

Kneser's Theorem and the Cauchy–Davenport Theorem

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February 6, 2026

Abstract

We formalise Kneser's Theorem in combinatorics [2, 3] following the proof from the 2014 paper "A short proof of Kneser's addition theorem for abelian groups" by Matt DeVos [1]. We also show a strict version of Kneser's Theorem as well as the Cauchy–Davenport Theorem as a corollary of Kneser's Theorem.

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Acknowledgements

Angeliki Koutsoukou-Argyraki is funded by the ERC Advanced Grant ALEXANDRIA (Project GA 742178) funded by the European Research Council and led by Lawrence C. Paulson (University of Cambridge, Department of Computer Science and Technology). Mantas Bakšys received funding for his internship supervised by Koutsoukou-Argyraki by the Cambridge Mathematics Placements (CMP) Programme and by the ALEXANDRIA Project. We wish to thank Manuel Eberl for his useful suggestion for treating induction when there is a type discrepancy between the induction hypothesis and the induction step.

1 Preliminaries

theory *Kneser-Cauchy-Davenport-preliminaries*

imports

Complex-Main

Pluenecke-Ruzsa-Inequality.Pluenecke-Ruzsa-Inequality

HOL-Number-Theory.Prime-Powers

begin

context *subgroup-of-group*

begin

interpretation *left: left-translations-of-group* $\langle proof \rangle$

interpretation *right: right-translations-of-group* $\langle proof \rangle$

interpretation *transformation-group left.translation* ‘ $H G$ ’ $\langle proof \rangle$

lemma *Right-Coset-eq-iff:*

assumes $x \in G$ **and** $y \in G$

shows $H \cdot x = (H \cdot y) \iff H \cdot x \cap (H \cdot y) \neq \{\}$

$\langle proof \rangle$

end

context *additive-abelian-group*

begin

1.1 Elementary lemmas on sumsets

lemma *sumset-translate-eq-right:*

assumes $A \subseteq G$ **and** $B \subseteq G$ **and** $x \in G$

shows $(\text{sumset } A \{x\} = \text{sumset } B \{x\}) \iff A = B$ $\langle proof \rangle$

lemma *sumset-translate-eq-left:*

assumes $A \subseteq G$ **and** $B \subseteq G$ **and** $x \in G$

shows $(\text{sumset } \{x\} A = \text{sumset } \{x\} B) \iff A = B$ $\langle proof \rangle$

lemma *differenceset-translate-eq-right:*

assumes $A \subseteq G$ **and** $B \subseteq G$ **and** $x \in G$

shows $(\text{differenceset } A \{x\} = \text{differenceset } B \{x\}) \iff A = B$ $\langle proof \rangle$

lemma *differenceset-translate-eq-left:*

assumes $A \subseteq G$ **and** $B \subseteq G$ **and** $x \in G$

shows $(\text{differenceset } \{x\} A = \text{differenceset } \{x\} B) \iff A = B$ $\langle proof \rangle$

lemma *sumset-inter-union-subset*:

sumset $(A \cap B) (A \cup B) \subseteq \text{sumset } A B$

<proof>

lemma *differenceset-group-eq*:

$G = \text{differenceset } G G$

<proof>

lemma *card-sumset-singleton-subset-eq*:

assumes $a \in G$ **and** $A \subseteq G$

shows $\text{card } (\text{sumset } \{a\} A) = \text{card } A$

<proof>

lemma *card-differenceset-singleton-mem-eq*:

assumes $a \in G$ **and** $A \subseteq G$

shows $\text{card } A = \text{card } (\text{differenceset } A \{a\})$

<proof>

lemma *card-singleton-differenceset-eq*:

assumes $a \in G$ **and** $A \subseteq G$

shows $\text{card } A = \text{card } (\text{differenceset } \{a\} A)$

<proof>

lemma *sumset-eq-Union-left*:

assumes $A \subseteq G$

shows $\text{sumset } A B = (\bigcup a \in A. \text{sumset } \{a\} B)$

<proof>

lemma *sumset-eq-Union-right*:

