

# A Verified Solver for Linear Recurrences

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## Abstract

Linear recurrences with constant coefficients are an interesting class of recurrence equations that can be solved explicitly. The most famous example are certainly the Fibonacci numbers with the equation  $f(n) = f(n - 1) + f(n - 2)$  and the quite non-obvious closed form

$$\frac{1}{\sqrt{5}}(\varphi^n - (-\varphi)^{-n})$$

where  $\varphi$  is the golden ratio.

In this work, I build on existing tools in Isabelle – such as formal power series and polynomial factorisation algorithms – to develop a theory of these recurrences and derive a fully executable solver for them that can be exported to programming languages like Haskell.

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# 1 Rational formal power series

```
theory RatFPS
imports
  Complex-Main
  HOL-Computational-Algebra.Computational-Algebra
  HOL-Computational-Algebra.Polynomial-Factorial
begin
```

## 1.1 Some auxiliary

```
abbreviation constant-term :: 'a poly  $\Rightarrow$  'a::zero
where constant-term p  $\equiv$  coeff p 0
```

```
lemma coeff-0-mult: coeff (p * q) 0 = coeff p 0 * coeff q 0
  <proof>
```

```
lemma coeff-0-div:
  assumes coeff p 0  $\neq$  0
  assumes (q :: 'a :: field poly) dvd p
  shows coeff (p div q) 0 = coeff p 0 div coeff q 0
  <proof>
```

```
lemma coeff-0-add-fract-nonzero:
  assumes coeff (snd (quot-of-fract x)) 0  $\neq$  0 coeff (snd (quot-of-fract y)) 0  $\neq$  0
  shows coeff (snd (quot-of-fract (x + y))) 0  $\neq$  0
  <proof>
```

```
lemma coeff-0-normalize-quot-nonzero [simp]:
  assumes coeff (snd x) 0  $\neq$  0
  shows coeff (snd (normalize-quot x)) 0  $\neq$  0
  <proof>
```

```
abbreviation numerator :: 'a fract  $\Rightarrow$  'a::{ring-gcd,idom-divide,semiring-gcd-mult-normalize}
where numerator x  $\equiv$  fst (quot-of-fract x)
```

```
abbreviation denominator :: 'a fract  $\Rightarrow$  'a::{ring-gcd,idom-divide,semiring-gcd-mult-normalize}
where denominator x  $\equiv$  snd (quot-of-fract x)
```

```
declare unit-factor-snd-quot-of-fract [simp]
  normalize-snd-quot-of-fract [simp]
```

```
lemma constant-term-denominator-nonzero-imp-constant-term-denominator-div-gcd-nonzero:
  constant-term (denominator x div gcd a (denominator x))  $\neq$  0
  if constant-term (denominator x)  $\neq$  0
  <proof>
```

## 1.2 The type of rational formal power series

```
typedef (overloaded) 'a :: field-gcd ratfps =
```

```

    {x :: 'a poly fract. constant-term (denominator x) ≠ 0}
    ⟨proof⟩

setup-lifting type-definition-ratfps

instantiation ratfps :: (field-gcd) idom
begin

lift-definition zero-ratfps :: 'a ratfps is 0 ⟨proof⟩

lift-definition one-ratfps :: 'a ratfps is 1 ⟨proof⟩

lift-definition uminus-ratfps :: 'a ratfps ⇒ 'a ratfps is uminus
    ⟨proof⟩

lift-definition plus-ratfps :: 'a ratfps ⇒ 'a ratfps ⇒ 'a ratfps is (+)
    ⟨proof⟩

lift-definition minus-ratfps :: 'a ratfps ⇒ 'a ratfps ⇒ 'a ratfps is (-)
    ⟨proof⟩

lift-definition times-ratfps :: 'a ratfps ⇒ 'a ratfps ⇒ 'a ratfps is (*)
    ⟨proof⟩

instance
    ⟨proof⟩

end

fun ratfps-nth-aux :: ('a::field) poly ⇒ nat ⇒ 'a
where
    ratfps-nth-aux p 0 = inverse (coeff p 0)
  | ratfps-nth-aux p n =
    - inverse (coeff p 0) * sum (λi. coeff p i * ratfps-nth-aux p (n - i)) {1..n}

lemma ratfps-nth-aux-correct: ratfps-nth-aux p n = natfun-inverse (fps-of-poly p)
    n
    ⟨proof⟩

lift-definition ratfps-nth :: 'a :: field-gcd ratfps ⇒ nat ⇒ 'a is
    λx n. let (a,b) = quot-of-fract x
          in (∑ i = 0..n. coeff a i * ratfps-nth-aux b (n - i)) ⟨proof⟩

lift-definition ratfps-subdegree :: 'a :: field-gcd ratfps ⇒ nat is
    λx. poly-subdegree (fst (quot-of-fract x)) ⟨proof⟩

context
includes lifting-syntax
begin

```

**lemma** *RatFPS-parametric*: (*rel-prod* (=) (=) ==> (=))  
 $(\lambda(p,q). \text{if } \text{coeff } q \ 0 = 0 \text{ then } 0 \text{ else } \text{quot-to-fract } (p, q))$   
 $(\lambda(p,q). \text{if } \text{coeff } q \ 0 = 0 \text{ then } 0 \text{ else } \text{quot-to-fract } (p, q))$   
 $\langle \text{proof} \rangle$

**end**

**lemma** *normalize-quot-quot-of-fract* [*simp*]:  
 $\text{normalize-quot } (\text{quot-of-fract } x) = \text{quot-of-fract } x$   
 $\langle \text{proof} \rangle$

**context**  
**assumes** *SORT-CONSTRAINT*('a::field-gcd)  
**begin**

**lift-definition** *quot-of-ratfps* :: 'a ratfps  $\Rightarrow$  ('a poly  $\times$  'a poly) **is**  
 $\text{quot-of-fract} :: 'a \text{ poly } \text{fract} \Rightarrow ('a \text{ poly} \times 'a \text{ poly}) \langle \text{proof} \rangle$

**lift-definition** *quot-to-ratfps* :: ('a poly  $\times$  'a poly)  $\Rightarrow$  'a ratfps **is**  
 $\lambda(x,y). \text{let } (x',y') = \text{normalize-quot } (x,y)$   
 $\text{in } \text{if } \text{coeff } y' \ 0 = 0 \text{ then } 0 \text{ else } \text{quot-to-fract } (x',y')$   
 $\langle \text{proof} \rangle$

**lemma** *quot-to-ratfps-quot-of-ratfps* [*code abstype*]:  
 $\text{quot-to-ratfps } (\text{quot-of-ratfps } x) = x$   
 $\langle \text{proof} \rangle$

**lemma** *coeff-0-snd-quot-of-ratfps-nonzero* [*simp*]:  
 $\text{coeff } (\text{snd } (\text{quot-of-ratfps } x)) \ 0 \neq 0$   
 $\langle \text{proof} \rangle$

**lemma** *quot-of-ratfps-quot-to-ratfps*:  
 $\text{coeff } (\text{snd } x) \ 0 \neq 0 \implies x \in \text{normalized-fracts} \implies \text{quot-of-ratfps } (\text{quot-to-ratfps } x) = x$   
 $\langle \text{proof} \rangle$

**lemma** *quot-of-ratfps-0* [*simp*, *code abstract*]:  $\text{quot-of-ratfps } 0 = (0, 1)$   
 $\langle \text{proof} \rangle$

**lemma** *quot-of-ratfps-1* [*simp*, *code abstract*]:  $\text{quot-of-ratfps } 1 = (1, 1)$   
 $\langle \text{proof} \rangle$

**lift-definition** *ratfps-of-poly* :: 'a poly  $\Rightarrow$  'a ratfps **is**  
 $\text{to-fract} :: 'a \text{ poly} \Rightarrow -$   
 $\langle \text{proof} \rangle$

**lemma** *ratfps-of-poly-code* [*code abstract*]:

$\text{quot-of-ratfps} (\text{ratfps-of-poly } p) = (p, 1)$   
 $\langle \text{proof} \rangle$

**lemmas**  $\text{zero-ratfps-code} = \text{quot-of-ratfps-0}$

**lemmas**  $\text{one-ratfps-code} = \text{quot-of-ratfps-1}$

**lemma**  $\text{uminus-ratfps-code}$  [code abstract]:  
 $\text{quot-of-ratfps} (-x) = (\text{let } (a, b) = \text{quot-of-ratfps } x \text{ in } (-a, b))$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{plus-ratfps-code}$  [code abstract]:  
 $\text{quot-of-ratfps} (x + y) =$   
 $(\text{let } (a, b) = \text{quot-of-ratfps } x; (c, d) = \text{quot-of-ratfps } y$   
 $\text{in } \text{normalize-quot} (a * d + b * c, b * d))$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{minus-ratfps-code}$  [code abstract]:  
 $\text{quot-of-ratfps} (x - y) =$   
 $(\text{let } (a, b) = \text{quot-of-ratfps } x; (c, d) = \text{quot-of-ratfps } y$   
 $\text{in } \text{normalize-quot} (a * d - b * c, b * d))$   
 $\langle \text{proof} \rangle$

**definition**  $\text{ratfps-cutoff} :: \text{nat} \Rightarrow 'a :: \text{field-gcd ratfps} \Rightarrow 'a \text{ poly}$  **where**  
 $\text{ratfps-cutoff } n \ x = \text{poly-of-list} (\text{map} (\text{ratfps-nth } x) [0..<n])$

**definition**  $\text{ratfps-shift} :: \text{nat} \Rightarrow 'a :: \text{field-gcd ratfps} \Rightarrow 'a \text{ ratfps}$  **where**  
 $\text{ratfps-shift } n \ x = (\text{let } (a, b) = \text{quot-of-ratfps} (x - \text{ratfps-of-poly} (\text{ratfps-cutoff } n$   
 $x))$   
 $\text{in } \text{quot-to-ratfps} (\text{poly-shift } n \ a, b))$

**lemma**  $\text{times-ratfps-code}$  [code abstract]:  
 $\text{quot-of-ratfps} (x * y) =$   
 $(\text{let } (a, b) = \text{quot-of-ratfps } x; (c, d) = \text{quot-of-ratfps } y;$   
 $(e, f) = \text{normalize-quot} (a, d); (g, h) = \text{normalize-quot} (c, b)$   
 $\text{in } (e * g, f * h))$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{ratfps-nth-code}$  [code]:  
 $\text{ratfps-nth } x \ n =$   
 $(\text{let } (a, b) = \text{quot-of-ratfps } x$   
 $\text{in } \sum_{i=0..n} \text{coeff } a \ i * \text{ratfps-nth-aux } b \ (n - i))$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{ratfps-subdegree-code}$  [code]:  
 $\text{ratfps-subdegree } x = \text{poly-subdegree} (\text{fst} (\text{quot-of-ratfps } x))$   
 $\langle \text{proof} \rangle$

**end**

**instantiation** *ratfps* :: (*field-gcd*) *inverse*  
**begin**

**lift-definition** *inverse-ratfps* :: 'a *ratfps*  $\Rightarrow$  'a *ratfps* **is**  
 $\lambda x.$  *let* (*a,b*) = *quot-of-fract* *x*  
     *in* *if* *coeff* *a* 0 = 0 *then* 0 *else* *inverse* *x*  
 ⟨*proof*⟩

**lift-definition** *divide-ratfps* :: 'a *ratfps*  $\Rightarrow$  'a *ratfps*  $\Rightarrow$  'a *ratfps* **is**  
 $\lambda f g.$  (*if* *g* = 0 *then* 0 *else*  
     *let* *n* = *ratfps-subdegree* *g*; *h* = *ratfps-shift* *n* *g*  
     *in* *ratfps-shift* *n* (*f* \* *inverse* *h*)) ⟨*proof*⟩

**instance** ⟨*proof*⟩  
**end**

**lemma** *ratfps-inverse-code* [*code abstract*]:  
*quot-of-ratfps* (*inverse* *x*) =  
 (*let* (*a,b*) = *quot-of-ratfps* *x*  
   *in* *if* *coeff* *a* 0 = 0 *then* (0, 1)  
   *else* *let* *u* = *unit-factor* *a* *in* (*b* *div* *u*, *a* *div* *u*))  
 ⟨*proof*⟩

**instantiation** *ratfps* :: (*equal*) *equal*  
**begin**

**definition** *equal-ratfps* :: 'a *ratfps*  $\Rightarrow$  'a *ratfps*  $\Rightarrow$  *bool* **where**  
 [*simp*]: *equal-ratfps* *x* *y*  $\longleftrightarrow$  *x* = *y*

**instance** ⟨*proof*⟩

**end**

**lemma** *quot-of-fract-eq-iff* [*simp*]: *quot-of-fract* *x* = *quot-of-fract* *y*  $\longleftrightarrow$  *x* = *y*  
 ⟨*proof*⟩

**lemma** *equal-ratfps-code* [*code*]: *HOL.equal* *x* *y*  $\longleftrightarrow$  *quot-of-ratfps* *x* = *quot-of-ratfps* *y*  
 ⟨*proof*⟩

