Negatively Associated Random Variables

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Abstract

Negative Association is a generalization of independence for random variables, that retains some of the key properties of independent random variables. In particular closure properties, such as composition with monotone functions, as well as, the well-known Chernoff-Hoeffding bounds.

This entry introduces the concept and verifies the most important closure properties, as well as, the concentration inequalities. It also verifies the FKG inequality, which is a generalization of Chebyshev's sum inequality for distributive lattices and a key tool for establishing negative association, but has also many applications beyond the context of negative association, in particular, statistical physics and graph theory.

As an example, permutation distributions are shown to be negatively associated, from which many more sets of negatively random variables can be derived, such as, e.g., n-subsets, or the the balls-into-bins process.

Finally, the entry derives a correct false-positive rate for Bloom filters using the library.

Contents

1 Preliminary Definitions and Lemmas

theory *Negative-Association-Util*
  imports
    Concentration-Inequalities.Concentration-Inequalities-Preliminary
    Universal-Hash-Families.Universal-Hash-Families-More-Product-PMF
begin

abbreviation (*input*) *flip* :: ‹$('a \Rightarrow 'b \Rightarrow 'c) \Rightarrow 'b \Rightarrow 'a \Rightarrow 'c$› **where**
  ‹*flip* $f\ x\ y \equiv f\ y\ x$›

Additional introduction rules for boundedness:

lemma *bounded-const-min*:
  fixes $f :: 'a \Rightarrow real$
  assumes *bdd-below* (f ‘ M)
  shows *bounded* (($\lambda x.$ *min* c ($f\ x$)) ‘ M)
⟨*proof*⟩

lemma *bounded-prod*:
  fixes $f :: 'i \Rightarrow 'a \Rightarrow real$
  assumes *finite* I
  assumes $\bigwedge i.\ i \in I \Longrightarrow$ *bounded* ($f\ i$ ‘ T)
  shows *bounded* (($\lambda x.$ ($\prod\ i \in I.\ f\ i\ x$)) ‘ T)
  ⟨*proof*⟩

lemma *bounded-vec-mult-comp*:
  fixes $f\ g :: 'a \Rightarrow real$
  assumes *bounded* (f ‘ T) *bounded* (g ‘ T)
  shows *bounded* (($\lambda x.$ ($f\ x$) $*_R$ ($g\ x$)) ‘ T)
  ⟨*proof*⟩

lemma *bounded-max*:
  fixes $f :: 'a \Rightarrow real$
  assumes *bounded* (($\lambda x.\ f\ x$) ‘ T)
  shows *bounded* (($\lambda x.$ *max* c ($f\ x$)) ‘ T)
⟨*proof*⟩

lemma *bounded-of-bool*: *bounded* (*range of-bool*) ⟨*proof*⟩

lemma *bounded-range-imp*:
  assumes *bounded* (*range* f)
  shows *bounded* (($\lambda\omega.\ f$ ($h\ \omega$)) ‘ S)
  ⟨*proof*⟩

The following allows to state integrability and conditions about the integral simultaneously, e.g. *has-int-that* $M\ f$ ($\lambda x.\ x \le c$) says f is integrable on M and the integral smaller or equal to c.

definition *has-int-that* **where**

has-int-that $M\, f\, P = (\mathit{integrable}\ M\, f \wedge (P\ (\int \omega.\ f\ \omega\ \partial M)))$

lemma *true-eq-iff*: $P \Longrightarrow \mathit{True} = P$ ⟨*proof*⟩
lemma *le-trans*: $y \leq z \Longrightarrow x \leq y \longrightarrow x \leq (z :: {'a} :: \mathit{order})$ ⟨*proof*⟩

lemma *has-int-that-mono*:
assumes $\bigwedge x.\ P\ x \longrightarrow Q\ x$
shows *has-int-that* $M\, f\, P \leq$ *has-int-that* $M\, f\, Q$
⟨*proof*⟩

lemma *has-int-thatD*:
assumes *has-int-that* $M\, f\, P$
shows *integrable* $M\, f\, P\ (\mathit{integral}^L\ M\, f)$
⟨*proof*⟩

This is useful to specify which components a functional depends on.

definition *depends-on* :: $(({'a} \Rightarrow {'b}) \Rightarrow {'c}) \Rightarrow {'a}\ \mathit{set} \Rightarrow \mathit{bool}$
where *depends-on* $f\, I = (\forall\, x\ y.\ \mathit{restrict}\ x\ I = \mathit{restrict}\ y\ I \longrightarrow f\, x = f\, y)$

lemma *depends-onI*:
assumes $\bigwedge x.\ \ f\, x = f\ (\lambda i.\ \mathit{if}\ i \in I\ \mathit{then}\ (x\ i)\ \mathit{else}\ \mathit{undefined})$
shows *depends-on* $f\, I$
⟨*proof*⟩

lemma *depends-on-comp*:
assumes *depends-on* $f\, I$
shows *depends-on* $(g \circ f)\ I$
⟨*proof*⟩

lemma *depends-on-comp-2*:
assumes *depends-on* $f\, I$
shows *depends-on* $(\lambda x.\ g\ (f\, x))\ I$
⟨*proof*⟩

lemma *depends-onD*:
assumes *depends-on* $f\, I$
shows $f\ \omega = f\ (\lambda i{\in}I.\ (\omega\ i))$
⟨*proof*⟩

lemma *depends-onD2*:
assumes *depends-on* $f\, I$ *restrict* $x\ I =$ *restrict* $y\ I$
shows $f\, x = f\, y$
⟨*proof*⟩

lemma *depends-on-empty*:
assumes *depends-on* $f\ \{\}$
shows $f\ \omega = f\ \mathit{undefined}$
⟨*proof*⟩

lemma *depends-on-mono*:
  assumes $I \subseteq J$ *depends-on* f I
  shows *depends-on* f J
  ⟨*proof*⟩

abbreviation *square-integrable* M $f \equiv$ *integrable* M $((power2 :: real \Rightarrow real) \circ f)$

There are many results in the field of negative association, where a statement is true for simultaneously monotone or anti-monotone functions. With the below construction, we introduce a mechanism where we can parameterize on the direction of a relation:

datatype *RelDirection* = *Fwd* | *Rev*

definition *dir-le* :: *RelDirection* ⇒ (('d::*order*) ⇒ ('d :: *order*) ⇒ *bool*) (**infixl** $\leq\geq_1$ *60*)
  where *dir-le* η = (*if* η = *Fwd then* ($\leq$) *else* ($\geq$))

lemma *dir-le*[*simp*]:
  $(\leq\geq_{Fwd}) = (\leq)$
  $(\leq\geq_{Rev}) = (\geq)$
  ⟨*proof*⟩

definition *dir-sign* :: *RelDirection* ⇒ 'a::{*one*,*uminus*} ($\pm_1$)
  where *dir-sign* η = (*if* η = *Fwd then 1 else* (−*1*))

lemma *dir-le-refl*: $x \leq\geq_\eta x$
  ⟨*proof*⟩

lemma *dir-sign*[*simp*]:
  $(\pm_{Fwd}) = (1)$
  $(\pm_{Rev}) = (-1)$
  ⟨*proof*⟩

lemma *conv-rel-to-sign*:
  fixes *f* :: 'a::*order* ⇒ *real*
  shows *monotone* ($\leq$) ($\leq\geq_\eta$) *f* = *mono* $((*)(\pm_\eta) \circ f)$
  ⟨*proof*⟩

instantiation *RelDirection* :: *times*
begin
definition *times-RelDirection* :: *RelDirection* ⇒ *RelDirection* ⇒ *RelDirection* **where**
  times-RelDirection-def: *times-RelDirection x y* = (*if x* = *y then Fwd else Rev*)

instance ⟨*proof*⟩
end

lemmas *rel-dir-mult*[*simp*] = *times-RelDirection-def*

lemma *dir-mult-hom*: $(\pm_{\sigma * \tau}) = (\pm_\sigma) * ((\pm_\tau)::real)$

⟨*proof*⟩

Additional lemmas about clamp for the specific case on reals.

lemma *clamp-eqI2*:
assumes $x \in \{a..b::real\}$
shows $x = clamp\ a\ b\ x$
⟨*proof*⟩

lemma *clamp-eqI*:
assumes $|x| \leq (a::real)$
shows $x = clamp\ (-a)\ a\ x$
⟨*proof*⟩

lemma *clamp-real-def*:
fixes $x :: real$
shows $clamp\ a\ b\ x = max\ a\ (min\ x\ b)$
⟨*proof*⟩

lemma *clamp-range*:
assumes $a \leq b$
shows $\bigwedge x.\ clamp\ a\ b\ x \geq a\ \bigwedge x.\ clamp\ a\ b\ x \leq b\ range\ (clamp\ a\ b) \subseteq \{a..b::real\}$
⟨*proof*⟩

lemma *clamp-abs-le*:
assumes $a \geq (0::real)$
shows $|clamp\ (-a)\ a\ x| \leq |x|$
⟨*proof*⟩

lemma *bounded-clamp*:
fixes $a\ b :: real$
shows $bounded\ ((clamp\ a\ b \circ f)\ `\ S)$
⟨*proof*⟩

lemma *bounded-clamp-alt*:
fixes $a\ b :: real$
shows $bounded\ ((\lambda x.\ clamp\ a\ b\ (f\ x))\ `\ S)$
⟨*proof*⟩

lemma *clamp-borel*[*measurable*]:
fixes $a\ b :: 'a::\{euclidean\text{-}space, second\text{-}countable\text{-}topology\}$
shows $clamp\ a\ b \in borel\text{-}measurable\ borel$
⟨*proof*⟩

lemma *monotone-clamp*:
assumes $monotone\ (\leq)\ (\leq\geq_\eta)\ f$
shows $monotone\ (\leq)\ (\leq\geq_\eta)\ (\lambda\omega.\ clamp\ a\ (b::real)\ (f\ \omega))$
⟨*proof*⟩

