

# The Nash-Williams Theorem

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## Abstract

In 1965, Nash-Williams [1] discovered a generalisation of the infinite form of Ramsey’s theorem. Where the latter concerns infinite sets of  $n$ -element sets for some fixed  $n$ , the Nash-Williams theorem concerns infinite sets of finite sets (or lists) subject to a “no initial segment” condition. The present formalisation follows Todorčević [2].

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## 1 The Pointwise Less-Than Relation Between Two Sets

**theory** *Nash-Extras*

**imports** *HOL-Library.Ramsey* *HOL-Library.Countable-Set*

**begin**

**definition** *less-sets* :: [*'a::order set, 'a::order set*]  $\Rightarrow$  *bool* (**infixr**  $\ll$  50)  
**where**  $A \ll B \equiv \forall x \in A. \forall y \in B. x < y$

**lemma** *less-sets-empty*[*iff*]:  $S \ll \{\} \{\} \ll T$   
*<proof>*

**lemma** *less-setsD*:  $\llbracket A \ll B; a \in A; b \in B \rrbracket \Longrightarrow a < b$   
*<proof>*

**lemma** *less-sets-irrefl* [*simp*]:  $A \ll A \longleftrightarrow A = \{\}$   
 ⟨*proof*⟩

**lemma** *less-sets-trans*:  $\llbracket A \ll B; B \ll C; B \neq \{\} \rrbracket \implies A \ll C$   
 ⟨*proof*⟩

**lemma** *less-sets-weaken1*:  $\llbracket A' \ll B; A \subseteq A' \rrbracket \implies A \ll B$   
 ⟨*proof*⟩

**lemma** *less-sets-weaken2*:  $\llbracket A \ll B'; B \subseteq B' \rrbracket \implies A \ll B$   
 ⟨*proof*⟩

**lemma** *less-sets-imp-disjnt*:  $A \ll B \implies \text{disjnt } A B$   
 ⟨*proof*⟩

**lemma** *less-sets-UN1*: *less-sets*  $(\bigcup \mathcal{A}) B \longleftrightarrow (\forall A \in \mathcal{A}. A \ll B)$   
 ⟨*proof*⟩

**lemma** *less-sets-UN2*: *less-sets*  $A (\bigcup \mathcal{B}) \longleftrightarrow (\forall B \in \mathcal{B}. A \ll B)$   
 ⟨*proof*⟩

**lemma** *less-sets-Un1*: *less-sets*  $(A \cup A') B \longleftrightarrow A \ll B \wedge A' \ll B$   
 ⟨*proof*⟩

**lemma** *less-sets-Un2*: *less-sets*  $A (B \cup B') \longleftrightarrow A \ll B \wedge A \ll B'$   
 ⟨*proof*⟩

**lemma** *strict-sorted-imp-less-sets*:  
*strict-sorted*  $(as @ bs) \implies (\text{list.set } as) \ll (\text{list.set } bs)$   
 ⟨*proof*⟩

**lemma** *Sup-nat-less-sets-singleton*:  
**fixes**  $n::\text{nat}$   
**assumes**  $\text{Sup } T < n$  *finite*  $T$   
**shows** *less-sets*  $T \{\! \{n\}\! \}$   
 ⟨*proof*⟩

end

## 2 The Nash-Williams Theorem

Following S. Todorćević, *Introduction to Ramsey Spaces*, Princeton University Press (2010), 11–12.

**theory** *Nash-Williams*  
**imports** *Nash-Extras*  
**begin**

**lemma** *finite-nat-Int-greaterThan-iff*:

**fixes**  $N :: \text{nat set}$   
**shows**  $\text{finite } (N \cap \{n < ..\}) \longleftrightarrow \text{finite } N$   
 $\langle \text{proof} \rangle$

## 2.1 Initial segments

**definition**  $\text{init-segment} :: \text{nat set} \Rightarrow \text{nat set} \Rightarrow \text{bool}$   
**where**  $\text{init-segment } S T \equiv \exists S'. T = S \cup S' \wedge S \ll S'$

**lemma**  $\text{init-segment-subset}: \text{init-segment } S T \Longrightarrow S \subseteq T$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{init-segment-refl}: \text{init-segment } S S$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{init-segment-antisym}: [\text{init-segment } S T; \text{init-segment } T S] \Longrightarrow S = T$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{init-segment-trans}: [\text{init-segment } S T; \text{init-segment } T U] \Longrightarrow \text{init-segment } S U$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{init-segment-empty2} [\text{iff}]: \text{init-segment } S \{\} \longleftrightarrow S = \{\}$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{init-segment-Un}: S \ll S' \Longrightarrow \text{init-segment } S (S \cup S')$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{init-segment-iff0}$ :  
**shows**  $\text{init-segment } S T \longleftrightarrow S \subseteq T \wedge S \ll (T - S)$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{init-segment-iff}$ :  
**shows**  $\text{init-segment } S T \longleftrightarrow S = T \vee (\exists m \in T. S = \{n \in T. n < m\})$  (is  
 $?\text{lhs} = ?\text{rhs}$ )  
 $\langle \text{proof} \rangle$

