Napoleon's Theorem in Isabelle/HOL

Arthur Freitas Ramos David Barros Hulak
Ruy J. G. B. de Queiroz

September 18, 2026

Abstract

We formalize Napoleon's theorem in Isabelle/HOL. Given a non-degenerate Euclidean triangle, equilateral triangles constructed externally on its three sides have centres that form an equilateral triangle. The construction is represented in complex coordinates, and both orientations of the equilateral rotation are covered by one parametrized development.

Contents

1 Introduction **1**

2 Definitions and construction correctness **2**

3 The centre calculation **2**

4 Formalization notes **3**

5 Napoleon's construction **3**

6 Algebra of the three centres **3**

7 External orientation and Napoleon's theorem **4**

1 Introduction

Napoleon's theorem is a classical triangle-geometry result [1]: the three equilateral triangles erected on the sides of an arbitrary triangle determine three centres which themselves are the vertices of an equilateral triangle. The orientation of the erected triangles matters. A coordinate formalization must therefore distinguish the two rotations through $\pm\pi/3$, and it must make explicit which orientation gives the external construction.

The exterior condition is part of the formal specification. For a directed side ab, the signed side orientation of a point x is $\mathrm{Im}((x-a)/(b-a))$. An apex is external precisely when its side orientation has strictly opposite sign to that of the third vertex.

The formalization uses the complex plane. If s is either $\mathrm{cis}(\pi/3)$ or $\mathrm{cis}(-\pi/3)$, the apex on the directed side from x to y is

$$E_s(x, y) = x + s(y - x).$$

The centre of the corresponding equilateral triangle is the average of its three vertices. The published AFP development of Morley's theorem [2] supplies the equilateral-triangle characterization used by the present entry; the present theory adds the cyclic centre calculation specific to Napoleon's configuration.

2 Definitions and construction correctness

The theory defines the oriented apex and the centre of an equilateral triangle on a directed side. For distinct endpoints, the rotation characterization immediately proves the two equal-length equations that certify the construction as equilateral. Keeping the rotation parameter explicit avoids duplicating the proof for clockwise and counterclockwise constructions. The predicate `external_side` states that two points lie strictly on opposite sides of the directed line. The theorem `napoleon_external_construction` proves this predicate for all three cyclic sides, in addition to the six equal-distance facts for the constructed equilateral triangles.

3 The centre calculation

Let N_{AB}, N_{BC}, and N_{CA} denote the three centres. The complex rotation parameter satisfies

$$s^2 = s - 1,$$

and the cyclic differences simplify to

$$N_{CA} - N_{AB} = (1 - s)(N_{BC} - N_{AB}).$$

Since $1 - s$ is the opposite $\pm\pi/3$ rotation, the two vectors from N_{AB} to the other two centres have equal lengths and meet at an equilateral angle. The final theorem is stated through the corresponding pairwise-distance characterization. The final bundled corollary `napoleon_external_theorem` packages this centre conclusion together with all three external equilateral constructions under the non-collinearity assumption, so the formal statement mirrors the complete English theorem.

4 Formalization notes

The entry imports the released `Morley_Theorem` session rather than duplicating its complex equilateral infrastructure. The `napoleon_centre` is the arithmetic mean of the three vertices, hence the centroid; on an equilateral triangle this agrees with the circumcenter, incenter, and orthocenter. The source theory is kept as a single top-level AFP theory, with the document containing the mathematical explanation and proof sketch.

theory *Napoleon*
  imports
    HOL−Decision-Procs.Commutative-Ring
    Morley-Theorem.Complex-Triangles
begin

5 Napoleon's construction

A complex number s equal to $cis\ (pi/3)$ or $cis\ (-pi/3)$ rotates a directed side through one of the two equilateral-triangle angles. The apex and the centre of the resulting equilateral triangle are defined parametrically so that the same algebraic lemmas cover both orientations.

