and verifications of properties for the embedding of HOML (S4) in HOL.

**theory** *TestsHOMLinS4* **imports** *HOMLinHOLonlyS4*  
**begin**

— Test for S5 modal logic

**lemma**  $axM$ :  $[\Box \varphi \supset \varphi]$   $\langle proof \rangle$   
**lemma**  $axD$ :  $[\Box \varphi \supset \Diamond \varphi]$   $\langle proof \rangle$   
**lemma**  $axB$ :  $[\varphi \supset \Box \Diamond \varphi]$   
**nitpick** $[expect=genuine]$   $\langle proof \rangle$

**lemma** *ax4*:  $[\Box\varphi \supset \Box\Box\varphi]$   $\langle$ *proof* $\rangle$   
**lemma** *ax5*:  $[\Diamond\varphi \supset \Box\Diamond\varphi]$   
**nitpick** $[expect=genuine]$   $\langle$ *proof* $\rangle$   
**lemma** *BarcanAct*:  $[(\forall^E x. \Box(\varphi x)) \supset \Box(\forall^E x. (\varphi x))]$   
**nitpick** $[expect=genuine]$   $\langle$ *proof* $\rangle$   
**lemma** *ConvBarcanAct*:  $[\Box(\forall^E x. (\varphi x)) \supset (\forall^E x. \Box(\varphi x))]$   
**nitpick** $[expect=genuine]$   $\langle$ *proof* $\rangle$   
**lemma** *BarcanPoss*:  $[(\forall x. \Box(\varphi x)) \supset \Box(\forall x. \varphi x)]$   $\langle$ *proof* $\rangle$   
**lemma** *ConvBarcanPoss*:  $[\Box(\forall x. (\varphi x)) \supset (\forall x. \Box(\varphi x))]$   $\langle$ *proof* $\rangle$   
**lemma** *Hilbert-A1*:  $[A \supset (B \supset A)]$   $\langle$ *proof* $\rangle$   
**lemma** *Hilbert-A2*:  $[(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))]$   $\langle$ *proof* $\rangle$   
**lemma** *Hilbert-MP*: **assumes**  $[A]$  **and**  $[A \supset B]$  **shows**  $[B]$   $\langle$ *proof* $\rangle$   
**lemma** *Quant-1*: **assumes**  $[A]$  **shows**  $[\forall x::'a. A]$   $\langle$ *proof* $\rangle$   
**lemma** *ExImPossibilist1*:  $[\exists x::e. x = x]$   $\langle$ *proof* $\rangle$   
**lemma** *ExImPossibilist2*:  $[\exists x::e. x \equiv x]$   $\langle$ *proof* $\rangle$   
**lemma** *ExImPossibilist3*:  $[\exists x::e. x = t]$   $\langle$ *proof* $\rangle$   
**lemma** *ExImPossibilist4*:  $[\exists x::'a. x \equiv t::'a]$   $\langle$ *proof* $\rangle$   
**lemma** *ExImPossibilist*:  $[\exists x::'a. \top]$   $\langle$ *proof* $\rangle$   
**lemma** *Quant-2*: **assumes**  $[A]$  **shows**  $[\forall^E x::e. A]$   $\langle$ *proof* $\rangle$   
**lemma** *ExImActualist1*:  $[\exists^E x::e. x = x]$   
**nitpick** $[card=1, expect=genuine]$   $\langle$ *proof* $\rangle$   
**lemma** *ExImActualist2*:  $[\exists^E x::e. x \equiv x]$   
**nitpick** $[card=1, expect=genuine]$   $\langle$ *proof* $\rangle$   
**lemma** *ExImActualist3*:  $[\exists^E x::e. x = t]$   
**nitpick** $[card=1, expect=genuine]$   $\langle$ *proof* $\rangle$   
**lemma** *ExImActualist*:  $[\exists^E x::e. \top]$   
**nitpick** $[card=1, expect=genuine]$   $\langle$ *proof* $\rangle$   
**lemma** *EqRefl*:  $[x = x]$   $\langle$ *proof* $\rangle$   
**lemma** *EqSym*:  $[(x = y) \leftrightarrow (y = x)]$   $\langle$ *proof* $\rangle$   
**lemma** *EqTrans*:  $[((x = y) \wedge (y = z)) \supset (x = z)]$   $\langle$ *proof* $\rangle$   
**lemma** *EQCong*:  $[(x = y) \supset ((\varphi x) = (\varphi y))]$   $\langle$ *proof* $\rangle$   
**lemma** *EQFuncExt*:  $[(\varphi = \psi) \supset (\forall x. ((\varphi x) = (\psi x)))]$   $\langle$ *proof* $\rangle$   
**lemma** *EQBoolExt1*:  $[(\varphi = \psi) \supset (\varphi \leftrightarrow \psi)]$   $\langle$ *proof* $\rangle$   
**lemma** *EQBoolExt2*:  $[(\varphi \leftrightarrow \psi) \supset (\varphi = \psi)]$   
**nitpick** $[card=2]$   $\langle$ *proof* $\rangle$   
**lemma** *EQBoolExt3*:  $[(\varphi \leftrightarrow \psi)] \longrightarrow [(\varphi = \psi)]$   $\langle$ *proof* $\rangle$   
**lemma** *EqPrimLeib*:  $[(x = y) \leftrightarrow (x \equiv y)]$   $\langle$ *proof* $\rangle$   
**lemma** *Comprehension1*:  $[\exists \varphi. \forall x. (\varphi x) \leftrightarrow A]$   $\langle$ *proof* $\rangle$   
**lemma** *Comprehension2*:  $[\exists \varphi. \forall x. (\varphi x) \leftrightarrow (A x)]$   $\langle$ *proof* $\rangle$   
**lemma** *Comprehension3*:  $[\exists \varphi. \forall x y. (\varphi x y) \leftrightarrow (A x y)]$   $\langle$ *proof* $\rangle$   
**lemma** *ModalCollapse*:  $[\forall \varphi. \varphi \supset \Box\varphi]$   
**nitpick** $[card=2, expect=genuine]$   $\langle$ *proof* $\rangle$   
**lemma** *TruePropertyAndSelfIdentity*:  $[(\lambda x::e. \top) = (\lambda x. x = x)]$   $\langle$ *proof* $\rangle$   
**lemma** *EmptyPropertyAndSelfDifference*:  $[(\lambda x::e. \perp) = (\lambda x. x \neq x)]$   $\langle$ *proof* $\rangle$   
**lemma** *EmptyProperty2*:  $[\exists x. \varphi x] \longrightarrow [\varphi \neq (\lambda x::e. \perp)]$   $\langle$ *proof* $\rangle$   
**lemma** *EmptyProperty3*:  $[\exists^E x. \varphi x] \longrightarrow [\varphi \neq (\lambda x::e. \perp)]$   $\langle$ *proof* $\rangle$   
**lemma** *EmptyProperty4*:  $[\varphi \neq (\lambda x::e. \perp)] \longrightarrow [\exists x. \varphi x]$   
**nitpick** $[expect=genuine]$   $\langle$ *proof* $\rangle$   
**lemma** *EmptyProperty5*:  $[\varphi \neq (\lambda x::e. \perp)] \longrightarrow [\exists^E x. \varphi x]$

