Laurent Series Expansions on an Annulus

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Abstract

This entry shows an important basic fact in complex analysis, namely that a complex-valued function f holomorphic on an open annulus $\{z \mid 0 \leq r < \|z - z_0\| \leq R\}$ has a (generalised) Laurent series expansion of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

that is valid on the entire annulus.

An important special case that is also developed is $r = 0$, i.e. the local behaviour of a function around an isolated singularity. For non-essential singularities, this reduces to the well-known simpler Laurent series expansions from `HOL-Complex_Analysis` of the form

$$\sum_{n=n_0}^{\infty} a_n (z - z_0)^n.$$

Contents

1 Laurent series expansions on an annulus

theory *Laurent_Annulus*
  imports *"HOL-Complex_Analysis.Complex_Analysis" "Path_Automation.Path_Automation"*
begin

1.1 Auxiliary material

lemma *contour_integral_affine:*
  assumes *"valid_path γ" "c $\neq$ 0"*
  shows *"contour_integral ((λx. c * x + b) $\circ$ γ) f = contour_integral γ (λw. c * f (c * w + b))"*
$\langle proof \rangle$

lemma *divide_pos_0_ereal [simp]: "x > 0 $\Longrightarrow$ x / (0 :: ereal) = ∞"*
  $\langle proof \rangle$

lemma *divide_pos_0_ereal' [simp]: "x > 0 $\Longrightarrow$ x / (ereal 0) = ∞"*
  $\langle proof \rangle$

lemma *analytic_on_empty: "f analytic_on {}"*
  $\langle proof \rangle$

lemma *power_series_analytic':*
  assumes *"conv_radius a $\geq$ r"*
  shows *"(λz. ($\sum$n. a n * (z - w) ^ n)) analytic_on eball w r"*
$\langle proof \rangle$

lemma *winding_number_circlepath':*
  assumes *"r $\geq$ 0" "dist w z $\neq$ r"*
  shows *"winding_number (circlepath z r) w = indicator (ball z r) w"*
$\langle proof \rangle$

lemma *contour_integral_rmul: "contour_integral g (λx. f x * c) = contour_integral g f * c"*
$\langle proof \rangle$

lemma *contour_integral_lmul: "contour_integral g (λx. c * f x) = c * contour_integral g f"*
  $\langle proof \rangle$

lemma *contour_integral_divide: "contour_integral g (λx. f x / c) = contour_integral g f / c"*
  $\langle proof \rangle$

lemma *uniform_limit_compose':*
  assumes *"uniform_limit A f g F"* **and** *"h ‘ B $\subseteq$ A"*
  shows *"uniform_limit B (λn x. f n (h x)) (λx. g (h x)) F"*
  $\langle proof \rangle$

lemma *uniform_limit_contour_integral_circlepath:*
  assumes *u: "uniform_limit (path_image (circlepath z r)) f g F"*
  assumes *c: "⋀n. continuous_on (path_image (circlepath z r)) (f n)"*
  assumes *[simp]: "F ≠ bot"*
  obtains *I J* **where**
    "⋀n. (f n has_contour_integral I n) (circlepath z r)"
    "(g has_contour_integral J) (circlepath z r)"
    "(I ⟶ J) F"
⟨*proof*⟩

lemma *contour_integral_sums_circlepath:*
  assumes *u: "uniform_limit (path_image (circlepath z r)) (λN w. ∑n<N. f n w) g sequentially"*
  assumes *c: "⋀n. continuous_on (path_image (circlepath z r)) (f n)"*
  obtains *J* **where**
    "(g has_contour_integral J) (circlepath z r)"
    "(λn. contour_integral (circlepath z r) (f n)) sums J"
  ⟨*proof*⟩

lemma *conv_radius_geI_ex'':*
  fixes *f :: "nat ⇒ 'a :: {banach, real_normed_div_algebra}"*
  assumes *"⋀r. c < r ⟹ ereal r < R ⟹ summable (λn. f n * of_real r ^ n)"*
  assumes *"ereal c < R"*
  shows *"conv_radius f ≥ R"*
⟨*proof*⟩

lemma *eq_loops_imp_contour_integral_eq:*
  assumes *"eq_loops p q" "valid_path p" "valid_path q"*
  assumes *"f analytic_on (path_image p ∩ path_image q)"*
  shows *"contour_integral p f = contour_integral q f"*
⟨*proof*⟩

1.2 Annuli

An open annulus is the set inbetween a small inner circle and a larger outer circle with the same centre. The circles themselve are not included.