assumes $B \subseteq G$

shows $\text{sumset } A B = (\bigcup b \in B. \text{sumset } A \{b\})$

<proof>

lemma *sumset-singletons-eq*:

assumes $a \in G$ **and** $b \in G$

shows $\text{sumset } \{a\} \{b\} = \{a \oplus b\}$

<proof>

lemma *sumset-eq-subset-differenceset*:

assumes $K \subseteq G$ **and** $K \neq \{\}$ **and** $A \subseteq G$ **and** $\text{sumset } A K = \text{sumset } B K$

shows $A \subseteq \text{differenceset } (\text{sumset } B K) K$

<proof>

end

locale *subgroup-of-additive-abelian-group* =

subgroup-of-abelian-group $H G (\oplus) \mathbf{0}$ + *additive-abelian-group* $G (\oplus) \mathbf{0}$

for $H G$ **and** *addition* (*infixl* $\langle \oplus \rangle$ 65) **and** *zero* ($\langle \mathbf{0} \rangle$)

begin

notation *Left-Coset* (**infixl** $\langle \cdot | \rangle$ 70)

lemma *Left-Coset-eq-sumset*:

assumes $x \in G$

shows $\text{sumset } \{x\} H = x \cdot | H$

<proof>

lemma *sumset-subgroup-eq-iff*:

assumes $a \in G$ **and** $b \in G$

shows $\text{sumset } \{a\} H = \text{sumset } \{b\} H \iff$

$(\text{sumset } \{a\} H) \cap (\text{sumset } \{b\} H) \neq \{\}$

<proof>

lemma *card-divide-sumset*:

assumes $A \subseteq G$

shows $\text{card } H \text{ dvd } \text{card } (\text{sumset } A H)$

<proof>

lemma *sumset-subgroup-eq-Class-Union*:

assumes $A \subseteq G$

shows $\text{sumset } A H = (\bigcup (\text{Class } 'A))$

<proof>

lemma *Class-image-sumset-subgroup-eq*:

assumes $A \subseteq G$

shows $\text{Class } ' (\text{sumset } A H) = \text{Class } ' A$

<proof>

lemma *Class-cover-imp-subset-or-disj*:

assumes $A = (\bigcup (\text{Class } ' C))$ **and** $x \in G$ **and** $C \subseteq G$

shows $\text{Class } x \subseteq A \vee \text{Class } x \cap A = \{\}$

<proof>

end

context *additive-abelian-group*

begin

1.2 Stabilizer and basic properties

We define the stabilizer or group of periods of a nonempty subset of an abelian group.

definition *stabilizer*::'a set \Rightarrow 'a set **where**

stabilizer $S \equiv \{x \in G. \text{sumset } \{x\} (S \cap G) = S \cap G\}$

lemma *stabilizer-is-subgroup*: **fixes** $S :: 'a$ set
shows *subgroup (stabilizer S) G (⊕) 0*
 ⟨proof⟩

interpretation *subgroup-of-additive-abelian-group stabilizer A G (⊕) 0*
 ⟨proof⟩

lemma *zero-mem-stabilizer*: $0 \in \text{stabilizer } A$ ⟨proof⟩

lemma *stabilizer-is-nonempty*:
shows *stabilizer S ≠ {}*
 ⟨proof⟩

lemma *Left-Coset-eq-sumset-stabilizer*:
assumes $x \in G$
shows *sumset {x} (stabilizer B) = x · | (stabilizer B)*
 ⟨proof⟩

lemma *stabilizer-subset-difference-singleton*:
assumes $S \subseteq G$ **and** $s \in S$
shows *stabilizer S ⊆ differenceset S {s}*
 ⟨proof⟩

lemma *stabilizer-subset-singleton-difference*:
assumes $S \subseteq G$ **and** $s \in S$
shows *stabilizer S ⊆ differenceset {s} S*
 ⟨proof⟩

