

Minimal, Maximal, Least, and Greatest Elements w.r.t. Restricted Ordering

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Abstract

This entry provides small, reusable, theories that specify the concepts of minimal, maximal, least, and greatest elements in sets, final sets, and final multisets. The concepts are uniformly specified as predicates parametrized by a binary relation. The binary relation is only required to be an ordering on the elements of the concrete collection considered. This is useful when working with a partial ordering, but some assumption or invariant proves that the ordering is total on all elements of the considered set.

Contents

1	Definitions	1
2	Conversions	2
3	Existence	3
4	Uniqueness	4
5	Existence of unique element	4
6	Transformations	5
7	Minimal and maximal elements	6
	7.1 Conversions	6
	7.2 Existence	6
	7.3 Miscellaneous	7
8	Least and greatest elements	7
	8.1 Conversions	8
	8.2 Uniqueness	8
	8.3 Existence	8

9	Hide stuff	9
10	Integration in type classes	9
11	Minimal and maximal elements	10
11.1	Conversions	11
11.2	Existence	11
11.3	Non-existence	11
11.4	Miscellaneous	11
12	Least and greatest elements	12
12.1	Conversions	12
12.2	Uniqueness	13
12.3	Existence	13
12.4	Nonexistence	14
12.5	Miscellaneous	14
13	Hide stuff	15
14	Integration in type classes	15
15	Minimal and maximal elements	17
15.1	Conversions	18
15.2	Existence	19
15.3	Miscellaneous	19
15.4	Nonuniqueness	21
16	Least and greatest elements	22
16.1	Conversions	22
16.2	Uniqueness	23
16.3	Miscellaneous	24
17	Examples of duplicate handling in set and multiset definitions	28
18	Hide stuff	29
19	Integration in type classes	29
	theory <i>Relation-Reachability</i>	
	imports <i>Main</i>	
	begin	

1 Definitions

When a binary relation hold for two values, i.e., $R\ x\ y$, we say that x reaches y and, conversely, that y is reachable by x .

definition non-reachable-wrt where

$$\text{non-reachable-wrt } R \ X \ x \longleftrightarrow x \in X \wedge (\forall y \in X - \{x\}. \neg (R \ y \ x))$$

definition non-reaching-wrt where

$$\text{non-reaching-wrt } R \ X \ x \longleftrightarrow x \in X \wedge (\forall y \in X - \{x\}. \neg (R \ x \ y))$$

definition reaching-all-wrt where

$$\text{reaching-all-wrt } R \ X \ x \longleftrightarrow x \in X \wedge (\forall y \in X - \{x\}. R \ x \ y)$$

definition reachable-by-all-wrt where

$$\text{reachable-by-all-wrt } R \ X \ x \longleftrightarrow x \in X \wedge (\forall y \in X - \{x\}. R \ y \ x)$$

2 Conversions

lemma non-reachable-wrt-iff:

$$\text{non-reachable-wrt } R \ X \ x \longleftrightarrow x \in X \wedge (\forall y \in X. y \neq x \longrightarrow \neg R \ y \ x)$$

unfolding non-reachable-wrt-def by blast

lemma non-reaching-wrt-iff:

$$\text{non-reaching-wrt } R \ X \ x \longleftrightarrow x \in X \wedge (\forall y \in X. y \neq x \longrightarrow \neg R \ x \ y)$$

unfolding non-reaching-wrt-def by blast

lemma reaching-all-wrt-iff:

$$\text{reaching-all-wrt } R \ X \ x \longleftrightarrow x \in X \wedge (\forall y \in X. y \neq x \longrightarrow R \ x \ y)$$

unfolding reaching-all-wrt-def by blast

lemma reachable-by-all-wrt-iff:

$$\text{reachable-by-all-wrt } R \ X \ x \longleftrightarrow x \in X \wedge (\forall y \in X. y \neq x \longrightarrow R \ y \ x)$$

unfolding reachable-by-all-wrt-def by blast

lemma non-reachable-wrt-filter-iff:

$$\text{non-reachable-wrt } R \ \{y \in X. P \ y\} \ x \longleftrightarrow x \in X \wedge P \ x \wedge (\forall y \in X - \{x\}. P \ y \longrightarrow \neg R \ y \ x)$$

by (auto simp: non-reachable-wrt-def)

lemma non-reachable-wrt-conversep[simp]:

$$\text{non-reachable-wrt } R^{-1-1} = \text{non-reaching-wrt } R$$

unfolding non-reaching-wrt-def non-reachable-wrt-def by simp

lemma non-reaching-wrt-conversep[simp]:

$$\text{non-reaching-wrt } R^{-1-1} = \text{non-reachable-wrt } R$$

unfolding non-reaching-wrt-def non-reachable-wrt-def by simp

lemma reaching-all-wrt-conversep[simp]:

$$\text{reaching-all-wrt } R^{-1-1} = \text{reachable-by-all-wrt } R$$

unfolding reaching-all-wrt-def reachable-by-all-wrt-def by simp

lemma reachable-by-all-wrt-conversep[simp]:

$$\text{reachable-by-all-wrt } R^{-1-1} = \text{reaching-all-wrt } R$$

unfolding *reaching-all-wrt-def reachable-by-all-wrt-def* **by** *simp*

lemma *non-reachable-wrt-eq-reaching-all-wrt:*

assumes *asym: asymp-on X R* **and** *tot: totalp-on X R*

shows *non-reachable-wrt R X = reaching-all-wrt R X*

proof (*intro ext iffI*)

fix *x*

from *tot* **show** *non-reachable-wrt R X x \implies reaching-all-wrt R X x*

unfolding *non-reachable-wrt-def reaching-all-wrt-def*

by (*metis Diff-iff insertCI totalp-onD*)

next

fix *x*

from *asym* **show** *reaching-all-wrt R X x \implies non-reachable-wrt R X x*

unfolding *reaching-all-wrt-def non-reachable-wrt-def*

by (*metis Diff-iff asymp-onD*)

qed

lemma *non-reaching-wrt-eq-reachable-by-all-wrt:*

assumes *asym: asymp-on X R* **and** *tot: totalp-on X R*

shows *non-reaching-wrt R X = reachable-by-all-wrt R X*

proof (*intro ext iffI*)

fix *x*

from *tot* **show** *non-reaching-wrt R X x \implies reachable-by-all-wrt R X x*

unfolding *non-reaching-wrt-def reachable-by-all-wrt-def*

by (*metis Diff-iff insertCI totalp-onD*)

next

fix *x*

from *asym* **show** *reachable-by-all-wrt R X x \implies non-reaching-wrt R X x*

unfolding *reachable-by-all-wrt-def non-reaching-wrt-def*

by (*metis Diff-iff asymp-onD*)

qed

lemma *non-reachable-wrt-reflclp[simp]:*

non-reachable-wrt R⁼⁼ = non-reachable-wrt R

by (*intro ext iffI*) (*simp-all add: non-reachable-wrt-iff*)

lemma *non-reaching-wrt-reflclp[simp]:*

non-reaching-wrt R⁼⁼ = non-reaching-wrt R

by (*intro ext iffI*) (*simp-all add: non-reaching-wrt-iff*)

lemma *reaching-all-wrt-reflclp[simp]:*

reaching-all-wrt R⁼⁼ = reaching-all-wrt R

by (*intro ext iffI*) (*simp-all add: reaching-all-wrt-iff*)

lemma *reachable-by-all-wrt-reflclp[simp]:*

reachable-by-all-wrt R⁼⁼ = reachable-by-all-wrt R

by (*intro ext iffI*) (*simp-all add: reachable-by-all-wrt-iff*)

3 Existence

lemma *ex-non-reachable-wrt:*

transp-on A R \implies asymp-on A R \implies finite A \implies A \neq {} \implies \exists m. non-reachable-wrt R A m
using *Finite-Set.bex-min-element*
by *(metis non-reachable-wrt-iff)*

lemma *ex-non-reaching-wrt:*

transp-on A R \implies asymp-on A R \implies finite A \implies A \neq {} \implies \exists m. non-reaching-wrt R A m
using *Finite-Set.bex-max-element*
by *(metis non-reaching-wrt-iff)*

lemma *ex-reaching-all-wrt:*

transp-on A R \implies totalp-on A R \implies finite A \implies A \neq {} \implies \exists g. reaching-all-wrt R A g
using *Finite-Set.bex-least-element[of A R]*
by *(metis reaching-all-wrt-iff)*

lemma *ex-reachable-by-all-wrt:*

transp-on A R \implies totalp-on A R \implies finite A \implies A \neq {} \implies \exists g. reachable-by-all-wrt R A g
using *Finite-Set.bex-greatest-element[of A R]*
by *(metis reachable-by-all-wrt-iff)*

lemma *not-ex-greatest-element-doubleton-if:*

assumes *x \neq y and \neg R x y and \neg R y x*
shows *\nexists g. reachable-by-all-wrt R {x, y} g*
proof *(rule notI)*
assume *\exists g. reachable-by-all-wrt R {x, y} g*
then obtain g where *reachable-by-all-wrt R {x, y} g ..*
then show *False*
unfolding *reachable-by-all-wrt-def*
using *assms(1) assms(2) assms(3) by blast*
qed

4 Uniqueness

lemma *Uniq-non-reachable-wrt:*

totalp-on X R \implies $\exists_{\leq 1} x$. non-reachable-wrt R X x
by *(rule Uniq-I) (metis insert-Diff insert-iff non-reachable-wrt-def totalp-onD)*

lemma *Uniq-non-reaching-wrt:*

totalp-on X R \implies $\exists_{\leq 1} x$. non-reaching-wrt R X x
by *(rule Uniq-I) (metis insert-Diff insert-iff non-reaching-wrt-def totalp-onD)*

lemma *Uniq-reaching-all-wrt:*

asymp-on X R \implies $\exists_{\leq 1} x$. reaching-all-wrt R X x

by (*rule Uniq-I*)
 (*metis antisymp-onD antisymp-on-if-asymp-on insertE insert-Diff reaching-all-wrt-def*)

lemma *Uniq-reachable-by-all-wrt*:
asymp-on X R $\implies \exists \leq_1 x. \text{reachable-by-all-wrt } R X x$
by (*rule Uniq-I*)
 (*metis antisymp-onD antisymp-on-if-asymp-on insertE insert-Diff reachable-by-all-wrt-def*)

