

# The Lévy-Prokhorov Metric

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## Abstract

We formalize the Lévy-Prokhorov metric, a metric on finite measures, mainly following the lecture notes by Gaans [4]. This entry includes the following formalization.

- Characterizations of closed sets, open sets, and topology by limit.
- A special case of Alaoglu's theorem.
- Weak convergence and the Portmanteau theorem.
- The Lévy-Prokhorov metric and its completeness and separability.
- The equivalence of the topology of weak convergence and the topology generated by the Lévy-Prokhorov metric.
- Prokhorov's theorem.
- Equality of two  $\sigma$ -algebras on the space of finite measures. One is the Borel algebra of the Lévy-Prokhorov metric and the other is the least  $\sigma$ -algebra that makes  $(\lambda\mu, \mu(A))$  measurable for all measurable sets  $A$ .
- The space of finite measures on a standard Borel space is also a standard Borel space.

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# 1 Preliminaries

```
theory Lemmas-Levy-Prokhorov
  imports Standard-Borel-Spaces.StandardBorel
begin
```

```
lemma(in Metric-space) [measurable]:
  shows mball-sets: mball x e ∈ sets (borel-of mtopology)
  and mcball-sets: mcball x e ∈ sets (borel-of mtopology)
  ⟨proof⟩
```

```
lemma Metric-space-eq-MCauchy:
  assumes Metric-space M d ∧ x y. x ∈ M ⇒ y ∈ M ⇒ d x y = d' x y
  and ∧ x y. d' x y = d' y x ∧ x y. d' x y ≥ 0
  shows Metric-space.MCauchy M d xn ↔ Metric-space.MCauchy M d' xn
  ⟨proof⟩
```

**lemma** *borel-of-compact*: Hausdorff-space  $X \implies$  compactin  $X$   $K \implies K \in$  sets  
(borel-of  $X$ )  
*<proof>*

**lemma** *prob-algebra-cong*: sets  $M =$  sets  $N \implies$  prob-algebra  $M =$  prob-algebra  $N$   
*<proof>*

**lemma** *topology-eq-closedin*:  $X = Y \iff (\forall C. \text{closedin } X C \iff \text{closedin } Y C)$   
*<proof>*

Another version of *finite-measure*  $?M \implies$  countable  $\{x. \text{Sigma-Algebra.measure } ?M \{x\} \neq 0\}$

**lemma**(in *finite-measure*) *countable-support-sets*:  
assumes *disjoint-family-on*  $A_i D$   
shows countable  $\{i \in D. \text{measure } M (A_i i) \neq 0\}$   
*<proof>*

## 1.1 Finite Sum of Measures

**definition** *sum-measure* :: 'b measure  $\Rightarrow$  'a set  $\Rightarrow$  ('a  $\Rightarrow$  'b measure)  $\Rightarrow$  'b measure  
**where**  
*sum-measure*  $M I M_i \equiv$  measure-of (space  $M$ ) (sets  $M$ )  $(\lambda A. \sum_{i \in I. \text{emeasure } (M_i i) A}$ )

**lemma** *sum-measure-cong*:  
assumes sets  $M =$  sets  $M' \wedge i. i \in I \implies N i = N' i$   
shows *sum-measure*  $M I N =$  *sum-measure*  $M' I N'$   
*<proof>*

**lemma** [*simp*]:  
shows *space-sum-measure*: space (*sum-measure*  $M I M_i$ ) = space  $M$   
and *sets-sum-measure*[*measurable-cong*]: sets (*sum-measure*  $M I M_i$ ) = sets  $M$   
*<proof>*

**lemma** *emeasure-sum-measure*:  
assumes [*measurable*]:  $A \in$  sets  $M$  and  $\wedge i. i \in I \implies$  sets  $(M_i i) =$  sets  $M$   
shows *emeasure* (*sum-measure*  $M I M_i$ )  $A = (\sum_{i \in I. M_i i A}$ )

*<proof>*

**lemma** *sum-measure-infinite*: infinite  $I \implies$  *sum-measure*  $M I M_i =$  null-measure  $M$   
*<proof>*

**lemma** *nn-integral-sum-measure*:  
assumes  $f \in$  borel-measurable  $M$  and [*measurable-cong*]:  $\wedge i. i \in I \implies$  sets  $(M_i i) =$  sets  $M$

**shows**  $(\int^{+x}. f x \partial \text{sum-measure } M I Mi) = (\sum i \in I. (\int^{+x}. f x \partial (Mi i)))$   
 ⟨proof⟩

**corollary** *integrable-sum-measure-iff-ne:*

**fixes**  $f :: 'a \Rightarrow 'b :: \{\text{banach, second-countable-topology}\}$   
**assumes**  $[\text{measurable-cong}]: \bigwedge i. i \in I \implies \text{sets } (Mi i) = \text{sets } M \text{ and finite } I \text{ and } I \neq \{\}$   
**shows**  $\text{integrable } (\text{sum-measure } M I Mi) f \longleftrightarrow (\forall i \in I. \text{integrable } (Mi i) f)$   
 ⟨proof⟩

**corollary** *integrable-sum-measure-iff:*

**fixes**  $f :: 'a \Rightarrow 'b :: \{\text{banach, second-countable-topology}\}$   
**assumes**  $[\text{measurable-cong}]: \bigwedge i. i \in I \implies \text{sets } (Mi i) = \text{sets } M \text{ and finite } I$   
**and**  $[\text{measurable}]: f \in \text{borel-measurable } M$   
**shows**  $\text{integrable } (\text{sum-measure } M I Mi) f \longleftrightarrow (\forall i \in I. \text{integrable } (Mi i) f)$   
 ⟨proof⟩

**lemma** *integral-sum-measure:*

**fixes**  $f :: 'a \Rightarrow 'b :: \{\text{banach, second-countable-topology}\}$   
**assumes**  $[\text{measurable-cong}]: \bigwedge i. i \in I \implies \text{sets } (Mi i) = \text{sets } M \bigwedge i. i \in I \implies \text{integrable } (Mi i) f$   
**shows**  $(\int x. f x \partial \text{sum-measure } M I Mi) = (\sum i \in I. (\int x. f x \partial (Mi i)))$   
 ⟨proof⟩

Lemmas related to scale measure

**lemma** *integrable-scale-measure:*

**fixes**  $f :: 'a \Rightarrow 'b :: \{\text{banach, second-countable-topology}\}$   
**assumes**  $\text{integrable } M f$   
**shows**  $\text{integrable } (\text{scale-measure } (\text{ennreal } r) M) f$   
 ⟨proof⟩

**lemma** *integral-scale-measure:*

**assumes**  $r \geq 0 \text{ integrable } M f$   
**shows**  $(\int x. f x \partial \text{scale-measure } (\text{ennreal } r) M) = r * (\int x. f x \partial M)$   
 ⟨proof⟩

**lemma**

**fixes**  $c :: \text{ereal}$   
**assumes**  $c: c \neq -\infty \text{ and } a: \bigwedge n. 0 \leq a n$   
**shows**  $\text{liminf-cadd: } \text{liminf } (\lambda n. c + a n) = c + \text{liminf } a$   
**and**  $\text{limsup-cadd: } \text{limsup } (\lambda n. c + a n) = c + \text{limsup } a$   
 ⟨proof⟩

**lemma**(in *Metric-space*) *frontier-measure-zero-balls:*

**assumes**  $\text{sets } N = \text{sets } (\text{borel-of } m\text{topology}) \text{ finite-measure } N M \neq \{\}$   
**and**  $e > 0 \text{ and separable-space } m\text{topology}$   
**obtains**  $ai ri \text{ where}$   
 $(\bigcup i::\text{nat. } m\text{ball } (ai i) (ri i)) = M (\bigcup i::\text{nat. } m\text{cball } (ai i) (ri i)) = M$   
 $\bigwedge i. ai i \in M \bigwedge i. ri i > 0 \bigwedge i. ri i < e$

$\bigwedge i. \text{measure } N (\text{mtopology frontier-of } (\text{mball } (ai\ i) (ri\ i))) = 0$   
 $\bigwedge i. \text{measure } N (\text{mtopology frontier-of } (\text{mcball } (ai\ i) (ri\ i))) = 0$   
 <proof>

**lemma** *finite-measure-integral-eq-dense*:

**assumes** *finite*: *finite-measure*  $N$  *finite-measure*  $M$   
**and** *sets-N*: *sets*  $N = \text{sets } (\text{borel-of } X)$  **and** *sets-M*: *sets*  $M = \text{sets } (\text{borel-of } X)$   
**and** *dense*: *dense-in* (*mtopology-of* (*cfunspace*  $X$  *euclidean-metric*))  $F$   
**and** *integ-eq*:  $\bigwedge f :: - \Rightarrow \text{real}. f \in F \implies (\int x. f\ x\ \partial N) = (\int x. f\ x\ \partial M)$   
**and** *f*: *continuous-map*  $X$  *euclideanreal* *f* *bounded* (*f* ' *topspace*  $X$ )  
**shows**  $(\int x. f\ x\ \partial N) = (\int x. f\ x\ \partial M)$   
 <proof>

