Notes on Gödel's and Scott's Variants of the Ontological Argument (Isabelle/HOL dataset)

Christoph Benzmüller and Dana S. Scott

January 7, 2025

Abstract

Experimental studies with Isabelle/HOL on Kurt Gödel's modal ontological argument and Dana Scott's variant of it are presented. They implicitly answer some questions that the authors have received over the last decade(s). In addition, some new results are reported.

Our contribution is explained in full detail in [2]. This document presents the corresponding Isabelle/HOL dataset (which is only slightly modified to meet AFP requirements).

Contents

1 Introduction

The Isabelle/HOL dataset associated with [2] is presented. Compared to previous work on Gödel's modal ontological argument as published in the Archive of Formal Proofs (AFP) [1, 4, 3], our dataset addresses several relevant and in some cases novel aspects, which are combined here in a single publication, including:

1. For the first time, Gödel's original manuscript [5] has been formalized as closely as possible.
2. The inconsistency of the postulates in Gödel's original manuscript is explained in detail.
3. Two different ways of eliminating this inconsistency are presented, one of which is novel.
4. Scott's variant [6] of Gödel's original manuscript is presented and compared with Gödel's original variant.
5. In addition to logics S5 and K, the above variants are also studied for logic S4.

6. The above variants are tested for various combinations of of possibilist and actualist quantifiers for individuals (which has not been done systematically in previous publications).

7. Concepts of evil are examined and the derivability of evil is critically questioned.

The purpose of this AFP publication is essentially twofold. One motivation is to make the data sources associated with [2] available in a sustainable, well-maintained way. The other motivation is to support university education in higher-order modal logic by providing a small dataset for reuse that illustrates a systematically explored philosophical argument, emphasizing in particular different notions of quantification.

Compared to [2], the Isabelle sources presented here have been slightly modified to meet some AFP requirements. This concerns the commenting out of calls to sledghemmer (to reduce computational resources) and some minor reformatting (e.g. insertion of new lines). The formalization code itself remains unchanged.

2 Interactive and automated theorem proving

2.1 SurjectiveCantor.thy (Figure 2 of [2])

The surjective Cantor theorem is used in [2] to illustrate some aspects of interactive and automated theorem proving in Isabelle/HOL as relevant for the paper. To keep the provided data material complete wrt. [2], we include these data sources also here.

```
theory SurjectiveCantor imports Main
begin
```

Surjective Cantor theorem: traditional interactive proof

```
theorem SurjectiveCantor: ¬(∃ G.∀ F::'a⇒bool.∃ X::'a. G X = F)
 proof
  assume 1: ∃ G.∀ F::'a⇒bool.∃ X::'a. G X = F
  obtain g::'a⇒('a⇒bool) where 2: ∀ F.∃ X. g X = F using 1 by auto
  let ?F = λX.¬ g X X
  have 3: ∃ Y. g Y = ?F using 2 by metis
  obtain a::'a where 4: g a = ?F using 3 by auto
  have 5: g a a = ?F a using 4 by metis
  have 6: g a a = (¬ g a a) using 5 by auto
  show False using 6 by auto
 qed
```

Avoiding proof by contradiction (Fuenmayor & Benzmüller, 2021)

```
theorem SurjectiveCantor': ¬(∃ G.∀ F::'a⇒bool.∃ X::'a. G X = F)
 proof –
```

{**fix** g :: $'a \Rightarrow ('a \Rightarrow bool)$
have *1*: $\forall X. \exists Y. (\neg g\ X\ Y) = (\neg g\ Y\ Y)$ **by** *auto*
have *2*: $\forall X. \exists Y. (\neg g\ X\ Y) = ((\lambda Z. \neg g\ Z\ Z)\ Y)$ **using** *1* **by** *auto*
have *3*: $\exists F. \forall X. \exists Y. (\neg g\ X\ Y) = (F\ Y)$ **using** *2* **by** *auto*
have *4*: $\exists F. \forall X. \neg(\forall Y. (g\ X\ Y) = (F\ Y))$ **using** *3* **by** *auto*
have $\exists F. \forall X. \neg(g\ X = F)$ **using** *4* **by** *metis*
}
hence *5*: $\forall G. \exists F{::}'a \Rightarrow bool. \forall X{::}'a. \neg(G\ X = F)$ **by** *auto*
have *6*: $\neg(\exists G. \forall F{::}'a \Rightarrow bool. \exists X{::}'a.\ G\ X = F)$ **using** *5* **by** *auto*
thus *?thesis* **.** — done, avoiding proof by contradiction
qed

Surjective Cantor theorem: automated proof by some internal/external theorem provers

theorem *SurjectiveCantor''*: $\neg(\exists G. \forall F{::}'a \Rightarrow bool. \exists X{::}'a.\ G\ X = F)$
nitpick[*expect=none*] — no counterexample found
— sledgehammer — most internal provers give up
— sledgehammer[remote_leo2 remote_leo3] — proof found: leo provers succeed
oops

Surjective Cantor theorem (wrong formalization attempt): the types are crucial

theorem *SurjectiveCantor'''*: $\neg(\exists G. \forall F{::}'b. \exists X{::}'a.\ G\ X = F)$
nitpick — counterexample found for card 'a = 1 and card 'b = 1: G=(λx. (a1 := b1)
nitpick[*satisfy*, *expect=genuine*] — model found for card 'a = 1 and card 'b = 2
nitpick[*card 'a=2*, *card 'b=3*, *expect=none*] — no counterexample found
oops
end

3 Mechanization of higher-order modal logic (HOML)

3.1 HOMLinHOL.thy (Figure 3 of [2])

Shallow embedding of higher-order modal logic (HOML) in the classical higher-order logic (HOL) of Isabelle/HOL utilizing the logic-pluralistic LogiKEy methodology. Here logic S5 is introduced.

theory *HOMLinHOL* **imports** *Main*
begin

— Global parameters setting for the model finder nitpick and the parser; unimport for the reader
nitpick-params[*user-axioms*,*expect=genuine*,*show-all*,*format=2*,*max-genuine=3*]
declare[[*syntax-ambiguity-warning=false*]]

— Type i is associated with possible worlds and type e with entities
typedecl *i* — Possible worlds

typedecl e — Individuals/entities
type-synonym $\sigma = i \Rightarrow bool$ — World-lifted propositions
type-synonym $\tau = e \Rightarrow \sigma$ — Modal properties

consts $R::i \Rightarrow i \Rightarrow bool$ (-**r**-) — Accessibility relation between worlds

axiomatization where
Rrefl: $\forall x.\ x\mathbf{r}x$ **and**
Rsymm: $\forall x\ y.\ x\mathbf{r}y \longrightarrow y\mathbf{r}x$ **and**
Rtrans: $\forall x\ y\ z.\ x\mathbf{r}y \wedge y\mathbf{r}z \longrightarrow x\mathbf{r}z$

— Logical connectives (operating on truth-sets)
abbreviation $Mbot::\sigma$ ($\bot$) **where** $\bot \equiv \lambda w.\ False$
abbreviation $Mtop::\sigma$ ($\top$) **where** $\top \equiv \lambda w.\ True$
abbreviation $Mneg::\sigma \Rightarrow \sigma$ ($\neg$- $[52]53$) **where** $\neg\varphi \equiv \lambda w.\ \neg(\varphi\ w)$
abbreviation $Mand::\sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixl** $\wedge$ 50) **where** $\varphi \wedge \psi \equiv \lambda w.\ \varphi\ w \wedge \psi\ w$
abbreviation $Mor::\sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixl** $\vee$ 49) **where** $\varphi \vee \psi \equiv \lambda w.\ \varphi\ w \vee \psi\ w$
abbreviation $Mimp::\sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixr** $\supset$ 48) **where** $\varphi \supset \psi \equiv \lambda w.\ \varphi\ w \longrightarrow \psi\ w$
abbreviation $Mequiv::\sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixl** $\leftrightarrow$ 47) **where** $\varphi \leftrightarrow \psi \equiv \lambda w.\ \varphi\ w \longleftrightarrow \psi\ w$
abbreviation $Mbox::\sigma \Rightarrow \sigma$ ($\Box$- $[54]55$) **where** $\Box\varphi \equiv \lambda w.\forall v.\ w\ \mathbf{r}\ v \longrightarrow \varphi\ v$
abbreviation $Mdia::\sigma \Rightarrow \sigma$ ($\Diamond$- $[54]55$) **where** $\Diamond\varphi \equiv \lambda w.\exists v.\ w\ \mathbf{r}\ v \wedge \varphi\ v$
abbreviation $Mprimeq::'a \Rightarrow 'a \Rightarrow \sigma$ (-$\mathbf{=}$-) **where** $x\mathbf{=}y \equiv \lambda w.\ x=y$
abbreviation $Mprimneg::'a \Rightarrow 'a \Rightarrow \sigma$ (-$\mathbf{\neq}$-) **where** $x\mathbf{\neq}y \equiv \lambda w.\ x \neq y$
abbreviation $Mnegpred::\tau \Rightarrow \tau$ ($\sim$-) **where** $\sim\Phi \equiv \lambda x.\lambda w.\ \neg\Phi\ x\ w$
abbreviation $Mconpred::\tau \Rightarrow \tau \Rightarrow \tau$ (**infixl .** 50) **where** $\Phi.\Psi \equiv \lambda x.\lambda w.\ \Phi\ x\ w \wedge \Psi\ x\ w$
abbreviation $Mexclor::\sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixl** $\vee^e$ 49) **where** $\varphi \vee^e \psi \equiv (\varphi \vee \psi) \wedge \neg(\varphi \wedge \psi)$

— Possibilist quantifiers (polymorphic)
abbreviation $Mallposs::('a \Rightarrow \sigma) \Rightarrow \sigma$ ($\forall$) **where** $\forall\Phi \equiv \lambda w.\forall x.\ \Phi\ x\ w$
abbreviation $Mallpossb$ (**binder** $\forall$ $[8]9$) **where** $\forall x.\ \varphi(x) \equiv \forall\varphi$
abbreviation $Mexiposs::('a \Rightarrow \sigma) \Rightarrow \sigma$ ($\exists$) **where** $\exists\Phi \equiv \lambda w.\exists x.\ \Phi\ x\ w$
abbreviation $Mexipossb$ (**binder** $\exists$ $[8]9$) **where** $\exists x.\ \varphi(x) \equiv \exists\varphi$

— Actualist quantifiers (for individuals/entities)
consts $existsAt::e \Rightarrow \sigma$ (-**@**-)
abbreviation $Mallact::(e \Rightarrow \sigma) \Rightarrow \sigma$ ($\forall^E$) **where** $\forall^E\Phi \equiv \lambda w.\forall x.\ x\mathbf{@}w \longrightarrow \Phi\ x\ w$
abbreviation $Mallactb$ (**binder** $\forall^E$ $[8]9$) **where** $\forall^E x.\ \varphi(x) \equiv \forall^E\varphi$
abbreviation $Mexiact::(e \Rightarrow \sigma) \Rightarrow \sigma$ ($\exists^E$) **where** $\exists^E\Phi \equiv \lambda w.\exists x.\ x\mathbf{@}w \wedge \Phi\ x\ w$
abbreviation $Mexiactb$ (**binder** $\exists^E$ $[8]9$) **where** $\exists^E x.\ \varphi(x) \equiv \exists^E\varphi$

— Leibniz equality (polymorphic)
abbreviation $Mleibeq::'a \Rightarrow 'a \Rightarrow \sigma$ (-$\mathbf{\equiv}$-) **where** $x\mathbf{\equiv}y \equiv \forall P.\ P\ x \supset P\ y$

— Meta-logical predicate for global validity
abbreviation $Mvalid::\sigma \Rightarrow bool$ ($\lfloor$-$\rfloor$) **where** $\lfloor\psi\rfloor \equiv \forall w.\ \psi\ w$

end

3.2 TestsHOML.thy (Figure 4 of [2])

Tests and verifications of properties for the embedding of HOML (S5) in HOL.

theory *TestsHOML* **imports** *HOMLinHOL*
begin

— Test for S5 modal logic
lemma *axM*: $\lfloor \Box\varphi \supset \varphi \rfloor$ **using** *Rrefl* **by** *blast*
lemma *axD*: $\lfloor \Box\varphi \supset \Diamond\varphi \rfloor$ **using** *Rrefl* **by** *blast*
lemma *axB*: $\lfloor \varphi \supset \Box\Diamond\varphi \rfloor$ **using** *Rsymm* **by** *blast*
lemma *ax4*: $\lfloor \Box\varphi \supset \Box\Box\varphi \rfloor$ **using** *Rtrans* **by** *blast*
lemma *ax5*: $\lfloor \Diamond\varphi \supset \Box\Diamond\varphi \rfloor$ **using** *Rsymm Rtrans* **by** *blast*

— Test for Barcan and converse Barcan formula:
lemma *BarcanAct*: $\lfloor (\forall^E x.\Box(\varphi\ x)) \supset \Box(\forall^E x.(\varphi\ x)) \rfloor$
nitpick[*expect=genuine*] **oops** — Countermodel found
lemma *ConvBarcanAct*: $\lfloor \Box(\forall^E x.(\varphi\ x)) \supset (\forall^E x.\Box(\varphi\ x)) \rfloor$
nitpick[*expect=genuine*] **oops** — Countermodel found
lemma *BarcanPoss*: $\lfloor (\forall x.\Box(\varphi\ x)) \supset \Box(\forall x.\ \varphi\ x) \rfloor$ **by** *blast*
lemma *ConvBarcanPoss*: $\lfloor \Box(\forall x.(\varphi\ x)) \supset (\forall x.\Box(\varphi\ x)) \rfloor$ **by** *blast*

— A simple Hilbert system for classical propositional logic is derived
lemma *Hilbert-A1*: $\lfloor A \supset (B \supset A) \rfloor$ **by** *blast*
lemma *Hilbert-A2*: $\lfloor (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C)) \rfloor$ **by** *blast*
lemma *Hilbert-MP*: **assumes** $\lfloor A \rfloor$ **and** $\lfloor A \supset B \rfloor$ **shows** $\lfloor B \rfloor$ **using** *assms* **by** *blast*

— We have a polymorphic possibilist quantifier for which existential import holds
lemma *Quant-1*: **assumes** $\lfloor A \rfloor$ **shows** $\lfloor \forall x{::}'a.\ A \rfloor$ **using** *assms* **by** *auto*

— Existential import holds for possibilist quantifiers
lemma *ExImPossibilist1*: $\lfloor \exists x{::}e.\ x = x \rfloor$ **by** *blast*
lemma *ExImPossibilist2*: $\lfloor \exists x{::}e.\ x \equiv x \rfloor$ **by** *blast*
lemma *ExImPossibilist3*: $\lfloor \exists x{::}e.\ x = t \rfloor$ **by** *blast*
lemma *ExImPossibilist4*: $\lfloor \exists x{::}'a.\ x \equiv t{::}'a \rfloor$ **by** *blast*
lemma *ExImPossibilist*: $\lfloor \exists x{::}'a.\ \top \rfloor$ **by** *blast*

— We have an actualist quantifier for individuals for which existential import does not hold
lemma *Quant-2*: **assumes** $\lfloor A \rfloor$ **shows** $\lfloor \forall^E x{::}e.\ A \rfloor$ **using** *assms* **by** *auto*

— Existential import does not hold for our actualist quantifiers (for individuals)
lemma *ExImActualist1*: $\lfloor \exists^E x{::}e.\ x = x \rfloor$
nitpick[*card=1,expect=genuine*] **oops** — Countermodel found
lemma *ExImActualist2*: $\lfloor \exists^E x{::}e.\ x \equiv x \rfloor$
nitpick[*card=1,expect=genuine*] **oops** — Countermodel found
lemma *ExImActualist3*: $\lfloor \exists^E x{::}e.\ x = t \rfloor$
nitpick[*card=1,expect=genuine*] **oops** — Countermodel found
lemma *ExImActualist*: $\lfloor \exists^E x{::}e.\ \top \rfloor$

nitpick[*card=1*,*expect=genuine*] **oops** — Countermodel found

— Properties of the embedded primitive equality, which coincides with Leibniz equality
lemma *EqRefl*: $\lfloor x = x \rfloor$ **by** *blast*
lemma *EqSym*: $\lfloor (x = y) \leftrightarrow (y = x) \rfloor$ **by** *blast*
lemma *EqTrans*: $\lfloor ((x = y) \wedge (y = z)) \supset (x = z) \rfloor$ **by** *blast*
lemma *EQCong*: $\lfloor (x = y) \supset ((\varphi\ x) = (\varphi\ y)) \rfloor$ **by** *blast*
lemma *EQFuncExt*: $\lfloor (\varphi = \psi) \supset (\forall x.\ ((\varphi\ x) = (\psi\ x))) \rfloor$ **by** *blast*
lemma *EQBoolExt1*: $\lfloor (\varphi = \psi) \supset (\varphi \leftrightarrow \psi) \rfloor$ **by** *blast*
lemma *EQBoolExt2*: $\lfloor (\varphi \leftrightarrow \psi) \supset (\varphi = \psi) \rfloor$
nitpick[*card=2*] **oops** — Countermodel found
lemma *EQBoolExt3*: $\lfloor (\varphi \leftrightarrow \psi) \rfloor \longrightarrow \lfloor (\varphi = \psi) \rfloor$ **by** *blast*
lemma *EqPrimLeib*: $\lfloor (x = y) \leftrightarrow (x \equiv y) \rfloor$ **by** *auto*

