The Kolmogorov-Chentsov Theorem

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Abstract

Continuous-time stochastic processes often carry the condition of having almost-surely continuous paths. If some process X satisfies certain bounds on its expectation, then the Kolmogorov-Chentsov theorem lets us construct a modification of X, i.e. a process X' such that $\forall t. X_t = X'_t$ almost surely, that has Hölder continuous paths.

In this work, we mechanise the Kolmogorov-Chentsov theorem. To get there, we develop a theory of stochastic processes, together with Hölder continuity, convergence in measure, and arbitrary intervals of dyadic rationals.

With this, we pave the way towards a construction of Brownian motion. The work is based on the exposition in Achim Klenke's probability theory text [1].

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1 Supporting lemmas

theory *Kolmogorov-Chentsov-Extras*
imports *HOL−Probability.Probability*
begin

lemma *atLeastAtMost-induct*[*consumes 1*, *case-names base Suc*]:
assumes $x \in \{n..m\}$
and $P\ n$
and $\bigwedge k.\ [\![k \geq n;\ k < m;\ P\ k]\!] \Longrightarrow P\ (Suc\ k)$
shows $P\ x$
⟨*proof*⟩

lemma *eventually-prodI'*:
assumes *eventually* $P\ F$ *eventually* $Q\ G\ \forall x\ y.\ P\ x \longrightarrow Q\ y \longrightarrow R\ (x,y)$
shows *eventually* $R\ (F \times_F G)$
⟨*proof*⟩

Analogous to $[\![$*almost-everywhere ?M ?P*; $\bigwedge N.\ [\![\bigwedge x.\ x \in$ *space ?M* $- N \Longrightarrow$ *?P x*; $N \in$ *null-sets ?M*$]\!] \Longrightarrow$ *?thesis*$]\!] \Longrightarrow$ *?thesis*

lemma *AE-I3*:
assumes $\bigwedge x.\ x \in$ *space* $M - N \Longrightarrow P\ x\ N \in$ *null-sets* M
shows *AE x in M*. $P\ x$
⟨*proof*⟩

Extends $[\![(?f \longrightarrow ?l)\ ?F;\ (?g \longrightarrow ?m)\ ?F]\!] \Longrightarrow ((\lambda x.\ dist\ (?f\ x)\ (?g\ x)) \longrightarrow dist\ ?l\ ?m)\ ?F$

lemma *tendsto-dist-prod*:
fixes $l\ m :: 'a ::$ *metric-space*
assumes f: $(f \longrightarrow l)\ F$
and g: $(g \longrightarrow m)\ G$
shows $((\lambda x.\ dist\ (f\ (fst\ x))\ (g\ (snd\ x))) \longrightarrow dist\ l\ m)\ (F \times_F G)$
⟨*proof*⟩

lemma *borel-measurable-at-within-metric* [*measurable*]:
fixes $f :: 'a ::$ *first-countable-topology* $\Rightarrow 'b \Rightarrow 'c ::$ *metric-space*
assumes [*measurable*]: $\bigwedge t.\ t \in S \Longrightarrow f\ t \in$ *borel-measurable* M
and *convergent*: $\bigwedge \omega.\ \omega \in$ *space* $M \Longrightarrow \exists\, l.\ ((\lambda t.\ f\ t\ \omega) \longrightarrow l)\ (at\ x\ within\ S)$
and *nontrivial*: *at x within* $S \neq \bot$
shows $(\lambda \omega.\ Lim\ (at\ x\ within\ S)\ (\lambda t.\ f\ t\ \omega)) \in$ *borel-measurable* M
⟨*proof*⟩

lemma *Max-finite-image-ex*:
assumes *finite* S $S \neq \{\}$ $P\ (MAX\ k{\in}S.\ f\ k)$
shows $\exists\, k \in S.\ P\ (f\ k)$
⟨*proof*⟩

lemma *measure-leq-emeasure-ennreal*: $0 \leq x \Longrightarrow$ *emeasure* $M\ A \leq$ *ennreal* $x \Longrightarrow$ *measure* $M\ A \leq x$

⟨*proof*⟩

lemma *Union-add-subset*: $(m :: nat) \leq n \Longrightarrow (\bigcup k.\ A\ (k + n)) \subseteq (\bigcup k.\ A\ (k + m))$
⟨*proof*⟩

lemma *floor-in-Nats* [*simp*]: $x \geq 0 \Longrightarrow \lfloor x \rfloor \in \mathbb{N}$
⟨*proof*⟩

lemma *triangle-ineq-list*:
fixes $l :: ('a :: metric\text{-}space)\ list$
assumes $l \neq []$
shows $dist\ (hd\ l)\ (last\ l) \leq (\sum\ i{=}1..length\ l - 1.\ dist\ (l!i)\ (l!(i-1)))$
⟨*proof*⟩

lemma *triangle-ineq-sum*:
fixes $f :: nat \Rightarrow 'a :: metric\text{-}space$
assumes $n \leq m$
shows $dist\ (f\ n)\ (f\ m) \leq (\sum\ i{=}Suc\ n..m.\ dist\ (f\ i)\ (f\ (i-1)))$
⟨*proof*⟩

lemma (**in** *product-prob-space*) *indep-vars-PiM-coordinate*:
assumes $I \neq \{\}$
shows *prob-space.indep-vars* $(\Pi_M\ i{\in}I.\ M\ i)\ M\ (\lambda x\ f.\ f\ x)\ I$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-indep-set*:
assumes *indep-sets* $F\ I\ i \in I\ j \in I\ i \neq j$
shows *indep-set* $(F\ i)\ (F\ j)$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-indep-var*:
assumes *indep-vars* $M'\ X\ I\ i \in I\ j \in I\ i \neq j$
shows *indep-var* $(M'\ i)\ (X\ i)\ (M'\ j)\ (X\ j)$
⟨*proof*⟩

end

2 Intervals of dyadic rationals

theory *Dyadic-Interval*
imports *HOL−Analysis.Analysis*
begin

In this file we describe intervals of dyadic numbers $S..T$ for reals S T. We use the floor and ceiling functions to approximate the numbers with increasing accuracy.

lemma *frac-floor*: $\lfloor x \rfloor = x - frac\ x$

⟨*proof*⟩

lemma *frac-ceil*: $\lceil x \rceil = x + frac\ (-\ x)$
⟨*proof*⟩

lemma *floor-pow2-lim*: $(\lambda n.\ \lfloor 2\hat{}n * T \rfloor\ /\ 2\ \hat{}\ n) \longrightarrow T$
⟨*proof*⟩

lemma *floor-pow2-leq*: $\lfloor 2\hat{}n * T \rfloor\ /\ 2\ \hat{}\ n \leq T$
⟨*proof*⟩

lemma *ceil-pow2-lim*: $(\lambda n.\ \lceil 2\hat{}n * T \rceil\ /\ 2\ \hat{}\ n) \longrightarrow T$
⟨*proof*⟩

lemma *ceil-pow2-geq*: $\lceil 2\hat{}n * T \rceil\ /\ 2\ \hat{}\ n \geq T$
⟨*proof*⟩

`dyadic_interval_step` $n\ S\ T$ is the collection of dyadic numbers in $\{S..T\}$ with denominator 2^n. As $n \to \infty$ this collection approximates $\{S..T\}$. Compare with $dyadics \equiv \bigcup_{k\ m} \{of\text{-}nat\ m\ /\ (2::?'a)^k\}$

definition *dyadic-interval-step* :: *nat* ⇒ *real* ⇒ *real* ⇒ *real set*
where *dyadic-interval-step n S T* ≡ $(\lambda k.\ k\ /\ (2\ \hat{}\ n))$ ‘ $\{\lceil 2\hat{}n * S \rceil .. \lfloor 2\hat{}n * T \rfloor\}$

definition *dyadic-interval* :: *real* ⇒ *real* ⇒ *real set*
where *dyadic-interval S T* ≡ $(\bigcup n.$ *dyadic-interval-step n S T*$)$

lemma *dyadic-interval-step-empty*[*simp*]: $T < S \Longrightarrow$ *dyadic-interval-step n S T* = {}
⟨*proof*⟩

lemma *dyadic-interval-step-singleton*[*simp*]: $X \in \mathbb{Z} \Longrightarrow$ *dyadic-interval-step n X X* = $\{X\}$
⟨*proof*⟩

lemma *dyadic-interval-step-zero* [*simp*]: *dyadic-interval-step 0 S T* = *real-of-int* ‘ $\{\lceil S \rceil\ ..\ \lfloor T \rfloor\}$
⟨*proof*⟩

lemma *dyadic-interval-step-mem* [*intro*]:
assumes*x* ≥ *0 T* ≥ *0 x* ≤ *T*
shows $\lfloor 2\hat{}n * x \rfloor\ /\ 2\ \hat{}\ n \in$ *dyadic-interval-step n 0 T*
⟨*proof*⟩

lemma *dyadic-interval-step-iff*:
x ∈ *dyadic-interval-step n S T* ⟷
$(\exists k.\ k \geq \lceil 2\hat{}n * S \rceil \wedge k \leq \lfloor 2\hat{}n * T \rfloor \wedge x = k/2\hat{}n)$
⟨*proof*⟩

lemma *dyadic-interval-step-memI* [*intro*]:

assumes $\exists\, k{::}int.\ x = k/2\hat{}n\ x \geq S\ x \leq T$
shows $x \in$ *dyadic-interval-step* $n\ S\ T$
⟨*proof*⟩

lemma *mem-dyadic-interval*: $x \in$ *dyadic-interval* $S\ T \longleftrightarrow (\exists\, n.\ x \in$ *dyadic-interval-step* $n\ S\ T)$
⟨*proof*⟩

lemma *mem-dyadic-intervalI*: $\exists\, n.\ x \in$ *dyadic-interval-step* $n\ S\ T \Longrightarrow x \in$ *dyadic-interval* $S\ T$
⟨*proof*⟩

lemma *dyadic-step-leq*: $x \in$ *dyadic-interval-step* $n\ S\ T \Longrightarrow x \leq T$
⟨*proof*⟩

lemma *dyadics-leq*: $x \in$ *dyadic-interval* $S\ T \Longrightarrow x \leq T$
⟨*proof*⟩

lemma *dyadic-step-geq*: $x \in$ *dyadic-interval-step* $n\ S\ T \Longrightarrow x \geq S$
⟨*proof*⟩

lemma *dyadics-geq*: $x \in$ *dyadic-interval* $S\ T \Longrightarrow x \geq S$
⟨*proof*⟩

corollary *dyadic-interval-subset-interval* [*simp*]: (*dyadic-interval* $0\ T) \subseteq \{0..T\}$
⟨*proof*⟩

lemma *zero-in-dyadics*: $T \geq 0 \Longrightarrow 0 \in$ *dyadic-interval-step* $n\ 0\ T$
⟨*proof*⟩