5 Existence of unique element

lemma *ex1-reaching-all-wrt*:
transp-on X R \implies asymp-on X R \implies totalp-on X R \implies finite X \implies X \neq {}
 \implies
 $\exists! x. \text{reaching-all-wrt } R X x$
using *ex1-iff-ex-Uniq ex-reaching-all-wrt Uniq-reaching-all-wrt* **by** *metis*

lemma *ex1-reachable-by-all-wrt*:
transp-on X R \implies asymp-on X R \implies totalp-on X R \implies finite X \implies X \neq {}
 \implies
 $\exists! x. \text{reachable-by-all-wrt } R X x$
using *ex1-iff-ex-Uniq ex-reachable-by-all-wrt Uniq-reachable-by-all-wrt* **by** *metis*

6 Transformations

lemma *non-reachable-wrt-insert-wrtI*:
assumes
trans: transp-on (insert y X) R and
asym: asymp-on (insert y X) R and
R y x and
x-non-reachable: non-reachable-wrt R X x
shows *non-reachable-wrt R (insert y X) y*
proof –
from *x-non-reachable* **have** *x-in: x \in X and x-min': $\forall y \in X - \{x\}. \neg R y x$*
by (*simp-all add: non-reachable-wrt-iff*)

have $\neg R z y$ **if** $z \in \text{insert } y X - \{y\}$ **for** z
proof –
from *that* **have** $z \in X$ **and** $z \neq y$
by *simp-all*

show $\neg R z y$
proof (*cases z = x*)
case *True*
thus *?thesis*
using $\langle R y x \rangle$ *asym x-in*
by (*metis asymp-onD insertI1 insertI2*)
next
case *False*

```

    hence  $\neg R z x$ 
    using x-min'[rule-format, of z, simplified]  $\langle z \in X \rangle$  by metis
    then show ?thesis
    using  $\langle R y x \rangle$  trans  $\langle z \in X \rangle$  x-in
    by (meson insertCI transp-onD)
  qed
qed
thus ?thesis
  by (simp add: non-reachable-wrt-def)
qed

end
theory Min-Max-Least-Greatest-Set
  imports
    Relation-Reachability
begin

```

7 Minimal and maximal elements

If the binary relation is a strict partial order, then non-reachability corresponds to minimality and non-reaching correspond to maximality.

definition *is-minimal-in-set-wrt* :: $('a \Rightarrow 'a \Rightarrow \text{bool}) \Rightarrow 'a \text{ set} \Rightarrow 'a \Rightarrow \text{bool}$ **where**
 $\text{transp-on } X R \Longrightarrow \text{asympt-on } X R \Longrightarrow \text{is-minimal-in-set-wrt } R X = \text{non-reachable-wrt } R X$

definition *is-maximal-in-set-wrt* :: $('a \Rightarrow 'a \Rightarrow \text{bool}) \Rightarrow 'a \text{ set} \Rightarrow 'a \Rightarrow \text{bool}$ **where**
 $\text{transp-on } X R \Longrightarrow \text{asympt-on } X R \Longrightarrow \text{is-maximal-in-set-wrt } R X = \text{non-reaching-wrt } R X$

```

context
  fixes  $X R$ 
  assumes
    trans: transp-on X R and
    asym: asympt-on X R
begin

```

7.1 Conversions

lemma *is-minimal-in-set-wrt-iff*:
 $\text{is-minimal-in-set-wrt } R X x \longleftrightarrow x \in X \wedge (\forall y \in X. y \neq x \longrightarrow \neg R y x)$
 using *trans asym is-minimal-in-set-wrt-def*[*unfolded non-reachable-wrt-iff*] by *metis*

lemma *is-maximal-in-set-wrt-iff*:
 $\text{is-maximal-in-set-wrt } R X x \longleftrightarrow x \in X \wedge (\forall y \in X. y \neq x \longrightarrow \neg R x y)$
 using *trans asym is-maximal-in-set-wrt-def*[*unfolded non-reaching-wrt-iff*] by *metis*

7.2 Existence

lemma *ex-minimal-in-set-wrt*:

finite X $\implies X \neq \{\}$ $\implies \exists x.$ *is-minimal-in-set-wrt R X x*

using *trans asym ex-non-reachable-wrt is-minimal-in-set-wrt-def* **by** *metis*

lemma *ex-maximal-in-set-wrt*:

finite X $\implies X \neq \{\}$ $\implies \exists m.$ *is-maximal-in-set-wrt R X m*

using *trans asym is-maximal-in-set-wrt-def ex-non-reaching-wrt* **by** *metis*

end

7.3 Miscellaneous

lemma *is-minimal-in-set-wrt-filter-iff*:

fixes *X R*

assumes *trans: transp-on {y ∈ X. P y} R* **and** *asym: asymp-on {y ∈ X. P y}*

R

shows *is-minimal-in-set-wrt R {y ∈ X. P y} x* $\longleftrightarrow x \in X \wedge P x \wedge (\forall y \in X - \{x\}. P y \longrightarrow \neg R y x)$

proof –

have *is-minimal-in-set-wrt R {y ∈ X. P y} x* \longleftrightarrow *non-reachable-wrt R {y ∈ X. P y} x*

using *trans asym is-minimal-in-set-wrt-def* **by** *metis*

also have $\dots \longleftrightarrow x \in X \wedge P x \wedge (\forall y \in X - \{x\}. P y \longrightarrow \neg R y x)$

unfolding *non-reachable-wrt-filter-iff* ..

finally show *?thesis* .

qed

lemma *is-minimal-in-set-wrt-insertI*:

assumes

trans: transp-on (insert y X) R **and**

asym: asymp-on (insert y X) R **and**

R y x **and**

x-min: is-minimal-in-set-wrt R X x

shows *is-minimal-in-set-wrt R (insert y X) y*

by (*metis assms(3) asym asymp-on-subset is-minimal-in-set-wrt-def trans non-reachable-wrt-insert-wrtI subset-insertI transp-on-subset x-min*)

8 Least and greatest elements

If the binary relation is a strict total ordering, then an element reaching all others is a least element and an element reachable by all others is a greatest element.

definition *is-least-in-set-wrt* :: $('a \Rightarrow 'a \Rightarrow \text{bool}) \Rightarrow 'a \text{ set} \Rightarrow 'a \Rightarrow \text{bool}$ **where**

transp-on X R \implies *asymp-on X R* \implies *totalp-on X R* \implies

is-least-in-set-wrt R X = reaching-all-wrt R X

definition *is-greatest-in-set-wrt* :: ('a ⇒ 'a ⇒ bool) ⇒ 'a set ⇒ 'a ⇒ bool **where**
transp-on X R ⇒ *asympt-on* X R ⇒ *totalp-on* X R ⇒
is-greatest-in-set-wrt R X = *reachable-by-all-wrt* R X

context

fixes X R

assumes

trans: *transp-on* X R **and**

asym: *asympt-on* X R **and**

tot: *totalp-on* X R

begin

8.1 Conversions

lemma *is-least-in-set-wrt-iff*:

is-least-in-set-wrt R X x ⇔ x ∈ X ∧ (∀ y ∈ X. y ≠ x → R x y)

using *trans asym tot is-least-in-set-wrt-def*[*unfolded reaching-all-wrt-iff*] **by** *metis*

lemma *is-greatest-in-set-wrt-iff*:

is-greatest-in-set-wrt R X x ⇔ x ∈ X ∧ (∀ y ∈ X. y ≠ x → R y x)

using *trans asym tot is-greatest-in-set-wrt-def*[*unfolded reachable-by-all-wrt-iff*]

by *metis*

lemma *is-minimal-in-set-wrt-eq-is-least-in-set-wrt*:

is-minimal-in-set-wrt R X = *is-least-in-set-wrt* R X

using *trans asym tot non-reachable-wrt-eq-reaching-all-wrt*

is-minimal-in-set-wrt-def is-least-in-set-wrt-def

by *metis*

lemma *is-maximal-in-set-wrt-eq-is-greatest-in-set-wrt*:

is-maximal-in-set-wrt R X = *is-greatest-in-set-wrt* R X

using *trans asym tot non-reaching-wrt-eq-reachable-by-all-wrt*

is-maximal-in-set-wrt-def is-greatest-in-set-wrt-def

by *metis*

8.2 Uniqueness

lemma *Uniq-is-least-in-set-wrt*:

∃_{≤1}x. *is-least-in-set-wrt* R X x

using *trans asym tot is-least-in-set-wrt-def Uniq-reaching-all-wrt* **by** *metis*

lemma *Uniq-is-greatest-in-set-wrt*:

∃_{≤1}x. *is-greatest-in-set-wrt* R X x

using *trans asym tot is-greatest-in-set-wrt-def Uniq-reachable-by-all-wrt* **by** *metis*

8.3 Existence

lemma *ex-least-in-set-wrt*:

finite X ⇒ X ≠ {} ⇒ ∃ x. *is-least-in-set-wrt* R X x

using *trans asym tot is-least-in-set-wrt-def ex-reaching-all-wrt* **by** *metis*

lemma *ex-greatest-in-set-wrt*:
finite X $\implies X \neq \{\}$ $\implies \exists x. \text{is-greatest-in-set-wrt } R \ X \ x$
using *trans asym tot is-greatest-in-set-wrt-def ex-reachable-by-all-wrt* **by** *metis*

lemma *ex1-least-in-set-wrt*:
finite X $\implies X \neq \{\}$ $\implies \exists!x. \text{is-least-in-set-wrt } R \ X \ x$
using *Uniq-is-least-in-set-wrt ex-least-in-set-wrt* **by** (*metis Uniq-def*)

lemma *ex1-greatest-in-set-wrt*:
finite X $\implies X \neq \{\}$ $\implies \exists!x. \text{is-greatest-in-set-wrt } R \ X \ x$
using *Uniq-is-greatest-in-set-wrt ex-greatest-in-set-wrt* **by** (*metis Uniq-def*)

end

9 Hide stuff

We restrict the public interface to ease future internal changes.

