MiniSail

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Abstract

MiniSail is a kernel language for Sail [1], an instruction set architecture (ISA) specification language. Sail is an imperative language with a light-weight dependent type system similar to refinement type systems such as [2]. From an ISA specification, the Sail compiler can generate theorem prover code and C (or OCaml) to give an executable emulator for an architecture. The idea behind MiniSail is to capture the key and novel features of Sail in terms of their syntax, typing rules and operational semantics, and to confirm that they work together by proving progress and preservation lemmas. We use the Nominal2 library to handle binding.

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Chapter 1

Prelude

Some useful Nominal lemmas. Many of these are from Launchbury.Nominal-Utils.

1.1 Lemmas helping with equivariance proofs

lemma *perm-rel-lemma*:
  assumes $\bigwedge \pi\ x\ y.\ r\ (\pi \cdot x)\ (\pi \cdot y) \Longrightarrow r\ x\ y$
  shows $r\ (\pi \cdot x)\ (\pi \cdot y) \longleftrightarrow r\ x\ y$ (**is** *?l* $\longleftrightarrow$ *?r*)
⟨*proof*⟩

lemma *perm-rel-lemma2*:
  assumes $\bigwedge \pi\ x\ y.\ r\ x\ y \Longrightarrow r\ (\pi \cdot x)\ (\pi \cdot y)$
  shows $r\ x\ y \longleftrightarrow r\ (\pi \cdot x)\ (\pi \cdot y)$ (**is** *?l* $\longleftrightarrow$ *?r*)
⟨*proof*⟩

lemma *fun-eqvtI*:
  assumes *f-eqvt*[*eqvt*]: $(\bigwedge\ p\ x.\ p \cdot (f\ x) = f\ (p \cdot x))$
  shows $p \cdot f = f$ ⟨*proof*⟩

lemma *eqvt-at-apply*:
  assumes *eqvt-at f x*
  shows $(p \cdot f)\ x = f\ x$
⟨*proof*⟩

lemma *eqvt-at-apply′*:
  assumes *eqvt-at f x*
  shows $p \cdot f\ x = f\ (p \cdot x)$
⟨*proof*⟩

lemma *eqvt-at-apply″*:
  assumes *eqvt-at f x*
  shows $(p \cdot f)\ (p \cdot x) = f\ (p \cdot x)$
⟨*proof*⟩

lemma *size-list-eqvt*[*eqvt*]: $p \cdot$ *size-list f x* = *size-list* $(p \cdot f)\ (p \cdot x)$
⟨*proof*⟩

1.2 Freshness via equivariance

lemma *eqvt-fresh-cong1*: $(\bigwedge p\ x.\ p \cdot (f\ x) = f\ (p \cdot x)) \Longrightarrow a \sharp x \Longrightarrow a \sharp f\ x$
⟨*proof*⟩

lemma *eqvt-fresh-cong2*:
assumes *eqvt*: $(\bigwedge p\ x\ y.\ p \cdot (f\ x\ y) = f\ (p \cdot x)\ (p \cdot y))$
and *fresh1*: $a \sharp x$ **and** *fresh2*: $a \sharp y$
shows $a \sharp f\ x\ y$
⟨*proof*⟩

lemma *eqvt-fresh-star-cong1*:
assumes *eqvt*: $(\bigwedge p\ x.\ p \cdot (f\ x) = f\ (p \cdot x))$
and *fresh1*: $a \sharp* x$
shows $a \sharp* f\ x$
⟨*proof*⟩

lemma *eqvt-fresh-star-cong2*:
assumes *eqvt*: $(\bigwedge p\ x\ y.\ p \cdot (f\ x\ y) = f\ (p \cdot x)\ (p \cdot y))$
and *fresh1*: $a \sharp* x$ **and** *fresh2*: $a \sharp* y$
shows $a \sharp* f\ x\ y$
⟨*proof*⟩

lemma *eqvt-fresh-cong3*:
assumes *eqvt*: $(\bigwedge p\ x\ y\ z.\ p \cdot (f\ x\ y\ z) = f\ (p \cdot x)\ (p \cdot y)\ (p \cdot z))$
and *fresh1*: $a \sharp x$ **and** *fresh2*: $a \sharp y$ **and** *fresh3*: $a \sharp z$
shows $a \sharp f\ x\ y\ z$
⟨*proof*⟩

lemma *eqvt-fresh-star-cong3*:
assumes *eqvt*: $(\bigwedge p\ x\ y\ z.\ p \cdot (f\ x\ y\ z) = f\ (p \cdot x)\ (p \cdot y)\ (p \cdot z))$
and *fresh1*: $a \sharp* x$ **and** *fresh2*: $a \sharp* y$ **and** *fresh3*: $a \sharp* z$
shows $a \sharp* f\ x\ y\ z$
⟨*proof*⟩

1.3 Additional simplification rules

lemma *not-self-fresh*[*simp*]: $atom\ x \sharp x \longleftrightarrow False$
⟨*proof*⟩

lemma *fresh-star-singleton*: $\{\ x\ \} \sharp* e \longleftrightarrow x \sharp e$
⟨*proof*⟩

1.4 Additional equivariance lemmas

lemma *eqvt-cases*:
fixes $f\ x\ \pi$
assumes *eqvt*: $\bigwedge x.\ \pi \cdot f\ x = f\ (\pi \cdot x)$
obtains $f\ x\ f\ (\pi \cdot x) \mid \neg f\ x \quad \neg f\ (\pi \cdot x)$
⟨*proof*⟩

lemma *range-eqvt*: $\pi \cdot range\ Y = range\ (\pi \cdot Y)$

⟨*proof*⟩

lemma *case-option-eqvt*[*eqvt*]:
$\pi \cdot$ *case-option* $d\ f\ x =$ *case-option* $(\pi \cdot d)\ (\pi \cdot f)\ (\pi \cdot x)$
⟨*proof*⟩

lemma *supp-option-eqvt*:
supp (*case-option* $d\ f\ x) \subseteq$ *supp* $d \cup$ *supp* $f \cup$ *supp* x
⟨*proof*⟩

lemma *funpow-eqvt*[*simp*,*eqvt*]:
$\pi \cdot ((f :: 'a \Rightarrow 'a{::}pt)\ \widehat{\ }\widehat{\ }\ n) = (\pi \cdot f)\ \widehat{\ }\widehat{\ }\ (\pi \cdot n)$
⟨*proof*⟩

lemma *delete-eqvt*[*eqvt*]:
$\pi \cdot$ *AList.delete* $x\ \Gamma =$ *AList.delete* $(\pi \cdot x)\ (\pi \cdot \Gamma)$
⟨*proof*⟩

lemma *restrict-eqvt*[*eqvt*]:
$\pi \cdot$ *AList.restrict* $S\ \Gamma =$ *AList.restrict* $(\pi \cdot S)\ (\pi \cdot \Gamma)$
⟨*proof*⟩

lemma *supp-restrict*:
supp (*AList.restrict* $S\ \Gamma) \subseteq$ *supp* Γ
⟨*proof*⟩

lemma *clearjunk-eqvt*[*eqvt*]:
$\pi \cdot$ *AList.clearjunk* $\Gamma =$ *AList.clearjunk* $(\pi \cdot \Gamma)$
⟨*proof*⟩

lemma *map-ran-eqvt*[*eqvt*]:
$\pi \cdot$ *map-ran* $f\ \Gamma =$ *map-ran* $(\pi \cdot f)\ (\pi \cdot \Gamma)$
⟨*proof*⟩

lemma *dom-perm*:
dom $(\pi \cdot f) = \pi \cdot$ (*dom* f)
⟨*proof*⟩

lemmas *dom-perm-rev*[*simp*,*eqvt*] = *dom-perm*[*symmetric*]

lemma *ran-perm*[*simp*]:
$\pi \cdot$ (*ran* f) = *ran* $(\pi \cdot f)$
⟨*proof*⟩

lemma *map-add-eqvt*[*eqvt*]:
$\pi \cdot$ (*m1* ++ *m2*) = $(\pi \cdot$ *m1*) ++ $(\pi \cdot$ *m2*)
⟨*proof*⟩

lemma *map-of-eqvt*[*eqvt*]:
$\pi \cdot$ *map-of* $l =$ *map-of* $(\pi \cdot l)$
⟨*proof*⟩

lemma *concat-eqvt*[*eqvt*]: $\pi \cdot concat\ l = concat\ (\pi \cdot l)$
⟨*proof*⟩

lemma *tranclp-eqvt*[*eqvt*]: $\pi \cdot tranclp\ P\ v_1\ v_2 = tranclp\ (\pi \cdot P)\ (\pi \cdot v_1)\ (\pi \cdot v_2)$
⟨*proof*⟩

lemma *rtranclp-eqvt*[*eqvt*]: $\pi \cdot rtranclp\ P\ v_1\ v_2 = rtranclp\ (\pi \cdot P)\ (\pi \cdot v_1)\ (\pi \cdot v_2)$
⟨*proof*⟩

lemma *Set-filter-eqvt*[*eqvt*]: $\pi \cdot Set.filter\ P\ S = Set.filter\ (\pi \cdot P)\ (\pi \cdot S)$
⟨*proof*⟩

lemma *Sigma-eqvt'*[*eqvt*]: $\pi \cdot Sigma = Sigma$
⟨*proof*⟩

lemma *override-on-eqvt*[*eqvt*]:
$\pi \cdot (override\text{-}on\ m1\ m2\ S) = override\text{-}on\ (\pi \cdot m1)\ (\pi \cdot m2)\ (\pi \cdot S)$
⟨*proof*⟩

lemma *card-eqvt*[*eqvt*]:
$\pi \cdot (card\ S) = card\ (\pi \cdot S)$
⟨*proof*⟩

lemma *Projl-permute*:
assumes *a*: $\exists\, y.\ f = Inl\ y$
shows $(p \cdot (Sum\text{-}Type.projl\ f)) = Sum\text{-}Type.projl\ (p \cdot f)$
⟨*proof*⟩

lemma *Projr-permute*:
assumes *a*: $\exists\, y.\ f = Inr\ y$
shows $(p \cdot (Sum\text{-}Type.projr\ f)) = Sum\text{-}Type.projr\ (p \cdot f)$
⟨*proof*⟩

1.5 Freshness lemmas

lemma *fresh-list-elem*:
assumes $a \sharp \Gamma$
and $e \in set\ \Gamma$
shows $a \sharp e$
⟨*proof*⟩

lemma *set-not-fresh*:
$x \in set\ L \Longrightarrow \neg(atom\ x \sharp L)$
⟨*proof*⟩

lemma *pure-fresh-star*[*simp*]: $a \sharp* (x :: 'a :: pure)$
⟨*proof*⟩

lemma *supp-set-mem*: $x \in set\ L \Longrightarrow supp\ x \subseteq supp\ L$
⟨*proof*⟩

lemma *set-supp-mono*: *set L* $\subseteq$ *set L2* $\Longrightarrow$ *supp L* $\subseteq$ *supp L2*
⟨*proof*⟩

lemma *fresh-star-at-base*:
fixes *x* :: *'a* :: *at-base*
shows *S* $\sharp*$ *x* $\longleftrightarrow$ *atom x* $\notin$ *S*
⟨*proof*⟩

1.6 Freshness and support for subsets of variables

lemma *supp-mono*: *finite* (*B*::*'a*::*fs set*) $\Longrightarrow$ *A* $\subseteq$ *B* $\Longrightarrow$ *supp A* $\subseteq$ *supp B*
⟨*proof*⟩

lemma *fresh-subset*:
finite B $\Longrightarrow$ *x* $\sharp$ (*B* :: *'a*::*at-base set*) $\Longrightarrow$ *A* $\subseteq$ *B* $\Longrightarrow$ *x* $\sharp$ *A*
⟨*proof*⟩

lemma *fresh-star-subset*:
finite B $\Longrightarrow$ *x* $\sharp*$ (*B* :: *'a*::*at-base set*) $\Longrightarrow$ *A* $\subseteq$ *B* $\Longrightarrow$ *x* $\sharp*$ *A*
⟨*proof*⟩

lemma *fresh-star-set-subset*:
x $\sharp*$ (*B* :: *'a*::*at-base list*) $\Longrightarrow$ *set A* $\subseteq$ *set B* $\Longrightarrow$ *x* $\sharp*$ *A*
⟨*proof*⟩

1.7 The set of free variables of an expression

definition *fv* :: *'a*::*pt* $\Rightarrow$ *'b*::*at-base set*
where *fv e* = {*v*. *atom v* $\in$ *supp e*}

lemma *fv-eqvt*[*simp*,*eqvt*]: $\pi \cdot$ (*fv e*) = *fv* ($\pi \cdot$ *e*)
⟨*proof*⟩

lemma *fv-Nil*[*simp*]: *fv* [] = {}
⟨*proof*⟩
lemma *fv-Cons*[*simp*]: *fv* (*x* # *xs*) = *fv x* $\cup$ *fv xs*
⟨*proof*⟩
lemma *fv-Pair*[*simp*]: *fv* (*x*, *y*) = *fv x* $\cup$ *fv y*
⟨*proof*⟩
lemma *fv-append*[*simp*]: *fv* (*x* @ *y*) = *fv x* $\cup$ *fv y*
⟨*proof*⟩
lemma *fv-at-base*[*simp*]: *fv a* = {*a*::*'a*::*at-base*}
⟨*proof*⟩
lemma *fv-pure*[*simp*]: *fv* (*a*::*'a*::*pure*) = {}
⟨*proof*⟩

lemma *fv-set-at-base*[*simp*]: *fv* (*l* :: (*'a* :: *at-base*) *list*) = *set l*
⟨*proof*⟩

lemma *flip-not-fv*: *a* $\notin$ *fv x* $\Longrightarrow$ *b* $\notin$ *fv x* $\Longrightarrow$ (*a* $\leftrightarrow$ *b*) $\cdot$ *x* = *x*

⟨*proof*⟩

lemma *fv-not-fresh*: *atom* $x \sharp e \longleftrightarrow x \notin fv\ e$
⟨*proof*⟩

lemma *fresh-fv*: *finite* (*fv* $e :: 'a\ set$) $\Longrightarrow$ *atom* $(x :: ('a::at\text{-}base)) \sharp (fv\ e :: 'a\ set) \longleftrightarrow atom\ x \sharp e$
⟨*proof*⟩

lemma *finite-fv*[*simp*]: *finite* (*fv* $(e::'a::fs) :: ('b::at\text{-}base)\ set$)
⟨*proof*⟩

definition *fv-list* :: $'a::fs \Rightarrow 'b::at\text{-}base\ list$
where *fv-list* $e = (SOME\ l.\ set\ l = fv\ e)$

lemma *set-fv-list*[*simp*]: *set* (*fv-list* e) = (*fv* $e :: ('b::at\text{-}base)\ set$)
⟨*proof*⟩

lemma *fresh-fv-list*[*simp*]:
$a \sharp (fv\text{-}list\ e :: 'b::at\text{-}base\ list) \longleftrightarrow a \sharp (fv\ e :: 'b::at\text{-}base\ set)$
⟨*proof*⟩

1.8 Other useful lemmas

lemma *pure-permute-id*: *permute* $p = (\lambda\ x.\ (x::'a::pure))$
⟨*proof*⟩

lemma *supp-set-elem-finite*:
assumes *finite* S
and $(m::'a::fs) \in S$
and $y \in supp\ m$
shows $y \in supp\ S$
⟨*proof*⟩

lemmas *fresh-star-Cons* = *fresh-star-list*(*2*)

lemma *mem-permute-set*:
shows $x \in p \cdot S \longleftrightarrow (-\ p \cdot x) \in S$
⟨*proof*⟩

lemma *flip-set-both-not-in*:
assumes $x \notin S$ **and** $x' \notin S$
shows $((x' \leftrightarrow x) \cdot S) = S$
⟨*proof*⟩

lemma *inj-atom*: *inj atom* ⟨*proof*⟩

lemmas *image-Int*[*OF inj-atom*, *simp*]

lemma *eqvt-uncurry*: *eqvt* $f \Longrightarrow$ *eqvt* (*case-prod* f)
⟨*proof*⟩

lemma *supp-fun-app-eqvt2*:

assumes *a*: *eqvt f*
shows *supp* (*f x y*) ⊆ *supp x* ∪ *supp y*
⟨*proof*⟩

lemma *supp-fun-app-eqvt3*:
assumes *a*: *eqvt f*
shows *supp* (*f x y z*) ⊆ *supp x* ∪ *supp y* ∪ *supp z*
⟨*proof*⟩

lemma *permute-0*[*simp*]: *permute 0* = (λ *x*. *x*)
⟨*proof*⟩
lemma *permute-comp*[*simp*]: *permute x* ∘ *permute y* = *permute* (*x* + *y*) ⟨*proof*⟩

lemma *map-permute*: *map* (*permute p*) = *permute p*
⟨*proof*⟩

lemma *fresh-star-restrictA*[*intro*]: *a* ♯∗ Γ ⟹ *a* ♯∗ *AList.restrict V* Γ
⟨*proof*⟩

lemma *Abs-lst-Nil-eq*[*simp*]: [[]]*lst*. (*x*::′*a*::*fs*) = [*xs*]*lst*. *x*′ ⟷ (([],*x*) = (*xs*, *x*′))
⟨*proof*⟩

lemma *Abs-lst-Nil-eq2*[*simp*]: [*xs*]*lst*. (*x*::′*a*::*fs*) = [[]]*lst*. *x*′ ⟷ ((*xs*,*x*) = ([], *x*′))
⟨*proof*⟩

lemma *prod-cases8* [*cases type*]:
obtains (*fields*) *a b c d e f g h* **where** *y* = (*a*, *b*, *c*, *d*, *e*, *f*, *g*,*h*)
⟨*proof*⟩

lemma *prod-induct8* [*case-names fields*, *induct type*]:
(⋀*a b c d e f g h*. *P* (*a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*)) ⟹ *P x*
⟨*proof*⟩

lemma *prod-cases9* [*cases type*]:
obtains (*fields*) *a b c d e f g h i* **where** *y* = (*a*, *b*, *c*, *d*, *e*, *f*, *g*,*h*,*i*)
⟨*proof*⟩

lemma *prod-induct9* [*case-names fields*, *induct type*]:
(⋀*a b c d e f g h i*. *P* (*a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*)) ⟹ *P x*
⟨*proof*⟩

named-theorems *nominal-prod-simps*

named-theorems *ms-fresh Facts for helping with freshness proofs*

lemma *fresh-prod2*[*nominal-prod-simps*,*ms-fresh*]: *x* ♯ (*a*,*b*) = (*x* ♯ *a* ∧ *x* ♯ *b*)
⟨*proof*⟩

lemma *fresh-prod3*[*nominal-prod-simps*,*ms-fresh*]: *x* ♯ (*a*,*b*,*c*) = (*x* ♯ *a* ∧ *x* ♯ *b* ∧ *x* ♯ *c*)
⟨*proof*⟩

lemma *fresh-prod4*[*nominal-prod-simps*,*ms-fresh*]: $x \sharp (a,b,c,d) = (x \sharp a \wedge x \sharp b \wedge x \sharp c \wedge x \sharp d)$
⟨*proof*⟩

lemma *fresh-prod5*[*nominal-prod-simps*,*ms-fresh*]: $x \sharp (a,b,c,d,e) = (x \sharp a \wedge x \sharp b \wedge x \sharp c \wedge x \sharp d \wedge x \sharp e)$
⟨*proof*⟩

lemma *fresh-prod6*[*nominal-prod-simps*,*ms-fresh*]: $x \sharp (a,b,c,d,e,f) = (x \sharp a \wedge x \sharp b \wedge x \sharp c \wedge x \sharp d \wedge x \sharp e \wedge x \sharp f)$
⟨*proof*⟩

lemma *fresh-prod7*[*nominal-prod-simps*,*ms-fresh*]: $x \sharp (a,b,c,d,e,f,g) = (x \sharp a \wedge x \sharp b \wedge x \sharp c \wedge x \sharp d \wedge x \sharp e \wedge x \sharp f \wedge x \sharp g)$
⟨*proof*⟩

lemma *fresh-prod8*[*nominal-prod-simps*,*ms-fresh*]: $x \sharp (a,b,c,d,e,f,g,h) = (x \sharp a \wedge x \sharp b \wedge x \sharp c \wedge x \sharp d \wedge x \sharp e \wedge x \sharp f \wedge x \sharp g \wedge x \sharp h)$
⟨*proof*⟩

lemma *fresh-prod9*[*nominal-prod-simps*,*ms-fresh*]: $x \sharp (a,b,c,d,e,f,g,h,i) = (x \sharp a \wedge x \sharp b \wedge x \sharp c \wedge x \sharp d \wedge x \sharp e \wedge x \sharp f \wedge x \sharp g \wedge x \sharp h \wedge x \sharp i)$
⟨*proof*⟩

lemma *fresh-prod10*[*nominal-prod-simps*,*ms-fresh*]: $x \sharp (a,b,c,d,e,f,g,h,i,j) = (x \sharp a \wedge x \sharp b \wedge x \sharp c \wedge x \sharp d \wedge x \sharp e \wedge x \sharp f \wedge x \sharp g \wedge x \sharp h \wedge x \sharp i \wedge x \sharp j)$
⟨*proof*⟩

lemma *fresh-prod12*[*nominal-prod-simps*,*ms-fresh*]: $x \sharp (a,b,c,d,e,f,g,h,i,j,k,l) = (x \sharp a \wedge x \sharp b \wedge x \sharp c \wedge x \sharp d \wedge x \sharp e \wedge x \sharp f \wedge x \sharp g \wedge x \sharp h \wedge x \sharp i \wedge x \sharp j \wedge x \sharp k \wedge x \sharp l)$
⟨*proof*⟩

lemmas *fresh-prodN = fresh-Pair fresh-prod3 fresh-prod4 fresh-prod5 fresh-prod6 fresh-prod7 fresh-prod8 fresh-prod9 fresh-prod10 fresh-prod12*

lemma *fresh-prod2I*:
fixes *x* **and** *x1* **and** *x2*
assumes $x \sharp x1$ **and** $x \sharp x2$
shows $x \sharp (x1,x2)$ ⟨*proof*⟩

lemma *fresh-prod3I*:
fixes *x* **and** *x1* **and** *x2* **and** *x3*
assumes $x \sharp x1$ **and** $x \sharp x2$ **and** $x \sharp x3$
shows $x \sharp (x1,x2,x3)$ ⟨*proof*⟩

lemma *fresh-prod4I*:
fixes *x* **and** *x1* **and** *x2* **and** *x3* **and** *x4*
assumes $x \sharp x1$ **and** $x \sharp x2$ **and** $x \sharp x3$ **and** $x \sharp x4$
shows $x \sharp (x1,x2,x3,x4)$ ⟨*proof*⟩

lemma *fresh-prod5I*:
fixes *x* **and** *x1* **and** *x2* **and** *x3* **and** *x4* **and** *x5*
assumes $x \sharp x1$ **and** $x \sharp x2$ **and** $x \sharp x3$ **and** $x \sharp x4$ **and** $x \sharp x5$

shows $x \sharp (x1,x2,x3,x4,x5)$ ⟨*proof*⟩

lemma *flip-collapse*[*simp*]:
fixes $b1::'a::pt$ **and** $bv1::'b::at$ **and** $bv2::'b::at$
assumes $atom\ bv2 \sharp b1$ **and** $atom\ c \sharp (bv1,bv2,b1)$ **and** $bv1 \neq bv2$
shows $(bv2 \leftrightarrow c) \cdot (bv1 \leftrightarrow bv2) \cdot b1 = (bv1 \leftrightarrow c) \cdot b1$
⟨*proof*⟩

lemma *triple-eqvt*[*simp*]:
$p \cdot (x, b,c) = (p \cdot x,\ \ p \cdot b\ ,\ p \cdot c)$
⟨*proof*⟩

lemma *lst-fst*:
fixes $x::'a::at$ **and** $t1::'b::fs$ **and** $x'::'a::at$ **and** $t2::'c::fs$
assumes $([[atom\ x]]lst.\ (t1,t2) = [[atom\ x']]lst.\ (t1',t2'))$
shows $([[atom\ x]]lst.\ t1 = [[atom\ x']]lst.\ t1')$
⟨*proof*⟩

lemma *lst-snd*:
fixes $x::'a::at$ **and** $t1::'b::fs$ **and** $x'::'a::at$ **and** $t2::'c::fs$
assumes $([[atom\ x]]lst.\ (t1,t2) = [[atom\ x']]lst.\ (t1',t2'))$
shows $([[atom\ x]]lst.\ t2 = [[atom\ x']]lst.\ t2')$
⟨*proof*⟩

lemma *lst-head-cons-pair*:
fixes $y1::'a::at$ **and** $y2::'a::at$ **and** $x1::'b::fs$ **and** $x2::'b::fs$ **and** $xs1::('b::fs)\ list$ **and** $xs2::('b::fs)\ list$
assumes $[[atom\ y1]]lst.\ (x1 \#\ xs1) = [[atom\ y2]]lst.\ (x2 \#\ xs2)$
shows $[[atom\ y1]]lst.\ (x1,xs1) = [[atom\ y2]]lst.\ (x2,xs2)$
⟨*proof*⟩

lemma *lst-head-cons-neq-nil*:
fixes $y1::'a::at$ **and** $y2::'a::at$ **and** $x1::'b::fs$ **and** $x2::'b::fs$ **and** $xs1::('b::fs)\ list$ **and** $xs2::('b::fs)\ list$
assumes $[[atom\ y1]]lst.\ (x1 \#\ xs1) = [[atom\ y2]]lst.\ (xs2)$
shows $xs2 \neq []$
⟨*proof*⟩

lemma *lst-head-cons*:
fixes $y1::'a::at$ **and** $y2::'a::at$ **and** $x1::'b::fs$ **and** $x2::'b::fs$ **and** $xs1::('b::fs)\ list$ **and** $xs2::('b::fs)\ list$
assumes $[[atom\ y1]]lst.\ (x1 \#\ xs1) = [[atom\ y2]]lst.\ (x2 \#\ xs2)$
shows $[[atom\ y1]]lst.\ x1\ = [[atom\ y2]]lst.\ x2$ **and** $[[atom\ y1]]lst.\ xs1\ = [[atom\ y2]]lst.\ xs2$
⟨*proof*⟩

lemma *lst-pure*:
fixes $x1::'a::at$ **and** $t1::'b::pure$ **and** $x2::'a::at$ **and** $t2::'b::pure$
assumes $[[atom\ x1]]lst.\ t1 = [[atom\ x2]]lst.\ t2$
shows $t1=t2$
⟨*proof*⟩

lemma *lst-supp*:
assumes $[[atom\ x1]]lst.\ t1 = [[atom\ x2]]lst.\ t2$
shows $supp\ t1 - \{atom\ x1\} = supp\ t2 - \{atom\ x2\}$
⟨*proof*⟩

lemma *lst-supp-subset*:
  assumes $[[atom\ x1]]lst.\ t1 = [[atom\ x2]]lst.\ t2$ **and** $supp\ t1 \subseteq \{atom\ x1\} \cup B$
  shows $supp\ t2 \subseteq \{atom\ x2\} \cup B$
⟨*proof*⟩

lemma *projl-inl-eqvt*:
  fixes $\pi :: perm$
  shows $\pi \cdot (projl\ (Inl\ x)) = projl\ (Inl\ (\pi \cdot x))$
⟨*proof*⟩

end

Chapter 2

Syntax

Syntax of MiniSail programs and the contexts we use in judgements.

2.1 Program Syntax

2.1.1 AST Datatypes

type-synonym *num-nat* = *nat*

atom-decl *x*
atom-decl *u*
atom-decl *bv*

type-synonym *f* = *string*
type-synonym *dc* = *string*
type-synonym *tyid* = *string*

Basic types. Types without refinement constraints

nominal-datatype *b* =
B-int | *B-bool* | *B-id tyid*
| *B-pair b b* (‹[- , -]^b›)
| *B-unit* | *B-bitvec* | *B-var bv*
| *B-app tyid b*

nominal-datatype *bit* = *BitOne* | *BitZero*

Literals

nominal-datatype *l* =
L-num int | *L-true* | *L-false* | *L-unit* | *L-bitvec bit list*

Values. We include a type identifier, tyid, in the literal for constructors to make typing and well-formedness checking easier

nominal-datatype *v* =
V-lit l (‹[-]^v›)
| *V-var x* (‹[-]^v›)
| *V-pair v v* (‹[- , -]^v›)
| *V-cons tyid dc v*

| *V-consp tyid dc b v*

Binary Operations

nominal-datatype *opp* = *Plus* (‹*plus*›) | *LEq* (‹*leq*›) | *Eq* (‹*eq*›)

Expressions

nominal-datatype *e* =
AE-val v (‹[-]^e›)
| *AE-app f v* (‹[- (-)]^e›)
| *AE-appP f b v* (‹[- [-] (-)]^e›)
| *AE-op opp v v* (‹[- - -]^e›)
| *AE-concat v v* (‹[- @@ -]^e›)
| *AE-fst v* (‹[#*1*-]^e›)
| *AE-snd v* (‹[#*2*-]^e›)
| *AE-mvar u* (‹[-]^e›)
| *AE-len v* (‹[| - |]^e›)
| *AE-split v v* (‹[- / -]^e›)

Expressions for constraints

nominal-datatype *ce* =
CE-val v (‹[-]^{ce}›)
| *CE-op opp ce ce* (‹[- - -]^{ce}›)
| *CE-concat ce ce* (‹[- @@ -]^{ce}›)
| *CE-fst ce* (‹[#*1*-]^{ce}›)
| *CE-snd ce* (‹[#*2*-]^{ce}›)
| *CE-len ce* (‹[| - |]^{ce}›)

Constraints

nominal-datatype *c* =
C-true (‹*TRUE*› [] *50*)
| *C-false* (‹*FALSE*› [] *50*)
| *C-conj c c* (‹- *AND* - › [*50*, *50*] *50*)
| *C-disj c c* (‹- *OR* - › [*50*,*50*] *50*)
| *C-not c* (‹¬ - › [] *50*)
| *C-imp c c* (‹- *IMP* - › [*50*, *50*] *50*)
| *C-eq ce ce* (‹- == - › [*50*, *50*] *50*)

Refined types

nominal-datatype *τ* =
T-refined-type x::*x b c*::*c* **binds** *x* **in** *c* (‹⦃ - : - | - ⦄› [*50*, *50*] *1000*)

Statements

nominal-datatype
s =
AS-val v (‹[-]^s›)
| *AS-let x*::*x e s*::*s* **binds** *x* **in** *s* (‹(*LET* - = - *IN* -)›)
| *AS-let2 x*::*x τ s s*::*s* **binds** *x* **in** *s* (‹(*LET* - : - = - *IN* -)›)
| *AS-if v s s* (‹(*IF* - *THEN* - *ELSE* -)› [*0*, *61*, *0*] *61*)
| *AS-var u*::*u τ v s*::*s* **binds** *u* **in** *s* (‹(*VAR* - : - = - *IN* -)›)
| *AS-assign u v* (‹(- ::= -)›)
| *AS-match v branch-list* (‹(*MATCH* - *WITH* { - })›)

```
  | AS-while s s                    ( ‹(WHILE - DO { - } )› [0, 0] 61)
  | AS-seq s s                      ( ‹( - ;; - )›  [1000, 61] 61)
  | AS-assert c s                   ( ‹(ASSERT - IN - )› )
    and branch-s =
    AS-branch dc x::x s::s binds x in s  ( ‹( - - ⇒ - )›)
    and branch-list =
    AS-final  branch-s               ( ‹{ - }› )
  | AS-cons  branch-s branch-list      ( ‹( - | -  )›)
```

Function and union type definitions

```
nominal-datatype fun-typ  =
  AF-fun-typ x::x b c::c τ::τ s::s binds x in c τ s

nominal-datatype fun-typ-q =
  AF-fun-typ-some bv::bv ft::fun-typ binds bv in ft
  | AF-fun-typ-none fun-typ

nominal-datatype fun-def =  AF-fundef f fun-typ-q

nominal-datatype type-def =
  AF-typedef string (string * τ) list
  |  AF-typedef-poly string bv::bv dclist::(string * τ) list  binds bv in dclist

lemma check-typedef-poly:
  AF-typedef-poly ''option'' bv [ (''None'', {| zz : B-unit | TRUE |}), (''Some'', {| zz : B-var bv | TRUE
|}) ] =
    AF-typedef-poly ''option'' bv2 [ (''None'', {| zz : B-unit | TRUE |}), (''Some'', {| zz : B-var bv2 |
TRUE |}) ]
  ⟨proof⟩

nominal-datatype var-def =  AV-def u τ v
```

Programs

```
nominal-datatype p =
  AP-prog type-def list fun-def list var-def list s (‹PROG - - - -›)

declare l.supp [simp] v.supp [simp]  e.supp [simp] s-branch-s-branch-list.supp [simp]  τ.supp [simp]
c.supp [simp] b.supp[simp]
```

2.1.2 Lemmas

These lemmas deal primarily with freshness and alpha-equivalence

Atoms

```
lemma x-not-in-u-atoms[simp]:
  fixes u::u and x::x and us::u set
  shows atom x ∉ atom‘us
  ⟨proof⟩

lemma x-fresh-u[simp]:
  fixes u::u and x::x
```

shows *atom x* ♯ *u*
⟨*proof*⟩

lemma *x-not-in-b-set*[*simp*]:
fixes *x*::*x* **and** *bs*::*bv fset*
shows *atom x* ∉ *supp bs*
⟨*proof*⟩

lemma *x-fresh-b*[*simp*]:
fixes *x*::*x* **and** *b*::*b*
shows *atom x* ♯ *b*
⟨*proof*⟩

lemma *x-fresh-bv*[*simp*]:
fixes *x*::*x* **and** *bv*::*bv*
shows *atom x* ♯ *bv*
⟨*proof*⟩

lemma *u-not-in-x-atoms*[*simp*]:
fixes *u*::*u* **and** *x*::*x* **and** *xs*::*x set*
shows *atom u* ∉ *atom*‘*xs*
⟨*proof*⟩

lemma *bv-not-in-x-atoms*[*simp*]:
fixes *bv*::*bv* **and** *x*::*x* **and** *xs*::*x set*
shows *atom bv* ∉ *atom*‘*xs*
⟨*proof*⟩

lemma *u-not-in-b-atoms*[*simp*]:
fixes *b* :: *b* **and** *u*::*u*
shows *atom u* ∉ *supp b*
⟨*proof*⟩

lemma *u-not-in-b-set*[*simp*]:
fixes *u*::*u* **and** *bs*::*bv fset*
shows *atom u* ∉ *supp bs*
⟨*proof*⟩

lemma *u-fresh-b*[*simp*]:
fixes *x*::*u* **and** *b*::*b*
shows *atom x* ♯ *b*
⟨*proof*⟩

lemma *supp-b-v-disjoint*:
fixes *x*::*x* **and** *bv*::*bv*
shows *supp* (*V-var x*) ∩ *supp* (*B-var bv*) = {}
⟨*proof*⟩

lemma *supp-b-u-disjoint*[*simp*]:
fixes *b*::*b* **and** *u*::*u*
shows *supp u* ∩ *supp b* = {}
⟨*proof*⟩

lemma *u-fresh-bv*[*simp*]:
fixes u::u **and** b::bv
shows $atom\ u \sharp b$
⟨*proof*⟩

Basic Types

nominal-function *b-of* :: $\tau \Rightarrow b$ **where**
$b\text{-}of\ \{\!|\ z : b \mid c\ |\!\} = b$
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

lemma *supp-b-empty*[*simp*]:
fixes $b :: b$ **and** x::x
shows $atom\ x \notin supp\ b$
⟨*proof*⟩

lemma *flip-b-id*[*simp*]:
fixes x::x **and** b::b
shows $(x \leftrightarrow x') \cdot b = b$
⟨*proof*⟩

lemma *flip-x-b-cancel*[*simp*]:
fixes x::x **and** y::x **and** b::b **and** bv::bv
shows $(x \leftrightarrow y) \cdot b = b$ **and** $(x \leftrightarrow y) \cdot bv = bv$
⟨*proof*⟩

lemma *flip-bv-x-cancel*[*simp*]:
fixes bv::bv **and** z::bv **and** x::x
shows $(bv \leftrightarrow z) \cdot x = x$ ⟨*proof*⟩

lemma *flip-bv-u-cancel*[*simp*]:
fixes bv::bv **and** z::bv **and** x::u
shows $(bv \leftrightarrow z) \cdot x = x$ ⟨*proof*⟩

Literals

lemma *supp-bitvec-empty*:
fixes bv::*bit list*
shows $supp\ bv = \{\}$
⟨*proof*⟩

lemma *bitvec-pure*[*simp*]:
fixes bv::*bit list* **and** x::x
shows $atom\ x \sharp bv$ ⟨*proof*⟩

lemma *supp-l-empty*[*simp*]:
fixes l:: l
shows $supp\ (V\text{-}lit\ l) = \{\}$
⟨*proof*⟩

lemma *type-l-nosupp*[*simp*]:

fixes $x::x$ **and** $l::l$
shows $atom\ x \notin supp\ (\{\!|\ z : b\ |\ [[z]^v]^{ce} ==\ [[l]^v]^{ce}\ |\!\})$
⟨*proof*⟩

lemma *flip-bitvec0*:
fixes $x::bit\ list$
assumes $atom\ c \sharp (z, x, z')$
shows $(z \leftrightarrow c) \cdot x = (z' \leftrightarrow c) \cdot x$
⟨*proof*⟩

lemma *flip-bitvec*:
assumes $atom\ c \sharp (z, L\text{-}bitvec\ x, z')$
shows $(z \leftrightarrow c) \cdot x = (z' \leftrightarrow c) \cdot x$
⟨*proof*⟩

lemma *type-l-eq*:
shows $\{\!|\ z : b\ |\ [[z]^v]^{ce} == [V\text{-}lit\ l]^{ce}\ |\!\} = (\{\!|\ z' : b\ |\ [[z']^v]^{ce} == [V\text{-}lit\ l]^{ce}\ |\!\})$
⟨*proof*⟩

lemma *flip-l-eq*:
fixes $x::l$
shows $(z \leftrightarrow c) \cdot x = (z' \leftrightarrow c) \cdot x$
⟨*proof*⟩

lemma *flip-l-eq1*:
fixes $x::l$
assumes $(z \leftrightarrow c) \cdot x = (z' \leftrightarrow c) \cdot x'$
shows $x' = x$
⟨*proof*⟩

Types

lemma *flip-base-eq*:
fixes $b::b$ **and** $x::x$ **and** $y::x$
shows $(x \leftrightarrow y) \cdot b = b$
⟨*proof*⟩

Obtain an alpha-equivalent type where the bound variable is fresh in some term t

lemma *has-fresh-z0*:
fixes $t::'b::fs$
shows $\exists z.\ atom\ z \sharp (c',t) \wedge (\{\!|z' : b\ |\ c'\ |\!\}) = (\{\!|\ z : b\ |\ (z \leftrightarrow z')\cdot c'\ |\!\})$
⟨*proof*⟩

lemma *has-fresh-z*:
fixes $t::'b::fs$
shows $\exists z\ b\ c.\ atom\ z \sharp t \wedge \tau = \{\!|\ z : b\ |\ c\ |\!\}$
⟨*proof*⟩

lemma *obtain-fresh-z*:
fixes $t::'b::fs$
obtains z **and** b **and** c **where** $atom\ z \sharp t \wedge \tau = \{\!|\ z : b\ |\ c\ |\!\}$
⟨*proof*⟩

lemma *has-fresh-z2*:
fixes *t*::'*b*::*fs*
shows ∃ *z c*. *atom z* ♯ *t* ∧ τ = {‖ *z* : *b-of* τ | *c* ‖}
⟨*proof*⟩

lemma *obtain-fresh-z2*:
fixes *t*::'*b*::*fs*
obtains *z* **and** *c* **where** *atom z* ♯ *t* ∧ τ = {‖ *z* : *b-of* τ | *c* ‖}
⟨*proof*⟩

Values

lemma *u-notin-supp-v*[*simp*]:
fixes *u*::*u* **and** *v*::*v*
shows *atom u* ∉ *supp v*
⟨*proof*⟩

lemma *u-fresh-xv*[*simp*]:
fixes *u*::*u* **and** *x*::*x* **and** *v*::*v*
shows *atom u* ♯ (*x*,*v*)
⟨*proof*⟩

Part of an effort to make the proofs across inductive cases more uniform by distilling the non-uniform parts into lemmas like this

lemma *v-flip-eq*:
fixes *v*::*v* **and** *va*::*v* **and** *x*::*x* **and** *c*::*x*
assumes *atom c* ♯ (*v*, *va*) **and** *atom c* ♯ (*x*, *xa*, *v*, *va*) **and** (*x* ↔ *c*) ∙ *v* = (*xa* ↔ *c*) ∙ *va*
shows ((*v* = *V-lit l* ⟶ (∃ *l'*. *va* = *V-lit l'* ∧ (*x* ↔ *c*) ∙ *l* = (*xa* ↔ *c*) ∙ *l'*))) ∧
((*v* = *V-var y* ⟶ (∃ *y'*. *va* = *V-var y'* ∧ (*x* ↔ *c*) ∙ *y* = (*xa* ↔ *c*) ∙ *y'*))) ∧
((*v* = *V-pair vone vtwo* ⟶ (∃ *v1'* *v2'*. *va* = *V-pair v1'* *v2'* ∧ (*x* ↔ *c*) ∙ *vone* = (*xa* ↔ *c*) ∙ *v1'* ∧ (*x* ↔ *c*) ∙ *vtwo* = (*xa* ↔ *c*) ∙ *v2'*))) ∧
((*v* = *V-cons tyid dc vone* ⟶ (∃ *v1'*. *va* = *V-cons tyid dc v1'* ∧ (*x* ↔ *c*) ∙ *vone* = (*xa* ↔ *c*) ∙ *v1'*))) ∧
((*v* = *V-consp tyid dc b vone* ⟶ (∃ *v1'*. *va* = *V-consp tyid dc b v1'* ∧ (*x* ↔ *c*) ∙ *vone* = (*xa* ↔ *c*) ∙ *v1'*)))
⟨*proof*⟩

lemma *flip-eq*:
fixes *x*::*x* **and** *xa*::*x* **and** *s*::'*a*::*fs* **and** *sa*::'*a*::*fs*
assumes (∀ *c*. *atom c* ♯ (*s*, *sa*) ⟶ *atom c* ♯ (*x*, *xa*, *s*, *sa*) ⟶ (*x* ↔ *c*) ∙ *s* = (*xa* ↔ *c*) ∙ *sa*) **and** *x* ≠ *xa*
shows (*x* ↔ *xa*) ∙ *s* = *sa*
⟨*proof*⟩

lemma *swap-v-supp*:
fixes *v*::*v* **and** *d*::*x* **and** *z*::*x*
assumes *atom d* ♯ *v*
shows *supp* ((*z* ↔ *d*) ∙ *v*) ⊆ *supp v* − { *atom z* } ∪ { *atom d*}
⟨*proof*⟩

Expressions

lemma *swap-e-supp*:

fixes $e::e$ **and** $d::x$ **and** $z::x$
assumes $atom\ d \sharp e$
shows $supp\ ((z \leftrightarrow d\ ) \cdot e) \subseteq supp\ e - \{\ atom\ z\ \} \cup \{\ atom\ d\}$
⟨*proof*⟩

lemma *swap-ce-supp*:
fixes $e::ce$ **and** $d::x$ **and** $z::x$
assumes $atom\ d \sharp e$
shows $supp\ ((z \leftrightarrow d\ ) \cdot e) \subseteq supp\ e - \{\ atom\ z\ \} \cup \{\ atom\ d\}$
⟨*proof*⟩

lemma *swap-c-supp*:
fixes $c::c$ **and** $d::x$ **and** $z::x$
assumes $atom\ d \sharp c$
shows $supp\ ((z \leftrightarrow d\ ) \cdot c) \subseteq supp\ c - \{\ atom\ z\ \} \cup \{\ atom\ d\}$
⟨*proof*⟩

lemma *type-e-eq*:
assumes $atom\ z \sharp e$ **and** $atom\ z' \sharp e$
shows $\{\!|\ z : b\ \mid\ [[z]^v]^{ce} == e\ |\!\} = (\{\!|\ z' : b\ \mid\ [[z']^v]^{ce} ==\ \ e\ |\!\})$
⟨*proof*⟩

lemma *type-e-eq2*:
assumes $atom\ z \sharp e$ **and** $atom\ z' \sharp e$ **and** $b{=}b'$
shows $\{\!|\ z : b\ \mid\ [[z]^v]^{ce} == e\ |\!\} = (\{\!|\ z' : b'\ \mid\ [[z']^v]^{ce} == e\ |\!\})$
⟨*proof*⟩

lemma *e-flip-eq*:
fixes $e::e$ **and** $ea::e$
assumes $atom\ c \sharp (e,\ ea)$ **and** $atom\ c \sharp (x,\ xa,\ e,\ ea)$ **and** $(x \leftrightarrow c) \cdot e = (xa \leftrightarrow c) \cdot ea$
shows $(e = AE\text{-}val\ w \longrightarrow (\exists w'.\ ea = AE\text{-}val\ w' \wedge (x \leftrightarrow c) \cdot w = (xa \leftrightarrow c) \cdot w')) \vee$
$(e = AE\text{-}op\ opp\ v1\ v2 \longrightarrow (\exists v1'\ v2'.\ ea = AS\text{-}op\ opp\ v1'\ v2' \wedge (x \leftrightarrow c) \cdot v1 = (xa \leftrightarrow c) \cdot v1')$
$\wedge (x \leftrightarrow c) \cdot v2 = (xa \leftrightarrow c) \cdot v2') \vee$
$(e = AE\text{-}fst\ v \longrightarrow (\exists v'.\ ea = AE\text{-}fst\ v' \wedge (x \leftrightarrow c) \cdot v = (xa \leftrightarrow c) \cdot v')) \vee$
$(e = AE\text{-}snd\ v \longrightarrow (\exists v'.\ ea = AE\text{-}snd\ v' \wedge (x \leftrightarrow c) \cdot v = (xa \leftrightarrow c) \cdot v')) \vee$
$(e = AE\text{-}len\ v \longrightarrow (\exists v'.\ ea = AE\text{-}len\ v' \wedge (x \leftrightarrow c) \cdot v = (xa \leftrightarrow c) \cdot v')) \vee$
$(e = AE\text{-}concat\ v1\ v2 \longrightarrow (\exists v1'\ v2'.\ ea = AS\text{-}concat\ v1'\ v2' \wedge (x \leftrightarrow c) \cdot v1 = (xa \leftrightarrow c) \cdot v1')$
$\wedge (x \leftrightarrow c) \cdot v2 = (xa \leftrightarrow c) \cdot v2') \vee$
$(e = AE\text{-}app\ f\ v \longrightarrow (\exists v'.\ ea = AE\text{-}app\ f\ \ v' \wedge (x \leftrightarrow c) \cdot v = (xa \leftrightarrow c) \cdot v'))$
⟨*proof*⟩

lemma *fresh-opp-all*:
fixes $opp::opp$
shows $z \sharp opp$
⟨*proof*⟩

lemma *fresh-e-opp-all*:
shows $(z \sharp v1 \wedge z \sharp v2) = z \sharp AE\text{-}op\ opp\ v1\ v2$
⟨*proof*⟩

lemma *fresh-e-opp*:
fixes $z::x$

assumes *atom z ♯ v1 ∧ atom z ♯ v2*
shows *atom z ♯ AE-op opp v1 v2*
⟨*proof*⟩

Statements

lemma *branch-s-flip-eq*:
fixes *v*::*v* **and** *va*::*v*
assumes *atom c ♯ (v, va)* **and** *atom c ♯ (x, xa, v, va)* **and** *(x ↔ c) · s = (xa ↔ c) · sa*
shows *(s = AS-val w ⟶ (∃ w'. sa = AS-val w' ∧ (x ↔ c) · w = (xa ↔ c) · w')) ∨*
(s = AS-seq s1 s2 ⟶ (∃ s1' s2'. sa = AS-seq s1' s2' ∧ (x ↔ c) · s1 = (xa ↔ c) · s1') ∧ (x ↔ c) · s2 = (xa ↔ c) · s2') ∨
(s = AS-if v s1 s2 ⟶ (∃ v' s1' s2'. sa = AS-if seq s1' s2' ∧ (x ↔ c) · s1 = (xa ↔ c) · s1') ∧ (x ↔ c) · s2 = (xa ↔ c) · s2' ∧ (x ↔ c) · c = (xa ↔ c) · v')
⟨*proof*⟩

2.2 Context Syntax

2.2.1 Datatypes

Type and function/type definition contexts

type-synonym Φ = *fun-def list*
type-synonym Θ = *type-def list*
type-synonym $\mathcal{B}$ = *bv fset*

datatype Γ =
GNil
| *GCons x*∗*b*∗*c* Γ (**infixr** ‹$\#_\Gamma$› *65*)

datatype Δ =
DNil (‹$[]_\Delta$›)
| *DCons u*∗*τ* Δ (**infixr** ‹$\#_\Delta$› *65*)

2.2.2 Functions and Lemmas

lemma Γ-*induct* [*case-names GNil GCons*] : *P GNil* ⟹ (⋀*x b c* Γ'. *P* Γ' ⟹ *P* ((*x*,*b*,*c*) $\#_\Gamma$ Γ')) ⟹ *P* Γ
⟨*proof*⟩

instantiation Δ :: *pt*
begin

primrec *permute*-Δ
where
DNil-eqvt: *p* · *DNil* = *DNil*
| *DCons-eqvt*: *p* · (*x* $\#_\Delta$ *xs*) = *p* · *x* $\#_\Delta$ *p* · (*xs*::Δ)

instance ⟨*proof*⟩
end

lemmas [*eqvt*] = *permute*-Δ.*simps*

lemma Δ-*induct* [*case-names DNil DCons*] : P *DNil* $\Longrightarrow$ ($\bigwedge u\ t\ \Delta'.\ P\ \Delta' \Longrightarrow P\ ((u,t)\ \#_\Delta\ \Delta')) \Longrightarrow P\ \Delta$
⟨*proof*⟩

lemma Φ-*induct* [*case-names PNil PConsNone PConsSome*] : $P\ [] \Longrightarrow (\bigwedge f\ x\ b\ c\ \tau\ s'\ \Phi'.\ P\ \Phi' \Longrightarrow P\ ((AF\text{-}fundef\ f\ (AF\text{-}fun\text{-}typ\text{-}none\ (AF\text{-}fun\text{-}typ\ x\ b\ c\ \tau\ s')))\ \#\ \Phi')) \Longrightarrow$
$(\bigwedge f\ bv\ x\ b\ c\ \tau\ s'\ \Phi'.\ P\ \Phi' \Longrightarrow P\ ((AF\text{-}fundef\ f\ (AF\text{-}fun\text{-}typ\text{-}some\ bv\ (AF\text{-}fun\text{-}typ\ x\ b\ c\ \tau\ s')))\ \#\ \Phi'))\ \Longrightarrow P\ \Phi$
⟨*proof*⟩

lemma Θ-*induct* [*case-names TNil AF-typedef AF-typedef-poly*] : $P\ [] \Longrightarrow (\bigwedge tid\ dclist\ \Theta'.\ P\ \Theta' \Longrightarrow P\ ((AF\text{-}typedef\ tid\ dclist)\ \#\ \Theta')) \Longrightarrow$
$(\bigwedge tid\ bv\ dclist\ \Theta'.\ P\ \Theta' \Longrightarrow P\ ((AF\text{-}typedef\text{-}poly\ tid\ bv\ dclist\ )\ \#\ \Theta'))\ \Longrightarrow P\ \Theta$
⟨*proof*⟩

instantiation Γ :: *pt*
begin

primrec *permute*-Γ
where
GNil-eqvt: $p \cdot GNil = GNil$
| *GCons-eqvt*: $p \cdot (x\ \#_\Gamma\ xs) = p \cdot x\ \#_\Gamma\ p \cdot (xs::\Gamma)$

instance ⟨*proof*⟩
end

lemmas [*eqvt*] = *permute*-Γ.*simps*

lemma *G-cons-eqvt*[*simp*]:
fixes $\Gamma::\Gamma$
shows $p \cdot ((x,b,c)\ \#_\Gamma\ \Gamma) = ((p \cdot x,\ \ p \cdot b\ ,\ p \cdot c)\ \#_\Gamma\ (p \cdot \Gamma\ ))$ (**is** *?A* = *?B*)
⟨*proof*⟩

lemma *G-cons-flip*[*simp*]:
fixes $x::x$ **and** $\Gamma::\Gamma$
shows $(x \leftrightarrow x') \cdot ((x'',b,c)\ \#_\Gamma\ \Gamma) = (((x \leftrightarrow x') \cdot x'',\ \ b\ ,\ (x \leftrightarrow x') \cdot c)\ \#_\Gamma\ ((x \leftrightarrow x') \cdot \Gamma\ ))$
⟨*proof*⟩

lemma *G-cons-flip-fresh*[*simp*]:
fixes $x::x$ **and** $\Gamma::\Gamma$
assumes *atom* $x \sharp (c,\Gamma)$ **and** *atom* $x' \sharp (c,\Gamma)$
shows $(x \leftrightarrow x') \cdot ((x',b,c)\ \#_\Gamma\ \Gamma) = ((x,\ \ b\ ,\ c)\ \#_\Gamma\ \Gamma\ )$
⟨*proof*⟩

lemma *G-cons-flip-fresh2*[*simp*]:
fixes $x::x$ **and** $\Gamma::\Gamma$
assumes *atom* $x \sharp (c,\Gamma)$ **and** *atom* $x' \sharp (c,\Gamma)$
shows $(x \leftrightarrow x') \cdot ((x,b,c)\ \#_\Gamma\ \Gamma) = ((x',\ \ b\ ,\ c)\ \#_\Gamma\ \Gamma\ )$
⟨*proof*⟩

lemma *G-cons-flip-fresh3*[*simp*]:

fixes $x::x$ **and** $\Gamma::\Gamma$
assumes *atom* $x \sharp \Gamma$ **and** *atom* $x' \sharp \Gamma$
shows $(x \leftrightarrow x') \cdot ((x',b,c) \#_\Gamma \Gamma) = ((x,\ \ b\ ,(x \leftrightarrow x') \cdot c) \#_\Gamma \Gamma\ )$
⟨*proof*⟩

lemma *neq-GNil-conv*: $(xs \neq GNil) = (\exists\, y\ ys.\ xs = y \#_\Gamma ys)$
⟨*proof*⟩

nominal-function *toList* :: $\Gamma \Rightarrow (x*b*c)$ *list* **where**
toList GNil = []
| *toList* (*GCons xbc G*) = *xbc*#(*toList G*)
⟨*proof*⟩
nominal-termination (*eqvt*)
⟨*proof*⟩

nominal-function *toSet* :: $\Gamma \Rightarrow (x*b*c)$ *set* **where**
toSet GNil = {}
| *toSet* (*GCons xbc G*) = {*xbc*} ∪ (*toSet G*)
⟨*proof*⟩
nominal-termination (*eqvt*)
⟨*proof*⟩

nominal-function *append-g* :: $\Gamma \Rightarrow \Gamma \Rightarrow \Gamma$ (**infixr** ‹@› *65*) **where**
append-g GNil g = *g*
| *append-g* (*xbc* $\#_\Gamma$ *g1*) *g2* = (*xbc* $\#_\Gamma$ (*g1*@*g2*))
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

nominal-function *dom* :: $\Gamma \Rightarrow x$ *set* **where**
dom Γ = (*fst*‘ (*toSet* Γ))
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

Use of this is sometimes mixed in with use of freshness and support for the context however it makes it clear that for immutable variables, the context is ‘self-supporting’

nominal-function *atom-dom* :: $\Gamma \Rightarrow$ *atom set* **where**
atom-dom Γ = *atom*‘(*dom* Γ)
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

2.2.3 Immutable Variable Context Lemmas

lemma *append-GNil*[*simp*]:
GNil @ *G* = *G*
⟨*proof*⟩

lemma *append-g-toSetU* [*simp*]: *toSet* (*G1*@*G2*) = *toSet G1* ∪ *toSet G2*
⟨*proof*⟩

lemma *supp-GNil*:
shows *supp GNil* = {}
⟨*proof*⟩

lemma *supp-GCons*:
fixes *xs*::Γ
shows *supp* (*x* #Γ *xs*) = *supp x* ∪ *supp xs*
⟨*proof*⟩

lemma *atom-dom-eq*[*simp*]:
fixes *G*::Γ
shows *atom-dom* ((*x*, *b*, *c*) #Γ *G*) = *atom-dom* ((*x*, *b*, *c*′) #Γ *G*)
⟨*proof*⟩

lemma *dom-append*[*simp*]:
atom-dom (Γ@Γ′) = *atom-dom* Γ ∪ *atom-dom* Γ′
⟨*proof*⟩

lemma *dom-cons*[*simp*]:
atom-dom ((*x*,*b*,*c*) #Γ *G*) = { *atom x* } ∪ *atom-dom G*
⟨*proof*⟩

lemma *fresh-GNil*[*ms-fresh*]:
shows *a* ♯ *GNil*
⟨*proof*⟩

lemma *fresh-GCons*[*ms-fresh*]:
fixes *xs*::Γ
shows *a* ♯ (*x* #Γ *xs*) ⟷ *a* ♯ *x* ∧ *a* ♯ *xs*
⟨*proof*⟩

lemma *dom-supp-g*[*simp*]:
atom-dom G ⊆ *supp G*
⟨*proof*⟩

lemma *fresh-append-g*[*ms-fresh*]:
fixes *xs*::Γ
shows *a* ♯ (*xs* @ *ys*) ⟷ *a* ♯ *xs* ∧ *a* ♯ *ys*
⟨*proof*⟩

lemma *append-g-assoc*:
fixes *xs*::Γ
shows (*xs* @ *ys*) @ *zs* = *xs* @ (*ys* @ *zs*)
⟨*proof*⟩

lemma *append-g-inside*:
fixes *xs*::Γ
shows *xs* @ (*x* #Γ *ys*) = (*xs* @ (*x* #Γ *GNil*)) @ *ys*
⟨*proof*⟩

lemma *finite*-Γ:
finite (*toSet* Γ)
⟨*proof*⟩

lemma *supp*-Γ:

supp Γ = *supp* (*toSet* Γ)
⟨*proof*⟩

lemma *supp-of-subset*:
fixes *G*::('*a*::*fs set*)
assumes *finite G* **and** *finite G'* **and** *G* ⊆ *G'*
shows *supp G* ⊆ *supp G'*
⟨*proof*⟩

lemma *supp-weakening*:
assumes *toSet G* ⊆ *toSet G'*
shows *supp G* ⊆ *supp G'*
⟨*proof*⟩

lemma *fresh-weakening*[*ms-fresh*]:
assumes *toSet G* ⊆ *toSet G'* **and** *x* ♯ *G'*
shows *x* ♯ *G*
⟨*proof*⟩

instance Γ :: *fs*
⟨*proof*⟩

lemma *fresh-gamma-elem*:
fixes Γ::Γ
assumes *a* ♯ Γ
and *e* ∈ *toSet* Γ
shows *a* ♯ *e*
⟨*proof*⟩

lemma *fresh-gamma-append*:
fixes *xs*::Γ
shows *a* ♯ (*xs* @ *ys*) ⟷ *a* ♯ *xs* ∧ *a* ♯ *ys*
⟨*proof*⟩

lemma *supp-triple*[*simp*]:
shows *supp* (*x*, *y*, *z*) = *supp x* ∪ *supp y* ∪ *supp z*
⟨*proof*⟩

lemma *supp-append-g*:
fixes *xs*::Γ
shows *supp* (*xs* @ *ys*) = *supp xs* ∪ *supp ys*
⟨*proof*⟩

lemma *fresh-in-g*[*simp*]:
fixes Γ::Γ **and** *x'*::*x*
shows *atom x'* ♯ Γ' @ (*x*, *b0*, *c0*) #Γ Γ = (*atom x'* ∉ *supp* Γ' ∪ *supp x* ∪ *supp b0* ∪ *supp c0* ∪ *supp* Γ)
⟨*proof*⟩

lemma *fresh-suffix*[*ms-fresh*]:
fixes Γ::Γ
assumes *atom x* ♯ Γ'@Γ

shows *atom x* ♯ Γ
⟨*proof*⟩

lemma *not-GCons-self* [*simp*]:
fixes *xs*::Γ
shows *xs* ≠ *x* #Γ *xs*
⟨*proof*⟩

lemma *not-GCons-self2* [*simp*]:
fixes *xs*::Γ
shows *x* #Γ *xs* ≠ *xs*
⟨*proof*⟩

lemma *fresh-restrict*:
fixes *y*::*x* **and** Γ::Γ
assumes *atom y* ♯ (Γ′ @ (*x*, *b*, *c*) #Γ Γ)
shows *atom y* ♯ (Γ′@Γ)
⟨*proof*⟩

lemma *fresh-dom-free*:
assumes *atom x* ♯ Γ
shows (*x*,*b*,*c*) ∉ *toSet* Γ
⟨*proof*⟩

lemma Γ-*set-intros*: *x* ∈ *toSet* (*x* #Γ *xs*) **and** *y* ∈ *toSet xs* ⟹ *y* ∈ *toSet* (*x* #Γ *xs*)
⟨*proof*⟩

lemma *fresh-dom-free2*:
assumes *atom x* ∉ *atom-dom* Γ
shows (*x*,*b*,*c*) ∉ *toSet* Γ
⟨*proof*⟩

2.2.4 Mutable Variable Context Lemmas

lemma *supp-DNil*:
shows *supp DNil* = {}
⟨*proof*⟩

lemma *supp-DCons*:
fixes *xs*::Δ
shows *supp* (*x* #Δ *xs*) = *supp x* ∪ *supp xs*
⟨*proof*⟩

lemma *fresh-DNil*[*ms-fresh*]:
shows *a* ♯ *DNil*
⟨*proof*⟩

lemma *fresh-DCons*[*ms-fresh*]:
fixes *xs*::Δ
shows *a* ♯ (*x* #Δ *xs*) ⟷ *a* ♯ *x* ∧ *a* ♯ *xs*
⟨*proof*⟩

instance Δ :: *fs*

⟨*proof*⟩

2.2.5 Lookup Functions

nominal-function *lookup* :: Γ ⇒ *x* ⇒ (*b*∗*c*) *option* **where**
lookup GNil x = *None*
| *lookup* ((*x*,*b*,*c*)#Γ *G*) *y* = (*if x*=*y then Some* (*b*,*c*) *else lookup G y*)
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

nominal-function *replace-in-g* :: Γ ⇒ *x* ⇒ *c* ⇒ Γ (‹-[-⟼-]› [*1000*,*0*,*0*] *200*) **where**
replace-in-g GNil - - = GNil
| *replace-in-g* ((*x*,*b*,*c*)#Γ *G*) *x′ c′* = (*if x*=*x′ then* ((*x*,*b*,*c′*)#Γ *G*) *else* (*x*,*b*,*c*)#Γ(*replace-in-g G x′ c′*))
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

Functions for looking up data-constructors in the Pi context

nominal-function *lookup-fun* :: Φ ⇒ *f* ⇒ *fun-def option* **where**
lookup-fun [] *g* = *None*
| *lookup-fun* ((*AF-fundef f ft*)#Π) *g* = (*if* (*f*=*g*) *then Some* (*AF-fundef f ft*) *else lookup-fun* Π *g*)
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

nominal-function *lookup-td* :: Θ ⇒ *string* ⇒ *type-def option* **where**
lookup-td [] *g* = *None*
| *lookup-td* ((*AF-typedef s lst*) # (Θ::Θ)) *g* = (*if* (*s* = *g*) *then Some* (*AF-typedef s lst*) *else lookup-td* Θ *g*)
| *lookup-td* ((*AF-typedef-poly s bv lst*) # (Θ::Θ)) *g* = (*if* (*s* = *g*) *then Some* (*AF-typedef-poly s bv lst*) *else lookup-td* Θ *g*)
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

nominal-function *name-of-type* ::*type-def* ⇒ *f* **where**
name-of-type (*AF-typedef f* -) = *f*
| *name-of-type* (*AF-typedef-poly f* - -) = *f*
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

nominal-function *name-of-fun* ::*fun-def* ⇒ *f* **where**
name-of-fun (*AF-fundef f ft*) = *f*
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

nominal-function *remove2* :: *'a*::*pt* ⇒ *'a list* ⇒ *'a list* **where**
remove2 x [] = [] |
remove2 x (*y* # *xs*) = (*if x* = *y then xs else y* # *remove2 x xs*)
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

nominal-function *base-for-lit* :: *l* ⇒ *b* **where**
base-for-lit (*L-true*) = *B-bool*
| *base-for-lit* (*L-false*) = *B-bool*
| *base-for-lit* (*L-num n*) = *B-int*

| *base-for-lit* (*L-unit*) = *B-unit*
| *base-for-lit* (*L-bitvec v*) = *B-bitvec*
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

lemma *neq-DNil-conv*: (*xs* ≠ *DNil*) = (∃ *y ys*. *xs* = *y* #$_\Delta$ *ys*)
⟨*proof*⟩

nominal-function *setD* :: Δ ⇒ (*u*∗τ) *set* **where**
setD DNil = {}
| *setD* (*DCons xbc G*) = {*xbc*} ∪ (*setD G*)
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

lemma *eqvt-triple*:
fixes *y*::′*a*::*at* **and** *ya*::′*a*::*at* **and** *xa*::′*c*::*at* **and** *va*::′*d*::*fs* **and** *s*::*s* **and** *sa*::*s* **and** *f*::*s*∗′*c*∗′*d* ⇒ *s*
assumes *atom y* ♯ (*xa*, *va*) **and** *atom ya* ♯ (*xa*, *va*) **and**
∀ *c*. *atom c* ♯ (*s*, *sa*) ⟶ *atom c* ♯ (*y*, *ya*, *s*, *sa*) ⟶ (*y* ↔ *c*) ⋅ *s* = (*ya* ↔ *c*) ⋅ *sa*
and *eqvt-at f* (*s*,*xa*,*va*) **and** *eqvt-at f* (*sa*,*xa*,*va*) **and**
atom c ♯ (*s*, *va*, *xa*, *sa*) **and** *atom c* ♯ (*y*, *ya*, *f* (*s*, *xa*, *va*), *f* (*sa*, *xa*, *va*))
shows (*y* ↔ *c*) ⋅ *f* (*s*, *xa*, *va*) = (*ya* ↔ *c*) ⋅ *f* (*sa*, *xa*, *va*)
⟨*proof*⟩

2.3 Functions for bit list/vectors

inductive *split* :: *int* ⇒ *bit list* ⇒ *bit list* ∗ *bit list* ⇒ *bool* **where**
split 0 xs ([], *xs*)
| *split m xs* (*ys*,*zs*) ⟹ *split* (*m*+1) (*x*#*xs*) ((*x* # *ys*), *zs*)
equivariance *split*
nominal-inductive *split* ⟨*proof*⟩

lemma *split-concat*:
assumes *split n v* (*v1*,*v2*)
shows *v* = *append v1 v2*
⟨*proof*⟩

lemma *split-n*:
assumes *split n v* (*v1*,*v2*)
shows *0* ≤ *n* ∧ *n* ≤ *int* (*length v*)
⟨*proof*⟩

lemma *split-length*:
assumes *split n v* (*v1*,*v2*)
shows *n* = *int* (*length v1*)
⟨*proof*⟩

lemma *obtain-split*:
assumes *0* ≤ *n* **and** *n* ≤ *int* (*length bv*)
shows ∃ *bv1 bv2*. *split n bv* (*bv1* , *bv2*)
⟨*proof*⟩

end

Chapter 3

Immutable Variable Substitution

Substitution involving immutable variables. We define a class and instances for all of the term forms

3.1 Class

class *has-subst-v* = *fs* +
  fixes *subst-v* :: *'a*::*fs* ⇒ *x* ⇒ *v* ⇒ *'a*::*fs* (‹-[-::=-]$_v$› [*1000*,*50*,*50*] *1000*)
  assumes *fresh-subst-v-if*: *y* ♯ (*subst-v a x v*) ⟷ (*atom x* ♯ *a* ∧ *y* ♯ *a*) ∨ (*y* ♯ *v* ∧ (*y* ♯ *a* ∨ *y* = *atom x*))
    and *forget-subst-v*[*simp*]: *atom x* ♯ *a* ⟹ *subst-v a* *x v* = *a*
    and *subst-v-id*[*simp*]: *subst-v a x* (*V-var x*) = *a*
    and *eqvt*[*simp*,*eqvt*]: (*p*::*perm*) ∙ (*subst-v a x v*) = (*subst-v* (*p* ∙ *a*) (*p* ∙*x*) (*p* ∙*v*))
    and *flip-subst-v*[*simp*]: *atom x* ♯ *c* ⟹ ((*x* ↔ *z*) ∙ *c*) = *c*[*z*::=[*x*]^v]$_v$
    and *subst-v-simple-commute*[*simp*]: *atom x* ♯ *c* ⟹(*c*[*z*::=[*x*]^v]$_v$)[*x*::=*b*]$_v$ = *c*[*z*::=*b*]$_v$
begin

lemma *subst-v-flip-eq-one*:
  fixes *z1*::*x* **and** *z2*::*x* **and** *x1*::*x* **and** *x2*::*x*
  assumes [[*atom z1*]]*lst. c1* = [[*atom z2*]]*lst. c2*
    and *atom x1* ♯ (*z1*,*z2*,*c1*,*c2*)
  shows (*c1*[*z1*::=[*x1*]^v]$_v$) = (*c2*[*z2*::=[*x1*]^v]$_v$)
⟨*proof*⟩

lemma *subst-v-flip-eq-two*:
  fixes *z1*::*x* **and** *z2*::*x* **and** *x1*::*x* **and** *x2*::*x*
  assumes [[*atom z1*]]*lst. c1* = [[*atom z2*]]*lst. c2*
  shows (*c1*[*z1*::=*b*]$_v$) = (*c2*[*z2*::=*b*]$_v$)
⟨*proof*⟩

lemma *subst-v-flip-eq-three*:
  assumes [[*atom z1*]]*lst. c1* = [[*atom z1'*]]*lst. c1'* **and** *atom x* ♯ *c1* **and** *atom x'* ♯ (*x*,*z1*,*z1'*, *c1*, *c1'*)
  shows (*x* ↔ *x'*) ∙ (*c1*[*z1*::=[*x*]^v]$_v$) = *c1'*[*z1'*::=[*x'*]^v]$_v$
⟨*proof*⟩

end

3.2 Values

nominal-function
subst-vv :: $v \Rightarrow x \Rightarrow v \Rightarrow v$ **where**
subst-vv (*V-lit l*) *x v* = *V-lit l*
| *subst-vv* (*V-var y*) *x v* = (*if x* = *y then v else V-var y*)
| *subst-vv* (*V-cons tyid c v'*) *x v* = *V-cons tyid c* (*subst-vv v' x v*)
| *subst-vv* (*V-consp tyid c b v'*) *x v* = *V-consp tyid c b* (*subst-vv v' x v*)
| *subst-vv* (*V-pair v1 v2*) *x v* = *V-pair* (*subst-vv v1 x v*) (*subst-vv v2 x v*)
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

abbreviation
subst-vv-abbrev :: $v \Rightarrow x \Rightarrow v \Rightarrow v$ ($\langle$-[-::=-]$_{vv}\rangle$ [*1000*,*50*,*50*] *1000*)
where
$v[x::=v']_{vv}$ ≡ *subst-vv v x v'*

lemma *fresh-subst-vv-if* [*simp*]:
$j \sharp t[i::=x]_{vv}$ = ((*atom i* ♯ *t* ∧ *j* ♯ *t*) ∨ (*j* ♯ *x* ∧ (*j* ♯ *t* ∨ *j* = *atom i*)))
⟨*proof*⟩

lemma *forget-subst-vv* [*simp*]: *atom a* ♯ *tm* ⟹ $tm[a::=x]_{vv} = tm$
⟨*proof*⟩

lemma *subst-vv-id* [*simp*]: $tm[a::=V\text{-}var\ a]_{vv} = tm$
⟨*proof*⟩

lemma *subst-vv-commute* [*simp*]:
atom j ♯ *tm* ⟹ $tm[i::=t]_{vv}[j::=u]_{vv} = tm[i::=t[j::=u]_{vv}]_{vv}$
⟨*proof*⟩

lemma *subst-vv-commute-full* [*simp*]:
atom j ♯ *t* ⟹ *atom i* ♯ *u* ⟹ *i* ≠ *j* ⟹ $tm[i::=t]_{vv}[j::=u]_{vv} = tm[j::=u]_{vv}[i::=t]_{vv}$
⟨*proof*⟩

lemma *subst-vv-var-flip*[*simp*]:
fixes *v*::*v*
assumes *atom y* ♯ *v*
shows (*y* ↔ *x*) ∙ *v* = *v* $[x::=V\text{-}var\ y]_{vv}$
⟨*proof*⟩

instantiation *v* :: *has-subst-v*
begin

definition
subst-v = *subst-vv*

instance ⟨*proof*⟩

end

3.3 Expressions

nominal-function *subst-ev* :: $e \Rightarrow x \Rightarrow v \Rightarrow e$ **where**
  subst-ev ((*AE-val v'*)) *x v* = ((*AE-val* (*subst-vv v' x v*)))
| *subst-ev* ((*AE-app f v'*)) *x v* = ((*AE-app f* (*subst-vv v' x v*)))
| *subst-ev* ((*AE-appP f b v'*)) *x v* = ((*AE-appP f b* (*subst-vv v' x v*)))
| *subst-ev* ((*AE-op opp v1 v2*)) *x v* = ((*AE-op opp* (*subst-vv v1 x v*) (*subst-vv v2 x v*)))
| *subst-ev* $[\#1\ v']^e$ *x v* = $[\#1$ (*subst-vv v' x v*)$]^e$
| *subst-ev* $[\#2\ v']^e$ *x v* = $[\#2$ (*subst-vv v' x v*)$]^e$
| *subst-ev* ((*AE-mvar u*)) *x v* = *AE-mvar u*
| *subst-ev* $[|\ v'\ |]^e$ *x v* = $[|$ (*subst-vv v' x v*) $|]^e$
| *subst-ev* (*AE-concat v1 v2*) *x v* = *AE-concat* (*subst-vv v1 x v*) (*subst-vv v2 x v*)
| *subst-ev* (*AE-split v1 v2*) *x v* = *AE-split* (*subst-vv v1 x v*) (*subst-vv v2 x v*)
  ⟨*proof*⟩

nominal-termination (*eqvt*) ⟨*proof*⟩

abbreviation
  subst-ev-abbrev :: $e \Rightarrow x \Rightarrow v \Rightarrow e$ ($\langle$-[-::=-]$_{ev}\rangle$ [*1000*,*50*,*50*] *500*)
  where
    $e[x::=v']_{ev}$ ≡ *subst-ev e x v'*

lemma *size-subst-ev* [*simp*]: *size* (*subst-ev A i x*) = *size A*
  ⟨*proof*⟩

lemma *forget-subst-ev* [*simp*]: *atom a* ♯ *A* ⟹ *subst-ev A a x* = *A*
  ⟨*proof*⟩

lemma *subst-ev-id* [*simp*]: *subst-ev A a* (*V-var a*) = *A*
  ⟨*proof*⟩

lemma *fresh-subst-ev-if* [*simp*]:
  j ♯ (*subst-ev A i x*) = ((*atom i* ♯ *A* ∧ *j* ♯ *A*) ∨ (*j* ♯ *x* ∧ (*j* ♯ *A* ∨ *j* = *atom i*)))
  ⟨*proof*⟩

lemma *subst-ev-commute* [*simp*]:
  atom j ♯ *A* ⟹ $(A[i::=t]_{ev})[j::=u]_{ev} = A[i::=t[j::=u]_{vv}]_{ev}$
  ⟨*proof*⟩

lemma *subst-ev-var-flip*[*simp*]:
  fixes *e*::*e* **and** *y*::*x* **and** *x*::*x*
  assumes *atom y* ♯ *e*
  shows (*y* ↔ *x*) ∙ *e* = *e* $[x::=$*V-var y*$]_{ev}$
  ⟨*proof*⟩

lemma *subst-ev-flip*:
  fixes *e*::*e* **and** *ea*::*e* **and** *c*::*x*
  assumes *atom c* ♯ (*e*, *ea*) **and** *atom c* ♯ (*x*, *xa*, *e*, *ea*) **and** (*x* ↔ *c*) ∙ *e* = (*xa* ↔ *c*) ∙ *ea*
  shows $e[x::=v']_{ev} = ea[xa::=v']_{ev}$
⟨*proof*⟩

lemma *subst-ev-var*[*simp*]:
  (*AE-val* (*V-var x*))$[x::=[z]^v]_{ev}$ = *AE-val* (*V-var z*)

⟨*proof*⟩

instantiation e :: *has-subst-v*
begin

definition
subst-v = *subst-ev*

instance ⟨*proof*⟩
end

lemma *subst-ev-commute-full*:
fixes e::e **and** w::v **and** v::v
assumes *atom z* ♯ *v* **and** *atom x* ♯ *w* **and** $x \neq z$
shows *subst-ev* $(e[z::=w]_{ev})$ *x v* = *subst-ev* $(e[x::=v]_{ev})$ *z w*
⟨*proof*⟩

lemma *subst-ev-v-flip1*[*simp*]:
fixes e::e
assumes *atom z1* ♯ (z,e) **and** *atom z1'* ♯ (z,e)
shows$(z1 \leftrightarrow z1') \cdot e[z::=v]_{ev} = e[z::= ((z1 \leftrightarrow z1') \cdot v)]_{ev}$
⟨*proof*⟩

3.4 Expressions in Constraints

nominal-function *subst-cev* :: *ce* ⇒ *x* ⇒ *v* ⇒ *ce* **where**
subst-cev ((*CE-val v'*)) *x v* = ((*CE-val* (*subst-vv v' x v*)))
| *subst-cev* ((*CE-op opp v1 v2*)) *x v* = ((*CE-op opp* (*subst-cev v1 x v*) (*subst-cev v2 x v*)))
| *subst-cev* ((*CE-fst v'*)) *x v* = *CE-fst* (*subst-cev v' x v*)
| *subst-cev* ((*CE-snd v'*)) *x v* = *CE-snd* (*subst-cev v' x v*)
| *subst-cev* ((*CE-len v'*)) *x v* = *CE-len* (*subst-cev v' x v*)
| *subst-cev* (*CE-concat v1 v2*) *x v* = *CE-concat* (*subst-cev v1 x v*) (*subst-cev v2 x v*)
⟨*proof*⟩

nominal-termination (*eqvt*) ⟨*proof*⟩

abbreviation
subst-cev-abbrev :: *ce* ⇒ *x* ⇒ *v* ⇒ *ce* (‹-[-::=-]$_{cev}$› [*1000*,*50*,*50*] *500*)
where
$e[x::=v']_{cev} \equiv$ *subst-cev e x v'*

lemma *size-subst-cev* [*simp*]: *size* (*subst-cev A i x*) = *size A*
⟨*proof*⟩

lemma *forget-subst-cev* [*simp*]: *atom a* ♯ *A* ⟹ *subst-cev A a x* = *A*
⟨*proof*⟩

lemma *subst-cev-id* [*simp*]: *subst-cev A a* (*V-var a*) = *A*
⟨*proof*⟩

lemma *fresh-subst-cev-if* [*simp*]:
j ♯ (*subst-cev A i x*) = ((*atom i* ♯ *A* ∧ *j* ♯ *A*) ∨ (*j* ♯ *x* ∧ (*j* ♯ *A* ∨ *j* = *atom i*)))

