Markov Models

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Abstract

This is a formalization of various Markov models in Isabelle/HOL. It builds on Isabelle's probability theory. The available models are currently discrete-time and continuous-time Markov chains as well as Markov decision processes. As application of these models we formalize probabilistic model checking of pCTL formulas, analysis of IPv4 address allocation in ZeroConf and an analysis of the anonymity of the Crowds protocol.

Contents

1 Introduction **3**

2 Auxiliary Theory **4**

3 Discrete-Time Markov Chain **11**
3.1 Discrete Markov Kernel 11
3.2 Trace Space for Discrete-Time Markov Chains 13
3.3 Fairness 15
3.4 First Hitting Time 16
3.5 Markov chain with Initial Distribution 17
3.6 Trace space with Restriction 19
3.7 Bisimulation 19
3.8 Reward Structure on Markov Chains 20
3.9 Bisimulation on a relation 21
3.10 Product Construction 22
3.11 Trace Space equal to Markov Chains 22

4 Classifying Markov Chain States **24**
4.1 Expected number of visits 24
4.2 Reachability probability 26
4.3 Recurrent states 28
4.4 Stationary distribution 32

5 Markov Decision Processes **35**
5.1 Configurations . . . 36
5.2 Configuration with Memoryless Scheduler . . . 37
5.3 MDP Kernel and Induced Configurations . . . 38
5.4 Trace Space . . . 39
5.5 Finite MDPs . . . 43

6 Discrete-time Markov Processes **51**
6.1 Constructing Discrete-Time Markov Processes . . . 52
6.2 Strong Markov Property for Discrete-Time Markov Processes 54

7 Continuous-time Markov chains **56**
7.1 Trace Operations: relate $('a \times real)$ *stream* and $real \Rightarrow 'a$ . . 56
7.2 Exponential Distribution . . . 56
7.3 Transition Rates . . . 58
7.4 Continuous-time Kernel . . . 59
7.5 Kernel equals Parallel Choice . . . 60
7.6 Markov Chain Property . . . 61
7.7 Explosion time . . . 62
7.8 Transition probability p_t . . . 62

8 Example A **65**
8.1 The essential classs $\{C1, C2, C3\}$ . . . 66
8.2 The stationary distribution n . . . 66

9 Example B **67**
9.1 Enabled, accessible and communicating states . . . 68
9.2 B is aperiodic . . . 68
9.3 The stationary distribution N . . . 68
9.4 Limit behavior and recurrence times . . . 68

10 Adapt Gauss-Jordan elimination to DTMCs **69**

11 pCTL model checking **70**
11.1 Syntax . . . 70
11.2 Semantics . . . 71
11.3 Implementation of *Sat* . . . 72
11.3.1 *Prob0* . . . 72
11.3.2 *Prob1* . . . 72
11.3.3 *ProbU*, *ExpCumm*, and *ExpState* . . . 73
11.3.4 *LES* . . . 73
11.3.5 *ProbUinfty*, compute unbounded until . . . 73
11.3.6 *ExpFuture*, compute unbounded reward . . . 73
11.3.7 *Sat* . . . 73
11.3.8 Finite expected reward . . . 75

11.3.9 The expected reward implies a unique LES 75
11.4 Soundness of *Sat* 76
11.5 Completeness of *Sat* 76
11.6 Completeness and Soundness *Sat* 76

12 Probabilistic Guarded Command Language (pGCL) 76
12.1 Syntax 76
12.2 Denotational Semantics 77
12.3 Operational Semantics 77
12.4 Equate Both Semantics 79

13 Formalization of the Crowds-Protocol 79
13.1 Definition of the Crowds-Protocol 79
13.2 Server gets no information 83
13.3 Probability that collaborators gain information 83
13.4 The probability that the sender hits a collaborator 84
13.5 Probability space of hitting a collaborator 85
13.6 Estimate the information to the collaborators 85
13.6.1 Setup random variables for mutual information 85

14 Formalizing the IPv4-address allocation in ZeroConf 86
14.1 Definition of a ZeroConf allocation run 87
14.2 The allocation run is a rewarded DTMC 88
14.3 Probability of a erroneous allocation 88
14.4 An allocation run terminates almost surely 88
14.5 Expected runtime of an allocation run 89

15 Formalization of the Gossip-Broadcast 90
15.1 Definition of the Gossip-Broadcast 90
15.2 The Gossip-Broadcast forms a DTMC 91

16 Certification of Reachability Problems on MDPs 91
16.1 Computable representation 92

1 Introduction

This is a formalization of probabilistic models in Isabelle/HOL. It builds on Isabelle's probability theory (HOL-Probability). It provides formalizations for the following models:

- Discrete-time Markov processes with measurable state spaces [2]
- Markov decision processes on discrete spaces [5]
- Continuous-time Markov chains on discrete spaces [2]

As application of these models we formalize

- a probabilistic model checking of pCTL formulas [4],
- an analysis of IPv4 address allocation in ZeroConf [3],
- an analysis of the anonymity of the Crowds protocol [3],
- the reachability analysis on finite-state MDPs [5], and
- expected running-time semantics for pGCL [1].

The formalization of rewarded DTMCs and pCTL model checking is discussed in detail in our paper.

2 Auxiliary Theory

Parts of it should be moved to the Isabelle repository

theory *Markov-Models-Auxiliary*
imports
HOL−Probability.Probability
HOL−Library.Rewrite
HOL−Library.Linear-Temporal-Logic-on-Streams
Coinductive.Coinductive-Stream
Coinductive.Coinductive-Nat
begin

lemma *lfp-upperbound*: $(\bigwedge y.\ x \leq f\ y) \Longrightarrow x \leq lfp\ f$
⟨*proof*⟩

lemma *lfp-arg*: $(\lambda t.\ lfp\ (F\ t)) = lfp\ (\lambda x\ t.\ F\ t\ (x\ t))$
⟨*proof*⟩

lemma *lfp-pair*: $lfp\ (\lambda f\ (a,\ b).\ F\ (\lambda a\ b.\ f\ (a,\ b))\ a\ b)\ (a,\ b) = lfp\ F\ a\ b$
⟨*proof*⟩

lemma *all-Suc-split*: $(\forall i.\ P\ i) \longleftrightarrow (P\ 0 \wedge (\forall i.\ P\ (Suc\ i)))$
⟨*proof*⟩

definition *with* $P\ f\ d = (if\ \exists x.\ P\ x\ then\ f\ (SOME\ x.\ P\ x)\ else\ d)$

lemma *withI*[*case-names default exists*]:
$((\bigwedge x.\ \neg\ P\ x) \Longrightarrow Q\ d) \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow Q\ (f\ x)) \Longrightarrow Q\ (with\ P\ f\ d)$
⟨*proof*⟩

context *order*
begin

definition
maximal f S = {*x*∈*S*. ∀ *y*∈*S*. *f y* ≤ *f x*}

lemma *maximalI*: *x* ∈ *S* ⟹ (⋀*y*. *y* ∈ *S* ⟹ *f y* ≤ *f x*) ⟹ *x* ∈ *maximal f S*
⟨*proof*⟩

lemma *maximalI-trans*: *x* ∈ *maximal f S* ⟹ *f x* ≤ *f y* ⟹ *y* ∈ *S* ⟹ *y* ∈ *maximal f S*
⟨*proof*⟩

lemma *maximalD1*: *x* ∈ *maximal f S* ⟹ *x* ∈ *S*
⟨*proof*⟩

lemma *maximalD2*: *x* ∈ *maximal f S* ⟹ *y* ∈ *S* ⟹ *f y* ≤ *f x*
⟨*proof*⟩

lemma *maximal-inject*: *x* ∈ *maximal f S* ⟹ *y* ∈ *maximal f S* ⟹ *f x* = *f y*
⟨*proof*⟩

lemma *maximal-empty*[*simp*]: *maximal f* {} = {}
⟨*proof*⟩

lemma *maximal-singleton*[*simp*]: *maximal f* {*x*} = {*x*}
⟨*proof*⟩

lemma *maximal-in-S*: *maximal f S* ⊆ *S*
⟨*proof*⟩

end

context *linorder*
begin

lemma *maximal-ne*:
assumes *finite S S* ≠ {}
shows *maximal f S* ≠ {}
⟨*proof*⟩

end

lemma *mono-les*:
fixes *s S N* **and** *l1 l2* :: *'a* ⇒ *real* **and** *K* :: *'a* ⇒ *'a pmf*
defines Δ *x* ≡ *l2 x* − *l1 x*
assumes *s*: *s* ∈ *S* **and** *S*: (⋃*s*∈*S*. *set-pmf* (*K s*)) ⊆ *S* ∪ *N*
assumes *int-l1*[*simp*]: ⋀*s*. *s* ∈ *S* ⟹ *integrable* (*K s*) *l1*
assumes *int-l2*[*simp*]: ⋀*s*. *s* ∈ *S* ⟹ *integrable* (*K s*) *l2*
assumes *to-N*: ⋀*s*. *s* ∈ *S* ⟹ ∃ *t*∈*N*. (*s*, *t*) ∈ (*SIGMA s*:*UNIV*. *K s*)*
assumes *l1*: ⋀*s*. *s* ∈ *S* ⟹ (∫ *t*. *l1 t* ∂*K s*) + *c s* ≤ *l1 s*
assumes *l2*: ⋀*s*. *s* ∈ *S* ⟹ *l2 s* ≤ (∫ *t*. *l2 t* ∂*K s*) + *c s*

assumes *eq*: $\bigwedge s.\ s \in N \Longrightarrow l2\ s \leq l1\ s$
assumes *finitary*: *finite* $(\Delta\ `\ (S \cup N))$
shows $l2\ s \leq l1\ s$
⟨*proof*⟩

lemma *unique-les*:
fixes $s\ S\ N$ **and** $l1\ l2 :: 'a \Rightarrow real$ **and** $K :: 'a \Rightarrow 'a\ pmf$
defines $\Delta\ x \equiv l2\ x - l1\ x$
assumes *s*: $s \in S$ **and** *S*: $(\bigcup s{\in}S.\ set\text{-}pmf\ (K\ s)) \subseteq S \cup N$
assumes $\bigwedge s.\ s \in S \Longrightarrow integrable\ (K\ s)\ l1$
assumes $\bigwedge s.\ s \in S \Longrightarrow integrable\ (K\ s)\ l2$
assumes $\bigwedge s.\ s \in S \Longrightarrow \exists\, t{\in}N.\ (s,\ t) \in (SIGMA\ s{:}UNIV.\ K\ s)^*$
assumes $\bigwedge s.\ s \in S \Longrightarrow l1\ s = (\int t.\ l1\ t\ \partial K\ s) + c\ s$
assumes $\bigwedge s.\ s \in S \Longrightarrow l2\ s = (\int t.\ l2\ t\ \partial K\ s) + c\ s$
assumes $\bigwedge s.\ s \in N \Longrightarrow l2\ s = l1\ s$
assumes *1*: *finite* $(\Delta\ `\ (S \cup N))$
shows $l2\ s = l1\ s$
⟨*proof*⟩

lemma *inf-continuous-suntil-disj*[*order-continuous-intros*]:
assumes *Q*: *inf-continuous Q*
assumes *disj*: $\bigwedge x\ \omega.\ \neg\ (P\ \omega \wedge Q\ x\ \omega)$
shows *inf-continuous* $(\lambda x.\ P\ suntil\ Q\ x)$
⟨*proof*⟩

lemma *inf-continuous-nxt*[*order-continuous-intros*]: *inf-continuous P* $\Longrightarrow$ *inf-continuous* $(\lambda x.\ nxt\ (P\ x)\ \omega)$
⟨*proof*⟩

lemma *sup-continuous-nxt*[*order-continuous-intros*]: *sup-continuous P* $\Longrightarrow$ *sup-continuous* $(\lambda x.\ nxt\ (P\ x)\ \omega)$
⟨*proof*⟩

lemma *mcont-ennreal-of-enat*: *mcont Sup* $(\leq)$ *Sup* $(\leq)$ *ennreal-of-enat*
⟨*proof*⟩

lemma *mcont2mcont-ennreal-of-enat*[*cont-intro*]:
mcont lub ord Sup $(\leq)$ *f* $\Longrightarrow$ *mcont lub ord Sup* $(\leq)$ $(\lambda x.\ ennreal\text{-}of\text{-}enat\ (f\ x))$
⟨*proof*⟩

declare *stream.exhaust*[*cases type*: *stream*]

lemma *scount-eq-emeasure*: *scount P* ω = *emeasure* (*count-space UNIV*) $\{i.\ P\ (sdrop\ i\ \omega)\}$
⟨*proof*⟩

lemma *measurable-scount*[*measurable*]:
assumes [*measurable*]: *Measurable.pred* (*stream-space M*) *P*
shows *scount P* $\in$ *measurable* (*stream-space M*) (*count-space UNIV*)

⟨*proof*⟩

lemma *measurable-sfirst2*:
assumes [*measurable*]: *Measurable.pred* ($N \bigotimes_M$ *stream-space* M) ($\lambda(x, \omega)$. $P\ x\ \omega$)
shows ($\lambda(x, \omega)$. *sfirst* ($P\ x$) ω) ∈ *measurable* ($N \bigotimes_M$ *stream-space* M) (*count-space UNIV*)
⟨*proof*⟩

lemma *measurable-sfirst2'*[*measurable* (*raw*)]:
assumes [*measurable* (*raw*)]: $f \in N \to_M$ *stream-space* M *Measurable.pred* ($N \bigotimes_M$ *stream-space* M) (λx. P (*fst* x) (*snd* x))
shows (λx. *sfirst* ($P\ x$) ($f\ x$)) ∈ *measurable* N (*count-space UNIV*)
⟨*proof*⟩

lemma *measurable-sfirst*[*measurable*]:
assumes [*measurable*]: *Measurable.pred* (*stream-space* M) P
shows *sfirst* P ∈ *measurable* (*stream-space* M) (*count-space UNIV*)
⟨*proof*⟩

lemma *measurable-epred*[*measurable*]: *epred* ∈ *count-space UNIV* $\to_M$ *count-space UNIV*
⟨*proof*⟩

lemma *nn-integral-stretch*:
$f \in$ *borel* $\to_M$ *borel* $\Longrightarrow$ $c \neq 0 \Longrightarrow$ ($\int^+ x.\ f\ (c * x)\ \partial$*lborel*) = (1 / |$c$|::*real*) * ($\int^+ x.\ f\ x\ \partial$*lborel*)
⟨*proof*⟩

lemma *prod-sum-distrib*:
fixes $f\ g$ :: $'a \Rightarrow 'b \Rightarrow 'c$::*comm-semiring-1*
assumes *finite* I **shows** ($\bigwedge i.\ i \in I \Longrightarrow$ *finite* ($J\ i$)) $\Longrightarrow$ ($\prod i{\in}I.\ \sum j{\in}J\ i.\ f\ i\ j$) = ($\sum m{\in}Pi_E\ I\ J.\ \prod i{\in}I.\ f\ i\ (m\ i)$)
⟨*proof*⟩

lemma *prod-add-distrib*:
fixes $f\ g$ :: $'a \Rightarrow 'b$::*comm-semiring-1*
assumes *finite* I **shows** ($\prod i{\in}I.\ f\ i + g\ i$) = ($\sum J{\in}Pow\ I.$ ($\prod i{\in}J.\ f\ i$) * ($\prod i{\in}I - J.\ g\ i$))
⟨*proof*⟩

subclass (**in** *linordered-nonzero-semiring*) *ordered-semiring-0*
⟨*proof*⟩

lemma (**in** *linordered-nonzero-semiring*) *prod-nonneg*: ($\forall\ a{\in}A.\ 0 \le f\ a$) $\Longrightarrow$ $0 \le$ *prod* f A
⟨*proof*⟩

lemma (**in** *linordered-nonzero-semiring*) *prod-mono*:

$\forall i{\in}A.\ 0 \le f\ i \wedge f\ i \le g\ i \Longrightarrow prod\ f\ A \le prod\ g\ A$
⟨*proof*⟩

lemma (**in** *linordered-nonzero-semiring*) *prod-mono2*:
assumes *finite J I* ⊆ *J* ⋀*i*. *i* ∈ *I* ⟹ *0* ≤ *g i* ∧ *g i* ≤ *f i* (⋀*i*. *i* ∈ *J* − *I* ⟹ *1* ≤ *f i*)
shows *prod g I* ≤ *prod f J*
⟨*proof*⟩

lemma (**in** *linordered-nonzero-semiring*) *prod-mono3*:
assumes *finite J I* ⊆ *J* ⋀*i*. *i* ∈ *J* ⟹ *0* ≤ *g i* ⋀*i*. *i* ∈ *I* ⟹ *g i* ≤ *f i* (⋀*i*. *i* ∈ *J* − *I* ⟹ *g i* ≤ *1*)
shows *prod g J* ≤ *prod f I*
⟨*proof*⟩

lemma (**in** *linordered-nonzero-semiring*) *one-le-prod*: (⋀*i*. *i* ∈ *I* ⟹ *1* ≤ *f i*) ⟹ *1* ≤ *prod f I*
⟨*proof*⟩

lemma *sum-plus-one-le-prod-plus-one*:
fixes *p* :: *'a* ⇒ *'b*::*linordered-nonzero-semiring*
assumes ⋀*i*. *i* ∈ *I* ⟹ *0* ≤ *p i*
shows (∑ *i*∈*I*. *p i*) + *1* ≤ (∏ *i*∈*I*. *p i* + *1*)
⟨*proof*⟩

lemma *summable-iff-convergent-prod*:
fixes *p* :: *nat* ⇒ *real* **assumes** *p*: ⋀*i*. *0* ≤ *p i*
shows *summable p* ⟷ *convergent* (λ*n*. ∏ *i*<*n*. *p i* + *1*)
⟨*proof*⟩

primrec *eexp* :: *ereal* ⇒ *ennreal*
where
eexp MInfty = *0*
| *eexp* (*ereal r*) = *ennreal* (*exp r*)
| *eexp PInfty* = *top*

lemma
shows *eexp-minus-infty*[*simp*]: *eexp* (−∞) = *0*
and *eexp-infty*[*simp*]: *eexp* ∞ = *top*
⟨*proof*⟩

lemma *eexp-0*[*simp*]: *eexp 0* = *1*
⟨*proof*⟩

lemma *eexp-inj*[*simp*]: *eexp x* = *eexp y* ⟷ *x* = *y*
⟨*proof*⟩

lemma *eexp-mono*[*simp*]: *eexp x* ≤ *eexp y* ⟷ *x* ≤ *y*
⟨*proof*⟩

lemma *eexp-strict-mono*[*simp*]: *eexp x* < *eexp y* ⟷ *x* < *y*
⟨*proof*⟩

lemma *exp-eq-0-iff*[*simp*]: *eexp x* = *0* ⟷ *x* = −∞
⟨*proof*⟩

lemma *eexp-surj*: *range eexp* = *UNIV*
⟨*proof*⟩

lemma *continuous-on-eexp'*: *continuous-on UNIV eexp*
⟨*proof*⟩

lemma *continuous-on-eexp*[*continuous-intros*]: *continuous-on A f* ⟹ *continuous-on A* (λ*x*. *eexp* (*f x*))
⟨*proof*⟩

lemma *tendsto-eexp*[*tendsto-intros*]: (*f* ⟶ *x*) *F* ⟹ ((λ*x*. *eexp* (*f x*)) ⟶ *eexp x*) *F*
⟨*proof*⟩

lemma *measurable-eexp*[*measurable*]: *eexp* ∈ *borel* →$_M$ *borel*
⟨*proof*⟩

lemma *eexp-add*: ¬ ((*x* = ∞ ∧ *y* = −∞) ∨ (*x* = −∞ ∧ *y* = ∞)) ⟹ *eexp* (*x* + *y*) = *eexp x* ∗ *eexp y*
⟨*proof*⟩

lemma *sum-Pinfty*:
fixes *f* :: *'a* ⇒ *ereal*
shows *sum f I* = ∞ ⟷ (*finite I* ∧ (∃ *i*∈*I*. *f i* = ∞))
⟨*proof*⟩

lemma *sum-Minfty*:
fixes *f* :: *'a* ⇒ *ereal*
shows *sum f I* = −∞ ⟷ (*finite I* ∧ ¬ (∃ *i*∈*I*. *f i* = ∞) ∧ (∃ *i*∈*I*. *f i* = −∞))
⟨*proof*⟩

lemma *eexp-sum*: ¬ (∃ *i*∈*I*. ∃ *j*∈*I*. *f i* = −∞ ∧ *f j* = ∞) ⟹ *eexp* (∑ *i*∈*I*. *f i*) = (∏ *i*∈*I*. *eexp* (*f i*))
⟨*proof*⟩

lemma *eexp-suminf*:
assumes *wf-f*: ¬ {−∞, ∞} ⊆ *range f* **and** *f*: *summable f*
shows (λ*n*. ∏ *i*<*n*. *eexp* (*f i*)) ⟶ *eexp* (∑ *i*. *f i*)
⟨*proof*⟩

lemma *continuous-onI-antimono*:
fixes *f* :: *'a*::*linorder-topology* ⇒ *'b*::{*dense-order*,*linorder-topology*}

assumes *open* (*f*‘*A*)
and *mono*: $\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow x \leq y \Longrightarrow f\ y \leq f\ x$
shows *continuous-on A f*
⟨*proof*⟩

lemma *minus-add-eq-ereal*: $\neg\ ((a = \infty \wedge b = -\infty) \vee (a = -\infty \wedge b = \infty)) \Longrightarrow -\ (a + b::ereal) = -a - b$
⟨*proof*⟩

lemma *setsum-negf-ereal*: $\neg\ \{-\infty, \infty\} \subseteq f`I \Longrightarrow (\sum i{\in}I.\ -\ f\ i) = -\ (\sum i{\in}I.\ f\ i::ereal)$
⟨*proof*⟩

lemma *convergent-minus-iff-ereal*: *convergent* $(\lambda x.\ -\ f\ x::ereal) \longleftrightarrow$ *convergent f*
⟨*proof*⟩

lemma *summable-minus-ereal*: $\neg\ \{-\infty, \infty\} \subseteq$ *range f* $\Longrightarrow$ *summable* $(\lambda n.\ f\ n) \Longrightarrow$ *summable* $(\lambda n.\ -\ f\ n::ereal)$
⟨*proof*⟩

lemma (**in** *product-prob-space*) *product-nn-integral-component*:
assumes $f \in$ *borel-measurable* $(M\ i) i \in I$
shows $integral^N\ (Pi_M\ I\ M)\ (\lambda x.\ f\ (x\ i)) = integral^N\ (M\ i)\ f$
⟨*proof*⟩

lemma *ennreal-inverse-le*[*simp*]: *inverse x* $\leq$ *inverse y* $\longleftrightarrow$ $y \leq (x::ennreal)$
⟨*proof*⟩

lemma *inverse-inverse-ennreal*[*simp*]: *inverse* (*inverse x*::*ennreal*) = *x*
⟨*proof*⟩

lemma *range-inverse-ennreal*: *range inverse* = (*UNIV*::*ennreal set*)
⟨*proof*⟩

lemma *continuous-on-inverse-ennreal*′: *continuous-on* (*UNIV* :: *ennreal set*) *inverse*
⟨*proof*⟩

lemma *sums-minus-ereal*: $\neg\ \{-\ \infty, \infty\} \subseteq f\ `\ UNIV \Longrightarrow (\lambda n.\ -\ f\ n::ereal)$ *sums* $x \Longrightarrow f$ *sums* $-\ x$
⟨*proof*⟩

lemma *suminf-minus-ereal*: $\neg\ \{-\ \infty, \infty\} \subseteq f\ `\ UNIV \Longrightarrow$ *summable f* $\Longrightarrow (\sum n.\ -\ f\ n :: ereal) = -\ suminf\ f$
⟨*proof*⟩

end

3 Discrete-Time Markov Chain

theory *Discrete-Time-Markov-Chain*
imports *Markov-Models-Auxiliary*
begin

Markov chain with discrete time steps and discrete state space.

lemma *sstart-eq'*: *sstart* Ω $(x \# xs) = \{\omega.\ shd\ \omega = x \wedge stl\ \omega \in sstart\ \Omega\ xs\}$
⟨*proof*⟩

lemma *measure-eq-stream-space-coinduct*[*consumes 1*, *case-names left right cont*]:
assumes *R N M*
assumes *R-1*: $\bigwedge N\ M.\ R\ N\ M \Longrightarrow N \in space\ (prob\text{-}algebra\ (stream\text{-}space\ (count\text{-}space\ UNIV)))$
and *R-2*: $\bigwedge N\ M.\ R\ N\ M \Longrightarrow M \in space\ (prob\text{-}algebra\ (stream\text{-}space\ (count\text{-}space\ UNIV)))$
and *cont*: $\bigwedge N\ M.\ R\ N\ M \Longrightarrow \exists\, N'\ M'\ p.\ (\forall\, y{\in}set\text{-}pmf\ p.\ R\ (N'\ y)\ (M'\ y)) \wedge$
$(\forall\, x.\ N'\ x \in space\ (prob\text{-}algebra\ (stream\text{-}space\ (count\text{-}space\ UNIV)))) \wedge (\forall\, x.\ M'\ x \in space\ (prob\text{-}algebra\ (stream\text{-}space\ (count\text{-}space\ UNIV)))) \wedge$
$N = (measure\text{-}pmf\ p \gg\!= (\lambda y.\ distr\ (N'\ y)\ (stream\text{-}space\ (count\text{-}space\ UNIV))\ ((\#\#)\ y))) \wedge$
$M = (measure\text{-}pmf\ p \gg\!= (\lambda y.\ distr\ (M'\ y)\ (stream\text{-}space\ (count\text{-}space\ UNIV))\ ((\#\#)\ y)))$
shows $N = M$
⟨*proof*⟩

3.1 Discrete Markov Kernel

locale *MC-syntax* =
fixes $K :: 's \Rightarrow 's\ pmf$
begin

abbreviation *acc* :: $('s \times 's)\ set$ **where**
$acc \equiv (SIGMA\ s{:}UNIV.\ K\ s)^*$

abbreviation *acc-on* :: $'s\ set \Rightarrow ('s \times 's)\ set$ **where**
$acc\text{-}on\ S \equiv (SIGMA\ s{:}UNIV.\ K\ s \cap S)^*$

lemma *countable-reachable*: *countable* $(acc\ ``\ \{s\})$
⟨*proof*⟩

lemma *countable-acc*: *countable* $X \Longrightarrow$ *countable* $(acc\ ``\ X)$
⟨*proof*⟩

context
notes [[*inductive-internals*]]
begin

coinductive *enabled* **where**
$enabled\ (shd\ \omega)\ (stl\ \omega) \Longrightarrow shd\ \omega \in K\ s \Longrightarrow enabled\ s\ \omega$

end

lemma *alw-enabled*: *enabled* (*shd* ω) (*stl* ω) $\Longrightarrow$ *alw* ($\lambda\omega$. *enabled* (*shd* ω) (*stl* ω)) ω
⟨*proof*⟩

abbreviation $S \equiv$ *stream-space* (*count-space UNIV*)

lemma *in-S* [*measurable* (*raw*)]: $x \in$ *space* S
⟨*proof*⟩

inductive-simps *enabled-iff*: *enabled* s ω

lemma *enabled-Stream*: *enabled* x (y ## ω) $\longleftrightarrow$ $y \in K\ x \wedge$ *enabled* y ω
⟨*proof*⟩

lemma *measurable-enabled*[*measurable*]:
Measurable.pred (*stream-space* (*count-space UNIV*)) (*enabled* s) (**is** *Measurable.pred ?S* -)
⟨*proof*⟩

lemma *enabled-iff-snth*: *enabled* s ω $\longleftrightarrow$ ($\forall\, i.\ \omega$!! $i \in K$ ((s ## ω) !! i))
⟨*proof*⟩

primcorec *force-enabled* **where**
force-enabled x ω =
(*let* y = *if shd* $\omega \in K\ x$ *then shd* ω *else* (*SOME* y. $y \in K\ x$) *in* y ## *force-enabled* y (*stl* ω))

lemma *force-enabled-in-set-pmf*[*simp*, *intro*]: *shd* (*force-enabled* x ω) $\in K\ x$
⟨*proof*⟩

lemma *enabled-force-enabled*: *enabled* x (*force-enabled* x ω)
⟨*proof*⟩

lemma *force-enabled*: *enabled* x $\omega \Longrightarrow$ *force-enabled* x $\omega = \omega$
⟨*proof*⟩

lemma *Ex-enabled*: $\exists\omega$. *enabled* x ω
⟨*proof*⟩

lemma *measurable-force-enabled*: *force-enabled* $x \in$ *measurable* S S
⟨*proof*⟩

abbreviation $D \equiv$ *stream-space* (Π_M $s \in$ *UNIV*. $K\ s$)

lemma *sets-D*: *sets* D = *sets* (*stream-space* (Π_M $s \in$ *UNIV*. *count-space UNIV*))
⟨*proof*⟩

lemma *space-D*: *space D* = *space* (*stream-space* (Π_M *s*∈*UNIV*. *count-space UNIV*))
⟨*proof*⟩

lemma *measurable-D-D*: *measurable D D* =
measurable (*stream-space* (Π_M *s*∈*UNIV*. *count-space UNIV*)) (*stream-space* (Π_M *s*∈*UNIV*. *count-space UNIV*))
⟨*proof*⟩

primcorec *walk* :: *'s* ⇒ (*'s* ⇒ *'s*) *stream* ⇒ *'s stream* **where**
shd (*walk s ω*) = (*if shd ω s* ∈ *K s then shd ω s else* (*SOME t*. *t* ∈ *K s*))
| *stl* (*walk s ω*) = *walk* (*if shd ω s* ∈ *K s then shd ω s else* (*SOME t*. *t* ∈ *K s*)) (*stl ω*)

lemma *enabled-walk*: *enabled s* (*walk s ω*)
⟨*proof*⟩

lemma *measurable-walk*[*measurable*]: *walk s* ∈ *measurable D S*
⟨*proof*⟩

3.2 Trace Space for Discrete-Time Markov Chains

definition *T* :: *'s* ⇒ *'s stream measure* **where**
T s = *distr* (*stream-space* (Π_M *s*∈*UNIV*. *K s*)) *S* (*walk s*)

lemma *space-T*[*simp*]: *space* (*T s*) = *space S*
⟨*proof*⟩

lemma *sets-T*[*simp*, *measurable-cong*]: *sets* (*T s*) = *sets S*
⟨*proof*⟩

lemma *measurable-T1*[*simp*]: *measurable* (*T s*) *M* = *measurable S M*
⟨*proof*⟩

lemma *measurable-T2*[*simp*]: *measurable M* (*T s*) = *measurable M S*
⟨*proof*⟩

lemma *in-measurable-T1*[*measurable* (*raw*)]: *f* ∈ *measurable S M* ⟹ *f* ∈ *measurable* (*T s*) *M*
⟨*proof*⟩

lemma *in-measurable-T2*[*measurable* (*raw*)]: *f* ∈ *measurable M S* ⟹ *f* ∈ *measurable M* (*T s*)
⟨*proof*⟩

lemma *AE-T-enabled*: *AE ω in T s*. *enabled s ω*
⟨*proof*⟩

sublocale *T*: *prob-space T s* **for** *s*

⟨*proof*⟩

lemma *emeasure-T-const*[*simp*]: *emeasure* (*T s*) (*space S*) = *1*
⟨*proof*⟩

lemma *nn-integral-T*:
assumes *f*[*measurable*]: *f* ∈ *borel-measurable S*
shows (∫⁺*X*. *f X* ∂*T s*) = (∫⁺*t*. (∫⁺ω. *f* (*t* ## ω) ∂*T t*) ∂*K s*)
⟨*proof*⟩

