The Kolmogorov-Chentsov Theorem

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Abstract

Continuous-time stochastic processes often carry the condition of having almost-surely continuous paths. If some process X satisfies certain bounds on its expectation, then the Kolmogorov-Chentsov theorem lets us construct a modification of X, i.e. a process X' such that $\forall t. X_t = X'_t$ almost surely, that has Hölder continuous paths.

In this work, we mechanise the Kolmogorov-Chentsov theorem. To get there, we develop a theory of stochastic processes, together with Hölder continuity, convergence in measure, and arbitrary intervals of dyadic rationals.

With this, we pave the way towards a construction of Brownian motion. The work is based on the exposition in Achim Klenke's probability theory text [1].

Contents

1 Supporting lemmas

theory *Kolmogorov-Chentsov-Extras*
imports *HOL−Probability.Probability*
begin

lemma *atLeastAtMost-induct*[*consumes 1*, *case-names base Suc*]:
assumes $x \in \{n..m\}$
and $P\ n$
and $\bigwedge k.\ [\![k \geq n;\ k < m;\ P\ k]\!] \Longrightarrow P\ (Suc\ k)$
shows $P\ x$
⟨*proof*⟩

lemma *eventually-prodI′*:
assumes $eventually\ P\ F\ eventually\ Q\ G\ \forall x\ y.\ P\ x \longrightarrow Q\ y \longrightarrow R\ (x,y)$
shows $eventually\ R\ (F \times_F\ G)$
⟨*proof*⟩

Analogous to $[\![almost\text{-}everywhere\ ?M\ ?P;\ \bigwedge N.\ [\![\bigwedge x.\ x \in space\ ?M - N \Longrightarrow ?P\ x;\ N \in null\text{-}sets\ ?M]\!] \Longrightarrow ?thesis]\!] \Longrightarrow ?thesis$

lemma *AE-I3*:
assumes $\bigwedge x.\ x \in space\ M - N \Longrightarrow P\ x\ N \in null\text{-}sets\ M$
shows $AE\ x\ in\ M.\ P\ x$
⟨*proof*⟩

Extends $[\![(?f \longrightarrow ?l)\ ?F;\ (?g \longrightarrow ?m)\ ?F]\!] \Longrightarrow ((\lambda x.\ dist\ (?f\ x)\ (?g\ x)) \longrightarrow dist\ ?l\ ?m)\ ?F$

lemma *tendsto-dist-prod*:
fixes $l\ m :: 'a :: metric\text{-}space$
assumes f: $(f \longrightarrow l)\ F$
and g: $(g \longrightarrow m)\ G$
shows $((\lambda x.\ dist\ (f\ (fst\ x))\ (g\ (snd\ x))) \longrightarrow dist\ l\ m)\ (F \times_F\ G)$
⟨*proof*⟩

lemma *borel-measurable-at-within-metric* [*measurable*]:
fixes $f :: 'a :: first\text{-}countable\text{-}topology \Rightarrow 'b \Rightarrow 'c :: metric\text{-}space$
assumes [*measurable*]: $\bigwedge t.\ t \in S \Longrightarrow f\ t \in borel\text{-}measurable\ M$
and *convergent*: $\bigwedge \omega.\ \omega \in space\ M \Longrightarrow \exists l.\ ((\lambda t.\ f\ t\ \omega) \longrightarrow l)\ (at\ x\ within\ S)$
and *nontrivial*: $at\ x\ within\ S \neq \bot$
shows $(\lambda\omega.\ Lim\ (at\ x\ within\ S)\ (\lambda t.\ f\ t\ \omega)) \in borel\text{-}measurable\ M$
⟨*proof*⟩

lemma *Max-finite-image-ex*:
assumes $finite\ S\ S \neq \{\}\ P\ (MAX\ k{\in}S.\ f\ k)$
shows $\exists k \in S.\ P\ (f\ k)$
⟨*proof*⟩

lemma *measure-leq-emeasure-ennreal*: $0 \leq x \Longrightarrow emeasure\ M\ A \leq ennreal\ x \Longrightarrow measure\ M\ A \leq x$

⟨*proof*⟩

lemma *Union-add-subset*: $(m :: nat) \leq n \Longrightarrow (\bigcup k.\ A\ (k + n)) \subseteq (\bigcup k.\ A\ (k + m))$
⟨*proof*⟩

lemma *floor-in-Nats* [*simp*]: $x \geq 0 \Longrightarrow \lfloor x \rfloor \in \mathbb{N}$
⟨*proof*⟩

lemma *triangle-ineq-list*:
fixes l :: (*'a* :: *metric-space*) *list*
assumes $l \neq []$
shows *dist* (*hd l*) (*last l*) $\leq (\sum\ i{=}1..length\ l - 1.\ dist\ (l!i)\ (l!(i-1)))$
⟨*proof*⟩

lemma *triangle-ineq-sum*:
fixes f :: $nat \Rightarrow$ *'a* :: *metric-space*
assumes $n \leq m$
shows *dist* (*f n*) (*f m*) $\leq (\sum\ i{=}Suc\ n..m.\ dist\ (f\ i)\ (f\ (i-1)))$
⟨*proof*⟩

lemma (**in** *product-prob-space*) *indep-vars-PiM-coordinate*:
assumes $I \neq \{\}$
shows *prob-space.indep-vars* $(\Pi_M\ i{\in}I.\ M\ i)\ M\ (\lambda x\ f.\ f\ x)\ I$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-indep-set*:
assumes *indep-sets* $F\ I\ i \in I\ j \in I\ i \neq j$
shows *indep-set* (*F i*) (*F j*)
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-indep-var*:
assumes *indep-vars* $M'\ X\ I\ i \in I\ j \in I\ i \neq j$
shows *indep-var* $(M'\ i)\ (X\ i)\ (M'\ j)\ (X\ j)$
⟨*proof*⟩

end

2 Intervals of dyadic rationals

theory *Dyadic-Interval*
imports *HOL−Analysis.Analysis*
begin

In this file we describe intervals of dyadic numbers $S..T$ for reals S T. We use the floor and ceiling functions to approximate the numbers with increasing accuracy.

lemma *frac-floor*: $\lfloor x \rfloor = x - frac\ x$

⟨*proof*⟩

lemma *frac-ceil*: $\lceil x \rceil = x + \mathit{frac}\ (- x)$
⟨*proof*⟩

lemma *floor-pow2-lim*: $(\lambda n.\ \lfloor 2\,\hat{}\,n * T \rfloor\ /\ 2\ \hat{}\ n) \longrightarrow T$
⟨*proof*⟩

lemma *floor-pow2-leq*: $\lfloor 2\,\hat{}\,n * T \rfloor\ /\ 2\ \hat{}\ n \leq T$
⟨*proof*⟩

lemma *ceil-pow2-lim*: $(\lambda n.\ \lceil 2\,\hat{}\,n * T \rceil\ /\ 2\ \hat{}\ n) \longrightarrow T$
⟨*proof*⟩

lemma *ceil-pow2-geq*: $\lceil 2\,\hat{}\,n * T \rceil\ /\ 2\ \hat{}\ n \geq T$
⟨*proof*⟩

`dyadic_interval_step` $n\ S\ T$ is the collection of dyadic numbers in $\{S..T\}$ with denominator 2^n. As $n \to \infty$ this collection approximates $\{S..T\}$. Compare with *dyadics* $\equiv \bigcup_{k\ m} \{\mathit{of\text{-}nat}\ m\ /\ (2::?'a)^k\}$

definition *dyadic-interval-step* :: *nat* ⇒ *real* ⇒ *real* ⇒ *real set*
where *dyadic-interval-step* $n\ S\ T \equiv (\lambda k.\ k\ /\ (2\ \hat{}\ n))$ ‘ $\{\lceil 2\,\hat{}\,n * S \rceil..\lfloor 2\,\hat{}\,n * T \rfloor\}$

definition *dyadic-interval* :: *real* ⇒ *real* ⇒ *real set*
where *dyadic-interval* $S\ T \equiv (\bigcup n.$ *dyadic-interval-step* $n\ S\ T)$

lemma *dyadic-interval-step-empty*[*simp*]: $T < S \Longrightarrow$ *dyadic-interval-step* $n\ S\ T = \{\}$
⟨*proof*⟩

lemma *dyadic-interval-step-singleton*[*simp*]: $X \in \mathbb{Z} \Longrightarrow$ *dyadic-interval-step* $n\ X\ X = \{X\}$
⟨*proof*⟩

lemma *dyadic-interval-step-zero* [*simp*]: *dyadic-interval-step* $0\ S\ T =$ *real-of-int* ‘ $\{\lceil S \rceil\ ..\ \lfloor T \rfloor\}$
⟨*proof*⟩

lemma *dyadic-interval-step-mem* [*intro*]:
assumes $x \geq 0\ T \geq 0\ x \leq T$
shows $\lfloor 2\,\hat{}\,n * x \rfloor\ /\ 2\ \hat{}\ n \in$ *dyadic-interval-step* $n\ 0\ T$
⟨*proof*⟩

lemma *dyadic-interval-step-iff*:
$x \in$ *dyadic-interval-step* $n\ S\ T \longleftrightarrow$
$(\exists k.\ k \geq \lceil 2\,\hat{}\,n * S \rceil \wedge k \leq \lfloor 2\,\hat{}\,n * T \rfloor \wedge x = k/2\,\hat{}\,n)$
⟨*proof*⟩

lemma *dyadic-interval-step-memI* [*intro*]:

assumes $\exists k::int.\ x = k/2\hat{\ }n\ x \geq S\ x \leq T$
shows $x \in$ *dyadic-interval-step n S T*
⟨*proof*⟩

lemma *mem-dyadic-interval*: $x \in$ *dyadic-interval S T* $\longleftrightarrow (\exists n.\ x \in$ *dyadic-interval-step n S T*)
⟨*proof*⟩

lemma *mem-dyadic-intervalI*: $\exists n.\ x \in$ *dyadic-interval-step n S T* $\Longrightarrow x \in$ *dyadic-interval S T*
⟨*proof*⟩

lemma *dyadic-step-leq*: $x \in$ *dyadic-interval-step n S T* $\Longrightarrow x \leq T$
⟨*proof*⟩

lemma *dyadics-leq*: $x \in$ *dyadic-interval S T* $\Longrightarrow x \leq T$
⟨*proof*⟩

lemma *dyadic-step-geq*: $x \in$ *dyadic-interval-step n S T* $\Longrightarrow x \geq S$
⟨*proof*⟩

lemma *dyadics-geq*: $x \in$ *dyadic-interval S T* $\Longrightarrow x \geq S$
⟨*proof*⟩

corollary *dyadic-interval-subset-interval* [*simp*]: (*dyadic-interval 0 T*) $\subseteq \{0..T\}$
⟨*proof*⟩

lemma *zero-in-dyadics*: $T \geq 0 \Longrightarrow 0 \in$ *dyadic-interval-step n 0 T*
⟨*proof*⟩

