Miquel's Theorem in Isabelle/HOL

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Abstract

We formalize Miquel's theorem, also known as the pivot theorem, in Isabelle/HOL. Let ABC be a triangle in the Euclidean plane and let P, Q, R be points on the side lines BC, CA, AB respectively. Then the circumcircles of the three triangles AQR, BRP, CPQ pass through a common point M, the *Miquel point* of the configuration.

The development uses complex coordinates.

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1 Introduction

Miquel's theorem is a classical incidence result about a triangle and three points chosen on its side lines [4, 2, 1]. Given a triangle ABC and points P, Q, R on the lines BC, CA, AB, the three circumcircles of AQR, BRP

and CPQ are concurrent. Their common point M is the Miquel point, and the theorem is sometimes called the pivot theorem because M is the centre of the spiral similarity that pivots the configuration.

Our formalization works in the complex plane. It reuses the shared circle, line, collinearity, and cross-ratio interface from the published Wallace–Simson entry [5], which in turn builds on the published *Complex Geometry* development [3]. Reducing the incidence statement to complex algebra then turns the theorem into a short computation.

2 The cross-ratio criterion

Four complex numbers a, b, c, d have cross ratio

$$(a, b; c, d) = \frac{(a-c)\,(b-d)}{(a-d)\,(b-c)}.$$

The criterion we use is that, for four suitably distinct points, being concyclic or collinear is equivalent to the cross ratio being real. We only need the concyclic case, in both directions.

For the forward direction, if a, b, c, d lie on a circle of centre o and radius $R > 0$, then each point satisfies $\overline{z-o} = R^2/(z-o)$. Substituting this into the conjugate of the cross ratio and simplifying, every factor R^2 and every factor $(z-o)$ cancels, leaving the cross ratio itself. Hence the cross ratio equals its own conjugate and is real.

For the converse, suppose c, p, q lie on the circle of centre o and radius R, and that the cross ratio of a fourth point m with them is real. Writing $M = m - o$ and using the same conjugate substitution for c, p, q, the reality of the cross ratio simplifies to

$$(K-P)\left(M\,\overline{M} - R^2\right) = 0,$$

where $P = p - o$ and $K = q - o$. Since $p \neq q$ we have $K \neq P$, so $M\,\overline{M} = R^2$, that is $|m - o| = R$ and m lies on the same circle.

3 The pivot theorem

Let M be a common point of the circumcircles of AQR and BRP. By the forward criterion the cross ratios $(m, a; q, r)$ and $(m, b; r, p)$ are real. A direct computation shows

$$(m, a; q, r)\,(m, b; r, p)\,(m, c; p, q) = \frac{(a-r)\,(b-p)\,(c-q)}{(a-q)\,(b-r)\,(c-p)},$$

the point m cancelling completely. The right-hand side factors as

$$\frac{a-r}{b-r} \cdot \frac{b-p}{c-p} \cdot \frac{c-q}{a-q},$$

and each of the three ratios is real because R lies on line AB, P on line BC, and Q on line CA. Therefore $(m, c; p, q)$ is real, and by the converse criterion M lies on the circumcircle of CPQ. The three circumcircles are thus concurrent at M.

4 Existence of the circumscribed circle

The converse criterion needs the circle through C, P, Q to exist. Rather than assuming it, we construct it whenever the three points are not collinear. Working relative to a, set $u = b - a$ and $v = c - a$ and

$$D = \overline{u}\,v - u\,\overline{v}.$$

A short conjugation calculation shows $D = 2\,\mathrm{Im}(\overline{u}\,v)\,i$, so D is purely imaginary and $D \neq 0$ is exactly the non-collinearity of a, b, c. The point

$$o = a + w, \qquad w = \frac{u\,v\,\overline{(u - v)}}{D},$$

then satisfies $|a - o| = |b - o| = |c - o| = |w| > 0$; indeed $\overline{w} = \overline{u}\,\overline{v}\,(v - u)/D$, and the equalities follow from the two identities $w\,\overline{u} + \overline{w}\,u = u\,\overline{u}$ and $w\,\overline{v} + \overline{w}\,v = v\,\overline{v}$. Thus any three non-collinear points are concyclic.

To apply this to C, P, Q we note that, with P on line BC and Q on line CA, the cross product $\mathrm{Im}(\overline{(P - C)}\,(Q - C))$ is a nonzero real multiple of $\mathrm{Im}(\overline{(B - C)}\,(A - C))$; the latter is nonzero because ABC is nondegenerate. Hence C, P, Q are non-collinear and their circumcircle exists. The same computation shows $P \neq Q$, which the converse criterion also requires.