hide-fact *is-minimal-in-set-wrt-def is-maximal-in-set-wrt-def*

hide-fact *is-least-in-set-wrt-def is-greatest-in-set-wrt-def*

10 Integration in type classes

abbreviation (**in** *order*) *is-minimal-in-set* **where**
is-minimal-in-set $\equiv \text{is-minimal-in-set-wrt } (<)$

abbreviation (**in** *order*) *is-maximal-in-set* **where**
is-maximal-in-set $\equiv \text{is-maximal-in-set-wrt } (<)$

lemmas (**in** *order*) *is-minimal-in-set-iff* =
is-minimal-in-set-wrt-iff[*OF transp-on-less asymp-on-less*]

lemmas (**in** *order*) *is-maximal-in-set-iff* =
is-maximal-in-set-wrt-iff[*OF transp-on-less asymp-on-less*]

lemmas (**in** *order*) *ex-minimal-in-set* =
ex-minimal-in-set-wrt[*OF transp-on-less asymp-on-less*]

lemmas (**in** *order*) *ex-maximal-in-set* =
ex-maximal-in-set-wrt[*OF transp-on-less asymp-on-less*]

lemmas (**in** *order*) *is-minimal-in-set-filter-iff* =
is-minimal-in-set-wrt-filter-iff[*OF transp-on-less asymp-on-less*]

abbreviation (**in** *linorder*) *is-least-in-set* **where**
is-least-in-set $\equiv \text{is-least-in-set-wrt } (<)$

abbreviation (in *linorder*) *is-greatest-in-set* **where**
is-greatest-in-set \equiv *is-greatest-in-set-wrt* ($<$)

lemmas (in *linorder*) *is-least-in-set-iff* =
is-least-in-set-wrt-iff[*OF transp-on-less asymp-on-less totalp-on-less*]

lemmas (in *linorder*) *is-greatest-in-set-iff* =
is-greatest-in-set-wrt-iff[*OF transp-on-less asymp-on-less totalp-on-less*]

lemmas (in *linorder*) *is-minimal-in-set-eq-is-least-in-set* =
is-minimal-in-set-wrt-eq-is-least-in-set-wrt[*OF transp-on-less asymp-on-less totalp-on-less*]

lemmas (in *linorder*) *is-maximal-in-set-eq-is-greatest-in-set* =
is-maximal-in-set-wrt-eq-is-greatest-in-set-wrt[*OF transp-on-less asymp-on-less totalp-on-less*]

lemmas (in *linorder*) *Uniq-is-least-in-set*[*intro*] =
Uniq-is-least-in-set-wrt[*OF transp-on-less asymp-on-less totalp-on-less*]

lemmas (in *linorder*) *Uniq-is-greatest-in-set*[*intro*] =
Uniq-is-greatest-in-set-wrt[*OF transp-on-less asymp-on-less totalp-on-less*]

lemmas (in *linorder*) *ex-least-in-set* =
ex-least-in-set-wrt[*OF transp-on-less asymp-on-less totalp-on-less*]

lemmas (in *linorder*) *ex-greatest-in-set* =
ex-greatest-in-set-wrt[*OF transp-on-less asymp-on-less totalp-on-less*]

lemmas (in *linorder*) *ex1-least-in-set* =
ex1-least-in-set-wrt[*OF transp-on-less asymp-on-less totalp-on-less*]

lemmas (in *linorder*) *ex1-greatest-in-set* =
ex1-greatest-in-set-wrt[*OF transp-on-less asymp-on-less totalp-on-less*]

end
theory *Min-Max-Least-Greatest-FSet*
imports
Min-Max-Least-Greatest-Set
HOL-Library.FSet
begin

11 Minimal and maximal elements

definition *is-minimal-in-fset-wrt* :: ($'a \Rightarrow 'a \Rightarrow \text{bool}$) \Rightarrow $'a \text{ fset} \Rightarrow 'a \Rightarrow \text{bool}$ **where**
transp-on (*fset* X) $R \Longrightarrow$ *asymp-on* (*fset* X) $R \Longrightarrow$
is-minimal-in-fset-wrt R $X =$ *is-minimal-in-set-wrt* R (*fset* X)

definition *is-maximal-in-fset-wrt* :: ('a ⇒ 'a ⇒ bool) ⇒ 'a fset ⇒ 'a ⇒ bool
where

transp-on (fset X) R ⇒⇒ *asympt-on* (fset X) R ⇒⇒
is-maximal-in-fset-wrt R X = *is-maximal-in-set-wrt* R (fset X)

context

fixes X R

assumes

trans: *transp-on* (fset X) R **and**

asym: *asympt-on* (fset X) R

begin

11.1 Conversions

lemma *is-minimal-in-fset-wrt-iff*:

is-minimal-in-fset-wrt R X x ⇔ x ∈| X ∧ fBall X (λy. y ≠ x → ¬ R y x)

using *is-minimal-in-set-wrt-iff*[OF *trans asym*]

using *is-minimal-in-fset-wrt-def*[OF *trans asym*]

by *simp*

lemma *is-maximal-in-fset-wrt-iff*:

is-maximal-in-fset-wrt R X x ⇔ x ∈| X ∧ fBall X (λy. y ≠ x → ¬ R x y)

using *is-maximal-in-set-wrt-iff*[OF *trans asym*]

using *is-maximal-in-fset-wrt-def*[OF *trans asym*]

by *simp*

11.2 Existence

lemma *ex-minimal-in-fset-wrt*:

X ≠ {} ⇒⇒ ∃ m. *is-minimal-in-fset-wrt* R X m

using *trans asym ex-minimal-in-set-wrt*[of fset X R] *is-minimal-in-fset-wrt-def*

by (*metis all-not-fin-conv empty-iff finite-fset*)

lemma *ex-maximal-in-fset-wrt*:

X ≠ {} ⇒⇒ ∃ m. *is-maximal-in-fset-wrt* R X m

using *trans asym ex-maximal-in-set-wrt*[of fset X R] *is-maximal-in-fset-wrt-def*

by (*metis all-not-fin-conv empty-iff finite-fset*)

end

11.3 Non-existence

lemma *not-is-minimal-in-fset-wrt-fempty*[*simp*]: $\bigwedge R x. \neg \text{is-minimal-in-fset-wrt } R \{\}\ x$

using *is-minimal-in-fset-wrt-iff*[of {}]

by (*simp add: transp-on-def asympt-on-def*)

lemma *not-is-maximal-in-fset-wrt-fempty*[*simp*]: $\bigwedge R x. \neg \text{is-maximal-in-fset-wrt } R \{\}\ x$

using *is-maximal-in-fset-wrt-iff*[of {}]

by (simp add: transp-on-def asym-on-def)

11.4 Miscellaneous

lemma *is-minimal-in-fset-wrt-ffilter-iff*:

assumes

tran: transp-on (fset (ffilter P X)) R **and**

asym: asym-on (fset (ffilter P X)) R

shows *is-minimal-in-fset-wrt R (ffilter P X) x* \longleftrightarrow

$(x \in X \wedge P x \wedge \text{fBall } (X - \{x\}) (\lambda y. P y \longrightarrow \neg R y x))$

proof –

have *is-minimal-in-fset-wrt R (ffilter P X) x* \longleftrightarrow *is-minimal-in-set-wrt R* ($\{y \in \text{fset } X. P y\}$) x

using *is-minimal-in-fset-wrt-iff*[OF *tran asym*]

using *is-minimal-in-set-wrt-iff*[OF *tran asym*]

by (simp only: *ffilter.rep-eq Set.filter-def*)

also have $\dots \longleftrightarrow x \in X \wedge P x \wedge (\forall y \in \text{fset } X - \{x\}. P y \longrightarrow \neg R y x)$

proof (rule *is-minimal-in-set-wrt-filter-iff*)

show transp-on $\{y. y \in X \wedge P y\}$ R

using *tran ffilter.rep-eq Set.filter-def* **by** *metis*

next

show asym-on $\{y. y \in X \wedge P y\}$ R

using *asym ffilter.rep-eq Set.filter-def* **by** *metis*

qed

finally show *?thesis*

by *simp*

qed

lemma *is-minimal-in-fset-wrt-finsertI*:

assumes *trans*: transp-on (fset (finsert y X)) R **and** *asym*: asym-on (fset (finsert y X)) R

shows $R y x \implies \text{is-minimal-in-fset-wrt } R X x \implies \text{is-minimal-in-fset-wrt } R (\text{finsert } y X) y$

using *trans asym is-minimal-in-set-wrt-insertI*[of - fset -, folded *fset-simps*]

by (smt (verit) *asym-on-def finsertCI finsertE is-minimal-in-fset-wrt-iff transp-on-def*)

12 Least and greatest elements

definition *is-least-in-fset-wrt* :: $('a \Rightarrow 'a \Rightarrow \text{bool}) \Rightarrow 'a \text{ fset} \Rightarrow 'a \Rightarrow \text{bool}$ **where**
transp-on (fset X) R \implies *asym-on (fset X) R* \implies *totalp-on (fset X) R* \implies
is-least-in-fset-wrt R X = is-least-in-set-wrt R (fset X)

definition *is-greatest-in-fset-wrt* :: $('a \Rightarrow 'a \Rightarrow \text{bool}) \Rightarrow 'a \text{ fset} \Rightarrow 'a \Rightarrow \text{bool}$ **where**
transp-on (fset X) R \implies *asym-on (fset X) R* \implies *totalp-on (fset X) R* \implies
is-greatest-in-fset-wrt R X = is-greatest-in-set-wrt R (fset X)

context

fixes X R

assumes

trans: *transp-on* (*fset* *X*) *R* **and**
asym: *asym-on* (*fset* *X*) *R* **and**
tot: *totalp-on* (*fset* *X*) *R*
begin

12.1 Conversions

lemma *is-least-in-fset-wrt-iff*:
is-least-in-fset-wrt *R* *X* $x \longleftrightarrow x \in X \wedge fBall\ X\ (\lambda y. y \neq x \longrightarrow R\ x\ y)$
using *is-least-in-set-wrt-iff*[*OF trans asym tot*]
using *is-least-in-fset-wrt-def*[*OF trans asym tot*]
by *simp*

lemma *is-greatest-in-fset-wrt-iff*:
is-greatest-in-fset-wrt *R* *X* $x \longleftrightarrow x \in X \wedge fBall\ X\ (\lambda y. y \neq x \longrightarrow R\ y\ x)$
using *is-greatest-in-set-wrt-iff*[*OF trans asym tot*]
using *is-greatest-in-fset-wrt-def*[*OF trans asym tot*]
by *simp*

lemma *is-minimal-in-fset-wrt-eq-is-least-in-fset-wrt*:
is-minimal-in-fset-wrt *R* *X* = *is-least-in-fset-wrt* *R* *X*
using *trans asym tot is-minimal-in-set-wrt-eq-is-least-in-set-wrt*
by (*metis is-least-in-fset-wrt-def is-minimal-in-fset-wrt-def*)

lemma *is-maximal-in-fset-wrt-eq-is-greatest-in-fset-wrt*:
is-maximal-in-fset-wrt *R* *X* = *is-greatest-in-fset-wrt* *R* *X*
using *trans asym tot is-maximal-in-set-wrt-eq-is-greatest-in-set-wrt*
by (*metis is-greatest-in-fset-wrt-def is-maximal-in-fset-wrt-def*)