The following theorem is useful for reasoning with at_within

lemma *dyadic-interval-converging-sequence*:
assumes $t \in \{0..T\}\ T \neq 0$
shows $\exists s.\ \forall n.\ s\ n \in$ *dyadic-interval 0 T* $- \{t\} \wedge s \longrightarrow t$
⟨*proof*⟩

lemma *dyadic-interval-dense*: *closure* (*dyadic-interval 0 T*) $= \{0..T\}$
⟨*proof*⟩

corollary *dyadic-interval-islimpt*:
assumes $T > 0\ t \in \{0..T\}$
shows *t islimpt dyadic-interval 0 T*
⟨*proof*⟩

corollary *at-within-dyadic-interval-nontrivial*[*simp*]:
assumes $T > 0\ t \in \{0..T\}$
shows (*at t within dyadic-interval 0 T*) $\neq$ *bot*
⟨*proof*⟩

lemma *dyadic-interval-step-finite*[*simp*]: *finite* (*dyadic-interval-step n S T*)
⟨*proof*⟩

lemma *dyadic-interval-countable*[*simp*]: *countable* (*dyadic-interval S T*)
⟨*proof*⟩

lemma *floor-pow2-add-leq*:
fixes *T* :: *real*
shows $\lfloor 2\hat{\ }n * T\rfloor$ / 2 ^ $n \leq \lfloor 2$ ^ $(n+k) * T\rfloor$ / 2 ^ $(n+k)$
⟨*proof*⟩

corollary *floor-pow2-mono*: *mono* ($\lambda n.$ $\lfloor 2\hat{\ }n * (T$ :: *real*$)\rfloor$ / 2 ^ n)
⟨*proof*⟩

lemma *dyadic-interval-step-Max*: $T \geq 0 \Longrightarrow$ *Max* (*dyadic-interval-step n 0 T*) = $\lfloor 2\hat{\ }n * T\rfloor$ / $2\hat{\ }n$
⟨*proof*⟩

lemma *dyadic-interval-step-subset*:
$n \leq m \Longrightarrow$ *dyadic-interval-step n 0 T* $\subseteq$ *dyadic-interval-step m 0 T*
⟨*proof*⟩

corollary *dyadic-interval-step-mono*:
assumes $x \in$ *dyadic-interval-step n 0 T* $n \leq m$
shows $x \in$ *dyadic-interval-step m 0 T*
⟨*proof*⟩

lemma *dyadic-as-natural*:
assumes $x \in$ *dyadic-interval-step n 0 T*
shows $\exists!k.$ $x =$ *real k* / 2 ^ n
⟨*proof*⟩

lemma *dyadic-of-natural*:
assumes *real k* / 2 ^ $n \leq T$
shows *real k* / 2 ^ n $\in$ *dyadic-interval-step n 0 T*
⟨*proof*⟩

lemma *dyadic-interval-minus*:
assumes $x \in$ *dyadic-interval-step n 0 T* $y \in$ *dyadic-interval-step n 0 T* $x \leq y$
shows $y - x \in$ *dyadic-interval-step n 0 T*
⟨*proof*⟩

lemma *dyadic-times-nat*: $x \in$ *dyadic-interval-step n 0 T* $\Longrightarrow$ $(x * 2$ ^ $n) \in \mathbb{N}$
⟨*proof*⟩

definition *dyadic-expansion x n b k* $\equiv$ *set b* $\subseteq \{0,1\}$
$\wedge$ *length b* $= n \wedge x =$ *real-of-int k* + ($\sum m\in\{1..n\}.$ *real* (*b* ! $(m-1)$)) / 2 ^ m)

lemma *dyadic-expansionI*:

assumes *set b* ⊆ {*0*,*1*} *length b* = *n x* = *k* + (∑ *m*∈{*1*..*n*}. (*b* ! (*m*−*1*)) / *2* ^ *m*)
shows *dyadic-expansion x n b k*
⟨*proof*⟩

lemma *dyadic-expansionD*:
assumes *dyadic-expansion x n b k*
shows *set b* ⊆ {*0*,*1*}
and *length b* = *n*
and *x* = *k* + (∑ *m*∈{*1*..*n*}. (*b* ! (*m*−*1*)) / *2* ^ *m*)
⟨*proof*⟩

lemma *dyadic-expansion-ex*:
assumes *x* ∈ *dyadic-interval-step n 0 T*
shows ∃ *b k*. *dyadic-expansion x n b k*
⟨*proof*⟩

lemma *dyadic-expansion-frac-le-1*:
assumes *dyadic-expansion x n b k*
shows (∑ *m*∈{*1*..*n*}. (*b* ! (*m*−*1*)) / *2* ^ *m*) < *1*
⟨*proof*⟩

lemma *dyadic-expansion-frac-range*:
assumes *dyadic-expansion x n b k m* ∈ {*1*..*n*}
shows *b* ! (*m*−*1*) ∈ {*0*,*1*}
⟨*proof*⟩

lemma *dyadic-expansion-interval*:
assumes *dyadic-expansion x n b k x* ∈ {*S*..*T*}
shows *x* ∈ *dyadic-interval-step n S T*
⟨*proof*⟩

lemma *dyadic-expansion-nth-geq*:
assumes *dyadic-expansion x n b k m* ∈ {*1*..*n*} *b* ! (*m*−*1*) = *1*
shows *x* ≥ *k* + *1*/*2*^*m*
⟨*proof*⟩

lemma *dyadic-expansion-frac-geq-0*:
assumes *dyadic-expansion x n b k*
shows (∑ *m*∈{*1*..*n*}. (*b* ! (*m*−*1*)) / *2* ^ *m*) ≥ *0*
⟨*proof*⟩

lemma *dyadic-expansion-frac*:
assumes *dyadic-expansion x n b k*
shows *frac x* = (∑ *m*∈{*1*..*n*}. (*b* ! (*m*−*1*))/ *2* ^ *m*)
⟨*proof*⟩

lemma *dyadic-expansion-floor*:
assumes *dyadic-expansion x n b k*

shows $k = \lfloor x \rfloor$
⟨*proof*⟩

lemma *sum-interval-pow2-inv*: $(\sum m{\in}\{Suc\ l..n\}.\ (1 :: real)\ /\ 2\ \hat{}\ m) = 1\ /\ 2\hat{}l - 1/2\hat{}n$ **if** $l < n$
⟨*proof*⟩

lemma *dyadic-expansion-unique*:
assumes *dyadic-expansion x n b k*
and *dyadic-expansion x n c j*
shows $b = c \wedge j = k$
⟨*proof*⟩

end

3 Hölder continuity

theory *Holder-Continuous*
imports *HOL−Analysis.Analysis*
begin

Hölder continuity is a weaker version of Lipschitz continuity.

definition *holder-at-within* :: *real* ⇒ *'a set* ⇒ *'a* ⇒ (*'a* :: *metric-space* ⇒ *'b* :: *metric-space*) ⇒ *bool* **where**
holder-at-within $\gamma\ D\ r\ \varphi \equiv \gamma \in \{0{<}..1\} \wedge$
$(\exists \varepsilon > 0.\ \exists C \geq 0.\ \forall s \in D.\ dist\ r\ s < \varepsilon \longrightarrow dist\ (\varphi\ r)\ (\varphi\ s) \leq C * dist\ r\ s\ powr\ \gamma)$

definition *local-holder-on* :: *real* ⇒ *'a* :: *metric-space set* ⇒ (*'a* ⇒ *'b* :: *metric-space*) ⇒ *bool* **where**
local-holder-on $\gamma\ D\ \varphi \equiv \gamma \in \{0{<}..1\} \wedge$
$(\forall t{\in}D.\ \exists \varepsilon > 0.\ \exists C \geq 0.\ (\forall r{\in}D.\ \forall s{\in}D.\ dist\ s\ t < \varepsilon \wedge dist\ r\ t < \varepsilon \longrightarrow dist\ (\varphi\ r)\ (\varphi\ s) \leq C * dist\ r\ s\ powr\ \gamma))$

definition *holder-on* :: *real* ⇒ *'a* :: *metric-space set* ⇒ (*'a* ⇒ *'b* :: *metric-space*) ⇒ *bool* (*-−holder'-on 1000*) **where**
$\gamma{-}holder\text{-}on\ D\ \varphi \longleftrightarrow \gamma \in \{0{<}..1\} \wedge (\exists C \geq 0.\ (\forall r{\in}D.\ \forall s{\in}D.\ dist\ (\varphi\ r)\ (\varphi\ s) \leq C * dist\ r\ s\ powr\ \gamma))$

lemma *holder-onI*:
assumes $\gamma \in \{0{<}..1\}\ \exists C \geq 0.\ (\forall r{\in}D.\ \forall s{\in}D.\ dist\ (\varphi\ r)\ (\varphi\ s) \leq C * dist\ r\ s\ powr\ \gamma)$
shows $\gamma{-}holder\text{-}on\ D\ \varphi$
⟨*proof*⟩

We prove various equivalent formulations of local holder continuity, using open and closed balls and inequalities.