lemma *nn-integral-T-gfp*:
fixes *g*
defines *l* ≡ λ*f* ω. *g* (*shd* ω) (*f* (*stl* ω))
assumes [*measurable*]: *case-prod g* ∈ *borel-measurable* (*count-space UNIV* ⨂*M* *borel*)
assumes *cont-g*[*THEN inf-continuous-compose*, *order-continuous-intros*]: ⋀*s*. *inf-continuous* (*g s*)
assumes *int-g*: ⋀*f s*. *f* ∈ *borel-measurable S* ⟹ (∫⁺ω. *g s* (*f* ω) ∂*T s*) = *g s* (∫⁺ω. *f* ω ∂*T s*)
assumes *bnd-g*: ⋀*f s*. *g s f* ≤ *b* 0 ≤ *b b* < ∞
shows (∫⁺ω. *gfp l* ω ∂*T s*) = *gfp* (λ*f s*. ∫⁺*t*. *g t* (*f t*) ∂*K s*) *s*
⟨*proof*⟩

lemma *nn-integral-T-lfp*:
fixes *g*
defines *l* ≡ λ*f* ω. *g* (*shd* ω) (*f* (*stl* ω))
assumes [*measurable*]: *case-prod g* ∈ *borel-measurable* (*count-space UNIV* ⨂*M* *borel*)
assumes *cont-g*[*THEN sup-continuous-compose*, *order-continuous-intros*]: ⋀*s*. *sup-continuous* (*g s*)
assumes *int-g*: ⋀*f s*. *f* ∈ *borel-measurable S* ⟹ (∫⁺ω. *g s* (*f* ω) ∂*T s*) = *g s* (∫⁺ω. *f* ω ∂*T s*)
shows (∫⁺ω. *lfp l* ω ∂*T s*) = *lfp* (λ*f s*. ∫⁺*t*. *g t* (*f t*) ∂*K s*) *s*
⟨*proof*⟩

lemma *emeasure-Collect-T*:
assumes *f*[*measurable*]: *Measurable.pred S P*
shows *emeasure* (*T s*) {*x*∈*space* (*T s*). *P x*} = (∫⁺*t*. *emeasure* (*T t*) {*x*∈*space* (*T t*). *P* (*t* ## *x*)} ∂*K s*)
⟨*proof*⟩

lemma *AE-T-iff*:
assumes [*measurable*]: *Measurable.pred S P*
shows (*AE* ω *in T x*. *P* ω) ⟷ (∀ *y*∈*K x*. *AE* ω *in T y*. *P* (*y* ## ω))
⟨*proof*⟩

lemma *AE-T-alw*:
assumes [*measurable*]: *Measurable.pred S P*
assumes *P*: ⋀*s*. (*x*, *s*) ∈ *acc* ⟹ *AE* ω *in T s*. *P* ω

shows *AE ω in T x. alw P ω*
⟨*proof*⟩

lemma *emeasure-suntil-disj*:
assumes [*measurable*]: *Measurable.pred S P*
assumes ∗: ⋀*t. AE ω in T t. ¬ (P ⊓ (HLD X ⊓ nxt (HLD X suntil P))) ω*
shows *emeasure (T s) {ω∈space (T s). (HLD X suntil P) ω} =*
lfp (λF s. emeasure (T s) {ω∈space (T s). P ω} + (∫⁺t. F t ∗ indicator X t ∂K s)) s
⟨*proof*⟩

lemma *emeasure-HLD-nxt*:
assumes [*measurable*]: *Measurable.pred S P*
shows *emeasure (T s) {ω∈space (T s). (X · P) ω} =*
(∫⁺x. emeasure (T x) {ω∈space (T x). P ω} ∗ indicator X x ∂K s)
⟨*proof*⟩

lemma *emeasure-HLD*:
emeasure (T s) {ω∈space (T s). HLD X ω} = emeasure (K s) X
⟨*proof*⟩

lemma *emeasure-suntil-HLD*:
assumes [*measurable*]: *Measurable.pred S P*
shows *emeasure (T s) {x∈space (T s). (not (HLD {t}) suntil (HLD {t} aand nxt P)) x} =*
emeasure (T s) {x∈space (T s). ev (HLD {t}) x} ∗ emeasure (T t) {x∈space (T t). P x}
⟨*proof*⟩

lemma *AE-suntil*:
assumes [*measurable*]: *Measurable.pred S P*
shows *(AE x in T s. (not (HLD {t}) suntil (HLD {t} aand nxt P)) x) ⟷*
(AE x in T s. ev (HLD {t}) x) ∧ (AE x in T t. P x)
⟨*proof*⟩

3.3 Fairness

definition *fair* :: *'s ⇒ 's ⇒ 's stream ⇒ bool* **where**
fair s t = alw (ev (HLD {s})) impl alw (ev (HLD {s} aand nxt (HLD {t})))

lemma *AE-T-fair*:
assumes *t′ ∈ K t*
shows *AE ω in T s. fair t t′ ω*
⟨*proof*⟩

lemma *enabled-imp-trancl*:
assumes *alw (HLD B) ω enabled s ω*
shows *alw (HLD (acc-on B `` {s})) ω*
⟨*proof*⟩

lemma *AE-T-reachable*: *AE* ω *in T s. alw* (*HLD* (*acc* `` $\{s\}$)) ω
⟨*proof*⟩

lemma *AE-T-all-fair*: *AE* ω *in T s.* $\forall (t,t') \in$ *SIGMA t*:*UNIV*. *K t. fair t t'* ω
⟨*proof*⟩

lemma *fair-imp*: **assumes** *fair t t'* ω *alw* (*ev* (*HLD* $\{t\}$)) ω **shows** *alw* (*ev* (*HLD* $\{t'\}$)) ω
⟨*proof*⟩

lemma *AE-T-ev-HLD*:
assumes *exiting*: $\bigwedge t.\ (s,\ t) \in$ *acc-on* $(-B) \Longrightarrow \exists\, t' \in B.\ (t,\ t') \in$ *acc*
assumes *fin*: *finite* (*acc-on* $(-B)$ `` $\{s\}$)
shows *AE* ω *in T s. ev* (*HLD B*) ω
⟨*proof*⟩

lemma *AE-T-ev-HLD'*:
assumes *exiting*: $\bigwedge s.\ s \notin X \Longrightarrow \exists\, t \in X.\ (s,\ t) \in$ *acc*
assumes *fin*: *finite* $(-X)$
shows *AE* ω *in T s. ev* (*HLD X*) ω
⟨*proof*⟩

lemma *AE-T-max-sfirst*:
assumes [*measurable*]: *Measurable.pred S X*
assumes *AE*: *AE* ω *in T c. sfirst X* $(c$ ## $\omega) < \infty$ **and** $0 < e$
shows $\exists\, N$::*nat*. $\mathcal{P}(\omega$ *in T c.* $N <$ *sfirst X* $(c$ ## $\omega)) < e$ (**is** $\exists N.$ *?P* $N < e$)
⟨*proof*⟩

3.4 First Hitting Time

lemma *nn-integral-sfirst-finite'*:
assumes $s \notin H$
assumes [*simp*]: *finite* (*acc-on* $(-H)$ `` $\{s\}$)
assumes *until*: *AE* ω *in T s. ev* (*HLD H*) ω
shows $(\int^+ \omega.$ *sfirst* (*HLD H*) $\omega\ \partial T\ s) \neq \infty$
⟨*proof*⟩

lemma *nn-integral-sfirst-finite*:
assumes [*simp*]: *finite* (*acc-on* $(-H)$ `` $\{s\}$)
assumes *until*: *AE* ω *in T s. ev* (*HLD H*) ω
shows $(\int^+ \omega.$ *sfirst* (*HLD H*) $(s$ ## $\omega)\ \partial T\ s) \neq \infty$
⟨*proof*⟩

lemma *prob-T*:
assumes *P*: *Measurable.pred S P*
shows $\mathcal{P}(\omega$ *in T s. P* $\omega) = (\int t.\ \mathcal{P}(\omega$ *in T t. P* $(t$ ## $\omega))\ \partial K\ s)$
⟨*proof*⟩

lemma *T-subprob*[*measurable*]: $T \in$ *measurable* (*measure-pmf I*) (*subprob-algebra S*)
⟨*proof*⟩

3.5 Markov chain with Initial Distribution

definition $T' :: 's\ pmf \Rightarrow 's\ stream\ measure$ **where**
$T'\ I = bind\ I\ (\lambda s.\ distr\ (T\ s)\ S\ ((\#\#)\ s))$

lemma *distr-Stream-subprob*:
$(\lambda s.\ distr\ (T\ s)\ S\ ((\#\#)\ s)) \in measurable\ (measure\text{-}pmf\ I)\ (subprob\text{-}algebra\ S)$
⟨*proof*⟩

lemma *sets-T'*: $sets\ (T'\ I) = sets\ S$
⟨*proof*⟩

lemma *prob-space-T'*: *prob-space* $(T'\ I)$
⟨*proof*⟩

lemma *AE-T'*:
assumes [*measurable*]: *Measurable.pred S P*
shows $(AE\ x\ in\ T'\ I.\ P\ x) \longleftrightarrow (\forall s \in I.\ AE\ x\ in\ T\ s.\ P\ (s\ \#\#\ x))$
⟨*proof*⟩

lemma *emeasure-T'*:
assumes [*measurable*]: $X \in sets\ S$
shows $emeasure\ (T'\ I)\ X = (\int^+ s.\ emeasure\ (T\ s)\ \{\omega \in space\ S.\ s\ \#\#\ \omega \in X\}\ \partial I)$
⟨*proof*⟩

lemma *prob-T'*:
assumes [*measurable*]: *Measurable.pred S P*
shows $\mathcal{P}(x\ in\ T'\ I.\ P\ x) = (\int s.\ \mathcal{P}(x\ in\ T\ s.\ P\ (s\ \#\#\ x))\ \partial I)$
⟨*proof*⟩

lemma *T-eq-T'*: $T\ s = T'\ (K\ s)$
⟨*proof*⟩

lemma *T-eq-bind*: $T\ s = (measure\text{-}pmf\ (K\ s) \gg\!= (\lambda t.\ distr\ (T\ t)\ S\ ((\#\#)\ t)))$
⟨*proof*⟩

lemma *T-split*:
$T\ s = (T\ s \gg\!= (\lambda \omega.\ distr\ (T\ ((s\ \#\#\ \omega)\ !!\ n))\ S\ (\lambda \omega'.\ stake\ n\ \omega\ @-\ \omega')))$
⟨*proof*⟩

lemma *nn-integral-T-split*:
assumes *f*[*measurable*]: $f \in borel\text{-}measurable\ S$
shows $(\int^+ \omega.\ f\ \omega\ \partial T\ s) = (\int^+ \omega.\ (\int^+ \omega'.\ f\ (stake\ n\ \omega\ @-\ \omega')\ \partial T\ ((s\ \#\#\ \omega)\ !!\ n))\ \partial T\ s)$

⟨*proof*⟩

lemma *emeasure-T-split*:
assumes *P*[*measurable*]: *Measurable.pred S P*
shows *emeasure* (*T s*) {*ω*∈*space* (*T s*). *P ω*} =
(∫⁺*ω*. *emeasure* (*T* ((*s* ## *ω*) !! *n*)) {*ω*′∈*space* (*T* ((*s* ## *ω*) !! *n*)). *P* (*stake n ω* @− *ω*′)} *∂T s*)
⟨*proof*⟩

lemma *prob-T-split*:
assumes *P*[*measurable*]: *Measurable.pred S P*
shows *P*(*ω in T s*. *P ω*) = (∫ *ω*. *P*(*ω*′ *in T* ((*s* ## *ω*) !! *n*). *P* (*stake n ω* @− *ω*′)) *∂T s*)
⟨*proof*⟩

lemma *enabled-imp-alw*:
(⋃*s*∈*X*. *set-pmf* (*K s*)) ⊆ *X* ⟹ *x* ∈ *X* ⟹ *enabled x ω* ⟹ *alw* (*HLD X*) *ω*
⟨*proof*⟩

lemma *alw-HLD-iff-sconst*:
alw (*HLD* {*x*}) *ω* ⟷ *ω* = *sconst x*
⟨*proof*⟩

lemma *enabled-iff-sconst*:
assumes [*simp*]: *set-pmf* (*K x*) = {*x*} **shows** *enabled x ω* ⟷ *ω* = *sconst x*
⟨*proof*⟩

lemma *AE-sconst*:
assumes [*simp*]: *set-pmf* (*K x*) = {*x*}
shows (*AE ω in T x*. *P ω*) ⟷ *P* (*sconst x*)
⟨*proof*⟩

lemma *ev-eq-lfp*: *ev P* = *lfp* (*λF ω*. *P ω* ∨ (¬ *P ω* ∧ *F* (*stl ω*)))
⟨*proof*⟩

lemma *INF-eq-zero-iff-ennreal*: ((∏ *i*∈*A*. *f i*) = (*0*::*ennreal*)) = (∀ *x*>*0*. ∃ *i*∈*A*. *f i* < *x*)
⟨*proof*⟩

lemma *inf-continuous-cmul*:
fixes *c* :: *ennreal*
assumes *f*: *inf-continuous f* **and** *c*: *c* < ⊤
shows *inf-continuous* (*λx*. *c* ∗ *f x*)
⟨*proof*⟩

lemma *AE-T-ev-HLD-infinite*:
fixes *X* :: *'s set* **and** *r* :: *real*
assumes *r* < *1*
assumes *r*: ⋀*x*. *x* ∈ *X* ⟹ *measure* (*K x*) *X* ≤ *r*

shows *AE* ω *in* *T x*. *ev* (*HLD* (− *X*)) ω
⟨*proof*⟩

3.6 Trace space with Restriction

definition *rT x* = *restrict-space* (*T x*) {ω. *enabled x* ω}

lemma *space-rT*: $\omega \in$ *space* (*rT x*) ⟷ *enabled x* ω
⟨*proof*⟩

lemma *Collect-enabled-S*[*measurable*]: *Collect* (*enabled x*) ∈ *sets S*
⟨*proof*⟩

lemma *space-rT-in-S*: *space* (*rT x*) ∈ *sets S*
⟨*proof*⟩

lemma *sets-rT*: *A* ∈ *sets* (*rT x*) ⟷ *A* ∈ *sets S* ∧ *A* ⊆ {ω. *enabled x* ω}
⟨*proof*⟩

lemma *prob-space-rT*: *prob-space* (*rT x*)
⟨*proof*⟩

lemma *measurable-force-enabled2*[*measurable*]: *force-enabled x* ∈ *measurable S* (*rT x*)
⟨*proof*⟩

lemma *space-rT-not-empty*[*simp*]: *space* (*rT x*) ≠ {}
⟨*proof*⟩

lemma *T-eq-bind'*: *T x* = *do* { *y* ← *measure-pmf* (*K x*) ; ω ← *T y* ; *return S* (*y* ## ω) }
⟨*proof*⟩

lemma *rT-eq-bind*: *rT x* = *do* { *y* ← *measure-pmf* (*K x*) ; ω ← *rT y* ; *return* (*rT x*) (*y* ## ω) }
⟨*proof*⟩

lemma *snth-rT*: (λ*x*. *x* !! *n*) ∈ *measurable* (*rT x*) (*count-space* (*acc* `` {*x*}))
⟨*proof*⟩

3.7 Bisimulation

lemma *T-coinduct*[*consumes 1*, *case-names prob sets cont*]:
assumes *R x M*
assumes *prob*: ⋀*x M*. *R x M* ⟹ *prob-space M*
and *sets*: ⋀*x M*. *R x M* ⟹ *sets M* = *sets S*
and *cont'*: ⋀*x M*. *R x M* ⟹ ∃ *M'*. (∀ *y*∈*K x*. *R y* (*M' y*)) ∧ (∀ *y*. *sets* (*M' y*) = *S* ∧ *prob-space* (*M' y*)) ∧
M = (*measure-pmf* (*K x*) ⤜ (λ*y*. *distr* (*M' y*) *S* ((##) *y*)))
shows *T x* = *M*

⟨*proof*⟩

lemma *T-bisim*:
assumes *M*: ⋀*x*. *prob-space* (*M x*) ⋀*x*. *sets* (*M x*) = *sets S*
and *M-eq*: ⋀*x*. *M x* = (*measure-pmf* (*K x*) ⤜ (λ*s*. *distr* (*M s*) *S* ((##) *s*)))
shows *T* = *M*
⟨*proof*⟩

lemma *T-subprob'*[*measurable*]: *T* ∈ *measurable* (*count-space UNIV*) (*subprob-algebra S*)
⟨*proof*⟩

lemma *T-subprob''*[*simp*]: *T a* ∈ *space* (*subprob-algebra S*)
⟨*proof*⟩

lemma *AE-not-suntil-coinduct* [*consumes 1*, *case-names ψ φ*]:
assumes *P s*
assumes *ψ*: ⋀*s*. *P s* ⟹ *s* ∉ *ψ*
assumes *φ*: ⋀*s t*. *P s* ⟹ *s* ∈ *φ* ⟹ *t* ∈ *K s* ⟹ *P t*
shows *AE ω in T s*. *not* (*HLD φ suntil HLD ψ*) (*s ## ω*)
⟨*proof*⟩

lemma *AE-not-suntil-coinduct-strong* [*consumes 1*, *case-names ψ φ*]:
assumes *P s*
assumes *P-ψ*: ⋀*s*. *P s* ⟹ *s* ∉ *ψ*
assumes *P-φ*: ⋀*s t*. *P s* ⟹ *s* ∈ *φ* ⟹ *t* ∈ *K s* ⟹ *P t* ∨
(*AE ω in T t*. *not* (*HLD φ suntil HLD ψ*) (*t ## ω*))
shows *AE ω in T s*. *not* (*HLD φ suntil HLD ψ*) (*s ## ω*) (**is** *?nuntil s*)
⟨*proof*⟩

end

3.8 Reward Structure on Markov Chains

locale *MC-with-rewards* = *MC-syntax K* **for** *K* :: *'s* ⇒ *'s pmf* +
fixes *ι* :: *'s* ⇒ *'s* ⇒ *ennreal* **and** *ϱ* :: *'s* ⇒ *ennreal*
assumes *ι-nonneg*: ⋀*s t*. *0* ≤ *ι s t* **and** *ϱ-nonneg*: ⋀*s*. *0* ≤ *ϱ s*
assumes *measurable-ι*[*measurable*]: (λ(*a*, *b*). *ι a b*) ∈ *borel-measurable* (*count-space UNIV* ⨂$_M$ *count-space UNIV*)
begin

definition *reward-until* :: *'s set* ⇒ *'s* ⇒ *'s stream* ⇒ *ennreal* **where**
reward-until X = *lfp* (λ*F s ω*. *if s* ∈ *X then 0 else ϱ s* + *ι s* (*shd ω*) + (*F* (*shd ω*) (*stl ω*)))

lemma *measurable-ϱ*[*measurable*]: *ϱ* ∈ *borel-measurable* (*count-space UNIV*)
⟨*proof*⟩

lemma *measurable-reward-until*[*measurable* (*raw*)]:

assumes [*measurable*]: $f \in$ *measurable* M (*count-space UNIV*)
assumes [*measurable*]: $g \in$ *measurable* M S
shows $(\lambda x.$ *reward-until* X $(f\ x)$ $(g\ x)) \in$ *borel-measurable* M
⟨*proof*⟩

lemma *continuous-reward-until*:
sup-continuous $(\lambda F\ s\ \omega.$ *if* $s \in X$ *then 0 else* $\varrho\ s + \iota\ s$ $(shd\ \omega) + (F$ $(shd\ \omega)$ $(stl\ \omega)))$
⟨*proof*⟩

lemma
shows *reward-until-unfold*: *reward-until* $X\ s\ \omega =$
(if $s \in X$ *then 0 else* $\varrho\ s + \iota\ s$ $(shd\ \omega) +$ *reward-until* X $(shd\ \omega)$ $(stl\ \omega))$
(**is** *?unfold*)
⟨*proof*⟩

lemma *reward-until-simps*[*simp*]:
shows $s \in X \Longrightarrow$ *reward-until* $X\ s\ \omega = 0$
and $s \notin X \Longrightarrow$ *reward-until* $X\ s\ \omega = \varrho\ s + \iota\ s$ $(shd\ \omega) +$ *reward-until* X $(shd\ \omega)$ $(stl\ \omega)$
⟨*proof*⟩

lemma *reward-until-SCons*[*simp*]:
reward-until $X\ s$ $(t\ \#\#\ \omega) =$ *(if* $s \in X$ *then 0 else* $\varrho\ s + \iota\ s\ t +$ *reward-until* X $t\ \omega)$
⟨*proof*⟩

lemma *nn-integral-reward-until-finite*:
assumes [*simp*]: *finite* $(acc\ ``\ \{s\})$ (**is** *finite* $(?R\ ``\ \{s\}))$
assumes ϱ: $\bigwedge t.\ (s,\ t) \in$ *acc-on* $(-H) \Longrightarrow \varrho\ t < \infty$
assumes ι: $\bigwedge t\ t'.\ (s,\ t) \in$ *acc-on* $(-H) \Longrightarrow t' \in K\ t \Longrightarrow \iota\ t\ t' < \infty$
assumes *ev*: $AE\ \omega$ *in* $T\ s.$ *ev* $(HLD\ H)\ \omega$
shows $(\int^+ \omega.$ *reward-until* $H\ s\ \omega\ \partial T\ s) \neq \infty$
⟨*proof*⟩

end

3.9 Bisimulation on a relation

definition *rel-set-strong* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow 'a\ set \Rightarrow 'b\ set \Rightarrow bool$
where *rel-set-strong* $R\ A\ B \longleftrightarrow (\forall x\ y.\ R\ x\ y \longrightarrow (x \in A \longleftrightarrow y \in B))$

lemma *T-eq-rel-half*[*consumes 4*, *case-names prob sets cont*]:
fixes $R :: 's \Rightarrow 't \Rightarrow bool$ **and** $f :: 's \Rightarrow 't$ **and** $S :: 's\ set$
assumes *R-def*: $\bigwedge s\ t.\ R\ s\ t \longleftrightarrow (s \in S \wedge f\ s = t)$
assumes A[*measurable*]: $A \in sets$ (*stream-space* (*count-space UNIV*))
and B[*measurable*]: $B \in sets$ (*stream-space* (*count-space UNIV*))
and *AB*: *rel-set-strong* (*stream-all2* R) $A\ B$ **and** *KL*: *rel-fun* R (*rel-pmf* R) $K\ L$ **and** *xy*: $R\ x\ y$

shows *MC-syntax.T K x A = MC-syntax.T L y B*
⟨*proof*⟩

3.10 Product Construction

locale *MC-pair* =
K1: *MC-syntax K1* + *K2*: *MC-syntax K2* **for** *K1 K2*
begin

definition *Kp* ≡ λ(*a*, *b*). *pair-pmf* (*K1 a*) (*K2 b*)

sublocale *MC-syntax Kp* ⟨*proof*⟩

definition
$szip_E$ *a b* ≡ λ(*ω1*, *ω2*). *szip* (*K1.force-enabled a ω1*) (*K2.force-enabled b ω2*)

lemma *szip-rT*[*measurable*]: (λ(*ω1*, *ω2*). *szip ω1 ω2*) ∈ *measurable* (*K1.rT x1* $\bigotimes_M$ *K2.rT x2*) *S*
⟨*proof*⟩

lemma *measurable-szipE*[*measurable*]: $szip_E$ *a b* ∈ *measurable* (*K1.S* $\bigotimes_M$ *K2.S*) *S*
⟨*proof*⟩

lemma *T-eq-prod*: *T* = (λ(*x1*, *x2*). *do* { *ω1* ← *K1.T x1* ; *ω2* ← *K2.T x2* ; *return S* ($szip_E$ *x1 x2* (*ω1*, *ω2*)) })
(**is** - = *?B*)
⟨*proof*⟩

lemma *nn-integral-pT*:
fixes *f* **assumes** [*measurable*]: *f* ∈ *borel-measurable S*
shows (∫⁺ω. *f ω* ∂*T* (*x*, *y*)) = (∫⁺*ω1*. ∫⁺*ω2*. *f* ($szip_E$ *x y* (*ω1*, *ω2*)) ∂*K2.T y* ∂*K1.T x*)
⟨*proof*⟩

lemma *prod-eq-prob-T*:
assumes [*measurable*]: *Measurable.pred K1.S P1 Measurable.pred K2.S P2*
shows *P*(*ω in K1.T x1*. *P1 ω*) ∗ *P*(*ω in K2.T x2*. *P2 ω*) =
P(*ω in T* (*x1*, *x2*). *P1* (*smap fst ω*) ∧ *P2* (*smap snd ω*))
⟨*proof*⟩

end

end

3.11 Trace Space equal to Markov Chains

theory *Trace-Space-Equals-Markov-Processes*
imports *Discrete-Time-Markov-Chain*
begin

We can construct for each time-homogeneous discrete-time Markov chain a corresponding probability space using *Markov-Models.Discrete-Time-Markov-Chain*. The constructed probability space has the same probabilities.

locale *Time-Homogeneous-Discrete-Markov-Process = M?: prob-space +*
fixes $S :: 's\ set$ **and** $X :: nat \Rightarrow 'a \Rightarrow 's$
assumes X [*measurable*]: $\bigwedge t.\ X\ t \in measurable\ M\ (count\text{-}space\ UNIV)$
assumes S: $countable\ S\ \bigwedge n.\ AE\ x\ in\ M.\ X\ n\ x \in S$
assumes MC: $\bigwedge n\ s\ s'.$
$\mathcal{P}(\omega\ in\ M.\ \forall\, t{\leq}n.\ X\ t\ \omega = s\ t\ ) \neq 0 \Longrightarrow$
$\mathcal{P}(\omega\ in\ M.\ X\ (Suc\ n)\ \omega = s' \mid \forall\, t{\leq}n.\ X\ t\ \omega = s\ t\ ) =$
$\mathcal{P}(\omega\ in\ M.\ X\ (Suc\ n)\ \omega = s' \mid X\ n\ \omega = s\ n\ )$
assumes TH: $\bigwedge n\ m\ s\ t.$
$\mathcal{P}(\omega\ in\ M.\ X\ n\ \omega = t) \neq 0 \Longrightarrow \mathcal{P}(\omega\ in\ M.\ X\ m\ \omega = t) \neq 0 \Longrightarrow$
$\mathcal{P}(\omega\ in\ M.\ X\ (Suc\ n)\ \omega = s \mid X\ n\ \omega = t) = \mathcal{P}(\omega\ in\ M.\ X\ (Suc\ m)\ \omega = s \mid X\ m\ \omega = t)$
begin

context
begin

interpretation *pmf-as-measure* ⟨*proof*⟩

lift-definition $I :: 's\ pmf$ **is** $distr\ M\ (count\text{-}space\ UNIV)\ (X\ 0)$
⟨*proof*⟩

lemma *I-in-S*:
assumes $pmf\ I\ s \neq 0$ **shows** $s \in S$
⟨*proof*⟩

lift-definition $K :: 's \Rightarrow 's\ pmf$ **is**
$\lambda s.\ with\ (\lambda n.\ \mathcal{P}(\omega\ in\ M.\ X\ n\ \omega = s) \neq 0)$
$(\lambda n.\ distr\ (uniform\text{-}measure\ M\ \{\omega{\in}space\ M.\ X\ n\ \omega = s\})\ (count\text{-}space\ UNIV)\ (X\ (Suc\ n)))$
$(uniform\text{-}measure\ (count\text{-}space\ UNIV)\ \{s\})$
⟨*proof*⟩

lemma *pmf-K*:
assumes n: $0 < \mathcal{P}(\omega\ in\ M.\ X\ n\ \omega = s)$
shows $pmf\ (K\ s)\ t = \mathcal{P}(\omega\ in\ M.\ X\ (Suc\ n)\ \omega = t \mid X\ n\ \omega = s)$
⟨*proof*⟩

lemma *pmf-K2*:
$(\bigwedge n.\ \mathcal{P}(\omega\ in\ M.\ X\ n\ \omega = s) = 0) \Longrightarrow pmf\ (K\ s)\ t = indicator\ \{t\}\ s$
⟨*proof*⟩

end

sublocale K: *MC-syntax K* ⟨*proof*⟩

lemma *bind-I-K-eq-M*: $K.T'\ I = distr\ M\ K.S\ (\lambda\omega.\ to\text{-}stream\ (\lambda n.\ X\ n\ \omega))$ (**is** *- = ?D*)
⟨*proof*⟩

end

lemma (**in** *MC-syntax*) *is-THDTMC*:
fixes $I :: 's\ pmf$
defines $U \equiv (SIGMA\ s{:}UNIV.\ K\ s)^*\ ``\ I$
shows *Time-Homogeneous-Discrete-Markov-Process* $(T'\ I)\ U\ (\lambda n\ \omega.\ \omega\ !!\ n)$
⟨*proof*⟩

end

4 Classifying Markov Chain States

theory *Classifying-Markov-Chain-States*
imports
HOL−Computational-Algebra.Group-Closure
Discrete-Time-Markov-Chain
begin

lemma *eventually-mult-Gcd*:
fixes $S :: nat\ set$
assumes S: $\bigwedge s\ t.\ s \in S \Longrightarrow t \in S \Longrightarrow s + t \in S$
assumes s: $s \in S\ s > 0$
shows *eventually* $(\lambda m.\ m * Gcd\ S \in S)$ *sequentially*
⟨*proof*⟩

context *MC-syntax*
begin

4.1 Expected number of visits

definition $G\ s\ t = (\int^+\omega.\ scount\ (HLD\ \{t\})\ (s\ \#\#\ \omega)\ \partial T\ s)$

lemma *G-eq*: $G\ s\ t = (\int^+\omega.\ emeasure\ (count\text{-}space\ UNIV)\ \{i.\ (s\ \#\#\ \omega)\ !!\ i = t\}\ \partial T\ s)$
⟨*proof*⟩

definition $p\ s\ t\ n = \mathcal{P}(\omega\ in\ T\ s.\ (s\ \#\#\ \omega)\ !!\ n = t)$

definition *gf-G* $s\ t\ z = (\sum n.\ p\ s\ t\ n *_R z \hat{}\ n)$

definition *convergence-G* $s\ t\ z \longleftrightarrow summable\ (\lambda n.\ p\ s\ t\ n * norm\ z \hat{}\ n)$

lemma *p-nonneg*[*simp*]: $0 \le p\ x\ y\ n$
⟨*proof*⟩

lemma *p-le-1*: $p\ x\ y\ n \leq 1$
⟨*proof*⟩

lemma *p-x-x-0*[*simp*]: $p\ x\ x\ 0 = 1$
⟨*proof*⟩

lemma *p-0*: $p\ x\ y\ 0 = (\mathit{if}\ x = y\ \mathit{then}\ 1\ \mathit{else}\ 0)$
⟨*proof*⟩

lemma *p-in-reachable*: **assumes** $(x, y) \notin (\mathit{SIGMA}\ x{:}\mathit{UNIV}.\ K\ x)^*$ **shows** $p\ x\ y\ n = 0$
⟨*proof*⟩

lemma *p-Suc*: $\mathit{ennreal}\ (p\ x\ y\ (\mathit{Suc}\ n)) = (\int^+ w.\ p\ w\ y\ n\ \partial K\ x)$
⟨*proof*⟩

lemma *p-Suc'*:
$p\ x\ y\ (\mathit{Suc}\ n) = (\int x'.\ p\ x'\ y\ n\ \partial K\ x)$
⟨*proof*⟩

lemma *p-add*: $p\ x\ y\ (n + m) = (\int^+ w.\ p\ x\ w\ n * p\ w\ y\ m\ \partial \mathit{count\text{-}space}\ \mathit{UNIV})$
⟨*proof*⟩

lemma *prob-reachable-le*:
assumes [*simp*]: $m \leq n$
shows $p\ x\ y\ m * p\ y\ w\ (n - m) \leq p\ x\ w\ n$
⟨*proof*⟩

lemma *G-eq-suminf*: $G\ x\ y = (\sum i.\ \mathit{ennreal}\ (p\ x\ y\ i))$
⟨*proof*⟩

lemma *G-eq-real-suminf*:
$\mathit{convergence\text{-}G}\ x\ y\ (1{::}\mathit{real}) \Longrightarrow G\ x\ y = \mathit{ennreal}\ (\sum i.\ p\ x\ y\ i)$
⟨*proof*⟩

lemma *convergence-norm-G*:
$\mathit{convergence\text{-}G}\ x\ y\ z \Longrightarrow \mathit{summable}\ (\lambda n.\ p\ x\ y\ n * \mathit{norm}\ z\ \hat{}\ n)$
⟨*proof*⟩

lemma *convergence-G*:
$\mathit{convergence\text{-}G}\ x\ y\ (z{::}'a{::}\{\mathit{banach}, \mathit{real\text{-}normed\text{-}div\text{-}algebra}\}) \Longrightarrow \mathit{summable}\ (\lambda n.\ p\ x\ y\ n *_R z\ \hat{}\ n)$
⟨*proof*⟩

lemma *convergence-G-less-1*:
fixes $z :: - :: \{\mathit{banach}, \mathit{real\text{-}normed\text{-}field}\}$
assumes z: $\mathit{norm}\ z < 1$ **shows** $\mathit{convergence\text{-}G}\ x\ y\ z$
⟨*proof*⟩

lemma *lim-gf-G*: $((\lambda z.\ ennreal\ (gf\text{-}G\ x\ y\ z)) \longrightarrow G\ x\ y)\ (at\text{-}left\ (1::real))$
⟨*proof*⟩