This part introduces the term *KL-div* as the Kullback-Leibler divergence between a pair of Bernoulli random variables. The expression is useful to express some of the Chernoff bounds more concisely [12, Th. 1].

lemma *radon-nikodym-pmf*:
  assumes *set-pmf p* $\subseteq$ *set-pmf q*
  defines $f \equiv (\lambda x.\ \mathit{ennreal}\ (\mathit{pmf}\ p\ x\ /\ \mathit{pmf}\ q\ x))$
  shows
    AE x in measure-pmf q. RN-deriv q p x $=$ *f x* (**is** *?R1*)
    AE x in measure-pmf p. RN-deriv q p x $=$ *f x* (**is** *?R2*)
⟨*proof*⟩

lemma *KL-divergence-pmf*:
  assumes *set-pmf q* $\subseteq$ *set-pmf p*
  shows *KL-divergence b* (*measure-pmf p*) (*measure-pmf q*) $=$ ($\int$ *x. log b* (*pmf q x* / *pmf p x*) ∂q)
  ⟨*proof*⟩

definition *KL-div* :: *real* $\Rightarrow$ *real* $\Rightarrow$ *real* **where**
  KL-div p q $=$ *KL-divergence* (*exp 1*) (*bernoulli-pmf q*) (*bernoulli-pmf p*)

lemma *KL-div-eq*:
  assumes $q \in \{0{<}..{<}1\}$ $p \in \{0..1\}$
  shows *KL-div p q* $= p * \mathit{ln}\ (p/q) + (1-p) * \mathit{ln}\ ((1-p)/(1-q))$ (**is** *?L* $=$ *?R*)
⟨*proof*⟩

lemma *KL-div-swap*:
  assumes $q \in \{0{<}..{<}1\}$ $p \in \{0..1\}$
  shows *KL-div p q* $=$ *KL-div* $(1-p)$ $(1-q)$
  ⟨*proof*⟩

A few results about independent random variables:

lemma (**in** *prob-space*) *indep-vars-const*:
  assumes $\bigwedge i.\ i \in I \Longrightarrow c\ i \in \mathit{space}\ (N\ i)$
  shows *indep-vars N* (λi -. *c i*) *I*
⟨*proof*⟩

lemma *indep-vars-map-pmf*:
  assumes *prob-space.indep-vars* (*measure-pmf p*) (λ-. *discrete*) (λi. *X i* $\circ$ *f*) *I*
  shows *prob-space.indep-vars* (*map-pmf f p*) (λ-. *discrete*) *X I*
  ⟨*proof*⟩

lemma *indep-var-pair-pmf*:
  fixes *x y* :: *'a pmf*
  shows *prob-space.indep-var* (*pair-pmf x y*) *discrete fst discrete snd*
⟨*proof*⟩

lemma *measure-pair-pmf*: *measure* (*pair-pmf p q*) $(A \times B) =$ *measure p A* $*$ *measure q B* (**is** *?L* $=$ *?R*)
⟨*proof*⟩

instance *bool* :: *second-countable-topology*
⟨*proof*⟩

end

2 Definition

This section introduces the concept of negatively associated random variables (RVs). The definition follows, as closely as possible, the original description by Joag-Dev and Proschan [13].

However, the following modifications have been made:

Singleton and empty sets of random variables are considered negatively associated. This is useful because it simplifies many of the induction proofs. The second modification is that the RV's don't have to be real valued. Instead the range can be into any linearly ordered space with the borel σ-algebra. This is a major enhancement compared to the original work, as well as results by following authors [6, 7, 8, 14, 17].

theory *Negative-Association-Definition*
  imports
    Concentration-Inequalities.Bienaymes-Identity
    Negative-Association-Util
begin

context *prob-space*
begin

definition *neg-assoc* :: $('i \Rightarrow 'a \Rightarrow 'c :: \{linorder\text{-}topology\}) \Rightarrow 'i\ set \Rightarrow bool$
  where $neg\text{-}assoc\ X\ I = ($
    $(\forall\, i \in I.\ random\text{-}variable\ borel\ (X\ i)) \wedge$
    $(\forall\, (f{::}nat \Rightarrow ('i \Rightarrow 'c) \Rightarrow real)\ J.\ J \subseteq I \wedge$
      $(\forall\, \iota{<}2.\ bounded\ (range\ (f\ \iota)) \wedge mono(f\ \iota) \wedge depends\text{-}on\ (f\ \iota)\ ([J,I{-}J]!\iota) \wedge$
      $f\ \iota \in PiM\ ([J,I{-}J]!\iota)\ (\lambda\text{-}.\ borel) \rightarrow_M borel) \longrightarrow$
      $covariance\ (f\ 0 \circ flip\ X)\ (f\ 1 \circ flip\ X) \leq 0))$

lemma *neg-assocI*:
  assumes $\bigwedge i.\ i \in I \Longrightarrow random\text{-}variable\ borel\ (X\ i)$
  assumes $\bigwedge f\ g\ J.\ J \subseteq I$
    $\Longrightarrow depends\text{-}on\ f\ J \Longrightarrow depends\text{-}on\ g\ (I{-}J)$
    $\Longrightarrow mono\ f \Longrightarrow mono\ g$
    $\Longrightarrow bounded\ (range\ f{::}real\ set) \Longrightarrow bounded\ (range\ g)$
    $\Longrightarrow f \in PiM\ J\ (\lambda\text{-}.\ borel) \rightarrow_M borel \Longrightarrow g \in PiM\ (I{-}J)\ (\lambda\text{-}.\ borel) \rightarrow_M borel$
    $\Longrightarrow covariance\ (f \circ flip\ X)\ (g \circ flip\ X) \leq 0$
  shows *neg-assoc X I*
  ⟨*proof*⟩

lemma *neg-assocI2*:

assumes [*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow$ *random-variable borel* $(X\ i)$
assumes $\bigwedge f\ g\ J.\ J \subseteq I$
$\Longrightarrow$ *depends-on* $f\ J \Longrightarrow$ *depends-on* $g\ (I-J)$
$\Longrightarrow$ *mono* $f \Longrightarrow$ *mono* g
$\Longrightarrow$ *bounded* (*range* f) $\Longrightarrow$ *bounded* (*range* g)
$\Longrightarrow f \in PiM\ J\ (\lambda\text{-}.\ borel) \rightarrow_M (borel :: real\ measure)$
$\Longrightarrow g \in PiM\ (I-J)\ (\lambda\text{-}.\ borel) \rightarrow_M (borel :: real\ measure)$
$\Longrightarrow (\int \omega.\ f(\lambda i.\ X\ i\ \omega) * g(\lambda i.\ X\ i\ \omega)\ \partial M) \leq (\int \omega.\ f(\lambda i.\ X\ i\ \omega) \partial M) * (\int \omega.\ g(\lambda i.\ X\ i\ \omega)\ \partial M)$
shows *neg-assoc* $X\ I$
⟨*proof*⟩

lemma *neg-assoc-empty*:
neg-assoc X {}
⟨*proof*⟩

lemma *neg-assoc-singleton*:
assumes *random-variable borel* $(X\ i)$
shows *neg-assoc* $X\ \{i\}$
⟨*proof*⟩

lemma *neg-assoc-imp-measurable*:
assumes *neg-assoc* $X\ I$
assumes $i \in I$
shows *random-variable borel* $(X\ i)$
⟨*proof*⟩

Even though the assumption was that defining property is true for pairs of monotone functions over the random variables, it is also true for pairs of anti-monotone functions.