lemma *local-holder-on-cball*:

local-holder-on γ *D* φ $\longleftrightarrow$ $\gamma \in \{0{<}..1\}$ $\wedge$
($\forall$ *t*$\in$*D*. $\exists\varepsilon > 0$. $\exists$ *C* ≥ 0. ($\forall$ *r*$\in$*cball t* ε $\cap$ *D*. $\forall$ *s*$\in$*cball t* ε $\cap$ *D*. *dist* (φ *r*) (φ *s*) $\leq$ *C* $*$ *dist r s powr* γ))
(**is** *?L* $\longleftrightarrow$ *?R*)
⟨*proof*⟩

corollary *local-holder-on-leq-def*: *local-holder-on* γ *D* φ $\longleftrightarrow$ $\gamma \in \{0{<}..1\}$ $\wedge$
($\forall$ *t*$\in$*D*. $\exists\varepsilon > 0$. $\exists$ *C* ≥ 0. ($\forall$ *r*$\in$*D*. $\forall$ *s*$\in$*D*. *dist s t* $\leq \varepsilon$ $\wedge$ *dist r t* $\leq \varepsilon$ $\longrightarrow$ *dist* (φ *r*) (φ *s*) $\leq$ *C* $*$ *dist r s powr* γ))
⟨*proof*⟩

corollary *local-holder-on-ball*: *local-holder-on* γ *D* φ $\longleftrightarrow$ $\gamma \in \{0{<}..1\}$ $\wedge$
($\forall$ *t*$\in$*D*. $\exists\varepsilon > 0$. $\exists$ *C* ≥ 0. ($\forall$ *r*$\in$*ball t* ε $\cap$ *D*. $\forall$ *s*$\in$*ball t* ε $\cap$ *D*. *dist* (φ *r*) (φ *s*) $\leq$ *C* $*$ *dist r s powr* γ))
⟨*proof*⟩

lemma *local-holder-on-altdef*:
assumes $D \neq \{\}$
shows *local-holder-on* γ *D* φ = ($\forall$ *t* $\in$ *D*. ($\exists\varepsilon > 0$. (γ*−holder-on* ((*cball t* ε) $\cap$ *D*) φ)))
⟨*proof*⟩

lemma *local-holder-on-cong*[*cong*]:
assumes $\gamma = \varepsilon$ *C* = *D* $\bigwedge x.\ x \in C \Longrightarrow \varphi\ x = \psi\ x$
shows *local-holder-on* γ *C* φ $\longleftrightarrow$ *local-holder-on* ε *D* ψ
⟨*proof*⟩

lemma *local-holder-onI*:
assumes $\gamma \in \{0{<}..1\}$ ($\forall$ *t*$\in$*D*. $\exists\varepsilon > 0$. $\exists$ *C* ≥ 0. ($\forall$ *r*$\in$*D*. $\forall$ *s*$\in$*D*. *dist s t* $< \varepsilon$ $\wedge$ *dist r t* $< \varepsilon$ $\longrightarrow$ *dist* (φ *r*) (φ *s*) $\leq$ *C* $*$ *dist r s powr* γ))
shows *local-holder-on* γ *D* φ
⟨*proof*⟩

lemma *local-holder-ballI*:
assumes $\gamma \in \{0{<}..1\}$
and $\bigwedge t.\ t \in D \Longrightarrow \exists\varepsilon > 0.\ \exists\ C \geq 0.$ $\forall$ *r*$\in$*ball t* ε $\cap$ *D*. $\forall$ *s*$\in$*ball t* ε $\cap$ *D*.
dist (φ *r*) (φ *s*) $\leq$ *C* $*$ *dist r s powr* γ
shows *local-holder-on* γ *D* φ
⟨*proof*⟩

lemma *local-holder-onE*:
assumes *local-holder*: *local-holder-on* γ *D* φ
and *gamma*: $\gamma \in \{0{<}..1\}$
and $t \in D$
obtains ε *C* **where** $\varepsilon > 0$ $C \geq 0$
$\bigwedge r\ s.\ r \in$ *ball t* ε $\cap$ *D* $\Longrightarrow$ *s* $\in$ *ball t* ε $\cap$ *D* $\Longrightarrow$ *dist* (φ *r*) (φ *s*) $\leq$ *C* $*$ *dist r s powr* γ
⟨*proof*⟩

Holder continuity matches up with the existing definitions in *HOL−Analysis.Lipschitz*

lemma *holder-1-eq-lipschitz*: *1−holder-on D φ* = (∃ *C*. *lipschitz-on C D φ*)
⟨*proof*⟩

lemma *local-holder-1-eq-local-lipschitz*:
assumes *T* ≠ {}
shows *local-holder-on 1 D φ* = *local-lipschitz T D* (λ-. *φ*)
⟨*proof*⟩

lemma *local-holder-refine*:
assumes *g*: *local-holder-on g D φ g* ≤ *1*
and *h*: *h* ≤ *g h* > *0*
shows *local-holder-on h D φ*
⟨*proof*⟩

lemma *holder-uniform-continuous*:
assumes *γ−holder-on X φ*
shows *uniformly-continuous-on X φ*
⟨*proof*⟩

corollary *holder-on-continuous-on*: *γ−holder-on X φ* ⟹ *continuous-on X φ*
⟨*proof*⟩

lemma *holder-implies-local-holder*: *γ−holder-on D φ* ⟹ *local-holder-on γ D φ*
⟨*proof*⟩

lemma *local-holder-imp-continuous*:
assumes *local-holder*: *local-holder-on γ X φ*
shows *continuous-on X φ*
⟨*proof*⟩

lemma *local-holder-compact-imp-holder*:
assumes *compact I local-holder-on γ I φ*
shows *γ−holder-on I φ*
⟨*proof*⟩

lemma *holder-const*: *γ−holder-on C* (λ-. *c*) ⟷ *γ* ∈ {*0*<..*1*}
⟨*proof*⟩

lemma *local-holder-const*: *local-holder-on γ C* (λ-. *c*) ⟷ *γ* ∈ {*0*<..*1*}
⟨*proof*⟩

end

4 Convergence in measure

theory *Measure-Convergence*
imports *HOL−Probability.Probability*
begin

We use measure rather than emeasure because ennreal is not a metric space, which we need to reason about convergence. By intersecting with the set of finite measure A, we don't run into issues where infinity is collapsed to 0.

For finite measures this definition is equal to the definition without set A – see below.

definition *tendsto-measure* :: $'b$ *measure* $\Rightarrow$ ($'a \Rightarrow 'b \Rightarrow$ ($'c$ :: {*second-countable-topology*,*metric-space*}))
$\Rightarrow$ ($'b \Rightarrow 'c$) $\Rightarrow$ $'a$ *filter* $\Rightarrow$ *bool*
where *tendsto-measure M X l F* $\equiv$ ($\forall$ *n*. *X n* $\in$ *borel-measurable M*) $\wedge$ *l* $\in$ *borel-measurable M* $\wedge$
($\forall \varepsilon > 0$. $\forall$ *A* $\in$ *fmeasurable M*.
((λn. *measure M* ({$\omega \in$ *space M*. *dist* (*X n* ω) (*l* ω) $> \varepsilon$} $\cap$ *A*)) $\longrightarrow$ *0*) *F*)

abbreviation (**in** *prob-space*) *tendsto-prob* (**infixr** $\longrightarrow_P$ *55*) **where**
(*f* $\longrightarrow_P$ *l*) *F* $\equiv$ *tendsto-measure M f l F*

lemma *tendsto-measure-measurable*[*measurable-dest*]:
tendsto-measure M X l F $\Longrightarrow$ *X n* $\in$ *borel-measurable M*
⟨*proof*⟩

lemma *tendsto-measure-measurable-lim*[*measurable-dest*]:
tendsto-measure M X l F $\Longrightarrow$ *l* $\in$ *borel-measurable M*
⟨*proof*⟩

lemma *tendsto-measure-mono*: $F \leq F'$ $\Longrightarrow$ *tendsto-measure M f l* F' $\Longrightarrow$ *tendsto-measure M f l F*
⟨*proof*⟩

lemma *tendsto-measureI*:
assumes [*measurable*]: $\bigwedge n$. *X n* $\in$ *borel-measurable M l* $\in$ *borel-measurable M*
and $\bigwedge \varepsilon$ *A*. $\varepsilon > 0$ $\Longrightarrow$ *A* $\in$ *fmeasurable M* $\Longrightarrow$
((λn. *measure M* ({$\omega \in$ *space M*. *dist* (*X n* ω) (*l* ω) $> \varepsilon$} $\cap$ *A*)) $\longrightarrow$ *0*) *F*
shows *tendsto-measure M X l F*
⟨*proof*⟩

lemma (**in** *finite-measure*) *finite-tendsto-measureI*:
assumes [*measurable*]: $\bigwedge n$. f' *n* $\in$ *borel-measurable M f* $\in$ *borel-measurable M*
and $\bigwedge \varepsilon$. $\varepsilon > 0$ $\Longrightarrow$ ((λn. *measure M* {$\omega \in$ *space M*. *dist* (f' *n* ω) (*f* ω) $> \varepsilon$}) $\longrightarrow$ *0*) *F*
shows *tendsto-measure M* f' *f F*
⟨*proof*⟩

lemma (**in** *finite-measure*) *finite-tendsto-measureD*:
assumes [*measurable*]: *tendsto-measure M* f' *f F*
shows ($\forall \varepsilon > 0$. ((λn. *measure M* {$\omega \in$ *space M*. *dist* (f' *n* ω) (*f* ω) $> \varepsilon$}) $\longrightarrow$ *0*) *F*)
⟨*proof*⟩

lemma (**in** *finite-measure*) *tendsto-measure-leq*:

assumes [*measurable*]: $\bigwedge n.\ f'\ n \in$ *borel-measurable M f* $\in$ *borel-measurable M*
shows *tendsto-measure M f' f F* $\longleftrightarrow$
$(\forall \varepsilon > 0.\ ((\lambda n.\ measure\ M\ \{\omega \in space\ M.\ dist\ (f'\ n\ \omega)\ (f\ \omega) \geq \varepsilon\}) \longrightarrow 0)$ $F)$ (**is** $?L \longleftrightarrow ?R$)
⟨*proof*⟩

abbreviation *LIMSEQ-measure M f l* $\equiv$ *tendsto-measure M f l sequentially*

lemma *LIMSEQ-measure-def*: *LIMSEQ-measure M f l* $\longleftrightarrow$
$(\forall n.\ f\ n \in borel\text{-}measurable\ M) \wedge (l \in borel\text{-}measurable\ M) \wedge$
$(\forall \varepsilon > 0.\ \forall A \in fmeasurable\ M.$
$(\lambda n.\ measure\ M\ (\{\omega \in space\ M.\ dist\ (f\ n\ \omega)\ (l\ \omega) > \varepsilon\} \cap A)) \longrightarrow 0)$
⟨*proof*⟩

lemma *LIMSEQ-measureD*:
assumes *LIMSEQ-measure M f l* $\varepsilon > 0$ $A \in$ *fmeasurable M*
shows $(\lambda n.\ measure\ M\ (\{\omega \in space\ M.\ dist\ (f\ n\ \omega)\ (l\ \omega) > \varepsilon\} \cap A)) \longrightarrow 0$
⟨*proof*⟩

lemma *fmeasurable-inter*: $[\![A \in sets\ M;\ B \in fmeasurable\ M]\!] \Longrightarrow A \cap B \in$ *fmeasurable M*
⟨*proof*⟩

lemma *LIMSEQ-measure-emeasure*:
assumes *LIMSEQ-measure M f l* $\varepsilon > 0$ $A \in$ *fmeasurable M*
and [*measurable*]: $\bigwedge i.\ f\ i \in$ *borel-measurable M l* $\in$ *borel-measurable M*
shows $(\lambda n.\ emeasure\ M\ (\{\omega \in space\ M.\ dist\ (f\ n\ \omega)\ (l\ \omega) > \varepsilon\} \cap A)) \longrightarrow 0$
⟨*proof*⟩

lemma *measure-Lim-within-LIMSEQ*:
fixes a :: *'a* :: *first-countable-topology*
assumes $\bigwedge t.\ X\ t \in$ *borel-measurable M L* $\in$ *borel-measurable M*
assumes $\bigwedge S.\ [\![(\forall n.\ S\ n \neq a \wedge S\ n \in T);\ S \longrightarrow a]\!] \Longrightarrow$ *LIMSEQ-measure M* $(\lambda n.\ X\ (S\ n))\ L$
shows *tendsto-measure M X L* (*at a within T*)
⟨*proof*⟩

definition *tendsto-AE* :: *'b measure* $\Rightarrow$ (*'a* $\Rightarrow$ *'b* $\Rightarrow$ *'c* :: *topological-space*) $\Rightarrow$ (*'b* $\Rightarrow$ *'c*) $\Rightarrow$ *'a filter* $\Rightarrow$ *bool* **where**
tendsto-AE M f' l F $\longleftrightarrow$ $(AE\ \omega\ in\ M.\ ((\lambda n.\ f'\ n\ \omega) \longrightarrow l\ \omega)\ F)$

lemma *LIMSEQ-ae-pointwise*: $(\bigwedge x.\ (\lambda n.\ f\ n\ x) \longrightarrow l\ x) \Longrightarrow$ *tendsto-AE M f l sequentially*
⟨*proof*⟩

lemma *tendsto-AE-within-LIMSEQ*:
fixes a :: *'a* :: *first-countable-topology*
assumes $\bigwedge S.\ [\![(\forall n.\ S\ n \neq a \wedge S\ n \in T);\ S \longrightarrow a]\!] \Longrightarrow$ *tendsto-AE M* $(\lambda n.\ X\ (S\ n))$ *L sequentially*

shows *tendsto-AE M X L* (*at a within T*)
⟨*proof*⟩

lemma *LIMSEQ-dominated-convergence*:
fixes X :: *nat* ⇒ *real*
assumes $X \longrightarrow L$ ($\bigwedge n.\ Y\ n \leq X\ n$) ($\bigwedge n.\ Y\ n \geq L$)
shows $Y \longrightarrow L$
⟨*proof*⟩

Klenke remark 6.4

lemma *measure-conv-imp-AE-sequentially*:
assumes [*measurable*]: $\bigwedge n.\ f'\ n \in$ *borel-measurable M f* ∈ *borel-measurable M*
and *tendsto-AE M f′ f sequentially*
shows *LIMSEQ-measure M f′ f*
⟨*proof*⟩

corollary *LIMSEQ-measure-pointwise*:
assumes $\bigwedge x.\ (\lambda n.\ f\ n\ x) \longrightarrow f'\ x$ $\bigwedge n.\ f\ n \in$ *borel-measurable M f′* ∈ *borel-measurable M*
shows *LIMSEQ-measure M f f′*
⟨*proof*⟩

lemma *Lim-measure-pointwise*:
fixes a :: *′a* :: *first-countable-topology*
assumes $\bigwedge x.\ ((\lambda n.\ f\ n\ x) \longrightarrow f'\ x)$(*at a within T*) $\bigwedge n.\ f\ n \in$ *borel-measurable M f′* ∈ *borel-measurable M*
shows *tendsto-measure M f f′* (*at a within T*)
⟨*proof*⟩

corollary *measure-conv-imp-AE-at-within*:
fixes x :: *′a* :: *first-countable-topology*
assumes [*measurable*]: $\bigwedge n.\ f'\ n \in$ *borel-measurable M f* ∈ *borel-measurable M*
and *tendsto-AE M f′ f* (*at x within S*)
shows *tendsto-measure M f′ f* (*at x within S*)
⟨*proof*⟩

Klenke remark 6.5

lemma (**in** *sigma-finite-measure*) *LIMSEQ-measure-unique-AE*:
fixes f :: *nat* ⇒ *′a* ⇒ *′b* :: {*second-countable-topology*,*metric-space*}
assumes [*measurable*]: $\bigwedge n.\ f\ n \in$ *borel-measurable M l* ∈ *borel-measurable M l′* ∈ *borel-measurable M*
and *LIMSEQ-measure M f l LIMSEQ-measure M f l′*
shows *AE x in M*. $l\ x = l'\ x$
⟨*proof*⟩

corollary (**in** *sigma-finite-measure*) *LIMSEQ-ae-unique-AE*:
fixes f :: *nat* ⇒ *′a* ⇒ *′b* :: {*second-countable-topology*,*metric-space*}
assumes $\bigwedge n.\ f\ n \in$ *borel-measurable M l* ∈ *borel-measurable M l′* ∈ *borel-measurable M*

and *tendsto-AE M f l sequentially tendsto-AE M f l′ sequentially*
shows *AE x in M. l x = l′ x*
⟨*proof*⟩

lemma (**in** *sigma-finite-measure*) *tendsto-measure-at-within-eq-AE*:
fixes *f* :: ′*b* :: *first-countable-topology* ⇒ ′*a* ⇒ ′*c* :: {*second-countable-topology*,*metric-space*}
assumes *f-measurable*: ⋀*x. x* ∈ *S* ⟹ *f x* ∈ *borel-measurable M*
and *l-measurable*: *l* ∈ *borel-measurable M l′* ∈ *borel-measurable M*
and *tendsto*: *tendsto-measure M f l* (*at t within S*) *tendsto-measure M f l′* (*at t within S*)
and *not-bot*: (*at t within S*) ≠ ⊥
shows *AE x in M. l x = l′ x*
⟨*proof*⟩
end

5 Stochastic processes

theory *Stochastic-Processes*
imports *Kolmogorov-Chentsov-Extras Dyadic-Interval*
begin

A stochastic process is an indexed collection of random variables. For compatibility with `product_prob_space` we don't enforce conditions on the index set I in the assumptions.

locale *stochastic-process* = *prob-space* +
fixes *M′* :: ′*b measure*
and *I* :: ′*t set*
and *X* :: ′*t* ⇒ ′*a* ⇒ ′*b*
assumes *random-process*[*measurable*]: ⋀*i. random-variable M′* (*X i*)

sublocale *stochastic-process* ⊆ *product*: *product-prob-space* (λ*t. distr M M′* (*X t*))
⟨*proof*⟩

lemma (**in** *prob-space*) *stochastic-processI*:
assumes ⋀*i. random-variable M′* (*X i*)
shows *stochastic-process M M′ X*
⟨*proof*⟩

typedef (′*t*, ′*a*, ′*b*) *stochastic-process* =
{(*M* :: ′*a measure*, *M′* :: ′*b measure*, *I* :: ′*t set*, *X* :: ′*t* ⇒ ′*a* ⇒ ′*b*).
stochastic-process M M′ X}
⟨*proof*⟩

setup-lifting *type-definition-stochastic-process*

lift-definition *proc-source* :: (′*t*,′*a*,′*b*) *stochastic-process* ⇒ ′*a measure*
is *fst* ⟨*proof*⟩

interpretation *proc-source*: *prob-space proc-source X*
⟨*proof*⟩

lift-definition *proc-target* :: (*'t,'a,'b*) *stochastic-process* ⇒ *'b measure*
is *fst* ∘ *snd* ⟨*proof*⟩

lift-definition *proc-index* :: (*'t,'a,'b*) *stochastic-process* ⇒ *'t set*
is *fst* ∘ *snd* ∘ *snd* ⟨*proof*⟩

lift-definition *process* :: (*'t,'a,'b*) *stochastic-process* ⇒ *'t* ⇒ *'a* ⇒ *'b*
is *snd* ∘ *snd* ∘ *snd* ⟨*proof*⟩

declare [[*coercion process*]]

lemma *stochastic-process-construct* [*simp*]: *stochastic-process* (*proc-source X*) (*proc-target X*) (*process X*)
⟨*proof*⟩

interpretation *stochastic-process proc-source X proc-target X proc-index X process X*
⟨*proof*⟩

lemma *stochastic-process-measurable* [*measurable*]: *process X t* ∈ (*proc-source X*) $\rightarrow_M$ (*proc-target X*)
⟨*proof*⟩