definition *napoleon-apex* :: *complex* ⇒ *complex* ⇒ *complex* ⇒ *complex*
  where *napoleon-apex s x y* = *x* + *s* ∗ (*y* − *x*)

definition *napoleon-centre* :: *complex* ⇒ *complex* ⇒ *complex* ⇒ *complex*
  where *napoleon-centre s x y* =
    (*x* + *y* + *napoleon-apex s x y*) / *3*

lemma *napoleon-apex-displacement*:
  napoleon-apex s x y − *x* = *s* ∗ (*y* − *x*)
  ⟨*proof*⟩

lemma *equilateral-on-side*:
  fixes *x y s* :: *complex*
  assumes *x* ≠ *y*
    and *s* = *cis* (*pi* / *3*) ∨ *s* = *cis* (−*pi* / *3*)
  shows
    cdist y x = *cdist* (*napoleon-apex s x y*) *x* ∧
     cdist (*napoleon-apex s x y*) *x* = *cdist* (*napoleon-apex s x y*) *y*
⟨*proof*⟩

6 Algebra of the three centres

lemma *cis-pi3-square*:
  (*cis* (*pi* / *3*) :: *complex*) ^ *2* = *cis* (*pi* / *3*) − *1*
  ⟨*proof*⟩

lemma *cis-minus-pi3-square*:
  $(cis\ (-pi\ /\ 3) :: complex)\ \hat{}\ 2 = cis\ (-pi\ /\ 3) - 1$
  ⟨*proof*⟩

lemma *one-minus-cis-pi3*:
  $(1 :: complex) - cis\ (pi\ /\ 3) = cis\ (-pi\ /\ 3)$
  ⟨*proof*⟩

lemma *one-minus-cis-minus-pi3*:
  $(1 :: complex) - cis\ (-pi\ /\ 3) = cis\ (pi\ /\ 3)$
  ⟨*proof*⟩

lemma *napoleon-centre-cycle*:
  fixes *a b c s* :: *complex*
  assumes $s\ \hat{}\ 2 = s - 1$
  shows
    napoleon-centre s c a − *napoleon-centre s a b* =
      (*1* − *s*) ∗ (*napoleon-centre s b c* − *napoleon-centre s a b*)
⟨*proof*⟩

lemma *napoleon-centres-equilateral*:
  fixes *a b c s* :: *complex*
  assumes *hsquare*: $s\ \hat{}\ 2 = s - 1$
    and *hroot*: $1 - s = cis\ (pi\ /\ 3) \lor 1 - s = cis\ (-pi\ /\ 3)$
  shows
    cdist (*napoleon-centre s b c*) (*napoleon-centre s a b*) =
      cdist (*napoleon-centre s c a*) (*napoleon-centre s a b*) ∧
    cdist (*napoleon-centre s c a*) (*napoleon-centre s a b*) =
      cdist (*napoleon-centre s c a*) (*napoleon-centre s b c*)
⟨*proof*⟩

7 External orientation and Napoleon's theorem

The sign of the oriented area of the triangle is the sign of the imaginary part of the quotient below. Choosing the opposite rotation makes the equilateral triangle on the directed side *a b* lie on the exterior side; the same choice works cyclically on *b c* and *c a*. This exteriority claim is formalized by *external-side*: the signed side orientations of the third vertex and the constructed apex have strictly negative product.

definition *napoleon-external-rotation* :: *complex* ⇒ *complex* ⇒ *complex* ⇒ *complex*
  where
    napoleon-external-rotation a b c =
     (*if* $0 < Im\ ((c - a)\ /\ (b - a))$
      then cis $(-pi\ /\ 3)$
      else cis $(pi\ /\ 3)$)

definition *side-orientation* :: *complex* ⇒ *complex* ⇒ *complex* ⇒ *real*
where *side-orientation a b x* = *Im* ((*x* − *a*) / (*b* − *a*))