**nitpick**[*expect=genuine*] *<proof>*

**end**

## 7.6 GoedelVariantHOML1inS4.thy

The same as GoedelVariantHOML1, but now in logic S4.

**theory** *GoedelVariantHOML1inS4* **imports** *HOMLinHOLonlyS4*  
**begin**

**consts** *PositiveProperty*:: $(e \Rightarrow \sigma) \Rightarrow \sigma$  (*P*)

**axiomatization where** *Ax1*:  $[P \ \varphi \wedge P \ \psi \supset P \ (\varphi \cdot \psi)]$

**axiomatization where** *Ax2a*:  $[P \ \varphi \vee^e P \ \sim\varphi]$

**definition** *God* (*G*) **where**  $G \ x \equiv \forall \varphi. P \ \varphi \supset \varphi \ x$

**abbreviation** *PropertyInclusion* ( $-\supset_N-$ ) **where**  $\varphi \supset_N \psi \equiv \Box(\forall^E y. \varphi \ y \supset \psi \ y)$

**definition** *Essence* ( $-Ess.-$ ) **where**  $\varphi \ Ess. \ x \equiv \forall \psi. \psi \ x \supset (\varphi \supset_N \psi)$

**axiomatization where** *Ax2b*:  $[P \ \varphi \supset \Box P \ \varphi]$

**lemma** *Ax2b'*:  $[\neg P \ \varphi \supset \Box(\neg P \ \varphi)]$  *<proof>*

**theorem** *Th1*:  $[G \ x \supset G \ Ess. \ x]$  *<proof>*

**definition** *NecExist* (*E*) **where**  $E \ x \equiv \forall \varphi. \varphi \ Ess. \ x \supset \Box(\exists^E x. \varphi \ x)$

**axiomatization where** *Ax3*:  $[P \ E]$

**theorem** *Th2*:  $[G \ x \supset \Box(\exists^E y. G \ y)]$  *<proof>*

**axiomatization where** *Ax4*:  $[P \ \varphi \wedge (\varphi \supset_N \psi) \supset P \ \psi]$

**theorem** *Th3*:  $[\Diamond(\exists^E x. G \ x) \supset \Box(\exists^E y. G \ y)]$  *<proof>*

**end**

## 7.7 GoedelVariantHOML2inS4.thy

The same as GoedelVariantHOML2, but now in logic S4, where the proof of theorem Th3 fails.

**theory** *GoedelVariantHOML2inS4* **imports** *HOMLinHOLonlyS4*  
**begin**

**consts** *PositiveProperty*:: $(e \Rightarrow \sigma) \Rightarrow \sigma$  (*P*)

**axiomatization where**  $Ax1$ :  $[P \varphi \wedge P \psi \supset P (\varphi \cdot \psi)]$

**abbreviation**  $PosProps \Phi \equiv \forall \varphi. \Phi \varphi \supset P \varphi$

**abbreviation**  $ConjOfPropsFrom \varphi \Phi \equiv \Box(\forall^E z. \varphi z \leftrightarrow (\forall \psi. \Phi \psi \supset \psi z))$

**axiomatization where**  $Ax1Gen$ :  $[(PosProps \Phi \wedge ConjOfPropsFrom \varphi \Phi) \supset P \varphi]$

**axiomatization where**  $Ax2a$ :  $[P \varphi \vee^e P \sim \varphi]$

**definition**  $God (G)$  **where**  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$

**abbreviation**  $PropertyInclusion (-\supset_N -)$  **where**  $\varphi \supset_N \psi \equiv \Box(\forall^E y. \varphi y \supset \psi y)$

**definition**  $Essence (-Ess.-)$  **where**  $\varphi Ess. x \equiv \varphi x \wedge (\forall \psi. \psi x \supset (\varphi \supset_N \psi))$

**axiomatization where**  $Ax2b$ :  $[P \varphi \supset \Box P \varphi]$

**lemma**  $Ax2b'$ :  $[\neg P \varphi \supset \Box(\neg P \varphi)]$   $\langle proof \rangle$

**theorem**  $Th1$ :  $[G x \supset G Ess. x]$   $\langle proof \rangle$

**definition**  $NecExist (E)$  **where**  $E x \equiv \forall \varphi. \varphi Ess. x \supset \Box(\exists^E x. \varphi x)$

**axiomatization where**  $Ax3$ :  $[P E]$

**axiomatization where**  $Ax4$ :  $[P \varphi \wedge (\varphi \supset_N \psi) \supset P \psi]$

**theorem**  $Th2$ :  $[G x \supset \Box(\exists^E y. G y)]$   $\langle proof \rangle$

**theorem**  $Th3$ :  $[\Diamond(\exists^E x. G x) \supset \Box(\exists^E y. G y)]$  — nitpick sledgehammer  $\langle proof \rangle$

**end**

## 7.8 GoedelVariantHOML2possInS4.thy

The same as GoedelVariantHOML2poss, but now in logic S4, where the proof of theorem Th3 fails.