⟨*proof*⟩

lemma *subst-cev-commute* [*simp*]:
  atom j ♯ *A* ⟹ (*subst-cev* (*subst-cev A i t*) *j u*) = *subst-cev A i* (*subst-vv t j u*)
  ⟨*proof*⟩

lemma *subst-cev-var-flip*[*simp*]:
  fixes *e*::*ce* **and** *y*::*x* **and** *x*::*x*
  assumes *atom y* ♯ *e*
  shows (*y* ↔ *x*) ∙ *e* = *e* [*x*::=*V-var y*]$_{cev}$
  ⟨*proof*⟩

lemma *subst-cev-flip*:
  fixes *e*::*ce* **and** *ea*::*ce* **and** *c*::*x*
  assumes *atom c* ♯ (*e*, *ea*) **and** *atom c* ♯ (*x*, *xa*, *e*, *ea*) **and** (*x* ↔ *c*) ∙ *e* = (*xa* ↔ *c*) ∙ *ea*
  shows *e*[*x*::=*v'*]$_{cev}$ = *ea*[*xa*::=*v'*]$_{cev}$
⟨*proof*⟩

lemma *subst-cev-var*[*simp*]:
  fixes *z*::*x* **and** *x*::*x*
  shows [[*x*]^{v}]^{ce} [*x*::=[*z*]^{v}]$_{cev}$ = [[*z*]^{v}]^{ce}
  ⟨*proof*⟩

instantiation *ce* :: *has-subst-v*
begin

definition
  subst-v = *subst-cev*

instance ⟨*proof*⟩

end

lemma *subst-cev-commute-full*:
  fixes *e*::*ce* **and** *w*::*v* **and** *v*::*v*
  assumes *atom z* ♯ *v* **and** *atom x* ♯ *w* **and** *x* ≠ *z*
  shows *subst-cev* (*e*[*z*::=*w*]$_{cev}$) *x v* = *subst-cev* (*e*[*x*::=*v*]$_{cev}$) *z w*
  ⟨*proof*⟩

lemma *subst-cev-v-flip1*[*simp*]:
  fixes *e*::*ce*
  assumes *atom z1* ♯ (*z*,*e*) **and** *atom z1'* ♯ (*z*,*e*)
  shows(*z1* ↔ *z1'*) ∙ *e*[*z*::=*v*]$_{cev}$ = *e*[*z*::= ((*z1* ↔ *z1'*) ∙ *v*)]$_{cev}$
  ⟨*proof*⟩

3.5 Constraints

nominal-function *subst-cv* :: *c* ⇒ *x* ⇒ *v* ⇒ *c* **where**
  subst-cv (*C-true*) *x v* = *C-true*
| *subst-cv* (*C-false*) *x v* = *C-false*
| *subst-cv* (*C-conj c1 c2*) *x v* = *C-conj* (*subst-cv c1 x v*) (*subst-cv c2 x v*)

| *subst-cv* (*C-disj c1 c2*) *x v* = *C-disj* (*subst-cv c1 x v*) (*subst-cv c2 x v*)
| *subst-cv* (*C-imp c1 c2*) *x v* = *C-imp* (*subst-cv c1 x v*) (*subst-cv c2 x v*)
| *subst-cv* (*e1* == *e2*) *x v* = ((*subst-cev e1 x v*) == (*subst-cev e2 x v*))
| *subst-cv* (*C-not c*) *x v* = *C-not* (*subst-cv c x v*)
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

abbreviation
subst-cv-abbrev :: *c* ⇒ *x* ⇒ *v* ⇒ *c* (‹-[-::=-]$_{cv}$› [*1000*,*50*,*50*] *1000*)
where
c[*x*::=*v'*]$_{cv}$ ≡ *subst-cv c x v'*

lemma *size-subst-cv* [*simp*]: *size* (*subst-cv A i x*) = *size A*
⟨*proof*⟩

lemma *forget-subst-cv* [*simp*]: *atom a* ♯ *A* ⟹ *subst-cv A a x* = *A*
⟨*proof*⟩

lemma *subst-cv-id* [*simp*]: *subst-cv A a* (*V-var a*) = *A*
⟨*proof*⟩

lemma *fresh-subst-cv-if* [*simp*]:
j ♯ (*subst-cv A i x*) ⟷ (*atom i* ♯ *A* ∧ *j* ♯ *A*) ∨ (*j* ♯ *x* ∧ (*j* ♯ *A* ∨ *j* = *atom i*))
⟨*proof*⟩

lemma *subst-cv-commute* [*simp*]:
atom j ♯ *A* ⟹ (*subst-cv* (*subst-cv A i t*) *j u*) = *subst-cv A i* (*subst-vv t j u*)
⟨*proof*⟩

lemma *let-s-size* [*simp*]: *size s* ≤ *size* (*AS-let x e s*)
⟨*proof*⟩

lemma *subst-cv-var-flip*[*simp*]:
fixes *c*::*c*
assumes *atom y* ♯ *c*
shows (*y* ↔ *x*) ∙ *c* = *c*[*x*::=*V-var y*]$_{cv}$
⟨*proof*⟩

instantiation *c* :: *has-subst-v*
begin

definition
subst-v = *subst-cv*

instance ⟨*proof*⟩

end

lemma *subst-cv-var-flip1*[*simp*]:
fixes *c*::*c*
assumes *atom y* ♯ *c*
shows (*x* ↔ *y*) ∙ *c* = *c*[*x*::=*V-var y*]$_{cv}$

⟨*proof*⟩

lemma *subst-cv-v-flip3*[*simp*]:
fixes *c*::*c*
assumes *atom z1* ♯ *c* **and** *atom z1'* ♯ *c*
shows(*z1* ↔ *z1'*) ∙ *c*[*z*::=[*z1*]^v]$_{cv}$ = *c*[*z*::=[*z1'*]^v]$_{cv}$
⟨*proof*⟩

lemma *subst-cv-v-flip*[*simp*]:
fixes *c*::*c*
assumes *atom x* ♯ *c*
shows ((*x* ↔ *z*) ∙ *c*)[*x*::=*v*]$_{cv}$ = *c* [*z*::=*v*]$_{cv}$
⟨*proof*⟩

lemma *subst-cv-commute-full*:
fixes *c*::*c*
assumes *atom z* ♯ *v* **and** *atom x* ♯ *w* **and** *x*≠*z*
shows (*c*[*z*::=*w*]$_{cv}$)[*x*::=*v*]$_{cv}$ = (*c*[*x*::=*v*]$_{cv}$)[*z*::=*w*]$_{cv}$
⟨*proof*⟩

lemma *subst-cv-eq*[*simp*]:
assumes *atom z1* ♯ *e1*
shows (*CE-val* (*V-var z1*) == *e1*)[*z1*::=[*x*]^v]$_{cv}$ = (*CE-val* (*V-var x*) == *e1*) (**is** *?A* = *?B*)
⟨*proof*⟩

3.6 Variable Context

The idea of this substitution is to remove x from the context. We really want to add the condition that x is fresh in v but this causes problems with proofs.

nominal-function *subst-gv* :: Γ ⇒ *x* ⇒ *v* ⇒ Γ **where**
subst-gv GNil x v = *GNil*
| *subst-gv* ((*y*,*b*,*c*) #$_\Gamma$ Γ) *x v* = (*if x* = *y then* Γ *else* ((*y*,*b*,*c*[*x*::=*v*]$_{cv}$)#$_\Gamma$ (*subst-gv* Γ *x v*)))
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

abbreviation
subst-gv-abbrev :: Γ ⇒ *x* ⇒ *v* ⇒ Γ (‹-[-::=-]$_{\Gamma v}$› [*1000*,*50*,*50*] *1000*)
where
g[*x*::=*v*]$_{\Gamma v}$ ≡ *subst-gv g x v*

lemma *size-subst-gv* [*simp*]: *size* (*subst-gv G i x*) ≤ *size G*
⟨*proof*⟩

lemma *forget-subst-gv* [*simp*]: *atom a* ♯ *G* ⟹ *subst-gv G a x* = *G*
⟨*proof*⟩

lemma *fresh-subst-gv*: *atom a* ♯ *G* ⟹ *atom a* ♯ *v* ⟹ *atom a* ♯ *subst-gv G x v*
⟨*proof*⟩

lemma *subst-gv-flip*:
fixes *x*::*x* **and** *xa*::*x* **and** *z*::*x* **and** *c*::*c* **and** *b*::*b* **and** Γ::Γ

assumes *atom xa* $\sharp$ $((x,\ b,\ c[z::=[x]^v]_{cv})\ \#_\Gamma\ \Gamma)$ **and** *atom xa* $\sharp$ Γ **and** *atom x* $\sharp$ Γ **and** *atom x* $\sharp$ $(z,\ c)$ **and** *atom xa* $\sharp$ $(z,\ c)$
shows $(x \leftrightarrow xa) \cdot\ ((x,\ b,\ c[z::=[x]^v]_{cv})\ \#_\Gamma\ \Gamma) = (xa,\ b,\ c[z::=V\text{-}var\ xa]_{cv})\ \#_\Gamma\ \Gamma$
⟨*proof*⟩

3.7 Types

nominal-function *subst-tv* :: $\tau \Rightarrow x \Rightarrow v \Rightarrow \tau$ **where**
atom z $\sharp$ $(x,v) \Longrightarrow$ *subst-tv* $\{\!|\ z : b \mid c\ |\!\}\ x\ v\ = \{\!|\ z : b \mid c[x::=v]_{cv}\ |\!\}$
⟨*proof*⟩

nominal-termination (*eqvt*) ⟨*proof*⟩

abbreviation
subst-tv-abbrev :: $\tau \Rightarrow x \Rightarrow v \Rightarrow \tau$ ($\langle$-[-::=-]$_{\tau v}\rangle$ [*1000*,*50*,*50*] *1000*)
where
$t[x::=v]_{\tau v}\ \equiv$ *subst-tv t x v*

lemma *size-subst-tv* [*simp*]: *size* (*subst-tv A i x*) = *size A*
⟨*proof*⟩

lemma *forget-subst-tv* [*simp*]: *atom a* $\sharp$ *A* $\Longrightarrow$ *subst-tv A a x* = *A*
⟨*proof*⟩

lemma *subst-tv-id* [*simp*]: *subst-tv A a* (*V-var a*) = *A*
⟨*proof*⟩

lemma *fresh-subst-tv-if* [*simp*]:
j $\sharp$ (*subst-tv A i x*) $\longleftrightarrow$ (*atom i* $\sharp$ *A* $\wedge$ *j* $\sharp$ *A*) $\vee$ (*j* $\sharp$ *x* $\wedge$ (*j* $\sharp$ *A* $\vee$ *j* = *atom i*))
⟨*proof*⟩

lemma *subst-tv-commute* [*simp*]:
atom y $\sharp$ $\tau \Longrightarrow (\tau[x::=\ t]_{\tau v})[y::=v]_{\tau v} = \tau[x::=\ t[y::=v]_{vv}]_{\tau v}$
⟨*proof*⟩

lemma *subst-tv-var-flip* [*simp*]:
fixes *x*::*x* **and** *xa*::*x* **and** τ::τ
assumes *atom xa* $\sharp$ τ
shows $(x \leftrightarrow xa) \cdot \tau = \tau[x::=V\text{-}var\ xa]_{\tau v}$
⟨*proof*⟩

instantiation τ :: *has-subst-v*
begin

definition
subst-v = *subst-tv*

instance ⟨*proof*⟩

end

lemma *subst-tv-commute-full*:

fixes $c::\tau$
assumes $atom\ z \sharp v$ **and** $atom\ x \sharp w$ **and** $x \neq z$
shows $(c[z::=w]_{\tau v})[x::=v]_{\tau v} = (c[x::=v]_{\tau v})[z::=w]_{\tau v}$
⟨*proof*⟩

lemma *type-eq-subst-eq*:
fixes $v::v$ **and** $c1::c$
assumes $\{\!|\ z1 : b1 \mid c1\ |\!\} = \{\!|\ z2 : b2 \mid c2\ |\!\}$
shows $c1[z1::=v]_{cv} = c2[z2::=v]_{cv}$
⟨*proof*⟩

Extract constraint from a type. We cannot just project out the constraint as this would mean alpha-equivalent types give different answers

nominal-function *c-of* :: $\tau \Rightarrow x \Rightarrow c$ **where**
$atom\ z \sharp x \Longrightarrow$ *c-of* (*T-refined-type z b c*) $x = c[z::=[x]^v]_{cv}$
⟨*proof*⟩

nominal-termination (*eqvt*) ⟨*proof*⟩

lemma *c-of-eq*:
shows *c-of* $\{\!|\ x : b \mid c\ |\!\}\ x = c$
⟨*proof*⟩

lemma *obtain-fresh-z-c-of*:
fixes $t::'b::fs$
obtains z **where** $atom\ z \sharp t \wedge \tau = \{\!|\ z : b\text{-}of\ \tau \mid c\text{-}of\ \tau\ z\ |\!\}$
⟨*proof*⟩

lemma *c-of-fresh*:
fixes $x::x$
assumes $atom\ x \sharp (t,z)$
shows $atom\ x \sharp c\text{-}of\ t\ z$
⟨*proof*⟩

lemma *c-of-switch*:
fixes $z::x$
assumes $atom\ z \sharp t$
shows $(c\text{-}of\ t\ z)[z::=V\text{-}var\ x]_{cv} = c\text{-}of\ t\ x$
⟨*proof*⟩

lemma *type-eq-subst-eq1*:
fixes $v::v$ **and** $c1::c$
assumes $\{\!|\ z1 : b1 \mid c1\ |\!\} = (\{\!|\ z2 : b2 \mid c2\ |\!\})$ **and** $atom\ z1 \sharp c2$
shows $c1[z1::=v]_{cv} = c2[z2::=v]_{cv}$ **and** $b1=b2$ **and** $c1 = (z1 \leftrightarrow z2) \cdot c2$
⟨*proof*⟩

lemma *type-eq-subst-eq2*:
fixes $v::v$ **and** $c1::c$
assumes $\{\!|\ z1 : b1 \mid c1\ |\!\} = (\{\!|\ z2 : b2 \mid c2\ |\!\})$
shows $c1[z1::=v]_{cv} = c2[z2::=v]_{cv}$ **and** $b1=b2$ **and** $[[atom\ z1]]lst.\ c1 = [[atom\ z2]]lst.\ c2$
⟨*proof*⟩

lemma *type-eq-subst-eq3*:
fixes *v*::*v* **and** *c1*::*c*
assumes $\{\!|\ z1 : b1 \mid c1\ |\!\} = (\{\!|\ z2 : b2 \mid c2\ |\!\})$ **and** *atom z1* $\sharp$ *c2*
shows *c1* = *c2*$[z2::=V\text{-}var\ z1]_{cv}$ **and** *b1*=*b2*
⟨*proof*⟩

lemma *type-eq-flip*:
assumes *atom x* $\sharp$ *c*
shows $\{\!|\ z : b \mid c\ |\!\} = \{\!|\ x : b \mid (x \leftrightarrow z) \cdot c\ |\!\}$
⟨*proof*⟩

lemma *c-of-true*:
c-of $\{\!|\ z' : B\text{-}bool \mid TRUE\ |\!\}$ *x* = *C-true*
⟨*proof*⟩

lemma *type-eq-subst*:
assumes *atom x* $\sharp$ *c*
shows $\{\!|\ z : b \mid c\ |\!\} = \{\!|\ x : b \mid c[z::=[x]^v]_{cv}\ |\!\}$
⟨*proof*⟩

lemma *type-e-subst-fresh*:
fixes *x*::*x* **and** *z*::*x*
assumes *atom z* $\sharp$ (*x*,*v*) **and** *atom x* $\sharp$ *e*
shows $\{\!|\ z : b \mid CE\text{-}val\ (V\text{-}var\ z) == e\ |\!\}[x::=v]_{\tau v} = \{\!|\ z : b \mid CE\text{-}val\ (V\text{-}var\ z) == e\ |\!\}$
⟨*proof*⟩

lemma *type-v-subst-fresh*:
fixes *x*::*x* **and** *z*::*x*
assumes *atom z* $\sharp$ (*x*,*v*) **and** *atom x* $\sharp$ *v'*
shows $\{\!|\ z : b \mid CE\text{-}val\ (V\text{-}var\ z) == CE\text{-}val\ v'\ |\!\}[x::=v]_{\tau v} = \{\!|\ z : b \mid CE\text{-}val\ (V\text{-}var\ z) == CE\text{-}val\ v'\ |\!\}$
⟨*proof*⟩

lemma *subst-tbase-eq*:
b-of τ = *b-of* $\tau[x::=v]_{\tau v}$
⟨*proof*⟩

lemma *subst-tv-if*:
assumes *atom z1* $\sharp$ (*x*,*v*) **and** *atom z'* $\sharp$ (*x*,*v*)
shows $\{\!|\ z1 : b \mid CE\text{-}val\ (v'[x::=v]_{vv}) == CE\text{-}val\ (V\text{-}lit\ l)\ \ IMP\ \ (c'[x::=v]_{cv})[z'::=[z1]^v]_{cv}\ |\!\} =$
$\{\!|\ z1 : b \mid CE\text{-}val\ v' == CE\text{-}val\ (V\text{-}lit\ l)\ \ IMP\ \ c'[z'::=[z1]^v]_{cv}\ |\!\}[x::=v]_{\tau v}$
⟨*proof*⟩

lemma *subst-tv-tid*:
assumes *atom za* $\sharp$ (*x*,*v*)
shows $\{\!|\ za : B\text{-}id\ tid \mid TRUE\ |\!\} = \{\!|\ za : B\text{-}id\ tid \mid TRUE\ |\!\}[x::=v]_{\tau v}$
⟨*proof*⟩

lemma *b-of-subst*:
b-of $(\tau[x::=v]_{\tau v})$ = *b-of* τ
⟨*proof*⟩

lemma *subst-tv-flip*:
  assumes $\tau'[x::=v]_{\tau v} = \tau$ **and** *atom* $x \sharp (v,\tau)$ **and** *atom* $x' \sharp (v,\tau)$
  shows $((x' \leftrightarrow x) \cdot \tau')[x'::=v]_{\tau v} = \tau$
⟨*proof*⟩

lemma *subst-cv-true*:
  $\{\!|\ z : B\text{-}id\ tid\ \mid TRUE\ |\!\} = \{\!|\ z : B\text{-}id\ tid\ \mid TRUE\ |\!\}[x::=v]_{\tau v}$
⟨*proof*⟩

lemma *t-eq-supp*:
  assumes $(\{\!|\ z : b \mid c\ |\!\}) = (\{\!|\ z1 : b1 \mid c1\ |\!\})$
  shows *supp* $c - \{$ *atom* z $\} =$ *supp* $c1 - \{$ *atom* $z1$ $\}$
⟨*proof*⟩

lemma *fresh-t-eq*:
  fixes $x::x$
  assumes $(\{\!|\ z : b \mid c\ |\!\}) = (\{\!|\ zz : b \mid cc\ |\!\})$ **and** *atom* $x \sharp c$ **and** $x \neq zz$
  shows *atom* $x \sharp cc$
⟨*proof*⟩

3.8 Mutable Variable Context

nominal-function *subst-dv* :: $\Delta \Rightarrow x \Rightarrow v \Rightarrow \Delta$ **where**
  subst-dv DNil x v = *DNil*
| *subst-dv* $((u,t)\ \#_\Delta\ \Delta)\ x\ v$ = $((u,t[x::=v]_{\tau v})\ \#_\Delta\ (subst\text{-}dv\ \Delta\ x\ v\ ))$
    ⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

abbreviation
  subst-dv-abbrev :: $\Delta \Rightarrow x \Rightarrow v \Rightarrow \Delta$ ($\langle$-[-::=-]$_{\Delta v}\rangle$ [1000,50,50] 1000)
  where
    $\Delta[x::=v]_{\Delta v} \equiv subst\text{-}dv\ \Delta\ x\ v$

nominal-function *dmap* :: $(u*\tau \Rightarrow u*\tau) \Rightarrow \Delta \Rightarrow \Delta$ **where**
  dmap f DNil = *DNil*
| *dmap f* $((u,t)\#_\Delta\Delta)$ = $(f\ (u,t)\ \#_\Delta\ (dmap\ f\ \Delta\ ))$
    ⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

lemma *subst-dv-iff*:
  $\Delta[x::=v]_{\Delta v} = dmap\ (\lambda(u,t).\ (u,\ t[x::=v]_{\tau v}))\ \Delta$
  ⟨*proof*⟩

lemma *size-subst-dv* [*simp*]: *size* (*subst-dv G i x*) $\leq$ *size G*
  ⟨*proof*⟩

lemma *forget-subst-dv* [*simp*]: *atom a* $\sharp$ *G* $\Longrightarrow$ *subst-dv G a x* = *G*
  ⟨*proof*⟩

lemma *subst-dv-member*:
  assumes $(u,\tau) \in setD\ \Delta$

shows $(u, \tau[x::=v]_{\tau v}) \in setD\ (\Delta[x::=v]_{\Delta v})$
⟨*proof*⟩

lemma *fresh-subst-dv*:
fixes $x::x$
assumes $atom\ xa \sharp \Delta$ **and** $atom\ xa \sharp v$
shows $atom\ xa \sharp \Delta[x::=v]_{\Delta v}$
⟨*proof*⟩

lemma *fresh-subst-dv-if*:
fixes $j::atom$ **and** $i::x$ **and** $x::v$ **and** $t::\Delta$
assumes $j \sharp t \wedge j \sharp x$
shows $(j \sharp subst\text{-}dv\ t\ i\ x)$
⟨*proof*⟩

3.9 Statements

Using ideas from proofs at top of AFP/Launchbury/Substitution.thy. Subproofs borrowed from there; hence the apply style proofs.

nominal-function (*default case-sum* (λ*x. Inl undefined*) (*case-sum* (λ*x. Inl undefined*) (λ*x. Inr undefined*)))
subst-sv :: $s \Rightarrow x \Rightarrow v \Rightarrow s$
and *subst-branchv* :: $branch\text{-}s \Rightarrow x \Rightarrow v \Rightarrow branch\text{-}s$
and *subst-branchlv* :: $branch\text{-}list \Rightarrow x \Rightarrow v \Rightarrow branch\text{-}list$ **where**
subst-sv ((*AS-val v′*)) *x v* = (*AS-val* (*subst-vv v′ x v*))
| *atom y* ♯ (*x*,*v*) ⟹ *subst-sv* (*AS-let y e s*) *x v* = (*AS-let y* ($e[x::=v]_{ev}$) (*subst-sv s x v*))
| *atom y* ♯ (*x*,*v*) ⟹ *subst-sv* (*AS-let2 y t s1 s2*) *x v* = (*AS-let2 y* ($t[x::=v]_{\tau v}$) (*subst-sv s1 x v*) (*subst-sv s2 x v*))
| *subst-sv* (*AS-match v′ cs*) *x v* = *AS-match* ($v'[x::=v]_{vv}$) (*subst-branchlv cs x v*)
| *subst-sv* (*AS-assign y v′*) *x v* = *AS-assign y* (*subst-vv v′ x v*)
| *subst-sv* ((*AS-if v′ s1 s2*)) *x v* = (*AS-if* (*subst-vv v′ x v*) (*subst-sv s1 x v*) (*subst-sv s2 x v*))
| *atom u* ♯ (*x*,*v*) ⟹ *subst-sv* (*AS-var u τ v′ s*) *x v* = *AS-var u* (*subst-tv τ x v*) (*subst-vv v′ x v*) (*subst-sv s x v*)
| *subst-sv* (*AS-while s1 s2*) *x v* = *AS-while* (*subst-sv s1 x v*) (*subst-sv s2 x v*)
| *subst-sv* (*AS-seq s1 s2*) *x v* = *AS-seq* (*subst-sv s1 x v*) (*subst-sv s2 x v*)
| *subst-sv* (*AS-assert c s*) *x v* = *AS-assert* (*subst-cv c x v*) (*subst-sv s x v*)
| *atom x1* ♯ (*x*,*v*) ⟹ *subst-branchv* (*AS-branch dc x1 s1*) *x v* = *AS-branch dc x1* (*subst-sv s1 x v*)

| *subst-branchlv* (*AS-final cs*) *x v* = *AS-final* (*subst-branchv cs x v*)
| *subst-branchlv* (*AS-cons cs css*) *x v* = *AS-cons* (*subst-branchv cs x v*) (*subst-branchlv css x v*)
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

abbreviation
subst-sv-abbrev :: $s \Rightarrow x \Rightarrow v \Rightarrow s$ (‹-[-::=-]$_{sv}$› [*1000*,*50*,*50*] *1000*)
where
$s[x::=v]_{sv} \equiv subst\text{-}sv\ s\ x\ v$

abbreviation
subst-branchv-abbrev :: $branch\text{-}s \Rightarrow x \Rightarrow v \Rightarrow branch\text{-}s$ (‹-[-::=-]$_{sv}$› [*1000*,*50*,*50*] *1000*)
where

$s[x::=v]_{sv} \equiv$ *subst-branchv s x v*

lemma *size-subst-sv* [*simp*]: *size* (*subst-sv A i x*) = *size A* **and** *size* (*subst-branchv B i x*) = *size B* **and** *size* (*subst-branchlv C i x*) = *size C*
⟨*proof*⟩

lemma *forget-subst-sv* [*simp*]: **shows** *atom a* ♯ *A* ⟹ *subst-sv A a x* = *A* **and** *atom a* ♯ *B* ⟹ *subst-branchv B a x* = *B* **and** *atom a* ♯ *C* ⟹ *subst-branchlv C a x* = *C*
⟨*proof*⟩

lemma *subst-sv-id* [*simp*]: *subst-sv A a* (*V-var a*) = *A* **and** *subst-branchv B a* (*V-var a*) = *B* **and** *subst-branchlv C a* (*V-var a*) = *C*
⟨*proof*⟩

lemma *fresh-subst-sv-if-rl*:
shows
(*atom x* ♯ *s* ∧ *j* ♯ *s*) ∨ (*j* ♯ *v* ∧ (*j* ♯ *s* ∨ *j* = *atom x*)) ⟹ *j* ♯ (*subst-sv s x v*) **and**
(*atom x* ♯ *cs* ∧ *j* ♯ *cs*) ∨ (*j* ♯ *v* ∧ (*j* ♯ *cs* ∨ *j* = *atom x*)) ⟹ *j* ♯ (*subst-branchv cs x v*) **and**
(*atom x* ♯ *css* ∧ *j* ♯ *css*) ∨ (*j* ♯ *v* ∧ (*j* ♯ *css* ∨ *j* = *atom x*)) ⟹ *j* ♯ (*subst-branchlv css x v*)
⟨*proof*⟩

lemma *fresh-subst-sv-if-lr*:
shows *j* ♯ (*subst-sv s x v*) ⟹ (*atom x* ♯ *s* ∧ *j* ♯ *s*) ∨ (*j* ♯ *v* ∧ (*j* ♯ *s* ∨ *j* = *atom x*)) **and**
j ♯ (*subst-branchv cs x v*) ⟹ (*atom x* ♯ *cs* ∧ *j* ♯ *cs*) ∨ (*j* ♯ *v* ∧ (*j* ♯ *cs* ∨ *j* = *atom x*)) **and**
j ♯ (*subst-branchlv css x v*) ⟹ (*atom x* ♯ *css* ∧ *j* ♯ *css*) ∨ (*j* ♯ *v* ∧ (*j* ♯ *css* ∨ *j* = *atom x*))
⟨*proof*⟩

lemma *fresh-subst-sv-if*[*simp*]:
fixes *x*::*x* **and** *v*::*v*
shows *j* ♯ (*subst-sv s x v*) ⟷ (*atom x* ♯ *s* ∧ *j* ♯ *s*) ∨ (*j* ♯ *v* ∧ (*j* ♯ *s* ∨ *j* = *atom x*)) **and**
j ♯ (*subst-branchv cs x v*) ⟷ (*atom x* ♯ *cs* ∧ *j* ♯ *cs*) ∨ (*j* ♯ *v* ∧ (*j* ♯ *cs* ∨ *j* = *atom x*))
⟨*proof*⟩

lemma *subst-sv-commute* [*simp*]:
fixes *A*::*s* **and** *t*::*v* **and** *j*::*x* **and** *i*::*x*
shows *atom j* ♯ *A* ⟹ (*subst-sv* (*subst-sv A i t*) *j u*) = *subst-sv A i* (*subst-vv t j u*) **and**
atom j ♯ *B* ⟹ (*subst-branchv* (*subst-branchv B i t*) *j u*) = *subst-branchv B i* (*subst-vv t j u*) **and**
atom j ♯ *C* ⟹ (*subst-branchlv* (*subst-branchlv C i t*) *j u*) = *subst-branchlv C i* (*subst-vv t j u*)
⟨*proof*⟩

lemma *c-eq-perm*:
assumes ((*atom z*) ⇌ (*atom z′*)) ∙ *c* = *c′* **and** *atom z′* ♯ *c*
shows ⦇ *z* : *b* | *c* ⦈ = ⦇ *z′* : *b* | *c′* ⦈
⟨*proof*⟩

lemma *subst-sv-flip*:
fixes *s*::*s* **and** *sa*::*s* **and** *v′*::*v*
assumes *atom c* ♯ (*s*, *sa*) **and** *atom c* ♯ (*v′*,*x*, *xa*, *s*, *sa*) *atom x* ♯ *v′* **and** *atom xa* ♯ *v′* **and** (*x* ↔ *c*) ∙ *s* = (*xa* ↔ *c*) ∙ *sa*
shows *s*[*x*::=*v′*]$_{sv}$ = *sa*[*xa*::=*v′*]$_{sv}$
⟨*proof*⟩

lemma *if-type-eq*:
  fixes Γ::Γ **and** *v*::*v* **and** *z1*::*x*
  assumes *atom z1'* $\sharp$ (*v*, *ca*, (*x*, *b*, *c*) $\#_\Gamma$ Γ, (*CE-val v == CE-val* (*V-lit ll*) *IMP ca*[*za*::=[*z1*]^v]$_{cv}$)) **and** *atom z1* $\sharp$ *v*
    and *atom z1* $\sharp$ (*za*,*ca*) **and** *atom z1'* $\sharp$ (*za*,*ca*)
  shows ({| *z1'* : *ba* | *CE-val v == CE-val* (*V-lit ll*) *IMP ca*[*za*::=[*z1'*]^v]$_{cv}$ |}) = {| *z1* : *ba* | *CE-val v == CE-val* (*V-lit ll*) *IMP ca*[*za*::=[*z1*]^v]$_{cv}$ |}
⟨*proof*⟩

lemma *subst-sv-var-flip*:
  fixes *x*::*x* **and** *s*::*s* **and** *z*::*x*
  shows *atom x* $\sharp$ *s* $\Longrightarrow$ ((*x* $\leftrightarrow$ *z*) $\cdot$ *s*) = *s*[*z*::=[*x*]^v]$_{sv}$ **and**
    atom x $\sharp$ *cs* $\Longrightarrow$ ((*x* $\leftrightarrow$ *z*) $\cdot$ *cs*) = *subst-branchv cs z* [*x*]^v **and**
    atom x $\sharp$ *css* $\Longrightarrow$ ((*x* $\leftrightarrow$ *z*) $\cdot$ *css*) = *subst-branchlv css z* [*x*]^v
    ⟨*proof*⟩

instantiation *s* :: *has-subst-v*
begin

definition
  subst-v = *subst-sv*

instance ⟨*proof*⟩
end

3.10 Type Definition

nominal-function *subst-ft-v* :: *fun-typ* $\Rightarrow$ *x* $\Rightarrow$ *v* $\Rightarrow$ *fun-typ* **where**
  atom z $\sharp$ (*x*,*v*) $\Longrightarrow$ *subst-ft-v* (*AF-fun-typ z b c t* (*s*::*s*)) *x v* = *AF-fun-typ z b c*[*x*::=*v*]$_{cv}$ *t*[*x*::=*v*]$_{\tau v}$ *s*[*x*::=*v*]$_{sv}$
    ⟨*proof*⟩

nominal-termination (*eqvt*) ⟨*proof*⟩

nominal-function *subst-ftq-v* :: *fun-typ-q* $\Rightarrow$ *x* $\Rightarrow$ *v* $\Rightarrow$ *fun-typ-q* **where**
  atom bv $\sharp$ (*x*,*v*) $\Longrightarrow$ *subst-ftq-v* (*AF-fun-typ-some bv ft*) *x v* = (*AF-fun-typ-some bv* (*subst-ft-v ft x v*))
| *subst-ftq-v* (*AF-fun-typ-none ft*) *x v* = (*AF-fun-typ-none* (*subst-ft-v ft x v*))
    ⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

lemma *size-subst-ft*[*simp*]: *size* (*subst-ft-v A x v*) = *size A*
  ⟨*proof*⟩

lemma *forget-subst-ft* [*simp*]: **shows** *atom x* $\sharp$ *A* $\Longrightarrow$ *subst-ft-v A x a* = *A*
  ⟨*proof*⟩

lemma *subst-ft-id* [*simp*]: *subst-ft-v A a* (*V-var a*) = *A*
  ⟨*proof*⟩

instantiation *fun-typ* :: *has-subst-v*
begin

definition
subst-v = *subst-ft-v*

instance ⟨*proof*⟩
end

instantiation *fun-typ-q* :: *has-subst-v*
begin

definition
subst-v = *subst-ftq-v*

instance ⟨*proof*⟩

end

3.11 Variable Context

lemma *subst-dv-fst-eq*:
fst ‘ *setD* ($\Delta[x::=v]_{\Delta v}$) = *fst* ‘ *setD* Δ
⟨*proof*⟩

lemma *subst-gv-member-iff*:
fixes $x'::x$ **and** $x::x$ **and** $v::v$ **and** $c'::c$
assumes (x',b',c') ∈ *toSet* Γ **and** *atom* $x \notin$ *atom-dom* Γ
shows (x',b',$c'[x::=v]_{cv}$) ∈ *toSet* $\Gamma[x::=v]_{\Gamma v}$
⟨*proof*⟩

lemma *fresh-subst-gv-if*:
fixes $j::atom$ **and** $i::x$ **and** $x::v$ **and** $t::\Gamma$
assumes $j \sharp t \wedge j \sharp x$
shows ($j \sharp$ *subst-gv t i x*)
⟨*proof*⟩

3.12 Lookup

lemma *set-GConsD*: $y \in$ *toSet* ($x \#_\Gamma$ *xs*) $\Longrightarrow$ $y{=}x \vee y \in$ *toSet xs*
⟨*proof*⟩

lemma *subst-g-assoc-cons*:
assumes $x \neq x'$
shows $(((x',\ b',\ c') \#_\Gamma \Gamma')[x::=v]_{\Gamma v}$ @ $G) = ((x',\ b',\ c'[x::=v]_{cv}) \#_\Gamma ((\Gamma'[x::=v]_{\Gamma v})$ @ $G))$
⟨*proof*⟩

end

Chapter 4

Basic Type Variable Substitution

4.1 Class

class *has-subst-b* = *fs* +
  fixes *subst-b* :: *'a*::*fs* ⇒ *bv* ⇒ *b* ⇒ *'a*::*fs* (‹-[-::=-]$_b$› [*1000*,*50*,*50*] *1000*)

assumes *fresh-subst-if*: *j* ♯ (*t*[*i*::=*x*]$_b$) ⟷ (*atom i* ♯ *t* ∧ *j* ♯ *t*) ∨ (*j* ♯ *x* ∧ (*j* ♯ *t* ∨ *j* = *atom i*))
  and *forget-subst*[*simp*]: *atom a* ♯ *tm* ⟹ *tm*[*a*::=*x*]$_b$ = *tm*
  and *subst-id*[*simp*]: *tm*[*a*::=(*B-var a*)]$_b$ = *tm*
  and *eqvt*[*simp*,*eqvt*]: (*p*::*perm*) ∙ (*subst-b t1 x1 v*) = (*subst-b* (*p* ∙*t1*) (*p* ∙*x1*) (*p* ∙*v*))
  and *flip-subst*[*simp*]: *atom bv* ♯ *c* ⟹ ((*bv* ↔ *z*) ∙ *c*) = *c*[*z*::=*B-var bv*]$_b$
  and *flip-subst-subst*[*simp*]: *atom bv* ♯ *c* ⟹ ((*bv* ↔ *z*) ∙ *c*)[*bv*::=*v*]$_b$ = *c*[*z*::=*v*]$_b$
begin

lemmas *flip-subst-b* = *flip-subst-subst*

lemma *subst-b-simple-commute*:
  fixes *x*::*bv*
  assumes *atom x* ♯ *c*
  shows (*c*[*z*::=*B-var x*]$_b$)[*x*::=*b*]$_b$ = *c*[*z*::=*b*]$_b$
⟨*proof*⟩

lemma *subst-b-flip-eq-one*:
  fixes *z1*::*bv* **and** *z2*::*bv* **and** *x1*::*bv* **and** *x2*::*bv*
  assumes [[*atom z1*]]*lst*. *c1* = [[*atom z2*]]*lst*. *c2*
    and *atom x1* ♯ (*z1*,*z2*,*c1*,*c2*)
  shows (*c1*[*z1*::=*B-var x1*]$_b$) = (*c2*[*z2*::=*B-var x1*]$_b$)
⟨*proof*⟩

lemma *subst-b-flip-eq-two*:
  fixes *z1*::*bv* **and** *z2*::*bv* **and** *x1*::*bv* **and** *x2*::*bv*
  assumes [[*atom z1*]]*lst*. *c1* = [[*atom z2*]]*lst*. *c2*
  shows (*c1*[*z1*::=*b*]$_b$) = (*c2*[*z2*::=*b*]$_b$)
⟨*proof*⟩

lemma *subst-b-fresh-x*:
  fixes *tm*::*'a*::*fs* **and** *x*::*x*
  shows *atom x* ♯ *tm* = *atom x* ♯ *tm*[*bv*::=*b'*]$_b$

⟨*proof*⟩

lemma *subst-b-x-flip*[*simp*]:
fixes $x'::x$ **and** $x::x$ **and** $bv::bv$
shows $((x' \leftrightarrow x) \cdot tm)[bv::=b']_b = (x' \leftrightarrow x) \cdot (tm[bv::=b']_b)$
⟨*proof*⟩

end

4.2 Base Type

nominal-function *subst-bb* :: $b \Rightarrow bv \Rightarrow b \Rightarrow b$ **where**
subst-bb (*B-var bv2*) *bv1 b* = (*if bv1* = *bv2 then b else* (*B-var bv2*))
| *subst-bb B-int bv1 b* = *B-int*
| *subst-bb B-bool bv1 b* = *B-bool*
| *subst-bb* (*B-id s*) *bv1 b* = *B-id s*
| *subst-bb* (*B-pair b1 b2*) *bv1 b* = *B-pair* (*subst-bb b1 bv1 b*) (*subst-bb b2 bv1 b*)
| *subst-bb B-unit bv1 b* = *B-unit*
| *subst-bb B-bitvec bv1 b* = *B-bitvec*
| *subst-bb* (*B-app s b2*) *bv1 b* = *B-app s* (*subst-bb b2 bv1 b*)
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

abbreviation
subst-bb-abbrev :: $b \Rightarrow bv \Rightarrow b \Rightarrow b$ (‹-[-::=-]$_{bb}$› [*1000*,*50*,*50*] *1000*)
where
$b[bv::=b']_{bb} \equiv$ *subst-bb b bv b'*

instantiation *b* :: *has-subst-b*
begin
definition *subst-b* = *subst-bb*

instance ⟨*proof*⟩
end

lemma *subst-bb-inject*:
assumes $b1 = b2[bv::=b]_{bb}$ **and** $b2 \neq$ *B-var bv*
shows
b1 = *B-int* $\Longrightarrow$ *b2* = *B-int* **and**
b1 = *B-bool* $\Longrightarrow$ *b2* = *B-bool* **and**
b1 = *B-unit* $\Longrightarrow$ *b2* = *B-unit* **and**
b1 = *B-bitvec* $\Longrightarrow$ *b2* = *B-bitvec* **and**
b1 = *B-pair b11 b12* $\Longrightarrow$ $(\exists\, b11'\ b12'.\ b11 = b11'[bv::=b]_{bb} \wedge b12 = b12'[bv::=b]_{bb} \wedge b2 =$ *B-pair b11' b12'*) **and**
b1 = *B-var bv'* $\Longrightarrow$ *b2* = *B-var bv'* **and**
b1 = *B-id tyid* $\Longrightarrow$ *b2* = *B-id tyid* **and**
b1 = *B-app tyid b11* $\Longrightarrow$ $(\exists\, b11'.\ b11 = b11'[bv::=b]_{bb} \wedge b2 =$ *B-app tyid b11'*)
⟨*proof*⟩

lemma *flip-b-subst4*:
fixes *b1*::*b* **and** *bv1*::*bv* **and** *c*::*bv* **and** *b*::*b*
assumes *atom c* ♯ (*b1*,*bv1*)

shows $b1[bv1::=b]_{bb} = ((bv1 \leftrightarrow c) \cdot b1)[c ::= b]_{bb}$
⟨*proof*⟩

lemma *subst-bb-flip-sym*:
fixes *b1*::*b* **and** *b2*::*b*
assumes *atom c* ♯ *b* **and** *atom c* ♯ (*bv1*,*bv2*, *b1*, *b2*) **and** $(bv1 \leftrightarrow c) \cdot b1 = (bv2 \leftrightarrow c) \cdot b2$
shows $b1[bv1::=b]_{bb} = b2[bv2::=b]_{bb}$
⟨*proof*⟩

4.3 Value

nominal-function *subst-vb* :: *v* ⇒ *bv* ⇒ *b* ⇒ *v* **where**
subst-vb (*V-lit l*) *x v* = *V-lit l*
| *subst-vb* (*V-var y*) *x v* = *V-var y*
| *subst-vb* (*V-cons tyid c v'*) *x v* = *V-cons tyid c* (*subst-vb v' x v*)
| *subst-vb* (*V-consp tyid c b v'*) *x v* = *V-consp tyid c* (*subst-bb b x v*) (*subst-vb v' x v*)
| *subst-vb* (*V-pair v1 v2*) *x v* = *V-pair* (*subst-vb v1 x v*) (*subst-vb v2 x v*)
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

abbreviation
subst-vb-abbrev :: *v* ⇒ *bv* ⇒ *b* ⇒ *v* (‹-[-::=-]$_{vb}$› [*1000*,*50*,*50*] *500*)
where
$e[bv::=b]_{vb} \equiv$ *subst-vb e bv b*

instantiation *v* :: *has-subst-b*
begin
definition *subst-b* = *subst-vb*

instance ⟨*proof*⟩
end

4.4 Constraints Expressions

nominal-function *subst-ceb* :: *ce* ⇒ *bv* ⇒ *b* ⇒ *ce* **where**
subst-ceb ((*CE-val v'*)) *bv b* = (*CE-val* (*subst-vb v' bv b*))
| *subst-ceb* ((*CE-op opp v1 v2*)) *bv b* = ((*CE-op opp* (*subst-ceb v1 bv b*)(*subst-ceb v2 bv b*)))
| *subst-ceb* ((*CE-fst v'*)) *bv b* = *CE-fst* (*subst-ceb v' bv b*)
| *subst-ceb* ((*CE-snd v'*)) *bv b* = *CE-snd* (*subst-ceb v' bv b*)
| *subst-ceb* ((*CE-len v'*)) *bv b* = *CE-len* (*subst-ceb v' bv b*)
| *subst-ceb* (*CE-concat v1 v2*) *bv b* = *CE-concat* (*subst-ceb v1 bv b*) (*subst-ceb v2 bv b*)
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

abbreviation
subst-ceb-abbrev :: *ce* ⇒ *bv* ⇒ *b* ⇒ *ce* (‹-[-::=-]$_{ceb}$› [*1000*,*50*,*50*] *500*)
where
$ce[bv::=b]_{ceb} \equiv$ *subst-ceb ce bv b*

instantiation *ce* :: *has-subst-b*
begin

definition *subst-b* = *subst-ceb*

instance ⟨*proof*⟩
end

4.5 Constraints

nominal-function *subst-cb* :: *c* ⇒ *bv* ⇒ *b* ⇒ *c* **where**
subst-cb (*C-true*) *x v* = *C-true*
| *subst-cb* (*C-false*) *x v* = *C-false*
| *subst-cb* (*C-conj c1 c2*) *x v* = *C-conj* (*subst-cb c1 x v*) (*subst-cb c2 x v*)
| *subst-cb* (*C-disj c1 c2*) *x v* = *C-disj* (*subst-cb c1 x v*) (*subst-cb c2 x v*)
| *subst-cb* (*C-imp c1 c2*) *x v* = *C-imp* (*subst-cb c1 x v*) (*subst-cb c2 x v*)
| *subst-cb* (*C-eq e1 e2*) *x v* = *C-eq* (*subst-ceb e1 x v*) (*subst-ceb e2 x v*)
| *subst-cb* (*C-not c*) *x v* = *C-not* (*subst-cb c x v*)
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

abbreviation
subst-cb-abbrev :: *c* ⇒ *bv* ⇒ *b* ⇒ *c* (‹-[-::=-]$_{cb}$› [*1000*,*50*,*50*] *500*)
where
c[*bv*::=*b*]$_{cb}$ ≡ *subst-cb c bv b*

instantiation *c* :: *has-subst-b*
begin
definition *subst-b* = *subst-cb*

instance ⟨*proof*⟩
end

4.6 Types

nominal-function *subst-tb* :: *τ* ⇒ *bv* ⇒ *b* ⇒ *τ* **where**
subst-tb (⦇ *z* : *b2* | *c* ⦈) *bv1 b1* = ⦇ *z* : *b2*[*bv1*::=*b1*]$_{bb}$ | *c*[*bv1*::=*b1*]$_{cb}$ ⦈
⟨*proof*⟩

nominal-termination (*eqvt*) ⟨*proof*⟩

abbreviation
subst-tb-abbrev :: *τ* ⇒ *bv* ⇒ *b* ⇒ *τ* (‹-[-::=-]$_{\tau b}$› [*1000*,*50*,*50*] *1000*)
where
t[*bv*::=*b'*]$_{\tau b}$ ≡ *subst-tb t bv b'*

instantiation *τ* :: *has-subst-b*
begin
definition *subst-b* = *subst-tb*

instance ⟨*proof*⟩
end

lemma *subst-bb-commute* [*simp*]:

atom j ♯ *A* ⟹ (*subst-bb* (*subst-bb A i t*) *j u*) = *subst-bb A i* (*subst-bb t j u*)
⟨*proof*⟩

lemma *subst-vb-commute* [*simp*]:
atom j ♯ *A* ⟹ (*subst-vb* (*subst-vb A i t*)) *j u* = *subst-vb A i* (*subst-bb t j u*)
⟨*proof*⟩

lemma *subst-ceb-commute* [*simp*]:
atom j ♯ *A* ⟹ (*subst-ceb* (*subst-ceb A i t*)) *j u* = *subst-ceb A i* (*subst-bb t j u*)
⟨*proof*⟩

lemma *subst-cb-commute* [*simp*]:
atom j ♯ *A* ⟹ (*subst-cb* (*subst-cb A i t*)) *j u* = *subst-cb A i* (*subst-bb t j u*)
⟨*proof*⟩

lemma *subst-tb-commute* [*simp*]:
atom j ♯ *A* ⟹ (*subst-tb* (*subst-tb A i t*)) *j u* = *subst-tb A i* (*subst-bb t j u*)
⟨*proof*⟩

4.7 Expressions

nominal-function *subst-eb* :: *e* ⇒ *bv* ⇒ *b* ⇒ *e* **where**
subst-eb ((*AE-val v′*)) *bv b* = (*AE-val* (*subst-vb v′ bv b*))
| *subst-eb* ((*AE-app f v′*)) *bv b* = ((*AE-app f* (*subst-vb v′ bv b*)))
| *subst-eb* ((*AE-appP f b′ v′*)) *bv b* = ((*AE-appP f* (*b′*[*bv*::=*b*]$_{bb}$) (*subst-vb v′ bv b*)))
| *subst-eb* ((*AE-op opp v1 v2*)) *bv b* = ((*AE-op opp* (*subst-vb v1 bv b*) (*subst-vb v2 bv b*)))
| *subst-eb* ((*AE-fst v′*)) *bv b* = *AE-fst* (*subst-vb v′ bv b*)
| *subst-eb* ((*AE-snd v′*)) *bv b* = *AE-snd* (*subst-vb v′ bv b*)
| *subst-eb* ((*AE-mvar u*)) *bv b* = *AE-mvar u*
| *subst-eb* ((*AE-len v′*)) *bv b* = *AE-len* (*subst-vb v′ bv b*)
| *subst-eb* (*AE-concat v1 v2*) *bv b* = *AE-concat* (*subst-vb v1 bv b*) (*subst-vb v2 bv b*)
| *subst-eb* (*AE-split v1 v2*) *bv b* = *AE-split* (*subst-vb v1 bv b*) (*subst-vb v2 bv b*)
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

abbreviation
subst-eb-abbrev :: *e* ⇒ *bv* ⇒ *b* ⇒ *e* (‹-[-::=-]$_{eb}$› [*1000*,*50*,*50*] *500*)
where
e[*bv*::=*b*]$_{eb}$ ≡ *subst-eb e bv b*

instantiation *e* :: *has-subst-b*
begin
definition *subst-b* = *subst-eb*

instance ⟨*proof*⟩
end

4.8 Statements

nominal-function (*default case-sum* (λ*x*. *Inl undefined*) (*case-sum* (λ*x*. *Inl undefined*) (λ*x*. *Inr undefined*)))

subst-sb :: $s \Rightarrow bv \Rightarrow b \Rightarrow s$
and *subst-branchb* :: *branch-s* $\Rightarrow bv \Rightarrow b \Rightarrow$ *branch-s*
and *subst-branchlb* :: *branch-list* $\Rightarrow bv \Rightarrow b \Rightarrow$ *branch-list*
where
subst-sb (*AS-val v'*) *bv b* = (*AS-val* (*subst-vb v' bv b*))
| *subst-sb* (*AS-let y e s*) *bv b* = (*AS-let y* ($e[bv::=b]_{eb}$) (*subst-sb s bv b*))
| *subst-sb* (*AS-let2 y t s1 s2*) *bv b* = (*AS-let2 y* (*subst-tb t bv b*) (*subst-sb s1 bv b*) (*subst-sb s2 bv b*))
| *subst-sb* (*AS-match v' cs*) *bv b* = *AS-match* (*subst-vb v' bv b*) (*subst-branchlb cs bv b*)
| *subst-sb* (*AS-assign y v'*) *bv b* = *AS-assign y* (*subst-vb v' bv b*)
| *subst-sb* (*AS-if v' s1 s2*) *bv b* = (*AS-if* (*subst-vb v' bv b*) (*subst-sb s1 bv b*) (*subst-sb s2 bv b*))
| *subst-sb* (*AS-var u* τ *v' s*) *bv b* = *AS-var u* (*subst-tb* τ *bv b*) (*subst-vb v' bv b*) (*subst-sb s bv b*)
| *subst-sb* (*AS-while s1 s2*) *bv b* = *AS-while* (*subst-sb s1 bv b*) (*subst-sb s2 bv b*)
| *subst-sb* (*AS-seq s1 s2*) *bv b* = *AS-seq* (*subst-sb s1 bv b*) (*subst-sb s2 bv b*)
| *subst-sb* (*AS-assert c s*) *bv b* = *AS-assert* (*subst-cb c bv b*) (*subst-sb s bv b*)

| *subst-branchb* (*AS-branch dc x1 s'*) *bv b* = *AS-branch dc x1* (*subst-sb s' bv b*)

| *subst-branchlb* (*AS-final sb*) *bv b* = *AS-final* (*subst-branchb sb bv b*)
| *subst-branchlb* (*AS-cons sb ssb*) *bv b* = *AS-cons* (*subst-branchb sb bv b*) (*subst-branchlb ssb bv b*)

⟨*proof*⟩

nominal-termination (*eqvt*) ⟨*proof*⟩

abbreviation
subst-sb-abbrev :: $s \Rightarrow bv \Rightarrow b \Rightarrow s$ ($\langle$-[-::=-]$_{sb}\rangle$ [*1000*,*50*,*50*] *1000*)
where
$b[bv::=b']_{sb} \equiv$ *subst-sb b bv b'*

lemma *fresh-subst-sb-if* [*simp*]:
($j \sharp$ (*subst-sb A i x*)) = ((*atom i* $\sharp$ *A* $\wedge$ $j \sharp A$) $\vee$ ($j \sharp x \wedge (j \sharp A \vee j =$ *atom i*))) **and**
($j \sharp$ (*subst-branchb B i x*)) = ((*atom i* $\sharp$ *B* $\wedge$ $j \sharp B$) $\vee$ ($j \sharp x \wedge (j \sharp B \vee j =$ *atom i*))) **and**
($j \sharp$ (*subst-branchlb C i x*)) = ((*atom i* $\sharp$ *C* $\wedge$ $j \sharp C$) $\vee$ ($j \sharp x \wedge (j \sharp C \vee j =$ *atom i*)))
⟨*proof*⟩

lemma
forget-subst-sb[*simp*]: *atom a* $\sharp$ *A* $\Longrightarrow$ *subst-sb A a x* = *A* **and**
forget-subst-branchb [*simp*]: *atom a* $\sharp$ *B* $\Longrightarrow$ *subst-branchb B a x* = *B* **and**
forget-subst-branchlb[*simp*]: *atom a* $\sharp$ *C* $\Longrightarrow$ *subst-branchlb C a x* = *C*
⟨*proof*⟩

lemma *subst-sb-id*: *subst-sb A a* (*B-var a*) = *A* **and**
subst-branchb-id [*simp*]: *subst-branchb B a* (*B-var a*) = *B* **and**
subst-branchlb-id: *subst-branchlb C a* (*B-var a*) = *C*
⟨*proof*⟩

lemma *flip-subst-s*:
fixes *bv*::*bv* **and** *s*::*s* **and** *cs*::*branch-s* **and** *z*::*bv*
shows *atom bv* $\sharp$ *s* $\Longrightarrow$ (($bv \leftrightarrow z$) $\cdot$ *s*) = $s[z::=$*B-var bv*$]_{sb}$ **and**
atom bv $\sharp$ *cs* $\Longrightarrow$ (($bv \leftrightarrow z$) $\cdot$ *cs*) = *subst-branchb cs z* (*B-var bv*) **and**
atom bv $\sharp$ *css* $\Longrightarrow$ (($bv \leftrightarrow z$) $\cdot$ *css*) = *subst-branchlb css z* (*B-var bv*)

⟨*proof*⟩

lemma *flip-subst-subst-s*:
fixes *bv*::*bv* **and** *s*::*s* **and** *cs*::*branch-s* **and** *z*::*bv*
shows *atom bv* ♯ *s* ⟹ ((*bv* ↔ *z*) ∙ *s*)[*bv*::=*v*]$_{sb}$ = *s*[*z*::=*v*]$_{sb}$ **and**
atom bv ♯ *cs* ⟹ *subst-branchb* ((*bv* ↔ *z*) ∙ *cs*) *bv v* = *subst-branchb cs z v* **and**
atom bv ♯ *css* ⟹ *subst-branchlb* ((*bv* ↔ *z*) ∙ *css*) *bv v* = *subst-branchlb css z v*
⟨*proof*⟩

instantiation *s* :: *has-subst-b*
begin
definition *subst-b* = (λ*s bv b*. *subst-sb s bv b*)

instance ⟨*proof*⟩
end

4.9 Function Type

nominal-function *subst-ft-b* :: *fun-typ* ⇒ *bv* ⇒ *b* ⇒ *fun-typ* **where**
subst-ft-b (*AF-fun-typ z b c t* (*s*::*s*)) *x v* = *AF-fun-typ z* (*subst-bb b x v*) (*subst-cb c x v*) *t*[*x*::=*v*]$_{\tau b}$ *s*[*x*::=*v*]$_{sb}$
⟨*proof*⟩

nominal-termination (*eqvt*) ⟨*proof*⟩

nominal-function *subst-ftq-b* :: *fun-typ-q* ⇒ *bv* ⇒ *b* ⇒ *fun-typ-q* **where**
atom bv ♯ (*x*,*v*) ⟹ *subst-ftq-b* (*AF-fun-typ-some bv ft*) *x v* = (*AF-fun-typ-some bv* (*subst-ft-b ft x v*))

| *subst-ftq-b* (*AF-fun-typ-none ft*) *x v* = (*AF-fun-typ-none* (*subst-ft-b ft x v*))
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

instantiation *fun-typ* :: *has-subst-b*
begin
definition *subst-b* = *subst-ft-b*

Note: Using just simp in the second apply unpacks and gives a single goal whereas auto gives 43 non-intuitive goals. These goals are easier to solve and tedious, however they that it clear if a mistake is made in the definition of the function. For example, I saw that one of the goals was going through with metis and the next wasn't. It turned out the definition of the function itself was wrong

instance ⟨*proof*⟩
end

instantiation *fun-typ-q* :: *has-subst-b*
begin
definition *subst-b* = *subst-ftq-b*

instance ⟨*proof*⟩
end

4.10 Contexts

4.10.1 Immutable Variables

nominal-function *subst-gb* :: $\Gamma \Rightarrow bv \Rightarrow b \Rightarrow \Gamma$ **where**
  subst-gb GNil - - = *GNil*
| *subst-gb* $((y,b',c)\#_\Gamma\Gamma)$ *bv b* = $((y,b'[bv::=b]_{bb},c[bv::=b]_{cb})\#_\Gamma$ (*subst-gb* Γ *bv b*))
  ⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

abbreviation
  subst-gb-abbrev :: $\Gamma \Rightarrow bv \Rightarrow b \Rightarrow \Gamma$ ($\langle$-[-::=-]$_{\Gamma b}\rangle$ [*1000*,*50*,*50*] *1000*)
  where
    $g[bv::=b']_{\Gamma b}$ ≡ *subst-gb g bv b'*

instantiation Γ :: *has-subst-b*
begin
definition *subst-b* = *subst-gb*

instance ⟨*proof*⟩
end

lemma *subst-b-base-for-lit*:
  (*base-for-lit l*)$[bv::=b]_{bb}$ = *base-for-lit l*
  ⟨*proof*⟩

lemma *subst-b-lookup*:
  assumes *Some* (*b*, *c*) = *lookup* Γ *x*
  shows *Some* ($b[bv::=b']_{bb}$, $c[bv::=b']_{cb}$) = *lookup* $\Gamma[bv::=b']_{\Gamma b}$ *x*
  ⟨*proof*⟩

lemma *subst-g-b-x-fresh*:
  fixes *x*::*x* **and** *b*::*b* **and** Γ::Γ **and** *bv*::*bv*
  assumes *atom x* ♯ Γ
  shows *atom x* ♯ $\Gamma[bv::=b]_{\Gamma b}$
  ⟨*proof*⟩

4.10.2 Mutable Variables

nominal-function *subst-db* :: $\Delta \Rightarrow bv \Rightarrow b \Rightarrow \Delta$ **where**
  subst-db $[]_\Delta$ *- -* = $[]_\Delta$
| *subst-db* $((u,t)\ \#_\Delta\ \Delta)$ *bv b* = $((u,t[bv::=b]_{\tau b})\ \#_\Delta$ (*subst-db* Δ *bv b*))
  ⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

abbreviation
  subst-db-abbrev :: $\Delta \Rightarrow bv \Rightarrow b \Rightarrow \Delta$ ($\langle$-[-::=-]$_{\Delta b}\rangle$ [*1000*,*50*,*50*] *1000*)
  where
    $\Delta[bv::=b]_{\Delta b}$ ≡ *subst-db* Δ *bv b*

instantiation Δ :: *has-subst-b*
begin
definition *subst-b* = *subst-db*

instance ⟨*proof*⟩
end

lemma *subst-d-b-member*:
assumes $(u, \tau) \in setD\ \Delta$
shows $(u, \tau[bv::=b]_{\tau b}) \in setD\ \Delta[bv::=b]_{\Delta b}$
⟨*proof*⟩

lemmas *ms-fresh-all* = *e.fresh s-branch-s-branch-list.fresh τ.fresh c.fresh ce.fresh v.fresh l.fresh fresh-at-base opp.fresh pure-fresh ms-fresh*

lemmas *fresh-intros*[*intro*] = *fresh-GNil x-not-in-b-set x-not-in-u-atoms x-fresh-b u-not-in-x-atoms bv-not-in-x-atoms u-not-in-b-atoms*

lemmas *subst-b-simps* = *subst-tb.simps subst-cb.simps subst-ceb.simps subst-vb.simps subst-bb.simps subst-eb.simps subst-branchb.simps subst-sb.simps*

lemma *subst-d-b-x-fresh*:
fixes $x::x$ **and** $b::b$ **and** $\Delta::\Delta$ **and** $bv::bv$
assumes $atom\ x \sharp \Delta$
shows $atom\ x \sharp \Delta[bv::=b]_{\Delta b}$
⟨*proof*⟩

lemma *subst-b-fresh-x*:
fixes $x::x$
shows $atom\ x \sharp v \Longrightarrow atom\ x \sharp v[bv::=b']_{vb}$ **and**
$atom\ x \sharp ce \Longrightarrow atom\ x \sharp ce[bv::=b']_{ceb}$ **and**
$atom\ x \sharp e \Longrightarrow atom\ x \sharp e[bv::=b']_{eb}$ **and**
$atom\ x \sharp c \Longrightarrow atom\ x \sharp c[bv::=b']_{cb}$ **and**
$atom\ x \sharp t \Longrightarrow atom\ x \sharp t[bv::=b']_{\tau b}$ **and**
$atom\ x \sharp d \Longrightarrow atom\ x \sharp d[bv::=b']_{\Delta b}$ **and**
$atom\ x \sharp g \Longrightarrow atom\ x \sharp g[bv::=b']_{\Gamma b}$ **and**
$atom\ x \sharp s \Longrightarrow atom\ x \sharp s[bv::=b']_{sb}$
⟨*proof*⟩

lemma *subst-b-fresh-u-cls*:
fixes $tm::'a::has\text{-}subst\text{-}b$ **and** $x::u$
shows $atom\ x \sharp tm = atom\ x \sharp tm[bv::=b']_b$
⟨*proof*⟩

lemma *subst-g-b-u-fresh*:
fixes $x::u$ **and** $b::b$ **and** $\Gamma::\Gamma$ **and** $bv::bv$
assumes $atom\ x \sharp \Gamma$
shows $atom\ x \sharp \Gamma[bv::=b]_{\Gamma b}$
⟨*proof*⟩

lemma *subst-d-b-u-fresh*:
fixes $x::u$ **and** $b::b$ **and** $\Gamma::\Delta$ **and** $bv::bv$
assumes $atom\ x \sharp \Gamma$
shows $atom\ x \sharp \Gamma[bv::=b]_{\Delta b}$
⟨*proof*⟩

lemma *subst-b-fresh-u*:
fixes $x::u$
shows $atom\ x \sharp v \Longrightarrow atom\ x \sharp v[bv::=b']_{vb}$ **and**
$atom\ x \sharp ce \Longrightarrow atom\ x \sharp ce[bv::=b']_{ceb}$ **and**
$atom\ x \sharp e \Longrightarrow atom\ x \sharp e[bv::=b']_{eb}$ **and**
$atom\ x \sharp c \Longrightarrow atom\ x \sharp c[bv::=b']_{cb}$ **and**
$atom\ x \sharp t \Longrightarrow atom\ x \sharp t[bv::=b']_{\tau b}$ **and**
$atom\ x \sharp d \Longrightarrow atom\ x \sharp d[bv::=b']_{\Delta b}$ **and**
$atom\ x \sharp g \Longrightarrow atom\ x \sharp g[bv::=b']_{\Gamma b}$ **and**
$atom\ x \sharp s \Longrightarrow atom\ x \sharp s[bv::=b']_{sb}$
⟨*proof*⟩

lemma *subst-db-u-fresh*:
fixes $u::u$ **and** $b::b$ **and** $D::\Delta$
assumes $atom\ u \sharp D$
shows $atom\ u \sharp D[bv::=b]_{\Delta b}$
⟨*proof*⟩

lemma *flip-bt-subst4*:
fixes $t::\tau$ **and** $bv::bv$
assumes $atom\ bv \sharp t$
shows $t[bv'::=b]_{\tau b} = ((bv' \leftrightarrow bv) \cdot t)[bv::=b]_{\tau b}$
⟨*proof*⟩

lemma *subst-bt-flip-sym*:
fixes $t1::\tau$ **and** $t2::\tau$
assumes $atom\ bv \sharp b$ **and** $atom\ bv \sharp (bv1,\ bv2,\ t1,\ t2)$ **and** $(bv1 \leftrightarrow bv) \cdot t1 = (bv2 \leftrightarrow bv) \cdot t2$
shows $t1[bv1::=b]_{\tau b} = t2[bv2::=b]_{\tau b}$
⟨*proof*⟩

end

Chapter 5

Wellformed Terms

We require that expressions and values are well-formed. This includes a notion of well-sortedness. We identify a sort with a basic type and define the judgement as two clusters of mutually recursive inductive predicates. Some of the proofs are across all of the predicates and although they seemed at first to be daunting, they have all worked out well.

named-theorems *ms-wb Facts for helping with well−sortedness*

5.1 Definitions

inductive $wfV :: \Theta \Rightarrow \mathcal{B} \Rightarrow \Gamma \Rightarrow v \Rightarrow b \Rightarrow bool$ ($\langle$ - ; - ; - $\vdash_{wf}$ - : - $\rangle$ [50,50,50] 50) **and**
$wfC :: \Theta \Rightarrow \mathcal{B} \Rightarrow \Gamma \Rightarrow c \Rightarrow bool$ ($\langle$ - ; - ; - $\vdash_{wf}$ - $\rangle$ [50,50] 50) **and**
$wfG :: \Theta \Rightarrow \mathcal{B} \Rightarrow \Gamma \Rightarrow bool$ ($\langle$ - ; - $\vdash_{wf}$ - $\rangle$ [50,50] 50) **and**
$wfT :: \Theta \Rightarrow \mathcal{B} \Rightarrow \Gamma \Rightarrow \tau \Rightarrow bool$ ($\langle$ - ; - ; - $\vdash_{wf}$ - $\rangle$ [50,50] 50) **and**
$wfTs :: \Theta \Rightarrow \mathcal{B} \Rightarrow \Gamma \Rightarrow (string*\tau)\ list \Rightarrow bool$ ($\langle$ - ; - ; - $\vdash_{wf}$ - $\rangle$ [50,50] 50) **and**
$wfTh :: \Theta \Rightarrow bool$ ($\langle$ $\vdash_{wf}$ - $\rangle$ [50] 50) **and**
$wfB :: \Theta \Rightarrow \mathcal{B} \Rightarrow b \Rightarrow bool$ ($\langle$ - ; - $\vdash_{wf}$ - $\rangle$ [50,50] 50) **and**
$wfCE :: \Theta \Rightarrow \mathcal{B} \Rightarrow \Gamma \Rightarrow ce \Rightarrow b \Rightarrow bool$ ($\langle$ - ; - ; - $\vdash_{wf}$ - : - $\rangle$ [50,50,50] 50) **and**
$wfTD :: \Theta \Rightarrow type\text{-}def \Rightarrow bool$ ($\langle$ - $\vdash_{wf}$ - $\rangle$ [50,50] 50)
where

wfB-intI: $\vdash_{wf} \Theta \Longrightarrow \Theta; \mathcal{B} \vdash_{wf} B\text{-}int$
| *wfB-boolI*: $\vdash_{wf} \Theta \Longrightarrow \Theta; \mathcal{B} \vdash_{wf} B\text{-}bool$
| *wfB-unitI*: $\vdash_{wf} \Theta \Longrightarrow \Theta; \mathcal{B} \vdash_{wf} B\text{-}unit$
| *wfB-bitvecI*: $\vdash_{wf} \Theta \Longrightarrow \Theta; \mathcal{B} \vdash_{wf} B\text{-}bitvec$
| *wfB-pairI*: $[\![\Theta; \mathcal{B} \vdash_{wf} b1 ; \Theta; \mathcal{B} \vdash_{wf} b2]\!] \Longrightarrow \Theta; \mathcal{B} \vdash_{wf} B\text{-}pair\ b1\ b2$

| *wfB-consI*: $[\![$
$\vdash_{wf} \Theta;$
$(AF\text{-}typedef\ s\ dclist) \in set\ \Theta$
$]\!] \Longrightarrow$
$\Theta; \mathcal{B} \vdash_{wf} B\text{-}id\ s$

| *wfB-appI*: $[\![$
$\vdash_{wf} \Theta;$
$\Theta; \mathcal{B} \vdash_{wf} b;$
$(AF\text{-}typedef\text{-}poly\ s\ bv\ dclist) \in set\ \Theta$
$]\!] \Longrightarrow$

Θ; $\mathcal{B} \vdash_{wf}$ *B-app s b*

| *wfV-varI*: $[\![\ \Theta;\ \mathcal{B} \vdash_{wf} \Gamma$; *Some* (b,c) = *lookup* Γ $x\ ]\!] \Longrightarrow \Theta;\ \mathcal{B};\ \Gamma \vdash_{wf}$ *V-var x* : *b*
| *wfV-litI*: $\Theta;\ \mathcal{B} \vdash_{wf} \Gamma \implies \Theta;\ \mathcal{B};\ \Gamma \vdash_{wf}$ *V-lit l* : *base-for-lit l*

| *wfV-pairI*: $[\![$
$\Theta;\ \mathcal{B};\ \Gamma \vdash_{wf}$ *v1* : *b1* ;
$\Theta;\ \mathcal{B};\ \Gamma \vdash_{wf}$ *v2* : *b2*
$]\!] \Longrightarrow$
$\Theta;\ \mathcal{B};\ \Gamma \vdash_{wf}$ (*V-pair v1 v2*) : *B-pair b1 b2*

| *wfV-consI*: $[\![$
AF-typedef s dclist $\in$ *set* Θ;
$(dc, \{\!|\ x : b'\ |\ c\ |\!\}) \in$ *set dclist* ;
$\Theta;\ \mathcal{B};\ \Gamma \vdash_{wf}$ *v* : b'
$]\!] \Longrightarrow$
$\Theta;\ \mathcal{B};\ \Gamma \vdash_{wf}$ *V-cons s dc v* : *B-id s*

| *wfV-conspI*: $[\![$
AF-typedef-poly s bv dclist $\in$ *set* Θ;
$(dc, \{\!|\ x : b'\ |\ c\ |\!\}) \in$ *set dclist* ;
Θ ; $\mathcal{B}\ \vdash_{wf} b$;
atom bv $\sharp$ $(\Theta, \mathcal{B}, \Gamma, b, v)$;
$\Theta;\ \mathcal{B};\ \Gamma \vdash_{wf} v : b'[bv::=b]_{bb}$
$]\!] \Longrightarrow$
$\Theta;\ \mathcal{B};\ \Gamma \vdash_{wf}$ *V-consp s dc b v* : *B-app s b*

| *wfCE-valI* : $[\![$
$\Theta;\ \mathcal{B};\ \Gamma \vdash_{wf} v$: *b*
$]\!] \Longrightarrow$
$\Theta;\ \mathcal{B};\ \Gamma\ \vdash_{wf}$ *CE-val v* : *b*

| *wfCE-plusI*: $[\![$
$\Theta;\ \mathcal{B};\ \Gamma \vdash_{wf}$ *v1* : *B-int*;
$\Theta;\ \mathcal{B};\ \Gamma \vdash_{wf}$ *v2* : *B-int*
$]\!] \Longrightarrow$
$\Theta;\ \mathcal{B};\ \Gamma\ \vdash_{wf}$ *CE-op Plus v1 v2* : *B-int*

| *wfCE-leqI*:$[\![$
$\Theta;\ \mathcal{B};\ \Gamma \vdash_{wf}$ *v1* : *B-int*;
$\Theta;\ \mathcal{B};\ \Gamma \vdash_{wf}$ *v2* : *B-int*
$]\!] \Longrightarrow$
$\Theta;\ \mathcal{B};\ \Gamma \vdash_{wf}$ *CE-op LEq v1 v2* : *B-bool*

| *wfCE-eqI*:$[\![$
$\Theta;\ \mathcal{B};\ \Gamma \vdash_{wf}$ *v1* : *b*;
$\Theta;\ \mathcal{B};\ \Gamma \vdash_{wf}$ *v2* : *b*
$]\!] \Longrightarrow$
$\Theta;\ \mathcal{B};\ \Gamma \vdash_{wf}$ *CE-op Eq v1 v2* : *B-bool*

| *wfCE-fstI*: $[\![$
$\Theta;\ \mathcal{B};\ \Gamma \vdash_{wf}$ *v1* : *B-pair b1 b2*