lemma *stabilizer-subset-nempty*:
assumes $S \neq \{\}$ **and** $S \subseteq G$
shows *stabilizer S ⊆ differenceset S S*
 ⟨proof⟩

lemma *stabilizer-coset-subset*:
assumes $A \subseteq G$ **and** $x \in A$
shows *sumset {x} (stabilizer A) ⊆ A*
 ⟨proof⟩

lemma *stabilizer-subset-stabilizer-dvd*:
assumes *stabilizer A ⊆ stabilizer B*
shows *card (stabilizer A) dvd card (stabilizer B)*
 ⟨proof⟩

lemma *stabilizer-coset-Un*:
assumes $A \subseteq G$
shows $(\bigcup x \in A. \text{sumset } \{x\} (\text{stabilizer } A)) = A$
 ⟨proof⟩

lemma *stabilizer-empty*: *stabilizer {} = G*

<proof>

lemma *stabilizer-finite:*

assumes $S \subseteq G$ **and** $S \neq \{\}$ **and** *finite S*

shows *finite (stabilizer S)*

<proof>

lemma *stabilizer-subset-group:*

shows *stabilizer S* $\subseteq G$ *<proof>*

lemma *sumset-stabilizer-eq-iff:*

assumes $a \in G$ **and** $b \in G$

shows $\text{sumset } \{a\} (\text{stabilizer } A) = \text{sumset } \{b\} (\text{stabilizer } A) \iff$
 $(\text{sumset } \{a\} (\text{stabilizer } A)) \cap (\text{sumset } \{b\} (\text{stabilizer } A)) \neq \{\}$

<proof>

lemma *sumset-stabilizer-eq-Class-Union:*

assumes $A \subseteq G$

shows $\text{sumset } A (\text{stabilizer } B) = (\bigcup (\text{Class } B \text{ ' } A))$

<proof>

lemma *card-stabilizer-divide-sumset:*

assumes $A \subseteq G$

shows $\text{card} (\text{stabilizer } B) \text{ dvd } \text{card} (\text{sumset } A (\text{stabilizer } B))$

<proof>

lemma *Class-image-sumset-stabilizer-eq:*

assumes $A \subseteq G$

shows $\text{Class } B \text{ ' } (\text{sumset } A (\text{stabilizer } B)) = \text{Class } B \text{ ' } A$

<proof>

lemma *Class-cover-imp-subset-or-disj:*

assumes $A = (\bigcup (\text{Class } B \text{ ' } C))$ **and** $x \in G$ **and** $C \subseteq G$

shows $\text{Class } B \ x \subseteq A \vee \text{Class } B \ x \cap A = \{\}$

<proof>

lemma *stabilizer-sumset-disjoint:*

fixes $S1 \ S2 :: 'a \text{ set}$

assumes $\text{stabilizer } S1 \cap \text{stabilizer } S2 = \{0\}$ **and** $S1 \subseteq G$ **and** $S2 \subseteq G$

and *finite S1* **and** *finite S2* **and** $S1 \neq \{\}$ **and** $S2 \neq \{\}$

shows $\text{card} (\text{sumset} (\text{stabilizer } S1) (\text{stabilizer } S2)) =$

$\text{card} (\text{stabilizer } S1) * \text{card} (\text{stabilizer } S2)$

<proof>

lemma *stabilizer-sub-sumset-left:*

$\text{stabilizer } A \subseteq \text{stabilizer} (\text{sumset } A \ B)$

<proof>

lemma *stabilizer-sub-sumset-right:*

stabilizer $B \subseteq \text{stabilizer} (\text{sumset } A B)$
<proof>

lemma *not-mem-stabilizer-obtain:*

assumes $A \neq \{\}$ **and** $x \notin \text{stabilizer } A$ **and** $x \in G$ **and** $A \subseteq G$ **and** *finite* A
obtains a **where** $a \in A$ **and** $x \oplus a \notin A$
<proof>

lemma *sumset-eq-sub-stabilizer:*

assumes $A \subseteq G$ **and** $B \subseteq G$ **and** *finite* B
shows $\text{sumset } A B = B \implies A \subseteq \text{stabilizer } B$
<proof>