**theory**  $GoedelVariantHOML2possInS4$  **imports**  $HOMLinHOLonlyS4$   
**begin**

**consts**  $PositiveProperty::(e \Rightarrow \sigma) \Rightarrow \sigma (P)$

**axiomatization where**  $Ax1$ :  $[P \varphi \wedge P \psi \supset P (\varphi \cdot \psi)]$

**abbreviation**  $PosProps \Phi \equiv \forall \varphi. \Phi \varphi \supset P \varphi$

**abbreviation**  $ConjOfPropsFrom \varphi \Phi \equiv \Box(\forall z. \varphi z \leftrightarrow (\forall \psi. \Phi \psi \supset \psi z))$

**axiomatization where** *Ax1Gen*:  $[(PosProps \Phi \wedge ConjOfPropsFrom \varphi \Phi) \supset P \varphi]$

**axiomatization where** *Ax2a*:  $[P \varphi \vee^e P \sim \varphi]$

**definition** *God* (*G*) **where**  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$

**abbreviation** *PropertyInclusion* ( $-\supset_N$ ) **where**  $\varphi \supset_N \psi \equiv \Box(\forall y::e. \varphi y \supset \psi y)$

**definition** *Essence* ( $-Ess.$ ) **where**  $\varphi Ess. x \equiv \varphi x \wedge (\forall \psi. \psi x \supset (\varphi \supset_N \psi))$

**axiomatization where** *Ax2b*:  $[P \varphi \supset \Box P \varphi]$

**lemma** *Ax2b'*:  $[\neg P \varphi \supset \Box(\neg P \varphi)]$  *<proof>*

**theorem** *Th1*:  $[G x \supset G Ess. x]$  *<proof>*

**definition** *NecExist* (*E*) **where**  $E x \equiv \forall \varphi. \varphi Ess. x \supset \Box(\exists x. \varphi x)$

**axiomatization where** *Ax3*:  $[P E]$

**axiomatization where** *Ax4*:  $[P \varphi \wedge (\varphi \supset_N \psi) \supset P \psi]$

**theorem** *Th2*:  $[G x \supset \Box(\exists y. G y)]$  *<proof>*

**theorem** *Th3*:  $[\Diamond(\exists x. G x) \supset \Box(\exists y. G y)]$  — nitpick sledgehammer *<proof>*

**end**

## 7.9 GoedelVariantHOML3inS4.thy

The same as GoedelVariantHOML3, but now in logic S4, where the proof of theorem Th3 fails.

**theory** *GoedelVariantHOML3inS4* **imports** *HOMLinHOLonlyS4*  
**begin**

**consts** *PositiveProperty*:  $(e \Rightarrow \sigma) \Rightarrow \sigma$  (*P*)

**axiomatization where** *Ax1*:  $[P \varphi \wedge P \psi \supset P (\varphi . \psi)]$

**abbreviation** *PosProps*  $\Phi \equiv \forall \varphi. \Phi \varphi \supset P \varphi$

**abbreviation** *ConjOfPropsFrom*  $\varphi \Phi \equiv \Box(\forall^E z. \varphi z \leftrightarrow (\forall \psi. \Phi \psi \supset \psi z))$

**axiomatization where** *Ax1Gen*:  $[(PosProps \Phi \wedge ConjOfPropsFrom \varphi \Phi) \supset P \varphi]$

**axiomatization where** *Ax2a*:  $[P \varphi \vee^e P \sim \varphi]$

**definition** *God* (*G*) **where**  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$

**abbreviation** *PropertyInclusion* ( $-\supset_N$ -) **where**  $\varphi \supset_N \psi \equiv \Box(\varphi \neq (\lambda x. \perp) \wedge (\forall^{E y}. \varphi y \supset \psi y))$