$\rrbracket \Longrightarrow$
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *CE-fst v1 : b1*

| *wfCE-sndI*: $[\![$
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf}$ *v1 : B-pair b1 b2*
$\rrbracket \Longrightarrow$
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *CE-snd v1 : b2*

| *wfCE-concatI*: $[\![$
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf}$ *v1 : B-bitvec* ;
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf}$ *v2 : B-bitvec*
$\rrbracket \Longrightarrow$
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *CE-concat v1 v2 : B-bitvec*

| *wfCE-lenI*: $[\![$
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf}$ *v1 : B-bitvec*
$\rrbracket \Longrightarrow$
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *CE-len v1 : B-int*

| *wfTI* : $[\![$
atom z $\sharp$ $(\Theta, \mathcal{B}, \Gamma)$;
Θ; $\mathcal{B} \vdash_{wf} b$;
Θ; $\mathcal{B}$; (*z*,*b*,*C-true*) $\#_\Gamma$ $\Gamma \vdash_{wf} c$
$\rrbracket \Longrightarrow$
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \{\!|\ z : b \mid c\ |\!\}$

| *wfC-eqI*: $[\![$
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *e1* : *b* ;
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *e2* : *b* $\rrbracket \Longrightarrow$
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf}$ *C-eq e1 e2*
| *wfC-trueI*: Θ; $\mathcal{B} \vdash_{wf} \Gamma \Longrightarrow \Theta$; $\mathcal{B}$; Γ $\vdash_{wf}$ *C-true*
| *wfC-falseI*: Θ; $\mathcal{B} \vdash_{wf} \Gamma \Longrightarrow \Theta$; $\mathcal{B}$; $\Gamma \vdash_{wf}$ *C-false*

| *wfC-conjI*: $[\![\ \Theta; \mathcal{B}; \Gamma \vdash_{wf}$ *c1* ; $\Theta; \mathcal{B}; \Gamma \vdash_{wf}$ *c2* $\rrbracket \Longrightarrow \Theta; \mathcal{B}; \Gamma \vdash_{wf}$ *C-conj c1 c2*
| *wfC-disjI*: $[\![\ \Theta; \mathcal{B}; \Gamma \vdash_{wf}$ *c1* ; $\Theta; \mathcal{B}; \Gamma \vdash_{wf}$ *c2* $\rrbracket \Longrightarrow \Theta; \mathcal{B}; \Gamma \vdash_{wf}$ *C-disj c1 c2*
| *wfC-notI*: $[\![\ \Theta; \mathcal{B}; \Gamma \vdash_{wf}$ *c1* $\rrbracket \Longrightarrow \Theta; \mathcal{B}; \Gamma \vdash_{wf}$ *C-not c1*
| *wfC-impI*: $[\![\ \Theta; \mathcal{B}; \Gamma \vdash_{wf}$ *c1* ;
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *c2* $\rrbracket \Longrightarrow \Theta$; $\mathcal{B}$; $\Gamma \vdash_{wf}$ *C-imp c1 c2*

| *wfG-nilI*: $\vdash_{wf} \Theta \Longrightarrow \Theta; \mathcal{B} \vdash_{wf}$ *GNil*
| *wfG-cons1I*: $[\![\ c \notin \{$ *TRUE*, *FALSE* $\}$;
Θ; $\mathcal{B}$ $\vdash_{wf} \Gamma$;
atom x $\sharp$ Γ ;
Θ ; $\mathcal{B}$; (*x*,*b*,*C-true*)$\#_\Gamma\Gamma \vdash_{wf} c$; *wfB* Θ $\mathcal{B}$ *b*
$\rrbracket \Longrightarrow$ Θ; $\mathcal{B}$ $\vdash_{wf}$ $((x,b,c)\#_\Gamma\Gamma)$
| *wfG-cons2I*: $[\![\ c \in \{$ *TRUE*, *FALSE* $\}$;
Θ; $\mathcal{B}$ $\vdash_{wf} \Gamma$;
atom x $\sharp$ Γ ;
wfB Θ $\mathcal{B}$ *b*
$\rrbracket \Longrightarrow$ Θ; $\mathcal{B} \vdash_{wf}$ $((x,b,c)\#_\Gamma\Gamma)$

| *wfTh-emptyI*: $\vdash_{wf}$ []

| *wfTh-consI*: ⟦
(*name-of-type tdef*) ∉ *name-of-type* ‘ *set* Θ ;
$\vdash_{wf}$ Θ ;
Θ $\vdash_{wf}$ *tdef* ⟧ ⟹ $\vdash_{wf}$ *tdef*#Θ

| *wfTD-simpleI*: ⟦
Θ ; {||} ; *GNil* $\vdash_{wf}$ *lst*
⟧ ⟹
Θ $\vdash_{wf}$ (*AF-typedef s lst*)

| *wfTD-poly*: ⟦
Θ ; {|*bv*|} ; *GNil* $\vdash_{wf}$ *lst*
⟧ ⟹
Θ $\vdash_{wf}$ (*AF-typedef-poly s bv lst*)

| *wfTs-nil*: Θ; $\mathcal{B}$ $\vdash_{wf}$ Γ ⟹ Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ []::(*string*∗τ) *list*
| *wfTs-cons*: ⟦ Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ τ ;
dc ∉ *fst* ‘ *set ts*;
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *ts*::(*string*∗τ) *list* ⟧ ⟹ Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ ((*dc*,τ)#*ts*)

inductive-cases *wfC-elims*:
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *C-true*
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *C-false*
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *C-eq e1 e2*
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *C-conj c1 c2*
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *C-disj c1 c2*
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *C-not c1*
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *C-imp c1 c2*

inductive-cases *wfV-elims*:
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *V-var x* : *b*
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *V-lit l* : *b*
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *V-pair v1 v2* : *b*
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *V-cons tyid dc v* : *b*
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *V-consp tyid dc b v* : *b*′

inductive-cases *wfCE-elims*:
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *CE-val v* : *b*
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *CE-op Plus v1 v2* : *b*
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *CE-op LEq v1 v2* : *b*
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *CE-fst v1* : *b*
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *CE-snd v1* : *b*
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *CE-concat v1 v2* : *b*
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *CE-len v1* : *b*
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *CE-op opp v1 v2* : *b*
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *CE-op Eq v1 v2* : *b*

inductive-cases *wfT-elims*:
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ τ::τ
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ {| *z* : *b* | *c* |}

inductive-cases *wfG-elims*:
$\Theta;\ \mathcal{B} \vdash_{wf} GNil$
$\Theta;\ \mathcal{B} \vdash_{wf} (x,b,c)\#_{\Gamma}\Gamma$
$\Theta;\ \mathcal{B} \vdash_{wf} (x,b,TRUE)\#_{\Gamma}\Gamma$
$\Theta;\ \mathcal{B} \vdash_{wf} (x,b,FALSE)\#_{\Gamma}\Gamma$

inductive-cases *wfTh-elims*:
$\vdash_{wf} []$
$\vdash_{wf} td\#\Pi$

inductive-cases *wfTD-elims*:
$\Theta \vdash_{wf} (AF\text{-}typedef\ s\ lst\ )$
$\Theta \vdash_{wf} (AF\text{-}typedef\text{-}poly\ s\ bv\ lst\ )$

inductive-cases *wfTs-elims*:
$\Theta;\ \mathcal{B}\ ;\ GNil \vdash_{wf} ([]::((string*\tau)\ list))$
$\Theta;\ \mathcal{B}\ ;\ GNil \vdash_{wf} ((t\#ts)::((string*\tau)\ list))$

inductive-cases *wfB-elims*:
$\Theta;\ \mathcal{B} \vdash_{wf} B\text{-}pair\ b1\ b2$
$\Theta;\ \mathcal{B} \vdash_{wf} B\text{-}id\ s$
$\Theta;\ \mathcal{B} \vdash_{wf} B\text{-}app\ s\ b$

equivariance *wfV*

This setup of 'avoiding' is not complete and for some of lemmas, such as weakening, do it the hard way

nominal-inductive *wfV*
avoids *wfV-conspI*: *bv* | *wfTI*: *z*
⟨*proof*⟩

inductive
wfE :: $\Theta \Rightarrow \Phi \Rightarrow \mathcal{B} \Rightarrow \Gamma \Rightarrow \Delta \Rightarrow e \Rightarrow b \Rightarrow bool\ (\langle$ - ; - ; - ; - ; - $\vdash_{wf}$ - : - $\rangle\ [50,50,50]\ 50)$ **and**
wfS :: $\Theta \Rightarrow \Phi \Rightarrow \mathcal{B} \Rightarrow \Gamma \Rightarrow \Delta \Rightarrow s \Rightarrow b \Rightarrow bool\ (\langle$ - ; - ; - ; - ; - $\vdash_{wf}$ - : - $\rangle\ [50,50,50]\ 50)$ **and**
wfCS :: $\Theta \Rightarrow \Phi \Rightarrow \mathcal{B} \Rightarrow \Gamma \Rightarrow \Delta \Rightarrow$ *tyid* $\Rightarrow$ *string* $\Rightarrow \tau \Rightarrow$ *branch-s* $\Rightarrow b \Rightarrow bool\ (\langle$ - ; - ; - ; - ; - ; - ; - ; - $\vdash_{wf}$ - : - $\rangle\ [50,50,50,50,50]\ 50)$ **and**
wfCSS :: $\Theta \Rightarrow \Phi \Rightarrow \mathcal{B} \Rightarrow \Gamma \Rightarrow \Delta \Rightarrow$ *tyid* $\Rightarrow$ $(string * \tau)\ list \Rightarrow$ *branch-list* $\Rightarrow b \Rightarrow bool\ (\langle$ - ; - ; - ; - ; - ; - ; - $\vdash_{wf}$ - : - $\rangle\ [50,50,50,50,50]\ 50)$ **and**
wfPhi :: $\Theta \Rightarrow \Phi \Rightarrow bool\ (\langle$ - $\vdash_{wf}$ - $\rangle\ [50,50]\ 50)$ **and**
wfD :: $\Theta \Rightarrow \mathcal{B} \Rightarrow \Gamma \Rightarrow \Delta \Rightarrow bool\ (\langle$ - ; - ; - $\vdash_{wf}$ - $\rangle\ [50,50]\ 50)$ **and**
wfFTQ :: $\Theta \Rightarrow \Phi \Rightarrow$ *fun-typ-q* $\Rightarrow bool\ (\langle$ - ; - $\vdash_{wf}$ - $\rangle\ [50]\ 50)$ **and**
wfFT :: $\Theta \Rightarrow \Phi \Rightarrow \mathcal{B} \Rightarrow$ *fun-typ* $\Rightarrow bool\ (\langle$ - ; - ; - $\vdash_{wf}$ - $\rangle\ [50]\ 50)$ **where**

wfE-valI : ⟦
$\Theta \vdash_{wf} \Phi;$
$\Theta;\ \mathcal{B};\ \Gamma \vdash_{wf} \Delta;$
$\Theta;\ \mathcal{B};\ \Gamma \vdash_{wf} v\ :\ b$
⟧ ⟹
$\Theta;\ \Phi;\ \mathcal{B};\ \Gamma;\ \Delta\ \vdash_{wf} AE\text{-}val\ v\ :\ b$

| *wfE-plusI*: ⟦
$\Theta \vdash_{wf} \Phi;$

Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$;
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf}$ *v1* : *B-int*;
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf}$ *v2* : *B-int*
$]\!] \Longrightarrow$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash_{wf}$ *AE-op Plus v1 v2* : *B-int*

| *wfE-leqI*:$[\![$
$\Theta \vdash_{wf} \Phi$;
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$;
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf}$ *v1* : *B-int*;
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf}$ *v2* : *B-int*
$]\!] \Longrightarrow$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash_{wf}$ *AE-op LEq v1 v2* : *B-bool*

| *wfE-eqI*:$[\![$
$\Theta \vdash_{wf} \Phi$;
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$;
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf}$ *v1* : *b*;
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf}$ *v2* : *b*;
b $\in$ { *B-bool*, *B-int*, *B-unit* }
$]\!] \Longrightarrow$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash_{wf}$ *AE-op Eq v1 v2* : *B-bool*

| *wfE-fstI*: $[\![$
$\Theta \vdash_{wf} \Phi$;
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$;
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf}$ *v1* : *B-pair b1 b2*
$]\!] \Longrightarrow$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash_{wf}$ *AE-fst v1* : *b1*

| *wfE-sndI*: $[\![$
$\Theta \vdash_{wf} \Phi$;
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$;
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf}$ *v1* : *B-pair b1 b2*
$]\!] \Longrightarrow$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash_{wf}$ *AE-snd v1* : *b2*

| *wfE-concatI*: $[\![$
$\Theta \vdash_{wf} \Phi$;
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$;
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf}$ *v1* : *B-bitvec*;
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf}$ *v2* : *B-bitvec*
$]\!] \Longrightarrow$
Θ ; Φ ; $\mathcal{B}$; Γ; $\Delta \vdash_{wf}$ *AE-concat v1 v2* : *B-bitvec*

| *wfE-splitI*: $[\![$
$\Theta \vdash_{wf} \Phi$;
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$;
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf}$ *v1* : *B-bitvec*;
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf}$ *v2* : *B-int*
$]\!] \Longrightarrow$
Θ ; Φ ; $\mathcal{B}$; Γ; $\Delta \vdash_{wf}$ *AE-split v1 v2* : *B-pair B-bitvec B-bitvec*

| *wfE-lenI*: ⟦
  $\Theta \vdash_{wf} \Phi$;
  Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$;
  Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} v1 : B\text{-}bitvec$
⟧ ⟹
  Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash_{wf} AE\text{-}len\ v1 : B\text{-}int$

| *wfE-appI*: ⟦
  $\Theta \vdash_{wf} \Phi$;
  Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$;
  $Some\ (AF\text{-}fundef\ f\ (AF\text{-}fun\text{-}typ\text{-}none\ (AF\text{-}fun\text{-}typ\ x\ b\ c\ \tau\ s))) = lookup\text{-}fun\ \Phi\ f$;
  Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} v : b$
⟧ ⟹
  Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash_{wf} AE\text{-}app\ f\ v : b\text{-}of\ \tau$

| *wfE-appPI*: ⟦
  $\Theta \vdash_{wf} \Phi$;
  Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$;
  Θ; $\mathcal{B} \vdash_{wf} b'$;
  $atom\ bv \sharp (\Phi, \Theta, \mathcal{B}, \Gamma, \Delta, b', v, (b\text{-}of\ \tau)[bv::=b']_b)$;
  $Some\ (AF\text{-}fundef\ f\ (AF\text{-}fun\text{-}typ\text{-}some\ bv\ (AF\text{-}fun\text{-}typ\ x\ b\ c\ \tau\ s))) = lookup\text{-}fun\ \Phi\ f$;
  Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} v : (b[bv::=b']_b)$
⟧ ⟹
  Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash_{wf} (AE\text{-}appP\ f\ b'\ v) : ((b\text{-}of\ \tau)[bv::=b']_b)$

| *wfE-mvarI*: ⟦
  $\Theta \vdash_{wf} \Phi$;
  Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$;
  $(u,\tau) \in setD\ \Delta$
⟧ ⟹
  Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash_{wf} AE\text{-}mvar\ u : b\text{-}of\ \tau$

| *wfS-valI*: ⟦
  $\Theta \vdash_{wf} \Phi$;
  Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} v : b$;
  Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$
⟧ ⟹
  Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash_{wf} (AS\text{-}val\ v) : b$

| *wfS-letI*: ⟦
  $wfE\ \Theta\ \Phi\ \mathcal{B}\ \Gamma\ \Delta\ \ e\ b'$;
  Θ ; Φ ; $\mathcal{B}$; $(x,b',C\text{-}true) \#_\Gamma \Gamma$; $\Delta \vdash_{wf} s : b$;
  Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$;
  $atom\ x \sharp (\Phi, \Theta, \mathcal{B}, \Gamma, \Delta, e, b)$
⟧ ⟹
  Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash_{wf} LET\ x = e\ IN\ s : b$

| *wfS-assertI*: ⟦
  Θ ; Φ ; $\mathcal{B}$; $(x,B\text{-}bool,c) \#_\Gamma \Gamma$; $\Delta \vdash_{wf} s : b$;
  Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} c$;
  Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$;

$$
\begin{array}{l}
\quad \mathit{atom}\ x \sharp (\Phi,\ \Theta,\ \mathcal{B},\ \Gamma,\ \Delta,\ c,\ b,\ s) \\
[\![\Longrightarrow \\
\quad \Theta;\ \Phi;\ \mathcal{B};\ \Gamma;\ \Delta \vdash_{wf} \mathit{ASSERT}\ c\ \mathit{IN}\ \ s : b \\
\\
\mid \mathit{wfS\text{-}let2I}: [\![\\
\quad \Theta;\ \Phi;\ \mathcal{B};\ \Gamma;\ \Delta\ \vdash_{wf} \mathit{s1} : \mathit{b\text{-}of}\ \tau\ \ ; \\
\quad \Theta;\ \mathcal{B};\ \Gamma \vdash_{wf} \tau; \\
\quad \Theta\ ;\ \Phi\ ;\ \mathcal{B}\ ;\ (x,\mathit{b\text{-}of}\ \tau,\mathit{C\text{-}true})\ \#_{\Gamma}\ \Gamma\ ;\ \Delta \vdash_{wf} \mathit{s2} : b\ ; \\
\quad \mathit{atom}\ x \sharp (\Phi,\ \Theta,\ \mathcal{B},\ \Gamma,\ \Delta,\ \mathit{s1},\ b,\tau) \\
]\!] \Longrightarrow \\
\quad \Theta;\ \Phi;\ \mathcal{B};\ \Gamma;\ \Delta \vdash_{wf} \mathit{LET}\ x : \tau = \mathit{s1}\ \mathit{IN}\ \mathit{s2} : b \\
\\
\mid \mathit{wfS\text{-}ifI}: [\![\\
\quad \Theta;\ \mathcal{B};\ \Gamma \vdash_{wf} v\ \ : \mathit{B\text{-}bool}; \\
\quad \Theta;\ \Phi;\ \mathcal{B};\ \Gamma;\ \Delta \vdash_{wf} \mathit{s1} : b\ ; \\
\quad \Theta;\ \Phi;\ \mathcal{B};\ \Gamma;\ \Delta \vdash_{wf} \mathit{s2} : b \\
]\!] \Longrightarrow \\
\quad \Theta;\ \Phi;\ \mathcal{B};\ \Gamma;\ \Delta \vdash_{wf} \mathit{IF}\ v\ \mathit{THEN}\ \mathit{s1}\ \mathit{ELSE}\ \mathit{s2} : b \\
\\
\mid \mathit{wfS\text{-}varI}\ : [\![\\
\quad \mathit{wfT}\ \Theta\ \mathcal{B}\ \Gamma\ \ \tau\ ; \\
\quad \Theta;\ \mathcal{B};\ \Gamma \vdash_{wf} v\ \ : \mathit{b\text{-}of}\ \tau; \\
\quad \mathit{atom}\ u \sharp (\Phi,\ \Theta,\ \mathcal{B},\ \Gamma,\ \Delta,\ \tau,\ v,\ b); \\
\quad \Theta\ ;\ \Phi\ ;\ \mathcal{B}\ ;\ \Gamma\ ;\ \ (u,\tau)\#_{\Delta}\Delta \vdash_{wf} s : b \\
]\!] \Longrightarrow \\
\quad \Theta;\ \Phi;\ \mathcal{B};\ \Gamma;\ \Delta \vdash_{wf} \mathit{VAR}\ u : \tau = v\ \mathit{IN}\ s : b \\
\\
\mid \mathit{wfS\text{-}assignI}: [\![\\
\quad (u,\tau) \in \mathit{setD}\ \Delta\ ; \\
\quad \Theta;\ \mathcal{B};\ \Gamma \vdash_{wf} \Delta\ ; \\
\quad \Theta \vdash_{wf} \Phi\ ; \\
\quad \Theta;\ \mathcal{B};\ \Gamma \vdash_{wf} v\ \ : \mathit{b\text{-}of}\ \tau \\
]\!] \Longrightarrow \\
\quad \Theta;\ \Phi;\ \mathcal{B};\ \Gamma;\ \Delta \vdash_{wf} u ::= v : \mathit{B\text{-}unit} \\
\\
\mid \mathit{wfS\text{-}whileI}: [\![\\
\quad \Theta;\ \Phi;\ \mathcal{B};\ \Gamma;\ \Delta \vdash_{wf} \mathit{s1} : \mathit{B\text{-}bool}\ ; \\
\quad \Theta;\ \Phi;\ \mathcal{B};\ \Gamma;\ \Delta \vdash_{wf} \mathit{s2} : b \\
]\!] \Longrightarrow \\
\quad \Theta;\ \Phi;\ \mathcal{B};\ \Gamma;\ \Delta \vdash_{wf} \mathit{WHILE}\ \mathit{s1}\ \mathit{DO}\ \{\ \mathit{s2}\ \} : b \\
\\
\mid \mathit{wfS\text{-}seqI}: [\![\\
\quad \Theta;\ \Phi;\ \mathcal{B};\ \Gamma;\ \Delta \vdash_{wf} \mathit{s1} : \mathit{B\text{-}unit}\ ; \\
\quad \Theta;\ \Phi;\ \mathcal{B};\ \Gamma;\ \Delta \vdash_{wf} \mathit{s2} : b \\
]\!] \Longrightarrow \\
\quad \Theta;\ \Phi;\ \mathcal{B};\ \Gamma;\ \Delta \vdash_{wf} \mathit{s1}\ ;;\ \mathit{s2} : b \\
\\
\mid \mathit{wfS\text{-}matchI}: [\![\\
\quad \mathit{wfV}\ \Theta\ \ \mathcal{B}\ \Gamma\ \ v\ \ (\mathit{B\text{-}id}\ \mathit{tid})\ ; \\
\quad (\mathit{AF\text{-}typedef}\ \mathit{tid}\ \mathit{dclist}\ ) \in \mathit{set}\ \Theta; \\
\quad \mathit{wfD}\ \Theta\ \ \mathcal{B}\ \Gamma\ \ \Delta\ ; \\
\quad \Theta \vdash_{wf} \Phi\ ;
\end{array}
$$

Θ ; Φ ; $\mathcal{B}$; Γ ; Δ ; *tid* ; *dclist* $\vdash_{wf}$ *cs* : *b*
$\rrbracket \Longrightarrow$
Θ; Φ; $\mathcal{B}$; Γ; Δ $\vdash_{wf}$ *AS-match v cs* : *b*

| *wfS-branchI*: $[\![$
Θ ; Φ ; $\mathcal{B}$; (*x*,*b-of* τ,*C-true*) $\#_\Gamma$ Γ ; $\Delta \vdash_{wf} s : b$;
atom x $\sharp$ (Φ, Θ, $\mathcal{B}$, Γ, Δ, Γ,τ);
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$
$\rrbracket \Longrightarrow$
Θ ; Φ ; $\mathcal{B}$; Γ ; Δ ; *tid* ; *dc* ; $\tau \vdash_{wf}$ *dc x* $\Rightarrow$ *s* : *b*

| *wfS-finalI*: $[\![$
Θ; Φ; $\mathcal{B}$; Γ; Δ ; *tid* ; *dc* ; *t* $\vdash_{wf}$ *cs* : *b*
$\rrbracket \Longrightarrow$
Θ ; Φ ; $\mathcal{B}$; Γ ; Δ ; *tid* ; [(*dc*,*t*)] $\vdash_{wf}$ *AS-final cs* : *b*

| *wfS-cons*: $[\![$
Θ; Φ; $\mathcal{B}$; Γ; Δ ; *tid* ; *dc* ; *t* $\vdash_{wf}$ *cs* : *b*;
Θ; Φ; $\mathcal{B}$; Γ; Δ ; *tid* ; *dclist* $\vdash_{wf}$ *css* : *b*
$\rrbracket \Longrightarrow$
Θ ; Φ ; $\mathcal{B}$; Γ ; Δ ; *tid* ; (*dc*,*t*)#*dclist* $\vdash_{wf}$ *AS-cons cs css* : *b*

| *wfD-emptyI*: Θ; $\mathcal{B} \vdash_{wf} \Gamma \Longrightarrow \Theta$; $\mathcal{B}$; $\Gamma \vdash_{wf} []_\Delta$

| *wfD-cons*: $[\![$
Θ ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$::$\Delta$;
Θ ; $\mathcal{B}$; $\Gamma \vdash_{wf} \tau$;
u $\notin$ *fst* ‘ *setD* Δ
$\rrbracket \Longrightarrow$
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf}$ ((u,τ) $\#_\Delta$ Δ)

| *wfPhi-emptyI*: $\vdash_{wf} \Theta \Longrightarrow \Theta \vdash_{wf} []$

| *wfPhi-consI*: $[\![$
f $\notin$ *name-of-fun* ‘ *set* Φ;
Θ ; $\Phi \vdash_{wf}$ *ft*;
$\Theta \vdash_{wf} \Phi$
$\rrbracket \Longrightarrow$
$\Theta \vdash_{wf}$ ((*AF-fundef f ft*)#Φ)

| *wfFTNone*: Θ ; Φ ; $\{||\} \vdash_{wf} ft \Longrightarrow \Theta$; $\Phi \vdash_{wf}$ *AF-fun-typ-none ft*
| *wfFTSome*: Θ ; Φ ; $\{|$ *bv* $|\} \vdash_{wf} ft \Longrightarrow \Theta$; $\Phi \vdash_{wf}$ *AF-fun-typ-some bv ft*

| *wfFTI*: $[\![$
Θ ; *B* $\vdash_{wf}$ *b*;
supp s $\subseteq$ {*atom x*} $\cup$ *supp B* ;
supp c $\subseteq$ { *atom x* } ;
Θ ; *B* ; (*x*,*b*,*c*) $\#_\Gamma$ *GNil* $\vdash_{wf} \tau$;
$\Theta \vdash_{wf} \Phi$
$\rrbracket \Longrightarrow$
Θ ; Φ ; *B* $\vdash_{wf}$ (*AF-fun-typ x b c* τ *s*)

inductive-cases *wfE-elims*:
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash_{wf}$ *AE-val v* : *b*
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash_{wf}$ *AE-op Plus v1 v2* : *b*
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash_{wf}$ *AE-op LEq v1 v2* : *b*
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash_{wf}$ *AE-fst v1* : *b*
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash_{wf}$ *AE-snd v1* : *b*
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash_{wf}$ *AE-concat v1 v2* : *b*
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash_{wf}$ *AE-len v1* : *b*
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash_{wf}$ *AE-op opp v1 v2* : *b*
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash_{wf}$ *AE-app f v*: *b*
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash_{wf}$ *AE-appP f b′ v*: *b*
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash_{wf}$ *AE-mvar u* : *b*
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash_{wf}$ *AE-op Eq v1 v2* : *b*

inductive-cases *wfCS-elims*:
Θ; Φ; $\mathcal{B}$; Γ; Δ ; *tid* ; *dc* ; *t* $\vdash_{wf}$ (*cs*::*branch-s*) : *b*
Θ; Φ; $\mathcal{B}$; Γ; Δ ; *tid* ; *dc* $\vdash_{wf}$ (*cs*::*branch-list*) : *b*

inductive-cases *wfPhi-elims*:
$\Theta \vdash_{wf} []$
$\Theta \vdash_{wf}$ ((*AF-fundef f ft*)#Π)
$\Theta \vdash_{wf}$ (*fd*#Φ::Φ)

declare[[*simproc del*: *alpha-lst*]]

inductive-cases *wfFTQ-elims*:
Θ ; Φ $\vdash_{wf}$ *AF-fun-typ-none ft*
Θ ; Φ $\vdash_{wf}$ *AF-fun-typ-some bv ft*
Θ ; Φ $\vdash_{wf}$ *AF-fun-typ-some bv* (*AF-fun-typ x b c* τ *s*)

inductive-cases *wfFT-elims*:
Θ ; Φ ; $\mathcal{B}$ $\vdash_{wf}$ *AF-fun-typ x b c* τ *s*

declare[[*simproc add*: *alpha-lst*]]

inductive-cases *wfD-elims*:
Π ; $\mathcal{B}$; (Γ::Γ) $\vdash_{wf}$ $[]_\Delta$
Π ; $\mathcal{B}$; (Γ::Γ) $\vdash_{wf}$ (*u*,τ) $\#_\Delta$ Δ::Δ

equivariance *wfE*

nominal-inductive *wfE*
avoids *wfE-appPI*: *bv* | *wfS-varI*: *u* | *wfS-letI*: *x* | *wfS-let2I*: *x* | *wfS-branchI*: *x* | *wfS-assertI*: *x*

⟨*proof*⟩

inductive *wfVDs* :: *var-def list* ⇒ *bool* **where**

wfVDs-nilI: *wfVDs* []

| *wfVDs-consI*: ⟦
atom u ♯ *ts*;

$wfV\ ([]::\Theta)\ \{||\}\ \ GNil\ \ v\ \ (b\text{-}of\ \tau);$
$wfT\ ([]::\Theta)\ \ \{||\}\ \ GNil\ \ \tau;$
$wfVDs\ ts$
$]\!] \Longrightarrow wfVDs\ \ ((AV\text{-}def\ u\ \tau\ v)\#ts)$

equivariance *wfVDs*
nominal-inductive *wfVDs* ⟨*proof*⟩

end

hide-const *Syntax.dom*

Chapter 6

Refinement Constraint Logic

Semantics for the logic we use in the refinement constraints. It is a multi-sorted, quantifier free logic with polymorphic datatypes and linear arithmetic. We could have modelled by using one of the encodings to FOL however we wanted to explore using a more direct model.

6.1 Evaluation and Satisfiability

6.1.1 Valuation

Refinement constraint logic values. SUt is a value for the uninterpreted sort that corresponds to basic type variables. For now we only need one of these universes. We wrap an smt_val inside it during a process we call 'boxing' which is introduced in the RCLogicL theory

nominal-datatype *rcl-val = SBitvec bit list | SNum int | SBool bool | SPair rcl-val rcl-val | SCons tyid string rcl-val | SConsp tyid string b rcl-val | SUnit | SUt rcl-val*

RCL sorts. Represent our domains. The universe is the union of all of the these. S_Ut is the single uninterpreted sort. These map almost directly to basic type but we have them to distinguish syntax (basic types) and semantics (RCL sorts)

nominal-datatype *rcl-sort = S-bool | S-int | S-unit | S-pair rcl-sort rcl-sort | S-id tyid | S-app tyid rcl-sort | S-bitvec | S-ut*

type-synonym *valuation = (x,rcl-val) map*

type-synonym *type-valuation = (bv,rcl-sort) map*

Well-sortedness for RCL values

inductive *wfRCV*:: $\Theta \Rightarrow$ *rcl-val* $\Rightarrow$ *b* $\Rightarrow$ *bool* (‹ - ⊢ - : -› [50,50] 50) **where**
wfRCV-BBitvecI: $P \vdash (SBitvec\ bv) : B\text{-}bitvec$
| *wfRCV-BIntI*: $P \vdash (SNum\ n) : B\text{-}int$
| *wfRCV-BBoolI*: $P \vdash (SBool\ b) : B\text{-}bool$
| *wfRCV-BPairI*: $[\![P \vdash s1 : b1 ; P \vdash s2 : b2]\!] \Longrightarrow P \vdash (SPair\ s1\ s2) : (B\text{-}pair\ b1\ b2)$
| *wfRCV-BConsI*: $[\![$ *AF-typedef s dclist* $\in$ *set* Θ;
$(dc, \{\!|\ x : b\ |\ c\ |\!\}) \in$ *set dclist* ;
$\Theta \vdash s1 : b\ ]\!] \Longrightarrow \Theta \vdash (SCons\ s\ dc\ s1) : (B\text{-}id\ s)$
| *wfRCV-BConsPI*:$[\![$ *AF-typedef-poly s bv dclist* $\in$ *set* Θ;

$(dc, \{\!|\ x : b\ \mid c\ |\!\}) \in set\ dclist$;
$atom\ bv \sharp (\Theta, SConsp\ s\ dc\ b'\ s1, B\text{-}app\ s\ b')$;
$\Theta \vdash s1 : b[bv::=b']_{bb}\ ]\!] \Longrightarrow \Theta \vdash (SConsp\ s\ dc\ b'\ s1) : (B\text{-}app\ s\ b')$
| *wfRCV-BUnitI*: $P \vdash SUnit : B\text{-}unit$
| *wfRCV-BVarI*: $P \vdash (SUt\ n) : (B\text{-}var\ bv)$
equivariance *wfRCV*
nominal-inductive *wfRCV*
avoids *wfRCV-BConsPI*: *bv*
⟨*proof*⟩

inductive-cases *wfRCV-elims* :
$wfRCV\ P\ s\ \ B\text{-}bitvec$
$wfRCV\ P\ s\ (B\text{-}pair\ b1\ b2)$
$wfRCV\ P\ s\ (B\text{-}int)$
$wfRCV\ P\ s\ (B\text{-}bool)$
$wfRCV\ P\ s\ (B\text{-}id\ ss)$
$wfRCV\ P\ s\ (B\text{-}var\ bv)$
$wfRCV\ P\ s\ (B\text{-}unit)$
$wfRCV\ P\ s\ (B\text{-}app\ tyid\ b)$
$wfRCV\ P\ (SBitvec\ bv)\ b$
$wfRCV\ P\ (SNum\ n)\ \ b$
$wfRCV\ P\ (SBool\ n)\ \ b$
$wfRCV\ P\ (SPair\ s1\ s2)\ b$
$wfRCV\ P\ (SCons\ s\ dc\ s1)\ b$
$wfRCV\ P\ (SConsp\ s\ dc\ b'\ s1)\ b$
$wfRCV\ P\ SUnit\ b$
$wfRCV\ P\ (SUt\ s1)\ b$

Sometimes we want to assert $P \vdash s \sim b[bv{=}b']$ and we want to know what b is however substitution is not injective so we can't write this in terms of *wfRCV*. So we define a relation that makes the components of the substitution explicit.

inductive *wfRCV-subst*:: $\Theta \Rightarrow rcl\text{-}val \Rightarrow b \Rightarrow (bv{*}b)\ option \Rightarrow bool$ **where**
$wfRCV\text{-}subst\text{-}BBitvecI:\ \ wfRCV\text{-}subst\ P\ (SBitvec\ bv)\ B\text{-}bitvec\ \ sub$
| $wfRCV\text{-}subst\text{-}BIntI:\ \ wfRCV\text{-}subst\ P\ (SNum\ n)\ \ B\text{-}int\ sub$
| $wfRCV\text{-}subst\text{-}BBoolI:\ wfRCV\text{-}subst\ P\ (SBool\ b)\ \ B\text{-}bool\ sub$
| $wfRCV\text{-}subst\text{-}BPairI:\ [\![\ wfRCV\text{-}subst\ P\ s1\ b1\ sub\ ;\ wfRCV\text{-}subst\ P\ s2\ b2\ sub\ ]\!] \Longrightarrow wfRCV\text{-}subst\ P\ (SPair\ s1\ s2)\ (B\text{-}pair\ b1\ b2)\ sub$
| $wfRCV\text{-}subst\text{-}BConsI:\ [\![\ AF\text{-}typedef\ s\ dclist \in set\ \Theta;$
$(dc, \{\!|\ x : b\ \mid c\ |\!\}) \in set\ dclist$;
$wfRCV\text{-}subst\ \Theta\ s1\ b\ None\ ]\!] \Longrightarrow wfRCV\text{-}subst\ \Theta\ (SCons\ s\ dc\ s1)\ (B\text{-}id\ s)\ sub$
| $wfRCV\text{-}subst\text{-}BConspI:\ [\![\ AF\text{-}typedef\text{-}poly\ s\ bv\ dclist \in set\ \Theta;$
$(dc, \{\!|\ x : b\ \mid c\ |\!\}) \in set\ dclist$;
$wfRCV\text{-}subst\ \Theta\ s1\ (b[bv::=b']_{bb})\ sub\ \ ]\!] \Longrightarrow wfRCV\text{-}subst\ \Theta\ (SConsp\ s\ dc\ b'\ s1)\ (B\text{-}app\ s\ b')\ sub$
| $wfRCV\text{-}subst\text{-}BUnitI:\ wfRCV\text{-}subst\ P\ SUnit\ B\text{-}unit\ sub$
| $wfRCV\text{-}subst\text{-}BVar1I:\ bvar \neq bv \Longrightarrow wfRCV\text{-}subst\ P\ (SUt\ n)\ (B\text{-}var\ bv)\ \ (Some\ (bvar,bin))$
| $wfRCV\text{-}subst\text{-}BVar2I:\ [\![\ bvar = bv;\ wfRCV\text{-}subst\ P\ s\ bin\ None\ ]\!] \Longrightarrow wfRCV\text{-}subst\ P\ s\ (B\text{-}var\ bv)\ (Some\ (bvar,bin))$
| $wfRCV\text{-}subst\text{-}BVar3I:\ wfRCV\text{-}subst\ P\ (SUt\ n)\ (B\text{-}var\ bv)\ \ None$
equivariance *wfRCV-subst*
nominal-inductive *wfRCV-subst* ⟨*proof*⟩

6.1.2 Evaluation base-types

inductive *eval-b* :: *type-valuation* ⇒ *b* ⇒ *rcl-sort* ⇒ *bool* (‹- ⟦ - ⟧ ~ - ›) **where**
  v ⟦ *B-bool* ⟧ ~ *S-bool*
| *v* ⟦ *B-int* ⟧ ~ *S-int*
| *Some s* = *v bv* ⟹ *v* ⟦ *B-var bv* ⟧ ~ *s*
equivariance *eval-b*
nominal-inductive *eval-b* ⟨*proof*⟩

6.1.3 Wellformed vvaluations

definition *wfI* :: Θ ⇒ Γ ⇒ *valuation* ⇒ *bool* (‹ - ; - ⊢ -›) **where**
  Θ ; Γ ⊢ *i* = (∀ (*x*,*b*,*c*) ∈ *toSet* Γ. ∃ *s*. *Some s* = *i x* ∧ Θ ⊢ *s* : *b*)

6.1.4 Evaluating Terms

nominal-function *eval-l* :: *l* ⇒ *rcl-val* (‹⟦ - ⟧ ›) **where**
  ⟦ *L-true* ⟧ = *SBool True*
| ⟦ *L-false* ⟧ = *SBool False*
| ⟦ *L-num n* ⟧ = *SNum n*
| ⟦ *L-unit* ⟧ = *SUnit*
| ⟦ *L-bitvec n* ⟧ = *SBitvec n*
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

inductive *eval-v* :: *valuation* ⇒ *v* ⇒ *rcl-val* ⇒ *bool* (‹- ⟦ - ⟧ ~ - ›) **where**
  eval-v-litI: *i* ⟦ *V-lit l* ⟧ ~ ⟦ *l* ⟧
| *eval-v-varI*: *Some sv* = *i x* ⟹ *i* ⟦ *V-var x* ⟧ ~ *sv*
| *eval-v-pairI*: ⟦ *i* ⟦ *v1* ⟧ ~ *s1* ; *i* ⟦ *v2* ⟧ ~ *s2* ⟧ ⟹ *i* ⟦ *V-pair v1 v2* ⟧ ~ *SPair s1 s2*
| *eval-v-consI*: *i* ⟦ *v* ⟧ ~ *s* ⟹ *i* ⟦ *V-cons tyid dc v* ⟧ ~ *SCons tyid dc s*
| *eval-v-conspI*: *i* ⟦ *v* ⟧ ~ *s* ⟹ *i* ⟦ *V-consp tyid dc b v* ⟧ ~ *SConsp tyid dc b s*
equivariance *eval-v*
nominal-inductive *eval-v* ⟨*proof*⟩

inductive-cases *eval-v-elims*:
  i ⟦ *V-lit l* ⟧ ~ *s*
  i ⟦ *V-var x* ⟧ ~ *s*
  i ⟦ *V-pair v1 v2* ⟧ ~ *s*
  i ⟦ *V-cons tyid dc v* ⟧ ~ *s*
  i ⟦ *V-consp tyid dc b v* ⟧ ~ *s*

inductive *eval-e* :: *valuation* ⇒ *ce* ⇒ *rcl-val* ⇒ *bool* (‹- ⟦ - ⟧ ~ - ›) **where**
  eval-e-valI: *i* ⟦ *v* ⟧ ~ *sv* ⟹ *i* ⟦ *CE-val v* ⟧ ~ *sv*
| *eval-e-plusI*: ⟦ *i* ⟦ *v1* ⟧ ~ *SNum n1*; *i* ⟦ *v2* ⟧ ~ *SNum n2* ⟧ ⟹ *i* ⟦ (*CE-op Plus v1 v2*) ⟧ ~ (*SNum* (*n1*+*n2*))
| *eval-e-leqI*: ⟦ *i* ⟦ *v1* ⟧ ~ (*SNum n1*); *i* ⟦ *v2* ⟧ ~ (*SNum n2*) ⟧ ⟹ *i* ⟦ (*CE-op LEq v1 v2*) ⟧ ~ (*SBool* (*n1* ≤ *n2*))
| *eval-e-eqI*: ⟦ *i* ⟦ *v1* ⟧ ~ *s1*; *i* ⟦ *v2* ⟧ ~ *s2* ⟧ ⟹ *i* ⟦ (*CE-op Eq v1 v2*) ⟧ ~ (*SBool* (*s1* = *s2*))
| *eval-e-fstI*: ⟦ *i* ⟦ *v* ⟧ ~ *SPair v1 v2* ⟧ ⟹ *i* ⟦ (*CE-fst v*) ⟧ ~ *v1*
| *eval-e-sndI*: ⟦ *i* ⟦ *v* ⟧ ~ *SPair v1 v2* ⟧ ⟹ *i* ⟦ (*CE-snd v*) ⟧ ~ *v2*
| *eval-e-concatI*:⟦ *i* ⟦ *v1* ⟧ ~ (*SBitvec bv1*); *i* ⟦ *v2* ⟧ ~ (*SBitvec bv2*) ⟧ ⟹ *i* ⟦ (*CE-concat v1 v2*) ⟧ ~ (*SBitvec* (*bv1*@*bv2*))
| *eval-e-lenI*:⟦ *i* ⟦ *v* ⟧ ~ (*SBitvec bv*) ⟧ ⟹ *i* ⟦ (*CE-len v*) ⟧ ~ (*SNum* (*int* (*List.length bv*)))

equivariance *eval-e*
nominal-inductive *eval-e* ⟨*proof*⟩

inductive-cases *eval-e-elims*:
i ⟦ (*CE-val v*) ⟧ ~ *s*
i ⟦ (*CE-op Plus v1 v2*) ⟧ ~ *s*
i ⟦ (*CE-op LEq v1 v2*) ⟧ ~ *s*
i ⟦ (*CE-op Eq v1 v2*) ⟧ ~ *s*
i ⟦ (*CE-fst v*) ⟧ ~ *s*
i ⟦ (*CE-snd v*) ⟧ ~ *s*
i ⟦ (*CE-concat v1 v2*) ⟧ ~ *s*
i ⟦ (*CE-len v*) ⟧ ~ *s*

inductive *eval-c* :: *valuation* ⇒ *c* ⇒ *bool* ⇒ *bool* (‹ - ⟦ - ⟧ ~ - ›) **where**
eval-c-trueI: *i* ⟦ *C-true* ⟧ ~ *True*
| *eval-c-falseI*:*i* ⟦ *C-false* ⟧ ~ *False*
| *eval-c-conjI*: ⟦ *i* ⟦ *c1* ⟧ ~ *b1* ; *i* ⟦ *c2* ⟧ ~ *b2* ⟧ ⟹ *i* ⟦ (*C-conj c1 c2*) ⟧ ~ (*b1* ∧ *b2*)
| *eval-c-disjI*: ⟦ *i* ⟦ *c1* ⟧ ~ *b1* ; *i* ⟦ *c2* ⟧ ~ *b2* ⟧ ⟹ *i* ⟦ (*C-disj c1 c2*) ⟧ ~ (*b1* ∨ *b2*)
| *eval-c-impI*:⟦ *i* ⟦ *c1* ⟧ ~ *b1* ; *i* ⟦ *c2* ⟧ ~ *b2* ⟧ ⟹ *i* ⟦ (*C-imp c1 c2*) ⟧ ~ (*b1* ⟶ *b2*)
| *eval-c-notI*:⟦ *i* ⟦ *c* ⟧ ~ *b* ⟧ ⟹ *i* ⟦ (*C-not c*) ⟧ ~ (¬ *b*)
| *eval-c-eqI*:⟦ *i* ⟦ *e1* ⟧ ~ *sv1*; *i* ⟦ *e2* ⟧ ~ *sv2* ⟧ ⟹ *i* ⟦ (*C-eq e1 e2*) ⟧ ~ (*sv1*=*sv2*)
equivariance *eval-c*
nominal-inductive *eval-c* ⟨*proof*⟩

inductive-cases *eval-c-elims*:
i ⟦ *C-true* ⟧ ~ *True*
i ⟦ *C-false* ⟧ ~ *False*
i ⟦ (*C-conj c1 c2*)⟧ ~ *s*
i ⟦ (*C-disj c1 c2*)⟧ ~ *s*
i ⟦ (*C-imp c1 c2*)⟧ ~ *s*
i ⟦ (*C-not c*) ⟧ ~ *s*
i ⟦ (*C-eq e1 e2*)⟧ ~ *s*
i ⟦ *C-true* ⟧ ~ *s*
i ⟦ *C-false* ⟧ ~ *s*

6.1.5 Satisfiability

inductive *is-satis* :: *valuation* ⇒ *c* ⇒ *bool* (‹ - ⊨ - ›) **where**
i ⟦ *c* ⟧ ~ *True* ⟹ *i* ⊨ *c*
equivariance *is-satis*
nominal-inductive *is-satis* ⟨*proof*⟩

nominal-function *is-satis-g* :: *valuation* ⇒ Γ ⇒ *bool* (‹ - ⊨ - ›) **where**
i ⊨ *GNil* = *True*
| *i* ⊨ ((*x*,*b*,*c*) #Γ *G*) = (*i* ⊨ *c* ∧ *i* ⊨ *G*)
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

6.2 Validity

nominal-function *valid* :: Θ ⇒ *B* ⇒ Γ ⇒ *c* ⇒ *bool* (‹- ; - ; - ⊨ - › [*50*, *50*] *50*) **where**
P ; *B* ; *G* ⊨ *c* = ((*P* ; *B* ; *G* ⊢$_{wf}$ *c*) ∧ (∀ *i*. (*P* ; *G* ⊢ *i*) ∧ *i* ⊨ *G* ⟶ *i* ⊨ *c*))

⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

6.3 Lemmas

Lemmas needed for Examples

lemma *valid-trueI* [*intro*]:
  fixes G::Γ
  assumes P ; $B \vdash_{wf}$ G
  shows P ; B ; $G \models$ *C-true*
⟨*proof*⟩

end

Chapter 7

Syntax Lemmas

7.1 Support, lookup and contexts

lemma *supp-v-tau* [*simp*]:
assumes *atom z* ♯ *v*
shows *supp* (⦃ *z* : *b* | *CE-val* (*V-var z*) == *CE-val v* ⦄) = *supp v* ∪ *supp b*
⟨*proof*⟩

lemma *supp-v-var-tau* [*simp*]:
assumes *z* ≠ *x*
shows *supp* (⦃ *z* : *b* | *CE-val* (*V-var z*) == *CE-val* (*V-var x*) ⦄) = { *atom x* } ∪ *supp b*
⟨*proof*⟩

Sometimes we need to work with a version of a binder where the variable is fresh in something else, such as a bigger context. I think these could be generated automatically

lemma *obtain-fresh-fun-def*:
fixes *t*::′*b*::*fs*
shows ∃ *y*::*x*. *atom y* ♯ (*s*,*c*,*τ*,*t*) ∧ (*AF-fundef f* (*AF-fun-typ-none* (*AF-fun-typ x b c τ s*)) = *AF-fundef f* (*AF-fun-typ-none* (*AF-fun-typ y b* ((*y* ↔ *x*) • *c*) ((*y* ↔ *x*) • *τ*) ((*y* ↔ *x*) • *s*))))
⟨*proof*⟩

lemma *lookup-fun-member*:
assumes *Some* (*AF-fundef f ft*) = *lookup-fun* Φ *f*
shows *AF-fundef f ft* ∈ *set* Φ
⟨*proof*⟩

lemma *rig-dom-eq*:
dom (*G*[*x* ⟼ *c*]) = *dom G*
⟨*proof*⟩

lemma *lookup-in-rig-eq*:
assumes *Some* (*b*,*c*) = *lookup* Γ *x*
shows *Some* (*b*,*c*′) = *lookup* (Γ[*x*⟼*c*′]) *x*
⟨*proof*⟩

lemma *lookup-in-rig-neq*:
assumes *Some* (*b*,*c*) = *lookup* Γ *y* **and** *x*≠*y*
shows *Some* (*b*,*c*) = *lookup* (Γ[*x*⟼*c*′]) *y*

⟨*proof*⟩

lemma *lookup-in-rig*:
assumes *Some* (*b*,*c*) = *lookup* Γ *y*
shows ∃ *c''*. *Some* (*b*,*c''*) = *lookup* (Γ[*x*↦*c'*]) *y*
⟨*proof*⟩

lemma *lookup-inside*[*simp*]:
assumes *x* ∉ *fst* ‘ *toSet* Γ′
shows *Some* (*b1*,*c1*) = *lookup* (Γ′@(*x*,*b1*,*c1*)#Γ) *x*
⟨*proof*⟩

lemma *lookup-inside2*:
assumes *Some* (*b1*,*c1*) = *lookup* (Γ′@((*x*,*b0*,*c0*)#Γ)) *y* **and** *x*≠*y*
shows *Some* (*b1*,*c1*) = *lookup* (Γ′@((*x*,*b0*,*c0′*)#Γ)) *y*
⟨*proof*⟩

fun *tail*:: ′*a list* ⇒ ′*a list* **where**
tail [] = []
| *tail* (*x*#*xs*) = *xs*

lemma *lookup-options*:
assumes *Some* (*b*,*c*) = *lookup* (*xt*#Γ *G*) *x*
shows ((*x*,*b*,*c*) = *xt*) ∨ (*Some* (*b*,*c*) = *lookup* *G* *x*)
⟨*proof*⟩

lemma *lookup-x*:
assumes *Some* (*b*,*c*) = *lookup* *G* *x*
shows *x* ∈ *fst* ‘ *toSet* *G*
⟨*proof*⟩

lemma *GCons-eq-appendI*:
fixes *xs1*::Γ
shows [| *x* #Γ *xs1* = *ys*; *xs* = *xs1* @ *zs* |] ==> *x* #Γ *xs* = *ys* @ *zs*
⟨*proof*⟩

lemma *split-G*: *x* : *toSet xs* ⟹ ∃ *ys zs*. *xs* = *ys* @ *x* #Γ *zs*
⟨*proof*⟩

lemma *lookup-not-empty*:
assumes *Some* τ = *lookup* *G* *x*
shows *G* ≠ *GNil*
⟨*proof*⟩

lemma *lookup-in-g*:
assumes *Some* (*b*,*c*) = *lookup* Γ *x*
shows (*x*,*b*,*c*) ∈ *toSet* Γ
⟨*proof*⟩

lemma *lookup-split*:
fixes Γ::Γ
assumes *Some* (*b*,*c*) = *lookup* Γ *x*

shows $\exists\, G\ G'\ .\ \Gamma =\ \ G'@(x,b,c)\#_\Gamma G$
⟨*proof*⟩

lemma *toSet-splitU*[*simp*]:
$(x',b',c') \in toSet\ (\Gamma'\ @\ (x,\ b,\ c)\ \#_\Gamma\ \Gamma) \longleftrightarrow (x',b',c') \in (toSet\ \Gamma' \cup \{(x,\ b,\ c)\} \cup toSet\ \Gamma)$
⟨*proof*⟩

lemma *toSet-splitP*[*simp*]:
$(\forall (x',\ b',\ c') \in toSet\ (\Gamma'\ @\ (x,\ b,\ c)\ \#_\Gamma\ \Gamma).\ P\ x'\ b'\ c') \longleftrightarrow (\forall\ (x',\ b',\ c') \in toSet\ \Gamma'.\ P\ x'\ b'\ c') \wedge P\ x\ b\ c \wedge (\forall\ (x',\ b',\ c') \in toSet\ \Gamma.\ P\ x'\ b'\ c')$ (**is** *?A* $\longleftrightarrow$ *?B*)
⟨*proof*⟩

lemma *lookup-restrict*:
assumes $Some\ (b',c') = lookup\ (\Gamma'@(x,b,c)\#_\Gamma\Gamma)\ y$ **and** $x \neq y$
shows $Some\ (b',c') = lookup\ (\Gamma'@\Gamma)\ y$
⟨*proof*⟩

lemma *supp-list-member*:
fixes $x::'a::fs$ **and** $l::'a\ list$
assumes $x \in set\ l$
shows $supp\ x \subseteq supp\ l$
⟨*proof*⟩

lemma *GNil-append*:
assumes $GNil = G1@G2$
shows $G1 = GNil \wedge G2 = GNil$
⟨*proof*⟩

lemma *GCons-eq-append-conv*:
fixes $xs::\Gamma$
shows $x\#_\Gamma xs = ys@zs = (ys = GNil \wedge x\#_\Gamma xs = zs \vee (\exists\, ys'.\ x\#_\Gamma ys' = ys \wedge xs = ys'@zs))$
⟨*proof*⟩

lemma *dclist-distinct-unique*:
assumes $(dc,\ const) \in set\ dclist2$ **and** $(cons,\ const1) \in set\ dclist2$ **and** $dc{=}cons$ **and** $distinct\ (List.map\ fst\ dclist2)$
shows $(const) = const1$
⟨*proof*⟩

lemma *fresh-d-fst-d*:
assumes $atom\ u \sharp \delta$
shows $u \notin fst\ `\ set\ \delta$
⟨*proof*⟩

lemma *bv-not-in-bset-supp*:
fixes $bv::bv$
assumes $bv\ |\notin|\ B$
shows $atom\ bv \notin supp\ B$
⟨*proof*⟩

lemma *u-fresh-d*:
assumes $atom\ u \sharp D$

shows $u \notin fst\ `\ setD\ D$
⟨*proof*⟩

7.2 Type Definitions

lemma *exist-fresh-bv*:
fixes $tm::'a::fs$
shows $\exists\, bva2\ dclist2.\ AF\text{-}typedef\text{-}poly\ tyid\ bva\ dclist = AF\text{-}typedef\text{-}poly\ tyid\ bva2\ dclist2 \wedge atom\ bva2 \sharp tm$
⟨*proof*⟩

lemma *obtain-fresh-bv*:
fixes $tm::'a::fs$
obtains $bva2::bv$ **and** $dclist2$ **where** $AF\text{-}typedef\text{-}poly\ tyid\ bva\ dclist = AF\text{-}typedef\text{-}poly\ tyid\ bva2\ dclist2 \wedge atom\ bva2 \sharp tm$
⟨*proof*⟩

7.3 Function Definitions

lemma *fun-typ-flip*:
fixes $bv1::bv$ **and** $c::bv$
shows $(bv1 \leftrightarrow c) \cdot AF\text{-}fun\text{-}typ\ x1\ b1\ c1\ \tau 1\ s1 = AF\text{-}fun\text{-}typ\ x1\ ((bv1 \leftrightarrow c) \cdot b1)\ ((bv1 \leftrightarrow c) \cdot c1)\ ((bv1 \leftrightarrow c) \cdot \tau 1)\ ((bv1 \leftrightarrow c) \cdot s1)$
⟨*proof*⟩

lemma *fun-def-eq*:
assumes $AF\text{-}fundef\ fa\ (AF\text{-}fun\text{-}typ\text{-}none\ (AF\text{-}fun\text{-}typ\ xa\ ba\ ca\ \tau a\ sa)) = AF\text{-}fundef\ f\ (AF\text{-}fun\text{-}typ\text{-}none\ (AF\text{-}fun\text{-}typ\ x\ b\ c\ \tau\ s))$
shows $f = fa$ **and** $b = ba$ **and** $[[atom\ xa]]lst.\ sa = [[atom\ x]]lst.\ s$ **and** $[[atom\ xa]]lst.\ \tau a = [[atom\ x]]lst.\ \tau$ **and**
$[[atom\ xa]]lst.\ ca = [[atom\ x]]lst.\ c$
⟨*proof*⟩

lemma *fun-arg-unique-aux*:
assumes $AF\text{-}fun\text{-}typ\ x1\ b1\ c1\ \tau 1'\ s1' = AF\text{-}fun\text{-}typ\ x2\ b2\ c2\ \tau 2'\ s2'$
shows $\{\!|\ x1 : b1 \mid c1\ |\!\} = \{\!|\ x2 : b2 \mid c2\ |\!\}$
⟨*proof*⟩

lemma *fresh-x-neq*:
fixes $x::x$ **and** $y::x$
shows $atom\ x \sharp y = (x \neq y)$
⟨*proof*⟩

lemma *obtain-fresh-z3*:
fixes $tm::'b::fs$
obtains $z::x$ **where** $\{\!|\ x : b \mid c\ |\!\} = \{\!|\ z : b \mid c[x::=V\text{-}var\ z]_{cv}\ |\!\} \wedge atom\ z \sharp tm \wedge atom\ z \sharp (x,c)$
⟨*proof*⟩

lemma *u-fresh-v*:
fixes $u::u$ **and** $t::v$

shows *atom u* ♯ *t*
⟨*proof*⟩

lemma *u-fresh-ce*:
fixes *u*::*u* **and** *t*::*ce*
shows *atom u* ♯ *t*
⟨*proof*⟩

lemma *u-fresh-c*:
fixes *u*::*u* **and** *t*::*c*
shows *atom u* ♯ *t*
⟨*proof*⟩

lemma *u-fresh-g*:
fixes *u*::*u* **and** *t*::Γ
shows *atom u* ♯ *t*
⟨*proof*⟩

lemma *u-fresh-t*:
fixes *u*::*u* **and** *t*::τ
shows *atom u* ♯ *t*
⟨*proof*⟩

lemma *b-of-c-of-eq*:
assumes *atom z* ♯ τ
shows {‖ *z* : *b-of* τ | *c-of* τ *z* ‖} = τ
⟨*proof*⟩

lemma *fresh-d-not-in*:
assumes *atom u2* ♯ Δ′
shows *u2* ∉ *fst* ‘ *setD* Δ′
⟨*proof*⟩

end

Chapter 8

Wellformedness Lemmas

8.1 Prelude

lemma *b-of-subst-bb-commute*:
$(b\text{-}of\ (\tau[bv::=b]_{\tau b})) = (b\text{-}of\ \tau)[bv::=b]_{bb}$
⟨*proof*⟩

lemmas *wf-intros* = *wfV-wfC-wfG-wfT-wfTs-wfTh-wfB-wfCE-wfTD.intros wfE-wfS-wfCS-wfCSS-wfPhi-wfD-wfFTQ-wfF*

lemmas *freshers* = *fresh-prodN b.fresh c.fresh v.fresh ce.fresh fresh-GCons fresh-GNil fresh-at-base*

8.2 Strong Elimination

Inversion/elimination for well-formed polymorphic constructors

lemma *wf-strong-elim*:
fixes $\Gamma::\Gamma$ **and** $\Gamma'::\Gamma$ **and** $v::v$ **and** $e::e$ **and** $c::c$ **and** $\tau::\tau$ **and** $ts::(string*\tau)\ list$
and $\Delta::\Delta$ **and** $b::b$ **and** $ftq::fun\text{-}typ\text{-}q$ **and** $ft::fun\text{-}typ$ **and** $ce::ce$ **and** $td::type\text{-}def$ **and** $s::s$
and $tm::'a::fs$
and $cs::branch\text{-}s$ **and** $css::branch\text{-}list$ **and** $\Theta::\Theta$
shows $\Theta;\ \mathcal{B};\ \Gamma\ \vdash_{wf} (V\text{-}consp\ tyid\ dc\ b\ v) : b'' \Longrightarrow (\exists\ bv\ dclist\ x\ b'\ c.\ b'' = B\text{-}app\ tyid\ b \wedge$
$AF\text{-}typedef\text{-}poly\ tyid\ bv\ dclist \in set\ \Theta \wedge$
$(dc, \{\!|\ x : b'\ |\ c\ |\!\}) \in set\ dclist \wedge$
$\Theta;\ \mathcal{B}\ \vdash_{wf} b\ \wedge\ atom\ bv \sharp (\Theta, \mathcal{B}, \Gamma, b, v) \wedge\ \Theta;\ \mathcal{B};\ \Gamma \vdash_{wf} v : b'[bv::=b]_{bb} \wedge\ atom\ bv \sharp tm)$
and
$\Theta;\ \mathcal{B};\ \Gamma\ \vdash_{wf} c \Longrightarrow True$ **and**
$\Theta;\ \mathcal{B} \vdash_{wf} \Gamma \Longrightarrow True$ **and**
$\Theta;\ \mathcal{B};\ \Gamma\ \vdash_{wf} \tau \Longrightarrow True$ **and**
$\Theta;\ \mathcal{B};\ \Gamma\ \vdash_{wf} ts \Longrightarrow True$ **and**
$\vdash_{wf} \Theta \Longrightarrow True$ **and**
$\Theta;\ \mathcal{B} \vdash_{wf} b \Longrightarrow True$ **and**
$\Theta;\ \mathcal{B};\ \Gamma \vdash_{wf} ce : b' \Longrightarrow True$ **and**
$\Theta\ \vdash_{wf} td \Longrightarrow True$
⟨*proof*⟩

8.3 Context Extension

definition *wfExt* :: $\Theta \Rightarrow \mathcal{B} \Rightarrow \Gamma \Rightarrow \Gamma \Rightarrow bool\ (\langle\ -\ ;\ -\ \vdash_{wf}\ -\ <\ -\ \rangle\ [50,50,50]\ 50)$ **where**

wfExt T B G1 G2 = (*wfG T B G2* ∧ *wfG T B G1* ∧ *toSet G1* ⊆ *toSet G2*)

8.4 Context

lemma *wfG-cons*[*ms-wb*]:
fixes Γ::Γ
assumes *P*; $\mathcal{B}$ $\vdash_{wf}$ (*z*,*b*,*c*) $\#_\Gamma\Gamma$
shows *P*; $\mathcal{B}$ $\vdash_{wf}$ Γ ∧ *atom z* ♯ Γ ∧ *wfB P* $\mathcal{B}$ *b*
⟨*proof*⟩

lemma *wfG-cons2*[*ms-wb*]:
fixes Γ::Γ
assumes *P*; $\mathcal{B}$ $\vdash_{wf}$ *zbc* $\#_\Gamma\Gamma$
shows *P*; $\mathcal{B}$ $\vdash_{wf}$ Γ
⟨*proof*⟩

lemma *wf-g-unique*:
fixes Γ::Γ
assumes Θ; $\mathcal{B}$ $\vdash_{wf}$ Γ **and** (*x*,*b*,*c*) ∈ *toSet* Γ **and** (*x*,*b*′,*c*′) ∈ *toSet* Γ
shows *b*=*b*′ ∧ *c*=*c*′
⟨*proof*⟩

lemma *lookup-if1*:
fixes Γ::Γ
assumes Θ; $\mathcal{B}$ $\vdash_{wf}$ Γ **and** *Some* (*b*,*c*) = *lookup* Γ *x*
shows (*x*,*b*,*c*) ∈ *toSet* Γ ∧ (∀ *b*′ *c*′. (*x*,*b*′,*c*′) ∈ *toSet* Γ ⟶ *b*′=*b* ∧ *c*′=*c*)
⟨*proof*⟩

lemma *lookup-if2*:
assumes *wfG P* $\mathcal{B}$ Γ **and** (*x*,*b*,*c*) ∈ *toSet* Γ ∧ (∀ *b*′ *c*′. (*x*,*b*′,*c*′) ∈ *toSet* Γ ⟶ *b*′=*b* ∧ *c*′=*c*)
shows *Some* (*b*,*c*) = *lookup* Γ *x*
⟨*proof*⟩

lemma *lookup-iff*:
fixes Θ::Θ **and** Γ::Γ
assumes Θ; $\mathcal{B}$ $\vdash_{wf}$ Γ
shows *Some* (*b*,*c*) = *lookup* Γ *x* ⟷ (*x*,*b*,*c*) ∈ *toSet* Γ ∧ (∀ *b*′ *c*′. (*x*,*b*′,*c*′) ∈ *toSet* Γ ⟶ *b*′=*b* ∧ *c*′=*c*)
⟨*proof*⟩

lemma *wfG-lookup-wf*:
fixes Θ::Θ **and** Γ::Γ **and** *b*::*b* **and** $\mathcal{B}$::$\mathcal{B}$
assumes Θ; $\mathcal{B}$ $\vdash_{wf}$ Γ **and** *Some* (*b*,*c*) = *lookup* Γ *x*
shows Θ; $\mathcal{B}$ $\vdash_{wf}$ *b*
⟨*proof*⟩

lemma *wfG-unique*:
fixes Γ::Γ
assumes *wfG B* Θ ((*x*, *b*, *c*) $\#_\Gamma$ Γ) **and** (*x1*,*b1*,*c1*) ∈ *toSet* ((*x*, *b*, *c*) $\#_\Gamma$ Γ) **and** *x1*=*x*
shows *b1* = *b* ∧ *c1* = *c*
⟨*proof*⟩

lemma *wfG-unique-full*:

fixes Γ::Γ
assumes *wfG* Θ B (Γ'@(x, b, c) #$_\Gamma$ Γ) **and** (*x1*,*b1*,*c1*) $\in$ *toSet* (Γ'@(x, b, c) #$_\Gamma$ Γ) **and** *x1*=x
shows *b1* = b $\wedge$ *c1* = c
⟨*proof*⟩

8.5 Converting between wb forms

We cannot prove wfB properties here for expressions and statements as need some more facts about Φ context which we can prove without this lemma. Trying to cram everything into a single large mutually recursive lemma is not a good idea

lemma *wfX-wfY1*:
fixes Γ::Γ **and** Γ'::Γ **and** v::v **and** e::e **and** c::c **and** τ::τ **and** *ts*::(*string*$*\tau$) *list* **and** Δ::Δ **and** s::s **and** b::b **and** *ftq*::*fun-typ-q* **and** *ft*::*fun-typ* **and** *ce*::*ce* **and** *td*::*type-def* **and** *cs*::*branch-s* **and** *css*::*branch-list*
shows *wfV-wf*: Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ v : b $\Longrightarrow$ Θ; $\mathcal{B}$ $\vdash_{wf}$ Γ $\wedge$ $\vdash_{wf}$ Θ **and**
wfC-wf: Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ c $\Longrightarrow$ Θ; $\mathcal{B}$ $\vdash_{wf}$ Γ $\wedge$ $\vdash_{wf}$ Θ **and**
wfG-wf :Θ; $\mathcal{B}$ $\vdash_{wf}$ Γ $\Longrightarrow$ $\vdash_{wf}$ Θ **and**
wfT-wf: Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ τ $\Longrightarrow$ Θ; $\mathcal{B}$ $\vdash_{wf}$ Γ $\wedge$ $\vdash_{wf}$ Θ $\wedge$ Θ; $\mathcal{B}$ $\vdash_{wf}$ *b-of* τ **and**
wfTs-wf:Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *ts* $\Longrightarrow$ Θ; $\mathcal{B}$ $\vdash_{wf}$ Γ $\wedge$ $\vdash_{wf}$ Θ **and**
$\vdash_{wf}$ Θ $\Longrightarrow$ *True* **and**
wfB-wf: Θ; $\mathcal{B}$ $\vdash_{wf}$ b $\Longrightarrow$ $\vdash_{wf}$ Θ **and**
wfCE-wf: Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *ce* : b $\Longrightarrow$ Θ; $\mathcal{B}$ $\vdash_{wf}$ Γ $\wedge$ $\vdash_{wf}$ Θ **and**
wfTD-wf: Θ $\vdash_{wf}$ *td* $\Longrightarrow$ $\vdash_{wf}$ Θ
⟨*proof*⟩

lemma *wfX-wfY2*:
fixes Γ::Γ **and** Γ'::Γ **and** v::v **and** e::e **and** c::c **and** τ::τ **and** *ts*::(*string*$*\tau$) *list* **and** Δ::Δ **and** s::s **and** b::b **and** *ftq*::*fun-typ-q* **and** *ft*::*fun-typ* **and** *ce*::*ce* **and** *td*::*type-def* **and** *cs*::*branch-s* **and** *css*::*branch-list*
shows
wfE-wf: Θ; Φ; $\mathcal{B}$; Γ; Δ $\vdash_{wf}$ e : b $\Longrightarrow$ Θ; $\mathcal{B}$ $\vdash_{wf}$ Γ $\wedge$ Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ Δ $\wedge$ $\vdash_{wf}$ Θ $\wedge$ Θ $\vdash_{wf}$ Φ **and**
wfS-wf: Θ; Φ; $\mathcal{B}$; Γ; Δ $\vdash_{wf}$ s : b $\Longrightarrow$ Θ; $\mathcal{B}$ $\vdash_{wf}$ Γ $\wedge$ Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ Δ $\wedge$ $\vdash_{wf}$ Θ $\wedge$ Θ $\vdash_{wf}$ Φ **and**
Θ; Φ; $\mathcal{B}$; Γ; Δ ; *tid* ; *dc* ; t $\vdash_{wf}$ *cs* : b $\Longrightarrow$ Θ; $\mathcal{B}$ $\vdash_{wf}$ Γ $\wedge$ Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ Δ $\wedge$ $\vdash_{wf}$ Θ $\wedge$ Θ $\vdash_{wf}$ Φ **and**
Θ; Φ; $\mathcal{B}$; Γ; Δ ; *tid* ; *dclist* $\vdash_{wf}$ *css* : b $\Longrightarrow$ Θ; $\mathcal{B}$ $\vdash_{wf}$ Γ $\wedge$ Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ Δ $\wedge$ $\vdash_{wf}$ Θ $\wedge$ Θ $\vdash_{wf}$ Φ **and**
wfPhi-wf: Θ $\vdash_{wf}$ (Φ::Φ) $\Longrightarrow$ $\vdash_{wf}$ Θ **and**
wfD-wf: Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ Δ $\Longrightarrow$ Θ; $\mathcal{B}$ $\vdash_{wf}$ Γ $\wedge$ $\vdash_{wf}$ Θ **and**
wfFTQ-wf: Θ ; Φ $\vdash_{wf}$ *ftq* $\Longrightarrow$ Θ $\vdash_{wf}$ Φ $\wedge$ $\vdash_{wf}$ Θ **and**
wfFT-wf: Θ ; Φ ; $\mathcal{B}$ $\vdash_{wf}$ *ft* $\Longrightarrow$ Θ $\vdash_{wf}$ Φ $\wedge$ $\vdash_{wf}$ Θ
⟨*proof*⟩

lemmas *wfX-wfY*=*wfX-wfY1 wfX-wfY2*

lemma *setD-ConsD*:
ut $\in$ *setD* (*ut'* #$_\Delta$ D) = (*ut* = *ut'* $\vee$ *ut* $\in$ *setD* D)
⟨*proof*⟩

lemma *wfD-wfT*:
fixes Δ::Δ **and** τ::τ
assumes Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ Δ

shows $\forall (u,\tau) \in setD\ \Delta.\ \Theta;\ \mathcal{B};\ \Gamma \vdash_{wf} \tau$
⟨*proof*⟩

lemma *subst-b-lookup-d*:
assumes $u \notin fst\ `\ setD\ \Delta$
shows $u \notin fst\ `\ setD\ \Delta[bv::=b]_{\Delta b}$
⟨*proof*⟩

lemma *wfG-cons-splitI*:
fixes $\Phi::\Phi$ **and** $\Gamma::\Gamma$
assumes $\Theta;\ \mathcal{B} \vdash_{wf} \Gamma$ **and** $atom\ x \sharp \Gamma$ **and** $wfB\ \Theta\ \mathcal{B}\ b$ **and**
$c \in \{\ TRUE,\ FALSE\ \} \longrightarrow \Theta;\ \mathcal{B} \vdash_{wf} \Gamma$ **and**
$c \notin \{\ TRUE,\ FALSE\ \} \longrightarrow \Theta\ \ ;\mathcal{B}\ ;\ (x,b,C\text{-}true)\ \ \#_\Gamma\Gamma \vdash_{wf} c$
shows $\Theta;\ \mathcal{B} \vdash_{wf} ((x,b,c)\ \ \#_\Gamma\Gamma)$
⟨*proof*⟩

lemma *wfG-consI*:
fixes $\Phi::\Phi$ **and** $\Gamma::\Gamma$ **and** $c::c$
assumes $\Theta;\ \mathcal{B} \vdash_{wf} \Gamma$ **and** $atom\ x \sharp \Gamma$ **and** $wfB\ \Theta\ \mathcal{B}\ b$ **and**
$\Theta\ \ ;\ \mathcal{B}\ ;\ (x,b,C\text{-}true)\ \ \#_\Gamma\Gamma \vdash_{wf} c$
shows $\Theta\ \ ;\ \mathcal{B} \vdash_{wf} ((x,b,c)\ \ \#_\Gamma\Gamma)$
⟨*proof*⟩

lemma *wfG-elim2*:
fixes $c::c$
assumes $wfG\ P\ \mathcal{B}\ \ ((x,b,c)\ \ \#_\Gamma\Gamma)$
shows $P;\ \mathcal{B}\ ;\ (x,\ b,\ TRUE)\ \ \#_\Gamma\ \Gamma\ \vdash_{wf} c \wedge wfB\ P\ \mathcal{B}\ b$
⟨*proof*⟩

lemma *wfG-cons-wfC*:
fixes $\Gamma::\Gamma$ **and** $c::c$
assumes $\Theta\ ;\ B\ \vdash_{wf} (x,\ b,\ c)\ \ \#_\Gamma\ \Gamma$
shows $\Theta\ ;\ B\ ;\ ((x,\ b,\ TRUE)\ \ \#_\Gamma\ \Gamma) \vdash_{wf} c$
⟨*proof*⟩

lemma *wfG-wfB*:
assumes $wfG\ P\ \mathcal{B}\ \Gamma$ **and** $b \in fst`snd`toSet\ \Gamma$
shows $wfB\ P\ \mathcal{B}\ b$
⟨*proof*⟩

lemma *wfG-cons-TRUE*:
fixes $\Gamma::\Gamma$ **and** $b::b$
assumes $P;\ \mathcal{B} \vdash_{wf} \Gamma$ **and** $atom\ z \sharp \Gamma$ **and** $P;\ \mathcal{B} \vdash_{wf} b$
shows $P\ \ ;\ \mathcal{B} \vdash_{wf} (z,\ b,\ TRUE)\ \ \#_\Gamma\ \Gamma$
⟨*proof*⟩

lemma *wfG-cons-TRUE2*:
assumes $P;\ \mathcal{B} \vdash_{wf} (z,b,c)\ \ \#_\Gamma\Gamma$ **and** $atom\ z \sharp \Gamma$
shows $P;\ \mathcal{B} \vdash_{wf} (z,\ b,\ TRUE)\ \ \#_\Gamma\ \Gamma$
⟨*proof*⟩

lemma *wfG-suffix*:

fixes Γ::Γ
assumes *wfG P* $\mathcal{B}$ (Γ'@Γ)
shows *wfG P* $\mathcal{B}$ Γ
⟨*proof*⟩

lemma *wfV-wfCE*:
fixes *v*::*v*
assumes Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *v* : *b*
shows Θ ; $\mathcal{B}$; Γ $\vdash_{wf}$ *CE-val v* : *b*
⟨*proof*⟩

8.6 Support

lemma *wf-supp1*:
fixes Γ::Γ **and** Γ'::Γ **and** *v*::*v* **and** *e*::*e* **and** *c*::*c* **and** τ::τ **and** *ts*::(*string*∗τ) *list* **and** Δ::Δ **and** *s*::*s* **and** *b*::*b* **and** *ftq*::*fun-typ-q* **and** *ft*::*fun-typ* **and** *ce*::*ce* **and** *td*::*type-def* **and** *cs*::*branch-s* **and** *css* ::*branch-list*

shows *wfV-supp*: Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *v* : *b* $\Longrightarrow$ *supp v* $\subseteq$ *atom-dom* Γ $\cup$ *supp* $\mathcal{B}$ **and**
wfC-supp: Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *c* $\Longrightarrow$ *supp c* $\subseteq$ *atom-dom* Γ $\cup$ *supp* $\mathcal{B}$ **and**
wfG-supp: Θ; $\mathcal{B}$ $\vdash_{wf}$ Γ $\Longrightarrow$ *atom-dom* Γ $\subseteq$ *supp* Γ **and**
wfT-supp: Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ τ $\Longrightarrow$ *supp* τ $\subseteq$ *atom-dom* Γ $\cup$ *supp* $\mathcal{B}$ **and**
wfTs-supp: Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *ts* $\Longrightarrow$ *supp ts* $\subseteq$ *atom-dom* Γ $\cup$ *supp* $\mathcal{B}$ **and**
wfTh-supp: $\vdash_{wf}$ Θ $\Longrightarrow$ *supp* Θ = {} **and**
wfB-supp: Θ; $\mathcal{B}$ $\vdash_{wf}$ *b* $\Longrightarrow$ *supp b* $\subseteq$ *supp* $\mathcal{B}$ **and**
wfCE-supp: Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *ce* : *b* $\Longrightarrow$ *supp ce* $\subseteq$ *atom-dom* Γ $\cup$ *supp* $\mathcal{B}$ **and**
wfTD-supp: Θ $\vdash_{wf}$ *td* $\Longrightarrow$ *supp td* $\subseteq$ {}
⟨*proof*⟩

lemma *wf-supp2*:
fixes Γ::Γ **and** Γ'::Γ **and** *v*::*v* **and** *e*::*e* **and** *c*::*c* **and** τ::τ **and**
ts::(*string*∗τ) *list* **and** Δ::Δ **and** *s*::*s* **and** *b*::*b* **and** *ftq*::*fun-typ-q* **and**
ft::*fun-typ* **and** *ce*::*ce* **and** *td*::*type-def* **and** *cs*::*branch-s* **and** *css* ::*branch-list*
shows
wfE-supp: Θ; Φ; $\mathcal{B}$; Γ; Δ $\vdash_{wf}$ *e* : *b* $\Longrightarrow$ (*supp e* $\subseteq$ *atom-dom* Γ $\cup$ *supp* $\mathcal{B}$ $\cup$ *atom* ‘ *fst* ‘ *setD* Δ)
and
wfS-supp: Θ; Φ; $\mathcal{B}$; Γ; Δ $\vdash_{wf}$ *s* : *b* $\Longrightarrow$ *supp s* $\subseteq$ *atom-dom* Γ $\cup$ *atom* ‘ *fst* ‘ *setD* Δ $\cup$ *supp* $\mathcal{B}$
and
Θ; Φ; $\mathcal{B}$; Γ; Δ ; *tid* ; *dc* ; *t* $\vdash_{wf}$ *cs* : *b* $\Longrightarrow$ *supp cs* $\subseteq$ *atom-dom* Γ $\cup$ *atom* ‘ *fst* ‘ *setD* Δ $\cup$ *supp* $\mathcal{B}$ **and**
Θ; Φ; $\mathcal{B}$; Γ; Δ ; *tid* ; *dclist* $\vdash_{wf}$ *css* : *b* $\Longrightarrow$ *supp css* $\subseteq$ *atom-dom* Γ $\cup$ *atom* ‘ *fst* ‘ *setD* Δ $\cup$ *supp* $\mathcal{B}$ **and**
wfPhi-supp: Θ $\vdash_{wf}$ (Φ::Φ) $\Longrightarrow$ *supp* Φ = {} **and**
wfD-supp: Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ Δ $\Longrightarrow$ *supp* Δ $\subseteq$ *atom*‘*fst*‘(*setD* Δ) $\cup$ *atom-dom* Γ $\cup$ *supp* $\mathcal{B}$ **and**
Θ ; Φ $\vdash_{wf}$ *ftq* $\Longrightarrow$ *supp ftq* = {} **and**
Θ ; Φ ; $\mathcal{B}$ $\vdash_{wf}$ *ft* $\Longrightarrow$ *supp ft* $\subseteq$ *supp* $\mathcal{B}$
⟨*proof*⟩

lemmas *wf-supp* = *wf-supp1 wf-supp2*

lemma *wfV-supp-nil*:
fixes *v*::*v*

assumes P ; $\{||\}$; $GNil \vdash_{wf} v : b$
shows $supp\ v = \{\}$
⟨*proof*⟩

lemma *wfT-TRUE-aux*:
assumes $wfG\ P\ \mathcal{B}\ \Gamma$ **and** $atom\ z \sharp (P, \mathcal{B}, \Gamma)$ **and** $wfB\ P\ \mathcal{B}\ b$
shows $wfT\ P\ \mathcal{B}\ \Gamma\ (\{\!|\ z : b\ |\ TRUE\ |\!\})$
⟨*proof*⟩

lemma *wfT-TRUE*:
assumes $wfG\ P\ \mathcal{B}\ \Gamma$ **and** $wfB\ P\ \mathcal{B}\ b$
shows $wfT\ P\ \mathcal{B}\ \Gamma\ (\{\!|\ z : b\ |\ TRUE\ |\!\})$
⟨*proof*⟩

lemma *phi-flip-eq*:
assumes $wfPhi\ T\ P$
shows $(x \leftrightarrow xa) \cdot P = P$
⟨*proof*⟩

lemma *wfC-supp-cons*:
fixes $c'::c$ **and** $G::\Gamma$
assumes $P; \mathcal{B}$; $(x',\ b'\ ,\ TRUE)\ \ \#_\Gamma G \vdash_{wf} c'$
shows $supp\ c' \subseteq atom\text{-}dom\ G \cup supp\ x' \cup supp\ \mathcal{B}$ **and** $supp\ c' \subseteq supp\ G \cup supp\ x' \cup supp\ \mathcal{B}$
⟨*proof*⟩

lemma *wfG-dom-supp*:
fixes $x::x$
assumes $wfG\ P\ \mathcal{B}\ G$
shows $atom\ x \in atom\text{-}dom\ G \longleftrightarrow atom\ x \in supp\ G$
⟨*proof*⟩

lemma *wfG-atoms-supp-eq* :
fixes $x::x$
assumes $wfG\ P\ \mathcal{B}\ G$
shows $atom\ x \in atom\text{-}dom\ G \longleftrightarrow atom\ x \in supp\ G$
⟨*proof*⟩

lemma *beta-flip-eq*:
fixes $x::x$ **and** $xa::x$ **and** $\mathcal{B}::\mathcal{B}$
shows $(x \leftrightarrow xa) \cdot \mathcal{B} = \mathcal{B}$
⟨*proof*⟩

lemma *theta-flip-eq2*:
assumes $\vdash_{wf} \Theta$
shows $(z \leftrightarrow za\ ) \cdot \Theta = \Theta$
⟨*proof*⟩

lemma *theta-flip-eq*:
assumes $wfTh\ \Theta$
shows $(x \leftrightarrow xa) \cdot \Theta = \Theta$
⟨*proof*⟩

lemma *wfT-wfC*:
fixes *c*::*c*
assumes Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ {| *z* : *b* | *c* |} **and** *atom z* ♯ Γ
shows Θ; $\mathcal{B}$; (*z*,*b*,*TRUE*) #Γ $\vdash_{wf}$ *c*
⟨*proof*⟩

lemma *u-not-in-dom-g*:
fixes *u*::*u*
shows *atom u* ∉ *atom-dom G*
⟨*proof*⟩

lemma *bv-not-in-dom-g*:
fixes *bv*::*bv*
shows *atom bv* ∉ *atom-dom G*
⟨*proof*⟩

An important lemma that confirms that Γ does not rely on mutable variables

lemma *u-not-in-g*:
fixes *u*::*u*
assumes *wfG* Θ *B G*
shows *atom u* ∉ *supp G*
⟨*proof*⟩