4.2 Reachability probability

definition $u\ x\ y\ n = \mathcal{P}(\omega\ in\ T\ x.\ ev\text{-}at\ (HLD\ \{y\})\ n\ \omega)$

definition $U\ s\ t = \mathcal{P}(\omega\ in\ T\ s.\ ev\ (HLD\ \{t\})\ \omega)$

definition $gf\text{-}U\ x\ y\ z = (\sum n.\ u\ x\ y\ n *_R z \mathbin{\hat{}} Suc\ n)$

definition $f\ x\ y\ n = \mathcal{P}(\omega\ in\ T\ x.\ ev\text{-}at\ (HLD\ \{y\})\ n\ (x\ \#\#\ \omega))$

definition $F\ s\ t = \mathcal{P}(\omega\ in\ T\ s.\ ev\ (HLD\ \{t\})\ (s\ \#\#\ \omega))$

definition $gf\text{-}F\ x\ y\ z = (\sum n.\ f\ x\ y\ n * z \mathbin{\hat{}} n)$

lemma *f-Suc*: $x \neq y \Longrightarrow f\ x\ y\ (Suc\ n) = u\ x\ y\ n$
⟨*proof*⟩

lemma *f-Suc-eq*: $f\ x\ x\ (Suc\ n) = 0$
⟨*proof*⟩

lemma *f-0*: $f\ x\ y\ 0 = (if\ x = y\ then\ 1\ else\ 0)$
⟨*proof*⟩

lemma shows *u-nonneg*: $0 \leq u\ x\ y\ n$ **and** *u-le-1*: $u\ x\ y\ n \leq 1$
⟨*proof*⟩

lemma shows *f-nonneg*: $0 \leq f\ x\ y\ n$ **and** *f-le-1*: $f\ x\ y\ n \leq 1$
⟨*proof*⟩

lemma *U-nonneg*[*simp*]: $0 \leq U\ x\ y$
⟨*proof*⟩

lemma *U-le-1*: $U\ s\ t \leq 1$
⟨*proof*⟩

lemma *U-cases*: $U\ s\ s = 1 \vee U\ s\ s < 1$
⟨*proof*⟩

lemma *u-sums-U*: $u\ x\ y\ sums\ U\ x\ y$
⟨*proof*⟩

lemma *gf-U-eq-U*: $gf\text{-}U\ x\ y\ 1 = U\ x\ y$
⟨*proof*⟩

lemma *f-sums-F*: $f\ x\ y\ sums\ F\ x\ y$
⟨*proof*⟩

lemma *F-nonneg*[*simp*]: $0 \leq F\ x\ y$
⟨*proof*⟩

lemma *F-le-1*: $F\ x\ y \leq 1$
⟨*proof*⟩

lemma *gf-F-eq-F*: $gf\text{-}F\ x\ y\ 1 = F\ x\ y$
⟨*proof*⟩

lemma *gf-F-le-1*:
fixes z :: *real*
assumes z: $0 \leq z\ z \leq 1$
shows $gf\text{-}F\ x\ y\ z \leq 1$
⟨*proof*⟩

lemma *u-le-p*: $u\ x\ y\ n \leq p\ x\ y\ (Suc\ n)$
⟨*proof*⟩

lemma *f-le-p*: $f\ x\ y\ n \leq p\ x\ y\ n$
⟨*proof*⟩

lemma *convergence-norm-U*:
fixes z :: - :: *real-normed-div-algebra*
assumes z: *convergence-G x y z*
shows *summable* $(\lambda n.\ u\ x\ y\ n * norm\ z \mathbin{\hat{}} Suc\ n)$
⟨*proof*⟩

lemma *convergence-norm-F*:
fixes z :: - :: *real-normed-div-algebra*
assumes z: *convergence-G x y z*
shows *summable* $(\lambda n.\ f\ x\ y\ n * norm\ z \mathbin{\hat{}} n)$
⟨*proof*⟩

lemma *gf-G-nonneg*:
fixes z :: *real*
shows $0 \leq z \Longrightarrow z < 1 \Longrightarrow 0 \leq gf\text{-}G\ x\ y\ z$
⟨*proof*⟩

lemma *gf-F-nonneg*:
fixes z :: *real*
shows $0 \leq z \Longrightarrow z < 1 \Longrightarrow 0 \leq gf\text{-}F\ x\ y\ z$
⟨*proof*⟩

lemma *convergence-U*:
fixes z :: - :: *banach*
shows *convergence-G x y z* $\Longrightarrow$ *summable* $(\lambda n.\ u\ x\ y\ n * z \mathbin{\hat{}} Suc\ n)$
⟨*proof*⟩

lemma *p-eq-sum-p-u*: $p\ x\ y\ (Suc\ n) = (\sum i{\leq}n.\ p\ y\ y\ (n - i) * u\ x\ y\ i)$
⟨*proof*⟩

lemma *p-eq-sum-p-f*: $p\ x\ y\ n = (\sum i{\leq}n.\ p\ y\ y\ (n - i) * f\ x\ y\ i)$
⟨*proof*⟩

lemma *gf-G-eq-gf-F*:
assumes *z*: $norm\ z < 1$
shows $gf\text{-}G\ x\ y\ z = gf\text{-}F\ x\ y\ z * gf\text{-}G\ y\ y\ z$
⟨*proof*⟩

lemma *gf-G-eq-gf-U*:
fixes $z :: 'z :: \{banach, real\text{-}normed\text{-}field\}$
assumes *z*: *convergence-G x x z*
shows $gf\text{-}G\ x\ x\ z = 1\ /\ (1 - gf\text{-}U\ x\ x\ z)\ gf\text{-}U\ x\ x\ z \neq 1$
⟨*proof*⟩

lemma *gf-U*: $(gf\text{-}U\ x\ y \longrightarrow U\ x\ y)\ (at\text{-}left\ 1)$
⟨*proof*⟩

lemma *gf-U-le-1*: **assumes** *z*: $0 < z\ z < 1$ **shows** $gf\text{-}U\ x\ y\ z \leq (1{::}real)$
⟨*proof*⟩

lemma *gf-F*: $(gf\text{-}F\ x\ y \longrightarrow F\ x\ y)\ (at\text{-}left\ 1)$
⟨*proof*⟩

lemma *U-bounded*: $0 \leq U\ x\ y\ U\ x\ y \leq 1$
⟨*proof*⟩

4.3 Recurrent states

definition *recurrent* :: $'s \Rightarrow bool$ **where**
$recurrent\ s \longleftrightarrow (AE\ \omega\ in\ T\ s.\ ev\ (HLD\ \{s\})\ \omega)$

lemma *recurrent-iff-U-eq-1*: $recurrent\ s \longleftrightarrow U\ s\ s = 1$
⟨*proof*⟩

definition $H\ s\ t = \mathcal{P}(\omega\ in\ T\ s.\ alw\ (ev\ (HLD\ \{t\}))\ \omega)$

lemma *H-eq*:
$recurrent\ s \longleftrightarrow H\ s\ s = 1$
$\neg\ recurrent\ s \longleftrightarrow H\ s\ s = 0$
$H\ s\ t = U\ s\ t * H\ t\ t$
⟨*proof*⟩

lemma *recurrent-iff-G-infinite*: $recurrent\ x \longleftrightarrow G\ x\ x = \infty$
⟨*proof*⟩

definition *communicating* :: $('s \times 's)\ set$ **where**

communicating = *acc* ∩ *acc*$^{-1}$

definition *essential-class* :: *'s set* ⇒ *bool* **where**
essential-class C ⟷ *C* ∈ *UNIV* // *communicating* ∧ *acc* `` *C* ⊆ *C*

lemma *accI-U*:
assumes *0* < *U x y* **shows** (*x*, *y*) ∈ *acc*
⟨*proof*⟩

lemma *accD-pos*:
assumes (*x*, *y*) ∈ *acc*
shows ∃ *n*. *0* < *p x y n*
⟨*proof*⟩

lemma *accI-pos*: *0* < *p x y n* ⟹ (*x*, *y*) ∈ *acc*
⟨*proof*⟩

lemma *recurrent-iffI-communicating*:
assumes (*x*, *y*) ∈ *communicating*
shows *recurrent x* ⟷ *recurrent y*
⟨*proof*⟩

lemma *recurrent-acc*:
assumes *recurrent x* (*x*, *y*) ∈ *acc*
shows *U y x* = *1 H y x* = *1 recurrent y* (*x*, *y*) ∈ *communicating*
⟨*proof*⟩

lemma *equiv-communicating*: *equiv UNIV communicating*
⟨*proof*⟩

lemma *recurrent-class*:
assumes *recurrent x*
shows *acc* `` {*x*} = *communicating* `` {*x*}
⟨*proof*⟩

lemma *irreduccible-recurrent-class*:
assumes *recurrent x* **shows** *acc* `` {*x*} ∈ *UNIV* // *communicating*
⟨*proof*⟩

lemma *essential-classI*:
assumes *C*: *C* ∈ *UNIV* // *communicating*
assumes *eq*: ⋀*x y*. *x* ∈ *C* ⟹ (*x*, *y*) ∈ *acc* ⟹ *y* ∈ *C*
shows *essential-class C*
⟨*proof*⟩

lemma *essential-recurrent-class*:
assumes *recurrent x* **shows** *essential-class* (*communicating* `` {*x*})
⟨*proof*⟩

lemma *essential-classD2*:
essential-class $C \Longrightarrow x \in C \Longrightarrow (x, y) \in acc \Longrightarrow y \in C$
⟨*proof*⟩

lemma *essential-classD3*:
essential-class $C \Longrightarrow x \in C \Longrightarrow y \in C \Longrightarrow (x, y) \in communicating$
⟨*proof*⟩

lemma *AE-acc*:
shows $AE\ \omega\ in\ T\ x.\ \forall m.\ (x, (x\ \#\#\ \omega)\ !!\ m) \in acc$
⟨*proof*⟩

lemma *finite-essential-class-imp-recurrent*:
assumes *C*: *essential-class C finite C* **and** *x*: $x \in C$
shows *recurrent x*
⟨*proof*⟩

lemma *irreducibleD*:
$C \in UNIV\ //\ communicating \Longrightarrow a \in C \Longrightarrow b \in C \Longrightarrow (a, b) \in communicating$
⟨*proof*⟩

lemma *irreducibleD2*:
$C \in UNIV\ //\ communicating \Longrightarrow a \in C \Longrightarrow (a, b) \in communicating \Longrightarrow b \in C$
⟨*proof*⟩

lemma *essential-class-iff-recurrent*:
$finite\ C \Longrightarrow C \in UNIV\ //\ communicating \Longrightarrow essential\text{-}class\ C \longleftrightarrow (\forall x{\in}C.\ recurrent\ x)$
⟨*proof*⟩

definition $U'\ x\ y = (\int{}^{+}\omega.\ eSuc\ (sfirst\ (HLD\ \{y\})\ \omega)\ \partial T\ x)$

lemma *U′-neq-zero*[*simp*]: $U'\ x\ y \neq 0$
⟨*proof*⟩

definition *gf-U′* $x\ y\ z = (\sum n.\ u\ x\ y\ n * Suc\ n * z\ \widehat{}\ n)$

definition *pos-recurrent* $x \longleftrightarrow recurrent\ x \wedge U'\ x\ x \neq \infty$

lemma *summable-gf-U′*:
assumes *z*: $norm\ z < 1$
shows *summable* $(\lambda n.\ u\ x\ y\ n * Suc\ n * z\ \widehat{}\ n)$
⟨*proof*⟩

lemma *gf-U′-nonneg*[*simp*]: $0 < z \Longrightarrow z < 1 \Longrightarrow 0 \leq gf\text{-}U'\ x\ y\ z$
⟨*proof*⟩

lemma *DERIV-gf-U*:

fixes $z :: real$ **assumes** z: $0 < z\ z < 1$
shows $DERIV\ (gf\text{-}U\ x\ y)\ z :> gf\text{-}U'\ x\ y\ z$
⟨*proof*⟩

lemma *sfirst-finiteI-recurrent*:
$recurrent\ x \Longrightarrow (x,\ y) \in acc \Longrightarrow AE\ \omega\ in\ T\ x.\ sfirst\ (HLD\ \{y\})\ \omega < \infty$
⟨*proof*⟩

lemma *U′-eq-suminf*:
assumes x: $recurrent\ x\ (x,\ y) \in acc$
shows $U'\ x\ y = (\sum i.\ ennreal\ (u\ x\ y\ i * Suc\ i))$
⟨*proof*⟩

lemma *gf-U′-tendsto-U′*:
assumes x: $recurrent\ x\ (x,\ y) \in acc$
shows $((\lambda z.\ ennreal\ (gf\text{-}U'\ x\ y\ z)) \longrightarrow U'\ x\ y)\ (at\text{-}left\ 1)$
⟨*proof*⟩

lemma *one-le-integral-t*:
assumes x: $recurrent\ x$ **shows** $1 \leq U'\ x\ x$
⟨*proof*⟩

lemma *gf-U′-pos*:
fixes $z :: real$
assumes z: $0 < z\ z < 1$ **and** $U\ x\ y \neq 0$
shows $0 < gf\text{-}U'\ x\ y\ z$
⟨*proof*⟩

lemma *inverse-gf-U′-tendsto*:
assumes $recurrent\ y$
shows $((\lambda x.\ -\ 1\ /\ -\ gf\text{-}U'\ y\ y\ x) \longrightarrow enn2real\ (1\ /\ U'\ y\ y))\ (at\text{-}left\ (1{::}real))$
⟨*proof*⟩

lemma *gf-G-pos*:
fixes $z :: real$
assumes z: $0 < z\ z < 1$ **and** $*$: $(x,\ y) \in acc$
shows $0 < gf\text{-}G\ x\ y\ z$
⟨*proof*⟩

lemma *pos-recurrentI-communicating*:
assumes y: $pos\text{-}recurrent\ y$ **and** x: $(y,\ x) \in communicating$
shows $pos\text{-}recurrent\ x$
⟨*proof*⟩

lemma *pos-recurrent-iffI-communicating*:
$(y,\ x) \in communicating \Longrightarrow pos\text{-}recurrent\ y \longleftrightarrow pos\text{-}recurrent\ x$
⟨*proof*⟩

lemma *U-le-F*: $U\ x\ y \leq F\ x\ y$

⟨*proof*⟩

lemma *not-empty-irreducible*: $C \in UNIV\ //\ communicating \Longrightarrow C \neq \{\}$
⟨*proof*⟩

4.4 Stationary distribution

definition *stat* :: *'s set* ⇒ *'s measure* **where**
stat C = *point-measure UNIV* (λ*x*. *indicator C x* / *U' x x*)

lemma *sets-stat*[*simp*]: *sets* (*stat C*) = *sets* (*count-space UNIV*)
⟨*proof*⟩

lemma *space-stat*[*simp*]: *space* (*stat C*) = *UNIV*
⟨*proof*⟩

lemma *stat-subprob*:
assumes *C*: *essential-class C* **and** *countable C* **and** *pos*: ∀ *c*∈*C*. *pos-recurrent c*
shows *emeasure* (*stat C*) *C* ≤ *1*
⟨*proof*⟩

lemma *emeasure-stat-not-C*:
assumes *y* ∉ *C*
shows *emeasure* (*stat C*) {*y*} = *0*
⟨*proof*⟩

definition *stationary-distribution* :: *'s pmf* ⇒ *bool* **where**
stationary-distribution N ⟷ *N* = *bind-pmf N K*

lemma *stationary-distributionI*:
assumes *le*: ⋀*y*. (∫ *x*. *pmf* (*K x*) *y* ∂*measure-pmf N*) ≤ *pmf N y*
shows *stationary-distribution N*
⟨*proof*⟩

lemma *stationary-distribution-iterate*:
assumes *N*: *stationary-distribution N*
shows *ennreal* (*pmf N y*) = (∫ $^+$*x*. *p x y n* ∂*N*)
⟨*proof*⟩

lemma *stationary-distribution-iterate'*:
assumes *stationary-distribution N*
shows *measure N* {*y*} = (∫ *x*. *p x y n* ∂*N*)
⟨*proof*⟩

lemma *stationary-distributionD*:
assumes *C*: *essential-class C countable C*
assumes *N*: *stationary-distribution N N* ⊆ *C*
shows ∀ *x*∈*C*. *pos-recurrent x measure-pmf N* = *stat C*
⟨*proof*⟩

lemma *measure-point-measure-singleton*:
$x \in A \Longrightarrow$ *measure* (*point-measure A X*) $\{x\}$ = *enn2real* (*X x*)
⟨*proof*⟩

lemma *stationary-distribution-imp-int-t*:
assumes *C*: *essential-class C countable C stationary-distribution N N* $\subseteq$ *C*
assumes *x*: $x \in C$ **shows** *U' x x* = *1 / ennreal* (*pmf N x*)
⟨*proof*⟩

definition *period-set x* = $\{i.\ 0 < i \wedge 0 < p\ x\ x\ i\ \}$
definition *period C* = (*SOME d*. $\forall x{\in}C.$ *d* = *Gcd* (*period-set x*))

lemma *Gcd-period-set-invariant*:
assumes *c*: $(x, y) \in$ *communicating*
shows *Gcd* (*period-set x*) = *Gcd* (*period-set y*)
⟨*proof*⟩

lemma *period-eq*:
assumes *C* $\in$ *UNIV // communicating x* $\in$ *C*
shows *period C* = *Gcd* (*period-set x*)
⟨*proof*⟩

definition *aperiodic C* $\longleftrightarrow$ *C* $\in$ *UNIV // communicating* $\wedge$ *period C* = *1*

definition *not-ephemeral C* $\longleftrightarrow$ *C* $\in$ *UNIV // communicating* $\wedge$ $\neg$ ($\exists x.$ *C* = $\{x\}$ $\wedge$ *p x x 1* = *0*)

lemma *not-ephemeralD*:
assumes *C*: *not-ephemeral C x* $\in$ *C*
shows $\exists\, n{>}0.$ *0* < *p x x n*
⟨*proof*⟩

lemma *not-ephemeralD-pos-period*:
assumes *C*: *not-ephemeral C*
shows *0* < *period C*
⟨*proof*⟩

lemma *period-posD*:
assumes *C*: *C* $\in$ *UNIV // communicating* **and** *0* < *period C x* $\in$ *C*
shows $\exists\, n{>}0.$ *0* < *p x x n*
⟨*proof*⟩

lemma *not-ephemeralD-pos-period'*:
assumes *C*: *C* $\in$ *UNIV // communicating*
shows *not-ephemeral C* $\longleftrightarrow$ *0* < *period C*
⟨*proof*⟩

lemma *eventually-periodic*:
assumes *C*: $C \in UNIV$ // *communicating* $0 < period\ C\ x \in C$
shows *eventually* $(\lambda m.\ 0 < p\ x\ x\ (m * period\ C))$ *sequentially*
⟨*proof*⟩

lemma *aperiodic-eventually-recurrent*:
$aperiodic\ C \longleftrightarrow C \in UNIV$ // $communicating \wedge (\forall x{\in}C.\ eventually\ (\lambda m.\ 0 < p\ x\ x\ m)\ sequentially)$
⟨*proof*⟩

lemma *stationary-distributionD-emeasure*:
assumes *N*: *stationary-distribution N*
shows $emeasure\ N\ A = (\int^+ s.\ emeasure\ (K\ s)\ A\ \partial N)$
⟨*proof*⟩

lemma *communicatingD1*:
$C \in UNIV$ // $communicating \Longrightarrow (a,\ b) \in communicating \Longrightarrow a \in C \Longrightarrow b \in C$
⟨*proof*⟩

lemma *communicatingD2*:
$C \in UNIV$ // $communicating \Longrightarrow (a,\ b) \in communicating \Longrightarrow b \in C \Longrightarrow a \in C$
⟨*proof*⟩

lemma *acc-iff*: $(x,\ y) \in acc \longleftrightarrow (\exists n.\ 0 < p\ x\ y\ n)$
⟨*proof*⟩

lemma *communicating-iff*: $(x,\ y) \in communicating \longleftrightarrow (\exists n.\ 0 < p\ x\ y\ n) \wedge (\exists n.\ 0 < p\ y\ x\ n)$
⟨*proof*⟩

end

context *MC-pair*
begin

lemma *p-eq-p1-p2*:
$p\ (x1,\ x2)\ (y1,\ y2)\ n = K1.p\ x1\ y1\ n * K2.p\ x2\ y2\ n$
⟨*proof*⟩

lemma *P-accD*:
assumes $((x1,\ x2),\ (y1,\ y2)) \in acc$**shows** $(x1,\ y1) \in K1.acc\ (x2,\ y2) \in K2.acc$
⟨*proof*⟩

lemma *aperiodicI-pair*:
assumes *C1*: *K1.aperiodic C1* **and** *C2*: *K2.aperiodic C2*

shows *aperiodic* (*C1* × *C2*)
⟨*proof*⟩

lemma *stationary-distributionI-pair*:
assumes *N1*: *K1.stationary-distribution N1*
assumes *N2*: *K2.stationary-distribution N2*
shows *stationary-distribution* (*pair-pmf N1 N2*)
⟨*proof*⟩

end

context *MC-syntax*
begin

lemma *stationary-distribution-imp-limit*:
assumes *C*: *aperiodic C essential-class C countable C* **and** *N*: *stationary-distribution N N* ⊆ *C*
assumes [*simp*]: *y* ∈ *C*
shows (λ*n*. ∫ *x*. |*p y x n* − *pmf N x*| ∂*count-space C*) ⟶ *0*
(**is** *?L* ⟶ *0*)
⟨*proof*⟩

lemma *stationary-distribution-imp-p-limit*:
assumes *aperiodic C essential-class C* **and** [*simp*]: *countable C*
assumes *N*: *stationary-distribution N N* ⊆ *C*
assumes [*simp*]: *x* ∈ *C y* ∈ *C*
shows *p x y* ⟶ *pmf N y*
⟨*proof*⟩

end

lemma (**in** *MC-syntax*) *essential-classI2*:
assumes *X* ≠ {}
assumes *accI*: ⋀*x y*. *x* ∈ *X* ⟹ *y* ∈ *X* ⟹ (*x*, *y*) ∈ *acc*
assumes *ED*: ⋀*x y*. *x* ∈ *X* ⟹ *y* ∈ *set-pmf* (*K x*) ⟹ *y* ∈ *X*
shows *essential-class X*
⟨*proof*⟩

end

5 Markov Decision Processes

theory *Markov-Decision-Process*
imports *Discrete-Time-Markov-Chain*
begin

lemma *some-elem-ne*: *s* ≠ {} ⟹ *some-elem s* ∈ *s*
⟨*proof*⟩

5.1 Configurations

We want to construct a *non-free* codatatype *'s cfg = Cfg (state: 's) (action: 's pmf) (cont: 's ⇒ 's cfg).* with the restriction *state (cont cfg s) = s*

hide-const *cont*

codatatype *'s scheduler = Scheduler (action-sch: 's pmf) (cont-sch: 's ⇒ 's scheduler)*

lemma *equivp-rel-prod: equivp R ⟹ equivp Q ⟹ equivp (rel-prod R Q)*
⟨*proof*⟩

coinductive *eq-scheduler :: 's scheduler ⇒ 's scheduler ⇒ bool*
where
⋀D. action-sch sc1 = D ⟹ action-sch sc2 = D ⟹
(∀ s∈D. eq-scheduler (cont-sch sc1 s) (cont-sch sc2 s)) ⟹ eq-scheduler sc1 sc2

lemma *eq-scheduler-refl[intro]: eq-scheduler sc sc*
⟨*proof*⟩

quotient-type *'s cfg = 's × 's scheduler / rel-prod (=) eq-scheduler*
⟨*proof*⟩

lift-definition *state :: 's cfg ⇒ 's* **is** *fst*
⟨*proof*⟩

lift-definition *action :: 's cfg ⇒ 's pmf* **is** *λ(s, sc). action-sch sc*
⟨*proof*⟩

lift-definition *cont :: 's cfg ⇒ 's ⇒ 's cfg* **is**
λ(s, sc) t. if t ∈ action-sch sc then (t, cont-sch sc t) else
(t, cont-sch sc (some-elem (action-sch sc)))
⟨*proof*⟩

lift-definition *Cfg :: 's ⇒ 's pmf ⇒ ('s ⇒ 's cfg) ⇒ 's cfg* **is**
λs D c. (s, Scheduler D (λt. snd (c t)))
⟨*proof*⟩

lift-definition *cfg-corec :: 's ⇒ ('a ⇒ 's pmf) ⇒ ('a ⇒ 's ⇒ 'a) ⇒ 'a ⇒ 's cfg* **is**
λs D C x. (s, corec-scheduler D (λx s. Inr (C x s)) x) ⟨*proof*⟩

lemma *state-cont[simp]: state (cont cfg s) = s*
⟨*proof*⟩

lemma *state-Cfg[simp]: state (Cfg s d' c') = s*
⟨*proof*⟩

lemma *action-Cfg[simp]: action (Cfg s d' c') = d'*

⟨*proof*⟩

lemma *cont-Cfg*[*simp*]: $t \in$ *set-pmf* $d' \Longrightarrow$ *state* $(c'\ t) = t \Longrightarrow$ *cont* $(Cfg\ s\ d'\ c')\ t = c'\ t$
⟨*proof*⟩

lemma *state-cfg-corec*[*simp*]: *state* $(cfg\text{-}corec\ s\ d\ c\ x) = s$
⟨*proof*⟩

lemma *action-cfg-corec*[*simp*]: *action* $(cfg\text{-}corec\ s\ d\ c\ x) = d\ x$
⟨*proof*⟩

lemma *cont-cfg-corec*[*simp*]: $t \in$ *set-pmf* $(d\ x) \Longrightarrow$ *cont* $(cfg\text{-}corec\ s\ d\ c\ x)\ t = cfg\text{-}corec\ t\ d\ c\ (c\ x\ t)$
⟨*proof*⟩

lemma *cfg-coinduct*[*consumes 1*, *case-names state action cont*, *coinduct pred*]:
$X\ c\ d \Longrightarrow (\bigwedge c\ d.\ X\ c\ d \Longrightarrow state\ c = state\ d) \Longrightarrow (\bigwedge c\ d.\ X\ c\ d \Longrightarrow action\ c = action\ d) \Longrightarrow$
$(\bigwedge c\ d\ t.\ X\ c\ d \Longrightarrow t \in set\text{-}pmf\ (action\ c) \Longrightarrow X\ (cont\ c\ t)\ (cont\ d\ t)) \Longrightarrow c = d$
⟨*proof*⟩

coinductive *rel-cfg* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow 'a\ cfg \Rightarrow 'b\ cfg \Rightarrow bool$ **for** $P :: 'a \Rightarrow 'b \Rightarrow bool$
where
$P\ (state\ cfg1)\ (state\ cfg2) \Longrightarrow$
rel-pmf $(\lambda s\ t.\ rel\text{-}cfg\ P\ (cont\ cfg1\ s)\ (cont\ cfg2\ t))\ (action\ cfg1)\ (action\ cfg2) \Longrightarrow$
rel-cfg P cfg1 cfg2

lemma *rel-cfg-state*: *rel-cfg P cfg1 cfg2* $\Longrightarrow$ *P* $(state\ cfg1)\ (state\ cfg2)$
⟨*proof*⟩

lemma *rel-cfg-cont*:
rel-cfg P cfg1 cfg2 $\Longrightarrow$
rel-pmf $(\lambda s\ t.\ rel\text{-}cfg\ P\ (cont\ cfg1\ s)\ (cont\ cfg2\ t))\ (action\ cfg1)\ (action\ cfg2)$
⟨*proof*⟩

lemma *rel-cfg-action*:
assumes *P*: *rel-cfg P cfg1 cfg2* **shows** *rel-pmf P* $(action\ cfg1)\ (action\ cfg2)$
⟨*proof*⟩

lemma *rel-cfg-eq*: *rel-cfg* $(=)$ *cfg1 cfg2* $\longleftrightarrow$ *cfg1* $=$ *cfg2*
⟨*proof*⟩

5.2 Configuration with Memoryless Scheduler

definition *memoryless-on f s* $=$ *cfg-corec s f* $(\lambda\text{-}\ t.\ t)\ s$

lemma
shows *state-memoryless-on*[*simp*]: *state* (*memoryless-on f s*) = *s*
and *action-memoryless-on*[*simp*]: *action* (*memoryless-on f s*) = *f s*
and *cont-memoryless-on*[*simp*]: *t* ∈ (*f s*) ⟹ *cont* (*memoryless-on f s*) *t* = *memoryless-on f t*
⟨*proof*⟩

definition *K-cfg* :: *'s cfg* ⇒ *'s cfg pmf* **where**
K-cfg cfg = *map-pmf* (*cont cfg*) (*action cfg*)

lemma *set-K-cfg*: *set-pmf* (*K-cfg cfg*) = *cont cfg* ‘ *set-pmf* (*action cfg*)
⟨*proof*⟩

lemma *nn-integral-K-cfg*: (∫⁺*cfg*. *f cfg* ∂*K-cfg cfg*) = (∫⁺*s*. *f* (*cont cfg s*) ∂*action cfg*)
⟨*proof*⟩

