

# Congruences of Bernoulli Numbers

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## Abstract

This entry provides proofs for two important congruences involving Bernoulli numbers. The proofs follow Cohen's textbook *Number Theory Volume II: Analytic and Modern Tools* [1]. In the following we write  $\mathcal{B}_k = N_k/D_k$  for the  $k$ -th Bernoulli number (with  $\gcd(N_k, D_k) = 1$ ).

The first result that I showed is *Voronoi's congruence*, which states that for any even integer  $k \geq 2$  and all positive coprime integers  $a, n$  we have:

$$(a^k - 1)N_k \equiv ka^{k-1}D_k \sum_{m=1}^{n-1} m^{k-1} \left\lfloor \frac{ma}{n} \right\rfloor \pmod{n}$$

Building upon this, I then derive *Kummer's congruence*. In its common form, it states that for a prime  $p$  and even integers  $k, k'$  with  $\min(k, k') \geq e + 1$  and  $(p - 1) \nmid k$  and  $k \equiv k' \pmod{\varphi(p^e)}$ , we have:

$$\frac{\mathcal{B}_k}{k} \equiv \frac{\mathcal{B}_{k'}}{k'} \pmod{p^e}$$

The version proved in my entry is slightly more general than this.

One application of these congruences is to prove that there are infinitely many irregular primes, which I formalised as well.

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# 1 Preliminary facts

```
theory Kummer_Library
imports
  "HOL-Number_Theory.Number_Theory"
  "Bernoulli.Bernoulli_Zeta"
begin
```

## 1.1 Miscellaneous facts

```
lemma fact_ge_monomial:
  fixes k :: "'a :: {linordered_semiodom, semiring_char_0}"
  assumes "n ≥ n0" "fact n0 ≥ c * k ^ n0" "of_nat n0 ≥ k" "k ≥ 0"
  shows "fact n ≥ c * k ^ n"
  ⟨proof⟩
```

```
lemma fact_ge_2pi_power:
  assumes "n ≥ 23"
  shows "fact n ≥ (2 * pi) ^ n * n"
  ⟨proof⟩
```

```
lemma Rats_power_int: "x ∈ ℚ ⇒ x powi n ∈ ℚ"
  ⟨proof⟩
```

```
lemma coprimeI_via_bezout:
  fixes x y :: "'a :: algebraic_semiodom"
  assumes "a * x + b * y = 1"
  shows "coprime x y"
  ⟨proof⟩
```

```
lemma quotient_of_eqI:
  assumes "coprime a b" "b > 0" "x = of_int a / of_int b"
  shows "quotient_of x = (a, b)"
  ⟨proof⟩
```

```
lemma quotient_of_of_nat [simp]: "quotient_of (of_nat n) = (int n, 1)"
  ⟨proof⟩
```

```
lemma quotient_of_of_int [simp]: "quotient_of (of_int n) = (n, 1)"
  ⟨proof⟩
```

```
lemma quotient_of_fraction_conv_normalize:
  "quotient_of (of_int a / of_int b) = Rat.normalize (a, b)"
  ⟨proof⟩
```

```
lemma dvd_imp_div_dvd: "(b :: 'a :: algebraic_semiodom) dvd a ⇒ a div
b dvd a"
  ⟨proof⟩
```

```
lemma dvd_rat_normalize:
```

```

    assumes "b ≠ 0"
    shows "fst (Rat.normalize (a, b)) dvd a" "snd (Rat.normalize (a, b))
dvd b"
  <proof>

```

```

lemma of_int_div: "b dvd a  $\implies$  of_int (a div b) = of_int a / (of_int
b :: 'a :: field_char_0)"
  <proof>

```

```

lemma coprime_lcm_left:
  fixes a b c :: "'a :: semiring_gcd"
  shows "coprime a c  $\implies$  coprime b c  $\implies$  coprime (lcm a b) c"
  <proof>

```

```

lemma coprime_Lcm_left:
  fixes x y :: "'a :: semiring_Gcd"
  assumes "finite A" " $\bigwedge x. x \in A \implies$  coprime x y"
  shows "coprime (Lcm A) y"
  <proof>

