

Mereology

Ben Blumson

February 6, 2026

Abstract

We use Isabelle/HOL to verify elementary theorems and alternative axiomatizations of classical extensional mereology.

Contents

1	Introduction	2
2	Premereology	2
2.1	Parthood	3
2.2	Overlap	3
2.3	Disjointness	4
3	Ground Mereology	5
3.1	Proper Parthood	6
4	Minimal Mereology	13
5	Extensional Mereology	18
6	Closed Mereology	22
6.1	Products	22
6.2	Differences	29
6.3	The Universe	30
6.4	Complements	32
7	Closed Extensional Mereology	37
7.1	Sums	38
7.2	Distributivity	45
7.3	Differences	51
7.4	The Universe	52
7.5	Complements	53
8	General Mereology	61
9	General Minimal Mereology	62

10 General Extensional Mereology	63
10.1 General Sums	63
10.2 General Products	74
10.3 Strong Fusion	75
10.4 Strong Sums	82

1 Introduction

In this paper, we use Isabelle/HOL to verify some elementary theorems and alternative axiomatizations of classical extensional mereology, as well as some of its weaker subtheories.¹ We mostly follow the presentations from [Simons, 1987], [Varzi, 1996] and [Casati and Varzi, 1999], with some important corrections from [Pontow, 2004] and [Hovda, 2009] as well as some detailed proofs adapted from [Pietruszczak, 2018].²

We will use the following notation throughout.³

```

typedecl i
consts part :: i ⇒ i ⇒ bool (⟨P⟩)
consts overlap :: i ⇒ i ⇒ bool (⟨O⟩)
consts proper-part :: i ⇒ i ⇒ bool (⟨PP⟩)
consts sum :: i ⇒ i ⇒ i (infix ⟨ $\oplus$ ⟩ 52)
consts product :: i ⇒ i ⇒ i (infix ⟨ $\otimes$ ⟩ 53)
consts difference :: i ⇒ i ⇒ i (infix ⟨ $\ominus$ ⟩ 51)
consts complement:: i ⇒ i (⟨ $\neg$ ⟩)
consts universe :: i (⟨u⟩)
consts general-sum :: (i ⇒ bool) ⇒ i (binder ⟨ $\sigma$ ⟩ 9)
consts general-product :: (i ⇒ bool) ⇒ i (binder ⟨ $\pi$ ⟩ [8] 9)

```

2 Premereology

The theory of *premereology* assumes parthood is reflexive and transitive.⁴ In other words, parthood is assumed to be a partial ordering relation.⁵ Overlap is defined as common parthood.⁶

```

locale PM =
  assumes part-reflexivity: P x x
  assumes part-transitivity : P x y ⇒ P y z ⇒ P x z

```

¹For similar developments see [Sen, 2017] and [Bittner, 2018].

²For help with this project I am grateful to Zach Barnett, Sam Baron, Bob Beddor, Olivier Danvy, Mark Goh, Jeremiah Joven Joaquin, Wang-Yen Lee, Kee Wei Loo, Bruno Woltzenlogel Paleo, Michael Pelczar, Hsueh Qu, Abelard Podgorski, Divyanshu Sharma, Manikaran Singh, Neil Sinhababu, Weng-Hong Tang and Zhang Jiang.

³See [Simons, 1987] pp. 99-100 for a helpful comparison of alternative notations.

⁴For discussion of reflexivity see [Kearns, 2011]. For transitivity see [Varzi, 2006].

⁵Hence the name *premereology*, from [Parsons, 2014] p. 6.

⁶See [Simons, 1987] p. 28, [Varzi, 1996] p. 261 and [Casati and Varzi, 1999] p. 36.

assumes *overlap-eq*: $O\ x\ y \longleftrightarrow (\exists\ z.\ P\ z\ x \wedge P\ z\ y)$
begin

2.1 Parthood

lemma *identity-implies-part* : $x = y \implies P\ x\ y$
proof –
 assume $x = y$
 moreover have $P\ x\ x$ **by** (*rule part-reflexivity*)
 ultimately show $P\ x\ y$ **by** (*rule subst*)
qed

2.2 Overlap

lemma *overlap-intro*: $P\ z\ x \implies P\ z\ y \implies O\ x\ y$
proof–
 assume $P\ z\ x$
 moreover assume $P\ z\ y$
 ultimately have $P\ z\ x \wedge P\ z\ y..$
 hence $\exists\ z.\ P\ z\ x \wedge P\ z\ y..$
 with *overlap-eq* **show** $O\ x\ y..$
qed

lemma *part-implies-overlap*: $P\ x\ y \implies O\ x\ y$
proof –
 assume $P\ x\ y$
 with *part-reflexivity* **have** $P\ x\ x \wedge P\ x\ y..$
 hence $\exists\ z.\ P\ z\ x \wedge P\ z\ y..$
 with *overlap-eq* **show** $O\ x\ y..$
qed

lemma *overlap-reflexivity*: $O\ x\ x$
proof –
 have $P\ x\ x \wedge P\ x\ x$ **using** *part-reflexivity part-reflexivity..*
 hence $\exists\ z.\ P\ z\ x \wedge P\ z\ x..$
 with *overlap-eq* **show** $O\ x\ x..$
qed

lemma *overlap-symmetry*: $O\ x\ y \implies O\ y\ x$
proof–
 assume $O\ x\ y$
 with *overlap-eq* **have** $\exists\ z.\ P\ z\ x \wedge P\ z\ y..$
 hence $\exists\ z.\ P\ z\ y \wedge P\ z\ x$ **by** *auto*
 with *overlap-eq* **show** $O\ y\ x..$
qed

lemma *overlap-monotonicity*: $P\ x\ y \implies O\ z\ x \implies O\ z\ y$
proof –
 assume $P\ x\ y$
 assume $O\ z\ x$

with *overlap-eq* have $\exists v. P v z \wedge P v x..$
then obtain v where $v: P v z \wedge P v x..$
hence $P v z..$
moreover from v have $P v x..$
hence $P v y$ **using** $\langle P x y \rangle$ **by** (*rule part-transitivity*)
ultimately have $P v z \wedge P v y..$
hence $\exists v. P v z \wedge P v y..$
with *overlap-eq* show $O z y..$
qed

The next lemma is from [Hovda, 2009] p. 66.

lemma *overlap-lemma*: $\exists x. (P x y \wedge O z x) \longrightarrow O y z$
proof –
fix x
have $P x y \wedge O z x \longrightarrow O y z$
proof
assume antecedent: $P x y \wedge O z x$
hence $O z x..$
with *overlap-eq* have $\exists v. P v z \wedge P v x..$
then obtain v where $v: P v z \wedge P v x..$
hence $P v x..$
moreover from antecedent have $P x y..$
ultimately have $P v y$ **by** (*rule part-transitivity*)
moreover from v have $P v z..$
ultimately have $P v y \wedge P v z..$
hence $\exists v. P v y \wedge P v z..$
with *overlap-eq* show $O y z..$
qed
thus $\exists x. (P x y \wedge O z x) \longrightarrow O y z..$
qed

2.3 Disjointness

lemma *disjoint-implies-distinct*: $\neg O x y \Longrightarrow x \neq y$
proof –
assume $\neg O x y$
show $x \neq y$
proof
assume $x = y$
hence $\neg O y y$ **using** $\langle \neg O x y \rangle$ **by** (*rule subst*)
thus *False* using *overlap-reflexivity*..
qed
qed

lemma *disjoint-implies-not-part*: $\neg O x y \Longrightarrow \neg P x y$
proof –
assume $\neg O x y$
show $\neg P x y$
proof

```

    assume  $P x y$ 
    hence  $O x y$  by (rule part-implies-overlap)
    with  $\langle \neg O x y \rangle$  show False..
  qed
qed

lemma disjoint-symmetry:  $\neg O x y \implies \neg O y x$ 
proof -
  assume  $\neg O x y$ 
  show  $\neg O y x$ 
  proof
    assume  $O y x$ 
    hence  $O x y$  by (rule overlap-symmetry)
    with  $\langle \neg O x y \rangle$  show False..
  qed
qed

lemma disjoint-demonotonicity:  $P x y \implies \neg O z y \implies \neg O z x$ 
proof -
  assume  $P x y$ 
  assume  $\neg O z y$ 
  show  $\neg O z x$ 
  proof
    assume  $O z x$ 
    with  $\langle P x y \rangle$  have  $O z y$ 
      by (rule overlap-monotonicity)
    with  $\langle \neg O z y \rangle$  show False..
  qed
qed
end

```

3 Ground Mereology

The theory of *ground mereology* adds to premereology the anti-symmetry of parthood, and defines proper parthood as nonidentical parthood.⁷ In other words, ground mereology assumes that parthood is a partial order.

```

locale  $M = PM +$ 
  assumes part-antisymmetry:  $P x y \implies P y x \implies x = y$ 
  assumes nip-eq:  $PP x y \longleftrightarrow P x y \wedge x \neq y$ 
begin

```

⁷For this axiomatization of ground mereology see, for example, [Varzi, 1996] p. 261 and [Casati and Varzi, 1999] p. 36. For discussion of the antisymmetry of parthood see, for example, [Cotnoir, 2010]. For the definition of proper parthood as nonidentical parthood, see for example, [Leonard and Goodman, 1940] p. 47.

3.1 Proper Parthood

lemma *proper-implies-part*: $PP\ x\ y \implies P\ x\ y$

proof –

assume $PP\ x\ y$

with *nip-eq* have $P\ x\ y \wedge x \neq y..$

thus $P\ x\ y..$

qed

lemma *proper-implies-distinct*: $PP\ x\ y \implies x \neq y$

proof –

assume $PP\ x\ y$

with *nip-eq* have $P\ x\ y \wedge x \neq y..$

thus $x \neq y..$

qed

lemma *proper-implies-not-part*: $PP\ x\ y \implies \neg P\ y\ x$

proof –

assume $PP\ x\ y$

hence $P\ x\ y$ by (*rule proper-implies-part*)

show $\neg P\ y\ x$

proof

from $\langle PP\ x\ y \rangle$ have $x \neq y$ by (*rule proper-implies-distinct*)

moreover assume $P\ y\ x$

with $\langle P\ y\ x \rangle$ have $x = y$ by (*rule part-antisymmetry*)

ultimately show *False..*

qed

qed

lemma *proper-part-asymmetry*: $PP\ x\ y \implies \neg PP\ y\ x$

proof –

assume $PP\ x\ y$

hence $P\ x\ y$ by (*rule proper-implies-part*)

from $\langle PP\ x\ y \rangle$ have $x \neq y$ by (*rule proper-implies-distinct*)

show $\neg PP\ y\ x$

proof

assume $PP\ y\ x$

hence $P\ y\ x$ by (*rule proper-implies-part*)

with $\langle P\ y\ x \rangle$ have $x = y$ by (*rule part-antisymmetry*)

with $\langle x \neq y \rangle$ show *False..*

qed

qed

lemma *proper-implies-overlap*: $PP\ x\ y \implies O\ x\ y$

proof –

assume $PP\ x\ y$

hence $P\ x\ y$ by (*rule proper-implies-part*)

thus $O\ x\ y$ by (*rule part-implies-overlap*)

qed

end

The rest of this section compares four alternative axiomatizations of ground mereology, and verifies their equivalence.

The first alternative axiomatization defines proper parthood as nonmutual instead of nonidentical parthood.⁸ In the presence of antisymmetry, the two definitions of proper parthood are equivalent.⁹

locale $M1 = PM +$
 assumes *nmp-eq*: $PP\ x\ y \longleftrightarrow P\ x\ y \wedge \neg P\ y\ x$
 assumes *part-antisymmetry*: $P\ x\ y \Longrightarrow P\ y\ x \Longrightarrow x = y$

sublocale $M \subseteq M1$

proof

fix $x\ y$

show *nmp-eq*: $PP\ x\ y \longleftrightarrow P\ x\ y \wedge \neg P\ y\ x$

proof

assume $PP\ x\ y$

with *nip-eq* **have** *nip*: $P\ x\ y \wedge x \neq y..$

hence $x \neq y..$

from *nip* **have** $P\ x\ y..$

moreover **have** $\neg P\ y\ x$

proof

assume $P\ y\ x$

with $\langle P\ x\ y \rangle$ **have** $x = y$ **by** (*rule part-antisymmetry*)

with $\langle x \neq y \rangle$ **show** *False..*

qed

ultimately **show** $P\ x\ y \wedge \neg P\ y\ x..$

next

assume *nmp*: $P\ x\ y \wedge \neg P\ y\ x$

hence $\neg P\ y\ x..$

from *nmp* **have** $P\ x\ y..$

moreover **have** $x \neq y$

proof

assume $x = y$

hence $\neg P\ y\ y$ **using** $\langle \neg P\ y\ x \rangle$ **by** (*rule subst*)

thus *False* **using** *part-reflexivity..*

qed

ultimately **have** $P\ x\ y \wedge x \neq y..$

with *nip-eq* **show** $PP\ x\ y..$

qed

show $P\ x\ y \Longrightarrow P\ y\ x \Longrightarrow x = y$ **using** *part-antisymmetry*.

qed

⁸See, for example, [Varzi, 1996] p. 261 and [Casati and Varzi, 1999] p. 36. For the distinction between nonmutual and nonidentical parthood, see [Parsons, 2014] pp. 6-8.

⁹See [Cotnoir, 2010] p. 398, [Donnelly, 2011] p. 233, [Cotnoir and Bacon, 2012] p. 191, [Obojska, 2013] p. 344, [Cotnoir, 2016] p. 128 and [Cotnoir, 2018].

```

sublocale  $M1 \subseteq M$ 
proof
  fix  $x y$ 
  show  $nip\text{-}eq: PP\ x\ y \longleftrightarrow P\ x\ y \wedge x \neq y$ 
  proof
    assume  $PP\ x\ y$ 
    with  $nmp\text{-}eq$  have  $nmp: P\ x\ y \wedge \neg P\ y\ x..$ 
    hence  $\neg P\ y\ x..$ 
    from  $nmp$  have  $P\ x\ y..$ 
    moreover have  $x \neq y$ 
    proof
      assume  $x = y$ 
      hence  $\neg P\ y\ y$  using  $\langle \neg P\ y\ x \rangle$  by (rule subst)
      thus False using part-reflexivity..
    qed
    ultimately show  $P\ x\ y \wedge x \neq y..$ 
  next
  assume  $nip: P\ x\ y \wedge x \neq y$ 
  hence  $x \neq y..$ 
  from  $nip$  have  $P\ x\ y..$ 
  moreover have  $\neg P\ y\ x$ 
  proof
    assume  $P\ y\ x$ 
    with  $\langle P\ x\ y \rangle$  have  $x = y$  by (rule part-antisymmetry)
    with  $\langle x \neq y \rangle$  show False..
  qed
  ultimately have  $P\ x\ y \wedge \neg P\ y\ x..$ 
  with  $nmp\text{-}eq$  show  $PP\ x\ y..$ 
qed
show  $P\ x\ y \implies P\ y\ x \implies x = y$  using part-antisymmetry.
qed

```

Conversely, assuming the two definitions of proper parthood are equivalent entails the antisymmetry of parthood, leading to the second alternative axiomatization, which assumes both equivalencies.¹⁰

```

locale  $M2 = PM +$ 
  assumes  $nip\text{-}eq: PP\ x\ y \longleftrightarrow P\ x\ y \wedge x \neq y$ 
  assumes  $nmp\text{-}eq: PP\ x\ y \longleftrightarrow P\ x\ y \wedge \neg P\ y\ x$ 

```

```

sublocale  $M \subseteq M2$ 
proof
  fix  $x y$ 
  show  $PP\ x\ y \longleftrightarrow P\ x\ y \wedge x \neq y$  using  $nip\text{-}eq.$ 
  show  $PP\ x\ y \longleftrightarrow P\ x\ y \wedge \neg P\ y\ x$  using  $nmp\text{-}eq.$ 
qed

```

¹⁰For this point see especially [Parsons, 2014] pp. 9-10.

```

sublocale  $M2 \subseteq M$ 
proof
  fix  $x y$ 
  show  $PP x y \longleftrightarrow P x y \wedge x \neq y$  using nip-eq.
  show  $P x y \implies P y x \implies x = y$ 
  proof –
    assume  $P x y$ 
    assume  $P y x$ 
    show  $x = y$ 
    proof (rule ccontr)
      assume  $x \neq y$ 
      with  $\langle P x y \rangle$  have  $P x y \wedge x \neq y..$ 
      with nip-eq have  $PP x y..$ 
      with nmp-eq have  $P x y \wedge \neg P y x..$ 
      hence  $\neg P y x..$ 
      thus False using  $\langle P y x \rangle..$ 
    qed
  qed
qed

```

In the context of the other axioms, antisymmetry is equivalent to the extensionality of parthood, which gives the third alternative axiomatization.¹¹

```

locale  $M3 = PM +$ 
  assumes nip-eq:  $PP x y \longleftrightarrow P x y \wedge x \neq y$ 
  assumes part-extensionality:  $x = y \longleftrightarrow (\forall z. P z x \longleftrightarrow P z y)$ 

```

```

sublocale  $M \subseteq M3$ 
proof
  fix  $x y$ 
  show  $PP x y \longleftrightarrow P x y \wedge x \neq y$  using nip-eq.
  show part-extensionality:  $x = y \longleftrightarrow (\forall z. P z x \longleftrightarrow P z y)$ 
  proof
    assume  $x = y$ 
    moreover have  $\forall z. P z x \longleftrightarrow P z x$  by simp
    ultimately show  $\forall z. P z x \longleftrightarrow P z y$  by (rule subst)
  next
    assume  $z: \forall z. P z x \longleftrightarrow P z y$ 
    show  $x = y$ 
    proof (rule part-antisymmetry)
      from  $z$  have  $P y x \longleftrightarrow P y y..$ 
      moreover have  $P y y$  by (rule part-reflexivity)
      ultimately show  $P y x..$ 
    next
      from  $z$  have  $P x x \longleftrightarrow P x y..$ 
      moreover have  $P x x$  by (rule part-reflexivity)
      ultimately show  $P x y..$ 
  qed

```

¹¹For this point see [Cotnoir, 2010] p. 401 and [Cotnoir and Bacon, 2012] p. 191-2.

```

    qed
  qed
  qed

  sublocale  $M_3 \subseteq M$ 
  proof
    fix  $x y$ 
    show  $PP\ x\ y \longleftrightarrow P\ x\ y \wedge x \neq y$  using nip-eq.
    show part-antisymmetry:  $P\ x\ y \Longrightarrow P\ y\ x \Longrightarrow x = y$ 
    proof -
      assume  $P\ x\ y$ 
      assume  $P\ y\ x$ 
      have  $\forall z. P\ z\ x \longleftrightarrow P\ z\ y$ 
      proof
        fix  $z$ 
        show  $P\ z\ x \longleftrightarrow P\ z\ y$ 
        proof
          assume  $P\ z\ x$ 
          thus  $P\ z\ y$  using  $\langle P\ x\ y \rangle$  by (rule part-transitivity)
        next
          assume  $P\ z\ y$ 
          thus  $P\ z\ x$  using  $\langle P\ y\ x \rangle$  by (rule part-transitivity)
        qed
      qed
    with part-extensionality show  $x = y$ ..
  qed
  qed
  qed

```

The fourth axiomatization adopts proper parthood as primitive.¹² Improper parthood is defined as proper parthood or identity.

```

  locale  $M_4 =$ 
    assumes part-eq:  $P\ x\ y \longleftrightarrow PP\ x\ y \vee x = y$ 
    assumes overlap-eq:  $O\ x\ y \longleftrightarrow (\exists z. P\ z\ x \wedge P\ z\ y)$ 
    assumes proper-part-asymmetry:  $PP\ x\ y \Longrightarrow \neg PP\ y\ x$ 
    assumes proper-part-transitivity:  $PP\ x\ y \Longrightarrow PP\ y\ z \Longrightarrow PP\ x\ z$ 
  begin

  lemma proper-part-irreflexivity:  $\neg PP\ x\ x$ 
  proof
    assume  $PP\ x\ x$ 
    hence  $\neg PP\ x\ x$  by (rule proper-part-asymmetry)
    thus False using  $\langle PP\ x\ x \rangle$ ..
  qed

  end

```

¹²See, for example, [Simons, 1987], p. 26 and [Casati and Varzi, 1999] p. 37.

```

sublocale M ⊆ M4
proof
  fix x y z
  show part-eq: P x y ⟷ (PP x y ∨ x = y)
  proof
    assume P x y
    show PP x y ∨ x = y
    proof cases
      assume x = y
      thus PP x y ∨ x = y..
    next
      assume x ≠ y
      with ⟨P x y⟩ have P x y ∧ x ≠ y..
      with nip-eq have PP x y..
      thus PP x y ∨ x = y..
    qed
  next
    assume PP x y ∨ x = y
    thus P x y
    proof
      assume PP x y
      thus P x y by (rule proper-implies-part)
    next
      assume x = y
      thus P x y by (rule identity-implies-part)
    qed
  qed
  show O x y ⟷ (∃ z. P z x ∧ P z y) using overlap-eq.
  show PP x y ⟹ ¬ PP y x using proper-part-asymmetry.
  show proper-part-transitivity: PP x y ⟹ PP y z ⟹ PP x z
  proof –
    assume PP x y
    assume PP y z
    have P x z ∧ x ≠ z
    proof
      from ⟨PP x y⟩ have P x y by (rule proper-implies-part)
      moreover from ⟨PP y z⟩ have P y z by (rule proper-implies-part)
      ultimately show P x z by (rule part-transitivity)
    next
      show x ≠ z
      proof
        assume x = z
        hence PP y x using ⟨PP y z⟩ by (rule ssubst)
        hence ¬ PP x y by (rule proper-part-asymmetry)
        thus False using ⟨PP x y⟩..
      qed
    qed
  with nip-eq show PP x z..
  qed

```

qed

sublocale $M_4 \subseteq M$

proof

fix $x y z$

show *proper-part-eq*: $PP\ x\ y \longleftrightarrow P\ x\ y \wedge x \neq y$

proof

assume $PP\ x\ y$

hence $PP\ x\ y \vee x = y..$

with *part-eq* have $P\ x\ y..$

moreover have $x \neq y$

proof

assume $x = y$

hence $PP\ y\ y$ using $\langle PP\ x\ y \rangle$ by (*rule subst*)

with *proper-part-irreflexivity* show *False*..

qed

ultimately show $P\ x\ y \wedge x \neq y..$

next

assume *rhs*: $P\ x\ y \wedge x \neq y$

hence $x \neq y..$

from *rhs* have $P\ x\ y..$

with *part-eq* have $PP\ x\ y \vee x = y..$

thus $PP\ x\ y$

proof

assume $PP\ x\ y$

thus $PP\ x\ y.$

next

assume $x = y$

with $\langle x \neq y \rangle$ show $PP\ x\ y..$

qed

qed

show $P\ x\ x$

proof –

have $x = x$ by (*rule refl*)

hence $PP\ x\ x \vee x = x..$

with *part-eq* show $P\ x\ x..$

qed

show $O\ x\ y \longleftrightarrow (\exists z. P\ z\ x \wedge P\ z\ y)$ using *overlap-eq*.

show $P\ x\ y \implies P\ y\ x \implies x = y$

proof –

assume $P\ x\ y$

assume $P\ y\ x$

from *part-eq* have $PP\ x\ y \vee x = y$ using $\langle P\ x\ y \rangle..$

thus $x = y$

proof

assume $PP\ x\ y$

hence $\neg PP\ y\ x$ by (*rule proper-part-asymmetry*)

from *part-eq* have $PP\ y\ x \vee y = x$ using $\langle P\ y\ x \rangle..$

thus $x = y$

```

proof
  assume  $PP\ y\ x$ 
  with  $\langle \neg\ PP\ y\ x \rangle$  show  $x = y..$ 
next
  assume  $y = x$ 
  thus  $x = y..$ 
qed
qed
show  $P\ x\ y \implies P\ y\ z \implies P\ x\ z$ 
proof –
  assume  $P\ x\ y$ 
  assume  $P\ y\ z$ 
  with part-eq have  $PP\ y\ z \vee y = z..$ 
  hence  $PP\ x\ z \vee x = z$ 
  proof
    assume  $PP\ y\ z$ 
    from part-eq have  $PP\ x\ y \vee x = y$  using  $\langle P\ x\ y \rangle..$ 
    hence  $PP\ x\ z$ 
    proof
      assume  $PP\ x\ y$ 
      thus  $PP\ x\ z$  using  $\langle PP\ y\ z \rangle$  by (rule proper-part-transitivity)
    next
      assume  $x = y$ 
      thus  $PP\ x\ z$  using  $\langle PP\ y\ z \rangle$  by (rule ssubst)
    qed
  thus  $PP\ x\ z \vee x = z..$ 
next
  assume  $y = z$ 
  moreover from part-eq have  $PP\ x\ y \vee x = y$  using  $\langle P\ x\ y \rangle..$ 
  ultimately show  $PP\ x\ z \vee x = z$  by (rule subst)
qed
with part-eq show  $P\ x\ z..$ 
qed
qed

```

4 Minimal Mereology

Minimal mereology adds to ground mereology the axiom of weak supplementation.¹³

locale $MM = M +$

assumes *weak-supplementation*: $PP\ y\ x \implies (\exists\ z.\ P\ z\ x \wedge \neg\ O\ z\ y)$

¹³See [Varzi, 1996] and [Casati and Varzi, 1999] p. 39. The name *minimal mereology* reflects the, controversial, idea that weak supplementation is analytic. See, for example, [Simons, 1987] p. 116, [Varzi, 2008] p. 110-1, and [Cotnoir, 2018]. For general discussion of weak supplementation see, for example [Smith, 2009] pp. 507 and [Donnelly, 2011].