5.3 MDP Kernel and Induced Configurations

locale *Markov-Decision-Process* =
fixes *K* :: *'s* ⇒ *'s pmf set*
assumes *K-wf*: ⋀*s*. *K s* ≠ {}
begin

definition *E* = (*SIGMA s*:*UNIV*. ⋃*D*∈*K s*. *set-pmf D*)

coinductive *cfg-onp* :: *'s* ⇒ *'s cfg* ⇒ *bool* **where**
⋀*s*. *state cfg* = *s* ⟹ *action cfg* ∈ *K s* ⟹ (⋀*t*. *t* ∈ *action cfg* ⟹ *cfg-onp t* (*cont cfg t*)) ⟹
cfg-onp s cfg

definition *cfg-on s* = {*cfg*. *cfg-onp s cfg*}

lemma
shows *cfg-onD-action*[*intro*, *simp*]: *cfg* ∈ *cfg-on s* ⟹ *action cfg* ∈ *K s*
and *cfg-onD-cont*[*intro*, *simp*]: *cfg* ∈ *cfg-on s* ⟹ *t* ∈ *action cfg* ⟹ *cont cfg t* ∈ *cfg-on t*
and *cfg-onD-state*[*simp*]: *cfg* ∈ *cfg-on s* ⟹ *state cfg* = *s*
and *cfg-onI*: *state cfg* = *s* ⟹ *action cfg* ∈ *K s* ⟹ (⋀*t*. *t* ∈ *action cfg* ⟹ *cont cfg t* ∈ *cfg-on t*) ⟹ *cfg* ∈ *cfg-on s*
⟨*proof*⟩

lemma *cfg-on-coinduct*[*coinduct set*: *cfg-on*]:
assumes *P s cfg*
assumes ⋀*cfg s*. *P s cfg* ⟹ *state cfg* = *s*
assumes ⋀*cfg s*. *P s cfg* ⟹ *action cfg* ∈ *K s*
assumes ⋀*cfg s t*. *P s cfg* ⟹ *t* ∈ *action cfg* ⟹ *P t* (*cont cfg t*)
shows *cfg* ∈ *cfg-on s*

⟨*proof*⟩

lemma *memoryless-on-cfg-onI*:
assumes $\bigwedge s.\ f\ s \in K\ s$
shows *memoryless-on f s* ∈ *cfg-on s*
⟨*proof*⟩

lemma *cfg-of-cfg-onI*:
$D \in K\ s \Longrightarrow (\bigwedge t.\ t \in D \Longrightarrow c\ t \in$ *cfg-on t*$) \Longrightarrow$ *Cfg s D c* ∈ *cfg-on s*
⟨*proof*⟩

definition *arb-act s* = (*SOME D*. $D \in K\ s$)

lemma *arb-actI*[*simp*]: *arb-act s* ∈ *K s*
⟨*proof*⟩

lemma *cfg-on-not-empty*[*intro*, *simp*]: *cfg-on s* ≠ {}
⟨*proof*⟩

sublocale *MC*: *MC-syntax K-cfg* ⟨*proof*⟩

abbreviation *St* :: *'s stream measure* **where**
St ≡ *stream-space* (*count-space UNIV*)

5.4 Trace Space

definition *T cfg* = *distr* (*MC.T cfg*) *St* (*smap state*)

sublocale *T*: *prob-space T cfg* **for** *cfg*
⟨*proof*⟩

lemma *space-T*[*simp*]: *space* (*T cfg*) = *space St*
⟨*proof*⟩

lemma *sets-T*[*simp*]: *sets* (*T cfg*) = *sets St*
⟨*proof*⟩

lemma *measurable-T1*[*simp*]: *measurable* (*T cfg*) *N* = *measurable St N*
⟨*proof*⟩

lemma *measurable-T2*[*simp*]: *measurable N* (*T cfg*) = *measurable N St*
⟨*proof*⟩

lemma *nn-integral-T*:
assumes [*measurable*]: *f* ∈ *borel-measurable St*
shows $(\int^+ X.\ f\ X\ \partial T\ cfg) = (\int^+ cfg'.\ (\int^+ x.\ f\ (state\ cfg' \#\#\ x)\ \partial T\ cfg')\ \partial K\text{-}cfg\ cfg)$
⟨*proof*⟩

lemma *T-eq*:
T cfg = (*measure-pmf* (*K-cfg cfg*) ⤜ (λ*cfg'*. *distr* (*T cfg'*) *St* (λω. *state cfg'* ## ω)))
⟨*proof*⟩

lemma *T-memoryless-on*: *T* (*memoryless-on ct s*) = *MC-syntax.T ct s*
⟨*proof*⟩

lemma *nn-integral-T-lfp*:
assumes [*measurable*]: *case-prod g* ∈ *borel-measurable* (*count-space UNIV* ⨂$_M$ *borel*)
assumes *cont-g*: ⋀*s*. *sup-continuous* (*g s*)
assumes *int-g*: ⋀*f cfg*. *f* ∈ *borel-measurable* (*stream-space* (*count-space UNIV*))
⟹
(∫⁺ω. *g* (*state cfg*) (*f* ω) ∂*T cfg*) = *g* (*state cfg*) (∫⁺ω. *f* ω ∂*T cfg*)
shows (∫⁺ω. *lfp* (λ*f* ω. *g* (*shd* ω) (*f* (*stl* ω))) ω ∂*T cfg*) =
lfp (λ*f cfg*. ∫⁺*t*. *g* (*state t*) (*f t*) ∂*K-cfg cfg*) *cfg*
⟨*proof*⟩

lemma *emeasure-Collect-T*:
assumes [*measurable*]: *Measurable.pred St P*
shows *emeasure* (*T cfg*) {*x*∈*space St*. *P x*} =
(∫⁺*cfg'*. *emeasure* (*T cfg'*) {*x*∈*space St*. *P* (*state cfg'* ## *x*)} ∂*K-cfg cfg*)
⟨*proof*⟩

definition *E-sup* :: *'s* ⇒ (*'s stream* ⇒ *ennreal*) ⇒ *ennreal*
where
E-sup s f = (⨆ *cfg*∈*cfg-on s*. ∫⁺*x*. *f x* ∂*T cfg*)

lemma *E-sup-const*: 0 ≤ *c* ⟹ *E-sup s* (λ-. *c*) = *c*
⟨*proof*⟩

lemma *E-sup-mult-right*:
assumes [*measurable*]: *f* ∈ *borel-measurable St* **and** [*simp*]: 0 ≤ *c*
shows *E-sup s* (λ*x*. *c* ∗ *f x*) = *c* ∗ *E-sup s f*
⟨*proof*⟩

lemma *E-sup-mono*:
(⋀ω. *f* ω ≤ *g* ω) ⟹ *E-sup s f* ≤ *E-sup s g*
⟨*proof*⟩

lemma *E-sup-add*:
assumes [*measurable*]: *f* ∈ *borel-measurable St g* ∈ *borel-measurable St*
shows *E-sup s* (λ*x*. *f x* + *g x*) ≤ *E-sup s f* + *E-sup s g*
⟨*proof*⟩

lemma *E-sup-add-left*:
assumes [*measurable*]: *f* ∈ *borel-measurable St*
shows *E-sup s* (λ*x*. *f x* + *c*) = *E-sup s f* + *c*

⟨*proof*⟩

lemma *E-sup-add-right*:
  $f \in$ *borel-measurable St* $\Longrightarrow$ *E-sup s* $(\lambda x.\ c + f\ x) = c +$ *E-sup s f*
  ⟨*proof*⟩

lemma *E-sup-SUP*:
  assumes [*measurable*]: $\bigwedge i.\ f\ i \in$ *borel-measurable St* **and** [*simp*]: *incseq f*
  shows *E-sup s* $(\lambda x.\ \bigsqcup i.\ f\ i\ x) = (\bigsqcup i.$ *E-sup s* $(f\ i))$
  ⟨*proof*⟩

lemma *E-sup-iterate*:
  assumes [*measurable*]: $f \in$ *borel-measurable St*
  shows *E-sup s f* $= (\bigsqcup D{\in}K\ s.\ \int^+ t.$ *E-sup t* $(\lambda\omega.\ f\ (t\ \#\#\ \omega))\ \partial$*measure-pmf D*)
⟨*proof*⟩

lemma *E-sup-bot*: *E-sup s* $\bot = 0$
  ⟨*proof*⟩

lemma *E-sup-lfp*:
  fixes *g*
  defines $l \equiv \lambda f\ \omega.\ g$ (*shd* ω) (f (*stl* ω))
  assumes *measurable-g*[*measurable*]: *case-prod g* $\in$ *borel-measurable* (*count-space UNIV* $\bigotimes_M$ *borel*)
  assumes *cont-g*: $\bigwedge s.$ *sup-continuous* ($g\ s$)
  assumes *int-g*: $\bigwedge f$ *cfg*. $f \in$ *borel-measurable St* $\Longrightarrow$
    $(\int^+ \omega.\ g$ (*state cfg*) ($f\ \omega$) ∂T *cfg*) $= g$ (*state cfg*) (*integral*^{N} (T *cfg*) f)
  shows $(\lambda s.$ *E-sup s* (*lfp l*)) $=$ *lfp* $(\lambda f\ s.\ \bigsqcup D{\in}K\ s.\ \int^+ t.\ g\ t\ (f\ t)\ \partial$*measure-pmf D*)
⟨*proof*⟩

definition *P-sup s P* $= (\bigsqcup$ *cfg*$\in$*cfg-on s. emeasure* (T *cfg*) $\{x{\in}$*space St*. $P\ x\})$

lemma *P-sup-eq-E-sup*:
  assumes [*measurable*]: *Measurable.pred St P*
  shows *P-sup s P* $=$ *E-sup s* (*indicator* $\{x{\in}$*space St*. $P\ x\}$)
  ⟨*proof*⟩

lemma *P-sup-True*[*simp*]: *P-sup t* $(\lambda\omega.$ *True*$) = 1$
  ⟨*proof*⟩

lemma *P-sup-False*[*simp*]: *P-sup t* $(\lambda\omega.$ *False*$) = 0$
  ⟨*proof*⟩

lemma *P-sup-SUP*:
  fixes P :: *nat* $\Rightarrow$ *'s stream* $\Rightarrow$ *bool*
  assumes *mono P* **and** P[*measurable*]: $\bigwedge i.$ *Measurable.pred St* ($P\ i$)
  shows *P-sup s* $(\lambda x.\ \exists i.\ P\ i\ x) = (\bigsqcup i.$ *P-sup s* $(P\ i))$

⟨*proof*⟩

lemma *P-sup-lfp*:
assumes *Q*: *sup-continuous Q*
assumes *f*: $f \in$ *measurable St M*
assumes *Q-m*: $\bigwedge P$. *Measurable.pred M P* $\Longrightarrow$ *Measurable.pred M* (*Q P*)
shows *P-sup s* (λx. *lfp Q* (*f x*)) = ($\bigsqcup i$. *P-sup s* (λx. (*Q* $\widehat{}\widehat{}$ *i*) $\bot$ (*f x*)))
⟨*proof*⟩

lemma *P-sup-iterate*:
assumes [*measurable*]: *Measurable.pred St P*
shows *P-sup s P* = ($\bigsqcup D{\in}K$ *s*. $\int^+$ *t*. *P-sup t* ($\lambda\omega$. *P* (*t* ## ω)) ∂*measure-pmf D*)
⟨*proof*⟩

definition *E-inf s f* = ($\bigsqcap$ *cfg*∈*cfg-on s*. $\int^+ x$. *f x* ∂*T cfg*)

lemma *E-inf-const*: *0* $\leq$ *c* $\Longrightarrow$ *E-inf s* (λ-. *c*) = *c*
⟨*proof*⟩

lemma *E-inf-mono*:
($\bigwedge\omega$. *f* ω $\leq$ *g* ω) $\Longrightarrow$ *E-inf s f* $\leq$ *E-inf s g*
⟨*proof*⟩

lemma *E-inf-iterate*:
assumes [*measurable*]: *f* $\in$ *borel-measurable St*
shows *E-inf s f* = ($\bigsqcap D{\in}K$ *s*. $\int^+$ *t*. *E-inf t* ($\lambda\omega$. *f* (*t* ## ω)) ∂*measure-pmf D*)
⟨*proof*⟩

lemma *emeasure-T-const*[*simp*]: *emeasure* (*T s*) (*space St*) = *1*
⟨*proof*⟩

lemma *E-inf-greatest*:
($\bigwedge$*cfg*. *cfg* $\in$ *cfg-on s* $\Longrightarrow$ *x* $\leq$ ($\int^+ x$. *f x* ∂*T cfg*)) $\Longrightarrow$ *x* $\leq$ *E-inf s f*
⟨*proof*⟩

lemma *E-inf-lower2*:
cfg $\in$ *cfg-on s* $\Longrightarrow$ ($\int^+ x$. *f x* ∂*T cfg*) $\leq$ *x* $\Longrightarrow$ *E-inf s f* $\leq$ *x*
⟨*proof*⟩

Maybe the following statement can be generalized to infinite *K s*.

lemma *E-inf-lfp*:
fixes *g*
defines *l* $\equiv$ $\lambda f\ \omega$. *g* (*shd* ω) (*f* (*stl* ω))
assumes *measurable-g*[*measurable*]: *case-prod g* $\in$ *borel-measurable* (*count-space UNIV* $\bigotimes_M$ *borel*)
assumes *cont-g*: $\bigwedge s$. *sup-continuous* (*g s*)
assumes *int-g*: $\bigwedge f$ *cfg*. *f* $\in$ *borel-measurable St* $\Longrightarrow$
($\int^+ \omega$. *g* (*state cfg*) (*f* ω) ∂*T cfg*) = *g* (*state cfg*) (*integral*^{N} (*T cfg*) *f*)

assumes *K-finite*: $\bigwedge s.$ *finite* $(K\ s)$
shows $(\lambda s.$ *E-inf* s $(lfp\ l)) = lfp\ (\lambda f\ s.\ \prod D{\in}K\ s.\ \int^+ t.\ g\ t\ (f\ t)\ \partial$*measure-pmf* $D)$
⟨*proof*⟩

definition *P-inf* $s\ P = (\prod cfg{\in}$*cfg-on* $s.$ *emeasure* $(T\ cfg)\ \{x{\in}$*space St*. $P\ x\})$

lemma *P-inf-eq-E-inf*:
assumes [*measurable*]: *Measurable.pred St P*
shows *P-inf* $s\ P =$ *E-inf* s (*indicator* $\{x{\in}$*space St*. $P\ x\})$
⟨*proof*⟩

lemma *P-inf-True*[*simp*]: *P-inf* $t\ (\lambda\omega.$ *True*$) = 1$
⟨*proof*⟩

lemma *P-inf-False*[*simp*]: *P-inf* $t\ (\lambda\omega.$ *False*$) = 0$
⟨*proof*⟩

lemma *P-inf-INF*:
fixes $P ::$ *nat* $\Rightarrow$ *'s stream* $\Rightarrow$ *bool*
assumes *decseq* P **and** P[*measurable*]: $\bigwedge i.$ *Measurable.pred St* $(P\ i)$
shows *P-inf* $s\ (\lambda x.\ \forall i.\ P\ i\ x) = (\prod i.$ *P-inf* $s\ (P\ i))$
⟨*proof*⟩

lemma *P-inf-gfp*:
assumes *Q*: *inf-continuous Q*
assumes *f*: $f \in$ *measurable St M*
assumes *Q-m*: $\bigwedge P.$ *Measurable.pred M* $P \Longrightarrow$ *Measurable.pred M* $(Q\ P)$
shows *P-inf* $s\ (\lambda x.$ *gfp* $Q\ (f\ x)) = (\prod i.$ *P-inf* $s\ (\lambda x.\ (Q\ \widehat{\ }\ i)\ \top\ (f\ x)))$
⟨*proof*⟩

lemma *P-inf-iterate*:
assumes [*measurable*]: *Measurable.pred St P*
shows *P-inf* $s\ P = (\prod D{\in}K\ s.\ \int^+ t.$ *P-inf* $t\ (\lambda\omega.\ P\ (t\ \#\#\ \omega))\ \partial$*measure-pmf* $D)$
⟨*proof*⟩

end

5.5 Finite MDPs

locale *Finite-Markov-Decision-Process* = *Markov-Decision-Process* K **for** $K ::$ *'s* $\Rightarrow$ *'s pmf set* +
fixes $S ::$ *'s set*
assumes *S-not-empty*: $S \neq \{\}$
assumes *S-finite*: *finite S*
assumes *K-closed*: $\bigwedge s.\ s \in S \Longrightarrow (\bigcup D{\in}K\ s.$ *set-pmf* $D) \subseteq S$
assumes *K-finite*: $\bigwedge s.\ s \in S \Longrightarrow$ *finite* $(K\ s)$
begin

lemma *action-closed*: $s \in S \Longrightarrow cfg \in cfg\text{-}on\ s \Longrightarrow t \in action\ cfg \Longrightarrow t \in S$
⟨*proof*⟩

lemma *set-pmf-closed*: $s \in S \Longrightarrow D \in K\ s \Longrightarrow t \in D \Longrightarrow t \in S$
⟨*proof*⟩

lemma *Pi-closed*: $ct \in Pi\ S\ K \Longrightarrow s \in S \Longrightarrow t \in ct\ s \Longrightarrow t \in S$
⟨*proof*⟩

lemma *E-closed*: $s \in S \Longrightarrow (s, t) \in E \Longrightarrow t \in S$
⟨*proof*⟩

lemma *set-pmf-finite*: $s \in S \Longrightarrow D \in K\ s \Longrightarrow finite\ D$
⟨*proof*⟩

definition *valid-cfg* = $(\bigcup s{\in}S.\ cfg\text{-}on\ s)$

lemma *valid-cfgI*: $s \in S \Longrightarrow cfg \in cfg\text{-}on\ s \Longrightarrow cfg \in valid\text{-}cfg$
⟨*proof*⟩

lemma *valid-cfgD*: $cfg \in valid\text{-}cfg \Longrightarrow cfg \in cfg\text{-}on\ (state\ cfg)$
⟨*proof*⟩

lemma
shows *valid-cfg-state-in-S*: $cfg \in valid\text{-}cfg \Longrightarrow state\ cfg \in S$
and *valid-cfg-action*: $cfg \in valid\text{-}cfg \Longrightarrow s \in action\ cfg \Longrightarrow s \in S$
and *valid-cfg-cont*: $cfg \in valid\text{-}cfg \Longrightarrow s \in action\ cfg \Longrightarrow cont\ cfg\ s \in valid\text{-}cfg$
⟨*proof*⟩

lemma *valid-K-cfg*[*intro*]: $cfg \in valid\text{-}cfg \Longrightarrow cfg' \in K\text{-}cfg\ cfg \Longrightarrow cfg' \in valid\text{-}cfg$
⟨*proof*⟩

definition *simple ct* = *memoryless-on* $(\lambda s.\ if\ s \in S\ then\ ct\ s\ else\ arb\text{-}act\ s)$

lemma *simple-cfg-on*[*simp*]: $ct \in Pi\ S\ K \Longrightarrow simple\ ct\ s \in cfg\text{-}on\ s$
⟨*proof*⟩

lemma *simple-valid-cfg*[*simp*]: $ct \in Pi\ S\ K \Longrightarrow s \in S \Longrightarrow simple\ ct\ s \in valid\text{-}cfg$
⟨*proof*⟩

lemma *cont-simple*[*simp*]: $s \in S \Longrightarrow t \in set\text{-}pmf\ (ct\ s) \Longrightarrow cont\ (simple\ ct\ s)\ t = simple\ ct\ t$
⟨*proof*⟩

lemma *state-simple*[*simp*]: $state\ (simple\ ct\ s) = s$
⟨*proof*⟩

lemma *action-simple*[*simp*]: $s \in S \Longrightarrow action\ (simple\ ct\ s) = ct\ s$
⟨*proof*⟩

lemma *simple-valid-cfg-iff*: $ct \in Pi\ S\ K \Longrightarrow simple\ ct\ s \in valid\text{-}cfg \longleftrightarrow s \in S$
⟨*proof*⟩

end

end
theory *MDP-Reachability-Problem*
imports *Markov-Decision-Process*
begin

inductive-set *directed-towards* :: $'a\ set \Rightarrow ('a \times 'a)\ set \Rightarrow 'a\ set$ **for** $A\ r$ **where**
start: $\bigwedge x.\ x \in A \Longrightarrow x \in directed\text{-}towards\ A\ r$
| *step*: $\bigwedge x\ y.\ y \in directed\text{-}towards\ A\ r \Longrightarrow (x, y) \in r \Longrightarrow x \in directed\text{-}towards\ A\ r$

hide-fact (**open**) *start step*

lemma *directed-towards-mono*:
assumes $s \in directed\text{-}towards\ A\ F$ $F \subseteq G$ **shows** $s \in directed\text{-}towards\ A\ G$
⟨*proof*⟩

lemma *directed-eq-rtrancl*: $x \in directed\text{-}towards\ A\ r \longleftrightarrow (\exists\, a{\in}A.\ (x, a) \in r^*)$
⟨*proof*⟩

lemma *directed-eq-rtrancl-Image*: $directed\text{-}towards\ A\ r = (r^*)^{-1}\ ``\ A$
⟨*proof*⟩

locale *Reachability-Problem* = *Finite-Markov-Decision-Process K S* **for** $K :: 's \Rightarrow 's\ pmf\ set$ **and** S +
fixes *S1 S2* :: *'s set*
assumes *S1*: $S1 \subseteq S$ **and** *S2*: $S2 \subseteq S$ **and** *S1-S2*: $S1 \cap S2 = \{\}$
begin

lemma [*measurable*]:
$S \in sets\ (count\text{-}space\ UNIV)$ $S1 \in sets\ (count\text{-}space\ UNIV)$ $S2 \in sets\ (count\text{-}space\ UNIV)$
⟨*proof*⟩

definition
$v = (\lambda cfg{\in}valid\text{-}cfg.\ emeasure\ (T\ cfg)\ \{x{\in}space\ St.\ (HLD\ S1\ suntil\ HLD\ S2)\ (state\ cfg\ \#\#\ x)\})$

lemma *v-eq*: $cfg \in valid\text{-}cfg \Longrightarrow$
$v\ cfg = emeasure\ (T\ cfg)\ \{x{\in}space\ St.\ (HLD\ S1\ suntil\ HLD\ S2)\ (state\ cfg\ \#\#\ x)\}$
⟨*proof*⟩

lemma *real-v*: $cfg \in valid\text{-}cfg \Longrightarrow enn2real\ (v\ cfg) = \mathcal{P}(\omega\ in\ T\ cfg.\ (HLD\ S1\ suntil\ HLD\ S2)\ (state\ cfg\ \#\#\ \omega))$

⟨*proof*⟩

lemma *v-le-1*: *cfg* ∈ *valid-cfg* ⟹ *v cfg* ≤ *1*
⟨*proof*⟩

lemma *v-neq-Pinf*[*simp*]: *cfg* ∈ *valid-cfg* ⟹ *v cfg* ≠ *top*
⟨*proof*⟩

lemma *v-1-AE*: *cfg* ∈ *valid-cfg* ⟹ *v cfg* = *1* ⟷ (*AE* ω *in T cfg*. (*HLD S1 suntil HLD S2*) (*state cfg ##* ω))
⟨*proof*⟩

lemma *v-0-AE*: *cfg* ∈ *valid-cfg* ⟹ *v cfg* = *0* ⟷ (*AE x in T cfg*. *not* (*HLD S1 suntil HLD S2*) (*state cfg ## x*))
⟨*proof*⟩

lemma *v-S2*[*simp*]: *cfg* ∈ *valid-cfg* ⟹ *state cfg* ∈ *S2* ⟹ *v cfg* = *1*
⟨*proof*⟩

lemma *v-nS12*[*simp*]: *cfg* ∈ *valid-cfg* ⟹ *state cfg* ∉ *S1* ⟹ *state cfg* ∉ *S2* ⟹ *v cfg* = *0*
⟨*proof*⟩

lemma *v-nS*[*simp*]: *cfg* ∉ *valid-cfg* ⟹ *v cfg* = *undefined*
⟨*proof*⟩

lemma *v-S1*:
assumes *cfg*[*simp*, *intro*]: *cfg* ∈ *valid-cfg* **and** *cfg-S1*[*simp*]: *state cfg* ∈ *S1*
shows *v cfg* = (∫⁺*s*. *v* (*cont cfg s*) ∂*action cfg*)
⟨*proof*⟩

lemma *real-v-integrable*:
integrable (*action cfg*) (λ*s*. *enn2real* (*v* (*cont cfg s*)))
⟨*proof*⟩

lemma *real-v-integral-eq*:
assumes *cfg*[*simp*]: *cfg* ∈ *valid-cfg*
shows *enn2real* (∫⁺ *s*. *v* (*cont cfg s*) ∂*action cfg*) = ∫ *s*. *enn2real* (*v* (*cont cfg s*)) ∂*action cfg*
⟨*proof*⟩

lemma *v-eq-0-coinduct*[*consumes 3*, *case-names valid nS2 cont*]:
assumes ∗: *P cfg*
assumes *valid*: ⋀*cfg*. *P cfg* ⟹ *cfg* ∈ *valid-cfg*
assumes *nS2*: ⋀*cfg*. *P cfg* ⟹ *state cfg* ∉ *S2*
assumes *cont*: ⋀*cfg cfg'*. *P cfg* ⟹ *state cfg* ∈ *S1* ⟹ *cfg'* ∈ *K-cfg cfg* ⟹ *P cfg'* ∨ *v cfg'* = *0*
shows *v cfg* = *0*
⟨*proof*⟩

definition $p = (\lambda s{\in}S.\ P\text{-}sup\ s\ (\lambda\omega.\ (HLD\ S1\ suntil\ HLD\ S2)\ (s\ \#\#\ \omega)))$

lemma *p-eq-SUP-v*: $s \in S \Longrightarrow p\ s = \bigsqcup\ (v\ `\ cfg\text{-}on\ s)$
⟨*proof*⟩

lemma *v-le-p*: $cfg \in valid\text{-}cfg \Longrightarrow v\ cfg \leq p\ (state\ cfg)$
⟨*proof*⟩

lemma *p-eq-0-imp*: $cfg \in valid\text{-}cfg \Longrightarrow p\ (state\ cfg) = 0 \Longrightarrow v\ cfg = 0$
⟨*proof*⟩

lemma *p-eq-0-iff*: $s \in S \Longrightarrow p\ s = 0 \longleftrightarrow (\forall\ cfg{\in}cfg\text{-}on\ s.\ v\ cfg = 0)$
⟨*proof*⟩

lemma *p-le-1*: $s \in S \Longrightarrow p\ s \leq 1$
⟨*proof*⟩

lemma *p-undefined*[*simp*]: $s \notin S \Longrightarrow p\ s = undefined$
⟨*proof*⟩

lemma *p-not-inf*[*simp*]: $s \in S \Longrightarrow p\ s \neq top$
⟨*proof*⟩

lemma *p-S1*: $s \in S1 \Longrightarrow p\ s = (\bigsqcup D{\in}K\ s.\ \int^+ t.\ p\ t\ \partial measure\text{-}pmf\ D)$
⟨*proof*⟩

lemma *p-S2*[*simp*]: $s \in S2 \Longrightarrow p\ s = 1$
⟨*proof*⟩

lemma *p-nS12*: $s \in S \Longrightarrow s \notin S1 \Longrightarrow s \notin S2 \Longrightarrow p\ s = 0$
⟨*proof*⟩

lemma *p-pos*:
assumes $(s,\ t) \in (SIGMA\ s{:}S1.\ \bigcup D{\in}K\ s.\ set\text{-}pmf\ D)^*\ t \in S2$ **shows** $0 < p\ s$
⟨*proof*⟩

definition *F-sup* :: $('s \Rightarrow ennreal) \Rightarrow 's \Rightarrow ennreal$ **where**
$F\text{-}sup\ f = (\lambda s{\in}S.\ if\ s \in S2\ then\ 1\ else\ if\ s \in S1\ then\ SUP\ D{\in}K\ s.\ \int^+ t.\ f\ t\ \partial measure\text{-}pmf\ D\ else\ 0)$

lemma *F-sup-cong*: $(\bigwedge s.\ s \in S \Longrightarrow f\ s = g\ s) \Longrightarrow F\text{-}sup\ f\ s = F\text{-}sup\ g\ s$
⟨*proof*⟩

lemma *continuous-F-sup*: *sup-continuous F-sup*
⟨*proof*⟩

lemma *mono-F-sup*: *mono F-sup*

⟨*proof*⟩

lemma *lfp-F-sup-iterate*: *lfp F-sup* = (*SUP i*. (*F-sup* ^^ *i*) (λ*x*∈*S*. *0*))
⟨*proof*⟩

lemma *p-eq-lfp-F-sup*: *p* = *lfp F-sup*
⟨*proof*⟩

definition S_e = {*s*∈*S*. *p s* = *0*}

lemma S_e: $S_e \subseteq S$
⟨*proof*⟩

lemma *v-*S_e: *cfg* ∈ *valid-cfg* ⟹ *state cfg* ∈ S_e ⟹ *v cfg* = *0*
⟨*proof*⟩

lemma S_e*-nS2*: $S_e \cap S2 = \{\}$
⟨*proof*⟩

lemma S_e*-E1*: *s* ∈ S_e ∩ *S1* ⟹ (*s*, *t*) ∈ *E* ⟹ *t* ∈ S_e
⟨*proof*⟩

lemma S_e*-E2*: *s* ∈ *S1* ⟹ (⋀*t*. (*s*, *t*) ∈ *E* ⟹ *t* ∈ S_e) ⟹ *s* ∈ S_e
⟨*proof*⟩

lemma S_e*-E-iff*: *s* ∈ *S1* ⟹ *s* ∈ S_e ⟷ (∀ *t*. (*s*, *t*) ∈ *E* ⟶ *t* ∈ S_e)
⟨*proof*⟩

definition S_r = *S* − (S_e ∪ *S2*)

lemma S_r: $S_r \subseteq S$
⟨*proof*⟩

lemma S_r*-S1*: $S_r \subseteq$ *S1*
⟨*proof*⟩

lemma S_r*-eq*: S_r = *S1* − S_e
⟨*proof*⟩

lemma *v-neq-0-imp*: *cfg* ∈ *valid-cfg* ⟹ *v cfg* ≠ *0* ⟹ *state cfg* ∈ S_r ∪ *S2*
⟨*proof*⟩

lemma *valid-cfg-action-in-K*: *cfg* ∈ *valid-cfg* ⟹ *action cfg* ∈ *K* (*state cfg*)
⟨*proof*⟩

lemma *K-cfg-E*: *cfg* ∈ *valid-cfg* ⟹ *cfg*′ ∈ *K-cfg cfg* ⟹ (*state cfg*, *state cfg*′) ∈ *E*
⟨*proof*⟩

lemma *S_r-directed-towards-S2*:
assumes *s*: $s \in S_r$
shows $s \in$ *directed-towards S2* $\{(s,\ t) \mid s\ t.\ s \in S_r \wedge (s,\ t) \in E\}$ (**is** $s \in ?D$)
⟨*proof*⟩

definition $proper\ ct \longleftrightarrow ct \in Pi_E\ S\ K \wedge (\forall\, s{\in}S_r.\ v\ (simple\ ct\ s) > 0)$