— Comprehension is natively supported in HOL (due to lambda-abstraction)
lemma *Comprehension1*: $\lfloor \exists \varphi.\ \forall x.\ (\varphi\ x) \leftrightarrow A \rfloor$ **by** *force*
lemma *Comprehension2*: $\lfloor \exists \varphi.\ \forall x.\ (\varphi\ x) \leftrightarrow (A\ x) \rfloor$ **by** *force*
lemma *Comprehension3*: $\lfloor \exists \varphi.\ \forall x\ y.\ (\varphi\ x\ y) \leftrightarrow (A\ x\ y) \rfloor$ **by** *force*

— Modal collapse does not hold
lemma *ModalCollapse*: $\lfloor \forall \varphi.\ \varphi \supset \Box\varphi \rfloor$
nitpick[*card=2*,*expect=genuine*] **oops** — Countermodel found

— Empty property and self-difference
lemma *TruePropertyAndSelfIdentity*: $\lfloor (\lambda x{::}e.\ \top) = (\lambda x.\ x = x) \rfloor$ **by** *blast*
lemma *EmptyPropertyAndSelfDifference*: $\lfloor (\lambda x{::}e.\ \bot) = (\lambda x.\ x \neq x) \rfloor$ **by** *blast*
lemma *EmptyProperty2*: $\lfloor \exists x.\ \varphi\ x \rfloor \longrightarrow \lfloor \varphi \neq (\lambda x{::}e.\ \bot) \rfloor$ **by** *blast*
lemma *EmptyProperty3*: $\lfloor \exists^E x.\ \varphi\ x \rfloor \longrightarrow \lfloor \varphi \neq (\lambda x{::}e.\ \bot) \rfloor$ **by** *blast*
lemma *EmptyProperty4*: $\lfloor \varphi \neq (\lambda x{::}e.\ \bot) \rfloor \longrightarrow \lfloor \exists x.\ \varphi\ x \rfloor$
nitpick[*expect=genuine*] **oops** — Countermodel found
lemma *EmptyProperty5*: $\lfloor \varphi \neq (\lambda x{::}e.\ \bot) \rfloor \longrightarrow \lfloor \exists^E x.\ \varphi\ x \rfloor$
nitpick[*expect=genuine*] **oops** — Countermodel found

end

3.3 ModalFilter.thy (Figure 5 of [2])

Set filter and ultrafilter formalized for our modal logic setting.

theory *ModalFilter* **imports** *HOMLinHOL*
begin

type-synonym $\tau = e \Rightarrow \sigma$
abbreviation *Element*$::\tau \Rightarrow (\tau \Rightarrow \sigma) \Rightarrow \sigma$ (**infix** $\in$ *90*) **where** $\varphi \in S \equiv S\ \varphi$
abbreviation *EmptySet*$::\tau$ ($\emptyset$) **where** $\emptyset \equiv \lambda x.\ \bot$
abbreviation *UniversalSet*$::\tau$ (**U**) **where** $\mathbf{U} \equiv \lambda x.\ \top$
abbreviation *Subset*$::\tau \Rightarrow \tau \Rightarrow \sigma$ (**infix** $\subseteq$ *80*)
where $\varphi \subseteq \psi \equiv \forall x.((\varphi\ x) \supset (\psi\ x))$
abbreviation *SubsetE*$::\tau \Rightarrow \tau \Rightarrow \sigma$ (**infix** $\subseteq^E$ *80*)

where $\varphi \subseteq^E \psi \equiv \forall^E x.((\varphi\ x) \supset (\psi\ x))$
abbreviation *Intersection*$::\tau \Rightarrow \tau \Rightarrow \tau$ (**infix** $\sqcap$ *91*)
where $\varphi \sqcap \psi \equiv \lambda x.((\varphi\ x) \wedge (\psi\ x))$
abbreviation *Inverse*$::\tau \Rightarrow \tau$ $({}^{-1})$
where ${}^{-1}\psi \equiv \lambda x.\ \neg(\psi\ x)$
abbreviation *Filter* $\Phi \equiv \mathbf{U} \in \Phi \wedge \neg(\emptyset \in \Phi) \wedge$
$(\forall \varphi\ \psi.\ \varphi \in \Phi \wedge \varphi \subseteq^E \psi \supset \psi \in \Phi) \wedge (\forall \varphi\ \psi.\ \varphi \in \Phi \wedge \psi \in \Phi \supset \varphi \sqcap \psi \in \Phi)$
abbreviation *UFilter* $\Phi \equiv$ *Filter* $\Phi \wedge (\forall \varphi.\ \varphi \in \Phi \vee ({}^{-1}\varphi) \in \Phi)$
abbreviation *FilterP* $\Phi \equiv \mathbf{U} \in \Phi \wedge \neg(\emptyset \in \Phi) \wedge (\forall \varphi\ \psi.\ \varphi \in \Phi \wedge \varphi \subseteq \psi \supset \psi \in \Phi) \wedge$
$(\forall \varphi\ \psi.\ \varphi \in \Phi \wedge \psi \in \Phi \supset \varphi \sqcap \psi \in \Phi)$
abbreviation *UFilterP* $\Phi \equiv$ *FilterP* $\Phi \wedge (\forall \varphi.\ \varphi \in \Phi \vee ({}^{-1}\varphi) \in \Phi)$

end

4 Gödels ontological argument – 1970 manuscript

4.1 GoedelVariantHOML1.thy (Figure 6 of [2])

Gödels axioms and definitions, as presented in the 1970 manuscript, are inconsistent. Actualist quantifiers (avoiding existential import) are used for quantification over entities, otherwise possibilist quantifiers are used.

theory *GoedelVariantHOML1* **imports** *HOMLinHOL*
begin

consts *PositiveProperty*$::(e \Rightarrow \sigma) \Rightarrow \sigma$ (P)

axiomatization where *Ax1*: $\lfloor P\ \varphi \wedge P\ \psi \supset P\ (\varphi\ \boldsymbol{.}\ \psi)\rfloor$

axiomatization where *Ax2a*: $\lfloor P\ \varphi \vee^e P\ {}^{\sim}\varphi\rfloor$

definition *God* (G) **where** $G\ x \equiv \forall \varphi.\ P\ \varphi \supset \varphi\ x$

abbreviation *PropertyInclusion* $(\text{-}\supset_N\text{-})$ **where** $\varphi \supset_N \psi \equiv \Box(\forall^E y.\ \varphi\ y \supset \psi\ y)$

definition *Essence* (*-Ess.-*) **where** φ *Ess.* $x \equiv \forall \psi.\ \psi\ x \supset (\varphi \supset_N \psi)$

axiomatization where *Ax2b*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi\rfloor$

lemma *Ax2b′*: $\lfloor \neg P\ \varphi \supset \Box(\neg P\ \varphi)\rfloor$ **using** *Ax2a Ax2b* **by** *blast*

theorem *Th1*: $\lfloor G\ x \supset G$ *Ess.* $x\rfloor$ **using** *Ax2a Ax2b Essence-def God-def* **by** (*smt* (*verit*))

definition *NecExist* (E) **where** $E\ x \equiv \forall \varphi.\ \varphi$ *Ess.* $x \supset \Box(\exists^E x.\ \varphi\ x)$

axiomatization where *Ax3*: $\lfloor P\ E\rfloor$

theorem *Th2*: $\lfloor G\ x \supset \Box(\exists^E y.\ G\ y)\rfloor$ **using** *Ax3 Th1 God-def NecExist-def* **by**

smt

theorem *Th3*: $\lfloor \Diamond(\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y)\rfloor$
— sledgehammer(Th2 Rsymm) — Proof found
proof –
have *1*: $\lfloor(\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y)\rfloor$ **using** *Th2* **by** *blast*
have *2*: $\lfloor\Diamond(\exists^E x.\ G\ x) \supset \Diamond\Box(\exists^E y.\ G\ y)\rfloor$ **using** *1* **by** *blast*
have *3*: $\lfloor\Diamond(\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y)\rfloor$ **using** *2 Rsymm* **by** *blast*
thus *?thesis* **by** *blast*
qed

axiomatization where *Ax4*: $\lfloor P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi\rfloor$

lemma *True* **nitpick**[*satisfy*,*expect=unknown*] **oops** — No model found

lemma *EmptyEssL*: $\lfloor(\lambda y.\bot)\ Ess.\ x\rfloor$ **using** *Essence-def* **by** *auto*

theorem *Inconsistency*: *False*
— sledgehammer(Ax2a Ax3 Ax4 EmptyEssL NecExist_def) — Proof found
proof –
have *1*: $\lfloor\neg(P\ (\lambda x.\bot))\rfloor$ **using** *Ax2a Ax4* **by** *blast*
have *2*: $\lfloor P\ (\lambda x.(\lambda y.\bot)\ Ess.\ x \supset \Box(\exists^E z.\ (\lambda y.\bot) z))\rfloor$ **using** *Ax3 Ax4 NecExist-def* **by** *smt*
have *3*: $\lfloor P\ (\lambda x.\Box(\exists^E z.\ (\lambda x.\bot)\ z))\rfloor$ **using** *2 EmptyEssL* **by** *simp*
have *4*: $\lfloor P\ (\lambda x.\Box\bot)\rfloor$ **using** *3* **by** *auto*
have *5*: $\lfloor P\ (\lambda x.\bot)\rfloor$ **using** *4 Ax2a Ax4* **by** *smt*
have *6*: $\lfloor\bot\rfloor$ **using** *1 5* **by** *blast*
thus *?thesis* **by** *blast*
qed

end

4.2 GoedelVariantHOML2.thy (Figure 7 of [2])

After an appropriate modification of the definition of essence in Gödels 1970 ontological proof, the inconsistency revealed in Figure 6 is avoided, and the argument can be successfully verified in modal logic S5 (indeed, as shown, only symmetry of the accessibility relation is actually needed). Actualist quantifiers (avoiding existential import) are used for quantification over entities, otherwise possibilist quantifiers are used.

theory *GoedelVariantHOML2* **imports** *HOMLinHOL ModalFilter*
begin

consts *PositiveProperty*::$(e\Rightarrow\sigma)\Rightarrow\sigma$ (P)

axiomatization where *Ax1*: $\lfloor P\ \varphi \wedge P\ \psi \supset P\ (\varphi \centerdot \psi)\rfloor$

axiomatization where *Ax2a*: $\lfloor P\ \varphi \vee^e P\ {}^{\sim}\varphi\rfloor$

definition *God* (G) **where** $G\ x \equiv \forall \varphi.\ P\ \varphi \supset \varphi\ x$

abbreviation *PropertyInclusion* $(\text{-}\supset_N\text{-})$ **where** $\varphi \supset_N \psi \equiv \Box(\forall^E y.\ \varphi\ y \supset \psi\ y)$

definition *Essence* $(\text{-}Ess.\text{-})$ **where** $\varphi\ Ess.\ x \equiv \varphi\ x \wedge (\forall \psi.\ \psi\ x \supset (\varphi \supset_N \psi))$

axiomatization where *Ax2b*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi \rfloor$

lemma *Ax2b′*: $\lfloor \neg P\ \varphi \supset \Box(\neg P\ \varphi)\rfloor$ **using** *Ax2a Ax2b* **by** *blast*

theorem *Th1*: $\lfloor G\ x \supset G\ Ess.\ x\rfloor$ **using** *Ax2a Ax2b Essence-def God-def* **by** (*smt* (*verit*))

definition *NecExist* (E) **where** $E\ x \equiv \forall \varphi.\ \varphi\ Ess.\ x \supset \Box(\exists^E x.\ \varphi\ x)$

axiomatization where *Ax3*: $\lfloor P\ E\rfloor$

theorem *Th2*: $\lfloor G\ x \supset \Box(\exists^E y.\ G\ y)\rfloor$ **using** *Ax3 Th1 God-def NecExist-def* **by** *smt*

theorem *Th3*: $\lfloor \Diamond(\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y)\rfloor$
— sledgehammer(Th2 Rsymm) — Proof found
proof −
  have *1*: $\lfloor(\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y)\rfloor$ **using** *Th2* **by** *blast*
  have *2*: $\lfloor\Diamond(\exists^E x.\ G\ x) \supset \Diamond\Box(\exists^E y.\ G\ y)\rfloor$ **using** *1* **by** *blast*
  have *3*: $\lfloor\Diamond(\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y)\rfloor$ **using** *2 Rsymm* **by** *blast*
  thus *?thesis* **by** *blast*
qed

axiomatization where *Ax4*: $\lfloor P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi\rfloor$

lemma *True* **nitpick**[*satisfy*,*card=1*,*eval*=$\lfloor P\ (\lambda x.\bot)\rfloor$] **oops** — One model found of cardinality one

abbreviation *PosProps* $\Phi \equiv \forall \varphi.\ \Phi\ \varphi \supset P\ \varphi$
abbreviation *ConjOfPropsFrom* $\varphi\ \Phi \equiv \Box(\forall^E z.\ \varphi\ z \leftrightarrow (\forall \psi.\ \Phi\ \psi \supset \psi\ z))$
axiomatization where *Ax1Gen*: $\lfloor(PosProps\ \Phi \wedge ConjOfPropsFrom\ \varphi\ \Phi) \supset P\ \varphi\rfloor$

lemma *L*: $\lfloor P\ G\rfloor$ **using** *Ax1Gen God-def* **by** (*smt* (*verit*))

theorem *Th4*: $\lfloor\Diamond(\exists^E x.\ G\ x)\rfloor$ **using** *Ax2a Ax4 L* **by** *blast*

theorem *Th5*: $\lfloor\Box(\exists^E x.\ G\ x)\rfloor$ **using** *Th3 Th4* **by** *blast*

lemma *MC*: $\lfloor\varphi \supset \Box\varphi\rfloor$
— sledgehammer(Ax2a Ax2b Th5 God_def Rsymm) — Proof found
proof −

{**fix** w **fix** Q
have *1*: $\forall x.(G\ x\ w \longrightarrow (\forall Z.\ Z\ x \supset \Box(\forall^E z.\ G\ z \supset Z\ z))\ w)$ **using** *Ax2a Ax2b God-def* **by** *smt*
have *2*: $(\exists x.\ G\ x\ w) \longrightarrow ((Q \supset \Box(\forall^E z.\ G\ z \supset Q))\ w)$ **using** *1* **by** *force*
have *3*: $(Q \supset \Box Q)\ w$ **using** *2 Th5 Rsymm* **by** *blast*}
thus *?thesis* **by** *auto*
qed

lemma *PosProps*: $\lfloor P\ (\lambda x.\top) \wedge P\ (\lambda x.\ x = x)\rfloor$ **using** *Ax2a Ax4* **by** *blast*
lemma *NegProps*: $\lfloor \neg P(\lambda x.\bot) \wedge \neg P(\lambda x.\ x \neq x)\rfloor$ **using** *Ax2a Ax4* **by** *blast*
lemma *UniqueEss1*: $\lfloor \varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\forall^E y.\ \varphi\ y \leftrightarrow \psi\ y)\rfloor$ **using** *Essence-def* **by** *smt*
lemma *UniqueEss2*: $\lfloor \varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\varphi \equiv \psi)\rfloor$ **nitpick**[*card i=1*] **oops**
— Countermodel found
lemma *UniqueEss3*: $\lfloor \varphi\ Ess.\ x \supset \Box(\forall^E y.\ \varphi\ y \supset y \equiv x)\rfloor$ **using** *Essence-def MC* **by** *auto*
lemma *Monotheism*: $\lfloor G\ x \wedge G\ y \supset x \equiv y\rfloor$ **using** *Ax2a God-def* **by** *smt*
lemma *Filter*: $\lfloor$*Filter* $P\rfloor$ **using** *Ax1 Ax4 MC NegProps PosProps Rsymm* **by** *smt*
lemma *UltraFilter*: $\lfloor$*UFilter* $P\rfloor$ **using** *Ax2a Filter* **by** *smt*
lemma *True* **nitpick**[*satisfy*,*card=1*,*eval*=$\lfloor P\ (\lambda x.\bot)\rfloor$] **oops** — One model found of cardinality one

end

4.3 GoedelVariantHOML3.thy (Figure 8 of [2])

After an appropriate modification of the notion of necessary property inclusion in Gödels 1970 ontological proof, the inconsistency revealed in Figure 6 is avoided, and the argument can be successfully verified in modal logic S5 (indeed, as shown, only symmetry of the accessibility relation is actually needed). Actualist quantifiers (avoiding existential import) are used for quantification over entities, otherwise possibilist quantifiers are used.