An important lemma that confirms that types only depend on immutable variables

lemma *u-not-in-t*:
fixes *u*::*u*
assumes *wfT* Θ *B G* τ
shows *atom u* ∉ *supp* τ
⟨*proof*⟩

lemma *wfT-supp-c*:
fixes $\mathcal{B}$::$\mathcal{B}$ **and** *z*::*x*
assumes *wfT P* $\mathcal{B}$ Γ ({| *z* : *b* | *c* |})
shows *supp c* − { *atom z* } ⊆ *atom-dom* Γ ∪ *supp* $\mathcal{B}$
⟨*proof*⟩

lemma *wfG-wfC*[*ms-wb*]:
assumes *wfG P* $\mathcal{B}$ ((*x*,*b*,*c*) #Γ Γ)
shows *wfC P* $\mathcal{B}$ ((*x*,*b*,*TRUE*) #Γ Γ) *c*
⟨*proof*⟩

lemma *wfT-wf-cons*:
assumes *wfT P* $\mathcal{B}$ Γ {| *z* : *b* | *c* |} **and** *atom z* ♯ Γ
shows *wfG P* $\mathcal{B}$ ((*z*,*b*,*c*) #Γ Γ)
⟨*proof*⟩

lemma *wfV-b-fresh*:
fixes *b*::*b* **and** *v*::*v* **and** *bv*::*bv*
assumes Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *v*: *b* **and** *bv* |∉| $\mathcal{B}$
shows *atom bv* ♯ *v*
⟨*proof*⟩

lemma *wfCE-b-fresh*:
fixes b::b **and** ce::ce **and** bv::bv
assumes Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} ce$: b **and** bv $|\notin|$ $\mathcal{B}$
shows *atom* $bv \sharp ce$
⟨*proof*⟩

8.7 Freshness

lemma *wfG-fresh-x*:
fixes Γ::Γ **and** z::x
assumes Θ; $\mathcal{B} \vdash_{wf} \Gamma$ **and** *atom* $z \sharp \Gamma$
shows *atom* $z \sharp (\Theta,\mathcal{B}, \Gamma)$
⟨*proof*⟩

lemma *wfG-wfT*:
assumes *wfG* P $\mathcal{B}$ $((x,\ b,\ c[z::=V\text{-}var\ x]_{cv})\quad \#_{\Gamma}\ G)$ **and** *atom* $x \sharp c$
shows P; $\mathcal{B}$; $G \vdash_{wf} \{\!|\ z : b \mid c\ |\!\}$
⟨*proof*⟩

lemma *wfT-wfT-if*:
assumes *wfT* Θ $\mathcal{B}$ Γ $(\{\!|\ z2 : b\ \mid CE\text{-}val\ v\ ==\ CE\text{-}val\ (V\text{-}lit\ L\text{-}false)\ IMP\ c[z::=V\text{-}var\ z2]_{cv}\ |\!\})$
and *atom* $z2 \sharp (c,\Gamma)$
shows *wfT* Θ $\mathcal{B}$ Γ $\{\!|\ z : b \mid\ c\ |\!\}$
⟨*proof*⟩

lemma *wfT-fresh-c*:
fixes x::x
assumes *wfT* P $\mathcal{B}$ Γ $\{\!|\ z : b \mid c\ |\!\}$ **and** *atom* $x \sharp \Gamma$ **and** $x \neq z$
shows *atom* $x \sharp c$
⟨*proof*⟩

lemma *wfG-x-fresh* [*simp*]:
fixes x::x
assumes *wfG* P $\mathcal{B}$ G
shows *atom* $x \notin$ *atom-dom* $G \longleftrightarrow$ *atom* $x \sharp G$
⟨*proof*⟩

lemma *wfD-x-fresh*:
fixes x::x
assumes *atom* $x \sharp \Gamma$ **and** *wfD* P B Γ Δ
shows *atom* $x \sharp \Delta$
⟨*proof*⟩

lemma *wfG-fresh-x2*:
fixes Γ::Γ **and** z::x **and** Δ::Δ **and** Φ::Φ
assumes Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$ **and** $\Theta\ \vdash_{wf} \Phi$ **and** *atom* $z \sharp \Gamma$
shows *atom* $z \sharp (\Theta,\Phi,\mathcal{B}, \Gamma,\Delta)$
⟨*proof*⟩

lemma *wfV-x-fresh*:
fixes v::v **and** b::b **and** Γ::Γ **and** x::x
assumes Θ; $\mathcal{B}$; $\Gamma\ \vdash_{wf} v : b$ **and** *atom* $x \sharp \Gamma$

shows *atom x* ♯ *v*
⟨*proof*⟩

lemma *wfE-x-fresh*:
fixes *e*::*e* **and** *b*::*b* **and** Γ::Γ **and** Δ::Δ **and** Φ::Φ **and** *x*::*x*
assumes Θ; Φ; $\mathcal{B}$; Γ ; Δ $\vdash_{wf}$ *e* : *b* **and** *atom x* ♯ Γ
shows *atom x* ♯ *e*
⟨*proof*⟩

lemma *wfT-x-fresh*:
fixes τ::τ **and** Γ::Γ **and** *x*::*x*
assumes Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ τ **and** *atom x* ♯ Γ
shows *atom x* ♯ τ
⟨*proof*⟩

lemma *wfS-x-fresh*:
fixes *s*::*s* **and** Δ::Δ **and** *x*::*x*
assumes Θ; Φ; $\mathcal{B}$; Γ; Δ $\vdash_{wf}$ *s* : *b* **and** *atom x* ♯ Γ
shows *atom x* ♯ *s*
⟨*proof*⟩

lemma *wfTh-fresh*:
fixes *x*
assumes *wfTh T*
shows *atom x* ♯ *T*
⟨*proof*⟩

lemmas *wfTh-x-fresh* = *wfTh-fresh*

lemma *wfPhi-fresh*:
fixes *x*
assumes *wfPhi T P*
shows *atom x* ♯ *P*
⟨*proof*⟩

lemmas *wfPhi-x-fresh* = *wfPhi-fresh*
lemmas *wb-x-fresh* = *wfTh-x-fresh wfPhi-x-fresh wfD-x-fresh wfT-x-fresh wfV-x-fresh*

lemma *wfG-inside-fresh*[*ms-fresh*]:
fixes Γ::Γ **and** *x*::*x*
assumes *wfG P* $\mathcal{B}$ (Γ′@(*x*,*b*,*c*) #Γ)
shows *atom x* ∉ *atom-dom* Γ′
⟨*proof*⟩

lemma *wfG-inside-x-in-atom-dom*:
fixes *c*::*c* **and** *x*::*x* **and** Γ::Γ
shows *atom x* ∈ *atom-dom* (Γ′@ (*x*, *b*, *c*[*z*::=*V-var x*]$_{cv}$) #Γ Γ)
⟨*proof*⟩

lemma *wfG-inside-x-neq*:
fixes *c*::*c* **and** *x*::*x* **and** Γ::Γ **and** *G*::Γ **and** *xa*::*x*
assumes *G*=(Γ′@ (*x*, *b*, *c*[*z*::=*V-var x*]$_{cv}$) #Γ Γ) **and** *atom xa* ♯ *G* **and** Θ; $\mathcal{B}$ $\vdash_{wf}$ *G*

shows $xa \neq x$
⟨*proof*⟩

lemma *wfG-inside-x-fresh*:
fixes c::c **and** x::x **and** Γ::Γ **and** G::Γ **and** xa::x
assumes G=(Γ'@ $(x,\ b,\ c[z::=V\text{-}var\ x]_{cv})$ $\#_\Gamma\ \Gamma$) **and** $atom\ xa\ \sharp\ G$ **and** $\Theta;\ \mathcal{B} \vdash_{wf}\ G$
shows $atom\ xa\ \sharp\ x$
⟨*proof*⟩

lemma *wfT-nil-supp*:
fixes t::τ
assumes $\Theta\ ;\ \{||\}\ ;\ GNil \vdash_{wf}\ t$
shows $supp\ t = \{\}$
⟨*proof*⟩

8.8 Misc

lemma *wfG-cons-append*:
fixes b'::b
assumes $\Theta;\ \mathcal{B} \vdash_{wf}\ ((x',\ b',\ c')\quad \#_\Gamma\ \Gamma')\ @\ (x,\ b,\ c)\quad \#_\Gamma\ \Gamma$
shows $\Theta;\ \mathcal{B} \vdash_{wf}\ (\Gamma'\ @\ (x,\ b,\ c)\quad \#_\Gamma\ \Gamma)\ \wedge\ atom\ x' \sharp (\Gamma'\ @\ (x,\ b,\ c)\quad \#_\Gamma\ \Gamma) \wedge \Theta;\ \mathcal{B} \vdash_{wf}\ b' \wedge x' \neq x$
⟨*proof*⟩

lemma *flip-u-eq*:
fixes u::u **and** u'::u **and** Θ::Θ **and** τ::τ
assumes $\Theta;\ \mathcal{B};\ \Gamma \vdash_{wf}\ \tau$
shows $(u \leftrightarrow u') \cdot \tau = \tau$ **and** $(u \leftrightarrow u') \cdot \Gamma = \Gamma$ **and** $(u \leftrightarrow u') \cdot \Theta = \Theta$ **and** $(u \leftrightarrow u') \cdot \mathcal{B} = \mathcal{B}$
⟨*proof*⟩

lemma *wfT-wf-cons-flip*:
fixes c::c **and** x::x
assumes $wfT\ P\ \mathcal{B}\ \Gamma\ \{\!|\ z : b\ \mid\ c\ |\!\}$ **and** $atom\ x\ \sharp\ (c,\Gamma)$
shows $wfG\ P\ \mathcal{B}\ ((x,b,c[z::=V\text{-}var\ x]_{cv})\quad \#_\Gamma\Gamma)$
⟨*proof*⟩

8.9 Context Strengthening

We can remove an entry for a variable from the context if the variable doesn't appear in the term and the variable is not used later in the context or any other context

lemma *fresh-restrict*:
fixes y::$'a$::*at-base* **and** Γ::Γ
assumes $atom\ y\ \sharp\ (\Gamma'\ @\ (x,\ b,\ c)\quad \#_\Gamma\ \Gamma)$
shows $atom\ y\ \sharp\ (\Gamma'@\Gamma)$
⟨*proof*⟩

lemma *wf-restrict1*:
fixes Γ::Γ **and** Γ'::Γ **and** v::v **and** e::e **and** c::c **and** τ::τ **and** ts::($string$*τ) *list* **and** Δ::Δ **and** s::s **and** b::b **and** *ftq*::*fun-typ-q* **and** *ft*::*fun-typ* **and** *ce*::*ce* **and** *td*::*type-def*
and *cs*::*branch-s* **and** *css*::*branch-list*
shows $\Theta;\ \mathcal{B};\ \Gamma\ \vdash_{wf}\ v : b \Longrightarrow \Gamma{=}\Gamma_1@((x,b',c')\ \#_\Gamma\Gamma_2) \Longrightarrow atom\ x\ \sharp\ v \Longrightarrow atom\ x\ \sharp\ \Gamma_1\ \Longrightarrow \Theta;\ \mathcal{B};\ \Gamma_1@\Gamma_2 \vdash_{wf}\ v : b$ **and**

Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ c $\Longrightarrow$ Γ=Γ_1@((x,b',c') $\#_\Gamma\Gamma_2$) $\Longrightarrow$ *atom* x $\sharp$ $c$$\Longrightarrow$ *atom* x $\sharp$ Γ_1 $\Longrightarrow$ Θ ; $\mathcal{B}$; Γ_1@Γ_2 $\vdash_{wf}$ c **and**
Θ; $\mathcal{B}$ $\vdash_{wf}$ Γ $\Longrightarrow$ Γ=Γ_1@((x,b',c') $\#_\Gamma\Gamma_2$) $\Longrightarrow$ *atom* x $\sharp$ Γ_1 $\Longrightarrow$ Θ; $\mathcal{B}$ $\vdash_{wf}$ Γ_1@Γ_2 **and**
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ τ $\Longrightarrow$ Γ=Γ_1@((x,b',c') $\#_\Gamma\Gamma_2$) $\Longrightarrow$ *atom* x $\sharp$ $\tau$$\Longrightarrow$ *atom* x $\sharp$ Γ_1 $\Longrightarrow$ Θ; $\mathcal{B}$; Γ_1@Γ_2 $\vdash_{wf}$ τ **and**
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *ts* $\Longrightarrow$ *True* **and**
$\vdash_{wf}$ Θ $\Longrightarrow$*True* **and**

Θ; $\mathcal{B}$ $\vdash_{wf}$ b $\Longrightarrow$ *True* **and**

Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *ce* : b $\Longrightarrow$ Γ=Γ_1@((x,b',c') $\#_\Gamma\Gamma_2$) $\Longrightarrow$ *atom* x $\sharp$ *ce* $\Longrightarrow$ *atom* x $\sharp$ Γ_1 $\Longrightarrow$ Θ; $\mathcal{B}$; Γ_1@Γ_2 $\vdash_{wf}$ *ce* : b **and**
Θ $\vdash_{wf}$ *td* $\Longrightarrow$ *True*
⟨*proof*⟩

lemma *wf-restrict2*:
fixes Γ::Γ **and** Γ'::Γ **and** v::v **and** e::e **and** c::c **and** τ::τ **and** *ts*::(*string*∗τ) *list* **and** Δ::Δ **and** s::s **and** b::b **and** *ftq*::*fun-typ-q* **and** *ft*::*fun-typ* **and** *ce*::*ce* **and** *td*::*type-def*
and *cs*::*branch-s* **and** *css*::*branch-list*
shows Θ; Φ; $\mathcal{B}$; Γ ; Δ $\vdash_{wf}$ e : b $\Longrightarrow$ Γ=Γ_1@((x,b',c') $\#_\Gamma\Gamma_2$) $\Longrightarrow$ *atom* x $\sharp$ e $\Longrightarrow$ *atom* x $\sharp$ Γ_1 $\Longrightarrow$ *atom* x $\sharp$ Δ $\Longrightarrow$ Θ; Φ; $\mathcal{B}$; Γ_1@Γ_2 ; Δ $\vdash_{wf}$ e : b **and**
Θ; Φ; $\mathcal{B}$; Γ ; Δ $\vdash_{wf}$ s : b $\Longrightarrow$ *True* **and**
Θ; Φ; $\mathcal{B}$; Γ ; Δ ; *tid* ; *dc* ; t $\vdash_{wf}$ *cs* : b $\Longrightarrow$ *True* **and**
Θ; Φ; $\mathcal{B}$; Γ ; Δ ; *tid* ; *dclist* $\vdash_{wf}$ *css* : b $\Longrightarrow$ *True* **and**
Θ $\vdash_{wf}$ (Φ::Φ) $\Longrightarrow$ *True* **and**
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ Δ $\Longrightarrow$ Γ=Γ_1@((x,b',c') $\#_\Gamma\Gamma_2$) $\Longrightarrow$ *atom* x $\sharp$ Γ_1 $\Longrightarrow$ *atom* x $\sharp$ Δ $\Longrightarrow$ Θ; $\mathcal{B}$; Γ_1@Γ_2 $\vdash_{wf}$ Δ **and**
Θ ; Φ $\vdash_{wf}$ *ftq* $\Longrightarrow$ *True* **and**
Θ ; Φ ; $\mathcal{B}$ $\vdash_{wf}$ *ft* $\Longrightarrow$ *True*

⟨*proof*⟩

lemmas *wf-restrict=wf-restrict1 wf-restrict2*

lemma *wfT-restrict2*:
fixes τ::τ
assumes *wfT* Θ $\mathcal{B}$ ((x, b, c) $\#_\Gamma$ Γ) τ **and** *atom* x $\sharp$ τ
shows Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ τ
⟨*proof*⟩

lemma *wfG-intros2*:
assumes *wfC* P $\mathcal{B}$ ((x,b,*TRUE*) $\#_\Gamma\Gamma$) c
shows *wfG* P $\mathcal{B}$ ((x,b,c) $\#_\Gamma\Gamma$)
⟨*proof*⟩

8.10 Type Definitions

lemma *wf-theta-weakening1*:
fixes Γ::Γ **and** Γ'::Γ **and** v::v **and** e::e **and** c::c **and** τ::τ **and** *ts*::(*string*∗τ) *list* **and** Δ::Δ **and** s::s **and** b::b **and** $\mathcal{B}$ :: $\mathcal{B}$ **and** *ftq*::*fun-typ-q* **and** *ft*::*fun-typ* **and** *ce*::*ce* **and** *td*::*type-def*
and *cs*::*branch-s* **and** *css*::*branch-list* **and** t::τ

shows $\Theta; \mathcal{B}; \Gamma \vdash_{wf} v : b \Longrightarrow \vdash_{wf} \Theta' \Longrightarrow set\ \Theta \subseteq set\ \Theta' \Longrightarrow \Theta'; \mathcal{B}; \Gamma \vdash_{wf} v : b$ **and**
$\Theta; \mathcal{B}; \Gamma \vdash_{wf} c \Longrightarrow \vdash_{wf} \Theta' \Longrightarrow set\ \Theta \subseteq set\ \Theta' \Longrightarrow \Theta'; \mathcal{B}; \Gamma \vdash_{wf} c$ **and**
$\Theta; \mathcal{B} \vdash_{wf} \Gamma \Longrightarrow \vdash_{wf} \Theta' \Longrightarrow set\ \Theta \subseteq set\ \Theta' \Longrightarrow \Theta'; \mathcal{B} \vdash_{wf} \Gamma$ **and**
$\Theta; \mathcal{B}; \Gamma \vdash_{wf} \tau \Longrightarrow \vdash_{wf} \Theta' \Longrightarrow set\ \Theta \subseteq set\ \Theta' \Longrightarrow \Theta'; \mathcal{B}; \Gamma \vdash_{wf} \tau$ **and**
$\Theta; \mathcal{B}; \Gamma \vdash_{wf} ts \Longrightarrow \vdash_{wf} \Theta' \Longrightarrow set\ \Theta \subseteq set\ \Theta' \Longrightarrow \Theta'; \mathcal{B}; \Gamma \vdash_{wf} ts$ **and**
$\vdash_{wf} P \Longrightarrow True$ **and**
$\Theta; \mathcal{B} \vdash_{wf} b \Longrightarrow \vdash_{wf} \Theta' \Longrightarrow set\ \Theta \subseteq set\ \Theta' \Longrightarrow \Theta'; \mathcal{B} \vdash_{wf} b$ **and**
$\Theta; \mathcal{B}; \Gamma \vdash_{wf} ce : b \Longrightarrow \vdash_{wf} \Theta' \Longrightarrow set\ \Theta \subseteq set\ \Theta' \Longrightarrow \Theta'; \mathcal{B}; \Gamma \vdash_{wf} ce : b$ **and**
$\Theta \vdash_{wf} td \Longrightarrow \vdash_{wf} \Theta' \Longrightarrow set\ \Theta \subseteq set\ \Theta' \Longrightarrow \Theta' \vdash_{wf} td$

⟨*proof*⟩

lemma *wf-theta-weakening2*:
fixes Γ::Γ **and** Γ'::Γ **and** *v*::*v* **and** *e*::*e* **and** *c*::*c* **and** τ::τ **and** *ts*::(*string*∗τ) *list* **and** Δ::Δ **and** *s*::*s* **and** *b*::*b* **and** $\mathcal{B}$ :: $\mathcal{B}$ **and** *ftq*::*fun-typ-q* **and** *ft*::*fun-typ* **and** *ce*::*ce* **and** *td*::*type-def*
and *cs*::*branch-s* **and** *css*::*branch-list* **and** *t*::τ
shows
$\Theta; \Phi; \mathcal{B}; \Gamma ; \Delta \vdash_{wf} e : b \Longrightarrow \vdash_{wf} \Theta' \Longrightarrow set\ \Theta \subseteq set\ \Theta' \Longrightarrow \Theta'; \Phi ; \mathcal{B} ; \Gamma ; \Delta \vdash_{wf} e : b$ **and**
$\Theta; \Phi; \mathcal{B}; \Gamma ; \Delta \vdash_{wf} s : b \Longrightarrow \vdash_{wf} \Theta' \Longrightarrow set\ \Theta \subseteq set\ \Theta' \Longrightarrow \Theta'; \Phi ; \mathcal{B} ; \Gamma ; \Delta \vdash_{wf} s : b$ **and**
$\Theta; \Phi; \mathcal{B}; \Gamma ; \Delta ; tid ; dc ; t \vdash_{wf} cs : b \Longrightarrow \vdash_{wf} \Theta' \Longrightarrow set\ \Theta \subseteq set\ \Theta' \Longrightarrow \Theta'; \Phi ; \mathcal{B} ; \Gamma ; \Delta ; tid ; dc ; t \vdash_{wf} cs : b$ **and**
$\Theta; \Phi; \mathcal{B}; \Gamma ; \Delta ; tid ; dclist \vdash_{wf} css : b \Longrightarrow \vdash_{wf} \Theta' \Longrightarrow set\ \Theta \subseteq set\ \Theta' \Longrightarrow \Theta'; \Phi ; \mathcal{B} ; \Gamma ; \Delta ; tid ; dclist \vdash_{wf} css : b$ **and**
$\Theta \vdash_{wf} (\Phi::\Phi) \Longrightarrow \vdash_{wf} \Theta' \Longrightarrow set\ \Theta \subseteq set\ \Theta' \Longrightarrow \Theta' \vdash_{wf} (\Phi::\Phi)$ **and**
$\Theta; \mathcal{B}; \Gamma \vdash_{wf} \Delta \Longrightarrow \vdash_{wf} \Theta' \Longrightarrow set\ \Theta \subseteq set\ \Theta' \Longrightarrow \Theta'; \mathcal{B}; \Gamma \vdash_{wf} \Delta$ **and**
$\Theta ; \Phi \vdash_{wf} ftq \Longrightarrow \vdash_{wf} \Theta' \Longrightarrow set\ \Theta \subseteq set\ \Theta' \Longrightarrow \Theta'; \Phi \vdash_{wf} ftq$ **and**
$\Theta ; \Phi ; \mathcal{B} \vdash_{wf} ft \Longrightarrow \vdash_{wf} \Theta' \Longrightarrow set\ \Theta \subseteq set\ \Theta' \Longrightarrow \Theta'; \Phi ; \mathcal{B} \vdash_{wf} ft$

⟨*proof*⟩

lemmas *wf-theta-weakening = wf-theta-weakening1 wf-theta-weakening2*

lemma *lookup-wfTD*:
fixes *td*::*type-def*
assumes $td \in set\ \Theta$ **and** $\vdash_{wf} \Theta$
shows $\Theta \vdash_{wf} td$
⟨*proof*⟩

8.10.1 Simple

lemma *wfTh-dclist-unique*:
assumes *wfTh* Θ **and** *AF-typedef tid dclist1* $\in set\ \Theta$ **and** *AF-typedef tid dclist2* $\in set\ \Theta$
shows *dclist1 = dclist2*
⟨*proof*⟩

lemma *wfTs-ctor-unique*:
fixes *dclist*::(*string*∗τ) *list*
assumes Θ ; {||} ; *GNil* $\vdash_{wf}$ *dclist* **and** (*c*, *t1*) $\in$ *set dclist* **and** (*c*,*t2*) $\in$ *set dclist*
shows *t1 = t2*
⟨*proof*⟩

lemma *wfTD-ctor-unique*:
assumes $\Theta \vdash_{wf}$ (*AF-typedef tid dclist*) **and** (*c*, *t1*) $\in$ *set dclist* **and** (*c*,*t2*) $\in$ *set dclist*

shows *t1 = t2*
⟨*proof*⟩

lemma *wfTh-ctor-unique*:
assumes *wfTh* Θ **and** *AF-typedef tid dclist* ∈ *set* Θ **and** (*c*, *t1*) ∈ *set dclist* **and** (*c*,*t2*) ∈ *set dclist*

shows *t1 = t2*
⟨*proof*⟩

lemma *wfTs-supp-t*:
fixes *dclist*::(*string*∗τ) *list*
assumes (*c*,*t*) ∈ *set dclist* **and** Θ ; *B* ; *GNil* ⊢$_{wf}$ *dclist*
shows *supp t* ⊆ *supp B*
⟨*proof*⟩

lemma *wfTh-lookup-supp-empty*:
fixes *t*::τ
assumes *AF-typedef tid dclist* ∈ *set* Θ **and** (*c*,*t*) ∈ *set dclist* **and** ⊢$_{wf}$ Θ
shows *supp t* = {}
⟨*proof*⟩

lemma *wfTh-supp-b*:
assumes *AF-typedef tid dclist* ∈ *set* Θ **and** (*dc*,{| *z* : *b* | *c* |}) ∈ *set dclist* **and** ⊢$_{wf}$ Θ
shows *supp b* = {}
⟨*proof*⟩

lemma *wfTh-b-eq-iff*:
fixes *bva1*::*bv* **and** *bva2*::*bv* **and** *dc*::*string*
assumes (*dc*, {| *x1* : *b1* | *c1* |}) ∈ *set dclist1* **and** (*dc*, {| *x2* : *b2* | *c2* |}) ∈ *set dclist2* **and**
wfTs P {|*bva1*|} *GNil dclist1* **and** *wfTs P* {|*bva2*|} *GNil dclist2*
[[*atom bva1*]]*lst.dclist1* = [[*atom bva2*]]*lst.dclist2*
shows [[*atom bva1*]]*lst.* (*dc*,{| *x1* : *b1* | *c1* |}) = [[*atom bva2*]]*lst.* (*dc*,{| *x2* : *b2* | *c2* |})
⟨*proof*⟩

8.10.2 Polymorphic

lemma *wfTh-wfTs-poly*:
fixes *dclist*::(*string* ∗ τ) *list*
assumes *AF-typedef-poly tyid bva dclist* ∈ *set P* **and** ⊢$_{wf}$ *P*
shows *P* ; {|*bva*|} ; *GNil* ⊢$_{wf}$ *dclist*
⟨*proof*⟩

lemma *wfTh-dclist-poly-unique*:
assumes *wfTh* Θ **and** *AF-typedef-poly tid bva dclist1* ∈ *set* Θ **and** *AF-typedef-poly tid bva2 dclist2* ∈ *set* Θ
shows [[*atom bva*]]*lst. dclist1* = [[*atom bva2*]]*lst.dclist2*
⟨*proof*⟩

lemma *wfTh-poly-lookup-supp*:
fixes *t*::τ
assumes *AF-typedef-poly tid bv dclist* ∈ *set* Θ **and** (*c*,*t*) ∈ *set dclist* **and** ⊢$_{wf}$ Θ
shows *supp t* ⊆ {*atom bv*}
⟨*proof*⟩

lemma *wfTh-poly-supp-b*:
assumes *AF-typedef-poly tid bv dclist* $\in$ *set* Θ **and** (*dc*,$\{\!|$ *z* : *b* | *c* $|\!\}$) $\in$ *set dclist* **and** $\vdash_{wf}$ Θ
shows *supp b* $\subseteq$ {*atom bv*}
⟨*proof*⟩

lemma *subst-g-inside*:
fixes *x*::*x* **and** *c*::*c* **and** Γ::Γ **and** Γ'::Γ
assumes *wfG P* $\mathcal{B}$ (Γ' @ (*x*, *b*, *c*[*z*::=*V-var x*]$_{cv}$) $\#_\Gamma$ Γ)
shows (Γ' @ (*x*, *b*, *c*[*z*::=*V-var x*]$_{cv}$) $\#_\Gamma$ Γ)[*x*::=*v*]$_{\Gamma v}$ = (Γ'[*x*::=*v*]$_{\Gamma v}$@Γ)
⟨*proof*⟩

lemma *wfTh-td-eq*:
assumes *td1* $\in$ *set* (*td2* # *P*) **and** *wfTh* (*td2* # *P*) **and** *name-of-type td1* = *name-of-type td2*
shows *td1* = *td2*
⟨*proof*⟩

lemma *wfTh-td-unique*:
assumes *td1* $\in$ *set P* **and** *td2* $\in$ *set P* **and** *wfTh P* **and** *name-of-type td1* = *name-of-type td2*
shows *td1* = *td2*
⟨*proof*⟩

lemma *wfTs-distinct*:
fixes *dclist*::(*string* $*$ τ) *list*
assumes Θ ; *B* ; *GNil* $\vdash_{wf}$ *dclist*
shows *distinct* (*map fst dclist*)
⟨*proof*⟩

lemma *wfTh-dclist-distinct*:
assumes *AF-typedef s dclist* $\in$ *set P* **and** *wfTh P*
shows *distinct* (*map fst dclist*)
⟨*proof*⟩

lemma *wfTh-dc-t-unique2*:
assumes *AF-typedef s dclist'* $\in$ *set P* **and** (*dc*,*tc'*) $\in$ *set dclist'* **and** *AF-typedef s dclist* $\in$ *set P* **and** *wfTh P* **and**
(*dc*, *tc*) $\in$ *set dclist*
shows *tc*= *tc'*
⟨*proof*⟩

lemma *wfTh-dc-t-unique*:
assumes *AF-typedef s dclist'* $\in$ *set P* **and** (*dc*, $\{\!|$ *x'* : *b'* | *c'* $|\!\}$) $\in$ *set dclist'* **and** *AF-typedef s dclist* $\in$ *set P* **and** *wfTh P* **and**
(*dc*, $\{\!|$ *x* : *b* | *c* $|\!\}$) $\in$ *set dclist*
shows $\{\!|$ *x'* : *b'* | *c'* $|\!\}$= $\{\!|$ *x* : *b* | *c* $|\!\}$
⟨*proof*⟩

lemma *wfTs-wfT*:
fixes *dclist*::(*string* $*\tau$) *list* **and** *t*::τ
assumes Θ; $\mathcal{B}$; *GNil* $\vdash_{wf}$ *dclist* **and** (*dc*,*t*) $\in$ *set dclist*
shows Θ; $\mathcal{B}$; *GNil* $\vdash_{wf}$ *t*
⟨*proof*⟩

lemma *wfTh-wfT*:
fixes $t::\tau$
assumes *wfTh P* **and** *AF-typedef tid dclist* $\in$ *set P* **and** $(dc,t) \in$ *set dclist*
shows P ; $\{||\}$; *GNil* $\vdash_{wf}$ t
⟨*proof*⟩

lemma *td-lookup-eq-iff*:
fixes *dc* :: *string* **and** *bva1*::*bv* **and** *bva2*::*bv*
assumes $[[atom\ bva1]]lst.\ dclist1 = [[atom\ bva2]]lst.\ dclist2$ **and** $(dc, \{\!|\ x : b\ |\ c\ |\!\}) \in$ *set dclist1*
shows $\exists\, x2\ b2\ c2.\ (dc, \{\!|\ x2 : b2\ |\ c2\ |\!\}) \in$ *set dclist2*
⟨*proof*⟩

lemma *lst-t-b-eq-iff*:
fixes *bva1*::*bv* **and** *bva2*::*bv*
assumes $[[atom\ bva1]]lst.\ \{\!|\ x1 : b1\ |\ c1\ |\!\} = [[atom\ bva2]]lst.\ \{\!|\ x2 : b2\ |\ c2\ |\!\}$
shows $[[atom\ bva1]]lst.\ b1 = [[atom\ bva2]]lst.b2$
⟨*proof*⟩

lemma *wfTh-typedef-poly-b-eq-iff*:
assumes *AF-typedef-poly tyid bva1 dclist1* $\in$ *set P* **and** $(dc, \{\!|\ x1 : b1\ |\ c1\ |\!\}) \in$ *set dclist1*
and *AF-typedef-poly tyid bva2 dclist2* $\in$ *set P* **and** $(dc, \{\!|\ x2 : b2\ |\ c2\ |\!\}) \in$ *set dclist2* **and** $\vdash_{wf}$ P
shows $b1[bva1::=b]_{bb} = b2[bva2::=b]_{bb}$
⟨*proof*⟩

8.11 Equivariance Lemmas

lemma *x-not-in-u-set*[*simp*]:
fixes $x::x$ **and** $us::u$ *fset*
shows *atom x* $\notin$ *supp us*
⟨*proof*⟩

lemma *wfS-flip-eq*:
fixes $s1::s$ **and** $x1::x$ **and** $s2::s$ **and** $x2::x$ **and** $\Delta::\Delta$
assumes $[[atom\ x1]]lst.\ s1 = [[atom\ x2]]lst.\ s2$ **and** $[[atom\ x1]]lst.\ t1 = [[atom\ x2]]lst.\ t2$ **and** $[[atom\ x1]]lst.\ c1 = [[atom\ x2]]lst.\ c2$ **and** *atom x2* $\sharp$ Γ **and**
$\Theta; \mathcal{B}; \Gamma \vdash_{wf} \Delta$ **and**
Θ ; Φ ; $\mathcal{B}$; $(x1,\ b,\ c1)$ $\#_\Gamma$ Γ ; $\Delta \vdash_{wf}$ *s1* : *b-of t1*
shows Θ ; Φ ; $\mathcal{B}$; $(x2,\ b,\ c2)$ $\#_\Gamma$ Γ ; Δ $\vdash_{wf}$ *s2* : *b-of t2*
⟨*proof*⟩

8.12 Lookup

lemma *wf-not-in-prefix*:
assumes Θ ; $B \vdash_{wf} (\Gamma'@(x,b1,c1)\ \#_\Gamma\Gamma)$
shows $x \notin$ *fst* ‘ *toSet* Γ'
⟨*proof*⟩

lemma *lookup-inside-wf*[*simp*]:
assumes Θ ; $B \vdash_{wf} (\Gamma'@(x,b1,c1)\ \#_\Gamma\Gamma)$
shows *Some* $(b1,c1)$ = *lookup* $(\Gamma'@(x,b1,c1)\ \#_\Gamma\Gamma)$ x

⟨*proof*⟩

lemma *lookup-weakening*:
fixes Θ::Θ **and** Γ::Γ **and** Γ′::Γ
assumes *Some* (*b*,*c*) = *lookup* Γ *x* **and** *toSet* Γ ⊆ *toSet* Γ′ **and** Θ; $\mathcal{B} \vdash_{wf}$ Γ′ **and** Θ; $\mathcal{B} \vdash_{wf}$ Γ
shows *Some* (*b*,*c*) = *lookup* Γ′ *x*
⟨*proof*⟩

lemma *wfPhi-lookup-fun-unique*:
fixes Φ::Φ
assumes Θ $\vdash_{wf}$ Φ **and** *AF-fundef f fd* ∈ *set* Φ
shows *Some* (*AF-fundef f fd*) = *lookup-fun* Φ *f*
⟨*proof*⟩

lemma *lookup-fun-weakening*:
fixes Φ′::Φ
assumes *Some fd* = *lookup-fun* Φ *f* **and** *set* Φ ⊆ *set* Φ′ **and** Θ $\vdash_{wf}$ Φ′
shows *Some fd* = *lookup-fun* Φ′ *f*
⟨*proof*⟩

lemma *fundef-poly-fresh-bv*:
assumes *atom bv2* ♯ (*bv1*,*b1*,*c1*,*τ1*,*s1*)
shows ∗ : (*AF-fun-typ-some bv2* (*AF-fun-typ x1* ((*bv1*↔*bv2*) · *b1*) ((*bv1*↔*bv2*) ·*c1*) ((*bv1*↔*bv2*) · *τ1*) ((*bv1*↔*bv2*) · *s1*)) = (*AF-fun-typ-some bv1* (*AF-fun-typ x1 b1 c1 τ1 s1*)))
(**is** (*AF-fun-typ-some ?bv ?fun-typ* = *AF-fun-typ-some ?bva ?fun-typa*))
⟨*proof*⟩

It is possible to collapse some of the easy to prove inductive cases into a single proof at the qed line but this makes it fragile under change. For example, changing the lemma statement might make one of the previously trivial cases non-trivial and so the collapsing needs to be unpacked. Is there a way to find which case has failed in the qed line?

lemma *wb-b-weakening1*:
fixes Γ::Γ **and** Γ′::Γ **and** *v*::*v* **and** *e*::*e* **and** *c*::*c* **and** *τ*::*τ* **and** *ts*::(*string*∗*τ*) *list* **and** Δ::Δ **and** *s*::*s* **and** $\mathcal{B}$::$\mathcal{B}$ **and** *ftq*::*fun-typ-q* **and** *ft*::*fun-typ* **and** *ce*::*ce* **and** *td*::*type-def*
and *cs*::*branch-s* **and** *css*::*branch-list*

shows Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *v* : *b* ⟹ $\mathcal{B}$ |⊆| $\mathcal{B}'$ ⟹ Θ; $\mathcal{B}'$; Γ $\vdash_{wf}$ *v* : *b* **and**
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *c* ⟹$\mathcal{B}$ |⊆| $\mathcal{B}'$ ⟹ Θ; $\mathcal{B}'$; Γ $\vdash_{wf}$ *c* **and**
Θ; $\mathcal{B}$ $\vdash_{wf}$ Γ ⟹$\mathcal{B}$ |⊆| $\mathcal{B}'$ ⟹ Θ; $\mathcal{B}'$ $\vdash_{wf}$ Γ **and**
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *τ* ⟹ $\mathcal{B}$ |⊆| $\mathcal{B}'$ ⟹ Θ; $\mathcal{B}'$; Γ $\vdash_{wf}$ *τ* **and**
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *ts* ⟹ $\mathcal{B}$ |⊆| $\mathcal{B}'$ ⟹ Θ; $\mathcal{B}'$; Γ $\vdash_{wf}$ *ts* **and**
$\vdash_{wf}$ *P* ⟹ *True* **and**
wfB Θ $\mathcal{B}$ *b* ⟹ $\mathcal{B}$ |⊆| $\mathcal{B}'$ ⟹ *wfB* Θ $\mathcal{B}'$ *b* **and**
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *ce* : *b* ⟹ $\mathcal{B}$ |⊆| $\mathcal{B}'$ ⟹ Θ; $\mathcal{B}'$; Γ $\vdash_{wf}$ *ce* : *b* **and**
Θ $\vdash_{wf}$ *td* ⟹ *True*
⟨*proof*⟩

lemma *wb-b-weakening2*:
fixes Γ::Γ **and** Γ′::Γ **and** *v*::*v* **and** *e*::*e* **and** *c*::*c* **and** *τ*::*τ* **and** *ts*::(*string*∗*τ*) *list* **and** Δ::Δ **and** *s*::*s* **and** $\mathcal{B}$::$\mathcal{B}$ **and** *ftq*::*fun-typ-q* **and** *ft*::*fun-typ* **and** *ce*::*ce* **and** *td*::*type-def*
and *cs*::*branch-s* **and** *css*::*branch-list*

shows
Θ; Φ; $\mathcal{B}$; Γ ; $\Delta \vdash_{wf} e : b \Longrightarrow \mathcal{B} \mid\subseteq\mid \mathcal{B}' \Longrightarrow \Theta$; Φ ; $\mathcal{B}'$; Γ ; $\Delta \vdash_{wf} e : b$ **and**
Θ; Φ; $\mathcal{B}$; Γ ; $\Delta \vdash_{wf} s : b \Longrightarrow \mathcal{B} \mid\subseteq\mid \mathcal{B}' \Longrightarrow \Theta$; Φ ; $\mathcal{B}'$; Γ ; $\Delta \vdash_{wf} s : b$ **and**
Θ ; Φ ; $\mathcal{B}$; Γ ; Δ ; *tid* ; *dc* ; $t \vdash_{wf}$ *cs* : $b \Longrightarrow \mathcal{B} \mid\subseteq\mid \mathcal{B}' \Longrightarrow \Theta$; Φ ; $\mathcal{B}'$; Γ ; Δ ; *tid* ; *dc* ; $t \vdash_{wf}$ *cs* : b **and**
Θ ; Φ ; $\mathcal{B}$; Γ ; Δ ; *tid* ; *dclist* $\vdash_{wf}$ *css* : $b \Longrightarrow \mathcal{B} \mid\subseteq\mid \mathcal{B}' \Longrightarrow \Theta$; Φ ; $\mathcal{B}'$; Γ ; Δ ; *tid* ; *dclist* $\vdash_{wf}$ *css* : b **and**
$\Theta \vdash_{wf} (\Phi::\Phi) \Longrightarrow$ *True* **and**
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta \Longrightarrow \mathcal{B} \mid\subseteq\mid \mathcal{B}' \Longrightarrow \Theta$; $\mathcal{B}'$; $\Gamma \vdash_{wf} \Delta$ **and**
Θ ; $\Phi \vdash_{wf}$ *ftq* $\Longrightarrow$ *True* **and**
Θ ; Φ ; $\mathcal{B} \vdash_{wf}$ *ft* $\Longrightarrow \mathcal{B} \mid\subseteq\mid \mathcal{B}' \Longrightarrow \Theta$; Φ ; $\mathcal{B}' \vdash_{wf}$ *ft*
⟨*proof*⟩

lemmas *wb-b-weakening = wb-b-weakening1 wb-b-weakening2*

lemma *wfG-b-weakening*:
fixes $\Gamma::\Gamma$
assumes $\mathcal{B} \mid\subseteq\mid \mathcal{B}'$ **and** Θ; $\mathcal{B} \vdash_{wf} \Gamma$
shows Θ; $\mathcal{B}' \vdash_{wf} \Gamma$
⟨*proof*⟩

lemma *wfT-b-weakening*:
fixes $\Gamma::\Gamma$ **and** $\Theta::\Theta$ **and** $\tau::\tau$
assumes $\mathcal{B} \mid\subseteq\mid \mathcal{B}'$ **and** Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \tau$
shows Θ; $\mathcal{B}'$; $\Gamma \vdash_{wf} \tau$
⟨*proof*⟩

lemma *wfB-subst-wfB*:
fixes $\tau::\tau$ **and** $b'::b$ **and** $b::b$
assumes Θ ; $\{|bv|\} \vdash_{wf} b$ **and** Θ; $\mathcal{B} \vdash_{wf} b'$
shows Θ; $\mathcal{B} \vdash_{wf} b[bv::=b']_{bb}$
⟨*proof*⟩

lemma *wfT-subst-wfB*:
fixes $\tau::\tau$ **and** $b'::b$
assumes Θ ; $\{|bv|\}$; $(x, b, c) \#_\Gamma$ *GNil* $\vdash_{wf} \tau$ **and** Θ; $\mathcal{B} \vdash_{wf} b'$
shows Θ; $\mathcal{B} \vdash_{wf}$ (*b-of* τ)$[bv::=b']_{bb}$
⟨*proof*⟩

lemma *wfG-cons-unique*:
assumes (*x1*,*b1*,*c1*) ∈ *toSet* (((*x*,*b*,*c*) $\#_\Gamma\Gamma$)) **and** *wfG* Θ $\mathcal{B}$ (((*x*,*b*,*c*) $\#_\Gamma\Gamma$)) **and** $x = x1$
shows $b1 = b \wedge c1 = c$
⟨*proof*⟩

lemma *wfG-member-unique*:
assumes (*x1*,*b1*,*c1*) ∈ *toSet* (Γ'@((*x*,*b*,*c*) $\#_\Gamma\Gamma$)) **and** *wfG* Θ $\mathcal{B}$ (Γ'@((*x*,*b*,*c*) $\#_\Gamma\Gamma$)) **and** $x = x1$
shows $b1 = b \wedge c1 = c$
⟨*proof*⟩

8.13 Function Definitions

lemma *wb-phi-weakening*:

fixes Γ::Γ **and** Γ'::Γ **and** v::v **and** e::e **and** c::c **and** τ::τ **and** ts::(*string*$*\tau$) *list* **and** Δ::Δ **and** s::s **and** $\mathcal{B}$::$\mathcal{B}$ **and** *ftq*::*fun-typ-q* **and** *ft*::*fun-typ* **and** *ce*::*ce* **and** *td*::*type-def*
and *cs*::*branch-s* **and** *css*::*branch-list* **and** Φ::Φ
shows
Θ; Φ; $\mathcal{B}$; Γ ; $\Delta \vdash_{wf} e : b \Longrightarrow \Theta \vdash_{wf} \Phi' \Longrightarrow set\ \Phi \subseteq set\ \Phi' \Longrightarrow \Theta$; Φ' ; $\mathcal{B}$; Γ ; $\Delta \vdash_{wf} e : b$
and
Θ; Φ; $\mathcal{B}$; Γ ; $\Delta \vdash_{wf} s : b \Longrightarrow \Theta \vdash_{wf} \Phi' \Longrightarrow set\ \Phi \subseteq set\ \Phi' \Longrightarrow \Theta$; Φ' ; $\mathcal{B}$; Γ ; $\Delta \vdash_{wf} s : b$
and
Θ ; Φ ; $\mathcal{B}$; Γ ; Δ ; *tid* ; *dc* ; $t \vdash_{wf} cs : b \Longrightarrow \Theta \vdash_{wf} \Phi' \Longrightarrow set\ \Phi \subseteq set\ \Phi' \Longrightarrow \Theta$; Φ' ; $\mathcal{B}$; Γ ; Δ ; *tid* ; *dc* ; $t \vdash_{wf} cs : b$ **and**
Θ ; Φ ; $\mathcal{B}$; Γ ; Δ ; *tid* ; *dclist* $\vdash_{wf} css : b \Longrightarrow \Theta \vdash_{wf} \Phi' \Longrightarrow set\ \Phi \subseteq set\ \Phi' \Longrightarrow \Theta$; Φ' ; $\mathcal{B}$; Γ ; Δ ; *tid* ; *dclist* $\vdash_{wf} css : b$ **and**
$\Theta \vdash_{wf} (\Phi::\Phi) \Longrightarrow True$ **and**
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta \Longrightarrow True$ **and**
Θ ; $\Phi \vdash_{wf} ftq \Longrightarrow \Theta \vdash_{wf} \Phi' \Longrightarrow set\ \Phi \subseteq set\ \Phi' \Longrightarrow \Theta$; $\Phi' \vdash_{wf} ftq$ **and**
Θ ; Φ ; $\mathcal{B} \vdash_{wf} ft \Longrightarrow \Theta \vdash_{wf} \Phi' \Longrightarrow set\ \Phi \subseteq set\ \Phi' \Longrightarrow \Theta$; Φ' ; $\mathcal{B} \vdash_{wf} ft$
⟨*proof*⟩

lemma *wfT-fun-return-t*:
fixes $\tau a'$::τ **and** τ'::τ
assumes Θ; $\mathcal{B}$; (*xa*, *b*, *ca*) $\#_\Gamma$ *GNil* $\vdash_{wf} \tau a'$ **and** (*AF-fun-typ x b c* τ' *s'*) = (*AF-fun-typ xa b ca* $\tau a'$ *sa'*)
shows Θ; $\mathcal{B}$; (*x*, *b*, *c*) $\#_\Gamma$ *GNil* $\vdash_{wf} \tau'$
⟨*proof*⟩

lemma *wfFT-wf-aux*:
fixes τ::τ **and** Θ::Θ **and** Φ::Φ **and** *ft* :: *fun-typ-q* **and** *s*::*s* **and** Δ::Δ
assumes Θ ; Φ ; $B \vdash_{wf}$ (*AF-fun-typ x b c* τ *s*)
shows Θ ; *B* ; (*x*,*b*,*c*) $\#_\Gamma$ *GNil* $\vdash_{wf} \tau \wedge \Theta \vdash_{wf} \Phi \wedge supp\ s \subseteq \{ atom\ x \} \cup supp\ B$
⟨*proof*⟩

lemma *wfFT-simple-wf*:
fixes τ::τ **and** Θ::Θ **and** Φ::Φ **and** *ft* :: *fun-typ-q* **and** *s*::*s* **and** Δ::Δ
assumes Θ ; $\Phi \vdash_{wf}$ (*AF-fun-typ-none* (*AF-fun-typ x b c* τ *s*))
shows Θ ; {||} ; (*x*,*b*,*c*) $\#_\Gamma$ *GNil* $\vdash_{wf} \tau \wedge \Theta \vdash_{wf} \Phi \wedge supp\ s \subseteq \{ atom\ x \}$
⟨*proof*⟩

lemma *wfFT-poly-wf*:
fixes τ::τ **and** Θ::Θ **and** Φ::Φ **and** *ftq* :: *fun-typ-q* **and** *s*::*s* **and** Δ::Δ
assumes Θ ; $\Phi \vdash_{wf}$ (*AF-fun-typ-some bv* (*AF-fun-typ x b c* τ *s*))
shows Θ ; {|*bv*|} ; (*x*,*b*,*c*) $\#_\Gamma$ *GNil* $\vdash_{wf} \tau \wedge \Theta \vdash_{wf} \Phi \wedge \Theta$; Φ ; {|*bv*|} $\vdash_{wf}$ (*AF-fun-typ x b c* τ *s*)
⟨*proof*⟩

lemma *wfFT-poly-wfT*:
fixes τ::τ **and** Θ::Θ **and** Φ::Φ **and** *ft* :: *fun-typ-q*
assumes Θ ; $\Phi \vdash_{wf}$ (*AF-fun-typ-some bv* (*AF-fun-typ x b c* τ *s*))
shows Θ ; {| *bv* |} ; (*x*,*b*,*c*) $\#_\Gamma$ *GNil* $\vdash_{wf} \tau$
⟨*proof*⟩

lemma *b-of-supp*:
supp (*b-of t*) $\subseteq$ *supp t*

⟨*proof*⟩

lemma *wfPhi-f-simple-wf*:
fixes τ::τ **and** Θ::Θ **and** Φ::Φ **and** *ft* :: *fun-typ-q* **and** *s*::*s* **and** Φ'::Φ
assumes *AF-fundef f* (*AF-fun-typ-none* (*AF-fun-typ x b c* τ *s*)) $\in$ *set* Φ **and** $\Theta \vdash_{wf} \Phi$ **and** *set* Φ $\subseteq$ *set* Φ' **and** $\Theta \vdash_{wf} \Phi'$
shows Θ ; {||} ; (*x*,*b*,*c*) $\#_\Gamma$ *GNil* $\vdash_{wf}$ τ $\wedge$ $\Theta \vdash_{wf} \Phi \wedge$ *supp s* $\subseteq$ { *atom x* }
⟨*proof*⟩

lemma *wfPhi-f-simple-wfT*:
fixes τ::τ **and** Θ::Θ **and** Φ::Φ **and** *ft* :: *fun-typ-q*
assumes *Some* (*AF-fundef f* (*AF-fun-typ-none* (*AF-fun-typ x b c* τ *s*))) = *lookup-fun* Φ *f* **and** $\Theta \vdash_{wf} \Phi$
shows Θ ; {||} ; (*x*,*b*,*c*) $\#_\Gamma$ *GNil* $\vdash_{wf}$ τ
⟨*proof*⟩

lemma *wfPhi-f-simple-supp-b*:
fixes τ::τ **and** Θ::Θ **and** Φ::Φ **and** *ft* :: *fun-typ-q*
assumes *Some* (*AF-fundef f* (*AF-fun-typ-none* (*AF-fun-typ x b c* τ *s*))) = *lookup-fun* Φ *f* **and** $\Theta \vdash_{wf} \Phi$
shows *supp b* = {}
⟨*proof*⟩

lemma *wfPhi-f-simple-supp-t*:
fixes τ::τ **and** Θ::Θ **and** Φ::Φ **and** *ft* :: *fun-typ-q*
assumes *Some* (*AF-fundef f* (*AF-fun-typ-none* (*AF-fun-typ x b c* τ *s*))) = *lookup-fun* Φ *f* **and** $\Theta \vdash_{wf} \Phi$
shows *supp* τ $\subseteq$ { *atom x* }
⟨*proof*⟩

lemma *wfPhi-f-simple-supp-c*:
fixes τ::τ **and** Θ::Θ **and** Φ::Φ **and** *ft* :: *fun-typ-q*
assumes *Some* (*AF-fundef f* (*AF-fun-typ-none* (*AF-fun-typ x b c* τ *s*))) = *lookup-fun* Φ *f* **and** $\Theta \vdash_{wf} \Phi$
shows *supp c* $\subseteq$ { *atom x* }
⟨*proof*⟩

lemma *wfPhi-f-simple-supp-s*:
fixes τ::τ **and** Θ::Θ **and** Φ::Φ **and** *ft* :: *fun-typ-q*
assumes *Some* (*AF-fundef f* (*AF-fun-typ-none* (*AF-fun-typ x b c* τ *s*))) = *lookup-fun* Φ *f* **and** $\Theta \vdash_{wf} \Phi$
shows *supp s* $\subseteq$ {*atom x*}
⟨*proof*⟩

lemma *wfPhi-f-poly-wf*:
fixes τ::τ **and** Θ::Θ **and** Φ::Φ **and** *ft* :: *fun-typ-q* **and** *s*::*s* **and** Φ'::Φ
assumes *AF-fundef f* (*AF-fun-typ-some bv* (*AF-fun-typ x b c* τ *s*)) $\in$ *set* Φ **and** $\Theta \vdash_{wf} \Phi$ **and** *set* Φ $\subseteq$ *set* Φ' **and** $\Theta \vdash_{wf} \Phi'$
shows Θ ; {|*bv*|} ; (*x*,*b*,*c*) $\#_\Gamma$ *GNil* $\vdash_{wf}$ τ $\wedge$ $\Theta \vdash_{wf} \Phi' \wedge$ Θ ; Φ' ; {|*bv*|} $\vdash_{wf}$ (*AF-fun-typ x b c* τ *s*)
⟨*proof*⟩

lemma *wfPhi-f-poly-wfT*:

fixes τ::τ **and** Θ::Θ **and** Φ::Φ **and** *ft* :: *fun-typ-q*
assumes *Some* (*AF-fundef f* (*AF-fun-typ-some bv* (*AF-fun-typ x b c* τ *s*))) = *lookup-fun* Φ *f* **and** Θ $\vdash_{wf}$ Φ
shows Θ ; {| *bv* |} ; (*x*,*b*,*c*) $\#_\Gamma$ *GNil* $\vdash_{wf}$ τ
⟨*proof*⟩

lemma *wfPhi-f-poly-supp-b*:
fixes τ::τ **and** Θ::Θ **and** Φ::Φ **and** *ft* :: *fun-typ-q*
assumes *Some* (*AF-fundef f* (*AF-fun-typ-some bv* (*AF-fun-typ x b c* τ *s*))) = *lookup-fun* Φ *f* **and** Θ $\vdash_{wf}$ Φ
shows *supp b* ⊆ *supp bv*
⟨*proof*⟩

lemma *wfPhi-f-poly-supp-t*:
fixes τ::τ **and** Θ::Θ **and** Φ::Φ **and** *ft* :: *fun-typ-q*
assumes *Some* (*AF-fundef f* (*AF-fun-typ-some bv* (*AF-fun-typ x b c* τ *s*))) = *lookup-fun* Φ *f* **and** Θ $\vdash_{wf}$ Φ
shows *supp* τ ⊆ { *atom x* , *atom bv* }
⟨*proof*⟩

lemma *wfPhi-f-poly-supp-b-of-t*:
fixes τ::τ **and** Θ::Θ **and** Φ::Φ **and** *ft* :: *fun-typ-q*
assumes *Some* (*AF-fundef f* (*AF-fun-typ-some bv* (*AF-fun-typ x b c* τ *s*))) = *lookup-fun* Φ *f* **and** Θ $\vdash_{wf}$ Φ
shows *supp* (*b-of* τ) ⊆ { *atom bv* }
⟨*proof*⟩

lemma *wfPhi-f-poly-supp-c*:
fixes τ::τ **and** Θ::Θ **and** Φ::Φ **and** *ft* :: *fun-typ-q*
assumes *Some* (*AF-fundef f* (*AF-fun-typ-some bv* (*AF-fun-typ x b c* τ *s*))) = *lookup-fun* Φ *f* **and** Θ $\vdash_{wf}$ Φ
shows *supp c* ⊆ { *atom x*, *atom bv* }
⟨*proof*⟩

lemma *wfPhi-f-poly-supp-s*:
fixes τ::τ **and** Θ::Θ **and** Φ::Φ **and** *ft* :: *fun-typ-q*
assumes *Some* (*AF-fundef f* (*AF-fun-typ-some bv* (*AF-fun-typ x b c* τ *s*))) = *lookup-fun* Φ *f* **and** Θ $\vdash_{wf}$ Φ
shows *supp s* ⊆ {*atom x*, *atom bv*}
⟨*proof*⟩

lemmas *wfPhi-f-supp* = *wfPhi-f-poly-supp-b wfPhi-f-simple-supp-b wfPhi-f-poly-supp-c wfPhi-f-simple-supp-t wfPhi-f-poly-supp-t wfPhi-f-simple-supp-t wfPhi-f-poly-wfT wfPhi-f-simple-wfT wfPhi-f-poly-supp-s wfPhi-f-simple-supp-s*

lemma *fun-typ-eq-ret-unique*:
assumes (*AF-fun-typ x1 b1 c1* τ*1' s1'*) = (*AF-fun-typ x2 b2 c2* τ*2' s2'*)
shows $\tau 1'[x1::=v]_{\tau v} = \tau 2'[x2::=v]_{\tau v}$
⟨*proof*⟩

lemma *fun-typ-eq-body-unique*:

fixes $v::v$ **and** $x1::x$ **and** $x2::x$ **and** $s1'::s$ **and** $s2'::s$
assumes $(AF\text{-}fun\text{-}typ\ x1\ b1\ c1\ \tau1'\ s1') = (AF\text{-}fun\text{-}typ\ x2\ b2\ c2\ \tau2'\ s2')$
shows $s1'[x1::=v]_{sv} = s2'[x2::=v]_{sv}$
⟨*proof*⟩

lemma *fun-ret-unique*:
assumes $Some\ (AF\text{-}fundef\ f\ (AF\text{-}fun\text{-}typ\text{-}none\ (AF\text{-}fun\text{-}typ\ x1\ b1\ c1\ \tau1'\ s1'))) = lookup\text{-}fun\ \Phi\ f$ **and** $Some\ (AF\text{-}fundef\ f\ (AF\text{-}fun\text{-}typ\text{-}none\ (AF\text{-}fun\text{-}typ\ x2\ b2\ c2\ \tau2'\ s2'))) = lookup\text{-}fun\ \Phi\ f$
shows $\tau1'[x1::=v]_{\tau v} = \tau2'[x2::=v]_{\tau v}$
⟨*proof*⟩

lemma *fun-poly-arg-unique*:
fixes $bv1::bv$ **and** $bv2::bv$ **and** $b::b$ **and** $\tau1::\tau$ **and** $\tau2::\tau$
assumes $[[atom\ bv1]]lst.\ (AF\text{-}fun\text{-}typ\ x1\ b1\ c1\ \tau1\ s1) = [[atom\ bv2]]lst.\ (AF\text{-}fun\text{-}typ\ x2\ b2\ c2\ \tau2\ s2)$ (**is** $[[atom\ ?x]]lst.\ ?a = [[atom\ ?y]]lst.\ ?b$)
shows $\{\!|\ x1 : b1[bv1::=b]_{bb} \mid c1[bv1::=b]_{cb}\ |\!\} = \{\!|\ x2 : b2[bv2::=b]_{bb} \mid c2[bv2::=b]_{cb}\ |\!\}$
⟨*proof*⟩

lemma *fun-poly-ret-unique*:
assumes $Some\ (AF\text{-}fundef\ f\ (AF\text{-}fun\text{-}typ\text{-}some\ bv1\ (AF\text{-}fun\text{-}typ\ x1\ b1\ c1\ \tau1'\ s1'))) = lookup\text{-}fun\ \Phi\ f$ **and** $Some\ (AF\text{-}fundef\ f\ (AF\text{-}fun\text{-}typ\text{-}some\ bv2\ (AF\text{-}fun\text{-}typ\ x2\ b2\ c2\ \tau2'\ s2'))) = lookup\text{-}fun\ \Phi\ f$
shows $\tau1'[bv1::=b]_{\tau b}[x1::=v]_{\tau v} = \tau2'[bv2::=b]_{\tau b}[x2::=v]_{\tau v}$
⟨*proof*⟩

lemma *fun-poly-body-unique*:
assumes $Some\ (AF\text{-}fundef\ f\ (AF\text{-}fun\text{-}typ\text{-}some\ bv1\ (AF\text{-}fun\text{-}typ\ x1\ b1\ c1\ \tau1'\ s1'))) = lookup\text{-}fun\ \Phi\ f$ **and** $Some\ (AF\text{-}fundef\ f\ (AF\text{-}fun\text{-}typ\text{-}some\ bv2\ (AF\text{-}fun\text{-}typ\ x2\ b2\ c2\ \tau2'\ s2'))) = lookup\text{-}fun\ \Phi\ f$
shows $s1'[bv1::=b]_{sb}[x1::=v]_{sv} = s2'[bv2::=b]_{sb}[x2::=v]_{sv}$
⟨*proof*⟩

lemma *funtyp-eq-iff-equalities*:
fixes $s'::s$ **and** $s::s$
assumes $[[atom\ x']]lst.\ ((c', \tau'), s') = [[atom\ x]]lst.\ ((c, \tau), s)$
shows $\{\!|\ x' : b\ \mid c'\ |\!\} = \{\!|\ x : b\ \mid c\ |\!\} \wedge\ s'[x'::=v]_{sv} = s[x::=v]_{sv} \wedge \tau'[x'::=v]_{\tau v} = \tau[x::=v]_{\tau v}$
⟨*proof*⟩

8.14 Weakening

lemma *wfX-wfB1*:
fixes $\Gamma::\Gamma$ **and** $\Gamma'::\Gamma$ **and** $v::v$ **and** $e::e$ **and** $c::c$ **and** $\tau::\tau$ **and** $ts::(string*\tau)\ list$ **and** $\Delta::\Delta$ **and** $s::s$ **and** $b::b$ **and** $\mathcal{B}::\mathcal{B}$ **and** $\Phi::\Phi$ **and** $ftq::fun\text{-}typ\text{-}q$ **and** $ft::fun\text{-}typ$ **and** $ce::ce$ **and** $td::type\text{-}def$ **and** $cs::branch\text{-}s$ **and** $css::branch\text{-}list$
shows *wfV-wfB*: $\Theta;\ \mathcal{B};\ \Gamma\ \vdash_{wf} v : b \Longrightarrow \Theta;\ \mathcal{B} \vdash_{wf} b$ **and**
$\Theta;\ \mathcal{B};\ \Gamma \vdash_{wf} c \Longrightarrow True$ **and**
$\Theta;\ \mathcal{B}\ \vdash_{wf} \Gamma \Longrightarrow\ True$ **and**
wfT-wfB: $\Theta;\ \mathcal{B};\ \Gamma \vdash_{wf} \tau \Longrightarrow\ \Theta;\ \mathcal{B} \vdash_{wf} b\text{-}of\ \tau$ **and**
$\Theta;\ \mathcal{B};\ \Gamma \vdash_{wf} ts \Longrightarrow\ True$ **and**
$\vdash_{wf} \Theta \Longrightarrow True$ **and**
$\Theta;\ \mathcal{B}\ \vdash_{wf} b \Longrightarrow\ True$ **and**
wfCE-wfB: $\Theta;\ \mathcal{B};\ \Gamma \vdash_{wf} ce : b \Longrightarrow \Theta;\ \mathcal{B}\ \vdash_{wf} b$ **and**
$\Theta\ \vdash_{wf} td \Longrightarrow True$
⟨*proof*⟩

lemma *wfX-wfB2*:
fixes Γ::Γ **and** Γ'::Γ **and** *v*::*v* **and** *e*::*e* **and** *c*::*c* **and** τ::τ **and** *ts*::(*string*$*\tau$) *list* **and** Δ::Δ **and** *s*::*s* **and** b::b **and** $\mathcal{B}$::$\mathcal{B}$ **and** Φ::Φ **and** *ftq*::*fun-typ-q* **and** *ft*::*fun-typ* **and** *ce*::*ce* **and** *td*::*type-def* **and** *cs*::*branch-s* **and** *css*::*branch-list*
shows
wfE-wfB: Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash_{wf} e : b \Longrightarrow \Theta$; $\mathcal{B} \vdash_{wf} b$ **and**
wfS-wfB: Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash_{wf} s : b \Longrightarrow \Theta$; $\mathcal{B} \vdash_{wf} b$ **and**
wfCS-wfB: Θ; Φ; $\mathcal{B}$; Γ; Δ ; *tid* ; *dc* ; $t \vdash_{wf} cs : b \Longrightarrow \Theta$; $\mathcal{B} \vdash_{wf} b$ **and**
wfCSS-wfB: Θ; Φ; $\mathcal{B}$; Γ; Δ ; *tid* ; *dclist* $\vdash_{wf} css : b \Longrightarrow \Theta$; $\mathcal{B} \vdash_{wf} b$ **and**
$\Theta \vdash_{wf} \Phi \Longrightarrow$ *True* **and**
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta \Longrightarrow$ *True* **and**
Θ ; $\Phi \vdash_{wf}$ *ftq* $\Longrightarrow$ *True* **and**
Θ ; Φ ; $\mathcal{B} \vdash_{wf} ft \Longrightarrow \mathcal{B} \mid\subseteq\mid \mathcal{B}' \Longrightarrow \Theta$; Φ ; $\mathcal{B}' \vdash_{wf} ft$
⟨*proof*⟩

lemmas *wfX-wfB* = *wfX-wfB1 wfX-wfB2*

lemma *wf-weakening1*:
fixes Γ::Γ **and** Γ'::Γ **and** *v*::*v* **and** *e*::*e* **and** *c*::*c* **and** τ::τ **and** *ts*::(*string*$*\tau$) *list* **and** Δ::Δ **and** *s*::*s* **and** $\mathcal{B}$::$\mathcal{B}$ **and** *ftq*::*fun-typ-q* **and** *ft*::*fun-typ* **and** *ce*::*ce* **and** *td*::*type-def* **and** *cs*::*branch-s* **and** *css*::*branch-list*

shows *wfV-weakening*: Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} v : b \Longrightarrow \Theta$; $\mathcal{B} \vdash_{wf} \Gamma' \Longrightarrow$ *toSet* $\Gamma \subseteq$ *toSet* $\Gamma' \Longrightarrow \Theta$; $\mathcal{B}$; $\Gamma' \vdash_{wf} v : b$ **and**
wfC-weakening: Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} c \Longrightarrow \Theta$; $\mathcal{B} \vdash_{wf} \Gamma' \Longrightarrow$ *toSet* $\Gamma \subseteq$ *toSet* $\Gamma' \Longrightarrow \Theta$; $\mathcal{B}$; $\Gamma' \vdash_{wf} c$ **and**
Θ; $\mathcal{B} \vdash_{wf} \Gamma \Longrightarrow$ *True* **and**
wfT-weakening: Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \tau \Longrightarrow \Theta$; $\mathcal{B} \vdash_{wf} \Gamma' \Longrightarrow$ *toSet* $\Gamma \subseteq$ *toSet* $\Gamma' \Longrightarrow \Theta$; $\mathcal{B}$; $\Gamma' \vdash_{wf} \tau$ **and**
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} ts \Longrightarrow$ *True* **and**
$\vdash_{wf} P \Longrightarrow$ *True* **and**
wfB-weakening: *wfB* Θ $\mathcal{B}$ $b \Longrightarrow \mathcal{B} \mid\subseteq\mid \mathcal{B}' \Longrightarrow$ *wfB* Θ $\mathcal{B}$ b **and**
wfCE-weakening: Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} ce : b \Longrightarrow \Theta$; $\mathcal{B} \vdash_{wf} \Gamma' \Longrightarrow$ *toSet* $\Gamma \subseteq$ *toSet* $\Gamma' \Longrightarrow \Theta$; $\mathcal{B}$; $\Gamma' \vdash_{wf} ce : b$ **and**
$\Theta \vdash_{wf} td \Longrightarrow$ *True*
⟨*proof*⟩

lemma *wf-weakening2*:
fixes Γ::Γ **and** Γ'::Γ **and** *v*::*v* **and** *e*::*e* **and** *c*::*c* **and** τ::τ **and** *ts*::(*string*$*\tau$) *list* **and** Δ::Δ **and** *s*::*s* **and** $\mathcal{B}$::$\mathcal{B}$ **and** *ftq*::*fun-typ-q* **and** *ft*::*fun-typ* **and** *ce*::*ce* **and** *td*::*type-def* **and** *cs*::*branch-s* **and** *css*::*branch-list*
shows
wfE-weakening: Θ; Φ; $\mathcal{B}$; Γ ; $\Delta \vdash_{wf} e : b \Longrightarrow \Theta$; $\mathcal{B} \vdash_{wf} \Gamma' \Longrightarrow$ *toSet* $\Gamma \subseteq$ *toSet* $\Gamma' \Longrightarrow \Theta$; Φ; $\mathcal{B}$; Γ' ; $\Delta \vdash_{wf} e : b$ **and**
wfS-weakening: Θ; Φ; $\mathcal{B}$; Γ ; $\Delta \vdash_{wf} s : b \Longrightarrow \Theta$; $\mathcal{B} \vdash_{wf} \Gamma' \Longrightarrow$ *toSet* $\Gamma \subseteq$ *toSet* $\Gamma' \Longrightarrow \Theta$; Φ; $\mathcal{B}$; Γ' ; $\Delta \vdash_{wf} s : b$ **and**
Θ ; Φ ; $\mathcal{B}$; Γ ; Δ ; *tid* ; *dc* ; $t \vdash_{wf} cs : b \Longrightarrow \Theta$; $\mathcal{B} \vdash_{wf} \Gamma' \Longrightarrow$ *toSet* $\Gamma \subseteq$ *toSet* $\Gamma' \Longrightarrow \Theta$; Φ; $\mathcal{B}$; Γ' ; Δ ; *tid* ; *dc* ; $t \vdash_{wf} cs : b$ **and**
Θ ; Φ ; $\mathcal{B}$; Γ ; Δ ; *tid* ; *dclist* $\vdash_{wf} css : b \Longrightarrow \Theta$; $\mathcal{B} \vdash_{wf} \Gamma' \Longrightarrow$ *toSet* $\Gamma \subseteq$ *toSet* $\Gamma' \Longrightarrow \Theta$; Φ; $\mathcal{B}$; Γ' ; Δ ; *tid* ; *dclist* $\vdash_{wf} css : b$ **and**
$\Theta \vdash_{wf} (\Phi::\Phi) \Longrightarrow$ *True* **and**

wfD-weakning: Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ Δ $\Longrightarrow$ Θ; $\mathcal{B}$ $\vdash_{wf}$ Γ' $\Longrightarrow$ *toSet* Γ $\subseteq$ *toSet* Γ' $\Longrightarrow$ Θ; $\mathcal{B}$; Γ' $\vdash_{wf}$ Δ
and
Θ ; Φ $\vdash_{wf}$ *ftq* $\Longrightarrow$ *True* **and**
Θ ; Φ ; $\mathcal{B}$ $\vdash_{wf}$ *ft* $\Longrightarrow$ *True*
⟨*proof*⟩

lemmas *wf-weakening = wf-weakening1 wf-weakening2*

lemma *wfV-weakening-cons*:
fixes Γ::Γ **and** Γ'::Γ **and** *v*::*v* **and** *c*::*c*
assumes Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *v* : *b* **and** *atom y* $\sharp$ Γ **and** Θ; $\mathcal{B}$; ((y,b',*TRUE*) $\#_\Gamma$ Γ) $\vdash_{wf}$ *c*
shows Θ; $\mathcal{B}$; (y,b',*c*) $\#_\Gamma$$\Gamma$ $\vdash_{wf}$ *v* : *b*
⟨*proof*⟩

lemma *wfG-cons-weakening*:
fixes Γ'::Γ
assumes Θ; $\mathcal{B}$ $\vdash_{wf}$ ((x, b, c) $\#_\Gamma$ Γ) **and** Θ; $\mathcal{B}$ $\vdash_{wf}$ Γ' **and** *toSet* Γ $\subseteq$ *toSet* Γ' **and** *atom x* $\sharp$ Γ'
shows Θ; $\mathcal{B}$ $\vdash_{wf}$ ((x, b, c) $\#_\Gamma$ Γ')
⟨*proof*⟩

lemma *wfT-weakening-aux*:
fixes Γ::Γ **and** Γ'::Γ **and** *c*::*c*
assumes Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ {| *z* : *b* | *c* |} **and** Θ; $\mathcal{B}$ $\vdash_{wf}$ Γ' **and** *toSet* Γ $\subseteq$ *toSet* Γ' **and** *atom z* $\sharp$ Γ'
shows Θ; $\mathcal{B}$; Γ' $\vdash_{wf}$ {| *z* : *b* | *c* |}
⟨*proof*⟩

lemma *wfT-weakening-all*:
fixes Γ::Γ **and** Γ'::Γ **and** τ::τ
assumes Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ τ **and** Θ; $\mathcal{B}'$ $\vdash_{wf}$ Γ' **and** *toSet* Γ $\subseteq$ *toSet* Γ' **and** $\mathcal{B}$ $|\subseteq|$ $\mathcal{B}'$
shows Θ; $\mathcal{B}'$; Γ' $\vdash_{wf}$ τ
⟨*proof*⟩

lemma *wfT-weakening-nil*:
fixes Γ::Γ **and** Γ'::Γ **and** τ::τ
assumes Θ ; {||} ; *GNil* $\vdash_{wf}$ τ **and** Θ; $\mathcal{B}'$ $\vdash_{wf}$ Γ'
shows Θ; $\mathcal{B}'$; Γ' $\vdash_{wf}$ τ
⟨*proof*⟩

lemma *wfTh-wfT2*:
fixes *x*::*x* **and** *v*::*v* **and** τ::τ **and** *G*::Γ
assumes *wfTh* Θ **and** *AF-typedef s dclist* $\in$ *set* Θ **and**
(*dc*, τ) $\in$ *set dclist* **and** Θ ; *B* $\vdash_{wf}$ *G*
shows *supp* τ = {} **and** τ[*x*::=*v*]$_{\tau v}$ = τ **and** *wfT* Θ *B G* τ
⟨*proof*⟩

lemma *wf-d-weakening*:
fixes Γ::Γ **and** Γ'::Γ **and** *v*::*v* **and** *e*::*e* **and** *c*::*c* **and** τ::τ **and** *ts*::(*string*∗τ) *list* **and** Δ::Δ **and** *s*::*s* **and** $\mathcal{B}$::$\mathcal{B}$ **and** *ftq*::*fun-typ-q* **and** *ft*::*fun-typ* **and** *ce*::*ce* **and** *td*::*type-def*
and *cs*::*branch-s* **and** *css*::*branch-list*
shows
Θ; Φ; $\mathcal{B}$; Γ ; Δ $\vdash_{wf}$ *e* : *b* $\Longrightarrow$ Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ Δ' $\Longrightarrow$ *setD* Δ $\subseteq$ *setD* Δ' $\Longrightarrow$ Θ; Φ; $\mathcal{B}$; Γ ; Δ' $\vdash_{wf}$ *e* : *b* **and**

Θ; Φ; $\mathcal{B}$; Γ ; $\Delta \vdash_{wf} s : b \Longrightarrow \Theta$; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta' \Longrightarrow setD\ \Delta \subseteq setD\ \Delta' \Longrightarrow \Theta$; Φ; $\mathcal{B}$; Γ ; $\Delta' \vdash_{wf} s : b$ **and**
Θ ; Φ ; $\mathcal{B}$; Γ ; Δ ; *tid* ; *dc* ; $t \vdash_{wf} cs : b \Longrightarrow \Theta$; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta' \Longrightarrow setD\ \Delta \subseteq setD\ \Delta' \Longrightarrow \Theta$; Φ; $\mathcal{B}$; Γ ; Δ' ; *tid* ; *dc* ; $t \vdash_{wf} cs : b$ **and**
Θ ; Φ ; $\mathcal{B}$; Γ ; Δ ; *tid* ; *dclist* $\vdash_{wf}$ *css* : $b \Longrightarrow \Theta$; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta' \Longrightarrow setD\ \Delta \subseteq setD\ \Delta' \Longrightarrow \Theta$; Φ; $\mathcal{B}$; Γ ; Δ' ; *tid* ; *dclist* $\vdash_{wf}$ *css* : b **and**
$\Theta \vdash_{wf} (\Phi::\Phi) \Longrightarrow True$ **and**
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta \Longrightarrow True$ **and**
Θ ; $\Phi \vdash_{wf} ftq \Longrightarrow True$ **and**
Θ ; Φ ; $\mathcal{B} \vdash_{wf} ft \Longrightarrow True$
⟨*proof*⟩

8.15 Useful well-formedness instances

Well-formedness for particular constructs that we will need later

lemma *wfC-e-eq*:
fixes *ce*::*ce* **and** Γ::Γ
assumes Θ ; $\mathcal{B}$; $\Gamma \vdash_{wf} ce : b$ **and** *atom* $x \sharp \Gamma$
shows Θ ; $\mathcal{B}$; $((x, b, TRUE) \#_\Gamma \Gamma) \vdash_{wf} (CE\text{-}val\ (V\text{-}var\ x) == ce)$
⟨*proof*⟩

lemma *wfC-e-eq2*:
fixes *e1*::*ce* **and** *e2*::*ce*
assumes Θ ; $\mathcal{B}$; $\Gamma \vdash_{wf} e1 : b$ **and** Θ ; $\mathcal{B}$; $\Gamma \vdash_{wf} e2 : b$ **and** $\vdash_{wf} \Theta$ **and** *atom* $x \sharp \Gamma$
shows Θ; $\mathcal{B}$; $(x, b, (CE\text{-}val\ (V\text{-}var\ x)) == e1) \#_\Gamma \Gamma \vdash_{wf} (CE\text{-}val\ (V\text{-}var\ x)) == e2$
⟨*proof*⟩

lemma *wfT-wfT-if-rev*:
assumes *wfV P* $\mathcal{B}$ Γ *v* (*base-for-lit l*) **and** *wfT P* $\mathcal{B}$ Γ *t* **and** ‹*atom z1* $\sharp$ Γ›
shows *wfT P* $\mathcal{B}$ Γ ({| *z1* : *b-of t* | *CE-val v* == *CE-val* (*V-lit l*) *IMP* (*c-of t z1*) |})
⟨*proof*⟩

lemma *wfT-eq-imp*:
fixes *zz*::*x* **and** *ll*::*l* **and** τ'::τ
assumes *base-for-lit ll* = *B-bool* **and** Θ ; {||} ; *GNil* $\vdash_{wf} \tau'$ **and**
Θ ; {||} $\vdash_{wf}$ (*x*, *b-of* {| *z'* : *B-bool* | *TRUE* |}, *c-of* {| *z'* : *B-bool* | *TRUE* |} *x*) $\#_\Gamma$ *GNil* **and** *atom zz* $\sharp$ *x*
shows Θ ; {||} ; (*x*, *b-of* {| *z'* : *B-bool* | *TRUE* |}, *c-of* {| *z'* : *B-bool* | *TRUE* |} *x*) $\#_\Gamma$
GNil $\vdash_{wf}$ {| *zz* : *b-of* τ' | [[*x*]^v]^{ce} == [[*ll*]^v]^{ce} *IMP* *c-of* τ' *zz* |}
⟨*proof*⟩

lemma *wfC-v-eq*:
fixes *ce*::*ce* **and** Γ::Γ **and** *v*::*v*
assumes Θ ; $\mathcal{B}$; $\Gamma \vdash_{wf} v : b$ **and** *atom* $x \sharp \Gamma$
shows Θ ; $\mathcal{B}$; $((x, b, TRUE) \#_\Gamma \Gamma) \vdash_{wf} (CE\text{-}val\ (V\text{-}var\ x) == CE\text{-}val\ v)$
⟨*proof*⟩

lemma *wfT-e-eq*:
fixes *ce*::*ce*
assumes Θ ; $\mathcal{B}$; $\Gamma \vdash_{wf} ce : b$ **and** *atom* $z \sharp \Gamma$
shows Θ; $\mathcal{B}$; $\Gamma \vdash_{wf}$ {| *z* : *b* | *CE-val* (*V-var z*) == *ce* |}