5 Concurrency: constructing the Miquel point

The pivot theorem assumes a point M already on two of the circumcircles. The classical statement instead asserts that such a point *exists*: the three circumcircles are concurrent. We prove this by constructing M.

The circumcircles of AQR and BRP, with centres o_1 and o_2, both pass through R. Two distinct circles meet in at most two points, and those points are symmetric about the line o_1o_2. We therefore take M to be the reflection of R in that line,

$$M = o_1 + \frac{(o_2 - o_1)\,\overline{(R - o_1)}}{\overline{(o_2 - o_1)}}.$$

Reflection in the line through o_1 and o_2 is an isometry that fixes both centres, so $|M - o_1| = |R - o_1|$ and $|M - o_2| = |R - o_2|$; as R lies on both circles, so does M. When R is not on the line o_1o_2 — that is, the two circles are not tangent at R — the reflection is genuinely distinct from R, and the pivot theorem places M on the third circumcircle as well. Should M happen to coincide with P or Q, concurrency is immediate because C, P, Q are non-collinear and hence concyclic. Thus the three circumcircles share the common point M.

6 Formalization notes

The development provides both forms. The pivot theorem *miquel-pivot* takes a point on two circumcircles and places it on the third. The concurrency theorem *miquel-concurrent* assumes no common point: it exhibits the Miquel point as the reflection above and shows it lies on all three circles, and the corollary *miquel-point-exists* states the pure existence form. The only structural hypothesis is that A, B, C are not collinear; from this the circumcircle of CPQ is constructed rather than assumed, and $P \neq Q$ is derived. The concurrency form additionally assumes the two circumcircles are not tangent at R (equivalently, R is not on the line through their centres), which is exactly the condition for a genuine second intersection. The remaining hypotheses are the three side-line conditions and general-position distinctness of the points involved, which guarantees that the cross ratios are defined and nonzero.

theory *Miquel*
imports *Simson.Simson-Complex-Geometry*
begin

7 Existence of the circumscribed circle

Any three non-collinear points lie on a common circle of positive radius. We exhibit the circumcentre in closed form. Writing $u = b - a$ and $v = c - a$, the centre is $a + w$ with $w = u * v * cnj\ (u - v)\ /\ D$ and $D = cnj\ u * v - u * cnj\ v$. Non-collinearity is exactly $D \neq 0$, and a short conjugation computation shows the centre is equidistant from the three points.

lemma *circumcircle-exists*:
assumes *ncol*: $\neg$ *collinear a b c*
shows *concyclic3 a b c*
⟨*proof*⟩

Antisymmetry of the planar cross product under a change of base point.

lemma *cross-base-change*:
fixes *a b c* :: *complex*
shows $Im\ (cnj\ (b - c) * (a - c)) = - Im\ (cnj\ (b - a) * (c - a))$
⟨*proof*⟩

If the triangle abc is nondegenerate and p, q lie on the side lines bc and ca without coinciding with c, then c, p, q are themselves non-collinear.

Hence their circumscribed circle exists.

lemma *cpq-ncol*:
  assumes *ncol*: ¬ *collinear a b c*
    and *lp*: *on-line b c p*
    and *lq*: *on-line c a q*
    and *pc*: $p \neq c$
    and *qc*: $q \neq c$
  shows ¬ *collinear c p q*
⟨*proof*⟩

8 Miquel's pivot theorem

The three cross ratios attached to the Miquel configuration multiply to a product of three ratios coming from the side lines.

lemma *miquel-complex-cross-ratio-product*:
  assumes *dmr*: $m \neq r$ **and** *daq*: $a \neq q$ **and** *dmp*: $m \neq p$
    and *dbr*: $b \neq r$ **and** *dmq*: $m \neq q$ **and** *dcp*: $c \neq p$
  shows *complex-cross-ratio m a q r* ∗ *complex-cross-ratio m b r p* ∗ *complex-cross-ratio m c p q*
    $= (a - r) * (b - p) * (c - q) \,/\, ((a - q) * (b - r) * (c - p))$
⟨*proof*⟩

The main theorem. With the side-line conditions on p, q, r, and general-position distinctness, any point m lying on the circumcircles of aqr and brp also lies on the circumcircle of cpq. Thus the three circumcircles are concurrent at the Miquel point m.

theorem *miquel-pivot*:
  assumes *lr*: *on-line a b r* **and** *lp*: *on-line b c p* **and** *lq*: *on-line c a q*
    and *h1*: *concyclic a q r m* **and** *h2*: *concyclic b r p m*
    and *ncol*: ¬ *collinear a b c*
    and *dmp*: $m \neq p$ **and** *dmq*: $m \neq q$ **and** *dmr*: $m \neq r$
    and *daq*: $a \neq q$ **and** *dar*: $a \neq r$ **and** *dbr*: $b \neq r$ **and** *dbp*: $b \neq p$
    and *dcp*: $c \neq p$ **and** *dcq*: $c \neq q$
  shows *concyclic c p q m*
⟨*proof*⟩