lemma *neg-assoc-imp-mult-mono-bounded*:
fixes $f\ g :: ('i \Rightarrow 'c::linorder\text{-}topology) \Rightarrow real$
assumes *neg-assoc* $X\ I$
assumes $J \subseteq I$
assumes *bounded* (*range* f) *bounded* (*range* g)
assumes *monotone* $(\leq)\ (\leq\geq_\eta)\ f$ *monotone* $(\leq)\ (\leq\geq_\eta)\ g$
assumes *depends-on* $f\ J$ *depends-on* $g\ (I-J)$
assumes [*measurable*]: $f \in borel\text{-}measurable\ (Pi_M\ J\ (\lambda\text{-}.\ borel))$
assumes [*measurable*]: $g \in borel\text{-}measurable\ (Pi_M\ (I-J)\ (\lambda\text{-}.\ borel))$
shows
covariance $(f \circ flip\ X)\ (g \circ flip\ X) \leq 0$
$(\int \omega.\ f\ (\lambda i.\ X\ i\ \omega) * g\ (\lambda i.\ X\ i\ \omega)\ \partial M) \leq expectation\ (\lambda x.\ f(\lambda y.\ X\ y\ x)) * expectation\ (\lambda x.\ g(\lambda y.\ X\ y\ x))$
(**is** $?L \leq ?R$)
⟨*proof*⟩

lemma *lim-min-n*: $(\lambda n.\ min\ (real\ n)\ x) \longrightarrow x$
⟨*proof*⟩

lemma *lim-clamp-n*: $(\lambda n.\ clamp\ (-real\ n)\ (real\ n)\ x) \longrightarrow x$
$\langle proof \rangle$

lemma *neg-assoc-imp-mult-mono*:
fixes $f\ g$ :: $('i \Rightarrow 'c::linorder\text{-}topology) \Rightarrow real$
assumes *neg-assoc X I*
assumes $J \subseteq I$
assumes *square-integrable* $M\ (f \circ flip\ X)$ *square-integrable* $M\ (g \circ flip\ X)$
assumes *monotone* $(\leq)\ (\leq\geq_\eta)\ f$ *monotone* $(\leq)\ (\leq\geq_\eta)\ g$
assumes *depends-on f J depends-on g* $(I-J)$
assumes [*measurable*]: $f \in$ *borel-measurable* $(Pi_M\ J\ (\lambda\text{-}.\ borel))$
assumes [*measurable*]: $g \in$ *borel-measurable* $(Pi_M\ (I-J)\ (\lambda\text{-}.\ borel))$
shows $(\int \omega.\ f\ (\lambda i.\ X\ i\ \omega) * g\ (\lambda i.\ X\ i\ \omega)\ \partial M) \leq (\int x.\ f(\lambda y.\ X\ y\ x)\partial M) * (\int x.\ g(\lambda y.\ X\ y\ x)\partial M)$
(**is** $?L \leq ?R$)
$\langle proof \rangle$

Property P4 [13]

lemma *neg-assoc-subset*:
assumes $J \subseteq I$
assumes *neg-assoc X I*
shows *neg-assoc X J*
$\langle proof \rangle$

lemma *neg-assoc-imp-mult-mono-nonneg*:
fixes $f\ g$ :: $('i \Rightarrow 'c::linorder\text{-}topology) \Rightarrow real$
assumes *neg-assoc X I* $J \subseteq I$
assumes *range* $f \subseteq \{0..\}$ *range* $g \subseteq \{0..\}$
assumes *integrable* $M\ (f \circ flip\ X)$ *integrable* $M\ (g \circ flip\ X)$
assumes *monotone* $(\leq)\ (\leq\geq_\eta)\ f$ *monotone* $(\leq)\ (\leq\geq_\eta)\ g$
assumes *depends-on f J depends-on g* $(I-J)$
assumes $f \in$ *borel-measurable* $(Pi_M\ J\ (\lambda\text{-}.\ borel))$ $g \in$ *borel-measurable* $(Pi_M\ (I-J)\ (\lambda\text{-}.\ borel))$
shows *has-int-that* $M\ (\lambda\omega.\ f\ (flip\ X\ \omega) * g\ (flip\ X\ \omega))$
$(\lambda r.\ r \leq expectation\ (f \circ flip\ X) * expectation\ (g \circ flip\ X))$
$\langle proof \rangle$

Property P2 [13]

lemma *neg-assoc-imp-prod-mono*:
fixes f :: $'i \Rightarrow ('c::linorder\text{-}topology) \Rightarrow real$
assumes *finite I*
assumes *neg-assoc X I*
assumes $\bigwedge i.\ i \in I \Longrightarrow integrable\ M\ (\lambda\omega.\ f\ i\ (X\ i\ \omega))$
assumes $\bigwedge i.\ i \in I \Longrightarrow monotone\ (\leq)\ (\leq\geq_\eta)\ (f\ i)$
assumes $\bigwedge i.\ i \in I \Longrightarrow range\ (f\ i) \subseteq \{0..\}$
assumes $\bigwedge i.\ i \in I \Longrightarrow f\ i \in borel\text{-}measurable\ borel$
shows *has-int-that* $M\ (\lambda\omega.\ (\prod i \in I.\ f\ i\ (X\ i\ \omega)))\ (\lambda r.\ r \leq (\prod i \in I.\ expectation\ (\lambda\omega.\ f\ i\ (X\ i\ \omega))))$
$\langle proof \rangle$

Property P5 [13]

lemma *neg-assoc-compose*:
  fixes $f :: 'j \Rightarrow ('i \Rightarrow ('c::linorder\text{-}topology)) \Rightarrow ('d ::linorder\text{-}topology)$
  assumes *finite I*
  assumes *neg-assoc X I*
  assumes $\bigwedge j.\ j \in J \Longrightarrow deps\ j \subseteq I$
  assumes $\bigwedge j1\ j2.\ j1 \in J \Longrightarrow j2 \in J \Longrightarrow j1 \neq j2 \Longrightarrow deps\ j1 \cap deps\ j2 = \{\}$
  assumes $\bigwedge j.\ j \in J \Longrightarrow monotone\ (\le)\ (\le\ge_\eta)\ (f\ j)$
  assumes $\bigwedge j.\ j \in J \Longrightarrow depends\text{-}on\ (f\ j)\ (deps\ j)$
  assumes $\bigwedge j.\ j \in J \Longrightarrow f\ j \in borel\text{-}measurable\ (PiM\ (deps\ j)\ (\lambda\text{-}.\ borel))$
  shows $neg\text{-}assoc\ (\lambda j\ \omega.\ f\ j\ (\lambda i.\ X\ i\ \omega))\ J$
⟨*proof*⟩

lemma *neg-assoc-compose-simple*:
  fixes $f :: 'i \Rightarrow ('c::linorder\text{-}topology) \Rightarrow ('d ::linorder\text{-}topology)$
  assumes *finite I*
  assumes *neg-assoc X I*
  assumes $\bigwedge i.\ i \in I \Longrightarrow monotone\ (\le)\ (\le\ge_\eta)\ (f\ i)$
  assumes [*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow f\ i \in borel\text{-}measurable\ borel$
  shows $neg\text{-}assoc\ (\lambda i\ \omega.\ f\ i\ (X\ i\ \omega))\ I$
⟨*proof*⟩

lemma *covariance-distr*:
  fixes $f\ g :: 'b \Rightarrow real$
  assumes [*measurable*]: $\varphi \in M \rightarrow_M N\ f \in borel\text{-}measurable\ N\ \ g \in borel\text{-}measurable\ N$
  shows $prob\text{-}space.covariance\ (distr\ M\ N\ \varphi)\ f\ g = covariance\ (f \circ \varphi)\ (g \circ \varphi)$ (**is** $?L = ?R$)
⟨*proof*⟩

lemma *neg-assoc-iff-distr*:
  assumes [*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow X\ i \in borel\text{-}measurable\ M$
  shows $neg\text{-}assoc\ X\ I \longleftrightarrow$
    $prob\text{-}space.neg\text{-}assoc\ (distr\ M\ (PiM\ I\ (\lambda\text{-}.\ borel))\ (\lambda\omega.\ \lambda i{\in}I.\ X\ i\ \omega))\ (flip\ id)\ I$
    (**is** $?L \longleftrightarrow ?R$)
⟨*proof*⟩

lemma *neg-assoc-cong*:
  assumes *finite I*
  assumes [*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow Y\ i \in borel\text{-}measurable\ M$
  assumes $neg\text{-}assoc\ X\ I\ \bigwedge i.\ i \in I \Longrightarrow AE\ \omega\ in\ M.\ X\ i\ \omega = Y\ i\ \omega$
  shows *neg-assoc Y I*
⟨*proof*⟩

lemma *neg-assoc-reindex-aux*:
  assumes *inj-on h I*
  assumes $neg\text{-}assoc\ X\ (h\ `\ I)$
  shows $neg\text{-}assoc\ (\lambda k.\ X\ (h\ k))\ I$
⟨*proof*⟩

lemma *neg-assoc-reindex*:
  assumes *inj-on h I finite I*
  shows *neg-assoc X* $(h$ ‘ $I) \longleftrightarrow$ *neg-assoc* $(\lambda k.\ X\ (h\ k))\ I$ (**is** *?L* $\longleftrightarrow$ *?R*)
⟨*proof*⟩

lemma *measurable-compose-merge-1*:
  assumes *depends-on h K*
  assumes $h \in PiM\ K\ M' \rightarrow_M N\ K \subseteq I \cup J$
  assumes $(\lambda x.\ restrict\ (fst\ (f\ x))\ (K \cap I)) \in A \rightarrow_M PiM\ (K \cap I)\ M'$
  assumes $(\lambda x.\ restrict\ (snd\ (f\ x))\ (K \cap J)) \in A \rightarrow_M PiM\ (K \cap J)\ M'$
  shows $(\lambda x.\ h(merge\ I\ J\ (f\ x))) \in A \rightarrow_M N$
⟨*proof*⟩

lemma *measurable-compose-merge-2*:
  assumes *depends-on h K* $h \in PiM\ K\ M' \rightarrow_M N\ K \subseteq I \cup J$
  assumes $(\lambda x.\ restrict\ (f\ x)\ (K \cap I)) \in A \rightarrow_M PiM\ (K \cap I)\ M'$
  assumes $(\lambda x.\ restrict\ (g\ x)\ (K \cap J)) \in A \rightarrow_M PiM\ (K \cap J)\ M'$
  shows $(\lambda x.\ h(merge\ I\ J\ (f\ x,\ g\ x))) \in A \rightarrow_M N$
  ⟨*proof*⟩

lemma *neg-assoc-combine*:
  fixes *I I1 I2* :: *'i set*
  fixes *X* :: *'i* ⇒ *'a* ⇒ (*'b*::*linorder-topology*)
  assumes *finite I* $I1 \cup I2 = I$ $I1 \cap I2 = \{\}$
  assumes *indep-var* (*PiM I1* (λ-. *borel*)) (λω. λi∈*I1*. *X i* ω) (*PiM I2* (λ-. *borel*))
(λω. λi∈*I2*. *X i* ω)
  assumes *neg-assoc X I1*
  assumes *neg-assoc X I2*
  shows *neg-assoc X I*
⟨*proof*⟩