```

```

lemma coprimeI_by_prime_factors:
  fixes x y :: "'a :: factorial_semiring"
  assumes " $\bigwedge p. p \in$  prime_factors x  $\implies$   $\neg p$  dvd y"
  assumes "x ≠ 0"
  shows "coprime x y"
  <proof>

```

```

lemma multiplicity_int: "multiplicity (int p) (int n) = multiplicity
p n"
  <proof>

```

```

lemma squarefree_int_iff [simp]: "squarefree (int n)  $\longleftrightarrow$  squarefree
n"
  <proof>

```

```

lemma squarefree_imp_multiplicity_prime_le_1:
  "squarefree n  $\implies$  n ≠ 0  $\implies$  prime p  $\implies$  multiplicity p n  $\leq$  1"
  <proof>

```

```

lemma residue_primroot_is_generator':
  assumes "m > 1" and "residue_primroot m g"
  shows "bij_betw ( $\lambda i. g ^ i \bmod m$ ) {1..totient m} (totatives m)"
  <proof>

```

## 1.2 Facts about congruence

```

lemma cong_modulus_mono:
  assumes "[a = b] (mod m)" "m' dvd m"
  shows "[a = b] (mod m')"

```

```

    <proof>

lemma cong_pow_totient:
  fixes x x' n k k' :: nat
  assumes "[x = x'] (mod n)" "[k = k'] (mod totient n)" "coprime x n"
  shows "[x ^ k = x' ^ k'] (mod n)"
  <proof>

lemma cong_modulus_power:
  assumes "[a = b] (mod (n ^ k))" "k > 0"
  shows "[a = b] (mod n)"
  <proof>

lemma cong_mult_cancel:
  assumes "[n * a = n * b] (mod (n * m))" "n ≠ 0"
  shows "[a = b] (mod m)"
  <proof>

lemma cong_mult_square:
  assumes "[a = 0] (mod n)" "[b = b'] (mod n)"
  shows "[a * b = a * b'] (mod (n^2))"
  <proof>

lemma sum_reindex_bij_betw_cong:
  assumes "∧a. a ∈ S ⇒ i (j a) = a"
  assumes "∧a. a ∈ S ⇒ j a ∈ T"
  assumes "∧b. b ∈ T ⇒ j (i b) = b"
  assumes "∧b. b ∈ T ⇒ i b ∈ S"
  assumes "∧a. a ∈ S ⇒ [h (j a) = g a] (mod m)"
  shows "[sum g S = sum h T] (mod m)"
  <proof>

lemma power_mult_cong:
  fixes a b :: "'a :: unique_euclidean_ring"
  assumes "[a = b] (mod n^k)" and "k' ≤ k + 1"
  shows "[n^1 * a = n^1 * b] (mod n^k)"
  <proof>

lemma residue_primroot_power_cong_neg1:
  fixes x :: nat and p :: nat
  assumes "prime p" "p ≠ 2" "residue_primroot p x"
  shows "[int x ^ ((p - 1) div 2) = -1] (mod p)"
  <proof>

lemma cong_mod_left: "[a = b] (mod p) ⇒ [a mod p = b] (mod p)"
  <proof>

lemma cong_mod_right: "[a = b] (mod p) ⇒ [a = b mod p] (mod p)"
  <proof>

```

```
lemma cong_mod: "[a = b] (mod p)  $\implies$  [a mod p = b mod p] (mod p)"
  <proof>
```