lemma *sumset-stabilizer-eq:*

shows $\text{sumset} (\text{stabilizer } A) (\text{stabilizer } A) = \text{stabilizer } A$
<proof>

lemma *differenceset-stabilizer-eq:*

shows $\text{differenceset} (\text{stabilizer } A) (\text{stabilizer } A) = \text{stabilizer } A$
<proof>

lemma *stabilizer2-sub-stabilizer:*

shows $\text{stabilizer}(\text{stabilizer } A) \subseteq \text{stabilizer } A$
<proof>

lemma *stabilizer-left-sumset-invariant:*

assumes $a \in G$ **and** $A \subseteq G$
shows $\text{stabilizer} (\text{sumset } \{a\} A) = \text{stabilizer } A$

<proof>

lemma *stabilizer-right-sumset-invariant:*

assumes $a \in G$ **and** $A \subseteq G$
shows $\text{stabilizer} (\text{sumset } A \{a\}) = \text{stabilizer } A$
<proof>

lemma *stabilizer-right-differenceset-invariant:*

assumes $b \in G$ **and** $A \subseteq G$
shows $\text{stabilizer} (\text{differenceset } A \{b\}) = \text{stabilizer } A$
<proof>

lemma *stabilizer-unchanged:*

assumes $a \in G$ **and** $b \in G$
shows $\text{stabilizer} (\text{sumset } A B) = \text{stabilizer} (\text{sumset } A (\text{sumset} (\text{differenceset } B \{b\}) \{a\}))$

<proof>

lemma *subset-stabilizer-of-subset-sumset:*

assumes $A \subseteq \text{sumset } \{x\}$ (*stabilizer B*) **and** $x \in G$ **and** $A \neq \{\}$ **and** $A \subseteq G$
shows $\text{stabilizer } A \subseteq \text{stabilizer } B$

<proof>

lemma *sumset-stabilizer-eq-self:*

assumes $A \subseteq G$

shows $\text{sumset } (\text{stabilizer } A) = A$

<proof>

lemma *stabilizer-neq-subset-sumset:*

assumes $A \subseteq \text{sumset } \{x\}$ (*stabilizer B*) **and** $x \in A$ **and** $\neg \text{sumset } \{x\}$ (*stabilizer B*) $\subseteq C$ **and**

$A \subseteq C$ **and** $C \subseteq G$

shows $\text{stabilizer } A \neq \text{stabilizer } B$

<proof>

lemma *subset-stabilizer-Un:*

shows $\text{stabilizer } A \cap \text{stabilizer } B \subseteq \text{stabilizer } (A \cup B)$

<proof>

lemma *mem-stabilizer-Un-and-left-imp-right:*

assumes *finite B* **and** $x \in \text{stabilizer } (A \cup B)$ **and** $x \in \text{stabilizer } A$ **and** *disjnt A B*

shows $x \in \text{stabilizer } B$

<proof>

lemma *mem-stabilizer-Un-and-right-imp-left:*

assumes *finite A* **and** $x \in \text{stabilizer } (A \cup B)$ **and** $x \in \text{stabilizer } B$ **and** *disjnt A B*

shows $x \in \text{stabilizer } A$

<proof>

lemma *Union-stabilizer-Class-eq:*

assumes $A \subseteq G$

shows $A = \bigcup (\text{Class } A \text{ ' } A)$ *<proof>*

lemma *card-stabilizer-sumset-divide-sumset:*

$\text{card } (\text{stabilizer } (\text{sumset } A \ B)) \ \text{dvd} \ \text{card } (\text{sumset } A \ B)$ *<proof>*

lemma *card-stabilizer-le:*

assumes $A \subseteq G$ **and** *finite A* **and** $A \neq \{\}$

shows $\text{card } (\text{stabilizer } A) \leq \text{card } A$ *<proof>*

lemma *sumset-Inter-subset-sumset:*

assumes $a \in G$ **and** $b \in G$

shows $\text{sumset } (A \cap \text{sumset } \{a\})$ (*stabilizer C*) $(B \cap \text{sumset } \{b\})$ (*stabilizer C*)