theory *GoedelVariantHOML3* **imports** *HOMLinHOL ModalFilter*
begin

consts *PositiveProperty*::$(e \Rightarrow \sigma) \Rightarrow \sigma$ (P)

axiomatization where *Ax1*: $\lfloor P\ \varphi \wedge P\ \psi \supset P\ (\varphi \centerdot \psi)\rfloor$

axiomatization where *Ax2a*: $\lfloor P\ \varphi \vee^e P\ {}^\sim\varphi\rfloor$

definition *God* (G) **where** $G\ x \equiv \forall \varphi.\ P\ \varphi \supset \varphi\ x$

abbreviation *PropertyInclusion* (-$\supset_N$-) **where** $\varphi \supset_N \psi \equiv \Box(\varphi \neq (\lambda x.\ \bot) \wedge (\forall^E y.\ \varphi\ y \supset \psi\ y))$

definition *Essence* (-*Ess.*-) **where** $\varphi\ Ess.\ x \equiv \forall \psi.\ \psi\ x \supset (\varphi \supset_N \psi)$

axiomatization where *Ax2b*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi\rfloor$

lemma *Ax2b′*: $\lfloor \neg P\ \varphi \supset \Box(\neg P\ \varphi)\rfloor$ **using** *Ax2a Ax2b* **by** *blast*

theorem *Th1*: $\lfloor G\ x \supset G\ Ess.\ x\rfloor$ **using** *Ax2a Ax2b Essence-def God-def* **by** (*smt* (*verit*))

definition *NecExist* (*E*) **where** $E\ x \equiv \forall\varphi.\ \varphi\ Ess.\ x \supset \Box(\exists^E x.\ \varphi\ x)$

axiomatization where *Ax3*: $\lfloor P\ E\rfloor$

theorem *Th2*: $\lfloor G\ x \supset \Box(\exists^E y.\ G\ y)\rfloor$ **using** *Ax3 Th1 God-def NecExist-def* **by** *smt*

theorem *Th3*: $\lfloor \Diamond(\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y)\rfloor$
— sledgehammer(Th2 Rsymm) — Proof found
proof –
have *1*: $\lfloor(\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y)\rfloor$ **using** *Th2* **by** *blast*
have *2*: $\lfloor\Diamond(\exists^E x.\ G\ x) \supset \Diamond\Box(\exists^E y.\ G\ y)\rfloor$ **using** *1* **by** *blast*
have *3*: $\lfloor\Diamond(\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y)\rfloor$ **using** *2 Rsymm* **by** *blast*
thus *?thesis* **by** *blast*
qed

axiomatization where *Ax4*: $\lfloor P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi\rfloor$

lemma *True* **nitpick**[*satisfy*,*card=1*,*eval*=$\lfloor P\ (\lambda x.\bot)\rfloor$] **oops** — Two models found of cardinality one

abbreviation *PosProps* $\Phi \equiv \forall\varphi.\ \Phi\ \varphi \supset P\ \varphi$
abbreviation *ConjOfPropsFrom* $\varphi\ \Phi \equiv \Box(\forall^E z.\ \varphi\ z \leftrightarrow (\forall\psi.\ \Phi\ \psi \supset \psi\ z))$
axiomatization where *Ax1Gen*: $\lfloor(PosProps\ \Phi \wedge ConjOfPropsFrom\ \varphi\ \Phi) \supset P\ \varphi\rfloor$

lemma *L*: $\lfloor P\ G\rfloor$ **using** *Ax1Gen God-def* **by** (*smt* (*verit*))

theorem *Th4*: $\lfloor\Diamond(\exists^E x.\ G\ x)\rfloor$
— sledgehammer[timeout=200](Ax2a L Ax1Gen) **oops** — sorry — Proof found
axiomatization where *Th4*: $\lfloor\Diamond(\exists^E x.\ G\ x)\rfloor$

theorem *Th5*: $\lfloor\Box(\exists^E x.\ G\ x)\rfloor$ **using** *Th3 Th4* **by** *blast*

lemma *MC*: $\lfloor\varphi \supset \Box\varphi\rfloor$
— sledgehammer(Ax2a Ax2b Th5 God_def Rsymm) — Proof found
proof – {**fix** *w* **fix** *Q*
have *1*: $\forall x.(G\ x\ w \longrightarrow (\forall Z.\ Z\ x \supset \Box(\forall^E z.\ G\ z \supset Z\ z))\ w)$ **using** *Ax2a Ax2b God-def* **by** *smt*
have *2*: $(\exists x.\ G\ x\ w) \longrightarrow ((Q \supset \Box(\forall^E z.\ G\ z \supset Q))\ w)$ **using** *1* **by** *force*
have *3*: $(Q \supset \Box Q)\ w$ **using** *2 Th5 Rsymm* **by** *blast*}
thus *?thesis* **by** *auto*

qed

lemma *PosProps*: $\lfloor P\ (\lambda x.\top) \wedge P(\lambda x.\ x = x)\rfloor$ **using** *Ax2a Ax4 L Th4* **by** *smt*
lemma *NegProps*: $\lfloor\neg P(\lambda x.\bot) \wedge \neg P(\lambda x.\ x \neq x)\rfloor$ **using** *Ax2a Ax4 L Th4* **by** *smt*
lemma *UniqueEss1*: $\lfloor\varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\forall^E y.\ \varphi\ y \leftrightarrow \psi\ y)\rfloor$ **oops** — Unclear, open question
lemma *UniqueEss2*: $\lfloor\varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\varphi \equiv \psi)\rfloor$ **oops** — Unclear, open question
lemma *UniqueEss3*: $\lfloor\varphi\ Ess.\ x \supset \Box(\forall^E y.\ \varphi\ y \supset y \equiv x)\rfloor$ **using** *Essence-def MC* **by** *auto*
lemma *Monotheism*: $\lfloor G\ x \wedge G\ y \supset x \equiv y\rfloor$ **using** *Ax2a God-def* **by** *smt*
lemma *Filter*: $\lfloor Filter\ P\rfloor$ **using** *Ax1 Ax4 MC NegProps PosProps Rsymm* **by** *smt*
lemma *UltraFilter*: $\lfloor UFilter\ P\rfloor$ **using** *Ax2a Filter* **by** *smt*
lemma *True* **nitpick**[*satisfy*,*card=1*,*eval*=$\lfloor P\ (\lambda x.\top)\rfloor$] **oops** — One model found of cardinality one

end

4.4 ThereIsNoEvil1.thy (Figure 10 of [2])

Importing Gödels modified axioms from Figure 7 we can prove that necessarily there exists no entity that possesses all non-positive (=negative) properties.

theory *ThereIsNoEvil1* **imports** *GoedelVariantHOML2*
begin

definition *Evil* (*Evil*) **where** $Evil\ x \equiv \forall\varphi.\ \neg\ P\ \varphi \supset \varphi\ x$

theorem *NecNoEvil*: $\lfloor\Box(\neg(\exists^E x.\ \ Evil\ x))\rfloor$
— sledgehammer(Ax2a Ax4 Evil_def) — Proof found
proof −
have $\lfloor\neg P(\lambda y.\bot)\rfloor$ **using** *Ax2a Ax4* **by** *blast*
hence $\lfloor(\forall^E x.\ \ Evil\ x \supset (\lambda y.\bot)\ x)\rfloor$ **using** *Evil-def* **by** *auto*
hence $\lfloor(\forall^E x.\ \ Evil\ x \supset \bot)\rfloor$ **by** *auto*
hence $\lfloor(\exists^E x.\ \ Evil\ x) \supset \bot\rfloor$ **by** *auto*
hence $\lfloor\neg(\exists^E x.\ Evil\ x)\rfloor$ **by** *blast*
thus *?thesis* **by** *blast*
qed

end

4.5 ThereIsNoEvil2.thy (Figure 11 of [2])

Importing Gödels modified axioms from Figure 8 we can prove that necessarily there exists no entity that possesses all non-positive (=negative) properties.

theory *ThereIsNoEvil2* **imports** *GoedelVariantHOML3*
begin

definition *Evil* (*Evil*) **where** $Evil\ x \equiv \forall \varphi.\ \neg\ P\ \varphi \supset \varphi\ x$
theorem *NecNoEvil*: $\lfloor \Box(\neg(\exists^E x.\ \ Evil\ x)) \rfloor$
— sledgehammer(Ax1Gen Ax2a Evil_def) **oops** — Proof found

end

5 Scotts variant

5.1 ScottVariantHOML.thy (Figure 12 of [2])

Verification of Scotts variant of Gödels ontological argument. Actualist quantifiers (avoiding existential import) are used for quantification over entities, otherwise possibilist quantifiers are used.

theory *ScottVariantHOML* **imports** *HOMLinHOL ModalFilter*
begin

consts $PositiveProperty::(e\Rightarrow\sigma)\Rightarrow\sigma\ (P)$

axiomatization where *A1*: $\lfloor \neg P\ \varphi \leftrightarrow P\ {}^{\sim}\varphi \rfloor$

axiomatization where *A2*: $\lfloor P\ \varphi \wedge \Box(\forall^E y.\ \varphi\ y \supset \psi\ y) \supset P\ \psi \rfloor$

theorem *T1*: $\lfloor P\ \varphi \supset \Diamond(\exists^E x.\ \varphi\ x) \rfloor$ **using** *A1 A2* **by** *blast*

definition *God* (*G*) **where** $G\ x \equiv \forall \varphi.\ P\ \varphi \supset \varphi\ x$

axiomatization where *A3*: $\lfloor P\ G \rfloor$

theorem *Coro*: $\lfloor \Diamond(\exists^E x.\ G\ x) \rfloor$ **using** *A3 T1* **by** *blast*

axiomatization where *A4*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi \rfloor$

definition *Ess* (*-Ess.-*) **where** $\varphi\ Ess.\ x \equiv \varphi\ x \wedge (\forall \psi.\ \psi\ x \supset \Box(\forall^E y.\ \varphi\ y \supset \psi\ y))$

theorem *T2*: $\lfloor G\ x \supset G\ Ess.\ x \rfloor$ **using** *A1 A4 Ess-def God-def* **by** *fastforce*

definition *NecExist* (*NE*) **where** $NE\ x \equiv \forall \varphi.\ \varphi\ Ess.\ x \supset \Box(\exists^E x.\ \varphi\ x)$

axiomatization where *A5*: $\lfloor P\ NE \rfloor$

lemma *True* **nitpick**[*satisfy*,*card=1*,*eval*=$\lfloor P\ (\lambda x.\top) \rfloor$] **oops** — One model found of cardinality one

theorem *T3*: $\lfloor \Box(\exists^E x.\ G\ x) \rfloor$
— sledgehammer(A5 Coro God_def NecExist_def Rsymm T2) — Proof found
proof −

have *1*: $\lfloor (G\ x \supset NE\ x) \wedge (G\ Ess.\ x \supset \Box(\exists^E x.\ G\ x)) \rfloor$ **using** *A5 Ess-def God-def NecExist-def* **by** *smt*
hence *2*: $\lfloor (\exists^E x.\ G\ x) \supset \Box(\exists^E x.\ G\ x) \rfloor$ **using** *A5 God-def NecExist-def T2* **by** *smt*
hence *3*: $\lfloor \Diamond(\exists^E x.\ G\ x) \supset (\Diamond(\Box(\exists^E x.\ G\ x)) \supset \Box(\exists^E x.\ G\ x)) \rfloor$ **using** *Rsymm* **by** *blast*
thus *?thesis* **using** *2 Coro* **by** *blast*
qed

lemma *MC*: $\lfloor \varphi \supset \Box\varphi \rfloor$
— sledgehammer(A1 A4 God_def Rsymm T3) — Proof found
proof − {**fix** *w* **fix** *Q*
have *1*: $\forall x.(G\ x\ w \longrightarrow (\forall Z.\ Z\ x \supset \Box(\forall^E z.((G\ z) \supset (Z\ z))))\ w)$ **using** *A1 A4 God-def* **by** *smt*
have *2*: $(\exists x.\ G\ x\ w) \longrightarrow ((Q \supset \Box(\forall^E z.((G\ z) \supset Q)))\ w)$ **using** *1* **by** *force*
have *3*: $(Q \supset \Box Q)\ w$ **using** *2 T3 Rsymm* **by** *blast*}
thus *?thesis* **by** *auto*
qed

lemma *PosProps*: $\lfloor P\ (\lambda x.\top) \wedge P\ (\lambda x.\ x = x) \rfloor$ **using** *A1 A2* **by** *blast*
lemma *NegProps*: $\lfloor \neg P(\lambda x.\bot) \wedge \neg P(\lambda x.\ x \neq x) \rfloor$ **using** *A1 A2* **by** *blast*
lemma *UniqueEss1*: $\lfloor \varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\forall^E y.\ \varphi\ y \leftrightarrow \psi\ y) \rfloor$ **using** *Ess-def* **by** *smt*
lemma *UniqueEss2*: $\lfloor \varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\varphi \equiv \psi) \rfloor$ **nitpick**[*card i=1*] **oops** — Countermodel found
lemma *UniqueEss3*: $\lfloor \varphi\ Ess.\ x \supset \Box(\forall^E y.\ \varphi\ y \supset y \equiv x) \rfloor$ **using** *Ess-def MC* **by** *auto*
lemma *Monotheism*: $\lfloor G\ x \wedge G\ y \supset x \equiv y \rfloor$ **using** *A1 God-def* **by** *smt*
lemma *Filter*: $\lfloor Filter\ P \rfloor$ **using** *A1 God-def Rsymm T1 T3* **by** (*smt* (*verit*, *best*))
lemma *UltraFilter*: $\lfloor UFilter\ P \rfloor$ **using** *Filter A1* **by** *blast*
lemma *True* **nitpick**[*satisfy*,*card=1*,*eval*=$\lfloor P\ (\lambda x.\bot) \rfloor$] **oops** — One model found of cardinality one

end

5.2 HOMLinHOLonlyK.thy (slight variation of Figure 3 of [2])

Shallow embedding of higher-order modal logic (HOML) in the classical higher-order logic (HOL) of Isabelle/HOL utilizing the LogiKEy methodology. Here logic K is introduced.

theory *HOMLinHOLonlyK* **imports** *Main*
begin

— Global parameters setting for the model finder nitpick and the parser; unimport for the reader
nitpick-params[*user-axioms*,*expect=genuine*,*show-all*,*format=2*,*max-genuine=3*]
declare[[*syntax-ambiguity-warning=false*]]

— Type i is associated with possible worlds and type e with entities
typedecl i — Possible worlds
typedecl e — Individuals/entities
type-synonym $\sigma = i \Rightarrow bool$ — World-lifted propositions
type-synonym $\tau = e \Rightarrow \sigma$ — modal properties

consts $R::i \Rightarrow i \Rightarrow bool$ (-**r**-) — Accessibility relation between worlds

— Logical connectives (operating on truth-sets)
abbreviation $Mbot::\sigma$ ($\bot$) **where** $\bot \equiv \lambda w.\ False$
abbreviation $Mtop::\sigma$ ($\top$) **where** $\top \equiv \lambda w.\ True$
abbreviation $Mneg::\sigma \Rightarrow \sigma$ ($\neg$- $[52]53$) **where** $\neg\varphi \equiv \lambda w.\ \neg(\varphi\ w)$
abbreviation $Mand::\sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixl** $\wedge$ 50) **where** $\varphi \wedge \psi \equiv \lambda w.\ \varphi\ w \wedge \psi\ w$
abbreviation $Mor::\sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixl** $\vee$ 49) **where** $\varphi \vee \psi \equiv \lambda w.\ \varphi\ w \vee \psi\ w$
abbreviation $Mimp::\sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixr** $\supset$ 48) **where** $\varphi \supset \psi \equiv \lambda w.\ \varphi\ w \longrightarrow \psi\ w$
abbreviation $Mequiv::\sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixl** $\leftrightarrow$ 47) **where** $\varphi \leftrightarrow \psi \equiv \lambda w.\ \varphi\ w \longleftrightarrow \psi\ w$
abbreviation $Mbox::\sigma \Rightarrow \sigma$ ($\Box$- $[54]55$) **where** $\Box\varphi \equiv \lambda w. \forall v.\ w\ \mathbf{r}\ v \longrightarrow \varphi\ v$
abbreviation $Mdia::\sigma \Rightarrow \sigma$ ($\Diamond$- $[54]55$) **where** $\Diamond\varphi \equiv \lambda w. \exists v.\ w\ \mathbf{r}\ v \wedge \varphi\ v$
abbreviation $Mprimeq::'a \Rightarrow 'a \Rightarrow \sigma$ (-$\mathbf{=}$-) **where** $x \mathbf{=} y \equiv \lambda w.\ x = y$
abbreviation $Mprimneg::'a \Rightarrow 'a \Rightarrow \sigma$ (-$\mathbf{\neq}$-) **where** $x \mathbf{\neq} y \equiv \lambda w.\ x \neq y$
abbreviation $Mnegpred::\tau \Rightarrow \tau$ ($\sim$-) **where** $\sim\Phi \equiv \lambda x. \lambda w.\ \neg\Phi\ x\ w$
abbreviation $Mconpred::\tau \Rightarrow \tau \Rightarrow \tau$ (**infixl** **.** 50) **where** $\Phi \mathbf{.} \Psi \equiv \lambda x. \lambda w.\ \Phi\ x\ w \wedge \Psi\ x\ w$
abbreviation $Mexclor::\sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixl** $\vee^e$ 49) **where** $\varphi \vee^e \psi \equiv (\varphi \vee \psi) \wedge \neg(\varphi \wedge \psi)$

— Possibilist quantifiers (polymorphic)
abbreviation $Mallposs::('a \Rightarrow \sigma) \Rightarrow \sigma$ ($\forall$) **where** $\forall \Phi \equiv \lambda w. \forall x.\ \Phi\ x\ w$
abbreviation $Mallpossb$ (**binder** $\forall$ $[8]9$) **where** $\forall x.\ \varphi(x) \equiv \forall \varphi$
abbreviation $Mexiposs::('a \Rightarrow \sigma) \Rightarrow \sigma$ ($\exists$) **where** $\exists \Phi \equiv \lambda w. \exists x.\ \Phi\ x\ w$
abbreviation $Mexipossb$ (**binder** $\exists$ $[8]9$) **where** $\exists x.\ \varphi(x) \equiv \exists \varphi$

— Actualist quantifiers (for individuals/entities)
consts $existsAt::e \Rightarrow \sigma$ (-**@**-)
abbreviation $Mallact::(e \Rightarrow \sigma) \Rightarrow \sigma$ ($\forall^E$) **where** $\forall^E \Phi \equiv \lambda w. \forall x.\ x \mathbf{@} w \longrightarrow \Phi\ x\ w$
abbreviation $Mallactb$ (**binder** $\forall^E$ $[8]9$) **where** $\forall^E x.\ \varphi(x) \equiv \forall^E \varphi$
abbreviation $Mexiact::(e \Rightarrow \sigma) \Rightarrow \sigma$ ($\exists^E$) **where** $\exists^E \Phi \equiv \lambda w. \exists x.\ x \mathbf{@} w \wedge \Phi\ x\ w$
abbreviation $Mexiactb$ (**binder** $\exists^E$ $[8]9$) **where** $\exists^E x.\ \varphi(x) \equiv \exists^E \varphi$