lemma *S_r-nS2*: $s \in S_r \Longrightarrow s \notin S2$
⟨*proof*⟩

lemma *properD1*: $proper\ ct \Longrightarrow ct \in Pi_E\ S\ K$
⟨*proof*⟩

lemma *proper-eq*:
assumes *ct*[*simp*, *intro*]: $ct \in Pi_E\ S\ K$
shows $proper\ ct \longleftrightarrow S_r \subseteq$ *directed-towards S2* $(SIGMA\ s{:}S_r.\ ct\ s)$
(**is** - ⟷ - ⊆ *?D*)
⟨*proof*⟩

lemma *exists-proper*:
obtains *ct* **where** *proper ct*
⟨*proof*⟩

definition $l\text{-}desc\ X\ ct\ l\ s \longleftrightarrow$
$s \in$ *directed-towards S2* $(SIGMA\ s : X.\ \{l\ s\}) \wedge$
$v\ (simple\ ct\ s) \leq v\ (simple\ ct\ (l\ s)) \wedge$
$l\ s \in maximal\ (\lambda s.\ v\ (simple\ ct\ s))\ (ct\ s)$

lemma *exists-l-desc*:
assumes *ct*: *proper ct*
shows $\exists\, l{\in}S_r \to S_r \cup S2.\ \forall\, s{\in}S_r.\ l\text{-}desc\ S_r\ ct\ l\ s$
⟨*proof*⟩

lemma *F-v-memoryless*:
obtains *ct* **where** $ct \in Pi_E\ S\ K\ v{\circ}simple\ ct = F\text{-}sup\ (v{\circ}simple\ ct)$
⟨*proof*⟩

lemma *p-v-memoryless*:
obtains *ct* **where** $ct \in Pi_E\ S\ K\ p = v{\circ}simple\ ct$
⟨*proof*⟩

definition $n = (\lambda s{\in}S.\ P\text{-}inf\ s\ (\lambda\omega.\ (HLD\ S1\ suntil\ HLD\ S2)\ (s\ \#\#\ \omega)))$

lemma *n-eq-INF-v*: $s \in S \Longrightarrow n\ s = (\prod cfg{\in}cfg\text{-}on\ s.\ v\ cfg)$
⟨*proof*⟩

lemma *n-le-v*: $s \in S \Longrightarrow cfg \in cfg\text{-}on\ s \Longrightarrow n\ s \leq v\ cfg$
⟨*proof*⟩

lemma *n-eq-1-imp*: $s \in S \Longrightarrow cfg \in cfg\text{-}on\ s \Longrightarrow n\ s = 1 \Longrightarrow v\ cfg = 1$
⟨*proof*⟩

lemma *n-eq-1-iff*: $s \in S \Longrightarrow n\ s = 1 \longleftrightarrow (\forall\, cfg{\in}cfg\text{-}on\ s.\ v\ cfg = 1)$
⟨*proof*⟩

lemma *n-le-1*: $s \in S \Longrightarrow n\ s \leq 1$
⟨*proof*⟩

lemma *n-undefined*[*simp*]: $s \notin S \Longrightarrow n\ s = undefined$
⟨*proof*⟩

lemma *n-eq-0*: $s \in S \Longrightarrow cfg \in cfg\text{-}on\ s \Longrightarrow v\ cfg = 0 \Longrightarrow n\ s = 0$
⟨*proof*⟩

lemma *n-not-inf*[*simp*]: $s \in S \Longrightarrow n\ s \neq top$
⟨*proof*⟩

lemma *n-S1*: $s \in S1 \Longrightarrow n\ s = (\prod D{\in}K\ s.\ \int^+ t.\ n\ t\ \partial measure\text{-}pmf\ D)$
⟨*proof*⟩

lemma *n-S2*[*simp*]: $s \in S2 \Longrightarrow n\ s = 1$
⟨*proof*⟩

lemma *n-nS12*: $s \in S \Longrightarrow s \notin S1 \Longrightarrow s \notin S2 \Longrightarrow n\ s = 0$
⟨*proof*⟩

lemma *n-pos*:
assumes $P\ s\ s \in S1\ wf\ R$
assumes *cont*: $\bigwedge s\ D.\ P\ s \Longrightarrow s \in S1 \Longrightarrow D \in K\ s \Longrightarrow \exists\, w{\in}D.\ ((w,\ s) \in R \wedge w \in S1 \wedge P\ w) \vee 0 < n\ w$
shows $0 < n\ s$
⟨*proof*⟩

definition *F-inf* :: $('s \Rightarrow ennreal) \Rightarrow ('s \Rightarrow ennreal)$ **where**
$F\text{-}inf\ f = (\lambda s{\in}S.\ if\ s \in S2\ then\ 1\ else\ if\ s \in S1\ then\ (\prod D{\in}K\ s.\ \int^+ t.\ f\ t\ \partial measure\text{-}pmf\ D)\ else\ 0)$

lemma *F-inf-n*: $F\text{-}inf\ n = n$
⟨*proof*⟩

lemma *F-inf-nS*[*simp*]: $s \notin S \Longrightarrow F\text{-}inf\ f\ s = undefined$
⟨*proof*⟩

lemma *mono-F-inf*: *mono F-inf*
⟨*proof*⟩

lemma *S1-nS2*: $s \in S1 \Longrightarrow s \notin S2$
⟨*proof*⟩

lemma *n-eq-lfp-F-inf*: $n = lfp\ F\text{-}inf$
⟨*proof*⟩

lemma *real-n*: $s \in S \Longrightarrow ennreal\ (enn2real\ (n\ s)) = n\ s$
⟨*proof*⟩

lemma *real-p*: $s \in S \Longrightarrow ennreal\ (enn2real\ (p\ s)) = p\ s$
⟨*proof*⟩

lemma *p-ub*:
fixes x
assumes $s \in S$
assumes *solution*: $\bigwedge s\ D.\ s \in S1 \Longrightarrow D \in K\ s \Longrightarrow (\sum t{\in}S.\ pmf\ D\ t * x\ t) \leq x\ s$
assumes *solution-0*: $\bigwedge s.\ s \in S \Longrightarrow p\ s = 0 \Longrightarrow x\ s = 0$
assumes *solution-S2*: $\bigwedge s.\ s \in S2 \Longrightarrow x\ s = 1$
shows $enn2real\ (p\ s) \leq x\ s$ (**is** *?y s* ≤ -)
⟨*proof*⟩

lemma *n-lb*:
fixes x
assumes $s \in S$
assumes *solution*: $\bigwedge s\ D.\ s \in S1 \Longrightarrow D \in K\ s \Longrightarrow x\ s \leq (\sum t{\in}S.\ pmf\ D\ t * x\ t)$
assumes *solution-n0*: $\bigwedge s.\ s \in S \Longrightarrow n\ s = 0 \Longrightarrow x\ s = 0$
assumes *solution-S2*: $\bigwedge s.\ s \in S2 \Longrightarrow x\ s = 1$
shows $x\ s \leq enn2real\ (n\ s)$ (**is** - ≤ *?y s*)
⟨*proof*⟩

end

end

6 Discrete-time Markov Processes

In this file we construct discrete-time Markov processes, e.g. with arbitrary state spaces.

theory *Discrete-Time-Markov-Process*
imports *Markov-Models-Auxiliary*
begin

lemma *measure-eqI-PiM-sequence*:
fixes $M :: nat \Rightarrow 'a\ measure$
assumes $*[simp]$: $sets\ P = PiM\ UNIV\ M\ sets\ Q = PiM\ UNIV\ M$
assumes *eq*: $\bigwedge A\ n.\ (\bigwedge i.\ A\ i \in sets\ (M\ i)) \Longrightarrow$
$P\ (prod\text{-}emb\ UNIV\ M\ \{..n\}\ (Pi_E\ \{..n\}\ A)) = Q\ (prod\text{-}emb\ UNIV\ M\ \{..n\}\ (Pi_E\ \{..n\}\ A))$
assumes *A*: *finite-measure P*

shows $P = Q$
⟨*proof*⟩

lemma *distr-cong-simp*:
$M = K \Longrightarrow sets\ N = sets\ L \Longrightarrow (\bigwedge x.\ x \in space\ M \Rightarrow f\ x = g\ x) \Longrightarrow distr\ M\ N\ f = distr\ K\ L\ g$
⟨*proof*⟩

6.1 Constructing Discrete-Time Markov Processes

locale *discrete-Markov-process* =
fixes $M :: 'a\ measure$ **and** $K :: 'a \Rightarrow 'a\ measure$
assumes $K[measurable]$: $K \in M \to_M prob\text{-}algebra\ M$
begin

lemma *space-K*: $x \in space\ M \Longrightarrow space\ (K\ x) = space\ M$
⟨*proof*⟩

lemma *sets-K*[*measurable-cong*]: $x \in space\ M \Longrightarrow sets\ (K\ x) = sets\ M$
⟨*proof*⟩

lemma *prob-space-K*: $x \in space\ M \Longrightarrow prob\text{-}space\ (K\ x)$
⟨*proof*⟩

definition $K' :: 'a \Rightarrow nat \Rightarrow (nat \Rightarrow 'a) \Rightarrow 'a\ measure$
where
$K'\ x\ n'\ \omega' = K\ (case\text{-}nat\ x\ \omega'\ n')$

lemma *IT-K'*:
assumes *x*: $x \in space\ M$ **shows** *Ionescu-Tulcea* $(K'\ x)\ (\lambda\text{-}.\ M)$
⟨*proof*⟩

definition *lim-sequence* :: $'a \Rightarrow (nat \Rightarrow 'a)\ measure$
where
lim-sequence x = *projective-family.lim UNIV* (*Ionescu-Tulcea.CI* $(K'\ x)\ (\lambda\text{-}.\ M)$) $(\lambda\text{-}.\ M)$

lemma
assumes *x*: $x \in space\ M$
shows *space-lim-sequence*: $space\ (lim\text{-}sequence\ x) = space\ (\Pi_M\ i \in UNIV.\ M)$
and *sets-lim-sequence*[*measurable-cong*]: $sets\ (lim\text{-}sequence\ x) = sets\ (\Pi_M\ i \in UNIV.\ M)$
and *emeasure-lim-sequence-emb*: $\bigwedge J\ X.\ finite\ J \Longrightarrow X \in sets\ (\Pi_M\ j \in J.\ M) \Longrightarrow$
$emeasure\ (lim\text{-}sequence\ x)\ (prod\text{-}emb\ UNIV\ (\lambda\text{-}.\ M)\ J\ X) =$
$emeasure\ (Ionescu\text{-}Tulcea.CI\ (K'\ x)\ (\lambda\text{-}.\ M)\ J)\ X$
and *emeasure-lim-sequence-emb-I0o*: $\bigwedge n\ X.\ X \in sets\ (\Pi_M\ i \in \{0..<n\}.\ M) \Longrightarrow$
$emeasure\ (lim\text{-}sequence\ x)\ (prod\text{-}emb\ UNIV\ (\lambda\text{-}.\ M)\ \{0..<n\}\ X) =$

emeasure (*Ionescu-Tulcea.C* (*K'* *x*) (λ-. *M*) *0* *n* (λ*x*. *undefined*)) *X*
⟨*proof*⟩

lemma *lim-sequence*[*measurable*]: *lim-sequence* ∈ *M* →$_M$ *prob-algebra* (Π$_M$ *i*∈*UNIV*. *M*)
⟨*proof*⟩

lemma *step-C*:
assumes *x*: *x* ∈ *space M*
shows *Ionescu-Tulcea.C* (*K'* *x*) (λ-. *M*) *0* *1* (λ-. *undefined*) ⤜ *Ionescu-Tulcea.C* (*K'* *x*) (λ-. *M*) *1* *n* =
K *x* ⤜ (λ*y*. *Ionescu-Tulcea.C* (*K'* *x*) (λ-. *M*) *1* *n* (*case-nat* *y* (λ-. *undefined*)))
⟨*proof*⟩

lemma *lim-sequence-eq*:
assumes *x*: *x* ∈ *space M*
shows *lim-sequence* *x* = *bind* (*K* *x*) (λ*y*. *distr* (*lim-sequence* *y*) (Π$_M$ *j*∈*UNIV*. *M*) (*case-nat* *y*))
(**is** - = *?B* *x*)
⟨*proof*⟩

lemma *AE-lim-sequence*:
assumes *x*[*simp*]: *x* ∈ *space M* **and** *P*[*measurable*]: *Measurable.pred* (Π$_M$ *i*∈*UNIV*. *M*) *P*
shows (*AE* ω *in lim-sequence* *x*. *P* ω) ⟷ (*AE* *y* *in* *K* *x*. *AE* ω *in lim-sequence* *y*. *P* (*case-nat* *y* ω))
⟨*proof*⟩

definition *lim-stream* :: *'a* ⇒ *'a stream measure*
where
lim-stream *x* = *distr* (*lim-sequence* *x*) (*stream-space* *M*) *to-stream*

lemma *space-lim-stream*: *space* (*lim-stream* *x*) = *streams* (*space* *M*)
⟨*proof*⟩

lemma *sets-lim-stream*[*measurable-cong*]: *sets* (*lim-stream* *x*) = *sets* (*stream-space* *M*)
⟨*proof*⟩

lemma *lim-stream*[*measurable*]: *lim-stream* ∈ *M* →$_M$ *prob-algebra* (*stream-space* *M*)
⟨*proof*⟩

lemma *space-stream-space-M-ne*: *x* ∈ *space M* ⟹ *space* (*stream-space* *M*) ≠ {}
⟨*proof*⟩

lemma *prob-space-lim-stream*: *x* ∈ *space M* ⟹ *prob-space* (*lim-stream* *x*)
⟨*proof*⟩

lemma *lim-stream-eq*:
assumes *x*: $x \in$ *space M*
shows *lim-stream x* = *do* { $y \leftarrow K\ x$; $\omega \leftarrow$ *lim-stream y*; *return* (*stream-space M*) (*y* ## ω) }
⟨*proof*⟩

lemma *AE-lim-stream*:
assumes *x*[*simp*]: $x \in$ *space M* **and** *P*[*measurable*]: *Measurable.pred* (*stream-space M*) *P*
shows (*AE* ω *in lim-stream x*. *P* ω) $\longleftrightarrow$ (*AE y in K x*. *AE* ω *in lim-stream y*. *P* (*y* ## ω))
⟨*proof*⟩

lemma *emeasure-lim-stream*:
assumes *x*[*measurable*, *simp*]: $x \in$ *space M* **and** *A*[*measurable*, *simp*]: *A* ∈ *sets* (*stream-space M*)
shows *lim-stream x A* = ($\int^+ y$. *emeasure* (*lim-stream y*) (((##) *y*) −‘ *A* ∩ *space* (*stream-space M*)) *∂K x*)
⟨*proof*⟩

lemma *lim-stream-eq-coinduct*[*case-names in-space step*]:
fixes *R* :: *'a* ⇒ *'a stream measure* ⇒ *bool*
assumes *x*: *R x B x* ∈ *space M*
assumes *R*: $\bigwedge x\ B.\ R\ x\ B \Longrightarrow \exists B' \in M \to_M$ *prob-algebra* (*stream-space M*).
(*AE y in K x*. *R y* (*B' y*) ∨ *lim-stream y* = *B' y*) ∧
B = *do* { *y* ← *K x*; $\omega \leftarrow$ *B' y*; *return* (*stream-space M*) (*y* ## ω) }
shows *lim-stream x* = *B*
⟨*proof*⟩

lemma *prob-space-lim-sequence*: *x* ∈ *space M* ⟹ *prob-space* (*lim-sequence x*)
⟨*proof*⟩

end

6.2 Strong Markov Property for Discrete-Time Markov Processes

The filtration adopted to streams, i.e. to the n-th projection.

definition *stream-filtration* :: *'a measure* ⇒ *enat* ⇒ *'a stream measure*
where *stream-filtration M n* = (*SUP i*∈{*i*::*nat*. *i* ≤ *n*}. *vimage-algebra* (*streams* (*space M*)) ($\lambda\omega$. ω !! *i*) *M*)

lemma *measurable-stream-filtration1*: *enat i* ≤ *n* ⟹ ($\lambda\omega$. ω !! *i*) ∈ *stream-filtration M n* →$_M$ *M*
⟨*proof*⟩

lemma *measurable-stream-filtration2*:
f ∈ *space N* → *streams* (*space M*) ⟹ ($\bigwedge i$. *enat i* ≤ *n* ⟹ (λx. *f x* !! *i*) ∈ *N* →$_M$ *M*) ⟹ *f* ∈ *N* →$_M$ *stream-filtration M n*

⟨*proof*⟩

lemma *space-stream-filtration*: *space* (*stream-filtration M n*) = *space* (*stream-space M*)
⟨*proof*⟩

lemma *sets-stream-filteration-le-stream-space*: *sets* (*stream-filtration M n*) ⊆ *sets* (*stream-space M*)
⟨*proof*⟩

interpretation *stream-filtration*: *filtration space* (*stream-space M*) *stream-filtration M*
⟨*proof*⟩

lemma *measurable-stopping-time-stream*:
stopping-time (*stream-filtration M*) *T* ⟹ *T* ∈ *stream-space M* →$_M$ *count-space UNIV*
⟨*proof*⟩

lemma *measurable-stopping-time-All-eq-0*:
assumes *T*: *stopping-time* (*stream-filtration M*) *T*
shows {*x*∈*space M*. ∀ω∈*streams* (*space M*). *T* (*x* ## ω) = *0*} ∈ *sets M*
⟨*proof*⟩

lemma *stopping-time-0*:
assumes *T*: *stopping-time* (*stream-filtration M*) *T*
and *x*: *x* ∈ *space M* **and** ω: ω ∈ *streams* (*space M*) *T* (*x* ## ω) > *0*
and ω′: ω′ ∈ *streams* (*space M*)
shows *T* (*x* ## ω′) > *0*
⟨*proof*⟩

lemma *stopping-time-epred-SCons*:
assumes *T*: *stopping-time* (*stream-filtration M*) *T*
and *x*: *x* ∈ *space M* **and** ω: ω ∈ *streams* (*space M*) *T* (*x* ## ω) > *0*
shows *stopping-time* (*stream-filtration M*) (λω. *epred* (*T* (*x* ## ω)))
⟨*proof*⟩

context *discrete-Markov-process*
begin

lemma *lim-stream-strong-Markov*:
assumes *x*: *x* ∈ *space M* **and** *T*: *stopping-time* (*stream-filtration M*) *T*
shows *lim-stream x* =
lim-stream x ⤜ (λω. *case T* ω *of*
enat i ⇒ *distr* (*lim-stream* (ω !! *i*)) (*stream-space M*) (λω′. *stake* (*Suc i*) ω @− ω′)
| ∞ ⇒ *return* (*stream-space M*) ω)
(**is** - = *?L T x*)
⟨*proof*⟩

end

end

7 Continuous-time Markov chains

theory *Continuous-Time-Markov-Chain*
imports *Discrete-Time-Markov-Process Discrete-Time-Markov-Chain*
begin

7.1 Trace Operations: relate (*'a* × *real*) *stream* and *real* ⇒ *'a*

partial-function (*tailrec*) *trace-at* :: *'a* ⇒ (*real* × *'a*) *stream* ⇒ *real* ⇒ *'a*
where
trace-at s ω j = (*case ω of* (*t'*, *s'*)##*ω* ⇒ *if t'* ≤ *j then trace-at s' ω j else s*)

lemma *trace-at-simp*[*simp*]: *trace-at s* ((*t'*, *s'*)##*ω*) *j* = (*if t'* ≤ *j then trace-at s' ω j else s*)
⟨*proof*⟩

lemma *trace-at-eq*:
trace-at s ω j = (*case sfirst* (λ*x*. *j* < *fst* (*shd x*)) *ω of* ∞ ⇒ *undefined* | *enat i* ⇒ (*s* ## *smap snd ω*) !! *i*)
⟨*proof*⟩

lemma *trace-at-shift*: *trace-at s* (*smap* (λ(*t*, *s'*). (*t* + *t'*, *s'*)) *ω*) *t* = *trace-at s ω* (*t* − *t'*)
⟨*proof*⟩

primcorec *merge-at* :: (*real* × *'a*) *stream* ⇒ *real* ⇒ (*real* × *'a*) *stream* ⇒ (*real* × *'a*) *stream*
where
merge-at ω j ω' = (*case ω of* (*t*, *s*) ## *ω* ⇒ *if t* ≤ *j then* (*t*, *s*)##*merge-at ω j ω' else ω'*)

lemma *merge-at-simp*[*simp*]: *merge-at* (*x*##*ω*) *j ω'* = (*if fst x* ≤ *j then x*##*merge-at ω j ω' else ω'*)
⟨*proof*⟩

7.2 Exponential Distribution

definition *exponential* :: *real* ⇒ *real measure*
where
exponential l = *density lborel* (*exponential-density l*)

lemma *space-exponential*: *space* (*exponential l*) = *UNIV*
⟨*proof*⟩

lemma *sets-exponential*[*measurable-cong*]: *sets* (*exponential l*) = *sets borel*
⟨*proof*⟩

lemma *prob-space-exponential*: *0* < *l* ⟹ *prob-space* (*exponential l*)
⟨*proof*⟩

lemma *AE-exponential*: *0* < *l* ⟹ *AE x in exponential l*. *0* < *x*
⟨*proof*⟩

lemma *emeasure-exponential-Ioi-cutoff*:
assumes *0* < *l*
shows *emeasure* (*exponential l*) {*x* <..} = *exp* (− (*max 0 x*) ∗ *l*)
⟨*proof*⟩

lemma *emeasure-exponential-Ioi*:
0 < *l* ⟹ *0* ≤ *x* ⟹ *emeasure* (*exponential l*) {*x* <..} = *exp* (− *x* ∗ *l*)
⟨*proof*⟩

lemma *exponential-eq-stretch*:
assumes *0* < *l*
shows *exponential l* = *distr* (*exponential 1*) *borel* (λ*x*. (*1*/*l*) ∗ *x*)
⟨*proof*⟩

lemma *uniform-measure-exponential*:
assumes *0* < *l 0* ≤ *t*
shows *uniform-measure* (*exponential l*) {*t* <..} = *distr* (*exponential l*) *borel* ((+) *t*) (**is** *?L* = *?R*)
⟨*proof*⟩

lemma *emeasure-PiM-exponential-Ioi-finite*:
assumes *J* ⊆ *I finite J* ⋀*i*. *i* ∈ *I* ⟹ *0* < *R i 0* ≤ *x*
shows *emeasure* (Π$_M$ *i*∈*I*. *exponential* (*R i*)) (*prod-emb I* (λ*i*. *exponential* (*R i*)) *J* (Π$_E$ *j*∈*J*. {*x*<..})) = *exp* (− *x* ∗ (∑ *i*∈*J*. *R i*))
⟨*proof*⟩

lemma *emeasure-PiM-exponential-Ioi-sequence*:
assumes *R*: *summable R* ⋀*i*. *0* < *R i 0* ≤ *x*
shows *emeasure* (Π$_M$ *i*∈*UNIV*. *exponential* (*R i*)) (Π *i*∈*UNIV*. {*x*<..}) = *exp* (− *x* ∗ *suminf R*)
⟨*proof*⟩

lemma *emeasure-PiM-exponential-Ioi-countable*:
assumes *R*: *J* ⊆ *I countable J* ⋀*i*. *i* ∈ *I* ⟹ *0* < *R i 0* ≤ *x* **and** *finite*: *integrable* (*count-space J*) *R*
shows *emeasure* (Π$_M$ *i*∈*I*. *exponential* (*R i*)) (*prod-emb I* (λ*i*. *exponential* (*R i*)) *J* (Π$_E$ *j*∈*J*. {*x*<..})) =
exp (− *x* ∗ (*LINT i*|*count-space J*. *R i*))
⟨*proof*⟩

lemma *AE-PiM-exponential-suminf-infty*:
fixes $R :: nat \Rightarrow real$
assumes *R*: $\bigwedge n.\ 0 < R\ n$ **and** *finite*: $(\sum n.\ ennreal\ (1\ /\ R\ n)) = top$
shows $AE\ \omega\ in\ \Pi_M\ n{\in}UNIV.\ exponential\ (R\ n).\ (\sum n.\ ereal\ (\omega\ n)) = \infty$
⟨*proof*⟩

7.3 Transition Rates

locale *transition-rates* =
fixes $R :: 'a \Rightarrow 'a \Rightarrow real$
assumes *R-nonneg*[*simp*]: $\bigwedge x\ y.\ 0 \leq R\ x\ y$
assumes *R-diagonal-0*[*simp*]: $\bigwedge x.\ R\ x\ x = 0$
assumes *finite-weight*: $\bigwedge x.\ (\int^+ y.\ R\ x\ y\ \partial count\text{-}space\ UNIV) < \infty$
assumes *positive-weight*: $\bigwedge x.\ 0 < (\int^+ y.\ R\ x\ y\ \partial count\text{-}space\ UNIV)$
begin

abbreviation $S :: (real \times 'a)\ measure$
where $S \equiv (borel \bigotimes_M count\text{-}space\ UNIV)$

abbreviation $T :: (real \times 'a)\ stream\ measure$
where $T \equiv stream\text{-}space\ S$

abbreviation $I :: 'a \Rightarrow 'a\ set$
where $I\ x \equiv \{y.\ 0 < R\ x\ y\}$

lemma *I-countable*: *countable* $(I\ x)$
⟨*proof*⟩

definition *escape-rate* $:: 'a \Rightarrow real$ **where**
$escape\text{-}rate\ x = \int y.\ R\ x\ y\ \partial count\text{-}space\ UNIV$

lemma *ennreal-escape-rate*: *ennreal* $(escape\text{-}rate\ x) = (\int^+ y.\ R\ x\ y\ \partial count\text{-}space\ UNIV)$
⟨*proof*⟩

lemma *escape-rate-pos*: $0 < escape\text{-}rate\ x$
⟨*proof*⟩

lemma *nonneg-escape-rate*[*simp*]: $0 \leq escape\text{-}rate\ x$
⟨*proof*⟩

lemma *prob-space-exponential-escape-rate*: *prob-space* $(exponential\ (escape\text{-}rate\ x))$
⟨*proof*⟩

lemma *measurable-escape-rate*[*measurable*]: $escape\text{-}rate \in count\text{-}space\ UNIV \rightarrow_M borel$
⟨*proof*⟩

lemma *measurable-exponential-escape-rate*[*measurable*]: $(\lambda x.\ exponential\ (escape\text{-}rate$

$x)) \in$ *count-space UNIV* $\rightarrow_M$ *prob-algebra borel*
⟨*proof*⟩

interpretation *pmf-as-function* ⟨*proof*⟩

lift-definition $J :: 'a \Rightarrow 'a$ *pmf* **is** $\lambda x\ y.\ \ R\ x\ y\ /$ *escape-rate* x
⟨*proof*⟩

lemma *set-pmf-J*: *set-pmf* $(J\ x) = I\ x$
⟨*proof*⟩

interpretation *exp-esc*: *pair-prob-space distr* (*exponential* (*escape-rate* x)) *borel*
$((+)\ t)\ J\ x$ **for** x
⟨*proof*⟩

7.4 Continuous-time Kernel

definition $K :: (real \times 'a) \Rightarrow (real \times 'a)$ *measure* **where**
$K = (\lambda(t, x).$ (*distr* (*exponential* (*escape-rate* x)) *borel* $((+)\ t)) \bigotimes_M\ J\ x)$

interpretation K: *discrete-Markov-process borel* $\bigotimes_M$ *count-space UNIV K*
⟨*proof*⟩

interpretation *DTMC*: *MC-syntax J* ⟨*proof*⟩

lemma *in-space-S*[*simp*]: $x \in$ *space S*
⟨*proof*⟩

lemma *in-space-T*[*simp*]: $x \in$ *space T*
⟨*proof*⟩

lemma *in-space-lim-stream*: $\omega \in$ *space* (*K.lim-stream* x)
⟨*proof*⟩

lemma *prob-space-K-lim*: *prob-space* (*K.lim-stream* x)
⟨*proof*⟩

definition *select-first* $:: 'a \Rightarrow ('a \Rightarrow real) \Rightarrow 'a \Rightarrow bool$
where *select-first* $x\ p\ y = (y \in I\ x \wedge (\forall\, y' \in I\ x - \{y\}.\ p\ y < p\ y'))$

lemma *select-firstD1*: *select-first* $x\ p\ y \Longrightarrow y \in I\ x$
⟨*proof*⟩

lemma *select-first-unique*:
assumes *y*: *select-first* $x\ p\ y1$ *select-first* $x\ p\ y2$ **shows** $y1 = y2$
⟨*proof*⟩

lemma *The-select-first*[*simp*]: *select-first* $x\ p\ y \Longrightarrow$ *The* (*select-first* $x\ p$) $= y$
⟨*proof*⟩

lemma *select-first-INF*:
select-first x p y $\Longrightarrow$ (*INF* $x{\in}I$ *x. p x*) = *p y*
⟨*proof*⟩

lemma *measurable-select-first*[*measurable*]:
(λ*p. select-first x p y*) ∈ (Π$_M$ $y{\in}I$ *x. borel*) →$_M$ *count-space UNIV*
⟨*proof*⟩

lemma *measurable-THE-select-first*[*measurable*]:
(λ*p. The* (*select-first x p*)) ∈ (Π$_M$ $y{\in}I$ *x. borel*) →$_M$ *count-space UNIV*
⟨*proof*⟩

lemma *sets-S-eq*: *sets S* = *sigma-sets UNIV* { {*t* ..} × *A* | *t A*. *A* ⊆ − *I x* ∨ (∃ $s{\in}I$ *x*. *A* = {*s*}) }
⟨*proof*⟩

7.5 Kernel equals Parallel Choice

abbreviation *PAR* :: *'a* ⇒ (*'a* ⇒ *real*) *measure*
where
PAR x ≡ (Π$_M$ $y{\in}I$ *x. exponential* (*R x y*))

lemma *PAR-least*:
assumes *y*: *y* ∈ *I x*
shows *PAR x* {*p*∈*space* (*PAR x*). *t* ≤ *p y* ∧ *select-first x p y*} =
emeasure (*exponential* (*escape-rate x*)) {*t* ..} ∗ *ennreal* (*pmf* (*J x*) *y*)
⟨*proof*⟩

lemma *AE-PAR-least*: *AE p in PAR x*. ∃ $y{\in}I$ *x. select-first x p y*
⟨*proof*⟩

lemma *K-alt*: *K* (*t*, *x*) = *distr* (Π$_M$ $y{\in}I$ *x. exponential* (*R x y*)) *S* (λ*p*. (*t* + (*INF* $y{\in}I$ *x. p y*), *The* (*select-first x p*))) (**is** - = *?R*)
⟨*proof*⟩

lemma *AE-K*: *AE y in K x. fst x* < *fst y* ∧ *snd y* ∈ *J* (*snd x*)
⟨*proof*⟩

lemma *AE-lim-stream*:
AE ω in K.lim-stream x. ∀ *i*. *snd* ((*x* ## *ω*) !! *i*) ∈ *DTMC.acc*‘‘{*snd x*} ∧ *snd* (*ω* !! *i*) ∈ *J* (*snd* ((*x* ## *ω*) !! *i*)) ∧ *fst* ((*x* ## *ω*) !! *i*) < *fst* (*ω* !! *i*)
(**is** *AE ω in K.lim-stream x*. ∀ *i*. *?P ω i*)
⟨*proof*⟩

lemma *measurable-merge-at*[*measurable*]: (λ(*ω*, *ω′*). *merge-at ω j ω′*) ∈ (*T* ⨂$_M$ *T*) →$_M$ *T*
⟨*proof*⟩

lemma *measurable-trace-at*[*measurable*]: $(\lambda(s,\ \omega).\ \textit{trace-at}\ s\ \omega\ j) \in (\textit{count-space}\ \textit{UNIV} \bigotimes_M T) \to_M \textit{count-space}\ \textit{UNIV}$
⟨*proof*⟩

lemma *measurable-trace-at'*: $(\lambda((s,\ j),\ \omega).\ \textit{trace-at}\ s\ \omega\ j) \in ((\textit{count-space}\ \textit{UNIV} \bigotimes_M \textit{borel}) \bigotimes_M T) \to_M \textit{count-space}\ \textit{UNIV}$
⟨*proof*⟩

lemma *K-time-split*:
assumes $t \le j$ **and** [*measurable*]: $f \in S \to_M \textit{borel}$
shows $(\int^+ x.\ f\ x * \textit{indicator}\ \{j <..\}\ (\textit{fst}\ x)\ \partial K\ (t,\ s)) = (\int^+ x.\ f\ x\ \partial K\ (j,\ s)) * \textit{exponential}\ (\textit{escape-rate}\ s)\ \{j - t <..\}$
⟨*proof*⟩

lemma *K-in-space*[*simp*]: $K\ x \in \textit{space}\ (\textit{prob-algebra}\ S)$
⟨*proof*⟩

lemma *L-in-space*[*simp*]: $K.\textit{lim-stream}\ x \in \textit{space}\ (\textit{prob-algebra}\ T)$
⟨*proof*⟩