We allow the inner circle to have radius 0 and the outer circle to have radius ∞. If the inner radius has radius 0 then we obtain a pointed ball, and if the outer circle has radius ∞ then we simply obtain the entire space minus a ball.

definition *annulus :: "'a::metric_space ⇒ real ⇒ ereal ⇒ 'a set"*
  where *"annulus x r R = eball x R - cball x r"*

lemma *annulus_0_conv_punctured_disc [simp]: "annulus x 0 R = eball x R - {x}"*
  ⟨*proof*⟩

lemma *open_annulus [intro]: "open (annulus x r R)"*
  ⟨*proof*⟩

lemma *sphere_subset_annulus: "r < r' ⟹ ereal r' < R ⟹ sphere z r' ⊆ annulus z r R"*
  ⟨*proof*⟩

An important obvious lemma: any two closed circles within an annulus are homotopic.

lemma *homotopic_loops_circlepath_annulus:*
  fixes *R :: ereal*
  assumes *"r < r1" "r1 < R" "r < r2" "r2 < R" "0 ≤ r"*
  shows *"homotopic_loops (annulus z r R) (circlepath z r1) (circlepath z r2)"*
  ⟨*proof*⟩

1.3 Basic properties of Laurent coefficients

We will show that any function that is holomorphic on some annulus has a Laurent series expansion that converges within that annulus. There is an explicit formula for the coefficients of this series by using the Cauchy integral formula on a circular path within the annulus (whose centre is the centre of the annulus). We use this to define the coefficients.

Generally speaking, the radius of this circle matters: if we have a function that is holomorphic within a ball around a point except at some isolated singularities, then the function is holomorphic on any annulus between two "adjacent" singularities (i.e. two singularities for which there is no other singularity in the annulus between them). Therefore, the function also has a Laurent series expansion in each annulus. Due to homotopy, the radius we pick does not change the series expansion as long as we stay within the same annulus; however, in two different annuli we will also have different series expansions. The radius parameter therefore allows us to "pick" the annulus we want.

definition *laurent_coeff :: "(complex ⇒ complex) ⇒ complex ⇒ real ⇒ int ⇒ complex"* **where**
  *"laurent_coeff f z0 r n = 1 / (2 * pi * i) *
    contour_integral (circlepath z0 r) (λw. f w * (w - z0) powi (-n-1))"*

The Laurent series coefficients thus defined clearly commute with translation and rotation.

lemma *laurent_coeff_translate:*
  "laurent_coeff (λw. f (z + w)) z' = laurent_coeff f (z + z')"

⟨proof⟩

lemma *laurent_coeff_translate_to_0:*
  "NO_MATCH 0 z ⟹ laurent_coeff f z = laurent_coeff (λw. f (z + w)) 0"
  ⟨proof⟩

lemma *laurent_coeff_rotate:*
  assumes *"norm c = 1" "f analytic_on sphere 0 r" "r ≥ 0"*
  shows *"laurent_coeff (λz. f (c * z)) 0 r n = laurent_coeff f 0 r n * c powi n"*
⟨proof⟩

Any two radii within the same annulus (in which the function has no singularities) gives the same Laurent series expansion:

lemma *laurent_coeff_radius_transfer:*
  assumes *"f holomorphic_on annulus z0 r R" "0 ≤ r" "r < r1" "r1 < R" "r < r2" "r2 < R"*
  shows *"laurent_coeff f z0 r1 n = laurent_coeff f z0 r2 n"*
  ⟨proof⟩