The following theorem is useful for reasoning with at_within

lemma *dyadic-interval-converging-sequence*:
assumes $t \in \{0..T\}\ T \neq 0$
shows $\exists\, s.\ \forall\, n.\ s\ n \in$ *dyadic-interval* $0\ T - \{t\} \wedge s \longrightarrow t$
⟨*proof*⟩

lemma *dyadic-interval-dense*: *closure* (*dyadic-interval* $0\ T) = \{0..T\}$
⟨*proof*⟩

corollary *dyadic-interval-islimpt*:
assumes $T > 0\ t \in \{0..T\}$
shows t *islimpt dyadic-interval* $0\ T$
⟨*proof*⟩

corollary *at-within-dyadic-interval-nontrivial*[*simp*]:
assumes $T > 0\ t \in \{0..T\}$
shows (*at* t *within dyadic-interval* $0\ T) \neq bot$
⟨*proof*⟩

lemma *dyadic-interval-step-finite*[*simp*]: *finite* (*dyadic-interval-step n S T*)
⟨*proof*⟩

lemma *dyadic-interval-countable*[*simp*]: *countable* (*dyadic-interval S T*)
⟨*proof*⟩

lemma *floor-pow2-add-leq*:
fixes *T* :: *real*
shows ⌊*2^n * T*⌋ / *2 ^ n* ≤ ⌊*2 ^ (n+k) * T*⌋ / *2 ^ (n+k)*
⟨*proof*⟩

corollary *floor-pow2-mono*: *mono* (λ*n*. ⌊*2^n * (T* :: *real)*⌋ / *2 ^ n*)
⟨*proof*⟩

lemma *dyadic-interval-step-Max*: *T* ≥ *0* ⟹ *Max* (*dyadic-interval-step n 0 T*) = ⌊*2^n * T*⌋ / *2^n*
⟨*proof*⟩

lemma *dyadic-interval-step-subset*:
n ≤ *m* ⟹ *dyadic-interval-step n 0 T* ⊆ *dyadic-interval-step m 0 T*
⟨*proof*⟩

corollary *dyadic-interval-step-mono*:
assumes *x* ∈ *dyadic-interval-step n 0 T n* ≤ *m*
shows *x* ∈ *dyadic-interval-step m 0 T*
⟨*proof*⟩

lemma *dyadic-as-natural*:
assumes *x* ∈ *dyadic-interval-step n 0 T*
shows ∃!*k*. *x* = *real k / 2 ^ n*
⟨*proof*⟩

lemma *dyadic-of-natural*:
assumes *real k / 2 ^ n* ≤ *T*
shows *real k / 2 ^ n* ∈ *dyadic-interval-step n 0 T*
⟨*proof*⟩

lemma *dyadic-interval-minus*:
assumes *x* ∈ *dyadic-interval-step n 0 T y* ∈ *dyadic-interval-step n 0 T x* ≤ *y*
shows *y − x* ∈ *dyadic-interval-step n 0 T*
⟨*proof*⟩

lemma *dyadic-times-nat*: *x* ∈ *dyadic-interval-step n 0 T* ⟹ (*x * 2 ^ n*) ∈ ℕ
⟨*proof*⟩

definition *dyadic-expansion x n b k* ≡ *set b* ⊆ {*0*,*1*}
∧ *length b* = *n* ∧ *x* = *real-of-int k* + (∑ *m*∈{*1..n*}. *real* (*b* ! (*m−1*)) / *2 ^ m*)

lemma *dyadic-expansionI*:

assumes *set b* $\subseteq \{0,1\}$ *length b* $= n$ $x = k + (\sum m{\in}\{1..n\}.\ (b\ !\ (m-1))\ /\ 2\ \hat{}\ m)$
shows *dyadic-expansion x n b k*
⟨*proof*⟩

lemma *dyadic-expansionD*:
assumes *dyadic-expansion x n b k*
shows *set b* $\subseteq \{0,1\}$
and *length b* $= n$
and $x = k + (\sum m{\in}\{1..n\}.\ (b\ !\ (m-1))\ /\ 2\ \hat{}\ m)$
⟨*proof*⟩

lemma *dyadic-expansion-ex*:
assumes $x \in$ *dyadic-interval-step n 0 T*
shows $\exists\, b\ k.$ *dyadic-expansion x n b k*
⟨*proof*⟩

lemma *dyadic-expansion-frac-le-1*:
assumes *dyadic-expansion x n b k*
shows $(\sum m{\in}\{1..n\}.\ (b\ !\ (m-1))\ /\ 2\ \hat{}\ m) < 1$
⟨*proof*⟩

lemma *dyadic-expansion-frac-range*:
assumes *dyadic-expansion x n b k* $m \in \{1..n\}$
shows $b\ !\ (m-1) \in \{0,1\}$
⟨*proof*⟩

lemma *dyadic-expansion-interval*:
assumes *dyadic-expansion x n b k* $x \in \{S..T\}$
shows $x \in$ *dyadic-interval-step n S T*
⟨*proof*⟩

lemma *dyadic-expansion-nth-geq*:
assumes *dyadic-expansion x n b k* $m \in \{1..n\}$ $b\ !\ (m-1) = 1$
shows $x \geq k + 1/2\hat{}m$
⟨*proof*⟩

lemma *dyadic-expansion-frac-geq-0*:
assumes *dyadic-expansion x n b k*
shows $(\sum m{\in}\{1..n\}.\ (b\ !\ (m-1))\ /\ 2\ \hat{}\ m) \geq 0$
⟨*proof*⟩

lemma *dyadic-expansion-frac*:
assumes *dyadic-expansion x n b k*
shows *frac* $x = (\sum m{\in}\{1..n\}.\ (b\ !\ (m-1))/\ 2\ \hat{}\ m)$
⟨*proof*⟩

lemma *dyadic-expansion-floor*:
assumes *dyadic-expansion x n b k*

shows $k = \lfloor x \rfloor$
⟨*proof*⟩

lemma *sum-interval-pow2-inv*: $(\sum m{\in}\{Suc\ l..n\}.\ (1 :: real)\ /\ 2\ \hat{}\ m) = 1\ /\ 2\hat{}l - 1/2\hat{}n$ **if** $l < n$
⟨*proof*⟩

lemma *dyadic-expansion-unique*:
assumes *dyadic-expansion x n b k*
and *dyadic-expansion x n c j*
shows $b = c \wedge j = k$
⟨*proof*⟩

end

3 Hölder continuity

theory *Holder-Continuous*
imports *HOL−Analysis.Analysis*
begin

Hölder continuity is a weaker version of Lipschitz continuity.

definition *holder-at-within* :: *real* ⇒ *'a set* ⇒ *'a* ⇒ (*'a* :: *metric-space* ⇒ *'b* :: *metric-space*) ⇒ *bool* **where**
holder-at-within $\gamma\ D\ r\ \varphi \equiv \gamma \in \{0<..1\} \wedge$
$(\exists \varepsilon > 0.\ \exists C \geq 0.\ \forall s \in D.\ dist\ r\ s < \varepsilon \longrightarrow dist\ (\varphi\ r)\ (\varphi\ s) \leq C * dist\ r\ s\ powr\ \gamma)$

definition *local-holder-on* :: *real* ⇒ *'a* :: *metric-space set* ⇒ (*'a* ⇒ *'b* :: *metric-space*) ⇒ *bool* **where**
local-holder-on $\gamma\ D\ \varphi \equiv \gamma \in \{0<..1\} \wedge$
$(\forall t{\in}D.\ \exists \varepsilon > 0.\ \exists C \geq 0.\ (\forall r{\in}D.\ \forall s{\in}D.\ dist\ s\ t < \varepsilon \wedge dist\ r\ t < \varepsilon \longrightarrow dist\ (\varphi\ r)\ (\varphi\ s) \leq C * dist\ r\ s\ powr\ \gamma))$

definition *holder-on* :: *real* ⇒ *'a* :: *metric-space set* ⇒ (*'a* ⇒ *'b* :: *metric-space*) ⇒ *bool* (*-−holder'-on 1000*) **where**
$\gamma{-}holder\text{-}on\ D\ \varphi \longleftrightarrow \gamma \in \{0<..1\} \wedge (\exists C \geq 0.\ (\forall r{\in}D.\ \forall s{\in}D.\ dist\ (\varphi\ r)\ (\varphi\ s) \leq C * dist\ r\ s\ powr\ \gamma))$

lemma *holder-onI*:
assumes $\gamma \in \{0<..1\}\ \exists C \geq 0.\ (\forall r{\in}D.\ \forall s{\in}D.\ dist\ (\varphi\ r)\ (\varphi\ s) \leq C * dist\ r\ s\ powr\ \gamma)$
shows $\gamma{-}holder\text{-}on\ D\ \varphi$
⟨*proof*⟩

We prove various equivalent formulations of local holder continuity, using open and closed balls and inequalities.