7.6 Markov Chain Property

lemma *lim-time-split*:
$t \le j \Longrightarrow K.\textit{lim-stream}\ (t,\ s) = \textit{do}\ \{\ \omega \leftarrow K.\textit{lim-stream}\ (t,\ s)\ ;\ \omega' \leftarrow K.\textit{lim-stream}\ (j,\ \textit{trace-at}\ s\ \omega\ j)\ ;\ \textit{return}\ T\ (\textit{merge-at}\ \omega\ j\ \omega')\}$
(**is** - $\Longrightarrow$ - = *?DO t s*)
⟨*proof*⟩

lemma *K-eq*: $K\ (t,\ s) = \textit{distr}\ (\textit{exponential}\ (\textit{escape-rate}\ s) \bigotimes_M J\ s)\ S\ (\lambda(t',\ s).\ (t + t',\ s))$
⟨*proof*⟩

lemma *K-shift*: $K\ (t + t',\ s) = \textit{distr}\ (K\ (t,\ s))\ S\ (\lambda(t,\ s).\ (t + t',\ s))$
⟨*proof*⟩

lemma *K-not-empty*: $\textit{space}\ (K\ x) \neq \{\}$
⟨*proof*⟩

lemma *lim-stream-not-empty*: $\textit{space}\ (K.\textit{lim-stream}\ x) \neq \{\}$
⟨*proof*⟩

lemma *lim-shift*: — Generalize to bijective function on *K.lim-stream* invariant on *K*
$K.\textit{lim-stream}\ (t + t',\ s) = \textit{distr}\ (K.\textit{lim-stream}\ (t,\ s))\ T\ (\textit{smap}\ (\lambda(t,\ s).\ (t + t',\ s)))$
(**is** - = *?D t s*)
⟨*proof*⟩

lemma *lim-0*: $K.\textit{lim-stream}\ (t,\ s) = \textit{distr}\ (K.\textit{lim-stream}\ (0,\ s))\ T\ (\textit{smap}\ (\lambda(t',$

$s).\ (t' + t,\ s)))$
⟨*proof*⟩

7.7 Explosion time

definition *explosion* :: $(real \times 'a)\ stream \Rightarrow ereal$
where $explosion\ \omega = (SUP\ i.\ ereal\ (fst\ (\omega\ !!\ i)))$

lemma *ball-less-Suc-eq*: $(\forall\, i{<}Suc\ n.\ P\ i) \longleftrightarrow (P\ 0 \wedge (\forall\, i{<}n.\ P\ (Suc\ i)))$
⟨*proof*⟩

lemma *lim-stream-timediff-eq-exponential-1*:
$distr\ (K.lim\text{-}stream\ ts)\ (PiM\ UNIV\ (\lambda\text{-}.\ borel))$
$(\lambda\omega\ i.\ escape\text{-}rate\ (snd\ ((ts\#\#\omega)\ !!\ i)) * (fst\ (\omega\ !!\ i) - fst\ ((ts\#\#\omega)\ !!\ i))) =$
$PiM\ UNIV\ (\lambda\text{-}.\ exponential\ 1)$
(**is** *?D* = *?P*)
⟨*proof*⟩

lemma *AE-explosion-infty*:
assumes *bdd*: *bdd-above* (*range escape-rate*)
shows $AE\ \omega\ in\ K.lim\text{-}stream\ x.\ explosion\ \omega = \infty$
⟨*proof*⟩

7.8 Transition probability p_t

context
begin

declare [[*inductive-internals* = *true*]]

inductive *trace-in* :: $'a\ set \Rightarrow real \Rightarrow 'a \Rightarrow (real \times 'a)\ stream \Rightarrow bool$ **for** $S\ t$
where
$t < t' \Longrightarrow s \in S \Longrightarrow trace\text{-}in\ S\ t\ s\ ((t',\ s')\#\#\omega)$
| $t \geq t' \Longrightarrow trace\text{-}in\ S\ t\ s'\ \omega \Longrightarrow trace\text{-}in\ S\ t\ s\ ((t',\ s')\#\#\omega)$

end

lemma *trace-in-simps*[*simp*]:
$trace\text{-}in\ ss\ t\ s\ (x\#\#\omega) = (if\ t < fst\ x\ then\ s \in ss\ else\ trace\text{-}in\ ss\ t\ (snd\ x)\ \omega)$
⟨*proof*⟩

lemma *trace-in-eq-lfp*:
$trace\text{-}in\ ss\ t = lfp\ (\lambda F\ s.\ \lambda(t',\ s')\#\#\omega \Rightarrow if\ t < t'\ then\ s \in ss\ else\ F\ s'\ \omega)$
⟨*proof*⟩

lemma *trace-in-shiftD*: $trace\text{-}in\ ss\ t\ s\ \omega \Longrightarrow trace\text{-}in\ ss\ (t + t')\ s\ (smap\ (\lambda(t,\ s').$
$(t + t',\ s'))\ \omega)$
⟨*proof*⟩

lemma *trace-in-shift*[*simp*]: *trace-in ss t s* (*smap* (λ(*t*, *s'*). (*t* + *t'*, *s'*)) *ω*) ⟷ *trace-in ss* (*t* − *t'*) *s ω*
⟨*proof*⟩

lemma *measurable-trace-in'*:
Measurable.pred (*borel* ⨂$_M$ *count-space UNIV* ⨂$_M$ *T*) (λ(*t*, *s*, *ω*). *trace-in ss t s ω*)
(**is** *?M* (λ(*t*, *s*, *ω*). *trace-in ss t s ω*))
⟨*proof*⟩

lemma *measurable-trace-in*[*measurable* (*raw*)]:
assumes [*measurable*]: *f* ∈ *M* →$_M$ *borel g* ∈ *M* →$_M$ *count-space UNIV h* ∈ *M* →$_M$ *T*
shows *Measurable.pred M* (λ*x*. *trace-in ss* (*f x*) (*g x*) (*h x*))
⟨*proof*⟩

definition *p* :: *'a* ⇒ *'a* ⇒ *real* ⇒ *real*
where *p s s' t* = *P*(*ω in K.lim-stream* (*0*, *s*). *trace-in* {*s'*} *t s ω*)

lemma *p*[*measurable*]: (λ(*s*, *t*). *p s s' t*) ∈ (*count-space UNIV* ⨂$_M$ *borel*) →$_M$ *borel*
⟨*proof*⟩

lemma *p-nonpos*: **assumes** *t* ≤ *0* **shows** *p s s' t* = *of-bool* (*s* = *s'*)
⟨*proof*⟩

lemma *p-0*: *p s s' 0* = *of-bool* (*s* = *s'*)
⟨*proof*⟩

lemma *in-sets-T*[*measurable* (*raw*)]: *Measurable.pred T P* ⟹ {*ω*. *P ω*} ∈ *sets T*
⟨*proof*⟩

lemma *distr-id'*: *sets M* = *sets N* ⟹ *distr M N* (λ*x*. *x*) = *M*
⟨*proof*⟩

lemma *p-nonneg*[*simp*]: *0* ≤ *p s s' t*
⟨*proof*⟩

lemma *p-le-1*[*simp*]: *p s s' t* ≤ *1*
⟨*proof*⟩

lemma *p-eq*:
assumes *0* ≤ *t*
shows *p s s'' t* = (*of-bool* (*s* = *s''*) + (*LINT u*:{*0*..*t*}|*lborel*. *escape-rate s* ∗ *exp* (*escape-rate s* ∗ *u*) ∗ (*LINT s'*|*J s*. *p s' s'' u*))) / *exp* (*t* ∗ *escape-rate s*)
⟨*proof*⟩

lemma *continuous-on-p*: *continuous-on A* (*p s s'*)
⟨*proof*⟩

lemma *p-vector-derivative*: — Backward equation
assumes $0 \leq t$
shows $(p\ s\ s'$ *has-vector-derivative* $(LINT\ s''|count\text{-}space\ UNIV.\ R\ s\ s'' * p\ s''\ s'\ t) - escape\text{-}rate\ s * p\ s\ s'\ t)$
$(at\ t\ within\ \{0..\})$
(**is** (- *has-vector-derivative ?A*) -)
⟨*proof*⟩

coinductive *wf-times* :: *real* ⇒ (*real* × *'a*) *stream* ⇒ *bool*
where
$t < t' \Longrightarrow wf\text{-}times\ t'\ \omega \Longrightarrow wf\text{-}times\ t\ ((t',\ s')\ \#\#\ \omega)$

lemma *wf-times-simp*[*simp*]: *wf-times t* (*x* ## *ω*) ⟷ *t* < *fst x* ∧ *wf-times* (*fst x*) *ω*
⟨*proof*⟩

lemma *trace-in-merge-at*:
assumes *ω'*: *wf-times t' ω'*
shows *trace-in ss t x* (*merge-at ω t' ω'*) ⟷
(*if t* < *t' then trace-in ss t x ω else* ∃ *y. trace-in* {*y*} *t' x ω* ∧ *trace-in ss t y ω'*)
(**is** *?merge* ⟷ *?cases*)
⟨*proof*⟩

lemma *AE-lim-wf-times*: *AE ω in K.lim-stream* (*t*, *s*). *wf-times t ω*
⟨*proof*⟩

lemma *wf-times-shiftD*: *wf-times t'* (*smap* (λ(*t'*, *y*). (*t'* + *t*, *y*)) *ω*) ⟹ *wf-times* (*t'* − *t*) *ω*
⟨*proof*⟩

lemma *wf-times-shift*[*simp*]: *wf-times t'* (*smap* (λ(*t'*, *y*). (*t'* + *t*, *y*)) *ω*) = *wf-times* (*t'* − *t*) *ω*
⟨*proof*⟩

lemma *trace-in-unique*: *trace-in* {*y1*} *t x ω* ⟹ *trace-in* {*y2*} *t x ω* ⟹ *y1* = *y2*
⟨*proof*⟩

lemma *trace-at-eq*: *trace-in* {*z*} *t x ω* ⟹ *trace-at x ω t* = *z*
⟨*proof*⟩

lemma *AE-lim-acc*: *AE ω in K.lim-stream* (*t*, *x*). ∀ *t z. trace-in* {*z*} *t x ω* ⟶ (*x*, *z*) ∈ *DTMC.acc*
⟨*proof*⟩

lemma *p-add*:
assumes $0 \leq t\ 0 \leq t'$
shows $p\ x\ y\ (t + t') = (LINT\ z|count\text{-}space\ (DTMC.acc`\{x\}).\ p\ x\ z\ t * p\ z\ y\ t')$

⟨*proof*⟩

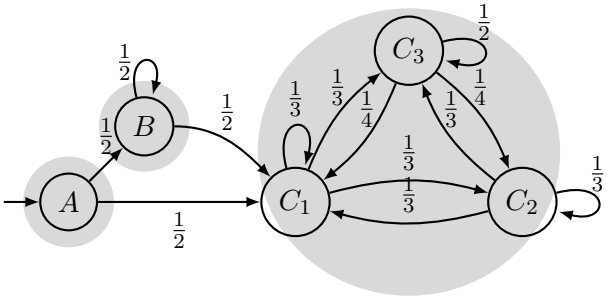
end

end
theory *Markov-Models*
imports
Markov-Models-Auxiliary
Discrete-Time-Markov-Chain
Trace-Space-Equals-Markov-Processes
Classifying-Markov-Chain-States
Markov-Decision-Process
MDP-Reachability-Problem
Discrete-Time-Markov-Process
Continuous-Time-Markov-Chain
begin

end
theory *Example-A*
imports *../Classifying-Markov-Chain-States*
begin

8 Example A

We formalize the following Markov chain:



First we define the state space as its own type:

datatype *state* = *A* | *B* | *C1* | *C2* | *C3*

Now the state space is *UNIV* :: *state set*

lemma *UNIV-state*: *UNIV* = {*A*, *B*, *C1*, *C2*, *C3*}
⟨*proof*⟩

instance *state* :: *finite*
⟨*proof*⟩

The transition function *tau* is easily defined using the case statement, this allows us to give a sparse specification as all *0* cases are collected at the end.

definition *tau* :: *state* ⇒ *state* ⇒ *real* **where**
tau s t = (*case* (*s*, *t*) *of*
(*A*, *B*) ⇒ *1 / 2* | (*A*, *C1*) ⇒ *1 / 2*
| (*B*, *B*) ⇒ *1 / 2* | (*B*, *C1*) ⇒ *1 / 2*
| (*C1*, *C1*) ⇒ *1 / 3* | (*C1*, *C2*) ⇒ *1 / 3* | (*C1*, *C3*) ⇒ *1 / 3*
| (*C2*, *C1*) ⇒ *1 / 3* | (*C2*, *C2*) ⇒ *1 / 3* | (*C2*, *C3*) ⇒ *1 / 3*
| (*C3*, *C1*) ⇒ *1 / 4* | (*C3*, *C2*) ⇒ *1 / 4* | (*C3*, *C3*) ⇒ *1 / 2*
| - ⇒ *0*)

lift-definition *K* :: *state* ⇒ *state pmf* **is** *tau*
⟨*proof*⟩

We use the *finite-pmf*-locale which introduces the point measure *tau.M*, and provides us with the necessary simplifier setup.

interpretation *A*: *MC-syntax K* ⟨*proof*⟩

8.1 The essential classs {*C1*, *C2*, *C3*}

context
begin

interpretation *pmf-as-function* ⟨*proof*⟩

lemma *A-E-eq*:
set-pmf (*K x*) = (*case x of A* ⇒ {*B*, *C1*} | *B* ⇒ {*B*, *C1*} | - ⇒ {*C1*, *C2*, *C3*})
⟨*proof*⟩

lemma *A-essential*: *A.essential-class* {*C1*, *C2*, *C3*}
⟨*proof*⟩

lemma *A-aperiodic*: *A.aperiodic* {*C1*, *C2*, *C3*}
⟨*proof*⟩

8.2 The stationary distribution *n*

Similar to *tau* we introduce *n* using the *finite-pmf*-locale.

lift-definition *n* :: *state pmf* **is** λ*C1* ⇒ *0.3* | *C2* ⇒ *0.3* | *C3* ⇒ *0.4* | - ⇒ *0*
⟨*proof*⟩

lemma *stationary-distribution-N*: *A.stationary-distribution n*
⟨*proof*⟩

lemma *exclusive-N*[*simp*]: *set-pmf n* = {*C1*, *C2*, *C3*}
⟨*proof*⟩

end

lemma *n-is-limit*:
assumes *x*: *x* ∈ {*C1*, *C2*, *C3*} **and** *y*: *y* ∈ {*C1*, *C2*, *C3*}

shows $(A.p\ x\ y) \longrightarrow pmf\ n\ y$
⟨*proof*⟩

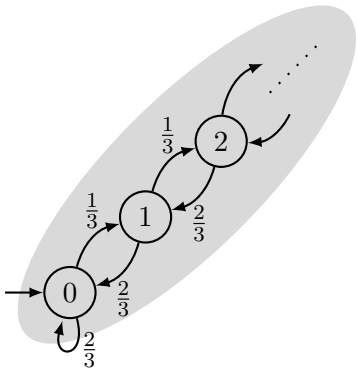
lemma *C-is-pos-recurrent*: $x \in \{C1,\ C2,\ C3\} \Longrightarrow A.pos\text{-}recurrent\ x$
⟨*proof*⟩

lemma *C-recurrence-time*:
assumes *x*: $x \in \{C1,\ C2,\ C3\}$
shows $A.U'\ x\ x = 1\ /\ pmf\ n\ x$
⟨*proof*⟩

end
theory *Example-B*
imports *../Classifying-Markov-Chain-States*
begin

9 Example B

We now formalize the following Markov chain:



As state space we have the set of natural numbers, the transition function *tau* has three cases:

definition $K :: nat \Rightarrow nat\ pmf$ **where**
$K\ x = map\text{-}pmf\ (\lambda True \Rightarrow x + 1 \mid False \Rightarrow x - 1)\ (bernoulli\text{-}pmf\ (1/3))$

For the special case when $x = 0$ we have $x - 1 = 0$ and hence $tau\ 0\ 0 = (2::'a)\ /\ (3::'a)$.

We pack this transition function into a discrete Markov kernel.

We call the locale of the Markov chain *B*, hence all constants and theorems from this Markov chain get a *B* prefix.

interpretation *B*: *MC-syntax K* ⟨*proof*⟩

9.1 Enabled, accessible and communicating states

For each step the predecessor and the successor are enabled (in the *0* case, the predecessor is again *0*. Hence every state is accessible from everywhere and every states is communicating with each other state. Finally we know that the state space is an essential class.

lemma *B-E-eq*: *set-pmf* $(K\ x) = \{x - 1, x + 1\}$
⟨*proof*⟩

lemma *B-E-Suc*: *Suc* $x \in$ *set-pmf* $(K\ x)$ $x \in$ *set-pmf* $(K\ (Suc\ x))$
⟨*proof*⟩

lemma *B-accessible*[*intro*]: $(i, j) \in B.acc$
⟨*proof*⟩

lemma *B-communicating*[*intro*]: $(i, j) \in B.communicating$
⟨*proof*⟩

lemma *B-essential*: *B.essential-class UNIV*
⟨*proof*⟩

9.2 B is aperiodic

lemma *B-aperiodic*: *B.aperiodic UNIV*
⟨*proof*⟩

9.3 The stationary distribution N

abbreviation N :: *nat pmf* **where**
$N \equiv$ *geometric-pmf* $(1\ /\ 2)$

lemma *stationary-distribution-N*: *B.stationary-distribution N*
⟨*proof*⟩

9.4 Limit behavior and recurrence times

lemma *limit*: $(B.p\ i\ j) \longrightarrow (1/2)\hat{}Suc\ j$
⟨*proof*⟩

lemma *pos-recurrent*: *B.pos-recurrent i*
⟨*proof*⟩

lemma *recurrence-time*: $B.U'\ i\ i = 2\hat{}Suc\ i$
⟨*proof*⟩

end

theory *PCTL*
imports

../Discrete-Time-Markov-Chain
Gauss−Jordan−Elim−Fun.Gauss-Jordan-Elim-Fun
HOL−Library.While-Combinator
HOL−Library.Monad-Syntax
begin

10 Adapt Gauss-Jordan elimination to DTMCs

locale *Finite-DTMC* =
fixes $K :: 's \Rightarrow 's\ pmf$ **and** $S :: 's\ set$ **and** $\varrho :: 's \Rightarrow real$ **and** $\iota :: 's \Rightarrow 's \Rightarrow real$
assumes *ι-nonneg*[*simp*]: $\bigwedge s\ t.\ 0 \leq \iota\ s\ t$ **and** *ϱ-nonneg*[*simp*]: $\bigwedge s.\ 0 \leq \varrho\ s$
assumes *measurable-ι*: $(\lambda(a,\ b).\ \iota\ a\ b) \in$ *borel-measurable* (*count-space UNIV* $\bigotimes_M$ *count-space UNIV*)
assumes *finite-S*[*simp*]: *finite S* **and** *S-not-empty*: $S \neq \{\}$
assumes *E-closed*: $(\bigcup s{\in}S.\ set\text{-}pmf\ (K\ s)) \subseteq S$
begin

lemma *measurable-ι'*[*measurable* (*raw*)]:
$f \in$ *measurable M* (*count-space UNIV*) $\Longrightarrow$ $g \in$ *measurable M* (*count-space UNIV*) $\Longrightarrow$
$(\lambda x.\ \iota\ (f\ x)\ (g\ x)) \in$ *borel-measurable M*
⟨*proof*⟩

lemma *measurable-ϱ*[*measurable*]: $\varrho \in$ *borel-measurable* (*count-space UNIV*)
⟨*proof*⟩

sublocale *R?*: *MC-with-rewards K ι ϱ*
⟨*proof*⟩

lemma *single-l*:
fixes *s* **and** *x* :: *real* **assumes** $s \in S$
shows $(\sum s'{\in}S.\ (if\ s' = s\ then\ 1\ else\ 0) * l\ s') = x \longleftrightarrow l\ s = x$
⟨*proof*⟩

definition *order* :: *nat* ⇒ *'s*
where *order* ≡ (*SOME f. bij-betw f* {..< *card S*} *S*)

lemma
shows *bij-order*[*simp*]: *bij-betw order* {..< *card S*} *S*
and *inj-order*[*simp*]: *inj-on order* {..<*card S*}
and *image-order*[*simp*]: *order* ‘ {..<*card S*} = *S*
and *order-S*[*simp*, *intro*]: $\bigwedge i.\ i < card\ S \Longrightarrow order\ i \in S$
⟨*proof*⟩

lemma *order-Ex*:
assumes $s \in S$ **obtains** *i* **where** $i < card\ S\ s = order\ i$
⟨*proof*⟩

definition *iorder* = *the-inv-into* {..<*card S*} *order*

lemma *bij-iorder*: *bij-betw iorder S* {..<*card S*}
⟨*proof*⟩

lemma *iorder-image-eq*: *iorder ' S* = {..<*card S*}
and *inj-iorder*: *inj-on iorder S*
⟨*proof*⟩

lemma *order-iorder*: ⋀*s*. *s* ∈ *S* ⟹ *order* (*iorder s*) = *s*
⟨*proof*⟩

definition *gauss-jordan'* :: ('*s* ⇒ '*s* ⇒ *real*) ⇒ ('*s* ⇒ *real*) ⇒ ('*s* ⇒ *real*) *option*
where
gauss-jordan' M a = *do* {
let M' = (λ*i j*. *if j* = *card S then a* (*order i*) *else M* (*order i*) (*order j*)) ;
sol ← *gauss-jordan M'* (*card S*) ;
Some (λ*i*. *sol* (*iorder i*) (*card S*))
}

lemma *gauss-jordan'-correct*:
assumes *gauss-jordan' M a* = *Some f*
shows ∀ *s*∈*S*. (∑ *s'*∈*S*. *M s s'* ∗ *f s'*) = *a s*
⟨*proof*⟩

lemma *gauss-jordan'-complete*:
assumes *exists*: ∀ *s*∈*S*. (∑ *s'*∈*S*. *M s s'* ∗ *x s'*) = *a s*
assumes *unique*: ⋀*y*. ∀ *s*∈*S*. (∑ *s'*∈*S*. *M s s'* ∗ *y s'*) = *a s* ⟹ ∀ *s*∈*S*. *y s* = *x s*
shows ∃ *y*. *gauss-jordan' M a* = *Some y*
⟨*proof*⟩

end

11 pCTL model checking

11.1 Syntax

datatype *realrel* = *LessEqual* | *Less* | *Greater* | *GreaterEqual* | *Equal*

datatype '*s sform* = *true*
| *Label* '*s set*
| *Neg* '*s sform*
| *And* '*s sform* '*s sform*
| *Prob realrel real* '*s pform*
| *Exp realrel real* '*s eform*
and '*s pform* = *X* '*s sform*
| *U nat* '*s sform* '*s sform*
| *UInfinity* '*s sform* '*s sform* (‹U^{∞}›)
and '*s eform* = *Cumm nat* (‹$C^{\leq}$›)
| *State nat* (‹$I^{=}$›)

| *Future 's sform*

primrec *bound-until* **where**
bound-until 0 φ ψ = ψ
| *bound-until* (*Suc n*) φ ψ = ψ *or* (φ *aand nxt* (*bound-until n* φ ψ))

lemma *measurable-bound-until*[*measurable*]:
assumes [*measurable*]: *Measurable.pred* (*stream-space M*) φ *Measurable.pred* (*stream-space M*) ψ
shows *Measurable.pred* (*stream-space M*) (*bound-until n* φ ψ)
⟨*proof*⟩

11.2 Semantics

primrec *inrealrel* :: *realrel* ⇒ 'a ⇒ ('a::*linorder*) ⇒ *bool* **where**
inrealrel LessEqual r q ⟷ $q \leq r$ |
inrealrel Less r q ⟷ $q < r$ |
inrealrel Greater r q ⟷ $q > r$ |
inrealrel GreaterEqual r q ⟷ $q \geq r$ |
inrealrel Equal r q ⟷ $q = r$

context *Finite-DTMC*
begin

abbreviation *prob s P* ≡ *measure* (*T s*) {*x*∈*space* (*T s*). *P x*}
abbreviation *E s* ≡ *set-pmf* (*K s*)

primrec *svalid* :: 's *sform* ⇒ 's *set*
and *pvalid* :: 's *pform* ⇒ 's *stream* ⇒ *bool*
and *reward* :: 's *eform* ⇒ 's *stream* ⇒ *ennreal* **where**
svalid true = *S* |
svalid (*Label L*) = {*s* ∈ *S*. *s* ∈ *L*} |
svalid (*Neg F*) = *S* − *svalid F* |
svalid (*And F1 F2*) = *svalid F1* ∩ *svalid F2* |
svalid (*Prob rel r F*) = {*s* ∈ *S*. *inrealrel rel r* $\mathcal{P}$(ω *in T s*. *pvalid F* (*s* ## ω)) } |
svalid (*Exp rel r F*) = {*s* ∈ *S*. *inrealrel rel* (*ennreal r*) ($\int^+$ ω. *reward F* (*s* ## ω) ∂*T s*) } |

pvalid (*X F*) = *nxt* (*HLD* (*svalid F*)) |
pvalid (*U k F1 F2*) = *bound-until k* (*HLD* (*svalid F1*)) (*HLD* (*svalid F2*)) |
pvalid (*U*$^\infty$ *F1 F2*) = *HLD* (*svalid F1*) *suntil HLD* (*svalid F2*) |

reward (*C*$^{\leq}$ *k*) = (λω. ($\sum$ *i*<*k*. ϱ (ω !! *i*) + ι (ω !! *i*) (ω !! (*Suc i*)))) |
reward (*I*$^{=}$ *k*) = (λω. ϱ (ω !! *k*)) |
reward (*Future F*) = (λω. *if ev* (*HLD* (*svalid F*)) ω *then reward-until* (*svalid F*) (*shd* ω) (*stl* ω) *else* ∞)

lemma *svalid-subset-S*: *svalid F* ⊆ *S*
⟨*proof*⟩

lemma *finite-svalid*[*simp*, *intro*]: *finite* (*svalid F*)
⟨*proof*⟩

lemma *svalid-sets*[*measurable*]: *svalid F* ∈ *sets* (*count-space S*)
⟨*proof*⟩

lemma *pvalid-sets*[*measurable*]: *Measurable.pred R.S* (*pvalid F*)
⟨*proof*⟩

lemma *reward-measurable*[*measurable*]: *reward F* ∈ *borel-measurable R.S*
⟨*proof*⟩

11.3 Implementation of *Sat*

11.3.1 *Prob0*

definition *Prob0* **where**
Prob0 Φ Ψ = *S* − *while* (λ*R*. ∃ *s*∈Φ. *R* ∩ *E s* ≠ {} ∧ *s* ∉ *R*) (λ*R*. *R* ∪ {*s*∈Φ. *R* ∩ *E s* ≠ {}}) Ψ

lemma *Prob0-subset-S*: *Prob0* Φ Ψ ⊆ *S*
⟨*proof*⟩

lemma *Prob0-iff-reachable*:
assumes Φ ⊆ *S* Ψ ⊆ *S*
shows *Prob0* Φ Ψ = {*s* ∈ *S*. ((*SIGMA x*:Φ. *E x*)* `` {*s*}) ∩ Ψ = {}} (**is** - = *?U*)
⟨*proof*⟩

lemma *Prob0-iff*:
assumes Φ ⊆ *S* Ψ ⊆ *S*
shows *Prob0* Φ Ψ = {*s*∈*S*. *AE* ω *in T s*. ¬ (*HLD* Φ *suntil HLD* Ψ) (*s* ## ω)} (**is** - = *?U*)
⟨*proof*⟩

lemma *E-rtrancl-closed*:
assumes *s* ∈ *S* (*s*, *t*) ∈ (*SIGMA x*:*A*. *B x*)* ⋀*x*. *x* ∈ *A* ⟹ *B x* ⊆ *E x* **shows** *t* ∈ *S*
⟨*proof*⟩

11.3.2 *Prob1*

definition *Prob1* **where**
Prob1 Y Φ Ψ = *Prob0* (Φ − Ψ) *Y*

lemma *Prob1-iff*:
assumes Φ ⊆ *S* Ψ ⊆ *S*
shows *Prob1* (*Prob0* Φ Ψ) Φ Ψ = {*s*∈*S*. *AE* ω *in T s*. (*HLD* Φ *suntil HLD* Ψ) (*s* ## ω)}
(**is** *Prob1 ?P0* - - = {*s*∈*S*. *?pU s*})

⟨*proof*⟩

11.3.3 *ProbU*, *ExpCumm*, and *ExpState*

abbreviation $\tau\ s\ t \equiv pmf\ (K\ s)\ t$

fun *ProbU* :: $'s \Rightarrow nat \Rightarrow 's\ set \Rightarrow 's\ set \Rightarrow real$ **where**
$ProbU\ q\ 0\ S1\ S2 \quad = (if\ q \in S2\ then\ 1\ else\ 0)\ |$
$ProbU\ q\ (Suc\ k)\ S1\ S2 =$
$(if\ q \in S1 - S2\ then\ (\sum q' \in S.\ \tau\ q\ q' * ProbU\ q'\ k\ S1\ S2)$
$\quad else\ if\ q \in S2\ then\ 1\ else\ 0)$

fun *ExpCumm* :: $'s \Rightarrow nat \Rightarrow ennreal$ **where**
$ExpCumm\ s\ 0 \quad = 0\ |$
$ExpCumm\ s\ (Suc\ k) = \varrho\ s + (\sum s' \in S.\ \tau\ s\ s' * (\iota\ s\ s' + ExpCumm\ s'\ k))$

fun *ExpState* :: $'s \Rightarrow nat \Rightarrow ennreal$ **where**
$ExpState\ s\ 0 \quad = \varrho\ s\ |$
$ExpState\ s\ (Suc\ k) = (\sum s' \in S.\ \tau\ s\ s' * ExpState\ s'\ k)$