— Leibniz equality (polymorphic)
abbreviation $Mleibeq::'a \Rightarrow 'a \Rightarrow \sigma$ (-$\mathbf{\equiv}$-) **where** $x \mathbf{\equiv} y \equiv \forall P.\ P\ x \supset P\ y$

— Meta-logical predicate for global validity
abbreviation $Mvalid::\sigma \Rightarrow bool$ ($\lfloor$-$\rfloor$) **where** $\lfloor \psi \rfloor \equiv \forall w.\ \psi\ w$

end

5.3 ScottVariantHOMLinK.thy (Figure 13 of [2])

Scotts variant of Gödels argument fails for base logic K (but only it the last step).

theory *ScottVariantHOMLinK* **imports** *HOMLinHOLonlyK*
begin

consts *PositiveProperty*::($e \Rightarrow \sigma$)$\Rightarrow \sigma$ (P)

axiomatization where *A1*: $\lfloor \neg P\ \varphi \leftrightarrow P\ {}^{\sim}\varphi \rfloor$

axiomatization where *A2*: $\lfloor P\ \varphi \wedge \Box(\forall^E y.\ \varphi\ y \supset \psi\ y) \supset P\ \psi \rfloor$

theorem *T1*: $\lfloor P\ \varphi \supset \Diamond(\exists^E x.\ \varphi\ x)\rfloor$ **using** *A1 A2* **by** *blast*

definition *God* (G) **where** $G\ x \equiv \forall \varphi.\ P\ \varphi \supset \varphi\ x$

axiomatization where *A3*: $\lfloor P\ G \rfloor$

theorem *Coro*: $\lfloor \Diamond(\exists^E x.\ G\ x)\rfloor$ **nitpick**[*satisfy*,*eval*=*G*] **using** *A3 T1* **by** *blast*

axiomatization where *A4*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi \rfloor$

definition *Ess* (-*Ess*.-) **where** φ *Ess*. $x \equiv \varphi\ x \wedge (\forall \psi.\ \psi\ x \supset \Box(\forall^E y.\ \varphi\ y \supset \psi\ y))$

theorem *T2*: $\lfloor G\ x \supset G$ *Ess*. $x \rfloor$ **using** *A1 A4 Ess-def God-def* **by** *fastforce*

definition *NecExist* (*NE*) **where** *NE* $x \equiv \forall \varphi.\ \varphi$ *Ess*. $x \supset \Box(\exists^E x.\ \varphi\ x)$

axiomatization where *A5*: $\lfloor P\ NE \rfloor$

lemma *True* **nitpick**[*satisfy*,*card*=*1*,*eval*=$\lfloor P\ (\lambda x.\top)\rfloor$] **oops** — One model found of cardinality one

theorem *T3*: $\lfloor \Box(\exists^E x.\ G\ x)\rfloor$ **nitpick**[*card e*=*1*, *card i*=*2*, *eval*=*G*] **oops** — Counterexample

lemma *MC*: $\lfloor \varphi \supset \Box \varphi \rfloor$ **nitpick**[*card e*=*1*, *card i*=*2*, *eval*=*G*] **oops** — Counterexample

end

5.4 ScottVariantHOMLAndersonQuant.thy (Figure 15 of [2])

Verification of Scotts variant of Gödels argument with a mixed use of actualist and possibilist quantifiers for entities; cf. Footnote 20 in [2].

theory *ScottVariantHOMLAndersonQuant* **imports** *HOMLinHOL ModalFilter*

begin

consts *PositiveProperty*::$(e{\Rightarrow}\sigma){\Rightarrow}\sigma$ (P)

axiomatization where *A1*: $\lfloor \neg P\ \varphi \leftrightarrow P\ {}^{\sim}\varphi \rfloor$

axiomatization where *A2*: $\lfloor P\ \varphi \wedge \Box(\forall y.\ \varphi\ y \supset \psi\ y) \supset P\ \psi \rfloor$

theorem *T1*: $\lfloor P\ \varphi \supset \Diamond(\exists x.\ \varphi\ x) \rfloor$ **using** *A1 A2* **by** *smt*

definition *God* (G) **where** $G\ x \equiv \forall \varphi.\ P\ \varphi \supset \varphi\ x$

axiomatization where *A3*: $\lfloor P\ G \rfloor$

theorem *Coro*: $\lfloor \Diamond(\exists x.\ G\ x) \rfloor$ **using** *A3 T1* **by** *blast*

axiomatization where *A4*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi \rfloor$

definition *Ess* (-*Ess.*-) **where** $\varphi\ Ess.\ x \equiv \varphi\ x \wedge (\forall \psi.\ \psi\ x \supset \Box(\forall y{::}e.\ \varphi\ y \supset \psi\ y))$

theorem *T2*: $\lfloor G\ x \supset G\ Ess.\ x \rfloor$ **using** *A1 A4 Ess-def God-def* **by** *smt*

definition *NecExist* (*NE*) **where** $NE\ x \equiv \forall \varphi.\ \varphi\ Ess.\ x \supset \Box(\exists^E x.\ \varphi\ x)$

axiomatization where *A5*: $\lfloor P\ NE \rfloor$

lemma *True* **nitpick**[*satisfy*,*card=1*,*eval*=$\lfloor P\ (\lambda x.\top) \rfloor$] **oops** — One model found of cardinality one

theorem *T3*: $\lfloor \Box(\exists^E x.\ G\ x) \rfloor$
  — sledgehammer(A5 Coro God_def NecExist_def Rsymm T2) — Proof found
  proof –
    have *1*: $\lfloor (G\ x \supset NE\ x) \wedge (G\ Ess.\ x \supset \Box(\exists^E x.\ G\ x)) \rfloor$ **using** *A5 Ess-def God-def NecExist-def* **by** *smt*
    hence *2*: $\lfloor (\exists x.\ G\ x) \supset \Box(\exists^E x.\ G\ x) \rfloor$ **using** *A5 God-def NecExist-def T2* **by** *smt*
    hence *3*: $\lfloor \Diamond(\exists x.\ G\ x) \supset (\Diamond(\Box(\exists x.\ G\ x)) \supset \Box(\exists^E x.\ G\ x)) \rfloor$ **using** *Rsymm* **by** *blast*
    thus *?thesis* **using** *2 Coro* **by** *blast*
  qed

lemma *MC*: $\lfloor \varphi \supset \Box\varphi \rfloor$
  — sledgehammer(A1 A4 God_def Rsymm T3) — Proof found
  proof – **{fix** *w* **fix** *Q*
    have *1*: $\forall x.(G\ x\ w \longrightarrow (\forall Z.\ Z\ x \supset \Box(\forall z.((G\ z) \supset (Z\ z))))\ w)$ **using** *A1 A4 God-def* **by** *smt*
    have *2*: $(\exists x.\ G\ x\ w) \longrightarrow ((Q \supset \Box(\forall z.((G\ z) \supset Q)))\ w)$ **using** *1* **by** *force*
    have *3*: $(Q \supset \Box Q)\ w$ **using** *2 T3 Rsymm* **by** *blast***}**

thus *?thesis* **by** *auto*
qed

lemma *PosProps*: $\lfloor P\ (\lambda x.\top) \wedge P\ (\lambda x.\ x = x)\rfloor$ **using** *A1 A2* **by** *blast*
lemma *NegProps*: $\lfloor \neg P(\lambda x.\bot) \wedge \neg P(\lambda x.\ x \neq x)\rfloor$ **using** *A1 A2* **by** *blast*
lemma *UniqueEss1*: $\lfloor \varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\forall y.\ \varphi\ y \leftrightarrow \psi\ y)\rfloor$ **using** *Ess-def* **by** *smt*
lemma *UniqueEss2*: $\lfloor \varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\varphi = \psi)\rfloor$ **nitpick**[*card i=2*] **oops** — Countermodel found
lemma *UniqueEss3*: $\lfloor \varphi\ Ess.\ x \supset \Box(\forall y.\ \varphi\ y \supset y \equiv x)\rfloor$ **using** *Ess-def MC* **by** *auto*
lemma *Monotheism*: $\lfloor G\ x \wedge G\ y \supset x \equiv y\rfloor$ **using** *A1 God-def* **by** *smt*
lemma *Filter*: $\lfloor Filter\ P\rfloor$ **using** *A1 God-def Rsymm T1 T3* **by** (*smt* (*verit*, *best*))
lemma *UltraFilter*: $\lfloor UFilter\ P\rfloor$ **using** *Filter A1* **by** *blast*
lemma *True* **nitpick**[*satisfy*,*card=1*,*eval=*$\lfloor P\ (\lambda x.\bot)\rfloor$] **oops** — One model found of cardinality one

end

6 Appendix

6.1 GoedelVariantHOML1poss.thy (Figure 16 of [2])

Gödels axioms and definitions, as presented in the 1970 manuscript, are inconsistent. In contrast to Figure 6 we here use only possibilist quantifiers and still derive falsity.

theory *GoedelVariantHOML1poss* **imports** *HOMLinHOL*
begin

consts *PositiveProperty*::$(e \Rightarrow \sigma) \Rightarrow \sigma$ (P)

axiomatization where *Ax1*: $\lfloor P\ \varphi \wedge P\ \psi \supset P\ (\varphi \mathbin{\boldsymbol{.}} \psi)\rfloor$

axiomatization where *Ax2a*: $\lfloor P\ \varphi \vee^e P\ {}^\sim\varphi\rfloor$

definition *God* (G) **where** $G\ x \equiv \forall \varphi.\ P\ \varphi \supset \varphi\ x$

abbreviation *PropertyInclusion* (-$\supset_N$-) **where** $\varphi \supset_N \psi \equiv \Box(\forall y{::}e.\ \varphi\ y \supset \psi\ y)$

definition *Essence* (-*Ess.*-) **where** $\varphi\ Ess.\ x \equiv \forall \psi.\ \psi\ x \supset (\varphi \supset_N \psi)$

axiomatization where *Ax2b*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi\rfloor$

lemma *Ax2b'*: $\lfloor \neg P\ \varphi \supset \Box(\neg P\ \varphi)\rfloor$ **using** *Ax2a Ax2b* **by** *blast*

theorem *Th1*: $\lfloor G\ x \supset G\ Ess.\ x\rfloor$ **using** *Ax2a Ax2b Essence-def God-def* **by** (*smt* (*verit*))

definition *NecExist* (*E*) **where** $E\ x \equiv \forall \varphi.\ \varphi\ Ess.\ x \supset \Box(\exists x.\ \varphi\ x)$

axiomatization where *Ax3*: $\lfloor P\ E \rfloor$

theorem *Th2*: $\lfloor G\ x \supset \Box(\exists y.\ G\ y)\rfloor$ **using** *Ax3 Th1 God-def NecExist-def* **by** *smt*

theorem *Th3*: $\lfloor \Diamond(\exists x.\ G\ x) \supset \Box(\exists y.\ G\ y)\rfloor$
— sledgehammer(Th2 Rsymm) — Proof found
proof −
have *1*: $\lfloor(\exists x.\ G\ x) \supset \Box(\exists y.\ G\ y)\rfloor$ **using** *Th2* **by** *blast*
have *2*: $\lfloor\Diamond(\exists x.\ G\ x) \supset \Diamond\Box(\exists y.\ G\ y)\rfloor$ **using** *1* **by** *blast*
have *3*: $\lfloor\Diamond(\exists x.\ G\ x) \supset \Box(\exists y.\ G\ y)\rfloor$ **using** *2 Rsymm* **by** *blast*
thus *?thesis* **by** *blast*
qed

axiomatization where *Ax4*: $\lfloor P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi\rfloor$

lemma *True* **nitpick**[*satisfy*,*expect=unknown*] **oops** — No model found

lemma *EmptyEssL*: $\lfloor(\lambda y.\bot)\ Ess.\ x\rfloor$ **using** *Essence-def* **by** *metis*

theorem *Inconsistency*: *False*
— sledgehammer(Ax2a Ax3 Ax4 EmptyEssL NecExist_def) — Proof found
proof −
have *1*: $\lfloor\neg(P\ (\lambda x.\bot))\rfloor$ **using** *Ax2a Ax4* **by** *blast*
have *2*: $\lfloor P\ (\lambda x.(\lambda y.\bot)\ Ess.\ x \supset \Box(\exists z{::}e.(\lambda y.\bot) z))\rfloor$ **using** *Ax3 Ax4 NecExist-def* **by** *smt*
have *3*: $\lfloor P\ (\lambda x.\Box(\exists z.\ (\lambda x.\bot)\ z))\rfloor$ **using** *2 EmptyEssL Ax4* **by** *smt*
have *4*: $\lfloor P\ (\lambda x.\Box\bot)\rfloor$ **using** *3* **by** *auto*
have *5*: $\lfloor P\ (\lambda x.\bot)\rfloor$ **using** *4 Ax2a Ax4* **by** *smt*
have *6*: $\lfloor\bot\rfloor$ **using** *1 5* **by** *blast*
thus *?thesis* **by** *blast*
qed

end

6.2 GoedelVariantHOML2poss.thy (Figure 17 of [2])

After an appropriate modification of the definition of essence, the inconsistency revealed in Figure 16 is avoided, and the argument can be successfully verified in modal logic S5 (indeed, as shown, only the modal schema B is actually needed). In contrast to Figure 7 we here use only possibilist quantifiers to obtain these results.

theory *GoedelVariantHOML2poss* **imports** *HOMLinHOL ModalFilter*
begin

consts *PositiveProperty*::$(e{\Rightarrow}\sigma){\Rightarrow}\sigma$ (P)

axiomatization where *Ax1*: $\lfloor P\ \varphi \wedge P\ \psi \supset P\ (\varphi \centerdot \psi)\rfloor$

axiomatization where *Ax2a*: $\lfloor P\ \varphi \vee^e P\ {}^\sim\varphi\rfloor$

definition *God* (G) **where** $G\ x \equiv \forall\varphi.\ P\ \varphi \supset \varphi\ x$

abbreviation *PropertyInclusion* (-$\supset_N$-) **where** $\varphi \supset_N \psi \equiv \Box(\forall y{::}e.\ \varphi\ y \supset \psi\ y)$

definition *Essence* (-*Ess.*-) **where** φ *Ess.* $x \equiv \varphi\ x \wedge (\forall\psi.\ \psi\ x \supset (\varphi \supset_N \psi))$

axiomatization where *Ax2b*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi\rfloor$

lemma *Ax2b′*: $\lfloor\neg P\ \varphi \supset \Box(\neg P\ \varphi)\rfloor$ **using** *Ax2a Ax2b* **by** *blast*

theorem *Th1*: $\lfloor G\ x \supset G$ *Ess.* $x\rfloor$ **using** *Ax2a Ax2b Essence-def God-def* **by** (*smt* (*verit*))

definition *NecExist* (E) **where** $E\ x \equiv \forall\varphi.\ \varphi$ *Ess.* $x \supset \Box(\exists x.\ \varphi\ x)$

axiomatization where *Ax3*: $\lfloor P\ E\rfloor$

theorem *Th2*: $\lfloor G\ x \supset \Box(\exists y.\ G\ y)\rfloor$ **using** *Ax3 Th1 God-def NecExist-def* **by** *smt*

theorem *Th3*: $\lfloor\Diamond(\exists x.\ G\ x) \supset \Box(\exists y.\ G\ y)\rfloor$
— sledgehammer(Th2 Rsymm) — Proof found
proof −
have *1*: $\lfloor(\exists x.\ G\ x) \supset \Box(\exists y.\ G\ y)\rfloor$ **using** *Th2* **by** *blast*
have *2*: $\lfloor\Diamond(\exists x.\ G\ x) \supset \Diamond\Box(\exists y.\ G\ y)\rfloor$ **using** *1* **by** *blast*
have *3*: $\lfloor\Diamond(\exists x.\ G\ x) \supset \Box(\exists y.\ G\ y)\rfloor$ **using** *2 Rsymm* **by** *blast*
thus *?thesis* **by** *blast*
qed

axiomatization where *Ax4*: $\lfloor P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi\rfloor$

lemma *True* **nitpick**[*satisfy*,*card=1*,*eval=*$\lfloor P\ (\lambda x.\bot)\rfloor$] **oops** — One model found of cardinality one

abbreviation *PosProps* $\Phi \equiv \forall\varphi.\ \Phi\ \varphi \supset P\ \varphi$
abbreviation *ConjOfPropsFrom* $\varphi\ \Phi \equiv \Box(\forall z.\ \varphi\ z \leftrightarrow (\forall\psi.\ \Phi\ \psi \supset \psi\ z))$
axiomatization where *Ax1Gen*: $\lfloor$(*PosProps* $\Phi \wedge$ *ConjOfPropsFrom* $\varphi\ \Phi) \supset P\ \varphi\rfloor$

lemma *L*: $\lfloor P\ G\rfloor$ **using** *Ax1Gen God-def* **by** *smt*

theorem *Th4*: $\lfloor\Diamond(\exists x.\ G\ x)\rfloor$ **using** *Ax2a Ax4 L* **by** *blast*

theorem *Th5*: $\lfloor\Box(\exists x.\ G\ x)\rfloor$ **using** *Th3 Th4* **by** *blast*

lemma *MC*: $\lfloor\varphi \supset \Box\varphi\rfloor$

— sledgehammer(Ax2a Ax2b Th5 God_def Rsymm) — Proof found
proof – {**fix** *w* **fix** *Q*
have *1*: $\forall x.(G\ x\ w \longrightarrow (\forall Z.\ Z\ x \supset \Box(\forall z.\ G\ z \supset Z\ z))\ w)$ **using** *Ax2a Ax2b God-def* **by** *smt*
have *2*: $(\exists x.\ G\ x\ w) \longrightarrow ((Q \supset \Box(\forall z.\ G\ z \supset Q))\ w)$ **using** *1* **by** *force*
have *3*: $(Q \supset \Box Q)\ w$**using** *2 Th5 Rsymm* **by** *blast*}
thus *?thesis* **by** *auto*
qed

lemma *PosProps*: $\lfloor P\ (\lambda x.\top) \wedge P\ (\lambda x.\ x = x)\rfloor$ **using** *Ax2a Ax4* **by** *blast*
lemma *NegProps*: $\lfloor\neg P(\lambda x.\bot) \wedge \neg P(\lambda x.\ x \neq x)\rfloor$ **using** *Ax2a Ax4* **by** *blast*
lemma *UniqueEss1*: $\lfloor\varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\forall y.\ \varphi\ y \leftrightarrow \psi\ y)\rfloor$ **using** *Essence-def* **by** *smt*
lemma *UniqueEss2*: $\lfloor\varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\varphi \equiv \psi)\rfloor$ **nitpick**[*card i=2*] **oops** — Countermodel found
lemma *UniqueEss3*: $\lfloor\varphi\ Ess.\ x \supset \Box(\forall y.\ \varphi\ y \supset y \equiv x)\rfloor$ **using** *Essence-def MC* **by** *auto*
lemma *Monotheism*: $\lfloor G\ x \wedge G\ y \supset x \equiv y\rfloor$ **using** *Ax2a God-def* **by** *smt*
lemma *Filter*: $\lfloor FilterP\ P\rfloor$ **using** *Ax1 Ax4 MC NegProps PosProps Rsymm* **by** *smt*
lemma *UltraFilter*: $\lfloor UFilterP\ P\rfloor$ **using** *Ax2a Filter* **by** *smt*
lemma *True* **nitpick**[*satisfy*,*card=1*,*eval*=$\lfloor P\ (\lambda x.\bot)\rfloor$] **oops** — One model found of cardinality one

end

6.3 GoedelVariantHOML3poss.thy (Figure 18 of [2])

After an appropriate modification of the definition of necessary property implication, the inconsistency shown in Figure 16 is avoided, and the argument can be successfully verified. As shown here, this still holds when using possibilist quantifiers only.