12.2 Uniqueness

lemma *Uniq-is-least-in-fset-wrt*[*intro*]:
 $\exists_{\leq 1} x. is-least-in-fset-wrt\ R\ X\ x$
using *trans asym tot Uniq-is-least-in-set-wrt*
by (*metis is-least-in-fset-wrt-def*)

lemma *Uniq-is-greatest-in-fset-wrt*[*intro*]:
 $\exists_{\leq 1} x. is-greatest-in-fset-wrt\ R\ X\ x$
using *trans asym tot Uniq-is-greatest-in-set-wrt*
by (*metis is-greatest-in-fset-wrt-def*)

12.3 Existence

lemma *ex-least-in-fset-wrt*:
 $X \neq \{\}\Longrightarrow \exists x. is-least-in-fset-wrt\ R\ X\ x$
using *trans asym tot ex-least-in-set-wrt*
by (*metis bot-fset.rep-eq finite-fset fset-cong is-least-in-fset-wrt-def*)

lemma *ex-greatest-in-fset-wrt*:
 $X \neq \{\}\Longrightarrow \exists x. is-greatest-in-fset-wrt\ R\ X\ x$

using *trans asym tot ex-greatest-in-set-wrt*
by (*metis bot-fset.rep-eq finite-fset fset-cong is-greatest-in-fset-wrt-def*)

lemma *ex1-least-in-fset-wrt*:
 $X \neq \{\mid\} \implies \exists!x. \text{is-least-in-fset-wrt } R \ X \ x$
using *Uniq-is-least-in-fset-wrt ex-least-in-fset-wrt*
by (*metis Uniq-def*)

lemma *ex1-greatest-in-fset-wrt*:
 $X \neq \{\mid\} \implies \exists!x. \text{is-greatest-in-fset-wrt } R \ X \ x$
using *Uniq-is-greatest-in-fset-wrt ex-greatest-in-fset-wrt*
by (*metis Uniq-def*)

end

12.4 Nonexistence

lemma *not-is-least-in-fset-wrt-fempty[simp]*: $\bigwedge R \ x. \neg \text{is-least-in-fset-wrt } R \ \{\mid\} \ x$
using *is-least-in-fset-wrt-iff[of \{\mid\}]*
by (*simp add: transp-on-def asymp-on-def*)

lemma *not-is-greatest-in-fset-wrt-fempty[simp]*: $\bigwedge R \ x. \neg \text{is-greatest-in-fset-wrt } R \ \{\mid\} \ x$
using *is-greatest-in-fset-wrt-iff[of \{\mid\}]*
by (*simp add: transp-on-def asymp-on-def*)

12.5 Miscellaneous

lemma *is-least-in-ffilter-wrt-iff*:
assumes
trans: transp-on (fset (ffilter P X)) R and
asym: asymp-on (fset (ffilter P X)) R and
tot: totalp-on (fset (ffilter P X)) R
shows *is-least-in-fset-wrt R (ffilter P X) x \longleftrightarrow*
(x \in X \wedge P x \wedge fBall X ($\lambda y. y \neq x \longrightarrow P y \longrightarrow R x y$))
unfolding *is-least-in-fset-wrt-iff[OF trans asym tot] by auto*

lemma *is-least-in-ffilter-wrt-swap-predicate*:
assumes
trans: transp-on (fset X) R and
asym: asymp-on (fset X) R and
tot: totalp-on (fset X) R
assumes
y-least: is-least-in-fset-wrt R (ffilter P X) y and
same-on-prefix: $\bigwedge x. x \in X \implies R \ x \ y \implies P x \longleftrightarrow Q x$
shows *is-least-in-fset-wrt R (ffilter Q X) y*
proof –
have $\bigwedge P. \text{fset } (\text{ffilter } P \ X) \subseteq \text{fset } X$
by *simp*

hence

linorder-wrt-P:

transp-on (*fset* (*ffilter* *P X*)) *R*

asympt-on (*fset* (*ffilter* *P X*)) *R*

totalp-on (*fset* (*ffilter* *P X*)) *R* **and**

linorder-wrt-Q:

transp-on (*fset* (*ffilter* *Q X*)) *R*

asympt-on (*fset* (*ffilter* *Q X*)) *R*

totalp-on (*fset* (*ffilter* *Q X*)) *R*

unfolding *atomize-conj*

using *trans asym tot* **by** (*metis transp-on-subset asympt-on-subset totalp-on-subset*)

have $y \in X$ **and** $P y$ **and** *y-lt*: $\forall z \in X. z \neq y \longrightarrow P z \longrightarrow R y z$

using *y-least* **unfolding** *is-least-in-ffilter-wrt-iff* [*OF linorder-wrt-P*] **by** *argo+*

show *?thesis*

unfolding *is-least-in-ffilter-wrt-iff* [*OF linorder-wrt-Q*]

proof (*intro conjI ballI impI*)

show $y \in X$

using $\langle y \in X \rangle$.

show $Q y$

using *same-on-prefix*[*of y*] $\langle y \in X \rangle \langle P y \rangle$ **by** *simp*

fix *z*

assume $z \in X$ **and** $z \neq y$ **and** $Q z$

then show $R y z$

using *y-lt*[*rule-format, of z*]

using *same-on-prefix*[*of z*]

by (*metis* $\langle y \in X \rangle$ *sup2I1 tot totalp-onD*)

qed

qed

lemma *ex-is-least-in-ffilter-wrt-iff*:

assumes

trans: *transp-on* (*fset* (*ffilter* *P X*)) *R* **and**

asym: *asympt-on* (*fset* (*ffilter* *P X*)) *R* **and**

tot: *totalp-on* (*fset* (*ffilter* *P X*)) *R*

shows $(\exists x. \text{is-least-in-fset-wrt } R \text{ (ffilter } P X) x) \longleftrightarrow (\exists x \in X. P x)$

by (*metis* *asym bot-fset.rep-eq empty-iff ex-least-in-fset-wrt ffilter-filter is-least-in-fset-wrt-iff local.trans tot*)

13 Hide stuff

We restrict the public interface to ease future internal changes.

hide-fact *is-minimal-in-fset-wrt-def is-maximal-in-fset-wrt-def*

hide-fact *is-least-in-fset-wrt-def is-greatest-in-fset-wrt-def*

14 Integration in type classes

abbreviation (in order) *is-minimal-in-fset* where
is-minimal-in-fset \equiv *is-minimal-in-fset-wrt* ($<$)

abbreviation (in order) *is-maximal-in-fset* where
is-maximal-in-fset \equiv *is-maximal-in-fset-wrt* ($<$)

lemmas (in order) *is-minimal-in-fset-iff* =
is-minimal-in-fset-wrt-iff[*OF transp-on-less asymp-on-less*]

lemmas (in order) *is-maximal-in-fset-iff* =
is-maximal-in-fset-wrt-iff[*OF transp-on-less asymp-on-less*]

lemmas (in order) *ex-minimal-in-fset* =
ex-minimal-in-fset-wrt[*OF transp-on-less asymp-on-less*]

lemmas (in order) *ex-maximal-in-fset* =
ex-maximal-in-fset-wrt[*OF transp-on-less asymp-on-less*]

lemmas (in order) *is-minimal-in-fset-ffilter-iff* =
is-minimal-in-fset-wrt-ffilter-iff[*OF transp-on-less asymp-on-less*]

lemmas (in order) *is-minimal-in-fset-finsertI* =
is-minimal-in-fset-wrt-finsertI[*OF transp-on-less asymp-on-less*]

abbreviation (in *linorder*) *is-least-in-fset* where
is-least-in-fset \equiv *is-least-in-fset-wrt* ($<$)

abbreviation (in *linorder*) *is-greatest-in-fset* where
is-greatest-in-fset \equiv *is-greatest-in-fset-wrt* ($<$)

lemmas (in *linorder*) *is-least-in-fset-iff* =
is-least-in-fset-wrt-iff[*OF transp-on-less asymp-on-less totalp-on-less*]

lemmas (in *linorder*) *is-greatest-in-fset-iff* =
is-greatest-in-fset-wrt-iff[*OF transp-on-less asymp-on-less totalp-on-less*]

lemmas (in *linorder*) *is-minimal-in-fset-eq-is-least-in-fset* =
is-minimal-in-fset-wrt-eq-is-least-in-fset-wrt[*OF transp-on-less asymp-on-less totalp-on-less*]

lemmas (in *linorder*) *is-maximal-in-fset-eq-is-greatest-in-fset* =
is-maximal-in-fset-wrt-eq-is-greatest-in-fset-wrt[*OF transp-on-less asymp-on-less totalp-on-less*]

lemmas (in *linorder*) *Uniq-is-least-in-fset*[*intro*] =
Uniq-is-least-in-fset-wrt[*OF transp-on-less asymp-on-less totalp-on-less*]

```

lemmas (in linorder) Uniq-is-greatest-in-fset[intro] =
  Uniq-is-greatest-in-fset-wrt[OF transp-on-less asymp-on-less totalp-on-less]

lemmas (in linorder) ex-least-in-fset =
  ex-least-in-fset-wrt[OF transp-on-less asymp-on-less totalp-on-less]

lemmas (in linorder) ex-greatest-in-fset =
  ex-greatest-in-fset-wrt[OF transp-on-less asymp-on-less totalp-on-less]

lemmas (in linorder) ex1-least-in-fset =
  ex1-least-in-fset-wrt[OF transp-on-less asymp-on-less totalp-on-less]

lemmas (in linorder) ex1-greatest-in-fset =
  ex1-greatest-in-fset-wrt[OF transp-on-less asymp-on-less totalp-on-less]

lemmas (in linorder) is-least-in-ffilter-iff =
  is-least-in-ffilter-wrt-iff[OF transp-on-less asymp-on-less totalp-on-less]

lemmas (in linorder) ex-is-least-in-ffilter-iff =
  ex-is-least-in-ffilter-wrt-iff[OF transp-on-less asymp-on-less totalp-on-less]

lemmas (in linorder) is-least-in-ffilter-swap-predicate =
  is-least-in-ffilter-wrt-swap-predicate[OF transp-on-less asymp-on-less totalp-on-less]

end
theory Min-Max-Least-Greatest-Multiset
  imports
    Relation-Reachability
    Min-Max-Least-Greatest-Set
    HOL-Library.Multiset
    HOL-Library.Multiset-Order
begin

```

15 Minimal and maximal elements

```

definition is-minimal-in-mset-wrt :: ('a ⇒ 'a ⇒ bool) ⇒ 'a multiset ⇒ 'a ⇒ bool
where
  transp-on (set-mset X) R ⇒ asymp-on (set-mset X) R ⇒
  is-minimal-in-mset-wrt R X = is-minimal-in-set-wrt R (set-mset X)

```

```

definition is-maximal-in-mset-wrt :: ('a ⇒ 'a ⇒ bool) ⇒ 'a multiset ⇒ 'a ⇒ bool
where
  transp-on (set-mset X) R ⇒ asymp-on (set-mset X) R ⇒
  is-maximal-in-mset-wrt R X = is-maximal-in-set-wrt R (set-mset X)