**definition** *Essence* ( $-Ess.$ -) **where**  $\varphi Ess. x \equiv \forall \psi. \psi x \supset (\varphi \supset_N \psi)$

**axiomatization where** *Ax2b*:  $[P \varphi \supset \Box P \varphi]$

**lemma** *Ax2b'*:  $[\neg P \varphi \supset \Box(\neg P \varphi)]$  *<proof>*

**theorem** *Th1*:  $[G x \supset G Ess. x]$  *<proof>*

**definition** *NecExist* ( $E$ ) **where**  $E x \equiv \forall \varphi. \varphi Ess. x \supset \Box(\exists^{E x}. \varphi x)$

**axiomatization where** *Ax3*:  $[P E]$

**theorem** *Th2*:  $[G x \supset \Box(\exists^{E y}. G y)]$  *<proof>*

**axiomatization where** *Ax4*:  $[P \varphi \wedge (\varphi \supset_N \psi) \supset P \psi]$

**theorem** *Th3*:  $[\Diamond(\exists^{E x}. G x) \supset \Box(\exists^{E y}. G y)]$  — nitpick sledgehammer *<proof>*

**end**

## 7.10 GoedelVariantHOML3possInS4.thy

The same as GoedelVariantHOML3poss, but now in logic S4, where the proof of theorem Th3 fails.

**theory** *GoedelVariantHOML3possInS4* **imports** *HOMLinHOLonlyS4*  
**begin**

**consts** *PositiveProperty*:: $(e \Rightarrow \sigma) \Rightarrow \sigma$  ( $P$ )

**axiomatization where** *Ax1*:  $[P \varphi \wedge P \psi \supset P (\varphi \cdot \psi)]$

**abbreviation** *PosProps*  $\Phi \equiv \forall \varphi. \Phi \varphi \supset P \varphi$

**abbreviation** *ConjOfPropsFrom*  $\varphi \Phi \equiv \Box(\forall z. \varphi z \leftrightarrow (\forall \psi. \Phi \psi \supset \psi z))$

**axiomatization where** *Ax1Gen*:  $[(PosProps \Phi \wedge ConjOfPropsFrom \varphi \Phi) \supset P \varphi]$

**axiomatization where** *Ax2a*:  $[P \varphi \vee^e P \sim \varphi]$

**definition** *God* ( $G$ ) **where**  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$

**abbreviation** *PropertyInclusion* ( $-\supset_N$ -) **where**  $\varphi \supset_N \psi \equiv \Box(\varphi \neq (\lambda x. \perp) \wedge (\forall y. \varphi y \supset \psi y))$

**definition** *Essence* ( $-Ess.$ -) **where**  $\varphi Ess. x \equiv \forall \psi. \psi x \supset (\varphi \supset_N \psi)$

**axiomatization where**  $Ax2b$ :  $[P \varphi \supset \Box P \varphi]$   
**lemma**  $Ax2b'$ :  $[\neg P \varphi \supset \Box(\neg P \varphi)]$   $\langle proof \rangle$   
**theorem**  $Th1$ :  $[G x \supset G \text{Ess. } x]$   $\langle proof \rangle$   
**definition**  $NecExist (E)$  **where**  $E x \equiv \forall \varphi. (\varphi \text{Ess. } x) \supset \Box(\exists x. \varphi x)$   
**axiomatization where**  $Ax3$ :  $[P E]$   
**axiomatization where**  $Ax4$ :  $[P \varphi \wedge (\varphi \supset_N \psi) \supset P \psi]$   
**theorem**  $Th2$ :  $[G x \supset \Box(\exists y. G y)]$   $\langle proof \rangle$   
**theorem**  $Th3$ :  $[\Diamond(\exists x. G x) \supset \Box(\exists y. G y)]$  — nitpick sledgehammer  $\langle proof \rangle$   
**end**