lemma *local-holder-on-cball*:

local-holder-on γ D $\varphi \longleftrightarrow \gamma \in \{0<..1\} \wedge$
$(\forall t{\in}D.\ \exists \varepsilon > 0.\ \exists C \geq 0.\ (\forall r{\in}cball\ t\ \varepsilon \cap D.\ \forall s{\in}cball\ t\ \varepsilon \cap D.\ dist\ (\varphi\ r)\ (\varphi\ s) \leq C * dist\ r\ s\ powr\ \gamma))$
(**is** *?L* $\longleftrightarrow$ *?R*)
⟨*proof*⟩

corollary *local-holder-on-leq-def*: *local-holder-on* γ D $\varphi \longleftrightarrow \gamma \in \{0<..1\} \wedge$
$(\forall t{\in}D.\ \exists \varepsilon > 0.\ \exists C \geq 0.\ (\forall r{\in}D.\ \forall s{\in}D.\ dist\ s\ t \leq \varepsilon \wedge dist\ r\ t \leq \varepsilon \longrightarrow dist\ (\varphi\ r)\ (\varphi\ s) \leq C * dist\ r\ s\ powr\ \gamma))$
⟨*proof*⟩

corollary *local-holder-on-ball*: *local-holder-on* γ D $\varphi \longleftrightarrow \gamma \in \{0<..1\} \wedge$
$(\forall t{\in}D.\ \exists \varepsilon > 0.\ \exists C \geq 0.\ (\forall r{\in}ball\ t\ \varepsilon \cap D.\ \forall s{\in}ball\ t\ \varepsilon \cap D.\ dist\ (\varphi\ r)\ (\varphi\ s) \leq C * dist\ r\ s\ powr\ \gamma))$
⟨*proof*⟩

lemma *local-holder-on-altdef*:
assumes $D \neq \{\}$
shows *local-holder-on* γ D φ = $(\forall t \in D.\ (\exists \varepsilon > 0.\ (\gamma{-}holder\text{-}on\ ((cball\ t\ \varepsilon) \cap D)\ \varphi)))$
⟨*proof*⟩

lemma *local-holder-on-cong*[*cong*]:
assumes $\gamma = \varepsilon$ $C = D$ $\bigwedge x.\ x \in C \Longrightarrow \varphi\ x = \psi\ x$
shows *local-holder-on* γ C $\varphi \longleftrightarrow$ *local-holder-on* ε D ψ
⟨*proof*⟩

lemma *local-holder-onI*:
assumes $\gamma \in \{0<..1\}$ $(\forall t{\in}D.\ \exists \varepsilon > 0.\ \exists C \geq 0.\ (\forall r{\in}D.\ \forall s{\in}D.\ dist\ s\ t < \varepsilon \wedge dist\ r\ t < \varepsilon \longrightarrow dist\ (\varphi\ r)\ (\varphi\ s) \leq C * dist\ r\ s\ powr\ \gamma))$
shows *local-holder-on* γ D φ
⟨*proof*⟩

lemma *local-holder-ballI*:
assumes $\gamma \in \{0<..1\}$
and $\bigwedge t.\ t \in D \Longrightarrow \exists \varepsilon > 0.\ \exists C \geq 0.\ \forall r{\in}ball\ t\ \varepsilon \cap D.\ \forall s{\in}ball\ t\ \varepsilon \cap D.$
$dist\ (\varphi\ r)\ (\varphi\ s) \leq C * dist\ r\ s\ powr\ \gamma$
shows *local-holder-on* γ D φ
⟨*proof*⟩

lemma *local-holder-onE*:
assumes *local-holder*: *local-holder-on* γ D φ
and *gamma*: $\gamma \in \{0<..1\}$
and $t \in D$
obtains ε C **where** $\varepsilon > 0$ $C \geq 0$
$\bigwedge r\ s.\ r \in ball\ t\ \varepsilon \cap D \Longrightarrow s \in ball\ t\ \varepsilon \cap D \Longrightarrow dist\ (\varphi\ r)\ (\varphi\ s) \leq C * dist\ r\ s\ powr\ \gamma$
⟨*proof*⟩

Holder continuity matches up with the existing definitions in *HOL−Analysis.Lipschitz*

lemma *holder-1-eq-lipschitz*: $1-$*holder-on* $D\ \varphi = (\exists\, C.$ *lipschitz-on* $C\ D\ \varphi)$
⟨*proof*⟩

lemma *local-holder-1-eq-local-lipschitz*:
assumes $T \neq \{\}$
shows *local-holder-on* $1\ D\ \varphi =$ *local-lipschitz* $T\ D\ (\lambda\text{-}.\ \varphi)$
⟨*proof*⟩

lemma *local-holder-refine*:
assumes g: *local-holder-on* $g\ D\ \varphi\ g \leq 1$
and h: $h \leq g\ h > 0$
shows *local-holder-on* $h\ D\ \varphi$
⟨*proof*⟩

lemma *holder-uniform-continuous*:
assumes $\gamma-$*holder-on* $X\ \varphi$
shows *uniformly-continuous-on* $X\ \varphi$
⟨*proof*⟩

corollary *holder-on-continuous-on*: $\gamma-$*holder-on* $X\ \varphi \Longrightarrow$ *continuous-on* $X\ \varphi$
⟨*proof*⟩

lemma *holder-implies-local-holder*: $\gamma-$*holder-on* $D\ \varphi \Longrightarrow$ *local-holder-on* $\gamma\ D\ \varphi$
⟨*proof*⟩

lemma *local-holder-imp-continuous*:
assumes *local-holder*: *local-holder-on* $\gamma\ X\ \varphi$
shows *continuous-on* $X\ \varphi$
⟨*proof*⟩

lemma *local-holder-compact-imp-holder*:
assumes *compact I local-holder-on* $\gamma\ I\ \varphi$
shows $\gamma-$*holder-on* $I\ \varphi$
⟨*proof*⟩

lemma *holder-const*: $\gamma-$*holder-on* $C\ (\lambda\text{-}.\ c) \longleftrightarrow \gamma \in \{0<..1\}$
⟨*proof*⟩

lemma *local-holder-const*: *local-holder-on* $\gamma\ C\ (\lambda\text{-}.\ c) \longleftrightarrow \gamma \in \{0<..1\}$
⟨*proof*⟩

end

4 Convergence in measure

theory *Measure-Convergence*
imports *HOL−Probability.Probability*
begin

We use measure rather than emeasure because ennreal is not a metric space, which we need to reason about convergence. By intersecting with the set of finite measure A, we don't run into issues where infinity is collapsed to 0. For finite measures this definition is equal to the definition without set A – see below.

definition *tendsto-measure* :: *'b measure* ⇒ (*'a* ⇒ *'b* ⇒ (*'c* :: {*second-countable-topology*,*metric-space*}))
⇒ (*'b* ⇒ *'c*) ⇒ *'a filter* ⇒ *bool*
where *tendsto-measure M X l F* ≡ (∀ *n*. *X n* ∈ *borel-measurable M*) ∧ *l* ∈ *borel-measurable M* ∧
(∀ *ε* > *0*. ∀ *A* ∈ *fmeasurable M*.
((λ*n*. *measure M* ({*ω* ∈ *space M*. *dist* (*X n ω*) (*l ω*) > *ε*} ∩ *A*)) ⟶ *0*) *F*)

abbreviation (**in** *prob-space*) *tendsto-prob* (**infixr** ⟶$_P$ *55*) **where**
(*f* ⟶$_P$ *l*) *F* ≡ *tendsto-measure M f l F*

lemma *tendsto-measure-measurable*[*measurable-dest*]:
tendsto-measure M X l F ⟹ *X n* ∈ *borel-measurable M*
⟨*proof*⟩

lemma *tendsto-measure-measurable-lim*[*measurable-dest*]:
tendsto-measure M X l F ⟹ *l* ∈ *borel-measurable M*
⟨*proof*⟩

lemma *tendsto-measure-mono*: *F* ≤ *F'* ⟹ *tendsto-measure M f l F'* ⟹ *tendsto-measure M f l F*
⟨*proof*⟩

lemma *tendsto-measureI*:
assumes [*measurable*]: ⋀*n*. *X n* ∈ *borel-measurable M l* ∈ *borel-measurable M*
and ⋀*ε A*. *ε* > *0* ⟹ *A* ∈ *fmeasurable M* ⟹
((λ*n*. *measure M* ({*ω* ∈ *space M*. *dist* (*X n ω*) (*l ω*) > *ε*} ∩ *A*)) ⟶ *0*) *F*
shows *tendsto-measure M X l F*
⟨*proof*⟩

lemma (**in** *finite-measure*) *finite-tendsto-measureI*:
assumes [*measurable*]: ⋀*n*. *f' n* ∈ *borel-measurable M f* ∈ *borel-measurable M*
and ⋀*ε*. *ε* > *0* ⟹ ((λ*n*. *measure M* {*ω* ∈ *space M*. *dist* (*f' n ω*) (*f ω*) > *ε*}) ⟶ *0*) *F*
shows *tendsto-measure M f' f F*
⟨*proof*⟩

lemma (**in** *finite-measure*) *finite-tendsto-measureD*:
assumes [*measurable*]: *tendsto-measure M f' f F*
shows (∀ *ε* > *0*. ((λ*n*. *measure M* {*ω* ∈ *space M*. *dist* (*f' n ω*) (*f ω*) > *ε*}) ⟶ *0*) *F*)
⟨*proof*⟩

lemma (**in** *finite-measure*) *tendsto-measure-leq*:

assumes [*measurable*]: $\bigwedge n.\ f'\ n \in$ *borel-measurable* $M\ f \in$ *borel-measurable* M
shows *tendsto-measure* $M\ f'\ f\ F \longleftrightarrow$
$(\forall \varepsilon > 0.\ ((\lambda n.\ \textit{measure}\ M\ \{\omega \in \textit{space}\ M.\ \textit{dist}\ (f'\ n\ \omega)\ (f\ \omega) \geq \varepsilon\}) \longrightarrow 0)$ F) (**is** $?L \longleftrightarrow ?R$)
⟨*proof*⟩

abbreviation *LIMSEQ-measure* $M\ f\ l \equiv$ *tendsto-measure* $M\ f\ l$ *sequentially*