definition *external-side* :: *complex* ⇒ *complex* ⇒ *complex* ⇒ *complex* ⇒ *bool*
where
external-side a b c x ⟷
side-orientation a b c ∗ *side-orientation a b x* < *0*

lemma *napoleon-external-rotation-cases*:
napoleon-external-rotation a b c = *cis* (*pi* / *3*) ∨
napoleon-external-rotation a b c = *cis* (−*pi* / *3*)
⟨*proof*⟩

lemma *napoleon-external-rotation-square*:
napoleon-external-rotation a b c ^ *2* =
napoleon-external-rotation a b c − *1*
⟨*proof*⟩

lemma *napoleon-external-rotation-one-minus*:
1 − *napoleon-external-rotation a b c* = *cis* (*pi* / *3*) ∨
1 − *napoleon-external-rotation a b c* = *cis* (−*pi* / *3*)
⟨*proof*⟩

lemma *oriented-area-cycle*:
Im ((*c* − *a*) ∗ *cnj* (*b* − *a*)) =
Im ((*a* − *b*) ∗ *cnj* (*c* − *b*))
⟨*proof*⟩

lemma *napoleon-external-rotation-cyclic*:
napoleon-external-rotation a b c =
napoleon-external-rotation b c a
⟨*proof*⟩

lemma *noncollinear-distinct*:
assumes ¬ *collinear a b c*
shows *a* ≠ *b* ∧ *b* ≠ *c* ∧ *c* ≠ *a*
⟨*proof*⟩

lemma *side-orientation-nonzero*:
assumes *hncol*: ¬ *collinear a b c*
shows *side-orientation a b c* ≠ *0*
⟨*proof*⟩

lemma *side-orientation-apex*:
fixes *a b s* :: *complex*
assumes *a* ≠ *b*
shows
side-orientation a b (*napoleon-apex s a b*) = *Im s*
⟨*proof*⟩

lemma *Im-cis-pi3-pos*:
  0 < *Im* (*cis* (*pi* / *3*) :: *complex*)
⟨*proof*⟩

lemma *Im-cis-minus-pi3-neg*:
  Im (*cis* (−*pi* / *3*) :: *complex*) < *0*
⟨*proof*⟩

lemma *napoleon-external-side*:
  fixes *a b c* :: *complex*
  assumes *hncol*: ¬ *collinear a b c*
  shows
    external-side a b c
      (*napoleon-apex* (*napoleon-external-rotation a b c*) *a b*)
⟨*proof*⟩

lemma *napoleon-external-construction*:
  assumes *hncol*: ¬ *collinear a b c*
  shows
    cdist b a =
      cdist (*napoleon-apex* (*napoleon-external-rotation a b c*) *a b*) *a* ∧
    cdist (*napoleon-apex* (*napoleon-external-rotation a b c*) *a b*) *a* =
      cdist (*napoleon-apex* (*napoleon-external-rotation a b c*) *a b*) *b* ∧
    cdist c b =
      cdist (*napoleon-apex* (*napoleon-external-rotation a b c*) *b c*) *b* ∧
    cdist (*napoleon-apex* (*napoleon-external-rotation a b c*) *b c*) *b* =
      cdist (*napoleon-apex* (*napoleon-external-rotation a b c*) *b c*) *c* ∧
    cdist a c =
      cdist (*napoleon-apex* (*napoleon-external-rotation a b c*) *c a*) *c* ∧
    cdist (*napoleon-apex* (*napoleon-external-rotation a b c*) *c a*) *c* =
      cdist (*napoleon-apex* (*napoleon-external-rotation a b c*) *c a*) *a* ∧
    external-side a b c
      (*napoleon-apex* (*napoleon-external-rotation a b c*) *a b*) ∧
    external-side b c a
      (*napoleon-apex* (*napoleon-external-rotation a b c*) *b c*) ∧
    external-side c a b
      (*napoleon-apex* (*napoleon-external-rotation a b c*) *c a*)
⟨*proof*⟩

theorem *napoleon-theorem*:
  fixes *a b c* :: *complex*
  assumes *hncol*: ¬ *collinear a b c*
  shows
    cdist (*napoleon-centre* (*napoleon-external-rotation a b c*) *b c*)
      (*napoleon-centre* (*napoleon-external-rotation a b c*) *a b*) =
      cdist (*napoleon-centre* (*napoleon-external-rotation a b c*) *c a*)
        (*napoleon-centre* (*napoleon-external-rotation a b c*) *a b*) ∧
    cdist (*napoleon-centre* (*napoleon-external-rotation a b c*) *c a*)