theory *GoedelVariantHOML3poss* **imports** *HOMLinHOL ModalFilter*
begin

consts *PositiveProperty*::$(e \Rightarrow \sigma) \Rightarrow \sigma$ (P)

axiomatization where *Ax1*: $\lfloor P\ \varphi \wedge P\ \psi \supset P\ (\varphi \centerdot \psi)\rfloor$

axiomatization where *Ax2a*: $\lfloor P\ \varphi \vee^e P\ {}^\sim\varphi\rfloor$

definition *God* (G) **where** $G\ x \equiv \forall\varphi.\ P\ \varphi \supset \varphi\ x$

abbreviation *PropertyInclusion* $(\text{-}\supset_N\text{-})$ **where** $\varphi \supset_N \psi \equiv \Box(\varphi \neq (\lambda x.\ \bot) \wedge (\forall y{::}e.\ \varphi\ y \supset \psi\ y))$

definition *Essence* (*-Ess.-*) **where** $\varphi\ Ess.\ x \equiv \forall\psi.\ \psi\ x \supset (\varphi \supset_N \psi)$

axiomatization where *Ax2b*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi\rfloor$

lemma *Ax2b′*: $\lfloor\neg P\ \varphi \supset \Box(\neg P\ \varphi)\rfloor$ **using** *Ax2a Ax2b* **by** *blast*

theorem *Th1*: $\lfloor G\ x \supset G\ Ess.\ x\rfloor$ **using** *Ax2a Ax2b Essence-def God-def* **by** (*smt* (*verit*))

definition *NecExist* (E) **where** $E\ x \equiv \forall\varphi.\ (\varphi\ Ess.\ x) \supset \Box(\exists x.\ \varphi\ x)$

axiomatization where *Ax3*: $\lfloor P\ E\rfloor$

theorem *Th2*: $\lfloor G\ x \supset \Box(\exists y.\ G\ y)\rfloor$ **using** *Ax3 Th1 God-def NecExist-def* **by** *smt*

theorem *Th3*: $\lfloor\Diamond(\exists x.\ G\ x) \supset \Box(\exists y.\ G\ y)\rfloor$
— sledgehammer(Th2 Rsymm) — Proof found
proof –
have *1*: $\lfloor(\exists x.\ G\ x) \supset \Box(\exists y.\ G\ y)\rfloor$ **using** *Th2* **by** *blast*
have *2*: $\lfloor\Diamond(\exists x.\ G\ x) \supset \Diamond\Box(\exists y.\ G\ y)\rfloor$ **using** *1* **by** *blast*
have *3*: $\lfloor\Diamond(\exists x.\ G\ x) \supset \Box(\exists y.\ G\ y)\rfloor$ **using** *2 Rsymm* **by** *blast*
thus *?thesis* **by** *blast*
qed

axiomatization where *Ax4*: $\lfloor P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi\rfloor$

lemma *True* **nitpick**[*satisfy*,*card=1*,*eval*=$\lfloor P\ (\lambda x.\bot)\rfloor$] **oops** — One model found of cardinality one

abbreviation *PosProps* $\Phi \equiv \forall\varphi.\ \Phi\ \varphi \supset P\ \varphi$
abbreviation *ConjOfPropsFrom* $\varphi\ \Phi \equiv \Box(\forall z.\ \varphi\ z \leftrightarrow (\forall\psi.\ \Phi\ \psi \supset \psi\ z))$
axiomatization where *Ax1Gen*: $\lfloor$(*PosProps* Φ $\wedge$ *ConjOfPropsFrom* $\varphi\ \Phi$) $\supset P\ \varphi\rfloor$

lemma *L*: $\lfloor P\ G\rfloor$ **using** *Ax1Gen God-def* **by** (*smt* (*verit*))

theorem *Th4*: $\lfloor\Diamond(\exists x.\ G\ x)\rfloor$
— sledgehammer[timeout=200](Ax2a L Ax1Gen) **oops** — sorry — Proof found

axiomatization where *Th4*: $\lfloor\Diamond(\exists x.\ G\ x)\rfloor$

theorem *Th5*: $\lfloor\Box(\exists x.\ G\ x)\rfloor$ **using** *Th3 Th4* **by** *blast*

lemma *MC*: $\lfloor\varphi \supset \Box\varphi\rfloor$
— sledgehammer(Ax2a Ax2b Th5 God_def Rsymm) — Proof found
proof – {**fix** *w* **fix** *Q*
have *1*: $\forall x.(G\ x\ w \longrightarrow (\forall Z.\ Z\ x \supset \Box(\forall z.\ G\ z \supset Z\ z))\ w)$ **using** *Ax2a Ax2b God-def* **by** *smt*
have *2*: $(\exists x.\ G\ x\ w) \longrightarrow ((Q \supset \Box(\forall z.\ G\ z \supset Q))\ w)$ **using** *1* **by** *force*
have *3*: $(Q \supset \Box Q)\ w$**using** *2 Th5 Rsymm* **by** *blast*}
thus *?thesis* **by** *auto*

qed

lemma *PosProps*: $\lfloor P\ (\lambda x.\top) \wedge P(\lambda x.\ x = x)\rfloor$ **using** *Ax2a Ax4 L Th4* **by** *smt*
lemma *NegProps*: $\lfloor\neg P(\lambda x.\bot) \wedge \neg P(\lambda x.\ x \neq x)\rfloor$ **using** *Ax2a Ax4 L Th4* **by** *smt*
lemma *UniqueEss1*: $\lfloor\varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\forall y.\ \varphi\ y \leftrightarrow \psi\ y)\rfloor$ **oops** — Unclear, open question
lemma *UniqueEss2*: $\lfloor\varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\varphi \equiv \psi)\rfloor$ **oops** — Unclear, open question
lemma *UniqueEss3*: $\lfloor\varphi\ Ess.\ x \supset \Box(\forall y.\ \varphi\ y \supset y \equiv x)\rfloor$ **using** *Essence-def MC* **by** *auto*
lemma *Monotheism*: $\lfloor G\ x \wedge G\ y \supset x \equiv y\rfloor$ **using** *Ax2a God-def* **by** *smt*
lemma *Filter*: $\lfloor FilterP\ P\rfloor$ **using** *Ax1 Ax4 MC NegProps PosProps Rsymm* **by** *smt*
lemma *UltraFilter*: $\lfloor UFilterP\ P\rfloor$ **using** *Ax2a Filter* **by** *smt*
lemma *True* **nitpick**[*satisfy*,*card=1*,*eval*=$\lfloor P\ (\lambda x.\top)\rfloor$] **oops** — One model found of cardinality one

end

6.4 ScottVariantHOMLposs.thy (Figure 19 of [2])

Scotts variant of Gödels ontological proof is still valid when using possibilist quantifiers only.

theory *ScottVariantHOMLposs* **imports** *HOMLinHOL ModalFilter*
begin

consts *PositiveProperty*::$(e\Rightarrow\sigma)\Rightarrow\sigma$ (P)

axiomatization where *A1*: $\lfloor\neg P\ \varphi \leftrightarrow P\ {}^{\sim}\varphi\rfloor$

axiomatization where *A2*: $\lfloor P\ \varphi \wedge \Box(\forall y.\ \varphi\ y \supset \psi\ y) \supset P\ \psi\rfloor$

theorem *T1*: $\lfloor P\ \varphi \supset \Diamond(\exists x.\ \varphi\ x)\rfloor$ **using** *A1 A2* **by** *blast*

definition *God* (G) **where** $G\ x \equiv \forall\varphi.\ P\ \varphi \supset \varphi\ x$

axiomatization where *A3*: $\lfloor P\ G\rfloor$

theorem *Coro*: $\lfloor\Diamond(\exists x.\ G\ x)\rfloor$ **using** *A3 T1* **by** *blast*

axiomatization where *A4*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi\rfloor$

definition *Ess* (*-Ess.-*) **where** $\varphi\ Ess.\ x \equiv \varphi\ x \wedge (\forall\psi.\ \psi\ x \supset \Box(\forall y{::}e.\ \varphi\ y \supset \psi\ y))$

theorem *T2*: $\lfloor G\ x \supset G\ Ess.\ x\rfloor$ **using** *A1 A4 Ess-def God-def* **by** *fastforce*

definition *NecExist* (NE) **where** $NE\ x \equiv \forall\varphi.\ \varphi\ Ess.\ x \supset \Box(\exists x.\ \varphi\ x)$

axiomatization where *A5*: $\lfloor P\ NE \rfloor$

lemma *True* **nitpick**[*satisfy*,*card=1*,*eval=*$\lfloor P\ (\lambda x.\bot)\rfloor$] **oops** — One model found of cardinality one

theorem *T3*: $\lfloor \Box(\exists\, x.\ G\ x)\rfloor$
— sledgehammer(A5 Coro God_def NecExist_def Rsymm T2) — Proof found
proof –
have *1*: $\lfloor (G\ x \supset NE\ x) \wedge (G\ Ess.\ x \supset \Box(\exists\, x.\ G\ x))\rfloor$ **using** *A5 Ess-def God-def NecExist-def* **by** *smt*
hence *2*: $\lfloor (\exists\, x.\ G\ x) \supset \Box(\exists\, x.\ G\ x)\rfloor$ **using** *A5 God-def NecExist-def T2* **by** *smt*
hence *3*: $\lfloor \Diamond(\exists\, x.\ G\ x) \supset (\Diamond(\Box(\exists\, x.\ G\ x)) \supset \Box(\exists\, x.\ G\ x))\rfloor$ **using** *Rsymm* **by** *blast*
thus *?thesis* **using** *2 Coro* **by** *blast*
qed

lemma *MC*: $\lfloor \varphi \supset \Box\varphi \rfloor$
— sledgehammer(A1 A4 God_def Rsymm T3) — Proof found
proof – {**fix** *w* **fix** *Q*
have *1*: $\forall\, x.(G\ x\ w \longrightarrow (\forall\, Z.\ Z\ x \supset \Box(\forall\, z.((G\ z) \supset (Z\ z))))\ w)$ **using** *A1 A4 God-def* **by** *smt*
have *2*: $(\exists\, x.\ G\ x\ w) \longrightarrow ((Q \supset \Box(\forall\, z.((G\ z) \supset Q)))\ w)$ **using** *1* **by** *force*
have *3*: $(Q \supset \Box Q)\ w$ **using** *2 T3 Rsymm* **by** *blast*}
thus *?thesis* **by** *auto*
qed

lemma *PosProps*: $\lfloor P\ (\lambda x.\top) \wedge P\ (\lambda x.\ x = x)\rfloor$ **using** *A1 A2* **by** *blast*
lemma *NegProps*: $\lfloor \neg P(\lambda x.\bot) \wedge \neg P(\lambda x.\ x \neq x)\rfloor$ **using** *A1 A2* **by** *blast*
lemma *UniqueEss1*: $\lfloor \varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\forall\, y.\ \varphi\ y \leftrightarrow \psi\ y)\rfloor$ **using** *Ess-def* **by** *smt*
lemma *UniqueEss2*: $\lfloor \varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\varphi \equiv \psi)\rfloor$ **nitpick**[*card i=2*] **oops** — Countermodel found
lemma *UniqueEss3*: $\lfloor \varphi\ Ess.\ x \supset \Box(\forall\, y.\ \varphi\ y \supset y \equiv x)\rfloor$ **using** *Ess-def MC* **by** *auto*
lemma *Monotheism*: $\lfloor G\ x \wedge G\ y \supset x \equiv y\rfloor$ **using** *A1 God-def* **by** *smt*
lemma *Filter*: $\lfloor$*FilterP P*$\rfloor$ **using** *A1 God-def Rsymm T1 T3* **by** (*smt* (*verit*, *best*))
lemma *UltraFilter*: $\lfloor$*UFilterP P*$\rfloor$ **using** *Filter A1* **by** *blast*
lemma *True* **nitpick**[*satisfy*,*card=1*,*eval=*$\lfloor P\ (\lambda x.\bot)\rfloor$] **oops** — One model found of cardinality one

end

6.5 EvilDerivable.thy (Figure 20 of [2])

The necessary existence of an Evil-like entity proved from (controversially) modified assumptions. By rejecting Gödels assumptions and instead postulating corresponding negative versions of them, as shown in the Figure 20, the necessary existence of Evil becomes derivable. The non-positive properties of this Evil-like entity are however identical to the positive properties of Gödels God-like entity.

theory *EvilDerivable* **imports** *HOMLinHOL ModalFilter*
begin

consts *PositiveProperty*::$(e \Rightarrow \sigma) \Rightarrow \sigma$ (P)

definition *Evil* (*Evil*) **where** $Evil\ x \equiv \forall \varphi.\ \neg\ P\ \varphi \supset \varphi\ x$

definition *Essence* (*-Ess.-*) **where** $\varphi\ Ess.\ x \equiv \varphi\ x \wedge (\forall \psi.\ \psi\ x \supset \Box(\forall^E y.\ \varphi\ y \supset \psi\ y))$

definition *NecExist* (E) **where** $E\ x \equiv \forall \varphi.\ \varphi\ Ess.\ x \supset \Box(\exists^E x.\ \varphi\ x)$

axiomatization where *A1*: $\lfloor \neg P\ \varphi \leftrightarrow P\ {}^{\sim}\varphi \rfloor$

axiomatization where *A2*: $\lfloor \neg P\ \varphi \wedge \Box(\forall^E y.\ \varphi\ y \supset \psi\ y) \supset \neg P\ \psi \rfloor$

axiomatization where *A4*: $\lfloor \neg P\ Evil \rfloor$

axiomatization where *A3*: $\lfloor \neg P\ \varphi \supset \Box\ (\neg P\ \varphi) \rfloor$

axiomatization where *A5*: $\lfloor \neg P\ E \rfloor$

lemma *True* **nitpick**[*satisfy*,*card i=1*,*eval*=$\lfloor P\ (\lambda x.\bot) \rfloor$,*eval*=$\lfloor P\ (\lambda x.\top) \rfloor$] **oops**
— Model found

theorem *T1*: $\lfloor \neg P\ \varphi \supset \Diamond(\exists^E x.\ \varphi\ x) \rfloor$ **using** *A1 A2* **by** *blast*

theorem *T2*: $\lfloor \Diamond(\exists^E x.\ Evil\ x) \rfloor$ **using** *A4 T1* **by** *blast*

theorem *T3*: $\lfloor Evil\ x \supset Evil\ Ess.\ x \rfloor$ **using** *A1 A3 Essence-def Evil-def* **by** (*smt* (*verit*, *best*))

theorem *T4*: $\lfloor \Diamond(\exists^E x.\ Evil\ x) \supset \Box(\exists^E y.\ Evil\ y) \rfloor$ **using** *A5 Evil-def NecExist-def Rsymm T3* **by** *smt*

theorem *T5*: $\lfloor \Box(\exists^E x.\ Evil\ x) \rfloor$ **using** *T2 T4* **by** *presburger*

lemma *MC*: $\lfloor \varphi \supset \Box\varphi \rfloor$
— sledgehammer(A1 A3 T5 Evil_def Rsymm) **oops** — proof found

lemma *PosProps*: $\lfloor P\ (\lambda x.\bot) \wedge P\ (\lambda x.\ x \neq x) \rfloor$ **using** *A1 A2* **by** *blast*
lemma *NegProps*: $\lfloor \neg P(\lambda x.\top) \wedge \neg P(\lambda x.\ x = x) \rfloor$ **using** *A1 A2* **by** *blast*
lemma *UniqueEss1*: $\lfloor \varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\forall^E y.\ \varphi\ y \leftrightarrow \psi\ y) \rfloor$ **using** *Essence-def* **by** *smt*
lemma *UniqueEss2*: $\lfloor \varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\varphi = \psi) \rfloor$ **nitpick**[*card i=2*] **oops**
— Countermodel found
lemma *Monoevilism*: $\lfloor Evil\ x \wedge Evil\ y \supset x \equiv y \rfloor$ **using** *A1 Evil-def* **by** *smt*

lemma *Filter*: $\lfloor Filter\ (\lambda\ \varphi.\ \neg P\ \varphi)\rfloor$ **using** *A1 Evil-def Rsymm T1 T5* **by** (*smt* (*verit*, *best*))
lemma *UltraFilter*: $\lfloor UFilter\ (\lambda\ \varphi.\ \neg P\ \varphi)\rfloor$ **using** *Filter A1* **by** *blast*

end

7 Further Appendices

7.1 GoedelVariantHOML1AndersonQuan.thy

The same as GoedelVariantHOML1, but now for a mixed use of actualist and possibilist quantifiers for entities; cf. Footnote 20 in [2].