11.3.4 *LES*

definition *LES* :: $'s\ set \Rightarrow 's \Rightarrow 's \Rightarrow real$ **where**
$LES\ F\ r\ c =$
$\quad (if\ r \in F\ then\ (if\ c = r\ then\ 1\ else\ 0)$
$\quad\quad else\ (if\ c = r\ then\ \tau\ r\ c - 1\ else\ \tau\ r\ c\ ))$

11.3.5 *ProbUinfty*, compute unbounded until

definition *ProbUinfty* :: $'s\ set \Rightarrow 's\ set \Rightarrow ('s \Rightarrow real)\ option$ **where**
$ProbUinfty\ S1\ S2 = gauss\text{-}jordan'\ (LES\ (Prob0\ S1\ S2 \cup S2))$
$\quad (\lambda i.\ if\ i \in S2\ then\ 1\ else\ 0)$

11.3.6 *ExpFuture*, compute unbounded reward

definition *ExpFuture* :: $'s\ set \Rightarrow ('s \Rightarrow ennreal)\ option$ **where**
$ExpFuture\ F = do\ \{$
$\quad let\ N = Prob0\ S\ F\ ;$
$\quad let\ Y = Prob1\ N\ S\ F\ ;$
$\quad sol \leftarrow gauss\text{-}jordan'\ (LES\ (S - Y \cup F))$
$\quad\quad (\lambda i.\ if\ i \in Y \wedge i \notin F\ then - \varrho\ i - (\sum s' \in S.\ \tau\ i\ s' * \iota\ i\ s')\ else\ 0)\ ;$
$\quad Some\ (\lambda s.\ if\ s \in Y\ then\ ennreal\ (sol\ s)\ else\ \infty)$
$\}$

11.3.7 *Sat*

fun *Sat* :: $'s\ sform \Rightarrow 's\ set\ option$ **where**
$Sat\ true \quad = Some\ S\ |$
$Sat\ (Label\ L) \quad = Some\ \{s \in S.\ s \in L\}\ |$
$Sat\ (Neg\ F) \quad = do\ \{\ F \leftarrow Sat\ F\ ;\ Some\ (S - F)\ \}\ |$

$Sat\ (And\ F1\ F2) \quad = do\ \{\ F1 \leftarrow Sat\ F1\ ;\ F2 \leftarrow Sat\ F2\ ;\ Some\ (F1 \cap F2)\ \}\ |$

$Sat\ (Prob\ rel\ r\ (X\ F)) \quad = do\ \{\ F \leftarrow Sat\ F\ ;\ Some\ \{q \in S.\ inrealrel\ rel\ r\ (\sum q' \in F.\ \tau\ q\ q')\}\ \}\ |$
$Sat\ (Prob\ rel\ r\ (U\ k\ F1\ F2)) = do\ \{\ F1 \leftarrow Sat\ F1\ ;\ F2 \leftarrow Sat\ F2\ ;\ Some\ \{q \in S.\ inrealrel\ rel\ r\ (ProbU\ q\ k\ F1\ F2)\ \}\ \}\ |$
$Sat\ (Prob\ rel\ r\ (U^{\infty}\ F1\ F2)) \quad = do\ \{\ F1 \leftarrow Sat\ F1\ ;\ F2 \leftarrow Sat\ F2\ ;\ P \leftarrow ProbUinfty\ F1\ F2\ ;\ Some\ \{q \in S.\ inrealrel\ rel\ r\ (P\ q)\ \}\ \}\ |$

$Sat\ (Exp\ rel\ r\ (Cumm\ k)) \quad = Some\ \{s \in S.\ inrealrel\ rel\ r\ (ExpCumm\ s\ k)\ \}\ |$
$Sat\ (Exp\ rel\ r\ (State\ k)) \quad = Some\ \{s \in S.\ inrealrel\ rel\ r\ (ExpState\ s\ k)\ \}\ |$
$Sat\ (Exp\ rel\ r\ (Future\ F)) \quad = do\ \{\ F \leftarrow Sat\ F\ ;\ E \leftarrow ExpFuture\ F\ ;\ Some\ \{q \in S.\ inrealrel\ rel\ (ennreal\ r)\ (E\ q)\ \}\ \}$

lemma *prob-sum*:
$s \in S \Longrightarrow Measurable.pred\ R.S\ P \Longrightarrow \mathcal{P}(\omega\ in\ T\ s.\ P\ \omega) = (\sum t \in S.\ \tau\ s\ t * \mathcal{P}(\omega\ in\ T\ t.\ P\ (t\ \#\#\ \omega)))$
⟨*proof*⟩

lemma *nn-integral-eq-sum*:
$s \in S \Longrightarrow f \in borel\text{-}measurable\ R.S \Longrightarrow (\int^{+} x.\ f\ x\ \partial T\ s) = (\sum t \in S.\ \tau\ s\ t * (\int^{+} x.\ f\ (t\ \#\#\ x)\ \partial T\ t))$
⟨*proof*⟩

lemma *T-space*[*simp*]: $measure\ (T\ s)\ (space\ R.S) = 1$
⟨*proof*⟩

lemma *emeasure-T-space*[*simp*]: $emeasure\ (T\ s)\ (space\ R.S) = 1$
⟨*proof*⟩

lemma τ*-distr*[*simp*]: $s \in S \Longrightarrow (\sum t \in S.\ \tau\ s\ t) = 1$
⟨*proof*⟩

lemma *ProbU*:
$q \in S \Longrightarrow ProbU\ q\ k\ (svalid\ F1)\ (svalid\ F2) = \mathcal{P}(\omega\ in\ T\ q.\ pvalid\ (U\ k\ F1\ F2)\ (q\ \#\#\ \omega))$
⟨*proof*⟩

lemma *Prob0-imp-not-Psi*:
assumes $\Phi \subseteq S\ \Psi \subseteq S\ s \in Prob0\ \Phi\ \Psi$ **shows** $s \notin \Psi$
⟨*proof*⟩

lemma *Psi-imp-not-Prob0*:
assumes $\Phi \subseteq S\ \Psi \subseteq S$ **shows** $s \in \Psi \Longrightarrow s \notin Prob0\ \Phi\ \Psi$
⟨*proof*⟩

11.3.8 Finite expected reward

abbreviation $s0 \equiv SOME\ s.\ s \in S$

lemma *s0-in-S*: $s0 \in S$
⟨*proof*⟩

lemma *nn-integral-reward-finite*:
assumes $s \in S$
assumes *until*: $AE\ \omega\ in\ T\ s.\ (HLD\ S\ suntil\ HLD\ (svalid\ F))\ (s\ \#\#\ \omega)$
shows $(\int^+ \omega.\ reward\ (Future\ F)\ (s\ \#\#\ \omega)\ \partial T\ s) \neq \infty$
⟨*proof*⟩

lemma *unique*:
assumes *in-S*: $\Phi \subseteq S\ \Psi \subseteq S\ N \subseteq S\ Prob0\ \Phi\ \Psi \subseteq N\ \Psi \subseteq N$
assumes *l1*: $\bigwedge s.\ s \in S \Longrightarrow s \notin N \Longrightarrow l1\ s - c\ s = (\sum s' \in S.\ \tau\ s\ s' * l1\ s')$
assumes *l2*: $\bigwedge s.\ s \in S \Longrightarrow s \notin N \Longrightarrow l2\ s - c\ s = (\sum s' \in S.\ \tau\ s\ s' * l2\ s')$
assumes *eq*: $\bigwedge s.\ s \in N \Longrightarrow l1\ s = l2\ s$
shows $\forall\, s \in S.\ l1\ s = l2\ s$
⟨*proof*⟩

lemma *uniqueness-of-ProbU*:
assumes *sol*:
$\forall\, s \in S.\ (\sum s' \in S.\ LES\ (Prob0\ (svalid\ F1)\ (svalid\ F2) \cup svalid\ F2)\ s\ s' * l\ s') =$
$(if\ s \in svalid\ F2\ then\ 1\ else\ 0)$
shows $\forall\, s \in S.\ l\ s = \mathcal{P}(\omega\ in\ T\ s.\ pvalid\ (U^\infty\ F1\ F2)\ (s\ \#\#\ \omega))$
⟨*proof*⟩

lemma *infinite-reward*:
fixes *s F*
defines $N \equiv Prob0\ S\ (svalid\ F)$ (**is** - ≡ *Prob0 S ?F*)
defines $Y \equiv Prob1\ N\ S\ (svalid\ F)$
assumes *s*: $s \in S\ s \notin Y$
shows $(\int^+ \omega.\ reward\ (Future\ F)\ (s\ \#\#\ \omega)\ \partial T\ s) = \infty$
⟨*proof*⟩

11.3.9 The expected reward implies a unique LES

lemma *existence-of-ExpFuture*:
fixes *s F*
assumes *N-def*: $N \equiv Prob0\ S\ (svalid\ F)$ (**is** - ≡ *Prob0 S ?F*)
assumes *Y-def*: $Y \equiv Prob1\ N\ S\ (svalid\ F)$
assumes *s*: $s \in S\ s \notin S - (Y - ?F)$
shows $enn2real\ (\int^+ \omega.\ reward\ (Future\ F)\ (s\ \#\#\ \omega)\ \partial T\ s) - (\varrho\ s + (\sum s' \in S.\ \tau\ s\ s' * \iota\ s\ s')) =$
$(\sum s' \in S.\ \tau\ s\ s' * enn2real\ (\int^+ \omega.\ reward\ (Future\ F)\ (s'\ \#\#\ \omega)\ \partial T\ s'))$
⟨*proof*⟩

lemma *uniqueness-of-ExpFuture*:
fixes *F*

assumes *N-def*: $N \equiv$ *Prob0 S* (*svalid F*) (**is** - $\equiv$ *Prob0 S ?F*)
assumes *Y-def*: $Y \equiv$ *Prob1 N S* (*svalid F*)
assumes *const-def*: *const* $\equiv \lambda s.$ *if* $s \in Y \wedge s \notin$ *svalid F then* $-\ \varrho\ s\ -\ (\sum s' \in S.\ \tau\ s\ s' * \iota\ s\ s')$ *else 0*
assumes *sol*: $\bigwedge s.\ s \in S \Longrightarrow (\sum s' \in S.\ LES\ (S\ -\ Y \cup ?F)\ s\ s' * l\ s') = const\ s$
shows $\forall s \in S.\ l\ s =$ *enn2real* $(\int^{+} \omega.\ reward\ (Future\ F)\ (s\ \#\#\ \omega)\ \partial T\ s)$
(**is** $\forall s \in S.\ l\ s =$ *enn2real* $(\int^{+} \omega.\ ?R\ (s\ \#\#\ \omega)\ \partial T\ s)$)
⟨*proof*⟩

11.4 Soundness of *Sat*

theorem *Sat-sound*:
Sat F $\neq$ *None* $\Longrightarrow$ *Sat F* = *Some* (*svalid F*)
⟨*proof*⟩

11.5 Completeness of *Sat*

theorem *Sat-complete*:
Sat F $\neq$ *None*
⟨*proof*⟩

11.6 Completeness and Soundness *Sat*

corollary *Sat*: *Sat* Φ = *Some* (*svalid* Φ)
⟨*proof*⟩

end

end

12 Probabilistic Guarded Command Language (pGCL)

theory *PGCL*
imports *../Markov-Decision-Process*
begin

12.1 Syntax

datatype *'s pgcl* =
Skip
| *Abort*
| *Assign 's* ⇒ *'s*
| *Seq 's pgcl 's pgcl*
| *Par 's pgcl 's pgcl*
| *If 's* ⇒ *bool 's pgcl 's pgcl*
| *Prob bool pmf 's pgcl 's pgcl*
| *While 's* ⇒ *bool 's pgcl*

12.2 Denotational Semantics

primrec *wp* :: *'s pgcl* ⇒ (*'s* ⇒ *ennreal*) ⇒ (*'s* ⇒ *ennreal*) **where**
  wp Skip f = *f*
| *wp Abort f* = (λ-. *0*)
| *wp* (*Assign u*) *f* = *f* ∘ *u*
| *wp* (*Seq* c_1 c_2) *f* = *wp* c_1 (*wp* c_2 *f*)
| *wp* (*If b* c_1 c_2) *f* = (λ*s*. *if b s then wp* c_1 *f s else wp* c_2 *f s*)
| *wp* (*Par* c_1 c_2) *f* = *wp* c_1 *f* ⊓ *wp* c_2 *f*
| *wp* (*Prob p* c_1 c_2) *f* = (λ*s*. *pmf p True* ∗ *wp* c_1 *f s* + *pmf p False* ∗ *wp* c_2 *f s*)
| *wp* (*While b c*) *f* = *lfp* (λ*X s*. *if b s then wp c X s else f s*)

lemma *wp-mono*: *mono* (*wp c*)
  ⟨*proof*⟩

abbreviation *det* :: *'s pgcl* ⇒ *'s* ⇒ (*'s pgcl* × *'s*) *pmf set* (‹≪ -, - ≫›) **where**
  det c s ≡ {*return-pmf* (*c*, *s*)}

12.3 Operational Semantics

fun *step* :: (*'s pgcl* × *'s*) ⇒ (*'s pgcl* × *'s*) *pmf set* **where**
  step (*Skip*, *s*) = ≪*Skip*, *s*≫
| *step* (*Abort*, *s*) = ≪*Abort*, *s*≫
| *step* (*Assign u*, *s*) = ≪*Skip*, *u s*≫
| *step* (*Seq* c_1 c_2, *s*) = (*map-pmf* (λ(*p1'*, *s'*). (*if p1'* = *Skip then* c_2 *else Seq p1'* c_2, *s'*))) ‘ *step* (c_1, *s*)
| *step* (*If b* c_1 c_2, *s*) = (*if b s then* ≪c_1, *s*≫ *else* ≪c_2, *s*≫)
| *step* (*Par* c_1 c_2, *s*) = ≪c_1, *s*≫ ∪ ≪c_2, *s*≫
| *step* (*Prob p* c_1 c_2, *s*) = {*map-pmf* (λ*b*. *if b then* (c_1, *s*) *else* (c_2, *s*)) *p*}
| *step* (*While b c*, *s*) = (*if b s then* ≪*Seq c* (*While b c*), *s*≫ *else* ≪*Skip*, *s*≫)

lemma *step-finite*: *finite* (*step x*)
  ⟨*proof*⟩

lemma *step-non-empty*: *step x* ≠ {}
  ⟨*proof*⟩

interpretation *step*: *Markov-Decision-Process step*
  ⟨*proof*⟩

definition *rF* :: (*'s* ⇒ *ennreal*) ⇒ ((*'s pgcl* × *'s*) *stream* ⇒ *ennreal*) ⇒ (*'s pgcl* × *'s*) *stream* ⇒ *ennreal* **where**
  rF f F ω = (*if fst* (*shd ω*) = *Skip then f* (*snd* (*shd ω*)) *else F* (*stl ω*))

abbreviation *r* :: (*'s* ⇒ *ennreal*) ⇒ (*'s pgcl* × *'s*) *stream* ⇒ *ennreal* **where**
  r f ≡ *lfp* (*rF f*)

lemma *continuous-rF*: *sup-continuous* (*rF f*)
  ⟨*proof*⟩

lemma *mono-rF*: *mono* (*rF f*)
⟨*proof*⟩

lemma *r-unfold*: *r f ω* = (*if fst* (*shd ω*) = *Skip then f* (*snd* (*shd ω*)) *else r f* (*stl ω*))
⟨*proof*⟩

lemma *mono-r*: *F* ≤ *G* ⟹ *r F ω* ≤ *r G ω*
⟨*proof*⟩

lemma *measurable-rF*:
assumes *F*[*measurable*]: *F* ∈ *borel-measurable step.St*
shows *rF f F* ∈ *borel-measurable step.St*
⟨*proof*⟩

lemma *measurable-r*[*measurable*]: *r f* ∈ *borel-measurable step.St*
⟨*proof*⟩

lemma *mono-r'*: *mono* (λ*F s*. ∏ *D*∈*step s*. ∫⁺ *t*. (*if fst t* = *Skip then f* (*snd t*) *else F t*) ∂*measure-pmf D*)
⟨*proof*⟩

lemma *E-inf-r*:
step.E-inf s (*r f*) =
lfp (λ*F s*. ∏ *D*∈*step s*. ∫⁺ *t*. (*if fst t* = *Skip then f* (*snd t*) *else F t*) ∂*measure-pmf D*) *s*
⟨*proof*⟩

lemma *E-inf-r-unfold*:
step.E-inf s (*r f*) = (∏ *D*∈*step s*. ∫⁺ *t*. (*if fst t* = *Skip then f* (*snd t*) *else step.E-inf t* (*r f*)) ∂*measure-pmf D*)
⟨*proof*⟩

lemma *E-inf-r-induct*[*consumes 1*, *case-names step*]:
assumes *P s y*
assumes ∗: ⋀*F s y*. *P s y* ⟹
(⋀*s y*. *P s y* ⟹ *F s* ≤ *y*) ⟹ (⋀*s*. *F s* ≤ *step.E-inf s* (*r f*)) ⟹
(∏ *D*∈*step s*. ∫⁺ *t*. (*if fst t* = *Skip then f* (*snd t*) *else F t*) ∂*measure-pmf D*) ≤ *y*
shows *step.E-inf s* (*r f*) ≤ *y*
⟨*proof*⟩

lemma *E-inf-Skip*: *step.E-inf* (*Skip*, *s*) (*r f*) = *f s*
⟨*proof*⟩

lemma *E-inf-Seq*:
assumes [*simp*]: ⋀*x*. *0* ≤ *f x*
shows *step.E-inf* (*Seq a b*, *s*) (*r f*) = *step.E-inf* (*a*, *s*) (*r* (λ*s*. *step.E-inf* (*b*, *s*) (*r f*)))

⟨*proof*⟩

lemma *E-inf-While*:
  step.E-inf (*While g c*, *s*) (*r f*) =
    lfp (λ*F s*. *if g s then step.E-inf* (*c*, *s*) (*r F*) *else f s*) *s*
⟨*proof*⟩

12.4 Equate Both Semantics

lemma *E-inf-r-eq-wp*: *step.E-inf* (*c*, *s*) (*r f*) = *wp c f s*
⟨*proof*⟩

end

13 Formalization of the Crowds-Protocol

theory *Crowds-Protocol*
  imports *../Discrete-Time-Markov-Chain*
begin

lemma *cond-prob-nonneg*[*simp*]: *0* ≤ *cond-prob M A B*
  ⟨*proof*⟩

lemma (**in** *MC-syntax*) *emeasure-suntil-geometric*:
  assumes [*measurable*]: *Measurable.pred S P*
  assumes *s* ∈ *X* **and** ∗[*simp*]: *0* ≤ *p 0* ≤ *r*
  assumes *r*: ⋀*s*. *s* ∈ *X* ⟹ *emeasure* (*T s*) {ω∈*space* (*T s*). *P* ω} = *ennreal r*
  assumes *p*: ⋀*s*. *s* ∈ *X* ⟹ *emeasure* (*K s*) *X* = *ennreal p p* < *1*
  assumes ⋀*t*. *AE* ω *in T t*. ¬ (*P* ⊓ (*HLD X* ⊓ *nxt* (*HLD X suntil P*))) ω
  shows *emeasure* (*T s*) {ω∈*space* (*T s*). (*HLD X suntil P*) ω} = *r* / (*1* − *p*)
⟨*proof*⟩

13.1 Definition of the Crowds-Protocol

datatype *'a state* = *Start* | *Init 'a* | *Mix 'a* | *End*

lemma *inj-Mix*[*simp*]: *inj-on Mix A*
  ⟨*proof*⟩

lemma *inj-Init*[*simp*]: *inj-on Init A*
  ⟨*proof*⟩

lemma *distinct-state-image*[*simp*]:
  Start ∉ *Mix* ‘ *A Init j* ∉ *Mix* ‘ *A End* ∉ *Mix* ‘ *A Mix j* ∈ *Mix* ‘ *A* ⟷ *j* ∈ *A*
  Start ∉ *Init* ‘ *A Mix j* ∉ *Init* ‘ *A End* ∉ *Init* ‘ *A Init j* ∈ *Init* ‘ *A* ⟷ *j* ∈ *A*
  ⟨*proof*⟩

lemma *Init-cut-Mix*[*simp*]:
  Init ‘ *H* ∩ *Mix* ‘ *J* = {}

⟨*proof*⟩

abbreviation *Jondo B* ≡ *Init'B* ∪ *Mix'B*

locale *Crowds-Protocol* =
  fixes *J* :: *'a set* **and** *C* :: *'a set* **and** *p-f* :: *real* **and** *p-i* :: *'a* ⇒ *real*
  assumes *J-not-empty*: *J* ≠ {} **and** *finite-J*[*simp*]: *finite J*
  assumes *C-smaller*: *C* ⊂ *J* **and** *C-non-empty*: *C* ≠ {}
  assumes *p-f*: *0* < *p-f p-f* < *1*
  assumes *p-i-nonneg*[*simp*]: ⋀*j*. *j* ∈ *J* ⟹ *0* ≤ *p-i j*
  assumes *p-i-distr*: (∑ *j*∈*J*. *p-i j*) = *1*
  assumes *p-i-C*: ⋀*j*. *j* ∈ *C* ⟹ *p-i j* = *0*
begin

abbreviation *H* :: *'a set* **where**
  H ≡ *J* − *C*

definition *p-j* = *1* / *card J*

lemma *p-f-nonneg*[*simp*]: *0* ≤ *p-f p-f* ≤ *1*
  ⟨*proof*⟩

lemma *p-j-nonneg*[*simp*]: *0* ≤ *p-j*
  ⟨*proof*⟩

definition *p-H* = *card H* / *card J*

lemma *p-H-nonneg*[*simp*]: *0* ≤ *p-H p-H* ≤ *1*
  ⟨*proof*⟩

definition *next-prob* :: *'a state* ⇒ *'a state* ⇒ *real* **where**
  next-prob s t = (*case* (*s*, *t*) *of* (*Start*, *Init j*) ⇒ *if j* ∈ *H then p-i j else 0*
  | (*Init j*, *Mix j'*) ⇒ *if j'* ∈ *J then p-j else 0*
  | (*Mix j*, *Mix j'*) ⇒ *if j'* ∈ *J then p-f* ∗ *p-j else 0*
  | (*Mix j*, *End*) ⇒ *1* − *p-f*
  | (*End*, *End*) ⇒ *1*
  | - ⇒ *0*)

definition *N s* = *embed-pmf* (*next-prob s*)

interpretation *MC-syntax N* ⟨*proof*⟩

abbreviation 𝔓 ≡ *T Start*

abbreviation *E s* ≡ *set-pmf* (*N s*)

lemma *finite-C*[*simp*]: *finite C*
  ⟨*proof*⟩

lemma *sum-p-i-C*[*simp*]: *sum p-i C = 0*
⟨*proof*⟩

lemma *sum-p-i-H*[*simp*]: *sum p-i H = 1*
⟨*proof*⟩

lemma *possible-jondo*:
obtains *j* **where** *j ∈ J j ∉ C p-i j ≠ 0*
⟨*proof*⟩

lemma *C-le-J*[*simp*]: *card C < card J*
⟨*proof*⟩

lemma *p-H*: *0 < p-H p-H < 1*
⟨*proof*⟩

lemma *p-H-p-f-pos*: *0 < p-H * p-f*
⟨*proof*⟩

lemma *p-H-p-f-less-1*: *p-H * p-f < 1*
⟨*proof*⟩

lemma *p-j-pos*: *0 < p-j*
⟨*proof*⟩

lemma *H-compl*: *1 − p-H = real (card C) / real (card J)*
⟨*proof*⟩

lemma *H-compl2*: *1 − p-H = card C * p-j*
⟨*proof*⟩

lemma *H-eq2*: *card H * p-j = p-H*
⟨*proof*⟩

lemma *pmf-next-pmf*[*simp*]: *pmf (N s) t = next-prob s t*
⟨*proof*⟩

lemma *next-prob-Start*[*simp*]: *next-prob Start (Init j) = (if j ∈ H then p-i j else 0)*
⟨*proof*⟩

lemma *next-prob-to-Init*[*simp*]: *j ∈ H ⟹ next-prob s (Init j) =*
(case s of Start ⇒ p-i j | - ⇒ 0)
⟨*proof*⟩

lemma *next-prob-to-Mix*[*simp*]: *j ∈ J ⟹ next-prob s (Mix j) =*
*(case s of Init j ⇒ p-j | Mix j ⇒ p-f * p-j | - ⇒ 0)*
⟨*proof*⟩

lemma *next-prob-to-End*[*simp*]: *next-prob s End* =
(*case s of Mix j* ⇒ *1* − *p-f* | *End* ⇒ *1* | - ⇒ *0*)
⟨*proof*⟩

lemma *next-prob-from-End*[*simp*]: *next-prob End s* = *0* ⟷ *s* ≠ *End*
⟨*proof*⟩

lemma *next-prob-Mix-MixI*: ∃ *j*. *s* = *Mix j* ⟹ ∃ *j*∈*J*. *s'* = *Mix j* ⟹ *next-prob s s'* = *p-f* ∗ *p-j*
⟨*proof*⟩

lemma *E-Start*: *E Start* = {*Init j* | *j*. *j* ∈ *H* ∧ *p-i j* ≠ *0* }
⟨*proof*⟩

lemma *E-Init*: *E* (*Init j*) = {*Mix j* | *j*. *j* ∈ *J* }
⟨*proof*⟩

lemma *E-Mix*: *E* (*Mix j*) = {*Mix j* | *j*. *j* ∈ *J* } ∪ {*End*}
⟨*proof*⟩

lemma *E-End*: *E End* = {*End*}
⟨*proof*⟩

lemma *enabled-End*:
enabled End ω ⟷ *ω* = *sconst End*
⟨*proof*⟩

lemma *AE-End*: (*AE ω in T End*. *P ω*) ⟷ *P* (*sconst End*)
⟨*proof*⟩

lemma *emeasure-Init-eq-Mix*:
assumes [*measurable*]: *Measurable.pred S P*
assumes *AE-End*: *AE x in T End*. ¬ *P* (*End* ## *x*)
shows *emeasure* (*T* (*Init j*)) {*x*∈*space* (*T* (*Init j*)). *P x*} =
emeasure (*T* (*Mix j*)) {*x*∈*space* (*T* (*Mix j*)). *P x*} / *p-f*
⟨*proof*⟩

What is the probability that the server sees a specific jondo (including the initiator) as sender.

definition *visit* :: *'a set* ⇒ *'a set* ⇒ *'a state stream* ⇒ *bool* **where**
visit I L = *Init*‘(*I* ∩ *H*) · (*HLD* (*Mix*‘*J*) *suntil* (*Mix*‘(*L* ∩ *J*) · *HLD* {*End*}))

lemma *visit-unique1*:
visit I1 L1 ω ⟹ *visit I2 L2 ω* ⟹ *I1* ∩ *I2* ≠ {}
⟨*proof*⟩

lemma *visit-unique2*:
assumes *visit I1 L1 ω visit I2 L2 ω*

shows *L1* ∩ *L2* ≠ {}
⟨*proof*⟩

lemma *visit-imp-in-H*: *visit* {*i*} *J* ω ⟹ *i* ∈ *H*
⟨*proof*⟩

lemma *emeasure-visit*:
assumes *I*: *I* ⊆ *H* **and** *L*: *L* ⊆ *J*
shows *emeasure* 𝔓 {ω∈*space* 𝔓. *visit I L* ω} = (∑ *i*∈*I*. *p-i i*) ∗ (*card L* ∗ *p-j*)
⟨*proof*⟩

lemma *measurable-visit*[*measurable*]: *Measurable.pred S* (*visit I L*)
⟨*proof*⟩

lemma *AE-visit*: *AE* ω *in* 𝔓. *visit H J* ω
⟨*proof*⟩

13.2 Server gets no information

lemma *server-view1*: *j* ∈ *J* ⟹ 𝒫(ω *in* 𝔓. *visit H* {*j*} ω) = *p-j*
⟨*proof*⟩

lemma *server-view-indep*:
L ⊆ *J* ⟹ *I* ⊆ *H* ⟹ 𝒫(ω *in* 𝔓. *visit I L* ω) = 𝒫(ω *in* 𝔓. *visit H L* ω) ∗ 𝒫(ω *in* 𝔓. *visit I J* ω)
⟨*proof*⟩

lemma *server-view*: 𝒫(ω *in* 𝔓. ∃*j*∈*H*. *visit* {*j*} {*j*} ω) = *p-j*
⟨*proof*⟩

13.3 Probability that collaborators gain information

definition *hit-C* = *Init*‘*H* · *ev* (*HLD* (*Mix*‘*C*))
definition *before-C B* = (*HLD* (*Jondo H*)) *suntil* ((*Jondo* (*B* ∩ *H*)) · *HLD* (*Mix* ‘ *C*))

lemma *measurable-hit-C*[*measurable*]: *Measurable.pred S hit-C*
⟨*proof*⟩

lemma *measurable-before-C*[*measurable*]: *Measurable.pred S* (*before-C B*)
⟨*proof*⟩

lemma *before-C*:
assumes ω: *enabled Start* ω
shows *before-C B* ω ⟷
((*Init*‘*H* · (*HLD* (*Mix*‘*H*) *suntil* (*Mix*‘(*B* ∩ *H*) · *HLD* (*Mix*‘*C*)))) *or* (*Init*‘(*B* ∩ *H*) · *HLD* (*Mix*‘*C*))) ω
⟨*proof*⟩

lemma *before-C-unique*:

assumes ω: *before-C I1* ω *before-C I2* ω **shows** $I1 \cap I2 \neq \{\}$
⟨*proof*⟩

lemma *hit-C-imp-before-C*:
assumes *enabled Start* ω *hit-C* ω **shows** *before-C H* ω
⟨*proof*⟩

lemma *before-C-single*:
assumes *before-C I* ω **shows** $\exists\, i{\in}I \cap H.$ *before-C* $\{i\}$ ω
⟨*proof*⟩

lemma *before-C-imp-in-H*: *before-C* $\{i\}$ $\omega \Longrightarrow i \in H$
⟨*proof*⟩

13.4 The probability that the sender hits a collaborator

lemma *Pr-hit-C*: $\mathcal{P}(\omega$ *in* $\mathfrak{P}.$ *hit-C* $\omega) = (1 - p\text{-}H) \,/\, (1 - p\text{-}H * p\text{-}f)$
⟨*proof*⟩

lemma *before-C-imp-hit-C*:
assumes *enabled Start* ω *before-C B* ω
shows *hit-C* ω
⟨*proof*⟩

lemma *negE*: $\neg\ P \Longrightarrow P \Longrightarrow$ *False*
⟨*proof*⟩

lemma *Pr-visit-before-C*:
assumes *L*: $L \subseteq H$ **and** *I*: $I \subseteq H$
shows $\mathcal{P}(\omega$ *in* $\mathfrak{P}.$ *visit I J* $\omega \wedge$ *before-C L* $\omega \mid$ *hit-C* $\omega\ ) =$
$(\sum i{\in}I.\ p\text{-}i\ i) * card\ L * p\text{-}j * p\text{-}f + (\sum i{\in}I \cap L.\ p\text{-}i\ i) * (1 - p\text{-}H * p\text{-}f)$
⟨*proof*⟩

lemma *Pr-visit-eq-before-C*:
$\mathcal{P}(\omega$ *in* $\mathfrak{P}.\ \exists\, j{\in}H.$ *visit* $\{j\}$ *J* $\omega \wedge$ *before-C* $\{j\}$ $\omega \mid$ *hit-C* $\omega\ ) = 1 - (p\text{-}H - p\text{-}j)$
$*\ p\text{-}f$
⟨*proof*⟩

lemma *probably-innocent*:
assumes *approx*: $1 \,/\, (2 * (p\text{-}H - p\text{-}j)) \leq p\text{-}f$ **and** $p\text{-}H \neq p\text{-}j$
shows $\mathcal{P}(\omega$ *in* $\mathfrak{P}.\ \exists\, j{\in}H.$ *visit* $\{j\}$ *J* $\omega \wedge$ *before-C* $\{j\}$ $\omega \mid$ *hit-C* $\omega\ ) \leq 1 \,/\, 2$
⟨*proof*⟩

lemma *Pr-before-C*:
assumes *L*: $L \subseteq H$
shows $\mathcal{P}(\omega$ *in* $\mathfrak{P}.$ *before-C L* $\omega \mid$ *hit-C* $\omega\ ) =$
$card\ L * p\text{-}j * p\text{-}f + (\sum l{\in}L.\ p\text{-}i\ l) * (1 - p\text{-}H * p\text{-}f)$
⟨*proof*⟩

lemma *P-visit*:
assumes *I*: $I \subseteq H$
shows $\mathcal{P}(\omega \ in \ \mathfrak{P}.\ visit\ I\ J\ \omega \mid hit\text{-}C\ \omega\ ) = (\sum i{\in}I.\ p\text{-}i\ i)$
⟨*proof*⟩