**lemma** *fps-of-poly-quot-normalize-quot* [*simp*]:  
*fps-of-poly* (*fst* (*normalize-quot* *x*)) / *fps-of-poly* (*snd* (*normalize-quot* *x*)) =  
*fps-of-poly* (*fst* *x*) / *fps-of-poly* (*snd* *x*)  
**if** (*snd* *x* :: 'a :: *field-gcd* *poly*)  $\neq$  0  
 ⟨*proof*⟩

**lemma** *fps-of-poly-quot-normalize-quot'* [*simp*]:  
*fps-of-poly* (*fst* (*normalize-quot* *x*)) / *fps-of-poly* (*snd* (*normalize-quot* *x*)) =

$\text{fps-of-poly } (\text{fst } x) / \text{fps-of-poly } (\text{snd } x)$   
**if**  $\text{coeff } (\text{snd } x) \neq 0 \neq (0 :: 'a :: \text{field-gcd})$   
 <proof>

**lift-definition**  $\text{fps-of-ratfps} :: 'a :: \text{field-gcd ratfps} \Rightarrow 'a \text{ fps is}$   
 $\lambda x. \text{fps-of-poly } (\text{numerator } x) / \text{fps-of-poly } (\text{denominator } x)$  <proof>

**lemma**  $\text{fps-of-ratfps-altdef}$ :  
 $\text{fps-of-ratfps } x = (\text{case quot-of-ratfps } x \text{ of } (a, b) \Rightarrow \text{fps-of-poly } a / \text{fps-of-poly } b)$   
 <proof>

**lemma**  $\text{fps-of-ratfps-ratfps-of-poly}$  [simp]:  $\text{fps-of-ratfps } (\text{ratfps-of-poly } p) = \text{fps-of-poly } p$   
 <proof>

**lemma**  $\text{fps-of-ratfps-0}$  [simp]:  $\text{fps-of-ratfps } 0 = 0$   
 <proof>

**lemma**  $\text{fps-of-ratfps-1}$  [simp]:  $\text{fps-of-ratfps } 1 = 1$   
 <proof>

**lemma**  $\text{fps-of-ratfps-uminus}$  [simp]:  $\text{fps-of-ratfps } (-x) = - \text{fps-of-ratfps } x$   
 <proof>

**lemma**  $\text{fps-of-ratfps-add}$  [simp]:  $\text{fps-of-ratfps } (x + y) = \text{fps-of-ratfps } x + \text{fps-of-ratfps } y$   
 <proof>

**lemma**  $\text{fps-of-ratfps-diff}$  [simp]:  $\text{fps-of-ratfps } (x - y) = \text{fps-of-ratfps } x - \text{fps-of-ratfps } y$   
 <proof>

**lemma**  $\text{is-unit-div-div-commute}$ :  $\text{is-unit } b \Longrightarrow \text{is-unit } c \Longrightarrow a \text{ div } b \text{ div } c = a \text{ div } c \text{ div } b$   
 <proof>

**lemma**  $\text{fps-of-ratfps-mult}$  [simp]:  $\text{fps-of-ratfps } (x * y) = \text{fps-of-ratfps } x * \text{fps-of-ratfps } y$   
 <proof>

**lemma**  $\text{div-const-unit-poly}$ :  $\text{is-unit } c \Longrightarrow p \text{ div } [:c] = \text{smult } (1 \text{ div } c) p$   
 <proof>

**lemma**  $\text{normalize-field}$ :  
 $\text{normalize } (x :: 'a :: \{\text{normalization-semidom,field}\}) = (\text{if } x = 0 \text{ then } 0 \text{ else } 1)$   
 <proof>

**lemma**  $\text{unit-factor-field}$  [simp]:  
 $\text{unit-factor } (x :: 'a :: \{\text{normalization-semidom,field}\}) = x$

*<proof>*

**lemma** *fps-of-poly-normalize-field:*

*fps-of-poly (normalize (p :: 'a :: {field, normalization-semidom} poly)) =*  
*fps-of-poly p \* fps-const (inverse (lead-coeff p))*

*<proof>*

**lemma** *unit-factor-poly-altdef:* *unit-factor p = monom (unit-factor (lead-coeff p))*  
*0*

*<proof>*

**lemma** *div-const-poly:* *p div [:c::'a::field:] = smult (inverse c) p*

*<proof>*

**lemma** *fps-of-ratfps-inverse [simp]:* *fps-of-ratfps (inverse x) = inverse (fps-of-ratfps*  
*x)*

*<proof>*

**context**

**includes** *fps-syntax*

**begin**

**lemma** *ratfps-nth-altdef:* *ratfps-nth x n = fps-of-ratfps x \$ n*

*<proof>*

**lemma** *fps-of-ratfps-is-unit:* *fps-of-ratfps a \$ 0 ≠ 0 ⟷ ratfps-nth a 0 ≠ 0*

*<proof>*

**lemma** *ratfps-nth-0 [simp]:* *ratfps-nth 0 n = 0*

*<proof>*

**lemma** *fps-of-ratfps-cases:*

**obtains** *p q where* *coeff q 0 ≠ 0 fps-of-ratfps f = fps-of-poly p / fps-of-poly q*

*<proof>*

**lemma** *fps-of-ratfps-cutoff [simp]:*

*fps-of-poly (ratfps-cutoff n x) = fps-cutoff n (fps-of-ratfps x)*

*<proof>*

**lemma** *subdegree-fps-of-ratfps:*

*subdegree (fps-of-ratfps x) = ratfps-subdegree x*

*<proof>*

**lemma** *ratfps-subdegree-altdef:*

*ratfps-subdegree x = subdegree (fps-of-ratfps x)*

*<proof>*

**end**

**code-datatype** *fps-of-ratfps*

**lemma** *fps-zero-code* [code]:  $0 = \text{fps-of-ratfps } 0$   $\langle \text{proof} \rangle$

**lemma** *fps-one-code* [code]:  $1 = \text{fps-of-ratfps } 1$   $\langle \text{proof} \rangle$

**lemma** *fps-const-code* [code]:  $\text{fps-const } c = \text{fps-of-poly } [:c:]$   $\langle \text{proof} \rangle$

**lemma** *fps-of-poly-code* [code]:  $\text{fps-of-poly } p = \text{fps-of-ratfps } (\text{ratfps-of-poly } p)$   $\langle \text{proof} \rangle$

**lemma** *fps-X-code* [code]:  $\text{fps-X} = \text{fps-of-ratfps } (\text{ratfps-of-poly } [:0,1:])$   $\langle \text{proof} \rangle$

**lemma** *fps-nth-code* [code]:  $\text{fps-nth } (\text{fps-of-ratfps } x) n = \text{ratfps-nth } x n$   
 $\langle \text{proof} \rangle$

**lemma** *fps-uminus-code* [code]:  $-\text{fps-of-ratfps } x = \text{fps-of-ratfps } (-x)$   $\langle \text{proof} \rangle$

**lemma** *fps-add-code* [code]:  $\text{fps-of-ratfps } x + \text{fps-of-ratfps } y = \text{fps-of-ratfps } (x + y)$   $\langle \text{proof} \rangle$

**lemma** *fps-diff-code* [code]:  $\text{fps-of-ratfps } x - \text{fps-of-ratfps } y = \text{fps-of-ratfps } (x - y)$   
 $\langle \text{proof} \rangle$

**lemma** *fps-mult-code* [code]:  $\text{fps-of-ratfps } x * \text{fps-of-ratfps } y = \text{fps-of-ratfps } (x * y)$   
 $\langle \text{proof} \rangle$

**lemma** *fps-inverse-code* [code]:  $\text{inverse } (\text{fps-of-ratfps } x) = \text{fps-of-ratfps } (\text{inverse } x)$   
 $\langle \text{proof} \rangle$

**lemma** *fps-cutoff-code* [code]:  $\text{fps-cutoff } n (\text{fps-of-ratfps } x) = \text{fps-of-poly } (\text{ratfps-cutoff } n x)$   
 $\langle \text{proof} \rangle$

**lemmas** *subdegree-code* [code] = *subdegree-fps-of-ratfps*

**lemma** *fractrel-normalize-quot*:

$\text{fractrel } p p \implies \text{fractrel } q q \implies$   
 $\text{fractrel } (\text{normalize-quot } p) (\text{normalize-quot } q) \iff \text{fractrel } p q$   
 $\langle \text{proof} \rangle$

**lemma** *fps-of-ratfps-eq-iff* [simp]:

$\text{fps-of-ratfps } p = \text{fps-of-ratfps } q \iff p = q$   
 $\langle \text{proof} \rangle$

**lemma** *fps-of-ratfps-eq-zero-iff* [simp]:

$\text{fps-of-ratfps } p = 0 \iff p = 0$

*<proof>*

**lemma** *unit-factor-snd-quot-of-ratfps* [simp]:  
unit-factor (snd (quot-of-ratfps x)) = 1  
*<proof>*

**lemma** *poly-shift-times-monom-le*:  
 $n \leq m \implies \text{poly-shift } n (\text{monom } c \ m * p) = \text{monom } c \ (m - n) * p$   
*<proof>*

**lemma** *poly-shift-times-monom-ge*:  
 $n \geq m \implies \text{poly-shift } n (\text{monom } c \ m * p) = \text{smult } c \ (\text{poly-shift } (n - m) \ p)$   
*<proof>*

**lemma** *poly-shift-times-monom*:  
 $\text{poly-shift } n (\text{monom } c \ n * p) = \text{smult } c \ p$   
*<proof>*

**lemma** *monom-times-poly-shift*:  
assumes *poly-subdegree*  $p \geq n$   
shows  $\text{monom } c \ n * \text{poly-shift } n \ p = \text{smult } c \ p$  (is ?lhs = ?rhs)  
*<proof>*

**lemma** *monom-times-poly-shift'*:  
assumes *poly-subdegree*  $p \geq n$   
shows  $\text{monom } (1 :: 'a :: \text{comm-semiring-1}) \ n * \text{poly-shift } n \ p = p$   
*<proof>*

**lemma** *subdegree-minus-cutoff-ge*:  
assumes  $f - \text{fps-cutoff } n$  ( $f :: 'a :: \text{ab-group-add fps}$ )  $\neq 0$   
shows  $\text{subdegree } (f - \text{fps-cutoff } n \ f) \geq n$   
*<proof>*

**lemma** *fps-shift-times-X-power''*:  $\text{fps-shift } n (\text{fps-X } ^n * f :: 'a :: \text{comm-ring-1} \ \text{fps}) = f$   
*<proof>*

**lemma**  
*ratfps-shift-code* [code abstract]:  
quot-of-ratfps (ratfps-shift n x) =  
 (let (a, b) = quot-of-ratfps (x - ratfps-of-poly (ratfps-cutoff n x))  
 in (poly-shift n a, b)) (is ?lhs1 = ?rhs1) and  
*fps-of-ratfps-shift* [simp]:  
fps-of-ratfps (ratfps-shift n x) = fps-shift n (fps-of-ratfps x)  
*<proof>*  
include *fps-syntax*  
*<proof>*

**lemma** *fps-shift-code* [code]:  $\text{fps-shift } n (\text{fps-of-ratfps } x) = \text{fps-of-ratfps } (\text{ratfps-shift } n \ x)$

$n\ x)$   
 $\langle proof \rangle$

**instantiation**  $fps :: (equal)\ equal$   
**begin**

**definition**  $equal-fps :: 'a\ fps \Rightarrow 'a\ fps \Rightarrow bool$  **where**  
 $[simp]: equal-fps\ f\ g \longleftrightarrow f = g$

**instance**  $\langle proof \rangle$

**end**

**lemma**  $equal-fps-code\ [code]: HOL.equal\ (fps-of-ratfps\ f)\ (fps-of-ratfps\ g) \longleftrightarrow f = g$   
 $\langle proof \rangle$

**lemma**  $fps-of-ratfps-divide\ [simp]:$   
 $fps-of-ratfps\ (f\ div\ g) = fps-of-ratfps\ f\ div\ fps-of-ratfps\ g$   
 $\langle proof \rangle$

**lemma**  $ratfps-eqI: fps-of-ratfps\ x = fps-of-ratfps\ y \Longrightarrow x = y\ \langle proof \rangle$

**instance**  $ratfps :: (field-gcd)\ algebraic-semidom$   
 $\langle proof \rangle$

**lemma**  $fps-of-ratfps-dvd\ [simp]:$   
 $fps-of-ratfps\ x\ dvd\ fps-of-ratfps\ y \longleftrightarrow x\ dvd\ y$   
 $\langle proof \rangle$

**lemma**  $is-unit-ratfps-iff\ [simp]:$   
 $is-unit\ x \longleftrightarrow ratfps-nth\ x\ 0 \neq 0$   
 $\langle proof \rangle$

**instantiation**  $ratfps :: (field-gcd)\ normalization-semidom$   
**begin**

**definition**  $unit-factor-ratfps :: 'a\ ratfps \Rightarrow 'a\ ratfps$  **where**  
 $unit-factor\ x = ratfps-shift\ (ratfps-subdegree\ x)\ x$