Property P7 [13]

lemma *neg-assoc-union*:
  fixes *I* :: *'i set*
  fixes *p* :: *'j* ⇒ *'i set*
  fixes *X* :: *'i* ⇒ *'a* ⇒ (*'b*::*linorder-topology*)
  assumes *finite I* $\bigcup (p$ ‘ $J) = I$
  assumes *indep-vars* (λj. *PiM* (*p j*) (λ-. *borel*)) (λj ω. λi ∈ *p j*. *X i* ω) *J*
  assumes $\bigwedge j.\ j \in J \Longrightarrow$ *neg-assoc X* (*p j*)
  assumes *disjoint-family-on p J*
  shows *neg-assoc X I*
⟨*proof*⟩

Property P5 [13]

lemma *indep-imp-neg-assoc*:
  assumes *finite I*
  assumes *indep-vars* (λ-. *borel*) *X I*
  shows *neg-assoc X I*

$\langle proof \rangle$

end

lemma *neg-assoc-map-pmf*:
shows *measure-pmf.neg-assoc* (*map-pmf f p*) *X I* = *measure-pmf.neg-assoc p* (λi ω. *X i* (*f* ω)) *I*
(**is** *?L* $\longleftrightarrow$ *?R*)
$\langle proof \rangle$

end

3 Chernoff-Hoeffding Bounds

This section shows that all the well-known Chernoff-Hoeffding bounds hold also for negatively associated random variables. The proofs follow the derivations by Hoeffding [11], as well as, Motwani and Raghavan [16, Ch. 4], with the modification that the crucial steps, where the classic proofs use independence, are replaced with the application of Property P2 for negatively associated RV's.

theory *Negative-Association-Chernoff-Bounds*
imports
Negative-Association-Definition
Concentration-Inequalities.McDiarmid-Inequality
Weighted-Arithmetic-Geometric-Mean.Weighted-Arithmetic-Geometric-Mean
begin

context *prob-space*
begin

context
fixes I :: *'i set*
fixes X :: *'i* $\Rightarrow$ *'a* $\Rightarrow$ *real*
assumes *na-X*: *neg-assoc X I*
assumes *fin-I*: *finite I*
begin

private lemma *transfer-to-clamped-vars*:
assumes ($\forall i \in I$. *AE* ω *in M*. $X\ i\ \omega \in \{a\ i..b\ i\} \wedge a\ i \leq b\ i$)
assumes $\mathcal{X}$*-def*: $\mathcal{X} = (\lambda i.\ clamp\ (a\ i)\ (b\ i) \circ X\ i)$
shows *neg-assoc* $\mathcal{X}$ *I* (**is** *?A*)
and $\bigwedge i.\ i \in I \Longrightarrow expectation\ (\mathcal{X}\ i) = expectation\ (X\ i)$
and $\mathcal{P}(\omega\ in\ M.\ (\sum i \in I.\ X\ i\ \omega) \leq\geq_\eta c) = \mathcal{P}(\omega\ in\ M.\ (\sum i \in I.\ \mathcal{X}\ i\ \omega) \leq\geq_\eta$ $c)$ (**is** *?C*)
and $\bigwedge i\ \omega.\ i \in I \Longrightarrow \mathcal{X}\ i\ \omega \in \{a\ i..b\ i\}$
and $\bigwedge i\ S.\ i \in I \Longrightarrow bounded\ (\mathcal{X}\ i\ `\ S)$
and $\bigwedge i.\ i \in I \Longrightarrow expectation\ (\mathcal{X}\ i) \in \{a\ i..b\ i\}$

⟨*proof*⟩

lemma *ln-one-plus-x-lower-bound*:
  assumes $x \geq (0::real)$
  shows $2*x/(2+x) \leq ln\ (1 + x)$
⟨*proof*⟩

Based on Theorem 4.1 by Motwani and Raghavan [16].

theorem *multiplicative-chernoff-bound-upper*:
  assumes $\delta > 0$
  assumes $\bigwedge i.\ i \in I \Longrightarrow AE\ \omega\ in\ M.\ X\ i\ \omega \in \{0..1\}$
  defines $\mu \equiv (\sum i \in I.\ expectation\ (X\ i))$
  shows $\mathcal{P}(\omega\ in\ M.\ (\sum i \in I.\ X\ i\ \omega) \geq (1{+}\delta) * \mu) \leq (exp\ \delta/((1{+}\delta)\ powr\ (1{+}\delta)))$ $powr\ \mu$ (**is** $?L \leq ?R$)
    and $\mathcal{P}(\omega\ in\ M.\ (\sum i \in I.\ X\ i\ \omega) \geq (1{+}\delta) * \mu) \leq exp\ (-(\delta\hat{\ }2) * \mu\ /\ (2{+}\delta))$ (**is** - $\leq ?R1$)
⟨*proof*⟩

lemma *ln-one-minus-x-lower-bound*:
  assumes $x \in \{(0::real)..{<}1\}$
  shows $(x\hat{\ }2/2{-}x)/(1{-}x) \leq ln\ (1 - x)$
⟨*proof*⟩

Based on Theorem 4.2 by Motwani and Raghavan [16].

theorem *multiplicative-chernoff-bound-lower*:
  assumes $\delta \in \{0{<}..{<}1\}$
  assumes $\bigwedge i.\ i \in I \Longrightarrow AE\ \omega\ in\ M.\ X\ i\ \omega \in \{0..1\}$
  defines $\mu \equiv (\sum i \in I.\ expectation\ (X\ i))$
  shows $\mathcal{P}(\omega\ in\ M.\ (\sum i \in I.\ X\ i\ \omega) \leq (1{-}\delta){*}\mu) \leq (exp\ ({-}\delta)/(1{-}\delta)\ powr\ (1{-}\delta))$ $powr\ \mu$ (**is** $?L \leq ?R$)
    and $\mathcal{P}(\omega\ in\ M.\ (\sum i \in I.\ X\ i\ \omega) \leq (1{-}\delta){*}\mu) \leq (exp\ (-(\delta\hat{\ }2){*}\mu/2))$ (**is** - $\leq$ $?R1$)
⟨*proof*⟩

theorem *multiplicative-chernoff-bound-two-sided*:
  assumes $\delta \in \{0{<}..{<}1\}$
  assumes $\bigwedge i.\ i \in I \Longrightarrow AE\ \omega\ in\ M.\ X\ i\ \omega \in \{0..1\}$
  defines $\mu \equiv (\sum i \in I.\ expectation\ (X\ i))$
  shows $\mathcal{P}(\omega\ in\ M.\ |(\sum i \in I.\ X\ i\ \omega) - \mu| \geq \delta{*}\mu) \leq 2{*}(exp\ (-(\delta\hat{\ }2){*}\mu/3))$ (**is** $?L \leq ?R$)
⟨*proof*⟩

lemma *additive-chernoff-bound-upper-aux*:
  assumes $\bigwedge i.\ i{\in}I \Longrightarrow AE\ \omega\ in\ M.\ X\ i\ \omega \in \{0..1\}\ I \neq \{\}$
  defines $\mu \equiv (\sum i{\in}I.\ expectation\ (X\ i))\ /\ real\ (card\ I)$
  assumes $\delta \in \{0{<}..{<}1{-}\mu\}\ \mu \in \{0{<}..{<}1\}$
  shows $\mathcal{P}(\omega\ in\ M.\ (\sum i{\in}I.\ X\ i\ \omega) \geq (\mu{+}\delta){*}real\ (card\ I)) \leq exp\ ({-}real\ (card\ I)$ $*\ KL\text{-}div\ (\mu{+}\delta)\ \mu)$
    (**is** $?L \leq ?R$)

⟨*proof*⟩

lemma *additive-chernoff-bound-upper-aux-2*:
assumes $\bigwedge i.\ i{\in}I \Longrightarrow AE\ \omega\ in\ M.\ X\ i\ \omega \in \{0..1\}\ I \neq \{\}$
defines $\mu \equiv (\sum i{\in}I.\ expectation\ (X\ i))\ /\ real\ (card\ I)$
assumes $\mu \in \{0{<}..{<}1\}$
shows $\mathcal{P}(\omega\ in\ M.\ (\sum i{\in}I.\ X\ i\ \omega) \geq real\ (card\ I)) \leq exp\ (-real\ (card\ I) * KL\text{-}div\ 1\ \mu)$
(**is** $?L \leq ?R$)
⟨*proof*⟩