Here we construct a process on a given index set. For this we need to produce measurable functions for indices outside the index set; we use the constant function, but it needs to point at an element of the target set to be measurable.

context *prob-space*
begin

lift-definition *process-of* :: *'b measure* ⇒ *'t set* ⇒ (*'t* ⇒ *'a* ⇒ *'b*) ⇒ *'b* ⇒ (*'t,'a,'b*) *stochastic-process*
is λ *M' I X ω*. *if* (∀ *t* ∈ *I*. *X t* ∈ *M* $\rightarrow_M$ *M'*) ∧ *ω* ∈ *space M'*
then (*M*, *M'*, *I*, (λ*t*. *if t* ∈ *I then X t else* (λ-. *ω*)))
else (*return* (*sigma UNIV* {{}, *UNIV*}) (*SOME x*. *True*), *sigma UNIV UNIV*, *I*, λ- -. *ω*)
⟨*proof*⟩

lemma *index-process-of*[*simp*]: *proc-index* (*process-of M' I X ω*) = *I*
⟨*proof*⟩

lemma
assumes ∀ *t* ∈ *I*. *X t* ∈ *M* $\rightarrow_M$ *M' ω* ∈ *space M'*
shows
source-process-of[*simp*]: *proc-source* (*process-of M' I X ω*) = *M* **and**
target-process-of[*simp*]: *proc-target* (*process-of M' I X ω*) = *M'* **and**

process-process-of[*simp*]: *process* (*process-of* M' *I X* ω) = (λt. *if* $t \in I$ *then X t else* (λ-. ω))
⟨*proof*⟩

lemma *process-of-apply*:
assumes $\forall t \in I.\ X\ t \in M \rightarrow_M M'$ $\omega \in$ *space* M' $t \in I$
shows *process* (*process-of* M' *I X* ω) $t = X\ t$
⟨*proof*⟩
end

We define the finite-dimensional distributions of our process.

lift-definition *distributions* :: ('*t*, '*a*, '*b*) *stochastic-process* ⇒ '*t set* ⇒ ('*t* ⇒ '*b*) *measure*
is λ(M, M', -, X) T. (Π_M $t{\in}T$. *distr M* M' (X t)) ⟨*proof*⟩

lemma *distributions-altdef*: *distributions X T* = (Π_M $t{\in}T$. *distr* (*proc-source X*) (*proc-target X*) (X t))
⟨*proof*⟩

lemma *prob-space-distributions*: *prob-space* (*distributions X J*)
⟨*proof*⟩

lemma *sets-distributions*: *sets* (*distributions X J*) = *sets* (*PiM J* (λ-. (*proc-target X*)))
⟨*proof*⟩

lemma *space-distributions*: *space* (*distributions X J*) = (Π_E $i{\in}J$. *space* (*proc-target X*))
⟨*proof*⟩

lemma *emeasure-distributions*:
assumes *finite J* $\bigwedge j.\ j{\in}J \Longrightarrow A\ j \in$ *sets* (*proc-target X*)
shows *emeasure* (*distributions X J*) (*Pi$_E$ J A*) = ($\prod j{\in}J$. *emeasure* (*distr* (*proc-source X*) (*proc-target X*) (X j)) (A j))
⟨*proof*⟩

interpretation *projective-family* (*proc-index X*) *distributions X* (λ-. *proc-target X*)
⟨*proof*⟩

locale *polish-stochastic* = *stochastic-process M borel* :: '*b*::*polish-space measure I X*
for *M* **and** *I* **and** *X*

lemma *distributed-cong-random-variable*:
assumes $M = K$ $N = L$ *AE x in M*. $X\ x = Y\ x$ $X \in M \rightarrow_M N$ $Y \in K \rightarrow_M L$ $f \in$ *borel-measurable N*
shows *distributed M N X f* ⟷ *distributed K L Y f*

⟨*proof*⟩

For all sorted lists of indices, the increments specified by this list are independent

lift-definition *indep-increments* :: (*'t* :: *linorder*, *'a*, *'b* :: *minus*) *stochastic-process* ⇒ *bool* **is**
λ(*M*, *M'*, *I*, *X*).
(∀ *l*. *set l* ⊆ *I* ∧ *sorted l* ∧ *length l* ≥ *2* ⟶
prob-space.indep-vars M (λ-. *M'*) (λ*k v*. *X* (*l*!*k*) *v* − *X* (*l*!(*k*−*1*)) *v*) {*1*..<*length l*}) ⟨*proof*⟩

lemma *indep-incrementsE*:
assumes *indep-increments X*
and *set l* ⊆ *proc-index X* ∧ *sorted l* ∧ *length l* ≥ *2*
shows *prob-space.indep-vars* (*proc-source X*) (λ-. *proc-target X*)
(λ*k v*. *X* (*l*!*k*) *v* − *X* (*l*!(*k*−*1*)) *v*) {*1*..<*length l*}
⟨*proof*⟩

lemma *indep-incrementsI*:
assumes ⋀*l*. *set l* ⊆ *proc-index X* ⟹ *sorted l* ⟹ *length l* ≥ *2* ⟹
prob-space.indep-vars (*proc-source X*) (λ-. *proc-target X*) (λ*k v*. *X* (*l*!*k*) *v* − *X* (*l*!(*k*−*1*)) *v*) {*1*..<*length l*}
shows *indep-increments X*
⟨*proof*⟩

lemma *indep-increments-indep-var*:
assumes *indep-increments X h* ∈ *proc-index X j* ∈ *proc-index X k* ∈ *proc-index X h* ≤ *j j* ≤ *k*
shows *prob-space.indep-var* (*proc-source X*) (*proc-target X*) (λ*v*. *X j v* − *X h v*) (*proc-target X*) (λ*v*. *X k v* − *X j v*)
⟨*proof*⟩

definition *stationary-increments X* ⟷ (∀ *t1 t2 k*. *t1* > *0* ∧ *t2* > *0* ∧ *k* > *0* ⟶

distr (*proc-source X*) (*proc-target X*) (λ*v*. *X* (*t1* + *k*) *v* − *X t1 v*) =
distr (*proc-source X*) (*proc-target X*) (λ*v*. *X* (*t2* + *k*) *v* − *X t2 v*))

Processes on the same source measure space, with the same index space, but not necessarily the same target measure since we only care about the measurable target space, not the measure

lift-definition *compatible* :: (*'t*,*'a*,*'b*) *stochastic-process* ⇒ (*'t*,*'a*,*'b*) *stochastic-process* ⇒ *bool*
is λ(*Mx*, *M'x*, *Ix*, *X*) (*My*, *M'y*, *Iy*, -). *Mx* = *My* ∧ *sets M'x* = *sets M'y* ∧ *Ix* = *Iy* ⟨*proof*⟩

lemma *compatibleI*:
assumes *proc-source X* = *proc-source Y sets* (*proc-target X*) = *sets* (*proc-target Y*)
proc-index X = *proc-index Y*

shows *compatible X Y*
⟨*proof*⟩

lemma
assumes *compatible X Y*
shows
compatible-source [*dest*]: *proc-source X* = *proc-source Y* **and**
compatible-target [*dest*]: *sets* (*proc-target X*) = *sets* (*proc-target Y*) **and**
compatible-index [*dest*]: *proc-index X* = *proc-index Y*
⟨*proof*⟩

lemma *compatible-refl* [*simp*]: *compatible X X*
⟨*proof*⟩

lemma *compatible-sym*: *compatible X Y* $\Longrightarrow$ *compatible Y X*
⟨*proof*⟩

lemma *compatible-trans*:
assumes *compatible X Y compatible Y Z*
shows *compatible X Z*
⟨*proof*⟩

lemma (**in** *prob-space*) *compatible-process-of*:
assumes *measurable*: $\forall t \in I.\ X\ t \in M \rightarrow_M M'$ $\forall t \in I.\ Y\ t \in M \rightarrow_M M'$
and $a \in$ *space* M' $b \in$ *space* M'
shows *compatible* (*process-of* M' *I X a*) (*process-of* M' *I Y b*)
⟨*proof*⟩

definition *modification* :: ('t,'a,'b) *stochastic-process* ⇒ ('t,'a,'b) *stochastic-process* ⇒ *bool* **where**
modification X Y $\longleftrightarrow$ *compatible X Y* ∧ (∀ *t* ∈ *proc-index X*. *AE x in proc-source X*. *X t x* = *Y t x*)

lemma *modificationI* [*intro*]:
assumes *compatible X Y* $\bigwedge t.\ t \in$ *proc-index X* $\Longrightarrow$ *AE x in proc-source X*. *X t x* = *Y t x*
shows *modification X Y*
⟨*proof*⟩

lemma *modificationD* [*dest*]:
assumes *modification X Y*
shows *compatible X Y*
and $\bigwedge t.\ t \in$ *proc-index X* $\Longrightarrow$ *AE x in proc-source X*. *X t x* = *Y t x*
⟨*proof*⟩

lemma *modification-null-set*:
assumes *modification X Y t* ∈ *proc-index X*
obtains *N* **where** {*x* ∈ *space* (*proc-source X*). *X t x* ≠ *Y t x*} ⊆ *N N* ∈ *null-sets* (*proc-source X*)