```

```

definition is-strictly-minimal-in-mset-wrt :: ('a ⇒ 'a ⇒ bool) ⇒ 'a multiset ⇒ 'a
⇒ bool where
  transp-on (set-mset X) R ⇒ asymp-on (set-mset X) R ⇒

```

is-strictly-minimal-in-mset-wrt $R X x \longleftrightarrow x \in \# X \wedge (\forall y \in \# X - \{\# x \# \}. \neg (R = y x))$

definition *is-strictly-maximal-in-mset-wrt* :: ('a \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a multiset \Rightarrow 'a \Rightarrow bool **where**
transp-on (set-mset X) R \Longrightarrow *asympt-on* (set-mset X) R \Longrightarrow
is-strictly-maximal-in-mset-wrt R X x $\longleftrightarrow x \in \# X \wedge (\forall y \in \# X - \{\# x \# \}. \neg (R = x y))$

context

fixes X R

assumes

trans: *transp-on* (set-mset X) R **and**

asym: *asympt-on* (set-mset X) R

begin

15.1 Conversions

lemma *is-minimal-in-mset-wrt-iff*:

is-minimal-in-mset-wrt R X x $\longleftrightarrow x \in \# X \wedge (\forall y \in \# X. y \neq x \longrightarrow \neg R y x)$

using *is-minimal-in-set-wrt-iff*[OF *trans asym*]

using *is-minimal-in-mset-wrt-def*[OF *trans asym*]

by *simp*

lemma *is-minimal-in-mset-wrt* R X x $\longleftrightarrow x \in \# X \wedge (\forall y \in \# X. \neg R y x)$

unfolding *is-minimal-in-mset-wrt-iff*

proof (rule *refl-conj-eq*, rule *ball-cong*)

show *set-mset* X = *set-mset* X ..

next

show $\bigwedge y. y \in \# X \Longrightarrow (y \neq x \longrightarrow \neg R y x) = (\neg R y x)$

using *asym*[THEN *asympt-onD*] **by** *metis*

qed

lemma *is-maximal-in-mset-wrt-iff*:

is-maximal-in-mset-wrt R X x $\longleftrightarrow x \in \# X \wedge (\forall y \in \# X. y \neq x \longrightarrow \neg R x y)$

using *is-maximal-in-set-wrt-iff*[OF *trans asym*]

using *is-maximal-in-mset-wrt-def*[OF *trans asym*]

by *simp*

lemma *is-maximal-in-mset-wrt* R X x $\longleftrightarrow x \in \# X \wedge (\forall y \in \# X. \neg R x y)$

unfolding *is-maximal-in-mset-wrt-iff*

proof (rule *refl-conj-eq*, rule *ball-cong*)

show *set-mset* X = *set-mset* X ..

next

show $\bigwedge y. y \in \# X \Longrightarrow (y \neq x \longrightarrow \neg R x y) = (\neg R x y)$

using *asym*[THEN *asympt-onD*] **by** *metis*

qed

lemma *is-strictly-minimal-in-mset-wrt-iff*:

is-strictly-minimal-in-mset-wrt $R X x \longleftrightarrow x \in \# X \wedge (\forall y \in \# X - \{\# x \# \}. \neg R^{==} y x)$
unfolding *is-strictly-minimal-in-mset-wrt-def*[*OF trans asym*]
by(*rule refl*)

lemma *is-strictly-maximal-in-mset-wrt-iff*:
is-strictly-maximal-in-mset-wrt $R X x \longleftrightarrow x \in \# X \wedge (\forall y \in \# X - \{\# x \# \}. \neg R^{==} x y)$
unfolding *is-strictly-maximal-in-mset-wrt-def*[*OF trans asym*]
by(*rule refl*)

lemma *is-minimal-in-mset-wrt-if-is-strictly-minimal-in-mset-wrt*:
is-strictly-minimal-in-mset-wrt $R X x \implies$ *is-minimal-in-mset-wrt* $R X x$
unfolding *is-minimal-in-mset-wrt-iff is-strictly-minimal-in-mset-wrt-iff*
using *multi-member-split by fastforce*

lemma *is-maximal-in-mset-wrt-if-is-strictly-maximal-in-mset-wrt*:
is-strictly-maximal-in-mset-wrt $R X x \implies$ *is-maximal-in-mset-wrt* $R X x$
unfolding *is-maximal-in-mset-wrt-iff is-strictly-maximal-in-mset-wrt-iff*
using *multi-member-split by fastforce*

15.2 Existence

lemma *ex-minimal-in-mset-wrt*:
 $X \neq \{\#\} \implies \exists m. \text{is-minimal-in-mset-wrt } R X m$
using *trans asym ex-minimal-in-set-wrt*[*of set-mset X R*] *is-minimal-in-mset-wrt-def*
by (*metis finite-set-mset set-mset-eq-empty-iff*)

lemma *ex-maximal-in-mset-wrt*:
 $X \neq \{\#\} \implies \exists m. \text{is-maximal-in-mset-wrt } R X m$
using *trans asym ex-maximal-in-set-wrt*[*of set-mset X R*] *is-maximal-in-mset-wrt-def*
by (*metis finite-set-mset set-mset-eq-empty-iff*)

15.3 Miscellaneous

lemma *explode-maximal-in-mset-wrt*:
assumes *max*: *is-maximal-in-mset-wrt* $R X x$
obtains $n :: \text{nat}$ **where** *replicate-mset* (*Suc n*) $x + \{\#y \in \# X. y \neq x\# \} = X$
using *max*[*unfolded is-maximal-in-mset-wrt-iff*]
by (*metis filter-eq-replicate-mset in-countE multiset-partition*)

lemma *explode-strictly-maximal-in-mset-wrt*:
assumes *max*: *is-strictly-maximal-in-mset-wrt* $R X x$
shows *add-mset* $x \{\#y \in \# X. y \neq x\# \} = X$
proof –
have $x \in \# X$ **and** $\forall y \in \# X - \{\#x\#\}. x \neq y$
using *max* **unfolding** *is-strictly-maximal-in-mset-wrt-iff* **by** *simp-all*
have *add-mset* $x (X - \{\#x\#\}) = X$
using $\langle x \in \# X \rangle$ **by** (*metis insert-DiffM*)

```

moreover have  $\{\#y \in\# X. y \neq x\# \} = X - \{\#x\# \}$ 
using  $\langle \forall y \in\# X - \{\#x\#\}. x \neq y \rangle$ 
by (smt (verit, best)  $\langle x \in\# X \rangle$  add-diff-cancel-left' diff-subset-eq-self filter-mset-eq-conv
      insert-DiffM2 set-mset-add-mset-insert set-mset-empty singletonD)

ultimately show ?thesis
by (simp only:)
qed

end

lemma is-minimal-in-filter-mset-wrt-iff:
assumes
  tran: transp-on (set-mset (filter-mset P X)) R and
  asym: asymp-on (set-mset (filter-mset P X)) R
shows is-minimal-in-mset-wrt R (filter-mset P X) x  $\longleftrightarrow$ 
  ( $x \in\# X \wedge P x \wedge (\forall y \in\# X - \{\#x\#\}. P y \longrightarrow \neg R y x)$ )
proof -
have is-minimal-in-mset-wrt R (filter-mset P X) x  $\longleftrightarrow$ 
  is-minimal-in-set-wrt R ( $\{y \in\# X. P y\}$ ) x
using is-minimal-in-mset-wrt-iff[OF tran asym]
using is-minimal-in-set-wrt-iff[OF tran asym]
by auto
also have  $\dots \longleftrightarrow x \in\# X \wedge P x \wedge (\forall y \in\# X - \{x\}. P y \longrightarrow \neg R y x)$ 
proof (rule is-minimal-in-set-wrt-filter-iff)
  show transp-on  $\{y. y \in\# X \wedge P y\} R$ 
    using tran by simp
next
  show asymp-on  $\{y. y \in\# X \wedge P y\} R$ 
    using asym by simp
qed
finally show ?thesis
by (metis (no-types, lifting) DiffD1 asym asymp-onD at-most-one-mset-mset-diff
      insertE
      insert-Diff is-minimal-in-mset-wrt-iff more-than-one-mset-mset-diff tran)
qed

lemma multp-if-maximal-of-lhs-is-less:
assumes
  trans: transp R and
  asym: asymp-on (set-mset M1) R and
  tot: totalp-on (set-mset M1  $\cup$  set-mset M2) R and
  x1  $\in\#$  M1 and x2  $\in\#$  M2 and
  is-maximal-in-mset-wrt R M1 x1 and R x1 x2
shows multp R M1 M2
proof (rule one-step-implies-multp[of - - -  $\{\#\}$ , simplified])
show M2  $\neq$   $\{\#\}$ 
  using  $\langle x2 \in\# M2 \rangle$  by auto

```

```

next
show  $\forall k \in \#M1. \exists j \in \#M2. R k j$ 
  using assms
  using is-maximal-in-mset-wrt-iff[OF transp-on-subset[OF trans subset-UNIV]
asym]
  by (metis Un-iff totalp-onD transpE)
qed

```

15.4 Nonuniqueness

lemma

fixes $x :: 'a$ and $y :: 'a$

assumes $x \neq y$

shows

not-Uniq-is-minimal-in-mset-if-two-distinct-elements:

$\neg (\forall (R :: 'a \Rightarrow 'a \Rightarrow \text{bool}) (X :: 'a \text{ multiset}).$
 $\text{transp-on } (\text{set-mset } X) R \longrightarrow \text{asym-on } (\text{set-mset } X) R \longrightarrow$
 $(\exists_{\leq 1} x. \text{is-minimal-in-mset-wrt } R X x))$ and

not-Uniq-is-maximal-in-mset-wrt-if-two-distinct-elements:

$\neg (\forall (R :: 'a \Rightarrow 'a \Rightarrow \text{bool}) (X :: 'a \text{ multiset}).$
 $\text{transp-on } (\text{set-mset } X) R \longrightarrow \text{asym-on } (\text{set-mset } X) R \longrightarrow$
 $(\exists_{\leq 1} x. \text{is-maximal-in-mset-wrt } R X x))$ and

not-Uniq-is-strictly-minimal-in-mset-if-two-distinct-elements:

$\neg (\forall (R :: 'a \Rightarrow 'a \Rightarrow \text{bool}) (X :: 'a \text{ multiset}).$
 $\text{transp-on } (\text{set-mset } X) R \longrightarrow \text{asym-on } (\text{set-mset } X) R \longrightarrow$
 $(\exists_{\leq 1} x. \text{is-strictly-minimal-in-mset-wrt } R X x))$ and

not-Uniq-is-strictly-maximal-in-mset-wrt-if-two-distinct-elements:

$\neg (\forall (R :: 'a \Rightarrow 'a \Rightarrow \text{bool}) (X :: 'a \text{ multiset}).$
 $\text{transp-on } (\text{set-mset } X) R \longrightarrow \text{asym-on } (\text{set-mset } X) R \longrightarrow$
 $(\exists_{\leq 1} x. \text{is-strictly-maximal-in-mset-wrt } R X x))$

proof -

let $?R = \lambda -. \text{False}$

let $?X = \{\#x, y\}$

have *trans*: $\text{transp-on } (\text{set-mset } ?X) ?R$ and *asym*: $\text{asym-on } (\text{set-mset } ?X) ?R$
 by (*simp-all add: transp-onI asym-onI*)

have *is-minimal-in-mset-wrt* $?R ?X x$ and *is-minimal-in-mset-wrt* $?R ?X y$
 using *is-minimal-in-mset-wrt-iff*[*OF trans asym*] by *simp-all*

thus $\neg (\forall (R :: 'a \Rightarrow 'a \Rightarrow \text{bool}) (X :: 'a \text{ multiset}).$

$\text{transp-on } (\text{set-mset } X) R \longrightarrow \text{asym-on } (\text{set-mset } X) R \longrightarrow$
 $(\exists_{\leq 1} x. \text{is-minimal-in-mset-wrt } R X x))$

using $\langle x \neq y \rangle$ *trans asym*

by (*metis Uniq-D*)

have *is-maximal-in-mset-wrt* $?R ?X x$ and *is-maximal-in-mset-wrt* $?R ?X y$
 using *is-maximal-in-mset-wrt-iff*[*OF trans asym*] by *simp-all*

thus $\neg (\forall (R :: 'a \Rightarrow 'a \Rightarrow \text{bool}) (X :: 'a \text{ multiset}).$
 $\text{transp-on } (\text{set-mset } X) R \longrightarrow \text{asympt-on } (\text{set-mset } X) R \longrightarrow$
 $(\exists_{\leq 1} x. \text{is-maximal-in-mset-wrt } R X x))$
using $\langle x \neq y \rangle \text{ trans asym}$
by (metis Uniq-D)

have $\text{is-strictly-minimal-in-mset-wrt } ?R ?X x$ **and** $\text{is-strictly-minimal-in-mset-wrt}$
 $?R ?X y$
using $\langle x \neq y \rangle \text{ is-strictly-minimal-in-mset-wrt-iff}[OF \text{ trans asym}]$ **by** simp-all

thus $\neg (\forall (R :: 'a \Rightarrow 'a \Rightarrow \text{bool}) (X :: 'a \text{ multiset}).$
 $\text{transp-on } (\text{set-mset } X) R \longrightarrow \text{asympt-on } (\text{set-mset } X) R \longrightarrow$
 $(\exists_{\leq 1} x. \text{is-strictly-minimal-in-mset-wrt } R X x))$
using $\langle x \neq y \rangle \text{ trans asym}$
by (metis Uniq-D)

have $\text{is-strictly-maximal-in-mset-wrt } ?R ?X x$ **and** $\text{is-strictly-maximal-in-mset-wrt}$
 $?R ?X y$
using $\langle x \neq y \rangle \text{ is-strictly-maximal-in-mset-wrt-iff}[OF \text{ trans asym}]$ **by** simp-all

thus $\neg (\forall (R :: 'a \Rightarrow 'a \Rightarrow \text{bool}) (X :: 'a \text{ multiset}).$
 $\text{transp-on } (\text{set-mset } X) R \longrightarrow \text{asympt-on } (\text{set-mset } X) R \longrightarrow$
 $(\exists_{\leq 1} x. \text{is-strictly-maximal-in-mset-wrt } R X x))$
using $\langle x \neq y \rangle \text{ trans asym}$
by (metis Uniq-D)

qed

16 Least and greatest elements

definition $\text{is-least-in-mset-wrt} :: ('a \Rightarrow 'a \Rightarrow \text{bool}) \Rightarrow 'a \text{ multiset} \Rightarrow 'a \Rightarrow \text{bool}$
where

$\text{transp-on } (\text{set-mset } X) R \Longrightarrow \text{asympt-on } (\text{set-mset } X) R \Longrightarrow \text{totalp-on } (\text{set-mset}$
 $X) R \Longrightarrow$
 $\text{is-least-in-mset-wrt } R X x \longleftrightarrow x \in \# X \wedge (\forall y \in \# X - \{\#x\}. R x y)$

definition $\text{is-greatest-in-mset-wrt} :: ('a \Rightarrow 'a \Rightarrow \text{bool}) \Rightarrow 'a \text{ multiset} \Rightarrow 'a \Rightarrow \text{bool}$
where

$\text{transp-on } (\text{set-mset } X) R \Longrightarrow \text{asympt-on } (\text{set-mset } X) R \Longrightarrow \text{totalp-on } (\text{set-mset}$
 $X) R \Longrightarrow$
 $\text{is-greatest-in-mset-wrt } R X x \longleftrightarrow x \in \# X \wedge (\forall y \in \# X - \{\#x\}. R y x)$

context

fixes $X R$

assumes

$\text{trans: transp-on } (\text{set-mset } X) R$ **and**

$\text{asym: asympt-on } (\text{set-mset } X) R$ **and**

$\text{tot: totalp-on } (\text{set-mset } X) R$

begin

16.1 Conversions

lemma *is-least-in-mset-wrt-iff*:

is-least-in-mset-wrt $R X x \longleftrightarrow x \in \# X \wedge (\forall y \in \# X - \{\#x\}. R x y)$
using *is-least-in-mset-wrt-def*[*OF trans asym tot*].

lemma *is-greatest-in-mset-wrt-iff*:

is-greatest-in-mset-wrt $R X x \longleftrightarrow x \in \# X \wedge (\forall y \in \# X - \{\#x\}. R y x)$
using *is-greatest-in-mset-wrt-def*[*OF trans asym tot*].

lemma *is-minimal-in-mset-wrt-if-is-least-in-mset-wrt*[*intro*]:

is-least-in-mset-wrt $R X x \implies is-minimal-in-mset-wrt R X x$

unfolding *is-minimal-in-mset-wrt-iff*[*OF trans asym*]

unfolding *is-least-in-mset-wrt-def*[*OF trans asym tot*]

by (*metis add-mset-remove-trivial-eq asym asymp-onD insert-noteq-member*)

lemma *is-maximal-in-mset-wrt-if-is-greatest-in-mset-wrt*[*intro*]:

is-greatest-in-mset-wrt $R X x \implies is-maximal-in-mset-wrt R X x$

unfolding *is-maximal-in-mset-wrt-iff*[*OF trans asym*]

unfolding *is-greatest-in-mset-wrt-def*[*OF trans asym tot*]

by (*metis add-mset-remove-trivial-eq asym asymp-onD insert-noteq-member*)

lemma *is-strictly-minimal-in-mset-wrt-iff-is-least-in-mset-wrt*:

is-strictly-minimal-in-mset-wrt $R X = is-least-in-mset-wrt R X$

unfolding *is-strictly-minimal-in-mset-wrt-iff*[*OF trans asym*] *is-least-in-mset-wrt-iff*

proof(*intro ext iffI*)

fix x

show $x \in \# X \wedge (\forall y \in \# X - \{\#x\}. \neg R^{==} y x) \implies x \in \# X \wedge (\forall y \in \# X - \{\#x\}. R x y)$

by (*metis (mono-tags, lifting) in-diffD sup2CI tot totalp-onD*)

next

fix x

show $x \in \# X \wedge (\forall y \in \# X - \{\#x\}. R x y) \implies x \in \# X \wedge (\forall y \in \# X - \{\#x\}. \neg R^{==} y x)$

by (*metis (full-types) asym asymp-onD in-diffD sup2E*)

qed

lemma *is-strictly-maximal-in-mset-wrt-iff-is-greatest-in-mset-wrt*:

is-strictly-maximal-in-mset-wrt $R X = is-greatest-in-mset-wrt R X$

unfolding *is-strictly-maximal-in-mset-wrt-iff*[*OF trans asym*] *is-greatest-in-mset-wrt-iff*

proof(*intro ext iffI*)

fix x

show $x \in \# X \wedge (\forall y \in \# X - \{\#x\}. \neg R^{==} x y) \implies x \in \# X \wedge (\forall y \in \# X - \{\#x\}. R y x)$

by (*metis (mono-tags, lifting) in-diffD sup2CI tot totalp-onD*)

next

fix x

show $x \in \# X \wedge (\forall y \in \# X - \{\#x\}. R y x) \implies x \in \# X \wedge (\forall y \in \# X - \{\#x\}. \neg R^{==} x y)$

$\neg R^{==} x y$
 by (metis (full-types) asym asymp-onD in-diffD sup2E)
 qed

16.2 Uniqueness

lemma *Uniq-is-minimal-in-mset-wrt*[intro]:
 $\exists_{\leq 1} x. \text{is-minimal-in-mset-wrt } R X x$
unfolding *is-minimal-in-mset-wrt-iff*[OF trans asym]
 by (smt (verit, best) Uniq-I tot totalp-onD)

lemma *Uniq-is-maximal-in-mset-wrt*[intro]:
 $\exists_{\leq 1} x. \text{is-maximal-in-mset-wrt } R X x$
unfolding *is-maximal-in-mset-wrt-iff*[OF trans asym]
 by (smt (verit, best) Uniq-I tot totalp-onD)

lemma *Uniq-is-least-in-mset-wrt*[intro]:
 $\exists_{\leq 1} x. \text{is-least-in-mset-wrt } R X x$
using *is-least-in-mset-wrt-iff*
 by (smt (verit, best) Uniq-I asym asymp-onD insert-DiffM insert-noteq-member)

lemma *Uniq-is-greatest-in-mset-wrt*[intro]:
 $\exists_{\leq 1} x. \text{is-greatest-in-mset-wrt } R X x$
unfolding *is-greatest-in-mset-wrt-iff*
 by (smt (verit, best) Uniq-I asym asymp-onD insert-DiffM insert-noteq-member)