(*napoleon-centre* (*napoleon-external-rotation* *a* *b* *c*) *a* *b*) =
cdist (*napoleon-centre* (*napoleon-external-rotation* *a* *b* *c*) *c* *a*)
(*napoleon-centre* (*napoleon-external-rotation* *a* *b* *c*) *b* *c*)
⟨*proof*⟩

theorem *napoleon-external-theorem*:
fixes *a* *b* *c* :: *complex*
assumes *hncol*: ¬ *collinear* *a* *b* *c*
shows
(*cdist* *b* *a* =
cdist (*napoleon-apex* (*napoleon-external-rotation* *a* *b* *c*) *a* *b*) *a* ∧
cdist (*napoleon-apex* (*napoleon-external-rotation* *a* *b* *c*) *a* *b*) *a* =
cdist (*napoleon-apex* (*napoleon-external-rotation* *a* *b* *c*) *a* *b*) *b* ∧
cdist *c* *b* =
cdist (*napoleon-apex* (*napoleon-external-rotation* *a* *b* *c*) *b* *c*) *b* ∧
cdist (*napoleon-apex* (*napoleon-external-rotation* *a* *b* *c*) *b* *c*) *b* =
cdist (*napoleon-apex* (*napoleon-external-rotation* *a* *b* *c*) *b* *c*) *c* ∧
cdist *a* *c* =
cdist (*napoleon-apex* (*napoleon-external-rotation* *a* *b* *c*) *c* *a*) *c* ∧
cdist (*napoleon-apex* (*napoleon-external-rotation* *a* *b* *c*) *c* *a*) *c* =
cdist (*napoleon-apex* (*napoleon-external-rotation* *a* *b* *c*) *c* *a*) *a* ∧
external-side *a* *b* *c*
(*napoleon-apex* (*napoleon-external-rotation* *a* *b* *c*) *a* *b*) ∧
external-side *b* *c* *a*
(*napoleon-apex* (*napoleon-external-rotation* *a* *b* *c*) *b* *c*) ∧
external-side *c* *a* *b*
(*napoleon-apex* (*napoleon-external-rotation* *a* *b* *c*) *c* *a*)) ∧
(*cdist* (*napoleon-centre* (*napoleon-external-rotation* *a* *b* *c*) *b* *c*)
(*napoleon-centre* (*napoleon-external-rotation* *a* *b* *c*) *a* *b*) =
cdist (*napoleon-centre* (*napoleon-external-rotation* *a* *b* *c*) *c* *a*)
(*napoleon-centre* (*napoleon-external-rotation* *a* *b* *c*) *a* *b*) ∧
cdist (*napoleon-centre* (*napoleon-external-rotation* *a* *b* *c*) *c* *a*)
(*napoleon-centre* (*napoleon-external-rotation* *a* *b* *c*) *a* *b*) =
cdist (*napoleon-centre* (*napoleon-external-rotation* *a* *b* *c*) *c* *a*)
(*napoleon-centre* (*napoleon-external-rotation* *a* *b* *c*) *b* *c*))
⟨*proof*⟩

end

References

[1] R. Honsberger. *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, volume 37 of *New Mathematical Library*. Mathematical Association of America, 1995.

[2] B. Puyobro. Morley's theorem. Archive of Formal Proofs. Available at https://isa-afp.org/entries/Morley_Theorem.html.