The rest of this section considers three alternative axiomatizations of minimal mereology. The first alternative axiomatization replaces improper with proper parthood in the consequent of weak supplementation.¹⁴

locale $MM1 = M +$

assumes *proper-weak-supplementation*:

$PP\ y\ x \implies (\exists\ z.\ PP\ z\ x \wedge \neg\ O\ z\ y)$

sublocale $MM \subseteq MM1$

proof

fix $x\ y$

show $PP\ y\ x \implies (\exists\ z.\ PP\ z\ x \wedge \neg\ O\ z\ y)$

proof –

assume $PP\ y\ x$

hence $\exists\ z.\ P\ z\ x \wedge \neg\ O\ z\ y$ **by** (*rule weak-supplementation*)

then obtain z **where** $z: P\ z\ x \wedge \neg\ O\ z\ y..$

hence $\neg\ O\ z\ y..$

from z **have** $P\ z\ x..$

hence $P\ z\ x \wedge z \neq x$

proof

show $z \neq x$

proof

assume $z = x$

hence $PP\ y\ z$

using $\langle PP\ y\ x \rangle$ **by** (*rule ssubst*)

hence $O\ y\ z$ **by** (*rule proper-implies-overlap*)

hence $O\ z\ y$ **by** (*rule overlap-symmetry*)

with $\langle \neg\ O\ z\ y \rangle$ **show** *False..*

qed

qed

with *nip-eq* **have** $PP\ z\ x..$

hence $PP\ z\ x \wedge \neg\ O\ z\ y$

using $\langle \neg\ O\ z\ y \rangle..$

thus $\exists\ z.\ PP\ z\ x \wedge \neg\ O\ z\ y..$

qed

qed

sublocale $MM1 \subseteq MM$

proof

fix $x\ y$

show *weak-supplementation*: $PP\ y\ x \implies (\exists\ z.\ P\ z\ x \wedge \neg\ O\ z\ y)$

proof –

assume $PP\ y\ x$

hence $\exists\ z.\ PP\ z\ x \wedge \neg\ O\ z\ y$ **by** (*rule proper-weak-supplementation*)

then obtain z **where** $z: PP\ z\ x \wedge \neg\ O\ z\ y..$

hence $PP\ z\ x..$

hence $P\ z\ x$ **by** (*rule proper-implies-part*)

¹⁴See [Simons, 1987] p. 28.

moreover from z have $\neg O z y..$
ultimately have $P z x \wedge \neg O z y..$
thus $\exists z. P z x \wedge \neg O z y..$

qed
qed

The following two corollaries are sometimes found in the literature.¹⁵

context MM
begin

corollary *weak-company*: $PP y x \implies (\exists z. PP z x \wedge z \neq y)$

proof –

assume $PP y x$
hence $\exists z. PP z x \wedge \neg O z y$ by (rule *proper-weak-supplementation*)
then obtain z where $z: PP z x \wedge \neg O z y..$
hence $PP z x..$
from z have $\neg O z y..$
hence $z \neq y$ by (rule *disjoint-implies-distinct*)
with $\langle PP z x \rangle$ have $PP z x \wedge z \neq y..$
thus $\exists z. PP z x \wedge z \neq y..$

qed

corollary *strong-company*: $PP y x \implies (\exists z. PP z x \wedge \neg P z y)$

proof –

assume $PP y x$
hence $\exists z. PP z x \wedge \neg O z y$ by (rule *proper-weak-supplementation*)
then obtain z where $z: PP z x \wedge \neg O z y..$
hence $PP z x..$
from z have $\neg O z y..$
hence $\neg P z y$ by (rule *disjoint-implies-not-part*)
with $\langle PP z x \rangle$ have $PP z x \wedge \neg P z y..$
thus $\exists z. PP z x \wedge \neg P z y..$

qed

end

If weak supplementation is formulated in terms of nonidentical parthood, then the antisymmetry of parthood is redundant, and we have the second alternative axiomatization of minimal mereology.¹⁶

locale $MM2 = PM +$

assumes *nip-eq*: $PP x y \longleftrightarrow P x y \wedge x \neq y$

assumes *weak-supplementation*: $PP y x \implies (\exists z. P z x \wedge \neg O z y)$

¹⁵See [Simons, 1987] p. 27. For the names *weak company* and *strong company* see [Cotnoir and Bacon, 2012] p. 192-3 and [Varzi, 2016].

¹⁶See [Cotnoir, 2010] p. 399, [Donnelly, 2011] p. 232, [Cotnoir and Bacon, 2012] p. 193 and [Obojska, 2013] pp. 235-6.

```

sublocale  $MM2 \subseteq MM$ 
proof
  fix  $x y$ 
  show  $PP\ x\ y \longleftrightarrow P\ x\ y \wedge x \neq y$  using nip-eq.
  show part-antisymmetry:  $P\ x\ y \implies P\ y\ x \implies x = y$ 
  proof –
    assume  $P\ x\ y$ 
    assume  $P\ y\ x$ 
    show  $x = y$ 
    proof (rule ccontr)
      assume  $x \neq y$ 
      with  $\langle P\ x\ y \rangle$  have  $P\ x\ y \wedge x \neq y..$ 
      with nip-eq have  $PP\ x\ y..$ 
      hence  $\exists z. P\ z\ y \wedge \neg O\ z\ x$  by (rule weak-supplementation)
      then obtain  $z$  where  $z: P\ z\ y \wedge \neg O\ z\ x..$ 
      hence  $\neg O\ z\ x..$ 
      hence  $\neg P\ z\ x$  by (rule disjoint-implies-not-part)
      from  $z$  have  $P\ z\ y..$ 
      hence  $P\ z\ x$  using  $\langle P\ y\ x \rangle$  by (rule part-transitivity)
      with  $\langle \neg P\ z\ x \rangle$  show False..
    qed
  qed
  show  $PP\ y\ x \implies \exists z. P\ z\ x \wedge \neg O\ z\ y$  using weak-supplementation.
qed

```

```

sublocale  $MM \subseteq MM2$ 
proof
  fix  $x y$ 
  show  $PP\ x\ y \longleftrightarrow (P\ x\ y \wedge x \neq y)$  using nip-eq.
  show  $PP\ y\ x \implies \exists z. P\ z\ x \wedge \neg O\ z\ y$  using weak-supplementation.
qed

```

Likewise, if proper parthood is adopted as primitive, then the asymmetry of proper parthood is redundant in the context of weak supplementation, leading to the third alternative axiomatization.¹⁷

```

locale  $MM3 =$ 
  assumes part-eq:  $P\ x\ y \longleftrightarrow PP\ x\ y \vee x = y$ 
  assumes overlap-eq:  $O\ x\ y \longleftrightarrow (\exists z. P\ z\ x \wedge P\ z\ y)$ 
  assumes proper-part-transitivity:  $PP\ x\ y \implies PP\ y\ z \implies PP\ x\ z$ 
  assumes weak-supplementation:  $PP\ y\ x \implies (\exists z. P\ z\ x \wedge \neg O\ z\ y)$ 
begin

  lemma part-reflexivity:  $P\ x\ x$ 
  proof –
    have  $x = x..$ 

```

¹⁷See [Donnelly, 2011] p. 232 and [Cotnoir, 2018].

hence $PP\ x\ x \vee x = x..$
with *part-eq* show $P\ x\ x..$
qed

lemma *proper-part-irreflexivity*: $\neg PP\ x\ x$

proof

assume $PP\ x\ x$
hence $\exists z. P\ z\ x \wedge \neg O\ z\ x$ by (*rule weak-supplementation*)
then obtain z where $z: P\ z\ x \wedge \neg O\ z\ x..$
hence $\neg O\ z\ x..$
from z have $P\ z\ x..$
with *part-reflexivity* have $P\ z\ z \wedge P\ z\ x..$
hence $\exists v. P\ v\ z \wedge P\ v\ x..$
with *overlap-eq* have $O\ z\ x..$
with $\langle \neg O\ z\ x \rangle$ show *False..*

qed

end

sublocale $MM3 \subseteq M4$

proof

fix $x\ y\ z$
show $P\ x\ y \longleftrightarrow PP\ x\ y \vee x = y$ using *part-eq*.
show $O\ x\ y \longleftrightarrow (\exists z. P\ z\ x \wedge P\ z\ y)$ using *overlap-eq*.
show *proper-part-irreflexivity*: $PP\ x\ y \Longrightarrow \neg PP\ y\ x$
proof –
assume $PP\ x\ y$
show $\neg PP\ y\ x$
proof
assume $PP\ y\ x$
hence $PP\ y\ y$ using $\langle PP\ x\ y \rangle$ by (*rule proper-part-transitivity*)
with *proper-part-irreflexivity* show *False..*
qed
qed
show $PP\ x\ y \Longrightarrow PP\ y\ z \Longrightarrow PP\ x\ z$ using *proper-part-transitivity*.

qed

sublocale $MM3 \subseteq MM$

proof

fix $x\ y$
show $PP\ y\ x \Longrightarrow (\exists z. P\ z\ x \wedge \neg O\ z\ y)$ using *weak-supplementation*.

qed

sublocale $MM \subseteq MM3$

proof

fix $x\ y\ z$
show $P\ x\ y \longleftrightarrow (PP\ x\ y \vee x = y)$ using *part-eq*.
show $O\ x\ y \longleftrightarrow (\exists z. P\ z\ x \wedge P\ z\ y)$ using *overlap-eq*.
show $PP\ x\ y \Longrightarrow PP\ y\ z \Longrightarrow PP\ x\ z$ using *proper-part-transitivity*.

show $PP\ y\ x \implies \exists z. P\ z\ x \wedge \neg O\ z\ y$ **using** *weak-supplementation*.
qed

5 Extensional Mereology

Extensional mereology adds to ground mereology the axiom of strong supplementation.¹⁸

locale $EM = M +$
assumes *strong-supplementation*:
 $\neg P\ x\ y \implies (\exists z. P\ z\ x \wedge \neg O\ z\ y)$
begin

Strong supplementation entails weak supplementation.¹⁹

lemma *weak-supplementation*: $PP\ x\ y \implies (\exists z. P\ z\ y \wedge \neg O\ z\ x)$
proof –
assume $PP\ x\ y$
hence $\neg P\ y\ x$ **by** (*rule proper-implies-not-part*)
thus $\exists z. P\ z\ y \wedge \neg O\ z\ x$ **by** (*rule strong-supplementation*)
qed

end

So minimal mereology is a subtheory of extensional mereology.²⁰

sublocale $EM \subseteq MM$
proof
fix $y\ x$
show $PP\ y\ x \implies \exists z. P\ z\ x \wedge \neg O\ z\ y$ **using** *weak-supplementation*.
qed

Strong supplementation also entails the proper parts principle.²¹

context EM
begin

lemma *proper-parts-principle*:
 $(\exists z. PP\ z\ x) \implies (\forall z. PP\ z\ x \longrightarrow P\ z\ y) \implies P\ x\ y$
proof –
assume $\exists z. PP\ z\ x$
then obtain v **where** $v: PP\ v\ x..$
hence $P\ v\ x$ **by** (*rule proper-implies-part*)
assume *antecedent*: $\forall z. PP\ z\ x \longrightarrow P\ z\ y$
hence $PP\ v\ x \longrightarrow P\ v\ y..$
hence $P\ v\ y$ **using** $\langle PP\ v\ x \rangle..$

¹⁸See [Simons, 1987] p. 29, [Varzi, 1996] p. 262 and [Casati and Varzi, 1999] p. 39-40.

¹⁹See [Simons, 1987] p. 29 and [Casati and Varzi, 1999] p. 40.

²⁰[Casati and Varzi, 1999] p. 40.

²¹See [Simons, 1987] pp. 28-9 and [Varzi, 1996] p. 263.

with $\langle P v x \rangle$ **have** $P v x \wedge P v y..$
hence $\exists v. P v x \wedge P v y..$
with *overlap-eq* **have** $O x y..$
show $P x y$
proof (*rule ccontr*)
 assume $\neg P x y$
 hence $\exists z. P z x \wedge \neg O z y$
 by (*rule strong-supplementation*)
 then obtain z **where** $z: P z x \wedge \neg O z y..$
 hence $P z x..$
 moreover have $z \neq x$
 proof
 assume $z = x$
 moreover from z **have** $\neg O z y..$
 ultimately have $\neg O x y$ **by** (*rule subst*)
 thus *False* **using** $\langle O x y \rangle..$
 qed
 ultimately have $P z x \wedge z \neq x..$
 with *nip-eq* **have** $PP z x..$
 from *antecedent* **have** $PP z x \longrightarrow P z y..$
 hence $P z y$ **using** $\langle PP z x \rangle..$
 hence $O z y$ **by** (*rule part-implies-overlap*)
 from z **have** $\neg O z y..$
 thus *False* **using** $\langle O z y \rangle..$
qed
qed

Which with antisymmetry entails the extensionality of proper parthood.²²

theorem *proper-part-extensionality*:

$(\exists z. PP z x \vee PP z y) \implies x = y \iff (\forall z. PP z x \iff PP z y)$

proof –

assume *antecedent*: $\exists z. PP z x \vee PP z y$

show $x = y \iff (\forall z. PP z x \iff PP z y)$

proof

assume $x = y$

moreover have $\forall z. PP z x \iff PP z x$ **by** *simp*

ultimately show $\forall z. PP z x \iff PP z y$ **by** (*rule subst*)

next

assume *right*: $\forall z. PP z x \iff PP z y$

have $\forall z. PP z x \longrightarrow P z y$

proof

fix z

show $PP z x \longrightarrow P z y$

proof

assume $PP z x$

from *right* **have** $PP z x \iff PP z y..$

²²See [Simons, 1987] p. 28, [Varzi, 1996] p. 263 and [Casati and Varzi, 1999] p. 40.

```

    hence  $PP\ z\ y$  using  $\langle PP\ z\ x \rangle..$ 
    thus  $P\ z\ y$  by (rule proper-implies-part)
  qed
qed
have  $\forall z. PP\ z\ y \longrightarrow P\ z\ x$ 
proof
  fix  $z$ 
  show  $PP\ z\ y \longrightarrow P\ z\ x$ 
  proof
    assume  $PP\ z\ y$ 
    from right have  $PP\ z\ x \longleftrightarrow PP\ z\ y..$ 
    hence  $PP\ z\ x$  using  $\langle PP\ z\ y \rangle..$ 
    thus  $P\ z\ x$  by (rule proper-implies-part)
  qed
qed
from antecedent obtain  $z$  where  $z: PP\ z\ x \vee PP\ z\ y..$ 
thus  $x = y$ 
proof (rule disjE)
  assume  $PP\ z\ x$ 
  hence  $\exists z. PP\ z\ x..$ 
  hence  $P\ x\ y$  using  $\langle \forall z. PP\ z\ x \longrightarrow P\ z\ y \rangle$ 
    by (rule proper-parts-principle)
  from right have  $PP\ z\ x \longleftrightarrow PP\ z\ y..$ 
  hence  $PP\ z\ y$  using  $\langle PP\ z\ x \rangle..$ 
  hence  $\exists z. PP\ z\ y..$ 
  hence  $P\ y\ x$  using  $\langle \forall z. PP\ z\ y \longrightarrow P\ z\ x \rangle$ 
    by (rule proper-parts-principle)
  with  $\langle P\ x\ y \rangle$  show  $x = y$ 
    by (rule part-antisymmetry)
next
  assume  $PP\ z\ y$ 
  hence  $\exists z. PP\ z\ y..$ 
  hence  $P\ y\ x$  using  $\langle \forall z. PP\ z\ y \longrightarrow P\ z\ x \rangle$ 
    by (rule proper-parts-principle)
  from right have  $PP\ z\ x \longleftrightarrow PP\ z\ y..$ 
  hence  $PP\ z\ x$  using  $\langle PP\ z\ y \rangle..$ 
  hence  $\exists z. PP\ z\ x..$ 
  hence  $P\ x\ y$  using  $\langle \forall z. PP\ z\ x \longrightarrow P\ z\ y \rangle$ 
    by (rule proper-parts-principle)
  thus  $x = y$ 
    using  $\langle P\ y\ x \rangle$  by (rule part-antisymmetry)
qed
qed
qed

```

It also follows from strong supplementation that parthood is definable in terms of overlap.²³

lemma part-overlap-eq: $P\ x\ y \longleftrightarrow (\forall z. O\ z\ x \longrightarrow O\ z\ y)$

²³See [Parsons, 2014] p. 4.

proof
 assume $P x y$
 show $(\forall z. O z x \longrightarrow O z y)$
proof
 fix z
 show $O z x \longrightarrow O z y$
proof
 assume $O z x$
 with $\langle P x y \rangle$ show $O z y$
 by (*rule overlap-monotonicity*)
 qed
 qed
 next
 assume *right*: $\forall z. O z x \longrightarrow O z y$
 show $P x y$
proof (*rule ccontr*)
 assume $\neg P x y$
 hence $\exists z. P z x \wedge \neg O z y$
 by (*rule strong-supplementation*)
 then obtain z where $z: P z x \wedge \neg O z y..$
 hence $\neg O z y..$
 from *right* have $O z x \longrightarrow O z y..$
 moreover from z have $P z x..$
 hence $O z x$ by (*rule part-implies-overlap*)
 ultimately have $O z y..$
 with $\langle \neg O z y \rangle$ show *False*..
 qed
 qed

Which entails the extensionality of overlap.

theorem *overlap-extensionality*: $x = y \longleftrightarrow (\forall z. O z x \longleftrightarrow O z y)$

proof
 assume $x = y$
 moreover have $\forall z. O z x \longleftrightarrow O z x$
proof
 fix z
 show $O z x \longleftrightarrow O z x..$
 qed
 ultimately show $\forall z. O z x \longleftrightarrow O z y$
 by (*rule subst*)
 next
 assume *right*: $\forall z. O z x \longleftrightarrow O z y$
 have $\forall z. O z y \longrightarrow O z x$
proof
 fix z
 from *right* have $O z x \longleftrightarrow O z y..$
 thus $O z y \longrightarrow O z x..$
 qed
 with *part-overlap-eq* have $P y x..$

```

have  $\forall z. O z x \longrightarrow O z y$ 
proof
  fix  $z$ 
  from right have  $O z x \longleftrightarrow O z y..$ 
  thus  $O z x \longrightarrow O z y..$ 
qed
with part-overlap-eq have  $P x y..$ 
thus  $x = y$ 
  using  $\langle P y x \rangle$  by (rule part-antisymmetry)
qed

end

```

6 Closed Mereology

The theory of *closed mereology* adds to ground mereology conditions guaranteeing the existence of sums and products.²⁴

```

locale  $CM = M +$ 
  assumes sum-eq:  $x \oplus y = (THE z. \forall v. O v z \longleftrightarrow O v x \vee O v y)$ 
  assumes sum-closure:  $\exists z. \forall v. O v z \longleftrightarrow O v x \vee O v y$ 
  assumes product-eq:
     $x \otimes y = (THE z. \forall v. P v z \longleftrightarrow P v x \wedge P v y)$ 
  assumes product-closure:
     $O x y \implies \exists z. \forall v. P v z \longleftrightarrow P v x \wedge P v y$ 
begin

```

6.1 Products

```

lemma product-intro:
   $(\forall w. P w z \longleftrightarrow (P w x \wedge P w y)) \implies x \otimes y = z$ 
proof –
  assume  $z: \forall w. P w z \longleftrightarrow (P w x \wedge P w y)$ 
  hence  $(THE v. \forall w. P w v \longleftrightarrow P w x \wedge P w y) = z$ 
  proof (rule the-equality)
    fix  $v$ 
    assume  $v: \forall w. P w v \longleftrightarrow (P w x \wedge P w y)$ 
    have  $\forall w. P w v \longleftrightarrow P w z$ 
    proof
      fix  $w$ 
      from  $z$  have  $P w z \longleftrightarrow (P w x \wedge P w y)..$ 
      moreover from  $v$  have  $P w v \longleftrightarrow (P w x \wedge P w y)..$ 
      ultimately show  $P w v \longleftrightarrow P w z$  by (rule ssubst)
    qed
  with part-extensionality show  $v = z..$ 

```

²⁴See [Masolo and Vieu, 1999] p. 238. [Varzi, 1996] p. 263 and [Casati and Varzi, 1999] p. 43 give a slightly weaker version of the sum closure axiom, which is equivalent given axioms considered later.

qed
 thus $x \otimes y = z$
 using *product-eq* by (*rule subst*)
 qed

lemma *product-idempotence*: $x \otimes x = x$
 proof –
 have $\forall w. P w x \longleftrightarrow P w x \wedge P w x$
 proof
 fix w
 show $P w x \longleftrightarrow P w x \wedge P w x$
 proof
 assume $P w x$
 thus $P w x \wedge P w x$ using $\langle P w x \rangle..$
 next
 assume $P w x \wedge P w x$
 thus $P w x..$
 qed
 qed
 thus $x \otimes x = x$ by (*rule product-intro*)
 qed

lemma *product-character*:
 $O x y \implies (\forall w. P w (x \otimes y) \longleftrightarrow (P w x \wedge P w y))$
 proof –
 assume $O x y$
 hence $\exists z. \forall w. P w z \longleftrightarrow (P w x \wedge P w y)$ by (*rule product-closure*)
 then obtain z where $z: \forall w. P w z \longleftrightarrow (P w x \wedge P w y)..$
 hence $x \otimes y = z$ by (*rule product-intro*)
 thus $\forall w. P w (x \otimes y) \longleftrightarrow P w x \wedge P w y$
 using z by (*rule ssubst*)
 qed

lemma *product-commutativity*: $O x y \implies x \otimes y = y \otimes x$
 proof –
 assume $O x y$
 hence $O y x$ by (*rule overlap-symmetry*)
 hence $\forall w. P w (y \otimes x) \longleftrightarrow (P w y \wedge P w x)$ by (*rule product-character*)
 hence $\forall w. P w (y \otimes x) \longleftrightarrow (P w x \wedge P w y)$ by *auto*
 thus $x \otimes y = y \otimes x$ by (*rule product-intro*)
 qed

lemma *product-in-factors*: $O x y \implies P (x \otimes y) x \wedge P (x \otimes y) y$
 proof –
 assume $O x y$
 hence $\forall w. P w (x \otimes y) \longleftrightarrow P w x \wedge P w y$ by (*rule product-character*)
 hence $P (x \otimes y) (x \otimes y) \longleftrightarrow P (x \otimes y) x \wedge P (x \otimes y) y..$

moreover have $P(x \otimes y)(x \otimes y)$ by (rule part-reflexivity)
ultimately show $P(x \otimes y)x \wedge P(x \otimes y)y..$
qed

lemma *product-in-first-factor*: $Oxy \implies P(x \otimes y)x$
proof –
assume Oxy
hence $P(x \otimes y)x \wedge P(x \otimes y)y$ by (rule product-in-factors)
thus $P(x \otimes y)x..$
qed

lemma *product-in-second-factor*: $Oxy \implies P(x \otimes y)y$
proof –
assume Oxy
hence $P(x \otimes y)x \wedge P(x \otimes y)y$ by (rule product-in-factors)
thus $P(x \otimes y)y..$
qed

lemma *nonpart-implies-proper-product*:
 $\neg Pxy \wedge Oxy \implies PP(x \otimes y)x$
proof –
assume *antecedent*: $\neg Pxy \wedge Oxy$
hence $\neg Pxy..$
from *antecedent* have $Oxy..$
hence $P(x \otimes y)x$ by (rule product-in-first-factor)
moreover have $(x \otimes y) \neq x$
proof
assume $(x \otimes y) = x$
hence $\neg P(x \otimes y)y$
using $\langle \neg Pxy \rangle$ by (rule ssubst)
moreover have $P(x \otimes y)y$
using $\langle Oxy \rangle$ by (rule product-in-second-factor)
ultimately show *False*..
qed
ultimately have $P(x \otimes y)x \wedge x \otimes y \neq x..$
with *nip-eq* show $PP(x \otimes y)x..$
qed

lemma *common-part-in-product*: $Pzx \wedge Pzy \implies Pz(x \otimes y)$
proof –
assume *antecedent*: $Pzx \wedge Pzy$
hence $\exists z. Pzx \wedge Pzy..$
with *overlap-eq* have $Oxy..$
hence $\forall w. Pw(x \otimes y) \iff (Pwx \wedge Pwy)$
by (rule product-character)
hence $Pz(x \otimes y) \iff (Pzx \wedge Pzy)..$
thus $Pz(x \otimes y)$
using $\langle Pzx \wedge Pzy \rangle..$
qed

lemma *product-part-in-factors*:

$$O x y \Longrightarrow P z (x \otimes y) \Longrightarrow P z x \wedge P z y$$

proof –

assume $O x y$

hence $\forall w. P w (x \otimes y) \longleftrightarrow (P w x \wedge P w y)$

by (*rule product-character*)

hence $P z (x \otimes y) \longleftrightarrow (P z x \wedge P z y)$..

moreover assume $P z (x \otimes y)$

ultimately show $P z x \wedge P z y$..

qed

corollary *product-part-in-first-factor*:

$$O x y \Longrightarrow P z (x \otimes y) \Longrightarrow P z x$$

proof –

assume $O x y$

moreover assume $P z (x \otimes y)$

ultimately have $P z x \wedge P z y$

by (*rule product-part-in-factors*)

thus $P z x$..