**definition**  $normalize-ratfps :: 'a\ ratfps \Rightarrow 'a\ ratfps$  **where**  
 $normalize\ x = (if\ x = 0\ then\ 0\ else\ ratfps-of-poly\ (monom\ 1\ (ratfps-subdegree\ x)))$

**lemma**  $fps-of-ratfps-unit-factor\ [simp]:$   
 $fps-of-ratfps\ (unit-factor\ x) = unit-factor\ (fps-of-ratfps\ x)$   
 $\langle proof \rangle$

**lemma**  $fps-of-ratfps-normalize\ [simp]:$

```

    fps-of-ratfps (normalize x) = normalize (fps-of-ratfps x)
    ⟨proof⟩

instance ⟨proof⟩

end

instance ratfps :: (field-gcd) normalization-semidom-multiplicative
⟨proof⟩

instantiation ratfps :: (field-gcd) semidom-modulo
begin

lift-definition modulo-ratfps :: 'a ratfps ⇒ 'a ratfps ⇒ 'a ratfps is
  λf g. if g = 0 then f else
    let n = ratfps-subdegree g; h = ratfps-shift n g
    in ratfps-of-poly (ratfps-cutoff n (f * inverse h)) * h ⟨proof⟩

lemma fps-of-ratfps-mod [simp]:
  fps-of-ratfps (f mod g :: 'a ratfps) = fps-of-ratfps f mod fps-of-ratfps g
  ⟨proof⟩

instance
  ⟨proof⟩

end

instantiation ratfps :: (field-gcd) euclidean-ring
begin

definition euclidean-size-ratfps :: 'a ratfps ⇒ nat where
  euclidean-size-ratfps x = (if x = 0 then 0 else 2 ^ ratfps-subdegree x)

lemma fps-of-ratfps-euclidean-size [simp]:
  euclidean-size x = euclidean-size (fps-of-ratfps x)
  ⟨proof⟩

instance ⟨proof⟩

end

instantiation ratfps :: (field-gcd) euclidean-ring-cancel
begin

instance
  ⟨proof⟩

end

```

**lemma** *quot-of-ratfps-eq-iff* [*simp*]:  $\text{quot-of-ratfps } x = \text{quot-of-ratfps } y \longleftrightarrow x = y$   
 ⟨*proof*⟩

**lemma** *ratfps-eq-0-code*:  $x = 0 \longleftrightarrow \text{fst } (\text{quot-of-ratfps } x) = 0$   
 ⟨*proof*⟩

**lemma** *fps-dvd-code* [*code-unfold*]:  
 $x \text{ dvd } y \longleftrightarrow y = 0 \vee ((x :: 'a :: \text{field-gcd } \text{fps}) \neq 0 \wedge \text{subdegree } x \leq \text{subdegree } y)$   
 ⟨*proof*⟩

**lemma** *ratfps-dvd-code* [*code-unfold*]:  
 $x \text{ dvd } y \longleftrightarrow y = 0 \vee (x \neq 0 \wedge \text{ratfps-subdegree } x \leq \text{ratfps-subdegree } y)$   
 ⟨*proof*⟩

**instance** *ratfps* :: (*field-gcd*) *normalization-euclidean-semiring* ⟨*proof*⟩

**instantiation** *ratfps* :: (*field-gcd*) *euclidean-ring-gcd*  
**begin**

**definition** *gcd-ratfps* = (*Euclidean-Algorithm.gcd* :: 'a *ratfps*  $\Rightarrow$  -)

**definition** *lcm-ratfps* = (*Euclidean-Algorithm.lcm* :: 'a *ratfps*  $\Rightarrow$  -)

**definition** *Gcd-ratfps* = (*Euclidean-Algorithm.Gcd* :: 'a *ratfps set*  $\Rightarrow$  -)

**definition** *Lcm-ratfps* = (*Euclidean-Algorithm.Lcm*:: 'a *ratfps set*  $\Rightarrow$  -)

**instance** ⟨*proof*⟩  
**end**

**lemma** *ratfps-eq-0-iff*:  $x = 0 \longleftrightarrow \text{fps-of-ratfps } x = 0$   
 ⟨*proof*⟩

**lemma** *ratfps-of-poly-eq-0-iff*:  $\text{ratfps-of-poly } x = 0 \longleftrightarrow x = 0$   
 ⟨*proof*⟩

**lemma** *ratfps-gcd*:  
**assumes** [*simp*]:  $f \neq 0 \ g \neq 0$   
**shows**  $\text{gcd } f \ g = \text{ratfps-of-poly } (\text{monom } 1 \ (\text{min } (\text{ratfps-subdegree } f) \ (\text{ratfps-subdegree } g)))$   
 ⟨*proof*⟩

**lemma** *ratfps-gcd-altdef*:  $\text{gcd } (f :: 'a :: \text{field-gcd } \text{ratfps}) \ g =$   
 (*if*  $f = 0 \wedge g = 0$  *then*  $0$  *else*  
   *if*  $f = 0$  *then*  $\text{ratfps-of-poly } (\text{monom } 1 \ (\text{ratfps-subdegree } g))$  *else*  
   *if*  $g = 0$  *then*  $\text{ratfps-of-poly } (\text{monom } 1 \ (\text{ratfps-subdegree } f))$  *else*  
      $\text{ratfps-of-poly } (\text{monom } 1 \ (\text{min } (\text{ratfps-subdegree } f) \ (\text{ratfps-subdegree } g)))$ )  
 ⟨*proof*⟩

**lemma** *ratfps-lcm*:

**assumes**  $[simp]: f \neq 0 \ g \neq 0$   
**shows**  $lcm\ f\ g = ratfps\text{-of-poly}\ (monom\ 1\ (max\ (ratfps\text{-subdegree}\ f)\ (ratfps\text{-subdegree}\ g)))$   
 $\langle proof \rangle$

**lemma**  $ratfps\text{-lcm-altdef}: lcm\ (f :: 'a :: field\text{-gcd}\ ratfps)\ g =$   
 $(if\ f = 0 \vee g = 0\ then\ 0\ else$   
 $\quad ratfps\text{-of-poly}\ (monom\ 1\ (max\ (ratfps\text{-subdegree}\ f)\ (ratfps\text{-subdegree}\ g))))$   
 $\langle proof \rangle$

**lemma**  $ratfps\text{-Gcd}$ :  
**assumes**  $A - \{0\} \neq \{\}$   
**shows**  $Gcd\ A = ratfps\text{-of-poly}\ (monom\ 1\ (INF\ f \in A - \{0\}.\ ratfps\text{-subdegree}\ f))$   
 $\langle proof \rangle$

**lemma**  $ratfps\text{-Gcd-altdef}: Gcd\ (A :: 'a :: field\text{-gcd}\ ratfps\ set) =$   
 $(if\ A \subseteq \{0\}\ then\ 0\ else\ ratfps\text{-of-poly}\ (monom\ 1\ (INF\ f \in A - \{0\}.\ ratfps\text{-subdegree}\ f)))$   
 $\langle proof \rangle$

**lemma**  $ratfps\text{-Lcm}$ :  
**assumes**  $A \neq \{\}\ 0 \notin A\ bdd\text{-above}\ (ratfps\text{-subdegree}\ 'A)$   
**shows**  $Lcm\ A = ratfps\text{-of-poly}\ (monom\ 1\ (SUP\ f \in A.\ ratfps\text{-subdegree}\ f))$   
 $\langle proof \rangle$

**lemma**  $ratfps\text{-Lcm-altdef}$ :  
 $Lcm\ (A :: 'a :: field\text{-gcd}\ ratfps\ set) =$   
 $(if\ 0 \in A \vee \neg bdd\text{-above}\ (ratfps\text{-subdegree}\ 'A)\ then\ 0\ else$   
 $\quad if\ A = \{\}\ then\ 1\ else\ ratfps\text{-of-poly}\ (monom\ 1\ (SUP\ f \in A.\ ratfps\text{-subdegree}\ f)))$   
 $\langle proof \rangle$

**lemma**  $fps\text{-of-ratfps-quot-to-ratfps}$ :  
 $coeff\ y\ 0 \neq 0 \implies fps\text{-of-ratfps}\ (quot\text{-to-ratfps}\ (x,y)) = fps\text{-of-poly}\ x / fps\text{-of-poly}\ y$   
 $\langle proof \rangle$

**lemma**  $fps\text{-of-ratfps-quot-to-ratfps-code-post1}$ :  
 $fps\text{-of-ratfps}\ (quot\text{-to-ratfps}\ (x,pCons\ 1\ y)) = fps\text{-of-poly}\ x / fps\text{-of-poly}\ (pCons\ 1\ y)$   
 $fps\text{-of-ratfps}\ (quot\text{-to-ratfps}\ (x,pCons\ (-1)\ y)) = fps\text{-of-poly}\ x / fps\text{-of-poly}\ (pCons\ (-1)\ y)$   
 $\langle proof \rangle$

**lemma**  $fps\text{-of-ratfps-quot-to-ratfps-code-post2}$ :  
 $fps\text{-of-ratfps}\ (quot\text{-to-ratfps}\ (x'::'a::\{field\text{-char}\ 0,\ field\text{-gcd}\}\ poly,pCons\ (numeral\ n)\ y')) =$   
 $\quad fps\text{-of-poly}\ x' / fps\text{-of-poly}\ (pCons\ (numeral\ n)\ y')$   
 $fps\text{-of-ratfps}\ (quot\text{-to-ratfps}\ (x'::'a::\{field\text{-char}\ 0,\ field\text{-gcd}\}\ poly,pCons\ (-numeral\ n)\ y')) =$

$\text{fps-of-poly } x' / \text{fps-of-poly } (\text{pCons } (-\text{numeral } n) y')$   
 $\langle \text{proof} \rangle$

**lemmas**  $\text{fps-of-ratfps-quot-to-ratfps-code-post}$  [code-post] =  
 $\text{fps-of-ratfps-quot-to-ratfps-code-post1}$   
 $\text{fps-of-ratfps-quot-to-ratfps-code-post2}$

**lemma**  $\text{fps-dehorner}$ :

**fixes**  $a b c :: 'a :: \text{semiring-1 fps}$  **and**  $d e f :: 'b :: \text{ring-1 fps}$

**shows**

$(b + c) * \text{fps-X} = b * \text{fps-X} + c * \text{fps-X}$   $(a * \text{fps-X}) * \text{fps-X} = a * \text{fps-X}^{\wedge} 2$   
 $a * \text{fps-X}^{\wedge} m * \text{fps-X} = a * \text{fps-X}^{\wedge} (\text{Suc } m)$   $a * \text{fps-X} * \text{fps-X}^{\wedge} m = a * \text{fps-X}^{\wedge} (\text{Suc } m)$   
 $a * \text{fps-X}^{\wedge} m * \text{fps-X}^{\wedge} n = a * \text{fps-X}^{\wedge} (m+n)$   $a + (b + c) = a + b + c$   $a * 1 = a$   
 $1 * a = a$   
 $d + - e = d - e$   $(-d) * e = - (d * e)$   $d + (e - f) = d + e - f$   
 $(d - e) * \text{fps-X} = d * \text{fps-X} - e * \text{fps-X}$   $\text{fps-X} * \text{fps-X} = \text{fps-X}^{\wedge} 2$   $\text{fps-X} * \text{fps-X}^{\wedge} m = \text{fps-X}^{\wedge} (\text{Suc } m)$   
 $\text{fps-X}^{\wedge} m * \text{fps-X} = \text{fps-X}^{\wedge} (\text{Suc } m)$   
 $\text{fps-X}^{\wedge} m * \text{fps-X}^{\wedge} n = \text{fps-X}^{\wedge} (m + n)$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{fps-divide-1}$ :  $(a :: 'a :: \text{field fps}) / 1 = a$   $\langle \text{proof} \rangle$

**lemmas**  $\text{fps-of-poly-code-post}$  [code-post] =  
 $\text{fps-of-poly-simps}$   $\text{fps-const-0-eq-0}$   $\text{fps-const-1-eq-1}$   $\text{numeral-fps-const}$  [symmetric]  
 $\text{fps-const-neg}$  [symmetric]  $\text{fps-const-divide}$  [symmetric]  
 $\text{fps-dehorner}$   $\text{Suc-numeral}$   $\text{arith-simps}$   $\text{fps-divide-1}$

**context**

**includes**  $\text{term-syntax}$

**begin**

**definition**

$\text{valterm-ratfps} ::$

$'a :: \{\text{field-gcd, typerep}\} \text{poly} \times (\text{unit} \Rightarrow \text{Code-Evaluation.term}) \Rightarrow$

$'a \text{poly} \times (\text{unit} \Rightarrow \text{Code-Evaluation.term}) \Rightarrow 'a \text{ratfps} \times (\text{unit} \Rightarrow \text{Code-Evaluation.term})$