\subseteq
sumset $\{a \oplus b\}$ (*stabilizer* C) (**is** *sumset* $?A ?B \subseteq -$)
 ⟨*proof*⟩

1.3 Convergent

definition *convergent* :: 'a set \Rightarrow 'a set \Rightarrow 'a set \Rightarrow bool **where**
convergent $C A B \equiv C \subseteq \text{sumset } A B \wedge C \neq \{\}$ \wedge
 $\text{card } C + \text{card } (\text{stabilizer } C) \geq \text{card } (A \cap B) + \text{card } (\text{sumset } (A \cup B) (\text{stabilizer } C))$

definition *convergent-set* :: 'a set \Rightarrow 'a set \Rightarrow 'a set set **where**
convergent-set $A B = \text{Collect } (\lambda C. \text{convergent } C A B)$

lemma *convergent-set-sub-powerset*:
convergent-set $A B \subseteq \text{Pow } (\text{sumset } A B)$ ⟨*proof*⟩

lemma *finite-convergent-set*:
assumes *finite* A **and** *finite* B
shows *finite* (*convergent-set* $A B$)
 ⟨*proof*⟩

1.4 Technical lemmas from DeVos's proof of Kneser's Theorem

The following lemmas correspond to intermediate arguments in the proof of Kneser's Theorem by DeVos that we will be following [1].

lemma *stabilizer-sumset-psubset-stabilizer*:
assumes $a \in G$ **and** $b \in G$ **and** $A \cap \text{sumset } \{a\} (\text{stabilizer } C) \neq \{\}$ **and**
 $B \cap \text{sumset } \{b\} (\text{stabilizer } C) \neq \{\}$ **and** *hnotsub*: $\neg \text{sumset } \{a \oplus b\} (\text{stabilizer } C) \subseteq \text{sumset } A B$
shows *stabilizer* (*sumset* ($A \cap \text{sumset } \{a\} (\text{stabilizer } C)$) ($B \cap \text{sumset } \{b\} (\text{stabilizer } C)$)) \subset
stabilizer C (**is** $?H \subset -$)
 ⟨*proof*⟩

lemma *stabilizer-eq-stabilizer-union*:
assumes $a \in G$ **and** $b \in G$ **and** $A \cap \text{sumset } \{a\} (\text{stabilizer } C) \neq \{\}$ **and**
 $B \cap \text{sumset } \{b\} (\text{stabilizer } C) \neq \{\}$ **and** *hnotsub*: $\neg \text{sumset } \{a \oplus b\} (\text{stabilizer } C) \subseteq \text{sumset } A B$ **and**
 $C \subseteq \text{sumset } A B$ **and** *finite* C **and**
 $C \cap \text{sumset } (A \cap \text{sumset } \{a\} (\text{stabilizer } C)) (B \cap \text{sumset } \{b\} (\text{stabilizer } C)) = \{\}$ **and** $C \neq \{\}$ **and**
finite A **and** *finite* B
shows *stabilizer* (*sumset* ($A \cap \text{sumset } \{a\} (\text{stabilizer } C)$) ($B \cap \text{sumset } \{b\} (\text{stabilizer } C)$)) =
stabilizer ($C \cup \text{sumset } (A \cap \text{sumset } \{a\} (\text{stabilizer } C)) (B \cap \text{sumset } \{b\} (\text{stabilizer } C))$) (**is** *stabilizer* $?H = \text{stabilizer } ?K$)
 ⟨*proof*⟩

lemma *sumset-inter-ineq*:

assumes $B \cap \text{sumset } \{a\} (\text{stabilizer } C) = \{\}$ **and** $\text{stabilizer } (\text{sumset } (A \cap \text{sumset } \{a\} (\text{stabilizer } C)) (B \cap \text{sumset } \{b\} (\text{stabilizer } C))) \subset \text{stabilizer } C$ **and**
 $a \in A$ **and** $a \in G$ **and** *finite* A **and** *finite* B **and** $A \neq \{\}$ **and** $B \neq \{\}$ **and** *finite* $(\text{stabilizer } C)$