If the function is holomorphic not only on an annulus but on a ball, the negative-index Laurent coefficients are all zero (since the function is then even analytic and therefore has a power series expansion).

lemma *laurent_coeff_neg_eq_0:*
  assumes *"f holomorphic_on ball z0 R" "r ≥ 0" "R > r" "n < 0"*
  shows *"laurent_coeff f z0 r n = 0"*
⟨proof⟩

1.4 Radius of convergence of the principal and non-principal parts

We show that if the annulus has inner radius r and outer radius R then the non-principal part of the Laurent series has radius of convergence at least R and the principal one (in terms of x^{-1}) has radius of convergence at least r^{-1}.

lemma *laurent_coeff_powser1:*
  fixes *R :: ereal*
  assumes *"f holomorphic_on annulus 0 r R" "z ∈ annulus 0 r R"*
  assumes *"r ≥ 0" "r < r0" "ereal r0 < R" "norm z < r0"*
  shows *"(λn. laurent_coeff f 0 r0 n * z ^ n) sums (contour_integral (circlepath 0 r0) (λw. f w / (w - z)) / (2 * pi * i))"*
⟨proof⟩

lemma *laurent_coeff_powser2:*
  assumes *"f holomorphic_on annulus 0 r R" "z ∈ annulus 0 r R"*

```
  assumes "r ≥ 0" "r < r0" "ereal r0 < R" "norm z > r0"
  shows   "(λn. laurent_coeff f 0 r0 (-int (Suc n)) / z ^ (Suc n)) sums
              (-contour_integral (circlepath 0 r0) (λw. f w / (w - z))
/ (2 * pi * i))"
⟨proof⟩

lemma conv_radius_laurent_coeff1:
  assumes "f holomorphic_on annulus z r R" "r ≥ 0" "r < r0" "ereal r0
< R"
  shows   "conv_radius (λn. laurent_coeff f z r0 (int n)) ≥ R"
⟨proof⟩

lemma conv_radius_laurent_coeff2:
  assumes "f holomorphic_on annulus z r R" "r ≥ 0" "r0 ∈ {r<..<R}"
  shows   "conv_radius (λn. laurent_coeff f z r0 (-int n)) ≥ 1 / ereal
r"
⟨proof⟩
```

1.5 Existence of the Laurent expansion in an annulus

We are now ready to show the main result: If a function is holomorphic on an annulus $\text{ann}(z_0; r, R) = \{z \mid 0 \leq r < \|z - z_0\| \leq R\}$, it has a Laurent series expansion around the centre z_0 of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

that is valid on the entire annulus.

Note however that this is not Laurent series in the classical sense since the sum may include infinitely many terms with a negative exponent (i.e. it can have subdegree $-\infty$).

If $r = 0$, i.e. if we the annulus is actually a punctured disc, the function has an isolated singularity at z_0. If that singularity is an essential one, the subdegree of the Laurent series will be $-\infty$. If, on the other hand, the singularity is removable or a pole, the subdegree will be $> -\infty$.

```
lemma laurent_expansion_annulus_aux:
  assumes "f holomorphic_on annulus 0 r R" "0 ≤ r" "r < r0" "ereal r0
< R"
  shows   "⋀z. z ∈ annulus 0 r R - ℝ ⟹
                  ((λn. laurent_coeff f 0 r0 n * z powi n) has_sum f z)
(UNIV :: int set)"
⟨proof⟩

theorem laurent_expansion_annulus:
  assumes "f holomorphic_on annulus z0 r R" "0 ≤ r" "r < r0" "r0 < R"
  shows   "⋀z. z ∈ annulus z0 r R ⟹
```

```
            ((λn. laurent_coeff f z0 r0 n * (z - z0) powi n) has_sum
f z) (UNIV :: int set)"
```
⟨*proof*⟩

1.6 Laurent expansion around an isolated singularity

We now move to the special case of $r = 0$, i.e. a function f with an isolated singularity z where we are interested in the behaviour of f in a pointed neighbourhood of z.