Based on Theorem 1 by Hoeffding [11].

lemma *additive-chernoff-bound-upper*:
assumes $\bigwedge i.\ i{\in}I \Longrightarrow AE\ \omega\ in\ M.\ X\ i\ \omega \in \{0..1\}\ I \neq \{\}$
defines $\mu \equiv (\sum i{\in}I.\ expectation\ (X\ i))\ /\ real\ (card\ I)$
assumes $\delta \in \{0..1{-}\mu\}\ \mu \in \{0{<}..{<}1\}$
shows $\mathcal{P}(\omega\ in\ M.\ (\sum i{\in}I.\ X\ i\ \omega) \geq (\mu{+}\delta){*}real\ (card\ I)) \leq exp\ (-real\ (card\ I) * KL\text{-}div\ (\mu{+}\delta)\ \mu)$
(**is** $?L \leq ?R$)
⟨*proof*⟩

Based on Theorem 2 by Hoeffding [11].

lemma *hoeffding-bound-upper*:
assumes $\bigwedge i.\ i{\in}I \Longrightarrow a\ i \leq b\ i$
assumes $\bigwedge i.\ i{\in}I \Longrightarrow AE\ \omega\ in\ M.\ X\ i\ \omega \in \{a\ i..b\ i\}$
defines $n \equiv real\ (card\ I)$
defines $\mu \equiv (\sum i{\in}I.\ expectation\ (X\ i))$
assumes $\delta \geq 0\ (\sum i{\in}I.\ (b\ i - a\ i)\hat{\ }2) > 0$
shows $\mathcal{P}(\omega\ in\ M.\ (\sum i{\in}I.\ X\ i\ \omega) \geq \mu + \delta * n) \leq exp\ (-2{*}(n{*}\delta)\hat{\ }2\ /\ (\sum i{\in}I.\ (b\ i - a\ i)\hat{\ }2))$
(**is** $?L \leq ?R$)
⟨*proof*⟩

end

Dual and two-sided versions of Theorem 1 and 2 by Hoeffding [11].

lemma *additive-chernoff-bound-lower*:
assumes *neg-assoc X I finite I*
assumes $\bigwedge i.\ i{\in}I \Longrightarrow AE\ \omega\ in\ M.\ X\ i\ \omega \in \{0..1\}\ I \neq \{\}$
defines $\mu \equiv (\sum i{\in}I.\ expectation\ (X\ i))\ /\ real\ (card\ I)$
assumes $\delta \in \{0..\mu\}\ \mu \in \{0{<}..{<}1\}$
shows $\mathcal{P}(\omega\ in\ M.\ (\sum i{\in}I.\ X\ i\ \omega) \leq (\mu{-}\delta){*}real\ (card\ I)) \leq exp\ (-real\ (card\ I) * KL\text{-}div\ (\mu{-}\delta)\ \mu)$
(**is** $?L \leq ?R$)
⟨*proof*⟩

lemma *hoeffding-bound-lower*:
assumes *neg-assoc X I finite I*
assumes $\bigwedge i.\ i{\in}I \Longrightarrow a\ i \leq b\ i$

assumes $\bigwedge i.\ i{\in}I \Longrightarrow AE\ \omega\ in\ M.\ X\ i\ \omega \in \{a\ i..b\ i\}$
defines $n \equiv real\ (card\ I)$
defines $\mu \equiv (\sum i{\in}I.\ expectation\ (X\ i))$
assumes $\delta \geq 0\ (\sum i{\in}I.\ (b\ i - a\ i)\hat{~}2) > 0$
shows $\mathcal{P}(\omega\ in\ M.\ (\sum i{\in}I.\ X\ i\ \omega) \leq \mu{-}\delta{*}n) \leq exp\ ({-}2{*}(n{*}\delta)\hat{~}2\ /\ (\sum i{\in}I.\ (b\ i - a\ i)\hat{~}2))$
(**is** $?L \leq ?R$)
⟨*proof*⟩

lemma *hoeffding-bound-two-sided*:
assumes *neg-assoc X I finite I*
assumes $\bigwedge i.\ i{\in}I \Longrightarrow a\ i \leq b\ i$
assumes $\bigwedge i.\ i{\in}I \Longrightarrow AE\ \omega\ in\ M.\ X\ i\ \omega \in \{a\ i..b\ i\}\ I \neq \{\}$
defines $n \equiv real\ (card\ I)$
defines $\mu \equiv (\sum i{\in}I.\ expectation\ (X\ i))$
assumes $\delta \geq 0\ (\sum i{\in}I.\ (b\ i - a\ i)\hat{~}2) > 0$
shows $\mathcal{P}(\omega\ in\ M.\ |(\sum i{\in}I.\ X\ i\ \omega){-}\mu| \geq \delta{*}n) \leq 2{*}exp\ ({-}2{*}(n{*}\delta)\hat{~}2\ /\ (\sum i{\in}I.\ (b\ i - a\ i)\hat{~}2))$
(**is** $?L \leq ?R$)
⟨*proof*⟩

end

end

4 The FKG inequality

The FKG inequality [9] is a generalization of Chebyshev's less known other inequality. It is sometimes referred to as Chebyshev's sum inequality. Although there is a also a continuous version, which can be stated as:

$$E[fg] \geq E[f]E[g]$$

where f, g are continuous simultaneously monotone or simultaneously antimonotone functions on the Lebesgue probability space $[a, b] \subseteq \mathbb{R}$. ($Ef$ denotes the expectation of the function.)

Note that the inequality is also true for totally ordered discrete probability spaces, for example: $\{1, \ldots, n\}$ with uniform probabilities.

The FKG inequality is essentially a generalization of the above to not necessarily totally ordered spaces, but finite distributive lattices.

The proof follows the derivation in the book by Alon and Spencer [2, Ch. 6].

theory *Negative-Association-FKG-Inequality*
imports
Negative-Association-Util
Birkhoff-Finite-Distributive-Lattices.Birkhoff-Finite-Distributive-Lattices

begin

theorem *four-functions-helper*:
fixes $\varphi :: nat \Rightarrow 'a\ set \Rightarrow real$
assumes *finite I*
assumes $\bigwedge i.\ i \in \{0..3\} \Longrightarrow \varphi\ i \in Pow\ I \rightarrow \{0..\}$
assumes $\bigwedge A\ B.\ A \subseteq I \Longrightarrow B \subseteq I \Longrightarrow \varphi\ 0\ A * \varphi\ 1\ B \leq \varphi\ 2\ (A \cup B) * \varphi\ 3\ (A \cap B)$
shows $(\sum A{\in}Pow\ I.\ \varphi\ 0\ A)*(\sum B{\in}Pow\ I.\ \varphi\ 1\ B) \leq (\sum C{\in}Pow\ I.\ \varphi\ 2\ C)*(\sum D{\in}Pow\ I.\ \varphi\ 3\ D)$
⟨*proof*⟩

The following is the Ahlswede-Daykin inequality [1] also referred to by Alon and Spencer as the four functions theorem [2, Th. 6.1.1].

theorem *four-functions*:
fixes $\alpha\ \beta\ \gamma\ \delta :: 'a\ set \Rightarrow real$
assumes *finite I*
assumes $\alpha \in Pow\ I \rightarrow \{0..\}\ \beta \in Pow\ I \rightarrow \{0..\}\ \gamma \in Pow\ I \rightarrow \{0..\}\ \delta \in Pow\ I \rightarrow \{0..\}$
assumes $\bigwedge A\ B.\ A \subseteq I \Longrightarrow B \subseteq I \Longrightarrow \alpha\ A * \beta\ B \leq \gamma\ (A \cup B) * \delta\ (A \cap B)$
assumes $M \subseteq Pow\ I\ N \subseteq Pow\ I$
shows $(\sum A{\in}M.\ \alpha\ A)*(\sum B{\in}N.\ \beta\ B) \leq (\sum C|\ \exists A{\in}M.\ \exists B{\in}N.\ C{=}A{\cup}B.\ \gamma\ C)*(\sum D|\ \exists A{\in}M.\ \exists B{\in}N.\ D{=}A{\cap}B.\ \delta\ D)$
(**is** $?L \leq ?R$)
⟨*proof*⟩

Using Birkhoff's Representation Theorem [3, 5] it is possible to generalize the previous to finite distributive lattices [2, Cor. 6.1.2].