**where**

[code-unfold]:  $\text{valterm-ratfps } k l =$

$\text{Code-Evaluation.valtermify } (/) \{\cdot\}$

$(\text{Code-Evaluation.valtermify } \text{ratfps-of-poly } \{\cdot\} k) \{\cdot\}$

$(\text{Code-Evaluation.valtermify } \text{ratfps-of-poly } \{\cdot\} l)$

**end**

**instantiation**  $\text{ratfps} :: (\{\text{field-gcd, random}\}) \text{random}$

**begin**

**context**

**includes**  $\text{state-combinator-syntax}$  **and**  $\text{term-syntax}$

**begin**

**definition**

```
Quickcheck-Random.random i =  
  Quickcheck-Random.random i ◦→ (λnum::'a poly × (unit ⇒ term)).  
  Quickcheck-Random.random i ◦→ (λdenom::'a poly × (unit ⇒ term)).  
  Pair (let denom = (if fst denom = 0 then Code-Evaluation.valtermify 1 else  
denom)  
      in valterm-ratfps num denom)))
```

**instance** ⟨proof⟩

**end**

**end**

**instantiation** ratfps :: ({field,factorial-ring-gcd,exhaustive}) exhaustive  
**begin**

**definition**

```
exhaustive-ratfps f d =  
  Quickcheck-Exhaustive.exhaustive (λnum.  
  Quickcheck-Exhaustive.exhaustive (λdenom. f (  
    let denom = if denom = 0 then 1 else denom  
    in ratfps-of-poly num / ratfps-of-poly denom)) d) d
```

**instance** ⟨proof⟩

**end**

**instantiation** ratfps :: ({field-gcd,full-exhaustive}) full-exhaustive  
**begin**

**definition**

```
full-exhaustive-ratfps f d =  
  Quickcheck-Exhaustive.full-exhaustive (λnum::'a poly × (unit ⇒ term)).  
  Quickcheck-Exhaustive.full-exhaustive (λdenom::'a poly × (unit ⇒ term)).  
  f (let denom = if fst denom = 0 then Code-Evaluation.valtermify 1 else  
denom  
      in valterm-ratfps num denom)) d) d
```

**instance** ⟨proof⟩

**end**

**quickcheck-generator** fps constructors: fps-of-ratfps

**end**

## 2 Falling factorial as a polynomial

**theory** *Pochhammer-Polynomials*

**imports**

*Complex-Main*

*HOL-Combinatorics.Stirling*

*HOL-Computational-Algebra.Polynomial*

**begin**

**definition** *pochhammer-poly* ::  $\text{nat} \Rightarrow 'a :: \{\text{comm-semiring-1}\}$  *poly* **where**  
*pochhammer-poly*  $n = \text{Poly} [\text{of-nat} (\text{stirling } n \ k), k \leftarrow [0..<\text{Suc } n]]$

**lemma** *pochhammer-poly-code* [*code abstract*]:  
*coeffs* (*pochhammer-poly*  $n$ ) = *map of-nat* (*stirling-row*  $n$ )  
(*proof*)

**lemma** *coeff-pochhammer-poly*: *coeff* (*pochhammer-poly*  $n$ )  $k = \text{of-nat} (\text{stirling } n \ k)$   
(*proof*)

**lemma** *degree-pochhammer-poly* [*simp*]: *degree* (*pochhammer-poly*  $n$ ) =  $n$   
(*proof*)

**lemma** *pochhammer-poly-0* [*simp*]: *pochhammer-poly*  $0 = 1$   
(*proof*)

**lemma** *pochhammer-poly-Suc*: *pochhammer-poly* (*Suc*  $n$ ) =  $[\text{of-nat } n, 1:] * \text{pochhammer-poly } n$   
(*proof*)

**lemma** *pochhammer-poly-altdef*: *pochhammer-poly*  $n = (\prod_{i < n}. [\text{of-nat } i, 1:])$   
(*proof*)

**lemma** *eval-pochhammer-poly*: *poly* (*pochhammer-poly*  $n$ )  $k = \text{pochhammer } k \ n$   
(*proof*)

**lemma** *pochhammer-poly-Suc'*:  
*pochhammer-poly* (*Suc*  $n$ ) = *pCons*  $0$  (*pcompose* (*pochhammer-poly*  $n$ )  $[\text{of-nat } n, 1:]$ )  
(*proof*)

**end**

## 3 Miscellaneous material required for linear recurrences

**theory** *Linear-Recurrences-Misc*

**imports**

*Complex-Main*

*HOL-Computational-Algebra.Computational-Algebra*  
*HOL-Computational-Algebra.Polynomial-Factorial*

**begin**

**fun** *zip-with* **where**

*zip-with* *f* (*x#xs*) (*y#ys*) = *f* *x* *y* # *zip-with* *f* *xs* *ys*  
| *zip-with* *f* - - = []

**lemma** *length-zip-with* [*simp*]: *length* (*zip-with* *f* *xs* *ys*) = *min* (*length* *xs*) (*length* *ys*)  
{*proof*}

**lemma** *zip-with-altdef*: *zip-with* *f* *xs* *ys* = *map* ( $\lambda(x,y). f\ x\ y$ ) (*zip* *xs* *ys*)  
{*proof*}

**lemma** *zip-with-nth* [*simp*]:  
 $n < \text{length } xs \implies n < \text{length } ys \implies \text{zip-with } f\ xs\ ys\ !\ n = f\ (xs!\ n)\ (ys!\ n)$   
{*proof*}

**lemma** *take-zip-with*: *take* *n* (*zip-with* *f* *xs* *ys*) = *zip-with* *f* (*take* *n* *xs*) (*take* *n* *ys*)  
{*proof*}

**lemma** *drop-zip-with*: *drop* *n* (*zip-with* *f* *xs* *ys*) = *zip-with* *f* (*drop* *n* *xs*) (*drop* *n* *ys*)  
{*proof*}

**lemma** *map-zip-with*: *map* *f* (*zip-with* *g* *xs* *ys*) = *zip-with* ( $\lambda x\ y. f\ (g\ x)\ y$ ) *xs* *ys*  
{*proof*}

**lemma** *zip-with-map*: *zip-with* *f* (*map* *g* *xs*) (*map* *h* *ys*) = *zip-with* ( $\lambda x\ y. f\ (g\ x)\ (h\ y)$ ) *xs* *ys*  
{*proof*}

**lemma** *zip-with-map-left*: *zip-with* *f* (*map* *g* *xs*) *ys* = *zip-with* ( $\lambda x\ y. f\ (g\ x)\ y$ ) *xs* *ys*  
{*proof*}

**lemma** *zip-with-map-right*: *zip-with* *f* *xs* (*map* *g* *ys*) = *zip-with* ( $\lambda x\ y. f\ x\ (g\ y)$ ) *xs* *ys*  
{*proof*}

**lemma** *zip-with-swap*: *zip-with* ( $\lambda x\ y. f\ y\ x$ ) *xs* *ys* = *zip-with* *f* *ys* *xs*  
{*proof*}

**lemma** *set-zip-with*: *set* (*zip-with* *f* *xs* *ys*) = ( $\lambda(x,y). f\ x\ y$ ) ‘ *set* (*zip* *xs* *ys*)  
{*proof*}

**lemma** *zip-with-Pair*: *zip-with* *Pair* (*xs* :: 'a list) (*ys* :: 'b list) = *zip* *xs* *ys*  
{*proof*}

**lemma** *zip-with-altdef'*:

$zip\text{-}with\ f\ xs\ ys = [f\ (xs!i)\ (ys!i).\ i \leftarrow [0..\<min\ (length\ xs)\ (length\ ys)]]$   
*<proof>*

**lemma** *zip-altdef*:  $zip\ xs\ ys = [(xs!i,\ ys!i).\ i \leftarrow [0..\<min\ (length\ xs)\ (length\ ys)]]$   
*<proof>*

**lemma** *card-poly-roots-bound*:

**fixes**  $p :: 'a :: \{comm\text{-}ring\text{-}1,\ ring\text{-}no\text{-}zero\text{-}divisors\}$  *poly*  
**assumes**  $p \neq 0$   
**shows**  $card\ \{x.\ poly\ p\ x = 0\} \leq degree\ p$   
*<proof>*

**lemma** *poly-eqI-degree*:

**fixes**  $p\ q :: 'a :: \{comm\text{-}ring\text{-}1,\ ring\text{-}no\text{-}zero\text{-}divisors\}$  *poly*  
**assumes**  $\bigwedge x.\ x \in A \implies poly\ p\ x = poly\ q\ x$   
**assumes**  $card\ A > degree\ p\ card\ A > degree\ q$   
**shows**  $p = q$   
*<proof>*

**lemma** *poly-root-order-induct* [*case-names 0 no-roots root*]:

**fixes**  $p :: 'a :: idom\ poly$   
**assumes**  $P\ 0 \bigwedge p.\ (\bigwedge x.\ poly\ p\ x \neq 0) \implies P\ p$   
 $\bigwedge p\ x\ n.\ n > 0 \implies poly\ p\ x \neq 0 \implies P\ p \implies P\ ([: -x,\ 1:] ^ n * p)$   
**shows**  $P\ p$   
*<proof>*

**lemma** *complex-poly-decompose*:

$smult\ (lead\text{-}coeff\ p)\ (\prod z.\ poly\ p\ z = 0.\ [: -z,\ 1:] ^ order\ z\ p) = (p :: complex\ poly)$   
*<proof>*

**lemma** *normalize-field*:

$normalize\ (x :: 'a :: \{normalization\text{-}semidom,\ field\}) = (if\ x = 0\ then\ 0\ else\ 1)$   
*<proof>*

**lemma** *unit-factor-field* [*simp*]:

$unit\text{-}factor\ (x :: 'a :: \{normalization\text{-}semidom,\ field\}) = x$   
*<proof>*

**lemma** *coprime-linear-poly*:

**fixes**  $c :: 'a :: field\text{-}gcd$   
**assumes**  $c \neq c'$   
**shows**  $coprime\ [:c,\ 1:]\ [:c',\ 1:]$   
*<proof>*

```

lemma coprime-linear-poly':
  fixes c :: 'a :: field-gcd
  assumes c ≠ c' c ≠ 0 c' ≠ 0
  shows coprime [1,c] [1,c']
  <proof>

end

```

## 4 Partial Fraction Decomposition

```

theory Partial-Fraction-Decomposition
imports
  Main
  HOL-Computational-Algebra.Computational-Algebra
  HOL-Computational-Algebra.Polynomial-Factorial
  HOL-Library.Sublist
  Linear-Recurrences-Misc
begin

```

### 4.1 Decomposition on general Euclidean rings

Consider elements  $x, y_1, \dots, y_n$  of a ring  $R$ , where the  $y_i$  are pairwise coprime. A *Partial Fraction Decomposition* of these elements (or rather the formal quotient  $x/(y_1 \dots y_n)$  that they represent) is a finite sum of summands of the form  $a/y_i^k$ . Obviously, the sum can be arranged such that there is at most one summand with denominator  $y_i^n$  for any combination of  $i$  and  $n$ ; in particular, there is at most one summand with denominator 1.

We can decompose the summands further by performing division with remainder until in all quotients, the numerator's Euclidean size is less than that of the denominator.

The following function performs the first step of the above process: it takes the values  $x$  and  $y_1, \dots, y_n$  and returns the numerators of the summands in the decomposition. (the denominators are simply the  $y_i$  from the input)

```

fun decompose :: ('a :: euclidean-ring-gcd) ⇒ 'a list ⇒ 'a list where
  decompose x [] = []
| decompose x [y] = [x]
| decompose x (y#ys) =
  (case bezout-coefficients y (prod-list ys) of
    (a, b) ⇒ (b*x) # decompose (a*x) ys)

```

```

lemma decompose-rec:
  ys ≠ [] ⇒ decompose x (y#ys) =
  (case bezout-coefficients y (prod-list ys) of
    (a, b) ⇒ (b*x) # decompose (a*x) ys)
  <proof>

```

**lemma** *length-decompose* [simp]:  $\text{length } (\text{decompose } x \text{ } ys) = \text{length } ys$   
 ⟨proof⟩

**fun** *decompose'* :: ('a :: euclidean-ring-gcd) ⇒ 'a list ⇒ 'a list ⇒ 'a list **where**  
*decompose'* x [] - = []  
 | *decompose'* x [y] - = [x]  
 | *decompose'* - - [] = []  
 | *decompose'* x (y#ys) (p#ps) =  
   (case bezout-coefficients y p of  
     (a, b) ⇒ (b\*x) # *decompose'* (a\*x) ys ps)

**primrec** *decompose-aux* :: 'a :: {ab-semigroup-mult, monoid-mult} ⇒ - **where**  
*decompose-aux* acc [] = [acc]  
 | *decompose-aux* acc (x#xs) = acc # *decompose-aux* (x \* acc) xs

**lemma** *decompose-code* [code]:  
*decompose* x ys = *decompose'* x ys (tl (rev (decompose-aux 1 (rev ys))))  
 ⟨proof⟩

The next function performs the second step: Given a quotient of the form  $x/y^n$ , it returns a list of  $x_0, \dots, x_n$  such that  $x/y^n = x_0/y^n + \dots + x_{n-1}/y + x_n$  and all  $x_i$  have a Euclidean size less than that of  $y$ .