The following definition gives us the Laurent series coefficients of f at z. These are obtained by using the above definition *laurent_coeff*, which gives us the coefficient at a given radius of integration, and letting that radius tend to 0.

```
definition laurent_coeff' :: "(complex ⇒ complex) ⇒ complex ⇒ int ⇒
complex" where
  "laurent_coeff' f z n = Lim (at_right 0) (λr. laurent_coeff f z r n)"
```

Any two functions that are locally the same around z have the same Laurent series coefficients at z.

```
lemma laurent_coeff'_cong:
  assumes "eventually (λw. f w = g w) (at z)" "z = z'" "n = n'"
  shows   "laurent_coeff' f z n = laurent_coeff' g z' n'"
```
⟨*proof*⟩

By homotopy, any *laurent_coeff* for any radius $r > 0$ gives the same result as *laurent_coeff'* as long there is a ball with radius $R > r$ in which the function is holomorphic.

```
lemma laurent_coeff'_conv_laurent_coeff:
  assumes "f holomorphic_on eball z R - {z}" "0 < r" "ereal r < R"
  shows   "laurent_coeff' f z n = laurent_coeff f z r n"
```
⟨*proof*⟩

If the function does not have a singularity at z (i.e. it is even analytic at z) then the negative-index Laurent series coefficients are all 0.

```
lemma laurent_coeff'_neg_eq_0:
  assumes "f analytic_on {z}" "n < 0"
  shows   "laurent_coeff' f z n = 0"
```
⟨*proof*⟩

```
context
  fixes f :: "complex ⇒ complex" and z :: complex
  assumes sing: "isolated_singularity_at f z"
begin
```

The index -1 coefficient is simply the residue.

lemma *laurent_coeff'_conv_residue: "laurent_coeff' f z (-1) = residue f z"*
⟨*proof*⟩

The non-principal part of the Laurent series expansion has positive radius of convergence. More precisely, the radius will be the distance to the nearest singularity of f, but we do not show this here.

```
lemma conv_radius_laurent_coeff'1:
  "conv_radius (λn. laurent_coeff' f z (int n)) > 0"
⟨proof⟩
```

The principal part converges everywhere.

```
lemma conv_radius_laurent_coeff'2:
  "conv_radius (λn. laurent_coeff' f z (-int n)) = ∞"
⟨proof⟩
```

end

We explicitly make a link between the Laurent series coefficients we just defined and the existing notions of Laurent expansions. Note that the previous notions only work if the isolated singularity at z is non-essential (i.e. removable or a pole), since the formal Laurent expansions used there are only allowed to have a finite number of negative-index terms.

```
lemma has_fps_expansion_laurent_coeff':
  assumes "(λw. f (z + w)) has_fps_expansion F"
  shows   "laurent_coeff' f z m = (if m ≥ 0 then fps_nth F (nat m) else
0)"
⟨proof⟩

lemma has_laurent_expansion_laurent_coeff':
  assumes "(λw. f (z + w)) has_laurent_expansion F"
  shows   "laurent_coeff' f z n = fls_nth F n"
⟨proof⟩

lemma laurent_coeff'_conv_laurent_expansion:
  assumes "f meromorphic_on {z}"
  shows   "laurent_coeff' f z n = fls_nth (laurent_expansion f z) n"
  ⟨proof⟩
```

1.7 Decomposition into a principal and non-principal part

One particular consequence of the above is that a function f holomorphic on an annulus $\text{ann}(0; r, R)$ can be decomposed into a sum $f(w) = g(w)+h(1/w)$, where g is holomorphic on the disc $B(0; R)$ and h is holomorphic on the disc $B(0; 1/r)$ and $h(0) = 0$. The functions g and h correspond to the non-principal and principal part of the Laurent series expansion around 0.