9 Concurrency of the three circumcircles

We now prove Miquel's theorem in its classical concurrency form: for a nondegenerate triangle the three circumcircles have a common point. The circumcircles of aqr and brp both pass through r. When they are not tangent there, they meet again in a second point, the *Miquel point*, which is the reflection of r in the line joining the two centres. Reflection in a line is an isometry fixing every point of that line, so this second point automatically lies on both circles; the pivot theorem then places it on the third.

definition *reflect-line* :: *complex* ⇒ *complex* ⇒ *complex* ⇒ *complex*
where *reflect-line o1 o2 z* = *o1* + (*o2* − *o1*) ∗ *cnj* (*z* − *o1*) / *cnj* (*o2* − *o1*)

Reflection in the line through *o1* and *o2* preserves the distance to *o1*.

lemma *reflect-line-dist1*:
assumes *oo*: *o1* ≠ *o2*
shows *cmod* (*reflect-line o1 o2 z* − *o1*) = *cmod* (*z* − *o1*)
⟨*proof*⟩

Reflection in the line through *o1* and *o2* also preserves the distance to *o2*.

lemma *reflect-line-dist2*:
assumes *oo*: *o1* ≠ *o2*
shows *cmod* (*reflect-line o1 o2 z* − *o2*) = *cmod* (*z* − *o2*)
⟨*proof*⟩

If *r* does not lie on the line through the two centres, i.e. the two circles are not tangent at *r*, then the reflected point is genuinely distinct from *r*.

lemma *reflect-line-neq*:
assumes *ncl*: ¬ *collinear o1 o2 r*
shows *reflect-line o1 o2 r* ≠ *r*
⟨*proof*⟩

The classical Miquel theorem. Let *abc* be a nondegenerate triangle with *r*, *p*, *q* on the side lines *ab*, *bc*, *ca*. Given the circumcircles of *aqr* and *brp*, not tangent at *r*, their second common point — the reflection of *r* in the centre line — lies on all three circumcircles, including that of *cpq*. No common point is assumed: it is constructed.

theorem *miquel-concurrent*:
assumes *lr*: *on-line a b r* **and** *lp*: *on-line b c p* **and** *lq*: *on-line c a q*
and *ncol*: ¬ *collinear a b c*
and *ca*: *oncircle o1 R1 a* **and** *cq*: *oncircle o1 R1 q*
and *car*: *oncircle o1 R1 r* **and** *R1pos*: *0* < *R1*
and *cbb*: *oncircle o2 R2 b* **and** *cbr*: *oncircle o2 R2 r*
and *cpp*: *oncircle o2 R2 p* **and** *R2pos*: *0* < *R2*
and *nonline*: ¬ *collinear o1 o2 r*
and *daq*: *a* ≠ *q* **and** *dar*: *a* ≠ *r* **and** *dbr*: *b* ≠ *r* **and** *dbp*: *b* ≠ *p*
and *dcp*: *c* ≠ *p* **and** *dcq*: *c* ≠ *q*
shows *concyclic a q r* (*reflect-line o1 o2 r*)
∧ *concyclic b r p* (*reflect-line o1 o2 r*)
∧ *concyclic c p q* (*reflect-line o1 o2 r*)
⟨*proof*⟩

The existence form: the three circumcircles have a common point.

corollary *miquel-point-exists*:
assumes *lr*: *on-line a b r* **and** *lp*: *on-line b c p* **and** *lq*: *on-line c a q*
and *ncol*: ¬ *collinear a b c*
and *ca*: *oncircle o1 R1 a* **and** *cq*: *oncircle o1 R1 q*

and *car*: *oncircle o1 R1 r* **and** *R1pos*: $0 < R1$
and *cbb*: *oncircle o2 R2 b* **and** *cbr*: *oncircle o2 R2 r*
and *cpp*: *oncircle o2 R2 p* **and** *R2pos*: $0 < R2$
and *nonline*: ¬ *collinear o1 o2 r*
and *daq*: $a \neq q$ **and** *dar*: $a \neq r$ **and** *dbr*: $b \neq r$ **and** *dbp*: $b \neq p$
and *dcp*: $c \neq p$ **and** *dcq*: $c \neq q$
shows ∃ *m. concyclic a q r m* ∧ *concyclic b r p m* ∧ *concyclic c p q m*
⟨*proof*⟩

end

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