⟨*proof*⟩

lemma *modification-refl* [*simp*]: *modification X X*
⟨*proof*⟩

lemma *modification-sym*: *modification X Y* $\Longrightarrow$ *modification Y X*
⟨*proof*⟩

lemma *modification-trans*:
assumes *modification X Y modification Y Z*
shows *modification X Z*
⟨*proof*⟩

lemma *modification-imp-identical-distributions*:
assumes *modification*: *modification X Y*
and *index*: $T \subseteq$ *proc-index X*
shows *distributions X T* = *distributions Y T*
⟨*proof*⟩

definition *indistinguishable* :: ($'t,'a,'b$) *stochastic-process* $\Rightarrow$ ($'t,'a,'b$) *stochastic-process* $\Rightarrow$ *bool* **where**
indistinguishable X Y $\longleftrightarrow$ *compatible X Y* $\wedge$
($\exists N \in$ *null-sets* (*proc-source X*). $\forall t \in$ *proc-index X*. $\{x \in$ *space* (*proc-source X*). $X\ t\ x \neq Y\ t\ x\} \subseteq N$)

lemma *indistinguishableI*:
assumes *compatible X Y*
and $\exists N \in$ *null-sets* (*proc-source X*). ($\forall t \in$ *proc-index X*. $\{x \in$ *space* (*proc-source X*). $X\ t\ x \neq Y\ t\ x\} \subseteq N$)
shows *indistinguishable X Y*
⟨*proof*⟩

lemma *indistinguishable-null-set*:
assumes *indistinguishable X Y*
obtains *N* **where**
$N \in$ *null-sets* (*proc-source X*)
$\bigwedge t.\ t \in$ *proc-index X* $\Longrightarrow \{x \in$ *space* (*proc-source X*). $X\ t\ x \neq Y\ t\ x\} \subseteq N$
⟨*proof*⟩

lemma *indistinguishableD*:
assumes *indistinguishable X Y*
shows *compatible X Y*
and $\exists N \in$ *null-sets* (*proc-source X*). ($\forall t \in$ *proc-index X*. $\{x \in$ *space* (*proc-source X*). $X\ t\ x \neq Y\ t\ x\} \subseteq N$)
⟨*proof*⟩

lemma *indistinguishable-eq-AE*:
assumes *indistinguishable X Y*
shows *AE x in proc-source X*. $\forall t \in$ *proc-index X*. $X\ t\ x = Y\ t\ x$

⟨*proof*⟩

lemma *indistinguishable-null-ex*:
assumes *indistinguishable X Y*
shows $\exists$ *N* $\in$ *null-sets* (*proc-source X*). {*x* $\in$ *space* (*proc-source X*).$\exists$ *t* $\in$ *proc-index X*. *X t x* $\neq$ *Y t x*} $\subseteq$ *N*
⟨*proof*⟩

lemma *indistinguishable-refl* [*simp*]: *indistinguishable X X*
⟨*proof*⟩

lemma *indistinguishable-sym*: *indistinguishable X Y* $\Longrightarrow$ *indistinguishable Y X*
⟨*proof*⟩

lemma *indistinguishable-trans*:
assumes *indistinguishable X Y indistinguishable Y Z*
shows *indistinguishable X Z*
⟨*proof*⟩

lemma *indistinguishable-modification*: *indistinguishable X Y* $\Longrightarrow$ *modification X Y*
⟨*proof*⟩

Klenke 21.5(i)

lemma *modification-countable*:
assumes *modification X Y countable* (*proc-index X*)
shows *indistinguishable X Y*
⟨*proof*⟩

Klenke 21.5(ii). The textbook statement is more general - we reduce right continuity to regular continuity

lemma *modification-continuous-indistinguishable*:
fixes *X* :: (*real*, *'a*, *'b* :: *metric-space*) *stochastic-process*
assumes *modification*: *modification X Y*
and *interval*: $\exists$ *T* > *0*. *proc-index X* = {*0*..*T*}
and *rc*: *AE* ω *in proc-source X*. *continuous-on* (*proc-index X*) (λ*t*. *X t* ω)
(**is** *AE* ω *in proc-source X*. *?cont-X* ω)
AE ω *in proc-source Y*. *continuous-on* (*proc-index Y*) (λ*t*. *Y t* ω)
(**is** *AE* ω *in proc-source Y*. *?cont-Y* ω)
shows *indistinguishable X Y*
⟨*proof*⟩

lift-definition *restrict-index* :: (*'t*, *'a*, *'b*) *stochastic-process* ⇒ *'t set* ⇒ (*'t*, *'a*, *'b*) *stochastic-process*
is λ(*M*, *M'*, *I*, *X*) *T*. (*M*, *M'*, *T*, *X*) ⟨*proof*⟩

lemma
shows

restrict-index-source[*simp*]: *proc-source* (*restrict-index X T*) = *proc-source X* **and**
restrict-index-target[*simp*]: *proc-target* (*restrict-index X T*) = *proc-target X* **and**
restrict-index-index[*simp*]: *proc-index* (*restrict-index X T*) = *T* **and**
restrict-index-process[*simp*]: *process* (*restrict-index X T*) = *process X*
⟨*proof*⟩

lemma *restrict-index-override*[*simp*]: *restrict-index* (*restrict-index X T*) *S* = *restrict-index X S*
⟨*proof*⟩

lemma *compatible-restrict-index*:
assumes *compatible X Y*
shows *compatible* (*restrict-index X S*) (*restrict-index Y S*)
⟨*proof*⟩

lemma *modification-restrict-index*:
assumes *modification X Y S* ⊆ *proc-index X*
shows *modification* (*restrict-index X S*) (*restrict-index Y S*)
⟨*proof*⟩

lemma *indistinguishable-restrict-index*:
assumes *indistinguishable X Y S* ⊆ *proc-index X*
shows *indistinguishable* (*restrict-index X S*) (*restrict-index Y S*)
⟨*proof*⟩

lemma *AE-eq-minus* [*intro*]:
fixes *a* :: *'a* ⇒ (*'b* :: *real-normed-vector*)
assumes *AE x in M. a x = b x AE x in M. c x = d x*
shows *AE x in M. a x − c x = b x − d x*
⟨*proof*⟩

lemma *modification-indep-increments*:
fixes *X Y* :: (*'a* :: *linorder*, *'b*, *'c* :: {*second-countable-topology*, *real-normed-vector*}) *stochastic-process*
assumes *modification X Y sets* (*proc-target Y*) = *sets borel*
shows *indep-increments X* ⟹ *indep-increments Y*
⟨*proof*⟩

end

6 The Kolomgorov-Chentsov theorem

theory *Kolmogorov-Chentsov*
imports *Stochastic-Processes Holder-Continuous Dyadic-Interval Measure-Convergence*
begin

6.1 Supporting lemmas

The main contribution of this file is the Kolmogorov-Chentsov theorem: given a stochastic process that satisfies some continuity properties, we can construct a Hölder continuous modification. We first prove some auxiliary lemmas before moving on to the main construction.

Klenke 5.11: Markov inequality. Compare with $[\![(\lambda x.\ ?u\ x * indicator\ ?A\ x) \in borel\text{-}measurable\ ?M;\ ?A \in sets\ ?M]\!] \Longrightarrow emeasure\ ?M\ \{x \in ?A.\ 1 \leq ?c * ?u\ x\} \leq ?c * set\text{-}nn\text{-}integral\ ?M\ ?A\ ?u$

lemma *nn-integral-Markov-inequality-extended*:
  fixes $f :: real \Rightarrow ennreal$ **and** $\varepsilon :: real$ **and** $X :: 'a \Rightarrow real$
  assumes *mono*: $mono\text{-}on\ (range\ X \cup \{0<..\})\ f$
    and *finite*: $\bigwedge x.\ f\ x < \infty$
    and *e*: $\varepsilon > 0\ f\ \varepsilon > 0$
    and [*measurable*]: $X \in borel\text{-}measurable\ M$
  shows $emeasure\ M\ \{p \in space\ M.\ (X\ p) \geq \varepsilon\} \leq (\int^{+} x.\ f\ (X\ x)\ \partial M)\ /\ f\ \varepsilon$
⟨*proof*⟩

lemma *nn-integral-Markov-inequality-extended-rnv*:
  fixes $f :: real \Rightarrow real$ **and** $\varepsilon :: real$ **and** $X :: 'a \Rightarrow 'b :: real\text{-}normed\text{-}vector$
  assumes [*measurable*]: $X \in borel\text{-}measurable\ M$
    and *mono*: $mono\text{-}on\ \{0..\}\ f$
    and *e*: $\varepsilon > 0\ f\ \varepsilon > 0$
  shows $emeasure\ M\ \{p \in space\ M.\ norm\ (X\ p) \geq \varepsilon\} \leq (\int^{+} x.\ f\ (norm\ (X\ x))\ \partial M)\ /\ f\ \varepsilon$
  ⟨*proof*⟩

6.2 Kolmogorov-Chentsov

Klenke theorem 21.6 - Kolmorogov-Chentsov

locale *Kolmogorov-Chentsov* =
  fixes $X :: (real, 'a, 'b :: polish\text{-}space)\ stochastic\text{-}process$
    and $a\ b\ C\ \gamma :: real$
  assumes *index*[*simp*]: $proc\text{-}index\ X = \{0..\}$
    and *target-borel*[*simp*]: $proc\text{-}target\ X = borel$
    and *gt-0*: $a > 0\ b > 0\ C > 0$
    and *b-leq-a*: $b \leq a$
    and *gamma*: $\gamma \in \{0<..<b/a\}$
    and *expectation*: $\bigwedge s\ t.\ [\![s \geq 0;\ t \geq 0]\!] \Longrightarrow$
        $(\int^{+} x.\ dist\ (X\ t\ x)\ (X\ s\ x)\ powr\ a\ \partial proc\text{-}source\ X) \leq C * dist\ t\ s\ powr\ (1+b)$
begin

lemma *gamma-0-1*[*simp*]: $\gamma \in \{0<..1\}$
  ⟨*proof*⟩

lemma *gamma-gt-0*[*simp*]: $\gamma > 0$

⟨*proof*⟩

lemma *gamma-le-1*[*simp*]: $\gamma \leq 1$
⟨*proof*⟩

abbreviation *source* ≡ *proc-source X*

lemma *X-borel-measurable*[*measurable*]: *X t* ∈ *borel-measurable source* **for** *t*
⟨*proof*⟩

lemma *markov*: $\mathcal{P}(x \text{ in source. } \varepsilon \leq dist\ (X\ t\ x)\ (X\ s\ x)) \leq (C * dist\ t\ s\ powr\ (1 + b))\ /\ \varepsilon\ powr\ a$
if $s \geq 0$ $t \geq 0$ $\varepsilon > 0$ **for** $s\ t\ \varepsilon$
⟨*proof*⟩

lemma *conv-in-prob*:
assumes $t \geq 0$
shows *tendsto-measure* (*proc-source X*) *X* (*X t*) (*at t within* {*0..*})
⟨*proof*⟩

lemma *conv-in-prob-finite*:
assumes $t \geq 0$
shows *tendsto-measure* (*proc-source X*) *X* (*X t*) (*at t within* {*0..T*})
⟨*proof*⟩

lemma *incr*: $\mathcal{P}(x \text{ in source. } 2\ powr\ (- \gamma * n) \leq dist\ (X\ ((k - 1) * 2\ powr - n)\ x)\ (X\ (k * 2\ powr - n)\ x))$
$\leq C * 2\ powr\ (-n * (1+b-a*\gamma))$
if $k \geq 1$ $n \geq 0$ **for** $k\ n$
⟨*proof*⟩

end

In order to construct the modification of X, it suffices to construct a modification of X on $\{0..T\}$ for all finite T, from which we construct the modification on $\{0..\}$ via a countable union.