**fun** *normalise-decomp* :: ('a :: semiring-modulo) ⇒ 'a ⇒ nat ⇒ 'a × ('a list) **where**  
*normalise-decomp* x y 0 = (x, [])  
 | *normalise-decomp* x y (Suc n) = (  
   case *normalise-decomp* (x div y) y n of  
     (z, rs) ⇒ (z, x mod y # rs))

**lemma** *length-normalise-decomp* [simp]:  $\text{length } (\text{snd } (\text{normalise-decomp } x \text{ } y \text{ } n)) = n$   
 ⟨proof⟩

The following constant implements the full process of partial fraction decomposition: The input is a quotient  $x/(y_1^{k_1} \dots y_n^{k_n})$  and the output is a sum of an entire element and terms of the form  $a/y_i^k$  where  $a$  has a Euclidean size less than  $y_i$ .

**definition** *partial-fraction-decomposition* ::  
 'a :: euclidean-ring-gcd ⇒ ('a × nat) list ⇒ 'a × 'a list list **where**  
*partial-fraction-decomposition* x ys = (if ys = [] then (x, []) else  
   (let zs = [let (y, n) = ys ! i  
     in *normalise-decomp* (decompose x (map (λ(y,n). y ^ Suc n) ys) ! i)  
   y (Suc n).  
     i ← [0..<length ys]]  
   in (sum-list (map fst zs), map snd zs)))

**lemma** *length-pfd1* [simp]:  
 $\text{length } (\text{snd } (\text{partial-fraction-decomposition } x \text{ } ys)) = \text{length } ys$

*<proof>*

**lemma** *length-pfd2* [simp]:

$i < \text{length } ys \implies \text{length } (\text{snd } (\text{partial-fraction-decomposition } x \text{ } ys) ! i) = \text{snd } (ys ! i) + 1$

*<proof>*

**lemma** *size-normalise-decomp*:

$a \in \text{set } (\text{snd } (\text{normalise-decomp } x \text{ } y \text{ } n)) \implies y \neq 0 \implies \text{euclidean-size } a < \text{euclidean-size } y$

*<proof>*

**lemma** *size-partial-fraction-decomposition*:

$i < \text{length } xs \implies \text{fst } (xs ! i) \neq 0 \implies x \in \text{set } (\text{snd } (\text{partial-fraction-decomposition } y \text{ } xs) ! i)$

$\implies \text{euclidean-size } x < \text{euclidean-size } (\text{fst } (xs ! i))$

*<proof>*

A homomorphism  $\varphi$  from a Euclidean ring  $R$  into another ring  $S$  with a notion of division. We will show that for any  $x, y \in R$  such that  $\phi(y)$  is a unit, we can perform partial fraction decomposition on the quotient  $\varphi(x)/\varphi(y)$ .

The obvious choice for  $S$  is the fraction field of  $R$ , but other choices may also make sense: If, for example,  $R$  is a ring of polynomials  $K[X]$ , then one could let  $S = K$  and  $\varphi$  the evaluation homomorphism. Or one could let  $S = K[[X]]$  (the ring of formal power series) and  $\varphi$  the canonical homomorphism from polynomials to formal power series.

**locale** *pfd-homomorphism* =

**fixes** *lift* :: ('a :: euclidean-ring-gcd)  $\Rightarrow$  ('b :: euclidean-semiring-cancel)

**assumes** *lift-add*:  $\text{lift } (a + b) = \text{lift } a + \text{lift } b$

**assumes** *lift-mult*:  $\text{lift } (a * b) = \text{lift } a * \text{lift } b$

**assumes** *lift-0* [simp]:  $\text{lift } 0 = 0$

**assumes** *lift-1* [simp]:  $\text{lift } 1 = 1$

**begin**

**lemma** *lift-power*:

$\text{lift } (a \wedge n) = \text{lift } a \wedge n$

*<proof>*

**definition** *from-decomp* :: 'a  $\Rightarrow$  'a  $\Rightarrow$  nat  $\Rightarrow$  'b **where**

*from-decomp*  $x \text{ } y \text{ } n = \text{lift } x \text{ div } \text{lift } y \wedge n$

**lemma** *decompose*:

**assumes**  $ys \neq []$  pairwise coprime (set ys) distinct ys

$\bigwedge y. y \in \text{set } ys \implies \text{is-unit } (\text{lift } y)$

**shows**  $(\sum i < \text{length } ys. \text{lift } (\text{decompose } x \text{ } ys ! i) \text{ div } \text{lift } (ys ! i)) = \text{lift } x \text{ div } \text{lift } (\text{prod-list } ys)$

*<proof>*

**lemma** *normalise-decomp*:  
**fixes**  $x\ y :: 'a$  **and**  $n :: nat$   
**assumes** *is-unit* (*lift*  $y$ )  
**defines**  $xs \equiv snd$  (*normalise-decomp*  $x\ y\ n$ )  
**shows**  $lift$  ( $fst$  (*normalise-decomp*  $x\ y\ n$ )) +  $(\sum_{i < n}. from-decomp\ (xs!i)\ y$   
 $(n-i)) =$   
 $lift\ x\ div\ lift\ y\ \hat{\ }n$   
 $\langle proof \rangle$

**lemma** *lift-prod-list*:  $lift$  (*prod-list*  $xs$ ) = *prod-list* (*map* *lift*  $xs$ )  
 $\langle proof \rangle$

**lemma** *lift-sum*:  $lift$  (*sum*  $f\ A$ ) = *sum* ( $\lambda x. lift$  ( $f\ x$ ))  $A$   
 $\langle proof \rangle$

**lemma** *partial-fraction-decomposition*:  
**fixes**  $ys :: ('a \times nat)$  *list*  
**defines**  $ys' \equiv map$  ( $\lambda(x,n). x\ \hat{\ }Suc\ n$ )  $ys :: 'a$  *list*  
**assumes** *unit*:  $\bigwedge y. y \in fst\ 'set\ ys \implies is-unit$  (*lift*  $y$ )  
**assumes** *coprime*: *pairwise coprime* (*set*  $ys'$ )  
**assumes** *distinct*: *distinct*  $ys'$   
**assumes** *partial-fraction-decomposition*  $x\ ys = (a, zs)$   
**shows**  $lift\ a + (\sum_{i < length\ ys}. \sum_{j \leq snd\ (ys!i)}. from-decomp\ (zs!i!j)\ (fst\ (ys!i))\ (snd\ (ys!i)+1 - j)) =$   
 $lift\ x\ div\ lift\ (prod-list\ ys')$   
 $\langle proof \rangle$

**end**

## 4.2 Specific results for polynomials

**definition** *divmod-field-poly*  $:: 'a :: field\ poly \Rightarrow 'a\ poly \Rightarrow 'a\ poly \times 'a\ poly$  **where**  
 $divmod-field-poly\ p\ q = (p\ div\ q, p\ mod\ q)$

**lemma** *divmod-field-poly-code* [*code*]:  
 $divmod-field-poly\ p\ q =$   
 $(let\ cg = coeffs\ q$   
 $in\ if\ cg = []\ then\ (0, p)$   
 $else\ let\ cf = coeffs\ p; ilc = inverse\ (last\ cg);$   
 $ch = map\ ((*)\ ilc)\ cg;$   
 $(q, r) =$   
 $divmod-poly-one-main-list\ []\ (rev\ cf)\ (rev\ ch)$   
 $(1 + length\ cf - length\ cg)$   
 $in\ (poly-of-list\ (map\ ((*)\ ilc)\ q), poly-of-list\ (rev\ r)))$   
 $\langle proof \rangle$

**definition** *normalise-decomp-poly*  $:: 'a :: field-gcd\ poly \Rightarrow 'a\ poly \Rightarrow nat \Rightarrow 'a\ poly$   
 $\times 'a\ poly\ list$

**where**  $[code-abbrev, simp]: normalise-decomp-poly (p :: - poly) q n = normalise-decomp p q n$

**lemma** *normalise-decomp-poly-code*  $[code]:$

$normalise-decomp-poly x y 0 = (x, [])$   
 $normalise-decomp-poly x y (Suc n) =$   
 let  $(x', r) = divmod-field-poly x y;$   
 $(z, rs) = normalise-decomp-poly x' y n$   
 in  $(z, r \# rs)$   
 $\langle proof \rangle$

**definition** *poly-pfd-simple where*

$poly-pfd-simple x cs = (if cs = [] then (x, []) else$   
 let  $zs = [let (c, n) = cs ! i$   
 in  $normalise-decomp-poly (decompose x$   
 $(map (\lambda(c,n). [:1, -c:] ^ Suc n) cs) ! i) [:1, -c:] (n+1).$   
 $i \leftarrow [0..<length cs]]$   
 in  $(sum-list (map fst zs), map (map (\lambda p. coeff p 0) \circ snd) zs)))$

**lemma** *poly-pfd-simple-code*  $[code]:$

$poly-pfd-simple x cs =$   
 (if  $cs = []$  then  $(x, [])$  else  
 let  $zs = zip-with (\lambda(c,n) decomp. normalise-decomp-poly decomp [:1, -c:]$   
 $(n+1))$   
 $cs (decompose x (map (\lambda(c,n). [:1, -c:] ^ Suc n) cs))$   
 in  $(sum-list (map fst zs), map (map (\lambda p. coeff p 0) \circ snd) zs)$   
 $\langle proof \rangle$

**lemma** *fst-poly-pfd-simple:*

$fst (poly-pfd-simple x cs) =$   
 $fst (partial-fraction-decomposition x (map (\lambda(c,n). ([:1, -c:], n)) cs))$   
 $\langle proof \rangle$

**lemma** *const-polyI:*  $degree p = 0 \implies [:coeff p 0:] = p$

$\langle proof \rangle$

**lemma** *snd-poly-pfd-simple:*

$map (map (\lambda c. [:c :: 'a :: field-gcd:])) (snd (poly-pfd-simple x cs)) =$   
 $(snd (partial-fraction-decomposition x (map (\lambda(c,n). ([:1, -c:], n)) cs)))$   
 $\langle proof \rangle$

**lemma** *poly-pfd-simple:*

$partial-fraction-decomposition x (map (\lambda(c,n). ([:1, -c:], n)) cs) =$   
 $(fst (poly-pfd-simple x cs), map (map (\lambda c. [:c:])) (snd (poly-pfd-simple x$   
 $cs)))$   
 $\langle proof \rangle$

**end**

## 5 Factorizations of polynomials

**theory** *Factorizations*

**imports**

*Complex-Main*

*Linear-Recurrences-Misc*

*HOL-Computational-Algebra.Computational-Algebra*

*HOL-Computational-Algebra.Polynomial-Factorial*

**begin**

We view a factorisation of a polynomial as a pair consisting of the leading coefficient and a list of roots with multiplicities. This gives us a factorization into factors of the form  $(X - c)^{n+1}$ .

**definition** *interp-factorization* **where**

*interp-factorization* =  $(\lambda(a, cs). \text{Polynomial.smult } a (\prod (c, n) \leftarrow cs. [:-c, 1:] \hat{\ } \text{Suc } n))$

An alternative way to factorise is as a pair of the leading coefficient and factors of the form  $(1 - cX)^{n+1}$ .

**definition** *interp-alt-factorization* **where**

*interp-alt-factorization* =  $(\lambda(a, cs). \text{Polynomial.smult } a (\prod (c, n) \leftarrow cs. [1, -c:] \hat{\ } \text{Suc } n))$

**definition** *is-factorization-of* **where**

*is-factorization-of* *fctrs* *p* =

$(\text{interp-factorization } fctrs = p \wedge \text{distinct } (\text{map } \text{fst } (\text{snd } fctrs)))$

**definition** *is-alt-factorization-of* **where**

*is-alt-factorization-of* *fctrs* *p* =

$(\text{interp-alt-factorization } fctrs = p \wedge 0 \notin \text{set } (\text{map } \text{fst } (\text{snd } fctrs)) \wedge \text{distinct } (\text{map } \text{fst } (\text{snd } fctrs)))$

Regular and alternative factorisations are related by reflecting the polynomial.

