The Mostowski Collapse Theorem

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Abstract

This entry formalizes the Mostowski collapse theorem in Isabelle/ZF [3, 4]. For a set equipped with a well-founded extensional relation, the development defines the collapsing map by well-founded recursion, proves that its range is transitive, and shows that the map is an order isomorphism between the original relation and membership on the transitive range. The construction is also proved unique among maps satisfying the same recursive equation. The result is a standard bridge from abstract well-founded extensional structures to transitive membership structures, and complements the existing Isabelle/ZF and AFP developments on ordinals, cardinals, forcing, and transitive models [2, 1]. AI assistance was used for proof engineering. The final definitions, statements, and proofs are checked by Isabelle.

Contents

theory *Mostowski-Collapse*
 imports *ZF*
begin

1 The Mostowski Collapse Theorem

The Mostowski collapse theorem says that every set-sized well-founded extensional relation is isomorphic to membership on a transitive set. The collapsing map is obtained by well-founded recursion: each object is sent to the set of the collapses of its predecessors. We work with a set A and collapse the restriction of a relation r to A. The assumptions are therefore $wf[A](r)$ and extensionality on A; the final isomorphism is stated for $r \cap A{*}A$.

definition *extensional-on* :: $[i, i] \Rightarrow o$ **where**
 $extensional\text{-}on(A,r) \equiv$
  $\forall x{\in}A.\ \forall y{\in}A.\ (\forall z{\in}A.\ \langle z,x\rangle \in r \longleftrightarrow \langle z,y\rangle \in r) \longrightarrow x = y$

definition *collapse* :: $[i,\ i,\ i] \Rightarrow i$ **where**
$\mathit{collapse}(A,r,x) \equiv$
$\mathit{wfrec}[A](r,\ x,\ \lambda y\ f.\ f\ ``\ ((r\ -``\ \{y\}) \cap A))$

definition *collapse-map* :: $[i,\ i] \Rightarrow i$ **where**
$\mathit{collapse\text{-}map}(A,r) \equiv (\lambda x{\in}A.\ \mathit{collapse}(A,r,x))$

definition *collapse-range* :: $[i,\ i] \Rightarrow i$ **where**
$\mathit{collapse\text{-}range}(A,r) \equiv \mathit{range}(\mathit{collapse\text{-}map}(A,r))$

lemma *collapse-unfold*:
assumes $\mathit{wf}[A](r)\ x \in A$
shows $\mathit{collapse}(A,r,x) = \{\mathit{collapse}(A,r,y).\ y \in (r\ -``\ \{x\}) \cap A\}$
⟨*proof*⟩

lemma *collapse-memI*:
assumes $\mathit{wf}[A](r)\ x \in A\ y \in A\ \langle y,x\rangle \in r$
shows $\mathit{collapse}(A,r,y) \in \mathit{collapse}(A,r,x)$
⟨*proof*⟩

lemma *collapse-memE*:
assumes $\mathit{wf}[A](r)\ x \in A\ u \in \mathit{collapse}(A,r,x)$
obtains y **where** $y \in A\ \langle y,x\rangle \in r\ u = \mathit{collapse}(A,r,y)$
⟨*proof*⟩

lemma *collapse-injective*:
assumes *wf*: $\mathit{wf}[A](r)$ **and** *ext*: $\mathit{extensional\text{-}on}(A,r)$
and *xA*: $x \in A$ **and** *yA*: $y \in A$
and *eq*: $\mathit{collapse}(A,r,x) = \mathit{collapse}(A,r,y)$
shows $x = y$
⟨*proof*⟩

lemma *collapse-map-type*:
$\mathit{collapse\text{-}map}(A,r) \in A \rightarrow \mathit{collapse\text{-}range}(A,r)$
⟨*proof*⟩

lemma *collapse-map-apply* [*simp*]:
$x \in A \Longrightarrow \mathit{collapse\text{-}map}(A,r)\ `\ x = \mathit{collapse}(A,r,x)$
⟨*proof*⟩

lemma *collapse-map-inj*:
assumes $\mathit{wf}[A](r)\ \mathit{extensional\text{-}on}(A,r)$
shows $\mathit{collapse\text{-}map}(A,r) \in \mathit{inj}(A,\ \mathit{collapse\text{-}range}(A,r))$
⟨*proof*⟩

lemma *collapse-map-bij*:
assumes $\mathit{wf}[A](r)\ \mathit{extensional\text{-}on}(A,r)$
shows $\mathit{collapse\text{-}map}(A,r) \in \mathit{bij}(A,\ \mathit{collapse\text{-}range}(A,r))$

⟨*proof*⟩

lemma *collapse-range-iff*:
$u \in$ *collapse-range*$(A,r) \longleftrightarrow (\exists\, x \in A.\ u = $ *collapse*$(A,r,x))$
⟨*proof*⟩

lemma *collapse-range-transitive*:
assumes *wf*: *wf*$[A](r)$
shows *Transset*(*collapse-range*(A,r))
⟨*proof*⟩

lemma *collapse-mem-iff*:
assumes *wf*: *wf*$[A](r)$ **and** *ext*: *extensional-on*(A,r)
and *xA*: $x \in A$ **and** *yA*: $y \in A$
shows *collapse*$(A,r,x) \in$ *collapse*$(A,r,y) \longleftrightarrow \langle x,y\rangle \in r$
⟨*proof*⟩

lemma *collapse-ord-iso*:
assumes *wf*: *wf*$[A](r)$ **and** *ext*: *extensional-on*(A,r)
shows *collapse-map*$(A,r) \in$
ord-iso$(A,\ r \cap A{*}A,$ *collapse-range*$(A,r),$ *Memrel*(*collapse-range*(A,r)))
⟨*proof*⟩

theorem *mostowski-collapse*:
assumes *wf*$[A](r)$ *extensional-on*(A,r)
shows *Transset*(*collapse-range*(A,r)) $\wedge$
collapse-map$(A,r) \in$
ord-iso$(A,\ r \cap A{*}A,$ *collapse-range*$(A,r),$ *Memrel*(*collapse-range*(A,r)))
⟨*proof*⟩

theorem *mostowski-collapse-exists*:
assumes *wf*$[A](r)$ *extensional-on*(A,r)
shows $\exists\, M\, f.$ *Transset*$(M) \wedge f \in$ *ord-iso*$(A,\ r \cap A{*}A,\ M,$ *Memrel*$(M))$
⟨*proof*⟩

lemma *collapse-unique*:
assumes *wf*: *wf*$[A](r)$
and *ftype*: $f \in A \to B$
and *frec*: $\bigwedge x.\ x \in A \Longrightarrow f`x = f `` ((r -`` \{x\}) \cap A)$
and *xA*: $x \in A$
shows $f`x =$ *collapse*(A,r,x)
⟨*proof*⟩

lemma *collapse-map-unique*:
assumes *wf*: *wf*$[A](r)$
and *ftype*: $f \in A \to B$
and *frec*: $\bigwedge x.\ x \in A \Longrightarrow f`x = f `` ((r -`` \{x\}) \cap A)$
shows $f =$ *collapse-map*(A,r)
⟨*proof*⟩

lemma *collapse-range-unique*:
assumes *wf*: $wf[A](r)$
and *ftype*: $f \in A \to B$
and *frec*: $\bigwedge x.\ x \in A \Longrightarrow f`x = f `` ((r -`` \{x\}) \cap A)$
and *M*: $M = range(f)$
shows $M = collapse\text{-}range(A,r)$
$\langle proof \rangle$

end

References

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