## 1.2 Sequentially Continuous Maps

**definition** *seq-continuous-map* ::  $'a$  *topology*  $\Rightarrow$   $'b$  *topology*  $\Rightarrow$  ( $'a \Rightarrow 'b$ )  $\Rightarrow$  *bool*  
**where**

*seq-continuous-map*  $X\ Y\ f \equiv (\forall xn\ x. \text{limitin } X\ xn\ x\ \text{sequentially} \longrightarrow \text{limitin } Y$   
 $(\lambda n. f\ (xn\ n))\ (f\ x)\ \text{sequentially})$

**lemma** *seq-continuous-map*:

*seq-continuous-map*  $X\ Y\ f \iff (\forall xn\ x. \text{limitin } X\ xn\ x\ \text{sequentially} \longrightarrow \text{limitin } Y$   
 $(\lambda n. f\ (xn\ n))\ (f\ x)\ \text{sequentially})$   
 <proof>

**lemma** *seq-continuous-map-funspace*:

**assumes** *seq-continuous-map*  $X\ Y\ f$   
**shows**  $f \in \text{topspace } X \rightarrow \text{topspace } Y$   
 <proof>

**lemma** *seq-continuous-iff-continuous-first-countable*:

**assumes** *first-countable*  $X$   
**shows** *seq-continuous-map*  $X\ Y = \text{continuous-map } X\ Y$   
 <proof>

## 1.3 Sequential Compactness

**definition** *seq-compactin* ::  $'a$  *topology*  $\Rightarrow$   $'a$  *set*  $\Rightarrow$  *bool* **where**

*seq-compactin*  $X\ S$

$\iff S \subseteq \text{topspace } X \wedge (\forall xn. (\forall n :: \text{nat}. xn\ n \in S) \longrightarrow (\exists l \in S. \exists a :: \text{nat} \Rightarrow \text{nat}.$   
 $\text{strict-mono } a \wedge \text{limitin } X\ (xn \circ a)\ l\ \text{sequentially}))$

**definition** *seq-compact-space*  $X \equiv \text{seq-compactin } X\ (\text{topspace } X)$

**lemma** *seq-compactin-subset-topspace*: *seq-compactin*  $X\ S \implies S \subseteq \text{topspace } X$

<proof>

**lemma** *seq-compactin-empty[simp]*: *seq-compactin*  $X\ \{\}$

<proof>

**lemma** *seq-compactin-seq-compact*[simp]: *seq-compactin euclidean S*  $\longleftrightarrow$  *seq-compact S*

*<proof>*

**lemma** *image-seq-compactin*:

**assumes** *seq-compactin X S seq-continuous-map X Y f*

**shows** *seq-compactin Y (f ` S)*

*<proof>*

**lemma** *closed-seq-compactin*:

**assumes** *seq-compactin X K C  $\subseteq$  K closedin X C*

**shows** *seq-compactin X C*

*<proof>*

**corollary** *closedin-seq-compact-space*:

*seq-compact-space X  $\implies$  closedin X C  $\implies$  seq-compactin X C*

*<proof>*

**lemma** *seq-compactin-subtopology*: *seq-compactin (subtopology X S) T*  $\longleftrightarrow$  *seq-compactin X T  $\wedge$  T  $\subseteq$  S*

*<proof>*

**corollary** *seq-compact-space-subtopology*: *seq-compactin X S  $\implies$  seq-compact-space (subtopology X S)*

*<proof>*

**lemma** *seq-compactin-PiED*:

**assumes** *seq-compactin (product-topology X I) (Pi\_E I S)*

**shows** *(Pi\_E I S = {}  $\vee$  ( $\forall i \in I$ . seq-compactin (X i) (S i)))*

*<proof>*

**lemma** *metrizable-seq-compactin-iff-compactin*:

**assumes** *metrizable-space X*

**shows** *seq-compactin X S  $\longleftrightarrow$  compactin X S*

*<proof>*

**corollary** *metrizable-seq-compact-space-iff-compact-space*:

**shows** *metrizable-space X  $\implies$  seq-compact-space X  $\longleftrightarrow$  compact-space X*

*<proof>*

## 1.4 Lemmas for Limsup and Liminf

**lemma** *real-less-add-ex-less-pair*:

**fixes** *x w v :: real*

**assumes** *x < w + v*

**shows**  $\exists y z. x = y + z \wedge y < w \wedge z < v$

*<proof>*

**lemma** *ereal-less-add-ex-less-pair*:

**fixes**  $x w v :: ereal$   
**assumes**  $-\infty < w - \infty < v x < w + v$   
**shows**  $\exists y z. x = y + z \wedge y < w \wedge z < v$   
 $\langle proof \rangle$

**lemma** *real-add-less*:  
**fixes**  $x w v :: real$   
**assumes**  $w + v < x$   
**shows**  $\exists y z. x = y + z \wedge w < y \wedge v < z$   
 $\langle proof \rangle$

**lemma** *ereal-add-less*:  
**fixes**  $x w v :: ereal$   
**assumes**  $w + v < x$   
**shows**  $\exists y z. x = y + z \wedge w < y \wedge v < z$   
 $\langle proof \rangle$

A generalized version of  $\neg (liminf ?u = \infty \wedge liminf ?v = -\infty \vee liminf ?u = -\infty \wedge liminf ?v = \infty) \implies liminf ?u + liminf ?v \leq liminf (\lambda n. ?u n + ?v n)$ .

**lemma** *ereal-Liminf-add-mono*:  
**fixes**  $u v :: 'a \Rightarrow ereal$   
**assumes**  $\neg((Liminf F u = \infty \wedge Liminf F v = -\infty) \vee (Liminf F u = -\infty \wedge Liminf F v = \infty))$   
**shows**  $Liminf F (\lambda n. u n + v n) \geq Liminf F u + Liminf F v$   
 $\langle proof \rangle$

A generalized version of  $limsup (\lambda n. ?u n + ?v n) \leq limsup ?u + limsup ?v$ .

**lemma** *ereal-Limsup-add-mono*:  
**fixes**  $u v :: 'a \Rightarrow ereal$   
**shows**  $Limsup F (\lambda n. u n + v n) \leq Limsup F u + Limsup F v$   
 $\langle proof \rangle$

## 1.5 A Characterization of Closed Sets by Limit

There is a net which characterize closed sets in terms of convergence. Since Isabelle/HOL's convergent is defined through filters, we transform the net to a filter. We refer to the lecture notes by Shi [3] for the conversion.

**definition** *derived-filter* ::  $['i \text{ set}, 'i \Rightarrow 'i \Rightarrow bool] \Rightarrow 'i \text{ filter}$  **where**  
 $derived-filter I op \equiv (\prod i \in I. principal \{j \in I. op i j\})$

**lemma** *eventually-derived-filter*:  
**assumes**  $A \neq \{\}$   
**and** *refl*:  $\bigwedge a. a \in A \implies rel a a$   
**and** *trans*:  $\bigwedge a b c. a \in A \implies b \in A \implies c \in A \implies rel a b \implies rel b c \implies rel a c$   
**and** *pair-bounded*:  $\bigwedge a b. a \in A \implies b \in A \implies \exists c \in A. rel a c \wedge rel b c$

**shows eventually  $P$  (derived-filter  $A$   $rel$ )**  $\longleftrightarrow (\exists i \in A. \forall n \in A. rel\ i\ n \longrightarrow P\ n)$   
 ⟨proof⟩

**definition  $nhdsin$ -sets** :: 'a topology  $\Rightarrow$  'a  $\Rightarrow$  'a set filter **where**  
 $nhdsin$ -sets  $X\ x \equiv$  derived-filter  $\{U. openin\ X\ U \wedge x \in U\}$  ( $\supseteq$ )

**lemma eventually- $nhdsin$ -sets:**

**assumes**  $x \in$  *topspace*  $X$   
**shows eventually  $P$  ( $nhdsin$ -sets  $X\ x$ )**  $\longleftrightarrow (\exists U. openin\ X\ U \wedge x \in U \wedge (\forall V. openin\ X\ V \longrightarrow x \in V \longrightarrow V \subseteq U \longrightarrow P\ V))$   
 ⟨proof⟩

**lemma eventually- $nhdsin$ -setsI:**

**assumes**  $x \in$  *topspace*  $X \wedge U. x \in U \Longrightarrow openin\ X\ U \Longrightarrow P\ U$   
**shows eventually  $P$  ( $nhdsin$ -sets  $X\ x$ )**  
 ⟨proof⟩

**lemma  $nhdsin$ -sets-bot**[*simp, intro*]:

**assumes**  $x \in$  *topspace*  $X$   
**shows  $nhdsin$ -sets  $X\ x \neq \perp$**   
 ⟨proof⟩

**corollary  $limitin$ - $nhdsin$ -sets:**  $limitin\ X\ xn\ x$  ( $nhdsin$ -sets  $X\ x$ )  $\longleftrightarrow x \in$  *topspace*  $X \wedge (\forall U. openin\ X\ U \longrightarrow x \in U \longrightarrow (\exists V. openin\ X\ V \wedge x \in V \wedge (\forall W. openin\ X\ W \longrightarrow x \in W \longrightarrow W \subseteq V \longrightarrow xn\ W \in U)))$   
 ⟨proof⟩

