Power Operator for Lists

Štěpán Holub, Martin Raška, Štěpán Starosta and Tobias Nipkow

February 23, 2025

Abstract

This entry defines the power operator `xs ^^ n`, the `n`-fold concatenation of `xs` with itself.

Much of the theory is taken from the AFP entry Combinatorics on Words Basics where the operator is called `^@`. This new entry uses the standard overloaded `^^` syntax and is aimed at becoming the central theory of the power operator for lists that can be extended easily.

1 The Power Operator $\widehat{}\widehat{}$ on Lists

theory *List-Power*
imports *Main*
begin

lemma *concat-replicate-single*[*simp*]: *concat* (*replicate m* [*a*]) = *replicate m a*
⟨*proof*⟩

overloading *pow-list* == *compow* :: *nat* ⇒ ′*a list* ⇒ ′*a list*
begin

primrec *pow-list* :: *nat* ⇒ ′*a list* ⇒ ′*a list* **where**
pow-list 0 xs = [] |
pow-list (*Suc n*) *xs* = *xs* @ *pow-list n xs*

end

context
begin

interpretation *monoid-mult* [] *append*
  rewrites *power u n* = *u* $\widehat{}\widehat{}$ *n*
⟨*proof*⟩

lemmas *pow-list-zero* = *power.power-0* **and**
  pow-list-one = *power-Suc0-right* **and**
  pow-list-1 = *power-one-right* **and**

pow-list-Nil = *power-one* **and**
pow-list-2 = *power2-eq-square* **and**
pow-list-Suc = *power-Suc* **and**
pow-list-Suc2 = *power-Suc2* **and**
pow-list-comm = *power-commutes* **and**
pow-list-add = *power-add* **and**
pow-list-eq-if = *power-eq-if* **and**
pow-list-mult = *power-mult* **and**
pow-list-commuting-commutes = *power-commuting-commutes*

end

lemma *pow-list-alt*: $xs^\frown\!\!^\frown n = concat\ (replicate\ n\ xs)$
⟨*proof*⟩

lemma *pow-list-single*: $[a]\ {}^\frown\!{}^\frown\ m = replicate\ m\ a$
⟨*proof*⟩

lemma *length-pow-list-single* [*simp*]: $length([a]\ {}^\frown\!{}^\frown\ n) = n$
⟨*proof*⟩

lemma *nth-pow-list-single*: $i < m \Longrightarrow ([a]\ {}^\frown\!{}^\frown\ m)\ !\ i = a$
⟨*proof*⟩

lemma *pow-list-not-NilD*: $xs\ {}^\frown\!{}^\frown\ m \neq [] \Longrightarrow 0 < m$
⟨*proof*⟩

lemma *length-pow-list*: $length(xs\ {}^\frown\!{}^\frown\ k) = k * length\ xs$
⟨*proof*⟩

lemma *pow-list-set*: $set\ (w\ {}^\frown\!{}^\frown\ Suc\ k) = set\ w$
⟨*proof*⟩

lemma *pow-list-slide*: $xs\ @\ (ys\ @\ xs)\ {}^\frown\!{}^\frown\ n\ \ @\ ys = (xs\ @\ ys)^\frown\!\!^\frown(Suc\ n)$
⟨*proof*⟩

lemma *hd-pow-list*: $0 < n \Longrightarrow hd(xs\ {}^\frown\!{}^\frown\ n) = hd\ xs$
⟨*proof*⟩

lemma *rev-pow-list*: $rev\ (xs\ {}^\frown\!{}^\frown\ m) = (rev\ xs)\ {}^\frown\!{}^\frown\ m$
⟨*proof*⟩

lemma *eq-pow-list-iff-eq-exp*[*simp*]: **assumes** $xs \neq []$ **shows** $xs\ {}^\frown\!{}^\frown\ k = xs\ {}^\frown\!{}^\frown\ m$
$\longleftrightarrow k = m$
⟨*proof*⟩

lemma *pow-list-Nil-iff-0*: $xs \neq [] \Longrightarrow xs\ {}^\frown\!{}^\frown\ m = [] \longleftrightarrow m = 0$
⟨*proof*⟩

lemma *pow-list-Nil-iff-Nil*: $0 < m \Longrightarrow xs$ ^^ $m = [] \longleftrightarrow xs = []$
⟨*proof*⟩

lemma *pow-eq-eq*:
assumes xs ^^ $k = ys$ ^^ k **and** $0 < k$
shows $(xs::'a\ list) = ys$
⟨*proof*⟩

lemma *map-pow-list*[*simp*]: $map\ f\ (xs$ ^^ $k) = (map\ f\ xs)$ ^^ k
⟨*proof*⟩

lemma *concat-pow-list*: $concat\ (xs$ ^^ $k) = (concat\ xs)$ ^^ k
⟨*proof*⟩

lemma *concat-pow-list-single*[*simp*]: $concat\ ([a]$ ^^ $k) = a$ ^^ k
⟨*proof*⟩

lemma *pow-list-single-Nil-iff*: $[a]$ ^^ $n = [] \longleftrightarrow n = 0$
⟨*proof*⟩

lemma *hd-pow-list-single*: $k \neq 0 \Longrightarrow hd\ ([a]$ ^^ $k) = a$
⟨*proof*⟩

lemma *index-pow-mod*: $i < length(xs$ ^^ $k) \Longrightarrow (xs$ ^^ $k)!i = xs!(i\ mod\ length\ xs)$
⟨*proof*⟩

lemma *unique-letter-word*: **assumes** $\bigwedge c.\ c \in set\ w \Longrightarrow c = a$ **shows** $w = [a]$ ^^ $length\ w$
⟨*proof*⟩

lemma *count-list-pow-list*: $count\text{-}list\ (w$ ^^ $k)\ a = k * (count\text{-}list\ w\ a)$
⟨*proof*⟩

lemma *sing-pow-lists*: $a \in A \Longrightarrow [a]$ ^^ $n \in lists\ A$
⟨*proof*⟩

lemma *one-generated-list-power*: $u \in lists\ \{x\} \Longrightarrow \exists\, k.\ concat\ u = x$ ^^ k
⟨*proof*⟩

lemma *pow-list-in-lists*: $0 < k \Longrightarrow u$ ^^ $k \in lists\ B \Longrightarrow u \in lists\ B$
⟨*proof*⟩

end