qed

corollary *product-part-in-second-factor*:

$$O x y \Longrightarrow P z (x \otimes y) \Longrightarrow P z y$$

proof –

assume $O x y$

moreover assume $P z (x \otimes y)$

ultimately have $P z x \wedge P z y$

by (*rule product-part-in-factors*)

thus $P z y$..

qed

lemma *part-product-identity*: $P x y \Longrightarrow x \otimes y = x$

proof –

assume $P x y$

with *part-reflexivity* have $P x x \wedge P x y$..

hence $P x (x \otimes y)$ by (*rule common-part-in-product*)

have $O x y$ using $\langle P x y \rangle$ by (*rule part-implies-overlap*)

hence $P (x \otimes y) x$ by (*rule product-in-first-factor*)

thus $x \otimes y = x$ using $\langle P x (x \otimes y) \rangle$ by (*rule part-antisymmetry*)

qed

lemma *product-overlap*: $P z x \Longrightarrow O z y \Longrightarrow O z (x \otimes y)$

proof –

assume $P z x$

assume $O z y$

with *overlap-eq* have $\exists v. P v z \wedge P v y$..

then obtain v where $v: P v z \wedge P v y$..

hence $P v y$..

from v **have** $P v z$..
hence $P v x$ **using** $\langle P z x \rangle$ **by** (*rule part-transitivity*)
hence $P v x \wedge P v y$ **using** $\langle P v y \rangle$..
hence $P v (x \otimes y)$ **by** (*rule common-part-in-product*)
with $\langle P v z \rangle$ **have** $P v z \wedge P v (x \otimes y)$..
hence $\exists v. P v z \wedge P v (x \otimes y)$..
with *overlap-eq* **show** $O z (x \otimes y)$..
qed

lemma *disjoint-from-second-factor*:

$$P x y \wedge \neg O x (y \otimes z) \implies \neg O x z$$

proof –

assume *antecedent*: $P x y \wedge \neg O x (y \otimes z)$

hence $\neg O x (y \otimes z)$..<

show $\neg O x z$

proof

from *antecedent* **have** $P x y$..<

moreover **assume** $O x z$

ultimately **have** $O x (y \otimes z)$

by (*rule product-overlap*)

with $\langle \neg O x (y \otimes z) \rangle$ **show** *False*..<

qed

qed

lemma *converse-product-overlap*:

$$O x y \implies O z (x \otimes y) \implies O z y$$

proof –

assume $O x y$

hence $P (x \otimes y) y$ **by** (*rule product-in-second-factor*)

moreover **assume** $O z (x \otimes y)$

ultimately **show** $O z y$

by (*rule overlap-monotonicity*)

qed

lemma *part-product-in-whole-product*:

$$O x y \implies P x v \wedge P y z \implies P (x \otimes y) (v \otimes z)$$

proof –

assume $O x y$

assume $P x v \wedge P y z$

have $\forall w. P w (x \otimes y) \longrightarrow P w (v \otimes z)$

proof

fix w

show $P w (x \otimes y) \longrightarrow P w (v \otimes z)$

proof

assume $P w (x \otimes y)$

with $\langle O x y \rangle$ **have** $P w x \wedge P w y$

by (*rule product-part-in-factors*)

have $P w v \wedge P w z$

proof

from $\langle P w x \wedge P w y \rangle$ **have** $P w x..$
moreover from $\langle P x v \wedge P y z \rangle$ **have** $P x v..$
ultimately show $P w v$ **by** (rule part-transitivity)
next
from $\langle P w x \wedge P w y \rangle$ **have** $P w y..$
moreover from $\langle P x v \wedge P y z \rangle$ **have** $P y z..$
ultimately show $P w z$ **by** (rule part-transitivity)
qed
thus $P w (v \otimes z)$ **by** (rule common-part-in-product)
qed
qed
hence $P (x \otimes y) (x \otimes y) \longrightarrow P (x \otimes y) (v \otimes z)..$
moreover have $P (x \otimes y) (x \otimes y)$ **by** (rule part-reflexivity)
ultimately show $P (x \otimes y) (v \otimes z)..$
qed

lemma right-associated-product: $(\exists w. P w x \wedge P w y \wedge P w z) \implies$
 $(\forall w. P w (x \otimes (y \otimes z)) \longleftrightarrow P w x \wedge (P w y \wedge P w z))$

proof –

assume antecedent: $(\exists w. P w x \wedge P w y \wedge P w z)$
then obtain w **where** $w: P w x \wedge P w y \wedge P w z..$
hence $P w x..$
from w **have** $P w y \wedge P w z..$
hence $\exists w. P w y \wedge P w z..$
with overlap-eq have $O y z..$
hence $yz: \forall w. P w (y \otimes z) \longleftrightarrow (P w y \wedge P w z)$
by (rule product-character)
hence $P w (y \otimes z) \longleftrightarrow (P w y \wedge P w z)..$
hence $P w (y \otimes z)$
using $\langle P w y \wedge P w z \rangle..$
with $\langle P w x \rangle$ **have** $P w x \wedge P w (y \otimes z)..$
hence $\exists w. P w x \wedge P w (y \otimes z)..$
with overlap-eq have $O x (y \otimes z)..$
hence $xyz: \forall w. P w (x \otimes (y \otimes z)) \longleftrightarrow P w x \wedge P w (y \otimes z)$
by (rule product-character)
show $\forall w. P w (x \otimes (y \otimes z)) \longleftrightarrow P w x \wedge (P w y \wedge P w z)$
proof
fix w
from yz **have** $wyz: P w (y \otimes z) \longleftrightarrow (P w y \wedge P w z)..$
moreover from xyz **have**
 $P w (x \otimes (y \otimes z)) \longleftrightarrow P w x \wedge P w (y \otimes z)..$
ultimately show
 $P w (x \otimes (y \otimes z)) \longleftrightarrow P w x \wedge (P w y \wedge P w z)$
by (rule subst)

qed

qed

lemma left-associated-product: $(\exists w. P w x \wedge P w y \wedge P w z) \implies$
 $(\forall w. P w ((x \otimes y) \otimes z) \longleftrightarrow (P w x \wedge P w y) \wedge P w z)$

proof –

assume *antecedent*: $(\exists w. P w x \wedge P w y \wedge P w z)$

then obtain w **where** $w: P w x \wedge P w y \wedge P w z..$

hence $P w y \wedge P w z..$

hence $P w y..$

have $P w z$

using $\langle P w y \wedge P w z \rangle..$

from w **have** $P w x..$

hence $P w x \wedge P w y$

using $\langle P w y \rangle..$

hence $\exists z. P z x \wedge P z y..$

with *overlap-eq* **have** $O x y..$

hence $xy: \forall w. P w (x \otimes y) \longleftrightarrow (P w x \wedge P w y)$

by (*rule product-character*)

hence $P w (x \otimes y) \longleftrightarrow (P w x \wedge P w y)..$

hence $P w (x \otimes y)$

using $\langle P w x \wedge P w y \rangle..$

hence $P w (x \otimes y) \wedge P w z$

using $\langle P w z \rangle..$

hence $\exists w. P w (x \otimes y) \wedge P w z..$

with *overlap-eq* **have** $O (x \otimes y) z..$

hence $xyz: \forall w. P w ((x \otimes y) \otimes z) \longleftrightarrow P w (x \otimes y) \wedge P w z$

by (*rule product-character*)

show $\forall w. P w ((x \otimes y) \otimes z) \longleftrightarrow (P w x \wedge P w y) \wedge P w z$

proof

fix v

from xy **have** $vxy: P v (x \otimes y) \longleftrightarrow (P v x \wedge P v y)..$

moreover from xyz **have**

$P v ((x \otimes y) \otimes z) \longleftrightarrow P v (x \otimes y) \wedge P v z..$

ultimately show $P v ((x \otimes y) \otimes z) \longleftrightarrow (P v x \wedge P v y) \wedge P v z$

by (*rule subst*)

qed

qed

theorem *product-associativity*:

$(\exists w. P w x \wedge P w y \wedge P w z) \implies x \otimes (y \otimes z) = (x \otimes y) \otimes z$

proof –

assume *ante*: $(\exists w. P w x \wedge P w y \wedge P w z)$

hence $(\forall w. P w (x \otimes (y \otimes z)) \longleftrightarrow P w x \wedge (P w y \wedge P w z))$

by (*rule right-associated-product*)

moreover from *ante* **have**

$(\forall w. P w ((x \otimes y) \otimes z) \longleftrightarrow (P w x \wedge P w y) \wedge P w z)$

by (*rule left-associated-product*)

ultimately have $\forall w. P w (x \otimes (y \otimes z)) \longleftrightarrow P w ((x \otimes y) \otimes z)$

by *simp*

with *part-extensionality* **show** $x \otimes (y \otimes z) = (x \otimes y) \otimes z..$

qed

end

6.2 Differences

Some writers also add to closed mereology the axiom of difference closure.²⁵

locale $CMD = CM +$

assumes *difference-eq*:

$$x \ominus y = (\text{THE } z. \forall w. P w z \longleftrightarrow P w x \wedge \neg O w y)$$

assumes *difference-closure*:

$$(\exists w. P w x \wedge \neg O w y) \implies (\exists z. \forall w. P w z \longleftrightarrow P w x \wedge \neg O w y)$$

begin

lemma *difference-intro*:

$$(\forall w. P w z \longleftrightarrow P w x \wedge \neg O w y) \implies x \ominus y = z$$

proof –

assume *antecedent*: $(\forall w. P w z \longleftrightarrow P w x \wedge \neg O w y)$

hence $(\text{THE } z. \forall w. P w z \longleftrightarrow P w x \wedge \neg O w y) = z$

proof (*rule the-equality*)

fix v

assume v : $(\forall w. P w v \longleftrightarrow P w x \wedge \neg O w y)$

have $\forall w. P w v \longleftrightarrow P w z$

proof

fix w

from *antecedent* have $P w z \longleftrightarrow P w x \wedge \neg O w y..$

moreover from v have $P w v \longleftrightarrow P w x \wedge \neg O w y..$

ultimately show $P w v \longleftrightarrow P w z$ by (*rule ssubst*)

qed

with *part-extensionality* show $v = z..$

qed

with *difference-eq* show $x \ominus y = z$ by (*rule ssubst*)

qed

lemma *difference-idempotence*: $\neg O x y \implies (x \ominus y) = x$

proof –

assume $\neg O x y$

hence $\neg O y x$ by (*rule disjoint-symmetry*)

have $\forall w. P w x \longleftrightarrow P w x \wedge \neg O w y$

proof

fix w

show $P w x \longleftrightarrow P w x \wedge \neg O w y$

proof

assume $P w x$

hence $\neg O y w$ using $\langle \neg O y x \rangle$

by (*rule disjoint-demonotonicity*)

hence $\neg O w y$ by (*rule disjoint-symmetry*)

with $\langle P w x \rangle$ show $P w x \wedge \neg O w y..$

next

²⁵See, for example, [Varzi, 1996] p. 263 and [Masolo and Vieu, 1999] p. 238.

assume $P w x \wedge \neg O w y$
thus $P w x..$
qed
qed
thus $(x \ominus y) = x$ **by** (rule *difference-intro*)
qed

lemma *difference-character*: $(\exists w. P w x \wedge \neg O w y) \implies$
 $(\forall w. P w (x \ominus y) \iff P w x \wedge \neg O w y)$

proof –
assume $\exists w. P w x \wedge \neg O w y$
hence $\exists z. \forall w. P w z \iff P w x \wedge \neg O w y$ **by** (rule *difference-closure*)
then obtain z **where** $z: \forall w. P w z \iff P w x \wedge \neg O w y..$
hence $(x \ominus y) = z$ **by** (rule *difference-intro*)
thus $\forall w. P w (x \ominus y) \iff P w x \wedge \neg O w y$ **using** z **by** (rule *ssubst*)
qed

lemma *difference-disjointness*:
 $(\exists z. P z x \wedge \neg O z y) \implies \neg O y (x \ominus y)$

proof –
assume $\exists z. P z x \wedge \neg O z y$
hence $xmy: \forall w. P w (x \ominus y) \iff (P w x \wedge \neg O w y)$
by (rule *difference-character*)
show $\neg O y (x \ominus y)$
proof
assume $O y (x \ominus y)$
with *overlap-eq* **have** $\exists v. P v y \wedge P v (x \ominus y)..$
then obtain v **where** $v: P v y \wedge P v (x \ominus y)..$
from xmy **have** $P v (x \ominus y) \iff (P v x \wedge \neg O v y)..$
moreover from v **have** $P v (x \ominus y)..$
ultimately have $P v x \wedge \neg O v y..$
hence $\neg O v y..$
moreover from v **have** $P v y..$
hence $O v y$ **by** (rule *part-implies-overlap*)
ultimately show *False..*
qed
qed

end

6.3 The Universe

Another closure condition sometimes considered is the existence of the universe.²⁶

locale $CMU = CM +$

²⁶See, for example, [Varzi, 1996] p. 264 and [Casati and Varzi, 1999] p. 45.

```

assumes universe-eq:  $u = (\text{THE } z. \forall w. P w z)$ 
assumes universe-closure:  $\exists y. \forall x. P x y$ 
begin

lemma universe-intro:  $(\forall w. P w z) \implies u = z$ 
proof –
  assume  $z: \forall w. P w z$ 
  hence  $(\text{THE } z. \forall w. P w z) = z$ 
  proof (rule the-equality)
    fix  $v$ 
    assume  $v: \forall w. P w v$ 
    have  $\forall w. P w v \longleftrightarrow P w z$ 
    proof
      fix  $w$ 
      show  $P w v \longleftrightarrow P w z$ 
      proof
        assume  $P w v$ 
        from  $z$  show  $P w z$ ..
      next
        assume  $P w z$ 
        from  $v$  show  $P w v$ ..
      qed
    qed
    with part-extensionality show  $v = z$ ..
  qed
thus  $u = z$  using universe-eq by (rule subst)
qed

```

```

lemma universe-character:  $P x u$ 
proof –
  from universe-closure obtain  $y$  where  $y: \forall x. P x y$ ..
  hence  $u = y$  by (rule universe-intro)
  hence  $\forall x. P x u$  using  $y$  by (rule ssubst)
  thus  $P x u$ ..
qed

```

```

lemma  $\neg PP u x$ 
proof
  assume  $PP u x$ 
  hence  $\neg P x u$  by (rule proper-implies-not-part)
  thus False using universe-character..
qed

```

```

lemma product-universe-implies-factor-universe:
   $O x y \implies x \otimes y = u \implies x = u$ 
proof –
  assume  $x \otimes y = u$ 
  moreover assume  $O x y$ 
  hence  $P (x \otimes y) x$ 

```

by (rule product-in-first-factor)
 ultimately have $P u x$
 by (rule subst)
 with universe-character show $x = u$
 by (rule part-antisymmetry)
 qed
 end

6.4 Complements

As is a condition ensuring the existence of complements.²⁷

locale $CMC = CM +$
 assumes complement-eq: $\neg x = (THE z. \forall w. P w z \longleftrightarrow \neg O w x)$
 assumes complement-closure:
 $(\exists z. \neg O w x) \implies (\exists z. \forall w. P w z \longleftrightarrow \neg O w x)$
 assumes difference-eq:
 $x \ominus y = (THE z. \forall w. P w z \longleftrightarrow P w x \wedge \neg O w y)$
 begin

lemma complement-intro:
 $(\forall w. P w z \longleftrightarrow \neg O w x) \implies \neg x = z$
 proof –
 assume antecedent: $\forall w. P w z \longleftrightarrow \neg O w x$
 hence $(THE z. \forall w. P w z \longleftrightarrow \neg O w x) = z$
 proof (rule the-equality)
 fix v
 assume $v: \forall w. P w v \longleftrightarrow \neg O w x$
 have $\forall w. P w v \longleftrightarrow P w z$
 proof
 fix w
 from antecedent have $P w z \longleftrightarrow \neg O w x..$
 moreover from v have $P w v \longleftrightarrow \neg O w x..$
 ultimately show $P w v \longleftrightarrow P w z$ by (rule ssubst)
 qed
 with part-extensionality show $v = z..$
 qed
 with complement-eq show $\neg x = z$ by (rule ssubst)
 qed

lemma complement-character:
 $(\exists w. \neg O w x) \implies (\forall w. P w (\neg x) \longleftrightarrow \neg O w x)$
 proof –
 assume $\exists w. \neg O w x$
 hence $(\exists z. \forall w. P w z \longleftrightarrow \neg O w x)$ by (rule complement-closure)
 then obtain z where $z: \forall w. P w z \longleftrightarrow \neg O w x..$
 hence $\neg x = z$ by (rule complement-intro)

²⁷See, for example, [Varzi, 1996] p. 264 and [Casati and Varzi, 1999] p. 45.

thus $\forall w. P w (-x) \longleftrightarrow \neg O w x$
using z **by** (rule *ssubst*)
qed

lemma *not-complement-part*: $\exists w. \neg O w x \implies \neg P x (-x)$

proof –

assume $\exists w. \neg O w x$
hence $\forall w. P w (-x) \longleftrightarrow \neg O w x$
by (rule *complement-character*)
hence $P x (-x) \longleftrightarrow \neg O x x..$
show $\neg P x (-x)$
proof
assume $P x (-x)$
with $\langle P x (-x) \longleftrightarrow \neg O x x \rangle$ **have** $\neg O x x..$
thus *False* **using** *overlap-reflexivity..*

qed

qed

lemma *complement-part*: $\neg O x y \implies P x (-y)$

proof –

assume $\neg O x y$
hence $\exists z. \neg O z y..$
hence $\forall w. P w (-y) \longleftrightarrow \neg O w y$
by (rule *complement-character*)
hence $P x (-y) \longleftrightarrow \neg O x y..$
thus $P x (-y)$ **using** $\langle \neg O x y \rangle..$

qed

lemma *complement-overlap*: $\neg O x y \implies O x (-y)$

proof –

assume $\neg O x y$
hence $P x (-y)$
by (rule *complement-part*)
thus $O x (-y)$
by (rule *part-implies-overlap*)

qed

lemma *or-complement-overlap*: $\forall y. O y x \vee O y (-x)$

proof

fix y
show $O y x \vee O y (-x)$
proof *cases*
assume $O y x$
thus $O y x \vee O y (-x)..$

next

assume $\neg O y x$
hence $O y (-x)$
by (rule *complement-overlap*)
thus $O y x \vee O y (-x)..$

qed
qed

lemma complement-disjointness: $\exists v. \neg O v x \implies \neg O x (-x)$

proof –

assume $\exists v. \neg O v x$
hence $w: \forall w. P w (-x) \longleftrightarrow \neg O w x$
by (rule complement-character)
show $\neg O x (-x)$
proof
assume $O x (-x)$
with *overlap-eq* have $\exists v. P v x \wedge P v (-x)$..
then obtain v where $v: P v x \wedge P v (-x)$..
from w have $P v (-x) \longleftrightarrow \neg O v x$..
moreover from v have $P v (-x)$..
ultimately have $\neg O v x$..
moreover from v have $P v x$..
hence $O v x$ by (rule part-implies-overlap)
ultimately show *False*..

qed
qed

lemma part-disjoint-from-complement:

$\exists v. \neg O v x \implies P y x \implies \neg O y (-x)$

proof

assume $\exists v. \neg O v x$
hence $\neg O x (-x)$ by (rule complement-disjointness)
assume $P y x$
assume $O y (-x)$
with *overlap-eq* have $\exists v. P v y \wedge P v (-x)$..
then obtain v where $v: P v y \wedge P v (-x)$..
hence $P v y$..
hence $P v x$ using $\langle P y x \rangle$ by (rule part-transitivity)
moreover from v have $P v (-x)$..
ultimately have $P v x \wedge P v (-x)$..
hence $\exists v. P v x \wedge P v (-x)$..
with *overlap-eq* have $O x (-x)$..
with $\langle \neg O x (-x) \rangle$ show *False*..

qed

lemma product-complement-character: $(\exists w. P w x \wedge \neg O w y) \implies$
 $(\forall w. P w (x \otimes (-y)) \longleftrightarrow (P w x \wedge (\neg O w y)))$

proof –

assume *antecedent*: $\exists w. P w x \wedge \neg O w y$
then obtain w where $w: P w x \wedge \neg O w y$..
hence $P w x$..
moreover from w have $\neg O w y$..
hence $P w (-y)$ by (rule complement-part)
ultimately have $P w x \wedge P w (-y)$..

hence $\exists w. P w x \wedge P w (-y)$..
with *overlap-eq* **have** $O x (-y)$..
hence *prod*: $(\forall w. P w (x \otimes (-y))) \longleftrightarrow (P w x \wedge P w (-y))$
by (*rule product-character*)
show $\forall w. P w (x \otimes (-y)) \longleftrightarrow (P w x \wedge (\neg O w y))$
proof
fix v
from w **have** $\neg O w y$..
hence $\exists w. \neg O w y$..
hence $\forall w. P w (-y) \longleftrightarrow \neg O w y$
by (*rule complement-character*)
hence $P v (-y) \longleftrightarrow \neg O v y$..
moreover **have** $P v (x \otimes (-y)) \longleftrightarrow (P v x \wedge P v (-y))$
using *prod*..
ultimately show $P v (x \otimes (-y)) \longleftrightarrow (P v x \wedge (\neg O v y))$
by (*rule subst*)
qed
qed

theorem *difference-closure*: $(\exists w. P w x \wedge \neg O w y) \implies$
 $(\exists z. \forall w. P w z \longleftrightarrow P w x \wedge \neg O w y)$

proof –
assume $\exists w. P w x \wedge \neg O w y$
hence $\forall w. P w (x \otimes (-y)) \longleftrightarrow P w x \wedge \neg O w y$
by (*rule product-complement-character*)
thus $(\exists z. \forall w. P w z \longleftrightarrow P w x \wedge \neg O w y)$ **by** (*rule exI*)
qed

end

sublocale $CMC \subseteq CMD$

proof
fix $x y$
show $x \ominus y = (THE z. \forall w. P w z = (P w x \wedge \neg O w y))$
using *difference-eq*.
show $(\exists w. P w x \wedge \neg O w y) \implies$
 $(\exists z. \forall w. P w z = (P w x \wedge \neg O w y))$
using *difference-closure*.
qed

corollary (**in** CMC) *difference-is-product-of-complement*:
 $(\exists w. P w x \wedge \neg O w y) \implies (x \ominus y) = x \otimes (-y)$

proof –
assume *antecedent*: $\exists w. P w x \wedge \neg O w y$
hence $\forall w. P w (x \otimes (-y)) \longleftrightarrow P w x \wedge \neg O w y$
by (*rule product-complement-character*)
thus $(x \ominus y) = x \otimes (-y)$ **by** (*rule difference-intro*)
qed

Universe and difference closure entail complement closure, since

the difference of an individual and the universe is the individual's complement.

locale $CMUD = CMU + CMD +$
assumes *complement-eq*: $-x = (THE\ z.\ \forall w.\ P\ w\ z \longleftrightarrow \neg O\ w\ x)$
begin

lemma *universe-difference*:

$(\exists w.\ \neg O\ w\ x) \implies (\forall w.\ P\ w\ (u \ominus x) \longleftrightarrow \neg O\ w\ x)$

proof –

assume $\exists w.\ \neg O\ w\ x$

then obtain w **where** $w:\ \neg O\ w\ x..$

from *universe-character* **have** $P\ w\ u.$

hence $P\ w\ u \wedge \neg O\ w\ x$ **using** $\langle \neg O\ w\ x \rangle..$

hence $\exists z.\ P\ z\ u \wedge \neg O\ z\ x..$

hence $ux:\ \forall w.\ P\ w\ (u \ominus x) \longleftrightarrow (P\ w\ u \wedge \neg O\ w\ x)$

by (*rule difference-character*)

show $\forall w.\ P\ w\ (u \ominus x) \longleftrightarrow \neg O\ w\ x$

proof

fix w

from ux **have** $wux:\ P\ w\ (u \ominus x) \longleftrightarrow (P\ w\ u \wedge \neg O\ w\ x)..$

show $P\ w\ (u \ominus x) \longleftrightarrow \neg O\ w\ x$

proof

assume $P\ w\ (u \ominus x)$

with wux **have** $P\ w\ u \wedge \neg O\ w\ x..$

thus $\neg O\ w\ x..$

next

assume $\neg O\ w\ x$

from *universe-character* **have** $P\ w\ u.$

hence $P\ w\ u \wedge \neg O\ w\ x$ **using** $\langle \neg O\ w\ x \rangle..$

with wux **show** $P\ w\ (u \ominus x)..$

qed

qed

qed

theorem *complement-closure*:

$(\exists w.\ \neg O\ w\ x) \implies (\exists z.\ \forall w.\ P\ w\ z \longleftrightarrow \neg O\ w\ x)$

proof –

assume $\exists w.\ \neg O\ w\ x$

hence $\forall w.\ P\ w\ (u \ominus x) \longleftrightarrow \neg O\ w\ x$

by (*rule universe-difference*)

thus $\exists z.\ \forall w.\ P\ w\ z \longleftrightarrow \neg O\ w\ x..$

qed

end

sublocale $CMUD \subseteq CMC$

proof

fix $x\ y$

show $-x = (THE\ z.\ \forall w.\ P\ w\ z \longleftrightarrow (\neg O\ w\ x))$

using *complement-eq.*
show $\exists w. \neg O w x \implies \exists z. \forall w. P w z \longleftrightarrow (\neg O w x)$
using *complement-closure.*
show $x \ominus y = (THE z. \forall w. P w z = (P w x \wedge \neg O w y))$
using *difference-eq.*
qed

corollary (in *CMUD*) *complement-universe-difference:*

$(\exists y. \neg O y x) \implies -x = (u \ominus x)$

proof –

assume $\exists w. \neg O w x$

hence $\forall w. P w (u \ominus x) \longleftrightarrow \neg O w x$

by (*rule universe-difference*)

thus $-x = (u \ominus x)$

by (*rule complement-intro*)

qed

7 Closed Extensional Mereology

Closed extensional mereology combines closed mereology with extensional mereology.²⁸

locale $CEM = CM + EM$

Likewise, closed minimal mereology combines closed mereology with minimal mereology.²⁹

locale $CMM = CM + MM$

But famously closed minimal mereology and closed extensional mereology are the same theory, because in closed minimal mereology product closure and weak supplementation entail strong supplementation.³⁰

sublocale $CMM \subseteq CEM$

proof

fix $x y$

show *strong-supplementation:* $\neg P x y \implies (\exists z. P z x \wedge \neg O z y)$

proof –

assume $\neg P x y$

show $\exists z. P z x \wedge \neg O z y$

proof *cases*

assume $O x y$

with $\langle \neg P x y \rangle$ **have** $\neg P x y \wedge O x y.$

hence $PP (x \otimes y) x$ **by** (*rule nonpart-implies-proper-product*)

²⁸See [Varzi, 1996] p. 263 and [Casati and Varzi, 1999] p. 43.