⟨*proof*⟩

lemma *wfT-v-eq*:
assumes *wfB* Θ $\mathcal{B}$ *b* **and** *wfV* Θ $\mathcal{B}$ Γ *v b* **and** *atom z* ♯ Γ
shows *wfT* Θ $\mathcal{B}$ Γ {| *z* : *b* | *C-eq* (*CE-val* (*V-var z*)) (*CE-val v*)|}
⟨*proof*⟩

lemma *wfC-wfG*:
fixes Γ::Γ **and** *c*::*c* **and** *b*::*b*
assumes Θ ; *B* ; Γ $\vdash_{wf}$ *c* **and** Θ ; *B* $\vdash_{wf}$ *b* **and** *atom x* ♯ Γ
shows Θ ; *B* $\vdash_{wf}$ (*x*,*b*,*c*)#$_\Gamma$ Γ
⟨*proof*⟩

8.16 Replacing the constraint on a variable in a context

lemma *wfG-cons-fresh2*:
fixes Γ′::Γ
assumes *wfG P* $\mathcal{B}$ (((*x*′,*b*′,*c*′) #$_\Gamma$ Γ′ @ (*x*, *b*, *c*) #$_\Gamma$ Γ))
shows *x*′≠*x*
⟨*proof*⟩

lemma *replace-in-g-inside*:
fixes Γ::Γ
assumes *wfG P* $\mathcal{B}$ (Γ′@((*x*,*b0*,*c0*′) #$_\Gamma$Γ))
shows *replace-in-g* (Γ′@((*x*,*b0*,*c0*′) #$_\Gamma$Γ)) *x c0* = (Γ′@((*x*,*b0*,*c0*) #$_\Gamma$Γ))
⟨*proof*⟩

lemma *wfG-supp-rig-eq*:
fixes Γ::Γ
assumes *wfG P* $\mathcal{B}$ (Γ″ @ (*x*, *b0*, *c0*) #$_\Gamma$ Γ) **and** *wfG P* $\mathcal{B}$ (Γ″ @ (*x*, *b0*, *c0*′) #$_\Gamma$ Γ)
shows *supp* (Γ″ @ (*x*, *b0*, *c0*′) #$_\Gamma$ Γ) ∪ *supp* $\mathcal{B}$ = *supp* (Γ″ @ (*x*, *b0*, *c0*) #$_\Gamma$ Γ) ∪ *supp* $\mathcal{B}$
⟨*proof*⟩

lemma *fresh-replace-inside*[*ms-fresh*]:
fixes *y*::*x* **and** Γ::Γ
assumes *wfG P* $\mathcal{B}$ (Γ″ @ (*x*, *b*, *c*) #$_\Gamma$ Γ) **and** *wfG P* $\mathcal{B}$ (Γ″ @ (*x*, *b*, *c*′) #$_\Gamma$ Γ)
shows *atom y* ♯ (Γ″ @ (*x*, *b*, *c*) #$_\Gamma$ Γ) = *atom y* ♯ (Γ″ @ (*x*, *b*, *c*′) #$_\Gamma$ Γ)
⟨*proof*⟩

lemma *wf-replace-inside1*:
fixes Γ::Γ **and** Φ::Φ **and** Θ::Θ **and** Γ′::Γ **and** *v*::*v* **and** *e*::*e* **and** *c*::*c* **and** *c*″::*c* **and** *c*′::*c* **and** τ::τ **and** *ts*::(*string*∗τ) *list* **and** Δ::Δ **and** *b*′::*b* **and** *b*::*b* **and** *s*::*s*
and *ftq*::*fun-typ-q* **and** *ft*::*fun-typ* **and** *ce*::*ce* **and** *td*::*type-def* **and** *cs*::*branch-s* **and** *css*::*branch-list*

shows *wfV-replace-inside*: Θ; $\mathcal{B}$; *G* $\vdash_{wf}$ *v* : *b*′ ⟹ *G* = (Γ′ @ (*x*, *b*, *c*′) #$_\Gamma$ Γ) ⟹ Θ; $\mathcal{B}$; ((*x*,*b*,*TRUE*) #$_\Gamma$Γ) $\vdash_{wf}$ *c* ⟹ Θ ; $\mathcal{B}$; (Γ′ @ (*x*, *b*, *c*) #$_\Gamma$ Γ) $\vdash_{wf}$ *v* : *b*′ **and**
wfC-replace-inside: Θ; $\mathcal{B}$; *G* $\vdash_{wf}$ *c*″ ⟹ *G* = (Γ′ @ (*x*, *b*, *c*′) #$_\Gamma$ Γ) ⟹ Θ; $\mathcal{B}$; ((*x*,*b*,*TRUE*) #$_\Gamma$Γ) $\vdash_{wf}$ *c* ⟹ Θ ; $\mathcal{B}$; (Γ′ @ (*x*, *b*, *c*) #$_\Gamma$ Γ) $\vdash_{wf}$ *c*″ **and**
wfG-replace-inside: Θ; $\mathcal{B}$ $\vdash_{wf}$ *G* ⟹ *G* = (Γ′ @ (*x*, *b*, *c*′) #$_\Gamma$ Γ) ⟹ Θ; $\mathcal{B}$; ((*x*,*b*,*TRUE*) #$_\Gamma$Γ) $\vdash_{wf}$ *c* ⟹ Θ; $\mathcal{B}$ $\vdash_{wf}$ (Γ′ @ (*x*, *b*, *c*) #$_\Gamma$ Γ) **and**
wfT-replace-inside: Θ; $\mathcal{B}$; *G* $\vdash_{wf}$ τ ⟹ *G* = (Γ′ @ (*x*, *b*, *c*′) #$_\Gamma$ Γ) ⟹ Θ; $\mathcal{B}$; ((*x*,*b*,*TRUE*) #$_\Gamma$Γ) $\vdash_{wf}$ *c* ⟹ Θ ; $\mathcal{B}$; (Γ′ @ (*x*, *b*, *c*) #$_\Gamma$ Γ) $\vdash_{wf}$ τ **and**

Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ *ts* $\Longrightarrow$ *True* **and**
$\vdash_{wf}$ *P* $\Longrightarrow$ *True* **and**
Θ; $\mathcal{B}$ $\vdash_{wf}$ *b* $\Longrightarrow$ *True* **and**
wfCE-replace-inside: Θ ; $\mathcal{B}$; *G* $\vdash_{wf}$ *ce* : *b'* $\Longrightarrow$ *G* = (Γ' @ (*x*, *b*, *c'*) #$_\Gamma$ Γ) $\Longrightarrow$ Θ; $\mathcal{B}$; ((*x*,*b*,*TRUE*) #$_\Gamma\Gamma$) $\vdash_{wf}$ *c* $\Longrightarrow$ Θ ; $\mathcal{B}$; (Γ' @ (*x*, *b*, *c*) #$_\Gamma$ Γ) $\vdash_{wf}$ *ce* : *b'* **and**
Θ $\vdash_{wf}$ *td* $\Longrightarrow$ *True*
⟨*proof*⟩

lemma *wf-replace-inside2*:
fixes Γ::Γ **and** Φ::Φ **and** Θ::Θ **and** Γ'::Γ **and** *v*::*v* **and** *e*::*e* **and** *c*::*c* **and** *c''*::*c* **and** *c'*::*c* **and** τ::τ **and** *ts*::(*string*∗τ) *list* **and** Δ::Δ **and** *b'*::*b* **and** *b*::*b* **and** *s*::*s*
and *ftq*::*fun-typ-q* **and** *ft*::*fun-typ* **and** *ce*::*ce* **and** *td*::*type-def* **and** *cs*::*branch-s* **and** *css*::*branch-list*
shows
Θ ; Φ ; $\mathcal{B}$; *G* ; *D* $\vdash_{wf}$ *e* : *b'* $\Longrightarrow$ *G* = (Γ' @ (*x*, *b*, *c'*) #$_\Gamma$ Γ) $\Longrightarrow$ Θ; $\mathcal{B}$; ((*x*,*b*,*TRUE*) #$_\Gamma\Gamma$) $\vdash_{wf}$ *c* $\Longrightarrow$ Θ ; Φ ; $\mathcal{B}$; (Γ' @ (*x*, *b*, *c*) #$_\Gamma$ Γ); *D* $\vdash_{wf}$ *e* : *b'* **and**
Θ; Φ; $\mathcal{B}$; Γ ; Δ $\vdash_{wf}$ *s* : *b* $\Longrightarrow$ *True* **and**
Θ; Φ; $\mathcal{B}$; Γ ; Δ ; *tid* ; *dc* ; *t* $\vdash_{wf}$ *cs* : *b* $\Longrightarrow$ *True* **and**
Θ; Φ; $\mathcal{B}$; Γ ; Δ ; *tid* ; *dclist* $\vdash_{wf}$ *css* : *b* $\Longrightarrow$ *True* **and**
Θ $\vdash_{wf}$ Φ $\Longrightarrow$ *True* **and**
Θ; $\mathcal{B}$; *G* $\vdash_{wf}$ Δ $\Longrightarrow$ *G* = (Γ' @ (*x*, *b*, *c'*) #$_\Gamma$ Γ) $\Longrightarrow$ Θ; $\mathcal{B}$; ((*x*,*b*,*TRUE*) #$_\Gamma\Gamma$) $\vdash_{wf}$ *c* $\Longrightarrow$ Θ ; $\mathcal{B}$; (Γ' @ (*x*, *b*, *c*) #$_\Gamma$ Γ) $\vdash_{wf}$ Δ **and**
Θ ; Φ $\vdash_{wf}$ *ftq* $\Longrightarrow$ *True* **and**
Θ ; Φ ; $\mathcal{B}$ $\vdash_{wf}$ *ft* $\Longrightarrow$ *True*
⟨*proof*⟩

lemmas *wf-replace-inside* = *wf-replace-inside1 wf-replace-inside2*

lemma *wfC-replace-cons*:
assumes *wfG P* $\mathcal{B}$ ((*x*,*b*,*c1*) #$_\Gamma\Gamma$) **and** *wfC P* $\mathcal{B}$ ((*x*,*b*,*TRUE*) #$_\Gamma\Gamma$) *c2*
shows *wfC P* $\mathcal{B}$ ((*x*,*b*,*c1*) #$_\Gamma\Gamma$) *c2*
⟨*proof*⟩

lemma *wfC-refl*:
assumes *wfG* Θ $\mathcal{B}$ ((*x*, *b'*, *c'*) #$_\Gamma\Gamma$)
shows *wfC* Θ $\mathcal{B}$ ((*x*, *b'*, *c'*) #$_\Gamma\Gamma$) *c'*
⟨*proof*⟩

lemma *wfG-wfC-inside*:
assumes (*x*, *b*, *c*) $\in$ *toSet G* **and** *wfG* Θ *B G*
shows *wfC* Θ *B G c*
⟨*proof*⟩

lemma *wfT-wf-cons3*:
assumes Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ {| *z* : *b* | *c* |} **and** *atom y* ♯ (*c*,Γ)
shows Θ; $\mathcal{B}$ $\vdash_{wf}$ (*y*, *b*, *c*[*z*::=*V-var y*]$_{cv}$) #$_\Gamma$ Γ
⟨*proof*⟩

lemma *wfT-wfC-cons*:
assumes *wfT P* $\mathcal{B}$ Γ {| *z1* : *b* | *c1* |} **and** *wfT P* $\mathcal{B}$ Γ {| *z2* : *b* | *c2* |} **and** *atom x* ♯ (*c1*,*c2*,Γ)
shows *wfC P* $\mathcal{B}$ ((*x*,*b*,*c1*[*z1*::=*V-var x*]$_v$) #$_\Gamma\Gamma$) (*c2*[*z2*::=*V-var x*]$_v$) (**is** *wfC P* $\mathcal{B}$ *?G ?c*)
⟨*proof*⟩

lemma *wfT-wfC2*:
fixes $c::c$ **and** $x::x$
assumes Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \{\!|\, z : b \mid c \,|\!\}$ **and** $atom\ x \sharp \Gamma$
shows Θ; $\mathcal{B}$; $(x,b,TRUE)\#_\Gamma\Gamma \vdash_{wf}\ c[z::=[x]^v]_v$
⟨*proof*⟩

lemma *wfT-wfG*:
fixes $x::x$ **and** $\Gamma::\Gamma$ **and** $z::x$ **and** $c::c$ **and** $b::b$
assumes Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \{\!|\, z : b \mid c \,|\!\}$ **and** $atom\ x \sharp \Gamma$
shows Θ; $\mathcal{B}\ \vdash_{wf}\ (x,b,\ c[z::=[\ x\ ]^v]_v)\ \#_\Gamma\ \Gamma$
⟨*proof*⟩

lemma *wfG-replace-inside2*:
fixes $\Gamma::\Gamma$
assumes $wfG\ P\ \mathcal{B}\ (\Gamma'\ @\ (x,\ b,\ c')\ \ \#_\Gamma\ \Gamma)$ **and** $wfG\ P\ \mathcal{B}\ ((x,b,c)\ \#_\Gamma\Gamma)$
shows $wfG\ P\ \mathcal{B}\ (\Gamma'\ @\ (x,\ b,\ c)\ \ \#_\Gamma\ \Gamma)$
⟨*proof*⟩

lemma *wfG-replace-inside-full*:
fixes $\Gamma::\Gamma$
assumes $wfG\ P\ \mathcal{B}\ (\Gamma'\ @\ (x,\ b,\ c')\ \ \#_\Gamma\ \Gamma)$ **and** $wfG\ P\ \mathcal{B}\ (\Gamma'@((x,b,c)\ \#_\Gamma\Gamma))$
shows $wfG\ P\ \mathcal{B}\ (\Gamma'\ @\ (x,\ b,\ c)\ \ \#_\Gamma\ \Gamma)$
⟨*proof*⟩

lemma *wfT-replace-inside2*:
assumes $wfT\ \Theta\ \mathcal{B}\ (\Gamma'\ @\ (x,\ b,\ c')\ \ \#_\Gamma\ \Gamma)\ t$ **and** $wfG\ \Theta\ \mathcal{B}\ (\Gamma'@((x,b,c)\ \#_\Gamma\Gamma))$
shows $wfT\ \Theta\ \mathcal{B}\ (\Gamma'\ @\ (x,\ b,\ c)\ \ \#_\Gamma\ \Gamma)\ t$
⟨*proof*⟩

lemma *wfD-unique*:
assumes $wfD\ P\ \ \mathcal{B}\ \Gamma\ \Delta$ **and** $\ (u,\tau')\in setD\ \Delta$ **and** $(u,\tau)\in setD\ \Delta$
shows $\tau'=\tau$
⟨*proof*⟩

lemma *replace-in-g-forget*:
fixes $x::x$
assumes $wfG\ P\ B\ G$
shows $atom\ x \notin atom\text{-}dom\ G \Longrightarrow (G[x\longmapsto c]) = G$ **and**
$atom\ x \sharp G \Longrightarrow\ (G[x\longmapsto c]) = G$
⟨*proof*⟩

lemma *replace-in-g-fresh-single*:
fixes $G::\Gamma$ **and** $x::x$
assumes $\langle\Theta;\ \mathcal{B} \vdash_{wf}\ G[x'\longmapsto c']\rangle$ **and** $atom\ x \sharp G$ **and** $\langle\Theta;\ \mathcal{B} \vdash_{wf}\ G\ \rangle$
shows $atom\ x \sharp G[x'\longmapsto c']$
⟨*proof*⟩

8.17 Preservation of well-formedness under substitution

lemma *wfC-cons-switch*:
fixes $c::c$ **and** $c'::c$
assumes Θ; $\mathcal{B}$; $(x,\ b,\ c)\ \ \#_\Gamma\ \Gamma\ \vdash_{wf}\ c'$

shows Θ; $\mathcal{B}$; $(x,\ b,\ c')\ \ \#_\Gamma\ \Gamma\ \vdash_{wf}\ c$
⟨*proof*⟩

lemma *subst-g-inside-simple*:
fixes Γ_1::Γ **and** Γ_2::Γ
assumes *wfG P* $\mathcal{B}$ $(\Gamma_1@((x,b,c)\ \#_\Gamma\Gamma_2))$
shows $(\Gamma_1@((x,b,c)\ \#_\Gamma\Gamma_2))[x::=v]_{\Gamma v} = \Gamma_1[x::=v]_{\Gamma v}@\Gamma_2$
⟨*proof*⟩

lemma *subst-c-TRUE-FALSE*:
fixes *c*::*c*
assumes $c \notin \{TRUE,FALSE\}$
shows $c[x::=v']_{cv} \notin \{TRUE,\ FALSE\}$
⟨*proof*⟩

lemma *lookup-subst*:
assumes *Some* $(b,\ c)$ = *lookup* $\Gamma\ x$ **and** $x \neq x'$
shows $\exists\, c'$. *Some* (b,c') = *lookup* $\Gamma[x'::=v']_{\Gamma v}\ x$
⟨*proof*⟩

lemma *lookup-subst2*:
assumes *Some* $(b,\ c)$ = *lookup* $(\Gamma'@((x',b_1,c0[z0::=[x']^v]_{cv})\#_\Gamma\Gamma))\ x$ **and** $x \neq x'$ **and**
Θ; $\mathcal{B} \vdash_{wf}\ (\Gamma'@((x',b_1,c0[z0::=[x']^v]_{cv})\#_\Gamma\Gamma))$
shows $\exists\, c'$. *Some* (b,c') = *lookup* $(\Gamma'[x'::=v']_{\Gamma v}@\Gamma)\ x$
⟨*proof*⟩

lemma *wf-subst1*:
fixes Γ::Γ **and** Γ'::Γ **and** *v*::*v* **and** *e*::*e* **and** *c*::*c* **and** τ::τ **and** *ts*::(*string*∗τ) *list* **and** Δ::Δ **and** *b*::*b* **and** *ftq*::*fun-typ-q* **and** *ft*::*fun-typ* **and** *ce*::*ce* **and** *td*::*type-def*
shows *wfV-subst*: Θ; $\mathcal{B}$; $\Gamma\ \vdash_{wf}\ v : b \Longrightarrow \Gamma$=$\Gamma_1@((x,b',c')\ \#_\Gamma\Gamma_2) \Longrightarrow \Theta$; $\mathcal{B}$;$\Gamma_2 \vdash_{wf}\ v' : b' \Longrightarrow \Theta\ ;\ \mathcal{B}\ ;\ \Gamma[x::=v']_{\Gamma v} \vdash_{wf}\ v[x::=v']_{vv} : b$ **and**
wfC-subst: Θ; $\mathcal{B}$; $\Gamma\ \vdash_{wf}\ c \Longrightarrow \Gamma$=$\Gamma_1@((x,b',c')\ \#_\Gamma\Gamma_2) \Longrightarrow \Theta$; $\mathcal{B}$; $\Gamma_2 \vdash_{wf}\ v' : b' \Longrightarrow \Theta$; $\mathcal{B}$; $\Gamma[x::=v']_{\Gamma v} \vdash_{wf}\ c[x::=v']_{cv}$ **and**
wfG-subst: Θ; $\mathcal{B} \vdash_{wf}\ \Gamma \Longrightarrow \Gamma$=$\Gamma_1@((x,b',c')\ \#_\Gamma\Gamma_2) \Longrightarrow \Theta$; $\mathcal{B}\ ;\ \Gamma_2 \vdash_{wf}\ v' : b' \Longrightarrow \Theta$; $\mathcal{B} \vdash_{wf}\ \Gamma[x::=v']_{\Gamma v}$ **and**
wfT-subst: Θ; $\mathcal{B}$; $\Gamma\ \vdash_{wf}\ \tau \Longrightarrow \Gamma$=$\Gamma_1@((x,b',c')\ \#_\Gamma\Gamma_2) \Longrightarrow \Theta$; $\mathcal{B}\ ;\ \Gamma_2 \vdash_{wf}\ v' : b' \Longrightarrow \Theta$; $\mathcal{B}$; $\Gamma[x::=v']_{\Gamma v} \vdash_{wf}\ \tau[x::=v']_{\tau v}$ **and**
Θ; $\mathcal{B}$; $\Gamma\ \vdash_{wf}\ ts \Longrightarrow$ *True* **and**
$\vdash_{wf}\ \Theta \Longrightarrow$ *True* **and**
Θ; $\mathcal{B} \vdash_{wf}\ b \Longrightarrow$ *True* **and**
wfCE-subst: Θ; $\mathcal{B}$; $\Gamma \vdash_{wf}\ ce : b \Longrightarrow \Gamma$=$\Gamma_1@((x,b',c')\ \#_\Gamma\Gamma_2) \Longrightarrow \Theta$; $\mathcal{B}\ ;\ \Gamma_2 \vdash_{wf}\ v' : b' \Longrightarrow \Theta\ ;\ \mathcal{B}\ ;\ \Gamma[x::=v']_{\Gamma v}\ \vdash_{wf}\ ce[x::=v']_{cev} : b$ **and**
$\Theta\ \vdash_{wf}\ td \Longrightarrow$ *True*
⟨*proof*⟩

lemma *wf-subst2*:
fixes Γ::Γ **and** Γ'::Γ **and** *v*::*v* **and** *e*::*e* **and** *c*::*c* **and** τ::τ **and** *ts*::(*string*∗τ) *list* **and** Δ::Δ **and** *b*::*b* **and** *ftq*::*fun-typ-q* **and** *ft*::*fun-typ* **and** *ce*::*ce* **and** *td*::*type-def*
shows Θ; Φ; $\mathcal{B}$; Γ ; $\Delta \vdash_{wf}\ e : b \Longrightarrow \Gamma$=$\Gamma_1@((x,b',c')\ \#_\Gamma\Gamma_2) \Longrightarrow \Theta$; $\mathcal{B}\ ;\ \Gamma_2 \vdash_{wf}\ v' : b' \Longrightarrow \Theta$; Φ ; $\mathcal{B}\ ;\ \Gamma[x::=v']_{\Gamma v}$; $\Delta[x::=v']_{\Delta v} \vdash_{wf}\ e[x::=v']_{ev} : b$ **and**
Θ; Φ; $\mathcal{B}$; Γ ; $\Delta \vdash_{wf}\ s : b \Longrightarrow \Gamma$=$\Gamma_1@((x,b',c')\ \#_\Gamma\Gamma_2) \Longrightarrow \Theta$;$\mathcal{B}\ ;\ \Gamma_2 \vdash_{wf}\ v' : b' \Longrightarrow \Theta$; Φ ; $\mathcal{B}\ ;\ \Gamma[x::=v']_{\Gamma v}$; $\Delta[x::=v']_{\Delta v} \vdash_{wf}\ s[x::=v']_{sv} : b$ **and**

Θ; Φ; $\mathcal{B}$; Γ ; Δ ; *tid* ; *dc* ; $t \vdash_{wf}$ *cs* : $b \Longrightarrow \Gamma{=}\Gamma_1@((x,b',c')\ \#_\Gamma\Gamma_2) \Longrightarrow \Theta$; $\mathcal{B}$; $\Gamma_2 \vdash_{wf} v' : b' \Longrightarrow \Theta$; Φ; $\mathcal{B}$; $\Gamma[x::=v']_{\Gamma v}$; $\Delta[x::=v']_{\Delta v}$; *tid* ; *dc* ; $t \vdash_{wf}$ *subst-branchv cs x v'* : b **and**
Θ; Φ; $\mathcal{B}$; Γ ; Δ ; *tid* ; *dclist* $\vdash_{wf}$ *css* : $b \Longrightarrow \Gamma{=}\Gamma_1@((x,b',c')\ \#_\Gamma\Gamma_2) \Longrightarrow \Theta$; $\mathcal{B}$; $\Gamma_2 \vdash_{wf} v' : b' \Longrightarrow \Theta$; Φ; $\mathcal{B}$; $\Gamma[x::=v']_{\Gamma v}$; $\Delta[x::=v']_{\Delta v}$; *tid* ; *dclist* $\vdash_{wf}$ *subst-branchlv css x v'* : b **and**
$\Theta \vdash_{wf} (\Phi::\Phi) \Longrightarrow$ *True* **and**
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta \Longrightarrow \Gamma{=}\Gamma_1@((x,b',c')\ \#_\Gamma\Gamma_2) \Longrightarrow \Theta$; $\mathcal{B}$; $\Gamma_2 \vdash_{wf} v' : b' \Longrightarrow \Theta$; $\mathcal{B}$; $\Gamma[x::=v']_{\Gamma v} \vdash_{wf} \Delta[x::=v']_{\Delta v}$ **and**
Θ ; $\Phi \vdash_{wf}$ *ftq* $\Longrightarrow$ *True* **and**
Θ ; Φ ; $\mathcal{B} \vdash_{wf}$ *ft* $\Longrightarrow$ *True*
⟨*proof*⟩

lemmas *wf-subst = wf-subst1 wf-subst2*

lemma *wfG-subst-wfV*:
assumes Θ; $\mathcal{B} \vdash_{wf} \Gamma' @ (x,\ b,\ c0[z0::=V\text{-}var\ x]_{cv})\ \#_\Gamma\ \Gamma$ **and** *wfV* Θ $\mathcal{B}$ Γ v b
shows Θ; $\mathcal{B} \vdash_{wf} \Gamma'[x::=v]_{\Gamma v} @ \Gamma$
⟨*proof*⟩

lemma *wfG-member-subst*:
assumes $(x1,b1,c1) \in$ *toSet* $(\Gamma'@\Gamma)$ **and** *wfG* Θ $\mathcal{B}$ $(\Gamma'@((x,b,c)\ \#_\Gamma\Gamma))$ **and** $x \neq x1$
shows $\exists\, c1'.\ (x1,b1,c1') \in$ *toSet* $((\Gamma'[x::=v]_{\Gamma v})@\Gamma)$
⟨*proof*⟩

lemma *wfG-member-subst2*:
assumes $(x1,b1,c1) \in$ *toSet* $(\Gamma'@((x,b,c)\ \#_\Gamma\Gamma))$ **and** *wfG* Θ $\mathcal{B}$ $(\Gamma'@((x,b,c)\ \#_\Gamma\Gamma))$ **and** $x \neq x1$
shows $\exists\, c1'.\ (x1,b1,c1') \in$ *toSet* $((\Gamma'[x::=v]_{\Gamma v})@\Gamma)$
⟨*proof*⟩

lemma *wbc-subst*:
fixes $\Gamma::\Gamma$ **and** $\Gamma'::\Gamma$ **and** $v::v$
assumes *wfC* Θ $\mathcal{B}$ $(\Gamma'@((x,b,c')\ \#_\Gamma\Gamma))$ c **and** Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} v : b$
shows Θ; $\mathcal{B}$; $((\Gamma'[x::=v]_{\Gamma v})@\Gamma) \vdash_{wf} c[x::=v]_{cv}$
⟨*proof*⟩

lemma *wfG-inside-fresh-suffix*:
assumes *wfG P B* $(\Gamma'@(x,b,c)\ \#_\Gamma\Gamma)$
shows *atom* $x \sharp \Gamma$
⟨*proof*⟩

lemmas *wf-b-subst-lemmas = subst-eb.simps wf-intros*
forget-subst subst-b-b-def subst-b-v-def subst-b-ce-def fresh-e-opp-all subst-bb.simps wfV-b-fresh ms-fresh-all(*6*)

lemma *wf-b-subst1*:
fixes $\Gamma::\Gamma$ **and** $\Gamma'::\Gamma$ **and** $v::v$ **and** $e::e$ **and** $c::c$ **and** $\tau::\tau$ **and** *ts*::(*string*$*\tau$) *list* **and** $\Delta::\Delta$ **and** $b::b$ **and** *ftq*::*fun-typ-q* **and** *ft*::*fun-typ* **and** *s*::*s* **and** $b'::b$ **and** *ce*::*ce* **and** *td*::*type-def*
and *cs*::*branch-s* **and** *css*::*branch-list*
shows Θ ; B' ; $\Gamma \vdash_{wf} v : b' \Longrightarrow \{|bv|\} = B' \Longrightarrow \Theta$; $B \vdash_{wf} b \Longrightarrow \Theta$; B ; $\Gamma[bv::=b]_{\Gamma b} \vdash_{wf} v[bv::=b]_{vb} : b'[bv::=b]_{bb}$ **and**
Θ ; B' ; $\Gamma \vdash_{wf} c \Longrightarrow \{|bv|\} = B' \Longrightarrow \Theta$; $B \vdash_{wf} b \Longrightarrow \Theta$; B ; $\Gamma[bv::=b]_{\Gamma b} \vdash_{wf} c[bv::=b]_{cb}$ **and**
Θ ; $B' \vdash_{wf} \Gamma \Longrightarrow \{|bv|\} = B' \Longrightarrow \Theta$; $B \vdash_{wf} b \Longrightarrow \Theta$; $B \vdash_{wf} \Gamma[bv::=b]_{\Gamma b}$ **and**
Θ ; B' ; $\Gamma \vdash_{wf} \tau \Longrightarrow \{|bv|\} = B' \Longrightarrow \Theta$; $B \vdash_{wf} b \Longrightarrow \Theta$; B ; $\Gamma[bv::=b]_{\Gamma b} \vdash_{wf}$

$\tau[bv::=b]_{\tau b}$ **and**
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ ts $\Longrightarrow$ *True* **and**
$\vdash_{wf}$ Θ $\Longrightarrow$ *True* **and**
Θ ; B' $\vdash_{wf}$ b' $\Longrightarrow$ $\{|bv|\} = B'$ $\Longrightarrow$ Θ ; B $\vdash_{wf}$ b $\Longrightarrow$ Θ ; B $\vdash_{wf}$ $b'[bv::=b]_{bb}$ **and**
Θ ; B' ; Γ $\vdash_{wf}$ $ce : b'$ $\Longrightarrow$ $\{|bv|\} = B'$ $\Longrightarrow$ Θ ; B $\vdash_{wf}$ b $\Longrightarrow$ Θ ; B ; $\Gamma[bv::=b]_{\Gamma b}$ $\vdash_{wf}$ $ce[bv::=b]_{ceb} : b'[bv::=b]_{bb}$ **and**
Θ $\vdash_{wf}$ td $\Longrightarrow$ *True*
⟨*proof*⟩

lemma *wf-b-subst2*:
fixes Γ::Γ **and** Γ'::Γ **and** v::v **and** e::e **and** c::c **and** τ::τ **and** ts::(*string*$*\tau$) *list* **and** Δ::Δ **and** b::b **and** *ftq*::*fun-typ-q* **and** *ft*::*fun-typ* **and** s::s **and** b'::b **and** *ce*::*ce* **and** *td*::*type-def* **and** *cs*::*branch-s* **and** *css*::*branch-list*
shows Θ ; Φ ; B' ; Γ ; Δ $\vdash_{wf}$ $e : b'$ $\Longrightarrow$ $\{|bv|\} = B'$ $\Longrightarrow$ Θ ; B $\vdash_{wf}$ b $\Longrightarrow$ Θ ; Φ ; B ; $\Gamma[bv::=b]_{\Gamma b}$; $\Delta[bv::=b]_{\Delta b}$ $\vdash_{wf}$ $e[bv::=b]_{eb} : b'[bv::=b]_{bb}$ **and**
Θ ; Φ ; $\mathcal{B}$; Γ ; Δ $\vdash_{wf}$ $s : b$ $\Longrightarrow$ *True* **and**
Θ ; Φ ; $\mathcal{B}$; Γ ; Δ ; *tid* ; *dc* ; t $\vdash_{wf}$ $cs : b$ $\Longrightarrow$ *True* **and**
Θ ; Φ ; $\mathcal{B}$; Γ ; Δ ; *tid* ; *dclist* $\vdash_{wf}$ *css* : b $\Longrightarrow$ *True* **and**
Θ $\vdash_{wf}$ $(\Phi::\Phi)$ $\Longrightarrow$ *True* **and**
Θ ; B' ; Γ $\vdash_{wf}$ Δ $\Longrightarrow$ $\{|bv|\} = B'$ $\Longrightarrow$ Θ ; B $\vdash_{wf}$ b $\Longrightarrow$ Θ ; B ; $\Gamma[bv::=b]_{\Gamma b}$ $\vdash_{wf}$ $\Delta[bv::=b]_{\Delta b}$ **and**
Θ ; Φ $\vdash_{wf}$ *ftq* $\Longrightarrow$ *True* **and**
Θ ; Φ ; $\mathcal{B}$ $\vdash_{wf}$ *ft* $\Longrightarrow$ *True*
⟨*proof*⟩

lemmas *wf-b-subst* = *wf-b-subst1* *wf-b-subst2*

lemma *wfT-subst-wfT*:
fixes τ::τ **and** b'::b **and** bv::bv
assumes Θ ; $\{|bv|\}$; (x,b,c) $\#_\Gamma$ *GNil* $\vdash_{wf}$ τ **and** Θ ; B $\vdash_{wf}$ b'
shows Θ ; B ; $(x,b[bv::=b']_{bb},c[bv::=b']_{cb})$ $\#_\Gamma$ *GNil* $\vdash_{wf}$ $(\tau[bv::=b']_{\tau b})$
⟨*proof*⟩

lemma *wf-trans*:
fixes Γ::Γ **and** Γ'::Γ **and** v::v **and** e::e **and** c::c **and** τ::τ **and** ts::(*string*$*\tau$) *list* **and** Δ::Δ **and** b::b **and** *ftq*::*fun-typ-q* **and** *ft*::*fun-typ* **and** *ce*::*ce* **and** *td*::*type-def* **and** s::s **and** *cs*::*branch-s* **and** *css*::*branch-list* **and** Θ::Θ
shows Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ $v : b'$ $\Longrightarrow$ $\Gamma = (x,\ b,\ c2)$ $\#_\Gamma$ G $\Longrightarrow$ Θ; $\mathcal{B}$; $(x,\ b,\ c1)$ $\#_\Gamma$ G $\vdash_{wf}$ $c2$ $\Longrightarrow$ Θ; $\mathcal{B}$; $(x,\ b,\ c1)$ $\#_\Gamma$ G $\vdash_{wf}$ $v : b'$ **and**
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ c $\Longrightarrow$ $\Gamma = (x,\ b,\ c2)$ $\#_\Gamma$ G $\Longrightarrow$ Θ; $\mathcal{B}$; $(x,\ b,\ c1)$ $\#_\Gamma$ G $\vdash_{wf}$ $c2$ $\Longrightarrow$ Θ; $\mathcal{B}$; $(x,\ b,\ c1)$ $\#_\Gamma$ G $\vdash_{wf}$ c **and**
Θ; $\mathcal{B}$ $\vdash_{wf}$ Γ $\Longrightarrow$ *True* **and**
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ τ $\Longrightarrow$ *True* **and**
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ ts $\Longrightarrow$ *True* **and**
$\vdash_{wf}$ Θ $\Longrightarrow$ *True* **and**
Θ; $\mathcal{B}$ $\vdash_{wf}$ b $\Longrightarrow$ *True* **and**
Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ $ce : b'$ $\Longrightarrow$ $\Gamma = (x,\ b,\ c2)$ $\#_\Gamma$ G $\Longrightarrow$ Θ; $\mathcal{B}$; $(x,\ b,\ c1)$ $\#_\Gamma$ G $\vdash_{wf}$ $c2$ $\Longrightarrow$ Θ; $\mathcal{B}$; $(x,\ b,\ c1)$ $\#_\Gamma$ G $\vdash_{wf}$ $ce : b'$ **and**
Θ $\vdash_{wf}$ td $\Longrightarrow$ *True*
⟨*proof*⟩

end

Chapter 9

Type System

The MiniSail type system. We define subtyping judgement first and then typing judgement for the term forms

9.1 Subtyping

Subtyping is defined on top of refinement constraint logic (RCL). A subtyping check is converted into an RCL validity check.

inductive *subtype* :: $\Theta \Rightarrow \mathcal{B} \Rightarrow \Gamma \Rightarrow \tau \Rightarrow \tau \Rightarrow$ *bool* ($\langle$- ; - ; - $\vdash$ - $\lesssim$ -$\rangle$ [*50*, *50*, *50*] *50*) **where**
  subtype-baseI: $[\![$
   atom $x \sharp (\Theta, \mathcal{B}, \Gamma, z, c, z', c')$;
   $\Theta; \mathcal{B}; \Gamma \vdash_{wf} \{\!|\ z : b \mid c\ |\!\}$;
   $\Theta; \mathcal{B}; \Gamma \vdash_{wf} \{\!|\ z' : b \mid c'\ |\!\}$;
   $\Theta; \mathcal{B}\ ;\ (x, b,\ c[z::=[x]^v]_v)\ \#_\Gamma\ \Gamma \models c'[z'::=[x]^v]_v$
$[\![\Longrightarrow$
   $\Theta; \mathcal{B}; \Gamma \vdash \{\!|\ z : b \mid c\ |\!\} \lesssim \{\!|\ z' : b \mid c'\ |\!\}$

equivariance *subtype*
nominal-inductive *subtype*
  avoids *subtype-baseI*: *x*
⟨*proof*⟩

inductive-cases *subtype-elims*:
  $\Theta; \mathcal{B}; \Gamma \vdash \{\!|\ z : b \mid c\ |\!\} \lesssim \{\!|\ z' : b \mid c'\ |\!\}$
  $\Theta; \mathcal{B}; \Gamma \vdash \tau_1 \lesssim \tau_2$

9.2 Literals

The type synthesised has the constraint that z equates to the literal

inductive *infer-l* :: $l \Rightarrow \tau \Rightarrow$ *bool* ($\langle \vdash$ - $\Rightarrow$ -$\rangle$ [*50*, *50*] *50*) **where**
  infer-trueI: $\vdash$ *L-true* $\Rightarrow \{\!|\ z : B\text{-}bool \mid [[z]^v]^{ce} == [[L\text{-}true]^v]^{ce}\ |\!\}$
| *infer-falseI*: $\vdash$ *L-false* $\Rightarrow \{\!|\ z : B\text{-}bool \mid [[z]^v]^{ce} == [[L\text{-}false]^v]^{ce}\ |\!\}$
| *infer-natI*: $\vdash$ *L-num n* $\Rightarrow \{\!|\ z : B\text{-}int \mid [[z]^v]^{ce} == [[L\text{-}num\ n]^v]^{ce}\ |\!\}$
| *infer-unitI*: $\vdash$ *L-unit* $\Rightarrow \{\!|\ z : B\text{-}unit \mid [[z]^v]^{ce} == [[L\text{-}unit]^v]^{ce}\ |\!\}$
| *infer-bitvecI*: $\vdash$ *L-bitvec bv* $\Rightarrow \{\!|\ z : B\text{-}bitvec \mid [[z]^v]^{ce} == [[L\text{-}bitvec\ bv]^v]^{ce}\ |\!\}$

nominal-inductive *infer-l* ⟨*proof*⟩
equivariance *infer-l*

inductive-cases *infer-l-elims*[*elim*!]:
⊢ *L-true* ⇒ τ
⊢ *L-false* ⇒ τ
⊢ *L-num n* ⇒ τ
⊢ *L-unit* ⇒ τ
⊢ *L-bitvec x* ⇒ τ
⊢ *l* ⇒ τ

lemma *infer-l-form2*[*simp*]:
shows ∃ *z*. ⊢ *l* ⇒ ({| *z* : *base-for-lit l* | $[[z]^v]^{ce}$ == $[[l]^v]^{ce}$ |})
⟨*proof*⟩

9.3 Values

inductive *infer-v* :: Θ ⇒ $\mathcal{B}$ ⇒ Γ ⇒ *v* ⇒ τ ⇒ *bool* (‹ - ; - ; - ⊢ - ⇒ -› [*50*, *50*, *50*] *50*) **where**

infer-v-varI: ⟦
Θ; $\mathcal{B}$ $\vdash_{wf}$ Γ ;
Some (*b*,*c*) = *lookup* Γ *x*;
atom z ♯ *x* ; *atom z* ♯ (Θ, $\mathcal{B}$, Γ)
⟧ ⟹
Θ; $\mathcal{B}$; Γ ⊢ $[x]^v$ ⇒ {| *z* : *b* | $[[z]^v]^{ce}$ == $[[x]^v]^{ce}$ |}

| *infer-v-litI*: ⟦
Θ; $\mathcal{B}$ $\vdash_{wf}$ Γ ;
⊢ *l* ⇒ τ
⟧ ⟹
Θ; $\mathcal{B}$; Γ ⊢ $[l]^v$ ⇒ τ

| *infer-v-pairI*: ⟦
atom z ♯ (*v1*, *v2*); *atom z* ♯ (Θ, $\mathcal{B}$, Γ) ;
Θ; $\mathcal{B}$; Γ ⊢ (*v1*::*v*) ⇒ *t1* ;
Θ; $\mathcal{B}$; Γ ⊢ (*v2*::*v*) ⇒ *t2*
⟧ ⟹
Θ; $\mathcal{B}$; Γ ⊢ *V-pair v1 v2* ⇒ ({| *z* : *B-pair* (*b-of t1*) (*b-of t2*) | $[[z]^v]^{ce}$ == $[[v1,v2]^v]^{ce}$ |})

| *infer-v-consI*: ⟦
AF-typedef s dclist ∈ *set* Θ;
(*dc*, *tc*) ∈ *set dclist* ;
Θ; $\mathcal{B}$; Γ ⊢ *v* ⇒ *tv* ;
Θ; $\mathcal{B}$; Γ ⊢ *tv* $\lesssim$ *tc* ;
atom z ♯ *v* ; *atom z* ♯ (Θ, $\mathcal{B}$, Γ)
⟧ ⟹
Θ; $\mathcal{B}$; Γ ⊢ *V-cons s dc v* ⇒ ({| *z* : *B-id s* | $[[z]^v]^{ce}$ == [*V-cons s dc v*]^{ce} |})

| *infer-v-conspI*: ⟦
AF-typedef-poly s bv dclist ∈ *set* Θ;
(*dc*, *tc*) ∈ *set dclist* ;

Θ; $\mathcal{B}$; $\Gamma \vdash v \Rightarrow tv$;
Θ; $\mathcal{B}$; $\Gamma \vdash tv \lesssim tc[bv::=b]_{\tau b}$;
atom $z \sharp (\Theta, \mathcal{B}, \Gamma, v, b)$;
atom $bv \sharp (\Theta, \mathcal{B}, \Gamma, v, b)$;
Θ; $\mathcal{B} \vdash_{wf} b$
$\rrbracket \Longrightarrow$
Θ; $\mathcal{B}$; $\Gamma \vdash$ *V-consp s dc b v* $\Rightarrow$ ($\{\!|$ *z* : *B-app s b* | $[[z]^v]^{ce} ==$ (*CE-val* (*V-consp s dc b v*)) $|\!\}$)

equivariance *infer-v*
nominal-inductive *infer-v*
avoids *infer-v-conspI*: *bv* **and** *z* | *infer-v-varI*: *z* | *infer-v-pairI*: *z* | *infer-v-consI*: *z*
⟨*proof*⟩

inductive-cases *infer-v-elims*[*elim*!]:
Θ; $\mathcal{B}$; $\Gamma \vdash$ *V-var x* $\Rightarrow \tau$
Θ; $\mathcal{B}$; $\Gamma \vdash$ *V-lit l* $\Rightarrow \tau$
Θ; $\mathcal{B}$; $\Gamma \vdash$ *V-pair v1 v2* $\Rightarrow \tau$
Θ; $\mathcal{B}$; $\Gamma \vdash$ *V-cons s dc v* $\Rightarrow \tau$
Θ; $\mathcal{B}$; $\Gamma \vdash$ *V-pair v1 v2* $\Rightarrow$ ($\{\!|$ *z* : *b* | *c* $|\!\}$)
Θ; $\mathcal{B}$; $\Gamma \vdash$ *V-pair v1 v2* $\Rightarrow$ ($\{\!|$ *z* : [*b1* , *b2*]^b | $[[z]^v]^{ce} == [[v1,v2]^v]^{ce}$ $|\!\}$)
Θ; $\mathcal{B}$; $\Gamma \vdash$ *V-consp s dc b v* $\Rightarrow \tau$

inductive *check-v* :: $\Theta \Rightarrow \mathcal{B} \Rightarrow \Gamma \Rightarrow v \Rightarrow \tau \Rightarrow$ *bool* (‹- ; - ; - ⊢ - ⇐ -› [*50*, *50*, *50*] *50*) **where**
check-v-subtypeI: $\rrbracket$ Θ; $\mathcal{B}$; $\Gamma \vdash \tau 1 \lesssim \tau 2$; Θ; $\mathcal{B}$; $\Gamma \vdash v \Rightarrow \tau 1$ $\llbracket \Longrightarrow \Theta$; $\mathcal{B}$; $\Gamma \vdash v \Leftarrow \tau 2$
equivariance *check-v*
nominal-inductive *check-v* ⟨*proof*⟩

inductive-cases *check-v-elims*[*elim*!]:
Θ; $\mathcal{B}$; $\Gamma \vdash v \Leftarrow \tau$

9.4 Expressions

Type synthesis for expressions

inductive *infer-e* :: $\Theta \Rightarrow \Phi \Rightarrow \mathcal{B} \Rightarrow \Gamma \Rightarrow \Delta \Rightarrow e \Rightarrow \tau \Rightarrow$ *bool* (‹- ; - ; - ; - ; - ⊢ - ⇒ -› [*50*, *50*, *50*,*50*] *50*) **where**

infer-e-valI: $\rrbracket$
(Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$) ;
($\Theta \vdash_{wf} (\Phi::\Phi)$) ;
(Θ; $\mathcal{B}$; $\Gamma \vdash v \Rightarrow \tau$) $\llbracket \Longrightarrow$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ (*AE-val v*) $\Rightarrow \tau$

| *infer-e-plusI*: $\rrbracket$
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$;
$\Theta \vdash_{wf} (\Phi::\Phi)$;
Θ; $\mathcal{B}$; $\Gamma \vdash$ *v1* $\Rightarrow$ $\{\!|$ *z1* : *B-int* | *c1* $|\!\}$;
Θ; $\mathcal{B}$; $\Gamma \vdash$ *v2* $\Rightarrow$ $\{\!|$ *z2* : *B-int* | *c2* $|\!\}$;
atom z3 $\sharp$ (*AE-op Plus v1 v2*); *atom z3* $\sharp \Gamma$ $\llbracket \Longrightarrow$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *AE-op Plus v1 v2* $\Rightarrow$ $\{\!|$ *z3* : *B-int* | $[[z3]^v]^{ce} ==$ (*CE-op Plus* $[v1]^{ce}$ $[v2]^{ce}$) $|\!\}$

| *infer-e-leqI*: $\rrbracket$

Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$;
$\Theta \vdash_{wf} (\Phi::\Phi)$;
Θ; $\mathcal{B}$; $\Gamma \vdash$ *v1* $\Rightarrow$ $\{\!|$ *z1* : *B-int* | *c1* $|\!\}$;
Θ; $\mathcal{B}$; $\Gamma \vdash$ *v2* $\Rightarrow$ $\{\!|$ *z2* : *B-int* | *c2* $|\!\}$;
atom z3 $\sharp$ (*AE-op LEq v1 v2*); *atom z3* $\sharp$ Γ
$]\!] \Longrightarrow$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *AE-op LEq v1 v2* $\Rightarrow$ $\{\!|$ *z3* : *B-bool* | $[[z3]^v]^{ce} == (CE\text{-}op\ LEq\ [v1]^{ce}\ [v2]^{ce})$ $|\!\}$

| *infer-e-eqI*: $[\![$
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$;
$\Theta \vdash_{wf} (\Phi::\Phi)$;
Θ; $\mathcal{B}$; $\Gamma \vdash$ *v1* $\Rightarrow$ $\{\!|$ *z1* : *b* | *c1* $|\!\}$;
Θ; $\mathcal{B}$; $\Gamma \vdash$ *v2* $\Rightarrow$ $\{\!|$ *z2* : *b* | *c2* $|\!\}$;
atom z3 $\sharp$ (*AE-op Eq v1 v2*); *atom z3* $\sharp$ Γ ;
b $\in$ { *B-bool*, *B-int*, *B-unit* }
$]\!] \Longrightarrow$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *AE-op Eq v1 v2* $\Rightarrow$ $\{\!|$ *z3* : *B-bool* | $[[z3]^v]^{ce} == (CE\text{-}op\ Eq\ [v1]^{ce}\ [v2]^{ce})$ $|\!\}$

| *infer-e-appI*: $[\![$
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$;
$\Theta \vdash_{wf} (\Phi::\Phi)$;
Some (*AF-fundef f* (*AF-fun-typ-none* (*AF-fun-typ x b c* τ' *s'*))) = *lookup-fun* Φ *f*;
Θ; $\mathcal{B}$; $\Gamma \vdash$ *v* $\Leftarrow$ $\{\!|$ *x* : *b* | *c* $|\!\}$;
atom x $\sharp$ (Θ, Φ, $\mathcal{B}$, Γ, Δ,*v* , τ);
$\tau'[x::=v]_v = \tau$
$]\!] \Longrightarrow$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *AE-app f v* $\Rightarrow \tau$

| *infer-e-appPI*: $[\![$
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$;
$\Theta \vdash_{wf} (\Phi::\Phi)$;
Θ; $\mathcal{B} \vdash_{wf} b'$;
Some (*AF-fundef f* (*AF-fun-typ-some bv* (*AF-fun-typ x b c* τ' *s'*))) = *lookup-fun* Φ *f*;
Θ; $\mathcal{B}$; $\Gamma \vdash$ *v* $\Leftarrow$ $\{\!|$ *x* : $b[bv::=b']_b$ | $c[bv::=b']_b$ $|\!\}$; *atom x* $\sharp$ Γ;
$(\tau'[bv::=b']_b[x::=v]_v) = \tau$;
atom bv $\sharp$ (Θ, Φ, $\mathcal{B}$, Γ, Δ, b', *v*, τ)
$]\!] \Longrightarrow$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *AE-appP f b' v* $\Rightarrow \tau$

| *infer-e-fstI*: $[\![$
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$;
$\Theta \vdash_{wf} (\Phi::\Phi)$;
Θ; $\mathcal{B}$; $\Gamma \vdash$ *v* $\Rightarrow$ $\{\!|$ *z'* : $[b1,b2]^b$ | *c* $|\!\}$;
atom z $\sharp$ *AE-fst v* ; *atom z* $\sharp$ Γ $]\!] \Longrightarrow$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *AE-fst v* $\Rightarrow$ $\{\!|$ *z* : *b1* | $[[z]^v]^{ce} == ((CE\text{-}fst\ [v]^{ce}))$ $|\!\}$

| *infer-e-sndI*: $[\![$
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$;
$\Theta \vdash_{wf} (\Phi::\Phi)$;
Θ; $\mathcal{B}$; $\Gamma \vdash$ *v* $\Rightarrow$ $\{\!|$ *z'* : [*b1*, *b2*]^b | *c* $|\!\}$;
atom z $\sharp$ *AE-snd v* ; *atom z* $\sharp$ Γ $]\!] \Longrightarrow$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *AE-snd v* $\Rightarrow$ $\{\!|$ *z* : *b2* | $[[z]^v]^{ce} == ((CE\text{-}snd\ [v]^{ce}))$ $|\!\}$

| *infer-e-lenI*: ⟦
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$;
$\Theta \vdash_{wf} (\Phi::\Phi)$;
Θ; $\mathcal{B}$; $\Gamma \vdash v \Rightarrow \{\!|\ z' : \textit{B-bitvec} \mid c\ |\!\}$;
atom $z \sharp$ *AE-len* v ; *atom* $z \sharp \Gamma$⟧ $\Longrightarrow$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *AE-len* $v \Rightarrow \{\!|\ z : \textit{B-int} \mid [[z]^v]^{ce} == ((\textit{CE-len}\ [v]^{ce}))\ |\!\}$

| *infer-e-mvarI*: ⟦
Θ; $\mathcal{B} \vdash_{wf} \Gamma$;
$\Theta \vdash_{wf} (\Phi::\Phi)$;
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$;
$(u,\tau) \in \textit{setD}\ \Delta$ ⟧ $\Longrightarrow$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *AE-mvar* $u \Rightarrow \tau$

| *infer-e-concatI*: ⟦
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$;
$\Theta \vdash_{wf} (\Phi::\Phi)$;
Θ; $\mathcal{B}$; $\Gamma \vdash$ *v1* $\Rightarrow \{\!|\ \textit{z1} : \textit{B-bitvec} \mid \textit{c1}\ |\!\}$;
Θ; $\mathcal{B}$; $\Gamma \vdash$ *v2* $\Rightarrow \{\!|\ \textit{z2} : \textit{B-bitvec} \mid \textit{c2}\ |\!\}$;
atom z3 $\sharp$ (*AE-concat v1 v2*); *atom z3* $\sharp \Gamma$ ⟧ $\Longrightarrow$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *AE-concat v1 v2* $\Rightarrow \{\!|\ \textit{z3} : \textit{B-bitvec} \mid [[\textit{z3}]^v]^{ce} == (\textit{CE-concat}\ [\textit{v1}]^{ce}\ [\textit{v2}]^{ce})\ |\!\}$

| *infer-e-splitI*: ⟦
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$;
$\Theta \vdash_{wf} (\Phi::\Phi)$;
Θ ; $\mathcal{B}$; $\Gamma \vdash$ *v1* $\Rightarrow \{\!|\ \textit{z1} : \textit{B-bitvec} \mid \textit{c1}\ |\!\}$;
Θ ; $\mathcal{B}$; $\Gamma \vdash$ *v2* $\Leftarrow \{\!|\ \textit{z2} : \textit{B-int} \mid (\textit{CE-op LEq}\ (\textit{CE-val}\ (\textit{V-lit}\ (\textit{L-num}\ 0)))\ (\textit{CE-val}\ (\textit{V-var z2}))$
$== (\textit{CE-val}\ (\textit{V-lit L-true}))$ *AND*
$(\textit{CE-op LEq}\ (\textit{CE-val}\ (\textit{V-var z2}))\ (\textit{CE-len}\ (\textit{CE-val}\ (\textit{v1}))) == (\textit{CE-val}$
$(\textit{V-lit L-true}))\ |\!\}$;
atom z1 $\sharp$ (*AE-split v1 v2*); *atom z1* $\sharp \Gamma$;
atom z2 $\sharp$ (*AE-split v1 v2*); *atom z2* $\sharp \Gamma$;
atom z3 $\sharp$ (*AE-split v1 v2*); *atom z3* $\sharp \Gamma$
⟧ $\Longrightarrow$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ (*AE-split v1 v2*) $\Rightarrow \{\!|\ \textit{z3} : \textit{B-pair B-bitvec B-bitvec}\ \mid$
$((\textit{CE-val v1}) == (\textit{CE-concat}\ (\textit{CE-fst}\ (\textit{CE-val}\ (\textit{V-var z3})))\ (\textit{CE-snd}\ (\textit{CE-val}\ (\textit{V-var}$
$\textit{z3}))))))$
AND $(((\textit{CE-len}\ (\textit{CE-fst}\ (\textit{CE-val}\ (\textit{V-var z3})))))) == (\textit{CE-val}\ (\ \textit{v2}))\ |\!\}$

equivariance *infer-e*
nominal-inductive *infer-e*
avoids *infer-e-appI*: *x* |*infer-e-appPI*: *bv* | *infer-e-splitI*: *z3* **and** *z1* **and** *z2*
⟨*proof*⟩

inductive-cases *infer-e-elims*[*elim*!]:
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash (\textit{AE-op Plus v1 v2}) \Rightarrow \{\!|\ \textit{z3} : \textit{B-int} \mid [[\textit{z3}]^v]^{ce} == (\textit{CE-op Plus}\ [\textit{v1}]^{ce}\ [\textit{v2}]^{ce})\ |\!\}$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash (\textit{AE-op LEq v1 v2}) \Rightarrow \{\!|\ \textit{z3} : \textit{B-bool} \mid [[\textit{z3}]^v]^{ce} == (\textit{CE-op LEq}\ [\textit{v1}]^{ce}\ [\textit{v2}]^{ce})\ |\!\}$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash (\textit{AE-op Plus v1 v2}) \Rightarrow \{\!|\ \textit{z3} : \textit{B-int} \mid \textit{c}\ |\!\}$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash (\textit{AE-op Plus v1 v2}) \Rightarrow \{\!|\ \textit{z3} : \textit{b} \mid \textit{c}\ |\!\}$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash (\textit{AE-op LEq v1 v2}) \Rightarrow \{\!|\ \textit{z3} : \textit{b} \mid \textit{c}\ |\!\}$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash (\textit{AE-app f v}\ ) \Rightarrow \tau$

Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash (AE\text{-}val\ v) \Rightarrow \tau$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash (AE\text{-}fst\ v) \Rightarrow \tau$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash (AE\text{-}snd\ v) \Rightarrow \tau$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash (AE\text{-}mvar\ u) \Rightarrow \tau$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash (AE\text{-}op\ Plus\ v1\ v2) \Rightarrow \tau$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash (AE\text{-}op\ LEq\ v1\ v2) \Rightarrow \tau$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash (AE\text{-}op\ LEq\ v1\ v2) \Rightarrow \{\!|\ z3 : B\text{-}bool \mid c\ |\!\}$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash (AE\text{-}app\ f\ v\ ) \Rightarrow \tau[x::=v]_{\tau v}$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash (AE\text{-}op\ opp\ v1\ v2) \Rightarrow \tau$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash (AE\text{-}len\ v) \Rightarrow \tau$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash (AE\text{-}len\ v) \Rightarrow \{\!|\ z : B\text{-}int \mid [[z]^v]^{ce} == ((CE\text{-}len\ [v]^{ce}))|\!\}$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash AE\text{-}concat\ v1\ v2 \Rightarrow \tau$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash AE\text{-}concat\ v1\ v2 \Rightarrow (\{\!|\ z : b \mid\ c\ |\!\})$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash AE\text{-}concat\ v1\ v2 \Rightarrow (\{\!|\ z : B\text{-}bitvec \mid\ [[z]^v]^{ce} == (CE\text{-}concat\ [v1]^{ce}\ [v1]^{ce})\ |\!\})$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash (AE\text{-}appP\ f\ b\ v\ ) \Rightarrow \tau$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash AE\text{-}split\ v1\ v2 \Rightarrow \tau$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash (AE\text{-}op\ Eq\ v1\ v2) \Rightarrow \{\!|\ z3 : b \mid c\ |\!\}$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash (AE\text{-}op\ Eq\ v1\ v2) \Rightarrow \{\!|\ z3 : B\text{-}bool \mid c\ |\!\}$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash (AE\text{-}op\ Eq\ v1\ v2) \Rightarrow \tau$
nominal-termination (*eqvt*) ⟨*proof*⟩

9.5 Statements

inductive *check-s* :: $\Theta \Rightarrow \Phi \Rightarrow \mathcal{B} \Rightarrow \Gamma \Rightarrow \Delta \Rightarrow s \Rightarrow \tau \Rightarrow bool$ ($\langle$ - ; - ; - ; - ; - $\vdash$ - $\Leftarrow$ -$\rangle$ [50, 50, 50,50,50] 50) **and**
check-branch-s :: $\Theta \Rightarrow \Phi \Rightarrow \mathcal{B} \Rightarrow \Gamma \Rightarrow \Delta \Rightarrow tyid \Rightarrow string \Rightarrow \tau \Rightarrow v \Rightarrow branch\text{-}s \Rightarrow \tau \Rightarrow bool$ ($\langle$ - ; - ; - ; - ; - ; - ; - ; - ; - ; - $\vdash$ - $\Leftarrow$ -$\rangle$ [50, 50, 50,50,50] 50) **and**
check-branch-list :: $\Theta \Rightarrow \Phi \Rightarrow \mathcal{B} \Rightarrow \Gamma \Rightarrow \Delta \Rightarrow tyid \Rightarrow (string * \tau)\ list \Rightarrow v \Rightarrow branch\text{-}list \Rightarrow \tau \Rightarrow bool$ ($\langle$ - ; - ; - ; - ; - ; - ; - ; - ; - $\vdash$ - $\Leftarrow$ -$\rangle$ [50, 50, 50,50,50] 50) **where**
check-valI: ⟦
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$;
$\Theta \vdash_{wf} \Phi$;
Θ; $\mathcal{B}$; $\Gamma \vdash v \Rightarrow \tau'$;
Θ; $\mathcal{B}$; $\Gamma \vdash \tau' \lesssim \tau$ ⟧ ⟹
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash (AS\text{-}val\ v) \Leftarrow \tau$

| *check-letI*: ⟦
atom $x \sharp (\Theta, \Phi, \mathcal{B}, \Gamma, \Delta, e, \tau)$;
atom $z \sharp (x, \Theta, \Phi, \mathcal{B}, \Gamma, \Delta, e, \tau, s)$;
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash e \Rightarrow \{\!|\ z : b \mid c\ |\!\}$;
Θ; Φ ; $\mathcal{B}$; $((x,b,c[z::=V\text{-}var\ x]_v)\#_\Gamma\Gamma)$; $\Delta \vdash s \Leftarrow \tau$
⟧ ⟹
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash (AS\text{-}let\ x\ e\ s) \Leftarrow \tau$

| *check-assertI*: ⟦
atom $x \sharp (\Theta, \Phi, \mathcal{B}, \Gamma, \Delta, c, \tau, s)$;
Θ; Φ ; $\mathcal{B}$; $((x,B\text{-}bool,c)\#_\Gamma\Gamma)$; $\Delta \vdash s \Leftarrow \tau$;
Θ; $\mathcal{B}$; $\Gamma \models c$;
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$
⟧ ⟹
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash (AS\text{-}assert\ c\ s) \Leftarrow \tau$

$|$*check-branch-s-branchI*: $[\![$
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$;
$\vdash_{wf} \Theta$;
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \tau$;
Θ ; $\{||\}$; *GNil* $\vdash_{wf}$ *const*;
atom $x \sharp (\Theta, \Phi, \mathcal{B}, \Gamma, \Delta,$ *tid*, *cons* , *const*, $v, \tau)$;
Θ; Φ; $\mathcal{B}$; $((x,$*b-of const*, $([v]^{ce} ==$ [*V-cons tid cons* $[x]^{v}]^{ce}$) *AND* (*c-of const* $x))\#_{\Gamma}\Gamma)$; $\Delta \vdash s \Leftarrow \tau$
$]\!] \Longrightarrow$
Θ; Φ; $\mathcal{B}$; Γ; Δ ; *tid* ; *cons* ; *const* ; $v \vdash$ (*AS-branch cons x s*) $\Leftarrow \tau$

$|$*check-branch-list-consI*: $[\![$
Θ; Φ; $\mathcal{B}$; Γ; Δ; *tid*; *cons*; *const*; $v \vdash$ *cs* $\Leftarrow \tau$;
Θ; Φ; $\mathcal{B}$; Γ; Δ; *tid*; *dclist*; $v \vdash$ *css* $\Leftarrow \tau$
$]\!] \Longrightarrow$
Θ; Φ; $\mathcal{B}$; Γ; Δ ; *tid* ; (*cons*,*const*)#*dclist* ; $v \vdash$ *AS-cons cs css* $\Leftarrow \tau$

$|$*check-branch-list-finalI*: $[\![$
Θ; Φ;$\mathcal{B}$; Γ; Δ; *tid* ; *cons* ; *const* ; $v \vdash$ *cs* $\Leftarrow \tau$
$]\!] \Longrightarrow$
Θ; Φ; $\mathcal{B}$; Γ; Δ ; *tid* ; [(*cons*,*const*)] ; $v \vdash$ *AS-final cs* $\Leftarrow \tau$

$|$ *check-ifI*: $[\![$
atom $z \sharp (\Theta, \Phi, \mathcal{B}, \Gamma, \Delta, v$, *s1* , *s2* , τ);
$(\Theta; \mathcal{B}; \Gamma \vdash v \Leftarrow (\{\!|\ z : B\text{-}bool \mid TRUE\ |\!\}))$;
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *s1* $\Leftarrow (\{\!|\ z :$ *b-of* $\tau \mid ([v]^{ce} == [[$*L-true*$]^{v}]^{ce})$ *IMP* (*c-of* τ $z)\ |\!\})$;
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *s2* $\Leftarrow (\{\!|\ z :$ *b-of* $\tau \mid ([v]^{ce} == [[$*L-false*$]^{v}]^{ce})$ *IMP* (*c-of* τ $z)\ |\!\})$
$]\!] \Longrightarrow$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *IF v THEN s1 ELSE s2* $\Leftarrow \tau$

$|$ *check-let2I*: $[\![$
atom $x \sharp (\Theta, \Phi, \mathcal{B}, G, \Delta, t,$ *s1*, $\tau)$;
Θ; Φ ; $\mathcal{B}$; G; $\Delta \vdash$ *s1* $\Leftarrow t$;
Θ; Φ ; $\mathcal{B}$; $((x,$*b-of t*,*c-of t* $x)\#_{\Gamma} G)$; $\Delta \vdash$ *s2* $\Leftarrow \tau$
$]\!] \Longrightarrow$
Θ; Φ ; $\mathcal{B}$; G ; $\Delta \vdash$ (*LET* $x : t$ $=$ *s1 IN s2*) $\Leftarrow \tau$

$|$ *check-varI*: $[\![$
atom $u \sharp (\Theta, \Phi, \mathcal{B}, \Gamma, \Delta, \tau', v, \tau)$;
Θ; $\mathcal{B}$; $\Gamma \vdash$ $v \Leftarrow \tau'$;
Θ; Φ; $\mathcal{B}$; Γ ; $((u,\tau')$ $\#_{\Delta}$ $\Delta)$ $\vdash s \Leftarrow \tau$
$]\!] \Longrightarrow$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ (*VAR* $u : \tau' = v$ *IN* $s) \Leftarrow \tau$

$|$ *check-assignI*: $[\![$
$\Theta \vdash_{wf} \Phi$;
Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \Delta$;
$(u,\tau) \in$ *setD* Δ ;
Θ; $\mathcal{B}$; $\Gamma \vdash$ $v \Leftarrow \tau$;
Θ; $\mathcal{B}$; $\Gamma \vdash (\{\!|\ z :$ *B-unit* $\mid$ *TRUE* $|\!\}) \lesssim \tau'$
$]\!] \Longrightarrow$
Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ $(u ::= v) \Leftarrow \tau'$

| *check-whileI*: ⟦
  Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *s1* $\Leftarrow$ {| *z* : *B-bool* | *TRUE* |};
  Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *s2* $\Leftarrow$ {| *z* : *B-unit* | *TRUE* |};
  Θ; $\mathcal{B}$; $\Gamma \vdash$ ({| *z* : *B-unit* | *TRUE* |}) $\lesssim \tau'$
⟧ $\Longrightarrow$
  Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *WHILE s1 DO* { *s2* } $\Leftarrow \tau'$

| *check-seqI*: ⟦
  Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *s1* $\Leftarrow$ {| *z* : *B-unit* | *TRUE* |};
  Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *s2* $\Leftarrow \tau$
⟧ $\Longrightarrow$
  Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *s1* ;; *s2* $\Leftarrow \tau$

| *check-caseI*: ⟦
  Θ; Φ; $\mathcal{B}$; Γ; Δ; *tid* ; *dclist* ; *v* $\vdash$ *cs* $\Leftarrow \tau$;
  (*AF-typedef tid dclist*) $\in$ *set* Θ ;
  Θ; $\mathcal{B}$; $\Gamma \vdash$ *v* $\Leftarrow$ {| *z* : *B-id tid* | *TRUE* |};
  $\vdash_{wf} \Theta$
⟧ $\Longrightarrow$
  Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *AS-match v cs* $\Leftarrow \tau$

equivariance *check-s*

We only need avoidance for cases where a variable is added to a context

nominal-inductive *check-s*
  avoids *check-letI*: *x* **and** *z* | *check-branch-s-branchI*: *x* | *check-let2I*: *x* | *check-varI*: *u* | *check-ifI*: *z* | *check-assertI*: *x*
⟨*proof*⟩

inductive-cases *check-s-elims*[*elim*!]:
  Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *AS-val v* $\Leftarrow \tau$
  Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *AS-let x e s* $\Leftarrow \tau$
  Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *AS-if v s1 s2* $\Leftarrow \tau$
  Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *AS-let2 x t s1 s2* $\Leftarrow \tau$
  Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *AS-while s1 s2* $\Leftarrow \tau$
  Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *AS-var u t v s* $\Leftarrow \tau$
  Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *AS-seq s1 s2* $\Leftarrow \tau$
  Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *AS-assign u v* $\Leftarrow \tau$
  Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *AS-match v cs* $\Leftarrow \tau$
  Θ; Φ; $\mathcal{B}$; Γ; $\Delta \vdash$ *AS-assert c s* $\Leftarrow \tau$

inductive-cases *check-branch-s-elims*[*elim*!]:
  Θ; Φ; $\mathcal{B}$; Γ; Δ; *tid* ; *dclist* ; *v* $\vdash$ (*AS-final cs*) $\Leftarrow \tau$
  Θ; Φ; $\mathcal{B}$; Γ; Δ; *tid* ; *dclist* ; *v* $\vdash$ (*AS-cons cs css*) $\Leftarrow \tau$
  Θ; Φ; $\mathcal{B}$; Γ; Δ; *tid* ; *cons* ; *const* ; *v* $\vdash$ (*AS-branch dc x s*) $\Leftarrow \tau$

9.6 Programs

Type check function bodies

inductive *check-funtyp* :: $\Theta \Rightarrow \Phi \Rightarrow \mathcal{B} \Rightarrow$ *fun-typ* $\Rightarrow$ *bool* (‹ - ; - ; - ⊢ - ›) **where**

check-funtypI: ⟦
atom x ♯ (Θ, Φ, *B* , *b*);
Θ; Φ ; *B* ; ((*x*,*b*,*c*) #$_\Gamma$ *GNil*) ; []$_\Delta$ ⊢ *s* ⇐ *τ*
⟧ ⟹
Θ; Φ ; *B* ⊢ (*AF-fun-typ x b c τ s*)

equivariance *check-funtyp*
nominal-inductive *check-funtyp*
avoids *check-funtypI*: *x*
⟨*proof*⟩

inductive *check-funtypq* :: Θ ⇒ Φ ⇒ *fun-typ-q* ⇒ *bool* (‹ - ; - ⊢ - ›) **where**
check-fundefq-simpleI: ⟦
Θ; Φ ; {||} ⊢ (*AF-fun-typ x b c t s*)
⟧ ⟹
Θ; Φ ⊢ ((*AF-fun-typ-none* (*AF-fun-typ x b c t s*)))

|*check-funtypq-polyI*: ⟦
atom bv ♯ (Θ, Φ, (*AF-fun-typ x b c t s*));
Θ; Φ; {|*bv*|} ⊢ (*AF-fun-typ x b c t s*)
⟧ ⟹
Θ; Φ ⊢ (*AF-fun-typ-some bv* (*AF-fun-typ x b c t s*))

equivariance *check-funtypq*
nominal-inductive *check-funtypq*
avoids *check-funtypq-polyI*: *bv*
⟨*proof*⟩

inductive *check-fundef* :: Θ ⇒ Φ ⇒ *fun-def* ⇒ *bool* (‹ - ; - ⊢ - ›) **where**
check-fundefI: ⟦
Θ; Φ ⊢ *ft*
⟧ ⟹
Θ; Φ ⊢ (*AF-fundef f ft*)

equivariance *check-fundef*
nominal-inductive *check-fundef* ⟨*proof*⟩

Temporarily remove this simproc as it produces untidy eliminations

declare[[*simproc del*: *alpha-lst*]]

inductive-cases *check-funtyp-elims*[*elim*!]:
check-funtyp Θ Φ *B ft*

inductive-cases *check-funtypq-elims*[*elim*!]:
check-funtypq Θ Φ (*AF-fun-typ-none* (*AF-fun-typ x b c τ s*))
check-funtypq Θ Φ (*AF-fun-typ-some bv* (*AF-fun-typ x b c τ s*))

inductive-cases *check-fundef-elims*[*elim*!]:
check-fundef Θ Φ (*AF-fundef f ftq*)

declare[[*simproc add*: *alpha-lst*]]

nominal-function Δ-*of* :: *var-def list* $\Rightarrow$ Δ **where**
  Δ-*of* [] = *DNil*
| Δ-*of* ((*AV-def u t v*)#*vs*) = (*u*,*t*) $\#_{\Delta}$ (Δ-*of vs*)
  ⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

inductive *check-prog* :: $p \Rightarrow \tau \Rightarrow$ *bool* (‹⊢ - ⇐ ->) **where**
  ⟦
   Θ; Φ; {||}; *GNil* ; Δ-*of* $\mathcal{G} \vdash s \Leftarrow \tau$
⟧ $\Longrightarrow$ $\vdash$ (*AP-prog* Θ Φ $\mathcal{G}$ *s*) $\Leftarrow \tau$

inductive-cases *check-prog-elims*[*elim*!]:
  $\vdash$ (*AP-prog* Θ Φ $\mathcal{G}$ *s*) $\Leftarrow \tau$

end

Chapter 10

Operational Semantics

Here we define the operational semantics in terms of a small-step reduction relation.