### 1.3 Modular inverses

```
definition modular_inverse where
  "modular_inverse p n = fst (bezout_coefficients n p) mod p"
```

```
lemma cong_modular_inverse1:
  assumes "coprime n p"
  shows "[n * modular_inverse p n = 1] (mod p)"
  <proof>
```

```
lemma cong_modular_inverse2:
  assumes "coprime n p"
  shows "[modular_inverse p n * n = 1] (mod p)"
  <proof>
```

```
lemma coprime_modular_inverse [simp, intro]:
  fixes n :: "'a :: {euclidean_ring_gcd, unique_euclidean_semiring}"
  assumes "coprime n p"
  shows "coprime (modular_inverse p n) p"
  <proof>
```

```
lemma modular_inverse_int_nonneg: "p > 0  $\implies$  modular_inverse p (n ::
int)  $\geq$  0"
  <proof>
```

```
lemma modular_inverse_int_less: "p > 0  $\implies$  modular_inverse p (n :: int)
< p"
  <proof>
```

```
lemma modular_inverse_int_eqI:
  fixes x y :: int
  assumes "y  $\in$  {0.. $m$ }" "[x * y = 1] (mod m)"
  shows "modular_inverse m x = y"
  <proof>
```

```
lemma modular_inverse_1 [simp]:
  assumes "m > (1 :: int)"
  shows "modular_inverse m 1 = 1"
  <proof>
```

```
lemma modular_inverse_int_mult:
  fixes x y :: int
  assumes "coprime x m" "coprime y m" "m > 0"
  shows "modular_inverse m (x * y) = (modular_inverse m y * modular_inverse
m x) mod m"
```

*<proof>*

```
lemma bij_betw_int_remainders_mult:
  fixes a n :: int
  assumes a: "coprime a n"
  shows "bij_betw ( $\lambda m. a * m \bmod n$ ) {1.. $n$ } {1.. $n$ }"
<proof>
```

## 1.4 Facts about Bernoulli numbers

```
definition bernoulli_rat :: "nat  $\Rightarrow$  rat"
  where "bernoulli_rat n = of_int (bernoulli_num n) / of_int (bernoulli_denom n)"
```

```
bundle bernoulli_syntax
begin
notation bernoulli_rat (< $\mathcal{B}$ >)
end
```

```
lemma bernoulli_num_eq_0_iff: "bernoulli_num n = 0  $\iff$  odd n  $\wedge$  n  $\neq$  1"
<proof>
```

```
lemma bernoulli_num_odd_eq_0: "odd k  $\implies$  k  $\neq$  1  $\implies$  bernoulli_num k = 0"
<proof>
```

```
lemma prime_dvd_bernoulli_denom_iff:
  assumes "prime p" "even k" "k > 0"
  shows "p dvd bernoulli_denom k  $\iff$  (p - 1) dvd k"
<proof>
```

```
lemma bernoulli_num_denom_eqI:
  assumes "bernoulli k = of_int a / of_nat b" "coprime a b" "b > 0"
  shows "bernoulli_num k = a" "bernoulli_denom k = b"
<proof>
```

```
lemma bernoulli_rat_eq_0_iff: "bernoulli_rat n = 0  $\iff$  odd n  $\wedge$  n  $\neq$  1"
<proof>
```

```
lemma bernoulli_rat_odd_eq_0: "odd n  $\implies$  n  $\neq$  1  $\implies$  bernoulli_rat n = 0"
<proof>
```

```
lemma bernoulli_rat_conv_bernoulli: "of_rat (bernoulli_rat n) = bernoulli n"
<proof>
```

```

lemma quotient_of_bernoulli_rat [simp]:
  "quotient_of (bernoulli_rat n) = (bernoulli_num n, int (bernoulli_denom
n))"
  <proof>

end

```

## 2 Congruence of rational numbers modulo an integer

```

theory Rat_Congruence
  imports Kummer_Library
begin

```

### 2.1 $p$ -adic valuation of a rational

The notion of the multiplicity  $\nu_p(n)$  of a prime  $p$  in an integer  $n$  can be generalised to rational numbers via  $\nu_p(a/b) = \nu_p(a) - \nu_p(b)$ . This is also called the  $p$ -adic valuation of  $a/b$ .

```

definition qmultiplicity :: "int  $\Rightarrow$  rat  $\Rightarrow$  int" where
  "qmultiplicity p x = (case quotient_of x of (a, b)  $\Rightarrow$  int (multiplicity
p a) - int (multiplicity p b))"

```

```

lemma qmultiplicity_of_int [simp]:
  "qmultiplicity p (of_int n) = int (multiplicity p n)"
  <proof>

```

```

lemma qmultiplicity_of_nat [simp]:
  "qmultiplicity p (of_nat n) = int (multiplicity p n)"
  <proof>