**lemma** *interp-factorization-reflect*:

**assumes**  $(0 :: 'a :: \text{idom}) \notin \text{fst } ' \text{set } (\text{snd } fctrs)$

**shows**  $\text{reflect-poly } (\text{interp-factorization } fctrs) = \text{interp-alt-factorization } fctrs$   
 $\langle \text{proof} \rangle$

**lemma** *interp-alt-factorization-reflect*:

**assumes**  $(0 :: 'a :: \text{idom}) \notin \text{fst } ' \text{set } (\text{snd } fctrs)$

**shows**  $\text{reflect-poly } (\text{interp-alt-factorization } fctrs) = \text{interp-factorization } fctrs$   
 $\langle \text{proof} \rangle$

**lemma** *coeff-0-interp-factorization*:

$\text{coeff } (\text{interp-factorization } fctrs) 0 = (0 :: 'a :: \text{idom}) \longleftrightarrow$

$\text{fst } fctrs = 0 \vee 0 \in \text{fst } ' \text{set } (\text{snd } fctrs)$

*<proof>*

**lemma** *reflect-factorization*:

**assumes**  $\text{coeff } p \ 0 \neq (0 :: 'a :: \text{idom})$

**assumes** *is-factorization-of fctrs*  $p$

**shows** *is-alt-factorization-of fctrs* (*reflect-poly*  $p$ )

*<proof>*

**lemma** *reflect-factorization'*:

**assumes**  $\text{coeff } p \ 0 \neq (0 :: 'a :: \text{idom})$

**assumes** *is-alt-factorization-of fctrs*  $p$

**shows** *is-factorization-of fctrs* (*reflect-poly*  $p$ )

*<proof>*

**lemma** *zero-in-factorization-iff*:

**assumes** *is-factorization-of fctrs*  $p$

**shows**  $\text{coeff } p \ 0 = 0 \iff p = 0 \vee (0 :: 'a :: \text{idom}) \in \text{fst } \text{' set } (\text{snd } \text{fctrs})$

*<proof>*

**lemma** *poly-prod-list [simp]*:  $\text{poly } (\text{prod-list } ps) \ x = \text{prod-list } (\text{map } (\lambda p. \text{poly } p \ x) \ ps)$

*<proof>*

**lemma** *is-factorization-of-roots*:

**fixes**  $a :: 'a :: \text{idom}$

**assumes** *is-factorization-of* ( $a, \text{fctrs}$ )  $p \ p \neq 0$

**shows**  $\text{set } (\text{map } \text{fst } \text{fctrs}) = \{x. \text{poly } p \ x = 0\}$

*<proof>*

**lemma** (**in** *monoid-mult*) *prod-list-prod-nth*:  $\text{prod-list } xs = (\prod_{i < \text{length } xs} xs \ ! \ i)$

*<proof>*

**lemma** *order-prod*:

**assumes**  $\bigwedge x. x \in A \implies f \ x \neq 0$

**assumes**  $\bigwedge x \ y. x \in A \implies y \in A \implies x \neq y \implies \text{coprime } (f \ x) \ (f \ y)$

**shows**  $\text{order } c \ (\text{prod } f \ A) = (\sum_{x \in A} \text{order } c \ (f \ x))$

*<proof>*

**lemma** *is-factorization-of-order*:

**fixes**  $p :: 'a :: \text{field-gcd poly}$

**assumes**  $p \neq 0$

**assumes** *is-factorization-of* ( $a, \text{fctrs}$ )  $p$

**assumes**  $(c, n) \in \text{set } \text{fctrs}$

**shows**  $\text{order } c \ p = \text{Suc } n$

*<proof>*

For complex polynomials, a factorisation in the above sense always exists.

**lemma** *complex-factorization-exists*:

$\exists \text{fctrs. } \text{is-factorization-of fctrs } (p :: \text{complex poly})$

*<proof>*

By reflecting the polynomial, this means that for complex polynomials with non-zero constant coefficient, the alternative factorisation also exists.

**corollary** *complex-alt-factorization-exists:*

**assumes** *coeff p 0 ≠ 0*

**shows**  $\exists$  *fctrs. is-alt-factorization-of fctrs (p :: complex poly)*

*<proof>*

**end**

## 6 Solver for rational formal power series

**theory** *Rational-FPS-Solver*

**imports**

*Complex-Main*

*Pochhammer-Polynomials*

*Partial-Fraction-Decomposition*

*Factorizations*

*HOL-Computational-Algebra.Field-as-Ring*

**begin**

We can determine the  $k$ -th coefficient of an FPS of the form  $d/(1 - cX)^n$ , which is an important step in solving linear recurrences. The  $k$ -th coefficient of such an FPS is always of the form  $p(k)c^k$  where  $p$  is the following polynomial:

**definition** *inverse-irred-power-poly :: 'a :: field-char-0 ⇒ nat ⇒ 'a poly where*

*inverse-irred-power-poly d n =*

*Poly [(d \* of-nat (stirling n (k+1))) / (fact (n - 1)). k ← [0..<n]]*

**lemma** *one-minus-const-fps-X-neg-power'':*

**fixes** *c :: 'a :: field-char-0*

**assumes** *n: n > 0*

**shows** *fps-const d / ((1 - fps-const (c :: 'a :: field-char-0) \* fps-X) ^ n) =*

*Abs-fps (λk. poly (inverse-irred-power-poly d n) (of-nat k) \* c ^ k) (is ?lhs*

*= ?rhs)*

*<proof>*

**include** *fps-syntax*

*<proof>*

**lemma** *inverse-irred-power-poly-code [code abstract]:*

*coeffs (inverse-irred-power-poly d n) =*

*(if n = 0 ∨ d = 0 then [] else*

*let e = d / (fact (n - 1))*

*in [e \* of-nat x. x ← tl (stirling-row n)])*

*<proof>*

**lemma** *solve-rat-fps-aux:*

**fixes**  $p :: 'a :: \{\text{field-char-0, field-gcd}\}$  **poly** **and**  $cs :: ('a \times \text{nat})$  **list**  
**assumes**  $\text{distinct: distinct (map fst cs)}$   
**assumes**  $\text{azs: (a, zs) = poly-pfd-simple p cs}$   
**assumes**  $\text{nz: } 0 \notin \text{fst ' set cs}$   
**shows**  $\text{fps-of-poly } p / \text{fps-of-poly } (\prod (c,n) \leftarrow cs. [:1, -c:] \wedge \text{Suc } n) =$   
 $\text{Abs-fps } (\lambda k. \text{coeff } a \ k + (\sum i < \text{length } cs. \text{poly } (\sum j \leq \text{snd } (cs ! i).$   
 $\text{inverse-irred-power-poly } (zs ! i ! j) (\text{snd } (cs ! i) + 1 - j)))$   
 $(\text{of-nat } k) * (\text{fst } (cs ! i)) \wedge k))$  (**is**  $- = ?\text{rhs}$ )  
 $\langle \text{proof} \rangle$

**definition**  $\text{solve-factored-ratfps} ::$   
 $('a :: \{\text{field-char-0, field-gcd}\})$  **poly**  $\Rightarrow ('a \times \text{nat})$  **list**  $\Rightarrow 'a$  **poly**  $\times ('a$  **poly**  $\times 'a)$   
**list** **where**  
 $\text{solve-factored-ratfps } p \ cs = (\text{let } n = \text{length } cs \text{ in case poly-pfd-simple } p \ cs \text{ of } (a,$   
 $zs) \Rightarrow$   
 $(a, \text{zip-with } (\lambda zs \ (c,n). ((\sum (z,j) \leftarrow \text{zip } zs \ [0..<\text{Suc } n].$   
 $\text{inverse-irred-power-poly } z \ (n + 1 - j)), c)) \ zs \ cs))$

**lemma**  $\text{length-snd-poly-pfd-simple [simp]: length (snd (poly-pfd-simple p cs)) =$   
 $\text{length } cs$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{length-nth-snd-poly-pfd-simple [simp]:}$   
 $i < \text{length } cs \implies \text{length (snd (poly-pfd-simple p cs) ! i) = snd (cs ! i) + 1}$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{solve-factored-ratfps-roots:}$   
 $\text{map snd (snd (solve-factored-ratfps p cs)) = map fst cs}$   
 $\langle \text{proof} \rangle$

**definition**  $\text{interp-ratfps-solution}$  **where**  
 $\text{interp-ratfps-solution} = (\lambda (p,cs) \ n. \text{coeff } p \ n + (\sum (q,c) \leftarrow cs. \text{poly } q \ (\text{of-nat } n) *$   
 $c \wedge n))$

**lemma**  $\text{solve-factored-ratfps:}$   
**fixes**  $p :: 'a :: \{\text{field-char-0, field-gcd}\}$  **poly** **and**  $cs :: ('a \times \text{nat})$  **list**  
**assumes**  $\text{distinct: distinct (map fst cs)}$   
**assumes**  $\text{nz: } 0 \notin \text{fst ' set cs}$   
**shows**  $\text{fps-of-poly } p / \text{fps-of-poly } (\prod (c,n) \leftarrow cs. [:1, -c:] \wedge \text{Suc } n) =$   
 $\text{Abs-fps (interp-ratfps-solution (solve-factored-ratfps p cs))}$  (**is**  $?lhs = ?rhs$ )  
 $\langle \text{proof} \rangle$

**definition**  $\text{solve-factored-ratfps'}$  **where**  
 $\text{solve-factored-ratfps'} = (\lambda p \ (a,cs). \text{solve-factored-ratfps (smult (inverse a) p) cs})$

**lemma** *solve-factored-ratfps'*:  
**assumes** *is-alt-factorization-of-fctrs*  $q \neq 0$   
**shows**  $Abs\text{-fps} (interp\text{-ratfps}\text{-solution} (solve\text{-factored}\text{-ratfps}' p \text{ fctrs})) =$   
 $fps\text{-of-poly } p / fps\text{-of-poly } q$   
 $\langle proof \rangle$

**lemma** *degree-Poly-eq*:  
**assumes**  $xs = [] \vee last\ xs \neq 0$   
**shows**  $degree (Poly\ xs) = length\ xs - 1$   
 $\langle proof \rangle$

**lemma** *degree-Poly'*:  $degree (Poly\ xs) \leq length\ xs - 1$   
 $\langle proof \rangle$

**lemma** *degree-inverse-irred-power-poly-le*:  
 $degree (inverse\text{-irred}\text{-power}\text{-poly } c\ n) \leq n - 1$   
 $\langle proof \rangle$

**lemma** *degree-inverse-irred-power-poly*:  
**assumes**  $c \neq 0$   
**shows**  $degree (inverse\text{-irred}\text{-power}\text{-poly } c\ n) = n - 1$   
 $\langle proof \rangle$

**lemma** *reflect-poly-0-iff [simp]*:  $reflect\text{-poly } p = 0 \iff p = 0$   
 $\langle proof \rangle$

**lemma** *degree-sum-list-le*:  $(\bigwedge p. p \in set\ ps \implies degree\ p \leq T) \implies degree (sum\text{-list}\ ps) \leq T$   
 $\langle proof \rangle$

**theorem** *ratfps-closed-form-exists*:  
**fixes**  $q :: complex\ poly$   
**assumes**  $nz: coeff\ q\ 0 \neq 0$   
**defines**  $q' \equiv reflect\text{-poly } q$   
**obtains**  $r\ rs$   
**where**  $\bigwedge n. fps\text{-nth} (fps\text{-of-poly } p / fps\text{-of-poly } q)\ n =$   
 $coeff\ r\ n + (\sum c \mid poly\ q'\ c = 0. poly (rs\ c) (of\text{-nat } n) * c \wedge n)$   
**and**  $\bigwedge z. poly\ q'\ z = 0 \implies degree (rs\ z) \leq order\ z\ q' - 1$   
 $\langle proof \rangle$

**end**

## 7 Material common to homogenous and inhomogenous linear recurrences

**theory** *Linear-Recurrences-Common*  
**imports**

*Complex-Main*  
*HOL-Computational-Algebra.Computational-Algebra*  
**begin**

**definition** *lr-fps-denominator* **where**  
*lr-fps-denominator cs = Poly (rev cs)*

**lemma** *lr-fps-denominator-code* [*code abstract*]:  
*coeffs (lr-fps-denominator cs) = rev (dropWhile ((=) 0) cs)*  
*<proof>*

**definition** *lr-fps-denominator'* **where**  
*lr-fps-denominator' cs = Poly cs*

**lemma** *lr-fps-denominator'-code* [*code abstract*]:  
*coeffs (lr-fps-denominator' cs) = strip-while ((=) 0) cs*  
*<proof>*

**lemma** *lr-fps-denominator-nz*: *last cs ≠ 0 ⇒ cs ≠ [] ⇒ lr-fps-denominator cs ≠ 0*  
*<proof>*

**lemma** *lr-fps-denominator'-nz*: *last cs ≠ 0 ⇒ cs ≠ [] ⇒ lr-fps-denominator' cs ≠ 0*  
*<proof>*

**end**

## 8 Homogenous linear recurrences

**theory** *Linear-Homogenous-Recurrences*

**imports**

*Complex-Main*  
*RatFPS*  
*Rational-FPS-Solver*  
*Linear-Recurrences-Common*

**begin**

The following is the numerator of the rational generating function of a linear homogenous recurrence.