locale *Kolmogorov-Chentsov-finite* = *Kolmogorov-Chentsov* +
fixes *T* :: *real*
assumes *zero-le-T*: $0 < T$
begin

A_n will characterise the set of states with increments that exceed the bounds required for Hölder continuity. As $n \longrightarrow \infty$, this approaches the set of states for which X is not Hölder continuous. We define N as this limit, and show that N is a null set. On $\omega \in \Omega - N$, we show that $X(\omega)$ is Hölder continuous (and therefore uniformly continuious) on the dyadic rationals, and construct a modification by taking the continuous extension on the reals.

definition $A \equiv \lambda n.$ *if* $2\ \hat{}\ n * T < 1$ *then space source else*

{$x \in$ *space source*.
Max {*dist* (*X* (*real-of-int* (*k* − *1*) ∗ *2* *powr* − *real* *n*) *x*) (*X* (*real-of-int* *k* ∗ *2* *powr* − *real* *n*) *x*)
| *k*. *k* ∈ {*1*..⌊*2*^*n* ∗ *T*⌋}} ≥ *2* *powr* (−γ ∗ *real* *n*)}

abbreviation *B* ≡ λ*n*. (⋃*m*. *A* (*m* + *n*))

abbreviation *N* ≡ ⋂(*range* *B*)

lemma *A-geq*: *2* ^ *n* ∗ *T* ≥ *1* ⟹ *A* *n* = {*x* ∈ *space source*.
Max {*dist* (*X* (*real-of-int* (*k* − *1*) ∗ *2* *powr* − *real* *n*) *x*) (*X* (*real-of-int* *k* ∗ *2* *powr* − *real* *n*) *x*)
| *k*. *k* ∈ {*1*..⌊*2*^*n* ∗ *T*⌋}} ≥ *2* *powr* (−γ ∗ *real* *n*)} **for** *n*
⟨*proof*⟩

lemma *A-measurable*[*measurable*]: *A* *n* ∈ *sets source*
⟨*proof*⟩

lemma *emeasure-A-leq*:
fixes *n* :: *nat*
assumes [*simp*]: *2*^*n* ∗ *T* ≥ *1*
shows *emeasure source* (*A* *n*) ≤ *C* ∗ *T* ∗ *2* *powr* (− *n* ∗ (*b* − *a* ∗ γ))
⟨*proof*⟩

lemma *measure-A-leq*:
assumes *2*^*n* ∗ *T* ≥ *1*
shows *measure source* (*A* *n*) ≤ *C* ∗ *T* ∗ *2* *powr* (− *n* ∗ (*b* − *a* ∗ γ))
⟨*proof*⟩

lemma *summable-A*: *summable* (λ*m*. *measure source* (*A* *m*))
⟨*proof*⟩

lemma *lim-B*: (λ*n*. *measure source* (*B* *n*)) ⟶ *0*
⟨*proof*⟩

lemma *N-null*: *N* ∈ *null-sets source*
⟨*proof*⟩

lemma *notin-N-index*:
assumes ω ∈ *space source* − *N*
obtains n_0 **where** ω ∉ (⋃*n*. *A* (*n* + n_0))
⟨*proof*⟩

context
fixes ω
assumes ω: ω ∈ *space source* − *N*
begin

definition n_0 ≡ *SOME* *m*. ω ∉ (⋃*n*. *A* (*n* + *m*)) ∧ *m* > *0*

lemma
  shows *n-zero*: $\omega \notin (\bigcup n.\ A\ (n + n_0))$
    and *n-zero-nonzero*: $n_0 > 0$
⟨*proof*⟩

lemma *nzero-ge*: $\bigwedge n.\ n \geq n_0 \Longrightarrow 2\hat{\ }n * T \geq 1$
⟨*proof*⟩

lemma *omega-notin*: $\bigwedge n.\ n \geq n_0 \Longrightarrow \omega \notin A\ n$
  ⟨*proof*⟩

Klenke 21.7

lemma *X-dyadic-incr*:
  assumes $k \in \{1..\lfloor 2\hat{\ }n * T\rfloor\}$ $n \geq n_0$
  shows *dist* $(X\ ((\textit{real-of-int}\ k-1)/2\hat{\ }n)\ \omega)\ (X\ (\textit{real-of-int}\ k/2\hat{\ }n)\ \omega) < 2\ \textit{powr}\ (-\ \gamma * n)$
⟨*proof*⟩

Klenke (21.8)

lemma *dist-dyadic-mn*:
  assumes *mn*: $n_0 \leq n$ $n \leq m$
    and *t-dyadic*: $t \in$ *dyadic-interval-step m 0 T*
    and *u-dyadic-n*: $u \in$ *dyadic-interval-step n 0 T*
    and *ut*: $u \leq t$ $t - u < 2/2\hat{\ }n$
  shows *dist* $(X\ u\ \omega)\ (X\ t\ \omega) \leq 2\ \textit{powr}\ (-\ \gamma * n)\ /\ (1 - 2\ \textit{powr}\ -\ \gamma)$
⟨*proof*⟩

lemma *dist-dyadic-fixed*:
  assumes *mn*: $n_0 \leq n$ $n \leq m$
    and *s-dyadic*: $s \in$ *dyadic-interval-step m 0 T*
    and *t-dyadic*: $t \in$ *dyadic-interval-step m 0 T*
    and *st*: $s \leq t$ $t - s \leq 1/2\hat{\ }n$
  shows *dist* $(X\ t\ \omega)\ (X\ s\ \omega) \leq 2 * 2\ \textit{powr}\ (-\ \gamma * n)\ /\ (1 - 2\ \textit{powr}\ -\ \gamma)$
⟨*proof*⟩

definition $C_0 \equiv 2 * 2\ \textit{powr}\ \gamma\ /\ (1 - 2\ \textit{powr}\ -\ \gamma)$

lemma *C-zero-ge*[*simp*]: $C_0 > 0$
  ⟨*proof*⟩

Klenke (21.9)

Let $s, t \in D$ with $|s - t| \leq \frac{1}{2_0^n}$. By choosing the minimal $n \geq n_0$ such that $|t - s| \geq 2^{-n}$, we obtain by $[\![n_0 \leq ?n;\ ?n \leq ?m;\ ?s \in \textit{dyadic-interval-step}\ ?m\ 0\ T;\ ?t \in \textit{dyadic-interval-step}\ ?m\ 0\ T;\ ?s \leq ?t;\ ?t - ?s \leq 1\ /\ 2^{?n}]\!] \Longrightarrow \textit{dist}\ (\textit{process}\ X\ ?t\ \omega)\ (\textit{process}\ X\ ?s\ \omega) \leq 2 * 2\ \textit{powr}\ (-\ \gamma * \textit{real}\ ?n)\ /\ (1 - 2\ \textit{powr}\ -\ \gamma)$:

$$|X_t(\omega) - X_s(\omega)| \leq C_0|t - s|^\gamma$$

lemma *dist-dyadic*:
  assumes *t*: $t \in$ *dyadic-interval* *0* *T*
    and *s*: $s \in$ *dyadic-interval* *0* *T*
    and *st-dist*: *dist t s* $\leq 1 \; / \; 2 \; \hat{} \; n_0$
  shows *dist* (*X t* ω) (*X s* ω) $\leq C_0 *$ (*dist t s*) *powr* γ
⟨*proof*⟩

definition $K \equiv C_0 * (2\hat{}nat \lceil 2\hat{}n_0 * T \rceil)$ *powr* $(1 - \gamma)$

lemma C_0*-le-K*: $C_0 \leq K$
  ⟨*proof*⟩

lemma *K-pos*: $0 < K$
  ⟨*proof*⟩

Klenke (21.10)

lemma *X-dyadic-le-K'*:
  assumes *dyadic*: $s \in$ *dyadic-interval* *0* *T* $t \in$ *dyadic-interval* *0* *T*
    and *st*: $s \leq t$
  shows *dist* (*X s* ω) (*X t* ω) $\leq K *$ *dist s t powr* γ
⟨*proof*⟩

lemma *X-dyadic-le-K*:
  assumes $s \in$ *dyadic-interval* *0* *T*
    and $t \in$ *dyadic-interval* *0* *T*
  shows *dist* (*X s* ω) (*X t* ω) $\leq K *$ *dist s t powr* γ
  ⟨*proof*⟩

corollary *holder-dyadic*: γ*−holder-on* (*dyadic-interval* *0* *T*) ($\lambda t.$ *X t* ω)
  ⟨*proof*⟩

lemma *uniformly-continuous-dyadic*: *uniformly-continuous-on* (*dyadic-interval* *0* *T*) ($\lambda t.$ *X t* ω)
  ⟨*proof*⟩