```

```

lemma qmultiplicity_numeral [simp]:
  "qmultiplicity p (numeral n) = int (multiplicity p (numeral n))"
  <proof>

```

```

lemma qmultiplicity_0 [simp]: "qmultiplicity p 0 = 0"
  <proof>

```

```

lemma qmultiplicity_1 [simp]: "qmultiplicity p 1 = 0"
  <proof>

```

```

lemma qmultiplicity_minus [simp]: "qmultiplicity p (-x) = qmultiplicity
p x"
  <proof>

```

```

lemma qmultiplicity_divide_of_int:
  assumes "x  $\neq$  0" "y  $\neq$  0" "prime_elem p"

```

```

  shows "qmultiplicity p (of_int x / of_int y) = int (multiplicity p
x) - int (multiplicity p y)"
⟨proof⟩

```

```

lemma qmultiplicity_mult [simp]:
  assumes "prime_elem p" "x ≠ 0" "y ≠ 0"
  shows "qmultiplicity p (x * y) = qmultiplicity p x + qmultiplicity
p y"
⟨proof⟩

```

```

lemma qmultiplicity_inverse [simp]:
  "qmultiplicity p (inverse x) = -qmultiplicity p x"
⟨proof⟩

```

```

lemma qmultiplicity_divide [simp]:
  assumes "prime_elem p" "x ≠ 0" "y ≠ 0"
  shows "qmultiplicity p (x / y) = qmultiplicity p x - qmultiplicity
p y"
⟨proof⟩

```

```

lemma qmultiplicity_nonneg_iff:
  assumes "a ≠ 0" "b ≠ 0" "coprime a b" "prime p"
  shows "qmultiplicity p (of_int a / of_int b) ≥ 0 ⟷ ¬p dvd b"
⟨proof⟩

```

```

lemma qmultiplicity_nonneg_imp_not_dvd_denom:
  assumes "qmultiplicity p x ≥ 0" "|p| ≠ 1"
  shows "¬p dvd snd (quotient_of x)"
⟨proof⟩

```

```

lemma qmultiplicity_prime_nonneg_imp_coprime_denom:
  assumes "qmultiplicity p x ≥ 0" "prime p"
  shows "coprime (snd (quotient_of x)) p"
⟨proof⟩

```

## 2.2 Rational modulo operation

Similarly, we can define  $(a/b) \bmod m$  whenever  $b$  and  $m$  are coprime by choosing to interpret  $(1/b) \bmod m$  as the modular inverse of  $b$  modulo  $m$ :

```

definition qmod :: "rat ⇒ int ⇒ int" (infixl <qmod> 70) where
  "x qmod m = (let (a, b) = quotient_of x in if coprime b m then (a *
modular_inverse m b) mod m else 0)"

```

```

lemma qmod_mod_absorb [simp]: "x qmod m mod m = x qmod m"
⟨proof⟩

```

```

lemma qmod_of_nat [simp]: "m > 1 ⇒ of_nat x qmod m = int x mod m"
⟨proof⟩

```

**lemma** `qmod_of_int [simp]: "m > 1  $\implies$  of_int x qmod m = x mod m"`  
`<proof>`

**lemma** `qmod_numeral [simp]: "m > 1  $\implies$  numeral n qmod m = numeral n mod m"`  
`<proof>`

**lemma** `qmod_0 [simp]: "0 qmod m = 0"`  
`<proof>`

**lemma** `qmod_1 [simp]: "m > 1  $\implies$  1 qmod m = 1"`  
`<proof>`

**lemma** `qmod_fraction_eq:`  
`assumes "coprime b m" "b  $\neq$  0" "m > 0"`  
`shows "(of_int a / of_int b) qmod m = a * modular_inverse m b mod m"`  
`<proof>`

## 2.3 Congruence relation

With this, it is now straightforward to define the congruence relation  $x = y \pmod{m}$  for rational  $x, y$ :

**definition** `qcong :: "rat  $\Rightarrow$  rat  $\Rightarrow$  int  $\Rightarrow$  bool" ( $\langle(1[_ = _] \text{'('} \text{qmod} \text{'})\rangle$ )`  
 where