**definition** *lhr-fps-numerator* **where**

*lhr-fps-numerator m cs f = (let N = length cs - 1 in*  
*Poly [(∑ i ≤ min N k. cs ! (N - i) \* f (k - i)). k ← [0..<N+m]])*

**lemma** *lhr-fps-numerator-code* [*code abstract*]:

*coeffs (lhr-fps-numerator m cs f) = (let N = length cs - 1 in*  
*strip-while ((=) 0) [(∑ i ≤ min N k. cs ! (N - i) \* f (k - i)). k ← [0..<N+m]])*  
*<proof>*

**lemma** *lhr-fps-aux*:  
**fixes**  $f :: \text{nat} \Rightarrow 'a :: \text{field}$   
**assumes**  $\bigwedge n. n \geq m \implies (\sum k \leq N. c \ k * f (n + k)) = 0$   
**assumes**  $cN: c \ N \neq 0$   
**defines**  $p \equiv \text{Poly} [c (N - k). k \leftarrow [0..< \text{Suc } N]]$   
**defines**  $q \equiv \text{Poly} [(\sum i \leq \min N \ k. c (N - i) * f (k - i)). k \leftarrow [0..< N+m]]$   
**shows**  $\text{Abs-fps } f = \text{fps-of-poly } q / \text{fps-of-poly } p$   
 $\langle \text{proof} \rangle$   
**include** *fps-syntax*  
 $\langle \text{proof} \rangle$

**lemma** *lhr-fps*:  
**fixes**  $f :: \text{nat} \Rightarrow 'a :: \text{field}$  **and**  $cs :: 'a \ \text{list}$   
**defines**  $N \equiv \text{length } cs - 1$   
**assumes**  $cs: cs \neq []$   
**assumes**  $\bigwedge n. n \geq m \implies (\sum k \leq N. cs ! k * f (n + k)) = 0$   
**assumes**  $cN: \text{last } cs \neq 0$   
**shows**  $\text{Abs-fps } f = \text{fps-of-poly } (\text{lhr-fps-numerator } m \ cs \ f) /$   
 $\text{fps-of-poly } (\text{lr-fps-denominator } cs)$   
 $\langle \text{proof} \rangle$

**fun** *lhr where*  
 $\text{lhr } cs \ fs \ n =$   
 $(\text{if } (cs :: 'a :: \text{field list}) = [] \vee \text{last } cs = 0 \vee \text{length } fs < \text{length } cs - 1 \text{ then}$   
 $\text{undefined else}$   
 $(\text{if } n < \text{length } fs \text{ then } fs ! n \text{ else}$   
 $(\sum k < \text{length } cs - 1. cs ! k * \text{lhr } cs \ fs (n + 1 - \text{length } cs + k)) / -\text{last}$   
 $cs))$

**declare** *lhr.simps* [*simp del*]

**lemma** *lhr-rec*:  
**assumes**  $cs \neq []$   $\text{last } cs \neq 0$   $\text{length } fs \geq \text{length } cs - 1$   $n \geq \text{length } fs$   
**shows**  $(\sum k < \text{length } cs. cs ! k * \text{lhr } cs \ fs (n + 1 - \text{length } cs + k)) = 0$   
 $\langle \text{proof} \rangle$

**lemma** *lhrI*:  
**assumes**  $cs \neq []$   $\text{last } cs \neq 0$   $\text{length } fs \geq \text{length } cs - 1$   
**assumes**  $\bigwedge n. n < \text{length } fs \implies f \ n = fs ! n$   
**assumes**  $\bigwedge n. n \geq \text{length } fs \implies (\sum k < \text{length } cs. cs ! k * f (n + 1 - \text{length } cs$   
 $+ k)) = 0$   
**shows**  $f \ n = \text{lhr } cs \ fs \ n$   
 $\langle \text{proof} \rangle$

**locale** *linear-homogenous-recurrence* =  
**fixes**  $f :: \text{nat} \Rightarrow 'a :: \text{comm-semiring-0}$  **and**  $cs \ fs :: 'a \ \text{list}$   
**assumes**  $\text{base}: n < \text{length } fs \implies f \ n = fs ! n$

```

assumes cs-not-null [simp]: cs ≠ [] and last-cs [simp]: last cs ≠ 0
and hd-cs [simp]: hd cs ≠ 0 and enough-base: length fs + 1 ≥ length cs
assumes rec:  $n \geq \text{length } fs - \text{length } cs \implies (\sum_{k < \text{length } cs} cs ! k * f (n + k))$ 
= 0
begin

```

```

lemma lhr-fps-numerator-altdef:
  lhr-fps-numerator (length fs + 1 - length cs) cs f =
  lhr-fps-numerator (length fs + 1 - length cs) cs (! fs)
⟨proof⟩

```

**end**

```

lemma solve-lhr-aux:
assumes linear-homogenous-recurrence f cs fs
assumes is-factorization-of-fctrs (lr-fps-denominator' cs)
shows  $f = \text{interp-ratfps-solution } (\text{solve-factored-ratfps}' (\text{lhr-fps-numerator}$ 
  (length fs + 1 - length cs) cs (! fs)) fctrs)
⟨proof⟩

```

**definition**

```

lhr-fps as fs = (
  let m = length fs + 1 - length as;
  p = lhr-fps-numerator m as ( $\lambda n. fs ! n$ );
  q = lr-fps-denominator as
  in ratfps-of-poly p / ratfps-of-poly q)

```

```

lemma lhr-fps-correct:
fixes  $f :: \text{nat} \Rightarrow 'a :: \{\text{field-char-0}, \text{field-gcd}\}$ 
assumes linear-homogenous-recurrence f cs fs
shows  $\text{fps-of-ratfps } (\text{lhr-fps } cs fs) = \text{Abs-fps } f$ 
⟨proof⟩

```

**end**

## 9 Eulerian polynomials

**theory** *Eulerian-Polynomials*

**imports**

*Complex-Main*

*HOL-Combinatorics.Stirling*

*HOL-Computational-Algebra.Computational-Algebra*

**begin**

The Eulerian polynomials are a sequence of polynomials that is related to

the closed forms of the power series

$$\sum_{n=0}^{\infty} n^k X^n$$

for a fixed  $k$ .

**primrec** *eulerian-poly* :: *nat*  $\Rightarrow$  '*a* :: *idom poly* **where**

*eulerian-poly* 0 = 1

| *eulerian-poly* (*Suc* *n*) = (let *p* = *eulerian-poly* *n* in  
[:0,1,-1:] \* *pderiv* *p* + *p* \* [:1, of-nat *n*:])

**lemmas** *eulerian-poly-Suc* [*simp del*] = *eulerian-poly.simps*(2)

**lemma** *poly-eulerian-poly-0* [*simp*]: *poly* (*eulerian-poly* *n*) 0 = 1  
<*proof*>

**lemma** *eulerian-poly-at-1* [*simp*]: *poly* (*eulerian-poly* *n*) 1 = *fact* *n*  
<*proof*>

**lemma** *eulerian-poly*:

*fps-of-poly* (*eulerian-poly* *k* :: '*a* :: *field poly*) =

*Abs-fps* ( $\lambda n. \text{of-nat } (n+1) \wedge k$ ) \* (1 - *fps-X*)  $\wedge$  (*k* + 1)

<*proof*>

**lemma** *eulerian-poly'*:

*Abs-fps* ( $\lambda n. \text{of-nat } (n+1) \wedge k$ ) =

*fps-of-poly* (*eulerian-poly* *k* :: '*a* :: *field poly*) / (1 - *fps-X*)  $\wedge$  (*k* + 1)

<*proof*>

**lemma** *eulerian-poly''*:

**assumes** *k*: *k* > 0

**shows** *Abs-fps* ( $\lambda n. \text{of-nat } n \wedge k$ ) =

*fps-of-poly* (*pCons* 0 (*eulerian-poly* *k* :: '*a* :: *field poly*)) / (1 - *fps-X*)  $\wedge$

(*k* + 1)

<*proof*>

**definition** *fps-monom-poly* :: '*a* :: *field*  $\Rightarrow$  *nat*  $\Rightarrow$  '*a* *poly*

**where** *fps-monom-poly* *c* *k* = (if *k* = 0 then 1 else *pcompose* (*pCons* 0 (*eulerian-poly* *k*)) [:0,*c*:])

**primrec** *fps-monom-poly-aux* :: '*a* :: *field*  $\Rightarrow$  *nat*  $\Rightarrow$  '*a* *poly* **where**

*fps-monom-poly-aux* *c* 0 = [:*c*:]

| *fps-monom-poly-aux* *c* (*Suc* *k*) =

(let *p* = *fps-monom-poly-aux* *c* *k*

in [:0,1,-*c*:] \* *pderiv* *p* + [:1, of-nat *k* \* *c*:] \* *p*)

**lemma** *fps-monom-poly-aux*:

*fps-monom-poly-aux* *c* *k* = *smult* *c* (*pcompose* (*eulerian-poly* *k*) [:0,*c*:])

<*proof*>

**lemma** *fps-monom-poly-code* [code]:

*fps-monom-poly*  $c$   $k = (\text{if } k = 0 \text{ then } 1 \text{ else } p\text{Cons } 0 \text{ (fps-monom-poly-aux } c \text{ } k))$   
 ⟨proof⟩

**lemma** *fps-monom-aux*:

*Abs-fps*  $(\lambda n. \text{ of-nat } n \text{ } ^k) = \text{fps-of-poly (fps-monom-poly } 1 \text{ } k) / (1 - \text{fps-X}) \text{ } ^{k+1}$   
 ⟨proof⟩

**lemma** *fps-monom*:

*Abs-fps*  $(\lambda n. \text{ of-nat } n \text{ } ^k * c \text{ } ^n) =$   
 $\text{fps-of-poly (fps-monom-poly } c \text{ } k) / (1 - \text{fps-const } c * \text{fps-X}) \text{ } ^{k+1}$   
 ⟨proof⟩

**end**

## 10 Inhomogenous linear recurrences

**theory** *Linear-Inhomogenous-Recurrences*

**imports**

*Complex-Main*

*Linear-Homogenous-Recurrences*

*Eulerian-Polynomials*

*RatFPS*

**begin**

**definition** *lir-fps-numerator* **where**

*lir-fps-numerator*  $m$   $cs$   $f$   $g = (\text{let } N = \text{length } cs - 1 \text{ in}$   
 $\text{Poly } [(\sum_{i \leq \text{min } N} k. cs ! (N - i) * f (k - i)) - g \text{ } k. k \leftarrow [0..<N+m]])$

**lemma** *lir-fps-numerator-code* [code abstract]:

*coeffs*  $(\text{lir-fps-numerator } m \text{ } cs \text{ } f \text{ } g) = (\text{let } N = \text{length } cs - 1 \text{ in}$   
 $\text{strip-while } ((=) 0) [(\sum_{i \leq \text{min } N} k. cs ! (N - i) * f (k - i)) - g \text{ } k. k \leftarrow$   
 $[0..<N+m]])$   
 ⟨proof⟩

**locale** *linear-inhomogenous-recurrence* =

**fixes**  $f$   $g :: \text{nat} \Rightarrow 'a :: \text{comm-ring}$  **and**  $cs$   $fs :: 'a \text{ list}$

**assumes** *base*:  $n < \text{length } fs \implies f \text{ } n = fs ! n$

**assumes** *cs-not-null* [simp]:  $cs \neq []$  **and** *last-cs* [simp]:  $\text{last } cs \neq 0$

**and** *hd-cs* [simp]:  $\text{hd } cs \neq 0$  **and** *enough-base*:  $\text{length } fs + 1 \geq \text{length } cs$

**assumes** *rec*:  $n \geq \text{length } fs + 1 - \text{length } cs \implies$

$(\sum_{k < \text{length } cs} k * f (n + k)) = g (n + \text{length } cs - 1)$

**begin**

**lemma** *coeff-0-lr-fps-denominator* [simp]: *coeff*  $(\text{lr-fps-denominator } cs) 0 = \text{last } cs$

*<proof>*

**lemma** *lir-fps-numerator-altdef*:

*lir-fps-numerator* (length fs + 1 - length cs) cs f g =  
*lir-fps-numerator* (length fs + 1 - length cs) cs (! fs) g

*<proof>*

**end**

**context**

**begin**

**private lemma** *lir-fps-aux*:

**fixes** f :: nat ⇒ 'a :: field

**assumes** rec:  $\bigwedge n. n \geq m \implies (\sum_{k \leq N}. c \ k * f (n + k)) = g (n + N)$

**assumes** cN:  $c \ N \neq 0$

**defines** p ≡ Poly [c (N - k). k ← [0..<Suc N]]

**defines** q ≡ Poly [( $\sum_{i \leq \min N \ k}. c (N - i) * f (k - i)$ ) - g k. k ← [0..<N+m]]

**shows** Abs-fps f = (fps-of-poly q + Abs-fps g) / fps-of-poly p

*<proof>*

**include** fps-syntax

*<proof>*

**lemma** *lir-fps*:

**fixes** f g :: nat ⇒ 'a :: field **and** cs :: 'a list

**defines** N ≡ length cs - 1

**assumes** cs: cs ≠ []

**assumes**  $\bigwedge n. n \geq m \implies (\sum_{k \leq N}. cs \ ! \ k * f (n + k)) = g (n + N)$

**assumes** cN: last cs ≠ 0

**shows** Abs-fps f = (fps-of-poly (lir-fps-numerator m cs f g) + Abs-fps g) /  
fps-of-poly (lir-fps-denominator cs)

*<proof>*

**end**

**type-synonym** 'a polyexp = ('a × nat × 'a) list

**definition** eval-polyexp :: ('a::semiring-1) polyexp ⇒ nat ⇒ 'a **where**

*eval-polyexp* xs = ( $\lambda n. \sum (a,k,b) \leftarrow xs. a * \text{of-nat } n \wedge k * b \wedge n$ )