²⁹See [Casati and Varzi, 1999] p. 43.

³⁰See [Simons, 1987] p. 31 and [Casati and Varzi, 1999] p. 44.

hence $\exists z. P z x \wedge \neg O z (x \otimes y)$ **by** (rule weak-supplementation)
then obtain z **where** $z: P z x \wedge \neg O z (x \otimes y)$..
hence $\neg O z y$ **by** (rule disjoint-from-second-factor)
moreover from z **have** $P z x$..
hence $P z x \wedge \neg O z y$
using $\langle \neg O z y \rangle$..
thus $\exists z. P z x \wedge \neg O z y$..
next
assume $\neg O x y$
with *part-reflexivity* **have** $P x x \wedge \neg O x y$..
thus $(\exists z. P z x \wedge \neg O z y)$..
qed
qed
qed

sublocale $CEM \subseteq CMM$..

7.1 Sums

context CEM

begin

lemma *sum-intro*:

$(\forall w. O w z \longleftrightarrow (O w x \vee O w y)) \implies x \oplus y = z$

proof –

assume *sum*: $\forall w. O w z \longleftrightarrow (O w x \vee O w y)$

hence (*THE* $v. \forall w. O w v \longleftrightarrow (O w x \vee O w y)$) = z

proof (rule *the-equality*)

fix a

assume a : $\forall w. O w a \longleftrightarrow (O w x \vee O w y)$

have $\forall w. O w a \longleftrightarrow O w z$

proof

fix w

from *sum* **have** $O w z \longleftrightarrow (O w x \vee O w y)$..

moreover from a **have** $O w a \longleftrightarrow (O w x \vee O w y)$..

ultimately show $O w a \longleftrightarrow O w z$ **by** (rule *ssubst*)

qed

with *overlap-extensionality* **show** $a = z$..

qed

thus $x \oplus y = z$

using *sum-eq* **by** (rule *subst*)

qed

lemma *sum-idempotence*: $x \oplus x = x$

proof –

have $\forall w. O w x \longleftrightarrow (O w x \vee O w x)$

proof

fix w

show $O w x \longleftrightarrow (O w x \vee O w x)$

```

proof (rule iffI)
  assume  $O w x$ 
  thus  $O w x \vee O w x..$ 
next
  assume  $O w x \vee O w x$ 
  thus  $O w x$  by (rule disjE)
qed
qed
thus  $x \oplus x = x$  by (rule sum-intro)
qed

```

lemma *part-sum-identity*: $P y x \implies x \oplus y = x$

```

proof -
  assume  $P y x$ 
  have  $\forall w. O w x \longleftrightarrow (O w x \vee O w y)$ 
  proof
    fix  $w$ 
    show  $O w x \longleftrightarrow (O w x \vee O w y)$ 
    proof
      assume  $O w x$ 
      thus  $O w x \vee O w y..$ 
    next
      assume  $O w x \vee O w y$ 
      thus  $O w x$ 
    proof
      assume  $O w x$ 
      thus  $O w x.$ 
    next
      assume  $O w y$ 
      with  $\langle P y x \rangle$  show  $O w x$ 
      by (rule overlap-monotonicity)
    qed
  qed
qed
thus  $x \oplus y = x$  by (rule sum-intro)
qed

```

lemma *sum-character*: $\forall w. O w (x \oplus y) \longleftrightarrow (O w x \vee O w y)$

```

proof -
  from sum-closure have  $(\exists z. \forall w. O w z \longleftrightarrow (O w x \vee O w y))..$ 
  then obtain  $a$  where  $a: \forall w. O w a \longleftrightarrow (O w x \vee O w y)..$ 
  hence  $x \oplus y = a$  by (rule sum-intro)
  thus  $\forall w. O w (x \oplus y) \longleftrightarrow (O w x \vee O w y)$ 
  using  $a$  by (rule ssubst)
qed

```

lemma *sum-overlap*: $O w (x \oplus y) \longleftrightarrow (O w x \vee O w y)$

using *sum-character*..

lemma *sum-part-character*:
 $P w (x \oplus y) \longleftrightarrow (\forall v. O v w \longrightarrow O v x \vee O v y)$
proof
 assume $P w (x \oplus y)$
 show $\forall v. O v w \longrightarrow O v x \vee O v y$
proof
 fix v
 show $O v w \longrightarrow O v x \vee O v y$
proof
 assume $O v w$
 with $\langle P w (x \oplus y) \rangle$ have $O v (x \oplus y)$
 by (*rule overlap-monotonicity*)
 with *sum-overlap* show $O v x \vee O v y$..
 qed
 qed
 next
 assume *right*: $\forall v. O v w \longrightarrow O v x \vee O v y$
 have $\forall v. O v w \longrightarrow O v (x \oplus y)$
proof
 fix v
 from *right* have $O v w \longrightarrow O v x \vee O v y$..
 with *sum-overlap* show $O v w \longrightarrow O v (x \oplus y)$
 by (*rule ssubst*)
 qed
 with *part-overlap-eq* show $P w (x \oplus y)$..
 qed

lemma *sum-commutativity*: $x \oplus y = y \oplus x$
proof –
 from *sum-character* have $\forall w. O w (y \oplus x) \longleftrightarrow O w y \vee O w x$.
 hence $\forall w. O w (y \oplus x) \longleftrightarrow O w x \vee O w y$ by *metis*
 thus $x \oplus y = y \oplus x$ by (*rule sum-intro*)
 qed

lemma *first-summand-overlap*: $O z x \Longrightarrow O z (x \oplus y)$
proof –
 assume $O z x$
 hence $O z x \vee O z y$..
 with *sum-overlap* show $O z (x \oplus y)$..
 qed

lemma *first-summand-disjointness*: $\neg O z (x \oplus y) \Longrightarrow \neg O z x$
proof –
 assume $\neg O z (x \oplus y)$
 show $\neg O z x$
proof
 assume $O z x$
 hence $O z (x \oplus y)$ by (*rule first-summand-overlap*)
 with $\langle \neg O z (x \oplus y) \rangle$ show *False*..
 qed

qed
qed

lemma *first-summand-in-sum*: $P x (x \oplus y)$

proof –

have $\forall w. O w x \longrightarrow O w (x \oplus y)$

proof

fix w

show $O w x \longrightarrow O w (x \oplus y)$

proof

assume $O w x$

thus $O w (x \oplus y)$

by (rule *first-summand-overlap*)

qed

qed

with *part-overlap-eq* show $P x (x \oplus y)$..

qed

lemma *common-first-summand*: $P x (x \oplus y) \wedge P x (x \oplus z)$

proof

from *first-summand-in-sum* show $P x (x \oplus y)$.

from *first-summand-in-sum* show $P x (x \oplus z)$.

qed

lemma *common-first-summand-overlap*: $O (x \oplus y) (x \oplus z)$

proof –

from *first-summand-in-sum* have $P x (x \oplus y)$.

moreover from *first-summand-in-sum* have $P x (x \oplus z)$.

ultimately have $P x (x \oplus y) \wedge P x (x \oplus z)$..

hence $\exists v. P v (x \oplus y) \wedge P v (x \oplus z)$..

with *overlap-eq* show *?thesis*..

qed

lemma *second-summand-overlap*: $O z y \implies O z (x \oplus y)$

proof –

assume $O z y$

from *sum-character* have $O z (x \oplus y) \longleftrightarrow (O z x \vee O z y)$..

moreover from $\langle O z y \rangle$ have $O z x \vee O z y$..

ultimately show $O z (x \oplus y)$..

qed

lemma *second-summand-disjointness*: $\neg O z (x \oplus y) \implies \neg O z y$

proof –

assume $\neg O z (x \oplus y)$

show $\neg O z y$

proof

assume $O z y$

hence $O z (x \oplus y)$

by (rule *second-summand-overlap*)

with $\langle \neg O z (x \oplus y) \rangle$ **show** *False..*
qed
qed

lemma *second-summand-in-sum*: $P y (x \oplus y)$
proof –
have $\forall w. O w y \longrightarrow O w (x \oplus y)$
proof
fix w
show $O w y \longrightarrow O w (x \oplus y)$
proof
assume $O w y$
thus $O w (x \oplus y)$
by (*rule second-summand-overlap*)
qed
qed
with *part-overlap-eq* **show** $P y (x \oplus y)..$
qed

lemma *second-summands-in-sums*: $P y (x \oplus y) \wedge P v (z \oplus v)$
proof
show $P y (x \oplus y)$ **using** *second-summand-in-sum.*
show $P v (z \oplus v)$ **using** *second-summand-in-sum.*
qed

lemma *disjoint-from-sum*: $\neg O z (x \oplus y) \longleftrightarrow \neg O z x \wedge \neg O z y$
proof –
from *sum-character* **have** $O z (x \oplus y) \longleftrightarrow (O z x \vee O z y)..$
thus *?thesis* **by** *simp*
qed

lemma *summands-part-implies-sum-part*:
 $P x z \wedge P y z \implies P (x \oplus y) z$
proof –
assume *antecedent*: $P x z \wedge P y z$
have $\forall w. O w (x \oplus y) \longrightarrow O w z$
proof
fix w
have $w: O w (x \oplus y) \longleftrightarrow (O w x \vee O w y)$
using *sum-character..*
show $O w (x \oplus y) \longrightarrow O w z$
proof
assume $O w (x \oplus y)$
with w **have** $O w x \vee O w y..$
thus $O w z$
proof
from *antecedent* **have** $P x z..$
moreover **assume** $O w x$
ultimately **show** $O w z$

by (rule overlap-monotonicity)
 next
 from antecedent have $P\ y\ z..$
 moreover assume $O\ w\ y$
 ultimately show $O\ w\ z$
 by (rule overlap-monotonicity)
 qed
 qed
 qed
 with part-overlap-eq show $P\ (x \oplus y)\ z..$
 qed

lemma *sum-part-implies-summands-part*:

$P\ (x \oplus y)\ z \implies P\ x\ z \wedge P\ y\ z$

proof –

assume antecedent: $P\ (x \oplus y)\ z$

show $P\ x\ z \wedge P\ y\ z$

proof

from first-summand-in-sum show $P\ x\ z$

using antecedent by (rule part-transitivity)

next

from second-summand-in-sum show $P\ y\ z$

using antecedent by (rule part-transitivity)

qed

qed

lemma *in-second-summand*: $P\ z\ (x \oplus y) \wedge \neg O\ z\ x \implies P\ z\ y$

proof –

assume antecedent: $P\ z\ (x \oplus y) \wedge \neg O\ z\ x$

hence $P\ z\ (x \oplus y)..$

show $P\ z\ y$

proof (rule ccontr)

assume $\neg P\ z\ y$

hence $\exists v. P\ v\ z \wedge \neg O\ v\ y$

by (rule strong-supplementation)

then obtain v where $v: P\ v\ z \wedge \neg O\ v\ y..$

hence $\neg O\ v\ y..$

from v have $P\ v\ z..$

hence $P\ v\ (x \oplus y)$

using $\langle P\ z\ (x \oplus y) \rangle$ by (rule part-transitivity)

hence $O\ v\ (x \oplus y)$ by (rule part-implies-overlap)

from sum-character have $O\ v\ (x \oplus y) \longleftrightarrow O\ v\ x \vee O\ v\ y..$

hence $O\ v\ x \vee O\ v\ y$ using $\langle O\ v\ (x \oplus y) \rangle..$

thus *False*

proof (rule disjE)

from antecedent have $\neg O\ z\ x..$

moreover assume $O\ v\ x$

hence $O\ x\ v$ by (rule overlap-symmetry)

with $\langle P\ v\ z \rangle$ have $O\ x\ z$

by (rule overlap-monotonicity)
 hence $O z x$ by (rule overlap-symmetry)
 ultimately show *False*..
 next
 assume $O v y$
 with $\langle \neg O v y \rangle$ show *False*..
 qed
 qed
 qed

lemma *disjoint-second-summands*:

$$P v (x \oplus y) \wedge P v (x \oplus z) \implies \neg O y z \implies P v x$$

proof –

assume *antecedent*: $P v (x \oplus y) \wedge P v (x \oplus z)$
 hence $P v (x \oplus z)$..
 assume $\neg O y z$
 show $P v x$
 proof (rule *ccontr*)
 assume $\neg P v x$
 hence $\exists w. P w v \wedge \neg O w x$ by (rule *strong-supplementation*)
 then obtain w where $w: P w v \wedge \neg O w x$..
 hence $\neg O w x$..
 from w have $P w v$..
 moreover from *antecedent* have $P v (x \oplus z)$..
 ultimately have $P w (x \oplus z)$ by (rule *part-transitivity*)
 hence $P w (x \oplus z) \wedge \neg O w x$ using $\langle \neg O w x \rangle$..
 hence $P w z$ by (rule *in-second-summand*)
 from *antecedent* have $P v (x \oplus y)$..
 with $\langle P w v \rangle$ have $P w (x \oplus y)$ by (rule *part-transitivity*)
 hence $P w (x \oplus y) \wedge \neg O w x$ using $\langle \neg O w x \rangle$..
 hence $P w y$ by (rule *in-second-summand*)
 hence $P w y \wedge P w z$ using $\langle P w z \rangle$..
 hence $\exists w. P w y \wedge P w z$..
 with *overlap-eq* have $O y z$..
 with $\langle \neg O y z \rangle$ show *False*..
 qed

qed

lemma *right-associated-sum*:

$$O w (x \oplus (y \oplus z)) \longleftrightarrow O w x \vee (O w y \vee O w z)$$

proof –

from *sum-character* have $O w (y \oplus z) \longleftrightarrow O w y \vee O w z$..
 moreover from *sum-character* have
 $O w (x \oplus (y \oplus z)) \longleftrightarrow (O w x \vee O w (y \oplus z))$..
 ultimately show *?thesis*
 by (rule *subst*)

qed

lemma *left-associated-sum*:

$O w ((x \oplus y) \oplus z) \longleftrightarrow (O w x \vee O w y) \vee O w z$
proof –
from *sum-character* **have** $O w (x \oplus y) \longleftrightarrow (O w x \vee O w y)$..
moreover from *sum-character* **have**
 $O w ((x \oplus y) \oplus z) \longleftrightarrow O w (x \oplus y) \vee O w z$..
ultimately show *?thesis*
by (*rule subst*)
qed

theorem *sum-associativity*: $x \oplus (y \oplus z) = (x \oplus y) \oplus z$
proof –
have $\forall w. O w (x \oplus (y \oplus z)) \longleftrightarrow O w ((x \oplus y) \oplus z)$
proof
fix w
have $O w (x \oplus (y \oplus z)) \longleftrightarrow (O w x \vee O w y) \vee O w z$
using *right-associated-sum* **by** *simp*
with *left-associated-sum* **show**
 $O w (x \oplus (y \oplus z)) \longleftrightarrow O w ((x \oplus y) \oplus z)$ **by** (*rule ssubst*)
qed
with *overlap-extensionality* **show** $x \oplus (y \oplus z) = (x \oplus y) \oplus z$..
qed

7.2 Distributivity

The proofs in this section are adapted from [Pietruszczak, 2018] pp. 102-4.

lemma *common-summand-in-product*: $P x ((x \oplus y) \otimes (x \oplus z))$
using *common-first-summand* **by** (*rule common-part-in-product*)

lemma *product-in-first-summand*:
 $\neg O y z \implies P ((x \oplus y) \otimes (x \oplus z)) x$
proof –
assume $\neg O y z$
have $\forall v. P v ((x \oplus y) \otimes (x \oplus z)) \longrightarrow P v x$
proof
fix v
show $P v ((x \oplus y) \otimes (x \oplus z)) \longrightarrow P v x$
proof
assume $P v ((x \oplus y) \otimes (x \oplus z))$
with *common-first-summand-overlap* **have**
 $P v (x \oplus y) \wedge P v (x \oplus z)$ **by** (*rule product-part-in-factors*)
thus $P v x$ **using** $\langle \neg O y z \rangle$ **by** (*rule disjoint-second-summands*)
qed
qed
hence $P ((x \oplus y) \otimes (x \oplus z)) ((x \oplus y) \otimes (x \oplus z)) \longrightarrow$
 $P ((x \oplus y) \otimes (x \oplus z)) x$..
thus $P ((x \oplus y) \otimes (x \oplus z)) x$ **using** *part-reflexivity*..
qed

lemma *product-is-first-summand*:
 $\neg O y z \implies (x \oplus y) \otimes (x \oplus z) = x$

proof –
assume $\neg O y z$
hence $P ((x \oplus y) \otimes (x \oplus z)) x$
by (*rule product-in-first-summand*)
thus $(x \oplus y) \otimes (x \oplus z) = x$
using *common-summand-in-product*
by (*rule part-antisymmetry*)

qed

lemma *sum-over-product-left*: $O y z \implies P (x \oplus (y \otimes z)) ((x \oplus y) \otimes (x \oplus z))$

proof –
assume $O y z$
hence $P (y \otimes z) ((x \oplus y) \otimes (x \oplus z))$ **using** *second-summands-in-sums*
by (*rule part-product-in-whole-product*)
with *common-summand-in-product* **have**
 $P x ((x \oplus y) \otimes (x \oplus z)) \wedge P (y \otimes z) ((x \oplus y) \otimes (x \oplus z))..$
thus $P (x \oplus (y \otimes z)) ((x \oplus y) \otimes (x \oplus z))$
by (*rule summands-part-implies-sum-part*)

qed

lemma *sum-over-product-right*:

$O y z \implies P ((x \oplus y) \otimes (x \oplus z)) (x \oplus (y \otimes z))$

proof –
assume $O y z$
show $P ((x \oplus y) \otimes (x \oplus z)) (x \oplus (y \otimes z))$
proof (*rule ccontr*)
assume $\neg P ((x \oplus y) \otimes (x \oplus z)) (x \oplus (y \otimes z))$
hence $\exists v. P v ((x \oplus y) \otimes (x \oplus z)) \wedge \neg O v (x \oplus (y \otimes z))$
by (*rule strong-supplementation*)
then obtain v **where** v :
 $P v ((x \oplus y) \otimes (x \oplus z)) \wedge \neg O v (x \oplus (y \otimes z))..$
hence $\neg O v (x \oplus (y \otimes z))..$
with *disjoint-from-sum* **have** $vd: \neg O v x \wedge \neg O v (y \otimes z)..$
hence $\neg O v (y \otimes z)..$
from vd **have** $\neg O v x..$
from v **have** $P v ((x \oplus y) \otimes (x \oplus z))..$
with *common-first-summand-overlap* **have**
 $vs: P v (x \oplus y) \wedge P v (x \oplus z)$ **by** (*rule product-part-in-factors*)
hence $P v (x \oplus y)..$
hence $P v (x \oplus y) \wedge \neg O v x$ **using** $\langle \neg O v x \rangle..$
hence $P v y$ **by** (*rule in-second-summand*)
moreover from vs **have** $P v (x \oplus z)..$
hence $P v (x \oplus z) \wedge \neg O v x$ **using** $\langle \neg O v x \rangle..$
hence $P v z$ **by** (*rule in-second-summand*)
ultimately have $P v y \wedge P v z..$
hence $P v (y \otimes z)$ **by** (*rule common-part-in-product*)

hence $O v (y \otimes z)$ by (rule *part-implies-overlap*)
 with $\langle \neg O v (y \otimes z) \rangle$ show *False..*

qed
 qed

Sums distribute over products.

theorem *sum-over-product:*

$$O y z \implies x \oplus (y \otimes z) = (x \oplus y) \otimes (x \oplus z)$$

proof –

assume $O y z$

hence $P (x \oplus (y \otimes z)) ((x \oplus y) \otimes (x \oplus z))$

by (rule *sum-over-product-left*)

moreover have $P ((x \oplus y) \otimes (x \oplus z)) (x \oplus (y \otimes z))$

using $\langle O y z \rangle$ by (rule *sum-over-product-right*)

ultimately show $x \oplus (y \otimes z) = (x \oplus y) \otimes (x \oplus z)$

by (rule *part-antisymmetry*)

qed

lemma *product-in-factor-by-sum:*

$$O x y \implies P (x \otimes y) (x \otimes (y \oplus z))$$

proof –

assume $O x y$

hence $P (x \otimes y) x$

by (rule *product-in-first-factor*)

moreover have $P (x \otimes y) y$

using $\langle O x y \rangle$ by (rule *product-in-second-factor*)

hence $P (x \otimes y) (y \oplus z)$

using *first-summand-in-sum* by (rule *part-transitivity*)

with $\langle P (x \otimes y) x \rangle$ have $P (x \otimes y) x \wedge P (x \otimes y) (y \oplus z)$..

thus $P (x \otimes y) (x \otimes (y \oplus z))$

by (rule *common-part-in-product*)

qed

lemma *product-of-first-summand:*

$$O x y \implies \neg O x z \implies P (x \otimes (y \oplus z)) (x \otimes y)$$

proof –

assume $O x y$

hence $O x (y \oplus z)$

by (rule *first-summand-overlap*)

assume $\neg O x z$

show $P (x \otimes (y \oplus z)) (x \otimes y)$

proof (rule *ccontr*)

assume $\neg P (x \otimes (y \oplus z)) (x \otimes y)$

hence $\exists v. P v (x \otimes (y \oplus z)) \wedge \neg O v (x \otimes y)$

by (rule *strong-supplementation*)

then obtain v where $v: P v (x \otimes (y \oplus z)) \wedge \neg O v (x \otimes y)$..

hence $P v (x \otimes (y \oplus z))$..

with $\langle O x (y \oplus z) \rangle$ have $P v x \wedge P v (y \oplus z)$

by (rule *product-part-in-factors*)

hence $P v x..$
 moreover from v have $\neg O v (x \otimes y)..$
 ultimately have $P v x \wedge \neg O v (x \otimes y)..$
 hence $\neg O v y$ by (rule disjoint-from-second-factor)
 from $\langle P v x \wedge P v (y \oplus z) \rangle$ have $P v (y \oplus z)..$
 hence $P v (y \oplus z) \wedge \neg O v y$ using $\langle \neg O v y \rangle..$
 hence $P v z$ by (rule in-second-summand)
 with $\langle P v x \rangle$ have $P v x \wedge P v z..$
 hence $\exists v. P v x \wedge P v z..$
 with overlap-eq have $O x z..$
 with $\langle \neg O x z \rangle$ show *False*..

qed
qed

theorem *disjoint-product-over-sum*:

$O x y \implies \neg O x z \implies x \otimes (y \oplus z) = x \otimes y$

proof –

assume $O x y$
 moreover assume $\neg O x z$
 ultimately have $P (x \otimes (y \oplus z)) (x \otimes y)$
 by (rule product-of-first-summand)
 moreover have $P (x \otimes y)(x \otimes (y \oplus z))$
 using $\langle O x y \rangle$ by (rule product-in-factor-by-sum)
 ultimately show $x \otimes (y \oplus z) = x \otimes y$
 by (rule part-antisymmetry)

qed

lemma *product-over-sum-left*:

$O x y \wedge O x z \implies P (x \otimes (y \oplus z))((x \otimes y) \oplus (x \otimes z))$

proof –

assume $O x y \wedge O x z$
 hence $O x y..$
 hence $O x (y \oplus z)$ by (rule first-summand-overlap)
 show $P (x \otimes (y \oplus z))((x \otimes y) \oplus (x \otimes z))$
 proof (rule ccontr)
 assume $\neg P (x \otimes (y \oplus z))((x \otimes y) \oplus (x \otimes z))$
 hence $\exists v. P v (x \otimes (y \oplus z)) \wedge \neg O v ((x \otimes y) \oplus (x \otimes z))$
 by (rule strong-supplementation)
 then obtain v where v :
 $P v (x \otimes (y \oplus z)) \wedge \neg O v ((x \otimes y) \oplus (x \otimes z))..$
 hence $\neg O v ((x \otimes y) \oplus (x \otimes z))..$
 with disjoint-from-sum have *oxyz*:
 $\neg O v (x \otimes y) \wedge \neg O v (x \otimes z)..$
 from v have $P v (x \otimes (y \oplus z))..$
 with $\langle O x (y \oplus z) \rangle$ have *pxyz*: $P v x \wedge P v (y \oplus z)$
 by (rule product-part-in-factors)
 hence $P v x..$
 moreover from *oxyz* have $\neg O v (x \otimes y)..$
 ultimately have $P v x \wedge \neg O v (x \otimes y)..$

hence $\neg O v y$ **by** (*rule disjoint-from-second-factor*)
from $oxyz$ **have** $\neg O v (x \otimes z)$..
with $\langle P v x \rangle$ **have** $P v x \wedge \neg O v (x \otimes z)$..
hence $\neg O v z$ **by** (*rule disjoint-from-second-factor*)
with $\langle \neg O v y \rangle$ **have** $\neg O v y \wedge \neg O v z$..
with *disjoint-from-sum* **have** $\neg O v (y \oplus z)$..
from $pxyz$ **have** $P v (y \oplus z)$..
hence $O v (y \oplus z)$ **by** (*rule part-implies-overlap*)
with $\langle \neg O v (y \oplus z) \rangle$ **show** *False*..

qed
qed

lemma *product-over-sum-right*:

$$O x y \wedge O x z \implies P((x \otimes y) \oplus (x \otimes z))(x \otimes (y \oplus z))$$

proof –

assume *antecedent*: $O x y \wedge O x z$
have $P (x \otimes y) (x \otimes (y \oplus z)) \wedge P (x \otimes z) (x \otimes (y \oplus z))$
proof
from *antecedent* **have** $O x y$..
thus $P (x \otimes y) (x \otimes (y \oplus z))$
by (*rule product-in-factor-by-sum*)

next

from *antecedent* **have** $O x z$..
hence $P (x \otimes z) (x \otimes (z \oplus y))$
by (*rule product-in-factor-by-sum*)
with *sum-commutativity* **show** $P (x \otimes z) (x \otimes (y \oplus z))$
by (*rule subst*)

qed

thus $P((x \otimes y) \oplus (x \otimes z))(x \otimes (y \oplus z))$
by (*rule summands-part-implies-sum-part*)

qed

theorem *product-over-sum*:

$$O x y \wedge O x z \implies x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$$

proof –

assume *antecedent*: $O x y \wedge O x z$
hence $P (x \otimes (y \oplus z))((x \otimes y) \oplus (x \otimes z))$
by (*rule product-over-sum-left*)
moreover **have** $P((x \otimes y) \oplus (x \otimes z))(x \otimes (y \oplus z))$
using *antecedent* **by** (*rule product-over-sum-right*)
ultimately **show** $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$
by (*rule part-antisymmetry*)

qed

lemma *joint-identical-sums*:

$$v \oplus w = x \oplus y \implies O x v \wedge O x w \implies ((x \otimes v) \oplus (x \otimes w)) = x$$

proof –

assume $v \oplus w = x \oplus y$
moreover **assume** $O x v \wedge O x w$

hence $x \otimes (v \oplus w) = x \otimes v \oplus x \otimes w$
by (*rule product-over-sum*)
ultimately have $x \otimes (x \oplus y) = x \otimes v \oplus x \otimes w$ **by** (*rule subst*)
moreover have $(x \otimes (x \oplus y)) = x$ **using** *first-summand-in-sum*
by (*rule part-product-identity*)
ultimately show $((x \otimes v) \oplus (x \otimes w)) = x$ **by** (*rule subst*)
qed

lemma *disjoint-identical-sums*:

$$v \oplus w = x \oplus y \implies \neg O y v \wedge \neg O w x \implies x = v \wedge y = w$$

proof –

assume *identical*: $v \oplus w = x \oplus y$

assume *disjoint*: $\neg O y v \wedge \neg O w x$

show $x = v \wedge y = w$

proof

from *disjoint* **have** $\neg O y v..$

hence $(x \oplus y) \otimes (x \oplus v) = x$

by (*rule product-is-first-summand*)

with *identical* **have** $(v \oplus w) \otimes (x \oplus v) = x$

by (*rule ssubst*)

moreover from *disjoint* **have** $\neg O w x..$

hence $(v \oplus w) \otimes (v \oplus x) = v$

by (*rule product-is-first-summand*)

with *sum-commutativity* **have** $(v \oplus w) \otimes (x \oplus v) = v$

by (*rule subst*)

ultimately show $x = v$ **by** (*rule subst*)

next

from *disjoint* **have** $\neg O w x..$

hence $(y \oplus w) \otimes (y \oplus x) = y$

by (*rule product-is-first-summand*)

moreover from *disjoint* **have** $\neg O y v..$

hence $(w \oplus y) \otimes (w \oplus v) = w$

by (*rule product-is-first-summand*)

with *sum-commutativity* **have** $(w \oplus y) \otimes (v \oplus w) = w$

by (*rule subst*)

with *identical* **have** $(w \oplus y) \otimes (x \oplus y) = w$

by (*rule subst*)

with *sum-commutativity* **have** $(w \oplus y) \otimes (y \oplus x) = w$

by (*rule subst*)

with *sum-commutativity* **have** $(y \oplus w) \otimes (y \oplus x) = w$

by (*rule subst*)

ultimately show $y = w$

by (*rule subst*)

qed

qed

end

7.3 Differences

locale $CEMD = CEM + CMD$

begin

lemma *plus-minus*: $PP\ y\ x \implies y \oplus (x \ominus y) = x$

proof –

assume $PP\ y\ x$

hence $\exists z. P\ z\ x \wedge \neg O\ z\ y$ by (*rule weak-supplementation*)

hence $xmy:\forall w. P\ w\ (x \ominus y) \longleftrightarrow (P\ w\ x \wedge \neg O\ w\ y)$

by (*rule difference-character*)

have $\forall w. O\ w\ x \longleftrightarrow (O\ w\ y \vee O\ w\ (x \ominus y))$

proof

fix w

from xmy have $w: P\ w\ (x \ominus y) \longleftrightarrow (P\ w\ x \wedge \neg O\ w\ y)..$

show $O\ w\ x \longleftrightarrow (O\ w\ y \vee O\ w\ (x \ominus y))$

proof

assume $O\ w\ x$

with *overlap-eq* have $\exists v. P\ v\ w \wedge P\ v\ x..$

then obtain v where $v: P\ v\ w \wedge P\ v\ x..$

hence $P\ v\ w..$

from v have $P\ v\ x..$

show $O\ w\ y \vee O\ w\ (x \ominus y)$

proof *cases*

assume $O\ v\ y$

hence $O\ y\ v$ by (*rule overlap-symmetry*)

with $\langle P\ v\ w \rangle$ have $O\ y\ w$ by (*rule overlap-monotonicity*)

hence $O\ w\ y$ by (*rule overlap-symmetry*)

thus $O\ w\ y \vee O\ w\ (x \ominus y)..$

next

from xmy have $P\ v\ (x \ominus y) \longleftrightarrow (P\ v\ x \wedge \neg O\ v\ y)..$

moreover assume $\neg O\ v\ y$

with $\langle P\ v\ x \rangle$ have $P\ v\ x \wedge \neg O\ v\ y..$

ultimately have $P\ v\ (x \ominus y)..$

with $\langle P\ v\ w \rangle$ have $P\ v\ w \wedge P\ v\ (x \ominus y)..$

hence $\exists v. P\ v\ w \wedge P\ v\ (x \ominus y)..$

with *overlap-eq* have $O\ w\ (x \ominus y)..$

thus $O\ w\ y \vee O\ w\ (x \ominus y)..$

qed

next

assume $O\ w\ y \vee O\ w\ (x \ominus y)$

thus $O\ w\ x$

proof

from $\langle PP\ y\ x \rangle$ have $P\ y\ x$

by (*rule proper-implies-part*)

moreover assume $O\ w\ y$

ultimately show $O\ w\ x$

by (*rule overlap-monotonicity*)

next

assume $O\ w\ (x \ominus y)$

with *overlap-eq* **have** $\exists v. P v w \wedge P v (x \ominus y)$..
then obtain v **where** $v: P v w \wedge P v (x \ominus y)$..
hence $P v w$..
from xmy **have** $P v (x \ominus y) \longleftrightarrow (P v x \wedge \neg O v y)$..
moreover from v **have** $P v (x \ominus y)$..
ultimately have $P v x \wedge \neg O v y$..
hence $P v x$..
with $\langle P v w \rangle$ **have** $P v w \wedge P v x$..
hence $\exists v. P v w \wedge P v x$..
with *overlap-eq* **show** $O w x$..
qed
qed
qed
thus $y \oplus (x \ominus y) = x$
by (*rule sum-intro*)
qed
end

7.4 The Universe

locale $CEMU = CEM + CMU$

begin

lemma *something-disjoint*: $x \neq u \implies (\exists v. \neg O v x)$

proof –

assume $x \neq u$

with *universe-character* **have** $P x u \wedge x \neq u$..
with *nip-eq* **have** $PP x u$..
hence $\exists v. P v u \wedge \neg O v x$
by (*rule weak-supplementation*)
then obtain v **where** $P v u \wedge \neg O v x$..
hence $\neg O v x$..
thus $\exists v. \neg O v x$..
qed

lemma *overlaps-universe*: $O x u$

proof –

from *universe-character* **have** $P x u$..
thus $O x u$ **by** (*rule part-implies-overlap*)
qed

lemma *universe-absorbing*: $x \oplus u = u$

proof –

from *universe-character* **have** $P (x \oplus u) u$..
thus $x \oplus u = u$ **using** *second-summand-in-sum*
by (*rule part-antisymmetry*)
qed

lemma *second-summand-not-universe*: $x \oplus y \neq u \implies y \neq u$
proof –

assume *antecedent*: $x \oplus y \neq u$
 show $y \neq u$
 proof
 assume $y = u$
 hence $x \oplus u \neq u$ **using** *antecedent* **by** (*rule subst*)
 thus *False* **using** *universe-absorbing*..
 qed
qed

lemma *first-summand-not-universe*: $x \oplus y \neq u \implies x \neq u$
proof –

assume $x \oplus y \neq u$
 with *sum-commutativity* **have** $y \oplus x \neq u$ **by** (*rule subst*)
 thus $x \neq u$ **by** (*rule second-summand-not-universe*)
qed

end

7.5 Complements

locale *CEMC* = *CEM* + *CMC* +
 assumes *universe-eq*: $u = (\text{THE } x. \forall y. P y x)$
begin

lemma *complement-sum-character*: $\forall y. P y (x \oplus (-x))$

proof
 fix y
 have $\forall v. O v y \longrightarrow O v x \vee O v (-x)$
 proof
 fix v
 show $O v y \longrightarrow O v x \vee O v (-x)$
 proof
 assume $O v y$
 show $O v x \vee O v (-x)$
 using *or-complement-overlap*..
 qed
 qed
 with *sum-part-character* **show** $P y (x \oplus (-x))$..
qed

lemma *universe-closure*: $\exists x. \forall y. P y x$
 using *complement-sum-character* **by** (*rule exI*)

end

sublocale *CEMC* \subseteq *CEMU*

proof

show $u = (\text{THE } z. \forall w. P w z)$ **using** *universe-eq*.
show $\exists x. \forall y. P y x$ **using** *universe-closure*.
qed

sublocale $CEMC \subseteq CEMD$
proof
qed

context $CEMC$
begin

corollary *universe-is-complement-sum*: $u = x \oplus (-x)$
using *complement-sum-character* **by** (*rule universe-intro*)

lemma *strong-complement-character*:
 $x \neq u \implies (\forall v. P v (-x) \longleftrightarrow \neg O v x)$
proof –
assume $x \neq u$
hence $\exists v. \neg O v x$ **by** (*rule something-disjoint*)
thus $\forall v. P v (-x) \longleftrightarrow \neg O v x$ **by** (*rule complement-character*)
qed

lemma *complement-part-not-part*: $x \neq u \implies P y (-x) \implies \neg P y x$
proof –
assume $x \neq u$
hence $\forall w. P w (-x) \longleftrightarrow \neg O w x$
by (*rule strong-complement-character*)
hence $y: P y (-x) \longleftrightarrow \neg O y x..$
moreover assume $P y (-x)$
ultimately have $\neg O y x..$
thus $\neg P y x$
by (*rule disjoint-implies-not-part*)
qed

lemma *complement-involution*: $x \neq u \implies x = -(-x)$
proof –
assume $x \neq u$
have $\neg P u x$
proof
assume $P u x$
with *universe-character* **have** $x = u$
by (*rule part-antisymmetry*)
with $\langle x \neq u \rangle$ **show** *False*..
qed
hence $\exists v. P v u \wedge \neg O v x$
by (*rule strong-supplementation*)
then obtain v **where** $v: P v u \wedge \neg O v x..$
hence $\neg O v x..$
hence $\exists v. \neg O v x..$

hence notx: $\forall w. P w (-x) \longleftrightarrow \neg O w x$
by (rule complement-character)
have $-x \neq u$
proof
assume $-x = u$
hence $\forall w. P w u \longleftrightarrow \neg O w x$ **using notx by** (rule subst)
hence $P x u \longleftrightarrow \neg O x x..$
hence $\neg O x x$ **using** universe-character..
thus False using overlap-reflexivity..
qed
have $\neg P u (-x)$
proof
assume $P u (-x)$
with universe-character **have** $-x = u$
by (rule part-antisymmetry)
with $\langle -x \neq u \rangle$ **show** False..
qed
hence $\exists v. P v u \wedge \neg O v (-x)$
by (rule strong-supplementation)
then obtain w **where** $w: P w u \wedge \neg O w (-x)..$
hence $\neg O w (-x)..$
hence $\exists v. \neg O v (-x)..$
hence notnotx: $\forall w. P w (-(-x)) \longleftrightarrow \neg O w (-x)$
by (rule complement-character)
hence $P x (-(-x)) \longleftrightarrow \neg O x (-x)..$
moreover have $\neg O x (-x)$
proof
assume $O x (-x)$
with overlap-eq **have** $\exists s. P s x \wedge P s (-x)..$
then obtain s **where** $s: P s x \wedge P s (-x)..$
hence $P s x..$
hence $O s x$ **by** (rule part-implies-overlap)
from notx have $P s (-x) \longleftrightarrow \neg O s x..$
moreover from s **have** $P s (-x)..$
ultimately have $\neg O s x..$
thus False using $\langle O s x \rangle..$
qed
ultimately have $P x (-(-x))..$
moreover have $P (-(-x)) x$
proof (rule ccontr)
assume $\neg P (-(-x)) x$
hence $\exists s. P s (-(-x)) \wedge \neg O s x$
by (rule strong-supplementation)
then obtain s **where** $s: P s (-(-x)) \wedge \neg O s x..$
hence $\neg O s x..$
from notnotx have $P s (-(-x)) \longleftrightarrow (\neg O s (-x))..$
moreover from s **have** $P s (-(-x))..$
ultimately have $\neg O s (-x)..$
from or-complement-overlap have $O s x \vee O s (-x)..$

thus $False$
proof
 assume $O\ s\ x$
 with $\langle \neg\ O\ s\ x \rangle$ **show** $False..$
next
 assume $O\ s\ (-x)$
 with $\langle \neg\ O\ s\ (-x) \rangle$ **show** $False..$
qed
qed
ultimately show $x = -(-x)$
 by $(rule\ part-antisymmetry)$
qed

lemma *part-complement-reversal*: $y \neq u \implies P\ x\ y \implies P\ (-y)\ (-x)$

proof –
 assume $y \neq u$
 hence ny : $\forall\ w.\ P\ w\ (-y) \longleftrightarrow \neg\ O\ w\ y$
 by $(rule\ strong-complement-character)$
 assume $P\ x\ y$
 have $x \neq u$
 proof
 assume $x = u$
 hence $P\ u\ y$ **using** $\langle P\ x\ y \rangle$ **by** $(rule\ subst)$
 with *universe-character* **have** $y = u$
 by $(rule\ part-antisymmetry)$
 with $\langle y \neq u \rangle$ **show** $False..$
 qed
 hence $\forall\ w.\ P\ w\ (-x) \longleftrightarrow \neg\ O\ w\ x$
 by $(rule\ strong-complement-character)$
 hence $P\ (-y)\ (-x) \longleftrightarrow \neg\ O\ (-y)\ x..$
 moreover have $\neg\ O\ (-y)\ x$
 proof
 assume $O\ (-y)\ x$
 with *overlap-eq* **have** $\exists\ v.\ P\ v\ (-y) \wedge P\ v\ x..$
 then obtain v **where** v : $P\ v\ (-y) \wedge P\ v\ x..$
 hence $P\ v\ (-y)..$
 from ny **have** $P\ v\ (-y) \longleftrightarrow \neg\ O\ v\ y..$
 hence $\neg\ O\ v\ y$ **using** $\langle P\ v\ (-y) \rangle..$
 moreover from v **have** $P\ v\ x..$
 hence $P\ v\ y$ **using** $\langle P\ x\ y \rangle$
 by $(rule\ part-transitivity)$
 hence $O\ v\ y$
 by $(rule\ part-implies-overlap)$
 ultimately show $False..$
 qed
 ultimately show $P\ (-y)\ (-x)..$
qed

lemma *complements-overlap*: $x \oplus y \neq u \implies O(-x)(-y)$

proof –

assume $x \oplus y \neq u$
hence $\exists z. \neg O z (x \oplus y)$
 by (rule something-disjoint)
then obtain z **where** $z: \neg O z (x \oplus y)$..
hence $\neg O z x$ **by** (rule first-summand-disjointness)
hence $P z (-x)$ **by** (rule complement-part)
moreover from z **have** $\neg O z y$
 by (rule second-summand-disjointness)
hence $P z (-y)$ **by** (rule complement-part)
ultimately show $O(-x)(-y)$
 by (rule overlap-intro)

qed

lemma *sum-complement-in-complement-product*:

$x \oplus y \neq u \implies P(-(x \oplus y))(-x \otimes -y)$

proof –

assume $x \oplus y \neq u$
hence $O(-x)(-y)$
 by (rule complements-overlap)
hence $\forall w. P w (-x \otimes -y) \longleftrightarrow (P w (-x) \wedge P w (-y))$
 by (rule product-character)
hence $P(-(x \oplus y))(-x \otimes -y) \longleftrightarrow (P(-(x \oplus y))(-x) \wedge P(-(x \oplus y))(-y))$..
moreover have $P(-(x \oplus y))(-x) \wedge P(-(x \oplus y))(-y)$
proof
 show $P(-(x \oplus y))(-x)$ **using** $\langle x \oplus y \neq u \rangle$ *first-summand-in-sum*
 by (rule part-complement-reversal)
next
 show $P(-(x \oplus y))(-y)$ **using** $\langle x \oplus y \neq u \rangle$ *second-summand-in-sum*
 by (rule part-complement-reversal)
qed
ultimately show $P(-(x \oplus y))(-x \otimes -y)$..
qed

lemma *complement-product-in-sum-complement*:

$x \oplus y \neq u \implies P(-x \otimes -y)(-(x \oplus y))$

proof –

assume $x \oplus y \neq u$
hence $\forall w. P w (-(x \oplus y)) \longleftrightarrow \neg O w (x \oplus y)$
 by (rule strong-complement-character)
hence $P(-x \otimes -y)(-(x \oplus y)) \longleftrightarrow (\neg O(-x \otimes -y)(x \oplus y))$..
moreover have $\neg O(-x \otimes -y)(x \oplus y)$
proof
 have $O(-x)(-y)$ **using** $\langle x \oplus y \neq u \rangle$ **by** (rule complements-overlap)
 hence $p: \forall v. P v ((-x) \otimes (-y)) \longleftrightarrow (P v (-x) \wedge P v (-y))$
 by (rule product-character)
 have $O(-x \otimes -y)(x \oplus y) \longleftrightarrow (O(-x \otimes -y) x \vee O(-x \otimes -y) y)$
 using *sum-character*..
qed

moreover assume $O(-x \otimes -y)(x \oplus y)$
ultimately have $O(-x \otimes -y) x \vee O(-x \otimes -y) y..$
thus *False*

proof

assume $O(-x \otimes -y) x$
with *overlap-eq* have $\exists v. P v(-x \otimes -y) \wedge P v x..$
then obtain v where $v: P v(-x \otimes -y) \wedge P v x..$
hence $P v(-x \otimes -y)..$
from p have $P v((-x) \otimes (-y)) \longleftrightarrow (P v(-x) \wedge P v(-y))..$
hence $P v(-x) \wedge P v(-y)$ using $\langle P v(-x \otimes -y) \rangle..$
hence $P v(-x)..$
have $x \neq u$ using $\langle x \oplus y \neq u \rangle$
by (*rule first-summand-not-universe*)
hence $\forall w. P w(-x) \longleftrightarrow \neg O w x$
by (*rule strong-complement-character*)
hence $P v(-x) \longleftrightarrow \neg O v x..$
hence $\neg O v x$ using $\langle P v(-x) \rangle..$
moreover from v have $P v x..$
hence $O v x$ by (*rule part-implies-overlap*)
ultimately show *False..*

next

assume $O(-x \otimes -y) y$
with *overlap-eq* have $\exists v. P v(-x \otimes -y) \wedge P v y..$
then obtain v where $v: P v(-x \otimes -y) \wedge P v y..$
hence $P v(-x \otimes -y)..$
from p have $P v((-x) \otimes (-y)) \longleftrightarrow (P v(-x) \wedge P v(-y))..$
hence $P v(-x) \wedge P v(-y)$ using $\langle P v(-x \otimes -y) \rangle..$
hence $P v(-y)..$
have $y \neq u$ using $\langle x \oplus y \neq u \rangle$
by (*rule second-summand-not-universe*)
hence $\forall w. P w(-y) \longleftrightarrow \neg O w y$
by (*rule strong-complement-character*)
hence $P v(-y) \longleftrightarrow \neg O v y..$
hence $\neg O v y$ using $\langle P v(-y) \rangle..$
moreover from v have $P v y..$
hence $O v y$ by (*rule part-implies-overlap*)
ultimately show *False..*

qed

qed

ultimately show $P(-x \otimes -y)(-(x \oplus y))..$

qed

theorem *sum-complement-is-complements-product:*

$$x \oplus y \neq u \implies -(x \oplus y) = (-x \otimes -y)$$

proof –

assume $x \oplus y \neq u$

show $-(x \oplus y) = (-x \otimes -y)$

proof (*rule part-antisymmetry*)

show $P(-(x \oplus y))(-x \otimes -y)$ using $\langle x \oplus y \neq u \rangle$

by (rule *sum-complement-in-complement-product*)
 show $P(-x \otimes -y)(-(x \oplus y))$ using $\langle x \oplus y \neq u \rangle$
 by (rule *complement-product-in-sum-complement*)
 qed
 qed

lemma *complement-sum-in-product-complement*:
 $Oxy \implies x \neq u \implies y \neq u \implies P((-x) \oplus (-y))(-(x \otimes y))$
proof –
 assume Oxy
 assume $x \neq u$
 assume $y \neq u$
 have $x \otimes y \neq u$
proof
 assume $x \otimes y = u$
 with $\langle Oxy \rangle$ have $x = u$
 by (rule *product-universe-implies-factor-universe*)
 with $\langle x \neq u \rangle$ show *False*..
 qed
 hence *notxty*: $\forall w. Pw(-(x \otimes y)) \longleftrightarrow \neg Ow(x \otimes y)$
 by (rule *strong-complement-character*)
 hence $P((-x) \oplus (-y))(-(x \otimes y)) \longleftrightarrow \neg O((-x) \oplus (-y))(x \otimes y)$..
 moreover have $\neg O((-x) \oplus (-y))(x \otimes y)$
proof
 from *sum-character* have
 $\forall w. Ow((-x) \oplus (-y)) \longleftrightarrow (Ow(-x) \vee Ow(-y))$.
 hence $O(x \otimes y)((-x) \oplus (-y)) \longleftrightarrow (O(x \otimes y)(-x) \vee O(x \otimes y)(-y))$..
 moreover assume $O((-x) \oplus (-y))(x \otimes y)$
 hence $O(x \otimes y)((-x) \oplus (-y))$ by (rule *overlap-symmetry*)
 ultimately have $O(x \otimes y)(-x) \vee O(x \otimes y)(-y)$..
 thus *False*
proof
 assume $O(x \otimes y)(-x)$
 with *overlap-eq* have $\exists v. Pv(x \otimes y) \wedge Pv(-x)$..
 then obtain v where $v: Pv(x \otimes y) \wedge Pv(-x)$..
 hence $Pv(-x)$..
 with $\langle x \neq u \rangle$ have $\neg Pv x$
 by (rule *complement-part-not-part*)
 moreover from v have $Pv(x \otimes y)$..
 with $\langle Oxy \rangle$ have $Pv x$ by (rule *product-part-in-first-factor*)
 ultimately show *False*..
 next
 assume $O(x \otimes y)(-y)$
 with *overlap-eq* have $\exists v. Pv(x \otimes y) \wedge Pv(-y)$..
 then obtain v where $v: Pv(x \otimes y) \wedge Pv(-y)$..
 hence $Pv(-y)$..
 with $\langle y \neq u \rangle$ have $\neg Pv y$
 by (rule *complement-part-not-part*)

moreover from v have $P v (x \otimes y)$.
 with $\langle O x y \rangle$ have $P v y$ by (rule product-part-in-second-factor)
 ultimately show *False*.
 qed
 qed
 ultimately show $P ((-x) \oplus (-y))(-x \otimes y)$.
 qed

lemma *product-complement-in-complements-sum*:
 $x \neq u \implies y \neq u \implies P(-(x \otimes y))((-x) \oplus (-y))$
proof –
 assume $x \neq u$
 hence $x = -(-x)$
 by (rule complement-involution)
 assume $y \neq u$
 hence $y = -(-y)$
 by (rule complement-involution)
 show $P(-(x \otimes y))((-x) \oplus (-y))$
proof cases
 assume $-x \oplus -y = u$
 thus $P(-(x \otimes y))((-x) \oplus (-y))$
 using *universe-character* by (rule *ssubst*)
 next
 assume $-x \oplus -y \neq u$
 hence $-x \oplus -y = -(-(-x \oplus -y))$
 by (rule complement-involution)
 moreover have $-(-x \oplus -y) = -(-x) \otimes -(-y)$
 using $\langle -x \oplus -y \neq u \rangle$
 by (rule *sum-complement-is-complements-product*)
 with $\langle x = -(-x) \rangle$ have $-(-x \oplus -y) = x \otimes -(-y)$
 by (rule *ssubst*)
 with $\langle y = -(-y) \rangle$ have $-(-x \oplus -y) = x \otimes y$
 by (rule *ssubst*)
 hence $P(-(x \otimes y))(-(-(-x \oplus -y)))$
 using *part-reflexivity* by (rule *subst*)
 ultimately show $P(-(x \otimes y))(-x \oplus -y)$
 by (rule *ssubst*)
 qed
 qed

theorem *complement-of-product-is-sum-of-complements*:
 $O x y \implies x \oplus y \neq u \implies -(x \otimes y) = (-x) \oplus (-y)$
proof –
 assume $O x y$
 assume $x \oplus y \neq u$
 show $-(x \otimes y) = (-x) \oplus (-y)$
proof (rule *part-antisymmetry*)
 have $x \neq u$ using $\langle x \oplus y \neq u \rangle$
 by (rule *first-summand-not-universe*)

```

have  $y \neq u$  using  $\langle x \oplus y \neq u \rangle$ 
  by (rule second-summand-not-universe)
show  $P(- (x \otimes y)) (- x \oplus - y)$ 
  using  $\langle x \neq u \rangle \langle y \neq u \rangle$  by (rule product-complement-in-complements-sum)
show  $P(- x \oplus - y) (- (x \otimes y))$ 
  using  $\langle O x y \rangle \langle x \neq u \rangle \langle y \neq u \rangle$  by (rule complement-sum-in-product-complement)
qed
qed

end