lemma *four-functions-in-lattice*:
fixes $\alpha\ \beta\ \gamma\ \delta :: 'a :: \textit{finite-distrib-lattice} \Rightarrow real$
assumes $range\ \alpha \subseteq \{0..\}\ range\ \beta \subseteq \{0..\}\ range\ \gamma \subseteq \{0..\}\ range\ \delta \subseteq \{0..\}$
assumes $\bigwedge x\ y.\ \alpha\ x * \beta\ y \leq \gamma\ (x \sqcup y) * \delta\ (x \sqcap y)$
shows $(\sum x{\in}M.\ \alpha\ x)*(\sum y{\in}N.\ \beta\ y) \leq (\sum c|\ \exists x{\in}M.\ \exists y{\in}N.\ c{=}x{\sqcup}y.\ \gamma\ c)*(\sum d|\ \exists x{\in}M.\ \exists y{\in}N.\ d{=}x{\sqcap}y.\ \delta\ d)$
(**is** $?L \leq ?R$)
⟨*proof*⟩

theorem *fkg-inequality*:
fixes $\mu :: 'a :: \textit{finite-distrib-lattice} \Rightarrow real$
assumes $range\ \mu \subseteq \{0..\}\ range\ f \subseteq \{0..\}\ range\ g \subseteq \{0..\}$
assumes $\bigwedge x\ y.\ \mu\ x * \mu\ y \leq \mu\ (x \sqcup y) * \mu\ (x \sqcap y)$
assumes *mono f mono g*
shows $(\sum x{\in}UNIV.\ \mu\ x{*}f\ x) * (\sum x{\in}UNIV.\ \mu\ x{*}g\ x) \leq (\sum x{\in}UNIV.\ \mu\ x{*}f\ x{*}g\ x) * sum\ \mu\ UNIV$
(**is** $?L \leq ?R$)
⟨*proof*⟩

theorem *fkg-inequality-gen*:
fixes $\mu :: 'a :: \textit{finite-distrib-lattice} \Rightarrow real$

assumes *range* $\mu \subseteq \{0..\}$
assumes $\bigwedge x\ y.\ \mu\ x * \mu\ y \leq \mu\ (x \sqcup y) * \mu\ (x \sqcap y)$
assumes *monotone* $(\leq)$ $(\leq\geq_\tau)$ f *monotone* $(\leq)$ $(\leq\geq_\sigma)$ g
shows $(\sum x{\in}UNIV.\ \mu\ x{*}f\ x) * (\sum x{\in}UNIV.\ \mu\ x{*}g\ x) \leq\geq_{\tau*\sigma} (\sum x{\in}UNIV.\ \mu\ x{*}f\ x{*}g\ x) * sum\ \mu\ UNIV$
(**is** $?L \leq\geq_{?x}$ $?R$)
$\langle proof \rangle$

theorem *fkg-inequality-pmf*:
fixes $M :: ('a :: \textit{finite-distrib-lattice})\ pmf$
fixes $f\ g :: 'a \Rightarrow real$
assumes $\bigwedge x\ y.\ pmf\ M\ x * pmf\ M\ y \leq pmf\ M\ (x \sqcup y) * pmf\ M\ (x \sqcap y)$
assumes *monotone* $(\leq)$ $(\leq\geq_\tau)$ f *monotone* $(\leq)$ $(\leq\geq_\sigma)$ g
shows $(\int x.\ f\ x\ \partial M) * (\int x.\ g\ x\ \partial M) \leq\geq_{\tau\ *\ \sigma} (\int x.\ f\ x * g\ x\ \partial M)$
(**is** $?L \leq\geq_{-}$ $?R$)
$\langle proof \rangle$

end

5 Preliminary Results on Lattices

This entry establishes a few missing lemmas for the set-based theory of lattices from "HOL-Algebra". In particular, it introduces the sublocale for distributive lattices.

More crucially, a transfer theorem which can be used in conjunction with the Types-To-Sets mechanism to be able to work with locally defined finite distributive lattices.

This is being needed for the verification of the negative association of permutation distributions in Section 6.

theory *Negative-Association-More-Lattices*
imports *HOL−Algebra.Lattice*
begin

Lemma 1 Birkhoff Lattice Theory, p.8, L3

lemma (**in** *lattice*) *meet-assoc-law*:
assumes $x \in carrier\ L$ $y \in carrier\ L$ $z \in carrier\ L$
shows $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$
$\langle proof \rangle$

Lemma 1 Birkhoff Lattice Theory, p.8, L3

lemma (**in** *lattice*) *join-assoc-law*:
assumes $x \in carrier\ L$ $y \in carrier\ L$ $z \in carrier\ L$
shows $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$
$\langle proof \rangle$

Lemma 1 Birkhoff Lattice Theory, p.8, L4

lemma (**in** *lattice*) *absorbtion-law*:
  assumes $x \in$ *carrier L* $y \in$ *carrier L*
  shows $x \sqcap (x \sqcup y) = x$ $x \sqcup (x \sqcap y) = x$
⟨*proof*⟩

Theorem 9 Birkhoff Lattice Theory, p.11

lemma (**in** *lattice*) *distrib-laws-equiv*:
  defines *meet-distrib* $\equiv$ ($\forall$ x y z. $\{x,y,z\}\subseteq$*carrier L* $\longrightarrow$ $(x \sqcap (y \sqcup z)) = (x \sqcap y) \sqcup (x \sqcap z)$)
  defines *join-distrib* $\equiv$ ($\forall$ x y z. $\{x,y,z\}\subseteq$*carrier L* $\longrightarrow$ $(x \sqcup (y \sqcap z)) = (x \sqcup y) \sqcap (x \sqcup z)$)
  shows *meet-distrib* $\longleftrightarrow$ *join-distrib*
⟨*proof*⟩

lemma (**in** *lattice*) *lub-unique-set*:
  assumes *is-lub L z S*
  shows $z = \bigsqcup S$
⟨*proof*⟩

lemma (**in** *lattice*) *lub-unique*:
  assumes *is-lub L z* $\{x,y\}$
  shows $z = x \sqcup y$
  ⟨*proof*⟩

lemma (**in** *lattice*) *glb-unique-set*:
  assumes *is-glb L z S*
  shows $z = \bigsqcap S$
⟨*proof*⟩

lemma (**in** *lattice*) *glb-unique*:
  assumes *is-glb L z* $\{x,y\}$
  shows $z = x \sqcap y$
  ⟨*proof*⟩

lemma (**in** *lattice*) *inf-lower*:
  assumes $S \subseteq$ *carrier L* $s \in S$ *finite S*
  shows $\bigsqcap S \sqsubseteq s$
⟨*proof*⟩

lemma (**in** *lattice*) *sup-upper*:
  assumes $S \subseteq$ *carrier L* $s \in S$ *finite S*
  shows $s \sqsubseteq \bigsqcup S$
⟨*proof*⟩

locale *distrib-lattice* = *lattice* +
  assumes *max-distrib*:
    $x \in$ *carrier L* $\Longrightarrow$ $y \in$ *carrier L* $\Longrightarrow$ $z \in$ *carrier L* $\Longrightarrow$ $(x \sqcap (y \sqcup z)) = (x \sqcap y) \sqcup (x \sqcap z)$
begin

lemma *min-distrib*:
  assumes $x \in$ *carrier* L $y \in$ *carrier* L $z \in$ *carrier* L
  shows $(x \sqcup (y \sqcap z)) = (x \sqcup y) \sqcap (x \sqcup z)$
⟨*proof*⟩

end

locale *finite-ne-distrib-lattice* = *distrib-lattice* +
  assumes *non-empty-carrier*: *carrier* $L \neq \{\}$
  assumes *finite-carrier*: *finite* (*carrier* L)
begin

lemma *bounded-lattice-axioms-1*: $\exists\, x.$ *least* L x (*carrier* L)
⟨*proof*⟩

lemma *bounded-lattice-axioms-2*: $\exists\, x.$ *greatest* L x (*carrier* L)
⟨*proof*⟩

sublocale *bounded-lattice*
  ⟨*proof*⟩

lemma *inf-empty*: $\bigsqcap\ \{\} = \top$
⟨*proof*⟩

lemma *inf-closed*: $S \subseteq$ *carrier* $L \Longrightarrow \bigsqcap S \in$ *carrier* L
  ⟨*proof*⟩

lemma *inf-insert*:
  assumes $x \in$ *carrier* L $S \subseteq$ *carrier* L
  shows $\bigsqcap$ (*insert* x S) $= x \sqcap (\bigsqcap S)$
⟨*proof*⟩

lemma *sup-empty*: $\bigsqcup\ \{\} = \bot$
⟨*proof*⟩

lemma *sup-closed*: $S \subseteq$ *carrier* $L \Longrightarrow \bigsqcup\ S \in$ *carrier* L
  ⟨*proof*⟩

lemma *sup-insert*:
  assumes $x \in$ *carrier* L $S \subseteq$ *carrier* L
  shows $\bigsqcup$ (*insert* x S) $= x \sqcup (\bigsqcup S)$
⟨*proof*⟩

lemma *inf-carrier*: $\bigsqcap$ (*carrier* L) $= \bot$
⟨*proof*⟩

lemma *sup-carrier*: $\bigsqcup$ (*carrier* L) $= \top$
⟨*proof*⟩

lemma *transfer-to-type*:
assumes *finite* (*carrier L*) *type-definition Rep Abs* (*carrier L*)
defines $inf' \equiv (\lambda M.\ Abs\ (\bigsqcap\ Rep\ `\ M))$
defines $sup' \equiv (\lambda M.\ Abs\ (\bigsqcup\ Rep\ `\ M))$
defines $join' \equiv (\lambda x\ y.\ Abs\ (Rep\ x \sqcap Rep\ y))$
defines $le' \equiv (\lambda x\ y.\ (Rep\ x \sqsubseteq Rep\ y))$
defines $less' \equiv (\lambda x\ y.\ (Rep\ x \sqsubset Rep\ y))$
defines $meet' \equiv (\lambda x\ y.\ (Abs\ (Rep\ x \sqcup Rep\ y)))$
defines $bot' \equiv (Abs\ \bot :: 'c)$
defines $top' \equiv Abs\ \top$
shows *class.finite-distrib-lattice inf′ sup′ join′ le′ less′ meet′ bot′ top′*
⟨*proof*⟩

end

end

6 Permutation Distributions

One of the fundamental examples for negatively associated random variables are permutation distributions.