**lemma  $closedin$ - $limitin$ :**

**assumes**  $T \subseteq$  *topspace*  $X \wedge xn\ x. x \in$  *topspace*  $X \Longrightarrow (\bigwedge U. x \in U \Longrightarrow openin\ X\ U \Longrightarrow xn\ U \neq x) \Longrightarrow (\bigwedge U. x \in U \Longrightarrow openin\ X\ U \Longrightarrow xn\ U \in T) \Longrightarrow (\bigwedge U. xn\ U \in$  *topspace*  $X) \Longrightarrow limitin\ X\ xn\ x$  ( $nhdsin$ -sets  $X\ x) \Longrightarrow x \in T$   
**shows  $closedin\ X\ T$**   
 ⟨proof⟩

**corollary  $closedin$ -iff- $limitin$ -eq:**

**fixes**  $X$  :: 'a topology  
**shows  $closedin\ X\ C$**   
 $\longleftrightarrow C \subseteq$  *topspace*  $X \wedge$   
 $(\forall xi\ x\ (F :: 'a\ set\ filter). (\forall i. xi\ i \in$  *topspace*  $X) \longrightarrow x \in$  *topspace*  $X$   
 $\longrightarrow (\forall_F\ i\ in\ F. xi\ i \in C) \longrightarrow F \neq \perp \longrightarrow limitin\ X\ xi\ x\ F \longrightarrow x \in C)$   
 ⟨proof⟩

**lemma  $closedin$ -iff- $limitin$ -sequentially:**

**assumes** *first-countable*  $X$   
**shows  $closedin\ X\ S \longleftrightarrow S \subseteq$  *topspace*  $X \wedge (\forall \sigma\ l. range\ \sigma \subseteq S \wedge limitin\ X\ \sigma\ l$  *sequentially*  $\longrightarrow l \in S)$  (is ?lhs=?rhs)**  
 ⟨proof⟩

## 1.6 A Characterization of Topology by Limit

**lemma** *topology-eq-filter:*

**fixes**  $X :: 'a \text{ topology}$   
**assumes**  $\text{topspace } X = \text{topspace } Y$   
**and**  $\bigwedge (F :: 'a \text{ set filter}) \ xi \ x. (\bigwedge i. \ xi \ i \in \text{topspace } X) \implies x \in \text{topspace } X \implies$   
 $\text{limitin } X \ xi \ x \ F \longleftrightarrow \text{limitin } Y \ xi \ x \ F$   
**shows**  $X = Y$   
 $\langle \text{proof} \rangle$

**lemma** *topology-eq-limit-sequentially:*

**assumes**  $\text{topspace } X = \text{topspace } Y$   
**and**  $\text{first-countable } X \ \text{first-countable } Y$   
**and**  $\bigwedge xn \ x. (\bigwedge n. \ xn \ i \in \text{topspace } X) \implies x \in \text{topspace } X \implies \text{limitin } X \ xn \ x$   
 $\text{sequentially} \longleftrightarrow \text{limitin } Y \ xn \ x \ \text{sequentially}$   
**shows**  $X = Y$   
 $\langle \text{proof} \rangle$

## 1.7 A Characterization of Open Sets by Limit

**corollary** *openin-limitin:*

**assumes**  $U \subseteq \text{topspace } X \ \bigwedge xi \ x. \ x \in \text{topspace } X \implies (\bigwedge i. \ xi \ i \in \text{topspace } X) \implies$   
 $\text{limitin } X \ xi \ x \ (\text{nhdsin-sets } X \ x) \implies x \in U \implies \forall_F \ i \ \text{in } (\text{nhdsin-sets } X \ x). \ xi$   
 $i \in U$   
**shows**  $\text{openin } X \ U$   
 $\langle \text{proof} \rangle$

**corollary** *openin-iff-limitin-eq:*

**fixes**  $X :: 'a \text{ topology}$   
**shows**  $\text{openin } X \ U \longleftrightarrow U \subseteq \text{topspace } X \wedge (\forall xi \ x \ (F :: 'a \text{ set filter}). \ (\forall i. \ xi \ i \in \text{topspace } X) \longrightarrow x \in U \longrightarrow$   
 $\text{limitin } X \ xi \ x \ F \longrightarrow (\forall_F \ i \ \text{in } F. \ xi \ i \in U))$   
 $\langle \text{proof} \rangle$

**corollary** *limitin-openin-sequentially:*

**assumes**  $\text{first-countable } X$   
**shows**  $\text{openin } X \ U \longleftrightarrow U \subseteq \text{topspace } X \wedge (\forall xn \ x. \ x \in U \longrightarrow \text{limitin } X \ xn \ x$   
 $\text{sequentially} \longrightarrow (\exists N. \ \forall n \geq N. \ xn \ n \in U))$   
 $\langle \text{proof} \rangle$

**lemma** *upper-semicontinuous-map-limsup-iff:*

**fixes**  $f :: 'a \Rightarrow ('b :: \{\text{complete-linorder, linorder-topology}\})$   
**assumes**  $\text{first-countable } X$   
**shows**  $\text{upper-semicontinuous-map } X \ f \longleftrightarrow (\forall xn \ x. \ \text{limitin } X \ xn \ x \ \text{sequentially}$   
 $\longrightarrow \text{limsup } (\lambda n. \ f \ (xn \ n)) \leq f \ x)$   
 $\langle \text{proof} \rangle$

## 1.8 Lemmas for Upper/Lower-Semi Continuous Maps

**lemma** *upper-semicontinuous-map-limsup-real:*

**fixes**  $f :: 'a \Rightarrow \text{real}$

**assumes** *first-countable X*  
**shows** *upper-semicontinuous-map X f*  $\longleftrightarrow$   $(\forall xn\ x. \text{limitin } X\ xn\ x\ \text{sequentially})$   
 $\longrightarrow$  *limsup*  $(\lambda n. f\ (xn\ n)) \leq f\ x$   
 $\langle$ *proof* $\rangle$

**lemma** *lower-semicontinuous-map-liminf-iff:*  
**fixes**  $f :: 'a \Rightarrow ('b :: \{complete-linorder, linorder-topology\})$   
**assumes** *first-countable X*  
**shows** *lower-semicontinuous-map X f*  $\longleftrightarrow$   $(\forall xn\ x. \text{limitin } X\ xn\ x\ \text{sequentially})$   
 $\longrightarrow$   $f\ x \leq \text{liminf}\ (\lambda n. f\ (xn\ n))$   
 $\langle$ *proof* $\rangle$

**lemma** *lower-semicontinuous-map-liminf-real:*  
**fixes**  $f :: 'a \Rightarrow \text{real}$   
**assumes** *first-countable X*  
**shows** *lower-semicontinuous-map X f*  $\longleftrightarrow$   $(\forall xn\ x. \text{limitin } X\ xn\ x\ \text{sequentially})$   
 $\longrightarrow$   $f\ x \leq \text{liminf}\ (\lambda n. f\ (xn\ n))$   
 $\langle$ *proof* $\rangle$

**end**

## 2 Alaoglu's Theorem

**theory** *Alaoglu-Theorem*  
**imports** *Lemmas-Levy-Prokhorov*  
*Riesz-Representation.Riesz-Representation*  
**begin**

We prove (a special case of) Alaoglu's theorem for the space of continuous functions. We refer to Section 9 of the lecture note by Heil [1].

### 2.1 Metrizable of the Space of Uniformly Bounded Positive Linear Functionals

**lemma** *metrizable-functional:*  
**fixes**  $X :: 'a\ \text{topology}$  **and**  $r :: \text{real}$   
**defines**  $\text{prod-space} \equiv \text{powertop-real}\ (\text{mspace}\ (\text{cfunspace}\ X\ \text{euclidean-metric}))$   
**defines**  $B \equiv \{\varphi \in \text{topspace}\ \text{prod-space}. \varphi\ (\lambda x \in \text{topspace}\ X. 1) \leq r \wedge \text{positive-linear-functional-on-}CX\ X\ \varphi\}$   
**defines**  $\Phi \equiv \text{subtopology}\ \text{prod-space}\ B$   
**assumes** *compact: compact-space X* **and** *met: metrizable-space X*  
**shows** *metrizable-space*  $\Phi$   
 $\langle$ *proof* $\rangle$

### 2.2 Alaoglu's Theorem

According to Alaoglu's theorem,  $\{\varphi \in C(X)^* \mid \|\varphi\| \leq r\}$  is compact. We show that  $\Phi = \{\varphi \in C(X)^* \mid \|\varphi\| \leq r \wedge \varphi\ \text{is positive}\}$  is compact. Note that

$\|\varphi\| = \varphi(1)$  when  $\varphi \in C(X)^*$  is positive.