lemma *LIMSEQ-measure-def*: *LIMSEQ-measure* $M\ f\ l \longleftrightarrow$
$(\forall n.\ f\ n \in$ *borel-measurable* $M) \wedge (l \in$ *borel-measurable* $M) \wedge$
$(\forall \varepsilon > 0.\ \forall A \in$ *fmeasurable* $M.$
$(\lambda n.\ \textit{measure}\ M\ (\{\omega \in \textit{space}\ M.\ \textit{dist}\ (f\ n\ \omega)\ (l\ \omega) > \varepsilon\} \cap A)) \longrightarrow 0)$
⟨*proof*⟩

lemma *LIMSEQ-measureD*:
assumes *LIMSEQ-measure* $M\ f\ l\ \varepsilon > 0\ A \in$ *fmeasurable* M
shows $(\lambda n.\ \textit{measure}\ M\ (\{\omega \in \textit{space}\ M.\ \textit{dist}\ (f\ n\ \omega)\ (l\ \omega) > \varepsilon\} \cap A)) \longrightarrow 0$
⟨*proof*⟩

lemma *fmeasurable-inter*: $[\![A \in$ *sets* $M;\ B \in$ *fmeasurable* $M]\!] \Longrightarrow A \cap B \in$ *fmeasurable* M
⟨*proof*⟩

lemma *LIMSEQ-measure-emeasure*:
assumes *LIMSEQ-measure* $M\ f\ l\ \varepsilon > 0\ A \in$ *fmeasurable* M
and [*measurable*]: $\bigwedge i.\ f\ i \in$ *borel-measurable* $M\ l \in$ *borel-measurable* M
shows $(\lambda n.\ \textit{emeasure}\ M\ (\{\omega \in \textit{space}\ M.\ \textit{dist}\ (f\ n\ \omega)\ (l\ \omega) > \varepsilon\} \cap A)) \longrightarrow 0$
⟨*proof*⟩

lemma *measure-Lim-within-LIMSEQ*:
fixes $a :: 'a ::$ *first-countable-topology*
assumes $\bigwedge t.\ X\ t \in$ *borel-measurable* $M\ L \in$ *borel-measurable* M
assumes $\bigwedge S.\ [\![(\forall n.\ S\ n \neq a \wedge S\ n \in T);\ S \longrightarrow a]\!] \Longrightarrow$ *LIMSEQ-measure* M $(\lambda n.\ X\ (S\ n))\ L$
shows *tendsto-measure* $M\ X\ L$ (*at* a *within* T)
⟨*proof*⟩

definition *tendsto-AE* $:: 'b$ *measure* $\Rightarrow$ $('a \Rightarrow 'b \Rightarrow 'c ::$ *topological-space*$) \Rightarrow$ $('b \Rightarrow 'c) \Rightarrow 'a$ *filter* $\Rightarrow$ *bool* **where**
tendsto-AE $M\ f'\ l\ F \longleftrightarrow$ (*AE* ω *in* $M.\ ((\lambda n.\ f'\ n\ \omega) \longrightarrow l\ \omega)\ F$)

lemma *LIMSEQ-ae-pointwise*: $(\bigwedge x.\ (\lambda n.\ f\ n\ x) \longrightarrow l\ x) \Longrightarrow$ *tendsto-AE* $M\ f\ l$ *sequentially*
⟨*proof*⟩

lemma *tendsto-AE-within-LIMSEQ*:
fixes $a :: 'a ::$ *first-countable-topology*
assumes $\bigwedge S.\ [\![(\forall n.\ S\ n \neq a \wedge S\ n \in T);\ S \longrightarrow a]\!] \Longrightarrow$ *tendsto-AE* M $(\lambda n.\ X\ (S\ n))\ L$ *sequentially*

shows *tendsto-AE M X L* (*at a within T*)
⟨*proof*⟩

lemma *LIMSEQ-dominated-convergence*:
fixes $X :: nat \Rightarrow real$
assumes $X \longrightarrow L$ $(\bigwedge n.\ Y\ n \leq X\ n)$ $(\bigwedge n.\ Y\ n \geq L)$
shows $Y \longrightarrow L$
⟨*proof*⟩

Klenke remark 6.4

lemma *measure-conv-imp-AE-sequentially*:
assumes [*measurable*]: $\bigwedge n.\ f'\ n \in$ *borel-measurable M f* $\in$ *borel-measurable M*
and *tendsto-AE M f' f sequentially*
shows *LIMSEQ-measure M f' f*
⟨*proof*⟩

corollary *LIMSEQ-measure-pointwise*:
assumes $\bigwedge x.\ (\lambda n.\ f\ n\ x) \longrightarrow f'\ x$ $\bigwedge n.\ f\ n \in$ *borel-measurable M f'* $\in$ *borel-measurable M*
shows *LIMSEQ-measure M f f'*
⟨*proof*⟩

lemma *Lim-measure-pointwise*:
fixes $a :: 'a ::$ *first-countable-topology*
assumes $\bigwedge x.\ ((\lambda n.\ f\ n\ x) \longrightarrow f'\ x)$(*at a within T*) $\bigwedge n.\ f\ n \in$ *borel-measurable M f'* $\in$ *borel-measurable M*
shows *tendsto-measure M f f'* (*at a within T*)
⟨*proof*⟩

corollary *measure-conv-imp-AE-at-within*:
fixes $x :: 'a ::$ *first-countable-topology*
assumes [*measurable*]: $\bigwedge n.\ f'\ n \in$ *borel-measurable M f* $\in$ *borel-measurable M*
and *tendsto-AE M f' f* (*at x within S*)
shows *tendsto-measure M f' f* (*at x within S*)
⟨*proof*⟩

Klenke remark 6.5

lemma (**in** *sigma-finite-measure*) *LIMSEQ-measure-unique-AE*:
fixes $f :: nat \Rightarrow 'a \Rightarrow 'b ::$ {*second-countable-topology*,*metric-space*}
assumes [*measurable*]: $\bigwedge n.\ f\ n \in$ *borel-measurable M l* $\in$ *borel-measurable M l'* $\in$ *borel-measurable M*
and *LIMSEQ-measure M f l LIMSEQ-measure M f l'*
shows *AE x in M*. $l\ x = l'\ x$
⟨*proof*⟩

corollary (**in** *sigma-finite-measure*) *LIMSEQ-ae-unique-AE*:
fixes $f :: nat \Rightarrow 'a \Rightarrow 'b ::$ {*second-countable-topology*,*metric-space*}
assumes $\bigwedge n.\ f\ n \in$ *borel-measurable M l* $\in$ *borel-measurable M l'* $\in$ *borel-measurable M*

and *tendsto-AE M f l sequentially tendsto-AE M f l′ sequentially*
shows *AE x in M. l x = l′ x*
⟨*proof*⟩

lemma (**in** *sigma-finite-measure*) *tendsto-measure-at-within-eq-AE*:
fixes *f* :: *′b* :: *first-countable-topology* ⇒ *′a* ⇒ *′c* :: {*second-countable-topology*,*metric-space*}
assumes *f-measurable*: ⋀*x*. *x* ∈ *S* ⟹ *f x* ∈ *borel-measurable M*
and *l-measurable*: *l* ∈ *borel-measurable M l′* ∈ *borel-measurable M*
and *tendsto*: *tendsto-measure M f l* (*at t within S*) *tendsto-measure M f l′* (*at t within S*)
and *not-bot*: (*at t within S*) ≠ ⊥
shows *AE x in M. l x = l′ x*
⟨*proof*⟩
end

5 Stochastic processes

theory *Stochastic-Processes*
imports *Kolmogorov-Chentsov-Extras Dyadic-Interval*
begin

A stochastic process is an indexed collection of random variables. For compatibility with `product_prob_space` we don't enforce conditions on the index set I in the assumptions.

locale *stochastic-process* = *prob-space* +
fixes *M′* :: *′b measure*
and *I* :: *′t set*
and *X* :: *′t* ⇒ *′a* ⇒ *′b*
assumes *random-process*[*measurable*]: ⋀*i*. *random-variable M′* (*X i*)

sublocale *stochastic-process* ⊆ *product*: *product-prob-space* (λ*t*. *distr M M′* (*X t*))
⟨*proof*⟩

lemma (**in** *prob-space*) *stochastic-processI*:
assumes ⋀*i*. *random-variable M′* (*X i*)
shows *stochastic-process M M′ X*
⟨*proof*⟩

typedef (*′t*, *′a*, *′b*) *stochastic-process* =
{(*M* :: *′a measure*, *M′* :: *′b measure*, *I* :: *′t set*, *X* :: *′t* ⇒ *′a* ⇒ *′b*).
stochastic-process M M′ X}
⟨*proof*⟩

setup-lifting *type-definition-stochastic-process*

lift-definition *proc-source* :: (*′t*,*′a*,*′b*) *stochastic-process* ⇒ *′a measure*
is *fst* ⟨*proof*⟩

interpretation *proc-source*: *prob-space proc-source X*
⟨*proof*⟩

lift-definition *proc-target* :: (*'t*,*'a*,*'b*) *stochastic-process* ⇒ *'b measure*
is *fst* ∘ *snd* ⟨*proof*⟩

lift-definition *proc-index* :: (*'t*,*'a*,*'b*) *stochastic-process* ⇒ *'t set*
is *fst* ∘ *snd* ∘ *snd* ⟨*proof*⟩

lift-definition *process* :: (*'t*,*'a*,*'b*) *stochastic-process* ⇒ *'t* ⇒ *'a* ⇒ *'b*
is *snd* ∘ *snd* ∘ *snd* ⟨*proof*⟩

declare [[*coercion process*]]

lemma *stochastic-process-construct* [*simp*]: *stochastic-process* (*proc-source X*) (*proc-target X*) (*process X*)
⟨*proof*⟩

interpretation *stochastic-process proc-source X proc-target X proc-index X process X*
⟨*proof*⟩

lemma *stochastic-process-measurable* [*measurable*]: *process X t* ∈ (*proc-source X*) →$_M$ (*proc-target X*)
⟨*proof*⟩