shows $\text{int } (\text{card } (\text{sumset } (A \cup B) (\text{stabilizer } C))) - \text{card } (\text{sumset } (A \cup B) (\text{stabilizer } (\text{sumset } (A \cap \text{sumset } \{a\} (\text{stabilizer } C)) (B \cap \text{sumset } \{b\} (\text{stabilizer } C)))))) \geq$

$\text{int } (\text{card } (\text{stabilizer } C)) - \text{card } (\text{sumset } (A \cap \text{sumset } \{a\} (\text{stabilizer } C)) (\text{stabilizer } (\text{sumset } (A \cap \text{sumset } \{a\} (\text{stabilizer } C)) (B \cap \text{sumset } \{b\} (\text{stabilizer } C))))))$

(**is** $\text{int } (\text{card } (\text{sumset } (A \cup B) (\text{stabilizer } C))) - \text{card } (\text{sumset } (A \cup B) ?H1) \geq$
 $\text{int } (\text{card } (\text{stabilizer } C)) - \text{card } (\text{sumset } ?A1 ?H1)$)

<proof>

lemma *exists-convergent-min-stabilizer*:

assumes *hind*: $\forall m < n. \forall C D. C \subseteq G \longrightarrow D \subseteq G \longrightarrow \text{finite } C \longrightarrow \text{finite } D \longrightarrow C \neq \{\} \longrightarrow$

$D \neq \{\} \longrightarrow \text{card } (\text{sumset } C D) + \text{card } C = m \longrightarrow$

$\text{card } (\text{sumset } C (\text{stabilizer } (\text{sumset } C D))) + \text{card } (\text{sumset } D (\text{stabilizer } (\text{sumset } C D))) -$

$\text{card } ((\text{stabilizer } (\text{sumset } C D)))$

$\leq \text{card } (\text{sumset } C D)$ **and** *hAG*: $A \subseteq G$ **and** *hBG*: $B \subseteq G$ **and** *hA*: *finite* A

and

hB: *finite* B **and** *hAne*: $A \neq \{\}$ **and** $A \cap B \neq \{\}$ **and**

hcardsum: $\text{card } (\text{sumset } A B) + \text{card } A = n$ **and** *hintercardA*: $\text{card } (A \cap B) < \text{card } A$

obtains X **where** *convergent* $X A B$ **and** $\bigwedge Y. Y \in \text{convergent-set } A B \implies$

$\text{card } (\text{stabilizer } Y) \geq \text{card } (\text{stabilizer } X)$

<proof>

end

context *normal-subgroup*

begin

1.5 A function that picks coset representatives randomly

definition $\varphi :: 'a \text{ set} \Rightarrow 'a$ **where**

$\varphi = (\lambda x. \text{if } x \in G // K \text{ then } (\text{SOME } a. a \in G \wedge x = a \cdot | K) \text{ else undefined})$

definition *quot-comp-alt* :: $'a \Rightarrow 'a \Rightarrow 'a$ **where** *quot-comp-alt* $a b = \varphi ((a \cdot b) \cdot | K)$

lemma *phi-eq-coset*:

assumes $\varphi x = a$ **and** $a \in G$ **and** $x \in G // K$

shows $x = a \cdot | K$

<proof>

lemma *phi-coset-mem*:

assumes $a \in G$

shows $\varphi (a \cdot | K) \in a \cdot | K$

\langle *proof* \rangle

lemma *phi-coset-eq*:

assumes $a \in G$ **and** $\varphi x = a$ **and** $x \in G // K$

shows $\varphi (a \cdot | K) = a$ \langle *proof* \rangle

lemma *phi-inverse-right*:

assumes $g \in G$

shows $\text{quot-comp-alt } g (\varphi (\text{inverse } g \cdot | K)) = \varphi K$