**lemma**  $\text{init-segment-empty} [\text{iff}]: \text{init-segment } \{\} S$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{init-segment-insert-iff}$ :  
**assumes**  $S n: S \ll \{n\}$  **and**  $TS: \bigwedge x. x \in T - S \Longrightarrow n \leq x$   
**shows**  $\text{init-segment } (\text{insert } n S) T \longleftrightarrow \text{init-segment } S T \wedge n \in T$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{init-segment-insert}$ :  
**assumes**  $\text{init-segment } S T$  **and**  $T: T \ll \{n\}$   
**shows**  $\text{init-segment } S (\text{insert } n T)$   
 $\langle \text{proof} \rangle$

## 2.2 Definitions and basic properties

**definition** *Ramsey* :: [nat set set, nat] ⇒ bool

**where** *Ramsey*  $\mathcal{F}$   $r \equiv \forall f \in \mathcal{F} \rightarrow \{..<r\}$ .  
 $\forall M. \text{infinite } M \longrightarrow$   
 $(\exists N i. N \subseteq M \wedge \text{infinite } N \wedge i < r \wedge$   
 $(\forall j < r. j \neq i \longrightarrow f - \{j\} \cap \mathcal{F} \cap \text{Pow } N = \{\}))$

Alternative, simpler definition suggested by a referee.

**lemma** *Ramsey-eq*:

*Ramsey*  $\mathcal{F}$   $r \longleftrightarrow (\forall f \in \mathcal{F} \rightarrow \{..<r\}$ .  
 $\forall M. \text{infinite } M \longrightarrow$   
 $(\exists N i. N \subseteq M \wedge \text{infinite } N \wedge i < r \wedge \mathcal{F} \cap \text{Pow } N \subseteq f - \{i\}))$   
 ⟨proof⟩

**definition** *thin-set* :: nat set set ⇒ bool

**where** *thin-set*  $\mathcal{F} \equiv \mathcal{F} \subseteq \text{Collect finite} \wedge (\forall S \in \mathcal{F}. \forall T \in \mathcal{F}. \text{init-segment } S \ T \longrightarrow S = T)$

**definition** *comparables* :: nat set ⇒ nat set ⇒ nat set set

**where** *comparables*  $S \ M \equiv \{T. \text{finite } T \wedge (\text{init-segment } T \ S \vee \text{init-segment } S \ T \wedge T - S \subseteq M)\}$

**lemma** *comparables-iff*:  $T \in \text{comparables } S \ M \longleftrightarrow \text{finite } T \wedge (\text{init-segment } T \ S \vee \text{init-segment } S \ T \wedge T \subseteq S \cup M)$

⟨proof⟩

**lemma** *comparables-subset*:  $\bigcup (\text{comparables } S \ M) \subseteq S \cup M$

⟨proof⟩

**lemma** *comparables-empty* [simp]: *comparables*  $\{\}$   $M = \text{Fpow } M$

⟨proof⟩

**lemma** *comparables-mono*:  $N \subseteq M \implies \text{comparables } S \ N \subseteq \text{comparables } S \ M$

⟨proof⟩

**definition** *rejects*  $\mathcal{F}$   $S \ M \equiv \text{comparables } S \ M \cap \mathcal{F} = \{\}$

**abbreviation** *accepts*

**where** *accepts*  $\mathcal{F}$   $S \ M \equiv \neg \text{rejects } \mathcal{F} \ S \ M$

**definition** *strongly-accepts*

**where** *strongly-accepts*  $\mathcal{F}$   $S \ M \equiv (\forall N \subseteq M. \text{rejects } \mathcal{F} \ S \ N \longrightarrow \text{finite } N)$

**definition** *decides*

**where** *decides*  $\mathcal{F}$   $S \ M \equiv \text{rejects } \mathcal{F} \ S \ M \vee \text{strongly-accepts } \mathcal{F} \ S \ M$

**definition** *decides-subsets*

**where** *decides-subsets*  $\mathcal{F} M \equiv \forall T. T \subseteq M \longrightarrow \text{finite } T \longrightarrow \text{decides } \mathcal{F} T M$

**lemma** *strongly-accepts-imp-accepts*:

$\llbracket \text{strongly-accepts } \mathcal{F} S M; \text{infinite } M \rrbracket \Longrightarrow \text{accepts } \mathcal{F} S M$   
*<proof>*