Here we construct a process on a given index set. For this we need to produce measurable functions for indices outside the index set; we use the constant function, but it needs to point at an element of the target set to be measurable.

context *prob-space*
begin

lift-definition *process-of* :: *'b measure* ⇒ *'t set* ⇒ (*'t* ⇒ *'a* ⇒ *'b*) ⇒ *'b* ⇒ (*'t*,*'a*,*'b*) *stochastic-process*
is λ *M' I X ω*. *if* (∀ *t* ∈ *I*. *X t* ∈ *M* →$_M$ *M'*) ∧ *ω* ∈ *space M'*
then (*M*, *M'*, *I*, (λ*t*. *if t* ∈ *I then X t else* (λ-. *ω*)))
else (*return* (*sigma UNIV* {{}, *UNIV*}) (*SOME x*. *True*), *sigma UNIV UNIV*, *I*, λ- -. *ω*)
⟨*proof*⟩

lemma *index-process-of*[*simp*]: *proc-index* (*process-of M' I X ω*) = *I*
⟨*proof*⟩

lemma
assumes ∀ *t* ∈ *I*. *X t* ∈ *M* →$_M$ *M' ω* ∈ *space M'*
shows
source-process-of[*simp*]: *proc-source* (*process-of M' I X ω*) = *M* **and**
target-process-of[*simp*]: *proc-target* (*process-of M' I X ω*) = *M'* **and**

process-process-of[*simp*]: *process* (*process-of M′ I X ω*) = (λ*t*. *if t* ∈ *I then X t else* (λ-. *ω*))
⟨*proof*⟩

lemma *process-of-apply*:
assumes ∀ *t* ∈ *I*. *X t* ∈ *M* →$_M$ *M′ ω* ∈ *space M′ t* ∈ *I*
shows *process* (*process-of M′ I X ω*) *t* = *X t*
⟨*proof*⟩
end

We define the finite-dimensional distributions of our process.

lift-definition *distributions* :: ('*t*, '*a*, '*b*) *stochastic-process* ⇒ '*t set* ⇒ ('*t* ⇒ '*b*) *measure*
is λ(*M*, *M′*, -, *X*) *T*. (Π$_M$ *t*∈*T*. *distr M M′* (*X t*)) ⟨*proof*⟩

lemma *distributions-altdef*: *distributions X T* = (Π$_M$ *t*∈*T*. *distr* (*proc-source X*) (*proc-target X*) (*X t*))
⟨*proof*⟩

lemma *prob-space-distributions*: *prob-space* (*distributions X J*)
⟨*proof*⟩

lemma *sets-distributions*: *sets* (*distributions X J*) = *sets* (*PiM J* (λ-. (*proc-target X*)))
⟨*proof*⟩

lemma *space-distributions*: *space* (*distributions X J*) = (Π$_E$ *i*∈*J*. *space* (*proc-target X*))
⟨*proof*⟩

lemma *emeasure-distributions*:
assumes *finite J* ⋀*j*. *j*∈*J* ⟹ *A j* ∈ *sets* (*proc-target X*)
shows *emeasure* (*distributions X J*) (*Pi*$_E$ *J A*) = (∏ *j*∈*J*. *emeasure* (*distr* (*proc-source X*) (*proc-target X*) (*X j*)) (*A j*))
⟨*proof*⟩

interpretation *projective-family* (*proc-index X*) *distributions X* (λ-. *proc-target X*)
⟨*proof*⟩

locale *polish-stochastic* = *stochastic-process M borel* :: '*b*::*polish-space measure I X*
for *M* **and** *I* **and** *X*

lemma *distributed-cong-random-variable*:
assumes *M* = *K N* = *L AE x in M*. *X x* = *Y x X* ∈ *M* →$_M$ *N Y* ∈ *K* →$_M$ *L f* ∈ *borel-measurable N*
shows *distributed M N X f* ⟷ *distributed K L Y f*

⟨*proof*⟩

For all sorted lists of indices, the increments specified by this list are independent

lift-definition *indep-increments* :: ('*t* :: *linorder*, '*a*, '*b* :: *minus*) *stochastic-process* ⇒ *bool* **is**
λ(*M*, *M'*, *I*, *X*).
(∀ *l*. *set l* ⊆ *I* ∧ *sorted l* ∧ *length l* ≥ *2* ⟶
prob-space.indep-vars M (λ-. *M'*) (λ*k v*. *X* (*l*!*k*) *v* − *X* (*l*!(*k*−*1*)) *v*) {*1*..<*length l*}) ⟨*proof*⟩

lemma *indep-incrementsE*:
assumes *indep-increments X*
and *set l* ⊆ *proc-index X* ∧ *sorted l* ∧ *length l* ≥ *2*
shows *prob-space.indep-vars* (*proc-source X*) (λ-. *proc-target X*)
(λ*k v*. *X* (*l*!*k*) *v* − *X* (*l*!(*k*−*1*)) *v*) {*1*..<*length l*}
⟨*proof*⟩

lemma *indep-incrementsI*:
assumes ⋀*l*. *set l* ⊆ *proc-index X* ⟹ *sorted l* ⟹ *length l* ≥ *2* ⟹
prob-space.indep-vars (*proc-source X*) (λ-. *proc-target X*) (λ*k v*. *X* (*l*!*k*) *v* − *X* (*l*!(*k*−*1*)) *v*) {*1*..<*length l*}
shows *indep-increments X*
⟨*proof*⟩

lemma *indep-increments-indep-var*:
assumes *indep-increments X h* ∈ *proc-index X j* ∈ *proc-index X k* ∈ *proc-index X h* ≤ *j j* ≤ *k*
shows *prob-space.indep-var* (*proc-source X*) (*proc-target X*) (λ*v*. *X j v* − *X h v*) (*proc-target X*) (λ*v*. *X k v* − *X j v*)
⟨*proof*⟩

definition *stationary-increments X* ⟷ (∀ *t1 t2 k*. *t1* > *0* ∧ *t2* > *0* ∧ *k* > *0* ⟶

distr (*proc-source X*) (*proc-target X*) (λ*v*. *X* (*t1* + *k*) *v* − *X t1 v*) =
distr (*proc-source X*) (*proc-target X*) (λ*v*. *X* (*t2* + *k*) *v* − *X t2 v*))

Processes on the same source measure space, with the same index space, but not necessarily the same target measure since we only care about the measurable target space, not the measure

lift-definition *compatible* :: ('*t*,'*a*,'*b*) *stochastic-process* ⇒ ('*t*,'*a*,'*b*) *stochastic-process* ⇒ *bool*
is λ(*Mx*, *M'x*, *Ix*, *X*) (*My*, *M'y*, *Iy*, -). *Mx* = *My* ∧ *sets M'x* = *sets M'y* ∧ *Ix* = *Iy* ⟨*proof*⟩

lemma *compatibleI*:
assumes *proc-source X* = *proc-source Y sets* (*proc-target X*) = *sets* (*proc-target Y*)
proc-index X = *proc-index Y*

shows *compatible X Y*
⟨*proof*⟩

lemma
assumes *compatible X Y*
shows
compatible-source [*dest*]: *proc-source X* = *proc-source Y* **and**
compatible-target [*dest*]: *sets* (*proc-target X*) = *sets* (*proc-target Y*) **and**
compatible-index [*dest*]: *proc-index X* = *proc-index Y*
⟨*proof*⟩

lemma *compatible-refl* [*simp*]: *compatible X X*
⟨*proof*⟩

lemma *compatible-sym*: *compatible X Y* ⟹ *compatible Y X*
⟨*proof*⟩

lemma *compatible-trans*:
assumes *compatible X Y compatible Y Z*
shows *compatible X Z*
⟨*proof*⟩

lemma (**in** *prob-space*) *compatible-process-of*:
assumes *measurable*: ∀ *t* ∈ *I*. *X t* ∈ *M* →$_M$ *M'* ∀ *t* ∈ *I*. *Y t* ∈ *M* →$_M$ *M'*
and *a* ∈ *space M' b* ∈ *space M'*
shows *compatible* (*process-of M' I X a*) (*process-of M' I Y b*)
⟨*proof*⟩

definition *modification* :: ('*t*,'*a*,'*b*) *stochastic-process* ⇒ ('*t*,'*a*,'*b*) *stochastic-process* ⇒ *bool* **where**
modification X Y ⟷ *compatible X Y* ∧ (∀ *t* ∈ *proc-index X*. *AE x in proc-source X*. *X t x* = *Y t x*)

lemma *modificationI* [*intro*]:
assumes *compatible X Y* ⋀*t*. *t* ∈ *proc-index X* ⟹ *AE x in proc-source X*. *X t x* = *Y t x*
shows *modification X Y*
⟨*proof*⟩

lemma *modificationD* [*dest*]:
assumes *modification X Y*
shows *compatible X Y*
and ⋀*t*. *t* ∈ *proc-index X* ⟹ *AE x in proc-source X*. *X t x* = *Y t x*
⟨*proof*⟩

lemma *modification-null-set*:
assumes *modification X Y t* ∈ *proc-index X*
obtains *N* **where** {*x* ∈ *space* (*proc-source X*). *X t x* ≠ *Y t x*} ⊆ *N N* ∈ *null-sets* (*proc-source X*)