**theorem** *Alaoglu-theorem-real-functional*:

**fixes**  $X :: 'a \text{ topology}$  **and**  $r :: \text{real}$

**defines** *prod-space*  $\equiv$  *powertop-real* (*mSPACE* (*cfunSPACE*  $X$  *euclidean-metric*))

**defines**  $B \equiv \{\varphi \in \text{topSPACE } \text{prod-space}. \varphi (\lambda x \in \text{topSPACE } X. 1) \leq r \wedge \text{positive-linear-functional-on-CX } X \varphi\}$

**assumes** *compact*: *compact-space*  $X$  **and** *ne*: *topSPACE*  $X \neq \{\}$

**shows** *compactin prod-space*  $B$

*<proof>*

**theorem** *Alaoglu-theorem-real-functional-seq*:

**fixes**  $X :: 'a \text{ topology}$  **and**  $r :: \text{real}$

**defines** *prod-space*  $\equiv$  *powertop-real* (*mSPACE* (*cfunSPACE*  $X$  *euclidean-metric*))

**defines**  $B \equiv \{\varphi \in \text{topSPACE } \text{prod-space}. \varphi (\lambda x \in \text{topSPACE } X. 1) \leq r \wedge \text{positive-linear-functional-on-CX } X \varphi\}$

**assumes** *compact*: *compact-space*  $X$  **and** *ne*: *topSPACE*  $X \neq \{\}$  **and** *met*: *metrizable-space*  $X$

**shows** *seq-compactin prod-space*  $B$

*<proof>*

**end**

### 3 General Weak Convergence

**theory** *General-Weak-Convergence*

**imports** *Lemmas-Levy-Prokhorov*

*Riesz-Representation.Regular-Measure*

**begin**

We formalize the notion of weak convergence and equivalent conditions. The formalization of weak convergence in HOL-Probability is restricted to probability measures on real numbers. Our formalization is generalized to finite measures on any metric spaces.

#### 3.1 Topology of Weak Convergence

**definition** *weak-conv-topology*  $:: 'a \text{ topology} \Rightarrow 'a \text{ measure topology}$  **where**

*weak-conv-topology*  $X \equiv$

*topology-generated-by*

$(\bigcup f \in \{f. \text{continuous-map } X \text{ euclideanreal } f \wedge (\exists B. \forall x \in \text{topSPACE } X. |f x| \leq B)\}.$

*Collect* (*openin* (*pullback-topology*  $\{N. \text{sets } N = \text{sets } (\text{borel-of } X) \wedge \text{finite-measure } N\}$

$(\lambda N. \int x. f x \partial N) \text{ euclideanreal}))$ )

**lemma** *topSPACE-weak-conv-topology[simp]*:

*topSPACE* (*weak-conv-topology*  $X$ ) =  $\{N. \text{sets } N = \text{sets } (\text{borel-of } X) \wedge \text{finite-measure } N\}$

*<proof>*

**lemma** *openin-weak-conv-topology-base*:

**assumes**  $f$ :continuous-map  $X$  euclideanreal  $f$  **and**  $B$ : $\bigwedge x. x \in \text{topspace } X \implies |f x| \leq B$   
**and**  $U$ :open  $U$   
**shows** openin (weak-conv-topology  $X$ ) (( $\lambda N. \int x. f x \partial N$ ) - '  $U$   
 $\cap \{N. \text{sets } N = \text{sets (borel-of } X) \wedge \text{finite-measure } N\}$ )  
*<proof>*

**lemma** *continuous-map-weak-conv-topology*:

**assumes**  $f$ :continuous-map  $X$  euclideanreal  $f$  **and**  $B$ : $\bigwedge x. x \in \text{topspace } X \implies |f x| \leq B$   
**shows** continuous-map (weak-conv-topology  $X$ ) euclideanreal ( $\lambda N. \int x. f x \partial N$ )  
*<proof>*

**lemma** *weak-conv-topology-minimal*:

**assumes**  $\text{topspace } Y = \{N. \text{sets } N = \text{sets (borel-of } X) \wedge \text{finite-measure } N\}$   
**and**  $\bigwedge B. \text{continuous-map } X \text{ euclideanreal } f$   
 $\implies (\bigwedge x. x \in \text{topspace } X \implies |f x| \leq B) \implies \text{continuous-map } Y$   
euclideanreal ( $\lambda N. \int x. f x \partial N$ )  
**shows** openin (weak-conv-topology  $X$ )  $U \implies \text{openin } Y U$   
*<proof>*

**lemma** *weak-conv-topology-continuous-map-integral*:

**assumes** continuous-map  $X$  euclideanreal  $f$   $\bigwedge x. x \in \text{topspace } X \implies |f x| \leq B$   
**shows** continuous-map (weak-conv-topology  $X$ ) euclideanreal ( $\lambda N. \int x. f x \partial N$ )  
*<proof>*

## 3.2 Weak Convergence

**abbreviation** *weak-conv-on* :: ('a  $\Rightarrow$  'b measure)  $\Rightarrow$  'b measure  $\Rightarrow$  'a filter  $\Rightarrow$  'b topology  $\Rightarrow$  bool

( ' ('(-)  $\Rightarrow_{WC}$  (-)') (-) / on (-)› [56, 55] 55) **where**  
*weak-conv-on*  $Ni N F X \equiv \text{limitin (weak-conv-topology } X) Ni N F$

**abbreviation** *weak-conv-on-seq* :: (nat  $\Rightarrow$  'b measure)  $\Rightarrow$  'b measure  $\Rightarrow$  'b topology  $\Rightarrow$  bool

( ' ('(-)  $\Rightarrow_{WC}$  (-)') on (-)› [56, 55] 55) **where**  
*weak-conv-on-seq*  $Ni N X \equiv \text{weak-conv-on } Ni N \text{ sequentially } X$

## 3.3 Limit in Topology of Weak Convergence

**lemma** *weak-conv-on-def*:

*weak-conv-on*  $Ni N F X \longleftrightarrow$   
 $(\forall F i \text{ in } F. \text{sets (} Ni i) = \text{sets (borel-of } X) \wedge \text{finite-measure (} Ni i)) \wedge \text{sets } N =$   
 $\text{sets (borel-of } X)$   
 $\wedge \text{finite-measure } N$   
 $\wedge (\forall f. \text{continuous-map } X \text{ euclideanreal } f \longrightarrow (\exists B. \forall x \in \text{topspace } X. |f x| \leq$   
 $B))$

$\longrightarrow ((\lambda i. \int x. f x \partial Ni i) \longrightarrow (\int x. f x \partial N)) F$   
 <proof>

**lemma** *weak-conv-on-def'*:

**assumes**  $\bigwedge i. \text{sets } (Ni i) = \text{sets } (\text{borel-of } X)$  **and**  $\bigwedge i. \text{finite-measure } (Ni i)$   
**and**  $\text{sets } N = \text{sets } (\text{borel-of } X)$  **and**  $\text{finite-measure } N$

**shows** *weak-conv-on*  $Ni N F X$

$\longleftrightarrow (\forall f. \text{continuous-map } X \text{ euclideanreal } f \longrightarrow (\exists B. \forall x \in \text{topspace } X. |f x| \leq B))$

$\longrightarrow ((\lambda i. \int x. f x \partial Ni i) \longrightarrow (\int x. f x \partial N)) F$

<proof>

**lemmas** *weak-conv-seq-def* = *weak-conv-on-def* [**where**  $F = \text{sequentially}$ ]

**lemma** *weak-conv-on-const*:

$(\bigwedge i. Ni i = N) \implies \text{sets } N = \text{sets } (\text{borel-of } X)$

$\implies \text{finite-measure } N \implies \text{weak-conv-on } Ni N F X$

<proof>

**lemmas** *weak-conv-on-seq-const* = *weak-conv-on-const* [**where**  $F = \text{sequentially}$ ]

**context** *Metric-space*

**begin**

**abbreviation** *mweak-conv*  $\equiv (\lambda Ni N F. \text{weak-conv-on } Ni N F \text{ mtopology})$

**abbreviation** *mweak-conv-seq*  $\equiv \lambda Ni N. \text{mweak-conv } Ni N \text{ sequentially}$

**lemmas** *mweak-conv-def* = *weak-conv-on-def* [**where**  $X = \text{mtopology, simplified}$ ]

**lemmas** *mweak-conv-seq-def* = *weak-conv-seq-def* [**where**  $X = \text{mtopology, simplified}$ ]

**end**

### 3.4 The Portmanteau Theorem

**locale** *mweak-conv-fin* = *Metric-space* +

**fixes**  $Ni :: 'b \Rightarrow 'a \text{ measure}$  **and**  $N :: 'a \text{ measure}$  **and**  $F$

**assumes**  $\text{sets-Ni}: \forall_F i \text{ in } F. \text{sets } (Ni i) = \text{sets } (\text{borel-of } \text{mtopology})$

**and**  $\text{sets-N}[\text{measurable-cong}]: \text{sets } N = \text{sets } (\text{borel-of } \text{mtopology})$

**and**  $\text{finite-measure-Ni}: \forall_F i \text{ in } F. \text{finite-measure } (Ni i)$

**and**  $\text{finite-measure-N}: \text{finite-measure } N$

**begin**

**interpretation**  $N: \text{finite-measure } N$

<proof>

**lemma** *space-N*:  $\text{space } N = M$

<proof>

**lemma** *space-Ni*:  $\forall_F i \text{ in } F. \text{space } (Ni i) = M$

*<proof>*

**lemma** *eventually-Ni*:  $\forall_F i \text{ in } F. \text{space } (Ni \ i) = M \wedge \text{sets } (Ni \ i) = \text{sets } (\text{borel-of } mtopology) \wedge \text{finite-measure } (Ni \ i)$