`"[a = b] (qmod m)  $\longleftrightarrow$`   
`coprime (snd (quotient_of a)) m  $\wedge$  coprime (snd (quotient_of b)) m`  
 `$\wedge$  a qmod m = b qmod m"`

**lemma** `qcong_of_int_iff [simp]:`  
`assumes "m > 1"`  
`shows "[of_int a = of_int b] (qmod m)  $\longleftrightarrow$  [a = b] (mod m)"`  
`<proof>`

**lemma** `cong_imp_qcong:`  
`assumes "[a = b] (mod m)" "m > 1"`  
`shows "[of_int a = of_int b] (qmod m)"`  
`<proof>`

**lemma** `cong_imp_qcong_of_nat:`  
`assumes "[a = b] (mod m)" "m > 1"`  
`shows "[of_nat a = of_nat b] (qmod m)"`  
`<proof>`

**lemma** `qcong_refl [intro]: "coprime (snd (quotient_of q)) m  $\implies$  [q = q] (qmod m)"`  
`<proof>`

**lemma** `qcong_sym_eq: "[q1 = q2] (qmod m)  $\longleftrightarrow$  [q2 = q1] (qmod m)"`

```

    <proof>

lemma qcong_sym: "[q1 = q2] (qmod m) ==> [q2 = q1] (qmod m)"
  <proof>

lemma qcong_trans [trans]:
  assumes "[q1 = q2] (qmod m)" "[q2 = q3] (qmod m)"
  shows "[q1 = q3] (qmod m)"
  <proof>

lemma qcong_0D:
  assumes "[x = 0] (qmod m)"
  shows "m dvd fst (quotient_of x)"
  <proof>

lemma qcong_0_iff:
  "[x = 0] (qmod m) <=> m dvd fst (quotient_of x) & coprime (snd (quotient_of
x)) m"
  <proof>

lemma qcong_1 [simp]: "[a = b] (qmod 1)"
  <proof>

lemma mod_minus_cong':
  fixes a b :: "'a :: euclidean_ring_cancel"
  assumes "(- a) mod b = (- a') mod b"
  shows "a mod b = a' mod b"
  <proof>

lemma qcong_minus_minus_iff:
  "[-b = -c] (qmod a) <=> [b = c] (qmod a)"
  <proof>

lemma qcong_minus: "[b = c] (qmod a) ==> [-b = -c] (qmod a)"
  <proof>

lemma qcong_fraction_iff:
  assumes "b ≠ 0" "d ≠ 0" "coprime b m" "coprime d m" "m > 0"
  shows "[of_int a / of_int b = of_int c / of_int d] (qmod m) <=> [a
* d = b * c] (mod m)"
  <proof>

lemma qcong_fractionI:
  assumes "x = of_int a / of_int b" "b ≠ 0" "coprime b m"
  shows "[x = of_int a / of_int b] (qmod m)"
  <proof>

lemma qcong_add:
  assumes "[x = x'] (qmod m)" "[y = y'] (qmod m)" "m > 0"