```

8 General Mereology

The theory of *general mereology* adds the axiom of fusion to ground mereology.³¹

```

locale  $GM = M +$ 
  assumes fusion:
     $\exists x. \varphi x \implies \exists z. \forall y. O y z \longleftrightarrow (\exists x. \varphi x \wedge O y x)$ 
begin

```

Fusion entails sum closure.

theorem *sum-closure*: $\exists z. \forall w. O w z \longleftrightarrow (O w a \vee O w b)$

proof –

```

  have  $a = a..$ 
  hence  $a = a \vee a = b..$ 
  hence  $\exists x. x = a \vee x = b..$ 
  hence  $(\exists z. \forall y. O y z \longleftrightarrow (\exists x. (x = a \vee x = b) \wedge O y x))$ 
    by (rule fusion)

```

then obtain z **where** z :

```

   $\forall y. O y z \longleftrightarrow (\exists x. (x = a \vee x = b) \wedge O y x)..$ 

```

```

have  $\forall w. O w z \longleftrightarrow (O w a \vee O w b)$ 

```

proof

```

  fix  $w$ 

```

```

  from  $z$  have  $w$ :  $O w z \longleftrightarrow (\exists x. (x = a \vee x = b) \wedge O w x)..$ 

```

```

  show  $O w z \longleftrightarrow (O w a \vee O w b)$ 

```

proof

```

  assume  $O w z$ 

```

```

  with  $w$  have  $\exists x. (x = a \vee x = b) \wedge O w x..$ 

```

```

  then obtain  $x$  where  $x$ :  $(x = a \vee x = b) \wedge O w x..$ 

```

```

  hence  $O w x..$ 

```

```

  from  $x$  have  $x = a \vee x = b..$ 

```

```

  thus  $O w a \vee O w b$ 

```

proof (rule disjE)

```

  assume  $x = a$ 

```

```

  hence  $O w a$  using  $\langle O w x \rangle$  by (rule subst)

```

³¹See [Simons, 1987] p. 36, [Varzi, 1996] p. 265 and [Casati and Varzi, 1999] p. 46.

```

      thus  $O w a \vee O w b..$ 
    next
      assume  $x = b$ 
      hence  $O w b$  using  $\langle O w x \rangle$  by (rule subst)
      thus  $O w a \vee O w b..$ 
    qed
  next
    assume  $O w a \vee O w b$ 
    hence  $\exists x. (x = a \vee x = b) \wedge O w x$ 
    proof (rule disjE)
      assume  $O w a$ 
      with  $\langle a = a \vee a = b \rangle$  have  $(a = a \vee a = b) \wedge O w a..$ 
      thus  $\exists x. (x = a \vee x = b) \wedge O w x..$ 
    next
      have  $b = b..$ 
      hence  $b = a \vee b = b..$ 
      moreover assume  $O w b$ 
      ultimately have  $(b = a \vee b = b) \wedge O w b..$ 
      thus  $\exists x. (x = a \vee x = b) \wedge O w x..$ 
    qed
  with  $w$  show  $O w z..$ 
qed
qed
thus  $\exists z. \forall w. O w z \longleftrightarrow (O w a \vee O w b)..$ 
qed

end

```

9 General Minimal Mereology

The theory of *general minimal mereology* adds general mereology to minimal mereology.³²

locale $GMM = GM + MM$
begin

It is natural to assume that just as closed minimal mereology and closed extensional mereology are the same theory, so are general minimal mereology and general extensional mereology.³³ But this is not the case, since the proof of strong supplementation in closed minimal mereology required the product closure axiom. However, in general minimal mereology, the fusion axiom does

³²See [Casati and Varzi, 1999] p. 46.

³³For this mistake see [Simons, 1987] p. 37 and [Casati and Varzi, 1999] p. 46. The mistake is corrected in [Pontow, 2004] and [Hovda, 2009]. For discussion of the significance of this issue see, for example, [Varzi, 2009] and [Cotnoir, 2016].

not entail the product closure axiom. So neither product closure nor strong supplementation are theorems.

lemma *product-closure*:

$$O x y \implies (\exists z. \forall v. P v z \longleftrightarrow P v x \wedge P v y)$$

nitpick [*expect = genuine*] **oops**

lemma *strong-supplementation*: $\neg P x y \implies (\exists z. P z x \wedge \neg O z y)$

nitpick [*expect = genuine*] **oops**

end

10 General Extensional Mereology

The theory of *general extensional mereology*, also known as *classical extensional mereology* adds general mereology to extensional mereology.³⁴

locale *GEM* = *GM* + *EM* +

assumes *sum-eq*: $x \oplus y = (\text{THE } z. \forall v. O v z \longleftrightarrow O v x \vee O v y)$

assumes *product-eq*:

$$x \otimes y = (\text{THE } z. \forall v. P v z \longleftrightarrow P v x \wedge P v y)$$

assumes *difference-eq*:

$$x \ominus y = (\text{THE } z. \forall w. P w z = (P w x \wedge \neg O w y))$$

assumes *complement-eq*: $\neg x = (\text{THE } z. \forall w. P w z \longleftrightarrow \neg O w x)$

assumes *universe-eq*: $u = (\text{THE } x. \forall y. P y x)$

assumes *fusion-eq*: $\exists x. F x \implies$

$$(\sigma x. F x) = (\text{THE } x. \forall y. O y x \longleftrightarrow (\exists z. F z \wedge O y z))$$

assumes *general-product-eq*: $(\pi x. F x) = (\sigma x. \forall y. F y \longrightarrow P x y)$

sublocale *GEM* \subseteq *GMM*

proof

qed

10.1 General Sums

context *GEM*

begin

lemma *fusion-intro*:

$$(\forall y. O y z \longleftrightarrow (\exists x. F x \wedge O y x)) \implies (\sigma x. F x) = z$$

proof –

assume *antecedent*: $(\forall y. O y z \longleftrightarrow (\exists x. F x \wedge O y x))$

hence $(\text{THE } x. \forall y. O y x \longleftrightarrow (\exists z. F z \wedge O y z)) = z$

proof (*rule the-equality*)

fix *a*

assume *a*: $(\forall y. O y a \longleftrightarrow (\exists x. F x \wedge O y x))$

³⁴For this axiomatization see [Varzi, 1996] p. 265 and [Casati and Varzi, 1999] p. 46.

have $\forall x. O x a \longleftrightarrow O x z$
proof
 fix b
 from antecedent have $O b z \longleftrightarrow (\exists x. F x \wedge O b x)..$
 moreover from a have $O b a \longleftrightarrow (\exists x. F x \wedge O b x)..$
 ultimately show $O b a \longleftrightarrow O b z$ **by** (rule *ssubst*)
qed
with overlap-extensionality show $a = z..$
qed
moreover from antecedent have $O z z \longleftrightarrow (\exists x. F x \wedge O z x)..$
hence $\exists x. F x \wedge O z x$ **using** *overlap-reflexivity..*
hence $\exists x. F x$ **by** *auto*
hence $(\sigma x. F x) = (THE x. \forall y. O y x \longleftrightarrow (\exists z. F z \wedge O y z))$
 by (rule *fusion-eq*)
ultimately show $(\sigma v. F v) = z$ **by** (rule *subst*)
qed

lemma fusion-idempotence: $(\sigma x. z = x) = z$

proof –
have $\forall y. O y z \longleftrightarrow (\exists x. z = x \wedge O y x)$
proof
 fix y
 show $O y z \longleftrightarrow (\exists x. z = x \wedge O y x)$
 proof
 assume $O y z$
 with refl have $z = z \wedge O y z..$
 thus $\exists x. z = x \wedge O y x..$
 next
 assume $\exists x. z = x \wedge O y x$
 then obtain x **where** $x: z = x \wedge O y x..$
 hence $z = x..$
 moreover from x have $O y x..$
 ultimately show $O y z$ **by** (rule *ssubst*)
 qed
qed
thus $(\sigma x. z = x) = z$
 by (rule *fusion-intro*)
qed

The whole is the sum of its parts.

lemma fusion-absorption: $(\sigma x. P x z) = z$

proof –
have $(\forall y. O y z \longleftrightarrow (\exists x. P x z \wedge O y x))$
proof
 fix y
 show $O y z \longleftrightarrow (\exists x. P x z \wedge O y x)$
 proof
 assume $O y z$
 with part-reflexivity have $P z z \wedge O y z..$

thus $\exists x. P x z \wedge O y x..$
next
assume $\exists x. P x z \wedge O y x$
then obtain x **where** $x: P x z \wedge O y x..$
hence $P x z..$
moreover from x **have** $O y x..$
ultimately show $O y z$ **by** (*rule overlap-monotonicity*)
qed
qed
thus $(\sigma x. P x z) = z$
by (*rule fusion-intro*)
qed

lemma part-fusion: $P w (\sigma v. P v x) \implies P w x$

proof –
assume $P w (\sigma v. P v x)$
with fusion-absorption show $P w x$ **by** (*rule subst*)
qed

lemma fusion-character:

$\exists x. F x \implies (\forall y. O y (\sigma v. F v) \longleftrightarrow (\exists x. F x \wedge O y x))$

proof –
assume $\exists x. F x$
hence $\exists z. \forall y. O y z \longleftrightarrow (\exists x. F x \wedge O y x)$
by (*rule fusion*)
then obtain z **where** $z: \forall y. O y z \longleftrightarrow (\exists x. F x \wedge O y x)..$
hence $(\sigma v. F v) = z$ **by** (*rule fusion-intro*)
thus $\forall y. O y (\sigma v. F v) \longleftrightarrow (\exists x. F x \wedge O y x)$ **using** z **by** (*rule ssubst*)
qed

The next lemma characterises fusions in terms of parthood.³⁵

lemma fusion-part-character: $\exists x. F x \implies$

$(\forall y. P y (\sigma v. F v) \longleftrightarrow (\forall w. P w y \longrightarrow (\exists v. F v \wedge O w v)))$

proof –
assume $(\exists x. F x)$
hence $F: \forall y. O y (\sigma v. F v) \longleftrightarrow (\exists x. F x \wedge O y x)$
by (*rule fusion-character*)
show $\forall y. P y (\sigma v. F v) \longleftrightarrow (\forall w. P w y \longrightarrow (\exists v. F v \wedge O w v))$
proof
fix y
show $P y (\sigma v. F v) \longleftrightarrow (\forall w. P w y \longrightarrow (\exists v. F v \wedge O w v))$
proof
assume $P y (\sigma v. F v)$
show $\forall w. P w y \longrightarrow (\exists v. F v \wedge O w v)$
proof
fix w
from F **have** $w: O w (\sigma v. F v) \longleftrightarrow (\exists x. F x \wedge O w x)..$

³⁵See [Pontow, 2004] pp. 202-9.

show $P w y \longrightarrow (\exists v. F v \wedge O w v)$
proof
 assume $P w y$
 hence $P w (\sigma v. F v)$ **using** $\langle P y (\sigma v. F v) \rangle$
 by (rule part-transitivity)
 hence $O w (\sigma v. F v)$ **by** (rule part-implies-overlap)
 with w **show** $\exists x. F x \wedge O w x..$

qed

qed

next

assume *right*: $\forall w. P w y \longrightarrow (\exists v. F v \wedge O w v)$

show $P y (\sigma v. F v)$

proof (rule ccontr)

assume $\neg P y (\sigma v. F v)$

hence $\exists v. P v y \wedge \neg O v (\sigma v. F v)$

by (rule strong-supplementation)

then obtain v where $v: P v y \wedge \neg O v (\sigma v. F v)..$

hence $\neg O v (\sigma v. F v)..$

from *right* have $P v y \longrightarrow (\exists w. F w \wedge O v w)..$

moreover from v have $P v y..$

ultimately have $\exists w. F w \wedge O v w..$

from F have $O v (\sigma v. F v) \longleftrightarrow (\exists x. F x \wedge O v x)..$

hence $O v (\sigma v. F v)$ **using** $\langle \exists w. F w \wedge O v w \rangle..$

with $\langle \neg O v (\sigma v. F v) \rangle$ **show** *False*..

qed

qed

qed

qed

lemma *fusion-part*: $F x \Longrightarrow P x (\sigma x. F x)$

proof –

assume $F x$

hence $\exists x. F x..$

hence $\forall y. P y (\sigma v. F v) \longleftrightarrow (\forall w. P w y \longrightarrow (\exists v. F v \wedge O w v))$

by (rule fusion-part-character)

hence $P x (\sigma v. F v) \longleftrightarrow (\forall w. P w x \longrightarrow (\exists v. F v \wedge O w v))..$

moreover have $\forall w. P w x \longrightarrow (\exists v. F v \wedge O w v)$

proof

fix w

show $P w x \longrightarrow (\exists v. F v \wedge O w v)$

proof

assume $P w x$

hence $O w x$ **by** (rule part-implies-overlap)

with $\langle F x \rangle$ **have** $F x \wedge O w x..$

thus $\exists v. F v \wedge O w v..$

qed

qed

ultimately show $P x (\sigma v. F v)..$

qed

lemma *common-part-fusion*:

$$O x y \implies (\forall w. P w (\sigma v. (P v x \wedge P v y)) \longleftrightarrow (P w x \wedge P w y))$$

proof –

assume $O x y$

with *overlap-eq* **have** $\exists z. (P z x \wedge P z y)..$

hence *sum*: $(\forall w. P w (\sigma v. (P v x \wedge P v y)) \longleftrightarrow$
 $(\forall z. P z w \longrightarrow (\exists v. (P v x \wedge P v y) \wedge O z v)))$

by (*rule fusion-part-character*)

show $\forall w. P w (\sigma v. (P v x \wedge P v y)) \longleftrightarrow (P w x \wedge P w y)$

proof

fix w

from *sum* **have** $w: P w (\sigma v. (P v x \wedge P v y))$

$\longleftrightarrow (\forall z. P z w \longrightarrow (\exists v. (P v x \wedge P v y) \wedge O z v))..$

show $P w (\sigma v. (P v x \wedge P v y)) \longleftrightarrow (P w x \wedge P w y)$

proof

assume $P w (\sigma v. (P v x \wedge P v y))$

with w **have** *bla*:

$(\forall z. P z w \longrightarrow (\exists v. (P v x \wedge P v y) \wedge O z v))..$

show $P w x \wedge P w y$

proof

show $P w x$

proof (*rule ccontr*)

assume $\neg P w x$

hence $\exists z. P z w \wedge \neg O z x$

by (*rule strong-supplementation*)

then obtain z **where** $z: P z w \wedge \neg O z x..$

hence $\neg O z x..$

from *bla* **have** $P z w \longrightarrow (\exists v. (P v x \wedge P v y) \wedge O z v)..$

moreover from z **have** $P z w..$

ultimately have $\exists v. (P v x \wedge P v y) \wedge O z v..$

then obtain v **where** $v: (P v x \wedge P v y) \wedge O z v..$

hence $P v x \wedge P v y..$

hence $P v x..$

moreover from v **have** $O z v..$

ultimately have $O z x$

by (*rule overlap-monotonicity*)

with $\langle \neg O z x \rangle$ **show** *False*..

qed

show $P w y$

proof (*rule ccontr*)

assume $\neg P w y$

hence $\exists z. P z w \wedge \neg O z y$

by (*rule strong-supplementation*)

then obtain z **where** $z: P z w \wedge \neg O z y..$

hence $\neg O z y..$

from *bla* **have** $P z w \longrightarrow (\exists v. (P v x \wedge P v y) \wedge O z v)..$

moreover from z **have** $P z w..$

ultimately have $\exists v. (P v x \wedge P v y) \wedge O z v..$

then obtain v where $v: (P v x \wedge P v y) \wedge O z v..$

hence $P v x \wedge P v y..$

hence $P v y..$

moreover from v have $O z v..$

ultimately have $O z y$

by (rule overlap-monotonicity)

with $\langle \neg O z y \rangle$ show *False..*

qed

qed

next

assume $P w x \wedge P w y$

thus $P w (\sigma v. (P v x \wedge P v y))$

by (rule fusion-part)

qed

qed

qed

theorem *product-closure*:

$O x y \implies (\exists z. \forall w. P w z \longleftrightarrow (P w x \wedge P w y))$

proof –

assume $O x y$

hence $(\forall w. P w (\sigma v. (P v x \wedge P v y)) \longleftrightarrow (P w x \wedge P w y))$

by (rule common-part-fusion)

thus $\exists z. \forall w. P w z \longleftrightarrow (P w x \wedge P w y)..$

qed

end

sublocale $GEM \subseteq CEM$

proof

fix $x y$

show $\exists z. \forall w. O w z = (O w x \vee O w y)$

using *sum-closure*.

show $x \oplus y = (THE z. \forall v. O v z \longleftrightarrow O v x \vee O v y)$

using *sum-eq*.

show $x \otimes y = (THE z. \forall v. P v z \longleftrightarrow P v x \wedge P v y)$

using *product-eq*.

show $O x y \implies (\exists z. \forall w. P w z = (P w x \wedge P w y))$

using *product-closure*.

qed

context GEM

begin

corollary $O x y \implies x \otimes y = (\sigma v. P v x \wedge P v y)$

proof –

assume $O x y$

hence $(\forall w. P w (\sigma v. (P v x \wedge P v y)) \longleftrightarrow (P w x \wedge P w y))$

by (rule common-part-fusion)

thus $x \otimes y = (\sigma v. P v x \wedge P v y)$ **by** (*rule product-intro*)
qed

lemma *disjoint-fusion*:

$\exists w. \neg O w x \implies (\forall w. P w (\sigma z. \neg O z x) \longleftrightarrow \neg O w x)$

proof –

assume *antecedent*: $\exists w. \neg O w x$

hence $\forall y. O y (\sigma v. \neg O v x) \longleftrightarrow (\exists v. \neg O v x \wedge O y v)$
by (*rule fusion-character*)

hence $x: O x (\sigma v. \neg O v x) \longleftrightarrow (\exists v. \neg O v x \wedge O x v)..$

show $\forall w. P w (\sigma z. \neg O z x) \longleftrightarrow \neg O w x$

proof

fix y

show $P y (\sigma z. \neg O z x) \longleftrightarrow \neg O y x$

proof

assume $P y (\sigma z. \neg O z x)$

moreover have $\neg O x (\sigma z. \neg O z x)$

proof

assume $O x (\sigma z. \neg O z x)$

with x **have** $(\exists v. \neg O v x \wedge O x v)..$

then obtain v **where** $v: \neg O v x \wedge O x v..$

hence $\neg O v x..$

from v **have** $O x v..$

hence $O v x$ **by** (*rule overlap-symmetry*)

with $\langle \neg O v x \rangle$ **show** *False*..

qed

ultimately have $\neg O x y$

by (*rule disjoint-demonotonicity*)

thus $\neg O y x$ **by** (*rule disjoint-symmetry*)

next

assume $\neg O y x$

thus $P y (\sigma v. \neg O v x)$

by (*rule fusion-part*)

qed

qed

qed

theorem *complement-closure*:

$\exists w. \neg O w x \implies (\exists z. \forall w. P w z \longleftrightarrow \neg O w x)$

proof –

assume $(\exists w. \neg O w x)$

hence $\forall w. P w (\sigma z. \neg O z x) \longleftrightarrow \neg O w x$

by (*rule disjoint-fusion*)

thus $\exists z. \forall w. P w z \longleftrightarrow \neg O w x..$

qed

end

sublocale $GEM \subseteq CEMC$

proof
fix $x y$
show $-x = (\text{THE } z. \forall w. P w z \longleftrightarrow \neg O w x)$
using *complement-eq.*
show $(\exists w. \neg O w x) \implies (\exists z. \forall w. P w z = (\neg O w x))$
using *complement-closure.*
show $x \ominus y = (\text{THE } z. \forall w. P w z = (P w x \wedge \neg O w y))$
using *difference-eq.*
show $u = (\text{THE } x. \forall y. P y x)$
using *universe-eq.*
qed

context *GEM*
begin

corollary *complement-is-disjoint-fusion:*

$\exists w. \neg O w x \implies -x = (\sigma z. \neg O z x)$

proof $-$

assume $\exists w. \neg O w x$

hence $\forall w. P w (\sigma z. \neg O z x) \longleftrightarrow \neg O w x$

by *(rule disjoint-fusion)*

thus $-x = (\sigma z. \neg O z x)$

by *(rule complement-intro)*

qed

theorem *strong-fusion:* $\exists x. F x \implies$

$\exists x. (\forall y. F y \longrightarrow P y x) \wedge (\forall y. P y x \longrightarrow (\exists z. F z \wedge O y z))$

proof $-$

assume $\exists x. F x$

have $(\forall y. F y \longrightarrow P y (\sigma v. F v)) \wedge$

$(\forall y. P y (\sigma v. F v) \longrightarrow (\exists z. F z \wedge O y z))$

proof

show $\forall y. F y \longrightarrow P y (\sigma v. F v)$

proof

fix y

show $F y \longrightarrow P y (\sigma v. F v)$

proof

assume $F y$

thus $P y (\sigma v. F v)$

by *(rule fusion-part)*

qed

qed

next

have $(\forall y. P y (\sigma v. F v) \longleftrightarrow$

$(\forall w. P w y \longrightarrow (\exists v. F v \wedge O w v)))$

using $\langle \exists x. F x \rangle$ **by** *(rule fusion-part-character)*

hence $P (\sigma v. F v) (\sigma v. F v) \longleftrightarrow (\forall w. P w (\sigma v. F v) \longrightarrow$