*<proof>*

**lemma** *measure-converge-bounded'*:

**assumes**  $((\lambda n. \text{measure } (Ni \ n) \ M) \longrightarrow \text{measure } N \ M) \ F$

**obtains**  $K \text{ where } \bigwedge A. \forall_F x \text{ in } F. \text{measure } (Ni \ x) \ A \leq K \wedge A. \text{measure } N \ A \leq K$

*<proof>*

**lemma**

**assumes**  $F \neq \perp \forall_F x \text{ in } F. \text{measure } (Ni \ x) \ A \leq K \text{ measure } N \ A \leq K$

**shows** *Liminf-measure-bounded*:  $\text{Liminf } F \ (\lambda i. \text{measure } (Ni \ i) \ A) < \infty \ 0 \leq \text{Liminf } F \ (\lambda i. \text{measure } (Ni \ i) \ A)$

**and** *Limsup-measure-bounded*:  $\text{Limsup } F \ (\lambda i. \text{measure } (Ni \ i) \ A) < \infty \ 0 \leq \text{Limsup } F \ (\lambda i. \text{measure } (Ni \ i) \ A)$

*<proof>*

**lemma** *mweak-conv1*:

**fixes**  $f:: 'a \Rightarrow \text{real}$

**assumes** *mweak-conv*  $Ni \ N \ F$

**and** *uniformly-continuous-map Self euclidean-metric*  $f$

**shows**  $(\exists B. \forall x \in M. |f \ x| \leq B) \implies ((\lambda n. \text{integral}^L (Ni \ n) \ f) \longrightarrow \text{integral}^L \ N \ f) \ F$

*<proof>*

**lemma** *mweak-conv2*:

**assumes**  $\bigwedge f:: 'a \Rightarrow \text{real}. \text{uniformly-continuous-map Self euclidean-metric } f \implies (\exists B. \forall x \in M. |f \ x| \leq B)$

$\implies ((\lambda n. \text{integral}^L (Ni \ n) \ f) \longrightarrow \text{integral}^L \ N \ f) \ F$

**and** *closedin mtopology*  $A$

**shows**  $\text{Limsup } F \ (\lambda x. \text{ereal } (\text{measure } (Ni \ x) \ A)) \leq \text{ereal } (\text{measure } N \ A)$

*<proof>*

**lemma** *mweak-conv3*:

**assumes**  $\bigwedge A. \text{closedin } mtopology \ A \implies \text{Limsup } F \ (\lambda n. \text{measure } (Ni \ n) \ A) \leq \text{measure } N \ A$

**and**  $((\lambda n. \text{measure } (Ni \ n) \ M) \longrightarrow \text{measure } N \ M) \ F$

**and** *openin mtopology*  $U$

**shows**  $\text{measure } N \ U \leq \text{Liminf } F \ (\lambda n. \text{measure } (Ni \ n) \ U)$

*<proof>*

**lemma** *mweak-conv3'*:

**assumes**  $\bigwedge U. \text{openin } mtopology \ U \implies \text{measure } N \ U \leq \text{Liminf } F \ (\lambda n. \text{measure } (Ni \ n) \ U)$

**and**  $((\lambda n. \text{measure } (Ni \ n) \ M) \longrightarrow \text{measure } N \ M) \ F$

**and** *closedin mtopology*  $A$

**shows**  $Limsup F (\lambda n. measure (Ni n) A) \leq measure N A$   
 ⟨proof⟩

**lemma** *mweak-conv4*:

**assumes**  $\bigwedge A. closedin\ mtopology\ A \implies Limsup F (\lambda n. measure (Ni n) A) \leq measure N A$

**and**  $\bigwedge U. openin\ mtopology\ U \implies measure N U \leq Liminf F (\lambda n. measure (Ni n) U)$

**and** [measurable]:  $A \in sets\ (borel-of\ mtopology)$

**and**  $measure N (mtopology\ frontier-of\ A) = 0$

**shows**  $((\lambda n. measure (Ni n) A) \longrightarrow measure N A) F$   
 ⟨proof⟩

**lemma** *mweak-conv5*:

**assumes**  $\bigwedge A. A \in sets\ (borel-of\ mtopology) \implies measure N (mtopology\ frontier-of\ A) = 0$

$\implies ((\lambda n. measure (Ni n) A) \longrightarrow measure N A) F$

**shows** *mweak-conv Ni N F*  
 ⟨proof⟩

**lemma** *mweak-conv-eq*: *mweak-conv Ni N F*

$\longleftrightarrow (\forall f::'a \Rightarrow real. continuous-map\ mtopology\ euclidean\ f \longrightarrow (\exists B. \forall x \in M. |f x| \leq B))$

$\longrightarrow ((\lambda n. \int x. f x \partial Ni n) \longrightarrow (\int x. f x \partial N)) F$

⟨proof⟩

**lemma** *mweak-conv-eq1*: *mweak-conv Ni N F*

$\longleftrightarrow (\forall f::'a \Rightarrow real. uniformly-continuous-map\ Self\ euclidean-metric\ f \longrightarrow (\exists B. \forall x \in M. |f x| \leq B))$

$\longrightarrow ((\lambda n. \int x. f x \partial Ni n) \longrightarrow (\int x. f x \partial N)) F$

⟨proof⟩

**lemma** *mweak-conv-eq2*: *mweak-conv Ni N F*

$\longleftrightarrow ((\lambda n. measure (Ni n) M) \longrightarrow measure N M) F \wedge (\forall A. closedin\ mtopology\ A$

$\longrightarrow Limsup F (\lambda n. measure (Ni n) A) \leq measure N A)$

⟨proof⟩

**lemma** *mweak-conv-eq3*: *mweak-conv Ni N F*

$\longleftrightarrow ((\lambda n. measure (Ni n) M) \longrightarrow measure N M) F \wedge$

$(\forall U. openin\ mtopology\ U \longrightarrow measure N U \leq Liminf F (\lambda n. measure (Ni n) U))$

⟨proof⟩

**lemma** *mweak-conv-eq4*: *mweak-conv Ni N F*

$\longleftrightarrow (\forall A \in sets\ (borel-of\ mtopology). measure N (mtopology\ frontier-of\ A) = 0$

$\longrightarrow ((\lambda n. measure (Ni n) A) \longrightarrow measure N A) F)$

⟨proof⟩

**corollary** *mweak-conv-imp-limit-space:*

**assumes** *mweak-conv Ni N F*

**shows**  $((\lambda i. \text{measure } (Ni\ i)\ M) \longrightarrow \text{measure } N\ M)\ F$

*<proof>*

**end**

**lemma**

**assumes** *metrizable-space X*

**and**  $\forall_F i \text{ in } F. \text{sets } (Ni\ i) = \text{sets } (\text{borel-of } X) \forall_F i \text{ in } F. \text{finite-measure } (Ni\ i)$

**and** *sets N = sets (borel-of X) finite-measure N*

**shows** *weak-conv-on-eq1:*

*weak-conv-on Ni N F X*

$\longleftrightarrow ((\lambda n. \text{measure } (Ni\ n)\ (\text{topspace } X)) \longrightarrow \text{measure } N\ (\text{topspace } X))\ F$

$\wedge (\forall A. \text{closedin } X\ A \longrightarrow \text{Limsup } F\ (\lambda n. \text{measure } (Ni\ n)\ A) \leq \text{measure } N$

*A) (is ?eq1)*

**and** *weak-conv-on-eq2:*

*weak-conv-on Ni N F X*

$\longleftrightarrow ((\lambda n. \text{measure } (Ni\ n)\ (\text{topspace } X)) \longrightarrow \text{measure } N\ (\text{topspace } X))\ F$

$\wedge (\forall U. \text{openin } X\ U \longrightarrow \text{measure } N\ U \leq \text{Liminf } F\ (\lambda n. \text{measure } (Ni\ n)$

*U)) (is ?eq2)*

**and** *weak-conv-on-eq3:*

*weak-conv-on Ni N F X*

$\longleftrightarrow (\forall A \in \text{sets } (\text{borel-of } X). \text{measure } N\ (X\ \text{frontier-of } A) = 0$

$\longrightarrow ((\lambda n. \text{measure } (Ni\ n)\ A) \longrightarrow \text{measure } N\ A)\ F)$  **(is ?eq3)**

*<proof>*

**end**

## 4 The Lévy-Prokhorov Metric

**theory** *Levy-Prokhorov-Distance*

**imports** *Lemmas-Levy-Prokhorov General-Weak-Convergence*

**begin**

### 4.1 The Lévy-Prokhorov Metric

**lemma** *LPm-ne'*:

**assumes** *finite-measure M finite-measure N*

**shows**  $\exists e > 0. \forall A\ B\ C\ D. \text{measure } M\ A \leq \text{measure } N\ (B\ A\ e) + e \wedge \text{measure } N\ C \leq \text{measure } M\ (D\ C\ e) + e$

*<proof>*

**locale** *Levy-Prokhorov = Metric-space*

**begin**

**definition**  $\mathcal{P} \equiv \{N. \text{sets } N = \text{sets } (\text{borel-of } \text{mtopology}) \wedge \text{finite-measure } N\}$