definition *nonprincipal_part :: "(complex ⇒ complex) ⇒ complex ⇒ real ⇒ complex ⇒ complex"* **where**
*"nonprincipal_part f z r w = (∑n. laurent_coeff f z r (int n) * (w - z) ^ n)"*

definition *principal_part :: "(complex ⇒ complex) ⇒ complex ⇒ real ⇒ complex ⇒ complex"* **where**
*"principal_part f z r w = (∑n. laurent_coeff f z r (-int (Suc n)) * w ^ Suc n)"*

definition *principal_part' :: "(complex ⇒ complex) ⇒ complex ⇒ complex ⇒ complex"* **where**
*"principal_part' f z w = (∑n. laurent_coeff' f z (-int (Suc n)) * w ^ Suc n)"*

definition *nonprincipal_part' :: "(complex ⇒ complex) ⇒ complex ⇒ complex ⇒ complex"* **where**
*"nonprincipal_part' f z w = (∑n. laurent_coeff' f z (int n) * (w - z) ^ n)"*

lemma *principal_part_0 [simp]: "principal_part f z r 0 = 0"*
⟨*proof*⟩

locale *holomorphic_on_annulus =*
fixes *f :: "complex ⇒ complex"* **and** *z :: complex* **and** *r r0 :: real* **and** *R :: ereal*
assumes *holo: "f holomorphic_on annulus z r R"* **and** *r: "0 ≤ r" "r < r0" "ereal r0 < R"*
begin

lemma *analytic_nonprincipal_part_aux:*
"nonprincipal_part f z r0 analytic_on eball z R"
⟨*proof*⟩

lemma *analytic_nonprincipal_part [analytic_intros]:*
assumes *"g analytic_on A" "⋀x. x ∈ A ⟹ g x ∈ eball z R"*
shows *"(λx. nonprincipal_part f z r0 (g x)) analytic_on A"*
⟨*proof*⟩

lemma *holomorphic_nonprincipal_part [holomorphic_intros]:*
assumes *"g holomorphic_on A" "⋀x. x ∈ A ⟹ g x ∈ eball z R"*
shows *"(λx. nonprincipal_part f z r0 (g x)) holomorphic_on A"*
⟨*proof*⟩

lemma *analytic_principal_part_aux:*
"principal_part f z r0 analytic_on eball 0 (1 / ereal r)"
⟨*proof*⟩

theorem *analytic_principal_part [analytic_intros]:*
  assumes *"g analytic_on A" "⋀x. x ∈ A ⟹ g x ∈ eball 0 (1 / ereal r)"*
  shows *"(λx. principal_part f z r0 (g x)) analytic_on A"*
  ⟨*proof*⟩

lemma *holomorphic_principal_part [holomorphic_intros]:*
  assumes *"g holomorphic_on A" "⋀x. x ∈ A ⟹ g x ∈ eball 0 (1 / ereal r)"*
  shows *"(λx. principal_part f z r0 (g x)) holomorphic_on A"*
  ⟨*proof*⟩

theorem *nonprincipal_part_principal_part:*
  assumes *"w ∈ annulus z r R"*
  shows *"f w = nonprincipal_part f z r0 w + principal_part f z r0 (1 / (w - z))"*
⟨*proof*⟩

end

context
  fixes *f :: "complex ⇒ complex"* **and** *z :: complex*
  assumes *sing: "isolated_singularity_at f z"*
begin

lemma *analytic_nonprincipal_part': "nonprincipal_part' f z analytic_on {z}"*
  and *analytic_principal_part'_aux: "principal_part' f z analytic_on A"*
  and *nonprincipal_part'_principal_part':*
    "eventually (λw. f w = nonprincipal_part' f z w + principal_part' f z (1 / (w - z))) (at z)"
⟨*proof*⟩

lemma *analytic_principal_part' [analytic_intros]:*
  assumes *"g analytic_on A"*
  shows *"(λx. principal_part' f z (g x)) analytic_on A"*
  ⟨*proof*⟩

lemma *holomorphic_principal_part' [holomorphic_intros]:*
  assumes *"g holomorphic_on A"*
  shows *"(λx. principal_part' f z (g x)) holomorphic_on A"*
  ⟨*proof*⟩

lemma *removable_singularity_principal_part':*
  "(λw. f w - principal_part' f z (1 / (w - z))) —z→ nonprincipal_part' f z z"
⟨*proof*⟩

end

end