10.1 Reduction Rules

The store for mutable variables

type-synonym $\delta = (u*v)\ list$

nominal-function *update-d* :: $\delta \Rightarrow u \Rightarrow v \Rightarrow \delta$ **where**
  update-d [] - - = []
| *update-d* ((u',v')#δ) u v = (*if* $u = u'$ *then* ((u,v)#δ) *else* ((u',v')# (*update-d* δ u v)))
  ⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

Relates constructor to the branch in the case and binding variable and statement

inductive *find-branch* :: *dc* ⇒ *branch-list* ⇒ *branch-s* ⇒ *bool* **where**
  find-branch-finalI: $dc' = dc$ ⟹ *find-branch* dc' (*AS-final* (*AS-branch dc x s*)) (*AS-branch dc x s*)
| *find-branch-branch-eqI*: $dc' = dc$ ⟹ *find-branch* dc' (*AS-cons* (*AS-branch dc x s*) *css*) (*AS-branch dc x s*)
| *find-branch-branch-neqI*: ⟦ $dc \neq dc'$; *find-branch* dc' *css cs* ⟧ ⟹ *find-branch* dc' (*AS-cons* (*AS-branch dc x s*) *css*) *cs*
equivariance *find-branch*
nominal-inductive *find-branch* ⟨*proof*⟩

inductive-cases *find-branch-elims*[*elim*!]:
  find-branch dc (*AS-final* cs') *cs*
  find-branch dc (*AS-cons* cs' *css*) *cs*

nominal-function *lookup-branch* :: *dc* ⇒ *branch-list* ⇒ *branch-s option* **where**
  lookup-branch dc (*AS-final* (*AS-branch* dc' x s)) = (*if* $dc = dc'$ *then* (*Some* (*AS-branch* dc' x s)) *else None*)
| *lookup-branch dc* (*AS-cons* (*AS-branch* dc' x s) *css*) = (*if* $dc = dc'$ *then* (*Some* (*AS-branch* dc' x s)) *else lookup-branch dc css*)
    ⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

Reduction rules

inductive *reduce-stmt* :: $\Phi \Rightarrow \delta \Rightarrow s \Rightarrow \delta \Rightarrow s \Rightarrow bool$ $(\langle\!\langle$ - $\vdash \langle$ - , -$\rangle \longrightarrow \langle$ - , -$\rangle\rangle$ $[50, 50, 50]$ $50)$
where
reduce-if-trueI: $\Phi \vdash \langle \delta,$ *AS-if* $[$*L-true*$]^v$ *s1 s2*$\rangle \longrightarrow \langle \delta,$ *s1*$\rangle$
| *reduce-if-falseI*: $\Phi \vdash \langle \delta,$ *AS-if* $[$*L-false*$]^v$ *s1 s2*$\rangle \longrightarrow \langle \delta,$ *s2*$\rangle$
| *reduce-let-valI*: $\Phi \vdash \langle \delta,$ *AS-let x* (*AE-val v*) *s*$\rangle \longrightarrow \langle \delta,$ *s*$[x::=v]_{sv}\rangle$
| *reduce-let-plusI*: $\Phi \vdash \langle \delta,$ *AS-let x* (*AE-op Plus* ((*V-lit* (*L-num n1*))) ((*V-lit* (*L-num n2*)))) *s*$\rangle \longrightarrow$
$\langle \delta,$ *AS-let x* (*AE-val* (*V-lit* (*L-num* (((*n1*)+(*n2*)))))) *s* $\rangle$
| *reduce-let-leqI*: *b* = (*if* (*n1* $\leq$ *n2*) *then L-true else L-false*) $\Longrightarrow$
$\Phi \vdash \langle \delta,$ *AS-let x* ((*AE-op LEq* (*V-lit* (*L-num n1*)) (*V-lit* (*L-num n2*)))) *s*$\rangle \longrightarrow$
$\langle \delta,$ *AS-let x* (*AE-val* (*V-lit b*)) *s*$\rangle$
| *reduce-let-eqI*: *b* = (*if* (*n1* = *n2*) *then L-true else L-false*) $\Longrightarrow$
$\Phi \vdash \langle \delta,$ *AS-let x* ((*AE-op Eq* (*V-lit n1*) (*V-lit n2*))) *s*$\rangle \longrightarrow$
$\langle \delta,$ *AS-let x* (*AE-val* (*V-lit b*)) *s*$\rangle$
| *reduce-let-appI*: *Some* (*AF-fundef f* (*AF-fun-typ-none* (*AF-fun-typ z b c* τ *s'*))) = *lookup-fun* Φ *f* $\Longrightarrow$
$\Phi \vdash \langle \delta,$ *AS-let x* ((*AE-app f v*)) *s*$\rangle \longrightarrow \langle \delta,$ *AS-let2 x* $\tau[z::=v]_{\tau v}$ *s'*$[z::=v]_{sv}$ *s*$\rangle$
| *reduce-let-appPI*: *Some* (*AF-fundef f* (*AF-fun-typ-some bv* (*AF-fun-typ z b c* τ *s'*))) = *lookup-fun* Φ *f* $\Longrightarrow$
$\Phi \vdash \langle \delta,$ *AS-let x* ((*AE-appP f b' v*)) *s*$\rangle \longrightarrow \langle \delta,$ *AS-let2 x* $\tau[bv::=b']_{\tau b}[z::=v]_{\tau v}$ *s'*$[bv::=b']_{sb}[z::=v]_{sv}$ *s*$\rangle$
| *reduce-let-fstI*: $\Phi \vdash \langle \delta,$ *AS-let x* (*AE-fst* (*V-pair v1 v2*)) *s*$\rangle \longrightarrow \langle \delta,$ *AS-let x* (*AE-val v1*) *s*$\rangle$
| *reduce-let-sndI*: $\Phi \vdash \langle \delta,$ *AS-let x* (*AE-snd* (*V-pair v1 v2*)) *s*$\rangle \longrightarrow \langle \delta,$ *AS-let x* (*AE-val v2*) *s*$\rangle$
| *reduce-let-concatI*: $\Phi \vdash \langle \delta,$ *AS-let x* (*AE-concat* (*V-lit* (*L-bitvec v1*)) (*V-lit* (*L-bitvec v2*))) *s*$\rangle \longrightarrow$
$\langle \delta,$ *AS-let x* (*AE-val* (*V-lit* (*L-bitvec* (*v1*@*v2*)))) *s*$\rangle$
| *reduce-let-splitI*: *split n v* (*v1* , *v2*) $\Longrightarrow \Phi \vdash \langle \delta,$ *AS-let x* (*AE-split* (*V-lit* (*L-bitvec v*)) (*V-lit* (*L-num n*))) *s*$\rangle \longrightarrow$
$\langle \delta,$ *AS-let x* (*AE-val* (*V-pair* (*V-lit* (*L-bitvec v1*)) (*V-lit* (*L-bitvec v2*)))) *s*$\rangle$
| *reduce-let-lenI*: $\Phi \vdash \langle \delta,$ *AS-let x* (*AE-len* (*V-lit* (*L-bitvec v*))) *s*$\rangle \longrightarrow$
$\langle \delta,$ *AS-let x* (*AE-val* (*V-lit* (*L-num* (*int* (*List.length v*))))) *s*$\rangle$
| *reduce-let-mvar*: $(u,v) \in$ *set* $\delta \Longrightarrow$ $\Phi \vdash \langle \delta,$ *AS-let x* (*AE-mvar u*) *s*$\rangle \longrightarrow \langle \delta,$ *AS-let x* (*AE-val v*) *s* $\rangle$
| *reduce-assert1I*: $\Phi \vdash \langle \delta,$ *AS-assert c* (*AS-val v*)$\rangle \longrightarrow \langle \delta,$ *AS-val v*$\rangle$
| *reduce-assert2I*: $\Phi \vdash \langle \delta,$ *s* $\rangle \longrightarrow \langle$ *d'*, *s'*$\rangle \Longrightarrow \Phi \vdash \langle \delta,$ *AS-assert c s*$\rangle \longrightarrow \langle$ *d'*, *AS-assert c s'*$\rangle$
| *reduce-varI*: *atom u* $\sharp$ $\delta \Longrightarrow$ $\Phi \vdash \langle \delta,$ *AS-var u* τ *v s* $\rangle \longrightarrow \langle$ ((*u*,*v*)#δ) , *s*$\rangle$
| *reduce-assignI*: $\Phi \vdash \langle \delta,$ *AS-assign u v* $\rangle \longrightarrow \langle$ *update-d* δ *u v* , *AS-val* (*V-lit L-unit*)$\rangle$
| *reduce-seq1I*: $\Phi \vdash \langle \delta,$ *AS-seq* (*AS-val* (*V-lit L-unit*)) *s* $\rangle \longrightarrow \langle \delta,$ *s*$\rangle$
| *reduce-seq2I*: $[\![$*s1* $\neq$ *AS-val v* ; $\Phi \vdash \langle \delta,$ *s1*$\rangle \longrightarrow \langle$ *d'*, *s1'*$\rangle$ $]\!] \Longrightarrow$
$\Phi \vdash \langle \delta,$ *AS-seq s1 s2*$\rangle \longrightarrow \langle$ *d'*, *AS-seq s1' s2*$\rangle$
| *reduce-let2-valI*: $\Phi \vdash \langle \delta,$ *AS-let2 x t* (*AS-val v*) *s*$\rangle \longrightarrow \langle \delta,$ *s*$[x::=v]_{sv}\rangle$
| *reduce-let2I*: $\Phi \vdash \langle \delta,$ *s1* $\rangle \longrightarrow \langle$ *d'*, *s1'*$\rangle \Longrightarrow \Phi \vdash \langle \delta,$ *AS-let2 x t s1 s2*$\rangle \longrightarrow \langle$ *d'*, *AS-let2 x t s1' s2*$\rangle$

| *reduce-caseI*: $[\![$ *Some* (*AS-branch dc x' s'*) = *lookup-branch dc cs* $]\!] \Longrightarrow$ $\Phi \vdash \langle \delta,$ *AS-match* (*V-cons tyid dc v*) *cs*$\rangle \longrightarrow \langle \delta,$ *s'*$[x'::=v]_{sv}\rangle$
| *reduce-whileI*: $[\![$ *atom x* $\sharp$ (*s1*,*s2*); *atom z* $\sharp$ *x* $]\!] \Longrightarrow$
$\Phi \vdash \langle \delta,$ *AS-while s1 s2* $\rangle \longrightarrow$
$\langle \delta,$ *AS-let2 x* ($\{\!|$ *z* : *B-bool* | *TRUE* $|\!\}$) *s1* (*AS-if* (*V-var x*) (*AS-seq s2* (*AS-while s1 s2*)) (*AS-val* (*V-lit L-unit*))) $\rangle$

equivariance *reduce-stmt*
nominal-inductive *reduce-stmt* $\langle$*proof*$\rangle$

inductive-cases *reduce-stmt-elims*[*elim*!]:
$\Phi \vdash \langle \delta,$ *AS-if* (*V-lit L-true*) *s1 s2*$\rangle \longrightarrow \langle \delta$, *s1*$\rangle$
$\Phi \vdash \langle \delta,$ *AS-if* (*V-lit L-false*) *s1 s2*$\rangle \longrightarrow \langle \delta$,*s2*$\rangle$
$\Phi \vdash \langle \delta,$ *AS-let x* (*AE-val v*) *s*$\rangle \longrightarrow \langle \delta$,*s'*$\rangle$
$\Phi \vdash \langle \delta,$ *AS-let x* (*AE-op Plus* ((*V-lit* (*L-num n1*))) ((*V-lit* (*L-num n2*)))) *s*$\rangle \longrightarrow$
$\langle \delta,$ *AS-let x* (*AE-val* (*V-lit* (*L-num* (((*n1*)+(*n2*)))))) *s* $\rangle$
$\Phi \vdash \langle \delta,$ *AS-let x* ((*AE-op LEq* (*V-lit* (*L-num n1*)) (*V-lit* (*L-num n2*)))) *s*$\rangle \longrightarrow \langle \delta,$ *AS-let x* (*AE-val* (*V-lit b*)) *s*$\rangle$
$\Phi \vdash \langle \delta,$ *AS-let x* ((*AE-app f v*)) *s*$\rangle \longrightarrow \langle \delta,$ *AS-let2 x* τ (*subst-sv s' x v*) *s*$\rangle$
$\Phi \vdash \langle \delta,$ *AS-let x* ((*AE-len v*)) *s*$\rangle \longrightarrow \langle \delta,$ *AS-let x v' s*$\rangle$
$\Phi \vdash \langle \delta,$ *AS-let x* ((*AE-concat v1 v2*)) *s*$\rangle \longrightarrow \langle \delta,$ *AS-let x v' s*$\rangle$
$\Phi \vdash \langle \delta,$ *AS-seq s1 s2*$\rangle \longrightarrow \langle$ *δ'* ,*s'*$\rangle$
$\Phi \vdash \langle \delta,$ *AS-let x* ((*AE-appP f b v*)) *s*$\rangle \longrightarrow \langle \delta,$ *AS-let2 x* τ (*subst-sv s' z v*) *s*$\rangle$
$\Phi \vdash \langle \delta,$ *AS-let x* ((*AE-split v1 v2*)) *s*$\rangle \longrightarrow \langle \delta,$ *AS-let x v' s*$\rangle$
$\Phi \vdash \langle \delta,$ *AS-assert c s* $\rangle \longrightarrow \langle \delta,$ *s'*$\rangle$
$\Phi \vdash \langle \delta,$ *AS-let x* ((*AE-op Eq* (*V-lit* (*n1*)) (*V-lit* (*n2*)))) *s*$\rangle \longrightarrow \langle \delta,$ *AS-let x* (*AE-val* (*V-lit b*)) *s*$\rangle$

inductive *reduce-stmt-many* :: $\Phi \Rightarrow \delta \Rightarrow s \Rightarrow \delta \Rightarrow s \Rightarrow$ *bool* ($\langle$- $\vdash \langle$ - , -$\rangle \longrightarrow^* \langle$ - , -$\rangle\rangle$ [*50*, *50*, *50*] *50*) **where**
reduce-stmt-many-oneI: $\Phi \vdash \langle \delta,$ *s*$\rangle \longrightarrow \langle \delta',$ *s'*$\rangle \Longrightarrow \Phi \vdash \langle \delta$, *s*$\rangle \longrightarrow^* \langle \delta',$ *s'*$\rangle$
| *reduce-stmt-many-manyI*: $[\![\Phi \vdash \langle \delta,$ *s*$\rangle \longrightarrow \langle \delta',$ *s'*$\rangle$; $\Phi \vdash \langle \delta',$ *s'*$\rangle \longrightarrow^* \langle \delta'',$ *s''*$\rangle]\!] \Longrightarrow \Phi \vdash \langle \delta,$ *s*$\rangle$ $\longrightarrow^* \langle \delta'',$ *s''*$\rangle$

nominal-function *convert-fds* :: *fun-def list* $\Rightarrow$ (*f*$*$*fun-def*) *list* **where**
convert-fds [] = []
| *convert-fds* ((*AF-fundeff* (*AF-fun-typ-none* (*AF-fun-typ x b c* τ *s*)))#*fs*) = ((*f*,*AF-fundeff* (*AF-fun-typ-none* (*AF-fun-typ x b c* τ *s*)))#*convert-fds fs*)
| *convert-fds* ((*AF-fundeff* (*AF-fun-typ-some bv* (*AF-fun-typ x b c* τ *s*)))#*fs*) = ((*f*,*AF-fundeff* (*AF-fun-typ-some bv* (*AF-fun-typ x b c* τ *s*)))#*convert-fds fs*)
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

nominal-function *convert-tds* :: *type-def list* $\Rightarrow$ (*f*$*$*type-def*) *list* **where**
convert-tds [] = []
| *convert-tds* ((*AF-typedef s dclist*)#*fs*) = ((*s*,*AF-typedef s dclist*)#*convert-tds fs*)
| *convert-tds* ((*AF-typedef-poly s bv dclist*)#*fs*) = ((*s*,*AF-typedef-poly s bv dclist*)#*convert-tds fs*)
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

inductive *reduce-prog* :: *p* $\Rightarrow$ *v* $\Rightarrow$ *bool* **where**
$[\![$ *reduce-stmt-many* Φ [] *s* δ (*AS-val v*) $]\!] \Longrightarrow$ *reduce-prog* (*AP-prog* Θ Φ [] *s*) *v*

10.2 Reduction Typing

Checks that the store is consistent with Δ

inductive *delta-sim* :: $\Theta \Rightarrow \delta \Rightarrow \Delta \Rightarrow$ *bool* (‹- ⊢ - ∼ - › [*50*,*50*] *50*) **where**
delta-sim-nilI: $\Theta \vdash [] \sim []_\Delta$
| *delta-sim-consI*: $[\![\Theta \vdash \delta \sim \Delta$; Θ ; {||} ; *GNil* $\vdash v \Leftarrow \tau$; $u \notin$ *fst* ‘ *set* δ $]\!] \Longrightarrow \Theta \vdash ((u,v)\#\delta) \sim ((u,\tau)\#_\Delta\Delta)$

equivariance *delta-sim*

nominal-inductive *delta-sim* ⟨*proof*⟩

inductive-cases *delta-sim-elims*[*elim*!]:
$\Theta \vdash [] \sim []_\Delta$
$\Theta \vdash ((u,v)\#ds) \sim (u,\tau) \#_\Delta D$
$\Theta \vdash ((u,v)\#ds) \sim D$

A typing judgement that combines typing of the statement, the store and the condition that definitions are well-typed

inductive *config-type* :: $\Theta \Rightarrow \Phi \Rightarrow \Delta \Rightarrow \delta \Rightarrow s \Rightarrow \tau \Rightarrow$ *bool* ($\langle$- ; - ; - $\vdash \langle$ - , -$\rangle \Leftarrow$ - $\rangle$ [*50*, *50*, *50*] *50*) **where**
config-typeI: $[\![\Theta ; \Phi ; \{||\} ; GNil ; \Delta \vdash s \Leftarrow \tau;$
$(\forall fd \in set\ \Phi.\ \Theta ; \Phi \vdash fd) ;$
$\Theta \vdash \delta \sim \Delta]\!]$
$\Longrightarrow \Theta ; \Phi ; \Delta \vdash \langle \delta , s\rangle \Leftarrow \tau$
equivariance *config-type*
nominal-inductive *config-type* ⟨*proof*⟩

inductive-cases *config-type-elims* [*elim*!]:
$\Theta ; \Phi ; \Delta \vdash \langle \delta , s\rangle \Leftarrow \tau$

nominal-function *δ-of* :: *var-def list* $\Rightarrow \delta$ **where**
δ-of [] = []
| *δ-of* ((*AV-def u t v*)#*vs*) = (*u*,*v*) # (*δ-of vs*)
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

inductive *config-type-prog* :: $p \Rightarrow \tau \Rightarrow$ *bool* ($\langle \vdash \langle$ -$\rangle \Leftarrow$ -$\rangle$) **where**
$[\![$
$\Theta ; \Phi ; \Delta$-*of* $\mathcal{G} \vdash \langle$ *δ-of* $\mathcal{G}$, $s\rangle \Leftarrow \tau$
$]\!] \Longrightarrow \vdash \langle$ *AP-prog* $\Theta\ \Phi\ \mathcal{G}\ s\rangle \Leftarrow \tau$

inductive-cases *config-type-prog-elims* [*elim*!]:
$\vdash \langle$ *AP-prog* $\Theta\ \Phi\ \mathcal{G}\ s\rangle \Leftarrow \tau$

end
theory *SubstMethods*

imports *IVSubst WellformedL HOL−Eisbach.Eisbach-Tools*
begin

See Eisbach/Examples.thy as well as Eisbach User Manual.

Freshness for various substitution situations. It seems that if undirected and we throw all the facts at them to try to solve in one shot, the automatic methods are *sometimes* unable to handle the different variants, so some guidance is needed. First we split into subgoals using fresh_prodN and intro conjI

The 'add', for example, will be induction premises that will contain freshness facts or freshness conditions from prior obtains

Use different arguments for different things or just lump into one bucket

method *fresh-subst-mth-aux* **uses** *add* = (

(*match* **conclusion in** *atom z* ♯ (Γ::Γ)[*x*::=*v*]$_{\Gamma v}$ **for** *z x v* Γ ⇒ ‹*auto simp add*: *fresh-subst-gv-if*[*of atom z* Γ *v x*] *add*›)
| (*match* **conclusion in** *atom z* ♯ (*v*′::*v*)[*x*::=*v*]$_{vv}$ **for** *z x v v*′ ⇒ ‹*auto simp add*: *v.fresh fresh-subst-v-if pure-fresh subst-v-v-def add*›)
| (*match* **conclusion in** *atom z* ♯ (*ce*::*ce*)[*x*::=*v*]$_{cev}$ **for** *z x v ce* ⇒ ‹*auto simp add*: *fresh-subst-v-if subst-v-ce-def add*›)
| (*match* **conclusion in** *atom z* ♯ (Δ::Δ)[*x*::=*v*]$_{\Delta v}$ **for** *z x v* Δ ⇒ ‹*auto simp add*: *fresh-subst-v-if fresh-subst-dv-if add*›)
| (*match* **conclusion in** *atom z* ♯ Γ′[*x*::=*v*]$_{\Gamma v}$ @ Γ **for** *z x v* Γ′ Γ ⇒ ‹*metis add* ›)
| (*match* **conclusion in** *atom z* ♯ (τ::τ)[*x*::=*v*]$_{\tau v}$ **for** *z x v* τ ⇒ ‹*auto simp add*: *v.fresh fresh-subst-v-if pure-fresh subst-v-τ-def add*›)
| (*match* **conclusion in** *atom z* ♯ ({||} :: *bv fset*) **for** *z* ⇒ ‹*auto simp add*: *fresh-empty-fset*›)

| (*auto simp add*: *add x-fresh-b pure-fresh*)
)

method *fresh-mth* **uses** *add* = (
(*unfold fresh-prodN*, *intro conjI*)*?*,
(*fresh-subst-mth-aux add*: *add*)+)

notepad
begin
⟨*proof*⟩

end

end

hide-const *Syntax.dom*

Chapter 11

Refinement Constraint Logic Lemmas

11.1 Lemmas

lemma *wfI-domi*:
assumes Θ ; $\Gamma \vdash i$
shows *fst* ' *toSet* $\Gamma \subseteq$ *dom* i
⟨*proof*⟩

lemma *wfI-lookup*:
fixes G::Γ **and** b::b
assumes *Some* (b,c) = *lookup* G x **and** P ; $G \vdash i$ **and** *Some* $s = i\ x$ **and** P ; $B \vdash_{wf} G$
shows $P \vdash s : b$
⟨*proof*⟩

lemma *wfI-restrict-weakening*:
assumes *wfI* Θ Γ' i' **and** $i =$ *restrict-map* i' (*fst* '*toSet* Γ) **and** *toSet* $\Gamma \subseteq$ *toSet* Γ'
shows Θ ; $\Gamma \vdash i$
⟨*proof*⟩

lemma *wfI-suffix*:
fixes G::Γ
assumes *wfI* P $(G'@G)$ i **and** P ; $B \vdash_{wf} G$
shows P ; $G \vdash i$
⟨*proof*⟩

lemma *wfI-replace-inside*:
assumes *wfI* Θ $(\Gamma' @ (x, b, c) \#_\Gamma \Gamma)$ i
shows *wfI* Θ $(\Gamma' @ (x, b, c') \#_\Gamma \Gamma)$ i
⟨*proof*⟩

11.2 Existence of evaluation

lemma *eval-l-base*:
$\Theta \vdash [\![l]\!]$: (*base-for-lit* l)
⟨*proof*⟩

lemma *obtain-fresh-bv-dclist*:
fixes *tm*::'*a*::*fs*
assumes (*dc*, {‖ *x* : *b* | *c* ‖}) ∈ *set dclist*
obtains *bv1*::*bv* **and** *dclist1 x1 b1 c1* **where** *AF-typedef-poly tyid bv dclist* = *AF-typedef-poly tyid bv1 dclist1*
∧ (*dc*, {‖ *x1* : *b1* | *c1* ‖}) ∈ *set dclist1* ∧ *atom bv1* ♯ *tm*
⟨*proof*⟩

lemma *obtain-fresh-bv-dclist-b-iff*:
fixes *tm*::'*a*::*fs*
assumes (*dc*, {‖ *x* : *b* | *c* ‖}) ∈ *set dclist* **and** *AF-typedef-poly tyid bv dclist* ∈ *set P* **and** ⊢$_{wf}$ *P*
obtains *bv1*::*bv* **and** *dclist1 x1 b1 c1* **where** *AF-typedef-poly tyid bv dclist* = *AF-typedef-poly tyid bv1 dclist1*
∧ (*dc*, {‖ *x1* : *b1* | *c1* ‖}) ∈ *set dclist1* ∧ *atom bv1* ♯ *tm* ∧ *b*[*bv*::=*b'*]$_{bb}$=*b1*[*bv1*::=*b'*]$_{bb}$
⟨*proof*⟩

lemma *eval-v-exist*:
fixes Γ::Γ **and** *v*::*v* **and** *b*::*b*
assumes *P* ; Γ ⊢ *i* **and** *P* ; *B* ; Γ ⊢$_{wf}$ *v* : *b*
shows ∃ *s*. *i* ⟦ *v* ⟧ ~ *s* ∧ *P* ⊢ *s* : *b*
⟨*proof*⟩

lemma *eval-v-uniqueness*:
fixes *v*::*v*
assumes *i* ⟦ *v* ⟧ ~ *s* **and** *i* ⟦ *v* ⟧ ~ *s'*
shows *s*=*s'*
⟨*proof*⟩

lemma *eval-v-base*:
fixes Γ::Γ **and** *v*::*v* **and** *b*::*b*
assumes *P* ; Γ ⊢ *i* **and** *P* ; *B* ; Γ ⊢$_{wf}$ *v* : *b* **and** *i* ⟦ *v* ⟧ ~ *s*
shows *P* ⊢ *s* : *b*
⟨*proof*⟩

lemma *eval-e-uniqueness*:
fixes *e*::*ce*
assumes *i* ⟦ *e* ⟧ ~ *s* **and** *i* ⟦ *e* ⟧ ~ *s'*
shows *s*=*s'*
⟨*proof*⟩

lemma *wfV-eval-bitvec*:
fixes *v*::*v*
assumes *P* ; *B* ; Γ ⊢$_{wf}$ *v* : *B-bitvec* **and** *P* ; Γ ⊢ *i*
shows ∃ *bv*. *eval-v i v* (*SBitvec bv*)
⟨*proof*⟩

lemma *wfV-eval-pair*:
fixes *v*::*v*
assumes *P* ; *B* ; Γ ⊢$_{wf}$ *v* : *B-pair b1 b2* **and** *P* ; Γ ⊢ *i*
shows ∃ *s1 s2*. *eval-v i v* (*SPair s1 s2*)
⟨*proof*⟩

lemma *wfV-eval-int*:
fixes v::v
assumes P ; B ; $\Gamma \vdash_{wf} v : B\text{-}int$ **and** P ; $\Gamma \vdash i$
shows $\exists n.\ eval\text{-}v\ i\ v\ (SNum\ n)$
⟨*proof*⟩

Well sorted value with a well sorted valuation evaluates

lemma *wfI-wfV-eval-v*:
fixes v::v **and** b::b
assumes Θ ; B ; $\Gamma \vdash_{wf} v : b$ **and** $wfI\ \Theta\ \Gamma\ i$
shows $\exists s.\ i \llbracket v \rrbracket \sim s \wedge \Theta \vdash s : b$
⟨*proof*⟩

lemma *wfI-wfCE-eval-e*:
fixes e::ce **and** b::b
assumes $wfCE\ P\ B\ G\ e\ b$ **and** P ; $G \vdash i$
shows $\exists s.\ i \llbracket e \rrbracket \sim s \wedge P \vdash s : b$
⟨*proof*⟩

lemma *eval-e-exist*:
fixes Γ::Γ **and** e::ce
assumes P ; $\Gamma \vdash i$ **and** P ; B ; $\Gamma \vdash_{wf} e : b$
shows $\exists s.\ i \llbracket e \rrbracket \sim s$
⟨*proof*⟩

lemma *eval-c-exist*:
fixes Γ::Γ **and** c::c
assumes P ; $\Gamma \vdash i$ **and** P ; B ; $\Gamma \vdash_{wf} c$
shows $\exists s.\ i \llbracket c \rrbracket \sim s$
⟨*proof*⟩

lemma *eval-c-uniqueness*:
fixes c::c
assumes $i \llbracket c \rrbracket \sim s$ **and** $i \llbracket c \rrbracket \sim s'$
shows s=s'
⟨*proof*⟩

lemma *wfI-wfC-eval-c*:
fixes c::c
assumes $wfC\ P\ B\ G\ c$ **and** P ; $G \vdash i$
shows $\exists s.\ i \llbracket c \rrbracket \sim s$
⟨*proof*⟩

11.3 Satisfiability

lemma *satis-reflI*:
fixes c::c
assumes $i \models ((x,\ b,\ c)\ \#_\Gamma\ G)$
shows $i \models c$
⟨*proof*⟩

lemma *is-satis-mp*:
fixes *c1*::*c* **and** *c2*::*c*
assumes $i \models (c1\ IMP\ c2)$ **and** $i \models c1$
shows $i \models c2$
⟨*proof*⟩

lemma *is-satis-imp*:
fixes *c1*::*c* **and** *c2*::*c*
assumes $i \models c1 \longrightarrow i \models c2$ **and** $i \llbracket\ c1\ \rrbracket \sim b1$ **and** $i \llbracket\ c2\ \rrbracket \sim b2$
shows $i \models (c1\ IMP\ c2)$
⟨*proof*⟩

lemma *is-satis-iff*:
$i \models G = (\forall x\ b\ c.\ (x,b,c) \in toSet\ G \longrightarrow i \models c)$
⟨*proof*⟩

lemma *is-satis-g-append*:
$i \models (G1@G2) = (i \models G1 \wedge i \models G2)$
⟨*proof*⟩

11.4 Substitution for Evaluation

lemma *eval-v-i-upd*:
fixes *v*::*v*
assumes $atom\ x \sharp v$ **and** $i \llbracket\ v\ \rrbracket \sim s'$
shows $eval\text{-}v\ ((i\ (\ x \mapsto s)))\ v\ s'$
⟨*proof*⟩

lemma *eval-e-i-upd*:
fixes *e*::*ce*
assumes $i \llbracket\ e\ \rrbracket \sim s'$ **and** $atom\ x \sharp e$
shows $(i\ (\ x \mapsto s)) \llbracket\ e\ \rrbracket \sim s'$
⟨*proof*⟩

lemma *eval-c-i-upd*:
fixes *c*::*c*
assumes $i \llbracket\ c\ \rrbracket \sim s'$ **and** $atom\ x \sharp c$
shows $((i\ (\ x \mapsto s))) \llbracket\ c\ \rrbracket \sim s'$
⟨*proof*⟩

lemma *subst-v-eval-v*[*simp*]:
fixes *v*::*v* **and** *v'*::*v*
assumes $i \llbracket\ v\ \rrbracket \sim s$ **and** $i \llbracket\ (v'[x::=v]_{vv})\ \rrbracket \sim s'$
shows $(i\ (\ x \mapsto s\ ))\ \llbracket\ v'\ \rrbracket \sim s'$
⟨*proof*⟩

lemma *subst-e-eval-v*[*simp*]:
fixes *y*::*x* **and** *e*::*ce* **and** *v*::*v* **and** *e'*::*ce*
assumes $i \llbracket\ e'\ \rrbracket \sim s'$ **and** $e'=(e[y::=v]_{cev})$ **and** $i \llbracket\ v\ \rrbracket \sim s$
shows $(i\ (\ y \mapsto s\ ))\ \llbracket\ e\ \rrbracket \sim s'$
⟨*proof*⟩

lemma *subst-c-eval-v*[*simp*]:
fixes $v::v$ **and** $c :: c$
assumes $i \llbracket v \rrbracket \sim s$ **and** $i \llbracket c[x::=v]_{cv} \rrbracket \sim s1$ **and**
$(i (x \mapsto s)) \llbracket c \rrbracket \sim s2$
shows $s1 = s2$
⟨*proof*⟩

lemma *wfI-upd*:
assumes $wfI\ \Theta\ \Gamma\ i$ **and** $wfRCV\ \Theta\ s\ b$ **and** $wfG\ \Theta\ B\ ((x,\ b,\ c)\ \#_\Gamma\ \Gamma)$
shows $wfI\ \Theta\ ((x,\ b,\ c)\ \#_\Gamma\ \Gamma)\ (i(x \mapsto s))$
⟨*proof*⟩

lemma *wfI-upd-full*:
fixes $v::v$
assumes $wfI\ \Theta\ G\ i$ **and** $G = ((\Gamma'[x::=v]_{\Gamma v})@\Gamma)$ **and** $wfRCV\ \Theta\ s\ b$ **and** $wfG\ \Theta\ B\ (\Gamma'@((x,b,c)\#_\Gamma\Gamma))$
and $\Theta\ ;\ B\ ;\ \Gamma \vdash_{wf} v : b$
shows $wfI\ \Theta\ (\Gamma'@((x,b,c)\#_\Gamma\Gamma))\ (i(x \mapsto s))$
⟨*proof*⟩

lemma *subst-c-satis*[*simp*]:
fixes $v::v$
assumes $i \llbracket v \rrbracket \sim s$ **and** $wfC\ \Theta\ B\ ((x,b,c')\#_\Gamma\Gamma)\ c$ **and** $wfI\ \Theta\ \Gamma\ i$ **and** $\Theta\ ;\ B\ ;\ \Gamma \vdash_{wf} v : b$
shows $i \models (c[x::=v]_{cv}) \longleftrightarrow (i (x \mapsto s)) \models c$
⟨*proof*⟩

Key theorem telling us we can replace a substitution with an update to the valuation

lemma *subst-c-satis-full*:
fixes $v::v$ **and** $\Gamma'::\Gamma$
assumes $i \llbracket v \rrbracket \sim s$ **and** $wfC\ \Theta\ B\ (\Gamma'@((x,b,c')\#_\Gamma\Gamma))\ c$ **and** $wfI\ \Theta\ ((\Gamma'[x::=v]_{\Gamma v})@\Gamma)\ i$ **and** $\Theta\ ;\ B\ ;\ \Gamma \vdash_{wf} v : b$
shows $i \models (c[x::=v]_{cv}) \longleftrightarrow (i (x \mapsto s)) \models c$
⟨*proof*⟩

11.5 Validity

lemma *validI*[*intro*]:
fixes $c::c$
assumes $wfC\ P\ B\ G\ c$ **and** $\forall i.\ P\ ;\ G \vdash i \wedge i \models G \longrightarrow i \models c$
shows $P\ ;\ B\ ;\ G \models c$
⟨*proof*⟩

lemma *valid-g-wf*:
fixes $c::c$ **and** $G::\Gamma$
assumes $P\ ;\ B\ ;\ G \models c$
shows $P\ ;\ B \vdash_{wf} G$
⟨*proof*⟩

lemma *valid-reflI* [*intro*]:
fixes $b::b$
assumes $P\ ;\ B\ ;\ ((x,b,c1)\#_\Gamma G) \vdash_{wf} c1$ **and** $c1 = c2$
shows $P\ ;\ B\ ;\ ((x,b,c1)\#_\Gamma G) \models c2$
⟨*proof*⟩

11.5.1 Weakening and Strengthening

Adding to the domain of a valuation doesn't change the result

lemma *eval-v-weakening*:
fixes c::v **and** B::*bv fset*
assumes $i = i' \lceil ` d$ **and** *supp* $c \subseteq$ *atom* ` $d \cup$ *supp* B **and** $i \llbracket c \rrbracket \sim s$
shows $i' \llbracket c \rrbracket \sim s$
⟨*proof*⟩

lemma *eval-v-restrict*:
fixes c::v **and** B::*bv fset*
assumes $i = i' \mid ` d$ **and** *supp* $c \subseteq$ *atom* ` $d \cup$ *supp* B **and** $i' \llbracket c \rrbracket \sim s$
shows $i \llbracket c \rrbracket \sim s$
⟨*proof*⟩

lemma *eval-e-weakening*:
fixes e::*ce* **and** B::*bv fset*
assumes $i \llbracket e \rrbracket \sim s$ **and** $i = i' \mid ` d$ **and** *supp* $e \subseteq$ *atom* ` $d \cup$ *supp* B
shows $i' \llbracket e \rrbracket \sim s$
⟨*proof*⟩

lemma *eval-e-restrict* :
fixes e::*ce* **and** B::*bv fset*
assumes $i' \llbracket e \rrbracket \sim s$ **and** $i = i' \mid ` d$ **and** *supp* $e \subseteq$ *atom* ` $d \cup$ *supp* B
shows $i \llbracket e \rrbracket \sim s$
⟨*proof*⟩

lemma *eval-c-i-weakening*:
fixes c::c **and** B::*bv fset*
assumes $i \llbracket c \rrbracket \sim s$ **and** $i = i' \mid ` d$ **and** *supp* $c \subseteq$ *atom* ` $d \cup$ *supp* B
shows $i' \llbracket c \rrbracket \sim s$
⟨*proof*⟩

lemma *eval-c-i-restrict*:
fixes c::c **and** B::*bv fset*
assumes $i' \llbracket c \rrbracket \sim s$ **and** $i = i' \mid ` d$ **and** *supp* $c \subseteq$ *atom* ` $d \cup$ *supp* B
shows $i \llbracket c \rrbracket \sim s$
⟨*proof*⟩

lemma *is-satis-i-weakening*:
fixes c::c **and** B::*bv fset*
assumes $i = i' \mid ` d$ **and** *supp* $c \subseteq$ *atom* ` $d \cup$ *supp* B **and** $i \models c$
shows $i' \models c$
⟨*proof*⟩

lemma *is-satis-i-restrict*:
fixes c::c **and** B::*bv fset*
assumes $i = i' \mid ` d$ **and** *supp* $c \subseteq$ *atom* ` $d \cup$ *supp* B **and** $i' \models c$
shows $i \models c$
⟨*proof*⟩

lemma *is-satis-g-restrict1*:

fixes $\Gamma'::\Gamma$ **and** $\Gamma::\Gamma$
assumes *toSet* $\Gamma \subseteq$ *toSet* Γ' **and** $i \models \Gamma'$
shows $i \models \Gamma$
⟨*proof*⟩

lemma *is-satis-g-restrict2*:
fixes $\Gamma'::\Gamma$ **and** $\Gamma::\Gamma$
assumes $i \models \Gamma$ **and** $i' = i \mid' d$ **and** *atom-dom* $\Gamma \subseteq$ *atom* ' d **and** Θ ; $B \vdash_{wf} \Gamma$
shows $i' \models \Gamma$
⟨*proof*⟩

lemma *is-satis-g-restrict*:
fixes $\Gamma'::\Gamma$ **and** $\Gamma::\Gamma$
assumes *toSet* $\Gamma \subseteq$ *toSet* Γ' **and** $i' \models \Gamma'$ **and** $i = i' \mid' (fst$ ' *toSet* $\Gamma)$ **and** Θ ; $B \vdash_{wf} \Gamma$
shows $i \models \Gamma$
⟨*proof*⟩

11.5.2 Updating valuation

lemma *is-satis-c-i-upd*:
fixes $c::c$
assumes *atom* $x \sharp c$ **and** $i \models c$
shows $((i\ (\ x \mapsto s))) \models c$
⟨*proof*⟩

lemma *is-satis-g-i-upd*:
fixes $G::\Gamma$
assumes *atom* $x \sharp G$ **and** $i \models G$
shows $((i\ (\ x \mapsto s))) \models G$
⟨*proof*⟩

lemma *valid-weakening*:
assumes Θ ; B ; $\Gamma \models c$ **and** *toSet* $\Gamma \subseteq$ *toSet* Γ' **and** *wfG* Θ B Γ'
shows Θ ; B ; $\Gamma' \models c$
⟨*proof*⟩

lemma *is-satis-g-suffix*:
fixes $G::\Gamma$
assumes $i \models (G'@G)$
shows $i \models G$
⟨*proof*⟩

lemma *wfG-inside-valid2*:
fixes $x::x$ **and** $\Gamma::\Gamma$ **and** $c0::c$ **and** $c0'::c$
assumes *wfG* Θ B $(\Gamma'@((x,b0,c0')\#_\Gamma\Gamma))$ **and**
Θ ; B ; $\Gamma'@(x,b0,c0)\#_\Gamma\Gamma \models c0'$
shows *wfG* Θ B $(\Gamma'@((x,b0,c0)\#_\Gamma\Gamma))$
⟨*proof*⟩

lemma *valid-trans*:
assumes Θ ; $\mathcal{B}$; $\Gamma \models c0[z::=v]_v$ **and** Θ ; $\mathcal{B}$; $(z,b,c0)\#_\Gamma\Gamma \models c1$ **and** *atom* $z \sharp \Gamma$ **and** *wfV* Θ $\mathcal{B}$ Γ v b
shows Θ ; $\mathcal{B}$; $\Gamma \models c1[z::=v]_v$

⟨*proof*⟩

lemma *valid-trans-full*:
assumes Θ ; $\mathcal{B}$; ((x, b, *c1*[*z1*::=*V-var x*]$_v$) $\#_\Gamma$ Γ) $\models$ *c2*[*z2*::=*V-var x*]$_v$ **and**
Θ ; $\mathcal{B}$; ((x, b, *c2*[*z2*::=*V-var x*]$_v$) $\#_\Gamma$ Γ) $\models$ *c3*[*z3*::=*V-var x*]$_v$
shows Θ ; $\mathcal{B}$; ((x, b, *c1*[*z1*::=*V-var x*]$_v$) $\#_\Gamma$ Γ) $\models$ *c3*[*z3*::=*V-var x*]$_v$
⟨*proof*⟩

lemma *eval-v-weakening-x*:
fixes *c*::*v*
assumes i' ⟦ c ⟧ $\sim$ s **and** *atom* x $\sharp$ c **and** $i = i'$ ($x \mapsto s'$)
shows i ⟦ c ⟧ $\sim$ s
⟨*proof*⟩

lemma *eval-e-weakening-x*:
fixes *c*::*ce*
assumes i' ⟦ c ⟧ $\sim$ s **and** *atom* x $\sharp$ c **and** $i = i'$ ($x \mapsto s'$)
shows i ⟦ c ⟧ $\sim$ s
⟨*proof*⟩

lemma *eval-c-weakening-x*:
fixes *c*::*c*
assumes i' ⟦ c ⟧ $\sim$ s **and** *atom* x $\sharp$ c **and** $i = i'$ ($x \mapsto s'$)
shows i ⟦ c ⟧ $\sim$ s
⟨*proof*⟩

lemma *is-satis-weakening-x*:
fixes *c*::*c*
assumes $i' \models c$ **and** *atom* x $\sharp$ c **and** $i = i'$ ($x \mapsto s$)
shows $i \models c$
⟨*proof*⟩

lemma *is-satis-g-weakening-x*:
fixes *G*::Γ
assumes $i' \models G$ **and** *atom* x $\sharp$ G **and** $i = i'$ ($x \mapsto s$)
shows $i \models G$
⟨*proof*⟩

11.6 Base Type Substitution

The idea of boxing is to take an smt val and its base type and at nodes in the smt val that correspond to type variables we wrap them in an SUt smt val node. Another way of looking at it is that s' where the node for the base type variable is an 'any node'. It is needed to prove subst_b_valid - the base-type variable substitution lemma for validity.

The first *rcl-val* is the expanded form (has type with base-variables replaced with base-type terms) ; the second is its corresponding form

We only have one variable so we need to ensure that in all of the *bs-boxed-BVarI* cases, the s has the same base type.

For example is an SMT value is (SPair (SInt 1) (SBool true)) and it has sort (BPair (BVar x) BBool)[x::=BInt] then the boxed version is SPair (SUt (SInt 1)) (SBool true) and is has sort

(BPair (BVar x) BBool). We need to do this so that we can obtain from a valuation i, that gives values like the first smt value, to a valuation i' that gives values like the second.

inductive *boxed-b* :: $\Theta \Rightarrow$ *rcl-val* $\Rightarrow$ *b* $\Rightarrow$ *bv* $\Rightarrow$ *b* $\Rightarrow$ *rcl-val* $\Rightarrow$ *bool* (‹ - ⊢ - ~ - [- ::= -] \ - › [50,50] 50) **where**
boxed-b-BVar1I: ⟦ *bv* = *bv'* ; *wfRCV P s b* ⟧ ⟹ *boxed-b P s* (*B-var bv'*) *bv b* (*SUt s*)
| *boxed-b-BVar2I*: ⟦ *bv* ≠ *bv'*; *wfRCV P s* (*B-var bv'*) ⟧ ⟹ *boxed-b P s* (*B-var bv'*) *bv b s*
| *boxed-b-BIntI*:*wfRCV P s B-int* ⟹ *boxed-b P s B-int - - s*
| *boxed-b-BBoolI*:*wfRCV P s B-bool* ⟹ *boxed-b P s B-bool - - s*
| *boxed-b-BUnitI*:*wfRCV P s B-unit* ⟹ *boxed-b P s B-unit - - s*
| *boxed-b-BPairI*:⟦ *boxed-b P s1 b1 bv b s1'* ; *boxed-b P s2 b2 bv b s2'* ⟧ ⟹ *boxed-b P* (*SPair s1 s2*) (*B-pair b1 b2*) *bv b* (*SPair s1' s2'*)

| *boxed-b-BConsI*:⟦
AF-typedef tyid dclist ∈ *set P*;
(*dc*, {⟦ *x* : *b* | *c* ⟧}) ∈ *set dclist* ;
boxed-b P s1 b bv b' s1'
⟧ ⟹
boxed-b P (*SCons tyid dc s1*) (*B-id tyid*) *bv b'* (*SCons tyid dc s1'*)

| *boxed-b-BConspI*:⟦ *AF-typedef-poly tyid bva dclist* ∈ *set P*;
atom bva ♯ (*b1*,*bv*,*b'*,*s1*,*s1'*);
(*dc*, {⟦ *x* : *b* | *c* ⟧}) ∈ *set dclist* ;
boxed-b P s1 (*b*[*bva*::=*b1*]$_{bb}$) *bv b' s1'*
⟧ ⟹
boxed-b P (*SConsp tyid dc b1*[*bv*::=*b'*]$_{bb}$ *s1*) (*B-app tyid b1*) *bv b'* (*SConsp tyid dc b1 s1'*)

| *boxed-b-Bbitvec*: *wfRCV P s B-bitvec* ⟹ *boxed-b P s B-bitvec bv b s*

equivariance *boxed-b*
nominal-inductive *boxed-b* ⟨*proof*⟩

inductive-cases *boxed-b-elims*:
boxed-b P s (*B-var bv*) *bv' b s'*
boxed-b P s B-int bv b s'
boxed-b P s B-bool bv b s'
boxed-b P s B-unit bv b s'
boxed-b P s (*B-pair b1 b2*) *bv b s'*
boxed-b P s (*B-id dc*) *bv b s'*
boxed-b P s B-bitvec bv b s'
boxed-b P s (*B-app dc b'*) *bv b s'*

lemma *boxed-b-wfRCV*:
assumes *boxed-b P s b bv b' s'* **and** $\vdash_{wf}$ *P*
shows *wfRCV P s b*[*bv*::=*b'*]$_{bb}$ ∧ *wfRCV P s' b*
⟨*proof*⟩

lemma *subst-b-var*:
assumes *B-var bv2* = *b*[*bv*::=*b'*]$_{bb}$
shows (*b* = *B-var bv* ∧ *b'* = *B-var bv2*) ∨ (*b*=*B-var bv2* ∧ *bv* ≠ *bv2*)
⟨*proof*⟩

Here the valuation i' is the conv wrap version of i. For every x in G, i' x is the conv wrap

version of i x. The next lemma for a clearer explanation of what this is. i produces values of sort b[bv::=b'] and i' produces values of sort b

inductive *boxed-i* :: $\Theta \Rightarrow \Gamma \Rightarrow b \Rightarrow bv \Rightarrow$ *valuation* $\Rightarrow$ *valuation* $\Rightarrow$ *bool* ($\langle$ - ; - ; - , - $\vdash$ - $\approx$ - $\rangle$ [*50*,*50*] *50*) **where**
  boxed-i-GNilI: Θ ; *GNil* ; *b* , *bv* $\vdash$ *i* $\approx$ *i*
| *boxed-i-GConsI*: $[\![$ *Some s* = *i x*; *boxed-b* Θ *s b bv b' s'* ; Θ ; Γ ; *b'* , *bv* $\vdash$ *i* $\approx$ *i'* $]\!]$ $\Longrightarrow$ Θ ; ((*x*,*b*,*c*)#$_\Gamma\Gamma$) ; *b'* , *bv* $\vdash$ *i* $\approx$ (*i'*(*x* $\mapsto$ *s'*))
equivariance *boxed-i*
nominal-inductive *boxed-i* ⟨*proof*⟩

inductive-cases *boxed-i-elims*:
  Θ ;*GNil* ; *b* , *bv* $\vdash$ *i* $\approx$ *i'*
  Θ ; ((*x*,*b*,*c*)#$_\Gamma\Gamma$) ; *b'* , *bv* $\vdash$ *i* $\approx$ *i'*

lemma *wfRCV-poly-elims*:
  fixes *tm*::'*a*::*fs* **and** *b*::*b*
  assumes *T* $\vdash$ *SConsp typid dc bdc s* : *b*
  obtains *bva dclist x1 b1 c1* **where** *b* = *B-app typid bdc* $\wedge$
   AF-typedef-poly typid bva dclist $\in$ *set T* $\wedge$ (*dc*, {$\!|$ *x1* : *b1* | *c1* $|\!$}) $\in$ *set dclist* $\wedge$ *T* $\vdash$ *s* : *b1*[*bva*::=*bdc*]$_{bb}$ $\wedge$ *atom bva* $\sharp$ *tm*
  ⟨*proof*⟩

lemma *boxed-b-ex*:
  assumes *wfRCV T s b*[*bv*::=*b'*]$_{bb}$ **and** *wfTh T*
  shows $\exists$ *s'*. *boxed-b T s b bv b' s'*
  ⟨*proof*⟩

lemma *boxed-i-ex*:
  assumes *wfI T* Γ[*bv*::=*b*]$_{\Gamma b}$ *i* **and** *wfTh T*
  shows $\exists$ *i'*. *T* ; Γ ; *b* , *bv* $\vdash$ *i* $\approx$ *i'*
  ⟨*proof*⟩

lemma *boxed-b-eq*:
  assumes *boxed-b* Θ *s1 b bv b' s1'* **and** $\vdash_{wf}$ Θ
  shows *wfTh* Θ $\Longrightarrow$ *boxed-b* Θ *s2 b bv b' s2'* $\Longrightarrow$ (*s1* = *s2*) = (*s1'* = *s2'*)
  ⟨*proof*⟩

lemma *bs-boxed-var*:
  assumes *boxed-i* Θ Γ *b' bv i i'*
  shows *Some* (*b*,*c*) = *lookup* Γ *x* $\Longrightarrow$ *Some s* = *i x* $\Longrightarrow$ *Some s'* = *i' x* $\Longrightarrow$ *boxed-b* Θ *s b bv b' s'*
  ⟨*proof*⟩

lemma *eval-l-boxed-b*:
  assumes $[\![$ *l* $]\!]$ = *s*
  shows *boxed-b* Θ *s* (*base-for-lit l*) *bv b' s*
  ⟨*proof*⟩

lemma *boxed-i-eval-v-boxed-b*:
  fixes *v*::*v*
  assumes *boxed-i* Θ Γ *b' bv i i'* **and** *i* $[\![$ *v*[*bv*::=*b'*]$_{vb}$ $]\!]$ $\sim$ *s* **and** *i'* $[\![$ *v* $]\!]$ $\sim$ *s'* **and** *wfV* Θ *B* Γ *v b* **and** *wfI* Θ Γ *i'*
  shows *boxed-b* Θ *s b bv b' s'*

⟨*proof*⟩

lemma *boxed-b-eq-eq*:
assumes *boxed-b* Θ *n1 b1 bv b′ n1′* **and** *boxed-b* Θ *n2 b1 bv b′ n2′* **and** $s = SBool\ (n1 = n2)$ **and** $\vdash_{wf} \Theta$
$s' = SBool\ (n1' = n2')$
shows $s{=}s'$
⟨*proof*⟩

lemma *boxed-i-eval-ce-boxed-b*:
fixes *e*::*ce*
assumes $i' \llbracket\ e\ \rrbracket \sim s'$ **and** $i \llbracket\ e[bv::=b']_{ceb}\ \rrbracket \sim s$ **and** *wfCE* Θ *B* Γ *e b* **and** *boxed-i* Θ Γ *b′ bv i i′* **and** *wfI* Θ Γ *i′*
shows *boxed-b* Θ *s b bv b′ s′*
⟨*proof*⟩

lemma *eval-c-eq-bs-boxed*:
fixes *c*::*c*
assumes $i \llbracket\ c[bv::=b]_{cb}\ \rrbracket \sim s$ **and** $i' \llbracket\ c\ \rrbracket \sim s'$ **and** *wfC* Θ *B* Γ *c* **and** *wfI* Θ Γ *i′* **and** $\Theta\ ;\ \Gamma[bv::=b]_{\Gamma b} \vdash i$
and *boxed-i* Θ Γ *b bv i i′*
shows $s = s'$
⟨*proof*⟩

lemma *is-satis-bs-boxed*:
fixes *c*::*c*
assumes *boxed-i* Θ Γ *b bv i i′* **and** *wfC* Θ *B* Γ *c* **and** *wfI* Θ $\Gamma[bv::=b]_{\Gamma b}$ *i* **and** $\Theta\ ;\ \Gamma \vdash i'$
and $(i \models c[bv::=b]_{cb})$
shows $(i' \models c)$
⟨*proof*⟩

lemma *is-satis-bs-boxed-rev*:
fixes *c*::*c*
assumes *boxed-i* Θ Γ *b bv i i′* **and** *wfC* Θ *B* Γ *c* **and** *wfI* Θ $\Gamma[bv::=b]_{\Gamma b}$ *i* **and** $\Theta\ ;\ \Gamma \vdash i'$ **and** *wfC* Θ $\{||\}$ $\Gamma[bv::=b]_{\Gamma b}$ $(c[bv::=b]_{cb})$
and $(i' \models c)$
shows $(i \models c[bv::=b]_{cb})$
⟨*proof*⟩

lemma *bs-boxed-wfi-aux*:
fixes *b*::*b* **and** *bv*::*bv* **and** Θ::Θ **and** *B*::$\mathcal{B}$
assumes *boxed-i* Θ Γ *b bv i i′* **and** *wfI* Θ $\Gamma[bv::=b]_{\Gamma b}$ *i* **and** $\vdash_{wf} \Theta$ **and** *wfG* Θ *B* Γ
shows $\Theta\ ;\ \Gamma \vdash i'$
⟨*proof*⟩

lemma *is-satis-g-bs-boxed-aux*:
fixes *G*::Γ
assumes *boxed-i* Θ *G1 b bv i i′* **and** *wfI* Θ $G1[bv::=b]_{\Gamma b}$ *i* **and** *wfI* Θ *G1 i′* **and** $G1 = (G2@G)$ **and** *wfG* Θ *B G1*
and $(i \models G[bv::=b]_{\Gamma b})$
shows $(i' \models G)$
⟨*proof*⟩

lemma *is-satis-g-bs-boxed*:
fixes G::Γ
assumes *boxed-i* Θ G b bv i i' **and** *wfI* Θ $G[bv::=b]_{\Gamma b}$ i **and** *wfI* Θ G i' **and** *wfG* Θ B G
and $(i \models G[bv::=b]_{\Gamma b})$
shows $(i' \models G)$
⟨*proof*⟩

lemma *subst-b-valid*:
fixes s::s **and** b::b
assumes Θ ; $\{||\} \vdash_{wf} b$ **and** $B = \{|bv|\}$ **and** Θ ; $\{|bv|\}$;$\Gamma \models c$
shows Θ ; $\{||\}$; $\Gamma[bv::=b]_{\Gamma b} \models c[bv::=b]_{cb}$
⟨*proof*⟩

11.7 Expression Operator Lemmas

lemma *is-satis-len-imp*:
assumes $i \models$ (*CE-val* (*V-var* x) == *CE-val* (*V-lit* (*L-num* (*int* (*length* v))))) (**is** *is-satis* i *?c1*)
shows $i \models$ (*CE-val* (*V-var* x) == *CE-len* [*V-lit* (*L-bitvec* v)]^{ce})
⟨*proof*⟩

lemma *is-satis-plus-imp*:
assumes $i \models$ (*CE-val* (*V-var* x) == *CE-val* (*V-lit* (*L-num* (*n1*+*n2*)))) (**is** *is-satis* i *?c1*)
shows $i \models$ (*CE-val* (*V-var* x) == *CE-op Plus* ([*V-lit* (*L-num n1*)]^{ce}) ([*V-lit* (*L-num n2*)]^{ce}))
⟨*proof*⟩

lemma *is-satis-leq-imp*:
assumes $i \models$ (*CE-val* (*V-var* x) == *CE-val* (*V-lit* (*if* (*n1* ≤ *n2*) *then L-true else L-false*))) (**is** *is-satis* i *?c1*)
shows $i \models$ (*CE-val* (*V-var* x) == *CE-op LEq* [(*V-lit* (*L-num n1*))]^{ce} [(*V-lit* (*L-num n2*))]^{ce})
⟨*proof*⟩

lemma *eval-lit-inj*:
fixes *n1*::l **and** *n2*::l
assumes ⟦ *n1* ⟧ = s **and** ⟦ *n2* ⟧ = s
shows *n1*=*n2*
⟨*proof*⟩

lemma *eval-e-lit-inj*:
fixes *n1*::l **and** *n2*::l
assumes i ⟦ [[*n1*]^{v}]^{ce} ⟧ ~ s **and** i ⟦ [[*n2*]^{v}]^{ce} ⟧ ~ s
shows *n1*=*n2*
⟨*proof*⟩

lemma *is-satis-eq-imp*:
assumes $i \models$ (*CE-val* (*V-var* x) == *CE-val* (*V-lit* (*if* (*n1* = *n2*) *then L-true else L-false*))) (**is** *is-satis* i *?c1*)
shows $i \models$ (*CE-val* (*V-var* x) == *CE-op Eq* [(*V-lit* (*n1*))]^{ce} [(*V-lit* (*n2*))]^{ce})
⟨*proof*⟩

lemma *valid-eq-e*:
assumes ∀ i *s1 s2*. *wfG P* $\mathcal{B}$ *GNil* ∧ *wfI P GNil i* ∧ *eval-e i e1 s1* ∧ *eval-e i e2 s2* ⟶ *s1* = *s2*

and *wfCE P* $\mathcal{B}$ *GNil e1 b* **and** *wfCE P* $\mathcal{B}$ *GNil e2 b*
shows *P* ; $\mathcal{B}$; (*x*, *b* , *CE-val* (*V-var x*) == *e1*)#$_\Gamma$ *GNil* $\models$ *CE-val* (*V-var x*) == *e2*
⟨*proof*⟩

lemma *valid-len*:
assumes $\vdash_{wf}$ Θ
shows Θ ; $\mathcal{B}$; (*x*, *B-int*, [[*x*]^v]^{ce} == [[*L-num* (*int* (*length v*))]^v]^{ce}) #$_\Gamma$ *GNil* $\models$ [[*x*]^v]^{ce} == *CE-len* [[*L-bitvec v*]^v]^{ce} (**is** Θ ; $\mathcal{B}$; *?G* $\models$ *?c*)
⟨*proof*⟩

lemma *valid-arith-bop*:
assumes *wfG* Θ $\mathcal{B}$ Γ **and** *opp* = *Plus* $\wedge$ *ll* = (*L-num* (*n1*+*n2*)) $\vee$ (*opp* = *LEq* $\wedge$ *ll* = (*if n1*≤*n2 then L-true else L-false*))
and (*opp* = *Plus* $\longrightarrow$ *b* = *B-int*) $\wedge$ (*opp* = *LEq* $\longrightarrow$ *b* = *B-bool*) **and**
atom x $\sharp$ Γ
shows Θ; $\mathcal{B}$; (*x*, *b*, (*CE-val* (*V-var x*) == *CE-val* (*V-lit* (*ll*)))) #$_\Gamma$ Γ
$\models$ (*CE-val* (*V-var x*) == *CE-op opp* ([*V-lit* (*L-num n1*)]^{ce}) ([*V-lit* (*L-num n2*)]^{ce})) (**is** Θ ; $\mathcal{B}$; *?G* $\models$ *?c*)
⟨*proof*⟩

lemma *valid-eq-bop*:
assumes *wfG* Θ $\mathcal{B}$ Γ **and** *atom x* $\sharp$ Γ **and** *base-for-lit l1* = *base-for-lit l2*
shows Θ; $\mathcal{B}$; (*x*, *B-bool*, (*CE-val* (*V-var x*) == *CE-val* (*V-lit* (*if l1* = *l2 then L-true else L-false*)))) #$_\Gamma$ Γ
$\models$ (*CE-val* (*V-var x*) == *CE-op Eq* ([*V-lit* (*l1*)]^{ce}) ([*V-lit* (*l2*)]^{ce})) (**is** Θ ; $\mathcal{B}$; *?G* $\models$ *?c*)
⟨*proof*⟩

lemma *valid-fst*:
fixes *x*::*x* **and** v_1::*v* **and** v_2::*v*
assumes *wfTh* Θ **and** *wfV* Θ $\mathcal{B}$ *GNil* (*V-pair* v_1 v_2) (*B-pair* b_1 b_2)
shows Θ ; $\mathcal{B}$; (*x*, b_1, [[*x*]^v]^{ce} == [v_1]^{ce}) #$_\Gamma$ *GNil* $\models$ [[*x*]^v]^{ce} == [#*1*[[v_1,v_2]^v]^{ce}]^{ce}
⟨*proof*⟩

lemma *valid-snd*:
fixes *x*::*x* **and** v_1::*v* **and** v_2::*v*
assumes *wfTh* Θ **and** *wfV* Θ $\mathcal{B}$ *GNil* (*V-pair* v_1 v_2) (*B-pair* b_1 b_2)
shows Θ ; $\mathcal{B}$; (*x*, b_2, [[*x*]^v]^{ce} == [v_2]^{ce}) #$_\Gamma$ *GNil* $\models$ [[*x*]^v]^{ce} == [#*2*[[v_1,v_2]^v]^{ce}]^{ce}
⟨*proof*⟩

lemma *valid-concat*:
fixes *v1*::*bit list* **and** *v2*::*bit list*
assumes $\vdash_{wf}$ Π
shows Π ; $\mathcal{B}$; (*x*, *B-bitvec*, (*CE-val* (*V-var x*) == *CE-val* (*V-lit* (*L-bitvec* (*v1*@ *v2*))))) #$_\Gamma$ *GNil* $\models$
(*CE-val* (*V-var x*) == *CE-concat* ([*V-lit* (*L-bitvec v1*)]^{ce}) ([*V-lit* (*L-bitvec v2*)]^{ce}))
⟨*proof*⟩

lemma *valid-ce-eq*:
fixes *ce*::*ce*
assumes Θ ; $\mathcal{B}$; Γ $\vdash_{wf}$ *ce* : *b*
shows ‹Θ ; $\mathcal{B}$; Γ $\models$ *ce* == *ce* ›
⟨*proof*⟩

lemma *valid-eq-imp*:
fixes *c1*::*c* **and** *c2*::*c*
assumes Θ ; $\mathcal{B}$; (*x*, *b*, *c2*) $\#_\Gamma$ Γ $\vdash_{wf}$ *c1 IMP c2*
shows Θ ; $\mathcal{B}$; (*x*, *b*, *c2*) $\#_\Gamma$ Γ $\models$ *c1 IMP c2*
⟨*proof*⟩

lemma *valid-range*:
assumes *0* $\leq$ *n* $\wedge$ *n* $\leq$ *m* **and** $\vdash_{wf}$ Θ
shows Θ ; {||} ; (*x*, *B-int* , (*C-eq* (*CE-val* (*V-var x*)) (*CE-val* (*V-lit* (*L-num n*))))) $\#_\Gamma$ *GNil* $\models$
(*C-eq* (*CE-op LEq* (*CE-val* (*V-var x*)) (*CE-val* (*V-lit* (*L-num m*)))) [[*L-true*]^v]^{ce}) *AND*
(*C-eq* (*CE-op LEq* (*CE-val* (*V-lit* (*L-num 0*))) (*CE-val* (*V-var x*))) [[*L-true*]^v]^{ce})
(**is** Θ ; {||} ; *?G* $\models$ *?c1 AND ?c2*)
⟨*proof*⟩

lemma *valid-range-length*:
fixes Γ::Γ
assumes *0* $\leq$ *n* $\wedge$ *n* $\leq$ *int* (*length v*) **and** Θ ; {||} $\vdash_{wf}$ Γ **and** *atom x* $\sharp$ Γ
shows Θ ; {||} ; (*x*, *B-int* , (*C-eq* (*CE-val* (*V-var x*)) (*CE-val* (*V-lit* (*L-num n*))))) $\#_\Gamma$ Γ $\models$
(*C-eq* (*CE-op LEq* (*CE-val* (*V-lit* (*L-num 0*))) (*CE-val* (*V-var x*))) [[*L-true*]^v]^{ce}) *AND*
(*C-eq* (*CE-op LEq* (*CE-val* (*V-var x*)) ([| [[*L-bitvec v*]^v]^{ce} |]^{ce})) [[*L-true*]^v]^{ce})

(**is** Θ ; {||} ; *?G* $\models$ *?c1 AND ?c2*)
⟨*proof*⟩

lemma *valid-range-length-inv-gnil*:
fixes Γ::Γ
assumes $\vdash_{wf}$ Θ
and Θ ; {||} ; (*x*, *B-int* , (*C-eq* (*CE-val* (*V-var x*)) (*CE-val* (*V-lit* (*L-num n*))))) $\#_\Gamma$ *GNil* $\models$
(*C-eq* (*CE-op LEq* (*CE-val* (*V-lit* (*L-num 0*))) (*CE-val* (*V-var x*))) [[*L-true*]^v]^{ce}) *AND*
(*C-eq* (*CE-op LEq* (*CE-val* (*V-var x*)) ([| [[*L-bitvec v*]^v]^{ce} |]^{ce})) [[*L-true*]^v]^{ce})

(**is** Θ ; {||} ; *?G* $\models$ *?c1 AND ?c2*)
shows *0* $\leq$ *n* $\wedge$ *n* $\leq$ *int* (*length v*)
⟨*proof*⟩

lemma *wfI-cons*:
fixes *i*::*valuation* **and** Γ::Γ
assumes *i'* $\models$ Γ **and** Θ ; Γ $\vdash$ *i'* **and** *i* = *i'* (*x* $\mapsto$ *s*) **and** Θ $\vdash$ *s* : *b* **and** *atom x* $\sharp$ Γ
shows Θ ; (*x*,*b*,*c*) $\#_\Gamma$ Γ $\vdash$ *i*
⟨*proof*⟩

lemma *valid-range-length-inv*:
fixes Γ::Γ
assumes Θ ; *B* $\vdash_{wf}$ Γ **and** *atom x* $\sharp$ Γ **and** $\exists$ *i*. *i* $\models$ Γ $\wedge$ Θ ; Γ $\vdash$ *i*
and Θ ; *B* ; (*x*, *B-int* , (*C-eq* (*CE-val* (*V-var x*)) (*CE-val* (*V-lit* (*L-num n*))))) $\#_\Gamma$ Γ $\models$
(*C-eq* (*CE-op LEq* (*CE-val* (*V-lit* (*L-num 0*))) (*CE-val* (*V-var x*))) [[*L-true*]^v]^{ce}) *AND*

$(C\text{-}eq\ (CE\text{-}op\ LEq\ (CE\text{-}val\ (V\text{-}var\ x))\ ([|\ [\ [\ L\text{-}bitvec\ v\ ]^v\ ]^{ce}\ |]^{ce}\ ))\ \ [[\ L\text{-}true\ ]^v\ ]^{ce})$

(**is** Θ ; *?B* ; *?G* $\models$ *?c1 AND ?c2*)
shows $0 \leq n \wedge n \leq int\ (length\ v)$
⟨*proof*⟩

lemma *eval-c-conj2I*[*intro*]:
assumes $i \llbracket\ c1\ \rrbracket \sim True$ **and** $i \llbracket\ c2\ \rrbracket \sim True$
shows $i \llbracket\ (C\text{-}conj\ c1\ c2)\ \rrbracket \sim True$
⟨*proof*⟩

lemma *valid-split*:
assumes *split n v* (*v1*,*v2*) **and** $\vdash_{wf} \Theta$
shows Θ ; {||} ; (*z* , [*B-bitvec* , *B-bitvec*]^b , [[*z*]^v]^{ce} == [[[*L-bitvec v1*]^v , [*L-bitvec v2*]^v]^v]^{ce}) $\#_\Gamma$ *GNil*
$\models$ ([[*L-bitvec v*]^v]^{ce} == [[*#1*[[*z*]^v]^{ce}]^{ce} @@ [*#2*[[*z*]^v]^{ce}]^{ce}]^{ce}) AND ([| [*#1*[[*z*]^v]^{ce}]^{ce} |]^{ce} == [[*L-num n*]^v]^{ce})
(**is** Θ ; {||} ; *?G* $\models$ *?c1 AND ?c2*)
⟨*proof*⟩

lemma *is-satis-eq*:
assumes *wfI* Θ *G i* **and** *wfCE* Θ $\mathcal{B}$ *G e b*
shows *is-satis i* (*e* == *e*)
⟨*proof*⟩

lemma *is-satis-g-i-upd2*:
assumes *eval-v i v s* **and** *is-satis* ((*i* (*x* $\mapsto$ *s*))) *c0* **and** *atom x* $\sharp$ *G* **and** *wfG* Θ $\mathcal{B}$ (*G3*@((*x*,*b*,*c0*)$\#_\Gamma$*G*)) **and** *wfV* Θ $\mathcal{B}$ *G v b* **and** *wfI* Θ (*G3*[*x*::=*v*]$_{\Gamma v}$@*G*) *i*
and *is-satis-g i* (*G3*[*x*::=*v*]$_{\Gamma v}$@*G*)
shows *is-satis-g* (*i* (*x* $\mapsto$ *s*)) (*G3*@((*x*,*b*,*c0*)$\#_\Gamma$*G*))
⟨*proof*⟩

end

Chapter 12

Typing Lemmas

12.1 Prelude

Needed as the typing elimination rules give us facts for an alpha-equivalent version of a term and so need to be able to 'jump back' to a typing judgement for the orginal term

lemma *τ-fresh-c*[*simp*]:
assumes *atom x* ♯ {| *z* : *b* | *c* |} **and** *atom z* ♯ *x*
shows *atom x* ♯ *c*
⟨*proof*⟩

lemmas *subst-defs* = *subst-b-b-def subst-b-c-def subst-b-τ-def subst-v-v-def subst-v-c-def subst-v-τ-def*

lemma *wfT-wfT-if1*:
assumes *wfT* Θ $\mathcal{B}$ Γ ({| *z* : *b-of t* | *CE-val v* == *CE-val* (*V-lit L-false*) *IMP c-of t z* |}) **and** *atom z* ♯ (Γ,*t*)
shows *wfT* Θ $\mathcal{B}$ Γ *t*
⟨*proof*⟩

lemma *fresh-u-replace-true*:
fixes *bv*::*bv* **and** Γ::Γ
assumes *atom bv* ♯ Γ′ @ (*x*, *b*, *c*) #$_\Gamma$ Γ
shows *atom bv* ♯ Γ′ @ (*x*, *b*, *TRUE*) #$_\Gamma$ Γ
⟨*proof*⟩

lemma *wf-replace-true1*:
fixes Γ::Γ **and** Φ::Φ **and** Θ::Θ **and** Γ′::Γ **and** *v*::*v* **and** *e*::*e* **and** *c*::*c* **and** *c″*::*c* **and** *c′*::*c* **and** *τ*::*τ* **and** *ts*::(*string*∗*τ*) *list* **and** Δ::Δ **and** *b′*::*b* **and** *b*::*b* **and** *s*::*s*
and *ftq*::*fun-typ-q* **and** *ft*::*fun-typ* **and** *ce*::*ce* **and** *td*::*type-def* **and** *cs*::*branch-s* **and** *css*::*branch-list*

shows Θ; $\mathcal{B}$; *G* ⊢$_{wf}$ *v* : *b′* ⟹ *G* = Γ′ @ (*x*, *b*, *c*) #$_\Gamma$ Γ ⟹ Θ ; $\mathcal{B}$; Γ′ @ ((*x*, *b*, *TRUE*) #$_\Gamma$ Γ) ⊢$_{wf}$ *v* : *b′* **and**
Θ; $\mathcal{B}$; *G* ⊢$_{wf}$ *c″* ⟹ *G* = Γ′ @(*x*, *b*, *c*) #$_\Gamma$ Γ ⟹ Θ ; $\mathcal{B}$; Γ′ @ ((*x*, *b*, *TRUE*) #$_\Gamma$ Γ) ⊢$_{wf}$ *c″* **and**
Θ ; $\mathcal{B}$ ⊢$_{wf}$ *G* ⟹ *G* = Γ′ @(*x*, *b*, *c*) #$_\Gamma$ Γ ⟹ Θ ; $\mathcal{B}$ ⊢$_{wf}$ Γ′ @ ((*x*, *b*, *TRUE*) #$_\Gamma$ Γ) **and**
Θ; $\mathcal{B}$; *G* ⊢$_{wf}$ *τ* ⟹ *G* = Γ′ @(*x*, *b*, *c*) #$_\Gamma$ Γ ⟹ Θ ; $\mathcal{B}$; Γ′ @ ((*x*, *b*, *TRUE*) #$_\Gamma$ Γ) ⊢$_{wf}$ *τ* **and**
Θ; $\mathcal{B}$; Γ ⊢$_{wf}$ *ts* ⟹ *True* **and**
⊢$_{wf}$ *P* ⟹ *True* **and**
Θ ; $\mathcal{B}$ ⊢$_{wf}$ *b* ⟹ *True* **and**

Θ ; $\mathcal{B}$; G $\vdash_{wf}$ ce : $b' \Longrightarrow$ $G =$ Γ' @$(x,\ b,\ c)$ $\#_\Gamma$ $\Gamma \Longrightarrow$ Θ ; $\mathcal{B}$; Γ' @ $((x,\ b,\ TRUE)$ $\#_\Gamma$ $\Gamma) \vdash_{wf}$ ce : b' **and**
Θ $\vdash_{wf}$ $td \Longrightarrow$ $True$
⟨*proof*⟩

lemma *wf-replace-true2*:
fixes Γ::Γ **and** Φ::Φ **and** Θ::Θ **and** Γ'::Γ **and** v::v **and** e::e **and** c::c **and** c''::c **and** c'::c **and** τ::τ **and** ts::(*string*$*\tau$) *list* **and** Δ::Δ **and** b'::b **and** b::b **and** s::s
and *ftq*::*fun-typ-q* **and** *ft*::*fun-typ* **and** *ce*::*ce* **and** *td*::*type-def* **and** *cs*::*branch-s* **and** *css*::*branch-list*

shows Θ ; Φ ; $\mathcal{B}$; G ; D $\vdash_{wf}$ e : $b' \Longrightarrow$ $G =$ Γ' @$(x,\ b,\ c)$ $\#_\Gamma$ $\Gamma \Longrightarrow$ Θ ; Φ ; $\mathcal{B}$; Γ' @ $((x,\ b,$ $TRUE)$ $\#_\Gamma$ $\Gamma)$; $D \vdash_{wf}$ e : b' **and**
Θ ; Φ ; $\mathcal{B}$; G ; $\Delta \vdash_{wf}$ s : $b' \Longrightarrow$ $G =$ Γ' @$(x,\ b,\ c)$ $\#_\Gamma$ $\Gamma \Longrightarrow \Theta$; Φ ; $\mathcal{B}$; Γ' @ $((x,\ b,\ TRUE)$ $\#_\Gamma$ $\Gamma)$; $\Delta \vdash_{wf}$ s : b' **and**
Θ ; Φ ; $\mathcal{B}$; G ; Δ ; tid ; dc ; $t \vdash_{wf}$ cs : $b' \Longrightarrow$ $G =$ Γ' @$(x,\ b,\ c)$ $\#_\Gamma$ $\Gamma \Longrightarrow \Theta$; Φ ; $\mathcal{B}$; Γ' @ $((x,$ $b,\ TRUE)$ $\#_\Gamma$ $\Gamma)$; Δ ; tid ; dc ; $t \vdash_{wf}$ cs : b' **and**
Θ ; Φ ; $\mathcal{B}$; G ; Δ ; tid ; $dclist \vdash_{wf}$ css : $b' \Longrightarrow$ $G =$ Γ' @$(x,\ b,\ c)$ $\#_\Gamma$ $\Gamma \Longrightarrow \Theta$; Φ ; $\mathcal{B}$; Γ' @ $((x,\ b,\ TRUE)$ $\#_\Gamma$ $\Gamma)$; Δ ; tid ; $dclist \vdash_{wf}$ css : b' **and**

$\Theta \vdash_{wf}$ $\Phi \Longrightarrow$ $True$ **and**
Θ; $\mathcal{B}$; G $\vdash_{wf}$ $\Delta \Longrightarrow$ $G =$ Γ' @$(x,\ b,\ c)$ $\#_\Gamma$ $\Gamma \Longrightarrow$ Θ ; $\mathcal{B}$; Γ' @ $((x,\ b,\ TRUE)$ $\#_\Gamma$ $\Gamma) \vdash_{wf}$ Δ **and**

Θ ; Φ $\vdash_{wf}$ $ftq \Longrightarrow$ $True$ **and**
Θ ; Φ ; $\mathcal{B} \vdash_{wf}$ $ft \Longrightarrow$ $True$
⟨*proof*⟩

lemmas *wf-replace-true* = *wf-replace-true1 wf-replace-true2*

12.2 Subtyping

lemma *subtype-reflI2*:
fixes τ::τ
assumes Θ; $\mathcal{B}$; $\Gamma \vdash_{wf}$ τ
shows Θ; $\mathcal{B}$; $\Gamma \vdash \tau \lesssim \tau$
⟨*proof*⟩

lemma *subtype-reflI*:
assumes $\{\!|\ z1 : b \mid c1\ |\!\}$ = $\{\!|\ z2 : b \mid c2\ |\!\}$ **and** *wf1*: Θ; $\mathcal{B}$;$\Gamma \vdash_{wf}$ $(\{\!|\ z1 : b \mid c1\ |\!\})$
shows Θ; $\mathcal{B}$; $\Gamma \vdash$ $(\{\!|\ z1 : b \mid c1\ |\!\})$ $\lesssim$ $(\{\!|\ z2 : b \mid c2\ |\!\})$
⟨*proof*⟩

nominal-function *base-eq* :: $\Gamma \Rightarrow \tau \Rightarrow \tau \Rightarrow$ *bool* **where**
base-eq - $\{\!|\ z1 : b1 \mid c1\ |\!\}$ $\{\!|\ z2 : b2 \mid c2\ |\!\}$ = $(b1 = b2)$
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

lemma *subtype-wfT*:
fixes *t1*::τ **and** *t2*::τ
assumes Θ; $\mathcal{B}$; $\Gamma \vdash$ *t1* $\lesssim$ *t2*
shows Θ; $\mathcal{B}$; $\Gamma \vdash_{wf}$ *t1* $\wedge$ Θ; $\mathcal{B}$; $\Gamma \vdash_{wf}$ *t2*
⟨*proof*⟩

lemma *subtype-eq-base*:
assumes Θ; $\mathcal{B}$; $\Gamma \vdash$ ($\{\!|$ *z1* : *b1*| *c1* $|\!\}$) $\lesssim$ ($\{\!|$ *z2* : *b2* | *c2* $|\!\}$)
shows *b1*=*b2*
⟨*proof*⟩

lemma *subtype-eq-base2*:
assumes Θ; $\mathcal{B}$; $\Gamma \vdash$ *t1* $\lesssim$ *t2*
shows *b-of t1* = *b-of t2*
⟨*proof*⟩

lemma *subtype-wf*:
fixes $\tau 1::\tau$ **and** $\tau 2::\tau$
assumes Θ; $\mathcal{B}$; $\Gamma \vdash$ $\tau 1$ $\lesssim$ $\tau 2$
shows Θ; $\mathcal{B}$; $\Gamma \vdash_{wf} \tau 1$ $\wedge\Theta$; $\mathcal{B}$; $\Gamma \vdash_{wf} \tau 2$
⟨*proof*⟩

lemma *subtype-g-wf*:
fixes $\tau 1::\tau$ **and** $\tau 2::\tau$ **and** $\Gamma::\Gamma$
assumes Θ; $\mathcal{B}$; $\Gamma \vdash$ $\tau 1$ $\lesssim$ $\tau 2$
shows Θ ; $\mathcal{B}\vdash_{wf} \Gamma$
⟨*proof*⟩