**lemma** *inP-D:*

**assumes**  $N \in \mathcal{P}$   
**shows** *finite-measure*  $N$  sets  $N =$  sets (borel-of mtopology) space  $N = M$   
 ⟨proof⟩

**declare** *inP-D(2)*[measurable-cong]

**lemma** *inP-I*: sets  $N =$  sets (borel-of mtopology)  $\implies$  *finite-measure*  $N \implies N \in \mathcal{P}$   
 ⟨proof⟩

**lemma** *inP-iff*:  $N \in \mathcal{P} \iff$  sets  $N =$  sets (borel-of mtopology)  $\wedge$  *finite-measure*  $N$   
 ⟨proof⟩

**lemma** *M-empty-P*:  
**assumes**  $M = \{\}$   
**shows**  $\mathcal{P} = \{\} \vee \mathcal{P} = \{\text{count-space } \{\}\}$   
 ⟨proof⟩

**lemma** *M-empty-P'*:  
**assumes**  $M = \{\}$   
**shows**  $\mathcal{P} = \{\} \vee \mathcal{P} = \{\text{null-measure (borel-of mtopology)}\}$   
 ⟨proof⟩

**lemma** *inP-mweak-conv-fin-all*:  
**assumes**  $\bigwedge i. Ni \ i \in \mathcal{P} \ N \in \mathcal{P}$   
**shows** *mweak-conv-fin*  $M \ d \ Ni \ N \ F$   
 ⟨proof⟩

**lemma** *inP-mweak-conv-fin*:  
**assumes**  $\forall_F \ i \ \text{in } F. Ni \ i \in \mathcal{P} \ N \in \mathcal{P}$   
**shows** *mweak-conv-fin*  $M \ d \ Ni \ N \ F$   
 ⟨proof⟩

**definition** *LPm* :: 'a measure  $\Rightarrow$  'a measure  $\Rightarrow$  real **where**  
 $LPm \ N \ L \equiv$   
 if  $N \in \mathcal{P} \wedge L \in \mathcal{P}$  then  
    $(\bigcap \{e. e > 0 \wedge (\forall A \in \text{sets (borel-of mtopology)}. \text{measure } N \ A \leq \text{measure } L (\bigcup_{a \in A. \text{mball } a \ e}) + e \wedge \text{measure } L \ A \leq \text{measure } N (\bigcup_{a \in A. \text{mball } a \ e}) + e)\})$   
 else 0

**lemma** *bdd-below-Levy-Prokhorov*:  
*bdd-below*  $\{e. e > 0 \wedge (\forall A \in \text{sets (borel-of mtopology)}. \text{measure } N \ A \leq \text{measure } L (\bigcup_{a \in A. \text{mball } a \ e}) + e \wedge \text{measure } L \ A \leq \text{measure } N (\bigcup_{a \in A. \text{mball } a \ e}) + e)\}$   
 ⟨proof⟩

**lemma** *LPm-ne*:  
**assumes**  $N \in \mathcal{P} \ L \in \mathcal{P}$

**shows**  $\{e. e > 0 \wedge (\forall A \in \text{sets } (\text{borel-of } \text{mtopology}))$   
 $\text{measure } N A \leq \text{measure } L (\bigcup a \in A. \text{mball } a e) + e \wedge$   
 $\text{measure } L A \leq \text{measure } N (\bigcup a \in A. \text{mball } a e) + e\}$   
 $\neq \{\}$   
 $\langle \text{proof} \rangle$

**lemma** *LPm-imp-le*:

**assumes**  $e > 0$   
**and**  $\bigwedge B. B \in \text{sets } (\text{borel-of } \text{mtopology}) \implies \text{measure } L B \leq \text{measure } N (\bigcup a \in B.$   
 $\text{mball } a e) + e$   
**and**  $\bigwedge B. B \in \text{sets } (\text{borel-of } \text{mtopology}) \implies \text{measure } N B \leq \text{measure } L (\bigcup a \in B.$   
 $\text{mball } a e) + e$   
**shows**  $\text{LPm } L N \leq e$   
 $\langle \text{proof} \rangle$

**lemma** *LPm-le-max-measure*:  $\text{LPm } L N \leq \max (\text{measure } L (\text{space } L)) (\text{measure } N (\text{space } N))$   
 $\langle \text{proof} \rangle$

**lemma** *LPm-less-then*:

**assumes**  $N \in \mathcal{P}$  **and**  $L \in \mathcal{P}$   
**and**  $\text{LPm } N L < e$   $A \in \text{sets } (\text{borel-of } \text{mtopology})$   
**shows**  $\text{measure } N A \leq \text{measure } L (\bigcup a \in A. \text{mball } a e) + e$   $\text{measure } L A \leq$   
 $\text{measure } N (\bigcup a \in A. \text{mball } a e) + e$   
 $\langle \text{proof} \rangle$

**lemma** *LPm-nonneg:0*  $\leq \text{LPm } N L$   
 $\langle \text{proof} \rangle$

**lemma** *LPm-open*:  $\text{LPm } L N = (\text{if } L \in \mathcal{P} \wedge N \in \mathcal{P} \text{ then}$   
 $(\prod \{e. e > 0 \wedge (\forall A \in \{U. \text{openin } \text{mtopology } U\}.$   
 $\text{measure } L A \leq \text{measure } N (\bigcup a \in A. \text{mball}$   
 $a e) + e \wedge$   
 $\text{measure } N A \leq \text{measure } L (\bigcup a \in A. \text{mball}$   
 $a e) + e\})$   
 $\text{else } 0)$   
 $\langle \text{proof} \rangle$

**lemma** *LPm-closed*:  $\text{LPm } L N = (\text{if } L \in \mathcal{P} \wedge N \in \mathcal{P} \text{ then}$   
 $(\prod \{e. e > 0 \wedge (\forall A \in \{U. \text{closedin } \text{mtopology } U\}.$   
 $\text{measure } L A \leq \text{measure } N (\bigcup a \in A. \text{mball}$   
 $a e) + e \wedge$   
 $\text{measure } N A \leq \text{measure } L (\bigcup a \in A. \text{mball}$   
 $a e) + e\})$   
 $\text{else } 0)$   
 $\langle \text{proof} \rangle$

**lemma** *LPm-compact*:

**assumes** *separable-space mtopology mcomplete*

**shows**  $LPm\ L\ N = (if\ L \in \mathcal{P} \wedge N \in \mathcal{P}\ then$   
 $(\prod \{e. e > 0 \wedge (\forall A \in \{U. compactin\ mtopology\ U}\.$   
 $measure\ L\ A \leq measure\ N\ (\bigcup_{a \in A. mball\ a\ e)$   
 $+ e \wedge$   
 $measure\ N\ A \leq measure\ L\ (\bigcup_{a \in A. mball\ a\ e)$   
 $+ e\})\}$   
 $else\ 0)$   
 $\langle proof \rangle$

**sublocale**  $LPm: Metric-space\ \mathcal{P}\ LPm$   
 $\langle proof \rangle$

## 4.2 Convergence and Weak Convergence

**lemma** *converge-imp-mweak-conv*:  
**assumes**  $limitin\ LPm.mtopology\ Ni\ N\ F$   
**shows**  $mweak-conv\ Ni\ N\ F$   
 $\langle proof \rangle$

**lemma** *mweak-conv-imp-converge*:  
**assumes**  $separable-space\ mtopology$   
**and**  $mweak-conv\ Ni\ N\ F$   
**shows**  $limitin\ LPm.mtopology\ Ni\ N\ F$   
 $\langle proof \rangle$

**corollary** *conv-iff-mweak-conv: separable-space mtopology  $\implies$  limitin LPm.mtopology*  
 $Ni\ N\ F \longleftrightarrow mweak-conv\ Ni\ N\ F$   
 $\langle proof \rangle$

## 4.3 Separability

**lemma** *LPm-countable-base*:  
**assumes**  $ai:mdense\ (range\ ai)$   
**shows**  $LPm.mdense$   
 $((\lambda(k,bi). sum-measure$   
 $(borel-of\ mtopology)\ \{..k\}$   
 $(\lambda i. scale-measure\ (ennreal\ (bi\ i))\ (return\ (borel-of\ mtopology)$   
 $(ai\ i))))$   
 $\langle (SIGMA\ k:(UNIV\ ::\ nat\ set). (\{..k\} \rightarrow_E\ \mathbb{Q} \cap \{0..\})) \rangle$  **(is**  $LPm.mdense$   
 $?D)$   
 $\langle proof \rangle$

**lemma** *separable-LPm*:  
**assumes**  $separable-space\ mtopology$   
**shows**  $separable-space\ LPm.mtopology$   
 $\langle proof \rangle$

**lemma** *closedin-bounded-measures*:  
 $closedin\ LPm.mtopology\ \{N. sets\ N = sets\ (borel-of\ mtopology) \wedge N\ (space\ N)$   
 $\leq\ ennreal\ r\}$

*<proof>*

**lemma** *closedin-subprobs:*

*closedin LPm.mtopology {N. subprob-space N  $\wedge$  sets N = sets (borel-of mtopology)}*