$(\exists v. F v \wedge O w v))..$

thus $\forall w. P w (\sigma v. F v) \longrightarrow (\exists v. F v \wedge O w v)$ **using** *part-reflexivity..*

qed
thus ?thesis..
qed

theorem *strong-fusion-eq*: $\exists x. F x \implies (\sigma x. F x) =$
 $(THE x. (\forall y. F y \longrightarrow P y x) \wedge (\forall y. P y x \longrightarrow (\exists z. F z \wedge O y z)))$

proof –
assume $\exists x. F x$
have $(THE x. (\forall y. F y \longrightarrow P y x) \wedge (\forall y. P y x \longrightarrow (\exists z. F z \wedge O y z))) = (\sigma x. F x)$
proof (*rule the-equality*)
show $(\forall y. F y \longrightarrow P y (\sigma x. F x)) \wedge (\forall y. P y (\sigma x. F x) \longrightarrow (\exists z. F z \wedge O y z))$
proof
show $\forall y. F y \longrightarrow P y (\sigma x. F x)$
proof
fix y
show $F y \longrightarrow P y (\sigma x. F x)$
proof
assume $F y$
thus $P y (\sigma x. F x)$
by (*rule fusion-part*)
qed
qed
next
show $(\forall y. P y (\sigma x. F x) \longrightarrow (\exists z. F z \wedge O y z))$
proof
fix y
show $P y (\sigma x. F x) \longrightarrow (\exists z. F z \wedge O y z)$
proof
have $\forall y. P y (\sigma v. F v) \longleftrightarrow (\forall w. P w y \longrightarrow (\exists v. F v \wedge O w v))$
using $\langle \exists x. F x \rangle$ **by** (*rule fusion-part-character*)
hence $P y (\sigma v. F v) \longleftrightarrow (\forall w. P w y \longrightarrow (\exists v. F v \wedge O w v))$
moreover assume $P y (\sigma x. F x)$
ultimately have $\forall w. P w y \longrightarrow (\exists v. F v \wedge O w v)$
hence $P y y \longrightarrow (\exists v. F v \wedge O y v)$
thus $\exists v. F v \wedge O y v$ **using** *part-reflexivity*
qed
qed
qed
next
fix x
assume $x: (\forall y. F y \longrightarrow P y x) \wedge (\forall y. P y x \longrightarrow (\exists z. F z \wedge O y z))$
have $\forall y. O y x \longleftrightarrow (\exists z. F z \wedge O y z)$
proof
fix y

show $O y x \longleftrightarrow (\exists z. F z \wedge O y z)$

proof

assume $O y x$

with *overlap-eq* **have** $\exists v. P v y \wedge P v x..$

then obtain v **where** $v: P v y \wedge P v x..$

from x **have** $\forall y. P y x \longrightarrow (\exists z. F z \wedge O y z)..$

hence $P v x \longrightarrow (\exists z. F z \wedge O v z)..$

moreover from v **have** $P v x..$

ultimately have $\exists z. F z \wedge O v z..$

then obtain z **where** $z: F z \wedge O v z..$

hence $F z..$

from v **have** $P v y..$

moreover from z **have** $O v z..$

hence $O z v$ **by** (*rule overlap-symmetry*)

ultimately have $O z y$ **by** (*rule overlap-monotonicity*)

hence $O y z$ **by** (*rule overlap-symmetry*)

with $\langle F z \rangle$ **have** $F z \wedge O y z..$

thus $\exists z. F z \wedge O y z..$

next

assume $\exists z. F z \wedge O y z$

then obtain z **where** $z: F z \wedge O y z..$

from x **have** $\forall y. F y \longrightarrow P y x..$

hence $F z \longrightarrow P z x..$

moreover from z **have** $F z..$

ultimately have $P z x..$

moreover from z **have** $O y z..$

ultimately show $O y x$

by (*rule overlap-monotonicity*)

qed

qed

hence $(\sigma x. F x) = x$

by (*rule fusion-intro*)

thus $x = (\sigma x. F x)..$

qed

thus *?thesis..*

qed

lemma *strong-sum-eq*: $x \oplus y = (THE z. (P x z \wedge P y z) \wedge (\forall w. P w z \longrightarrow O w x \vee O w y))$

proof –

have $(THE z. (P x z \wedge P y z) \wedge (\forall w. P w z \longrightarrow O w x \vee O w y))$
 $= x \oplus y$

proof (*rule the-equality*)

show $(P x (x \oplus y) \wedge P y (x \oplus y)) \wedge (\forall w. P w (x \oplus y) \longrightarrow O w x \vee O w y)$

proof

show $P x (x \oplus y) \wedge P y (x \oplus y)$

proof

show $P x (x \oplus y)$ **using** *first-summand-in-sum.*

show $P y (x \oplus y)$ **using** *second-summand-in-sum*.
qed
show $\forall w. P w (x \oplus y) \longrightarrow O w x \vee O w y$
proof
fix w
show $P w (x \oplus y) \longrightarrow O w x \vee O w y$
proof
assume $P w (x \oplus y)$
hence $O w (x \oplus y)$ **by** (*rule part-implies-overlap*)
with *sum-overlap* **show** $O w x \vee O w y$.
qed
qed
qed
fix z
assume $z: (P x z \wedge P y z) \wedge (\forall w. P w z \longrightarrow O w x \vee O w y)$
hence $P x z \wedge P y z$.
have $\forall w. O w z \longleftrightarrow (O w x \vee O w y)$
proof
fix w
show $O w z \longleftrightarrow (O w x \vee O w y)$
proof
assume $O w z$
with *overlap-eq* **have** $\exists v. P v w \wedge P v z$.
then obtain v **where** $v: P v w \wedge P v z$.
hence $P v w$.
from z **have** $\forall w. P w z \longrightarrow O w x \vee O w y$.
hence $P v z \longrightarrow O v x \vee O v y$.
moreover from v **have** $P v z$.
ultimately have $O v x \vee O v y$.
thus $O w x \vee O w y$
proof
assume $O v x$
hence $O x v$ **by** (*rule overlap-symmetry*)
with $\langle P v w \rangle$ **have** $O x w$ **by** (*rule overlap-monotonicity*)
hence $O w x$ **by** (*rule overlap-symmetry*)
thus $O w x \vee O w y$.
next
assume $O v y$
hence $O y v$ **by** (*rule overlap-symmetry*)
with $\langle P v w \rangle$ **have** $O y w$ **by** (*rule overlap-monotonicity*)
hence $O w y$ **by** (*rule overlap-symmetry*)
thus $O w x \vee O w y$.
qed
next
assume $O w x \vee O w y$
thus $O w z$
proof
from $\langle P x z \wedge P y z \rangle$ **have** $P x z$.
moreover assume $O w x$

ultimately show $O w z$
by (*rule overlap-monotonicity*)
next
from $\langle P x z \wedge P y z \rangle$ **have** $P y z..$
moreover assume $O w y$
ultimately show $O w z$
by (*rule overlap-monotonicity*)
qed
qed
qed
hence $x \oplus y = z$ **by** (*rule sum-intro*)
thus $z = x \oplus y..$
qed
thus *?thesis..*
qed

10.2 General Products

lemma *general-product-intro*: $(\forall y. O y x \longleftrightarrow (\exists z. (\forall y. F y \longrightarrow P z y) \wedge O y z)) \implies (\pi x. F x) = x$

proof –
assume $\forall y. O y x \longleftrightarrow (\exists z. (\forall y. F y \longrightarrow P z y) \wedge O y z)$
hence $(\sigma x. \forall y. F y \longrightarrow P x y) = x$ **by** (*rule fusion-intro*)
with *general-product-eq* **show** $(\pi x. F x) = x$ **by** (*rule ssubst*)
qed

lemma *general-product-idempotence*: $(\pi z. z = x) = x$

proof –
have $\forall y. O y x \longleftrightarrow (\exists z. (\forall y. y = x \longrightarrow P z y) \wedge O y z)$
by (*meson overlap-eq part-reflexivity part-transitivity*)
thus $(\pi z. z = x) = x$ **by** (*rule general-product-intro*)
qed

lemma *general-product-absorption*: $(\pi z. P x z) = x$

proof –
have $\forall y. O y x \longleftrightarrow (\exists z. (\forall y. P x y \longrightarrow P z y) \wedge O y z)$
by (*meson overlap-eq part-reflexivity part-transitivity*)
thus $(\pi z. P x z) = x$ **by** (*rule general-product-intro*)
qed

lemma *general-product-character*: $\exists z. \forall y. F y \longrightarrow P z y \implies \forall y. O y (\pi x. F x) \longleftrightarrow (\exists z. (\forall y. F y \longrightarrow P z y) \wedge O y z)$

proof –
assume $(\exists z. \forall y. F y \longrightarrow P z y)$
hence $(\exists x. \forall y. O y x \longleftrightarrow (\exists z. (\forall y. F y \longrightarrow P z y) \wedge O y z))$
by (*rule fusion*)
then obtain x **where** x :
 $\forall y. O y x \longleftrightarrow (\exists z. (\forall y. F y \longrightarrow P z y) \wedge O y z)..$
hence $(\pi x. F x) = x$ **by** (*rule general-product-intro*)

thus $(\forall y. O y (\pi x. F x) \longleftrightarrow (\exists z. (\forall y. F y \longrightarrow P z y) \wedge O y z))$
 using x by (rule *ssubst*)

qed

corollary $\neg (\exists x. F x) \implies u = (\pi x. F x)$

proof –

assume *antecedent*: $\neg (\exists x. F x)$

have $\forall y. P y (\pi x. F x)$

proof

fix y

show $P y (\pi x. F x)$

proof (rule *ccontr*)

assume $\neg P y (\pi x. F x)$

hence $\exists z. P z y \wedge \neg O z (\pi x. F x)$ by (rule *strong-supplementation*)

then obtain z where $z: P z y \wedge \neg O z (\pi x. F x)$..

hence $\neg O z (\pi x. F x)$..

from *antecedent* have *bla*: $\forall y. F y \longrightarrow P z y$ by *simp*

hence $\exists v. \forall y. F y \longrightarrow P v y$..

hence $(\forall y. O y (\pi x. F x) \longleftrightarrow (\exists z. (\forall y. F y \longrightarrow P z y) \wedge O y z))$ by (rule *general-product-character*)

hence $O z (\pi x. F x) \longleftrightarrow (\exists v. (\forall y. F y \longrightarrow P v y) \wedge O z v)$..

moreover from *bla* have $(\forall y. F y \longrightarrow P z y) \wedge O z z$

using *overlap-reflexivity*..

hence $\exists v. (\forall y. F y \longrightarrow P v y) \wedge O z v$..

ultimately have $O z (\pi x. F x)$..

with $\langle \neg O z (\pi x. F x) \rangle$ show *False*..

qed

qed

thus $u = (\pi x. F x)$

by (rule *universe-intro*)

qed

end

10.3 Strong Fusion

An alternative axiomatization of general extensional mereology adds a stronger version of the fusion axiom to minimal mereology, with correspondingly stronger definitions of sums and general sums.³⁶

locale *GEM1* = *MM* +

assumes *strong-fusion*: $\exists x. F x \implies \exists x. (\forall y. F y \longrightarrow P y x) \wedge (\forall y. P y x \longrightarrow (\exists z. F z \wedge O y z))$

assumes *strong-sum-eq*: $x \oplus y = (THE z. (P x z \wedge P y z) \wedge (\forall w. P w z \longrightarrow O w x \vee O w y))$

assumes *product-eq*:

$x \otimes y = (THE z. \forall v. P v z \longleftrightarrow P v x \wedge P v y)$

³⁶See [Tarski, 1983] p. 25. The proofs in this section are adapted from [Hovda, 2009].

assumes *difference-eq*:
 $x \ominus y = (\text{THE } z. \forall w. P w z = (P w x \wedge \neg O w y))$
assumes *complement-eq*: $\neg x = (\text{THE } z. \forall w. P w z \longleftrightarrow \neg O w x)$
assumes *universe-eq*: $u = (\text{THE } x. \forall y. P y x)$
assumes *strong-fusion-eq*: $\exists x. F x \implies (\sigma x. F x) = (\text{THE } x. (\forall y. F y \longrightarrow P y x) \wedge (\forall y. P y x \longrightarrow (\exists z. F z \wedge O y z)))$
assumes *general-product-eq*: $(\pi x. F x) = (\sigma x. \forall y. F y \longrightarrow P x y)$
begin

theorem *fusion*:

$\exists x. \varphi x \implies (\exists z. \forall y. O y z \longleftrightarrow (\exists x. \varphi x \wedge O y x))$

proof –

assume $\exists x. \varphi x$

hence $\exists x. (\forall y. \varphi y \longrightarrow P y x) \wedge (\forall y. P y x \longrightarrow (\exists z. \varphi z \wedge O y z))$ *by (rule strong-fusion)*

then obtain x **where** x :

$(\forall y. \varphi y \longrightarrow P y x) \wedge (\forall y. P y x \longrightarrow (\exists z. \varphi z \wedge O y z))..$

have $\forall y. O y x \longleftrightarrow (\exists v. \varphi v \wedge O y v)$

proof

fix y

show $O y x \longleftrightarrow (\exists v. \varphi v \wedge O y v)$

proof

assume $O y x$

with *overlap-eq* **have** $\exists z. P z y \wedge P z x..$

then obtain z **where** z : $P z y \wedge P z x..$

hence $P z x..$

from x **have** $\forall y. P y x \longrightarrow (\exists v. \varphi v \wedge O y v)..$

hence $P z x \longrightarrow (\exists v. \varphi v \wedge O z v)..$

hence $\exists v. \varphi v \wedge O z v$ **using** $\langle P z x \rangle..$

then obtain v **where** v : $\varphi v \wedge O z v..$

hence $O z v..$

with *overlap-eq* **have** $\exists w. P w z \wedge P w v..$

then obtain w **where** w : $P w z \wedge P w v..$

hence $P w z..$

moreover from z **have** $P z y..$

ultimately have $P w y$

by *(rule part-transitivity)*

moreover from w **have** $P w v..$

ultimately have $P w y \wedge P w v..$

hence $\exists w. P w y \wedge P w v..$

with *overlap-eq* **have** $O y v..$

from v **have** $\varphi v..$

hence $\varphi v \wedge O y v$ **using** $\langle O y v \rangle..$

thus $\exists v. \varphi v \wedge O y v..$

next

assume $\exists v. \varphi v \wedge O y v$

then obtain v **where** v : $\varphi v \wedge O y v..$

hence $O y v..$

with *overlap-eq* **have** $\exists z. P z y \wedge P z v..$

then obtain z **where** $z: P z y \wedge P z v..$

hence $P z v..$

from x **have** $\forall y. \varphi y \longrightarrow P y x..$

hence $\varphi v \longrightarrow P v x..$

moreover from v **have** $\varphi v..$

ultimately have $P v x..$

with $\langle P z v \rangle$ **have** $P z x$

by (*rule part-transitivity*)

from z **have** $P z y..$

thus $O y x$ **using** $\langle P z x \rangle$

by (*rule overlap-intro*)

qed

qed

thus $(\exists z. \forall y. O y z \longleftrightarrow (\exists x. \varphi x \wedge O y x))..$

qed

lemma pair: $\exists v. (\forall w. (w = x \vee w = y) \longrightarrow P w v) \wedge (\forall w. P w v \longrightarrow (\exists z. (z = x \vee z = y) \wedge O w z))$

proof –

have $x = x..$

hence $x = x \vee x = y..$

hence $\exists v. v = x \vee v = y..$

thus *?thesis*

by (*rule strong-fusion*)

qed

lemma or-id: $(v = x \vee v = y) \wedge O w v \implies O w x \vee O w y$

proof –

assume $v: (v = x \vee v = y) \wedge O w v$

hence $O w v..$

from v **have** $v = x \vee v = y..$

thus $O w x \vee O w y$

proof

assume $v = x$

hence $O w x$ **using** $\langle O w v \rangle$ **by** (*rule subst*)

thus $O w x \vee O w y..$

next

assume $v = y$

hence $O w y$ **using** $\langle O w v \rangle$ **by** (*rule subst*)

thus $O w x \vee O w y..$

qed

qed

lemma strong-sum-closure:

$\exists z. (P x z \wedge P y z) \wedge (\forall w. P w z \longrightarrow O w x \vee O w y)$

proof –

from pair obtain z **where** $z: (\forall w. (w = x \vee w = y) \longrightarrow P w z) \wedge (\forall w. P w z \longrightarrow (\exists v. (v = x \vee v = y) \wedge O w v))..$

have $(P x z \wedge P y z) \wedge (\forall w. P w z \longrightarrow O w x \vee O w y)$

proof
from z **have** $allw: \forall w. (w = x \vee w = y) \longrightarrow P w z..$
hence $x = x \vee x = y \longrightarrow P x z..$
moreover have $x = x \vee x = y$ **using** *refl..*
ultimately have $P x z..$
from $allw$ **have** $y = x \vee y = y \longrightarrow P y z..$
moreover have $y = x \vee y = y$ **using** *refl..*
ultimately have $P y z..$
with $\langle P x z \rangle$ **show** $P x z \wedge P y z..$
next
show $\forall w. P w z \longrightarrow O w x \vee O w y$
proof
fix w
show $P w z \longrightarrow O w x \vee O w y$
proof
assume $P w z$
from z **have** $\forall w. P w z \longrightarrow (\exists v. (v = x \vee v = y) \wedge O w v)..$
hence $P w z \longrightarrow (\exists v. (v = x \vee v = y) \wedge O w v)..$
hence $\exists v. (v = x \vee v = y) \wedge O w v$ **using** $\langle P w z \rangle..$
then obtain v **where** $v: (v = x \vee v = y) \wedge O w v..$
thus $O w x \vee O w y$ **by** (*rule or-id*)
qed
qed
qed
thus *?thesis..*
qed

end

sublocale $GEM1 \subseteq GMM$

proof
fix $x y \varphi$
show $(\exists x. \varphi x) \implies (\exists z. \forall y. O y z \longleftrightarrow (\exists x. \varphi x \wedge O y x))$ **using**
fusion.
qed

context $GEM1$

begin

lemma *least-upper-bound:*

assumes $sf:$

$((\forall y. F y \longrightarrow P y x) \wedge (\forall y. P y x \longrightarrow (\exists z. F z \wedge O y z)))$

shows *lub:*

$(\forall y. F y \longrightarrow P y x) \wedge (\forall z. (\forall y. F y \longrightarrow P y z) \longrightarrow P x z)$

proof

from sf **show** $\forall y. F y \longrightarrow P y x..$

next

show $(\forall z. (\forall y. F y \longrightarrow P y z) \longrightarrow P x z)$

proof

fix z
show $(\forall y. F y \longrightarrow P y z) \longrightarrow P x z$
proof
assume $z: \forall y. F y \longrightarrow P y z$
from pair obtain v **where** $v: (\forall w. (w = x \vee w = z) \longrightarrow P w v)$
 $\wedge (\forall w. P w v \longrightarrow (\exists y. (y = x \vee y = z) \wedge O w y))..$
hence left: $(\forall w. (w = x \vee w = z) \longrightarrow P w v)..$
hence $(x = x \vee x = z) \longrightarrow P x v..$
moreover have $x = x \vee x = z$ **using** *refl.*
ultimately have $P x v..$
have $z = v$
proof (*rule ccontr*)
assume $z \neq v$
from left have $z = x \vee z = z \longrightarrow P z v..$
moreover have $z = x \vee z = z$ **using** *refl.*
ultimately have $P z v..$
hence $P z v \wedge z \neq v$ **using** $\langle z \neq v \rangle..$
with nip-eq have $PP z v..$
hence $\exists w. P w v \wedge \neg O w z$ **by** (*rule weak-supplementation*)
then obtain w **where** $w: P w v \wedge \neg O w z..$
hence $P w v..$
from v **have right:**
 $\forall w. P w v \longrightarrow (\exists y. (y = x \vee y = z) \wedge O w y)..$
hence $P w v \longrightarrow (\exists y. (y = x \vee y = z) \wedge O w y)..$
hence $\exists y. (y = x \vee y = z) \wedge O w y$ **using** $\langle P w v \rangle..$
then obtain s **where** $s: (s = x \vee s = z) \wedge O w s..$
hence $s = x \vee s = z..$
thus *False*
proof
assume $s = x$
moreover from s **have** $O w s..$
ultimately have $O w x$ **by** (*rule subst*)
with overlap-eq have $\exists t. P t w \wedge P t x..$
then obtain t **where** $t: P t w \wedge P t x..$
hence $P t x..$
from sf have $(\forall y. P y x \longrightarrow (\exists z. F z \wedge O y z))..$
hence $P t x \longrightarrow (\exists z. F z \wedge O t z)..$
hence $\exists z. F z \wedge O t z$ **using** $\langle P t x \rangle..$
then obtain a **where** $a: F a \wedge O t a..$
hence $F a..$
from sf have $ub: \forall y. F y \longrightarrow P y x..$
hence $F a \longrightarrow P a x..$
hence $P a x$ **using** $\langle F a \rangle..$
moreover from a **have** $O t a..$
ultimately have $O t x$
by (*rule overlap-monotonicity*)
from t **have** $P t w..$
moreover have $O z t$
proof –

from z **have** $F a \longrightarrow P a z$..
moreover from a **have** $F a$..
ultimately have $P a z$..
moreover from a **have** $O t a$..
ultimately have $O t z$
by (*rule overlap-monotonicity*)
thus $O z t$ **by** (*rule overlap-symmetry*)
qed
ultimately have $O z w$
by (*rule overlap-monotonicity*)
hence $O w z$ **by** (*rule overlap-symmetry*)
from w **have** $\neg O w z$..
thus *False* **using** $\langle O w z \rangle$..
next
assume $s = z$
moreover from s **have** $O w s$..
ultimately have $O w z$ **by** (*rule subst*)
from w **have** $\neg O w z$..
thus *False* **using** $\langle O w z \rangle$..
qed
qed
thus $P x z$ **using** $\langle P x v \rangle$ **by** (*rule ssubst*)
qed
qed
qed

corollary *strong-fusion-intro*: $(\forall y. F y \longrightarrow P y x) \wedge (\forall y. P y x \longrightarrow (\exists z. F z \wedge O y z)) \implies (\sigma x. F x) = x$

proof –

assume antecedent: $(\forall y. F y \longrightarrow P y x) \wedge (\forall y. P y x \longrightarrow (\exists z. F z \wedge O y z))$

with least-upper-bound have *lubx*:

$(\forall y. F y \longrightarrow P y x) \wedge (\forall z. (\forall y. F y \longrightarrow P y z) \longrightarrow P x z)$.

from antecedent have $\forall y. P y x \longrightarrow (\exists z. F z \wedge O y z)$..

hence $P x x \longrightarrow (\exists z. F z \wedge O x z)$..

hence $\exists z. F z \wedge O x z$ **using** *part-reflexivity*..

then obtain z **where** $z: F z \wedge O x z$..

hence $F z$..

hence $\exists z. F z$..

hence $(\sigma x. F x) = (\text{THE } x. (\forall y. F y \longrightarrow P y x) \wedge (\forall y. P y x \longrightarrow (\exists z. F z \wedge O y z)))$ **by** (*rule strong-fusion-eq*)

moreover have $(\text{THE } x. (\forall y. F y \longrightarrow P y x) \wedge (\forall y. P y x \longrightarrow (\exists z. F z \wedge O y z))) = x$

proof (*rule the-equality*)

show $(\forall y. F y \longrightarrow P y x) \wedge (\forall y. P y x \longrightarrow (\exists z. F z \wedge O y z))$

using *antecedent*.

next

fix w

assume w :

$(\forall y. F y \longrightarrow P y w) \wedge (\forall y. P y w \longrightarrow (\exists z. F z \wedge O y z))$
with *least-upper-bound* **have** *lubw*:
 $(\forall y. F y \longrightarrow P y w) \wedge (\forall z. (\forall y. F y \longrightarrow P y z) \longrightarrow P w z)$.
hence $(\forall z. (\forall y. F y \longrightarrow P y z) \longrightarrow P w z)$..
hence $(\forall y. F y \longrightarrow P y x) \longrightarrow P w x$..
moreover from antecedent have $\forall y. F y \longrightarrow P y x$..
ultimately have $P w x$..
from lubx have $(\forall z. (\forall y. F y \longrightarrow P y z) \longrightarrow P x z)$..
hence $(\forall y. F y \longrightarrow P y w) \longrightarrow P x w$..
moreover from lubw have $(\forall y. F y \longrightarrow P y w)$..
ultimately have $P x w$..
with $\langle P w x \rangle$ **show** $w = x$
by (*rule part-antisymmetry*)
qed
ultimately show $(\sigma x. F x) = x$ **by** (*rule ssubst*)
qed

lemma strong-fusion-character: $\exists x. F x \implies ((\forall y. F y \longrightarrow P y (\sigma x. F x)) \wedge (\forall y. P y (\sigma x. F x) \longrightarrow (\exists z. F z \wedge O y z)))$
proof –
assume $\exists x. F x$
hence $(\exists x. (\forall y. F y \longrightarrow P y x) \wedge (\forall y. P y x \longrightarrow (\exists z. F z \wedge O y z)))$ **by** (*rule strong-fusion*)
then obtain x where x:
 $(\forall y. F y \longrightarrow P y x) \wedge (\forall y. P y x \longrightarrow (\exists z. F z \wedge O y z))$..
hence $(\sigma x. F x) = x$ **by** (*rule strong-fusion-intro*)
thus ?thesis using x by (*rule ssubst*)
qed

lemma F-in: $\exists x. F x \implies (\forall y. F y \longrightarrow P y (\sigma x. F x))$
proof –
assume $\exists x. F x$
hence $((\forall y. F y \longrightarrow P y (\sigma x. F x)) \wedge (\forall y. P y (\sigma x. F x) \longrightarrow (\exists z. F z \wedge O y z)))$ **by** (*rule strong-fusion-character*)
thus $\forall y. F y \longrightarrow P y (\sigma x. F x)$..
qed

lemma parts-overlap-Fs:
 $\exists x. F x \implies (\forall y. P y (\sigma x. F x) \longrightarrow (\exists z. F z \wedge O y z))$
proof –
assume $\exists x. F x$
hence $((\forall y. F y \longrightarrow P y (\sigma x. F x)) \wedge (\forall y. P y (\sigma x. F x) \longrightarrow (\exists z. F z \wedge O y z)))$ **by** (*rule strong-fusion-character*)
thus $(\forall y. P y (\sigma x. F x) \longrightarrow (\exists z. F z \wedge O y z))$..
qed

lemma in-strong-fusion: $P z (\sigma x. z = x)$
proof –
have $\exists y. z = y$ **using** *refl*..

hence $\forall y. z = y \longrightarrow P y (\sigma x. z = x)$
by (rule *F-in*)
hence $z = z \longrightarrow P z (\sigma x. z = x)$..
thus $P z (\sigma x. z = x)$ **using** *refl.*
qed

lemma *strong-fusion-in*: $P (\sigma x. z = x) z$

proof –

have $\exists y. z = y$ **using** *refl.*

hence *sf*:

$(\forall y. z = y \longrightarrow P y (\sigma x. z = x)) \wedge (\forall y. P y (\sigma x. z = x) \longrightarrow (\exists v. z = v \wedge O y v))$

by (rule *strong-fusion-character*)

with *least-upper-bound* **have** *lub*: $(\forall y. z = y \longrightarrow P y (\sigma x. z = x)) \wedge (\forall v. (\forall y. z = y \longrightarrow P y v) \longrightarrow P (\sigma x. z = x) v)$.

hence $(\forall v. (\forall y. z = y \longrightarrow P y v) \longrightarrow P (\sigma x. z = x) v)$..
hence $(\forall y. z = y \longrightarrow P y z) \longrightarrow P (\sigma x. z = x) z$..
moreover **have** $(\forall y. z = y \longrightarrow P y z)$

proof

fix y

show $z = y \longrightarrow P y z$

proof

assume $z = y$

thus $P y z$ **using** *part-reflexivity* **by** (rule *subst*)

qed

qed

ultimately show $P (\sigma x. z = x) z$..
qed

lemma *strong-fusion-idempotence*: $(\sigma x. z = x) = z$

using *strong-fusion-in in-strong-fusion* **by** (rule *part-antisymmetry*)

10.4 Strong Sums

lemma *pair-fusion*: $(P x z \wedge P y z) \wedge (\forall w. P w z \longrightarrow O w x \vee O w y) \longrightarrow (\sigma z. z = x \vee z = y) = z$

proof

assume $z: (P x z \wedge P y z) \wedge (\forall w. P w z \longrightarrow O w x \vee O w y)$

have $(\forall v. v = x \vee v = y \longrightarrow P v z) \wedge (\forall v. P v z \longrightarrow (\exists z. (z = x \vee z = y) \wedge O v z))$

proof

show $\forall v. v = x \vee v = y \longrightarrow P v z$

proof

fix w

from z **have** $P x z \wedge P y z$..
show $w = x \vee w = y \longrightarrow P w z$

proof

assume $w = x \vee w = y$

thus $P w z$

proof
 assume $w = x$
 moreover from $\langle P x z \wedge P y z \rangle$ have $P x z..$
 ultimately show $P w z$ by (rule *ssubst*)
 next
 assume $w = y$
 moreover from $\langle P x z \wedge P y z \rangle$ have $P y z..$
 ultimately show $P w z$ by (rule *ssubst*)
 qed
 qed
 qed
 show $\forall v. P v z \longrightarrow (\exists z. (z = x \vee z = y) \wedge O v z)$
proof
 fix v
 show $P v z \longrightarrow (\exists z. (z = x \vee z = y) \wedge O v z)$
proof
 assume $P v z$
 from z have $\forall w. P w z \longrightarrow O w x \vee O w y..$
 hence $P v z \longrightarrow O v x \vee O v y..$
 hence $O v x \vee O v y$ using $\langle P v z \rangle..$
 thus $\exists z. (z = x \vee z = y) \wedge O v z$
proof
 assume $O v x$
 have $x = x \vee x = y$ using *refl..*
 hence $(x = x \vee x = y) \wedge O v x$ using $\langle O v x \rangle..$
 thus $\exists z. (z = x \vee z = y) \wedge O v z..$
 next
 assume $O v y$
 have $y = x \vee y = y$ using *refl..*
 hence $(y = x \vee y = y) \wedge O v y$ using $\langle O v y \rangle..$
 thus $\exists z. (z = x \vee z = y) \wedge O v z..$
 qed
 qed
 qed
 thus $(\sigma z. z = x \vee z = y) = z$
 by (rule *strong-fusion-intro*)
 qed

corollary strong-sum-fusion: $x \oplus y = (\sigma z. z = x \vee z = y)$
proof –
 have (THE $z. (P x z \wedge P y z) \wedge$
 $(\forall w. P w z \longrightarrow O w x \vee O w y)) = (\sigma z. z = x \vee z = y)$
proof (rule *the-equality*)
 have $x = x \vee x = y$ using *refl..*
 hence *exz:* $\exists z. z = x \vee z = y..$
 hence *allw:* $(\forall w. w = x \vee w = y \longrightarrow P w (\sigma z. z = x \vee z = y))$
 by (rule *F-in*)
 show $(P x (\sigma z. z = x \vee z = y) \wedge P y (\sigma z. z = x \vee z = y)) \wedge$

$(\forall w. P w (\sigma z. z = x \vee z = y) \longrightarrow O w x \vee O w y)$
proof
show $(P x (\sigma z. z = x \vee z = y) \wedge P y (\sigma z. z = x \vee z = y))$
proof
from *allw* **have** $x = x \vee x = y \longrightarrow P x (\sigma z. z = x \vee z = y)$..
thus $P x (\sigma z. z = x \vee z = y)$
using $\langle x = x \vee x = y \rangle$..
next
from *allw* **have** $y = x \vee y = y \longrightarrow P y (\sigma z. z = x \vee z = y)$..
moreover **have** $y = x \vee y = y$
using *refl*..
ultimately **show** $P y (\sigma z. z = x \vee z = y)$..
qed
next
show $\forall w. P w (\sigma z. z = x \vee z = y) \longrightarrow O w x \vee O w y$
proof
fix w
show $P w (\sigma z. z = x \vee z = y) \longrightarrow O w x \vee O w y$
proof
have $\forall v. P v (\sigma z. z = x \vee z = y) \longrightarrow (\exists z. (z = x \vee z = y) \wedge O v z)$ **using** *exz* **by** (*rule parts-overlap-Fs*)
hence $P w (\sigma z. z = x \vee z = y) \longrightarrow (\exists z. (z = x \vee z = y) \wedge O w z)$..
moreover **assume** $P w (\sigma z. z = x \vee z = y)$
ultimately **have** $(\exists z. (z = x \vee z = y) \wedge O w z)$..
then **obtain** z **where** $z: (z = x \vee z = y) \wedge O w z$..
thus $O w x \vee O w y$ **by** (*rule or-id*)
qed
qed
qed
next
fix z
assume $z: (P x z \wedge P y z) \wedge (\forall w. P w z \longrightarrow O w x \vee O w y)$
with *pair-fusion* **have** $(\sigma z. z = x \vee z = y) = z$..
thus $z = (\sigma z. z = x \vee z = y)$..
qed
with *strong-sum-eq* **show** $x \oplus y = (\sigma z. z = x \vee z = y)$
by (*rule ssubst*)
qed

corollary *strong-sum-intro*:
 $(P x z \wedge P y z) \wedge (\forall w. P w z \longrightarrow O w x \vee O w y) \longrightarrow x \oplus y = z$
proof
assume $z: (P x z \wedge P y z) \wedge (\forall w. P w z \longrightarrow O w x \vee O w y)$
with *pair-fusion* **have** $(\sigma z. z = x \vee z = y) = z$..
with *strong-sum-fusion* **show** $(x \oplus y) = z$
by (*rule ssubst*)
qed

corollary strong-sum-character: $(P x (x \oplus y) \wedge P y (x \oplus y)) \wedge (\forall w. P w (x \oplus y) \longrightarrow O w x \vee O w y)$

proof –

from *strong-sum-closure* obtain z where z :

$(P x z \wedge P y z) \wedge (\forall w. P w z \longrightarrow O w x \vee O w y)$..

with *strong-sum-intro* have $x \oplus y = z$..

thus *thesis* using z by (rule *ssubst*)

qed

corollary summands-in: $(P x (x \oplus y) \wedge P y (x \oplus y))$

using *strong-sum-character*..

corollary first-summand-in: $P x (x \oplus y)$ using *summands-in*..

corollary second-summand-in: $P y (x \oplus y)$ using *summands-in*..

corollary sum-part-overlap: $(\forall w. P w (x \oplus y) \longrightarrow O w x \vee O w y)$

using *strong-sum-character*..

lemma strong-sum-absorption: $y = (x \oplus y) \Longrightarrow P x y$

proof –

assume $y = (x \oplus y)$

thus $P x y$ using *first-summand-in* by (rule *ssubst*)

qed

theorem strong-supplementation: $\neg P x y \Longrightarrow (\exists z. P z x \wedge \neg O z y)$

proof –

assume $\neg P x y$

have $\neg (\forall z. P z x \longrightarrow O z y)$

proof

assume $z: \forall z. P z x \longrightarrow O z y$

have $(\forall v. y = v \longrightarrow P v (x \oplus y)) \wedge$

$(\forall v. P v (x \oplus y) \longrightarrow (\exists z. y = z \wedge O v z))$

proof

show $\forall v. y = v \longrightarrow P v (x \oplus y)$

proof

fix v

show $y = v \longrightarrow P v (x \oplus y)$

proof

assume $y = v$

thus $P v (x \oplus y)$

using *second-summand-in* by (rule *subst*)

qed

qed

show $\forall v. P v (x \oplus y) \longrightarrow (\exists z. y = z \wedge O v z)$

proof

fix v

show $P v (x \oplus y) \longrightarrow (\exists z. y = z \wedge O v z)$

proof

assume $P v (x \oplus y)$
moreover from *sum-part-overlap* **have**
 $P v (x \oplus y) \longrightarrow O v x \vee O v y..$
ultimately have $O v x \vee O v y$ **by** (*rule rev-mp*)
hence $O v y$
proof
assume $O v x$
with *overlap-eq* **have** $\exists w. P w v \wedge P w x..$
then obtain w **where** $w: P w v \wedge P w x..$
from z **have** $P w x \longrightarrow O w y..$
moreover from w **have** $P w x..$
ultimately have $O w y..$
with *overlap-eq* **have** $\exists t. P t w \wedge P t y..$
then obtain t **where** $t: P t w \wedge P t y..$
hence $P t w..$
moreover from w **have** $P w v..$
ultimately have $P t v$
by (*rule part-transitivity*)
moreover from t **have** $P t y..$
ultimately show $O v y$
by (*rule overlap-intro*)
next
assume $O v y$
thus $O v y.$
qed
with *refl* **have** $y = y \wedge O v y..$
thus $\exists z. y = z \wedge O v z..$
qed
qed
qed
hence $(\sigma z. y = z) = (x \oplus y)$ **by** (*rule strong-fusion-intro*)
with *strong-fusion-idempotence* **have** $y = x \oplus y$ **by** (*rule subst*)
hence $P x y$ **by** (*rule strong-sum-absorption*)
with $\langle \neg P x y \rangle$ **show** *False..*
qed
thus $\exists z. P z x \wedge \neg O z y$ **by** *simp*
qed

lemma *sum-character*: $\forall v. O v (x \oplus y) \longleftrightarrow (O v x \vee O v y)$

proof

fix v

show $O v (x \oplus y) \longleftrightarrow (O v x \vee O v y)$

proof

assume $O v (x \oplus y)$

with *overlap-eq* **have** $\exists w. P w v \wedge P w (x \oplus y)..$

then obtain w **where** $w: P w v \wedge P w (x \oplus y)..$

hence $P w v..$

have $P w (x \oplus y) \longrightarrow O w x \vee O w y$ **using** *sum-part-overlap..*

moreover from w **have** $P w (x \oplus y)..$

ultimately have $O w x \vee O w y..$

thus $O v x \vee O v y$

proof

assume $O w x$

hence $O x w$

by (rule overlap-symmetry)

with $\langle P w v \rangle$ have $O x v$

by (rule overlap-monotonicity)

hence $O v x$

by (rule overlap-symmetry)

thus $O v x \vee O v y..$

next

assume $O w y$

hence $O y w$

by (rule overlap-symmetry)

with $\langle P w v \rangle$ have $O y v$

by (rule overlap-monotonicity)

hence $O v y$ by (rule overlap-symmetry)

thus $O v x \vee O v y..$

qed

next

assume $O v x \vee O v y$

thus $O v (x \oplus y)$

proof

assume $O v x$

with *overlap-eq* have $\exists w. P w v \wedge P w x..$

then obtain w where $w: P w v \wedge P w x..$

hence $P w v..$

moreover from w have $P w x..$

hence $P w (x \oplus y)$ using *first-summand-in*

by (rule part-transitivity)

ultimately show $O v (x \oplus y)$

by (rule overlap-intro)

next

assume $O v y$

with *overlap-eq* have $\exists w. P w v \wedge P w y..$

then obtain w where $w: P w v \wedge P w y..$

hence $P w v..$

moreover from w have $P w y..$

hence $P w (x \oplus y)$ using *second-summand-in*

by (rule part-transitivity)

ultimately show $O v (x \oplus y)$

by (rule overlap-intro)

qed

qed

qed

lemma *sum-eq*: $x \oplus y = (\text{THE } z. \forall v. O v z = (O v x \vee O v y))$

proof –

have (*THE* $z. \forall v. O v z \longleftrightarrow (O v x \vee O v y) = x \oplus y$)
proof (*rule the-equality*)
show $\forall v. O v (x \oplus y) \longleftrightarrow (O v x \vee O v y)$ **using** *sum-character*.
next
fix z
assume $z: \forall v. O v z \longleftrightarrow (O v x \vee O v y)$
have $(P x z \wedge P y z) \wedge (\forall w. P w z \longrightarrow O w x \vee O w y)$
proof
show $P x z \wedge P y z$
proof
show $P x z$
proof (*rule ccontr*)
assume $\neg P x z$
hence $\exists v. P v x \wedge \neg O v z$
by (*rule strong-supplementation*)
then obtain v **where** $v: P v x \wedge \neg O v z$.
hence $\neg O v z$.
from z **have** $O v z \longleftrightarrow (O v x \vee O v y)$.
moreover from v **have** $P v x$.
hence $O v x$ **by** (*rule part-implies-overlap*)
hence $O v x \vee O v y$.
ultimately have $O v z$.
with $\langle \neg O v z \rangle$ **show** *False*.
qed
next
show $P y z$
proof (*rule ccontr*)
assume $\neg P y z$
hence $\exists v. P v y \wedge \neg O v z$
by (*rule strong-supplementation*)
then obtain v **where** $v: P v y \wedge \neg O v z$.
hence $\neg O v z$.
from z **have** $O v z \longleftrightarrow (O v x \vee O v y)$.
moreover from v **have** $P v y$.
hence $O v y$ **by** (*rule part-implies-overlap*)
hence $O v x \vee O v y$.
ultimately have $O v z$.
with $\langle \neg O v z \rangle$ **show** *False*.
qed
qed
show $\forall w. P w z \longrightarrow (O w x \vee O w y)$
proof
fix w
show $P w z \longrightarrow (O w x \vee O w y)$
proof
from z **have** $O w z \longleftrightarrow O w x \vee O w y$.
moreover assume $P w z$
hence $O w z$ **by** (*rule part-implies-overlap*)
ultimately show $O w x \vee O w y$.

qed
 qed
 qed
 with *strong-sum-intro* have $x \oplus y = z..$
 thus $z = x \oplus y..$
 qed
 thus *?thesis..*
 qed

theorem *fusion-eq*: $\exists x. F x \implies$
 $(\sigma x. F x) = (THE x. \forall y. O y x \longleftrightarrow (\exists z. F z \wedge O y z))$
proof –
 assume $\exists x. F x$
 hence *bla*: $\forall y. P y (\sigma x. F x) \longrightarrow (\exists z. F z \wedge O y z)$
 by (*rule parts-overlap-Fs*)
 have $(THE x. \forall y. O y x \longleftrightarrow (\exists z. F z \wedge O y z)) = (\sigma x. F x)$
proof (*rule the-equality*)
 show $\forall y. O y (\sigma x. F x) \longleftrightarrow (\exists z. F z \wedge O y z)$
proof
 fix *y*
 show $O y (\sigma x. F x) \longleftrightarrow (\exists z. F z \wedge O y z)$
proof
 assume $O y (\sigma x. F x)$
 with *overlap-eq* have $\exists v. P v y \wedge P v (\sigma x. F x)..$
 then obtain *v* where $v: P v y \wedge P v (\sigma x. F x)..$
 hence $P v y..$
 from *bla* have $P v (\sigma x. F x) \longrightarrow (\exists z. F z \wedge O v z)..$
 moreover from *v* have $P v (\sigma x. F x)..$
 ultimately have $(\exists z. F z \wedge O v z)..$
 then obtain *z* where $z: F z \wedge O v z..$
 hence $F z..$
 moreover from *z* have $O v z..$
 hence $O z v$ by (*rule overlap-symmetry*)
 with $\langle P v y \rangle$ have $O z y$ by (*rule overlap-monotonicity*)
 hence $O y z$ by (*rule overlap-symmetry*)
 ultimately have $F z \wedge O y z..$
 thus $(\exists z. F z \wedge O y z)..$
next
 assume $\exists z. F z \wedge O y z$
 then obtain *z* where $z: F z \wedge O y z..$
 from $\langle \exists x. F x \rangle$ have $(\forall y. F y \longrightarrow P y (\sigma x. F x))$
 by (*rule F-in*)
 hence $F z \longrightarrow P z (\sigma x. F x)..$
 moreover from *z* have $F z..$
 ultimately have $P z (\sigma x. F x)..$
 moreover from *z* have $O y z..$
 ultimately show $O y (\sigma x. F x)$
 by (*rule overlap-monotonicity*)
 qed

qed
next
fix x
assume $x: \forall y. O y x \longleftrightarrow (\exists v. F v \wedge O y v)$
have $(\forall y. F y \longrightarrow P y x) \wedge (\forall y. P y x \longrightarrow (\exists z. F z \wedge O y z))$
proof
show $\forall y. F y \longrightarrow P y x$
proof
fix y
show $F y \longrightarrow P y x$
proof
assume $F y$
show $P y x$
proof (*rule ccontr*)
assume $\neg P y x$
hence $\exists z. P z y \wedge \neg O z x$
by (*rule strong-supplementation*)
then obtain z **where** $z: P z y \wedge \neg O z x..$
hence $\neg O z x..$
from x **have** $O z x \longleftrightarrow (\exists v. F v \wedge O z v)..$
moreover from z **have** $P z y..$
hence $O z y$ **by** (*rule part-implies-overlap*)
with $\langle F y \rangle$ **have** $F y \wedge O z y..$
hence $\exists y. F y \wedge O z y..$
ultimately have $O z x..$
with $\langle \neg O z x \rangle$ **show** *False*..
qed
qed
qed
show $\forall y. P y x \longrightarrow (\exists z. F z \wedge O y z)$
proof
fix y
show $P y x \longrightarrow (\exists z. F z \wedge O y z)$
proof
from x **have** $O y x \longleftrightarrow (\exists z. F z \wedge O y z)..$
moreover assume $P y x$
hence $O y x$ **by** (*rule part-implies-overlap*)
ultimately show $\exists z. F z \wedge O y z..$
qed
qed
qed
hence $(\sigma x. F x) = x$
by (*rule strong-fusion-intro*)
thus $x = (\sigma x. F x)..$
qed
thus $(\sigma x. F x) = (THE x. \forall y. O y x \longleftrightarrow (\exists z. F z \wedge O y z))..$
qed
end

sublocale $GEM1 \subseteq GEM$

proof

fix $x\ y\ F$

show $\neg P\ x\ y \implies \exists z. P\ z\ x \wedge \neg O\ z\ y$

using *strong-supplementation*.

show $x \oplus y = (THE\ z. \forall v. O\ v\ z \longleftrightarrow (O\ v\ x \vee O\ v\ y))$

using *sum-eq*.

show $x \otimes y = (THE\ z. \forall v. P\ v\ z \longleftrightarrow P\ v\ x \wedge P\ v\ y)$

using *product-eq*.

show $x \ominus y = (THE\ z. \forall w. P\ w\ z = (P\ w\ x \wedge \neg O\ w\ y))$

using *difference-eq*.

show $-x = (THE\ z. \forall w. P\ w\ z \longleftrightarrow \neg O\ w\ x)$

using *complement-eq*.

show $u = (THE\ x. \forall y. P\ y\ x)$

using *universe-eq*.

show $\exists x. F\ x \implies (\sigma\ x. F\ x) = (THE\ x. \forall y. O\ y\ x \longleftrightarrow (\exists z. F\ z \wedge O\ y\ z))$ **using** *fusion-eq*.

show $(\pi\ x. F\ x) = (\sigma\ x. \forall y. F\ y \longrightarrow P\ x\ y)$

using *general-product-eq*.

qed

sublocale $GEM \subseteq GEM1$

proof

fix $x\ y\ F$

show $\exists x. F\ x \implies (\exists x. (\forall y. F\ y \longrightarrow P\ y\ x) \wedge (\forall y. P\ y\ x \longrightarrow (\exists z. F\ z \wedge O\ y\ z)))$ **using** *strong-fusion*.

show $\exists x. F\ x \implies (\sigma\ x. F\ x) = (THE\ x. (\forall y. F\ y \longrightarrow P\ y\ x) \wedge (\forall y. P\ y\ x \longrightarrow (\exists z. F\ z \wedge O\ y\ z)))$ **using** *strong-fusion-eq*.

show $(\pi\ x. F\ x) = (\sigma\ x. \forall y. F\ y \longrightarrow P\ x\ y)$ **using** *general-product-eq*.

show $x \oplus y = (THE\ z. (P\ x\ z \wedge P\ y\ z) \wedge (\forall w. P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y))$ **using** *strong-sum-eq*.

show $x \otimes y = (THE\ z. \forall v. P\ v\ z \longleftrightarrow P\ v\ x \wedge P\ v\ y)$

using *product-eq*.

show $x \ominus y = (THE\ z. \forall w. P\ w\ z = (P\ w\ x \wedge \neg O\ w\ y))$

using *difference-eq*.

show $-x = (THE\ z. \forall w. P\ w\ z \longleftrightarrow \neg O\ w\ x)$ **using** *complement-eq*.

show $u = (THE\ x. \forall y. P\ y\ x)$ **using** *universe-eq*.

qed

References

- [Bittner, 2018] Bittner, T. (2018). Formal ontology of space, time, and physical entities in classical mechanics. *Applied Ontology*, 13(2):135–179.
- [Casati and Varzi, 1999] Casati, R. and Varzi, A. C. (1999). *Parts and Places. The Structures of Spatial Representation*. MIT Press,

Cambridge, Mass.

- [Cotnoir, 2010] Cotnoir, A. J. (2010). Anti-Symmetry and Non-Extensional Mereology. *The Philosophical Quarterly*, 60(239):396–405.
- [Cotnoir, 2016] Cotnoir, A. J. (2016). Does Universalism Entail Extensionalism? *Noûs*, 50(1):121–132.
- [Cotnoir, 2018] Cotnoir, A. J. (2018). Is Weak Supplementation analytic? *Synthese*.
- [Cotnoir and Bacon, 2012] Cotnoir, A. J. and Bacon, A. (2012). Non-Wellfounded Mereology. *The Review of Symbolic Logic*, 5(2):187–204.
- [Donnelly, 2011] Donnelly, M. (2011). Using Mereological Principles to Support Metaphysics. *The Philosophical Quarterly*, 61(243):225–246.
- [Hovda, 2009] Hovda, P. (2009). What is Classical Mereology? *Journal of Philosophical Logic*, 38(1):55–82.
- [Kearns, 2011] Kearns, S. (2011). Can a Thing be Part of Itself? *American Philosophical Quarterly*, 48(1):87–93.
- [Leonard and Goodman, 1940] Leonard, H. S. and Goodman, N. (1940). The Calculus of Individuals and Its Uses. *The Journal of Symbolic Logic*, 5(2):45–55.
- [Masolo and Vieu, 1999] Masolo, C. and Vieu, L. (1999). Atomicity vs. Infinite Divisibility of Space. In *Spatial Information Theory. Cognitive and Computational Foundations of Geographic Information Science*, Lecture Notes in Computer Science, pages 235–250, Berlin. Springer.
- [Obojska, 2013] Obojska, L. (2013). Some Remarks on Supplementation Principles in the Absence of Antisymmetry. *The Review of Symbolic Logic*, 6(2):343–347.
- [Parsons, 2014] Parsons, J. (2014). The Many Primitives of Mereology. In *Mereology and Location*. Oxford University Press, Oxford.
- [Pietruszczak, 2018] Pietruszczak, A. (2018). *Metamereology*. Nicolaus Copernicus University Scientific Publishing House, Turun.
- [Pontow, 2004] Pontow, C. (2004). A note on the axiomatics of theories in parthood. *Data & Knowledge Engineering*, 50(2):195–213.
- [Sen, 2017] Sen, A. (2017). *Computational Axiomatic Science*. PhD thesis, Rensselaer Polytechnic Institute.
- [Simons, 1987] Simons, P. (1987). *Parts: A Study in Ontology*. Oxford University Press, Oxford.
- [Smith, 2009] Smith, D. (2009). Mereology without Weak Supplementation. *Australasian Journal of Philosophy*, 87(3):505–511.
- [Tarski, 1983] Tarski, A. (1983). Foundations of the Geometry of Solids. In *Logic, Semantics, Metamathematics*, pages 24–29. Hackett Publishing, Indianapolis, second edition.

- [Varzi, 1996] Varzi, A. C. (1996). Parts, wholes, and part-whole relations: The prospects of mereotopology. *Data & Knowledge Engineering*, 20(3):259–286.
- [Varzi, 2006] Varzi, A. C. (2006). A Note on the Transitivity of Parthood. *Applied Ontology*, 1(2):141–146.
- [Varzi, 2008] Varzi, A. C. (2008). The Extensionality of Parthood and Composition. *The Philosophical Quarterly*, 58(230):108–133.
- [Varzi, 2009] Varzi, A. C. (2009). Universalism entails Extensionalism. *Analysis*, 69(4):599–604.
- [Varzi, 2016] Varzi, A. C. (2016). Mereology. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, winter 2016 edition.