For when we have a particular y that satisfies the freshness conditions that we want the validity check to use

lemma *valid-flip-simple*:
assumes Θ; $\mathcal{B}$; (z, b, c) $\#_\Gamma$ Γ $\models c'$ **and** *atom* $z \sharp \Gamma$ **and** *atom* $x \sharp (z, c, z, c', \Gamma)$
shows Θ; $\mathcal{B}$; (x, b, ($z \leftrightarrow x$) $\cdot$ c) $\#_\Gamma$ Γ $\models$ ($z \leftrightarrow x$) $\cdot$ c'
⟨*proof*⟩

lemma *valid-wf-all*:
assumes Θ; $\mathcal{B}$; (*z0*,*b*,*c0*)$\#_\Gamma G \models c$
shows *wfG* Θ $\mathcal{B}$ *G* **and** *wfC* Θ $\mathcal{B}$ ((*z0*,*b*,*c0*)$\#_\Gamma G$) *c* **and** *atom z0* $\sharp$ *G*
⟨*proof*⟩

lemma *valid-wfT*:
fixes *z*::*x*
assumes Θ; $\mathcal{B}$; (*z0*,*b*,*c0*[*z*::=*V-var z0*]$_v$)$\#_\Gamma G \models c$[*z*::=*V-var z0*]$_v$ **and** *atom z0* $\sharp$ (Θ, $\mathcal{B}$, *G*,*c*,*c0*)
shows Θ; $\mathcal{B}$; *G* $\vdash_{wf}$ $\{\!|$ *z* : *b* | *c0* $|\!\}$ **and** Θ; $\mathcal{B}$; *G* $\vdash_{wf}$ $\{\!|$ *z* : *b* | *c* $|\!\}$
⟨*proof*⟩

lemma *valid-flip*:
fixes *c*::*c* **and** *z*::*x* **and** *z0*::*x* **and** *xx2*::*x*
assumes Θ; $\mathcal{B}$; (*xx2*, *b*, *c0*[*z0*::=*V-var xx2*]$_v$) $\#_\Gamma$ Γ $\models c$[*z*::=*V-var xx2*]$_v$ **and**
atom xx2 $\sharp$ (*c0*,Γ,*c*,*z*) **and** *atom z0* $\sharp$ (Γ,*c*,*z*)
shows Θ; $\mathcal{B}$; (*z0*, *b*, *c0*) $\#_\Gamma$ Γ $\models c$[*z*::=*V-var z0*]$_v$
⟨*proof*⟩

lemma *subtype-valid*:
assumes Θ; $\mathcal{B}$; $\Gamma \vdash$ *t1* $\lesssim$ *t2* **and** *atom y* $\sharp \Gamma$ **and** *t1* = $\{\!|$ *z1* : *b* | *c1* $|\!\}$ **and** *t2* = $\{\!|$ *z2* : *b* | *c2* $|\!\}$
shows Θ; $\mathcal{B}$; ((*y*, *b*, *c1*[*z1*::=*V-var y*]$_v$) $\#_\Gamma$ Γ) $\models$ *c2*[*z2*::=*V-var y*]$_v$
⟨*proof*⟩

lemma *subtype-valid-simple*:
assumes Θ; $\mathcal{B}$; $\Gamma \vdash t1 \lesssim t2$ **and** $atom\ z \sharp \Gamma$ **and** $t1 = \{\!|\ z : b\ |\ c1\ |\!\}$ **and** $t2 = \{\!|\ z : b\ |\ c2\ |\!\}$
shows Θ; $\mathcal{B}$; $((z, b, c1) \#_\Gamma \Gamma) \models c2$
⟨*proof*⟩

lemma *obtain-for-t-with-fresh*:
assumes $atom\ x \sharp t$
shows $\exists b\ c.\ t = \{\!|\ x : b \mid c\ |\!\}$
⟨*proof*⟩

lemma *subtype-trans*:
assumes Θ; $\mathcal{B}$; $\Gamma \vdash \tau 1 \lesssim \tau 2$ **and** Θ; $\mathcal{B}$; $\Gamma \vdash \tau 2 \lesssim \tau 3$
shows Θ; $\mathcal{B}$; $\Gamma \vdash \tau 1 \lesssim \tau 3$
⟨*proof*⟩

lemma *subtype-eq-e*:
assumes $\forall i\ s1\ s2\ G.\ wfG\ P\ \mathcal{B}\ G \wedge wfI\ P\ G\ i \wedge eval\text{-}e\ i\ e1\ s1 \wedge eval\text{-}e\ i\ e2\ s2 \longrightarrow s1 = s2$ **and** $atom\ z1 \sharp e1$ **and** $atom\ z2 \sharp e2$ **and** $atom\ z1 \sharp \Gamma$ **and** $atom\ z2 \sharp \Gamma$
and $wfCE\ P\ \mathcal{B}\ \Gamma\ e1\ b$ **and** $wfCE\ P\ \mathcal{B}\ \Gamma\ e2\ b$
shows P; $\mathcal{B}$; $\Gamma \vdash \{\!|\ z1 : b\ |\ CE\text{-}val\ (V\text{-}var\ z1) == e1\ |\!\} \lesssim (\{\!|\ z2 : b\ |\ CE\text{-}val\ (V\text{-}var\ z2) == e2\ |\!\})$
⟨*proof*⟩

lemma *subtype-eq-e-nil*:
assumes $\forall i\ s1\ s2\ G.\ wfG\ P\ \mathcal{B}\ G \wedge wfI\ P\ G\ i \wedge eval\text{-}e\ i\ e1\ s1 \wedge eval\text{-}e\ i\ e2\ s2 \longrightarrow s1 = s2$ **and** $supp\ e1 = \{\}$ **and** $supp\ e2 = \{\}$ **and** $wfTh\ P$
and $wfCE\ P\ \mathcal{B}\ GNil\ e1\ b$ **and** $wfCE\ P\ \mathcal{B}\ GNil\ e2\ b$ **and** $atom\ z1 \sharp GNil$ **and** $atom\ z2 \sharp GNil$
shows P; $\mathcal{B}$; $GNil \vdash \{\!|\ z1 : b\ |\ CE\text{-}val\ (V\text{-}var\ z1) == e1\ |\!\} \lesssim (\{\!|\ z2 : b\ |\ CE\text{-}val\ (V\text{-}var\ z2) == e2\ |\!\})$
⟨*proof*⟩

lemma *subtype-gnil-fst-aux*:
assumes $supp\ v_1 = \{\}$ **and** $supp\ (V\text{-}pair\ v_1\ v_2) = \{\}$ **and** $wfTh\ P$ **and** $wfCE\ P\ \mathcal{B}\ GNil\ (CE\text{-}val\ v_1)\ b$ **and** $wfCE\ P\ \mathcal{B}\ GNil\ (CE\text{-}fst\ [V\text{-}pair\ v_1\ v_2]^{ce})\ b$ **and**
$wfCE\ P\ \mathcal{B}\ GNil\ (CE\text{-}val\ v_2)\ b2$ **and** $atom\ z1 \sharp GNil$ **and** $atom\ z2 \sharp GNil$
shows P; $\mathcal{B}$; $GNil \vdash (\{\!|\ z1 : b \mid CE\text{-}val\ (V\text{-}var\ z1) == CE\text{-}val\ v_1\ |\!\}) \lesssim (\{\!|\ z2 : b \mid CE\text{-}val\ (V\text{-}var\ z2) == CE\text{-}fst\ [V\text{-}pair\ v_1\ v_2]^{ce}\ |\!\})$
⟨*proof*⟩

lemma *subtype-gnil-snd-aux*:
assumes $supp\ v_2 = \{\}$ **and** $supp\ (V\text{-}pair\ v_1\ v_2) = \{\}$ **and** $wfTh\ P$ **and** $wfCE\ P\ \mathcal{B}\ GNil\ (CE\text{-}val\ v_2)\ b$ **and**
$wfCE\ P\ \mathcal{B}\ GNil\ (CE\text{-}snd\ [(V\text{-}pair\ v_1\ v_2)]^{ce})\ b$ **and**
$wfCE\ P\ \mathcal{B}\ GNil\ (CE\text{-}val\ v_1)\ b2$ **and** $atom\ z1 \sharp GNil$ **and** $atom\ z2 \sharp GNil$
shows P; $\mathcal{B}$; $GNil \vdash (\{\!|\ z1 : b \mid CE\text{-}val\ (V\text{-}var\ z1) == CE\text{-}val\ v_2\ |\!\}) \lesssim (\{\!|\ z2 : b \mid CE\text{-}val\ (V\text{-}var\ z2) == CE\text{-}snd\ [(V\text{-}pair\ v_1\ v_2)]^{ce}\ |\!\})$
⟨*proof*⟩

lemma *subtype-gnil-fst*:
assumes Θ ; $\{||\}$; $GNil \vdash_{wf} [\#1[[v_1,v_2]^v]^{ce}]^{ce} : b$
shows Θ ; $\{||\}$; $GNil \vdash (\{\!|\ z_1 : b \mid [[z_1]^v]^{ce} == [v_1]^{ce}\ |\!\}) \lesssim$
$(\{\!|\ z_2 : b \mid [[z_2]^v]^{ce} == [\#1[[v_1, v_2]^v]^{ce}]^{ce}\ |\!\})$

⟨*proof*⟩

lemma *subtype-gnil-snd*:
assumes *wfCE P* {||} *GNil* (*CE-snd* ([*V-pair* v_1 v_2]^{ce})) *b*
shows *P* ; {||} ; *GNil* ⊢ (⦃ *z1* : *b* | *CE-val* (*V-var z1*) == *CE-val* v_2 ⦄) ≲ (⦃ *z2* : *b* | *CE-val* (*V-var z2*) == *CE-snd* [(*V-pair* v_1 v_2)]^{ce} ⦄)
⟨*proof*⟩

lemma *subtype-fresh-tau*:
fixes *x*::*x*
assumes *atom x* ♯ *t1* **and** *atom x* ♯ Γ **and** *P*; *B*; Γ ⊢ *t1* ≲ *t2*
shows *atom x* ♯ *t2*
⟨*proof*⟩

lemma *subtype-if-simp*:
assumes *wfT P B GNil* (⦃ *z1* : *b* | *CE-val* (*V-lit l*) == *CE-val* (*V-lit l*) *IMP* *c*[*z*::=*V-var z1*]$_v$ ⦄) **and**
wfT P B GNil (⦃ *z* : *b* | *c* ⦄) **and** *atom z1* ♯ *c*
shows *P*; *B* ; *GNil* ⊢ (⦃ *z1* : *b* | *CE-val* (*V-lit l*) == *CE-val* (*V-lit l*) *IMP* *c*[*z*::=*V-var z1*]$_v$ ⦄) ≲ (⦃ *z* : *b* | *c* ⦄)
⟨*proof*⟩

lemma *subtype-if*:
assumes *P*; *B*; Γ ⊢ ⦃ *z* : *b* | *c* ⦄ ≲ ⦃ *z'* : *b* | *c'* ⦄ **and**
wfT P B Γ (⦃ *z1* : *b* | *CE-val v* == *CE-val* (*V-lit l*) *IMP* *c*[*z*::=*V-var z1*]$_v$ ⦄) **and**
wfT P B Γ (⦃ *z2* : *b* | *CE-val v* == *CE-val* (*V-lit l*) *IMP* *c'*[*z'*::=*V-var z2*]$_v$ ⦄) **and**
atom z1 ♯ *v* **and** *atom z* ♯ Γ **and** *atom z1* ♯ *c* **and** *atom z2* ♯ *c'* **and** *atom z2* ♯ *v*
shows *P*; *B* ; Γ ⊢ ⦃ *z1* : *b* | *CE-val v* == *CE-val* (*V-lit l*) *IMP* *c*[*z*::=*V-var z1*]$_v$ ⦄ ≲ ⦃ *z2* : *b* | *CE-val v* == *CE-val* (*V-lit l*) *IMP* *c'*[*z'*::=*V-var z2*]$_v$ ⦄
⟨*proof*⟩

lemma *eval-e-concat-eq*:
assumes *wfI* Θ Γ *i*
shows ∃ *s*. *eval-e i* (*CE-val* (*V-lit* (*L-bitvec* (*v1* @ *v2*)))) *s* ∧ *eval-e i* (*CE-concat* [(*V-lit* (*L-bitvec v1*))]^{ce} [(*V-lit* (*L-bitvec v2*))]^{ce}) *s*
⟨*proof*⟩

lemma *is-satis-eval-e-eq-imp*:
assumes *wfI* Θ Γ *i* **and** *eval-e i e1 s* **and** *eval-e i e2 s*
and *is-satis i* (*CE-val* (*V-var x*) == *e1*) (**is** *is-satis i ?c1*)
shows *is-satis i* (*CE-val* (*V-var x*) == *e2*)
⟨*proof*⟩

lemma *valid-eval-e-eq*:
fixes *e1*::*ce* **and** *e2*::*ce*
assumes ∀ Γ *i*. *wfI* Θ Γ *i* ⟶ (∃ *s*. *eval-e i e1 s* ∧ *eval-e i e2 s*) **and** Θ; *B*; *GNil* ⊢$_{wf}$ *e1* : *b* **and** Θ; *B*; *GNil* ⊢$_{wf}$ *e2* : *b*
shows Θ; *B*; (*x*, *b*, (*CE-val* (*V-var x*) == *e1*)) #$_\Gamma$ *GNil* ⊨ (*CE-val* (*V-var x*) == *e2*)
⟨*proof*⟩

lemma *subtype-concat*:
assumes ⊢$_{wf}$ Θ

shows Θ; $\mathcal{B}$; *GNil* $\vdash$ $\{\!|$ z : *B-bitvec* | *CE-val* (*V-var* z) == *CE-val* (*V-lit* (*L-bitvec* (*v1* @ *v2*))) $|\!\}$ $\lesssim$
$\{\!|$ z : *B-bitvec* | *CE-val* (*V-var* z) == *CE-concat* [(*V-lit* (*L-bitvec v1*))]^{ce} [(*V-lit* (*L-bitvec v2*))]^{ce} $|\!\}$ (**is** Θ; $\mathcal{B}$; *GNil* $\vdash$ *?t1* $\lesssim$ *?t2*)
⟨*proof*⟩

lemma *subtype-len*:
assumes $\vdash_{wf}$ Θ
shows Θ; $\mathcal{B}$; *GNil* $\vdash$ $\{\!|$ z' : *B-int* | *CE-val* (*V-var* z') == *CE-val* (*V-lit* (*L-num* (*int* (*length* v)))) $|\!\}$ $\lesssim$
$\{\!|$ z : *B-int* | *CE-val* (*V-var* z) == *CE-len* [(*V-lit* (*L-bitvec* v))]^{ce} $|\!\}$ (**is** Θ; $\mathcal{B}$; *GNil* $\vdash$ *?t1* $\lesssim$ *?t2*)
⟨*proof*⟩

lemma *subtype-base-fresh*:
assumes Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ $\{\!|$ z : b | c $|\!\}$ **and** Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ $\{\!|$ z : b | c' $|\!\}$ **and**
atom z $\sharp$ Γ **and** Θ; $\mathcal{B}$; (z, b, c) $\#_\Gamma$ Γ $\models$ c'
shows Θ; $\mathcal{B}$; Γ $\vdash$ $\{\!|$ z : b | c $|\!\}$ $\lesssim$ $\{\!|$ z : b | c' $|\!\}$
⟨*proof*⟩

lemma *subtype-bop-arith*:
assumes *wfG* Θ $\mathcal{B}$ Γ **and** (*opp* = *Plus* $\wedge$ *ll* = (*L-num* (*n1*+*n2*))) $\vee$ (*opp* = *LEq* $\wedge$ *ll* = (*if n1*≤*n2 then L-true else L-false*))
and (*opp* = *Plus* $\longrightarrow$ *b* = *B-int*) $\wedge$ (*opp* = *LEq* $\longrightarrow$ *b* = *B-bool*)
shows Θ; $\mathcal{B}$; Γ $\vdash$ ($\{\!|$ z : b | *C-eq* (*CE-val* (*V-var* z)) (*CE-val* (*V-lit* (*ll*))) $|\!\}$) $\lesssim$
$\{\!|$ z : b | *C-eq* (*CE-val* (*V-var* z)) (*CE-op opp* [(*V-lit* (*L-num n1*))]^{ce} [(*V-lit* (*L-num n2*))]^{ce}) $|\!\}$ (**is** Θ; $\mathcal{B}$; Γ $\vdash$ *?T1* $\lesssim$ *?T2*)
⟨*proof*⟩

lemma *subtype-bop-eq*:
assumes *wfG* Θ $\mathcal{B}$ Γ **and** *base-for-lit l1* = *base-for-lit l2*
shows Θ; $\mathcal{B}$; Γ $\vdash$ ($\{\!|$ z : *B-bool* | *C-eq* (*CE-val* (*V-var* z)) (*CE-val* (*V-lit* (*if l1* = *l2 then L-true else L-false*))) $|\!\}$) $\lesssim$
$\{\!|$ z : *B-bool* | *C-eq* (*CE-val* (*V-var* z)) (*CE-op Eq* [(*V-lit l1*)]^{ce} [(*V-lit l2*)]^{ce}) $|\!\}$ (**is** Θ; $\mathcal{B}$; Γ $\vdash$ *?T1* $\lesssim$ *?T2*)
⟨*proof*⟩

lemma *subtype-top*:
assumes *wfT* Θ $\mathcal{B}$ *G* ($\{\!|$ z : b | c $|\!\}$)
shows Θ; $\mathcal{B}$; *G* $\vdash$ ($\{\!|$ z : b | c $|\!\}$) $\lesssim$ ($\{\!|$ z : b | *TRUE* $|\!\}$)
⟨*proof*⟩

lemma *if-simp*:
(*if* x = x *then e1 else e2*) = *e1*
⟨*proof*⟩

lemma *subtype-split*:
assumes *split n v* (*v1*,*v2*) **and** $\vdash_{wf}$ Θ
shows Θ ; {||} ; *GNil* $\vdash$ $\{\!|$ z : [*B-bitvec* , *B-bitvec*]^b | [[z]^v]^{ce} == [[[*L-bitvec v1*]^v , [*L-bitvec v2*]^v]^v]^{ce} $|\!\}$ $\lesssim$ $\{\!|$ z : [*B-bitvec* , *B-bitvec*]^b | [[*L-bitvec* v]^v]^{ce} == [[#*1*[[z]^v]^{ce}]^{ce} @@ [#*2*[[z]^v]^{ce}]^{ce}]^{ce} *AND* [| [#*1*[[z]^v]^{ce}]^{ce} |]^{ce} == [

[*L-num*
n]^{v}]^{ce} ⦈
(**is** Θ ;*?B* ; *GNil* ⊢ ⦇ *z* : [*B-bitvec* , *B-bitvec*]^{b} | *?c1* ⦈ ≲ ⦇ *z* : [*B-bitvec* , *B-bitvec*]^{b} | *?c2* ⦈)
⟨*proof*⟩

lemma *subtype-range*:
fixes *n*::*int* **and** Γ::Γ
assumes *0* ≤ *n* ∧ *n* ≤ *int* (*length v*) **and** Θ ; {||} ⊢$_{wf}$ Γ
shows Θ ; {||} ; Γ ⊢ ⦇ *z* : *B-int* | [[*z*]^{v}]^{ce} == [[*L-num n*]^{v}]^{ce} ⦈ ≲
⦇ *z* : *B-int* | ([*leq* [[*L-num 0*]^{v}]^{ce} [[*z*]^{v}]^{ce}]^{ce} == [[*L-true*]^{v}]^{ce}) *AND* (
[*leq* [[*z*]^{v}]^{ce} [|| [[*L-bitvec v*]^{v}]^{ce} ||]^{ce}]^{ce} == [[*L-true*]^{v}]^{ce}) ⦈
(**is** Θ ; *?B* ; Γ ⊢ ⦇ *z* : *B-int* | *?c1* ⦈ ≲ ⦇ *z* : *B-int* | *?c2 AND ?c3* ⦈)
⟨*proof*⟩

lemma *check-num-range*:
assumes *0* ≤ *n* ∧ *n* ≤ *int* (*length v*) **and** ⊢$_{wf}$ Θ
shows Θ ; {||} ; *GNil* ⊢ ([*L-num n*]^{v}) ⇐ ⦇ *z* : *B-int* | ([*leq* [[*L-num 0*]^{v}]^{ce} [[*z*]^{v}]^{ce}]^{ce} == [
[*L-true*]^{v}]^{ce}) *AND*
[*leq* [[*z*]^{v}]^{ce} [|| [[*L-bitvec v*]^{v}]^{ce} ||]^{ce}]^{ce} == [[*L-true*]^{v}]^{ce} ⦈
⟨*proof*⟩

12.3 Literals

nominal-function *type-for-lit* :: *l* ⇒ τ **where**
type-for-lit (*L-true*) = (⦇ *z* : *B-bool* | [[*z*]^{v}]^{ce} == [*V-lit L-true*]^{ce} ⦈)
| *type-for-lit* (*L-false*) = (⦇ *z* : *B-bool* | [[*z*]^{v}]^{ce} == [*V-lit L-false*]^{ce} ⦈)
| *type-for-lit* (*L-num n*) = (⦇ *z* : *B-int* | [[*z*]^{v}]^{ce} == [*V-lit* (*L-num n*)]^{ce} ⦈)
| *type-for-lit* (*L-unit*) = (⦇ *z* : *B-unit* | [[*z*]^{v}]^{ce} == [*V-lit* (*L-unit*)]^{ce} ⦈)
| *type-for-lit* (*L-bitvec v*) = (⦇ *z* : *B-bitvec* | [[*z*]^{v}]^{ce} == [*V-lit* (*L-bitvec v*)]^{ce} ⦈)
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

nominal-function *type-for-var* :: Γ ⇒ τ ⇒ *x* ⇒ τ **where**
type-for-var G τ *x* = (*case lookup G x of*
None ⇒ τ
| *Some* (*b*,*c*) ⇒ (⦇ *x* : *b* | *c* ⦈))
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

lemma *infer-l-form*:
fixes *l*::*l* **and** *tm*::'*a*::*fs*
assumes ⊢ *l* ⇒ τ
shows ∃ *z b*. τ = (⦇ *z* : *b* | *C-eq* (*CE-val* (*V-var z*)) (*CE-val* (*V-lit l*)) ⦈) ∧ *atom z* ♯ *tm*
⟨*proof*⟩

lemma *infer-l-form3*:
fixes *l*::*l*
assumes ⊢ *l* ⇒ τ
shows ∃ *z*. τ = (⦇ *z* : *base-for-lit l* | *C-eq* (*CE-val* (*V-var z*)) (*CE-val* (*V-lit l*)) ⦈)
⟨*proof*⟩

lemma *infer-l-form4* [*simp*]:
fixes Γ::Γ
assumes Θ ; $\mathcal{B} \vdash_{wf}$ Γ
shows ∃ *z*. ⊢ *l* ⇒ ({| *z* : *base-for-lit l* | *C-eq* (*CE-val* (*V-var z*)) (*CE-val* (*V-lit l*)) |})
⟨*proof*⟩

lemma *infer-v-unit-form*:
fixes *v*::*v*
assumes *P* ; $\mathcal{B}$; Γ ⊢ *v* ⇒ ({| *z1* : *B-unit* | *c1* |}) **and** *supp v* = {}
shows *v* = *V-lit L-unit*
⟨*proof*⟩

lemma *base-for-lit-wf*:
assumes $\vdash_{wf}$ Θ
shows Θ ; $\mathcal{B}$ $\vdash_{wf}$ *base-for-lit l*
⟨*proof*⟩

lemma *infer-l-t-wf*:
fixes Γ::Γ
assumes Θ ; $\mathcal{B}$ $\vdash_{wf}$ Γ ∧ *atom z* ♯ Γ
shows Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ {| *z* : *base-for-lit l* | *C-eq* (*CE-val* (*V-var z*)) (*CE-val* (*V-lit l*)) |}
⟨*proof*⟩

lemma *infer-l-wf*:
fixes *l*::*l* **and** Γ::Γ **and** τ::τ **and** Θ::Θ
assumes ⊢ *l* ⇒ τ **and** Θ ; $\mathcal{B}$ $\vdash_{wf}$ Γ
shows $\vdash_{wf}$ Θ **and** Θ ; $\mathcal{B}$ $\vdash_{wf}$ Γ **and** Θ; $\mathcal{B}$; Γ $\vdash_{wf}$ τ
⟨*proof*⟩

lemma *infer-l-uniqueness*:
fixes *l*::*l*
assumes ⊢ *l* ⇒ τ **and** ⊢ *l* ⇒ τ′
shows τ = τ′
⟨*proof*⟩

12.4 Values

lemma *type-v-eq*:
assumes {| *z1* : *b1* | *c1* |} = {| *z* : *b* | *C-eq* (*CE-val* (*V-var z*)) (*CE-val* (*V-var x*)) |} **and** *atom z* ♯ *x*
shows *b* = *b1* **and** *c1* = *C-eq* (*CE-val* (*V-var z1*)) (*CE-val* (*V-var x*))
⟨*proof*⟩

lemma *infer-var2* [*elim*]:
assumes *P*; $\mathcal{B}$; *G* ⊢ *V-var x* ⇒ τ
shows ∃ *b c*. *Some* (*b*,*c*) = *lookup G x*
⟨*proof*⟩

lemma *infer-var3* [*elim*]:
assumes Θ; $\mathcal{B}$; Γ ⊢ *V-var x* ⇒ τ
shows ∃ *z b c*. *Some* (*b*,*c*) = *lookup* Γ *x* ∧ τ = ({| *z* : *b* | *C-eq* (*CE-val* (*V-var z*)) (*CE-val* (*V-var x*)) |}) ∧ *atom z* ♯ *x* ∧ *atom z* ♯ (Θ, $\mathcal{B}$, Γ)
⟨*proof*⟩

lemma *infer-bool-options2*:
fixes v::v
assumes Θ; $\mathcal{B}$; $\Gamma \vdash v \Rightarrow \{\!|\ z : b \mid c\ |\!\}$ **and** *supp* $v = \{\} \wedge b = B\text{-}bool$
shows $v = V\text{-}lit\ L\text{-}true \vee (v = (V\text{-}lit\ L\text{-}false))$
⟨*proof*⟩

lemma *infer-bool-options*:
fixes v::v
assumes Θ; $\mathcal{B}$; $\Gamma\ \vdash v \Rightarrow \{\!|\ z : B\text{-}bool\ \mid c\ |\!\}$ **and** *supp* $v = \{\}$
shows $v = V\text{-}lit\ L\text{-}true \vee (v = (V\text{-}lit\ L\text{-}false))$
⟨*proof*⟩

lemma *infer-int2*:
fixes v::v
assumes Θ; $\mathcal{B}$; $\Gamma \vdash v \Rightarrow \{\!|\ z : b \mid c\ |\!\}$
shows *supp* $v = \{\}\ \wedge b = B\text{-}int \longrightarrow (\exists\, n.\ v = V\text{-}lit\ (L\text{-}num\ n))$
⟨*proof*⟩

lemma *infer-bitvec*:
fixes Θ::Θ **and** v::v
assumes Θ; $\mathcal{B}$; $\Gamma \vdash v \Rightarrow \{\!|\ z' : B\text{-}bitvec \mid c'\ |\!\}$ **and** *supp* $v = \{\}$
shows $\exists\, bv.\ v = V\text{-}lit\ (L\text{-}bitvec\ bv)$
⟨*proof*⟩

lemma *infer-int*:
assumes *infer-v* $\Theta\ \mathcal{B}\ \Gamma\ v\ \ (\{\!|\ z : B\text{-}int \mid c\ |\!\})$ **and** *supp* v= $\{\}$
shows $\exists\, n.\ V\text{-}lit\ (L\text{-}num\ n) = v$
⟨*proof*⟩

lemma *infer-lit*:
assumes *infer-v* $\Theta\ \mathcal{B}\ \Gamma\ v\ \ (\{\!|\ z : b \mid c\ |\!\})$ **and** *supp* v= $\{\}$ **and** $b \in \{\ B\text{-}bool\ ,\ B\text{-}int\ ,\ B\text{-}unit\ \}$
shows $\exists\, l.\ V\text{-}lit\ l = v$
⟨*proof*⟩

lemma *infer-v-form*[*simp*]:
fixes v::v
assumes Θ; $\mathcal{B}$; $\Gamma \vdash v \Rightarrow \tau$
shows $\exists\, z\ b.\ \tau = (\{\!|\ z : b \mid C\text{-}eq\ (CE\text{-}val\ (V\text{-}var\ z))\ (CE\text{-}val\ v)|\!\}) \wedge atom\ z \sharp v \wedge atom\ z \sharp (\Theta, \mathcal{B}, \Gamma)$
⟨*proof*⟩

lemma *infer-v-form2*:
fixes v::v
assumes Θ; $\mathcal{B}$; $\Gamma \vdash v \Rightarrow (\{\!|\ z : b \mid c\ |\!\})$ **and** $atom\ z \sharp v$
shows $c = C\text{-}eq\ (CE\text{-}val\ (V\text{-}var\ z))\ (CE\text{-}val\ v)$
⟨*proof*⟩

lemma *infer-v-form3*:
fixes v::v
assumes Θ; $\mathcal{B}$; $\Gamma \vdash v \Rightarrow \tau$ **and** $atom\ z \sharp (v,\Gamma)$
shows Θ; $\mathcal{B}$; $\Gamma \vdash v \Rightarrow \{\!|\ z : b\text{-}of\ \tau \mid C\text{-}eq\ (CE\text{-}val\ (V\text{-}var\ z))\ (CE\text{-}val\ v)|\!\}$
⟨*proof*⟩

lemma *infer-v-form4*:
fixes $v::v$
assumes Θ; $\mathcal{B}$; $\Gamma \vdash v \Rightarrow \tau$ **and** *atom* $z \sharp (v,\Gamma)$ **and** b = *b-of* τ
shows Θ; $\mathcal{B}$; $\Gamma \vdash v \Rightarrow \{\!|\ z : b \mid$ *C-eq* (*CE-val* (*V-var* z)) (*CE-val* v)$|\!\}$
⟨*proof*⟩

lemma *infer-v-v-wf*:
fixes $v::v$
shows Θ; $\mathcal{B}$; $G \vdash v \Rightarrow \tau \Longrightarrow$ Θ; $\mathcal{B}$; $G \vdash_{wf} v : (b\text{-}of\ \tau)$
⟨*proof*⟩

lemma *infer-v-t-form-wf*:
assumes *wfB* Θ $\mathcal{B}$ b **and** *wfV* Θ $\mathcal{B}$ Γ v b **and** *atom* $z \sharp \Gamma$
shows *wfT* Θ $\mathcal{B}$ Γ $\{\!|\ z : b \mid$ *C-eq* (*CE-val* (*V-var* z)) (*CE-val* v)$|\!\}$
⟨*proof*⟩

lemma *infer-v-t-wf*:
fixes $v::v$
assumes Θ; $\mathcal{B}$; $G \vdash v \Rightarrow \tau$
shows *wfT* Θ $\mathcal{B}$ G $\tau \wedge$ *wfB* Θ $\mathcal{B}$ (*b-of* τ)
⟨*proof*⟩

lemma *infer-v-wf*:
fixes $v::v$
assumes Θ; $\mathcal{B}$; $G \vdash v \Rightarrow \tau$
shows Θ; $\mathcal{B}$; $G \vdash_{wf} v : (b\text{-}of\ \tau)$ **and** *wfT* Θ $\mathcal{B}$ G τ **and** *wfTh* Θ **and** *wfG* Θ $\mathcal{B}$ G
⟨*proof*⟩

lemma *check-bool-options*:
assumes Θ; $\mathcal{B}$; $\Gamma \vdash v \Leftarrow \{\!|\ z : B\text{-}bool \mid TRUE\ |\!\}$ **and** *supp* $v = \{\}$
shows $v =$ *V-lit L-true* $\vee$ $v =$ *V-lit L-false*
⟨*proof*⟩

lemma *check-v-wf*:
fixes $v::v$ **and** $\Gamma::\Gamma$ **and** $\tau::\tau$
assumes Θ; $\mathcal{B}$; $\Gamma \vdash v \Leftarrow \tau$
shows Θ ; $\mathcal{B} \vdash_{wf} \Gamma$ **and** Θ; $\mathcal{B}$;$\Gamma \vdash_{wf} v : b\text{-}of\ \tau$ **and** Θ; $\mathcal{B}$;$\Gamma \vdash_{wf} \tau$
⟨*proof*⟩

lemma *infer-v-form-fresh*:
fixes $v::v$ **and** $t::'a::fs$
assumes Θ; $\mathcal{B}$; $\Gamma \vdash v \Rightarrow \tau$
shows $\exists\, z\ b.\ \tau = \{\!|\ z : b \mid$ *C-eq* (*CE-val* (*V-var* z)) (*CE-val* v)$|\!\} \wedge$ *atom* $z \sharp (t,v)$
⟨*proof*⟩

More generally, if support of a term is empty then any G will do

lemma *infer-v-form-consp*:
assumes Θ; $\mathcal{B}$; $\Gamma \vdash$ *V-consp* s dc b $v \Rightarrow \tau$
shows *b-of* τ = *B-app* s b
⟨*proof*⟩

lemma *lookup-in-rig-b*:
assumes *Some* (*b2*, *c2*) = *lookup* (Γ[*x*⟼*c'*]) *x'* **and**
Some (*b1*, *c1*) = *lookup* Γ *x'*
shows *b1* = *b2*
⟨*proof*⟩

lemma *infer-v-uniqueness-rig*:
fixes *x*::*x* **and** *c*::*c*
assumes *infer-v P B G v τ* **and** *infer-v P B* (*replace-in-g G x c'*) *v τ'*
shows *τ* = *τ'*
⟨*proof*⟩

lemma *infer-v-uniqueness*:
assumes *infer-v P B G v τ* **and** *infer-v P B G v τ'*
shows *τ* = *τ'*
⟨*proof*⟩

lemma *infer-v-tid-form*:
fixes *v*::*v*
assumes Θ ; *B* ; Γ ⊢ *v* ⇒ {| *z* : *B-id tid* | *c* |} **and** *AF-typedef tid dclist* ∈ *set* Θ **and** *supp v* = {}
shows ∃ *dc v' t*. *v* = *V-cons tid dc v'* ∧ (*dc* , *t*) ∈ *set dclist*
⟨*proof*⟩

lemma *check-v-tid-form*:
assumes Θ ; *B* ; Γ ⊢ *v* ⇐ {| *z* : *B-id tid* | *TRUE* |} **and** *AF-typedef tid dclist* ∈ *set* Θ **and** *supp v* = {}
shows ∃ *dc v' t*. *v* = *V-cons tid dc v'* ∧ (*dc* , *t*) ∈ *set dclist*
⟨*proof*⟩

lemma *check-v-num-leq*:
fixes *n*::*int* **and** Γ::Γ
assumes *0* ≤ *n* ∧ *n* ≤ *int* (*length v*) **and** ⊢$_{wf}$ Θ **and** Θ ; {||} ⊢$_{wf}$ Γ
shows Θ ; {||} ; Γ ⊢ [*L-num n*]^{v} ⇐ {| *z* : *B-int* | ([*leq* [[*L-num 0*]^{v}]^{ce} [[*z*]^{v}]^{ce}]^{ce} == [[*L-true*]^{v}]^{ce})
AND ([*leq* [[*z*]^{v}]^{ce} [| [[*L-bitvec v*]^{v}]^{ce} |]^{ce}]^{ce} == [[*L-true*]^{v}]^{ce}) |}
⟨*proof*⟩

lemma *check-int*:
assumes *check-v* Θ *B* Γ *v* ({| *z* : *B-int* | *c* |}) **and** *supp v* = {}
shows ∃ *n*. *V-lit* (*L-num n*) = *v*
⟨*proof*⟩

definition *sble* :: Θ ⇒ Γ ⇒ *bool* **where**
sble Θ Γ = (∃ *i*. *i* ⊨ Γ ∧ Θ ; Γ ⊢ *i*)

lemma *check-v-range*:
assumes Θ ; *B* ; Γ ⊢ *v2* ⇐ {| *z* : *B-int* | [*leq* [[*L-num 0*]^{v}]^{ce} [[*z*]^{v}]^{ce}]^{ce} == [[*L-true*]^{v}]^{ce}
AND
[*leq* [[*z*]^{v}]^{ce} [| [*v1*]^{ce} |]^{ce}]^{ce} == [[*L-true*]^{v}]^{ce} |}
(**is** Θ ; *?B* ; Γ ⊢ *v2* ⇐ {| *z* : *B-int* | *?c1* |})
and *v1* = *V-lit* (*L-bitvec bv*) ∧ *v2* = *V-lit* (*L-num n*) **and** *atom z* ♯ Γ **and** *sble* Θ Γ
shows *0* ≤ *n* ∧ *n* ≤ *int* (*length bv*)

⟨*proof*⟩

12.5 Expressions

lemma *infer-e-plus*[*elim*]:
fixes *v1*::*v* **and** *v2*::*v*
assumes Θ ; Φ ; $\mathcal{B}$; Γ ; Δ ⊢ *AE-op Plus v1 v2* ⇒ τ
shows ∃ *z* . ({‖ *z* : *B-int* | *C-eq* (*CE-val* (*V-var z*)) (*CE-op Plus* [*v1*]^{ce} [*v2*]^{ce}) ‖} = τ)
⟨*proof*⟩

lemma *infer-e-leq*[*elim*]:
assumes Θ ; Φ ; $\mathcal{B}$; Γ ; Δ ⊢ *AE-op LEq v1 v2* ⇒ τ
shows ∃ *z* . ({‖ *z* : *B-bool* | *C-eq* (*CE-val* (*V-var z*)) (*CE-op LEq* [*v1*]^{ce} [*v2*]^{ce}) ‖} = τ)
⟨*proof*⟩

lemma *infer-e-eq*[*elim*]:
assumes Θ ; Φ ; $\mathcal{B}$; Γ ; Δ ⊢ *AE-op Eq v1 v2* ⇒ τ
shows ∃ *z* . ({‖ *z* : *B-bool* | *C-eq* (*CE-val* (*V-var z*)) (*CE-op Eq* [*v1*]^{ce} [*v2*]^{ce}) ‖} = τ)
⟨*proof*⟩

lemma *infer-e-e-wf*:
fixes *e*::*e*
assumes Θ ; Φ ; $\mathcal{B}$; Γ ; Δ ⊢ *e* ⇒ τ
shows Θ ; Φ ; $\mathcal{B}$; Γ ; Δ ⊢$_{wf}$ *e* : *b-of* τ
⟨*proof*⟩

lemma *infer-e-t-wf*:
fixes *e*::*e* **and** Γ::Γ **and** τ::τ **and** Δ::Δ **and** Φ::Φ
assumes Θ ; Φ ; $\mathcal{B}$; Γ ; Δ ⊢ *e* ⇒ τ
shows Θ; $\mathcal{B}$;Γ ⊢$_{wf}$ τ ∧ Θ ⊢$_{wf}$ Φ
⟨*proof*⟩

lemma *infer-e-wf*:
fixes *e*::*e* **and** Γ::Γ **and** τ::τ **and** Δ::Δ **and** Φ::Φ
assumes Θ ; Φ ; $\mathcal{B}$; Γ ; Δ ⊢ *e* ⇒ τ
shows Θ; $\mathcal{B}$; Γ ⊢$_{wf}$ τ **and** Θ ; $\mathcal{B}$ ⊢$_{wf}$ Γ **and** Θ; $\mathcal{B}$; Γ ⊢$_{wf}$ Δ **and** Θ ⊢$_{wf}$ Φ **and** Θ ; Φ ; $\mathcal{B}$; Γ ; Δ ⊢$_{wf}$ *e* : (*b-of* τ)
⟨*proof*⟩

lemma *infer-e-fresh*:
fixes *x*::*x*
assumes Θ ; Φ ; $\mathcal{B}$; Γ ; Δ ⊢ *e* ⇒ τ **and** *atom x* ♯ Γ
shows *atom x* ♯ (*e*,τ)
⟨*proof*⟩

inductive *check-e* :: Θ ⇒ Φ ⇒ $\mathcal{B}$ ⇒ Γ ⇒ Δ ⇒ *e* ⇒ τ ⇒ *bool* (‹ - ; - ; - ; - ; - ⊢ - ⇐ -› [*50*, *50*, *50*] *50*) **where**
check-e-subtypeI: ⟦ *infer-e T P B G D e τ'* ; *subtype T B G τ' τ* ⟧ ⟹ *check-e T P B G D e τ*
equivariance *check-e*
nominal-inductive *check-e* ⟨*proof*⟩

inductive-cases *check-e-elims*[*elim*!]:

check-e F D B G Θ (AE-val v) τ
check-e F D B G Θ e τ

lemma *infer-e-fst-pair*:
fixes *v1*::*v*
assumes Θ ; Φ ; {||} ; *GNil* ; Δ ⊢ [#*1*[*v1* , *v2*]^v]^e ⇒ τ
shows ∃τ′. Θ ; Φ ; {||} ; *GNil* ; Δ ⊢ [*v1*]^e ⇒ τ′ ∧
Θ ; {||} ; *GNil* ⊢ τ′ ≲ τ
⟨*proof*⟩

lemma *infer-e-snd-pair*:
assumes Θ ; Φ ; {||} ; *GNil* ; Δ ⊢ *AE-snd* (*V-pair v1 v2*) ⇒ τ
shows ∃τ′. Θ ; Φ ; {||} ; *GNil* ; Δ ⊢ *AE-val v2* ⇒ τ′ ∧ Θ ; {||} ; *GNil* ⊢ τ′ ≲ τ
⟨*proof*⟩

12.6 Statements

lemma *check-s-v-unit*:
assumes Θ; $\mathcal{B}$; Γ ⊢ ({| *z* : *B-unit* | *TRUE* |}) ≲ τ **and** *wfD* Θ $\mathcal{B}$ Γ Δ **and** *wfPhi* Θ Φ
shows Θ ; Φ ; $\mathcal{B}$; Γ ; Δ ⊢ *AS-val* (*V-lit L-unit*) ⇐ τ
⟨*proof*⟩

lemma *check-s-check-branch-s-wf*:
fixes *s*::*s* **and** *cs*::*branch-s* **and** Θ::Θ **and** Φ::Φ **and** Γ::Γ **and** Δ::Δ **and** *v*::*v* **and** τ::τ **and** *css*::*branch-list*
shows Θ ; Φ ; *B* ; Γ ; Δ ⊢ *s* ⇐ τ ⟹ Θ ; *B* ⊢$_{wf}$ Γ ∧ *wfTh* Θ ∧ *wfD* Θ *B* Γ Δ ∧ *wfT* Θ *B* Γ τ ∧ *wfPhi* Θ Φ **and**
check-branch-s Θ Φ *B* Γ Δ *tid cons const v cs* τ ⟹ Θ ; *B* ⊢$_{wf}$ Γ ∧ *wfTh* Θ ∧ *wfD* Θ *B* Γ Δ ∧ *wfT* Θ *B* Γ τ ∧ *wfPhi* Θ Φ
check-branch-list Θ Φ *B* Γ Δ *tid dclist v css* τ ⟹ Θ ; *B* ⊢$_{wf}$ Γ ∧ *wfTh* Θ ∧ *wfD* Θ *B* Γ Δ ∧ *wfT* Θ *B* Γ τ ∧ *wfPhi* Θ Φ
⟨*proof*⟩

lemma *check-s-check-branch-s-wfS*:
fixes *s*::*s* **and** *cs*::*branch-s* **and** Θ::Θ **and** Φ::Φ **and** Γ::Γ **and** Δ::Δ **and** *v*::*v* **and** τ::τ **and** *css*::*branch-list*
shows Θ ; Φ ; *B* ; Γ ; Δ ⊢ *s* ⇐ τ ⟹ Θ ; Φ ; *B* ; Γ ; Δ ⊢$_{wf}$ *s* : *b-of* τ **and**
check-branch-s Θ Φ *B* Γ Δ *tid cons const v cs* τ ⟹ *wfCS* Θ Φ *B* Γ Δ *tid cons const cs* (*b-of* τ)
check-branch-list Θ Φ *B* Γ Δ *tid dclist v css* τ ⟹ *wfCSS* Θ Φ *B* Γ Δ *tid dclist css* (*b-of* τ)
⟨*proof*⟩

lemma *check-s-wf*:
fixes *s*::*s*
assumes Θ ; Φ ; *B* ; Γ ; Δ ⊢ *s* ⇐ τ
shows Θ ; *B* ⊢$_{wf}$ Γ ∧ *wfT* Θ *B* Γ τ ∧ *wfPhi* Θ Φ ∧ *wfTh* Θ ∧ *wfD* Θ *B* Γ Δ ∧ *wfS* Θ Φ *B* Γ Δ *s* (*b-of* τ)
⟨*proof*⟩

lemma *check-s-flip-u1*:
fixes *s*::*s* **and** *u*::*u* **and** *u*′::*u*
assumes Θ ; Φ ; $\mathcal{B}$; Γ ; Δ ⊢ *s* ⇐ τ
shows Θ ; Φ ; $\mathcal{B}$; Γ ; (*u* ↔ *u*′) · Δ ⊢ (*u* ↔ *u*′) · *s* ⇐ τ
⟨*proof*⟩

lemma *check-s-flip-u2*:
fixes $s::s$ **and** $u::u$ **and** $u'::u$
assumes Θ ; Φ ; B ; Γ ; ($u \leftrightarrow u'$) $\cdot$ Δ $\vdash$ ($u \leftrightarrow u'$) $\cdot$ $s \Leftarrow \tau$
shows Θ ; Φ ; B ; Γ ; Δ $\vdash$ $s \Leftarrow \tau$
⟨*proof*⟩

lemma *check-s-flip-u*:
fixes $s::s$ **and** $u::u$ **and** $u'::u$
shows Θ ; Φ ; B ; Γ ; ($u \leftrightarrow u'$) $\cdot$ Δ $\vdash$ ($u \leftrightarrow u'$) $\cdot$ $s \Leftarrow \tau$ = (Θ ; Φ ; B ; Γ ; $\Delta \vdash s \Leftarrow \tau$)
⟨*proof*⟩

lemma *check-s-abs-u*:
fixes $s::s$ **and** $s'::s$ **and** $u::u$ **and** $u'::u$ **and** $\tau'::\tau$
assumes $[[atom\ u]]lst.\ s = [[atom\ u']]lst.\ s'$ **and** $atom\ u \sharp \Delta$ **and** $atom\ u' \sharp \Delta$
and Θ ; B ; $\Gamma \vdash_{wf} \tau'$
and Θ ; Φ ; B ; Γ ; (u , τ') $\#_\Delta \Delta$ $\vdash$ $s \Leftarrow \tau$
shows Θ ; Φ ; B ; Γ ; (u', τ') $\#_\Delta \Delta \vdash s' \Leftarrow \tau$
⟨*proof*⟩

12.7 Additional Elimination and Intros

12.7.1 Values

nominal-function *b-for* :: *opp* ⇒ *b* **where**
b-for Plus = *B-int*
| *b-for LEq* = *B-bool* | *b-for Eq* = *B-bool*
⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

lemma *infer-v-pair2I*:
fixes $v_1::v$ **and** $v_2::v$
assumes Θ; $\mathcal{B}$; $\Gamma \vdash v_1 \Rightarrow \tau_1$ **and** Θ; $\mathcal{B}$; $\Gamma \vdash v_2 \Rightarrow \tau_2$
shows $\exists \tau.$ Θ; $\mathcal{B}$; $\Gamma \vdash$ *V-pair* v_1 $v_2 \Rightarrow \tau \wedge$ *b-of* τ = *B-pair* (*b-of* τ_1) (*b-of* τ_2)
⟨*proof*⟩

lemma *infer-v-pair2I-zbc*:
fixes $v_1::v$ **and** $v_2::v$
assumes Θ; $\mathcal{B}$; $\Gamma \vdash v_1 \Rightarrow \tau_1$ **and** Θ; $\mathcal{B}$; $\Gamma \vdash v_2 \Rightarrow \tau_2$
shows $\exists z\ \tau.$ Θ; $\mathcal{B}$; $\Gamma \vdash$ *V-pair* v_1 $v_2 \Rightarrow \tau \wedge \tau$ = ({| z : *B-pair* (*b-of* τ_1) (*b-of* τ_2) | *C-eq* (*CE-val* (*V-var* z)) (*CE-val* (*V-pair* v_1 v_2)) |}) $\wedge$ *atom* $z \sharp (v_1,v_2) \wedge$ *atom* $z \sharp \Gamma$
⟨*proof*⟩

lemma *infer-v-pair2E*:
assumes Θ; $\mathcal{B}$; $\Gamma \vdash$ *V-pair* v_1 $v_2 \Rightarrow \tau$
shows $\exists \tau_1\ \tau_2\ z\ .$ Θ; $\mathcal{B}$; $\Gamma \vdash v_1 \Rightarrow \tau_1 \wedge \Theta$; $\mathcal{B}$; $\Gamma \vdash v_2 \Rightarrow \tau_2 \wedge$
τ = ({| z : *B-pair* (*b-of* τ_1) (*b-of* τ_2) | *C-eq* (*CE-val* (*V-var* z)) (*CE-val* (*V-pair* v_1 v_2)) |}) $\wedge$ *atom* $z \sharp$ (v_1, v_2)
⟨*proof*⟩

12.7.2 Expressions

lemma *infer-e-app2E*:
fixes Φ::Φ **and** Θ::Θ
assumes Θ ; Φ ; $\mathcal{B}$; Γ ; $\Delta \vdash$ *AE-app f v* $\Rightarrow \tau$
shows $\exists\, x\ b\ c\ s'\ \tau'.$ *wfD* $\Theta\ \mathcal{B}\ \Gamma\ \Delta \wedge$ *Some* (*AF-fundef f* (*AF-fun-typ-none* (*AF-fun-typ x b c* τ' s'))) = *lookup-fun* Φ *f* $\wedge$ $\Theta \vdash_{wf} \Phi \wedge$
Θ; $\mathcal{B}$; $\Gamma \vdash v \Leftarrow \{\!|\ x : b\ \mid c\ |\!\} \wedge \tau = \tau'[x::=v]_{\tau v} \wedge$ *atom x* $\sharp$ (Θ, Φ, $\mathcal{B}$, Γ, Δ, *v*, τ)
⟨*proof*⟩

lemma *infer-e-appP2E*:
fixes Φ::Φ **and** Θ::Θ
assumes Θ ; Φ ; $\mathcal{B}$; Γ ; $\Delta \vdash$ *AE-appP f b v* $\Rightarrow \tau$
shows $\exists$ *bv x ba c s'* τ'. *wfD* $\Theta\ \mathcal{B}\ \Gamma\ \Delta \wedge$ *Some* (*AF-fundef f* (*AF-fun-typ-some bv* (*AF-fun-typ x ba c* τ' s'))) = *lookup-fun* Φ *f* $\wedge$ $\Theta \vdash_{wf} \Phi \wedge$ Θ ; $\mathcal{B}$ $\vdash_{wf}$ *b* $\wedge$
(Θ; $\mathcal{B}$; $\Gamma \vdash v \Leftarrow \{\!|\ x : ba[bv::=b]_{bb}\ \mid c[bv::=b]_{cb}\ |\!\}) \wedge (\tau = \tau'[bv::=b]_{\tau b}[x::=v]_{\tau v}) \wedge$ *atom x* $\sharp\ \Gamma \wedge$ *atom bv* $\sharp$ *v*
⟨*proof*⟩

12.8 Weakening

Lemmas showing that typing judgements hold when a context is extended

lemma *subtype-weakening*:
fixes Γ'::Γ
assumes Θ; $\mathcal{B}$; $\Gamma \vdash \tau 1 \lesssim \tau 2$ **and** *toSet* $\Gamma \subseteq$ *toSet* Γ' **and** Θ ; $\mathcal{B} \vdash_{wf} \Gamma'$
shows Θ; $\mathcal{B}$; $\Gamma' \vdash \tau 1 \lesssim \tau 2$
⟨*proof*⟩

method *many-rules* **uses** *add* = ((*rule*+),((*simp add*: *add*)+)?)

lemma *infer-v-g-weakening*:
fixes *e*::*e* **and** Γ'::Γ **and** *v*::*v*
assumes Θ; $\mathcal{B}$; $\Gamma \vdash v \Rightarrow \tau$ **and** *toSet* $\Gamma \subseteq$ *toSet* Γ' **and** Θ ; $\mathcal{B} \vdash_{wf} \Gamma'$
shows Θ; $\mathcal{B}$; $\Gamma' \vdash v \Rightarrow \tau$
⟨*proof*⟩

lemma *check-v-g-weakening*:
fixes *e*::*e* **and** Γ'::Γ
assumes Θ; $\mathcal{B}$; $\Gamma \vdash v \Leftarrow \tau$ **and** *toSet* $\Gamma \subseteq$ *toSet* Γ' **and** Θ ; $\mathcal{B} \vdash_{wf} \Gamma'$
shows Θ; $\mathcal{B}$; $\Gamma' \vdash v \Leftarrow \tau$
⟨*proof*⟩

lemma *infer-e-g-weakening*:
fixes *e*::*e* **and** Γ'::Γ
assumes Θ ; Φ ; $\mathcal{B}$; Γ ; $\Delta \vdash e \Rightarrow \tau$ **and** *toSet* $\Gamma \subseteq$ *toSet* Γ' **and** Θ ; $\mathcal{B} \vdash_{wf} \Gamma'$
shows Θ ; Φ ; $\mathcal{B}$; Γ'; $\Delta \vdash e \Rightarrow \tau$
⟨*proof*⟩

Special cases proved explicitly, other cases at the end with method +

lemma *infer-e-d-weakening*:
fixes *e*::*e*

assumes Θ ; Φ ; $\mathcal{B}$; Γ ; $\Delta \vdash e \Rightarrow \tau$ **and** *setD* $\Delta \subseteq$ *setD* Δ' **and** *wfD* Θ $\mathcal{B}$ Γ Δ'
shows Θ; Φ ; $\mathcal{B}$; Γ ; $\Delta' \vdash e \Rightarrow \tau$
⟨*proof*⟩

lemma *wfG-x-fresh-in-v-simple*:
fixes *x*::*x* **and** *v* :: *v*
assumes Θ; $\mathcal{B}$; $\Gamma \vdash v \Rightarrow \tau$ **and** *atom x* $\sharp$ Γ
shows *atom x* $\sharp$ *v*
⟨*proof*⟩

lemma *check-s-g-weakening*:
fixes *v*::*v* **and** *s*::*s* **and** *cs*::*branch-s* **and** *x*::*x* **and** *c*::*c* **and** *b*::*b* **and** Γ'::Γ **and** Θ::Θ **and** *css*::*branch-list*
shows *check-s* Θ Φ $\mathcal{B}$ Γ Δ *s t* $\Longrightarrow$ *toSet* $\Gamma \subseteq$ *toSet* $\Gamma' \Longrightarrow$ Θ ; $\mathcal{B} \vdash_{wf} \Gamma' \Longrightarrow$ *check-s* Θ Φ $\mathcal{B}$ Γ' Δ *s t* **and**
check-branch-s Θ Φ $\mathcal{B}$ Γ Δ *tid cons const v cs t* $\Longrightarrow$ *toSet* $\Gamma \subseteq$ *toSet* $\Gamma' \Longrightarrow$ Θ ; $\mathcal{B} \vdash_{wf} \Gamma' \Longrightarrow$ *check-branch-s* Θ Φ $\mathcal{B}$ Γ' Δ *tid cons const v cs t* **and**
check-branch-list Θ Φ $\mathcal{B}$ Γ Δ *tid dclist v css t* $\Longrightarrow$ *toSet* $\Gamma \subseteq$ *toSet* $\Gamma' \Longrightarrow$ Θ ; $\mathcal{B} \vdash_{wf} \Gamma' \Longrightarrow$ *check-branch-list* Θ Φ $\mathcal{B}$ Γ' Δ *tid dclist v css t*
⟨*proof*⟩

lemma *wfG-xa-fresh-in-v*:
fixes *c*::*c* **and** Γ::Γ **and** *G*::Γ **and** *v*::*v* **and** *xa*::*x*
assumes Θ; $\mathcal{B}$; $\Gamma \vdash v \Rightarrow \tau$ **and** *G*=(Γ'@ (*x*, *b*, *c*[*z*::=*V-var x*]$_v$) $\#_\Gamma$ Γ) **and** *atom xa* $\sharp$ *G* **and** Θ ; $\mathcal{B}$ $\vdash_{wf}$ *G*
shows *atom xa* $\sharp$ *v*
⟨*proof*⟩

lemma *fresh-z-subst-g*:
fixes *G*::Γ
assumes *atom z'* $\sharp$ (*x*,*v*) **and** ‹*atom z'* $\sharp$ *G*›
shows *atom z'* $\sharp$ *G*[*x*::=*v*]$_{\Gamma v}$
⟨*proof*⟩

lemma *wfG-xa-fresh-in-subst-v*:
fixes *c*::*c* **and** *v*::*v* **and** *x*::*x* **and** Γ::Γ **and** *G*::Γ **and** *xa*::*x*
assumes Θ; $\mathcal{B}$; $\Gamma \vdash v \Rightarrow \tau$ **and** *G*=(Γ'@ (*x*, *b*, *c*[*z*::=*V-var x*]$_v$) $\#_\Gamma$ Γ) **and** *atom xa* $\sharp$ *G* **and** Θ ; $\mathcal{B}$ $\vdash_{wf}$ *G*
shows *atom xa* $\sharp$ (*subst-gv G x v*)
⟨*proof*⟩

12.8.1 Weakening Immutable Variable Context

declare *check-s-check-branch-s-check-branch-list.intros*[*simp*]
declare *check-s-check-branch-s-check-branch-list.intros*[*intro*]

lemma *check-s-d-weakening*:
fixes *s*::*s* **and** *v*::*v* **and** *cs*::*branch-s* **and** *css*::*branch-list*
shows Θ ; Φ ; $\mathcal{B}$; Γ ; $\Delta \vdash s \Leftarrow \tau \Longrightarrow$ *setD* $\Delta \subseteq$ *setD* $\Delta' \Longrightarrow$ *wfD* Θ $\mathcal{B}$ Γ $\Delta' \Longrightarrow \Theta$; Φ ; $\mathcal{B}$; Γ ; $\Delta' \vdash s \Leftarrow \tau$ **and**
check-branch-s Θ Φ $\mathcal{B}$ Γ Δ *tid cons const v cs* $\tau \Longrightarrow$ *setD* $\Delta \subseteq$ *setD* $\Delta' \Longrightarrow$ *wfD* Θ $\mathcal{B}$ Γ $\Delta' \Longrightarrow$ *check-branch-s* Θ Φ $\mathcal{B}$ Γ Δ' *tid cons const v cs* τ **and**
check-branch-list Θ Φ $\mathcal{B}$ Γ Δ *tid dclist v css* $\tau \Longrightarrow$ *setD* $\Delta \subseteq$ *setD* $\Delta' \Longrightarrow$ *wfD* Θ $\mathcal{B}$ Γ $\Delta' \Longrightarrow$ *check-branch-list* Θ Φ $\mathcal{B}$ Γ Δ' *tid dclist v css* τ

⟨*proof*⟩

lemma *valid-ce-eq*:
fixes *v*::*v* **and** *ce2*::*ce*
assumes *ce1* = *ce2*[*x*::=*v*]$_{cev}$ **and** *wfV* Θ $\mathcal{B}$ *GNil v b* **and** *wfCE* Θ $\mathcal{B}$ ((*x*, *b*, *TRUE*) #$_\Gamma$ *GNil*) *ce2 b*′ **and** *wfCE* Θ $\mathcal{B}$ *GNil ce1 b*′
shows ‹Θ; $\mathcal{B}$; (*x*, *b*, ([[*x*]*v*]*ce* == [*v*]*ce*)) #$_\Gamma$ *GNil* ⊨ *ce1* == *ce2* ›
⟨*proof*⟩

lemma *check-v-top*:
fixes *v*::*v*
assumes Θ; $\mathcal{B}$; *GNil* ⊢ *v* ⇐ *τ* **and** *ce1* = *ce2*[*z*::=*v*]$_{cev}$ **and** Θ; $\mathcal{B}$; *GNil* ⊢$_{wf}$ {| *z* : *b-of τ* | *ce1* == *ce2* |}
and *supp ce1* ⊆ *supp* $\mathcal{B}$
shows Θ; $\mathcal{B}$; *GNil* ⊢ *v* ⇐ {| *z* : *b-of τ* | *ce1* == *ce2* |}
⟨*proof*⟩

end

declare *freshers*[*simp del*]

Chapter 13

Context Subtyping Lemmas

Lemmas allowing us to replace the type of a variable in the context with a subtype and have the judgement remain valid. Also known as narrowing.

13.1 Replace or exchange type of variable in a context

Because the G-context is extended by the statements like let, we will need a generalised substitution lemma for statements. For this we setup a function that replaces in G (rig) for a particular x the constraint for it. We also define a well-formedness relation for RIGs that ensures that each new constraint implies the old one

nominal-function *replace-in-g-many* :: $\Gamma \Rightarrow (x*c)\ list \Rightarrow \Gamma$ **where**
  replace-in-g-many G xcs = List.foldr $(\lambda(x,c)\ G.\ G[x \longmapsto c])$ *xcs G*
  ⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

inductive *replace-in-g-subtyped* :: $\Theta \Rightarrow \mathcal{B} \Rightarrow \Gamma \Rightarrow (x*c)\ list \Rightarrow \Gamma \Rightarrow bool$ (‹ - ; - ⊢ - ⟨ - ⟩ ⇝ -› [*100*,*50*,*50*] *50*) **where**
  replace-in-g-subtyped-nilI: $\Theta;\ \mathcal{B} \vdash G \langle\ []\ \rangle \rightsquigarrow G$
| *replace-in-g-subtyped-consI*: ⟦
    Some $(b,c') = lookup\ G\ x$;
    $\Theta;\ \mathcal{B};\ G \vdash_{wf} c$;
    $\Theta;\ \mathcal{B};\ G[x \longmapsto c] \models c'$;
    $\Theta;\ \mathcal{B} \vdash G[x \longmapsto c] \langle\ xcs\ \rangle \rightsquigarrow G'$; $x \notin fst$ ‘ *set xcs* ⟧ ⟹
    $\Theta;\ \mathcal{B} \vdash G \langle\ (x,c)\#xcs\ \rangle \rightsquigarrow G'$
equivariance *replace-in-g-subtyped*
nominal-inductive *replace-in-g-subtyped* ⟨*proof*⟩

inductive-cases *replace-in-g-subtyped-elims*[*elim*!]:
  $\Theta;\ \mathcal{B} \vdash G \langle\ []\ \rangle \rightsquigarrow G'$
  $\Theta;\ \mathcal{B} \vdash ((x,b,c)\#_\Gamma \Gamma\ G) \langle\ acs\ \rangle \rightsquigarrow ((x,b,c)\#_\Gamma G')$
  $\Theta;\ \mathcal{B} \vdash G' \langle\ (x,c)\#\ acs\ \rangle \rightsquigarrow G$

lemma *rigs-atom-dom-eq*:
  assumes $\Theta;\ \mathcal{B} \vdash G \langle\ xcs\ \rangle \rightsquigarrow G'$
  shows *atom-dom G* = *atom-dom G′*
  ⟨*proof*⟩

lemma *replace-in-g-wfG*:
assumes Θ; $\mathcal{B} \vdash G \langle xcs \rangle \rightsquigarrow G'$ **and** *wfG* Θ $\mathcal{B}$ G
shows *wfG* Θ $\mathcal{B}$ G'
⟨*proof*⟩

lemma *wfD-rig-single*:
fixes Δ::Δ **and** x::x **and** c::c **and** G::Γ
assumes Θ; $\mathcal{B}$; $G \vdash_{wf} \Delta$ **and** *wfG* Θ $\mathcal{B}$ $(G[x \longmapsto c])$
shows Θ; $\mathcal{B}$; $G[x \longmapsto c] \vdash_{wf} \Delta$
⟨*proof*⟩

lemma *wfD-rig*:
assumes Θ; $\mathcal{B} \vdash G \langle xcs \rangle \rightsquigarrow G'$ **and** *wfD* Θ $\mathcal{B}$ G Δ
shows *wfD* Θ $\mathcal{B}$ G' Δ
⟨*proof*⟩

lemma *replace-in-g-fresh*:
fixes x::x
assumes Θ; $\mathcal{B} \vdash \Gamma \langle xcs \rangle \rightsquigarrow \Gamma'$ **and** *wfG* Θ $\mathcal{B}$ Γ **and** *wfG* Θ $\mathcal{B}$ Γ' **and** *atom* $x \sharp \Gamma$
shows *atom* $x \sharp \Gamma'$
⟨*proof*⟩

lemma *replace-in-g-fresh1*:
fixes x::x
assumes Θ; $\mathcal{B} \vdash \Gamma \langle xcs \rangle \rightsquigarrow \Gamma'$ **and** *wfG* Θ $\mathcal{B}$ Γ **and** *atom* $x \sharp \Gamma$
shows *atom* $x \sharp \Gamma'$
⟨*proof*⟩

Wellscoping for an eXchange list

inductive *wsX*:: $\Gamma \Rightarrow (x*c)$ *list* $\Rightarrow$ *bool* **where**
wsX-NilI: *wsX G* []
| *wsX-ConsI*: ⟦ *wsX G xcs* ; *atom x* ∈ *atom-dom G* ; $x \notin$ *fst* ‘ *set xcs* ⟧ ⟹ *wsX G* ((*x*,*c*)#*xcs*)
equivariance *wsX*
nominal-inductive *wsX* ⟨*proof*⟩

lemma *wsX-if1*:
assumes *wsX G xcs*
shows ((*atom* ‘ *fst* ‘ *set xcs*) ⊆ *atom-dom G*) ∧ *List.distinct* (*List.map fst xcs*)
⟨*proof*⟩

lemma *wsX-if2*:
assumes ((*atom* ‘ *fst* ‘ *set xcs*) ⊆ *atom-dom G*) ∧ *List.distinct* (*List.map fst xcs*)
shows *wsX G xcs*
⟨*proof*⟩

lemma *wsX-iff*:
wsX G xcs = (((*atom* ‘ *fst* ‘ *set xcs*) ⊆ *atom-dom G*) ∧ *List.distinct* (*List.map fst xcs*))
⟨*proof*⟩

inductive-cases *wsX-elims*[*elim*!]:
wsX G []

wsX G ((x,c)#xcs)

lemma *wsX-cons*:
assumes *wsX* Γ *xcs* **and** $x \notin$ *fst ' set xcs*
shows *wsX* $((x,\ b,\ c1)\ \#_\Gamma\ \Gamma)\ ((x,\ c2)\ \#\ xcs)$
⟨*proof*⟩

lemma *wsX-cons2*:
assumes *wsX* Γ *xcs* **and** $x \notin$ *fst ' set xcs*
shows *wsX* $((x,\ b,\ c1)\ \#_\Gamma\ \Gamma)\ \ xcs$
⟨*proof*⟩

lemma *wsX-cons3*:
assumes *wsX* Γ *xcs*
shows *wsX* $((x,\ b,\ c1)\ \#_\Gamma\ \Gamma)\ \ xcs$
⟨*proof*⟩

lemma *wsX-fresh*:
assumes *wsX G xcs* **and** *atom x* $\sharp$ *G* **and** *wfG* $\Theta\ \mathcal{B}\ G$
shows $x \notin$ *fst ' set xcs*
⟨*proof*⟩

lemma *replace-in-g-dist*:
assumes $x' \neq x$
shows *replace-in-g* $((x,\ b,c)\ \#_\Gamma\ G)\ x'\ c'' = ((x,\ b,c)\ \#_\Gamma$ (*replace-in-g G x' c''*)) ⟨*proof*⟩

lemma *wfG-replace-inside-rig*:
fixes $c''::c$
assumes $\langle\Theta;\ \mathcal{B} \vdash_{wf}\ G[x'\longmapsto c'']\rangle$ $\langle\Theta;\ \mathcal{B} \vdash_{wf}\ (x,\ b,\ c)\ \#_\Gamma\ G\rangle$
shows $\Theta;\ \mathcal{B} \vdash_{wf}\ (x,\ b,\ c)\ \#_\Gamma\ G[x'\longmapsto c'']$
⟨*proof*⟩

lemma *replace-in-g-valid-weakening*:
assumes $\Theta;\ \mathcal{B};\ \Gamma[x'\longmapsto c''] \models c'$ **and** $x' \neq x$ **and** $\Theta;\ \mathcal{B} \vdash_{wf}\ (x,\ b,\ c)\ \#_\Gamma\ \Gamma[x'\longmapsto c'']$
shows $\Theta;\ \mathcal{B};\ ((x,\ b,c)\ \#_\Gamma\ \Gamma)[x'\longmapsto\ c'']\ \models c'$
⟨*proof*⟩

lemma *replace-in-g-subtyped-cons*:
assumes *replace-in-g-subtyped* $\Theta\ \mathcal{B}\ G\ xcs\ G'$ **and** *wfG* $\Theta\ \mathcal{B}\ ((x,b,c)\#_\Gamma G)$
shows $x \notin$ *fst ' set xcs* $\Longrightarrow$ *replace-in-g-subtyped* $\Theta\ \mathcal{B}\ ((x,b,c)\#_\Gamma G)\ xcs\ ((x,b,c)\#_\Gamma G')$
⟨*proof*⟩

lemma *replace-in-g-split*:
fixes $G::\Gamma$
assumes $\Gamma =$ *replace-in-g* $\Gamma'\ x\ c$ **and** $\Gamma' = \ G'@(x,b,c')\#_\Gamma G$ **and** *wfG* $\Theta\ \mathcal{B}\ \Gamma'$
shows $\Gamma = \ G'@(x,b,c)\#_\Gamma G$
⟨*proof*⟩

lemma *replace-in-g-subtyped-split0*:
fixes $G::\Gamma$
assumes *replace-in-g-subtyped* $\Theta\ \mathcal{B}\ \Gamma'[(x,c)]\ \Gamma$ **and** $\Gamma' = \ G'@(x,b,c')\#_\Gamma G$ **and** *wfG* $\Theta\ \mathcal{B}\ \Gamma'$
shows $\Gamma = \ G'@(x,b,c)\#_\Gamma G$

⟨*proof*⟩

lemma *replace-in-g-subtyped-split*:
assumes *Some* $(b,\ c') = lookup\ G\ x$ **and** $\Theta;\ \mathcal{B};\ replace\text{-}in\text{-}g\ G\ x\ c\ \models c'$ **and** $wfG\ \Theta\ \mathcal{B}\ G$
shows $\exists\ \Gamma\ \Gamma'.\ G = \Gamma'@(x,b,c')\#_\Gamma\Gamma \wedge \Theta;\ \mathcal{B};\ \Gamma'@(x,b,c)\#_\Gamma\Gamma \models c'$
⟨*proof*⟩

13.2 Validity and Subtyping

lemma *wfC-replace-in-g*:
fixes *c*::*c* **and** *c0*::*c*
assumes $\Theta;\ \mathcal{B};\ \Gamma'@(x,b,c0')\#_\Gamma\Gamma \vdash_{wf} c$ **and** $\Theta;\ \mathcal{B};\ (x,b,TRUE)\#_\Gamma\Gamma \vdash_{wf} c0$
shows $\Theta;\ \mathcal{B};\ \Gamma'\ @\ (x,\ b,\ c0)\ \#_\Gamma\ \Gamma \vdash_{wf} c$
⟨*proof*⟩

lemma *ctx-subtype-valid*:
assumes $\Theta;\ \mathcal{B};\ \Gamma'@(x,b,c0')\#_\Gamma\Gamma \models c$ **and**
$\Theta;\ \mathcal{B};\ \Gamma'@(x,b,c0)\#_\Gamma\Gamma \models c0'$
shows $\Theta;\ \mathcal{B};\ \Gamma'@(x,b,c0)\#_\Gamma\Gamma \models c$
⟨*proof*⟩

lemma *ctx-subtype-subtype*:
fixes Γ::Γ
shows $\Theta;\ \mathcal{B};\ G \vdash t1 \lesssim t2 \Longrightarrow G = \Gamma'@(x,b0,c0')\#_\Gamma\Gamma \Longrightarrow \Theta;\ \mathcal{B};\ \Gamma'@(x,b0,c0)\#_\Gamma\Gamma \models c0' \Longrightarrow \Theta;\ \mathcal{B};\ \Gamma'@(x,b0,c0)\#_\Gamma\Gamma \vdash t1 \lesssim t2$
⟨*proof*⟩

lemma *ctx-subtype-subtype-rig*:
assumes *replace-in-g-subtyped* $\Theta\ \ \mathcal{B}\ \Gamma'\ [(x,c0)]\ \Gamma$ **and** $\Theta;\ \mathcal{B};\ \Gamma' \vdash t1 \lesssim t2$
shows $\Theta;\ \mathcal{B};\ \Gamma \vdash t1 \lesssim t2$
⟨*proof*⟩

We now prove versions of the *ctx-subtype* lemmas above using *replace-in-g*. First we do case where the replace is just for a single variable (indicated by suffix rig) and then the general case for multiple replacements (indicated by suffix rigs)

lemma *ctx-subtype-subtype-rigs*:
assumes *replace-in-g-subtyped* $\Theta\ \mathcal{B}\ \Gamma'\ xcs\ \Gamma$ **and** $\Theta;\ \mathcal{B};\ \Gamma' \vdash t1 \lesssim t2$
shows $\Theta;\ \mathcal{B};\ \Gamma \vdash t1 \lesssim t2$
⟨*proof*⟩

lemma *replace-in-g-inside-valid*:
assumes *replace-in-g-subtyped* $\Theta\ \mathcal{B}\ \Gamma'\ [(x,c0)]\ \Gamma$ **and** $wfG\ \Theta\ \mathcal{B}\ \Gamma'$
shows $\exists\ b\ c0'\ G\ G'.\ \Gamma' = G'\ @\ (x,b,c0')\#_\Gamma G \wedge\ \ \Gamma = G'\ @\ (x,b,c0)\#_\Gamma G \wedge \Theta;\ \mathcal{B};\ G'@\ (x,b,c0)\#_\Gamma G \models c0'$
⟨*proof*⟩

lemma *replace-in-g-valid*:
assumes $\Theta;\ \mathcal{B}\ \vdash G\ \langle\ xcs\ \rangle \rightsquigarrow G'$ **and** $\Theta;\ \mathcal{B};\ G\ \models c$
shows $\langle\Theta;\ \mathcal{B};\ G'\ \models c\ \rangle$
⟨*proof*⟩

13.3 Literals

13.4 Values

lemma *lookup-inside-unique-b*[*simp*]:
assumes Θ ; $B \vdash_{wf} (\Gamma'@(x,b0,c0)\#_\Gamma\Gamma)$ **and** Θ ; $B \vdash_{wf} (\Gamma'@(x,b0,c0')\#_\Gamma\Gamma)$
and *Some* $(b, c) =$ *lookup* $(\Gamma' @ (x, b0, c0') \#_\Gamma \Gamma)$ y **and** *Some* $(b0,c0) =$ *lookup* $(\Gamma'@((x,b0,c0))\#_\Gamma\Gamma)$ x **and** $x{=}y$
shows $b = b0$
⟨*proof*⟩

lemma *ctx-subtype-v-aux*:
fixes $v{::}v$
assumes $\Theta; \mathcal{B}; \Gamma'@((x,b0,c0')\#_\Gamma\Gamma) \vdash v \Rightarrow t1$ **and** $\Theta; \mathcal{B}; \Gamma'@(x,b0,c0)\#_\Gamma\Gamma \models c0'$
shows $\Theta; \mathcal{B}; \Gamma'@((x,b0,c0)\#_\Gamma\Gamma) \vdash v \Rightarrow t1$
⟨*proof*⟩

lemma *ctx-subtype-v*:
fixes $v{::}v$
assumes $\Theta; \mathcal{B}; \Gamma'@((x,b0,c0')\#_\Gamma\Gamma) \vdash v \Rightarrow t1$ **and** $\Theta; \mathcal{B}; \Gamma'@(x,b0,c0)\#_\Gamma\Gamma \models c0'$
shows $\exists\, t2.\ \Theta; \mathcal{B}; \Gamma'@((x,b0,c0)\#_\Gamma\Gamma) \vdash v \Rightarrow t2 \wedge \Theta; \mathcal{B}; \Gamma'@((x,b0,c0)\#_\Gamma\Gamma) \vdash t2 \lesssim t1$
⟨*proof*⟩

lemma *ctx-subtype-v-eq*:
fixes $v{::}v$
assumes
$\Theta; \mathcal{B}; \Gamma'@((x,b0,c0')\#_\Gamma\Gamma) \vdash v \Rightarrow t1$ **and**
$\Theta; \mathcal{B}; \Gamma'@(x,b0,c0)\#_\Gamma\Gamma \models c0'$
shows $\Theta; \mathcal{B}; \Gamma'@((x,b0,c0)\#_\Gamma\Gamma) \vdash v \Rightarrow t1$
⟨*proof*⟩

lemma *ctx-subtype-check-v-eq*:
assumes $\Theta; \mathcal{B}; \Gamma'@((x,b0,c0')\#_\Gamma\Gamma) \vdash v \Leftarrow t1$ **and** $\Theta; \mathcal{B}; \Gamma'@(x,b0,c0)\#_\Gamma\Gamma \models c0'$
shows $\Theta; \mathcal{B}; \Gamma'@((x,b0,c0)\#_\Gamma\Gamma) \vdash v \Leftarrow t1$
⟨*proof*⟩

Basically the same as *ctx-subtype-v-eq* but in a different form

lemma *ctx-subtype-v-rig-eq*:
fixes $v{::}v$
assumes *replace-in-g-subtyped* $\Theta\ \mathcal{B}\ \Gamma'\ [(x,c0)]\ \Gamma$ **and**
$\Theta; \mathcal{B}; \Gamma' \vdash v \Rightarrow t1$
shows $\Theta; \mathcal{B}; \Gamma \vdash v \Rightarrow t1$
⟨*proof*⟩

lemma *ctx-subtype-v-rigs-eq*:
fixes $v{::}v$
assumes *replace-in-g-subtyped* $\Theta\ \mathcal{B}\ \Gamma'\ xcs\ \Gamma$ **and**
$\Theta; \mathcal{B}; \Gamma' \vdash v \Rightarrow t1$
shows $\Theta; \mathcal{B}; \Gamma \vdash v \Rightarrow t1$
⟨*proof*⟩

lemma *ctx-subtype-check-v-rigs-eq*:

assumes *replace-in-g-subtyped* $\Theta\ \mathcal{B}\ \Gamma'\ xcs\ \Gamma$ **and**
Θ; $\mathcal{B}$; $\Gamma'\ \vdash v \Leftarrow t1$
shows Θ; $\mathcal{B}$; $\Gamma \vdash v \Leftarrow t1$
⟨*proof*⟩

13.5 Expressions

lemma *valid-wfC*:
fixes *c0*::*c*
assumes Θ; $\mathcal{B}$; $\Gamma'@(x,b0,c0)\#_\Gamma\Gamma \models c0'$
shows Θ; $\mathcal{B}$; $(x,\ b0,\ TRUE)\ \#_\Gamma\ \Gamma\ \vdash_{wf}\ c0$
⟨*proof*⟩

lemma *ctx-subtype-e-eq*:
fixes *G*::Γ
assumes
$\Theta\ ;\ \Phi\ ;\ \mathcal{B}\ ;\ G\ ;\ \Delta \vdash e \Rightarrow t1$ **and** $G = \ \Gamma'@((x,b0,c0')\#_\Gamma\Gamma)$
Θ; $\mathcal{B}$; $\Gamma'@(x,b0,c0)\#_\Gamma\Gamma \models c0'$
shows $\Theta\ ;\ \Phi\ ;\ \mathcal{B}\ ;\ \Gamma'@((x,b0,c0)\#_\Gamma\Gamma)\ ;\ \Delta \vdash e \Rightarrow t1$
⟨*proof*⟩

lemma *ctx-subtype-e-rig-eq*:
assumes *replace-in-g-subtyped* $\Theta\ \mathcal{B}\ \Gamma'\ [(x,c0)]\ \Gamma$ **and**
$\Theta\ ;\ \Phi\ ;\ \mathcal{B}\ ;\ \Gamma'\ ;\ \Delta \vdash e \Rightarrow t1$
shows $\Theta\ ;\ \Phi\ ;\ \mathcal{B}\ ;\ \Gamma\ ;\ \Delta \vdash e \Rightarrow t1$
⟨*proof*⟩

lemma *ctx-subtype-e-rigs-eq*:
assumes *replace-in-g-subtyped* $\Theta\ \mathcal{B}\ \Gamma'\ xcs\ \Gamma$ **and**
$\Theta\ ;\ \Phi\ ;\ \mathcal{B}\ ;\ \Gamma';\ \Delta \vdash e \Rightarrow t1$
shows $\Theta\ ;\ \Phi\ ;\ \mathcal{B}\ ;\ \Gamma\ ;\ \Delta \vdash e \Rightarrow t1$
⟨*proof*⟩

13.6 Statements

lemma *ctx-subtype-s-rigs*:
fixes *c0*::*c* **and** *s*::*s* **and** *G'*::Γ **and** *xcs* :: $(x*c)$ *list* **and** *css*::*branch-list*
shows
check-s $\Theta\ \Phi\ \mathcal{B}\ G\ \Delta\ \ s\ \ t1 \Longrightarrow wsX\ G\ xcs \Longrightarrow$ *replace-in-g-subtyped* $\Theta\ \mathcal{B}\ G\ xcs\ G' \Longrightarrow$ *check-s* $\Theta\ \Phi\ \mathcal{B}\ G'\ \Delta\ \ s\ \ t1$ **and**
check-branch-s $\Theta\ \Phi\ \mathcal{B}\ G\ \Delta$ *tid cons const v cs* $\ t1 \Longrightarrow\ wsX\ G\ xcs \Longrightarrow$ *replace-in-g-subtyped* $\Theta\ \mathcal{B}\ G\ xcs\ G' \Longrightarrow$ *check-branch-s* $\Theta\ \Phi\ \mathcal{B}\ G'\ \Delta$ *tid cons const v cs t1*
check-branch-list $\Theta\ \Phi\ \mathcal{B}\ G\ \Delta$ *tid dclist v css* $\ t1 \Longrightarrow\ wsX\ G\ xcs \Longrightarrow$ *replace-in-g-subtyped* $\Theta\ \mathcal{B}\ G\ xcs\ G' \Longrightarrow$ *check-branch-list* $\Theta\ \Phi\ \mathcal{B}\ G'\ \Delta$ *tid dclist v css t1*
⟨*proof*⟩

lemma *replace-in-g-subtyped-empty*:
assumes *wfG* $\Theta\ \mathcal{B}\ (\Gamma'\ @\ (x,\ b,\ c[z::=V\text{-}var\ x]_{cv})\ \#_\Gamma\ \Gamma)$
shows *replace-in-g-subtyped* $\Theta\ \mathcal{B}$ (*replace-in-g* $(\Gamma'\ @\ (x,\ b,\ c[z::=V\text{-}var\ x]_{cv})\ \#_\Gamma\ \Gamma)\ x\ (c'[z'::=V\text{-}var\ x]_{cv}))\ []\ (\Gamma'\ @\ (x,\ b,\ c'[z'::=V\text{-}var\ x]_{cv})\ \#_\Gamma\ \Gamma)$
⟨*proof*⟩

lemma *ctx-subtype-s*:
  fixes $s::s$
  assumes Θ ; Φ ; $\mathcal{B}$; $\Gamma'@((x,b,c[z::=V\text{-}var\ x]_{cv})\#_\Gamma\Gamma)$; $\Delta \vdash s \Leftarrow \tau$ **and**
    Θ; $\mathcal{B}$; $\Gamma \vdash \{\!|\ z' : b \mid c'\ |\!\} \lesssim \{\!|\ z : b \mid c\ |\!\}$ **and**
    $atom\ x \sharp (z,z',c,c')$
  shows Θ ; Φ ; $\mathcal{B}$; $\Gamma'@(x,b,c'[z'::=V\text{-}var\ x]_{cv})\#_\Gamma\Gamma$; $\Delta \vdash s \Leftarrow \tau$
⟨*proof*⟩

end

Chapter 14

Immutable Variable Substitution Lemmas

Lemmas that show that types are preserved, in some way, under immutable variable substitution