*<proof>*

**lemma** *closedin-probs: closedin LPm.mtopology {N. prob-space N  $\wedge$  sets N = sets (borel-of mtopology)}*

*<proof>*

#### 4.4 The Lévy-Prokhorov Metric and Topology of Weak Convergence

**lemma** *weak-conv-topology-le-LPm-topology:*

**assumes** *openin (weak-conv-topology mtopology) S*

**shows** *openin LPm.mtopology S*

*<proof>*

**lemma** *LPm-topology-eq-weak-conv-topology:*

**assumes** *separable-space mtopology*

**shows** *LPm.mtopology = weak-conv-topology mtopology*

*<proof>*

**end**

**corollary**

**assumes** *metrizable-space X separable-space X*

**shows** *metrizable-weak-conv-topology:metrizable-space (weak-conv-topology X)*

**and** *separable-weak-conv-topology:separable-space (weak-conv-topology X)*

*<proof>*

**end**

## 5 Prokhorov's Theorem

**theory** *Prokhorov-Theorem*

**imports** *Levy-Prokhorov-Distance*

*Alaoglu-Theorem*

**begin**

### 5.1 Prokhorov's Theorem

**context** *Levy-Prokhorov*

**begin**

**lemma** *relatively-compact-imp-tight-LP:*

**assumes**  $\Gamma \subseteq \mathcal{P}$  *separable-space mtopology mcomplete*

**and** *compactin LPm.mtopology (LPm.mtopology closure-of  $\Gamma$ )*

**shows** *tight-on-set mtopology*  $\Gamma$   
 ⟨*proof*⟩

**lemma** *mcompact-imp-LPmcompact:*

**assumes** *compact-space mtopology*  
**shows** *compactin LPm.mtopology*  $\{N. \text{sets } N = \text{sets (borel-of mtopology)} \wedge N$   
*(space }  $N) \leq \text{ennreal } r\}$   
*(is compactin - ?N)*  
 ⟨*proof*⟩*

**lemma** *tight-imp-relatively-compact-LP:*

**assumes**  $\Gamma \subseteq \{N. \text{sets } N = \text{sets (borel-of mtopology)} \wedge N$  *(space }  $N) \leq \text{ennreal } r\}$   
*separable-space mtopology*  
**and** *tight-on-set mtopology*  $\Gamma$   
**shows** *compactin LPm.mtopology* *(LPm.mtopology closure-of*  $\Gamma)$   
 ⟨*proof*⟩*

**corollary** *Prokhorov-theorem-LP:*

**assumes**  $\Gamma \subseteq \{N. \text{sets } N = \text{sets (borel-of mtopology)} \wedge \text{emeasure } N$  *(space }  $N) \leq \text{ennreal } r\}$   
**and** *separable-space mtopology mcomplete*  
**shows** *compactin LPm.mtopology* *(LPm.mtopology closure-of*  $\Gamma) \longleftrightarrow$  *tight-on-set*  
*mtopology*  $\Gamma$   
 ⟨*proof*⟩*

## 5.2 Completeness of the Lévy-Prokhorov Metric

**lemma** *mcomplete-tight-on-set:*

**assumes**  $\Gamma \subseteq \mathcal{P}$  *mcomplete*  
**and**  $\bigwedge e f. e > 0 \implies f > 0$   
 $\implies \exists \text{an } n. \text{an } \{..n::\text{nat}\} \subseteq M \wedge (\forall N \in \Gamma. \text{measure } N (M - (\bigcup_{i \leq n.}$   
*mball (an }  $i) f)) \leq e)$   
**shows** *tight-on-set mtopology*  $\Gamma$   
 ⟨*proof*⟩*

**lemma** *mcomplete-LPmcomplete:*

**assumes** *mcomplete separable-space mtopology*  
**shows** *LPm.mcomplete*  
 ⟨*proof*⟩

## 5.3 Equivalence of Separability, Completeness, and Compactness

**lemma** *return-inP[simp]:return (borel-of mtopology) x*  $\in \mathcal{P}$   
 ⟨*proof*⟩

**lemma** *LPm-return-eq:*

**assumes**  $x \in M$   $y \in M$   
**shows** *LPm (return (borel-of mtopology) x) (return (borel-of mtopology) y) =*

*min 1 (d x y)*  
*<proof>*

**corollary** *LPm-return-eq-capped-dist:*

**assumes**  $x \in M \ y \in M$   
**shows**  $LPm \ (return \ (borel\text{-of} \ mtopology) \ x)(return \ (borel\text{-of} \ mtopology) \ y) =$   
*capped-dist 1 x y*  
*<proof>*

**corollary** *MCauchy-iff-MCauchy-return:*

**assumes**  $range \ xn \subseteq M$   
**shows**  $MCauchy \ xn \longleftrightarrow LPm.MCauchy \ (\lambda n. \ return \ (borel\text{-of} \ mtopology) \ (xn \ n))$   
*<proof>*

**lemma** *conv-conv-return:*

**assumes** *limitin mtopology xn x sequentially*  
**shows** *limitin LPm.mtopology (λn. return (borel-of mtopology) (xn n)) (return*  
*(borel-of mtopology) x) sequentially*  
*<proof>*

**lemma** *conv-iff-conv-return:*

**assumes**  $range \ xn \subseteq M \ x \in M$   
**shows** *limitin mtopology xn x sequentially*  
 $\longleftrightarrow \ limitin \ LPm.mtopology \ (\lambda n. \ return \ (borel\text{-of} \ mtopology) \ (xn \ n))$   
 $\quad \quad \quad (return \ (borel\text{-of} \ mtopology) \ x) \ sequentially$   
*<proof>*

**lemma** *continuous-map-return: continuous-map mtopology LPm.mtopology (λx.*  
*return (borel-of mtopology) x)*  
*<proof>*

**lemma** *homeomorphic-map-return:*

*homeomorphic-map mtopology*  
 $(subtopology \ LPm.mtopology \ ((\lambda x. \ return \ (borel\text{-of} \ mtopology) \ x) \ 'M))$   
 $(\lambda x. \ return \ (borel\text{-of} \ mtopology) \ x)$   
*<proof>*

**corollary** *homeomorphic-space-mtopology-return:*

*mtopology homeomorphic-space (subtopology LPm.mtopology ((λx. return (borel-of*  
*mtopology) x) ' M))*  
*<proof>*

**lemma** *closedin-returnM: closedin LPm.mtopology ((λx. return (borel-of mtopol-*  
*ogy) x) ' M)*  
*<proof>*

**corollary** *separable-iff-LPm-separable: separable-space mtopology  $\longleftrightarrow$  separable-space*  
*LPm.mtopology*

*<proof>*

**corollary** *LPmcomplete-mcomplete:*

**assumes** *LPm.mcomplete*

**shows** *mcomplete*

*<proof>*

**corollary** *mcomplete-iff-LPmcomplete: separable-space mtopology  $\implies$  mcomplete*

*$\longleftarrow$  LPm.mcomplete*

*<proof>*

**lemma** *LPmcompact-imp-mcompact: compact-space LPm.mtopology  $\implies$  compact-space mtopology*

*<proof>*

**end**

**corollary** *Polish-space-weak-conv-topology:*

**assumes** *Polish-space X*

**shows** *Polish-space (weak-conv-topology X)*

*<proof>*

## 5.4 Prokhorov Theorem for Topology of Weak Convergence

**lemma** *relatively-compact-imp-tight:*

**assumes** *Polish-space X  $\Gamma \subseteq \{N. \text{sets } N = \text{sets (borel-of } X) \wedge \text{finite-measure } N\}$*

**and** *compactin (weak-conv-topology X) (weak-conv-topology X closure-of  $\Gamma$ )*

**shows** *tight-on-set X  $\Gamma$*

*<proof>*

**lemma** *tight-imp-relatively-compact:*

**assumes** *metrizable-space X separable-space X*

*$\Gamma \subseteq \{N. N (\text{space } N) \leq \text{ennreal } r \wedge \text{sets } N = \text{sets (borel-of } X)\}$*

**and** *tight-on-set X  $\Gamma$*

**shows** *compactin (weak-conv-topology X) (weak-conv-topology X closure-of  $\Gamma$ )*

*<proof>*

**lemma** *Prokhorov:*

**assumes** *Polish-space X  $\Gamma \subseteq \{N. N (\text{space } N) \leq \text{ennreal } r \wedge \text{sets } N = \text{sets (borel-of } X)\}$*

**shows** *tight-on-set X  $\Gamma \longleftrightarrow$  compactin (weak-conv-topology X) (weak-conv-topology X closure-of  $\Gamma$ )*

*<proof>*

**corollary** *tight-on-set-imp-convergent-subsequence:*

**fixes** *Ni :: nat  $\Rightarrow$  - measure*

**assumes** *metrizable-space X separable-space X*

**and** *tight-on-set X (range Ni)  $\bigwedge i. (Ni i) (\text{space } (Ni i)) \leq \text{ennreal } r$*

**shows**  $\exists a N. \text{strict-mono } a \wedge \text{finite-measure } N \wedge \text{sets } N = \text{sets (borel-of } X)$   
 $\wedge N (\text{space } N) \leq \text{ennreal } r \wedge \text{weak-conv-on } (Ni \circ a) N \text{ sequentially } X$   
 ⟨proof⟩

**end**

**theory** *Space-of-Finite-Measures*  
**imports** *Prokhorov-Theorem*  
**begin**

## 6 Measurable Space of Finite Measures

### 6.1 Measurable Space of Finite Measures

We define the measurable space of all finite measures in the same way as *subprob-algebra*.