\langle *proof* \rangle

lemma *phi-inverse-left*:

assumes $g \in G$

shows $\text{quot-comp-alt } (\varphi (\text{inverse } g \cdot | K)) g = \varphi K$

\langle *proof* \rangle

lemma *phi-mem-coset-eq*:

assumes $a \in G // K$ **and** $b \in G$

shows $\varphi a \in b \cdot | K \implies a = (b \cdot | K)$

\langle *proof* \rangle

lemma *forall-unique-repr*:

$\forall x \in G // K. \exists! k \in \varphi '(G // K). x = k \cdot | K$

\langle *proof* \rangle

lemma *phi-inj-on*:

shows *inj-on* $\varphi (G // K)$

\langle *proof* \rangle

lemma *phi-coset-eq-self*:

assumes $a \in G // K$

shows $\varphi a \cdot | K = a$

\langle *proof* \rangle

lemma *phi-coset-comp-eq*:

assumes $a \in G // K$ **and** $b \in G // K$

shows $\varphi a \cdot \varphi b \cdot | K = a [.] b$ \langle *proof* \rangle

lemma *phi-comp-eq*:

assumes $a \in G // K$ **and** $b \in G // K$

shows $\varphi (a [.] b) = \text{quot-comp-alt } (\varphi a) (\varphi b)$

\langle *proof* \rangle

lemma *phi-image-subset*:

$\varphi '(G // K) \subseteq G$
<proof>

lemma *phi-image-group:*

Group-Theory.group ($\varphi '(G // K)$) *quot-comp-alt* (φK)
<proof>

lemma *phi-map: Set-Theory.map* φ *Partition* ($\varphi ' Partition$)
<proof>

lemma *phi-image-isomorphic:*

group-isomorphism $\varphi (G // K)$ ($[.]$) (*Class 1*) ($\varphi '(G // K)$) *quot-comp-alt* (φK)
<proof>

end

context *subgroup-of-additive-abelian-group*

begin

lemma *Union-Coset-card-eq:*

assumes *hSG*: $S \subseteq G$ **and** *hSU*: $(\bigcup (Class ' S)) = S$
shows $card S = card H * card (Class ' S)$
<proof>

end

context *subgroup-of-abelian-group*

begin

interpretation *GH: additive-abelian-group* $G // H$ ($[.]$) *Class 1*
<proof>

interpretation *GH-repr: additive-abelian-group* $\varphi '(G // H)$ *quot-comp-alt* φH
<proof>

lemma *phi-image-sumset-eq:*

assumes $A \subseteq G // H$ **and** $B \subseteq G // H$
shows $\varphi '(GH.sumset A B) = GH-repr.sumset (\varphi ' A) (\varphi ' B)$
<proof>

lemma *phi-image-stabilizer-eq:*

assumes $A \subseteq G // H$
shows $\varphi '(GH.stabilizer A) = GH-repr.stabilizer (\varphi ' A)$
<proof>

end

1.6 Useful group-theoretic results

lemma *residue-group: abelian-group* $\{0..(m :: nat)-1\}$ $(\lambda x y. ((x + y) \text{ mod } m))$
 $(0 :: int)$
 $\langle \text{proof} \rangle$

lemma *(in subgroup-of-group) prime-order-simple:*
assumes *prime* $(\text{card } G)$
shows $H = \{1\} \vee H = G$
 $\langle \text{proof} \rangle$

lemma *residue-group-simple:*
assumes *prime* p **and** *subgroup* H $\{0..(p :: nat)-1\}$ $(\lambda x y. ((x + y) \text{ mod } p))$
 $(0 :: int)$
shows $H = \{0\} \vee H = \{0..int(p-1)\}$
 $\langle \text{proof} \rangle$

end

2 Kneser's Theorem and the Cauchy–Davenport Theorem: main proofs

theory *Kneser-Cauchy-Davenport-main-proofs*
imports
Kneser-Cauchy-Davenport-preliminaries

begin

context *additive-abelian-group*

begin

2.1 Proof of Kneser's Theorem

The proof we formalise follows the paper [1]. This version of Kneser's Theorem corresponds to Theorem 3.2 in [3], or to Theorem 4.3 in [2].