Let $x_1, \ldots, x_n$ be n (not-necessarily) distinct values from a totally ordered set, then we choose a permutation $\sigma : \{0, \ldots, n-1\} \to \{0, \ldots, n-1\}$ uniformly at random Then the random variables defined by $X_i(\sigma) = x_{\sigma(i)}$ are negatively associated.

An important special case is the case where x consists of 1 one and $(n-1)$ zeros, modelling randomly putting a ball into one of n bins. Of course the process can be repeated independently, the resulting distribution is also referred to as the balls into bins process. Because of the closure properties established before, it is possible to conclude that the number of hits of each bin in such a process are also negatively associated random variables.

In this section, we will derive that permutation distributions are negatively associated. The proof follows Dubashi [8, Th. 10] closely. A very short proof was presented in the work by Joag-Dev [13], however after close inspection that proof seemed to missing a lot of details. In fact, I don't think it is correct.

theory *Negative-Association-Permutation-Distributions*
imports
Negative-Association-Definition
Negative-Association-FKG-Inequality
Negative-Association-More-Lattices
Finite-Fields.Finite-Fields-More-PMF
HOL−Types-To-Sets.Types-To-Sets

Executable-Randomized-Algorithms.Randomized-Algorithm
Twelvefold-Way.Card-Bijections
begin

The following introduces a lattice for n-element subsets of a finite set (with size larger or equal to n.) A subset x is smaller or equal to y, if the smallest element of x is smaller or equal to the smallest element of y, the second smallest element of x is smaller or equal to the second smallest element of y, etc.)

The lattice is introduced without name by Dubashi [**?**, Example 7].

definition *le-ordered-set-lattice* :: ('*a*::*linorder*) *set* ⇒ '*a set* ⇒ *bool*
where *le-ordered-set-lattice S T* = *list-all2* (≤) (*sorted-list-of-set S*) (*sorted-list-of-set T*)

definition *ordered-set-lattice* :: ('*a* :: *linorder*) *set* ⇒ *nat* ⇒ '*a set gorder*
where *ordered-set-lattice S n* =
⦇ *carrier* = {*T*. *T* ⊆ *S* ∧ *finite T* ∧ *card T* = *n*},
eq = (=),
le = *le-ordered-set-lattice* ⦈

definition *osl-repr* :: ('*a* :: *linorder*) *set* ⇒ *nat* ⇒ '*a set* ⇒ *nat* ⇒ '*a*
where *osl-repr S n e* = (λ*i* ∈ {..<*n*}. *sorted-list-of-set e* ! *i*)

lemma *osl-carr-sorted-list-of-set*:
assumes *finite S n* ≤ *card S*
assumes *s* ∈ *carrier* (*ordered-set-lattice S n*)
defines *t* ≡ *sorted-list-of-set s*
shows *finite s card s* = *n s* ⊆ *S length t* = *n set t* = *s sorted-wrt* (<) *t*
⟨*proof*⟩

lemma *ordered-set-lattice-carrier-intro*:
assumes *finite S n* ≤ *card S*
assumes *set s* ⊆ *S distinct s length s* = *n*
shows *set s* ∈ *carrier* (*ordered-set-lattice S n*)
⟨*proof*⟩

lemma *osl-list-repr-inj*:
assumes *finite S n* ≤ *card S*
assumes *s* ∈ *carrier* (*ordered-set-lattice S n*)
assumes *t* ∈ *carrier* (*ordered-set-lattice S n*)
assumes ⋀*i*. *osl-repr S n s i* = *osl-repr S n t i*
shows *s* = *t*
⟨*proof*⟩

lemma *osl-leD*:
assumes *finite S n* ≤ *card S*
assumes *e* ∈ *carrier* (*ordered-set-lattice S n*)
assumes *f* ∈ *carrier* (*ordered-set-lattice S n*)

shows $e \sqsubseteq_{\mathit{ordered\text{-}set\text{-}lattice}\ S\ n} f \longleftrightarrow (\forall\, i.\ \mathit{osl\text{-}repr}\ S\ n\ e\ i \le \mathit{osl\text{-}repr}\ S\ n\ f\ i)$ (**is** $?L = ?R$)
$\langle \mathit{proof} \rangle$

lemma *ordered-set-lattice-partial-order*:
fixes $S :: ('a :: \mathit{linorder})\ \mathit{set}$
assumes *finite S* $n \le$ *card S*
shows *partial-order* (*ordered-set-lattice S n*)
$\langle \mathit{proof} \rangle$

lemma *map2-max-mono*:
fixes $xs :: ('a :: \mathit{linorder})\ \mathit{list}$
assumes *length xs* = *length ys*
assumes *sorted-wrt* (<) *xs sorted-wrt* (<) *ys*
shows *sorted-wrt* (<) (*map2 max xs ys*)
$\langle \mathit{proof} \rangle$

lemma *map2-min-mono*:
fixes $xs :: ('a :: \mathit{linorder})\ \mathit{list}$
assumes *length xs* = *length ys*
assumes *sorted-wrt* (<) *xs sorted-wrt* (<) *ys*
shows *sorted-wrt* (<) (*map2 min xs ys*)
$\langle \mathit{proof} \rangle$

lemma *ordered-set-lattice-carrier-finite-ne*:
assumes *finite S* $n \le$ *card S*
shows *carrier* (*ordered-set-lattice S n*) $\neq \{\}$ *finite* (*carrier* (*ordered-set-lattice S n*))
$\langle \mathit{proof} \rangle$

lemma *ordered-set-lattice-lattice*:
fixes $S :: ('a :: \mathit{linorder})\ \mathit{set}$
assumes *finite S* $n \le$ *card S*
shows *finite-ne-distrib-lattice* (*ordered-set-lattice S n*)
$\langle \mathit{proof} \rangle$

lemma *insort-eq*:
fixes $xs :: ('a :: \mathit{linorder})\ \mathit{list}$
assumes *sorted xs*
shows $\exists\, ys\ zs.\ \mathit{insort}\ e\ xs = ys@e\#zs \wedge ys@zs{=}xs \wedge \mathit{set}\ ys \subseteq \{..{<}e\} \wedge \mathit{set}\ zs \subseteq \{e..\}$
$\langle \mathit{proof} \rangle$

lemma *list-all2-insort*:
fixes $xs\ ys :: ('a :: \mathit{linorder})\ \mathit{list}$
assumes *length xs* = *length ys sorted xs sorted ys*
shows *list-all2* (≤) *xs ys* ⟷ *list-all2* (≤) (*insort e xs*) (*insort e ys*)
$\langle \mathit{proof} \rangle$

lemma *le-ordered-set-lattice-diff*:
  fixes $x\ y :: ('a :: linorder)\ set$
  assumes *finite x finite y card x = card y*
  shows $le\text{-}ordered\text{-}set\text{-}lattice\ x\ y \longleftrightarrow le\text{-}ordered\text{-}set\text{-}lattice\ (x - y)\ (y - x)$
⟨*proof*⟩

lemma *ordered-set-lattice-carrier*:
  assumes $T \in carrier\ (ordered\text{-}set\text{-}lattice\ S\ n)$
  shows *finite T card T = n* $T \subseteq S$
  ⟨*proof*⟩

lemma *ordered-set-lattice-dual*:
  assumes *finite S* $n \le card\ S$
  defines $L \equiv ordered\text{-}set\text{-}lattice\ S\ n$
  defines $M \equiv ordered\text{-}set\text{-}lattice\ S\ (card\ S - n)$
  shows
    $\bigwedge x.\ x \in carrier\ L \Longrightarrow (S{-}x) \in carrier\ M$
    $\bigwedge x.\ x \in carrier\ M \Longrightarrow (S{-}x) \in carrier\ L$
    $\bigwedge x\ y.\ x \in carrier\ L \wedge y \in carrier\ L \Longrightarrow x \sqsubseteq_L y \longleftrightarrow (S{-}y) \sqsubseteq_M (S{-}x)$
⟨*proof*⟩

lemma *bij-betw-ord-set-lattice-pairs*:
  assumes *finite S* $n \le card\ S$
  defines $L \equiv ordered\text{-}set\text{-}lattice\ S\ n$
  assumes $x \in carrier\ L\ y \in carrier\ L\ x \sqsubseteq_L y$
  shows $\exists \varphi.\ bij\text{-}betw\ \varphi\ x\ y \wedge strict\text{-}mono\text{-}on\ x\ \varphi \wedge (\forall e.\ \varphi\ e \ge e)$
⟨*proof*⟩

definition $bij\text{-}pmf\ I\ F = pmf\text{-}of\text{-}set\ \{f.\ bij\text{-}betw\ f\ I\ F \wedge f \in extensional\ I\}$

lemma *card-bijections′*:
  assumes *finite A finite B card A = card B*
  shows $card\ \{f.\ bij\text{-}betw\ f\ A\ B \wedge f \in extensional\ A\} = fact\ (card\ A)$ (**is** *?L = ?R*)
⟨*proof*⟩