**lemma** *eval-polyexp-Nil [simp]*: *eval-polyexp* [] = ( $\lambda -. 0$ )

*<proof>*

**lemma** *eval-polyexp-Cons*:

*eval-polyexp* (x#xs) = ( $\lambda n. (\text{case } x \text{ of } (a,k,b) \Rightarrow a * \text{of-nat } n \wedge k * b \wedge n) +$   
*eval-polyexp* xs n)

*<proof>*

**definition**  $\text{polyexp-fps} :: ('a :: \text{field}) \text{polyexp} \Rightarrow 'a \text{fps}$  **where**

$$\text{polyexp-fps } xs = \left( \sum (a,k,b) \leftarrow xs. \text{fps-of-poly } (\text{Polynomial.smult } a \text{ (fps-monom-poly } b \text{ } k)) / (1 - \text{fps-const } b * \text{fps-X}) ^ (k + 1) \right)$$

**lemma**  $\text{polyexp-fps-Nil}$  [simp]:  $\text{polyexp-fps } [] = 0$   
 ⟨proof⟩

**lemma**  $\text{polyexp-fps-Cons}$ :

$$\text{polyexp-fps } (x\#xs) = (\text{case } x \text{ of } (a,k,b) \Rightarrow \text{fps-of-poly } (\text{Polynomial.smult } a \text{ (fps-monom-poly } b \text{ } k)) / (1 - \text{fps-const } b * \text{fps-X}) ^ (k + 1)) + \text{polyexp-fps } xs$$

⟨proof⟩

**definition**  $\text{polyexp-ratfps} :: ('a :: \text{field-gcd}) \text{polyexp} \Rightarrow 'a \text{ratfps}$  **where**

$$\text{polyexp-ratfps } xs = \left( \sum (a,k,b) \leftarrow xs. \text{ratfps-of-poly } (\text{Polynomial.smult } a \text{ (fps-monom-poly } b \text{ } k)) / \text{ratfps-of-poly } ([:1, -b:] ^ (k + 1)) \right)$$

**lemma**  $\text{polyexp-ratfps-Nil}$  [simp]:  $\text{polyexp-ratfps } [] = 0$   
 ⟨proof⟩

**lemma**  $\text{polyexp-ratfps-Cons}$ :  $\text{polyexp-ratfps } (x\#xs) = (\text{case } x \text{ of } (a,k,b) \Rightarrow \text{ratfps-of-poly } (\text{Polynomial.smult } a \text{ (fps-monom-poly } b \text{ } k)) / \text{ratfps-of-poly } ([:1, -b:] ^ (k + 1))) + \text{polyexp-ratfps } xs$   
 ⟨proof⟩

**lemma**  $\text{polyexp-fps}$ :  $\text{Abs-fps } (\text{eval-polyexp } xs) = \text{polyexp-fps } xs$   
 ⟨proof⟩

**lemma**  $\text{polyexp-ratfps}$  [simp]:  $\text{fps-of-ratfps } (\text{polyexp-ratfps } xs) = \text{polyexp-fps } xs$   
 ⟨proof⟩

**definition**  $\text{lir-fps} ::$

$'a :: \text{field-gcd}$   $\text{list} \Rightarrow 'a \text{list} \Rightarrow 'a \text{polyexp} \Rightarrow ('a \text{ratfps}) \text{option}$  **where**  
 $\text{lir-fps } cs \text{ fs } g = (\text{if } cs = [] \vee \text{length } fs < \text{length } cs - 1 \text{ then } \text{None} \text{ else}$   
 let  $m = \text{length } fs + 1 - \text{length } cs;$   
 $p = \text{lir-fps-numerator } m \text{ cs } (\lambda n. \text{fs } ! \text{ } n) (\text{eval-polyexp } g);$   
 $q = \text{lr-fps-denominator } cs$   
 in  $\text{Some } ((\text{ratfps-of-poly } p + \text{polyexp-ratfps } g) * \text{inverse } (\text{ratfps-of-poly } q))$ )

**lemma**  $\text{lir-fps-correct}$ :

**fixes**  $f :: \text{nat} \Rightarrow 'a :: \text{field-gcd}$

**assumes**  $\text{linear-inhomogenous-recurrence } f (\text{eval-polyexp } g) \text{ cs } fs$

**shows**  $\text{map-option } \text{fps-of-ratfps } (\text{lir-fps } cs \text{ fs } g) = \text{Some } (\text{Abs-fps } f)$

*<proof>*

**end**

**theory** *Rational-FPS-Asymptotics*

**imports**

*HOL-Library.Landau-Symbols*  
*Polynomial-Factorization.Square-Free-Factorization*  
*HOL-Real-Asymp.Real-Asymp*  
*Count-Complex-Roots.Count-Complex-Roots*  
*Linear-Homogenous-Recurrences*  
*Linear-Inhomogenous-Recurrences*  
*RatFPS*  
*Rational-FPS-Solver*  
*HOL-Library.Code-Target-Numeral*

**begin**

**lemma** *poly-asymp-equiv:*

**assumes**  $p \neq 0$  **and**  $F \leq at\text{-infinity}$

**shows**  $poly\ p \sim [F] (\lambda x. lead\text{-coeff}\ p * x^{\wedge} degree\ p)$

*<proof>*

**lemma** *poly-bigtheta:*

**assumes**  $p \neq 0$  **and**  $F \leq at\text{-infinity}$

**shows**  $poly\ p \in \Theta[F](\lambda x. x^{\wedge} degree\ p)$

*<proof>*

**lemma** *poly-bigo:*

**assumes**  $F \leq at\text{-infinity}$  **and**  $degree\ p \leq k$

**shows**  $poly\ p \in O[F](\lambda x. x^{\wedge} k)$

*<proof>*

**lemma** *reflect-poly-dvdI:*

**fixes**  $p\ q :: 'a :: \{comm\text{-semiring-1}, semiring\text{-no-zero-divisors}\}$  *poly*

**assumes**  $p\ dvd\ q$

**shows**  $reflect\text{-poly}\ p\ dvd\ reflect\text{-poly}\ q$

*<proof>*

**lemma** *smult-altdef:*  $smult\ c\ p = [:c:] * p$

*<proof>*

**lemma** *smult-power:*  $smult\ (c^{\wedge} n)\ (p^{\wedge} n) = (smult\ c\ p)^{\wedge} n$

*<proof>*

**lemma** *order-reflect-poly-ge:*

**fixes**  $c :: 'a :: field$

**assumes**  $c \neq 0$  **and**  $p \neq 0$

**shows**  $order\ c\ (reflect\text{-poly}\ p) \geq order\ (1 / c)\ p$

*<proof>*

**lemma** *order-reflect-poly*:

**fixes**  $c :: 'a :: field$

**assumes**  $c \neq 0$  **and**  $coeff\ p\ 0 \neq 0$

**shows**  $order\ c\ (reflect\text{-}poly\ p) = order\ (1 / c)\ p$

*<proof>*

**lemma** *poly-reflect-eq-0-iff*:

$poly\ (reflect\text{-}poly\ p)\ (x :: 'a :: field) = 0 \longleftrightarrow p = 0 \vee x \neq 0 \wedge poly\ p\ (1 / x) = 0$

*<proof>*

**theorem** *ratfps-nth-bigo*:

**fixes**  $q :: complex\ poly$

**assumes**  $R > 0$

**assumes** *roots1*:  $\bigwedge z. z \in ball\ 0\ (1 / R) \implies poly\ q\ z \neq 0$

**assumes** *roots2*:  $\bigwedge z. z \in sphere\ 0\ (1 / R) \implies poly\ q\ z = 0 \implies order\ z\ q \leq Suc\ k$

**shows**  $fps\text{-}nth\ (fps\text{-}of\text{-}poly\ p / fps\text{-}of\text{-}poly\ q) \in O(\lambda n. of\text{-}nat\ n^k * of\text{-}real\ R^{\wedge} n)$

*<proof>*

**lemma** *order-power*:  $p \neq 0 \implies order\ c\ (p^{\wedge} n) = n * order\ c\ p$

*<proof>*

**lemma** *same-root-imp-not-coprime*:

**assumes**  $poly\ p\ x = 0$  **and**  $poly\ q\ (x :: 'a :: \{factorial\text{-}ring\text{-}gcd, semiring\text{-}gcd\text{-}mult\text{-}normalize\}) = 0$

**shows**  $\neg coprime\ p\ q$

*<proof>*

**lemma** *ratfps-nth-bigo-square-free-factorization*:

**fixes**  $p :: complex\ poly$

**assumes** *square-free-factorization*  $q\ (b, cs)$

**assumes**  $q \neq 0$  **and**  $R > 0$

**assumes** *roots1*:  $\bigwedge c\ l. (c, l) \in set\ cs \implies \forall x \in ball\ 0\ (1 / R). poly\ c\ x \neq 0$

**assumes** *roots2*:  $\bigwedge c\ l. (c, l) \in set\ cs \implies l > Suc\ k \implies \forall x \in sphere\ 0\ (1 / R). poly\ c\ x \neq 0$

**shows**  $fps\text{-}nth\ (fps\text{-}of\text{-}poly\ p / fps\text{-}of\text{-}poly\ q) \in O(\lambda n. of\text{-}nat\ n^k * of\text{-}real\ R^{\wedge} n)$

*<proof>*

**lemma** *proots-within-card-zero-iff*:

**assumes**  $p \neq (0 :: 'a :: idom\ poly)$

**shows**  $card\ (proots\text{-}within\ p\ A) = 0 \longleftrightarrow (\forall x \in A. poly\ p\ x \neq 0)$

*<proof>*

**lemma** *ratfps-nth-bigo-square-free-factorization'*:

**fixes**  $p :: \text{complex poly}$

**assumes** *square-free-factorization*  $q (b, cs)$

**assumes**  $q \neq 0$  **and**  $R > 0$

**assumes** *roots1*: *list-all*  $(\lambda cl. \text{proots-ball-card } (fst\ cl)\ 0\ (1 / R) = 0)\ cs$

**assumes** *roots2*: *list-all*  $(\lambda cl. \text{proots-sphere-card } (fst\ cl)\ 0\ (1 / R) = 0)$

*(filter*  $(\lambda cl. \text{snd } cl > Suc\ k)\ cs)$

**shows**  $\text{fps-nth } (\text{fps-of-poly } p / \text{fps-of-poly } q) \in O(\lambda n. \text{of-nat } n^k * \text{of-real } R^n)$

*<proof>*

**definition** *ratfps-has-asymptotics where*

*ratfps-has-asymptotics*  $q\ k\ R \longleftrightarrow q \neq 0 \wedge R > 0 \wedge$

*(let*  $cs = \text{snd } (\text{yun-factorization } gcd\ q)$

*in* *list-all*  $(\lambda cl. \text{proots-ball-card } (fst\ cl)\ 0\ (1 / R) = 0)\ cs \wedge$

*list-all*  $(\lambda cl. \text{proots-sphere-card } (fst\ cl)\ 0\ (1 / R) = 0)\ (\text{filter } (\lambda cl. \text{snd } cl > Suc\ k)\ cs))$

**lemma** *ratfps-has-asymptotics-correct*:

**assumes** *ratfps-has-asymptotics*  $q\ k\ R$

**shows**  $\text{fps-nth } (\text{fps-of-poly } p / \text{fps-of-poly } q) \in O(\lambda n. \text{of-nat } n^k * \text{of-real } R^n)$

*<proof>*

**value** *map*  $(\text{fps-nth } (\text{fps-of-poly } [:0, 1:] / \text{fps-of-poly } [:1, -1, -1 :: \text{real:}]))\ [0..<5]$

**method** *ratfps-bigo* = *(rule ratfps-has-asymptotics-correct; eval)*

**lemma**  $\text{fps-nth } (\text{fps-of-poly } [:0, 1:] / \text{fps-of-poly } [:1, -1, -1 :: \text{complex:}]) \in$

$O(\lambda n. \text{of-nat } n^0 * \text{complex-of-real } 1.618034^n)$

*<proof>*

**lemma**  $\text{fps-nth } (\text{fps-of-poly } 1 / \text{fps-of-poly } [:1, -3, 3, -1 :: \text{complex:}]) \in$

$O(\lambda n. \text{of-nat } n^2 * \text{complex-of-real } 1^n)$

*<proof>*

**lemma**  $\text{fps-nth } (\text{fps-of-poly } f / \text{fps-of-poly } [:5, 4, 3, 2, 1 :: \text{complex:}]) \in$

$O(\lambda n. \text{of-nat } n^0 * \text{complex-of-real } 0.69202^n)$

*<proof>*

**end**