13.5 Probability space of hitting a collaborator

definition $hC = uniform\text{-}measure\ \mathfrak{P}\ \{\omega{\in}space\ \mathfrak{P}.\ hit\text{-}C\ \omega\}$

lemma *emeasure-hit-C-not-0*: $emeasure\ \mathfrak{P}\ \{\omega \in space\ \mathfrak{P}.\ hit\text{-}C\ \omega\} \neq 0$
⟨*proof*⟩

lemma *measurable-hC*[*measurable* (*raw*)]:
$A \in sets\ S \Longrightarrow A \in sets\ hC$
$f \in measurable\ M\ S \Longrightarrow f \in measurable\ M\ hC$
$g \in measurable\ S\ M \Longrightarrow g \in measurable\ hC\ M$
$A \cap space\ S \in sets\ S \Longrightarrow A \cap space\ hC \in sets\ S$
⟨*proof*⟩

lemma *vimage-Int-space-C*[*simp*]:
$f -` \{x\} \cap space\ hC = \{\omega{\in}space\ S.\ f\ \omega = x\}$
⟨*proof*⟩

sublocale *hC*: *information-space hC 2*
⟨*proof*⟩

abbreviation
mutual-information-Pow-CP (‹$\mathcal{I}'$(- ; -')›) **where**
$\mathcal{I}(X\ ;\ Y) \equiv hC.mutual\text{-}information\ 2\ (count\text{-}space\ (X`space\ hC))\ (count\text{-}space\ (Y`space\ hC))\ X\ Y$

lemma *simple-functionI*:
assumes *finite* (*range f*)
assumes [*measurable*]: $\bigwedge x.\ \{\omega{\in}space\ S.\ f\ \omega = x\} \in sets\ S$
shows *simple-function hC f*
⟨*proof*⟩

13.6 Estimate the information to the collaborators

lemma *measure-hC*[*simp*]:
assumes *A*[*measurable*]: $A \in sets\ S$
shows $measure\ hC\ A = \mathcal{P}(\omega\ in\ \mathfrak{P}.\ \omega \in A \mid hit\text{-}C\ \omega\ )$
⟨*proof*⟩

13.6.1 Setup random variables for mutual information

definition $first\text{-}J\ \omega = (THE\ i.\ visit\ \{i\}\ J\ \omega)$

lemma *first-J-eq*:
$visit\ \{i\}\ J\ \omega \Longrightarrow first\text{-}J\ \omega = i$

⟨*proof*⟩

lemma *AE-first-J*:
AE ω *in* $\mathfrak{P}$. *visit* $\{i\}$ *J* $\omega \longleftrightarrow$ *first-J* $\omega = i$
⟨*proof*⟩

lemma *measurbale-first-J*[*measurable*]: *first-J* $\in$ *measurable S* (*count-space UNIV*)
⟨*proof*⟩

definition *last-H* ω = (*THE i*. *before-C* $\{i\}$ ω)

lemma *measurbale-last-H*[*measurable*]: *last-H* $\in$ *measurable S* (*count-space UNIV*)
⟨*proof*⟩

lemma *last-H-eq*:
before-C $\{i\}$ $\omega \Longrightarrow$ *last-H* $\omega = i$
⟨*proof*⟩

lemma *last-H*:
assumes *enabled Start* ω *hit-C* ω
shows *before-C* $\{$*last-H* $\omega\}$ ω *last-H* $\omega \in H$
⟨*proof*⟩

lemma *AE-last-H*:
AE ω *in* $\mathfrak{P}$. *hit-C* $\omega \longrightarrow$ *before-C* $\{i\}$ $\omega \longleftrightarrow$ *last-H* $\omega = i$
⟨*proof*⟩

lemma *information-flow*:
defines $h \equiv$ *real* (*card H*)
assumes *init-uniform*: $\bigwedge i.\ i \in H \Longrightarrow$ *p-i i* = *1* / *h*
shows $\mathcal{I}$(*first-J* ; *last-H*) $\leq$ (*1* − (*h* − *1*) ∗ *p-j* ∗ *p-f*) ∗ *log 2 h*
⟨*proof*⟩

end

end

14 Formalizing the IPv4-address allocation in ZeroConf

theory *Zeroconf-Analysis*
imports *../Discrete-Time-Markov-Chain*
begin

declare *UNIV-bool*[*simp*]

14.1 Definition of a ZeroConf allocation run

datatype *zc-state* = *start*
| *probe nat*
| *ok*
| *error*

lemma *inj-probe*: *inj-on probe X*
⟨*proof*⟩

Countability of *zc-state* simplifies measurability of functions on *zc-state*.

instance *zc-state* :: *countable*
⟨*proof*⟩

locale *Zeroconf-Analysis* =
fixes *N* :: *nat* **and** *p q r e* :: *real*
assumes *p*: *0* < *p p* < *1* **and** *q*: *0* < *q q* < *1*
assumes *r*[*simp*]: *0* ≤ *r* **and** *e*[*simp*]: *0* ≤ *e*
begin

lemma *p-bounds*[*simp*]: *0* ≤ *p p* ≤ *1*
⟨*proof*⟩

lemma *q-bounds*[*simp*]: *0* ≤ *q q* ≤ *1*
⟨*proof*⟩

abbreviation *states* **where**
states ≡ *probe* ' {.. *N*} ∪ {*start*, *ok*, *error*}

primrec *τ* :: *zc-state* ⇒ *zc-state pmf* **where**
τ start = *map-pmf* (λ*True* ⇒ *probe 0* | *False* ⇒ *ok*) (*bernoulli-pmf q*)
| *τ* (*probe n*) = *map-pmf* (λ*True* ⇒ (*if n* < *N then probe* (*Suc n*) *else error*) |
False ⇒ *start*) (*bernoulli-pmf p*)
| *τ ok* = *return-pmf ok*
| *τ error* = *return-pmf error*

primrec *ϱ* :: *zc-state* ⇒ *zc-state* ⇒ *real* **where**
ϱ start = (λ-. *0*) (*probe 0* := *r*, *ok* := *r* ∗ (*N* + *1*))
| *ϱ* (*probe n*) = (*if n* < *N then* (λ-. *0*) (*probe* (*Suc n*) := *r*) *else* (λ-. *0*) (*error* :=
e))
| *ϱ ok* = (λ-. *0*) (*ok* := *0*)
| *ϱ error* = (λ-. *0*) (*error* := *0*)

lemma *ϱ-nonneg'*[*simp*]: *0* ≤ *ϱ s t*
⟨*proof*⟩

sublocale *MC-with-rewards τ ϱ* λ*s*. *0*
⟨*proof*⟩

14.2 The allocation run is a rewarded DTMC

abbreviation $E\ s \equiv \mathit{set\text{-}pmf}\ (\tau\ s)$

lemma *enabled-ok*: $\mathit{enabled}\ \mathit{ok}\ \omega \longleftrightarrow \omega = \mathit{sconst}\ \mathit{ok}$
⟨*proof*⟩

lemma *finite-E*[*intro*, *simp*]: *finite* $(E\ s)$
⟨*proof*⟩

lemma *E-closed*: $s \in \mathit{states} \Longrightarrow E\ s \subseteq \mathit{states}$
⟨*proof*⟩

lemma *enabled-error*: $\mathit{enabled}\ \mathit{error}\ \omega \longleftrightarrow \omega = \mathit{sconst}\ \mathit{error}$
⟨*proof*⟩

lemma *pos-neg-q-pn*: $0 < 1 - q * (1 - p \hat{} \mathit{Suc}\ N)$
⟨*proof*⟩

lemma *to-error*: **assumes** $n \leq N$ **shows** $(\mathit{probe}\ n, \mathit{error}) \in \mathit{acc}$
⟨*proof*⟩

14.3 Probability of a erroneous allocation

definition $\mathit{P\text{-}err}\ s = \mathcal{P}(\omega\ \mathit{in}\ T\ s.\ \mathit{ev}\ (\mathit{HLD}\ \{\mathit{error}\})\ (s\ \#\#\ \omega))$

lemma *P-err*:
defines $\mathit{p\text{-}start} == (q * p \hat{} \mathit{Suc}\ N)\ /\ (1 - q * (1 - p \hat{} \mathit{Suc}\ N))$
defines $\mathit{p\text{-}probe} == (\lambda n.\ p \hat{} \mathit{Suc}\ (N - n) + (1 - p \hat{} \mathit{Suc}\ (N - n)) * \mathit{p\text{-}start})$
assumes *s*: $s \in \mathit{states} - \{\mathit{ok}, \mathit{error}\}$
shows $\mathit{P\text{-}err}\ s = (\mathit{case}\ s\ \mathit{of}\ \mathit{ok} \Rightarrow 0 \mid \mathit{error} \Rightarrow 1 \mid \mathit{probe}\ n \Rightarrow \mathit{p\text{-}probe}\ n \mid \mathit{start} \Rightarrow \mathit{p\text{-}start})$
(**is** $\ldots = ?E\ s$)
⟨*proof*⟩

lemma *P-err-start*: $\mathit{P\text{-}err}\ \mathit{start} = (q * p \hat{} \mathit{Suc}\ N)\ /\ (1 - q * (1 - p \hat{} \mathit{Suc}\ N))$
⟨*proof*⟩

14.4 An allocation run terminates almost surely

lemma *states-closed*:
assumes $s \in \mathit{states}$
assumes $(s, t) \in \mathit{acc\text{-}on}\ (- \{\mathit{error}, \mathit{ok}\})$
shows $t \in \mathit{states}$
⟨*proof*⟩

lemma *finite-reached*:
assumes *s*: $s \in \mathit{states}$ **shows** $\mathit{finite}\ (\mathit{acc\text{-}on}\ (- \{\mathit{error}, \mathit{ok}\})\ ``\ \{s\})$
⟨*proof*⟩

lemma *AE-reaches-error-or-ok*:
assumes *s*: $s \in states$
shows $AE\ \omega\ in\ T\ s.\ ev\ (HLD\ \{error,\ ok\})\ \omega$
⟨*proof*⟩

14.5 Expected runtime of an allocation run

definition $R\ s = (\int^+ \omega.\ reward\text{-}until\ \{error,\ ok\}\ s\ \omega\ \partial T\ s)$

definition $R'\ s = enn2real\ (R\ s)$

lemma *R-iter*: $s \neq error \Longrightarrow s \neq ok \Longrightarrow R\ s = (\int^+ t.\ ennreal\ (\varrho\ s\ t) + R\ t\ \partial\tau\ s)$
⟨*proof*⟩

lemma *R-finite*:
assumes *s*: $s \in states$
shows $R\ s \neq \infty$
⟨*proof*⟩

lemma *R-less-top*: $s \in states \Longrightarrow R\ s < top$
⟨*proof*⟩

lemma *R'-iter*: **assumes** *s*: $s \in states\ s \neq error\ s \neq ok$ **shows** $R'\ s = (\int t.\ \varrho\ s\ t + R'\ t\ \partial\tau\ s)$
⟨*proof*⟩

lemma *cost-from-start*:
$R'\ start =$
$(q * (r + p\hat{}Suc\ N * e + r * p * (1 - p\hat{}N) / (1 - p)) + (1 - q) * (r * Suc\ N)) /$
$(1 - q + q * p\hat{}Suc\ N)$
⟨*proof*⟩

end

interpretation *ZC*: *Zeroconf-Analysis 2 16 / 65024* :: *real 0.01 0.002 3600*
⟨*proof*⟩

lemma $ZC.P\text{-}err\ start \leq 1 / 10\hat{}12$
⟨*proof*⟩

lemma $ZC.R'\ start \leq 0.007$
⟨*proof*⟩

end

15 Formalization of the Gossip-Broadcast

theory *Gossip-Broadcast*
imports *../Discrete-Time-Markov-Chain*
begin

lemma *inj-on-upd-PiE*:
assumes $i \notin I$ **shows** *inj-on* $(\lambda(x,f).\ f(i := x))$ $(M \times (\Pi_E\ i{\in}I.\ A\ i))$
⟨*proof*⟩

lemma *sum-folded-product*:
fixes I :: *'i set* **and** f :: *'s* ⇒ *'i* ⇒ *'a*::{*semiring-0*, *comm-monoid-mult*}
assumes *finite* I $\bigwedge i.\ i \in I \Longrightarrow$ *finite* $(S\ i)$
shows $(\sum x{\in}Pi_E\ I\ S.\ \prod i{\in}I.\ f\ (x\ i)\ i) = (\prod i{\in}I.\ \sum s{\in}S\ i.\ f\ s\ i)$
⟨*proof*⟩

15.1 Definition of the Gossip-Broadcast

datatype *state* = *listening* | *sending* | *sleeping*

type-synonym *sys-state* = (*nat* × *nat*) ⇒ *state*

lemma *state-UNIV*: *UNIV* = {*listening*, *sending*, *sleeping*}
⟨*proof*⟩

locale *gossip-broadcast* =
fixes *size* :: *nat* **and** *p* :: *real*
assumes *size*: *0* < *size*
assumes *p*: *0* < *p* *p* < *1*
begin

interpretation *pmf-as-function* ⟨*proof*⟩

definition *states* :: *sys-state set* **where**
states = ({..< *size*} × {..< *size*}) →$_E$ {*listening*, *sending*, *sleeping*}

definition *start* :: *sys-state* **where**
start = $(\lambda x{\in}\{..<$ *size*$\}{\times}\{..<$ *size*$\}.$ *listening*$)((0,\ 0) :=$ *sending*$)$

definition *neighbour-sending* **where**
neighbour-sending s = $(\lambda(x,y).$
$(x > 0 \wedge s\ (x - 1,\ y) =$ *sending*$) \vee$
$(x <$ *size* $\wedge\ s\ (x + 1,\ y) =$ *sending*$) \vee$
$(y > 0 \wedge s\ (x,\ y - 1) =$ *sending*$) \vee$
$(y <$ *size* $\wedge\ s\ (x,\ y + 1) =$ *sending*$))$

definition *node-trans* :: *sys-state* ⇒ (*nat* × *nat*) ⇒ *state* ⇒ *state* ⇒ *real* **where**
node-trans g x s = (*case s of*
listening ⇒ (*if neighbour-sending g x*
then $(\lambda$-.*0*$)$ (*sending* := *p*, *sleeping* := *1* − *p*)

else (λ-.*0*) (*listening* := *1*))
| *sending* ⇒ (λ-.*0*) (*sleeping* := *1*)
| *sleeping* ⇒ (λ-.*0*) (*sleeping* := *1*))

lemma *node-trans-sum-eq-1*[*simp*]:
node-trans g x s′ listening + (*node-trans g x s′ sending* + *node-trans g x s′ sleeping*) = *1*
⟨*proof*⟩

lemma *node-trans-nonneg*[*simp*]: *0* ≤ *node-trans s x i j*
⟨*proof*⟩

lift-definition *proto-trans* :: *sys-state* ⇒ *sys-state pmf* **is**
λ*s s′*. *if s′* ∈ *states then* (∏ *x*∈{..< *size*}×{..< *size*}. *node-trans s x* (*s x*) (*s′ x*)) *else 0*
⟨*proof*⟩

end

15.2 The Gossip-Broadcast forms a DTMC

sublocale *gossip-broadcast* ⊆ *MC-syntax proto-trans* ⟨*proof*⟩

end

16 Certification of Reachability Problems on MDPs

theory *MDP-RP-Certification*
imports
../MDP-Reachability-Problem
HOL−Library.IArray
HOL−Library.Code-Target-Numeral
begin

context *Reachability-Problem*
begin

lemma *p-ub′*:
fixes *x*
assumes *1*: *s* ∈ *S* ⋀*s D*. *s* ∈ *S1* ⟹ *D* ∈ *K s* ⟹ (∑ *t*∈*S*. *pmf D t* ∗ *x t*) ≤ *x s*
assumes *2*: ⋀*s*. *s* ∈ *S1* ⟹ *x s* ≠ *0* ⟹ (∃ *t*∈*S2*. (*s*, *t*) ∈ (*SIGMA s*:*S1*. ⋃ *D*∈*K s*. *set-pmf D*)*)
assumes *3*: ⋀*s*. *s* ∈ *S* − *S1* − *S2* ⟹ *x s* = *0*
assumes *4*: ⋀*s*. *s* ∈ *S2* ⟹ *x s* = *1*
shows *enn2real* (*p s*) ≤ *x s*
⟨*proof*⟩

lemma *n-lb′*:
fixes *x*

assumes *wf R*
assumes *1*: $s \in S$ $\bigwedge s$ *D*. $s \in S1 \Longrightarrow D \in K\ s \Longrightarrow x\ s \leq (\sum t{\in}S.\ pmf\ D\ t * x\ t)$
assumes *2*: $\bigwedge s$ *D*. $s \in S1 \Longrightarrow D \in K\ s \Longrightarrow x\ s \neq 0 \Longrightarrow \exists\, t{\in}D.\ ((t,\ s) \in R \wedge t \in S1 \wedge x\ t \neq 0) \vee t \in S2$
assumes *3*: $\bigwedge s.\ s \in S - S1 - S2 \Longrightarrow x\ s = 0$
assumes *4*: $\bigwedge s.\ s \in S2 \Longrightarrow x\ s = 1$
shows $x\ s \leq enn2real\ (n\ s)$
⟨*proof*⟩

end

no-notation *Stream.snth* (**infixl** ‹!!› *100*) — we use !! for IArray

16.1 Computable representation

record *mdp-reachability-problem* =
state-count :: *nat*
distrs :: (*nat* × *rat*) *list list iarray*
states1 :: *bool iarray*
states2 :: *bool iarray*

record *'a RP-sub-cert* =
solution :: *rat iarray*
witness :: (*'a* × *nat*) *iarray*

record *RP-cert* =
pos-cert :: (*nat* × *nat*) *RP-sub-cert*
neg-cert :: *nat list RP-sub-cert*

definition *sparse-mult sx y* = *sum-list* (*map* (λ(*n*, *x*). *x* ∗ *y* !! *n*) *sx*)

primrec *lookup* **where**
lookup d [] *x* = *d*
| *lookup d* (*y*#*ys*) *x* = (*if fst y* = *x then snd y else lookup d ys x*)

lemma *lookup-eq-map-of*: *lookup d xs x* = (*case map-of xs x of Some x* ⇒ *x* | *None* ⇒ *d*)
⟨*proof*⟩

lemma *lookup-in-set*:
distinct (*map fst xs*) ⟹ *x* ∈ *set xs* ⟹ *lookup d xs* (*fst x*) = *snd x*
⟨*proof*⟩

lemma *lookup-not-in-set*:
x ∉ *fst* ‘ *set xs* ⟹ *lookup d xs x* = *d*
⟨*proof*⟩

lemma *lookup-nonneg*:
(⋀*x v*. (*x*, *v*) ∈ *set xs* ⟹ 0 ≤ *v*) ⟹ (*0::'a::ordered-comm-monoid-add*) ≤ *lookup*

0 xs x
⟨*proof*⟩

lemma *sparse-mult-eq-sum-lookup*:
fixes *xs* :: (*nat* × *'a*::*comm-semiring-1*) *list*
assumes *list-all* (λ(*n*, *x*). *n* < *M*) *xs distinct* (*map fst xs*)
shows *sparse-mult xs y* = (∑ *i*<*M*. *lookup 0 xs i* ∗ *y* !! *i*)
⟨*proof*⟩

lemma *sum-list-eq-sum-lookup*:
fixes *xs* :: (*nat* × *'a*::*comm-semiring-1*) *list*
assumes *list-all* (λ(*n*, *x*). *n* < *M*) *xs distinct* (*map fst xs*)
shows *sum-list* (*map snd xs*) = (∑ *i*<*M*. *lookup 0 xs i*)
⟨*proof*⟩

definition
valid-mdp-rp mdp ⟷
0 < *state-count mdp* ∧
IArray.length (*distrs mdp*) = *state-count mdp* ∧
IArray.length (*states1 mdp*) = *state-count mdp* ∧
IArray.length (*states2 mdp*) = *state-count mdp* ∧
(∀ *i*<*state-count mdp*. ¬ (*states1 mdp* !! *i* ∧ *states2 mdp* !! *i*) ∧
list-all (λ*ds*. *distinct* (*map fst ds*) ∧ *list-all* (λ(*n*, *x*). *0* ≤ *x* ∧ *n* < *state-count mdp*) *ds* ∧
sum-list (*map snd ds*) = *1*) (*distrs mdp* !! *i*) ∧
¬ *List.null* (*distrs mdp* !! *i*))

definition
valid-sub-cert mdp c ord check ⟷
IArray.length (*witness c*) = *state-count mdp* ∧
IArray.length (*solution c*) = *state-count mdp* ∧
(∀ *i*<*state-count mdp*.
if states2 mdp !! *i then solution c* !! *i* = *1*
else if states1 mdp !! *i then 0* ≤ *solution c* !! *i* ∧
(*list-all* (λ*ds*. *ord* (*sparse-mult ds* (*solution c*)) (*solution c* !! *i*)) (*distrs mdp* !! *i*)) ∧
(*0* < *solution c* !! *i* ⟶ *check* (*distrs mdp* !! *i*) (*witness c* !! *i*))
else solution c !! *i* = *0*)

definition
valid-pos-cert mdp c ⟷
valid-sub-cert mdp c (≤)
(λ*D* ((*j*, *a*), *n*). *j* < *state-count mdp* ∧ *snd* (*witness c* !! *j*) < *n* ∧ *0* < *solution c* !! *j* ∧
a < *length D* ∧ *lookup 0* (*D* ! *a*) *j* ≠ *0*)

definition
valid-neg-cert mdp c ⟷
valid-sub-cert mdp c (≥)

$(\lambda D\ (J,\ n).\ \mathit{list\text{-}all2}\ (\lambda j\ d.\ j < \mathit{state\text{-}count}\ \mathit{mdp} \wedge \mathit{snd}\ (\mathit{witness}\ c\ !!\ j) < n \wedge \mathit{lookup}\ 0\ d\ j \neq 0 \wedge 0 < \mathit{solution}\ c\ !!\ j)\ J\ D)$

definition
$\mathit{valid\text{-}cert}\ \mathit{mdp}\ c \longleftrightarrow \mathit{valid\text{-}pos\text{-}cert}\ \mathit{mdp}\ (\mathit{pos\text{-}cert}\ c) \wedge \mathit{valid\text{-}neg\text{-}cert}\ \mathit{mdp}\ (\mathit{neg\text{-}cert}\ c)$

lemma *valid-mdp-rpD-length*:
assumes $\mathit{valid\text{-}mdp\text{-}rp}\ \mathit{mdp}$
shows $0 < \mathit{state\text{-}count}\ \mathit{mdp}\ \mathit{IArray.length}\ (\mathit{distrs}\ \mathit{mdp}) = \mathit{state\text{-}count}\ \mathit{mdp}$
$\mathit{IArray.length}\ (\mathit{states1}\ \mathit{mdp}) = \mathit{state\text{-}count}\ \mathit{mdp}\ \mathit{IArray.length}\ (\mathit{states2}\ \mathit{mdp}) = \mathit{state\text{-}count}\ \mathit{mdp}$
⟨*proof*⟩

lemma *valid-mdp-rpD*:
assumes $\mathit{valid\text{-}mdp\text{-}rp}\ \mathit{mdp}\ i < \mathit{state\text{-}count}\ \mathit{mdp}$
shows $\neg\ (\mathit{states1}\ \mathit{mdp}\ !!\ i \wedge \mathit{states2}\ \mathit{mdp}\ !!\ i)$
and $\bigwedge \mathit{ds}\ n\ x.\ \mathit{ds} \in \mathit{set}\ (\mathit{distrs}\ \mathit{mdp}\ !!\ i) \Longrightarrow (n,\ x) \in \mathit{set}\ \mathit{ds} \Longrightarrow n < \mathit{state\text{-}count}\ \mathit{mdp}$
and $\bigwedge \mathit{ds}\ n\ x.\ \mathit{ds} \in \mathit{set}\ (\mathit{distrs}\ \mathit{mdp}\ !!\ i) \Longrightarrow (n,\ x) \in \mathit{set}\ \mathit{ds} \Longrightarrow 0 \leq x$
and $\bigwedge \mathit{ds}.\ \mathit{ds} \in \mathit{set}\ (\mathit{distrs}\ \mathit{mdp}\ !!\ i) \Longrightarrow \mathit{sum\text{-}list}\ (\mathit{map}\ \mathit{snd}\ \mathit{ds}) = 1$
and $\bigwedge \mathit{ds}.\ \mathit{ds} \in \mathit{set}\ (\mathit{distrs}\ \mathit{mdp}\ !!\ i) \Longrightarrow \mathit{distinct}\ (\mathit{map}\ \mathit{fst}\ \mathit{ds})$
and $\mathit{distrs}\ \mathit{mdp}\ !!\ i \neq []$
⟨*proof*⟩

lemma *valid-mdp-rp-sparse-mult*:
assumes $\mathit{valid\text{-}mdp\text{-}rp}\ \mathit{mdp}\ i < \mathit{state\text{-}count}\ \mathit{mdp}\ \mathit{ds} \in \mathit{set}\ (\mathit{distrs}\ \mathit{mdp}\ !!\ i)$
shows $\mathit{sparse\text{-}mult}\ \mathit{ds}\ y = (\sum i{<}\mathit{state\text{-}count}\ \mathit{mdp}.\ \mathit{lookup}\ 0\ \mathit{ds}\ i * y\ !!\ i)$
⟨*proof*⟩

lemma *valid-sub-certD*:
assumes $\mathit{valid\text{-}mdp\text{-}rp}\ \mathit{mdp}\ \mathit{valid\text{-}sub\text{-}cert}\ \mathit{mdp}\ c\ \mathit{ord}\ \mathit{check}\ i < \mathit{state\text{-}count}\ \mathit{mdp}$
shows $\neg\ \mathit{states1}\ \mathit{mdp}\ !!\ i \Longrightarrow \neg\ \mathit{states2}\ \mathit{mdp}\ !!\ i \Longrightarrow \mathit{solution}\ c\ !!\ i = 0$
and $\mathit{states2}\ \mathit{mdp}\ !!\ i \Longrightarrow \mathit{solution}\ c\ !!\ i = 1$
and $\mathit{states1}\ \mathit{mdp}\ !!\ i \Longrightarrow 0 \leq \mathit{solution}\ c\ !!\ i$
and $\bigwedge \mathit{ds}.\ \mathit{states1}\ \mathit{mdp}\ !!\ i \Longrightarrow \mathit{ds} \in \mathit{set}\ (\mathit{distrs}\ \mathit{mdp}\ !!\ i) \Longrightarrow \mathit{ord}\ (\mathit{sparse\text{-}mult}\ \mathit{ds}\ (\mathit{solution}\ c))\ (\mathit{solution}\ c\ !!\ i)$
and $\bigwedge \mathit{ds}.\ \mathit{states1}\ \mathit{mdp}\ !!\ i \Longrightarrow 0 < \mathit{solution}\ c\ !!\ i \longrightarrow \mathit{check}\ (\mathit{distrs}\ \mathit{mdp}\ !!\ i)\ (\mathit{witness}\ c\ !!\ i)$
⟨*proof*⟩

lemma *valid-pos-certD*:
assumes $\mathit{valid\text{-}mdp\text{-}rp}\ \mathit{mdp}\ \mathit{valid\text{-}pos\text{-}cert}\ \mathit{mdp}\ c\ i < \mathit{state\text{-}count}\ \mathit{mdp}\ \mathit{states1}\ \mathit{mdp}\ !!\ i$
$0 < \mathit{solution}\ c\ !!\ i\ \mathit{witness}\ c\ !!\ i = ((j,\ a),\ n)$
shows $\mathit{snd}\ (\mathit{witness}\ c\ !!\ j) < n \wedge j < \mathit{state\text{-}count}\ \mathit{mdp} \wedge a < \mathit{length}\ (\mathit{distrs}\ \mathit{mdp}\ !!\ i) \wedge$
$\mathit{lookup}\ 0\ ((\mathit{distrs}\ \mathit{mdp}\ !!\ i)\ !\ a)\ j \neq 0 \wedge 0 < \mathit{solution}\ c\ !!\ j$
⟨*proof*⟩

lemma *valid-neg-certD*:
assumes *valid-mdp-rp mdp valid-neg-cert mdp c i* < *state-count mdp states1 mdp* !! *i*
0 < *solution c* !! *i witness c* !! *i* = (*js*, *n*)
shows *list-all2* (λ*j ds*. *j* < *state-count mdp* ∧ *snd* (*witness c* !! *j*) < *n* ∧ *lookup* 0 *ds j* ≠ 0 ∧ 0 < *solution c* !! *j*) *js* (*distrs mdp* !! *i*)
⟨*proof*⟩

context
fixes *mdp c*
assumes *rp*: *valid-mdp-rp mdp*
assumes *cert*: *valid-cert mdp c*
begin

interpretation *pmf-as-function* ⟨*proof*⟩

abbreviation *S* ≡ {..< *state-count mdp*}
abbreviation *S1* ≡ {*i*. *i* < *state-count mdp* ∧ (*states1 mdp*) !! *i*}
abbreviation *S2* ≡ {*i*. *i* < *state-count mdp* ∧ (*states2 mdp*) !! *i*}

lift-definition *K* :: *nat* ⇒ *nat pmf set* **is**
λ*i*. *if i* < *state-count mdp then*
{ (λ*j*. *of-rat* (*lookup* 0 *D j*) :: *real*) | *D*. *D* ∈ *set* (*distrs mdp* !! *i*) }
else { *indicator* {0} }
⟨*proof*⟩

interpretation *MDP*: *Reachability-Problem K S S1 S2*
⟨*proof*⟩

definition *P-max s* = *enn2real* (*MDP.p s*)
definition *P-min s* = *enn2real* (*MDP.n s*)

lemma
assumes *i* < *state-count mdp*
shows *P-max*: *P-max i* ≤ *real-of-rat* (*solution* (*pos-cert c*) !! *i*) (**is** *?max*)
and *P-min*: *P-min i* ≥ *real-of-rat* (*solution* (*neg-cert c*) !! *i*) (**is** *?min*)
⟨*proof*⟩

end

end

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