16.3 Miscellaneous

lemma *is-least-in-mset-wrt-iff-is-minimal-and-count-eq-one*:
 $\text{is-least-in-mset-wrt } R X x \longleftrightarrow \text{is-minimal-in-mset-wrt } R X x \wedge \text{count } X x = 1$
proof (rule iffI)
assume *is-least-in-mset-wrt* $R X x$
thus *is-minimal-in-mset-wrt* $R X x \wedge \text{count } X x = 1$
unfolding *is-least-in-mset-wrt-iff is-minimal-in-mset-wrt-iff*[OF trans asym]
by (metis One-nat-def add-mset-remove-trivial-eq asym asymp-onD count-add-mset not-in-iff)
next
assume *is-minimal-in-mset-wrt* $R X x \wedge \text{count } X x = 1$
then show *is-least-in-mset-wrt* $R X x$
unfolding *is-least-in-mset-wrt-iff is-minimal-in-mset-wrt-iff*[OF trans asym]
by (metis count-single in-diffD in-diff-count nat-less-le tot totalp-onD)
 qed

lemma *is-greatest-in-mset-wrt-iff-is-maximal-and-count-eq-one*:
 $\text{is-greatest-in-mset-wrt } R X x \longleftrightarrow \text{is-maximal-in-mset-wrt } R X x \wedge \text{count } X x = 1$
proof (rule iffI)
assume *is-greatest-in-mset-wrt* $R X x$
thus *is-maximal-in-mset-wrt* $R X x \wedge \text{count } X x = 1$
unfolding *is-greatest-in-mset-wrt-iff is-maximal-in-mset-wrt-iff*[OF trans asym]

by (*metis One-nat-def add-mset-remove-trivial-eq asym asymp-onD count-add-mset not-in-iff*)
next
assume *is-maximal-in-mset-wrt R X x* \wedge *count X x = 1*
then show *is-greatest-in-mset-wrt R X x*
unfolding *is-greatest-in-mset-wrt-iff is-maximal-in-mset-wrt-iff*[*OF trans asym*]
by (*metis count-single in-diffD in-diff-count nat-less-le tot totalp-onD*)
qed

lemma *count-ge-2-if-minimal-in-mset-wrt-and-not-least-in-mset-wrt:*
assumes *is-minimal-in-mset-wrt R X x* **and** \neg *is-least-in-mset-wrt R X x*
shows *count X x \geq 2*
using *assms*
unfolding *is-least-in-mset-wrt-iff-is-minimal-and-count-eq-one*
by (*metis Suc-1 Suc-le-eq antisym-conv1 asym count-greater-eq-one-iff is-minimal-in-mset-wrt-iff trans*)

lemma *count-ge-2-if-maximal-in-mset-wrt-and-not-greatest-in-mset-wrt:*
assumes *is-maximal-in-mset-wrt R X x* **and** \neg *is-greatest-in-mset-wrt R X x*
shows *count X x \geq 2*
using *assms*
unfolding *is-greatest-in-mset-wrt-iff-is-maximal-and-count-eq-one*
by (*metis Suc-1 Suc-le-eq antisym-conv1 asym count-greater-eq-one-iff is-maximal-in-mset-wrt-iff trans*)

lemma *explode-greatest-in-mset-wrt:*
assumes *max: is-greatest-in-mset-wrt R X x*
shows *add-mset x {#y \in # X. y \neq x#} = X*
using *max*[*folded is-strictly-maximal-in-mset-wrt-iff-is-greatest-in-mset-wrt*]
using *explode-strictly-maximal-in-mset-wrt*[*OF trans asym*]
by *metis*

end

lemma *multp_{HO}-if-maximal-wrt-less-than-maximal-wrt:*

assumes

trans: transp-on (set-mset M1 \cup set-mset M2) R **and**

asym: asymp-on (set-mset M1 \cup set-mset M2) R **and**

tot: totalp-on (set-mset M1 \cup set-mset M2) R **and**

x1-maximal: is-maximal-in-mset-wrt R M1 x1 **and**

x2-maximal: is-maximal-in-mset-wrt R M2 x2 **and**

R x1 x2

shows *multp_{HO} R M1 M2*

proof –

have

trans1: transp-on (set-mset M1) R **and** *trans2: transp-on (set-mset M2) R* **and**

asym1: asymp-on (set-mset M1) R **and** *asym2: asymp-on (set-mset M2) R*

and

tot1: *totalp-on* (set-mset $M1$) R **and** **tot2**: *totalp-on* (set-mset $M2$) R
using *trans*[*THEN transp-on-subset*] *asym*[*THEN asymp-on-subset*] *tot*[*THEN totalp-on-subset*]
by *simp-all*

have *x1-in*: $x1 \in \# M1$ **and** *x1-gr*: $\forall y \in \# M1. y \neq x1 \longrightarrow \neg R x1 y$
using *x1-maximal*[*unfolded is-maximal-in-mset-wrt-iff*[*OF trans1 asym1*]] **by**
argo+

have *x2-in*: $x2 \in \# M2$ **and** *x2-gr*: $\forall y \in \# M2. y \neq x2 \longrightarrow \neg R x2 y$
using *x2-maximal*[*unfolded is-maximal-in-mset-wrt-iff*[*OF trans2 asym2*]] **by**
argo+

show *multp_{HO}* $R M1 M2$
unfolding *multp_{HO}-def*
proof (*intro conjI*)
show $M1 \neq M2$
using *x1-in x2-in x1-gr*
by (*metis* $\langle R x1 x2 \rangle$ *asym2 asymp-onD*)
next
show $\forall y. \text{count } M2 y < \text{count } M1 y \longrightarrow (\exists x. R y x \wedge \text{count } M1 x < \text{count } M2 x)$
using *x1-in x2-in x1-gr x2-gr*
by (*smt* (*verit, best*) *assms*(6) *asym1 asymp-onD count-greater-zero-iff count-inI dual-order.strict-trans local.trans subsetD sup-ge1 sup-ge2 tot1 totalp-onD transp-onD*)
qed
qed

lemma *multp_{DM}-if-maximal-wrt-less-that-maximal-wrt*:

assumes

trans: *transp-on* (set-mset $M1 \cup$ set-mset $M2$) R **and**
asym: *asymp-on* (set-mset $M1 \cup$ set-mset $M2$) R **and**
tot: *totalp-on* (set-mset $M1 \cup$ set-mset $M2$) R **and**
x1-maximal: *is-maximal-in-mset-wrt* $R M1 x1$ **and**
x2-maximal: *is-maximal-in-mset-wrt* $R M2 x2$ **and**
 $R x1 x2$

shows *multp_{DM}* $R M1 M2$

using *multp_{HO}-if-maximal-wrt-less-that-maximal-wrt*[*OF assms, THEN multp_{HO}-imp-multp_{DM}*]

lemma *multp-if-maximal-wrt-less-that-maximal-wrt*:

assumes

trans: *transp-on* (set-mset $M1 \cup$ set-mset $M2$) R **and**
asym: *asymp-on* (set-mset $M1 \cup$ set-mset $M2$) R **and**
tot: *totalp-on* (set-mset $M1 \cup$ set-mset $M2$) R **and**
x1-maximal: *is-maximal-in-mset-wrt* $R M1 x1$ **and**
x2-maximal: *is-maximal-in-mset-wrt* $R M2 x2$ **and**
 $R x1 x2$

shows $\text{multp } R \ M1 \ M2$
using $\text{multp}_{DM}\text{-if-maximal-wrt-less-that-maximal-wrt}[OF \ \text{assms}, \ \text{THEN } \text{multp}_{DM}\text{-imp-multp}]$

lemma $\text{multp}_{HO}\text{-if-same-maximal-wrt-and-count-lt}$:

assumes

trans : $\text{transp-on } (\text{set-mset } M1 \cup \text{set-mset } M2) \ R$ **and**

asym : $\text{asyp-on } (\text{set-mset } M1 \cup \text{set-mset } M2) \ R$ **and**

tot : $\text{totalp-on } (\text{set-mset } M1 \cup \text{set-mset } M2) \ R$ **and**

$x1\text{-maximal}$: $\text{is-maximal-in-mset-wrt } R \ M1 \ x$ **and**

$x2\text{-maximal}$: $\text{is-maximal-in-mset-wrt } R \ M2 \ x$ **and**

$\text{count } M1 \ x < \text{count } M2 \ x$

shows $\text{multp}_{HO} \ R \ M1 \ M2$

by ($\text{metis } \text{assms}(6) \ \text{asym } \text{asyp-on-subset } \text{count-inI } \text{is-greatest-in-set-wrt-iff}$

$\text{is-maximal-in-mset-wrt-def } \text{is-maximal-in-set-wrt-eq-is-greatest-in-set-wrt-less-zeroE}$

$\text{local.trans } \text{multp}_{HO}\text{-def } \text{order-less-imp-not-less } \text{sup-ge1 } \text{tot } \text{totalp-on-subset}$

transp-on-subset

$x1\text{-maximal}$)

lemma $\text{multp}\text{-if-same-maximal-wrt-and-count-lt}$:

assumes

trans : $\text{transp-on } (\text{set-mset } M1 \cup \text{set-mset } M2) \ R$ **and**

asym : $\text{asyp-on } (\text{set-mset } M1 \cup \text{set-mset } M2) \ R$ **and**

tot : $\text{totalp-on } (\text{set-mset } M1 \cup \text{set-mset } M2) \ R$ **and**

$x1\text{-maximal}$: $\text{is-maximal-in-mset-wrt } R \ M1 \ x$ **and**

$x2\text{-maximal}$: $\text{is-maximal-in-mset-wrt } R \ M2 \ x$ **and**

$\text{count } M1 \ x < \text{count } M2 \ x$

shows $\text{multp } R \ M1 \ M2$

using $\text{multp}_{HO}\text{-if-same-maximal-wrt-and-count-lt}$

$OF \ \text{assms}, \ \text{THEN } \text{multp}_{HO}\text{-imp-multp}_{DM}, \ \text{THEN } \text{multp}_{DM}\text{-imp-multp}]$.