lemma *bij-betw-non-empty-finite*:
  assumes *finite I finite F card I = card F*
  shows
    $finite\ \{f.\ bij\text{-}betw\ f\ I\ F \wedge f \in extensional\ I\}$ (**is** *?T1*)
    $\{f.\ bij\text{-}betw\ f\ I\ F \wedge f \in extensional\ I\} \neq \{\}$ (**is** *?T2*)
⟨*proof*⟩

lemma *bij-pmf*:
  assumes *finite I finite F card I = card F*
  shows
    $set\text{-}pmf\ (bij\text{-}pmf\ I\ F) = \{f.\ bij\text{-}betw\ f\ I\ F \wedge f \in extensional\ I\}$
    $finite\ (set\text{-}pmf\ (bij\text{-}pmf\ I\ F))$
  ⟨*proof*⟩

lemma *expectation-ge-eval-at-point*:
  assumes $\bigwedge y.\ y \in \textit{set-pmf}\ p \Longrightarrow f\ y \geq (0{::}\textit{real})$
  assumes *integrable p f*
  shows $\textit{pmf}\ p\ x * f\ x \leq (\int x.\ f\ x\ \partial p)$ (**is** $?L \leq ?R$)
$\langle \textit{proof} \rangle$

lemma *split-bij-pmf*:
  assumes $\textit{finite}\ I\ \textit{finite}\ F\ \textit{card}\ I = \textit{card}\ F\ J \subseteq I$
  shows $\textit{bij-pmf}\ I\ F =$
    do {
      $S \leftarrow \textit{pmf-of-set}\ \{S.\ \textit{card}\ S = \textit{card}\ J \wedge S \subseteq F\}$;
      $\varphi \leftarrow \textit{bij-pmf}\ J\ S$;
      $\psi \leftarrow \textit{bij-pmf}\ (I-J)\ (F-S)$;
      $\textit{return-pmf}\ (\textit{merge}\ J\ (I-J)\ (\varphi, \psi))$
    } (**is** $?L = ?R$)
$\langle \textit{proof} \rangle$

lemma *map-bij-pmf*:
  assumes $\textit{finite}\ I\ \textit{finite}\ F\ \textit{card}\ I = \textit{card}\ F\ \textit{inj-on}\ \varphi\ F$
  shows $\textit{map-pmf}\ (\lambda f.\ (\lambda x{\in}I.\ \varphi(f\ x)))\ (\textit{bij-pmf}\ I\ F) = \textit{bij-pmf}\ I\ (\varphi\ `\ F)$
$\langle \textit{proof} \rangle$

lemma *pmf-of-multiset-eq-pmf-of-setI*:
  assumes $c > 0\ \ x \neq \{\#\}$
  assumes $\bigwedge i.\ i \in y \Longrightarrow \textit{count}\ x\ i = c$
  assumes $\bigwedge i.\ i \in\#\ x \Longrightarrow i \in y$
  shows $\textit{pmf-of-multiset}\ x = \textit{pmf-of-set}\ y$
$\langle \textit{proof} \rangle$

lemma *card-multi-bij*:
  assumes *finite J*
  assumes $I = \bigcup (A\ `\ J)\ \textit{disjoint-family-on}\ A\ J$
  assumes $\bigwedge j.\ j \in J \Longrightarrow \textit{finite}\ (A\ j) \wedge \textit{finite}\ (B\ j) \wedge \textit{card}\ (A\ j) = \textit{card}\ (B\ j)$
    shows $\textit{card}\ \{f.\ (\forall j{\in}J.\ \textit{bij-betw}\ f\ (A\ j)\ (B\ j)) \wedge f{\in}\textit{extensional}\ I\} = (\prod i{\in}J.$
$\textit{fact}\ (\textit{card}\ (A\ i)))$
      (**is** $\textit{card}\ ?L = ?R$)
$\langle \textit{proof} \rangle$

lemma *map-bij-pmf-non-inj*:
  fixes $I :: 'a\ \textit{set}$
  fixes $F :: 'b\ \textit{set}$
  fixes $\varphi :: 'b \Rightarrow 'c$
  assumes $\textit{finite}\ I\ \textit{finite}\ F\ \textit{card}\ I = \textit{card}\ F$
  defines $q \equiv \{f.\ f \in \textit{extensional}\ I \wedge \{\#f\ x.\ x \in\#\ \textit{mset-set}\ I\#\} = \{\#\varphi\ x.\ x{\in}\#$
$\textit{mset-set}\ F\#\}\}$
  shows $\textit{map-pmf}\ (\lambda f.\ (\lambda x{\in}I.\ \varphi(f\ x)))\ (\textit{bij-pmf}\ I\ F) = \textit{pmf-of-set}\ q$ (**is** $?L = \text{-}$)
$\langle \textit{proof} \rangle$

lemmas *fkg-inequality-pmf-internalized* = *fkg-inequality-pmf*[*unoverload-type* $'a$]

lemma *permutation-distributions-are-neg-associated*:
  fixes F :: ($'a$ :: *linorder-topology*) *set*
  fixes I :: $'b$ *set*
  assumes *finite F finite I card I* = *card F*
  shows *measure-pmf.neg-assoc* (*bij-pmf I F*) ($\lambda i\ \omega.\ \omega\ i$) I
⟨*proof*⟩

lemma *multiset-permutation-distributions-are-neg-associated*:
  fixes F :: ($'a$ :: *linorder-topology*) *multiset*
  fixes I :: $'b$ *set*
  assumes *finite I card I* = *size F*
  defines $p \equiv$ *pmf-of-set* $\{\varphi.\ \varphi \in$ *extensional I* $\wedge$ *image-mset* φ (*mset-set I*) $= F\}$
  shows *measure-pmf.neg-assoc p* ($\lambda i\ \omega.\ \omega\ i$) I
⟨*proof*⟩

lemma *n-subsets-prob*:
  assumes $d \leq$ *card S finite S* $s \in S$
  shows
    measure-pmf.prob (*pmf-of-set* $\{a.\ a \subseteq S \wedge$ *card* $a = d\}$) $\{\omega.\ s \notin \omega\}$ = (*1* − *real* *d*/*card S*)
    measure-pmf.prob (*pmf-of-set* $\{a.\ a \subseteq S \wedge$ *card* $a = d\}$) $\{\omega.\ s \in \omega\}$ = *real* *d*/*card S*
⟨*proof*⟩

lemma *n-subsets-distribution-neg-assoc*:
  assumes *finite S* $k \leq$ *card S*
  defines $p \equiv$ *pmf-of-set* $\{T.\ T \subseteq S \wedge$ *card* $T = k\}$
  shows *measure-pmf.neg-assoc p* ($\in$) S
⟨*proof*⟩

end

7 Application: Bloom Filters

The false positive probability of Bloom Filters is a case where negative association is really useful. Traditionally it is derived only approximately. Bloom [4] first derives the expected number of bits set to true given the number of elements inserted, then the false positive probability is computed, pretending that the expected number of bits is the actual number of bits.

Both Blooms original derivation and Mitzenmacher and Upfal [15] use this method.

A more correct approach would be to derive a tail bound for the number of set bits and derive a false-positive probability based on that, which unfortunately leads to a complex formula.

An exact result has later been derived using combinatorial methods by Gopinathan and Sergey [10]. However their formula is less useful, as it consists of a sum with Stirling numbers and binomial coefficients.

It is however easy to see that the original bound derived by Bloom is a correct upper bound for the false positive probability using negative association. (This is pointed out by Bao et al. [**?**].)

In this section, we derive the same bound using this library as an example for the applicability of this library.

theory *Negative-Association-Bloom-Filters*
  imports *Negative-Association-Permutation-Distributions*
begin

fun *bloom-filter-pmf* **where**
  $bloom\text{-}filter\text{-}pmf\ 0\ d\ N = return\text{-}pmf\ \{\}\ |$
  $bloom\text{-}filter\text{-}pmf\ (Suc\ n)\ d\ N = do\ \{$
    $h \leftarrow bloom\text{-}filter\text{-}pmf\ n\ d\ N;$
    $a \leftarrow pmf\text{-}of\text{-}set\ \{a.\ a \subseteq \{..<(N::nat)\} \wedge card\ a = d\};$
    $return\text{-}pmf\ (a \cup h)$
  $\}$

lemma *bloom-filter-neg-assoc*:
  assumes $d \leq N$
  shows $measure\text{-}pmf.neg\text{-}assoc\ (bloom\text{-}filter\text{-}pmf\ n\ d\ N)\ (\lambda i\ \omega.\ i \in \omega)\ \{..<N\}$
⟨*proof*⟩

lemma *bloom-filter-cell-prob*:
  assumes $d \leq N\ i < N$
  shows $measure\ (bloom\text{-}filter\text{-}pmf\ n\ d\ N)\ \{\omega.\ i \in \omega\} = 1 - (1 - real\ d/real\ N)\hat{}n$
⟨*proof*⟩

lemma *bloom-filter-false-positive-prob*:
  assumes $d \leq N\ T \subseteq \{..<N\}\ card\ T = d$
  shows $measure\ (bloom\text{-}filter\text{-}pmf\ n\ d\ N)\ \{\omega.\ T \subseteq \omega\} \leq (1 - (1 - real\ d/real\ N)\hat{}n)\hat{}d$
    (**is** $?L \leq ?R$)
⟨*proof*⟩

end

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