**lemma** *rejects-trivial*:  $\llbracket \text{rejects } \mathcal{F} S M; \text{thin-set } \mathcal{F}; \text{init-segment } F S; F \in \mathcal{F} \rrbracket \Longrightarrow \text{False}$

*<proof>*

**lemma** *rejects-subset*:  $\llbracket \text{rejects } \mathcal{F} S M; N \subseteq M \rrbracket \Longrightarrow \text{rejects } \mathcal{F} S N$

*<proof>*

**lemma** *strongly-accepts-subset*:  $\llbracket \text{strongly-accepts } \mathcal{F} S M; N \subseteq M \rrbracket \Longrightarrow \text{strongly-accepts } \mathcal{F} S N$

*<proof>*

**lemma** *decides-subset*:  $\llbracket \text{decides } \mathcal{F} S M; N \subseteq M \rrbracket \Longrightarrow \text{decides } \mathcal{F} S N$

*<proof>*

**lemma** *decides-subsets-subset*:  $\llbracket \text{decides-subsets } \mathcal{F} M; N \subseteq M \rrbracket \Longrightarrow \text{decides-subsets } \mathcal{F} N$

*<proof>*

**lemma** *rejects-empty [simp]*:  $\text{rejects } \mathcal{F} \{\} M \longleftrightarrow \text{Fpow } M \cap \mathcal{F} = \{\}$

*<proof>*

**lemma** *strongly-accepts-empty [simp]*:  $\text{strongly-accepts } \mathcal{F} \{\} M \longleftrightarrow (\forall N \subseteq M. \text{Fpow } N \cap \mathcal{F} = \{\} \longrightarrow \text{finite } N)$

*<proof>*

**lemma** *ex-infinite-decides-1*:

**assumes** *infinite*  $M$

**obtains**  $N$  **where**  $N \subseteq M$  *infinite*  $N$  *decides*  $\mathcal{F} S N$

*<proof>*

**proposition** *ex-infinite-decides-finite*:

**assumes** *infinite*  $M$  *finite*  $S$

**obtains**  $N$  **where**  $N \subseteq M$  *infinite*  $N \wedge T. T \subseteq S \Longrightarrow \text{decides } \mathcal{F} T N$

*<proof>*

**lemma** *sorted-wrt-subset*:  $\llbracket X \in \text{list.set } l; \text{sorted-wrt } (\leq) l \rrbracket \Longrightarrow \text{hd } l \subseteq X$

*<proof>*

Todorčević's Lemma 1.18

**proposition** *ex-infinite-decides-subsets*:

**assumes** *thin-set*  $\mathcal{F}$  *infinite*  $M$

**obtains**  $N$  **where**  $N \subseteq M$  *infinite*  $N$  *decides-subsets*  $\mathcal{F} N$

*<proof>*

Todorčević's Lemma 1.19

**proposition** *strongly-accepts-1-19*:

**assumes** *acc*: *strongly-accepts*  $\mathcal{F}$   $S$   $M$

**and** *thin-set*  $\mathcal{F}$  *infinite*  $M$   $S \subseteq M$  *finite*  $S$

**and** *dsM*: *decides-subsets*  $\mathcal{F}$   $M$

**shows** *finite*  $\{n \in M. \neg \text{strongly-accepts } \mathcal{F} (\text{insert } n \ S) \ M\}$

*<proof>*

Much work is needed for this slight strengthening of the previous result!

**proposition** *strongly-accepts-1-19-plus*:

**assumes** *thin-set*  $\mathcal{F}$  *infinite*  $M$

**and** *dsM*: *decides-subsets*  $\mathcal{F}$   $M$

**obtains**  $N$  **where**  $N \subseteq M$  *infinite*  $N$

$\bigwedge S \ n. \llbracket S \subseteq N; \text{finite } S; \text{strongly-accepts } \mathcal{F} \ S \ N; n \in N; S \ll \{n\} \rrbracket$

$\implies \text{strongly-accepts } \mathcal{F} (\text{insert } n \ S) \ N$

*<proof>*

## 2.3 Main Theorem

**lemma** *Nash-Williams-1: Ramsey*  $\mathcal{F}$  1

*<proof>*

**theorem** *Nash-Williams-2*:

**assumes** *thin-set*  $\mathcal{F}$  **shows** *Ramsey*  $\mathcal{F}$  2

*<proof>*

**theorem** *Nash-Williams*:

**assumes**  $\mathcal{F}$ : *thin-set*  $\mathcal{F}$   $r > 0$  **shows** *Ramsey*  $\mathcal{F}$   $r$

*<proof>*

**end**

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## References

- [1] C. S. J. A. Nash-Williams. On well-quasi-ordering transfinite sequences. *Mathematical Proceedings of the Cambridge Philosophical Society*, 61(1):33–39, 1965.

- [2] S. Todorčević. *Introduction to Ramsey Spaces*. Princeton University Press, 2010.