lemma $\text{less-than-maximal-wrt-if-multp}_{HO}$:

assumes

trans : $\text{transp-on } (\text{set-mset } M1 \cup \text{set-mset } M2) \ R$ **and**

asym : $\text{asyp-on } (\text{set-mset } M2) \ R$ **and**

tot : $\text{totalp-on } (\text{set-mset } M2) \ R$ **and**

$x2\text{-maximal}$: $\text{is-maximal-in-mset-wrt } R \ M2 \ x2$ **and**

$\text{multp}_{HO} \ R \ M1 \ M2$ **and**

$x1 \in \# \ M1$

shows $R = x1 \ x2$

proof –

have

trans2 : $\text{transp-on } (\text{set-mset } M2) \ R$ **and**

asym2 : $\text{asyp-on } (\text{set-mset } M2) \ R$ **and**

tot2 : $\text{totalp-on } (\text{set-mset } M2) \ R$

using $\text{trans}[\text{THEN } \text{transp-on-subset}] \ \text{asym}[\text{THEN } \text{asyp-on-subset}] \ \text{tot}[\text{THEN } \text{totalp-on-subset}]$

by simp-all

have $x2\text{-in}: x2 \in\# M2$ **and** $x2\text{-gr}: \forall y \in\# M2. y \neq x2 \longrightarrow \neg R\ x2\ y$
using $x2\text{-maximal}[unfolding\ is\ maximal\ in\ mset\ wrt\ iff[OF\ trans2\ asym2]]$ **by**
argo+

show *?thesis*

proof (*cases* $x1 \in\# M2$)

case *True*

thus *?thesis*

using $x2\text{-gr}$ **by** (*metis* (*mono-tags*) *sup2CI tot2 totalp-onD x2-in*)

next

case *False*

hence $x1 \in\# M1 - M2$

using $\langle x1 \in\# M1 \rangle$ **by** (*simp add: in-diff-count not-in-iff*)

moreover have $\forall k \in\# M1 - M2. \exists x \in\# M2 - M1. R\ k\ x$

using *multp_{HO}-implies-one-step-strong(2)[OF $\langle multp_{HO}\ R\ M1\ M2 \rangle$]*.

ultimately obtain x **where** $x \in\# M2 - M1$ **and** $R\ x1\ x$

by *metis*

hence $x \neq x2 \longrightarrow \neg R\ x2\ x$

using $x2\text{-gr}$ **by** (*metis in-diffD*)

hence $x \neq x2 \longrightarrow R\ x\ x2$

by (*metis* $\langle x \in\# M2 - M1 \rangle$ *in-diffD tot2 totalp-onD x2-in*)

thus *?thesis*

using $\langle R\ x1\ x \rangle$

by (*meson Un-iff* $\langle x \in\# M2 - M1 \rangle$ *assms(6) in-diffD local.trans sup2I1 transp-onD x2-in*)

qed

qed

17 Examples of duplicate handling in set and multiset definitions

lemma

fixes $M :: nat\ multiset$

defines $M \equiv \{\#0, 0, 1, 1, 2, 2\# \}$

shows

is-minimal-in-set-wrt ($<$) (*set-mset* M) 0

is-minimal-in-mset-wrt ($<$) $M\ 0$

is-least-in-set-wrt ($<$) (*set-mset* M) 0

$\nexists y. is-least-in-mset-wrt\ (<)\ M\ y$

by (*auto simp: M-def is-minimal-in-set-wrt-iff is-minimal-in-mset-wrt-def*

is-least-in-set-wrt-iff is-least-in-mset-wrt-def)

lemma
fixes $x\ y :: 'a$ **and** $M :: 'a$ *multiset*
defines $M \equiv \{\#x, y, y\#\}$
defines $R \equiv \lambda - . False$
assumes $x \neq y$
shows
is-maximal-in-mset-wrt $R\ M\ x$
is-maximal-in-mset-wrt $R\ M\ y$
is-strictly-maximal-in-mset-wrt $R\ M\ x$
 \neg *is-strictly-maximal-in-mset-wrt* $R\ M\ y$
proof –
have *transp-on-False*[simp]: $\bigwedge A. \text{transp-on } A (\lambda - . False)$
by (*simp add: transp-onI*)

have *asympt-on-False*[simp]: $\bigwedge A. \text{asympt-on } A (\lambda - . False)$
by (*simp add: asympt-onI*)

show
is-maximal-in-mset-wrt $R\ M\ x$
is-maximal-in-mset-wrt $R\ M\ y$
is-strictly-maximal-in-mset-wrt $R\ M\ x$
 \neg *is-strictly-maximal-in-mset-wrt* $R\ M\ y$
unfolding *is-maximal-in-mset-wrt-iff*[of $M\ R$, *unfolded R-def*, *simplified*, *folded R-def*]
unfolding *is-strictly-maximal-in-mset-wrt-iff*[of $M\ R$, *unfolded R-def*, *simplified*, *folded R-def*]
unfolding *atomize-conj*
using $\langle x \neq y \rangle$
by (*simp add: M-def*)
qed

18 Hide stuff

We restrict the public interface to ease future internal changes.

hide-fact *is-minimal-in-mset-wrt-def is-maximal-in-mset-wrt-def*

hide-fact *is-strictly-minimal-in-mset-wrt-def is-strictly-maximal-in-mset-wrt-def*

hide-fact *is-least-in-mset-wrt-def is-greatest-in-mset-wrt-def*

19 Integration in type classes

abbreviation (in *order*) *is-minimal-in-mset* **where**
is-minimal-in-mset \equiv *is-minimal-in-mset-wrt* ($<$)

abbreviation (in *order*) *is-maximal-in-mset* **where**
is-maximal-in-mset \equiv *is-maximal-in-mset-wrt* ($<$)

lemmas (in order) *is-minimal-in-mset-iff* =
is-minimal-in-mset-wrt-iff[*OF transp-on-less asymp-on-less*]

lemmas (in order) *is-maximal-in-mset-iff* =
is-maximal-in-mset-wrt-iff[*OF transp-on-less asymp-on-less*]

lemmas (in order) *ex-minimal-in-mset* =
ex-minimal-in-mset-wrt[*OF transp-on-less asymp-on-less*]

lemmas (in order) *ex-maximal-in-mset* =
ex-maximal-in-mset-wrt[*OF transp-on-less asymp-on-less*]

lemmas (in order) *explode-maximal-in-mset* =
explode-maximal-in-mset-wrt[*OF transp-on-less asymp-on-less*]

lemmas (in order) *explode-strictly-maximal-in-mset* =
explode-strictly-maximal-in-mset-wrt[*OF transp-on-less asymp-on-less*]

lemmas (in order) *is-minimal-in-filter-mset-iff* =
is-minimal-in-filter-mset-wrt-iff[*OF transp-on-less asymp-on-less*]

abbreviation (in *linorder*) *is-least-in-mset* **where**
is-least-in-mset \equiv *is-least-in-mset-wrt* ($<$)

abbreviation (in *linorder*) *is-greatest-in-mset* **where**
is-greatest-in-mset \equiv *is-greatest-in-mset-wrt* ($<$)

lemmas (in *linorder*) *is-least-in-mset-iff* =
is-least-in-mset-wrt-iff[*OF transp-on-less asymp-on-less totalp-on-less*]

lemmas (in *linorder*) *is-greatest-in-mset-iff* =
is-greatest-in-mset-wrt-iff[*OF transp-on-less asymp-on-less totalp-on-less*]

lemmas (in *linorder*) *is-minimal-in-mset-if-is-least-in-mset*[*intro*] =
is-minimal-in-mset-wrt-if-is-least-in-mset-wrt[*OF transp-on-less asymp-on-less totalp-on-less*]

lemmas (in *linorder*) *is-maximal-in-mset-if-is-greatest-in-mset*[*intro*] =
is-maximal-in-mset-wrt-if-is-greatest-in-mset-wrt[*OF transp-on-less asymp-on-less totalp-on-less*]

lemmas (in *linorder*) *Uniq-is-minimal-in-mset*[*intro*] =
Uniq-is-minimal-in-mset-wrt[*OF transp-on-less asymp-on-less totalp-on-less*]

lemmas (in *linorder*) *Uniq-is-maximal-in-mset*[*intro*] =
Uniq-is-maximal-in-mset-wrt[*OF transp-on-less asymp-on-less totalp-on-less*]

lemmas (in *linorder*) *Uniq-is-least-in-mset*[*intro*] =

Uniq-is-least-in-mset-wrt[OF transp-on-less asymp-on-less totalp-on-less]

lemmas (in *linorder*) *Uniq-is-greatest-in-mset[intro] =*
Uniq-is-greatest-in-mset-wrt[OF transp-on-less asymp-on-less totalp-on-less]

lemmas (in *linorder*) *is-least-in-mset-iff-is-minimal-and-count-eq-one =*
is-least-in-mset-wrt-iff-is-minimal-and-count-eq-one[OF transp-on-less asymp-on-less
totalp-on-less]

lemmas (in *linorder*) *is-greatest-in-mset-iff-is-maximal-and-count-eq-one =*
is-greatest-in-mset-wrt-iff-is-maximal-and-count-eq-one[OF transp-on-less asymp-on-less
totalp-on-less]

lemmas (in *linorder*) *count-ge-2-if-minimal-in-mset-and-not-least-in-mset =*
count-ge-2-if-minimal-in-mset-wrt-and-not-least-in-mset-wrt[OF transp-on-less asymp-on-less
totalp-on-less]

lemmas (in *linorder*) *count-ge-2-if-maximal-in-mset-and-not-greatest-in-mset =*
count-ge-2-if-maximal-in-mset-wrt-and-not-greatest-in-mset-wrt[OF transp-on-less
asymp-on-less
totalp-on-less]

lemmas (in *linorder*) *explode-greatest-in-mset =*
explode-greatest-in-mset-wrt[OF transp-on-less asymp-on-less totalp-on-less]

lemmas (in *linorder*) *multp_{HO}-if-maximal-less-that-maximal =*
multp_{HO}-if-maximal-wrt-less-that-maximal-wrt[OF transp-on-less asymp-on-less
totalp-on-less]

lemmas (in *linorder*) *multp_{DM}-if-maximal-less-that-maximal =*
multp_{DM}-if-maximal-wrt-less-that-maximal-wrt[OF transp-on-less asymp-on-less
totalp-on-less]

lemmas (in *linorder*) *multp-if-maximal-less-that-maximal =*
multp-if-maximal-wrt-less-that-maximal-wrt[OF transp-on-less asymp-on-less
totalp-on-less]

lemmas (in *linorder*) *multp_{HO}-if-same-maximal-and-count-lt =*
multp_{HO}-if-same-maximal-wrt-and-count-lt[OF transp-on-less asymp-on-less to-
talp-on-less]

lemmas (in *linorder*) *multp-if-same-maximal-and-count-lt =*
multp-if-same-maximal-wrt-and-count-lt[OF transp-on-less asymp-on-less totalp-on-less]

lemmas (in *linorder*) *less-than-maximal-if-multp_{HO} =*
less-than-maximal-wrt-if-multp_{HO}[OF transp-on-less asymp-on-less totalp-on-less]

lemma (in *linorder*)
assumes is-greatest-in-mset C L


```

shows  $C - \{L\} = \{K \in C. K \neq L\}$ 
using assms
by (smt (verit, del-insts) add-diff-cancel-left' diff-subset-eq-self diff-zero filter-empty-mset
      filter-mset-add-mset filter-mset-eq-conv insert-DiffM2 local.is-greatest-in-mset-iff
      local.not-less-iff-gr-or-eq)
end

```