14.1 Proof Methods

method *subst-mth* = (*metis subst-g-inside infer-e-wf infer-v-wf infer-v-wf*)

method *subst-tuple-mth* **uses** *add* = (
(*unfold fresh-prodN*), (*simp add*: *add*)+,
(*rule*,*metis fresh-z-subst-g add fresh-Pair*),
(*metis fresh-subst-dv add fresh-Pair*))

14.2 Prelude

lemma *subst-top-eq*:
$\{\!|\ z : b\ |\ TRUE\ |\!\} = \{\!|\ z : b\ |\ TRUE\ |\!\}[x::=v]_{\tau v}$
⟨*proof*⟩

lemma *wfD-subst*:
fixes $\tau_1::\tau$ **and** $v::v$ **and** $\Delta::\Delta$ **and** $\Theta::\Theta$ **and** $\Gamma::\Gamma$
assumes $\Theta\ ;\ \mathcal{B}\ ;\ \Gamma \vdash v \Rightarrow \tau_1$ **and** *wfD* $\Theta\ \ \mathcal{B}\ (\Gamma'@((x,b_1,c0[z0::=[x]^v]_{cv})\ \#_\Gamma\ \Gamma))\ \ \Delta$ **and** *b-of* $\tau_1=b_1$
shows $\Theta\ ;\ \mathcal{B}\ ;\ \Gamma'[x::=v]_{\Gamma v}\ @\ \Gamma\ \vdash_{wf}\ \Delta[x::=v]_{\Delta v}$
⟨*proof*⟩

lemma *subst-v-c-of*:
assumes *atom xa* ♯ (*v*,*x*)
shows *c-of* $t[x::=v]_{\tau v}$ *xa* = (*c-of t xa*)$[x::=v]_{cv}$
⟨*proof*⟩

14.3 Context

lemma *subst-lookup*:
assumes *Some* (*b*,*c*) = *lookup* $(\Gamma'@((x,b_1,c_1)\#_\Gamma\Gamma))$ *y* **and** $x \neq y$ **and** *wfG* $\Theta\ \mathcal{B}\ (\Gamma'@((x,b_1,c_1)\#_\Gamma\Gamma))$
shows ∃ *d*. *Some* (*b*,*d*) = *lookup* $((\Gamma'[x::=v]_{\Gamma v})@\Gamma)$ *y*

⟨*proof*⟩

14.4 Validity

lemma *subst-self-valid*:
fixes $v::v$
assumes $\Theta\ ;\ \mathcal{B}\ ;\ G \vdash v \Rightarrow \{\!|\ z : b \mid c\ |\!\}$ **and** $atom\ z \sharp v$
shows $\Theta\ ;\ \mathcal{B}\ ;\ G \models c[z::=v]_{cv}$
⟨*proof*⟩

lemma *subst-valid-simple*:
fixes $v::v$
assumes $\Theta\ ;\ \mathcal{B}\ ;\ G \vdash v \Rightarrow \{\!|\ z0 : b \mid c0\ |\!\}$ **and**
$atom\ z0 \sharp c$ **and** $atom\ z0 \sharp v$
$\Theta;\ \mathcal{B}\ ;\ (z0,b,c0)\#_\Gamma G \models c[z::=V\text{-}var\ z0]_{cv}$
shows $\Theta\ ;\ \mathcal{B}\ ;\ G \models c[z::=v]_{cv}$
⟨*proof*⟩

lemma *wfI-subst1*:
assumes $wfI\ \Theta\ (G'[x::=v]_{\Gamma v}\ @\ G)\ i$ **and** $wfG\ \Theta\ \mathcal{B}\ (G'\ @\ (x,\ b,\ c[z::=[x]^v]_{cv})\ \#_\Gamma\ G)$ **and** $eval\text{-}v\ i\ v\ sv$ **and** $wfRCV\ \Theta\ sv\ b$
shows $wfI\ \Theta\ (G'\ @\ (x,\ b,\ c[z::=[x]^v]_{cv})\ \#_\Gamma\ G)\ (\ i(\ x \mapsto sv))$
⟨*proof*⟩

lemma *subst-valid*:
fixes $v::v$ **and** $c'::c$ **and** $\Gamma::\Gamma$
assumes $\Theta\ ;\ \mathcal{B}\ ;\ \Gamma \models c[z::=v]_{cv}$ **and** $\Theta\ ;\ \mathcal{B}\ ;\ \Gamma \vdash_{wf} v : b$ **and**
$\Theta\ ;\ \mathcal{B} \vdash_{wf} \Gamma$ **and** $atom\ x \sharp c$ **and** $atom\ x \sharp \Gamma$ **and**
$\Theta\ ;\ \mathcal{B} \vdash_{wf} (\Gamma'@(x,b,c[z::=[x]^v]_{cv}\ )\ \#_\Gamma\ \Gamma)$ **and**
$\Theta\ ;\ \mathcal{B}\ ;\ \Gamma'@(x,b,\ c[z::=[x]^v]_{cv}\ )\ \#_\Gamma\ \Gamma \models c'$ (**is** $\Theta\ ;\ \mathcal{B};\ \ ?G \models c'$)
shows $\Theta\ ;\ \mathcal{B}\ ;\ \Gamma'[x::=v]_{\Gamma v}@\Gamma \models c'[x::=v]_{cv}$
⟨*proof*⟩

lemma *subst-valid-infer-v*:
fixes $v::v$ **and** $c'::c$
assumes $\Theta\ ;\ \mathcal{B}\ ;\ G \vdash v \Rightarrow \{\!|\ z0 : b \mid c0\ |\!\}$ **and** $atom\ x \sharp c$ **and** $atom\ x \sharp G$ **and** $wfG\ \Theta\ \mathcal{B}$ $(G'@(x,b,c[z::=[x]^v]_{cv}\ )\ \#_\Gamma\ G)$ **and** $atom\ z0 \sharp v$
$\Theta;\mathcal{B};(z0,b,c0)\#_\Gamma G \models c[z::=V\text{-}var\ z0]_{cv}$ **and** $atom\ z0 \sharp c$ **and**
$\Theta;\mathcal{B};G'@(x,b,\ c[z::=[x]^v]_{cv}\ )\ \#_\Gamma\ G \models c'$ (**is** $\Theta\ ;\ \mathcal{B};\ \ ?G \models c'$)
shows $\Theta;\mathcal{B};G'[x::=v]_{\Gamma v}@G \models c'[x::=v]_{cv}$
⟨*proof*⟩

14.5 Subtyping

lemma *subst-subtype*:
fixes $v::v$
assumes $\Theta\ ;\ \mathcal{B}\ ;\ \Gamma \vdash v \Rightarrow (\{\!|z0{:}b|c0|\!\})$ **and**
$\Theta;\mathcal{B};\Gamma \vdash (\{\!|z0{:}b|c0|\!\}) \lesssim\ (\{\!|\ z : b \mid c\ |\!\})$ **and**
$\Theta;\mathcal{B};\Gamma'@((x,b,c[z::=[x]^v]_{cv})\#_\Gamma\Gamma) \vdash (\{\!|\ z1 : b1 \mid c1\ |\!\}) \lesssim (\{\!|\ z2 : b1 \mid c2\ |\!\})$ (**is** $\Theta\ ;\ \mathcal{B};\ ?G1 \vdash ?t1 \lesssim ?t2\ )$ **and**
$atom\ z \sharp (x,v) \wedge atom\ z0 \sharp (c,x,v,z,\Gamma) \wedge atom\ z1 \sharp (x,v) \wedge atom\ z2 \sharp (x,v)$ **and** $wsV\ \Theta\ \mathcal{B}\ \Gamma\ v$

shows $\Theta;\mathcal{B};\Gamma'[x::=v]_{\Gamma v}@\Gamma \vdash \{\!|\ z1 : b1 \mid c1\ |\!\}[x::=v]_{\tau v} \lesssim \{\!|\ z2 : b1 \mid c2\ |\!\}[x::=v]_{\tau v}$
⟨*proof*⟩

lemma *subst-subtype-tau*:
fixes $v::v$
assumes $\Theta\ ;\ \mathcal{B}\ ;\ \Gamma \vdash v \Rightarrow \tau$ **and**
$\Theta\ ;\ \mathcal{B}\ ;\ \Gamma \vdash \tau \lesssim (\{\!|\ z : b \mid c\ |\!\})$
$\Theta\ ;\ \mathcal{B}\ ;\ \Gamma'@((x,b,c[z::=[x]^v]_{cv})\#_\Gamma\Gamma) \vdash \tau 1 \lesssim \tau 2$ **and**
atom $z \sharp (x,v)$
shows $\Theta\ ;\ \mathcal{B}\ ;\ \Gamma'[x::=v]_{\Gamma v}@\Gamma \vdash \tau 1[x::=v]_{\tau v} \lesssim \tau 2[x::=v]_{\tau v}$
⟨*proof*⟩

lemma *subtype-if1*:
fixes $v::v$
assumes $P\ ;\ \mathcal{B}\ ;\ \Gamma\ \vdash t1 \lesssim t2$ **and** *wfV* $P\ \mathcal{B}\ \Gamma\ v$ (*base-for-lit* l) **and**
atom $z1 \sharp v$ **and** *atom* $z2 \sharp v$ **and** *atom* $z1 \sharp t1$ **and** *atom* $z2 \sharp t2$ **and** *atom* $z1 \sharp \Gamma$ **and** *atom* $z2 \sharp \Gamma$
shows $P\ ;\ \mathcal{B}\ ;\ \Gamma \vdash \{\!|\ z1 : b\text{-}of\ t1\ |\ CE\text{-}val\ v\ ==\ CE\text{-}val\ (V\text{-}lit\ l)\ \ IMP\ \ (c\text{-}of\ t1\ z1)\ |\!\} \lesssim \{\!|\ z2 : b\text{-}of\ t2\ |\ CE\text{-}val\ v\ ==\ CE\text{-}val\ (V\text{-}lit\ l)\ IMP\ \ (c\text{-}of\ t2\ z2)\ |\!\}$
⟨*proof*⟩

14.6 Values

lemma *subst-infer-aux*:
fixes $\tau_1::\tau$ **and** $v'::v$
assumes $\Theta\ ;\ \mathcal{B}\ ;\ \Gamma \vdash v'[x::=v]_{vv} \Rightarrow \tau_1$ **and** $\Theta\ ;\ \mathcal{B}\ ;\ \Gamma' \vdash v' \Rightarrow \tau_2$ **and** $b\text{-}of\ \tau_1 = b\text{-}of\ \tau_2$
shows $\tau_1 = (\tau_2[x::=v]_{\tau v})$
⟨*proof*⟩

lemma *subst-t-b-eq*:
fixes $x::x$ **and** $v::v$
shows $b\text{-}of\ (\tau[x::=v]_{\tau v}) = b\text{-}of\ \tau$
⟨*proof*⟩

lemma *fresh-g-fresh-v*:
fixes $x::x$
assumes *atom* $x \sharp \Gamma$ **and** *wfV* $\Theta\ \mathcal{B}\ \Gamma\ v\ b$
shows *atom* $x \sharp v$
⟨*proof*⟩

lemma *infer-v-fresh-g-fresh-v*:
fixes $x::x$ **and** $\Gamma::\Gamma$ **and** $v::v$
assumes *atom* $x \sharp \Gamma'@\Gamma$ **and** $\Theta\ ;\ \mathcal{B}\ ;\ \Gamma \vdash v \Rightarrow \tau$
shows *atom* $x \sharp v$
⟨*proof*⟩

lemma *infer-v-fresh-g-fresh-xv*:
fixes $xa::x$ **and** $v::v$ **and** $\Gamma::\Gamma$
assumes *atom* $xa \sharp \Gamma'@((x,b,c[z::=[x]^v]_{cv})\#_\Gamma\Gamma)$ **and** $\Theta\ ;\ \mathcal{B}\ ;\ \Gamma \vdash v \Rightarrow \tau$
shows *atom* $xa \sharp (x,v)$
⟨*proof*⟩

lemma *wfG-subst-infer-v*:
fixes $v::v$
assumes $\Theta\ ;\ \mathcal{B} \vdash_{wf} \Gamma'\ @\ (x,\ b,\ c0[z0::=[x]^v]_{cv})\ \#_\Gamma\ \Gamma$ **and** $\Theta\ ;\ \mathcal{B}\ ;\ \Gamma \vdash v \Rightarrow \tau$ **and** *b-of* $\tau = b$
shows $\Theta\ ;\ \mathcal{B} \vdash_{wf} \Gamma'[x::=v]_{\Gamma v}\ @\ \Gamma$
⟨*proof*⟩

lemma *fresh-subst-gv-inside*:
fixes $\Gamma::\Gamma$
assumes *atom* $z \sharp \Gamma'\ @\ (x,\ b_1,\ c0[z0::=[\ x\ ]^v]_{cv})\ \#_\Gamma\ \Gamma$ **and** *atom* $z \sharp v$
shows *atom* $z \sharp \Gamma'[x::=v]_{\Gamma v} @ \Gamma$
⟨*proof*⟩

lemma *subst-t*:
fixes $x::x$ **and** $b::b$ **and** $xa::x$
assumes *atom* $z \sharp x$ **and** *atom* $z \sharp v$
shows $(\{\!|\ z : b\ |\ [\ [\ z\ ]^v\ ]^{ce}\ ==\ [\ v'[x::=v]_{vv}\ ]^{ce}\ |\!\}) = (\{\!|\ z : b\ |\ [\ [\ z\ ]^v\ ]^{ce}\ ==\ [\ v']^{ce}\ |\!\}[x::=v]_{\tau v})$
⟨*proof*⟩

lemma *infer-l-fresh*:
assumes $\vdash l \Rightarrow \tau$
shows *atom* $x \sharp \tau$
⟨*proof*⟩

lemma *subst-infer-v*:
fixes $v::v$ **and** $v'::v$
assumes $\Theta\ ;\ \mathcal{B}\ ;\ \Gamma'@((x,b_1,c0[z0::=[x]^v]_{cv})\#_\Gamma\Gamma) \vdash v' \Rightarrow \tau_2$ **and**
$\Theta\ ;\ \mathcal{B}\ ;\ \Gamma \vdash v \Rightarrow \tau_1$ **and**
$\Theta\ ;\ \mathcal{B}\ ;\ \Gamma \vdash \tau_1 \lesssim (\{\!|\ z0 : b_1\ |\ c0\ |\!\})$ **and** *atom* $z0 \sharp (x,v)$
shows $\Theta\ ;\ \mathcal{B}\ ;\ (\Gamma'[x::=v]_{\Gamma v})@\Gamma\ \vdash v'[x::=v]_{vv}\ \Rightarrow \tau_2[x::=v]_{\tau v}$
⟨*proof*⟩

lemma *subst-infer-check-v*:
fixes $v::v$ **and** $v'::v$
assumes $\Theta\ ;\ \mathcal{B}\ ;\ \Gamma \vdash v \Rightarrow \tau_1$ **and**
check-v $\Theta\ \mathcal{B}\ (\Gamma'@((x,b_1,c0[z0::=[x]^v]_{cv})\#_\Gamma\Gamma))\ \ v'\ \tau_2$ **and**
$\Theta\ ;\ \mathcal{B}\ ;\ \Gamma \vdash \tau_1 \lesssim \{\!|\ z0 : b_1\ |\ c0\ |\!\}$ **and** *atom* $z0 \sharp (x,v)$
shows *check-v* $\Theta\ \mathcal{B}\ ((\Gamma'[x::=v]_{\Gamma v})@\Gamma)\ \ (v'[x::=v]_{vv})\ (\tau_2[x::=v]_{\tau v})$
⟨*proof*⟩

lemma *type-veq-subst*[*simp*]:
assumes *atom* $z \sharp (x,v)$
shows $\{\!|\ z : b\ |\ \textit{CE-val}\ (\textit{V-var}\ z)\ ==\ \textit{CE-val}\ v'\ |\!\}[x::=v]_{\tau v} = \{\!|\ z : b\ |\ \textit{CE-val}\ (\textit{V-var}\ z)\ ==\ \textit{CE-val}\ v'[x::=v]_{vv}\ |\!\}$
⟨*proof*⟩

lemma *subst-infer-v-form*:
fixes $v::v$ **and** $v'::v$ **and** $\Gamma::\Gamma$
assumes $\Theta\ ;\ \mathcal{B}\ ;\ \Gamma \vdash v \Rightarrow \tau_1$ **and**
$\Theta\ ;\ \mathcal{B}\ ;\ \Gamma'@((x,b_1,c0[z0::=[x]^v]_{cv})\#_\Gamma\Gamma) \vdash v' \Rightarrow \tau_2$ **and** $b = b\text{-}of\ \tau_2$
$\Theta\ ;\ \mathcal{B}\ ;\ \Gamma \vdash \tau_1 \lesssim (\{\!|\ z0 : b_1\ |\ c0\ |\!\})$ **and** *atom* $z0 \sharp (x,v)$ **and** *atom* $z3' \sharp (x,v,v',\Gamma'@((x,b_1,c0[z0::=[x]^v]_{cv})\#_\Gamma\Gamma)$
)
shows $\langle\ \Theta\ ;\ \mathcal{B}\ ;\ \Gamma'[x::=v]_{\Gamma v}\ @\ \Gamma\ \vdash v'[x::=v]_{vv} \Rightarrow \{\!|\ z3' : b\ |\ \textit{CE-val}\ (\textit{V-var}\ z3')\ ==\ \textit{CE-val}$

$v'[x::=v]_{vv}$ $\}\!\rangle$
⟨*proof*⟩

14.7 Expressions

For operator, fst and snd cases, we use elimination to get one or more values, apply the substitution lemma for values. The types always have the same form and are equal under substitution. For function application, the subst value is a subtype of the value which is a subtype of the argument. The return of the function is the same under substitution.

Observe a similar pattern for each case

lemma *subst-infer-e*:
fixes $v::v$ **and** $e::e$ **and** $\Gamma'::\Gamma$
assumes
Θ ; Φ ; $\mathcal{B}$; G ; $\Delta \vdash e \Rightarrow \tau_2$ **and** $G = (\Gamma'@((x,b_1,$*subst-cv c0 z0* $(V\text{-}var\ x))\#_\Gamma\Gamma))$
Θ ; $\mathcal{B}$; $\Gamma \vdash v \Rightarrow \tau_1$ **and**
Θ; $\mathcal{B}$; $\Gamma \vdash \tau_1 \lesssim \{\!|\ z0 : b_1 \mid c0\ |\!\}$ **and** *atom z0* $\sharp$ (x,v)
shows Θ ; Φ ; $\mathcal{B}$; $((\Gamma'[x::=v]_{\Gamma v})@\Gamma)$; $(\Delta[x::=v]_{\Delta v}) \vdash$ (*subst-ev e x v*) $\Rightarrow \tau_2[x::=v]_{\tau v}$
⟨*proof*⟩

lemma *infer-e-uniqueness*:
assumes Θ ; Φ ; $\mathcal{B}$; *GNil* ; $\Delta \vdash e_1 \Rightarrow \tau_1$ **and** Θ ; Φ ; $\mathcal{B}$; *GNil* ; $\Delta \vdash e_1 \Rightarrow \tau_2$
shows $\tau_1 = \tau_2$
⟨*proof*⟩

14.8 Statements

lemma *subst-infer-check-v1*:
fixes $v::v$ **and** $v'::v$ **and** $\Gamma::\Gamma$
assumes $\Gamma = \Gamma_1@((x,b_1,c0[z0::=[x]^v]_{cv})\#_\Gamma\Gamma_2)$ **and**
Θ ; $\mathcal{B}$; $\Gamma_2 \vdash v \Rightarrow \tau_1$ **and**
Θ ; $\mathcal{B}$; $\Gamma \vdash v' \Leftarrow \tau_2$ **and**
Θ ; $\mathcal{B}$; $\Gamma_2 \vdash \tau_1 \lesssim \{\!|\ z0 : b_1 \mid c0\ |\!\}$ **and** *atom z0* $\sharp$ (x,v)
shows Θ ; $\mathcal{B}$; $\Gamma[x::=v]_{\Gamma v} \vdash v'[x::=v]_{vv} \Leftarrow \tau_2[x::=v]_{\tau v}$
⟨*proof*⟩

lemma *infer-v-c-valid*:
assumes Θ ; $\mathcal{B}$; $\Gamma \vdash v \Rightarrow \tau$ **and** Θ ; $\mathcal{B}$; $\Gamma \vdash \tau \lesssim \{\!|\ z : b \mid c\ |\!\}$
shows $\langle\!|\Theta$; $\mathcal{B}$; $\Gamma \models c[z::=v]_{cv}\ |\!\rangle$
⟨*proof*⟩

Substitution Lemma for Statements

lemma *subst-infer-check-s*:
fixes $v::v$ **and** $s::s$ **and** *cs*::*branch-s* **and** $x::x$ **and** $c::c$ **and** $b::b$ **and**
$\Gamma_1::\Gamma$ **and** $\Gamma_2::\Gamma$ **and** *css*::*branch-list*
assumes Θ ; $\mathcal{B}$; $\Gamma_1 \vdash v \Rightarrow \tau$ **and** Θ ; $\mathcal{B}$; $\Gamma_1 \vdash \tau \lesssim \{\!|\ z : b \mid c\ |\!\}$ **and**
atom $z \sharp (x, v)$
shows Θ ; Φ ; $\mathcal{B}$; Γ; $\Delta \vdash s \Leftarrow \tau' \Longrightarrow$
$\Gamma = (\Gamma_2@((x,b,c[z::=[x]^v]_{cv})\#_\Gamma\Gamma_1)) \Longrightarrow$
Θ ; Φ ; $\mathcal{B}$; $\Gamma[x::=v]_{\Gamma v}$; $\Delta[x::=v]_{\Delta v} \vdash s[x::=v]_{sv} \Leftarrow \tau'[x::=v]_{\tau v}$

and
Θ ; Φ ; $\mathcal{B}$; Γ; Δ; *tid* ; *cons* ; *const* ; $v' \vdash$ *cs* $\Leftarrow \tau' \Longrightarrow$
$\Gamma = (\Gamma_2 @ ((x,b,c[z::=[x]^v]_{cv}) \#_\Gamma \Gamma_1)) \Longrightarrow$
Θ ; Φ ; $\mathcal{B}$; $\Gamma[x::=v]_{\Gamma v}$; $\Delta[x::=v]_{\Delta v}$;
tid ; *cons* ; *const* ; $v'[x::=v]_{vv} \vdash$ *cs*$[x::=v]_{sv} \Leftarrow \tau'[x::=v]_{\tau v}$
and
Θ ; Φ ; $\mathcal{B}$; Γ; Δ; *tid* ; *dclist* ; $v' \vdash$ *css* $\Leftarrow \tau' \Longrightarrow$
$\Gamma = (\Gamma_2 @ ((x,b,c[z::=[x]^v]_{cv}) \#_\Gamma \Gamma_1)) \Longrightarrow$
Θ ; Φ ; $\mathcal{B}$; $\Gamma[x::=v]_{\Gamma v}$; $\Delta[x::=v]_{\Delta v}$; *tid* ; *dclist* ; $v'[x::=v]_{vv} \vdash$
subst-branchlv css x v $\Leftarrow \tau'[x::=v]_{\tau v}$

⟨*proof*⟩

lemma *subst-check-check-s*:
fixes *v*::*v* **and** *s*::*s* **and** *cs*::*branch-s* **and** *x*::*x* **and** *c*::*c* **and** *b*::*b* **and** Γ_1::Γ **and** Γ_2::Γ
assumes Θ ; $\mathcal{B}$; $\Gamma_1 \vdash v \Leftarrow \{\!|\ z : b \mid c\ |\!\}$ **and** *atom z* $\sharp$ (*x*, *v*)
and *check-s* Θ Φ $\mathcal{B}$ Γ Δ *s* τ' **and** Γ $= (\Gamma_2 @ ((x,b,c[z::=[x]^v]_{cv}) \#_\Gamma \Gamma_1))$
shows *check-s* Θ Φ $\mathcal{B}$ (*subst-gv* Γ *x v*) $\Delta[x::=v]_{\Delta v}$ (*s*$[x::=v]_{sv}$) (*subst-tv* τ' *x v*)
⟨*proof*⟩

If a statement checks against a type τ then it checks against a supertype of τ

lemma *check-s-supertype*:
fixes *v*::*v* **and** *s*::*s* **and** *cs*::*branch-s* **and** *x*::*x* **and** *c*::*c* **and** *b*::*b* **and** Γ::Γ **and** Γ'::Γ **and** *css*::*branch-list*
shows *check-s* Θ Φ $\mathcal{B}$ *G* Δ *s t1* $\Longrightarrow \Theta$; $\mathcal{B}$; *G* $\vdash$ *t1* $\lesssim$ *t2* $\Longrightarrow$ *check-s* Θ Φ $\mathcal{B}$ *G* Δ *s t2* **and**
check-branch-s Θ Φ $\mathcal{B}$ *G* Δ *tid cons const v cs t1* $\Longrightarrow \Theta$; $\mathcal{B}$; *G* $\vdash$ *t1* $\lesssim$ *t2* $\Longrightarrow$ *check-branch-s* Θ Φ $\mathcal{B}$ *G* Δ *tid cons const v cs t2* **and**
check-branch-list Θ Φ $\mathcal{B}$ *G* Δ *tid dclist v css t1* $\Longrightarrow \Theta$; $\mathcal{B}$; *G* $\vdash$ *t1* $\lesssim$ *t2* $\Longrightarrow$ *check-branch-list* Θ Φ $\mathcal{B}$ *G* Δ *tid dclist v css t2*
⟨*proof*⟩

lemma *subtype-let*:
fixes *s*′::*s* **and** *cs*::*branch-s* **and** *css*::*branch-list* **and** *v*::*v*
shows Θ ; Φ ; $\mathcal{B}$; *GNil* ; Δ $\vdash$ *AS-let x* e_1 *s* $\Leftarrow \tau \Longrightarrow \Theta$; Φ ; $\mathcal{B}$; *GNil* ; Δ $\vdash e_1 \Rightarrow \tau_1 \Longrightarrow$
Θ ; Φ ; $\mathcal{B}$; *GNil* ; Δ $\vdash e_2 \Rightarrow \tau_2 \Longrightarrow \Theta$; $\mathcal{B}$; *GNil* $\vdash \tau_2 \lesssim \tau_1 \Longrightarrow \Theta$; Φ ; $\mathcal{B}$; *GNil* ; Δ $\vdash$ *AS-let x* e_2 *s* $\Leftarrow \tau$ **and**
check-branch-s Θ Φ $\{||\}$ *GNil* Δ *tid dc const v cs* $\tau \Longrightarrow$ *True* **and**
check-branch-list Θ Φ $\{||\}$ Γ Δ *tid dclist v css* $\tau \Longrightarrow$ *True*
⟨*proof*⟩

end

Chapter 15

Basic Type Variable Substitution Lemmas

Lemmas that show that types are preserved, in some way, under basic type variable substitution

lemma *subst-vv-subst-bb-commute*:
fixes $bv::bv$ **and** $b::b$ **and** $x::x$ **and** $v::v$
assumes *atom* $bv \sharp v$
shows $(v'[x::=v]_{vv})[bv::=b]_{vb} = (v'[bv::=b]_{vb})[x::=v]_{vv}$
⟨*proof*⟩

lemma *subst-cev-subst-bb-commute*:
fixes $bv::bv$ **and** $b::b$ **and** $x::x$ **and** $v::v$
assumes *atom* $bv \sharp v$
shows $(ce[x::=v]_v)[bv::=b]_{ceb} = (ce[bv::=b]_{ceb})[x::=v]_v$
⟨*proof*⟩

lemma *subst-cv-subst-bb-commute*:
fixes $bv::bv$ **and** $b::b$ **and** $x::x$ **and** $v::v$
assumes *atom* $bv \sharp v$
shows $c[x::=v]_{cv}[bv::=b]_{cb} = (c[bv::=b]_{cb})[x::=v]_{cv}$
⟨*proof*⟩

lemma *subst-b-c-of*:
$(c\text{-}of\ \tau\ z)[bv::=b]_{cb} = \ c\text{-}of\ (\tau[bv::=b]_{\tau b})\ z$
⟨*proof*⟩

lemma *subst-b-b-of*:
$(b\text{-}of\ \tau)[bv::=b]_{bb} = \ b\text{-}of\ (\tau[bv::=b]_{\tau b})$
⟨*proof*⟩

lemma *subst-b-if*:
$\{\!|\ z : b\text{-}of\ \tau[bv::=b]_{\tau b}\ |\ CE\text{-}val\ (v[bv::=b]_{vb})\ ==\ CE\text{-}val\ (V\text{-}lit\ ll)\ \ IMP\ \ c\text{-}of\ \tau[bv::=b]_{\tau b}\ z\ |\!\} = \{\!|\ z : b\text{-}of\ \tau\ |\ CE\text{-}val\ (v)\ ==\ CE\text{-}val\ (V\text{-}lit\ ll)\ \ IMP\ \ c\text{-}of\ \tau\ z\ |\!\}[bv::=b]_{\tau b}$
⟨*proof*⟩

lemma *subst-b-top-eq*:
$\{\!|\ z : B\text{-}unit\ |\ TRUE\ |\!\}[bv::=b]_{\tau b} = \{\!|\ z : B\text{-}unit\ |\ TRUE\ |\!\}$ **and** $\{\!|\ z : B\text{-}bool\ |\ TRUE\ |\!\}[bv::=b]_{\tau b} =$

$\{\!|\ z : B\text{-}bool\ |\ TRUE\ |\!\}$ **and** $\{\!|\ z : B\text{-}id\ tid\ |\ TRUE\ |\!\} = \{\!|\ z : B\text{-}id\ tid\ |\ TRUE\ |\!\}[bv::=b]_{\tau b}$
⟨*proof*⟩

lemmas *subst-b-eq = subst-b-c-of subst-b-b-of subst-b-if subst-b-top-eq*

lemma *subst-cx-subst-bb-commute*[*simp*]:
fixes $bv::bv$ **and** $b::b$ **and** $x::x$ **and** $v'::c$
shows $(v'[x::=V\text{-}var\ y]_{cv})[bv::=b]_{cb} = (v'[bv::=b]_{cb})[x::=V\text{-}var\ y]_{cv}$
⟨*proof*⟩

lemma *subst-b-infer-b*:
fixes $l::l$ **and** $b::b$
assumes $\vdash l \Rightarrow \tau$ **and** $\Theta\ ;\ \{||\} \vdash_{wf} b$ **and** $B = \{|bv|\}$
shows $\vdash l \Rightarrow (\tau[bv::=b]_{\tau b})$
⟨*proof*⟩

lemma *subst-b-subtype*:
fixes $s::s$ **and** $b'::b$
assumes $\Theta\ ;\ \{|bv|\}\ ;\ \Gamma\ \vdash \tau 1 \lesssim \tau 2$ **and** $\Theta\ ;\ \{||\} \vdash_{wf} b'$ **and** $B = \{|bv|\}$
shows $\Theta\ ;\ \{||\}\ ;\ \Gamma[bv::=b']_{\Gamma b}\ \vdash \tau 1[bv::=b']_{\tau b} \lesssim \tau 2[bv::=b']_{\tau b}$
⟨*proof*⟩

lemma *b-of-subst-bv*:
$(b\text{-}of\ \tau)[x::=v]_{bb} = b\text{-}of\ (\tau[x::=v]_{\tau b})$
⟨*proof*⟩

lemma *subst-b-infer-v*:
fixes $v::v$ **and** $b::b$
assumes $\Theta\ ;\ B\ ;\ G \vdash v \Rightarrow \tau$ **and** $\Theta\ ;\ \{||\} \vdash_{wf} b$ **and** $B = \{|bv|\}$
shows $\Theta\ ;\ \{||\}\ ;\ G[bv::=b]_{\Gamma b} \vdash v[bv::=b]_{vb} \Rightarrow (\tau[bv::=b]_{\tau b})$
⟨*proof*⟩

lemma *subst-b-check-v*:
fixes $v::v$ **and** $b::b$
assumes $\Theta\ ;\ B\ ;\ G \vdash v \Leftarrow \tau$ **and** $\Theta\ ;\ \{||\} \vdash_{wf} b$ **and** $B = \{|bv|\}$
shows $\Theta\ ;\ \{||\}\ ;\ G[bv::=b]_{\Gamma b} \vdash v[bv::=b]_{vb} \Leftarrow (\tau[bv::=b]_{\tau b})$
⟨*proof*⟩

lemma *subst-vv-subst-vb-switch*:
shows $(v'[bv::=b']_{vb})[x::=v[bv::=b']_{vb}]_{vv} = v'[x::=v]_{vv}[bv::=b']_{vb}$
⟨*proof*⟩

lemma *subst-cev-subst-vb-switch*:
shows $(ce[bv::=b']_{ceb})[x::=v[bv::=b']_{vb}]_{cev} = (ce[x::=v]_{cev})[bv::=b']_{ceb}$
⟨*proof*⟩

lemma *subst-cv-subst-vb-switch*:
shows $(c[bv::=b']_{cb})[x::=v[bv::=b']_{vb}]_{cv} = c[x::=v]_{cv}[bv::=b']_{cb}$
⟨*proof*⟩

lemma *subst-tv-subst-vb-switch*:
shows $(\tau[bv::=b']_{\tau b})[x::=v[bv::=b']_{vb}]_{\tau v} = \tau[x::=v]_{\tau v}[bv::=b']_{\tau b}$

⟨*proof*⟩

lemma *subst-tb-triple*:
assumes *atom bv* ♯ τ'
shows $\tau'[bv'::=b'[bv::=b]_{bb}]_{\tau b}[x'::=v'[bv::=b]_{vb}]_{\tau v} = \tau'[bv'::=b']_{\tau b}[x'::=v']_{\tau v}[bv::=b]_{\tau b}$
⟨*proof*⟩

lemma *subst-b-infer-e*:
fixes *s*::*s* **and** *b*::*b*
assumes Θ ; Φ ; B ; G; $D \vdash e \Rightarrow \tau$ **and** Θ ; $\{||\} \vdash_{wf} b$ **and** $B = \{|bv|\}$
shows Θ ; Φ ; $\{||\}$; $G[bv::=b]_{\Gamma b}$; $D[bv::=b]_{\Delta b} \vdash (e[bv::=b]_{eb}) \Rightarrow (\tau[bv::=b]_{\tau b})$
⟨*proof*⟩

This is needed for preservation. When we apply a function "f [b] v" we need to substitute into the body of the function f replacing type-variable with b

lemma *subst-b-c-of-forget*:
assumes *atom bv* ♯ *const*
shows (*c-of const x*)$[bv::=b]_{cb}$ = *c-of const x*
⟨*proof*⟩

lemma *subst-b-check-s*:
fixes *s*::*s* **and** *b*::*b* **and** *cs*::*branch-s* **and** *css*::*branch-list* **and** *v*::*v* **and** τ::τ
assumes Θ ; $\{||\} \vdash_{wf} b$ **and** $B = \{|bv|\}$
shows Θ ; Φ ; B ; G; $D \vdash s \Leftarrow \tau \Longrightarrow \Theta$; Φ ; $\{||\}$; $G[bv::=b]_{\Gamma b}$; $D[bv::=b]_{\Delta b} \vdash (s[bv::=b]_{sb}) \Leftarrow (\tau[bv::=b]_{\tau b})$ **and**
Θ ; Φ ; B ; G; D ; *tid* ; *cons* ; *const* ; $v \vdash cs \Leftarrow \tau \Longrightarrow \Theta$; Φ ; $\{||\}$; $G[bv::=b]_{\Gamma b}$; $D[bv::=b]_{\Delta b}$; *tid* ; *cons* ; *const* ; $v[bv::=b]_{vb} \vdash$ (*subst-branchb cs bv b*) $\Leftarrow (\tau[bv::=b]_{\tau b})$ **and**
Θ ; Φ ; B ; G; D ; *tid* ; *dclist* ; $v \vdash css \Leftarrow \tau \Longrightarrow \Theta$; Φ ; $\{||\}$; $G[bv::=b]_{\Gamma b}$; $D[bv::=b]_{\Delta b}$; *tid* ; *dclist* ; $v[bv::=b]_{vb} \vdash$ (*subst-branchlb css bv b*) $\Leftarrow (\tau[bv::=b]_{\tau b})$
⟨*proof*⟩

end

method *supp-calc* = (*metis* (*mono-tags*, *opaque-lifting*) *pure-supp* *c.supp e.supp v.supp supp-l-empty opp.supp sup-bot.right-neutral supp-at-base*)
declare *infer-e.intros*[*simp*]
declare *infer-e.intros*[*intro*]

Chapter 16

Safety

Lemmas about the operational semantics leading up to progress and preservation and then safety.

16.1 Store Lemmas

abbreviation *delta-ext* (‹ - $\sqsubseteq$ - ›) **where**
  delta-ext Δ Δ' $\equiv$ (*setD* Δ $\subseteq$ *setD* Δ')

nominal-function *dc-of* :: *branch-s* $\Rightarrow$ *string* **where**
  dc-of (*AS-branch dc* - -) = *dc*
  ⟨*proof*⟩
nominal-termination (*eqvt*) ⟨*proof*⟩

lemma *delta-sim-fresh*:
  assumes $\Theta \vdash \delta \sim \Delta$ **and** *atom u* $\sharp$ δ
  shows *atom u* $\sharp$ Δ
  ⟨*proof*⟩

lemma *delta-sim-v*:
  fixes Δ::Δ
  assumes $\Theta \vdash \delta \sim \Delta$ **and** (*u*,*v*) $\in$ *set* δ **and** (*u*,τ) $\in$ *setD* Δ **and** Θ ; {||} ; *GNil* $\vdash_{wf}$ Δ
  shows Θ ; {||} ; *GNil* $\vdash$ *v* $\Leftarrow$ τ
  ⟨*proof*⟩

lemma *delta-sim-delta-lookup*:
  assumes $\Theta \vdash \delta \sim \Delta$ **and** (*u*, {$\!|$ *z* : *b* | *c* $|\!$}) $\in$ *setD* Δ
  shows $\exists$ *v*. (*u*,*v*) $\in$ *set* δ
  ⟨*proof*⟩

lemma *update-d-stable*:
  fst ‘ *set* δ = *fst* ‘ *set* (*update-d* δ *u v*)
⟨*proof*⟩

lemma *update-d-sim*:
  fixes Δ::Δ
  assumes $\Theta \vdash \delta \sim \Delta$ **and** Θ ; {||} ; *GNil* $\vdash$ *v* $\Leftarrow$ τ **and** (*u*,τ) $\in$ *setD* Δ **and** Θ ; {||} ; *GNil* $\vdash_{wf}$ Δ

shows $\Theta \vdash (\mathit{update\text{-}d}\ \delta\ u\ v) \sim \Delta$
⟨*proof*⟩

16.2 Preservation

Types are preserved under reduction step. Broken down into lemmas about different operations

16.2.1 Function Application

lemma *check-s-x-fresh*:
fixes $x::x$ **and** $s::s$
assumes $\Theta\ ;\ \Phi\ ;\ B\ ;\ \ GNil\ ;\ D\ \vdash s \Leftarrow \tau$
shows $\mathit{atom}\ x \sharp s \wedge \mathit{atom}\ x \sharp \tau \wedge \mathit{atom}\ x \sharp D$
⟨*proof*⟩

lemma *check-funtyp-subst-b*:
fixes $b'::b$
assumes *check-funtyp* $\Theta\ \Phi\ \{|bv|\}\ (\mathit{AF\text{-}fun\text{-}typ}\ x\ b\ c\ \tau\ s)$ **and** $\langle\ \Theta\ ;\ \{||\}\ \vdash_{wf}\ b'\ \rangle$
shows *check-funtyp* $\Theta\ \Phi\ \ \{||\}\ (\mathit{AF\text{-}fun\text{-}typ}\ x\ b[bv::=b']_{bb}\ (c[bv::=b']_{cb})\ \tau[bv::=b']_{\tau b}\ s[bv::=b']_{sb})$
⟨*proof*⟩

lemma *funtyp-simple-check*:
fixes $s::s$ **and** $\Delta::\Delta$ **and** $\tau::\tau$ **and** $v::v$
assumes *check-funtyp* $\Theta\ \Phi\ \ (\{||\}::bv\ \mathit{fset})\ (\mathit{AF\text{-}fun\text{-}typ}\ x\ b\ c\ \tau\ s)$ **and**
$\Theta\ ;\ \{||\}\ ;\ GNil \vdash v \Leftarrow \{\!|\ x : b \mid c\ |\!\}$
shows $\Theta\ ;\ \Phi\ ;\ \ \{||\}\ ;\ GNil\ ;\ DNil \vdash s[x::=v]_{sv} \Leftarrow \tau[x::=v]_{\tau v}$
⟨*proof*⟩

lemma *funtypq-simple-check*:
fixes $s::s$ **and** $\Delta::\Delta$ **and** $\tau::\tau$ **and** $v::v$
assumes *check-funtypq* $\Theta\ \Phi\ \ \ (\mathit{AF\text{-}fun\text{-}typ\text{-}none}\ (\mathit{AF\text{-}fun\text{-}typ}\ x\ b\ c\ t\ s))$ **and**
$\Theta\ ;\ \{||\}\ ;\ GNil \vdash v \Leftarrow \{\!|\ x : b \mid c\ |\!\}$
shows $\Theta;\ \Phi;\ \{||\};\ GNil;\ DNil \vdash s[x::=v]_{sv} \Leftarrow t[x::=v]_{\tau v}$
⟨*proof*⟩

lemma *funtyp-poly-eq-iff-equalities*:
assumes $[[\mathit{atom}\ bv']]\mathit{lst}.\ \mathit{AF\text{-}fun\text{-}typ}\ x'\ b''\ c'\ t'\ s' = [[\mathit{atom}\ bv]]\mathit{lst}.\ \mathit{AF\text{-}fun\text{-}typ}\ x\ b\ c\ t\ s$
shows $\{\!|\ x' : b''[bv'::=b']_{bb}\ \mid c'[bv'::=b']_{cb}\ |\!\} = \{\!|\ x : b[bv::=b']_{bb}\ \mid c[bv::=b']_{cb}\ |\!\} \wedge$
$s'[bv'::=b']_{sb}[x'::=v]_{sv} = s[bv::=b']_{sb}[x::=v]_{sv} \wedge t'[bv'::=b']_{\tau b}[x'::=v]_{\tau v} = t[bv::=b']_{\tau b}[x::=v]_{\tau v}$
⟨*proof*⟩

lemma *funtypq-poly-check*:
fixes $s::s$ **and** $\Delta::\Delta$ **and** $\tau::\tau$ **and** $v::v$ **and** $b'::b$
assumes *check-funtypq* $\Theta\ \Phi\ \ \ (\mathit{AF\text{-}fun\text{-}typ\text{-}some}\ bv\ (\mathit{AF\text{-}fun\text{-}typ}\ x\ b\ c\ t\ s))$ **and**
$\Theta\ ;\ \{||\}\ ;\ GNil \vdash v \Leftarrow \{\!|\ x : b[bv::=b']_{bb} \mid c[bv::=b']_{cb}\ |\!\}$ **and**
$\Theta\ ;\ \{||\} \vdash_{wf}\ b'$
shows $\Theta;\ \Phi;\ \{||\};\ GNil;\ DNil \vdash s[bv::=b']_{sb}[x::=v]_{sv} \Leftarrow t[bv::=b']_{\tau b}[x::=v]_{\tau v}$
⟨*proof*⟩

lemma *fundef-simple-check*:
fixes $s::s$ **and** $\Delta::\Delta$ **and** $\tau::\tau$ **and** $v::v$

assumes *check-fundef* Θ Φ (*AF-fundef f* (*AF-fun-typ-none* (*AF-fun-typ x b c t s*))) **and**
Θ ; {||} ; *GNil* $\vdash$ *v* $\Leftarrow$ {| *x* : *b* | *c* |} **and** Θ ; {||} ; *GNil* $\vdash_{wf}$ Δ
shows Θ; Φ; {||}; *GNil*; Δ $\vdash$ $s[x::=v]_{sv}$ $\Leftarrow$ $t[x::=v]_{\tau v}$
⟨*proof*⟩

lemma *fundef-poly-check*:
fixes *s*::*s* **and** Δ::Δ **and** τ::τ **and** *v*::*v* **and** *b'*::*b*
assumes *check-fundef* Θ Φ (*AF-fundef f* (*AF-fun-typ-some bv* (*AF-fun-typ x b c t s*))) **and**
Θ ; {||} ; *GNil* $\vdash$ *v* $\Leftarrow$ {| *x* : *b*[*bv*::=*b'*]$_{bb}$ | *c*[*bv*::=*b'*]$_{cb}$ |} **and** Θ ; {||} ; *GNil* $\vdash_{wf}$ Δ **and** Θ ; {||} $\vdash_{wf}$ *b'*
shows Θ; Φ; {||}; *GNil*; Δ $\vdash$ $s[bv::=b']_{sb}[x::=v]_{sv}$ $\Leftarrow$ $t[bv::=b']_{\tau b}[x::=v]_{\tau v}$
⟨*proof*⟩

lemma *preservation-app*:
assumes
Some (*AF-fundef f* (*AF-fun-typ-none* (*AF-fun-typ x1 b1 c1* $\tau 1'$ *s1'*))) = *lookup-fun* Φ *f* **and** ($\forall$ *fd*$\in$*set* Φ. *check-fundef* Θ Φ *fd*)
shows Θ ; Φ ; *B* ; *G* ; Δ $\vdash$ *ss* $\Leftarrow$ τ $\Longrightarrow$ *B* = {||} $\Longrightarrow$ *G* = *GNil* $\Longrightarrow$ *ss* = *LET x* = (*AE-app f v*) *IN s* $\Longrightarrow$
Θ; Φ; {||}; *GNil*; Δ $\vdash$ *LET x* : ($\tau 1'[x1::=v]_{\tau v}$) = ($s1'[x1::=v]_{sv}$) *IN s* $\Leftarrow$ τ **and**
check-branch-s Θ Φ $\mathcal{B}$ *GNil* Δ *tid dc const v cs* τ $\Longrightarrow$ *True* **and**
check-branch-list Θ Φ $\mathcal{B}$ Γ Δ *tid dclist v css* τ $\Longrightarrow$ *True*
⟨*proof*⟩

lemma *fresh-subst-v-subst-b*:
fixes *x2*::*x* **and** *tm*::'*a*::{*has-subst-v*,*has-subst-b*} **and** *x*::*x*
assumes *supp tm* $\subseteq$ { *atom bv*, *atom x* } **and** *atom x2* $\sharp$ *v*
shows *atom x2* $\sharp$ *tm*[*bv*::=*b*]$_b$[*x*::=*v*]$_v$
⟨*proof*⟩

lemma *preservation-poly-app*:
assumes
Some (*AF-fundef f* (*AF-fun-typ-some bv1* (*AF-fun-typ x1 b1 c1* $\tau 1'$ *s1'*))) = *lookup-fun* Φ *f* **and** ($\forall$ *fd*$\in$*set* Φ. *check-fundef* Θ Φ *fd*)
shows Θ ; Φ ; *B* ; *G* ; Δ $\vdash$ *ss* $\Leftarrow$ τ $\Longrightarrow$ *B* = {||} $\Longrightarrow$ *G* = *GNil* $\Longrightarrow$ *ss* = *LET x* = (*AE-appP f b'* *v*) *IN s* $\Longrightarrow$ Θ ; {||} $\vdash_{wf}$ *b'* $\Longrightarrow$
Θ; Φ; {||}; *GNil*; Δ $\vdash$ *LET x* : ($\tau 1'[bv1::=b']_{\tau b}[x1::=v]_{\tau v}$) = ($s1'[bv1::=b']_{sb}[x1::=v]_{sv}$) *IN s* $\Leftarrow$ τ **and**
check-branch-s Θ Φ $\mathcal{B}$ *GNil* Δ *tid dc const v cs* τ $\Longrightarrow$ *True* **and**
check-branch-list Θ Φ $\mathcal{B}$ Γ Δ *tid dclist v css* τ $\Longrightarrow$ *True*
⟨*proof*⟩

lemma *check-s-plus*:
assumes Θ; Φ; {||}; *GNil*; Δ $\vdash$ *LET x* = (*AE-op Plus* (*V-lit* (*L-num n1*)) (*V-lit* (*L-num n2*))) *IN s'* $\Leftarrow$ τ
shows Θ; Φ; {||}; *GNil*; Δ $\vdash$ *LET x* = (*AE-val* (*V-lit* (*L-num* (*n1*+*n2*)))) *IN s'* $\Leftarrow$ τ
⟨*proof*⟩

lemma *check-s-leq*:
assumes Θ ; Φ ; {||} ; *GNil* ; Δ $\vdash$ *LET x* = (*AE-op LEq* (*V-lit* (*L-num n1*)) (*V-lit* (*L-num n2*))) *IN s'* $\Leftarrow$ τ
shows Θ; Φ; {||}; *GNil*; Δ $\vdash$ *LET x* = (*AE-val* (*V-lit* (*if* (*n1* $\leq$ *n2*) *then L-true else L-false*))) *IN*

$s' \Leftarrow \tau$
⟨*proof*⟩

lemma *check-s-eq*:
assumes Θ ; Φ ; $\{||\}$; *GNil* ; Δ $\vdash$ *LET x* = (*AE-op Eq* (*V-lit* (*n1*)) (*V-lit* (*n2*))) *IN s'* $\Leftarrow \tau$
shows Θ; Φ; $\{||\}$; *GNil*; Δ $\vdash$ *LET x* = (*AE-val* (*V-lit* (*if* (*n1* = *n2*) *then L-true else L-false*))) *IN s'* $\Leftarrow \tau$
⟨*proof*⟩

16.2.2 Operators

lemma *preservation-plus*:
assumes Θ; Φ; $\Delta \vdash \langle\ \delta$, *LET x* = (*AE-op Plus* (*V-lit* (*L-num n1*)) (*V-lit* (*L-num n2*))) *IN s'* $\rangle \Leftarrow \tau$

shows Θ; Φ; Δ $\vdash \langle\ \delta$, *LET x* = (*AE-val* (*V-lit* (*L-num* (*n1*+*n2*)))) *IN s'* $\rangle \Leftarrow \tau$
⟨*proof*⟩

lemma *preservation-leq*:
assumes Θ; Φ; $\Delta \vdash \langle\ \delta$, *AS-let x* (*AE-op LEq* (*V-lit* (*L-num n1*)) (*V-lit* (*L-num n2*))) *s'* $\rangle \Leftarrow \tau$
shows Θ; Φ; Δ $\vdash \langle\ \delta$, *AS-let x* (*AE-val* (*V-lit* (((*if* (*n1* $\leq$ *n2*) *then L-true else L-false*))))) *s'* $\rangle \Leftarrow \tau$
⟨*proof*⟩

lemma *preservation-eq*:
assumes Θ; Φ; $\Delta \vdash \langle\ \delta$, *AS-let x* (*AE-op Eq* (*V-lit* (*n1*)) (*V-lit* (*n2*))) *s'* $\rangle \Leftarrow \tau$
shows Θ; Φ; Δ $\vdash \langle\ \delta$, *AS-let x* (*AE-val* (*V-lit* (((*if* (*n1* = *n2*) *then L-true else L-false*))))) *s'* $\rangle \Leftarrow \tau$
⟨*proof*⟩

16.2.3 Let Statements

lemma *subst-s-abs-lst*:
fixes *s*::*s* **and** *sa*::*s* **and** *v'*::*v*
assumes [[*atom x*]]*lst*. *s* = [[*atom xa*]]*lst*. *sa* **and** *atom xa* $\sharp$ *v* $\wedge$ *atom x* $\sharp$ *v*
shows $s[x::=v]_{sv} = sa[xa::=v]_{sv}$
⟨*proof*⟩

lemma *check-let-val*:
fixes *v*::*v* **and** *s*::*s*
shows Θ ; Φ ; *B* ; *G* ; Δ $\vdash$ *ss* $\Leftarrow \tau \Longrightarrow$ *B* = $\{||\} \Longrightarrow$ *G* = *GNil* $\Longrightarrow$
ss = *AS-let x* (*AE-val v*) *s* $\vee$ *ss* = *AS-let2 x t* (*AS-val v*) *s* $\Longrightarrow$ Θ; Φ; $\{||\}$; *GNil*; Δ $\vdash$ $(s[x::=v]_{sv}) \Leftarrow \tau$ **and**
check-branch-s Θ Φ $\mathcal{B}$ *GNil* Δ *tid dc const v cs* $\tau \Longrightarrow$ *True* **and**
check-branch-list Θ Φ $\mathcal{B}$ Γ Δ *tid dclist v css* $\tau \Longrightarrow$ *True*
⟨*proof*⟩

lemma *preservation-let-val*:
assumes Θ; Φ; $\Delta \vdash \langle\ \delta$, *AS-let x* (*AE-val v*) *s* $\rangle \Leftarrow \tau \vee \Theta$; Φ; $\Delta \vdash \langle\ \delta$, *AS-let2 x t* (*AS-val v*) *s* $\rangle \Leftarrow \tau$ (**is** *?A* $\vee$ *?B*)
shows $\exists \Delta'.\ \Theta$; Φ; $\Delta' \vdash \langle\ \delta$, $s[x::=v]_{sv}$ $\rangle \Leftarrow \tau\ \wedge \Delta \sqsubseteq \Delta'$
⟨*proof*⟩

lemma *check-s-fst-snd*:
assumes *fst-snd* = *AE-fst* $\wedge$ *v*=*v1* $\vee$ *fst-snd* = *AE-snd* $\wedge$ *v*=*v2*
and Θ; Φ; $\{||\}$; *GNil*; $\Delta \vdash$ *AS-let x* (*fst-snd* (*V-pair v1 v2*)) *s'* $\Leftarrow \tau$

shows Θ; Φ; {||}; *GNil*; $\Delta \vdash$ *AS-let x* (*AE-val v*) $s' \Leftarrow \tau$
⟨*proof*⟩

lemma *preservation-fst-snd*:
assumes Θ; Φ; $\Delta \vdash \langle\ \delta$, *LET x* = (*fst-snd* (*V-pair v1 v2*)) *IN* $s'\ \rangle \Leftarrow \tau$ **and**
fst-snd = *AE-fst* $\wedge$ *v*=*v1* $\vee$ *fst-snd* = *AE-snd* $\wedge$ *v*=*v2*
shows $\exists \Delta'.\ \Theta$; Φ; $\Delta \vdash \langle\ \delta$, *LET x* = (*AE-val v*) *IN* $s'\ \rangle \Leftarrow \tau\ \wedge \Delta \sqsubseteq \Delta'$
⟨*proof*⟩

inductive-cases *check-branch-s-elims2*[*elim*!]:
Θ ; Φ ; $\mathcal{B}$; Γ ; Δ; *tid* ; *cons* ; *const* ; *v* $\vdash$ *cs* $\Leftarrow \tau$

lemmas *freshers* = *freshers atom-dom.simps toSet.simps fresh-def x-not-in-b-set*
declare *freshers* [*simp*]

lemma *subtype-eq-if*:
fixes *t*::τ **and** *va*::*v*
assumes Θ ; $\mathcal{B}$; $\Gamma \vdash_{wf}$ $\{\!|$ *z* : *b-of t* | *c-of t z* $|\!\}$ **and** Θ ; $\mathcal{B}$; $\Gamma \vdash_{wf} \{\!|$ *z* : *b-of t* | *c IMP c-of t z* $|\!\}$
shows Θ ; $\mathcal{B}$; $\Gamma \vdash \{\!|$ *z* : *b-of t* | *c-of t z* $|\!\} \lesssim \{\!|$ *z* : *b-of t* | *c IMP c-of t z* $|\!\}$
⟨*proof*⟩

lemma *subtype-eq-if-*τ:
fixes *t*::τ **and** *va*::*v*
assumes Θ ; $\mathcal{B}$; $\Gamma \vdash_{wf}$ *t* **and** Θ ; $\mathcal{B}$; $\Gamma \vdash_{wf} \{\!|$ *z* : *b-of t* | *c IMP c-of t z* $|\!\}$ **and** *atom z* $\sharp$ *t*
shows Θ ; $\mathcal{B}$; $\Gamma \vdash t \lesssim \{\!|$ *z* : *b-of t* | *c IMP c-of t z* $|\!\}$
⟨*proof*⟩

lemma *valid-conj*:
assumes Θ ; $\mathcal{B}$; $\Gamma \models$ *c1* **and** Θ ; $\mathcal{B}$; $\Gamma \models$ *c2*
shows Θ ; $\mathcal{B}$; $\Gamma \models$ *c1 AND c2*
⟨*proof*⟩

16.2.4 Other Statements

lemma *check-if*:
fixes *s'*::*s* **and** *cs*::*branch-s* **and** *css*::*branch-list* **and** *v*::*v*
shows Θ; Φ; *B*; *G*; $\Delta \vdash$ $s' \Leftarrow \tau \Longrightarrow s' =$ *IF* (*V-lit ll*) *THEN s1 ELSE s2* $\Longrightarrow$
Θ ; {||} ; *GNil* $\vdash_{wf} \tau \Longrightarrow G =$ *GNil* $\Longrightarrow B = \{||\} \Longrightarrow ll =$ *L-true* $\wedge$ *s* = *s1* $\vee$ *ll* = *L-false* $\wedge$ *s* = *s2* $\Longrightarrow$
Θ; Φ; {||}; *GNil*; Δ $\vdash$ *s* $\Leftarrow \tau$ **and**
check-branch-s Θ Φ {||} *GNil* Δ *tid dc const v cs* $\tau \Longrightarrow$ *True* **and**
check-branch-list Θ Φ {||} Γ Δ *tid dclist v css* $\tau \Longrightarrow$ *True*
⟨*proof*⟩

lemma *preservation-if*:
assumes Θ; Φ; $\Delta \vdash \langle\ \delta$, *IF* (*V-lit ll*) *THEN s1 ELSE s2* $\rangle \Leftarrow \tau$ **and**
ll = *L-true* $\wedge$ *s* = *s1* $\vee$ *ll* = *L-false* $\wedge$ *s* = *s2*
shows Θ; Φ; Δ $\vdash \langle \delta, s \rangle \Leftarrow \tau\ \wedge$ *setD* $\Delta \subseteq$ *setD* Δ
⟨*proof*⟩

lemma *wfT-conj*:
assumes Θ ; $\mathcal{B}$; *GNil* $\vdash_{wf} \{\!|$ *z* : *b* | *c1* $|\!\}$ **and** Θ ; $\mathcal{B}$; *GNil* $\vdash_{wf} \{\!|$ *z* : *b* | *c2* $|\!\}$
shows Θ ; $\mathcal{B}$; *GNil* $\vdash_{wf} \{\!|$ *z* : *b* | *c1 AND c2* $|\!\}$

⟨*proof*⟩

lemma *subtype-conj*:
assumes $\Theta\ ;\ \mathcal{B}\ ;\ GNil\ \vdash\ t \lesssim \{\!|\ z : b\ |\ c1\ |\!\}$ **and** $\Theta\ ;\ \mathcal{B}\ ;\ GNil\ \vdash\ t \lesssim \{\!|\ z : b\ |\ c2\ |\!\}$
shows $\Theta\ ;\ \mathcal{B}\ ;\ GNil\ \vdash\ \{\!|\ z : b\ |\ c\text{-}of\ t\ z\ |\!\}\ \lesssim \{\!|\ z : b\ |\ c1\ AND\ c2\ |\!\}$
⟨*proof*⟩

lemma *infer-v-conj*:
assumes $\Theta\ ;\ \mathcal{B}\ ;\ GNil\ \vdash\ v \Leftarrow \{\!|\ z : b\ |\ c1\ |\!\}$ **and** $\Theta\ ;\ \mathcal{B}\ ;\ GNil\ \vdash\ v \Leftarrow \{\!|\ z : b\ |\ c2\ |\!\}$
shows $\Theta\ ;\ \mathcal{B}\ ;\ GNil\ \vdash\ v \Leftarrow \{\!|\ z : b\ |\ c1\ AND\ c2\ |\!\}$
⟨*proof*⟩

lemma *wfG-conj*:
fixes *c1*::*c*
assumes $\Theta\ ;\ \mathcal{B}\ \vdash_{wf}\ (x,\ b,\ c1\ AND\ c2)\ \#_\Gamma\ \Gamma$
shows $\Theta\ ;\ \mathcal{B}\ \vdash_{wf}\ (x,\ b,\ c1)\ \#_\Gamma\ \Gamma$
⟨*proof*⟩

lemma *check-match*:
fixes *s′*::*s* **and** *s*::*s* **and** *css*::*branch-list* **and** *cs*::*branch-s*
shows $\Theta\ ;\ \Phi\ ;\ \mathcal{B}\ ;\ \Gamma\ ;\ \Delta \vdash\ s \Leftarrow \tau \Longrightarrow True$ **and**
$\Theta\ ;\ \Phi\ ;\ B\ ;\ G\ ;\ \Delta\ ;\ tid\ ;\ dc\ ;\ const\ \ ;\ \ vcons \vdash\ \ cs \Leftarrow\ \ \tau \Longrightarrow$
$vcons = V\text{-}cons\ tid\ dc\ v \Longrightarrow B =\ \{||\} \Longrightarrow G = GNil \Longrightarrow cs =\ \ (dc\ x' \Rightarrow s') \Longrightarrow$
$\Theta\ ;\ \{||\}\ ;\ GNil \vdash v \Leftarrow const \Longrightarrow$
$\Theta;\ \Phi;\ \{||\};\ GNil;\ \Delta\ \vdash\ s'[x'::=v]_{sv} \Leftarrow \tau$ **and**
$\Theta\ ;\ \Phi\ ;\ B\ ;\ G\ ;\ \Delta\ ;\ tid\ ;\ dclist\ ;\ vcons\ \vdash\ css \Leftarrow \tau \Longrightarrow distinct\ (map\ fst\ dclist) \Longrightarrow$
$vcons = V\text{-}cons\ tid\ dc\ v \Longrightarrow B =\ \{||\} \Longrightarrow (dc,\ const) \in set\ dclist \Longrightarrow G = GNil \Longrightarrow$
$Some\ (AS\text{-}branch\ dc\ x'\ s') = lookup\text{-}branch\ dc\ css \Longrightarrow \Theta\ ;\ \{||\}\ ;\ GNil \vdash v \Leftarrow const \Longrightarrow$
$\Theta;\ \Phi;\ \{||\};\ GNil;\ \Delta\ \vdash\ s'[x'::=v]_{sv} \Leftarrow \tau$
⟨*proof*⟩

Lemmas for while reduction. Making these separate lemmas allows flexibility in wiring them into the main proof and robustness if we change it

lemma *check-unit*:
fixes τ::τ **and** Φ::Φ **and** Δ::Δ **and** *G*::Γ
assumes $\Theta\ ;\ \{||\}\ ;\ GNil\ \vdash\ \{\!|\ z : B\text{-}unit\ \ |\ \ TRUE\ |\!\} \lesssim \tau'$ **and** $\Theta\ ;\ \{||\}\ ;\ GNil \vdash_{wf}\ \Delta$ **and** $\Theta\ \vdash_{wf}\ \Phi$ **and** $\Theta\ ;\ \{||\} \vdash_{wf}\ G$
shows $\langle\ \Theta\ ;\ \Phi\ ;\ \{||\}\ ;\ G\ ;\ \Delta\ \vdash\ [[\ L\text{-}unit\ ]^v]^s \Leftarrow \tau' \rangle$
⟨*proof*⟩

lemma *preservation-var*:
shows $\Theta;\ \Phi;\ \{||\};\ GNil;\ \Delta\ \vdash\ VAR\ u : \tau' = v\ IN\ s \Leftarrow \tau \Longrightarrow \Theta \vdash \delta \sim \Delta \Longrightarrow atom\ u \sharp \delta \Longrightarrow atom\ u \sharp \Delta \Longrightarrow$
$\Theta;\ \Phi;\ \{||\};\ GNil;\ (u,\tau')\#_\Delta\Delta\ \vdash\ s \Leftarrow \tau \wedge \Theta \vdash (u,v)\#\delta \sim (u,\tau')\#_\Delta\Delta$
and
check-branch-s $\Theta\ \Phi\ \ \{||\}\ GNil\ \Delta\ tid\ dc\ const\ v\ cs\ \tau \Longrightarrow True$ **and**
check-branch-list $\Theta\ \Phi\ \ \{||\}\ \Gamma\ \Delta\ tid\ dclist\ v\ css\ \tau \Longrightarrow True$
⟨*proof*⟩

lemma *check-while*:
shows $\Theta;\ \Phi;\ \{||\};\ GNil;\ \Delta\ \vdash\ WHILE\ s1\ DO\ \{\ s2\ \}\ \Leftarrow \tau \Longrightarrow atom\ x \sharp (s1,\ s2) \Longrightarrow\ atom\ z' \sharp x \Longrightarrow$
$\Theta;\ \Phi;\ \{||\};\ GNil;\ \Delta\ \vdash\ LET\ x : (\{\!|\ z' : B\text{-}bool\ \ |\ \ TRUE\ |\!\}) = s1\ IN\ (IF\ (V\text{-}var\ x)\ THEN\ (s2\ ;;$

(*WHILE s1 DO* {*s2*}))
ELSE ([*V-lit L-unit*]^s)) $\Leftarrow \tau$ **and**
check-branch-s Θ Φ {||} *GNil* Δ *tid dc const v cs* $\tau \Longrightarrow$ *True* **and**
check-branch-list Θ Φ {||} Γ Δ *tid dclist v css* $\tau \Longrightarrow$ *True*
⟨*proof*⟩

lemma *check-s-narrow*:
fixes *s*::*s* **and** *x*::*x*
assumes *atom x* ♯ (Θ, Φ, $\mathcal{B}$, Γ, Δ, *c*, τ, *s*) **and** Θ ; Φ ; $\mathcal{B}$; (*x*, *B-bool*, *c*) #$_\Gamma$ Γ ; Δ $\vdash$ *s* $\Leftarrow$ τ **and**
Θ ; $\mathcal{B}$; Γ $\models$ *c*
shows Θ ; Φ ; $\mathcal{B}$; Γ ; Δ $\vdash$ *s* $\Leftarrow$ τ
⟨*proof*⟩

lemma *check-assert-s*:
fixes *s*::*s* **and** *x*::*x*
assumes Θ; Φ; {||}; *GNil*; Δ $\vdash$ *AS-assert c s* $\Leftarrow$ τ
shows Θ; Φ; {||}; *GNil*; Δ $\vdash$ *s* $\Leftarrow$ τ $\wedge$ Θ ; {||} ; *GNil* $\models$ *c*
⟨*proof*⟩

lemma *infer-v-pair2I*:
atom z ♯ (*v1*, *v2*) $\Longrightarrow$
atom z ♯ (Θ, $\mathcal{B}$, Γ) $\Longrightarrow$
Θ ; $\mathcal{B}$; Γ $\vdash$ *v1* $\Rightarrow$ *t1* $\Longrightarrow$
Θ ; $\mathcal{B}$; Γ $\vdash$ *v2* $\Rightarrow$ *t2* $\Longrightarrow$
b1 = *b-of t1* $\Longrightarrow$ *b2* = *b-of t2* $\Longrightarrow$
Θ ; $\mathcal{B}$; Γ $\vdash$ [*v1* , *v2*]^v $\Rightarrow$ {| *z* : [*b1* , *b2*]^b | [[*z*]^v]^{ce} == [[*v1* , *v2*]^v]^{ce} |}
⟨*proof*⟩

16.2.5 Main Lemma

lemma *preservation*:
assumes $\Phi \vdash \langle \delta, s \rangle \longrightarrow \langle \delta', s' \rangle$ **and** Θ; Φ; $\Delta \vdash \langle \delta, s \rangle \Leftarrow \tau$
shows $\exists \Delta'.$ Θ; Φ; $\Delta' \vdash \langle \delta', s' \rangle \Leftarrow \tau \wedge \Delta \sqsubseteq \Delta'$
⟨*proof*⟩

lemma *preservation-many*:
assumes $\Phi \vdash \langle \delta, s \rangle \longrightarrow^* \langle \delta', s' \rangle$
shows Θ; Φ; $\Delta \vdash \langle \delta, s \rangle \Leftarrow \tau \Longrightarrow \exists \Delta'.$ Θ; Φ; $\Delta' \vdash \langle \delta', s' \rangle \Leftarrow \tau \wedge \Delta \sqsubseteq \Delta'$
⟨*proof*⟩

16.3 Progress

We prove that a well typed program is either a value or we can make a step

lemma *check-let-op-infer*:
assumes Θ; Φ; {||}; Γ; Δ $\vdash$ *LET x* = (*AE-op opp v1 v2*) *IN s* $\Leftarrow$ τ **and** *supp* (*LET x* = (*AE-op opp v1 v2*) *IN s*) $\subseteq$ *atom*‘*fst*‘*setD* Δ
shows $\exists$ *z b c*. Θ; Φ; {||}; Γ; $\Delta \vdash$ (*AE-op opp v1 v2*) $\Rightarrow$ {|*z*:*b*|*c*|}
⟨*proof*⟩

lemma *infer-pair*:
assumes Θ ; *B*; Γ $\vdash$ *v* $\Rightarrow$ {| *z* : *B-pair b1 b2* | *c* |} **and** *supp v* = {}

obtains *v1* **and** *v2* **where** $v = V\text{-}pair\ v1\ v2$
⟨*proof*⟩

lemma *progress-fst*:
assumes $\Theta;\ \Phi;\ \{||\};\ \Gamma;\ \Delta\ \vdash LET\ x = (AE\text{-}fst\ v)\ IN\ s \Leftarrow \tau$ **and** $\Theta \vdash \delta \sim \Delta$ **and**
$supp\ (LET\ x = (AE\text{-}fst\ v)\ IN\ s)\ \subseteq atom`fst`setD\ \Delta$
shows $\exists\, \delta'\ s'.\ \Phi\ \vdash \langle\ \delta\ ,\ LET\ x = (AE\text{-}fst\ v)\ IN\ s\ \rangle \longrightarrow \langle \delta',\ s' \rangle$
⟨*proof*⟩

lemma *progress-let*:
assumes $\Theta;\ \Phi;\ \{||\};\ \Gamma;\ \Delta \vdash LET\ x = e\ IN\ s \Leftarrow \tau$ **and** $\Theta \vdash \delta \sim \Delta$ **and**
$supp\ (LET\ x = e\ IN\ s) \subseteq atom\ `\ fst\ `\ setD\ \Delta$ **and** $sble\ \Theta\ \Gamma$
shows $\exists\, \delta'\ s'.\ \Phi\ \vdash \langle\ \delta\ ,\ LET\ x = e\ IN\ s \rangle \longrightarrow \langle\ \delta'\ ,\ s' \rangle$
⟨*proof*⟩

lemma *check-css-lookup-branch-exist*:
fixes *s*::*s* **and** *cs*::*branch-s* **and** *css*::*branch-list* **and** *v*::*v*
shows
$\Theta;\ \Phi;\ B;\ G;\ \Delta \vdash\ s \Leftarrow \tau \Longrightarrow True$ **and**
$check\text{-}branch\text{-}s\ \Theta\ \Phi\ \{||\}\ GNil\ \Delta\ tid\ dc\ const\ v\ cs\ \tau \Longrightarrow True$ **and**
$\Theta;\ \Phi;\ \mathcal{B};\ \Gamma;\ \Delta;\ tid;\ dclist;\ v \vdash css \Leftarrow \tau \Longrightarrow (dc,\ t) \in set\ dclist \Longrightarrow$
$\exists\, x'\ s'.\ Some\ (AS\text{-}branch\ dc\ x'\ s') = lookup\text{-}branch\ dc\ css$
⟨*proof*⟩

lemma *progress-aux*:
shows $\Theta;\ \Phi;\ \mathcal{B};\ \Gamma;\ \Delta \vdash s \Leftarrow \tau \Longrightarrow \mathcal{B} = \{||\} \Longrightarrow\ sble\ \Theta\ \Gamma \Longrightarrow supp\ s \subseteq atom\ `\ fst\ `\ setD\ \Delta \Longrightarrow$
$\Theta \vdash \delta \sim \Delta \Longrightarrow$
$(\exists\, v.\ s = [v]^s) \vee (\exists\, \delta'\ s'.\ \Phi \vdash \langle \delta,\ s \rangle \longrightarrow \langle \delta',\ s' \rangle)$ **and**
$\Theta;\ \Phi;\ \{||\};\ \Gamma;\ \Delta;\ tid;\ dc;\ const;\ v2 \vdash cs \Leftarrow \tau\ \Longrightarrow supp\ cs = \{\} \Longrightarrow True$
$\Theta;\ \Phi;\ \{||\};\ \Gamma;\ \Delta;\ tid;\ dclist;\ v2 \vdash css \Leftarrow \tau \Longrightarrow supp\ css = \{\} \Longrightarrow True$
⟨*proof*⟩

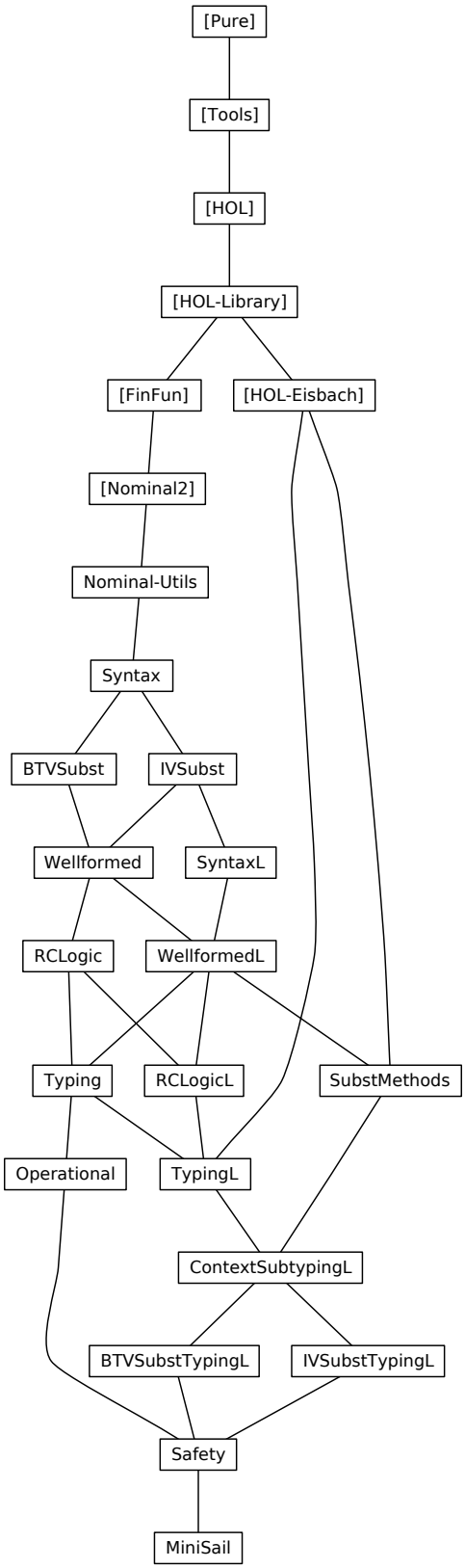
lemma *progress*:
assumes $\Theta;\ \Phi;\ \Delta \vdash \langle \delta,\ s \rangle \Leftarrow \tau$
shows $(\exists\, v.\ s = [v]^s) \vee (\exists\, \delta'\ s'.\ \Phi\ \vdash \langle \delta,\ s \rangle \longrightarrow \langle \delta',\ s' \rangle)$
⟨*proof*⟩

16.4 Safety

lemma *safety-stmt*:
assumes $\Phi \vdash \langle \delta,\ s \rangle \longrightarrow^* \langle \delta',\ s' \rangle$ **and** $\Theta;\ \Phi;\ \Delta \vdash\ \langle \delta,\ s \rangle \Leftarrow \tau$
shows $(\exists\, v.\ s' = [v]^s) \vee (\exists\, \delta''\ s''.\ \Phi\ \vdash \langle \delta',\ s' \rangle \longrightarrow \langle \delta'',\ s'' \rangle)$
⟨*proof*⟩

lemma *safety*:
assumes $\vdash \langle PROG\ \Theta\ \Phi\ \mathcal{G}\ s \rangle \Leftarrow \tau$ **and** $\Phi\ \vdash \langle \delta\text{-}of\ \mathcal{G},\ s \rangle \longrightarrow^* \langle \delta',\ s' \rangle$
shows $(\exists\, v.\ s' = [v]^s) \vee (\exists\, \delta''\ s''.\ \Phi\ \vdash \langle \delta',\ s' \rangle \longrightarrow \langle \delta'',\ s'' \rangle)$
⟨*proof*⟩

end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[HOL-Eisbach]
[Nominal2]
Nominal-Utils
Syntax
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IVSubst
Wellformed
SyntaxL
RCLogic
WellformedL
Typing
RCLogicL
SubstMethods
Operational
TypingL
ContextSubtypingL
BTVSubstTypingL
IVSubstTypingL
Safety
MiniSail

Bibliography

[1] A. Armstrong, C. Pulte, S. Flur, I. Stark, N. Krishnaswami, P. Sewell, T. Bauereiss, B. Campbell, A. Reid, K. E. Gray, R. M. Norton, P. Mundkur, M. Wassell, and J. French. ISA semantics for ARMv8-A, RISC-V, and CHERI-MIPS. *Proceedings of the ACM on Programming Languages*, 3(POPL):1–31, 2019.

[2] N. Vazou, E. L. Seidel, and S. Peyton-jones. Refinement Types For Haskell. *ICFP '14 Proceedings of the 19th ACM SIGPLAN international conference on Functional programming*, 2014.