theory *GoedelVariantHOML1AndersonQuant* **imports** *HOMLinHOL*
begin

consts *PositiveProperty*::$(e\Rightarrow\sigma)\Rightarrow\sigma$ (P)

axiomatization where *Ax1*: $\lfloor P\ \varphi \wedge P\ \psi \supset P\ (\varphi\ \mathbf{.}\ \psi)\rfloor$

axiomatization where *Ax2a*: $\lfloor P\ \varphi \vee^e P\ {}^\sim\varphi\rfloor$

definition *God* (G) **where** $G\ x \equiv \forall\varphi.\ P\ \varphi \supset \varphi\ x$

abbreviation *PropertyInclusion* $(\text{-}\supset_N\text{-})$ **where** $\varphi \supset_N \psi \equiv \Box(\forall y{::}e.\ \varphi\ y \supset \psi\ y)$

definition *Essence* (*-Ess.-*) **where** $\varphi\ Ess.\ x \equiv \forall\psi.\ \psi\ x \supset (\varphi \supset_N \psi)$

axiomatization where *Ax2b*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi\rfloor$

lemma *Ax2b'*: $\lfloor\neg P\ \varphi \supset \Box(\neg P\ \varphi)\rfloor$ **using** *Ax2a Ax2b* **by** *blast*

theorem *Th1*: $\lfloor G\ x \supset G\ Ess.\ x\rfloor$ **using** *Ax2a Ax2b Essence-def God-def* **by** (*smt* (*verit*))

definition *NecExist* (E) **where** $E\ x \equiv \forall\varphi.\ \varphi\ Ess.\ x \supset \Box(\exists^E x.\ \varphi\ x)$

axiomatization where *Ax3*: $\lfloor P\ E\rfloor$

theorem *Th2*: $\lfloor G\ x \supset \Box(\exists^E y.\ G\ y)\rfloor$ **using** *Ax3 Th1 God-def NecExist-def* **by** *smt*

theorem *Th3*: $\lfloor\Diamond(\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y)\rfloor$
— sledgehammer(Th2 Rsymm) — Proof found
proof –
have *1*: $\lfloor(\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y)\rfloor$ **using** *Th2* **by** *blast*
have *2*: $\lfloor\Diamond(\exists^E x.\ G\ x) \supset \Diamond\Box(\exists^E y.\ G\ y)\rfloor$ **using** *1* **by** *blast*
have *3*: $\lfloor\Diamond(\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y)\rfloor$ **using** *2 Rsymm* **by** *blast*
thus *?thesis* **by** *blast*

qed

axiomatization where *Ax4*: $\lfloor P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi \rfloor$

lemma *True* **nitpick**[*satisfy*,*expect*=*unknown*] **oops** — No model found

lemma *EmptyEssL*: $\lfloor (\lambda y.\bot)\ Ess.\ x \rfloor$ **using** *Essence-def* **by** *auto*

theorem *Inconsistency*: *False*
— sledgehammer(Ax2a Ax3 Ax4 EmptyEssL NecExist_def) — Proof found
proof –
have *1*: $\lfloor \neg(P\ (\lambda x.\bot)) \rfloor$ **using** *Ax2a Ax4* **by** *blast*
have *2*: $\lfloor P\ (\lambda x.(\lambda y.\bot)\ Ess.\ x \supset \Box(\exists^E z.\ (\lambda y.\bot) z)) \rfloor$ **using** *Ax3 Ax4 NecExist-def* **by** (*smt* (*verit*))
have *3*: $\lfloor P\ (\lambda x.\Box(\exists^E z.\ (\lambda x.\bot)\ z)) \rfloor$ **using** *2 EmptyEssL* **by** *simp*
have *4*: $\lfloor P\ (\lambda x.\Box\bot) \rfloor$ **using** *3* **by** *auto*
have *5*: $\lfloor P\ (\lambda x.\bot) \rfloor$ **using** *4 Ax2a Ax4* **by** *smt*
have *6*: $\lfloor \bot \rfloor$ **using** *1 5* **by** *blast*
thus *?thesis* **by** *blast*
qed

end

7.2 GoedelVariantHOML2AndersonQuan.thy

The same as GoedelVariantHOML2, but now for a mixed use of actualist and possibilist quantifiers for entities; cf. Footnote 20 in [2].

theory *GoedelVariantHOML2AndersonQuant* **imports** *HOMLinHOL ModalFilter*
begin

consts *PositiveProperty*::$(e \Rightarrow \sigma) \Rightarrow \sigma$ (P)

axiomatization where *Ax1*: $\lfloor P\ \varphi \wedge P\ \psi \supset P\ (\varphi \centerdot \psi) \rfloor$

axiomatization where *Ax2a*: $\lfloor P\ \varphi \vee^e P\ {}^\sim\varphi \rfloor$

definition *God* (G) **where** $G\ x \equiv \forall \varphi.\ P\ \varphi \supset \varphi\ x$

abbreviation *PropertyInclusion* ($\text{-}\supset_N\text{-}$) **where** $\varphi \supset_N \psi \equiv \Box(\forall y{::}e.\ \varphi\ y \supset \psi\ y)$

definition *Essence* (*-Ess.-*) **where** $\varphi\ Ess.\ x \equiv \varphi\ x \wedge (\forall \psi.\ \psi\ x \supset (\varphi \supset_N \psi))$

axiomatization where *Ax2b*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi \rfloor$

lemma *Ax2b'*: $\lfloor \neg P\ \varphi \supset \Box(\neg P\ \varphi) \rfloor$ **using** *Ax2a Ax2b* **by** *blast*

theorem *Th1*: $\lfloor G\ x \supset G\ Ess.\ x \rfloor$ **using** *Ax2a Ax2b Essence-def God-def* **by** (*smt* (*verit*))

definition *NecExist* (*E*) **where** $E\ x \equiv \forall\varphi.\ \varphi\ Ess.\ x \supset \Box(\exists^E x.\ \varphi\ x)$

axiomatization where *Ax3*: $\lfloor P\ E\rfloor$

theorem *Th2*: $\lfloor G\ x \supset \Box(\exists^E y.\ G\ y)\rfloor$ **using** *Ax3 Th1 God-def NecExist-def* **by** *smt*

theorem *Th3*: $\lfloor\Diamond(\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y)\rfloor$
— sledgehammer(Th2 Rsymm) — Proof found
proof –
have *1*: $\lfloor(\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y)\rfloor$ **using** *Th2* **by** *blast*
have *2*: $\lfloor\Diamond(\exists^E x.\ G\ x) \supset \Diamond\Box(\exists^E y.\ G\ y)\rfloor$ **using** *1* **by** *blast*
have *3*: $\lfloor\Diamond(\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y)\rfloor$ **using** *2 Rsymm* **by** *blast*
thus *?thesis* **by** *blast*
qed

axiomatization where *Ax4*: $\lfloor P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi\rfloor$

lemma *True* **nitpick**[*satisfy*,*card=1*,*eval=*$\lfloor P\ (\lambda x.\bot)\rfloor$] **oops** — One model found of cardinality one

abbreviation *PosProps* $\Phi \equiv \forall\varphi.\ \Phi\ \varphi \supset P\ \varphi$
abbreviation *ConjOfPropsFrom* $\varphi\ \Phi \equiv \Box(\forall z.\ \varphi\ z \leftrightarrow (\forall\psi.\ \Phi\ \psi \supset \psi\ z))$
axiomatization where *Ax1Gen*: $\lfloor(PosProps\ \Phi \wedge ConjOfPropsFrom\ \varphi\ \Phi) \supset P\ \varphi\rfloor$

lemma *L*: $\lfloor P\ G\rfloor$ **using** *Ax1Gen God-def* **by** *smt*

theorem *Th4*: $\lfloor\Diamond(\exists^E x.\ G\ x)\rfloor$ **using** *Ax2a Ax4 L Rsymm Th2* **by** *metis*

theorem *Th5*: $\lfloor\Box(\exists^E x.\ G\ x)\rfloor$ **using** *Th3 Th4* **by** *blast*

lemma *MC*: $\lfloor\varphi \supset \Box\varphi\rfloor$
— sledgehammer(Ax2a Ax2b Th5 God_def Rsymm) — proof found
proof – {**fix** *w* **fix** *Q*
have *1*: $\forall x.(G\ x\ w \longrightarrow (\forall Z.\ Z\ x \supset \Box(\forall z.\ G\ z \supset Z\ z))\ w)$ **using** *Ax2a Ax2b God-def* **by** *smt*
have *2*: $(\exists x.\ G\ x\ w)\longrightarrow((Q \supset \Box(\forall z.\ G\ z \supset Q))\ w)$ **using** *1* **by** *force*
have *3*: $(Q \supset \Box Q)\ w$ **using** *2 Th5 Rsymm* **by** *blast*}
thus *?thesis* **by** *auto*
qed

lemma *PosProps*: $\lfloor P\ (\lambda x.\top) \wedge P\ (\lambda x.\ x = x)\rfloor$ **using** *Ax2a Ax4* **by** *blast*
lemma *NegProps*: $\lfloor\neg P(\lambda x.\bot) \wedge \neg P(\lambda x.\ x \neq x)\rfloor$ **using** *Ax2a Ax4* **by** *blast*
lemma *UniqueEss1*: $\lfloor\varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\forall y.\ \varphi\ y \leftrightarrow \psi\ y)\rfloor$ **using** *Essence-def* **by** *smt*
lemma *UniqueEss2*: $\lfloor\varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\varphi \equiv \psi)\rfloor$ **nitpick**[*card i=2*] **oops** — Countermodel found
lemma *UniqueEss3*: $\lfloor\varphi\ Ess.\ x \supset \Box(\forall y.\ \varphi\ y \supset y \equiv x)\rfloor$ **using** *Essence-def MC*

by *auto*
lemma *Monotheism*: $\lfloor G\ x \wedge G\ y \supset x \equiv y \rfloor$ **using** *Ax2a God-def* **by** *smt*
lemma *Filter*: $\lfloor Filter\ P \rfloor$ **using** *Ax1 Ax4 MC NegProps PosProps Rsymm Ax2a God-def Th5* **by** (*smt* (*verit*, *ccfv-threshold*))
lemma *UltraFilter*: $\lfloor UFilter\ P \rfloor$ **using** *Ax2a Filter* **by** *smt*
lemma *True* **nitpick**[*satisfy*,*card=1*,*eval*=$\lfloor P\ (\lambda x.\bot) \rfloor$] **oops** — One model found of cardinality one

end

7.3 GoedelVariantHOML3AndersonQuan.thy

The same as GoedelVariantHOML3, but now for a mixed use of actualist and possibilist quantifiers for entities; cf. Footnote 20 in [2].

theory *GoedelVariantHOML3AndersonQuant* **imports** *HOMLinHOL ModalFilter*
begin

consts *PositiveProperty*::$(e \Rightarrow \sigma) \Rightarrow \sigma$ (P)

axiomatization where *Ax1*: $\lfloor P\ \varphi \wedge P\ \psi \supset P\ (\varphi \boldsymbol{.}\ \psi) \rfloor$

axiomatization where *Ax2a*: $\lfloor P\ \varphi \vee^e P\ {}^\sim\varphi \rfloor$

definition *God* (G) **where** $G\ x \equiv \forall \varphi.\ P\ \varphi \supset \varphi\ x$

abbreviation *PropertyInclusion* $(\text{-}\supset_N\text{-})$ **where** $\varphi \supset_N \psi \equiv \Box(\varphi \neq (\lambda x.\ \bot) \wedge (\forall y{::}e.\ \varphi\ y \supset \psi\ y))$

definition *Essence* (-*Ess.*-) **where** $\varphi\ Ess.\ x \equiv \forall \psi.\ \psi\ x \supset (\varphi \supset_N \psi)$

axiomatization where *Ax2b*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi \rfloor$

lemma *Ax2b′*: $\lfloor \neg P\ \varphi \supset \Box(\neg P\ \varphi) \rfloor$ **using** *Ax2a Ax2b* **by** *blast*

theorem *Th1*: $\lfloor G\ x \supset G\ Ess.\ x \rfloor$ **using** *Ax2a Ax2b Essence-def God-def* **by** (*smt* (*verit*))

definition *NecExist* (E) **where** $E\ x \equiv \forall \varphi.\ \varphi\ Ess.\ x \supset \Box(\exists^E x.\ \varphi\ x)$

axiomatization where *Ax3*: $\lfloor P\ E \rfloor$

theorem *Th2*: $\lfloor G\ x \supset \Box(\exists^E y.\ G\ y) \rfloor$ **using** *Ax3 Th1 God-def NecExist-def* **by** *smt*

theorem *Th3*: $\lfloor \Diamond(\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y) \rfloor$
— sledgehammer(Th2 Rsymm) — Proof found
proof −
have *1*: $\lfloor (\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y) \rfloor$ **using** *Th2* **by** *blast*
have *2*: $\lfloor \Diamond(\exists^E x.\ G\ x) \supset \Diamond\Box(\exists^E y.\ G\ y) \rfloor$ **using** *1* **by** *blast*

have *3*: $\lfloor\Diamond(\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y)\rfloor$ **using** *2 Rsymm* **by** *blast*
thus *?thesis* **by** *blast*
qed

axiomatization where *Ax4*: $\lfloor P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi\rfloor$

lemma *True* **nitpick**[*satisfy,card=1,eval=*$\lfloor P\ (\lambda x.\bot)\rfloor$] **oops** — Two models found of cardinality one

abbreviation *PosProps* $\Phi \equiv \forall\varphi.\ \Phi\ \varphi \supset P\ \varphi$
abbreviation *ConjOfPropsFrom* $\varphi\ \Phi \equiv \Box(\forall^E z.\ \varphi\ z \leftrightarrow (\forall\psi.\ \Phi\ \psi \supset \psi\ z))$
axiomatization where *Ax1Gen*: $\lfloor($*PosProps* $\Phi \wedge$ *ConjOfPropsFrom* $\varphi\ \Phi) \supset P\ \varphi\rfloor$

lemma *L*: $\lfloor P\ G\rfloor$ **using** *Ax1Gen God-def* **by** (*smt* (*verit*))

theorem *Th4*: $\lfloor\Diamond(\exists^E x.\ G\ x)\rfloor$
— sledgehammer[timeout=200](Ax2a L Ax1Gen) **oops** — sorry — Proof found

axiomatization where *Th4*: $\lfloor\Diamond(\exists^E x.\ G\ x)\rfloor$

theorem *Th5*: $\lfloor\Box(\exists^E x.\ G\ x)\rfloor$ **using** *Th3 Th4* **by** *blast*

lemma *MC*: $\lfloor\varphi \supset \Box\varphi\rfloor$
— sledgehammer(Ax2a Ax2b Th5 God_def Rsymm) — Proof found
proof – {**fix** *w* **fix** *Q*
have *1*: $\forall x.(G\ x\ w \longrightarrow (\forall Z.\ Z\ x \supset \Box(\forall z.\ G\ z \supset Z\ z))\ w)$ **using** *Ax2a Ax2b God-def* **by** *smt*
have *2*: $(\exists x.\ G\ x\ w) \longrightarrow ((Q \supset \Box(\forall z.\ G\ z \supset Q))\ w)$ **using** *1* **by** *force*
have *3*: $(Q \supset \Box Q)\ w$ **using** *2 Th5 Rsymm* **by** *blast*}
thus *?thesis* **by** *auto*
qed

lemma *PosProps*: $\lfloor P\ (\lambda x.\top) \wedge P(\lambda x.\ x = x)\rfloor$ **using** *Ax2a Ax4 L Th4* **by** *smt*
lemma *NegProps*: $\lfloor\neg P(\lambda x.\bot) \wedge \neg P(\lambda x.\ x \neq x)\rfloor$ **using** *Ax2a Ax4 L Th4* **by** *smt*
lemma *UniqueEss1*: $\lfloor\varphi$ *Ess.* $x \wedge \psi$ *Ess.* $x \supset \Box(\forall y.\ \varphi\ y \leftrightarrow \psi\ y)\rfloor$ **oops** — Unclear, open question
lemma *UniqueEss2*: $\lfloor\varphi$ *Ess.* $x \wedge \psi$ *Ess.* $x \supset \Box(\varphi \equiv \psi)\rfloor$ **oops** — Unclear, open question
lemma *UniqueEss3*: $\lfloor\varphi$ *Ess.* $x \supset \Box(\forall y.\ \varphi\ y \supset y \equiv x)\rfloor$ **using** *Essence-def MC* **by** *auto*
lemma *Monotheism*: $\lfloor G\ x \wedge G\ y \supset x \equiv y\rfloor$ **using** *Ax2a God-def* **by** *smt*
lemma *Filter*: $\lfloor$*Filter P*$\rfloor$ **using** *Ax1 Ax4 MC NegProps PosProps Rsymm Ax2a God-def Th5* **by** (*smt* (*verit*, *ccfv-threshold*))
lemma *UltraFilter*: $\lfloor$*UFilter P*$\rfloor$ **using** *Ax2a Filter* **by** *blast*
lemma *True* **nitpick**[*satisfy,card=1,eval=*$\lfloor P\ (\lambda x.\top)\rfloor$] **oops** — One model found of cardinality one

end

7.4 HOMLinHOLonlyS4.thy (slight variation of Figure 3 of [2])

Shallow embedding of higher-order modal logic (HOML) in the classical higher-order logic (HOL) of Isabelle/HOL utilizing the LogiKEy methodology. Here logic S4 is introduced.