**definition** *finite-measure-algebra* :: 'a measure  $\Rightarrow$  'a measure measure **where**  
*finite-measure-algebra*  $K =$   
 $(\text{SUP } A \in \text{sets } K. \text{vimage-algebra } \{M. \text{finite-measure } M \wedge \text{sets } M = \text{sets } K\}$   
 $(\lambda M. \text{emeasure } M A) \text{ borel})$

**lemma** *space-finite-measure-algebra*:  
 $\text{space (finite-measure-algebra } A) = \{M. \text{finite-measure } M \wedge \text{sets } M = \text{sets } A\}$   
 ⟨proof⟩

**lemma** *finite-measure-algebra-cong*:  $\text{sets } M = \text{sets } N \implies \text{finite-measure-algebra } M = \text{finite-measure-algebra } N$   
 ⟨proof⟩

**lemma** *measurable-emeasure-finite-measure-algebra[measurable]*:  
 $a \in \text{sets } A \implies (\lambda M. \text{emeasure } M a) \in \text{borel-measurable (finite-measure-algebra } A)$   
 ⟨proof⟩

**lemma** *measurable-measure-finite-measure-algebra[measurable]*:  
 $a \in \text{sets } A \implies (\lambda M. \text{measure } M a) \in \text{borel-measurable (finite-measure-algebra } A)$   
 ⟨proof⟩

**lemma** *finite-measure-measurableD*:  
**assumes**  $N: N \in \text{measurable } M (\text{finite-measure-algebra } S)$  **and**  $x: x \in \text{space } M$   
**shows**  $\text{space } (N x) = \text{space } S$   
**and**  $\text{sets } (N x) = \text{sets } S$   
**and**  $\text{measurable } (N x) K = \text{measurable } S K$   
**and**  $\text{measurable } K (N x) = \text{measurable } K S$   
 ⟨proof⟩

⟨ML⟩

**context**

**fixes**  $K M N$  **assumes**  $K$ :  $K \in \text{measurable } M \text{ (finite-measure-algebra } N)$   
**begin**

**lemma** *finite-measure-space-kernel*:  $a \in \text{space } M \implies \text{finite-measure } (K a)$   
*<proof>*

**lemma** *sets-finite-kernel*:  $a \in \text{space } M \implies \text{sets } (K a) = \text{sets } N$   
*<proof>*

**lemma** *measurable-emeasure-finite-kernel*[*measurable*]:  
 $A \in \text{sets } N \implies (\lambda a. \text{emeasure } (K a) A) \in \text{borel-measurable } M$   
*<proof>*

**end**

**lemma** *measurable-finite-measure-algebra*:  
 $(\bigwedge a. a \in \text{space } M \implies \text{finite-measure } (K a)) \implies$   
 $(\bigwedge a. a \in \text{space } M \implies \text{sets } (K a) = \text{sets } N) \implies$   
 $(\bigwedge A. A \in \text{sets } N \implies (\lambda a. \text{emeasure } (K a) A) \in \text{borel-measurable } M) \implies$   
 $K \in \text{measurable } M \text{ (finite-measure-algebra } N)$   
*<proof>*

**lemma** *measurable-finite-markov*:  
 $K \in \text{measurable } M \text{ (finite-measure-algebra } M) \longleftrightarrow$   
 $(\forall x \in \text{space } M. \text{finite-measure } (K x) \wedge \text{sets } (K x) = \text{sets } M) \wedge$   
 $(\forall A \in \text{sets } M. (\lambda x. \text{emeasure } (K x) A) \in \text{measurable } M \text{ borel})$   
*<proof>*

**lemma** *measurable-finite-measure-algebra-generated*:  
**assumes** *eq*:  $\text{sets } N = \text{sigma-sets } \Omega G$  **and** *Int-stable*  $G G \subseteq \text{Pow } \Omega$   
**assumes** *subsp*:  $\bigwedge a. a \in \text{space } M \implies \text{finite-measure } (K a)$   
**assumes** *sets*:  $\bigwedge a. a \in \text{space } M \implies \text{sets } (K a) = \text{sets } N$   
**assumes**  $\bigwedge A. A \in G \implies (\lambda a. \text{emeasure } (K a) A) \in \text{borel-measurable } M$   
**assumes**  $\Omega$ :  $(\lambda a. \text{emeasure } (K a) \Omega) \in \text{borel-measurable } M$   
**shows**  $K \in \text{measurable } M \text{ (finite-measure-algebra } N)$   
*<proof>*

**lemma** *space-finite-measure-algebra-empty*:  $\text{space } N = \{\} \implies \text{space (finite-measure-algebra } N) = \{\text{null-measure } N\}$   
*<proof>*

**lemma** *sets-subprob-algebra-restrict*:  
 $\text{sets (subprob-algebra } M) = \text{sets (restrict-space (finite-measure-algebra } M) \{N. \text{subprob-space } N\})$   
**(is**  $\text{sets } ?L = \text{sets } ?R)$   
*<proof>*

## 6.2 Equivalence between Spaces of Finite Measures

Corollary 17.21 [2].

**lemma**(in *Levy-Prokhorov*) *openin-lower-semicontinuous*:  
**assumes** *openin mtopology U*  
**shows** *lower-semicontinuous-map LPm.mtopology ( $\lambda N$ . measure N U)*  
 ⟨*proof*⟩

**lemma**(in *Levy-Prokhorov*) *closedin-upper-semicontinuous*:  
**assumes** *closedin mtopology A*  
**shows** *upper-semicontinuous-map LPm.mtopology ( $\lambda N$ . measure N A)*  
 ⟨*proof*⟩

**context** *Levy-Prokhorov*  
**begin**

We show that the measurable space generated from *LPm.mtopology* is equal to *finite-measure-algebra (borel-of LPm.mtopology)*.

**lemma** *sets-LPm1*: *sets (finite-measure-algebra (borel-of mtopology))*  
 $\subseteq$  *sets (borel-of LPm.mtopology)* (**is** *sets ?Giry*  $\subseteq$  *sets ?Levy*)  
 ⟨*proof*⟩

**lemma** *sets-LPm2*:  
**assumes** *mcomplete separable-space mtopology*  
**shows** *sets (borel-of LPm.mtopology)  $\subseteq$  sets (finite-measure-algebra (borel-of mtopology))*  
 (**is** *sets ?Levy*  $\subseteq$  *sets ?Giry*)  
 ⟨*proof*⟩

**corollary** *sets-LPm-eq-sets-finite-measure-algebra*:  
**assumes** *mcomplete separable-space mtopology*  
**shows** *sets (borel-of LPm.mtopology) = sets (finite-measure-algebra (borel-of mtopology))*  
 ⟨*proof*⟩

**end**

**corollary** *weak-conv-topology-eq-finite-measure-algebra*:  
**assumes** *Polish-space X*  
**shows** *sets (borel-of (weak-conv-topology X)) = sets (finite-measure-algebra (borel-of X))*  
 ⟨*proof*⟩

**corollary** *weak-conv-topology-eq-subprob-algebra*:  
**assumes** *Polish-space X*  
**shows** *sets (borel-of (subtopology (weak-conv-topology X) {N. subprob-space N  $\wedge$  sets N = sets (borel-of X)}))*  
 $=$  *sets (subprob-algebra (borel-of X))* (**is** *?lhs = ?rhs*)  
 ⟨*proof*⟩

**corollary** *weak-conv-topology-eq-prob-algebra:*

**assumes** *Polish-space X*

**shows** *sets (borel-of (subtopology (weak-conv-topology X) {N. prob-space N ∧ sets N = sets (borel-of X)}))*

*= sets (prob-algebra (borel-of X)) (is ?lhs = ?rhs)*

*<proof>*

### 6.3 Standardness

**lemma** *closedin-weak-conv-topology-r:*

*closedin (weak-conv-topology X) {N. sets N = sets (borel-of X) ∧ N (space N) ≤ ennreal r}*

*<proof>*

**lemma** *closedin-weak-conv-topology-subprob:*

*closedin (weak-conv-topology X) {N. subprob-space N ∧ sets N = sets (borel-of X)}*

*<proof>*

**lemma** *closedin-weak-conv-topology-prob:*

*closedin (weak-conv-topology X) {N. prob-space N ∧ sets N = sets (borel-of X)}*

*<proof>*

**corollary**

**assumes** *standard-borel M*

**shows** *standard-borel-finite-measure-algebra: standard-borel (finite-measure-algebra M)*

**and** *standard-borel-ne-finite-measure-algebra: standard-borel-ne (finite-measure-algebra M)*

**and** *standard-borel-subprob-algebra: standard-borel (subprob-algebra M)*

**and** *standard-borel-prob-algebra: standard-borel (prob-algebra M)*

*<proof>*

**corollary**

**assumes** *standard-borel-ne M*

**shows** *standard-borel-ne-subprob-algebra: standard-borel-ne (subprob-algebra M)*

**and** *standard-borel-ne-prob-algebra: standard-borel-ne (prob-algebra M)*

*<proof>*

**end**

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