```

```

  shows "[x + y = x' + y'] (qmod m)"
⟨proof⟩

lemma qcong_diff:
  assumes "[x = x'] (qmod m)" "[y = y'] (qmod m)" "m > 0"
  shows "[x - y = x' - y'] (qmod m)"
⟨proof⟩

lemma qcong_mult:
  assumes "[x = x'] (qmod m)" "[y = y'] (qmod m)" "m > 0"
  shows "[x * y = x' * y'] (qmod m)"
⟨proof⟩

lemma qcong_divide_of_int:
  assumes "[x = x'] (qmod m)" "[c = c'] (mod m)" "coprime c m" "c ≠ 0"
  "c' ≠ 0" "m > 0"
  shows "[x / of_int c = x' / of_int c'] (qmod m)"
⟨proof⟩

lemma qcong_mult_of_int_cancel_left:
  assumes "[of_int a * b = of_int a * c] (qmod m)" "coprime a m" "a ≠
  0" "m > 0"
  shows "[b = c] (qmod m)"
⟨proof⟩

lemma qcong_pow:
  assumes "[a = b] (qmod m)" "m > 0"
  shows "[a ^ n = b ^ n] (qmod m)"
⟨proof⟩

lemma qcong_sum:
  "[sum f A = sum g A] (qmod m)" if "∧x. x ∈ A ⇒ [f x = g x] (qmod
  m)" "m > 0"
  ⟨proof⟩

lemma qcong_prod:
  "[prod f A = prod g A] (qmod m)" if "(∧x. x ∈ A ⇒ [f x = g x] (qmod
  m))" "m > 0"
  ⟨proof⟩

lemma qcong_modulus_abs_1:
  assumes "|n| = 1"
  shows "[a = b] (qmod n)"
⟨proof⟩

lemma qcong_divide_of_int_left_iff:
  assumes "coprime c n" "c ≠ 0" <n > 0>
  shows "[a / of_int c = b] (qmod n) ↔ [a = b * of_int c] (qmod n)"
⟨proof⟩

```

```

lemma qcong_divide_of_nat_left_iff:
  assumes "coprime (int c) n" "c ≠ 0" "n > 0"
  shows "[a / of_nat c = b] (qmod n) ⟷ [a = b * of_nat c] (qmod n)"
  ⟨proof⟩

lemma qcong_divide_of_int_right_iff:
  assumes "coprime c n" "c ≠ 0" "n > 0"
  shows "[a = b / of_int c] (qmod n) ⟷ [a * of_int c = b] (qmod n)"
  ⟨proof⟩

lemma qcong_divide_of_nat_right_iff:
  assumes "coprime (int c) n" "c ≠ 0" "n > 0"
  shows "[a = b / of_nat c] (qmod n) ⟷ [a * of_nat c = b] (qmod n)"
  ⟨proof⟩

lemma qcong_qmultiplicity_pos_transfer:
  assumes "[x = y] (qmod m)" "qmultiplicity m x > 0"
  shows "y = 0 ∨ qmultiplicity m y > 0"
  ⟨proof⟩

end

```

### 3 The Voronoi congruence

```

theory Voronoi_Congruence
  imports Kummer_Library Rat_Congruence
begin

unbundle bernoulli_syntax

lemma sum_of_powers_mod_prime:
  assumes p: "prime p"
  shows "[ $(\sum_{x=1..<p. \text{int } x^m) = (\text{if } (p - 1) \text{ dvd } m \text{ then } -1 \text{ else } 0)]$ 
(mod p)"
  ⟨proof⟩

lemma sum_of_powers_mod_prime':
  fixes p m :: nat
  assumes p: "prime p" "¬(p - 1) dvd m"
  shows "[ $(\sum_{x=1..<p. x^m) = 0]$  (mod p)"
  ⟨proof⟩

lemma voronoi_congruence_aux1:
  assumes "prime p" "j ≥ 4"
  shows "multiplicity p (j + 1) ≤ (if p ∈ {2, 3} then 1 else 0) + j - 2"
  ⟨proof⟩