theory *HOMLinHOLonlyS4* **imports** *Main*
begin

— Global parameters setting for the model finder nitpick and the parser; unimport for the reader
nitpick-params[*user-axioms*,*expect=genuine*,*show-all*,*format=2*,*max-genuine=3*]
declare[[*syntax-ambiguity-warning=false*]]

— Type i is associated with possible worlds and type e with entities:
typedecl *i* — Possible worlds
typedecl *e* — Individuals/entities
type-synonym $\sigma = i \Rightarrow bool$ — World-lifted propositions
type-synonym $\tau = e \Rightarrow \sigma$ — modal properties

consts $R::i\Rightarrow i\Rightarrow bool$ (-**r**-) — Accessibility relation between worlds
axiomatization where
Rrefl: $\forall x.\ x\mathbf{r}x$ **and**
Rtrans: $\forall x\ y\ z.\ x\mathbf{r}y \wedge y\mathbf{r}z \longrightarrow x\mathbf{r}z$

— Logical connectives (operating on truth-sets)
abbreviation $Mbot::\sigma$ ($\bot$) **where** $\bot \equiv \lambda w.\ False$
abbreviation $Mtop::\sigma$ ($\top$) **where** $\top \equiv \lambda w.\ True$
abbreviation $Mneg::\sigma\Rightarrow\sigma$ ($\neg$- [52]53) **where** $\neg\varphi \equiv \lambda w.\ \neg(\varphi\ w)$
abbreviation $Mand::\sigma\Rightarrow\sigma\Rightarrow\sigma$ (**infixl** $\wedge$ 50) **where** $\varphi\wedge\psi \equiv \lambda w.\ \varphi\ w \wedge \psi\ w$
abbreviation $Mor::\sigma\Rightarrow\sigma\Rightarrow\sigma$ (**infixl** $\vee$ 49) **where** $\varphi\vee\psi \equiv \lambda w.\ \varphi\ w \vee \psi\ w$
abbreviation $Mimp::\sigma\Rightarrow\sigma\Rightarrow\sigma$ (**infixr** $\supset$ 48) **where** $\varphi\supset\psi \equiv \lambda w.\ \varphi\ w \longrightarrow \psi\ w$
abbreviation $Mequiv::\sigma\Rightarrow\sigma\Rightarrow\sigma$ (**infixl** $\leftrightarrow$ 47) **where** $\varphi\leftrightarrow\psi \equiv \lambda w.\ \varphi\ w \longleftrightarrow \psi\ w$
abbreviation $Mbox::\sigma\Rightarrow\sigma$ ($\Box$- [54]55) **where** $\Box\varphi \equiv \lambda w.\forall v.\ w\ \mathbf{r}\ v \longrightarrow \varphi\ v$
abbreviation $Mdia::\sigma\Rightarrow\sigma$ ($\Diamond$- [54]55) **where** $\Diamond\varphi \equiv \lambda w.\exists v.\ w\ \mathbf{r}\ v \wedge \varphi\ v$
abbreviation $Mprimeq::'a\Rightarrow'a\Rightarrow\sigma$ (-**=**-) **where** $x\mathbf{=}y \equiv \lambda w.\ x=y$
abbreviation $Mprimneg::'a\Rightarrow'a\Rightarrow\sigma$ (-**≠**-) **where** $x\mathbf{\neq}y \equiv \lambda w.\ x\neq y$
abbreviation $Mnegpred::\tau\Rightarrow\tau$ (~-) **where** $\sim\Phi \equiv \lambda x.\lambda w.\ \neg\Phi\ x\ w$
abbreviation $Mconpred::\tau\Rightarrow\tau\Rightarrow\tau$ (**infixl .** 50) **where** $\Phi.\Psi \equiv \lambda x.\lambda w.\ \Phi\ x\ w \wedge \Psi\ x\ w$
abbreviation $Mexclor::\sigma\Rightarrow\sigma\Rightarrow\sigma$ (**infixl** $\vee^e$ 49) **where** $\varphi\vee^e\psi \equiv (\varphi \vee \psi) \wedge \neg(\varphi \wedge \psi)$

— Possibilist quantifiers (polymorphic)
abbreviation $Mallposs::('a\Rightarrow\sigma)\Rightarrow\sigma$ ($\forall$) **where** $\forall\Phi \equiv \lambda w.\forall x.\ \Phi\ x\ w$
abbreviation *Mallpossb* (**binder** $\forall$ [8]9) **where** $\forall x.\ \varphi(x) \equiv \forall\varphi$
abbreviation $Mexiposs::('a\Rightarrow\sigma)\Rightarrow\sigma$ ($\exists$) **where** $\exists\Phi \equiv \lambda w.\exists x.\ \Phi\ x\ w$
abbreviation *Mexipossb* (**binder** $\exists$ [8]9) **where** $\exists x.\ \varphi(x) \equiv \exists\varphi$

— Actualist quantifiers (for individuals/entities)
consts *existsAt*::$e \Rightarrow \sigma$ (-@-)
abbreviation *Mallact*::$(e \Rightarrow \sigma) \Rightarrow \sigma$ ($\forall^E$) **where** $\forall^E \Phi \equiv \lambda w. \forall x.\ x@w \longrightarrow \Phi\ x\ w$
abbreviation *Mallactb* (**binder** $\forall^E$ [*8*]*9*) **where** $\forall^E x.\ \varphi(x) \equiv \forall^E \varphi$
abbreviation *Mexiact*::$(e \Rightarrow \sigma) \Rightarrow \sigma$ ($\exists^E$) **where** $\exists^E \Phi \equiv \lambda w. \exists x.\ x@w \wedge \Phi\ x\ w$
abbreviation *Mexiactb* (**binder** $\exists^E$ [*8*]*9*) **where** $\exists^E x.\ \varphi(x) \equiv \exists^E \varphi$

— Leibniz equality (polymorphic)
abbreviation *Mleibeq*::$'a \Rightarrow 'a \Rightarrow \sigma$ (-≡-) **where** $x \equiv y \equiv \forall P.\ P\ x \supset P\ y$

— Meta-logical predicate for global validity
abbreviation *Mvalid*::$\sigma \Rightarrow$ *bool* ($\lfloor$-$\rfloor$) **where** $\lfloor \psi \rfloor \equiv \forall w.\ \psi\ w$

end

7.5 TestsHOMLinS4.thy

Tests and verifications of properties for the embedding of HOML (S4) in HOL.

theory *TestsHOMLinS4* **imports** *HOMLinHOLonlyS4*
begin

— Test for S5 modal logic
lemma *axM*: $\lfloor \Box\varphi \supset \varphi \rfloor$ **using** *Rrefl* **by** *blast*
lemma *axD*: $\lfloor \Box\varphi \supset \Diamond\varphi \rfloor$ **using** *Rrefl* **by** *blast*
lemma *axB*: $\lfloor \varphi \supset \Box\Diamond\varphi \rfloor$
  nitpick[*expect=genuine*] **oops** — Countermodel found
lemma *ax4*: $\lfloor \Box\varphi \supset \Box\Box\varphi \rfloor$ **using** *Rtrans* **by** *blast*
lemma *ax5*: $\lfloor \Diamond\varphi \supset \Box\Diamond\varphi \rfloor$
  nitpick[*expect=genuine*] **oops** — Countermodel found

— Test for Barcan and converse Barcan formula:
lemma *BarcanAct*: $\lfloor (\forall^E x.\Box(\varphi\ x)) \supset \Box(\forall^E x.(\varphi\ x)) \rfloor$
  nitpick[*expect=genuine*] **oops** — Countermodel found
lemma *ConvBarcanAct*: $\lfloor \Box(\forall^E x.(\varphi\ x)) \supset (\forall^E x.\Box(\varphi\ x)) \rfloor$
  nitpick[*expect=genuine*] **oops** — Countermodel found
lemma *BarcanPoss*: $\lfloor (\forall x.\Box(\varphi\ x)) \supset \Box(\forall x.\ \varphi\ x) \rfloor$ **by** *blast*
lemma *ConvBarcanPoss*: $\lfloor \Box(\forall x.(\varphi\ x)) \supset (\forall x.\Box(\varphi\ x)) \rfloor$ **by** *blast*

— A simple Hilbert system for classical propositional logic is derived
lemma *Hilbert-A1*: $\lfloor A \supset (B \supset A) \rfloor$ **by** *blast*
lemma *Hilbert-A2*: $\lfloor (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C)) \rfloor$ **by** *blast*
lemma *Hilbert-MP*: **assumes** $\lfloor A \rfloor$ **and** $\lfloor A \supset B \rfloor$ **shows** $\lfloor B \rfloor$ **using** *assms* **by** *blast*

— We have a polymorphic possibilist quantifier for which existential import holds
lemma *Quant-1*: **assumes** $\lfloor A \rfloor$ **shows** $\lfloor \forall x{::}'a.\ A \rfloor$ **using** *assms* **by** *auto*

— Existential import holds for possibilist quantifiers

lemma *ExImPossibilist1*: $\lfloor \exists\, x{::}e.\ x = x \rfloor$ **by** *blast*
lemma *ExImPossibilist2*: $\lfloor \exists\, x{::}e.\ x \equiv x \rfloor$ **by** *blast*
lemma *ExImPossibilist3*: $\lfloor \exists\, x{::}e.\ x = t \rfloor$ **by** *blast*
lemma *ExImPossibilist4*: $\lfloor \exists\, x{::}'a.\ x \equiv t{::}'a \rfloor$ **by** *blast*
lemma *ExImPossibilist*: $\lfloor \exists\, x{::}'a.\ \top \rfloor$ **by** *blast*

— We have an actualist quantifier for individuals for which existential import does not hold
lemma *Quant-2*: **assumes** $\lfloor A \rfloor$ **shows** $\lfloor \forall^E x{::}e.\ A \rfloor$ **using** *assms* **by** *auto*

— Existential import does not hold for our actualist quantifiers (for individuals)
lemma *ExImActualist1*: $\lfloor \exists^E x{::}e.\ x = x \rfloor$
nitpick[*card=1*,*expect=genuine*] **oops** — Countermodel found
lemma *ExImActualist2*: $\lfloor \exists^E x{::}e.\ x \equiv x \rfloor$
nitpick[*card=1*,*expect=genuine*] **oops** — Countermodel found
lemma *ExImActualist3*: $\lfloor \exists^E x{::}e.\ x = t \rfloor$
nitpick[*card=1*,*expect=genuine*] **oops** — Countermodel found
lemma *ExImActualist*: $\lfloor \exists^E x{::}e.\ \top \rfloor$
nitpick[*card=1*,*expect=genuine*] **oops** — Countermodel found

— Properties of the embedded primitive equality, which coincides with Leibniz equality
lemma *EqRefl*: $\lfloor x = x \rfloor$ **by** *blast*
lemma *EqSym*: $\lfloor (x = y) \leftrightarrow (y = x) \rfloor$ **by** *blast*
lemma *EqTrans*: $\lfloor ((x = y) \wedge (y = z)) \supset (x = z) \rfloor$ **by** *blast*
lemma *EQCong*: $\lfloor (x = y) \supset ((\varphi\ x) = (\varphi\ y)) \rfloor$ **by** *blast*
lemma *EQFuncExt*: $\lfloor (\varphi = \psi) \supset (\forall\, x.\ ((\varphi\ x) = (\psi\ x))) \rfloor$ **by** *blast*
lemma *EQBoolExt1*: $\lfloor (\varphi = \psi) \supset (\varphi \leftrightarrow \psi) \rfloor$ **by** *blast*
lemma *EQBoolExt2*: $\lfloor (\varphi \leftrightarrow \psi) \supset (\varphi = \psi) \rfloor$
nitpick[*card=2*] **oops** — Countermodel found
lemma *EQBoolExt3*: $\lfloor (\varphi \leftrightarrow \psi) \rfloor \longrightarrow \lfloor (\varphi = \psi) \rfloor$ **by** *blast*
lemma *EqPrimLeib*: $\lfloor (x = y) \leftrightarrow (x \equiv y) \rfloor$ **by** *auto*

— Comprehension is natively supported in HOL (due to lambda-abstraction)
lemma *Comprehension1*: $\lfloor \exists\, \varphi.\ \forall\, x.\ (\varphi\ x) \leftrightarrow A \rfloor$ **by** *force*
lemma *Comprehension2*: $\lfloor \exists\, \varphi.\ \forall\, x.\ (\varphi\ x) \leftrightarrow (A\ x) \rfloor$ **by** *force*
lemma *Comprehension3*: $\lfloor \exists\, \varphi.\ \forall\, x\ y.\ (\varphi\ x\ y) \leftrightarrow (A\ x\ y) \rfloor$ **by** *force*

— Modal collapse does not hold
lemma *ModalCollapse*: $\lfloor \forall\, \varphi.\ \varphi \supset \Box\varphi \rfloor$
nitpick[*card=2*,*expect=genuine*] **oops** — Countermodel found

— Empty property and self-difference
lemma *TruePropertyAndSelfIdentity*: $\lfloor (\lambda x{::}e.\ \top) = (\lambda x.\ x = x) \rfloor$ **by** *blast*
lemma *EmptyPropertyAndSelfDifference*: $\lfloor (\lambda x{::}e.\ \bot) = (\lambda x.\ x \neq x) \rfloor$ **by** *blast*
lemma *EmptyProperty2*: $\lfloor \exists\, x.\ \varphi\ x \rfloor \longrightarrow \lfloor \varphi \neq (\lambda x{::}e.\ \bot) \rfloor$ **by** *blast*
lemma *EmptyProperty3*: $\lfloor \exists^E x.\ \varphi\ x \rfloor \longrightarrow \lfloor \varphi \neq (\lambda x{::}e.\ \bot) \rfloor$ **by** *blast*
lemma *EmptyProperty4*: $\lfloor \varphi \neq (\lambda x{::}e.\ \bot) \rfloor \longrightarrow \lfloor \exists\, x.\ \varphi\ x \rfloor$
nitpick[*expect=genuine*] **oops** — Countermodel found

lemma *EmptyProperty5*: $\lfloor\varphi \neq (\lambda x{::}e.\ \bot)\rfloor \longrightarrow \lfloor\exists^E x.\ \varphi\ x\rfloor$
nitpick[*expect=genuine*] **oops** — Countermodel found

end

7.6 GoedelVariantHOML1inS4.thy

The same as GoedelVariantHOML1, but now in logic S4.

theory *GoedelVariantHOML1inS4* **imports** *HOMLinHOLonlyS4*
begin

consts *PositiveProperty*::$(e{\Rightarrow}\sigma){\Rightarrow}\sigma$ (P)

axiomatization where *Ax1*: $\lfloor P\ \varphi \wedge P\ \psi \supset P\ (\varphi \centerdot \psi)\rfloor$

axiomatization where *Ax2a*: $\lfloor P\ \varphi \vee^e P\ {}^\sim\varphi\rfloor$

definition *God* (G) **where** $G\ x \equiv \forall\varphi.\ P\ \varphi \supset \varphi\ x$

abbreviation *PropertyInclusion* $(\text{-}\supset_N\text{-})$ **where** $\varphi \supset_N \psi \equiv \Box(\forall^E y.\ \varphi\ y \supset \psi\ y)$

definition *Essence* (*-Ess.-*) **where** φ *Ess.* $x \equiv \forall\psi.\ \psi\ x \supset (\varphi \supset_N \psi)$

axiomatization where *Ax2b*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi\rfloor$

lemma *Ax2b'*: $\lfloor\neg P\ \varphi \supset \Box(\neg P\ \varphi)\rfloor$ **using** *Ax2a Ax2b* **by** *blast*

theorem *Th1*: $\lfloor G\ x \supset G$ *Ess.* $x\rfloor$ **using** *Ax2a Ax2b Essence-def God-def* **by** (*smt* (*verit*))

definition *NecExist* (E) **where** $E\ x \equiv \forall\varphi.\ \varphi$ *Ess.* $x \supset \Box(\exists^E x.\ \varphi\ x)$

axiomatization where *Ax3*: $\lfloor P\ E\rfloor$

theorem *Th2*: $\lfloor G\ x \supset \Box(\exists^E y.\ G\ y)\rfloor$ **using** *Ax3 Th1 God-def NecExist-def* **by** *smt*

axiomatization where *Ax4*: $\lfloor P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi\rfloor$

theorem *Th3*: $\lfloor\Diamond(\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y)\rfloor$ **using** *Ax3 Essence-def God-def NecExist-def Rrefl* **by** *fastforce*

end

7.7 GoedelVariantHOML2inS4.thy

The same as GoedelVariantHOML2, but now in logic S4, where the proof of theorem Th3 fails.