## 7.11 ScottVariantHOMLinS4.thy

**theory**  $ScottVariantHOMLinS4$  **imports**  $HOMLinHOLonlyS4$   
**begin**

**consts**  $PositiveProperty::(e \Rightarrow \sigma) \Rightarrow \sigma (P)$   
**axiomatization where**  $A1$ :  $[\neg P \varphi \leftrightarrow P \sim \varphi]$   
**axiomatization where**  $A2$ :  $[P \varphi \wedge \Box(\forall^E y. \varphi y \supset \psi y) \supset P \psi]$   
**theorem**  $T1$ :  $[P \varphi \supset \Diamond(\exists^E x. \varphi x)]$   $\langle proof \rangle$   
**definition**  $God (G)$  **where**  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$   
**axiomatization where**  $A3$ :  $[P G]$   
**theorem**  $Coro$ :  $[\Diamond(\exists^E x. G x)]$   $\langle proof \rangle$   
**axiomatization where**  $A4$ :  $[P \varphi \supset \Box P \varphi]$   
**definition**  $Ess (-Ess.)$  **where**  $\varphi \text{Ess. } x \equiv \varphi x \wedge (\forall \psi. \psi x \supset \Box(\forall^E y. \varphi y \supset \psi y))$   
**theorem**  $T2$ :  $[G x \supset G \text{Ess. } x]$   $\langle proof \rangle$   
**definition**  $NecExist (NE)$  **where**  $NE x \equiv \forall \varphi. \varphi \text{Ess. } x \supset \Box(\exists^E x. \varphi x)$   
**axiomatization where**  $A5$ :  $[P NE]$

**lemma** *True* **nitpick**[*satisfy,card=1,eval=[P (λx.⊤)]*] ⟨*proof*⟩

**theorem** *T3*:  $[\Box(\exists^E x. G x)]$  **nitpick**[*card e=1, card i=2*] ⟨*proof*⟩

**lemma** *MC*:  $[\varphi \supset \Box\varphi]$  **nitpick**[*card e=1, card i=2*] ⟨*proof*⟩

**end**

## 7.12 ScottVariantHOMLpossInS4.thy

**theory** *ScottVariantHOMLpossInS4* **imports** *HOMLinHOLonlyS4*  
**begin**

**consts** *PositiveProperty*:: $(e \Rightarrow \sigma) \Rightarrow \sigma$  (*P*)

**axiomatization** **where** *A1*:  $[\neg P \varphi \leftrightarrow P \sim \varphi]$

**axiomatization** **where** *A2*:  $[P \varphi \wedge \Box(\forall y. \varphi y \supset \psi y) \supset P \psi]$

**theorem** *T1*:  $[P \varphi \supset \Diamond(\exists x. \varphi x)]$  ⟨*proof*⟩

**definition** *God* (*G*) **where**  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$

**axiomatization** **where** *A3*:  $[P G]$

**theorem** *Coro*:  $[\Diamond(\exists x. G x)]$  ⟨*proof*⟩

**axiomatization** **where** *A4*:  $[P \varphi \supset \Box P \varphi]$

**definition** *Ess* (*-Ess.*) **where**  $\varphi \text{Ess. } x \equiv \varphi x \wedge (\forall \psi. \psi x \supset \Box(\forall y::e. \varphi y \supset \psi y))$

**theorem** *T2*:  $[G x \supset G \text{Ess. } x]$  ⟨*proof*⟩

**definition** *NecExist* (*NE*) **where**  $NE x \equiv \forall \varphi. \varphi \text{Ess. } x \supset \Box(\exists x. \varphi x)$

**axiomatization** **where** *A5*:  $[P NE]$

**lemma** *True* **nitpick**[*satisfy,card=1,eval=[P (λx.⊥)]*] ⟨*proof*⟩

**theorem** *T3*:  $[\Box(\exists x. G x)]$  **nitpick**[*card e=1, card i=2*] ⟨*proof*⟩

**lemma** *MC*:  $[\varphi \supset \Box\varphi]$  **nitpick**[*card e=1, card i=2*] ⟨*proof*⟩

**end**

## References

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