```

```

context
  fixes S :: "nat ⇒ nat ⇒ nat" and D :: "nat ⇒ nat" and N :: "nat
⇒ int"
  defines "S ≡ (λk n. ∑ r<n. r ^ k)"
  defines "N ≡ bernoulli_num" and "D ≡ bernoulli_denom"
begin

lemma voronoi_congruence_aux2:
  fixes k n :: nat
  assumes k: "even k" "k ≥ 2" and n: "n > 0"
  shows "real (S k n) = (∑ j≤k. real (k choose j) / real (j + 1) *
bernoulli (k - j) * real (n ^ (j + 1)))"
  <proof>

lemma voronoi_congruence_aux3:
  fixes k n :: nat
  assumes k: "even k" "k ≥ 2" and n: "n > 0"
  shows "[D k * S k n = N k * n] (mod (n^2))"
  <proof>

Proposition 9.5.20

theorem voronoi_congruence:
  fixes k n :: nat and a :: int
  assumes k: "even k" "k ≥ 2" and n: "n > 0" and a: "coprime a n"
  shows "[ (a^{k-1}) * N k = k * a^{(k-1)} * D k * (∑ m=1..<n. m^{(k-1)} *
((m * a) div n)) ] (mod n)"
  <proof>

corollary voronoi_congruence':
  fixes k p :: nat and a :: int
  assumes k: "even k" "k ≥ 2" and p: "prime p" "¬(p - 1) dvd k" and
a: "¬p dvd a" "[a ^ k ≠ 1] (mod p)"
  shows "[B k = of_int (k * a^{(k-1)}) / of_int (a^k - 1) *
of_int (∑ m=1..<p. m^{(k-1)} * ((m * a) div p))] (qmod
p)"
  <proof>

corollary voronoi_congruence_harvey:
  fixes k p :: nat and c a :: int and h :: "nat ⇒ rat"
  assumes k: "even k" "k ∈ {2..p-3}" and p: "prime p" "p ≥ 5" and c:
"c ∈ {0<..corollary voronoi_congruence_harvey':

```

```

fixes k p :: nat and g :: nat and h :: "nat ⇒ rat" and a :: int
assumes k: "even k" "k ∈ {2..p-3}" and p: "prime p" "p ≥ 5"
assumes g: "residue_primroot p g" "g ∈ {0<..

```

## 4 Kummer's Congruence

```

theory Kummer_Congruence
  imports Voronoi_Congruence
begin

```

```

unbundle bernoulli_syntax

```

```

context

```

```

  fixes S :: "nat ⇒ nat ⇒ nat" and D :: "nat ⇒ nat" and N :: "nat
⇒ int"

```

```

  defines "S ≡ (λk n. ∑ r<n. r ^ k)"

```

```

  defines "N ≡ bernoulli_num" and "D ≡ bernoulli_denom"

```

```

begin

```

Auxiliary lemma for Proposition 9.5.23: if  $k$  is even and  $(p-1) \nmid k$ , then  $\nu_p(N_k) \geq \nu_p(k)$ .

```

lemma multiplicity_prime_bernoulli_num_ge:

```

```

  fixes p k :: nat

```

```

  assumes p: "prime p" "¬(p - 1) dvd k" and k: "even k"

```

```

  shows "multiplicity p (N k) ≥ multiplicity p k"

```

```

⟨proof⟩

```

Proposition 9.5.23: if  $k$  is even and  $(p-1) \nmid k$ , then  $B_k/k$  is  $p$ -integral.

```

lemma bernoulli_k_over_k_is_p_integral:

```

```

  fixes p k :: nat

```

```

  assumes p: "prime p" "¬(p - 1) dvd k" and k: "k ≠ 1"

```

```

  shows "qmultiplicity p (B k / of_nat k) ≥ 0"

```

```

⟨proof⟩

```

```

lemma kummer_congruence_aux:
  fixes k p a :: nat
  assumes k: "even k" "k ≥ 2" and p: "¬(p - 1) dvd k" "prime p"
  assumes a: "¬p dvd a"
  assumes s: "s ≥ multiplicity p k"
  shows "[of_int ((1 - int p^(k-1)) * (int a^k - 1)) * B k / of_nat k
  =
    of_int (int a^(k-1) *
      (∑ m∈{m∈{1..<p^(s+e)}. ¬p dvd m}. m^(k-1) * (int m * a
div p ^ (e + s))))] (qmod p^e)"
  <proof>

```

```

theorem kummer_congruence:
  fixes k k' p :: nat
  assumes k: "even k" "k ≥ 2" and k': "even k'" "k' ≥ 2" and p: "¬(p
- 1) dvd k" "prime p"
  assumes cong: "[k = k'] (mod totient (p ^ e))"
  shows "[[of_nat p^(k-1)-1) * B k / of_nat k =
  (of_nat p^(k'-1)-1) * B k' / of_nat k'] (qmod (p^e))"
  <proof>