⟨*proof*⟩

lemma *modification-refl* [*simp*]: *modification X X*
⟨*proof*⟩

lemma *modification-sym*: *modification X Y* ⟹ *modification Y X*
⟨*proof*⟩

lemma *modification-trans*:
assumes *modification X Y modification Y Z*
shows *modification X Z*
⟨*proof*⟩

lemma *modification-imp-identical-distributions*:
assumes *modification*: *modification X Y*
and *index*: *T* ⊆ *proc-index X*
shows *distributions X T* = *distributions Y T*
⟨*proof*⟩

definition *indistinguishable* :: ('*t*,'*a*,'*b*) *stochastic-process* ⇒ ('*t*,'*a*,'*b*) *stochastic-process* ⇒ *bool* **where**
indistinguishable X Y ⟷ *compatible X Y* ∧
(∃ *N* ∈ *null-sets* (*proc-source X*). ∀ *t* ∈ *proc-index X*. {*x* ∈ *space* (*proc-source X*). *X t x* ≠ *Y t x*} ⊆ *N*)

lemma *indistinguishableI*:
assumes *compatible X Y*
and ∃ *N* ∈ *null-sets* (*proc-source X*). (∀ *t* ∈ *proc-index X*. {*x* ∈ *space* (*proc-source X*). *X t x* ≠ *Y t x*} ⊆ *N*)
shows *indistinguishable X Y*
⟨*proof*⟩

lemma *indistinguishable-null-set*:
assumes *indistinguishable X Y*
obtains *N* **where**
N ∈ *null-sets* (*proc-source X*)
⋀*t*. *t* ∈ *proc-index X* ⟹ {*x* ∈ *space* (*proc-source X*). *X t x* ≠ *Y t x*} ⊆ *N*
⟨*proof*⟩

lemma *indistinguishableD*:
assumes *indistinguishable X Y*
shows *compatible X Y*
and ∃ *N* ∈ *null-sets* (*proc-source X*). (∀ *t* ∈ *proc-index X*. {*x* ∈ *space* (*proc-source X*). *X t x* ≠ *Y t x*} ⊆ *N*)
⟨*proof*⟩

lemma *indistinguishable-eq-AE*:
assumes *indistinguishable X Y*
shows *AE x in proc-source X*. ∀ *t* ∈ *proc-index X*. *X t x* = *Y t x*

⟨*proof*⟩

lemma *indistinguishable-null-ex*:
assumes *indistinguishable X Y*
shows ∃ *N* ∈ *null-sets* (*proc-source X*). {*x* ∈ *space* (*proc-source X*).∃ *t* ∈ *proc-index X*. *X t x* ≠ *Y t x*} ⊆ *N*
⟨*proof*⟩

lemma *indistinguishable-refl* [*simp*]: *indistinguishable X X*
⟨*proof*⟩

lemma *indistinguishable-sym*: *indistinguishable X Y* ⟹ *indistinguishable Y X*
⟨*proof*⟩

lemma *indistinguishable-trans*:
assumes *indistinguishable X Y indistinguishable Y Z*
shows *indistinguishable X Z*
⟨*proof*⟩

lemma *indistinguishable-modification*: *indistinguishable X Y* ⟹ *modification X Y*
⟨*proof*⟩

Klenke 21.5(i)

lemma *modification-countable*:
assumes *modification X Y countable* (*proc-index X*)
shows *indistinguishable X Y*
⟨*proof*⟩

Klenke 21.5(ii). The textbook statement is more general - we reduce right continuity to regular continuity

lemma *modification-continuous-indistinguishable*:
fixes *X* :: (*real*, *'a*, *'b* :: *metric-space*) *stochastic-process*
assumes *modification*: *modification X Y*
and *interval*: ∃ *T* > *0*. *proc-index X* = {*0*..*T*}
and *rc*: *AE ω in proc-source X*. *continuous-on* (*proc-index X*) (λ*t*. *X t ω*)
(**is** *AE ω in proc-source X*. *?cont-X ω*)
AE ω in proc-source Y. *continuous-on* (*proc-index Y*) (λ*t*. *Y t ω*)
(**is** *AE ω in proc-source Y*. *?cont-Y ω*)
shows *indistinguishable X Y*
⟨*proof*⟩

lift-definition *restrict-index* :: (*'t*, *'a*, *'b*) *stochastic-process* ⇒ *'t set* ⇒ (*'t*, *'a*, *'b*) *stochastic-process*
is λ(*M*, *M'*, *I*, *X*) *T*. (*M*, *M'*, *T*, *X*) ⟨*proof*⟩

lemma
shows

restrict-index-source[*simp*]: *proc-source* (*restrict-index X T*) = *proc-source X*
and
restrict-index-target[*simp*]: *proc-target* (*restrict-index X T*) = *proc-target X* **and**
restrict-index-index[*simp*]: *proc-index* (*restrict-index X T*) = *T* **and**
restrict-index-process[*simp*]: *process* (*restrict-index X T*) = *process X*
⟨*proof*⟩

lemma *restrict-index-override*[*simp*]: *restrict-index* (*restrict-index X T*) *S* = *restrict-index X S*
⟨*proof*⟩

lemma *compatible-restrict-index*:
assumes *compatible X Y*
shows *compatible* (*restrict-index X S*) (*restrict-index Y S*)
⟨*proof*⟩

lemma *modification-restrict-index*:
assumes *modification X Y S* ⊆ *proc-index X*
shows *modification* (*restrict-index X S*) (*restrict-index Y S*)
⟨*proof*⟩

lemma *indistinguishable-restrict-index*:
assumes *indistinguishable X Y S* ⊆ *proc-index X*
shows *indistinguishable* (*restrict-index X S*) (*restrict-index Y S*)
⟨*proof*⟩

lemma *AE-eq-minus* [*intro*]:
fixes *a* :: *'a* ⇒ (*'b* :: *real-normed-vector*)
assumes *AE x in M. a x* = *b x AE x in M. c x* = *d x*
shows *AE x in M. a x* − *c x* = *b x* − *d x*
⟨*proof*⟩

lemma *modification-indep-increments*:
fixes *X Y* :: (*'a* :: *linorder*, *'b*, *'c* :: {*second-countable-topology*, *real-normed-vector*}) *stochastic-process*
assumes *modification X Y sets* (*proc-target Y*) = *sets borel*
shows *indep-increments X* ⟹ *indep-increments Y*
⟨*proof*⟩

end

6 The Kolomgorov-Chentsov theorem

theory *Kolmogorov-Chentsov*
imports *Stochastic-Processes Holder-Continuous Dyadic-Interval Measure-Convergence*
begin

6.1 Supporting lemmas

The main contribution of this file is the Kolmogorov-Chentsov theorem: given a stochastic process that satisfies some continuity properties, we can construct a Hölder continuous modification. We first prove some auxiliary lemmas before moving on to the main construction.

Klenke 5.11: Markov inequality. Compare with $[\![(\lambda x.\ ?u\ x * \mathit{indicator}\ ?A\ x) \in \mathit{borel\text{-}measurable}\ ?M;\ ?A \in \mathit{sets}\ ?M]\!] \Longrightarrow \mathit{emeasure}\ ?M\ \{x \in ?A.\ 1 \leq ?c * ?u\ x\} \leq ?c * \mathit{set\text{-}nn\text{-}integral}\ ?M\ ?A\ ?u$

lemma *nn-integral-Markov-inequality-extended*:
  fixes $f :: \mathit{real} \Rightarrow \mathit{ennreal}$ **and** $\varepsilon :: \mathit{real}$ **and** $X :: 'a \Rightarrow \mathit{real}$
  assumes *mono*: *mono-on* $(\mathit{range}\ X \cup \{0<..\})\ f$
    and *finite*: $\bigwedge x.\ f\ x < \infty$
    and *e*: $\varepsilon > 0\ f\ \varepsilon > 0$
    and [*measurable*]: $X \in \mathit{borel\text{-}measurable}\ M$
  shows $\mathit{emeasure}\ M\ \{p \in \mathit{space}\ M.\ (X\ p) \geq \varepsilon\} \leq (\int^+ x.\ f\ (X\ x)\ \partial M)\ /\ f\ \varepsilon$
⟨*proof*⟩

lemma *nn-integral-Markov-inequality-extended-rnv*:
  fixes $f :: \mathit{real} \Rightarrow \mathit{real}$ **and** $\varepsilon :: \mathit{real}$ **and** $X :: 'a \Rightarrow 'b :: \mathit{real\text{-}normed\text{-}vector}$
  assumes [*measurable*]: $X \in \mathit{borel\text{-}measurable}\ M$
    and *mono*: *mono-on* $\{0..\}\ f$
    and *e*: $\varepsilon > 0\ f\ \varepsilon > 0$
  shows $\mathit{emeasure}\ M\ \{p \in \mathit{space}\ M.\ \mathit{norm}\ (X\ p) \geq \varepsilon\} \leq (\int^+ x.\ f\ (\mathit{norm}\ (X\ x))\ \partial M)\ /\ f\ \varepsilon$
  ⟨*proof*⟩

6.2 Kolmogorov-Chentsov

Klenke theorem 21.6 - Kolmorogov-Chentsov

locale *Kolmogorov-Chentsov* =
  fixes $X :: (\mathit{real}, 'a, 'b :: \mathit{polish\text{-}space})\ \mathit{stochastic\text{-}process}$
    and $a\ b\ C\ \gamma :: \mathit{real}$
  assumes *index*[*simp*]: $\mathit{proc\text{-}index}\ X = \{0..\}$
    and *target-borel*[*simp*]: $\mathit{proc\text{-}target}\ X = \mathit{borel}$
    and *gt-0*: $a > 0\ b > 0\ C > 0$
    and *b-leq-a*: $b \leq a$
    and *gamma*: $\gamma \in \{0<..<b/a\}$
    and *expectation*: $\bigwedge s\ t.\ [\![s \geq 0;\ t \geq 0]\!] \Longrightarrow$
      $(\int^+ x.\ \mathit{dist}\ (X\ t\ x)\ (X\ s\ x)\ \mathit{powr}\ a\ \partial \mathit{proc\text{-}source}\ X) \leq C * \mathit{dist}\ t\ s\ \mathit{powr}\ (1+b)$
begin

lemma *gamma-0-1*[*simp*]:$\gamma \in \{0<..1\}$
  ⟨*proof*⟩

lemma *gamma-gt-0*[*simp*]: $\gamma > 0$

⟨*proof*⟩

lemma *gamma-le-1*[*simp*]: $\gamma \leq 1$
⟨*proof*⟩

abbreviation *source* ≡ *proc-source X*

lemma *X-borel-measurable*[*measurable*]: *X t* ∈ *borel-measurable source* **for** *t*
⟨*proof*⟩