theory *GoedelVariantHOML2inS4* **imports** *HOMLinHOLonlyS4*
begin

consts *PositiveProperty*::$(e \Rightarrow \sigma) \Rightarrow \sigma$ (P)

axiomatization where *Ax1*: $\lfloor P\ \varphi \wedge P\ \psi \supset P\ (\varphi \centerdot \psi)\rfloor$

abbreviation *PosProps* $\Phi \equiv \forall\varphi.\ \Phi\ \varphi \supset P\ \varphi$
abbreviation *ConjOfPropsFrom* $\varphi\ \Phi \equiv \Box(\forall^E z.\ \varphi\ z \leftrightarrow (\forall\psi.\ \Phi\ \psi \supset \psi\ z))$
axiomatization where *Ax1Gen*: $\lfloor(PosProps\ \Phi \wedge ConjOfPropsFrom\ \varphi\ \Phi) \supset P\ \varphi\rfloor$

axiomatization where *Ax2a*: $\lfloor P\ \varphi \vee^e P\ {\sim}\varphi\rfloor$

definition *God* (G) **where** $G\ x \equiv \forall\varphi.\ P\ \varphi \supset \varphi\ x$

abbreviation *PropertyInclusion* $(\text{-}\supset_N\text{-})$ **where** $\varphi \supset_N \psi \equiv \Box(\forall^E y.\ \varphi\ y \supset \psi\ y)$

definition *Essence* $(\text{-}Ess.\text{-})$ **where** $\varphi\ Ess.\ x \equiv \varphi\ x \wedge (\forall\psi.\ \psi\ x \supset (\varphi \supset_N \psi))$

axiomatization where *Ax2b*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi\rfloor$

lemma *Ax2b′*: $\lfloor\neg P\ \varphi \supset \Box(\neg P\ \varphi)\rfloor$ **using** *Ax2a Ax2b* **by** *blast*

theorem *Th1*: $\lfloor G\ x \supset G\ Ess.\ x\rfloor$ **using** *Ax2a Ax2b Essence-def God-def* **by** (*smt* (*verit*))

definition *NecExist* (E) **where** $E\ x \equiv \forall\varphi.\ \varphi\ Ess.\ x \supset \Box(\exists^E x.\ \varphi\ x)$

axiomatization where *Ax3*: $\lfloor P\ E\rfloor$

axiomatization where *Ax4*: $\lfloor P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi\rfloor$

theorem *Th2*: $\lfloor G\ x \supset \Box(\exists^E y.\ G\ y)\rfloor$ **using** *Ax3 Th1 God-def NecExist-def* **by** *smt*

theorem *Th3*: $\lfloor\Diamond(\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y)\rfloor$ — nitpick sledgehammer **oops** — Open problem

end

7.8 GoedelVariantHOML2possInS4.thy

The same as GoedelVariantHOML2poss, but now in logic S4, where the proof of theorem Th3 fails.

theory *GoedelVariantHOML2possInS4* **imports** *HOMLinHOLonlyS4*
begin

consts *PositiveProperty*::$(e \Rightarrow \sigma) \Rightarrow \sigma$ (P)

axiomatization where *Ax1*: $\lfloor P\ \varphi \wedge P\ \psi \supset P\ (\varphi \mathbin{\boldsymbol{.}} \psi)\rfloor$

abbreviation *PosProps* $\Phi \equiv \forall \varphi.\ \Phi\ \varphi \supset P\ \varphi$

abbreviation *ConjOfPropsFrom* $\varphi\ \Phi \equiv \Box(\forall z.\ \varphi\ z \leftrightarrow (\forall \psi.\ \Phi\ \psi \supset \psi\ z))$

axiomatization where *Ax1Gen*: $\lfloor(\mathit{PosProps}\ \Phi \wedge \mathit{ConjOfPropsFrom}\ \varphi\ \Phi) \supset P\ \varphi\rfloor$

axiomatization where *Ax2a*: $\lfloor P\ \varphi \vee^e P\ {}^\sim\varphi\rfloor$

definition *God* (G) **where** $G\ x \equiv \forall \varphi.\ P\ \varphi \supset \varphi\ x$

abbreviation *PropertyInclusion* $(\text{-}\supset_N\text{-})$ **where** $\varphi \supset_N \psi \equiv \Box(\forall y{::}e.\ \varphi\ y \supset \psi\ y)$

definition *Essence* $(\text{-}\mathit{Ess.}\text{-})$ **where** $\varphi\ \mathit{Ess.}\ x \equiv \varphi\ x \wedge (\forall \psi.\ \psi\ x \supset (\varphi \supset_N \psi))$

axiomatization where *Ax2b*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi\rfloor$

lemma *Ax2b′*: $\lfloor\neg P\ \varphi \supset \Box(\neg P\ \varphi)\rfloor$ **using** *Ax2a Ax2b* **by** *blast*

theorem *Th1*: $\lfloor G\ x \supset G\ \mathit{Ess.}\ x\rfloor$ **using** *Ax2a Ax2b Essence-def God-def* **by** (*smt* (*verit*))

definition *NecExist* (E) **where** $E\ x \equiv \forall \varphi.\ \varphi\ \mathit{Ess.}\ x \supset \Box(\exists x.\ \varphi\ x)$

axiomatization where *Ax3*: $\lfloor P\ E\rfloor$

axiomatization where *Ax4*: $\lfloor P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi\rfloor$

theorem *Th2*: $\lfloor G\ x \supset \Box(\exists y.\ G\ y)\rfloor$ **using** *Ax3 Th1 God-def NecExist-def* **by** *smt*

theorem *Th3*: $\lfloor\Diamond(\exists x.\ G\ x) \supset \Box(\exists y.\ G\ y)\rfloor$ — nitpick sledgehammer **oops** — Open problem

end

7.9 GoedelVariantHOML3inS4.thy

The same as GoedelVariantHOML3, but now in logic S4, where the proof of theorem Th3 fails.

theory *GoedelVariantHOML3inS4* **imports** *HOMLinHOLonlyS4*
begin

consts *PositiveProperty*::$(e \Rightarrow \sigma) \Rightarrow \sigma$ (P)

axiomatization where *Ax1*: $\lfloor P\ \varphi \wedge P\ \psi \supset P\ (\varphi \mathbin{\boldsymbol{.}} \psi)\rfloor$

abbreviation *PosProps* $\Phi \equiv \forall \varphi.\ \Phi\ \varphi \supset P\ \varphi$
abbreviation *ConjOfPropsFrom* $\varphi\ \Phi \equiv \Box(\forall^E z.\ \varphi\ z \leftrightarrow (\forall \psi.\ \Phi\ \psi \supset \psi\ z))$
axiomatization where *Ax1Gen*: $\lfloor(\mathit{PosProps}\ \Phi \wedge \mathit{ConjOfPropsFrom}\ \varphi\ \Phi) \supset P\ \varphi\rfloor$

axiomatization where *Ax2a*: $\lfloor P\ \varphi \vee^e P\ {\sim}\varphi\rfloor$

definition *God* (G) **where** $G\ x \equiv \forall \varphi.\ P\ \varphi \supset \varphi\ x$

abbreviation *PropertyInclusion* $(\text{-}\supset_N\text{-})$ **where** $\varphi \supset_N \psi \equiv \Box(\varphi \neq (\lambda x.\ \bot) \wedge (\forall^E y.\ \varphi\ y \supset \psi\ y))$

definition *Essence* (-*Ess.*-) **where** $\varphi\ \mathit{Ess.}\ x \equiv \forall \psi.\ \psi\ x \supset (\varphi \supset_N \psi)$

axiomatization where *Ax2b*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi\rfloor$

lemma *Ax2b′*: $\lfloor \neg P\ \varphi \supset \Box(\neg P\ \varphi)\rfloor$ **using** *Ax2a Ax2b* **by** *blast*

theorem *Th1*: $\lfloor G\ x \supset G\ \mathit{Ess.}\ x\rfloor$ **using** *Ax2a Ax2b Essence-def God-def* **by** (*smt* (*verit*))

definition *NecExist* (E) **where** $E\ x \equiv \forall \varphi.\ \varphi\ \mathit{Ess.}\ x \supset \Box(\exists^E x.\ \varphi\ x)$

axiomatization where *Ax3*: $\lfloor P\ E\rfloor$

theorem *Th2*: $\lfloor G\ x \supset \Box(\exists^E y.\ G\ y)\rfloor$ **using** *Ax3 Th1 God-def NecExist-def* **by** *smt*

axiomatization where *Ax4*: $\lfloor P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi\rfloor$

theorem *Th3*: $\lfloor \Diamond(\exists^E x.\ G\ x) \supset \Box(\exists^E y.\ G\ y)\rfloor$ — nitpick sledgehammer **oops** — Open problem

end

7.10 GoedelVariantHOML3possInS4.thy

The same as GoedelVariantHOML3poss, but now in logic S4, where the proof of theorem Th3 fails.

theory *GoedelVariantHOML3possInS4* **imports** *HOMLinHOLonlyS4*
begin

consts *PositiveProperty*::$(e \Rightarrow \sigma) \Rightarrow \sigma$ (P)

axiomatization where *Ax1*: $\lfloor P\ \varphi \wedge P\ \psi \supset P\ (\varphi \cdot \psi)\rfloor$

abbreviation *PosProps* $\Phi \equiv \forall \varphi.\ \Phi\ \varphi \supset P\ \varphi$

abbreviation *ConjOfPropsFrom* $\varphi\ \Phi \equiv \Box(\forall z.\ \varphi\ z \leftrightarrow (\forall \psi.\ \Phi\ \psi \supset \psi\ z))$

axiomatization where *Ax1Gen*: $\lfloor(PosProps\ \Phi \wedge ConjOfPropsFrom\ \varphi\ \Phi) \supset P\ \varphi\rfloor$

axiomatization where *Ax2a*: $\lfloor P\ \varphi \vee^e P\ {}^\sim\varphi\rfloor$

definition *God* (G) **where** $G\ x \equiv \forall \varphi.\ P\ \varphi \supset \varphi\ x$

abbreviation *PropertyInclusion* $(\text{-}\supset_N\text{-})$ **where** $\varphi \supset_N \psi \equiv \Box(\varphi \neq (\lambda x.\ \bot) \wedge (\forall y.\ \varphi\ y \supset \psi\ y))$

definition *Essence* (-*Ess.*-) **where** $\varphi\ Ess.\ x \equiv \forall \psi.\ \psi\ x \supset (\varphi \supset_N \psi)$

axiomatization where *Ax2b*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi\rfloor$

lemma *Ax2b′*: $\lfloor\neg P\ \varphi \supset \Box(\neg P\ \varphi)\rfloor$ **using** *Ax2a Ax2b* **by** *blast*

theorem *Th1*: $\lfloor G\ x \supset G\ Ess.\ x\rfloor$ **using** *Ax2a Ax2b Essence-def God-def* **by** (*smt* (*verit*))

definition *NecExist* (E) **where** $E\ x \equiv \forall \varphi.\ (\varphi\ Ess.\ x) \supset \Box(\exists x.\ \varphi\ x)$

axiomatization where *Ax3*: $\lfloor P\ E\rfloor$

axiomatization where *Ax4*: $\lfloor P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi\rfloor$

theorem *Th2*: $\lfloor G\ x \supset \Box(\exists y.\ G\ y)\rfloor$ **using** *Ax3 Th1 God-def NecExist-def* **by** *smt*

theorem *Th3*: $\lfloor\Diamond(\exists x.\ G\ x) \supset \Box(\exists y.\ G\ y)\rfloor$ — nitpick sledgehammer **oops** — Open problem

end

7.11 ScottVariantHOMLinS4.thy

theory *ScottVariantHOMLinS4* **imports** *HOMLinHOLonlyS4*
begin

consts *PositiveProperty*::$(e \Rightarrow \sigma) \Rightarrow \sigma$ (P)

axiomatization where *A1*: $\lfloor\neg P\ \varphi \leftrightarrow P\ {}^\sim\varphi\rfloor$

axiomatization where *A2*: $\lfloor P\ \varphi \wedge \Box(\forall^E y.\ \varphi\ y \supset \psi\ y) \supset P\ \psi\rfloor$

theorem *T1*: $\lfloor P\ \varphi \supset \Diamond(\exists^E x.\ \varphi\ x)\rfloor$ **using** *A1 A2* **by** *blast*

definition *God* (G) **where** $G\ x \equiv \forall \varphi.\ P\ \varphi \supset \varphi\ x$

axiomatization where *A3*: $\lfloor P\ G\rfloor$

theorem *Coro*: $\lfloor \Diamond(\exists^E x.\ G\ x) \rfloor$ **using** *A3 T1* **by** *blast*

axiomatization where *A4*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi \rfloor$

definition *Ess* (-*Ess.*-) **where** φ *Ess.* $x \equiv \varphi\ x \wedge (\forall \psi.\ \psi\ x \supset \Box(\forall^E y.\ \varphi\ y \supset \psi\ y))$

theorem *T2*: $\lfloor G\ x \supset G\ Ess.\ x \rfloor$ **using** *A1 A4 Ess-def God-def* **by** *fastforce*

definition *NecExist* (*NE*) **where** $NE\ x \equiv \forall \varphi.\ \varphi\ Ess.\ x \supset \Box(\exists^E x.\ \varphi\ x)$

axiomatization where *A5*: $\lfloor P\ NE \rfloor$

lemma *True* **nitpick**[*satisfy*,*card=1*,*eval=*$\lfloor P\ (\lambda x.\top)\rfloor$] **oops** — One model found of cardinality one

theorem *T3*: $\lfloor \Box(\exists^E x.\ G\ x) \rfloor$ **nitpick**[*card e=1*, *card i=2*] **oops** — Countermodel found

lemma *MC*: $\lfloor \varphi \supset \Box\varphi \rfloor$ **nitpick**[*card e=1*, *card i=2*] **oops** — Countermodel found

end

7.12 ScottVariantHOMLpossInS4.thy

theory *ScottVariantHOMLpossInS4* **imports** *HOMLinHOLonlyS4*
begin

consts *PositiveProperty*::$(e \Rightarrow \sigma) \Rightarrow \sigma$ (P)

axiomatization where *A1*: $\lfloor \neg P\ \varphi \leftrightarrow P\ {}^{\sim}\varphi \rfloor$

axiomatization where *A2*: $\lfloor P\ \varphi \wedge \Box(\forall y.\ \varphi\ y \supset \psi\ y) \supset P\ \psi \rfloor$

theorem *T1*: $\lfloor P\ \varphi \supset \Diamond(\exists x.\ \varphi\ x) \rfloor$ **using** *A1 A2* **by** *blast*

definition *God* (G) **where** $G\ x \equiv \forall \varphi.\ P\ \varphi \supset \varphi\ x$

axiomatization where *A3*: $\lfloor P\ G \rfloor$

theorem *Coro*: $\lfloor \Diamond(\exists x.\ G\ x) \rfloor$ **using** *A3 T1* **by** *blast*

axiomatization where *A4*: $\lfloor P\ \varphi \supset \Box\ P\ \varphi \rfloor$

definition *Ess* (-*Ess.*-) **where** φ *Ess.* $x \equiv \varphi\ x \wedge (\forall \psi.\ \psi\ x \supset \Box(\forall y{::}e.\ \varphi\ y \supset \psi\ y))$

theorem *T2*: $\lfloor G\ x \supset G\ Ess.\ x \rfloor$ **using** *A1 A4 Ess-def God-def* **by** *fastforce*

definition *NecExist* (*NE*) **where** *NE* $x \equiv \forall \varphi.\ \varphi\ Ess.\ x \supset \Box(\exists x.\ \varphi\ x)$

axiomatization where *A5*: $\lfloor P\ NE \rfloor$

lemma *True* **nitpick**[*satisfy*,*card=1*,*eval*=$\lfloor P\ (\lambda x.\bot)\rfloor$] **oops** — One model found of cardinality one

theorem *T3*: $\lfloor \Box(\exists x.\ G\ x)\rfloor$ **nitpick**[*card e=1*, *card i=2*] **oops** — Countermodel found

lemma *MC*: $\lfloor \varphi \supset \Box\varphi \rfloor$ **nitpick**[*card e=1*, *card i=2*] **oops** — Countermodel found

end

References

[1] C. Benzmüller. Exploring simplified variants of Gödels ontological argument in Isabelle/HOL. *Archive of Formal Proofs*, November 2021. https://isa-afp.org/entries/SimplifiedOntologicalArgument.html, Formal proof development.

[2] C. Benzmüller and D. S. Scott. Notes on Gödel's and Scott's Variants of the ontological argument. *Monatshefte für Mathematik*, 2024. Forthcoming/accepted.

[3] C. Benzmüller and B. Woltzenlogel Paleo. Gödel's God in Isabelle/HOL. *Archive of Formal Proofs*, November 2013. https://isa-afp.org/entries/GoedelGod.html, Formal proof development.

[4] D. Fuenmayor and C. Benzmüller. Types, Tableaus and Gödels God in Isabelle/HOL. *Archive of Formal Proofs*, May 2017. https://isa-afp.org/entries/Types_Tableaus_and_Goedels_God.html, Formal proof development.

[5] K. Gödel. Appendix A. Notes in Kurt Gödel's Hand. In Sobel [7], pages 144–145.

[6] D. S. Scott. Appendix B: Notes in Dana Scott's Hand. In Sobel [7], pages 145–146.

[7] J. H. Sobel. *Logic and Theism: Arguments for and Against Beliefs in God*. Cambridge University Press, Cambridge, 2004.