```

```

corollary kummer_congruence':
  assumes kk': "even k" "even k'" "k ≥ e+1" "k' ≥ e+1"
  assumes cong: "[k = k'] (mod totient (p ^ e))"
  assumes p: "prime p" "¬(p-1) dvd k"
  shows "[B k / of_nat k = B k' / of_nat k'] (qmod (p^e))"
  <proof>

```

```

corollary kummer_congruence'_prime:
  assumes kk': "even k" "even k'" "k > 0" "k' > 0"
  assumes cong: "[k = k'] (mod (p - 1))"
  assumes p: "prime p" "¬(p-1) dvd k"
  shows "[B k / of_nat k = B k' / of_nat k'] (qmod p)"
  <proof>

```

end

unbundle no bernoulli\_syntax

end

## 5 Regular primes

```

theory Regular_Primes
  imports Kummer_Congruence Zeta_Function.Zeta_Function
begin

```

```

definition regular_prime :: "nat ⇒ bool" where

```

```
"regular_prime p  $\longleftrightarrow$  prime p  $\wedge$  (p = 2  $\vee$  ( $\forall k \in \{2..p-3\}$ . even k  $\longrightarrow$ 
 $\neg$ p dvd bernoulli_num k))"
```

```
definition irregular_prime :: "nat  $\Rightarrow$  bool" where
  "irregular_prime p  $\longleftrightarrow$  prime p  $\wedge$  (p  $\neq$  2  $\wedge$  ( $\exists k \in \{2..p-3\}$ . even k  $\wedge$ 
  p dvd bernoulli_num k))"
```

```
lemma irregular_primeI:
  assumes "prime p" "p  $\neq$  2" "p dvd bernoulli_num k" "even k" "k  $\in$  {2..p-3}"
  shows "irregular_prime p"
  <proof>
```

```
lemma bernoulli_32: "bernoulli 32 = -7709321041217 / 510"
  <proof>
```

The smallest irregular prime is 37.

```
lemma irregular_prime_37: "irregular_prime 37"
  <proof>
```

Irregularity of primes can be certified relatively easily with the code generator:

```
experiment
begin
```

```
lemma irregular_59: "irregular_prime 59"
  <proof>
```

```
lemma irregular_67: "irregular_prime 67"
  <proof>
```

```
end
```

```
end
```

## 6 Infinitude of irregular primes

```
theory Irregular_Primes_Infinite
  imports Regular_Primes
begin
```

One consequence of Kummer's congruence is that there are infinitely many irregular primes. We shall derive this here.

```
lemma zeta_real_gt_1:
  assumes "x > 1"
  shows "Re (zeta (of_real x)) > 1"
  <proof>
```

```
lemma zeta_real_gt_1':
```

```

assumes "Re s > 1" "s ∈ ℝ"
shows "Re (zeta s) > 1"
⟨proof⟩

lemma bernoulli_even_conv_zeta:
  "complex_of_real (bernoulli (2*n)) = (-1)^Suc n * 2 * fact (2*n) / (2*pi)^(2*n)
  * zeta (2 * of_nat n)"
  ⟨proof⟩

lemma bernoulli_even_conv_zeta':
  "bernoulli (2*n) = (-1)^Suc n * 2 * fact (2*n) / (2*pi)^(2*n) * Re (zeta
  (2 * of_nat n))"
  ⟨proof⟩

lemma abs_bernoulli_even_conv_zeta:
  assumes "even n" "n > 0"
  shows " |bernoulli n| = 2 * fact n / (2*pi)^n * Re (zeta (of_nat n))"
  ⟨proof⟩

lemma abs_bernoulli_over_n_ge_2:
  assumes "n ≥ 23" "even n"
  shows " |bernoulli n / n| ≥ 2"
  ⟨proof⟩

lemma infinite_irregular_primes_aux:
  assumes "finite P" "∀p∈P. irregular_prime p" "37 ∈ P"
  shows "∃p. irregular_prime p ∧ p ∉ P"
  ⟨proof⟩

theorem infinite_irregular_primes: "infinite {p. irregular_prime p}"
  ⟨proof⟩

end

```

## References

- [1] H. Cohen. *Number Theory: Volume II: Analytic and Modern Tools*. Graduate Texts in Mathematics. Springer New York, 2007.