lemma *markov*: $\mathcal{P}(x \text{ in source. } \varepsilon \leq dist\ (X\ t\ x)\ (X\ s\ x)) \leq (C * dist\ t\ s\ powr\ (1 + b)) / \varepsilon\ powr\ a$
if $s \geq 0\ t \geq 0\ \varepsilon > 0$ **for** *s t* ε
⟨*proof*⟩

lemma *conv-in-prob*:
assumes $t \geq 0$
shows *tendsto-measure* (*proc-source X*) *X* (*X t*) (*at t within* {*0*..}*)*
⟨*proof*⟩

lemma *conv-in-prob-finite*:
assumes $t \geq 0$
shows *tendsto-measure* (*proc-source X*) *X* (*X t*) (*at t within* {*0*..*T*}*)*
⟨*proof*⟩

lemma *incr*: $\mathcal{P}(x \text{ in source. } 2\ powr\ (-\gamma * n) \leq dist\ (X\ ((k-1) * 2\ powr\ -\ n)\ x)\ (X\ (k * 2\ powr\ -\ n)\ x))$
$\leq C * 2\ powr\ (-n * (1+b-a*\gamma))$
if $k \geq 1\ n \geq 0$ **for** *k n*
⟨*proof*⟩

end

In order to construct the modification of X, it suffices to construct a modification of X on $\{0..T\}$ for all finite T, from which we construct the modification on $\{0..\}$ via a countable union.

locale *Kolmogorov-Chentsov-finite* = *Kolmogorov-Chentsov* +
fixes *T* :: *real*
assumes *zero-le-T*: $0 < T$
begin

A_n will characterise the set of states with increments that exceed the bounds required for Hölder continuity. As $n \longrightarrow \infty$, this approaches the set of states for which X is not Hölder continuous. We define N as this limit, and show that N is a null set. On $\omega \in \Omega - N$, we show that $X(\omega)$ is Hölder continuous (and therefore uniformly continuious) on the dyadic rationals, and construct a modification by taking the continuous extension on the reals.

definition $A \equiv \lambda n.$ *if* $2 \hat{} n * T < 1$ *then space source else*

$\{x \in$ *space source*.
Max $\{$*dist* (*X* (*real-of-int* $(k - 1) * 2$ *powr* $-$ *real* n) x) (*X* (*real-of-int* $k * 2$ *powr* $-$ *real* n) x)
$\mid k.\ k \in \{1..\lfloor 2\hat{}n * T\rfloor\}\} \geq 2$ *powr* $(-\gamma *$ *real* $n)\}$

abbreviation $B \equiv \lambda n.\ (\bigcup m.\ A\ (m + n))$

abbreviation $N \equiv \bigcap (range\ B)$

lemma *A-geq*: $2\ \hat{}\ n * T \geq 1 \Longrightarrow A\ n = \{x \in$ *space source*.
Max $\{$*dist* (*X* (*real-of-int* $(k - 1) * 2$ *powr* $-$ *real* n) x) (*X* (*real-of-int* $k * 2$ *powr* $-$ *real* n) x)
$\mid k.\ k \in \{1..\lfloor 2\hat{}n * T\rfloor\}\} \geq 2$ *powr* $(-\gamma *$ *real* $n)\}$ **for** n
⟨*proof*⟩

lemma *A-measurable*[*measurable*]: $A\ n \in$ *sets source*
⟨*proof*⟩

lemma *emeasure-A-leq*:
fixes n :: *nat*
assumes [*simp*]: $2\hat{}n * T \geq 1$
shows *emeasure source* $(A\ n) \leq C * T * 2$ *powr* $(-\ n * (b - a * \gamma))$
⟨*proof*⟩

lemma *measure-A-leq*:
assumes $2\hat{}n * T \geq 1$
shows *measure source* $(A\ n) \leq C * T * 2$ *powr* $(-\ n * (b - a * \gamma))$
⟨*proof*⟩

lemma *summable-A*: *summable* $(\lambda m.$ *measure source* $(A\ m))$
⟨*proof*⟩

lemma *lim-B*: $(\lambda n.$ *measure source* $(B\ n)) \longrightarrow 0$
⟨*proof*⟩

lemma *N-null*: $N \in$ *null-sets source*
⟨*proof*⟩

lemma *notin-N-index*:
assumes $\omega \in$ *space source* $- N$
obtains n_0 **where** $\omega \notin (\bigcup n.\ A\ (n + n_0))$
⟨*proof*⟩

context
fixes ω
assumes ω: $\omega \in$ *space source* $- N$
begin

definition $n_0 \equiv$ *SOME* $m.\ \omega \notin (\bigcup n.\ A\ (n + m)) \wedge m > 0$

lemma
shows *n-zero*: $\omega \notin (\bigcup n.\ A\ (n + n_0))$
and *n-zero-nonzero*: $n_0 > 0$
⟨*proof*⟩

lemma *nzero-ge*: $\bigwedge n.\ n \geq n_0 \Longrightarrow 2\hat{}n * T \geq 1$
⟨*proof*⟩

lemma *omega-notin*: $\bigwedge n.\ n \geq n_0 \Longrightarrow \omega \notin A\ n$
⟨*proof*⟩

Klenke 21.7

lemma *X-dyadic-incr*:
assumes $k \in \{1..\lfloor 2\hat{}n * T \rfloor\}$ $n \geq n_0$
shows *dist* (*X* ((*real-of-int k*−1)/2^*n*) *ω*) (*X* (*real-of-int k*/2^*n*) *ω*) < *2 powr* (− *γ* ∗ *n*)
⟨*proof*⟩

Klenke (21.8)

lemma *dist-dyadic-mn*:
assumes *mn*: $n_0 \leq n$ $n \leq m$
and *t-dyadic*: $t \in$ *dyadic-interval-step m 0 T*
and *u-dyadic-n*: $u \in$ *dyadic-interval-step n 0 T*
and *ut*: $u \leq t$ $t - u < 2/2\hat{}n$
shows *dist* (*X u ω*) (*X t ω*) ≤ *2 powr* (− *γ* ∗ *n*) / (*1* − *2 powr* − *γ*)
⟨*proof*⟩

lemma *dist-dyadic-fixed*:
assumes *mn*: $n_0 \leq n$ $n \leq m$
and *s-dyadic*: $s \in$ *dyadic-interval-step m 0 T*
and *t-dyadic*: $t \in$ *dyadic-interval-step m 0 T*
and *st*: $s \leq t$ $t - s \leq 1/2\hat{}n$
shows *dist* (*X t ω*) (*X s ω*) ≤ *2* ∗ *2 powr* (− *γ* ∗ *n*) / (*1* − *2 powr* − *γ*)
⟨*proof*⟩

definition $C_0 \equiv$ *2* ∗ *2 powr γ* / (*1* − *2 powr* − *γ*)

lemma *C-zero-ge*[*simp*]: $C_0 > 0$
⟨*proof*⟩

Klenke (21.9)

Let $s, t \in D$ with $|s - t| \leq \frac{1}{2_0^n}$. By choosing the minimal $n \geq n_0$ such that $|t - s| \geq 2^{-n}$, we obtain by ⟦$n_0 \leq$ *?n*; *?n* ≤ *?m*; *?s* ∈ *dyadic-interval-step ?m 0 T*; *?t* ∈ *dyadic-interval-step ?m 0 T*; *?s* ≤ *?t*; *?t* − *?s* ≤ *1* / *2*^*?n*⟧ ⟹ *dist* (*process X ?t ω*) (*process X ?s ω*) ≤ *2* ∗ *2 powr* (− *γ* ∗ *real ?n*) / (*1* − *2 powr* − *γ*):

$$|X_t(\omega) - X_s(\omega)| \leq C_0 |t - s|^\gamma$$

lemma *dist-dyadic*:
assumes *t*: $t \in$ *dyadic-interval 0 T*
and *s*: $s \in$ *dyadic-interval 0 T*
and *st-dist*: *dist t s* $\leq$ *1 / 2* $\widehat{}$ n_0
shows *dist* (*X t* ω) (*X s* ω) $\leq$ C_0 $*$ (*dist t s*) *powr* γ
⟨*proof*⟩

definition $K \equiv C_0 * (2\widehat{}nat\ \lceil 2\widehat{}n_0 * T \rceil)$ *powr* (*1* $-$ γ)

lemma C_0*-le-K*: $C_0 \leq K$
⟨*proof*⟩

lemma *K-pos*: *0* $<$ K
⟨*proof*⟩

Klenke (21.10)

lemma *X-dyadic-le-K′*:
assumes *dyadic*: $s \in$ *dyadic-interval 0 T* $t \in$ *dyadic-interval 0 T*
and *st*: $s \leq t$
shows *dist* (*X s* ω) (*X t* ω) $\leq$ K $*$ *dist s t powr* γ
⟨*proof*⟩

lemma *X-dyadic-le-K*:
assumes $s \in$ *dyadic-interval 0 T*
and $t \in$ *dyadic-interval 0 T*
shows *dist* (*X s* ω) (*X t* ω) $\leq$ K $*$ *dist s t powr* γ
⟨*proof*⟩

corollary *holder-dyadic*: $\gamma-$*holder-on* (*dyadic-interval 0 T*) ($\lambda t.$ *X t* ω)
⟨*proof*⟩

lemma *uniformly-continuous-dyadic*: *uniformly-continuous-on* (*dyadic-interval 0 T*) ($\lambda t.$ *X t* ω)
⟨*proof*⟩

lemma *Lim-exists*: $\exists$ *L.* (($\lambda s.$ *X s* ω) $\longrightarrow$ *L*) (*at t within* (*dyadic-interval 0 T*))
if $t \in \{0..T\}$
⟨*proof*⟩

lemma *Lim-unique*: $\exists!$*L.* (($\lambda s.$ *X s* ω) $\longrightarrow$ *L*) (*at t within* (*dyadic-interval 0 T*))