theorem *Kneser:*
assumes $A \subseteq G$ **and** $B \subseteq G$ **and** *finite* A **and** *finite* B **and** $hAne: A \neq \{\}$ **and**
 $hBne: B \neq \{\}$
shows $\text{card } (\text{sumset } A B) \geq \text{card } (\text{sumset } A (\text{stabilizer } (\text{sumset } A B))) +$
 $\text{card } (\text{sumset } B (\text{stabilizer } (\text{sumset } A B))) - \text{card } (\text{stabilizer } (\text{sumset } A B))$
 $\langle \text{proof} \rangle$

2.2 Strict version of Kneser's Theorem

We show a strict version of Kneser's Theorem as presented in Theorem 3.2 of [3].

theorem *Kneser-strict-aux*: **fixes** A **and** B **assumes** $hAG: A \subseteq G$ **and** $hBG: B \subseteq G$ **and** hA : *finite* A
and hB : *finite* B **and** $hAne: A \neq \{\}$ **and** $hBne: B \neq \{\}$ **and**
 $hineq: \text{card} (\text{sumset } A \ B) > \text{card} (\text{sumset } A \ (\text{stabilizer} (\text{sumset } A \ B))) +$
 $\text{card} (\text{sumset } B \ (\text{stabilizer} (\text{sumset } A \ B))) - \text{card} (\text{stabilizer} (\text{sumset } A \ B))$
shows $\text{card} (\text{sumset } A \ B) \geq \text{card } A + \text{card } B$

<proof>

theorem *Kneser-strict*: **fixes** A **and** B **assumes** $A \subseteq G$ **and** $B \subseteq G$ **and** *finite* A **and** *finite* B
and $\text{stabilizer} (\text{sumset } A \ B) = H$ **and** $A \neq \{\}$ **and** $B \neq \{\}$ **and** $\text{card} (\text{sumset } A \ B) < \text{card } A + \text{card } B$
shows $\text{card} (\text{sumset } A \ B) = \text{card} (\text{sumset } A \ (\text{stabilizer} (\text{sumset } A \ B))) +$
 $\text{card} (\text{sumset } B \ (\text{stabilizer} (\text{sumset } A \ B))) - \text{card} (\text{stabilizer} (\text{sumset } A \ B))$
<proof>

2.3 The Cauchy–Davenport Theorem

We show the Cauchy–Davenport Theorem as a corollary of Kneser’s Theorem, following a comment on Theorem 3.2 in [3].

interpretation *Z-p: additive-abelian-group* $\{0..int ((p :: nat)-1)\}$ $(\lambda x y. ((x + y) \text{ mod } int p))$ $0::int$
<proof>

theorem *Cauchy-Davenport*:
fixes $p :: nat$
assumes *prime* p **and** $A \neq \{\}$ **and** $B \neq \{\}$ **and** *finite* A **and** *finite* B **and**
 $A \subseteq \{0..p-1\}$ **and** $B \subseteq \{0..p-1\}$
shows $\text{card} (Z\text{-}p.\text{sumset } p \ A \ B) \geq \text{Min} \{p, \text{card } A + \text{card } B - 1\}$

<proof>

end
end

References

- [1] M. DeVos. A short proof of kneser’s addition theorem for abelian groups. In M. B. Nathanson, editor, *Combinatorial and Additive Number Theory*, pages 39–41, New York, NY, 2014. Springer New York.
- [2] M. B. Nathanson. *Additive Number Theory: Inverse Problems and the Geometry of Sumsets*, volume 165 of *Graduate Texts in Mathematics*. Springer-Verlag, 1996.

- [3] I. Z. Ruzsa. Sunsets and structure, 2008. Course notes, available on <https://www.math.cmu.edu/users/af1p/Teaching/AdditiveCombinatorics/Additive-Combinatorics.pdf>.