lemma *Lim-exists*: $\exists L.$ (($\lambda s.$ *X s* ω) $\longrightarrow L$) (*at t within* (*dyadic-interval* *0* *T*))
  if $t \in \{0..T\}$
  ⟨*proof*⟩

lemma *Lim-unique*: $\exists! L.$ (($\lambda s.$ *X s* ω) $\longrightarrow L$) (*at t within* (*dyadic-interval* *0* *T*))

  if $t \in \{0..T\}$
  ⟨*proof*⟩

definition $L \equiv$ ($\lambda t.$ (*Lim* (*at t within dyadic-interval* *0* *T*) ($\lambda s.$ *X s* ω)))

lemma *X-tendsto-L*:
  assumes $t \in \{0..T\}$

shows $((\lambda s.\ X\ s\ \omega) \longrightarrow L\ t)$ (*at t within* (*dyadic-interval 0 T*))
⟨*proof*⟩

lemma *L-dist-K*:
assumes *s*: $s \in \{0..T\}$
and *t*: $t \in \{0..T\}$
shows *dist* $(L\ s)\ (L\ t) \leq K * $ *dist s t powr* γ
⟨*proof*⟩

corollary *L-holder*: γ*−holder-on* $\{0..T\}$ *L*
⟨*proof*⟩

corollary *L-local-holder*: *local-holder-on* γ $\{0..T\}$ *L*
⟨*proof*⟩

lemma *X-dyadic-eq-L*:
assumes $t \in$ *dyadic-interval 0 T*
shows $X\ t\ \omega = L\ t$
⟨*proof*⟩
end

definition *default* :: *'b* **where** *default* = (*SOME x. True*)

definition *X-tilde* :: *real* ⇒ *'a* ⇒ *'b* **where**
X-tilde ≡ ($\lambda t\ \omega$. *if* $\omega \in N$ *then default else* (*Lim* (*at t within dyadic-interval 0 T*) ($\lambda s.\ X\ s\ \omega$)))

lemma *X-tilde-not-N-Lim*:
assumes $\omega \in$ *space source* − *N*
shows *X-tilde t* ω = *Lim* (*at t within dyadic-interval 0 T*) ($\lambda s.\ X\ s\ \omega$)
⟨*proof*⟩

lemma *X-tilde-not-N-L*:
assumes $\omega \in$ *space source* − *N*
shows *X-tilde t* $\omega = L\ \omega\ t$
⟨*proof*⟩

lemma *local-holder-X-tilde*: *local-holder-on* γ $\{0..T\}$ (λt. *X-tilde t* ω)
if $\omega \in$ *space source* **for** ω
⟨*proof*⟩

corollary *X-tilde-eq-L-AE*: *AE* ω *in source. X-tilde t* $\omega = L\ \omega\ t$
⟨*proof*⟩

corollary *X-tilde-eq-Lim-AE*:
AE ω *in source. X-tilde t* ω = *Lim* (*at t within dyadic-interval 0 T*) ($\lambda s.\ X\ s\ \omega$)
⟨*proof*⟩

lemma *X-tilde-tendsto-AE*: $t \in \{0..T\} \Longrightarrow$ *tendsto-AE source X* (*X-tilde t*) (*at t*

within dyadic-interval 0 T)
⟨*proof*⟩

end

context *Kolmogorov-Chentsov-finite*
begin

By (21.5) $0 \leq ?t \Longrightarrow$ *tendsto-measure source* (*process X*) (*process X ?t*) (*at ?t within* $\{0..?T\}$) and (21.11) $?\omega \in$ *space source* $- (\bigcap_n \bigcup_m A\ (m + n))$ $\Longrightarrow L\ ?\omega \equiv \lambda t.$ *Lim* (*at t within dyadic-interval 0 T*) ($\lambda s.$ *process X s* $?\omega$), $P[X \neq \tilde{X}] = 0$

lemma *X-tilde-measurable*[*measurable*]:
assumes $t \in \{0..T\}$
shows *X-tilde t* $\in$ *borel-measurable source*
⟨*proof*⟩

lemma *X-eq-X-tilde-AE*: *AE* ω *in source. X t* ω = *X-tilde t* ω **if** $t \in \{0..T\}$ **for** t
⟨*proof*⟩

lemma *X-tilde-modification*: *modification* (*restrict-index X* $\{0..T\}$)
(*prob-space.process-of source* (*proc-target X*) $\{0..T\}$ *X-tilde default*)
⟨*proof*⟩
end

We have now shown that we can construct a modification of X for any interval $\{0..T\}$. We want to extend this result to construct a modification on the interval $\{0..\}$ - this can be constructed by gluing together all modifications with natural-valued T which results in a countable union of modifications, which itself is a modification.

context *Kolmogorov-Chentsov*
begin

lemma *Kolmogorov-Chentsov-finite*: $T > 0 \Longrightarrow$ *Kolmogorov-Chentsov-finite X a b C* γ *T*
⟨*proof*⟩

definition *Mod* $\equiv \lambda T.$ *SOME Y. modification* (*restrict-index X* $\{0..T\}$) $Y \wedge$
($\forall x \in$*space source. local-holder-on* γ $\{0..T\}$ ($\lambda t.$ *Y t x*))

lemma *Mod*: *modification* (*restrict-index X* $\{0..T\}$) (*Mod T*)
($\forall x \in$*space source. local-holder-on* γ $\{0..T\}$ ($\lambda t.$ (*Mod T*) *t x*)) **if** $0 < T$ **for** T
⟨*proof*⟩

lemma *compatible-Mod*: *compatible* (*restrict-index X* $\{0..T\}$) (*Mod T*) **if** $0 < T$ **for** T
⟨*proof*⟩

lemma *Mod-source*[*simp*]: *proc-source* (*Mod T*) = *source* **if** *0* < *T* **for** *T*
⟨*proof*⟩

lemma *Mod-target*: *sets* (*proc-target* (*Mod T*)) = *sets* (*proc-target X*) **if** *0* < *T* **for** *T*
⟨*proof*⟩

lemma *Mod-index*[*simp*]: *0* < *T* ⟹ *proc-index* (*Mod T*) = {*0*..*T*}
⟨*proof*⟩

lemma *indistinguishable-mod*:
indistinguishable (*restrict-index* (*Mod S*) {*0*..*min S T*}) (*restrict-index* (*Mod T*) {*0*..*min S T*})
if *S* > *0* *T* > *0* **for** *S T*
⟨*proof*⟩

definition *N S T* ≡ *SOME N*. *N* ∈ *null-sets source* ∧ {ω ∈ *space source*. ∃ *t* ∈ {*0*..*min S T*}. (*Mod S*) *t* ω ≠ (*Mod T*) *t* ω} ⊆ *N*

lemma *N*:
assumes *S* > *0* *T* > *0*
shows *N S T* ∈ *null-sets source* ∧ {ω ∈ *space source*. ∃ *t* ∈ {*0*..*min S T*}. (*Mod S*) *t* ω ≠ (*Mod T*) *t* ω} ⊆ *N S T*
⟨*proof*⟩

definition *N-inf* **where** *N-inf* ≡ (⋃ *S* ∈ ℕ − {*0*}. (⋃ *T* ∈ ℕ − {*0*}. *N S T*))

lemma *N-inf-null*: *N-inf* ∈ *null-sets source*
⟨*proof*⟩

lemma *Mod-eq-N-inf*: (*Mod S*) *t* ω = (*Mod T*) *t* ω
if ω ∈ *space source* − *N-inf* *t* ∈ {*0*..*min S T*} *S* ∈ ℕ − {*0*} *T* ∈ ℕ − {*0*} **for** ω *t S T*
⟨*proof*⟩

definition *default* :: ′*b* **where** *default* = (*SOME x*. *True*)

definition *X-mod* ≡ λ*t* ω. *if* ω ∈ *space source* − *N-inf* *then* (*Mod* ⌊*t*+*1*⌋) *t* ω *else default*

definition *X-mod-process* ≡ *prob-space.process-of source* (*proc-target X*) {*0*..} *X-mod default*

lemma *Mod-measurable*[*measurable*]: ∀ *t*∈{*0*..}. *X-mod t* ∈ *source* $\rightarrow_M$ *proc-target X*
⟨*proof*⟩

lemma *modification-X-mod-process*: *modification X X-mod-process*
⟨*proof*⟩

lemma *local-holder-X-mod*: *local-holder-on* γ $\{0..\}$ ($\lambda t.$ *X-mod* t ω) **for** ω
$\langle proof \rangle$

lemma *local-holder-X-mod-process*: *local-holder-on* γ $\{0..\}$ ($\lambda t.$ *X-mod-process* t ω)
for ω
$\langle proof \rangle$

theorem *continuous-modification*:
$\exists X'.$ *modification* X X' $\wedge$ ($\forall \omega.$ *local-holder-on* γ $\{0..\}$ ($\lambda t.$ X' t ω))
$\langle proof \rangle$
end

theorem *Kolmogorov-Chentsov*:
fixes X :: (*real*, $'a$, $'b$ :: *polish-space*) *stochastic-process*
and a b C γ :: *real*
assumes *index*[*simp*]: *proc-index* $X = \{0..\}$
and *target-borel*[*simp*]: *proc-target* X = *borel*
and *gt-0*: $a > 0$ $b > 0$ $C > 0$
and *b-leq-a*: $b \leq a$
and *gamma*: $\gamma \in \{0<..<b/a\}$
and *expectation*: $\bigwedge s\ t.\ [\![s \geq 0;\ t \geq 0]\!] \Longrightarrow$
($\int^{+}$ $x.$ *dist* (X t x) (X s x) *powr* a ∂*proc-source* X) $\leq$ C $*$ *dist* t s *powr*
$(1+b)$
shows $\exists X'.$ *modification* X X' $\wedge$ ($\forall \omega.$ *local-holder-on* γ $\{0..\}$ ($\lambda t.$ X' t ω))
$\langle proof \rangle$
end

References

[1] A. Klenke. *Probability theory: a comprehensive course*. Springer Science & Business Media, 2020.