if $t \in \{0..T\}$
⟨*proof*⟩

definition $L \equiv (\lambda t.$ (*Lim* (*at t within dyadic-interval 0 T*) ($\lambda s.$ *X s* ω)))

lemma *X-tendsto-L*:
assumes $t \in \{0..T\}$

shows $((\lambda s.\ X\ s\ \omega) \longrightarrow L\ t)$ $(at\ t\ within\ (dyadic\text{-}interval\ 0\ T))$
⟨*proof*⟩

lemma *L-dist-K*:
assumes *s*: $s \in \{0..T\}$
and *t*: $t \in \{0..T\}$
shows $dist\ (L\ s)\ (L\ t) \leq K * dist\ s\ t\ powr\ \gamma$
⟨*proof*⟩

corollary *L-holder*: γ−*holder-on* $\{0..T\}$ *L*
⟨*proof*⟩

corollary *L-local-holder*: *local-holder-on* γ $\{0..T\}$ *L*
⟨*proof*⟩

lemma *X-dyadic-eq-L*:
assumes $t \in dyadic\text{-}interval\ 0\ T$
shows $X\ t\ \omega = L\ t$
⟨*proof*⟩
end

definition *default* :: *'b* **where** $default = (SOME\ x.\ True)$

definition *X-tilde* :: $real \Rightarrow 'a \Rightarrow 'b$ **where**
$X\text{-}tilde \equiv (\lambda t\ \omega.\ if\ \omega \in N\ then\ default\ else\ (Lim\ (at\ t\ within\ dyadic\text{-}interval\ 0\ T)\ (\lambda s.\ X\ s\ \omega)))$

lemma *X-tilde-not-N-Lim*:
assumes $\omega \in space\ source - N$
shows $X\text{-}tilde\ t\ \omega = Lim\ (at\ t\ within\ dyadic\text{-}interval\ 0\ T)\ (\lambda s.\ X\ s\ \omega)$
⟨*proof*⟩

lemma *X-tilde-not-N-L*:
assumes $\omega \in space\ source - N$
shows $X\text{-}tilde\ t\ \omega = L\ \omega\ t$
⟨*proof*⟩

lemma *local-holder-X-tilde*: *local-holder-on* γ $\{0..T\}$ $(\lambda t.\ X\text{-}tilde\ t\ \omega)$
if $\omega \in space\ source$ **for** ω
⟨*proof*⟩

corollary *X-tilde-eq-L-AE*: $AE\ \omega\ in\ source.\ X\text{-}tilde\ t\ \omega = L\ \omega\ t$
⟨*proof*⟩

corollary *X-tilde-eq-Lim-AE*:
$AE\ \omega\ in\ source.\ X\text{-}tilde\ t\ \omega = Lim\ (at\ t\ within\ dyadic\text{-}interval\ 0\ T)\ (\lambda s.\ X\ s\ \omega)$
⟨*proof*⟩

lemma *X-tilde-tendsto-AE*: $t \in \{0..T\} \Longrightarrow tendsto\text{-}AE\ source\ X\ (X\text{-}tilde\ t)\ (at\ t$

within dyadic-interval 0 T)
⟨*proof*⟩

end

context *Kolmogorov-Chentsov-finite*
begin

By (21.5) $0 \leq ?t \Longrightarrow$ *tendsto-measure source* (*process X*) (*process X ?t*) (*at ?t within* $\{0..?T\}$) and (21.11) $?\omega \in$ *space source* $- (\bigcap n \bigcup m\ A\ (m + n)) \Longrightarrow L\ ?\omega \equiv \lambda t.$ *Lim* (*at t within dyadic-interval 0 T*) ($\lambda s.$ *process X s* $?\omega$), $P[X \neq \tilde{X}] = 0$

lemma *X-tilde-measurable*[*measurable*]:
assumes $t \in \{0..T\}$
shows *X-tilde t* ∈ *borel-measurable source*
⟨*proof*⟩

lemma *X-eq-X-tilde-AE*: *AE ω in source. X t ω* = *X-tilde t ω* **if** $t \in \{0..T\}$ **for** *t*
⟨*proof*⟩

lemma *X-tilde-modification*: *modification* (*restrict-index X* $\{0..T\}$)
(*prob-space.process-of source* (*proc-target X*) $\{0..T\}$ *X-tilde default*)
⟨*proof*⟩
end

We have now shown that we can construct a modification of X for any interval $\{0..T\}$. We want to extend this result to construct a modification on the interval $\{0..\}$ - this can be constructed by gluing together all modifications with natural-valued T which results in a countable union of modifications, which itself is a modification.

context *Kolmogorov-Chentsov*
begin

lemma *Kolmogorov-Chentsov-finite*: $T > 0 \Longrightarrow$ *Kolmogorov-Chentsov-finite X a b C γ T*
⟨*proof*⟩

definition *Mod* ≡ $\lambda T.$ *SOME Y. modification* (*restrict-index X* $\{0..T\}$) *Y* ∧
($\forall x \in$ *space source. local-holder-on γ* $\{0..T\}$ ($\lambda t.$ *Y t x*))

lemma *Mod*: *modification* (*restrict-index X* $\{0..T\}$) (*Mod T*)
($\forall x \in$ *space source. local-holder-on γ* $\{0..T\}$ ($\lambda t.$ (*Mod T*) *t x*)) **if** $0 < T$ **for** *T*
⟨*proof*⟩

lemma *compatible-Mod*: *compatible* (*restrict-index X* $\{0..T\}$) (*Mod T*) **if** $0 < T$ **for** *T*
⟨*proof*⟩

lemma *Mod-source*[*simp*]: *proc-source* (*Mod T*) = *source* **if** *0* < *T* **for** *T*
⟨*proof*⟩

lemma *Mod-target*: *sets* (*proc-target* (*Mod T*)) = *sets* (*proc-target X*) **if** *0* < *T* **for** *T*
⟨*proof*⟩

lemma *Mod-index*[*simp*]: *0* < *T* ⟹ *proc-index* (*Mod T*) = {*0*..*T*}
⟨*proof*⟩

lemma *indistinguishable-mod*:
indistinguishable (*restrict-index* (*Mod S*) {*0*..*min S T*}) (*restrict-index* (*Mod T*) {*0*..*min S T*})
if *S* > *0* *T* > *0* **for** *S T*
⟨*proof*⟩

definition *N S T* ≡ *SOME N*. *N* ∈ *null-sets source* ∧ {*ω* ∈ *space source*. ∃ *t* ∈ {*0*..*min S T*}. (*Mod S*) *t ω* ≠ (*Mod T*) *t ω*} ⊆ *N*

lemma *N*:
assumes *S* > *0* *T* > *0*
shows *N S T* ∈ *null-sets source* ∧ {*ω* ∈ *space source*. ∃ *t* ∈ {*0*..*min S T*}. (*Mod S*) *t ω* ≠ (*Mod T*) *t ω*} ⊆ *N S T*
⟨*proof*⟩

definition *N-inf* **where** *N-inf* ≡ (⋃ *S* ∈ ℕ − {*0*}. (⋃ *T* ∈ ℕ − {*0*}. *N S T*))

lemma *N-inf-null*: *N-inf* ∈ *null-sets source*
⟨*proof*⟩

lemma *Mod-eq-N-inf*: (*Mod S*) *t ω* = (*Mod T*) *t ω*
if *ω* ∈ *space source* − *N-inf t* ∈ {*0*..*min S T*} *S* ∈ ℕ − {*0*} *T* ∈ ℕ − {*0*} **for** *ω t S T*
⟨*proof*⟩

definition *default* :: *'b* **where** *default* = (*SOME x*. *True*)

definition *X-mod* ≡ λ*t ω*. *if ω* ∈ *space source* − *N-inf then* (*Mod* ⌊*t*+*1*⌋) *t ω else default*

definition *X-mod-process* ≡ *prob-space.process-of source* (*proc-target X*) {*0*..} *X-mod default*

lemma *Mod-measurable*[*measurable*]: ∀ *t*∈{*0*..}. *X-mod t* ∈ *source* →$_M$ *proc-target X*
⟨*proof*⟩

lemma *modification-X-mod-process*: *modification X X-mod-process*
⟨*proof*⟩

lemma *local-holder-X-mod*: *local-holder-on* γ $\{0..\}$ $(\lambda t.\ X\text{-}mod\ t\ \omega)$ **for** ω
⟨*proof*⟩

lemma *local-holder-X-mod-process*: *local-holder-on* γ $\{0..\}$ $(\lambda t.\ X\text{-}mod\text{-}process\ t\ \omega)$
for ω
⟨*proof*⟩

theorem *continuous-modification*:
$\exists X'.\ modification\ X\ X' \wedge (\forall \omega.\ local\text{-}holder\text{-}on\ \gamma\ \{0..\}\ (\lambda t.\ X'\ t\ \omega))$
⟨*proof*⟩

end

theorem *Kolmogorov-Chentsov*:
fixes $X :: (real, 'a, 'b :: polish\text{-}space)\ stochastic\text{-}process$
and $a\ b\ C\ \gamma :: real$
assumes *index*[*simp*]: *proc-index* $X = \{0..\}$
and *target-borel*[*simp*]: *proc-target* $X = borel$
and *gt-0*: $a > 0\ b > 0\ C > 0$
and *b-leq-a*: $b \leq a$
and *gamma*: $\gamma \in \{0<..<b/a\}$
and *expectation*: $\bigwedge s\ t.\ [\![s \geq 0;\ t \geq 0]\!] \Longrightarrow$
$(\int^+ x.\ dist\ (X\ t\ x)\ (X\ s\ x)\ powr\ a\ \partial proc\text{-}source\ X) \leq C * dist\ t\ s\ powr$
$(1+b)$
shows $\exists X'.\ modification\ X\ X' \wedge (\forall \omega.\ local\text{-}holder\text{-}on\ \gamma\ \{0..\}\ (\lambda t.\ X'\ t\ \omega))$
⟨*proof*⟩
end

References

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